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# Different perspectives on Multiple $q$-Zeta Values

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"*An innocent looking problem often gives no hint as to its true nature.*"

Paul Erdős

# Contents

*Für meine Eltern und meine Schwester*

# Chapter 1

# Different perspectives on Multiple $q$-Zeta Values

In this thesis, we investigate $q$-analogues of Multiple Zeta Values algebraically, combinatorially, and analytically. It is a cumulative thesis, consisting of three works [16], a revised version of [15], and [12] that can be found in Chapters 2, 3, and 4, respectively. They consider $q$-analogues of Multiple Zeta Values from an algebraic, combinatorial, and analytic perspective, respectively. Although we assume the reader is familiar with Multiple Zeta Values, we provide in Section 1.1 all the basic knowledge about Multiple Zeta Values and their $q$-analogues one might need to understand this thesis. Sections 1.2, 1.3, and 1.4 then introduce the three works of this thesis and contain their main results, while Section 1.5 gives a conclusion of the three works and how they are connected. This chapter is an overview and summary of the results obtained in the mentioned works. As this chapter is intended as an introduction, the proofs of the main results are mainly omitted.

Multiple Zeta Values (MZVs for short) are real numbers. They are defined for integers $\ell_1 \geq 2,\ \ell_2, \dots, \ell_r \geq 1$ as follows:

$$\zeta(\ell_1, \dots, \ell_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{\ell_1} \cdots m_r^{\ell_r}},$$

see also Definition 1.10. They have a long history (Euler [27] already studied them in the 18th century!) but were temporarily pushed into the background. In the last few decades, however, Multiple Zeta Values have emerged in various areas of mathematics and theoretical physics, so their study has regained importance. Of particular interest in current research is their algebraic structure. Compared to the importance of Multiple Zeta Values in research, only little is known about their algebraic structure, e.g., already statements about the irrationality of single Zeta Values (which are $\zeta(\ell)$ for $\ell \in \mathbb{Z}_{\geq 2}$) is possible only for very few of them. Nevertheless, a lot of linear relations among Multiple Zeta Values are known. For example, we have

$$\zeta(4) = 4\zeta(3,1) \quad \text{and} \quad \zeta(3) = \zeta(2,1).$$

In general, studying the algebraic behaviour of real numbers is a hard task. A common approach to expose the algebraic structure of real numbers is to study so-called $q$-analogues of them. These are objects depending on an extra parameter $q$ that give back the original object in the limit $q \to 1$ (after possible minor modification such as multiplying with an appropriate power of $(1-q)$) and inheriting parts of the algebraic structure.

For example,

$$\zeta_q^{\mathrm{SZ}}(\ell_1,\dots,\ell_r) := \sum_{m_1>\dots>m_r>0} \frac{q^{m_1\ell_1}}{(1-q^{m_1})^{\ell_1}} \cdots \frac{q^{m_r\ell_r}}{(1-q^{m_r})^{\ell_r}}$$

is a $q$-analogue of $\zeta(\ell_1,\dots,\ell_r)$ since it gives back the Multiple Zeta Value after multiplication with $(1-q)^{\ell_1+\dots+\ell_r}$ and taking the limit $q \to 1$ then (see Definitions 1.23 and 1.26 for a general definition). We will call such objects *Multiple $q$-Zeta Values* ($q$MZVs for short); the particular ones from above are referred to as *Schlesinger–Zudilin Multiple $q$-Zeta Values.* Note that Schlesinger–Zudilin $q$MZVs are defined for all $\ell_1 \in \mathbb{Z}_{>0}$, $\ell_2,\dots,\ell_r \in \mathbb{Z}_{\geq 0}$, where $r \in \mathbb{Z}_{\geq 0}$, in contrast to MZVs.

Both MZVs and Schlesinger–Zudilin $q$MZVs satisfy the *stuffle product* which describes the product of MZVs, respectively Schlesinger–Zudilin $q$MZVs, as linear combination of MZVs, respectively Schlesinger–Zudilin $q$MZVs, again and arises from the multiplication of iterated sums. We refer to Definition 1.3 for the precise definition of the stuffle product and to Proposition 1.12, respectively Proposition 1.28, for the statement that the product of MZVs, respectively Schlesinger–Zudilin $q$MZVs, indeed can be described by the stuffle product.

Besides the stuffle product, *duality* (of Schlesinger–Zudilin $q$MZVs) is of importance for this thesis. By duality, we mean, in this thesis, the relations

$$\zeta_q^{\mathrm{SZ}}(k_1,\underbrace{0,\dots,0}_{z_1},\dots,k_d,\underbrace{0,\dots,0}_{z_d}) = \zeta_q^{\mathrm{SZ}}(z_d+1,\underbrace{0,\dots,0}_{k_d-1},\dots,z_1+1,\underbrace{0,\dots,0}_{k_1-1}),$$

where $k_1,\dots,k_d \in \mathbb{Z}_{>0}$ and $z_1,\dots,z_d \in \mathbb{Z}_{\geq 0}$. For the general statement, we refer to Definition 1.6/Theorem 1.29. Together with the stuffle product, one obtains many more linear relations among Schlesinger–Zudilin $q$MZVs. Moreover, conjecturally those are all:

**Conjecture** (Bachmann, [1])**.** All $\mathbb{Q}$-linear relations among Schlesinger–Zudilin $q$MZVs are obtained by the stuffle product and duality.

Large parts of this thesis were motivated by this conjecture, and the results are consistent with this conjecture.

**The algebraic perspective on $q$MZVs.** As mentioned, Section 1.2 introduces the work that builds Chapter 2, which contains an algebraic view on Multiple $q$-Zeta Values. We focus on the $\mathbb{Q}$-algebra $\mathcal{Z}_q$ of all Schlesinger–Zudilin $q$MZVs, the subalgebra $\mathcal{Z}_q^{\circ}$, generated by those Schlesinger–Zudilin $q$MZVs with $\ell_1,\dots,\ell_r \geq 1$, and a conjecture by Bachmann [2] stating that they coincide.

**Conjecture** (Bachmann, [2])**.** We have $\mathcal{Z}_q = \mathcal{Z}_q^{\circ}$.

Partial results already exist by works of Bachmann [3], Burmester [21], and Vleeshouwers [44]. Theorem 1.52 will extend those partial results by Bachmann and Burmester again by using the stuffle product and duality only. The motivation for the latter conjecture due to Bachmann comes from viewing (Schlesinger–Zudilin) $q$MZVs as $q$-series: By geometric series expansion, one obtains that every (Schlesinger–Zudilin) $q$MZV is of shape

$$\sum_{d=0}^{H} \sum_{\substack{m_1>\dots>m_d>0\\ n_1,\dots,n_d>0}} Q_d(m_1,\dots,m_d,n_1,\dots,n_d) q^{m_1n_1+\dots+m_dn_d},$$

where the $Q_d \in \mathbb{Q}[X_1, \ldots, X_d, Y_1, \ldots, Y_d]$ are polynomials and $H \in \mathbb{Z}_{\geq 0}$ depends on the $q$MZV. Bachmann's $\mathcal{Z}_q = \mathcal{Z}_q^\circ$-Conjecture now states that it is possible to find polynomials $\widetilde{Q}_d \in \mathbb{Q}[X_1, \ldots, X_d]$ such that every $q$MZV is of shape

$$\sum_{d=0}^{\widetilde{H}} \sum_{\substack{m_1 > \cdots > m_d > 0 \\ n_1, \ldots, n_d > 0}} \widetilde{Q}_d(m_1, \ldots, m_d) q^{m_1 n_1 + \cdots + m_d n_d}$$

with some $\widetilde{H} \in \mathbb{Z}_{\geq 0}$ depending on the $q$MZV. Note that one could let the polynomials $\widetilde{Q}_d$ depend on $n_1, \ldots, n_d$ only as well instead of $m_1, \ldots, m_d$ by using duality.

A main result of this thesis is that Bachmann's $\mathcal{Z}_q = \mathcal{Z}_q^\circ$-Conjecture is true in some small cases. Expressed in terms of Schlesinger–Zudilin $q$MZVs, it is the following.

**Theorem** (Theorem 2.75)**.** For all $\ell_1, \ldots, \ell_r \in \mathbb{Z}_{\geq 0}$ with $\ell_1 \geq 1$ and $r \leq 6$, we have

$$\zeta_q^{\mathrm{SZ}}(\ell_1, \ldots, \ell_r) \in \mathcal{Z}_q^\circ.$$

The result builds Theorem 2.75 and is obtained using the stuffle product and duality only. Hence, consistent with the first conjecture above, we will work mainly with *formal qMZVs*, which are formal objects satisfying the stuffle product and duality only. Furthermore, we will refine Bachmann's conjecture $\mathcal{Z}_q = \mathcal{Z}_q^\circ$, see Conjecture 2.10, and we will give evidence for small cases, see Theorem 2.76.

While playing around with relations induced by the stuffle product and duality, it seemed useful to develop the *box product*, which gives a specific part of the stuffle product that is of main interest when studying such relations. In this way, we will refine the already refined $\mathcal{Z}_q = \mathcal{Z}_q^\circ$-conjecture again for "half" of the cases occurring, see Conjecture 2.39. More precisely, if Conjecture 2.39 is true, one can write every $\zeta_q^{\mathrm{SZ}}(\ell_1, \ldots, \ell_r)$, satisfying $z := \#\{\ell_j = 0\} \geq \#\{\ell_j \neq 0\}$, as a $\mathbb{Q}$-linear combination of Schlesinger–Zudilin $q$MZVs with less than $z$ zero-entries. For the other "half" of the cases, we present a promising approach that also works in small cases, as the proof of our main results will show.

**The combinatorial perspective on $q$MZVs.** Section 1.3 introduces the work that builds Chapter 3, which contains a combinatorial perspective on Multiple Zeta Values. It arises from writing Multiple $q$-Zeta Values as (formal) power series in the parameter $q$ via geometric series expansion

$$\frac{q^{m\ell}}{(1-q^m)^\ell} = \sum_{n>0} \binom{n-1}{\ell-1} q^{mn},$$

yielding, for $k_1, \ldots, k_d \in \mathbb{Z}_{>0}$ and $z_1, \ldots, z_d \in \mathbb{Z}_{\geq 0}$, that

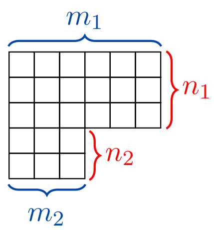
$$\zeta_q^{\mathrm{SZ}}(k_1, \underbrace{0, \ldots, 0}_{z_1}, \ldots, k_d, \underbrace{0, \ldots, 0}_{z_d}) \\ = \sum_{\substack{m_1 > \cdots > m_d > 0 \\ n_1, \ldots, n_d > 0}} \left[ \prod_{j=1}^{d} \binom{m_j - m_{j+1} - 1}{z_j} \binom{n_j - 1}{k_j - 1} \right] q^{m_1 n_1 + \cdots + m_d n_d}$$

(with $m_{d+1} := 0$). In this way, one can consider Schlesinger–Zudilin $q$MZVs as the generating series of a distinguished set of partitions of non-negative integers counted with some multiplicity. For this, one visualizes the occurring exponents $N := m_1 n_1 + \cdots + m_d n_d$

via the *Young Tableau* of the partition of $N$ that is given in *Stanley Coordinates* by

$$(m_1, \dots, m_d; n_1, \dots, n_d).$$

As usual, one calls the $m_j$'s the *parts* and the $n_j$'s the *multiplicities* (of the $m_j$-part) of the partition. We refer to [42] for further details about Stanley Coordinates. For example,



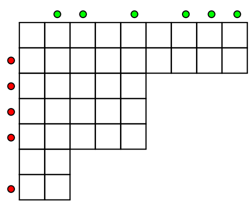

is the Young Tableau of the partition of $N = 24$, given in Stanley Coordinates by

$$(m_1, m_2; n_1, n_2) = (6, 3; 3, 2).$$

The multiplicity

$$\prod_{j=1}^{d} \binom{m_j - m_{j+1} - 1}{z_j} \binom{n_j - 1}{k_j - 1}$$

in front of $q^N$ can be visualized by marking rows and columns of the Young Tableau in a specific way, leading to the notion of *marked partitions* as introduced in [14]. I.e., Multiple $q$-Zeta Values will be considered as generating series of distinguished marked partitions. More precisely, we interpret the binomial coefficient $\binom{n_j-1}{k_j-1}$ as the number of marking exactly $k_j - 1$ rows in the Young Tableau between the rows containing the $(j-1)$-th and $j$-th corner, counted top to bottom; similarly, we interpret $\binom{m_j - m_{j+1} - 1}{z_j}$ as the number of marking exactly $z_j$ columns between columns containing the $j$-th and $(j+1)$-th rightmost corner. For convenience, we mark rows and columns containing corners by default. For example,



is a marked partition that we will associate with $\zeta_q^{\mathrm{SZ}}(1, 0, 0, 3, 0, 1)$; see also Example 1.75.

In [14], the following was already proven regarding marked partitions.

**Theorem.** The duality of $q$MZVs can be described via an explicit bijection among corresponding sets of marked partitions.

The main theorem of Chapter 3 is the following.

**Theorem** (Theorem 3.17, weak version)**.** The stuffle product of $q$MZVs can be described explicitly as pairing on marked partitions.

The refined version of this statement is Theorem 3.17. The proof consists of showing that particular numbers of marked partitions satisfy the same recursion as multiplicities occurring in the stuffle product. Recalling the above conjecture by Bachmann that the stuffle product and duality of Multiple $q$-Zeta Values already give all linear relations among Multiple $q$-Zeta Values, the main result of Chapter 3 is that (under the assumption of this conjecture) now all linear relations among $q$MZVs can be described on the level of marked partitions. In this way, we have created a combinatorial approach to the algebraic structure of $q$MZVs, giving rise to plenty of future projects. For example, using marked partitions, one can try to make progress in proving Bachmann's $\mathcal{Z}_q = \mathcal{Z}_q^\circ$-conjecture.

**The analytic perspective on $q$MZVs.** Section 1.4 then introduces the joint paper that builds Chapter 4, which mainly contains an analytic view on Multiple $q$-Zeta Values. We use Wright's Circle Method to focus on asymptotic formulas for $q$-series which is a tool from complex analysis to study the asymptotic behaviour of a given sequence $(c(n))_{n\in\mathbb{N}_0}$ of (complex) numbers having moderate growth, see Section 1.4.2 for a brief introduction to the Circle Method.

For roughly stating our main theorem of Chapter 4, let $f : \mathbb{N} \to \mathbb{N}_0$ be a function and set $\Lambda := \mathbb{N} \setminus f^{-1}(\{0\})$. For $q = e^{-z}$ ($z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$), define

$$G_f(z) := \sum_{n\geq 0} p_f(n)q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)^{f(n)}}, \qquad L_f(s) := \sum_{n\geq 1} \frac{f(n)}{n^s}.$$

**Theorem** (Theorem 4.5)**.** Under assumption of certain conditions the on density of $\Lambda$ in $\mathbb{N}$, the meromorphic continuation of $L_f$, and a growth condition on $L_f$ (see (P1), (P2), and (P3) for the precise assumption), for some integers $M, N \in \mathbb{N}$, we have

$$p_f(n) = \frac{C}{n^b} \exp\left(A_1 n^{\frac{\alpha}{\alpha+1}} + \sum_{j=2}^{M} A_j n^{\alpha_j}\right)\left(1 + \sum_{j=2}^{N} \frac{B_j}{n^{\beta_j}} + O_{L,R}\left(n^{-\min\left\{\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\right\}}\right)\right),$$

where $L \in \mathbb{N}$, $R > 0$ come from the assumptions (P1) and (P2), $\alpha$ the largest pole of $L_f$, $0 \leq \alpha_M < \alpha_{M-1} < \cdots \alpha_2 < \alpha_1 = \frac{\alpha}{\alpha+1}$ are given by $\mathcal{L}$, and $0 < \beta_2 < \beta_3 < \cdots$ are given by $\mathcal{M} + \mathcal{N}$, where $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ are sets depending on the poles of $L_f$. Moreover, if $\alpha$ is the only positive pole of $L_f$, then we have $M = 1$.

For the concrete definition of $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$, we refer to (4.6.1), (4.6.2), and (4.6.3), respectively. Furthermore, the coefficients $A_j$ and $B_j$ can be calculated explicitly; the constants $A_1$, $C$, and $b$ are provided in (4.6.4) and (4.6.5).

The theorem gives asymptotic formulas for a large class of $q$-series, including several (infinite) sums of $q$-analogues of Multiple Zeta Values, such as

$$\sum_{n\geq 0} \zeta_q^{\mathrm{SZ}}(\underbrace{1,\ldots,1}_{n}).$$

The analytic study of Multiple $q$-Zeta Values is of interest since the coefficients occurring in such asymptotic expansions often (maybe always; this is current research) are $\mathbb{Q}$-linear combinations of Multiple Zeta Values. Hence, by comparing coefficients, a relation among Multiple $q$-Zeta Values gives a set of $\mathbb{Q}$-linear relations among Multiple Zeta Values. In this way, the analytic study of Multiple $q$-Zeta Values has an impact on the algebraic study of Multiple Zeta Values. It is also current research, whether the $\mathbb{Q}$-linear relations among MZVs obtained in this way give all $\mathbb{Q}$-linear relations among MZVs.

## 1.1 Introduction to Multiple ($q$-)Zeta Values

In this section, we introduce the basic knowledge about Multiple Zeta Values and their $q$-analogues one might need to understand this thesis and the related works. For this, we introduce in Section 1.1.1 the algebraic setup. In particular, there, we introduce quasi-shuffle products, of which the stuffle product mentioned above is a distinguished one. Section 1.1.2 introduces Multiple Zeta Values, while Section 1.1.3 introduces their $q$-analogues in the common understanding (as in [2, 11, 37, 38, 40, 43, 46, 48]). Last, Section 1.1.4 contains particular so-called *models* of Multiple $q$-Zeta Values that will be of importance for the thesis.

### 1.1.1 Algebraic setup

For the algebraic setup, we introduce some notation on quasi-shuffle algebras and related algebraic objects. Notably, we will need the stuffle product, a special quasi-shuffle product, and duality mainly in Sections 1.2 and 1.3, and in the corresponding Chapters 2 and 3 to obtain new results on the structure of Multiple $q$-Zeta Values.

Let us fix some general notation first.

**Definition 1.1.** Given a field $F$ and a countable set $\mathcal{A}$. We call $\mathcal{A}$ also an *alphabet*, and elements of $\mathcal{A}$ are referred to as *letters*. Denote by $\operatorname{span}_F \mathcal{A}$ the $F$-vector space spanned by elements of $\mathcal{A}$. Furthermore, monomials of elements in $\mathcal{A}$ (with respect to concatenation) are called *words*. Usually, the neutral element with respect to concatenation is denoted by $\mathbf{1}$ and called the *empty word*. Let $\mathcal{A}^*$ denote the set of words with letters in $\mathcal{A}$, then we write $F\langle\mathcal{A}\rangle$ for the $F$-vector space $\operatorname{span}_F \mathcal{A}^*$, equipped with the non-commutative, but associative multiplication, given by concatenation.

We will also need the notion of *quasi-shuffle* algebras, studied, e.g., by Hoffman [32], to describe the product structure of ($q$)MZVs.

**Definition 1.2** (Quasi-shuffle product)**.** Let $F$ be a field, $\mathcal{A}$ an alphabet, and $\diamond$ a $F$-bilinear, associative and commutative product on $\operatorname{span}_F \mathcal{A}$. Then we define the *quasi-shuffle product* $*_\diamond\colon F\langle\mathcal{A}\rangle \times F\langle\mathcal{A}\rangle \to F\langle\mathcal{A}\rangle$ as the $F$-bilinear product, which is defined via $\mathbf{1} *_\diamond \mathtt{W} := \mathtt{W} *_\diamond \mathbf{1} := \mathtt{W}$ for any $\mathtt{W} \in \mathcal{A}^*$ and recursively through

$$a\mathtt{W}_1 *_\diamond b\mathtt{W}_2 := a(\mathtt{W}_1 *_\diamond b\mathtt{W}_2) + b(a\mathtt{W}_1 *_\diamond \mathtt{W}_2) + (a \diamond b)(\mathtt{W}_1 *_\diamond \mathtt{W}_2)$$

for any $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{A}^*$ and $a, b \in \mathcal{A}$.

In [32] it is shown that $(F\langle\mathcal{A}\rangle, *_\diamond)$ is a commutative algebra. For more details, we refer to [32] and [33]. The following example of a quasi-shuffle product, the *stuffle product*, will be important for almost all parts of this thesis.

**Definition 1.3** (Stuffle product)**.** Choose $F = \mathbb{Q}$, $\mathcal{A} = \mathcal{U} := \{u_j \mid j \in \mathbb{Z}_{\geq 0}\}$, and define $u_{k_1} \diamond u_{k_2} := u_{k_1+k_2}$. We call the induced quasi-shuffle product the *stuffle product* and write $*$ for short instead of $*_\diamond$. I.e., we have $\mathtt{W} * \mathbf{1} := \mathbf{1} * \mathtt{W} := \mathtt{W}$ for all $\mathtt{W} \in \mathcal{U}^*$. Furthermore, for all $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ and $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^*$, by definition, we have

$$u_{k_1}\mathtt{W}_1 * u_{k_2}\mathtt{W}_2 = u_{k_1}\left(\mathtt{W}_1 * u_{k_2}\mathtt{W}_2\right) + u_{k_2}\left(u_{k_1}\mathtt{W}_1 * \mathtt{W}_2\right) + u_{k_1+k_2}\left(\mathtt{W}_1 * \mathtt{W}_2\right).$$

**Remark 1.4.** We write $\mathbb{Q}\langle\mathcal{U}\rangle^\circ$ for the $\mathbb{Q}$-span of words with letters in $\mathcal{U}$, not starting with $u_0$. Furthermore, the set of such words is denoted by $\mathcal{U}^{*,\circ} := \mathcal{U}^* \backslash u_0\mathcal{U}^*$. Note that the stuffle product restricts to a map

$$*\colon \mathbb{Q}\langle\mathcal{U}\rangle^\circ \times \mathbb{Q}\langle\mathcal{U}\rangle^\circ \longrightarrow \mathbb{Q}\langle\mathcal{U}\rangle^\circ.$$

Hence, $(\mathbb{Q}\langle\mathcal{U}\rangle^\circ, *)$ is a commutative $\mathbb{Q}$-algebra.

Let us fix some basic notations we will need throughout the whole thesis.

**Definition 1.5.** Let be $r \in \mathbb{Z}_{\geq 0}$ and $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}_{\geq 0}^r$ an *index*.

(i) We write $u_\emptyset := \mathbf{1}$ for $\mathbf{k} = \emptyset$ and $u_\mathbf{k} := u_{k_1} \cdots u_{k_r}$ if $r > 0$.

(ii) We define

$$\begin{aligned} \operatorname{len}(u_\mathbf{k}) &:= \operatorname{len}(\mathbf{k}) := r, \\ \operatorname{zero}(u_\mathbf{k}) &:= \operatorname{zero}(\mathbf{k}) := \#\{j \mid k_j = 0\}, \\ \operatorname{depth}(u_\mathbf{k}) &:= \operatorname{depth}(\mathbf{k}) := \#\{j \mid k_j \neq 0\}, \\ \operatorname{wt}(u_\mathbf{k}) &:= \operatorname{wt}(\mathbf{k}) := k_1 + \cdots + k_r + \operatorname{zero}(\mathbf{k}) \end{aligned}$$

to be the *length*, the *number of zeros*, the *depth*, the *weight*, respectively, of $u_\mathbf{k}$ and $\mathbf{k}$, respectively.

The following definition of *duality* will be important when considering the algebraic structure of ($q$)MZVs. We will use the (in the following defined) duality $\tau$ for investigating $q$MZVs, while the similar looking map $\tilde{\tau}$ is important for MZVs.

**Definition 1.6.** (i) Define the $\mathbb{Q}$-linear map $\tau\colon \mathbb{Q}\langle\mathcal{U}\rangle^\circ \to \mathbb{Q}\langle\mathcal{U}\rangle^\circ$ by $\tau(\mathbf{1}) := \mathbf{1}$ and

$$\tau\left(u_{k_1} u_0^{z_1} \cdots u_{k_d} u_0^{z_d}\right) := u_{z_d+1} u_0^{k_d-1} \cdots u_{z_1+1} u_0^{k_1-1}$$

for all $k_1, \ldots, k_d \in \mathbb{Z}_{>0}$ and $z_1, \ldots, z_d \in \mathbb{Z}_{\geq 0}$. Furthermore, for $\mathtt{W} \in \mathbb{Q}\langle\mathcal{U}\rangle^\circ$, we call $\tau(\mathtt{W})$ the *dual* of $\mathtt{W}$.

(ii) Write $\mathbb{Q}\langle\mathcal{U}\rangle^1 \subset \mathbb{Q}\langle\mathcal{U}\rangle^\circ$ for the $\mathbb{Q}$-span of words with letters in $\mathcal{U}\backslash\{u_0\}$, not starting with $u_1$. Furthermore, write $\mathcal{U}^{*,1} := (\mathcal{U}\backslash\{u_0\})^* \backslash u_1 (\mathcal{U}\backslash\{u_0\})^*$ for such words. We define the $\mathbb{Q}$-linear map $\tilde{\tau}\colon \mathbb{Q}\langle\mathcal{U}\rangle^1 \to \mathbb{Q}\langle\mathcal{U}\rangle^1$ by $\tilde{\tau}(\mathbf{1}) := \mathbf{1}$ and

$$\tilde{\tau}\left(u_{k_1} u_1^{z_1} \cdots u_{k_d} u_1^{z_d}\right) := u_{z_d+2} u_1^{k_d-2} \cdots u_{z_1+2} u_1^{k_1-2}$$

for all $k_1, \ldots, k_d \in \mathbb{Z}_{\geq 2}$ and $z_1, \ldots, z_d \in \mathbb{Z}_{\geq 0}$.

Since duality will be of significant impact, particularly in Section 1.2 and Chapter 2, we collect some basic properties.

**Remark 1.7.** (i) We remark that $*$ restricts to a map $\mathbb{Q}\langle\mathcal{U}\rangle^1 \times \mathbb{Q}\langle\mathcal{U}\rangle^1 \to \mathbb{Q}\langle\mathcal{U}\rangle^1$, giving rise to a commutative $\mathbb{Q}$-algebra $(\mathbb{Q}\langle\mathcal{U}\rangle^1, *)$.

(ii) Note that $\tau$ is an involution on $\mathbb{Q}\langle\mathcal{U}\rangle^\circ$. Furthermore, $\tilde{\tau}$ is an involution on $\mathbb{Q}\langle\mathcal{U}\rangle^1$.

(iii) Depth and weight are invariant under $\tau$ while the number of zeros and the length generally are not.

(iv) With $\tilde{\tau}$ and duality $\tau$, we can describe particular linear relations that Multiple Zeta Values, respectively their $q$-analogues, satisfy, see Theorems 1.18 and 1.29.

Let us consider a small example.

**Example 1.8.** We have

$$\tau(u_3 u_0 u_1) = \tau\left(u_{2+1} u_0^{1} u_{0+1} u_0^{0}\right) = u_{0+1} u_0^{0} u_{1+1} u_0^{2} = u_1 u_2 u_0 u_0$$

and

$$\tilde{\tau}(u_3) = \tilde{\tau}\left(u_3 u_1^{0}\right) = u_{0+2} u_1^{3-2} = u_2 u_1.$$

For investigating the multiplicative structure of Multiple Zeta Values, besides the stuffle product, it will be necessary to have the notion of the shuffle product ([26]) defined in the following.

**Definition 1.9** (Shuffle product)**.** We define the *shuffle product* to be the $\mathbb{Q}$-bilinear map $⧢\colon \mathbb{Q}\langle\{p,y\}\rangle \times \mathbb{Q}\langle\{p,y\}\rangle \to \mathbb{Q}\langle\{p,y\}\rangle$, given by $\mathtt{W}*\mathbf{1} := \mathbf{1}*\mathtt{W} := \mathtt{W}$ for all $\mathtt{W} \in \mathbb{Q}\langle\{p,y\}\rangle$ and

$$a\mathtt{W}_1 ⧢ b\mathtt{W}_2 := a(\mathtt{W}_1 ⧢ b\mathtt{W}_2) + b(a\mathtt{W}_1 ⧢ \mathtt{W}_2)$$

for all $\mathtt{W}_1, \mathtt{W}_2 \in \mathbb{Q}\langle\{p,y\}\rangle$ and $a, b \in \{p,y\}$.

Denoting by $\mathbb{Q}\langle\{p,y\}\rangle^\circ$ the $\mathbb{Q}$-vector space spanned by words not starting with $y$, the shuffle product restricts to a map

$$⧢\colon \mathbb{Q}\langle\{p,y\}\rangle^\circ \times \mathbb{Q}\langle\{p,y\}\rangle^\circ \longrightarrow \mathbb{Q}\langle\{p,y\}\rangle^\circ.$$

In the following, we will use the translation

$$\begin{aligned} \iota\colon \mathbb{Q}\langle\{p,y\}\rangle^\circ &\longrightarrow \mathbb{Q}\langle\mathcal{U}\rangle^1, \\ p^{k_1-1}y\cdots p^{k_r-1}y &\longmapsto u_{k_1}\cdots u_{k_r} \end{aligned}$$

for all $r$, $k_1,\dots,k_r \in \mathbb{Z}_{\geq 1}$ with $k_1 \geq 2$, and $\iota(\mathbf{1}) := \mathbf{1}$. By abuse of notation, we will also write $⧢$ for the map

$$\mathbb{Q}\langle\mathcal{U}\rangle^1 \times \mathbb{Q}\langle\mathcal{U}\rangle^1 \longrightarrow \mathbb{Q}\langle\mathcal{U}\rangle^1, \quad (\mathtt{W}_1,\mathtt{W}_2) \longmapsto \iota(\iota^{-1}(\mathtt{W}_1) ⧢ \iota^{-1}(\mathtt{W}_2)).$$

### 1.1.2 Basics on Multiple Zeta Values

Multiple Zeta Values (MZVs) are real numbers defined as iterated sums. From this representation, one may see that their product again is a sum of MZVs. Besides this representation, MZVs can be described as iterated integrals. Also, using this representation gives rise to the observation that their product is a sum of MZVs again. Surprisingly, these two representations of the product of MZVs generally do not coincide. I.e., one obtains linear relations among MZVs, the *double shuffle relations*. The name comes from the fact that both representations of the product can be described via quasi-shuffle products. Besides the double shuffle relations, there is another class of linear relations among MZVs, the *MZV duality*. We refer to [20, 31, 35] for more details on the basic properties of MZVs.

With the notion of $\mathcal{U}^{*,1}$ (see Definition 1.6(ii)), we are prepared to define Multiple Zeta Values.

**Definition 1.10.** For all words $\mathtt{W} = u_{k_1}\cdots u_{k_r} \in \mathcal{U}^{*,1}$, the *Multiple Zeta Value* (MZV) of $\mathtt{W}$ is defined as

$$\zeta(\mathtt{W}) := \sum_{m_1>\cdots>m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}.$$

For $\mathtt{W} = \mathbf{1}$, we set $\zeta(\mathbf{1}) := 1$. Furthermore, we consider $\zeta$ as map $\mathbb{Q}\langle\mathcal{U}\rangle^1 \to \mathbb{R}$ via $\mathbb{Q}$-linear continuation.

With MZVs, we are able to express products of (single) Zeta Values as integer linear combinations of MZVs, induced by handling iterated sums. In particular, we can describe the product of MZVs using the stuffle product.

**Example 1.11.** Considering for $k_1, k_2 \in \mathbb{Z}_{\geq 2}$ the product of $\zeta(u_{k_1})$ and $\zeta(u_{k_2})$,

$$\begin{aligned}\zeta(u_{k_1}) \cdot \zeta(u_{k_2}) &= \sum_{n\geq 1} \frac{1}{n^{k_1}} \sum_{m\geq 1} \frac{1}{m^{k_2}} \\ &= \sum_{n>m>0} \frac{1}{n^{k_1} m^{k_2}} + \sum_{m>n>0} \frac{1}{m^{k_2} n^{k_1}} + \sum_{n=m>0} \frac{1}{n^{k_1} m^{k_2}} \\ &= \zeta(u_{k_1} u_{k_2}) + \zeta(u_{k_2} u_{k_1}) + \zeta(u_{k_1+k_2}) \\ &= \zeta(u_{k_1} * u_{k_2}),\end{aligned}$$

one obtains that this product indeed is a sum of MZVs. In particular, all words of MZVs occurring are of the same weight $k_1+k_2$, which is the sum of the weights of the length-one words $u_{k_1}$ and $u_{k_2}$ we started with.

The observation that the product of MZVs is $\zeta$ of the stuffle product of the corresponding words is always the case.

**Proposition 1.12** ([31, Theorem 4.2])**.** *The map $\zeta\colon (\mathbb{Q}\langle\mathcal{U}\rangle^1, *) \to (\mathbb{R}, \cdot)$ is an algebra homomorphism. In particular, for all $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^{*,1}$, we have*

$$\zeta(\mathtt{W}_1) \cdot \zeta(\mathtt{W}_2) = \zeta(\mathtt{W}_1 * \mathtt{W}_2).$$

Besides the definition of MZVs as iterated sums, MZVs have a remarkable representation as iterated integrals, the so-called *Kontsevich integrals*.

**Proposition 1.13** (Kontsevich integral, [31, Theorem 6.1])**.** *Let $\mathtt{W} = u_{k_1} \cdots u_{k_r}$ be a word in $\mathcal{U}^{*,1}$. Then we have*

$$\zeta(\mathtt{W}) = \int\limits_{1>t_1>\cdots>t_k>0} \omega_1(t_1) \cdots \omega_k(t_k),$$

*where $k := \mathrm{wt}(\mathtt{W})$ and*

$$\omega_i(t) := \begin{cases} \frac{dt}{1-t}, & \text{if } i \in \{k_1, k_1+k_2, \ldots, k_1+\cdots+k_r\}, \\ \frac{dt}{t}, & \text{else.} \end{cases}$$

*Proof.* For proof, see [20]. □

One can see that the product of such iterated integrals again is an integer linear combination of such iterated integrals, i.e., multiplying MZVs represented as Kontsevich integrals leads to a linear combination of MZVs again. One can describe it using the shuffle product.

**Proposition 1.14** ([34, Theorem 4.1])**.** *The map $\zeta\colon (\mathbb{Q}\langle\mathcal{U}\rangle^1, ⧢) \to (\mathbb{R}, \cdot)$ is an algebra homomorphism. In particular, for all $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^{*,1}$, we have*

$$\zeta(\mathtt{W}_1) \cdot \zeta(\mathtt{W}_2) = \zeta(\mathtt{W}_1 ⧢ \mathtt{W}_2).$$

What is remarkable about the shuffle product representation of MZVs is that it is generally different from the stuffle product representation.

**Example 1.15.** Considering for integers $k_1, k_2 \in \mathbb{Z}_{\geq 2}$ the product of $\zeta(u_{k_1})$ and $\zeta(u_{k_2})$, both represented as Kontsevich integrals, one obtains

$$\zeta(u_{k_1}) \cdot \zeta(u_{k_2}) = \sum_{\substack{a+b=k_1+k_2 \\ a\geq 2}} \left( \binom{a-1}{k-1} + \binom{a-1}{\ell-1} \right) \zeta(u_a u_b) = \zeta(u_{k_1} ⧢ u_{k_2}).$$

In particular, we see that, in general, the right sight is represented by different MZVs than in $\zeta(u_{k_1} * u_{k_2})$; see Example 1.11.

Again, we observe that the words occurring in the shuffle product have weight equals the sum of the weight of the words $u_{k_1}$ and $u_{k_2}$. This holds for the general case; details can be seen in [20].

**Linear relations among MZVs** Since stuffle and shuffle product representation, in general, are different, one obtains via

$$\zeta\left(\mathtt{W}_1 * \mathtt{W}_2 - \mathtt{W}_1 \sqcup\!\sqcup \mathtt{W}_2\right) = 0$$

for $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^{*,1}$ non-trivial linear relations among MZVs. They are called *double shuffle relations.*

**Example 1.16.** Using the the stuffle product representation, one observes

$$\zeta(u_2) \cdot \zeta(u_2) = \zeta(u_2 * u_2) = 2\zeta(u_2 u_2) + \zeta(u_4).$$

The shuffle product representation gives

$$\zeta(u_2) \cdot \zeta(u_2) = \zeta(u_2 \sqcup\!\sqcup u_2) = 2\zeta(u_2 u_2) + 4\zeta(u_3 u_1).$$

Hence, we obtain the linear relation

$$\zeta(u_4) = 4\zeta(u_3 u_1).$$

**Remark 1.17.** There are lots of proofs in the literature for the famous so-called Euler identity ([27])

$$\zeta(u_3) = \zeta(u_2 u_1). \tag{1.17.1}$$

Recall that for every word $\mathtt{W}$ occurring in $\mathtt{W}_1 * \mathtt{W}_2$ or $\mathtt{W}_1 \sqcup\!\sqcup \mathtt{W}_2$ (with $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^{*,\circ}$, we have $\mathrm{wt}(\mathtt{W}) = \mathrm{wt}(\mathtt{W}_1) + \mathrm{wt}(\mathtt{W}_2)$. Hence, double shuffle relations cannot obtain the Euler identity since $\zeta(u_1)$ is not defined for convergence reasons. There are three ways to deal with that. One possibility is to *regularize* both stuffle and shuffle product, leading to *extended double shuffle relations* from which relations such as (1.17.1) can be obtained. Those extended double shuffle relations conjecturally imply all $\mathbb{Q}$-linear relations among MZVs. Proving this is still an open problem and one of the most famous ones in the broad field of MZVs. For a detailed study of (extended) double shuffle relations, we refer to [35]. Another way to obtain identities like (1.17.1) is via MZV duality, presented in the following theorem. The third way to deal with the fact that $\zeta(u_1)$ is not defined in considering $q$-analogues of them as we will do in Section 1.1.3 since on this level we can give a well-defined $q$-analogue for $\zeta(u_1)$. Among $q$-analogues, one can study $\mathbb{Q}$-linear relations similarly to the (extended) double shuffle relations that give back relations among MZVs when taking the limit $q \to 1$ after some small modification; see the paragraph after Definition 1.26.

**Theorem 1.18** (MZV duality, [31, Corollary 6.2])**.** *On $\mathbb{Q}\langle\mathcal{U}\rangle^1$, we have $\zeta \circ \tilde{\tau} = \zeta$.*

When speaking about MZV duality (Theorem 1.18), we will always speak indeed about *MZV duality* in contrast to just *duality* which is a similar looking Theorem for $q$-analogues of MZVs (see Theorem 1.29). Let us consider a small example.

**Example 1.19.** By Example 1.8, Theorem 1.18 gives $\zeta(u_3) = \zeta(u_2 u_1)$, i.e., the Euler identity (1.17.1).

A natural question is what a basis of $\mathcal{Z}$ looks like. Finding such a basis is still an open problem, so one focuses on finding generating systems as small as possible. Considering the Euler identity and its generalization (obtained via MZV duality)

$$\zeta(u_k) = \zeta\left(u_2 u_1^{k-2}\right)$$

for $k \in \mathbb{Z}_{\geq 2}$, it could be reasonable that $\mathcal{Z}$ is spanned by $\zeta(\mathtt{W})$'s for $\mathtt{W} \in (\mathcal{U}\backslash\{u_0, u_1\})^*$'s, i.e., for words with letters in $\{u_2, u_3, \dots\}$. Indeed, this statement is true and can be deduced directly from Theorem 1.22.

**Theorem 1.20.** *We have* $\mathcal{Z} = \operatorname{span}_{\mathbb{Q}} \{\zeta(\mathtt{W}) \,|\, \mathtt{W} \in (\mathcal{U}\backslash\{u_0, u_1\})^*\}$.

Although a stronger version of this theorem is already proven (see Theorem 1.22 below), this statement is still interesting for what we will do in Section 1.2/Chapter 2 since we investigate there Conjecture 1.49 which can be seen as the analogous statement to Theorem 1.20 for the space $q$MZVs span.

Regarding the question of how a basis of $\mathcal{Z}$ looks like, Hoffman [32] conjectured a stronger statement of Theorem 1.20.

**Conjecture 1.21** ([32])**.** *A basis of the* $\mathbb{Q}$*-vector space* $\mathcal{Z}$ *is given by the set of MZVs* $\zeta(\mathtt{W})$ *satisfying* $\mathtt{W} \in \{u_2, u_3\}^*$.

The current status is that one is not able to prove the $\mathbb{Q}$-linear independence of the MZVs occurring in Hoffman's Conjecture 1.21. A partial result was obtained by Brown.

**Theorem 1.22** ([19])**.** *A spanning set of the* $\mathbb{Q}$*-vector space* $\mathcal{Z}$ *is given by the set of MZVs* $\zeta(\mathtt{W})$ *satisfying* $\mathtt{W} \in \{u_2, u_3\}^*$.

### 1.1.3 Basics on Multiple $q$-Zeta Values

Multiple Zeta Values are real numbers, which makes investigating their algebraic structure sometimes hard. A common strategy, not only used for MZVs, to avoid parts of this problem is to consider *q-analogues* of the objects one investigates. These are modified objects with an extra parameter $q$ such that they inherit (parts) of the algebraic structure of the original objects. Furthermore, they have the property that one gets back the original objects when taking the limit $q \to 1$, maybe after some modification such as multiplying with a specific power of $(1-q)$, for example. In Section 1.1.3, we introduce $q$-analogues of MZVs ($q$MZVs for short) and their $\mathbb{Q}$-algebra $\mathcal{Z}_q$ (defined in [10]).

One of the most used $q$-analogues are the expressions

$$[n]_q := \frac{1-q^n}{1-q},$$

seen as $q$-analogues of positive integers $n$. Indeed, one has, as $q$ approaches 1,

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1} \longrightarrow n.$$

Now, an intuitive way to obtain $q$-analogues of MZVs would be to replace in the definition of MZVs (Definition 1.10) every $m_j$ with $[m_j]_q$. To avoid convergence issues, one has to be more careful, but it is the main idea leading to the definition of $q$MZVs.

**Definition 1.23** ([10, Equation 1])**.** (i) Define for all $\mathtt{W} = u_{k_1} \cdots u_{k_r} \in \mathcal{U}^{*,\circ}$ and polynomials $Q_1 \in X\mathbb{Q}[X]$, $Q_2, \dots, Q_r \in \mathbb{Q}[X]$, the *Multiple q-Zeta Value* ($q$MZV)

$$\zeta_q(\mathtt{W}; Q_1, \dots, Q_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{Q_1(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{Q_r(q^{m_r})}{(1-q^{m_r})^{k_r}} \in \mathbb{Q}[\![q]\!],$$

where we set $\zeta_q(\mathbf{1},\emptyset) := 1$ in case $r=0$.

(ii) Define $\mathcal{Z}_q$ as

$$\operatorname{span}_{\mathbb{Q}}\left\{\zeta_q(\mathtt{W};Q_1,\ldots,Q_r)\,\middle|\, \substack{\mathtt{W}=u_{k_1}\cdots u_{k_r}\in\mathcal{U}^{*,\circ},\, r\in\mathbb{Z}_{\geq 0},\\ Q_1\in X\mathbb{Q}[X],\, \deg(Q_j)\leq k_j\ (1\leq j\leq r)}\right\}.$$

Note that $Q_1 \in X\mathbb{Q}[X]$ is necessary for convergence reasons. Furthermore, for a word $\mathtt{W}\in\mathcal{U}^{*,1}$, with $r:=\operatorname{len}(\mathtt{W})$, one has

$$\lim_{q\to 1}(1-q)^{\operatorname{wt}(\mathtt{W})}\zeta_q(\mathtt{W};Q_1,\ldots,Q_r)=\zeta(\mathtt{W})\prod_{j=1}^{r}Q_j(1).$$

In this general definition of $q$MZVs, there is no obvious notion of weight as for MZVs. A canonical choice would be $\operatorname{wt}(\mathtt{W})$ leading to uniqueness issues since, e.g., for every $k_1\in\mathbb{Z}_{>0}$ and polynomials $Q_1\in X\mathbb{Q}[X]$, we have

$$\zeta_q(u_{k_1};Q_1)=\zeta(u_{k_1+1};Q_1\cdot(1-X)).$$

Therefore, one usually chooses the polynomials $Q_j$ such that $(1-X)\nmid Q_j(X)$.

In the definition of $q$MZVs, one can also say that the words $\mathtt{W}\in\mathcal{U}^{*,\circ}$ should satisfy $\operatorname{zero}(\mathtt{W})=0$ when one allows an additional polynomial factor (in $m_1,\ldots,m_r$) in the summand, see Proposition 1.25 below. Let us consider an example first.

**Example 1.24.** Choose $k_1\in\mathbb{Z}_{>0}$ and $Q_1\in X\mathbb{Q}[X]$ with $\deg(Q_1)\leq k_1$. Furthermore, let $Q_2=Q_3=1$ and $k_2=k_3=0$. Then, by definition, $\zeta_q\left(u_{k_1}u_0u_0;Q_1,1,1\right)$ is an element of $\mathcal{Z}_q$. But we also have

$$\zeta_q\left(u_{k_1}u_0u_0;Q_1,1,1\right)=\sum_{m_1>m_2>m_3>0}\frac{Q_1(q^{m_1})}{(1-q^{m_1})^{k_1}}=\sum_{m_1>0}\binom{m_1-1}{2}\frac{Q_1(q^{m_1})}{(1-q^{m_1})^{k_1}}.$$

In particular, the latter sum is of the same shape as in the original definition of $q$MZVs with an additional polynomial factor in $m_1$ in the summand.

In the following, we show that the observation of Example 1.24 is true in general; we will use it (implicitly) in Chapter 2 to validate that our definition of Schlesinger–Zudilin $q$MZVs (see Section 1.1.4) is equivalent to the usual one.

**Proposition 1.25.** *Choose a word $\mathtt{W}=u_{k_1}\cdots u_{k_r}\in\mathcal{U}^{*,\circ}$ and polynomials $Q_j$ satisfying $Q_1\in X\mathbb{Q}[X]$, $\deg(Q_j)\leq k_j$ for $1\leq j\leq r=\operatorname{len}(\mathtt{W})$. For every polynomial $Q_0\in\mathbb{Q}[Y_1,\ldots,Y_r]$, we have*

$$\sum_{m_1>\cdots>m_r>0}Q_0(m_1,\ldots,m_r)\frac{Q_1(q^{m_1})}{(1-q^{m_1})^{k_1}}\cdots\frac{Q_r(q^{m_r})}{(1-q^{m_r})^{k_r}}\in\mathcal{Z}_q.$$

*Proof.* We sketch a variation of the proof provided in [10]. For this, choose an arbitrary word $\mathtt{W}=u_{k_1}u_0^{z_1}\cdots u_{k_d}u_0^{z_d}\in\mathcal{U}^{*,\circ}$ and abbreviate $f_j:=z_1+\cdots+z_{j-1}+j$ for $1\leq j\leq d$. Furthermore, choose $Q_j\in\mathbb{Q}[X]$ with $\deg(Q_{f_j})\leq f_j$ for $1\leq j\leq d$ and $Q_j=1$ else. We obtain that

$$\sum_{m_1>\cdots>m_{\operatorname{len}(\mathtt{W})}>0}\frac{Q_1(q^{m_1})}{(1-q^{m_1})^{k_1}}\cdots\frac{Q_{\operatorname{len}(\mathtt{W})}(q^{m_{\operatorname{len}(\mathtt{W})}})}{(1-q^{m_{\operatorname{len}(\mathtt{W})}})^{k_{\operatorname{len}(\mathtt{W})}}}$$
$$=\sum_{m_1>\cdots>m_d>0}\binom{m_1-m_2-1}{z_1}\cdots\binom{m_{d-1}-m_d-1}{z_{d-1}}\binom{m_d-1}{z_d}$$

$$\times \frac{Q_1(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{Q_{f_d}(q^{m_d})}{(1-q^{m_d})^{k_{f_d}}} \in \mathcal{Z}_q.$$

Note that

$$\left\{ \binom{Y_1 - Y_2 - 1}{z_1} \cdots \binom{Y_{d-1} - Y_d - 1}{z_{d-1}} \binom{Y_d - 1}{z_d} \,\middle|\, z_1, \ldots, z_d \in \mathbb{Z}_{\geq 0} \right\}$$

builds a basis of $\mathbb{Q}[Y_1, \ldots, Y_d]$. Hence, we indeed have

$$\sum_{m_1 > \cdots > m_r > 0} Q_0(m_1, \ldots, m_r) \frac{Q_1(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{Q_r(q^{m_r})}{(1-q^{m_r})^{k_r}} \in \mathcal{Z}_q$$

for every $u_{k_1} \cdots u_{k_r} \in \mathcal{U}^{*,\circ}$, polynomials $Q_j$ with $Q_1 \in X\mathbb{Q}[X]$, $\deg(Q_j) \leq k_j$ for all integers $1 \leq j \leq r$, and for every polynomial $Q_0 \in \mathbb{Q}[Y_1, \ldots, Y_r]$. $\square$

### 1.1.4 Models of $q$MZVs

Similar to $\mathcal{Z}$, a natural question is what a basis of $\mathcal{Z}_q$ looks like. Finding such a basis is one of the open problems regarding $q$MZVs and seems very difficult to answer. Therefore, one focuses on finding (small) generating sets of $\mathcal{Z}_q$, which we call in distinguished cases *models (of Multiple q-Zeta Values)*. We refer to the original works [2, 11, 37, 38, 40, 43, 46, 48] for details on often used models and to [14] for an overview. Every model has its advantage when studying $q$MZVs and their structure. Schlesinger–Zudilin's model, e.g., inherits the stuffle product, while Bradley–Zhao's model satisfies the same duality relation as the one MZVs satisfy. Important as well is Bachmann's model given by bi-brackets since it gives a direct connection to quasi-modular forms playing an essential role in the theory of MZVs as Gangl, Kaneko, and Zagier [29] have shown. We present the considered models' main facts about their algebraic structure. In particular, we will focus on relations for $q$MZVs similar to the duality of MZVs.

For a particular choice of the polynomials $Q_j$ (in dependence of the $k_j$'s), we say that the corresponding $q$MZVs build a *model* (of $q$MZVs) if they span $\mathcal{Z}_q$. We now present two models with which we will work in this thesis.

**Schlesinger–Zudilin model.** A particular generating system of $\mathcal{Z}_q$ is the following.

**Definition 1.26** (Schlesinger–Zudilin $q$MZVs)**.** For all $\mathtt{W} = u_{\ell_1} \cdots u_{\ell_r} \in \mathcal{U}^{*,\circ}$, we define the *Schlesinger–Zudilin qMZV* (SZ-$q$MZV for short) by $\zeta_q^{\mathrm{SZ}}(\mathbf{1}) := 1$ and, for positive $r$, by

$$\zeta_q^{\mathrm{SZ}}(\mathtt{W}) := \zeta_q(\mathtt{W}; X^{\ell_1}, \ldots, X^{\ell_r}).$$

By definition of $\zeta_q$, note that for all $\mathtt{W} = u_{k_1} \cdots u_{k_r} \in \mathcal{U}^{*,\circ}$, we have

$$\zeta_q^{\mathrm{SZ}}(\mathtt{W}) = \sum_{m_1 > \cdots > m_r > 0} \frac{q^{m_1 \ell_1}}{(1-q^{m_1})^{\ell_1}} \cdots \frac{q^{m_r \ell_r}}{(1-q^{m_r})^{\ell_r}}.$$

Furthermore, writing $\mathtt{W} = u_{k_1}u_0^{z_1}\cdots u_{k_d}u_0^{z_d}$ with $d, z_1, \cdots, z_d \in \mathbb{Z}_{\geq 0}$, $k_1, \ldots, k_d \in \mathbb{Z}_{>0}$ uniquely determined, we obtain the following representation of SZ-$q$MZVs, which is consistent with Proposition 1.25,

$$\zeta_q^{\mathrm{SZ}}(\mathtt{W}) = \sum_{m_1>\cdots>m_d>0}\sum_{m_1>\cdots>m_d>0}\prod_{j=1}^{d}\binom{m_j - m_{j+1} - 1}{z_j}\frac{q^{m_j k_j}}{(1-q^{m_j})^{k_j}}.$$

Also, when $k_1 > 1$ and $z_1 = \cdots = z_d = 0$, i.e., $\mathtt{W} \in \mathcal{U}^{*,1}$, we observe

$$\lim_{q\to 1}(1-q)^{\mathrm{wt}(\mathtt{W})}\zeta_q^{\mathrm{SZ}}(\mathtt{W}) = \zeta(\mathtt{W}) \in \mathcal{Z}.$$

By [8, Theorem 1.2], it is known that this limit exists (after possible regularization) for all words $\mathtt{W} \in \mathcal{U}^{*,\circ}$ and is always element of $\mathcal{Z}$. Moreover, SZ-$q$MZVs span $\mathcal{Z}_q$. This justifies calling $q$MZVs as introduced in Definition 1.23 indeed $q$-analogues of MZVs.

For describing SZ-$q$MZVs algebraically, we need the following evaluation map.

**Definition 1.27.** We define the map $\zeta_q^{\mathrm{SZ}}\colon \mathbb{Q}\langle\mathcal{U}\rangle^\circ \longrightarrow \mathcal{Z}_q$ via

$$\mathtt{W} \longmapsto \zeta_q^{\mathrm{SZ}}(\mathtt{W})$$

for every word $\mathtt{W} \in \mathcal{U}^{*,\circ}$ and extend it to $\mathbb{Q}\langle\mathcal{U}\rangle^\circ$ by $\mathbb{Q}$-linearity.

Similar to MZVs (see Proposition 1.12), SZ-$q$MZVs respect the product structure induced by the stuffle product.

**Proposition 1.28** ([41, Theorem 3.3])**.** *The map $\zeta_q^{\mathrm{SZ}} : (\mathbb{Q}\langle\mathcal{U}\rangle^\circ, *) \longrightarrow (\mathcal{Z}_q, \cdot)$ is an algebra homomorphism. In particular, for all $\mathtt{W}_1, \mathtt{W}_2 \in \mathbb{Q}\langle\mathcal{U}\rangle^\circ$, we have*

$$\zeta_q^{\mathrm{SZ}}(\mathtt{W}_1)\cdot\zeta_q^{\mathrm{SZ}}(\mathtt{W}_2) = \zeta_q^{\mathrm{SZ}}(\mathtt{W}_1 * \mathtt{W}_2).$$

Besides the stuffle product, another fact on SZ-$q$MZVs makes it interesting to study their structure. For this, we need the involution $\tau$ from Definition 1.6. One can obtain the following identity of $q$MZVs, e.g., using marked partitions (see Section 1.3 and Chapter 3).

**Theorem 1.29** (Duality, [46, Theorem 8.3])**.** *On $\mathbb{Q}\langle\mathcal{U}\rangle^\circ$, we have $\zeta_q^{\mathrm{SZ}} \circ \tau = \zeta_q^{\mathrm{SZ}}$.*

Recall that $\tau$ is an involution on $\mathbb{Q}\langle\mathcal{U}\rangle^\circ$. The name of duality comes from the fact that it looks very similar to MZV duality. Nevertheless, it is not the same in the sense that it is an open problem to show that $\tau$ and $*$ imply MZV duality already. One approach for the connection of $\tau$ and $\tilde{\tau}$ can be found in [13], where so-called *connected sums* are used to prove both at once. As one can see from duality, the role of $u_0$ is special for the map $\zeta_q^{\mathrm{SZ}}$. Hence, we introduce $\mathcal{Z}_q^\circ$ as the subalgebra of $\mathcal{Z}_q$ generated by SZ-$q$MZVs of words containing no $u_0$.

**Definition 1.30.** We define

$$\mathcal{Z}_q^\circ := \mathrm{span}_{\mathbb{Q}}\left\{\zeta_q^{\mathrm{SZ}}(\mathtt{W}) \,\middle|\, \mathtt{W} \in (\mathcal{U}\backslash\{u_0\})^*\right\}.$$

Note that the stuffle product is closed on $\mathcal{Z}_q^\circ$ making $(\mathcal{Z}_q^\circ, *)$ indeed a subalgebra of $(\mathcal{Z}_q, *)$. Besides the similarity between $\tau$ and $\tilde{\tau}$, a second connection exists between SZ-$q$MZVs and MZVs. Namely, the structure of $\mathcal{Z}_q$ seems to be similar to the one of $\mathcal{Z}$ in the sense that $u_0$'s seem to be 'unnecessary', similar to Theorem 1.20 which stated that $u_1$'s in words are not needed to obtain a generating system of $\mathcal{Z}$. More precisely, we have the following.

**Conjecture 1.31** ([3, Conjecture 4.3])**.** *We have* $\mathcal{Z}_q = \mathcal{Z}_q^\circ$.

We will discuss this conjecture in Section 1.2 and Chapter 2 in more detail and give new partial results.

**Bi-brackets.** Every quasi-modular form is an element of $\mathcal{Z}_q$. Therefore, $q$MZVs can be seen as a generalization of quasi-modular forms. A model of $q$MZVs depicting this property was introduced by Bachmann in his Thesis ([2]). The $q$MZVs in this model are called *bi-brackets*.

**Definition 1.32** ([3, Definition 2.1])**.** (i) For all integers $d \in \mathbb{Z}_{\geq 0}$, $k_1, \ldots, k_d \in \mathbb{Z}_{>0}$, and $z_1, \ldots, z_d \in \mathbb{Z}_{\geq 0}$, the *bi-bracket* is $\mathrm{g}\binom{\emptyset}{\emptyset} := 1$ for $d = 0$ and for $d > 0$ it is

$$\mathrm{g}\begin{pmatrix} k_1, \ldots, k_d \\ z_1, \ldots, z_d \end{pmatrix} := \sum_{m_1 > \cdots > m_d > 0} \prod_{j=1}^{d} \frac{m_j^{d_j}}{d_j!} \frac{P_{k_j}(q^{m_j})}{(1 - q^{m_j})^{k_j}},$$

where $P_k$ is the $k$-th *Eulerian polynomial*,

$$P_k(X) := (1 - X)^k \sum_{n > 0} \frac{n^{k-1}}{(k-1)!} X^n.$$

Furthermore, we define

$$\begin{aligned} \mathrm{zero}\begin{pmatrix} k_1, \ldots, k_d \\ z_1, \ldots, z_d \end{pmatrix} &:= z_1 + \cdots + z_d, \\ \mathrm{depth}\begin{pmatrix} k_1, \ldots, k_d \\ z_1, \ldots, z_d \end{pmatrix} &:= d, \\ \mathrm{wt}\begin{pmatrix} k_1, \ldots, k_d \\ z_1, \ldots, z_d \end{pmatrix} &:= k_1 + \cdots + k_d + z_1 + \cdots + z_d. \end{aligned}$$

(ii) We define $\mathrm{g}(\emptyset) := 1$, and for any $d \in \mathbb{Z}_{>0}$ and any $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}_{>0}^d$, we define the *bracket* of $\mathbf{k}$ as

$$\mathrm{g}(\mathbf{k}) := \sum_{m_1 > \cdots > m_d > 0} \prod_{j=1}^{d} \frac{P_{k_j}(q^{m_j})}{(1 - q^{m_j})^{k_j}}.$$

Note that by Proposition 1.25, bi-brackets indeed are elements of $\mathcal{Z}_q$. Moreover, we have[1]

$$\mathcal{Z}_q = \mathrm{span}_{\mathbb{Q}} \left\{ \mathrm{g}\begin{pmatrix} k_1, \ldots, k_d \\ z_1, \ldots, z_d \end{pmatrix} \middle|\, k_j \in \mathbb{Z}_{>0},\, d, z_j \in \mathbb{Z}_{\geq 0} \ (1 \leq j \leq d) \right\}$$

and

$$\mathcal{Z}_q^\circ = \mathrm{span}_{\mathbb{Q}} \left\{ \mathrm{g}(k_1, \ldots, k_d) \ \middle|\ d \in \mathbb{Z}_{\geq 0},\, k_j \in \mathbb{Z}_{>0} \ (1 \leq j \leq d) \right\}.$$

[1] In earlier works regarding (bi-)brackets such as [3], $\mathcal{Z}_q$ was denoted by $\mathcal{BD}$ and $\mathcal{Z}_q^\circ$ by $\mathcal{MD}$.

Both, $\mathcal{Z}_q^\circ$ and $\mathcal{Z}_q$ are $\mathbb{Q}$-algebras (with the usual product of $q$-series via multiplication) as shown in [3]. Furthermore, note that every bracket is a bi-bracket since

$$\mathrm{g}(k_1,\ldots,k_d)=\mathrm{g}\binom{k_1,\ldots,k_d}{0,\ldots,0}$$

for all $k_1,\ldots,k_d\in\mathbb{Z}_{>0}$.

**Definition 1.33.** (i) For $(\mathrm{N},\mathrm{op})\in\{(\mathrm{Z},\mathrm{zero}),(\mathrm{D},\mathrm{depth}),(\mathrm{W},\mathrm{wt})\}$, $n\in\mathbb{Z}$, and for a set $\mathcal{S}\subset\mathcal{Z}_q$, we write

$$\mathrm{Fil}_n^{\mathrm{N}}\,\mathcal{S}:=\mathrm{span}_{\mathbb{Q}}\left\{\mathrm{g}\binom{\mathbf{k}}{\mathbf{z}}\;\middle|\;\mathrm{op}\binom{\mathbf{k}}{\mathbf{z}}\leq n\right\}\cap\mathcal{S}.$$

(ii) For $\mathcal{S}\subset\mathcal{Z}_q$, $\mathrm{N}_1,\ldots,\mathrm{N}_m\in\{\mathrm{Z},\mathrm{D},\mathrm{W}\}$ ($m\in\mathbb{Z}_{>0}$), and integers $n_1,\ldots,n_m\in\mathbb{Z}$, we abbreviate

$$\mathrm{Fil}_{n_1,\ldots,n_m}^{\mathrm{N}_1,\ldots,\mathrm{N}_m}\,\mathcal{S}:=\bigcap_{j=1}^m\mathrm{Fil}_{n_j}^{\mathrm{N}_j}\,\mathcal{S}.$$

In particular, we have $\mathcal{Z}_q^\circ=\mathrm{Fil}_0^{\mathrm{Z}}\,\mathcal{Z}_q$. Similarly to duality, bi-brackets satisfy a relation analogous to the duality relation (Theorem 1.29). For this, we introduce for every $d\in\mathbb{Z}_{\geq0}$ the generating series

$$\mathfrak{g}\binom{X_1,\ldots,X_d}{Y_1,\ldots,Y_d}:=\sum_{\substack{k_1,\ldots,k_d>0\\ z_1,\ldots,z_d\geq0}}\mathrm{g}\binom{k_1,\ldots,k_d}{z_1,\ldots,z_d}\prod_{j=1}^d X_j^{k_j-1}Y_j^{z_j}.$$

**Theorem 1.34** (Partition relation, [3, Theorem 2.3])**.** *For all $d\in\mathbb{Z}_{>0}$ we have*

$$\mathfrak{g}\binom{X_1,\ldots,X_d}{Y_1,\ldots,Y_d}=\mathfrak{g}\binom{Y_1+\cdots+Y_d,\ldots,Y_1+Y_2,Y_1}{X_d,X_{d-1}-X_d,\ldots,X_1-X_2}.$$

Note that in the case $d=1$, one can express the partitions relation with bi-brackets of depth 1 as follows.

**Corollary 1.35.** *For all $k\in\mathbb{Z}_{>0}$ and $z\in\mathbb{Z}_{\geq0}$, we have*

$$\mathrm{g}\binom{k}{z}=\mathrm{g}\binom{z+1}{k-1}.$$

The name of the partition relation comes from the combinatorial interpretation of $q$MZVs as generating series of particular partitions, similar to what we will do in Section 1.3. The partition relation comes from the invariance of the considered $q$-series under transposing the Young Tableau of each partition.

**Theorem 1.36** ([14, Theorem 14])**.** *Under the translation of bi-brackets to the SZ-model [48, Proposition 3], duality and the partition relation are equivalent.*

In Section 1.4, we will need the following fact about bi-brackets.

**Theorem 1.37** ([9, Theorem 1.7])**.** *We have $q\frac{d}{dq}\mathcal{Z}_q^\circ\subset\mathcal{Z}_q^\circ$.*

Using generating series of bi-brackets, Bachmann obtained the following explicit result for depth 1.

**Proposition 1.38** ([3, Proposition 4.2])**.** *For $k \in \mathbb{Z}_{>0},\ z \in \mathbb{Z}_{\geq 0}$, we have*

$$q\frac{d}{dq}\,\mathrm{g}\binom{k}{z} = k(z+1)\,\mathrm{g}\binom{k+1}{z+1}.$$

Using Theorem 1.37, the following is an immediate consequence of Proposition 1.38.

**Corollary 1.39** ([3, Proposition 4.4])**.** *For all integers $k \in \mathbb{Z}_{>0}$, and $z \in \mathbb{Z}_{\geq 0}$, we have $\mathrm{g}\binom{k}{z} \in \mathcal{Z}_q^{\circ}$.*

**Bradley–Zhao $q$MZVs.** Although the Multiple $q$-Zeta Values, introduced by Bradley and Zhao, do not span $\mathcal{Z}_q$ (but a proper subspace), we mention them here since they are of importance for the theory of Multiple ($q$-)Zeta Values. This is, for example, due to the fact that they satisfy the same duality relation as Multiple Zeta Values.

**Definition 1.40** (Bradley–Zhao $q$MZVs)**.** For all words $\mathtt{W} = u_{k_1}\cdots u_{k_r} \in \mathcal{U}^{*,1}$, the *Bradley–Zhao Multiple $q$-Zeta Value* (BZ-$q$MZV) of $\mathtt{W}$ is defined as

$$\zeta_q^{\mathrm{BZ}}(\mathtt{W}) := \sum_{m_1>\cdots>m_r>0} \frac{q^{m_1(k_1-1)}}{(1-q_1^m)^{k_1}}\cdots\frac{q^{m_r(k_r-1)}}{(1-q_r^m)^{k_r}}.$$

For $\mathtt{W} = \mathbf{1}$, we set $\zeta_q^{\mathrm{BZ}}(\mathbf{1}) := 1$. Furthermore, we consider $\zeta_q^{\mathrm{BZ}}$ as map $\mathbb{Q}\langle\mathcal{U}\rangle^1 \to \mathbb{R}$ via $\mathbb{Q}$-linear continuation.

Note that for all $\mathtt{W} \in \mathcal{U}^{*,1}$, we have

$$\lim_{q\to 1}(1-q)^{\mathrm{wt}(\mathtt{W})}\zeta_q^{\mathrm{BZ}}(\mathtt{W}) = \zeta(\mathtt{W}).$$

Therefore, Bradley-Zhao $q$MZVs indeed are $q$-analogues of MZVs. In particular, they are in the sense of Definition 1.23. Furthermore, Bradley-Zhao $q$MZVs are of interest since a part of the structure MZVs have transfers.

**Theorem 1.41** (BZ duality, [11, Theorem 5])**.** *On $\mathbb{Q}\langle\mathcal{U}\rangle^1$, we have $\zeta_q^{\mathrm{BZ}} \circ \tilde{\tau} = \zeta_q^{\mathrm{BZ}}$.*

I.e., BZ-$q$MZVs satisfy the same duality relation as MZVs (Theorem 1.18). We should note that BZ-$q$MZVs do not span $\mathcal{Z}_q$ as mentioned at the beginning of this paragraph since, e.g., $\zeta_q(u_1; X) \in \mathcal{Z}_q$ can not be written as a linear combination of BZ-$q$MZVs.

**Remark 1.42.** (i) If Conjectures 1.43 and 1.44 below are true, one can show (via induction on the weight) that BZ-$q$MZVs satisfy the analogue of Theorem 1.20 in the sense

$$\mathrm{span}_{\mathbb{Q}}\left\{\zeta_q^{\mathrm{BZ}}(\mathtt{W}) \,|\, \mathtt{W} \in \mathcal{U}^{*,1}\right\} = \mathrm{span}_{\mathbb{Q}}\left\{\zeta_q^{\mathrm{BZ}}(\mathtt{W}) \,|\, \mathtt{W} \in (\mathcal{U}\backslash\{u_0, u_1\})^*\right\}.$$

(ii) However, BZ-$q$MZVs do not satisfy the analogue of Theorem 1.22, i.e.,

$$\mathrm{span}_{\mathbb{Q}}\left\{\zeta_q^{\mathrm{BZ}}(\mathtt{W}) \,|\, \mathtt{W} \in \mathcal{U}^{*,1}\right\} \supsetneq \mathrm{span}_{\mathbb{Q}}\left\{\zeta_q^{\mathrm{BZ}}(\mathtt{W}) \,|\, \mathtt{W} \in \{u_2, u_3\}^*\right\}.$$

## 1.2 The algebraic side of Multiple $q$-Zeta Values considered in Paper I

In this section, we introduce the work from Chapter 2 that considers $q$MZVs algebraically. For this, we will focus on the conjecture that the stuffle product and duality already imply all linear relations among $q$MZVs. Motivated by this conjecture, we present *formal Multiple q-Zeta Values*, similar as in [21]. These are algebraic objects satisfying the stuffle product and duality by definition but not having linear relations among them that are not implied by stuffle product or duality. We first present in Section 1.2.1 known results and conjectures about $\mathcal{Z}_q$. One of the most well-known conjectures is the one by Bachmann that bi-brackets and brackets span the same space. Furthermore, we present the main results of Chapter 2, which are partial results towards Bachmann's conjecture. In Section 1.2.2, we introduce and motivate the main ideas leading to the proof of those new results.

Section 1.2.1 then is about various conjectures and known results about the structure of $q$MZVs. As part of this thesis, we give new partial results in Section 1.2.2 for one of the main conjectures (Bachmann's Conjecture 1.49) among $q$MZVs and their structure. Furthermore, we develop the main ideas for the proof, which will be given in Chapter 2. These ideas, in particular, then lead to a refinement of Bachmann's Conjecture 1.49.

### 1.2.1 Known statements about the algebraic structure of $q$MZVs

This section will gather well-known results and conjectures about the structure of $\mathcal{Z}_q$. One folklore conjecture is the following by Bachmann (see [1]) that can be found in [48, Conjecture 1].

**Conjecture 1.43** (Bachmann). *All $\mathbb{Q}$-linear relations among elements in $\mathcal{Z}_q$ are obtained by the stuffle product $*$ and duality $\tau$.*

According to Conjecture 1.43, when investigating the structure of $\mathcal{Z}_q$ (see Section 1.2.2 and Chapter 2), we will use the stuffle product and duality only. More precisely, we will use relations only that are of shape

$$\zeta_q^{\mathrm{SZ}}(\mathtt{W}_1 * (\mathtt{W}_2 - \tau(\mathtt{W}_2))) = 0$$

for any words $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^{*,\circ}$.

Furthermore, the conjecture of paramount importance for this section and Chapter 2 is the following one by Bachmann.

**Conjecture 1.44** ([3, Conjecture 4.3]). *We have $\mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w}\, \mathcal{Z}_q = \mathrm{Fil}^{\mathrm{D,W}}_{z+d,w}\, \mathcal{Z}_q^\circ$ for all integers $z, d, w \in \mathbb{Z}_{\geq 0}$.*

We will use the SZ-model of $q$MZVs in Chapter 2. For this, we need the following notation of formal $q$MZVs as introduced in [21].

**Definition 1.45.** The algebra of *formal qMZV* is

$$\mathcal{Z}_q^f := {(\mathbb{Q}\langle\mathcal{U}\rangle^\circ, *)}\big/{T},$$

where $T$ is the $*$-ideal in $\mathbb{Q}\langle\mathcal{U}\rangle^\circ$ generated by $\{\tau(\mathtt{W}) - \mathtt{W} \mid \mathtt{W} \in \mathbb{Q}\langle\mathcal{U}\rangle^\circ\}$.

**Remark 1.46.** Note that the notion of *formal multiple Eisenstein series* [7] exists, which is equivalent to our notion but inspired by considering $q$MZVs as so-called *multiple Eisenstein series*. In contrast, the definition of $\mathcal{Z}_q^f$ is inspired by considering $q$MZVs as Schlesinger–Zudilin $q$MZVs.

For a more detailed description of what we are doing, it will be necessary to have filtrations by number of zeros, depth, and weight respectively on $\mathcal{Z}_q^f$ (and $\mathbb{Q}\langle\mathcal{U}\rangle^\circ$). We introduce them in the following definition. First, note that depth and weight are invariant under $\tau$, while the number of zeros, in general, is not as one may see, e.g., in the example

$$\tau(u_k) = u_1 u_0^{k-1}$$

for $k \in \mathbb{Z}_{>0}$. Therefore, we must be careful (compared to the analogous filtration on $\mathbb{Q}\langle\mathcal{U}\rangle^\circ$) when defining the filtration by the number of zeros on $\mathcal{Z}_q^f$. Recall that we write $\mathcal{U}^{*,\circ} = \mathcal{U}^* \backslash u_0 \mathcal{U}^*$ for the set of words not starting with $u_0$.

**Definition 1.47.** (i) For $(\mathrm{N}, \mathrm{op}) \in \{(\mathrm{Z}, \mathrm{zero}), (\mathrm{D}, \mathrm{depth}), (\mathrm{W}, \mathrm{wt})\}$, $n \in \mathbb{Z}$, and for sets $\mathcal{S} \subset \mathbb{Q}\langle\mathcal{U}\rangle^\circ$, $\mathcal{S}' \subset \mathcal{Z}_q^f$, we write

$$\begin{aligned}
\mathrm{Fil}_n^{\mathrm{N}} \mathcal{S} &:= \mathrm{span}_{\mathbb{Q}} \{\mathtt{W} \in \mathcal{U}^{*,\circ} \mid \mathrm{op}(\mathtt{W}) \leq n\} \cap \mathcal{S}, \\
\mathrm{Fil}_n^{\mathrm{N}} \mathcal{S}' &:= \mathrm{span}_{\mathbb{Q}} \left\{\zeta_q^{\mathrm{f}}(\mathtt{W}) \in \mathcal{Z}_q^f \mid \mathtt{W} \in \mathcal{U}^{*,\circ}, \mathrm{op}(\mathtt{W}) \leq n\right\} \cap \mathcal{S}'
\end{aligned}$$

for the filtration by number of zeros (if N = Z), depth (if N = D), and weight (if N = W) respectively on $\mathcal{S}$ and $\mathcal{S}'$ respectively. Furthermore, we define

$$\mathcal{Z}_q^{f,\circ} := \mathrm{Fil}_0^{\mathrm{Z}} \mathcal{Z}_q^f.$$

(ii) For $\mathcal{S} \subset \mathbb{Q}\langle\mathcal{U}\rangle^\circ$ or $\mathcal{S} \subset \mathcal{Z}_q^f$, $\mathrm{N}_1, \ldots, \mathrm{N}_m \in \{\mathrm{Z}, \mathrm{D}, \mathrm{W}\}$ ($m \in \mathbb{Z}_{>0}$), and integers $n_1, \ldots, n_m \in \mathbb{Z}$, we abbreviate

$$\mathrm{Fil}_{n_1,\ldots,n_m}^{\mathrm{N}_1,\ldots,\mathrm{N}_m} \mathcal{S} := \bigcap_{j=1}^{m} \mathrm{Fil}_{n_j}^{\mathrm{N}_j} \mathcal{S}.$$

**Remark 1.48.** Note that $\mathcal{Z}_q^{f,\circ}$ is a subalgebra of $\mathcal{Z}_q^f$ and that we can consider $\zeta_q^{\mathrm{SZ}}$ also as map $\mathcal{Z}_q^f \to \mathcal{Z}_q$ due to Theorem 1.29. Furthermore, referring to the translation from bi-brackets to the SZ-model (see [14, Theorem 13]), for all $z, d, w \in \mathbb{Z}_{\geq 0}$, we have

$$\zeta_q^{\mathrm{SZ}}\left(\mathrm{Fil}_{z,d,w}^{\mathrm{Z,D,W}} \mathcal{Z}_q^f\right) = \mathrm{Fil}_{z,d,w}^{\mathrm{Z,D,W}} \mathcal{Z}_q.$$

We refer to the translation from bi-brackets to the SZ-model (see [14, Theorem 13]) to obtain the statement of Conjecture 1.44 written in the SZ-model, and strengthened under consideration of Conjecture 1.43.

**Conjecture 1.49** (Bachmann, Conjecture 1.44 strengthened)**.** *For all $z, d, w \in \mathbb{Z}_{>0}$, we have*

$$\mathrm{Fil}_{z,d,w}^{\mathrm{Z,D,W}} \mathcal{Z}_q^f \subset \mathrm{Fil}_{z+d,w}^{\mathrm{D,W}} \mathcal{Z}_q^{f,\circ}.$$

*In particular, we have $\mathcal{Z}_q^f = \mathcal{Z}_q^{f,\circ}$.*

Note that Conjecture 1.49 indeed is a strengthened version of Conjecture 1.44 by definition of formal $q$MZVs since they fulfill - by definition - no other relations than SZ-$q$MZVs do, i.e., proving Conjecture 1.49 would imply a proof of Conjecture 1.44 directly.

In the following, we give an overview of known results regarding Conjecture 1.44 and what their analogue regarding Bachmann's Conjecture 1.49 looks like. First, we give the statements using Bachmann's model of bi-brackets since most of them originally were stated in this model.

**Theorem 1.50.** *(i) [3, Proposition 4.4] For all $k \in \mathbb{Z}_{>0}$, $z \in \mathbb{Z}_{\geq 0}$, we have*

$$\mathrm{g}\binom{k}{z} \in \mathrm{Fil}^{\mathrm{D,W}}_{z+1,k+z}\,\mathcal{Z}_q^{\circ}.$$

*(ii) [3, Proposition 5.9] For all $k_1, k_2 \in \mathbb{Z}_{>0}$, we have*

$$\mathrm{g}\binom{k_1,k_2}{1,0},\ \mathrm{g}\binom{k_1,k_2}{0,1} \in \mathrm{Fil}^{\mathrm{D,W}}_{3,k_1+k_2+1}\,\mathcal{Z}_q^{\circ}.$$

*(iii) [44, Theorem 5.3] For all $k_1, k_2 \in \mathbb{Z}_{>0}$, $z_1, z_2 \in \mathbb{Z}_{\geq 0}$ with $k_1+k_2+z_1+z_2$ odd, we have*

$$\mathrm{g}\binom{k_1,k_2}{z_1,z_2} \in \mathcal{Z}_q^{\circ}.$$

*(iv) [21, Theorem 6.4] For all $k_1, \dots, k_d \in \mathbb{Z}_{>0}$, and for all $1 \leq j \leq d$, we have*

$$\mathrm{g}\binom{k_1,\dots,k_j,\dots,k_d}{0,\dots,1,\dots,0} \in \mathrm{Fil}^{\mathrm{D,W}}_{d+1,k_1+\cdots+k_d+1}\,\mathcal{Z}_q^{\circ}.$$

As mentioned, we will rephrase these known results using the model introduced by Schlesinger and Zudilin and the translation from [14, Theorem 13].

**Theorem 1.51** (Theorem 1.50 rewritten)**.** *(i) For all $k \in \mathbb{Z}_{>0}$, $z \in \mathbb{Z}_{\geq 0}$, we have*

$$\zeta_q^{\mathrm{f}}\left(u_k u_0^z\right) \in \mathrm{Fil}^{\mathrm{D,W}}_{z+1,k+z}\,\mathcal{Z}_q^{f,\circ}.$$

*(ii) For all $k_1, k_2 \in \mathbb{Z}_{>0}$, we have*

$$\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}\right),\ \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0\right) \in \mathrm{Fil}^{\mathrm{D,W}}_{3,k_1+k_2+1}\,\mathcal{Z}_q^{f,\circ}.$$

*(iii) For all $k_1, k_2 \in \mathbb{Z}_{>0}$, $z_1, z_2 \in \mathbb{Z}_{\geq 0}$ with $k_1+k_2+z_1+z_2$ odd, we have that the formal qMZV $\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0^{z_1}u_{k_2}u_0^{z_2}\right)$ is an element of $\mathcal{Z}_q^{f,\circ}$ up to lower weight terms in depth 2 with at most $z_1+z_2$ zeros each.*

*(iv) For all $k_1, \dots, k_d \in \mathbb{Z}_{>0}$ and for all $1 \leq j \leq d$, we have*

$$\zeta_q^{\mathrm{f}}\left(u_{k_1}\cdots u_{k_j}u_0u_{k_{j+1}}\cdots u_{k_d}\right) \in \mathrm{Fil}^{\mathrm{D,W}}_{d+1,k_1+\cdots+k_d+1}\,\mathcal{Z}_q^{f,\circ}.$$

For a translation of bi-brackets to the SZ-model, we refer to [14]. From there, the lower weight terms mentioned in Theorem 1.51(iii) can be deduced explicitly.

### 1.2.2 Statement of results

We present in the following the main results of the work in Chapter 2. The first contributes to the general idea of an approach to proving Bachmann's Conjecture 1.49, which is showing $\zeta_q^{\mathrm{f}}(\mathtt{W}) \in \mathrm{Fil}^{\mathrm{Z}}_{\mathrm{zero}(\mathtt{W})-1}\,\mathcal{Z}_q^f$ for every $\mathtt{W} \in \mathcal{U}^{*,\circ}$ with $\mathrm{zero}(\mathtt{W}) \geq 1$. Furthermore, it generalizes results by Bachmann ([3, Proposition 4.4], see Proposition 2.21) and Burmester ([21, Theorem 6.4], see Corollary 2.28).

**Theorem 1.52** (Theorem 2.6)**.** *Let be $z, d \in \mathbb{Z}_{>0}$, $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}_{>0}^d$, and let be $1 \leq j_1 \leq j_2 \leq d$. Deconcatenate $\mathbf{k}$ as*

$$\mathbf{k}_{(1;j_1)} = (k_1, \ldots, k_{j_1}),\ \mathbf{k}_{(j_1+1;j_2)} = (k_{j_1+1}, \ldots, k_{j_2}),\ \mathbf{k}_{(j_2+1;d)} = (k_{j_2+1}, \ldots, k_d).$$

*We have*

$$\zeta_q^{\mathrm{f}}\left(u_{\mathbf{k}_{(1;j_1)}}\left(u_{\mathbf{k}_{(j_1+1;j_2)}} * u_{\mathbf{k}_{(j_2+1;d)}} u_0^z\right)\right) \in \sum_{s=1}^{z} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w} \mathcal{Z}_q^f,$$

*where $w = |\mathbf{k}| + z$. In particular, for all $1 \leq j \leq d$, deconcatenating the index $\mathbf{k}$ as $\mathbf{k}_{(1;j-1)} = (k_1, \ldots, k_{j-1})$, $(k_j, \ldots, k_d)$, we have*

$$\sum_{\substack{\ell_j, \ldots, \ell_d \geq 0 \\ \ell_j + \cdots + \ell_d = z}} \zeta_q^{\mathrm{f}}\left(u_{\mathbf{k}_{(1;j-1)}} u_{k_j} u_0^{\ell_j} \cdots u_{k_d} u_0^{\ell_d}\right) \in \sum_{s=1}^{z} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w} \mathcal{Z}_q^f.$$

Extending our methods used for the proof of Theorem 1.52, we observe the following result.

**Theorem 1.53** (Theorem 2.8)**.** *Bachmann's Conjecture 1.49 is true for all triples of positive integers $(z, d, w) \in \mathbb{Z}_{>0}^3$ with $z + d \leq 6$.*

**Remark 1.54.** Note that Theorem 1.53 is indeed independent of the weight $w$. Therefore, Theorem 1.53 is a generalization of Theorem 1.51(ii) and, in parts, a generalization of (iii) and (iv) of the same theorem. We refer to Example 1.64 for a first example regarding Theorem 1.53.

Theorem 1.53, e.g., will be proven in several steps. In this way, we obtain the following conjecture that strengthens Bachmann's Conjecture 1.49 (see Lemma 2.68 for proof of this statement). To state it, we denote

$$\mathrm{F}_{z,d,w} := \mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w-1} \mathcal{Z}_q^f + \sum_{\substack{z'+d'=z+d-1 \\ 0 \leq z' \leq z}} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z',d',w} \mathcal{Z}_q^f$$

for all $z, d, w \in \mathbb{Z}_{>0}$.

**Conjecture 1.55** (Refined Bachmann Conjecture, Conjecture 2.10)**.** *For all triples of positive integers $(z, d, w) \in \mathbb{Z}_{>0}^3$, we have*

$$\mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w} \mathcal{Z}_q^f \subset \mathrm{F}_{z,d,w}.$$

The following is a particular result regarding the refined Bachmann Conjecture 1.55.

**Theorem 1.56** (Theorem 2.12)**.** *The refined Bachmann Conjecture 1.55 is true for all $(z, d, w) \in \mathbb{Z}_{>0}$ with $1 \leq d \leq 4$.*

Together with Lemma 2.68, the impact of Theorem 1.56 is that one can prove Bachmann's Conjecture 1.49 now almost for $z + d \leq 7$ (extending Theorem 1.53) in the sense that just the proof for $(2, 5, w) \in \mathbb{Z}_{>0}^3$ of the refined Bachmann Conjecture 1.55 remains to be done.

### 1.2.3 Our approach to the refined Bachmann Conjecture

In the following, we present our approach to the refined Bachmann Conjecture 1.55. In general, we will use $\mathbb{Q}$-linear relations only that are implied by

$$\zeta_q^{\mathrm{f}}\left(\mathtt{W}_1 * (\mathtt{W}_2 - \tau(\mathtt{W}_2))\right) = 0 \tag{1.56.1}$$

for any words $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^{*,\circ}$. At this point, note that

$$\mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w} \mathbb{Q}\langle\mathcal{U}\rangle^\circ * \mathrm{Fil}^{\mathrm{Z,D,W}}_{z',d',w'} \mathbb{Q}\langle\mathcal{U}\rangle^\circ \subset \mathrm{Fil}^{\mathrm{Z,D,W}}_{z+z',d+d',w+w'} \mathbb{Q}\langle\mathcal{U}\rangle^\circ$$

and

$$\tau\left(\mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w} \mathbb{Q}\langle\mathcal{U}\rangle^\circ\right) = \mathrm{Fil}^{\mathrm{Z,D,W}}_{w-z-d,d,w} \mathbb{Q}\langle\mathcal{U}\rangle^\circ$$

for all $z, z', d, d', w, w' \in \mathbb{Z}_{\geq 0}$. Hence, considering (1.56.1), $\mathtt{W}_1 * \mathtt{W}_2$ and $\mathtt{W}_1 * \tau(\mathtt{W}_2)$ are, in general, in different filtrations of $\mathbb{Q}\langle\mathcal{U}\rangle^\circ$ regarding the number of zeros since, in general $z \neq w - z - d$ (we precise this observation in the following proposition). Therefore, for given $\mathtt{W} \in \mathcal{U}^{*,\circ}$, it is difficult to find the minimal $z \in \mathbb{Z}_{\geq 0}$ such that $\zeta_q^{\mathrm{f}}(\mathtt{W}) \in \mathrm{Fil}^{\mathrm{Z}}_z \mathcal{Z}_q^f$.

**Proposition 1.57** (Proposition 2.13)**.** *Let be $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^{*,\circ}$ and write*

$$z = \mathrm{zero}(\tau(\mathtt{W}_1)) + \mathrm{zero}(\tau(\mathtt{W}_2)),\ d_1 = \mathrm{depth}(\mathtt{W}_1),\ d_2 = \mathrm{depth}(\mathtt{W}_2),\ w = \mathrm{wt}(\mathtt{W}_1) + \mathrm{wt}(\mathtt{W}_2).$$

*Then, for $0 \leq s \leq \min\{d_1, d_2\}$, there are uniquely determined*

$$\mathcal{L}_{\max\{d_1,d_2\}+s} \in \mathrm{span}_{\mathbb{Q}} \left\{\mathtt{W} \in \mathcal{U}^{*,\circ} \mid \mathrm{depth}(\mathtt{W}) = s + \max\{d_1, d_2\}\right\}$$

*such that*

$$\mathtt{W}_1 * \mathtt{W}_2 = \sum_{s=0}^{\min\{d_1,d_2\}} \mathcal{L}_{\max\{d_1,d_2\}+s}.$$

*Furthermore, for all $0 \leq s \leq \min\{d_1, d_2\}$, we have*

$$\tau\left(\mathcal{L}_{\max\{d_1,d_2\}+s}\right) \in \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,\max\{d_1,d_2\}+s,w} \mathbb{Q}\langle\mathcal{U}\rangle^\circ.$$

*In particular, $\tau\left(\mathcal{L}_{\max\{d_1,d_2\}}\right)$ is the part of $\tau(\mathtt{W}_1 * \mathtt{W}_2)$ having the maximum number of zeros and we have*

$$\tau(\mathtt{W}_1 * \mathtt{W}_2) \in \sum_{s=0}^{\min\{d_1,d_2\}} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,\max\{d_1,d_2\}+s,w} \mathbb{Q}\langle\mathcal{U}\rangle^\circ.$$

Let us consider an example to point out the statement of Proposition 1.57.

**Example 1.58** (Example 2.14)**.** Choose $\mathtt{W}_1 = u_2$, $\mathtt{W}_2 = u_1u_2$, i.e., $d = 2$ in the notion of Proposition 1.57. We have

$$\mathtt{W}_1 * \mathtt{W}_2 = \underbrace{u_3u_2 + u_1u_4}_{=\mathcal{L}_2} + \underbrace{u_2u_1u_2 + 2u_1u_2u_2}_{=\mathcal{L}_3}.$$

Observe

$$\tau(\mathcal{L}_2) = u_1u_0u_1u_0u_0 + u_1u_0u_0u_0u_1, \quad \tau(\mathcal{L}_3) = u_1u_0u_1u_1u_0 + 2u_1u_0u_1u_0u_1.$$

We see that $\tau(\mathcal{L}_2)$ indeed is the part of $\tau(u_2 * u_1u_2)$ having the maximum number of zeros.

Since - regarding Bachmann's Conjecture 1.49 and regarding the refined Bachmann Conjecture 1.55 - we want to 'reduce the number of zeros', we often will be interested in the part of the stuffle product only that has the maximum number of zeros. Therefore, Proposition 1.57 motivates the definition of the *box product* that extracts this part of the stuffle product.

**Definition 1.59** (Box product, Definition 2.15)**.** We define the $\mathbb{Q}$-bilinear *box product* $\circledast\colon \mathbb{Q}\langle\mathcal{U}\rangle^\circ \times \mathbb{Q}\langle\mathcal{U}\rangle^\circ \to \mathbb{Q}\langle\mathcal{U}\rangle^\circ$ as follows: For $\mathtt{W}_j \in \mathcal{U}^{*,\circ}$ with $\mathrm{depth}(\mathtt{W}_j) = d_j$, where $j \in \{1,2\}$, we set

$$\mathtt{W}_1 \boxast \mathtt{W}_2 := \mathcal{L}_{\max\{d_1,d_2\}}$$

in the notion of Proposition 1.57.

We continue Example 1.58 regarding the box product.

**Example 1.60** (Example 2.16)**.** Choose $\mathtt{W}_1 = u_2$, $\mathtt{W}_2 = u_1u_2$. We have

$$\mathtt{W}_1 \boxast \mathtt{W}_2 = u_2 \boxast u_1u_2 = u_3u_2 + u_1u_4,$$

which is exactly $\mathcal{L}_2$ of Example 1.58, i.e., after applying $\tau$, one indeed obtains the part of the stuffle product $u_2 * u_1u_2$ having maximum number of zeros.

We refer to Section 2.4 of Chapter 2 for a detailed investigation. Let us consider an example of how the box product may help us concerning the refined Bachmann Conjecture 1.55.

**Example 1.61.** Fix $k_1,k_2,k_3 \in \mathbb{Z}_{>0}$ and set $\mathtt{W}_1 := u_1u_1$, $\mathtt{W}_2 := u_{k_1}u_{k_2}u_{k_3}$. Clearly, we have $\zeta_q^{\mathrm{f}}(\mathtt{W}_1 * \mathtt{W}_2) \in \mathrm{Fil}^{\mathrm{D,W}}_{5,w}\,\mathcal{Z}_q^{f,\circ} \subset \mathrm{F}_{1,5,w}$ (where $w := k_1+k_2+k_3+2$). Using $\tau$-invariance of formal $q$MZVs, we have, modulo $\mathrm{Fil}^{\mathrm{Z,D,W}}_{1,5,w}\,\mathcal{Z}_q^f$,

$$\begin{aligned}
&\zeta_q^{\mathrm{f}}(\mathtt{W}_1 * \mathtt{W}_2)\\
\equiv\,&\zeta_q^{\mathrm{f}}\left(u_1u_1 * u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right)\\
\equiv\,&\zeta_q^{\mathrm{f}}\left(u_2u_0^{k_3-1}u_2u_0^{k_2-1}u_1u_0^{k_1-1} + u_2u_0^{k_3-1}u_1u_0^{k_2-1}u_2u_0^{k_1-1} + u_1u_0^{k_3-1}u_2u_0^{k_2-1}u_2u_0^{k_1-1}\right)\\
\equiv\,&\zeta_q^{\mathrm{f}}(u_{k_1}u_{k_2}u_0u_{k_3}u_0) + \zeta_q^{\mathrm{f}}(u_{k_1}u_0u_{k_2}u_{k_3}u_0) + \zeta_q^{\mathrm{f}}(u_{k_1}u_0u_{k_2}u_0u_{k_3}) \qquad (1.61.1)\\
\equiv\,&\Psi_{(k_1,k_2,k_3)}(u_1u_1 \boxast u_1u_1u_1),
\end{aligned}$$

where we set (see Definition 2.54 for a generalized definition), for all $\ell_1,\ell_2,\ell_3 \in \mathbb{Z}_{>0}$,

$$\Psi_{(k_1,k_2,k_3)}(u_{\ell_1}u_{\ell_2}u_{\ell_3}) := u_{\ell_1}u_0^{k_3-1}u_{\ell_2}u_0^{k_2-1}u_{\ell_3}u_0^{k_1-1}.$$

We see in this way that the linear combination from (1.61.1) is an element of $\mathrm{F}_{1,5,w}$ already, although all the three words displayed there are in $\mathrm{Fil}^{\mathrm{Z,D,W}}_{2,3,w}\,\mathbb{Q}\langle\mathcal{U}\rangle^\circ$.

Example 1.61 shows that (non-trivial) box-products will be important. With the notion of

$$\mathcal{P} := \mathrm{span}_{\mathbb{Q}}\left\{\mathtt{W}_1 \boxast \mathtt{W}_2 \mid \mathtt{W}_1,\, \mathtt{W}_2 \in (\mathcal{U}\backslash\{u_0\})^*,\, \mathtt{W}_1,\mathtt{W}_2 \neq \mathbf{1}\right\} \subset \mathbb{Q}\langle\mathcal{U}\backslash\{u_0\}\rangle$$

(see also Definition 2.30), we obtain the following generalization of Example 1.61 which is our main approach towards the refined Bachmann Conjecture 1.55.

**Lemma 1.62** (Lemma 2.56). *Let be* $\mathbf{z} = (z_d, \dots, z_1) \in \mathbb{Z}_{>0}^d$ *such that* $u_{\mathbf{z}} \in \mathcal{P}$. *Then, for every word* $\mathtt{W} = u_{k_1} u_0^{z_1-1} \cdots u_{k_d} u_0^{z_d-1}$ *with* $k_1, \dots, k_d \in \mathbb{Z}_{>0}$ *arbitrary, we have*

$$\zeta_q^{\mathrm{f}}(\mathtt{W}) \in \sum_{1 \le s \le \min\{z,d\}} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w} \mathcal{Z}_q^f \subset \mathrm{F}_{z,d,w},$$

*where* $z = \mathrm{zero}(\mathtt{W})$ *and* $w = \mathrm{wt}(\mathtt{W})$.

**Approach to the refined Bachmann Conjecture 1.55 in case $z \ge d$.** In the case $z \ge d$, the approach to the refined Bachmann Conjecture 1.55 melts down to investigate the box products. More precisely, with the notion of

$$\mathfrak{T}_{z,d} := \mathrm{span}_{\mathbb{Q}} \left\{ u_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathbb{Z}_{>0}^d,\ |\boldsymbol{\mu}| = z + d \right\}.$$

for all $z, d \in \mathbb{Z}_{>0}$, we need to prove Conjecture 2.39 which claims in case $z \ge d$ that

$$\mathfrak{T}_{z,d} = \mathfrak{T}_{z,d} \cap \mathcal{P}.$$

I.e., if the Conjecture 2.39 is true for $z \ge d$, then we obtain $\mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w} \mathcal{Z}_q^f \subset \mathrm{F}_{z,d,w}$ for all $w \in \mathbb{Z}_{>0}$ immediately from Lemma 1.62. What is remarkable about this approach is that it is independent of $w$.

**Approach to the refined Bachmann Conjecture 1.55 in case $z < d$.** In the case $z < d$, we will extend our approach towards the refined Bachmann Conjecture 1.55 since then, we conjecturally (see Conjecture 2.39) have $\mathfrak{T}_{z,d} \cap \mathcal{P} \subsetneq \mathfrak{T}_{z,d}$. We will present in the Outlook of Chapter 2 (Section 2.7) an approach that conjecturally works. To prove our main results, we will not use this approach in its abstract form. However, we will consider some $\mathbb{Q}$-linear combinations of formal Multiple Zeta Values in $\mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w} \mathcal{Z}_q^f$ explicitly that arise from stuffle products again.

More precisely, we fix $z, d, w \in \mathbb{Z}_{\ge 0}$ with $z < d$ in the following and assume that

$$\mathrm{Fil}^{\mathrm{Z,D,W}}_{\tilde{z},\tilde{d},\tilde{w}} \mathcal{Z}_q^f \subset \mathrm{F}_{\tilde{z},\tilde{d},\tilde{w}}$$

for $\tilde{z} \le z,\ \tilde{d} < d,\ \tilde{w} < w$ is proven already. Let us consider an example of such a linear combination of formal $q$MZVs arising from a stuffle product we consider in case $z < d$ additionally to the ones from the approach for case $z \ge d$.

**Example 1.63.** Let be $k_1, k_2, k_3 \in \mathbb{Z}_{>0}$ and denote $w = k_1 + k_2 + k_3 + 2$ in the following. First, we see that $\zeta_q^{\mathrm{f}}(u_2 * u_{k_1} u_{k_2} u_{k_3}) \in \mathrm{F}_{1,3,w}$. Furthermore, we have

$$\begin{aligned}
&\zeta_q^{\mathrm{f}}(\tau(\tau(u_2) * \tau(u_{k_1} u_{k_2} u_{k_3}))) \\
=&\zeta_q^{\mathrm{f}}\left(\tau\left(u_1 u_0 * u_1 u_0^{k_3-1} u_1 u_0^{k_2-1} u_1 u_0^{k_1-1}\right)\right) \\
\equiv&\zeta_q^{\mathrm{f}}\Big(\tau\Big(k_3 u_2 u_0^{k_3} u_1 u_0^{k_2-1} u_1 u_0^{k_1-1} + k_2 u_2 u_0^{k_3-1} u_1 u_0^{k_2} u_1 u_0^{k_1-1} \\
&\quad + k_2 u_1 u_0^{k_3-1} u_2 u_0^{k_2} u_1 u_0^{k_1-1} + k_1 u_2 u_0^{k_3-1} u_1 u_0^{k_2-1} u_1 \\
&\quad + k_1 u_1 u_0^{k_3-1} u_2 u_0^{k_2-1} u_1 + k_1 u_1 u_0^{k_3-1} u_1 u_0^{k_2-1} u_2 u_0^{k_1}\Big)\Big) \mod \mathrm{F}_{1,3,w} \\
\equiv& k_3 \zeta_q^{\mathrm{f}}(u_{k_1} u_{k_2} u_{k_3+1} u_0) + k_2 \zeta_q^{\mathrm{f}}(u_{k_1} u_{k_2+1} u_{k_3} u_0) + k_2 \zeta_q^{\mathrm{f}}(u_{k_1} u_{k_2+1} u_0 u_{k_3}) \\
&\quad + k_1 \zeta_q^{\mathrm{f}}(u_{k_1+1} u_{k_2} u_{k_3} u_0) + k_1 \zeta_q^{\mathrm{f}}(u_{k_1+1} u_{k_2} u_0 u_{k_3}) + k_1 \zeta_q^{\mathrm{f}}(u_{k_1+1} u_0 u_{k_2} u_{k_3}) \mod \mathrm{F}_{1,3,w}.
\end{aligned}$$

I.e., the latter linear combination of formal $q$MZVs is in $\mathrm{F}_{1,3,w}$. In comparison to (1.61.1), it stands out that the shape of the latter linear combination does dependent on $(k_1, k_2, k_3)$ in the sense that the coefficients are not independent of $(k_1, k_2, k_3)$ and that occurring words do not have the same non-$u_0$-letters in the same order. Nevertheless, all occurring words $u_{k_1'} u_0^{z_1} u_{k_2'} u_0^{z_2} u_{k_3'} u_0^{z_3}$ satisfy $k_j' \geq k_j$ and $\sum_{j=1}^{3} (k_j' - k_j) = 1 \leq 2 = d - z$, when $z = 1$ and $d = 3$ denote the number of zeros and depth, respectively, of each of such words. We will generalize this observation briefly in Section 2.7.

Let us consider a small example of how the proof of Theorem 1.53 works in case $z < d$ using the ideas described.

**Example 1.64.** Let be $k_1, k_2, k_3 \in \mathbb{Z}_{>0}$. Consider $\zeta_q^{\mathrm{f}}(u_{k_1} u_0 u_{k_2} u_0 u_{k_3}) \in \mathrm{Fil}^{\mathrm{Z,D,W}}_{2,3,w} \mathcal{Z}_q^f$, where $w = k_1 + k_2 + k_3 + 2$. In the following, we prove that this formal $q$MZV is an element of $\mathcal{F} := \mathrm{Fil}^{\mathrm{Z,D,W}}_{0,5,w} \mathcal{Z}_q^f + \mathrm{Fil}^{\mathrm{Z,D,W}}_{1,4,w} \mathcal{Z}_q^f \subset \mathrm{F}_{2,3,w}$. Particularly, by Theorem 1.53(iv), then it will be proven that $\zeta_q^{\mathrm{f}}(u_{k_1} u_0 u_{k_2} u_0 u_{k_3}) \in \mathrm{Fil}^{\mathrm{D,W}}_{5,w} \mathcal{Z}_q^{f,\circ}$. Note that this is not a direct consequence of the known results (Theorem 1.53) since $u_{k_1} u_0 u_{k_2} u_0 u_{k_3}$ has depth 3 and more than one $u_0$ which will be also the case after applying $\tau$ (if $k_1 + k_2 + k_3 > 4$). In the following, we will use the calculation from Example 1.61 and similar ones. We use $\tau$-invariance in each of the following steps to obtain

$$
\begin{aligned}
&\zeta_q^{\mathrm{f}}(u_{k_1} u_0 u_{k_2} u_0 u_{k_3}) = \zeta_q^{\mathrm{f}}\left(u_1 u_0^{k_3-1} u_2 u_0^{k_2-1} u_2 u_0^{k_1-1}\right) \\
\equiv\, &\zeta_q^{\mathrm{f}}\left(u_1 u_1 * u_1 u_0^{k_3-1} u_1 u_0^{k_2-1} u_1 u_0^{k_1-1}\right) - \zeta_q^{\mathrm{f}}\left(u_1 * u_2 u_0^{k_3-1} u_1 u_0^{k_2-1} u_1 u_0^{k_1-1}\right) \\
&+ \zeta_q^{\mathrm{f}}\left(u_3 u_0^{k_3-1} u_1 u_0^{k_2-1} u_1 u_0^{k_1-1}\right) && \mod \mathcal{F} \\
\equiv\, &\zeta_q^{\mathrm{f}}(u_1 u_1 * u_{k_1} u_{k_2} u_{k_3}) - \zeta_q^{\mathrm{f}}(u_1 * u_{k_1} u_{k_2} u_{k_3} u_0) \\
&+ \zeta_q^{\mathrm{f}}(u_{k_1} u_{k_2} u_{k_3} u_0 u_0) && \mod \mathcal{F}.
\end{aligned}
$$

Now, the first two summands are in $\mathcal{F}$ since in the stuffle product, the number of $u_0$'s does not increase. Hence,

$$
\zeta_q^{\mathrm{f}}(u_{k_1} u_0 u_{k_2} u_0 u_{k_3}) \equiv \zeta_q^{\mathrm{f}}(u_{k_1} u_{k_2} u_{k_3} u_0 u_0) \mod \mathcal{F}. \tag{1.64.1}
$$

Now, note that $\zeta_q^{\mathrm{f}}(u_{k_3} u_0 u_0) \in \mathrm{Fil}^{\mathrm{D,W}}_{3,k_3+2} \mathcal{Z}_q^{f,\circ}$ due to Theorem 1.51(i), i.e., we already have $\zeta_q^{\mathrm{f}}(u_{k_1} u_{k_2} * u_{k_3} u_0 u_0) \in \mathrm{Fil}^{\mathrm{D,W}}_{5,w} \mathcal{Z}_q^{f,\circ} \subset \mathcal{F}$. Therefore, we have

$$
\begin{aligned}
&\zeta_q^{\mathrm{f}}(u_{k_1} u_{k_2} u_{k_3} u_0 u_0) \\
=\, &\zeta_q^{\mathrm{f}}(u_{k_1} u_{k_2} * u_{k_3} u_0 u_0) - \zeta_q^{\mathrm{f}}(u_{k_3}(u_{k_1} u_{k_2} * u_0 u_0)) \\
&- \zeta_q^{\mathrm{f}}(u_{k_1} u_{k_3}(u_{k_2} * u_0 u_0)) - \zeta_q^{\mathrm{f}}(u_{k_1+k_3}(u_{k_2} * u_0 u_0)) - \zeta_q^{\mathrm{f}}(u_{k_1} u_{k_2+k_3} u_0 u_0) \\
\equiv\, &\zeta_q^{\mathrm{f}}\left(u_{k_1} u_{k_2} * u_3 u_0^{k_3-1}\right) - \zeta_q^{\mathrm{f}}\left((u_2 + u_1 u_1) * u_1 u_0^{k_2-1} u_1 u_0^{k_1-1} u_1 u_0^{k_3-1}\right) \\
&- \zeta_q^{\mathrm{f}}\left((u_2 + u_1 u_1) * u_1 u_0^{k_2-1} u_1 u_0^{k_3-1} u_1 u_0^{k_1-1} - u_1 * u_1 u_0^{k_2-1} u_1 u_0^{k_3-1} u_2 u_0^{k_1-1}\right) \\
&- \zeta_q^{\mathrm{f}}\left((u_2 + u_1 u_1) * u_1 u_0^{k_2-1} u_1 u_0^{k_1+k_3-1}\right) - \zeta_q^{\mathrm{f}}\left(u_3 u_0^{k_2+k_3-1} u_1 u_0^{k_1-1}\right) && \mod \mathcal{F} \\
\equiv\, &\zeta_q^{\mathrm{f}}\left(u_{k_1} u_{k_2} * u_2 * u_1 u_0^{k_3-1}\right) - \zeta_q^{\mathrm{f}}((u_2 + u_1 u_1) * u_{k_3} u_{k_1} u_{k_2}) \\
&- \zeta_q^{\mathrm{f}}((u_2 + u_1 u_1) * u_{k_2} u_{k_3} u_{k_1} - u_1 * u_{k_2} u_0 u_{k_3} u_{k_1}) - \zeta_q^{\mathrm{f}}((u_2 + u_1 u_1) * u_{k_1+k_3} u_{k_2}) \\
&- \zeta_q^{\mathrm{f}}\left(u_1 * u_2 u_0^{k_2+k_3-1} u_1 u_0^{k_1-1} - u_1 u_1 * u_1 u_0^{k_2+k_3-1} u_1 u_0^{k_1-1}\right) && \mod \mathcal{F} \\
\equiv\, &\zeta_q^{\mathrm{f}}(u_{k_1} u_{k_2} * u_2 * u_{k_3}) - \zeta_q^{\mathrm{f}}(u_1 * u_{k_1} u_{k_2+k_3} u_0) + \zeta_q^{\mathrm{f}}(u_1 u_1 * u_{k_1} u_{k_2+k_3}) && \mod \mathcal{F}.
\end{aligned}
$$

The latter linear combination of formal $q$MZVs clearly is in $\mathcal{F}$. I.e., we have proven that $\zeta_q^{\mathrm{f}}(u_{k_1}u_0u_{k_2}u_0u_{k_3}) \in \mathcal{F}$. Now, note the inclusion $\mathrm{Fil}^{\mathrm{Z,D,W}}_{1,4,w}\,\mathcal{Z}_q^f \subset \mathrm{Fil}^{\mathrm{D,W}}_{5,w}\,\mathcal{Z}_q^{f,\circ}$ (by Burmester's Theorem 1.51(iv)), we have $\zeta_q^{\mathrm{f}}(u_{k_1}u_0u_{k_2}u_0u_{k_3}) \in \mathrm{Fil}^{\mathrm{D,W}}_{5,w}\,\mathcal{Z}_q^{f,\circ}$. Moreover, due to (1.64.1), we also have shown that $\zeta_q^{\mathrm{f}}(u_{k_1}u_{k_2}u_{k_3}u_0u_0) \in \mathrm{Fil}^{\mathrm{D,W}}_{5,w}\,\mathcal{Z}_q^{f,\circ}$.

### 1.2.4 Outlook

Besides the connection of the box product with $\mathcal{Z}_q^f$ and its structure, the box product seems interesting due to its straightforward definition. Nevertheless, it seems it did not occur so far in the literature. In particular, for future work, it would be interesting to investigate where the box product can be used, e.g., in combinatorics. Furthermore, for future work, it would be of interest to understand the box product completely in the sense that one could prove Conjecture 2.39 (and its refinement, Conjecture 2.58). Partial results regarding Conjecture 2.39 will be presented in Chapter 2. Similar to Example 1.64 and Proposition 2.21, our approach to the refined Bachmann Conjecture 1.55 is usable to derive explicit formulas for every $\zeta_q^{\mathrm{f}}(\mathtt{W})$ with $\mathtt{W} \in \mathcal{U}^{*,\circ}$ and $\mathrm{zero}(\mathtt{W}) \geq 1$ as linear combination of (products of) elements in $\mathcal{Z}_q^{f,\circ}$.

## 1.3 The combinatorial side of Multiple $q$-Zeta Values considered in Paper II

This section introduces the work that builds Chapter 3. It is the combinatorial view on Multiple $q$-Zeta Values of this thesis. Section 1.3.1 introduces the notion of partitions and Stanley coordinates needed in Section 1.3.2 to introduce marked partitions and their connection to Multiple $q$-Zeta Values. Furthermore, in Section 1.3.3, the results of Chapter 3 are presented.

By using geometric series expansion, for appropriate $a_{n,k,\ell} \in \mathbb{Z}_{\geq 0}$, one obtains

$$\frac{q^{m\ell}}{(1-q^m)^k} = \sum_{n\geq 0} a_{n,k,\ell}\, q^{nm}$$

for all positive integers $m$ and non-negative integers $k$ and $\ell$. Therefore, every $q$MZV, seen as a formal $q$-series, is of shape

$$\sum_{\substack{m_1>\cdots>m_d>0\\ n_1,\ldots,n_d>0}} c_{m_1,\ldots,m_d}\, q^{m_1n_1+\cdots+m_dn_d}.$$

Hence, we can view the $q$MZV as the generating series of partitions, given in Stanley coordinates (see Section 1.3.1), with multiplicities $c_{m_1,\ldots,m_d}$, depending on the several parts $m_1,\ldots,m_d$ of the partition, but not on their multiplicities $n_1,\ldots,n_d$. This is part of Section 1.3.2 and is mainly based on [14]. The main result of this section will be Theorem 1.84 describing the stuffle product on the level of marked partitions and can be found in Section 1.3.3.

### 1.3.1 Partitions and Stanley coordinates

This section aims to introduce Stanley coordinates as described in [42]. For this, we first clarify what a partition of a positive integer is.

**Definition 1.65.** Given a positive integer $N$. A partition $\lambda$ of $N$ is a non-increasing sequence $(\lambda_1, \dots, \lambda_r)$ of positive integers summing to $N$, i.e.,

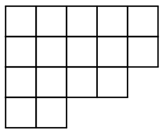
$$\lambda_1 \geq \dots \geq \lambda_r, \qquad |\lambda| := \sum_{j=1}^{r} \lambda_j = N.$$

For $N = 0$, the unique partition is $\lambda = \emptyset$.

A visualization of a partition is given by the Young Tableau, also known as the Ferres diagram. For a reference and more details, see [28].

**Definition 1.66.** Given a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of $N \in \mathbb{Z}_{>0}$. The Young Tableau of $\lambda$ is obtained by drawing, left aligned and each other, $\lambda_1, \dots, \lambda_r$ boxes to the right.

**Example 1.67.** Consider the partition $\lambda = (5, 5, 4, 2)$ of $N = 5 + 5 + 4 + 2 = 16$. Then,

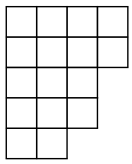


is the Young Tableau of $\lambda$.

We need the following definition since we will describe $\tau$ on a combinatorial level.

**Definition 1.68.** Given a partition $\lambda$ of some positive integer $N$. The *conjugated partition* of $\lambda$ has the Young Tableau of $\lambda$ reflected at the diagonal as the Young Tableau.

**Example 1.69.** Take the partition $\lambda = (5, 5, 4, 2)$ from Example 1.67. The Young Tableau reflected at the main diagonal is the following.



I.e., the conjugated partition of $\lambda$ is $(4, 4, 3, 3, 2)$.

Next, we define Stanley coordinates of a partition. With those, one can describe the conjugated partition very well; see Proposition 1.72 below.

**Definition 1.70.** Given a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of a positive integer $N$. Denote by $m_j \in \mathbb{Z}_{>0}$ the $j$-th largest value of $\lambda$ and by $n_j \in \mathbb{Z}_{>0}$ the multiplicity of $m_j$ occurring in $\lambda$. I.e., if $\lambda$ consists of $d$ different integers, we have

$$m_1 > \dots > m_d > 0, \qquad \sum_{j=1}^{d} m_j n_j = N.$$

The pair of indices $((m_1, \dots, m_d), (n_1, \dots, n_d))$ is called *Stanley coordinates* of the partition $N$.

**Example 1.71.** The Stanley coordinates of the partition $\lambda = (5, 5, 4, 2)$, considered in Example 1.67, are

$$(m_1, m_2, m_3) = (5, 4, 2), \qquad (n_1, n_2, n_3) = (2, 1, 1).$$

In Stanley coordinates, it is easy to express the conjugated partition, as we will see now.

**Proposition 1.72.** *Given a partition $\lambda$ with Stanley coordinates $(\mathbf{m},\mathbf{n}) \in \mathbb{Z}^d_{>0} \times \mathbb{Z}^d_{>0}$. The conjugated partition of $\lambda$ has Stanley coordinates*

$$((n_1+\cdots+n_d, n_1+\cdots+n_{d-1},\ldots,n_1),(m_d, m_{d-1}-m_d,\ldots,m_1-m_2)) .$$

### 1.3.2 $q$MZVs as generating series of marked partitions

As mentioned in the introduction of this section, we can interpret $q$MZVs as generating series of partitions, particularly when considering their Stanley coordinates. In the following, we will extend Young Tableaus by marking rows and columns specifically, leading to marked partitions as introduced in [14]. The first aim of the procedure is to interpret Schlesinger–Zudilin $q$MZVs as generating series of specific marked partitions.

The following can be obtained using geometric series expansion and is a variation of a statement from the proof of [25, Lemma 5.1].

**Proposition 1.73.** *For any word $\mathtt{W} = u_{k_1}u_0^{z_1}\cdots u_{k_d}u_0^{z_d} \in \mathcal{U}^{*,\circ}$, we have*

$$\zeta_q^{\mathrm{SZ}}(\mathtt{W}) = \sum_{\substack{m_1>\cdots>m_d>0\\ n_1,\ldots,n_d>0}} \left(\prod_{j=1}^{d}\binom{m_j-m_{j+1}-1}{z_j}\binom{n_j-1}{k_j-1}\right) q^{m_1n_1+\cdots+m_dn_d}.$$

In the following, we introduce the marked partitions regarding the Schlesinger–Zudilin model (see also Chapter 3).

**Definition 1.74.** Let $\lambda$ be a partition of $N$ and denote by $(\mathbf{m},\mathbf{n}) \in \mathbb{Z}^d_{>0} \times \mathbb{Z}^d_{>0}$ the Stanley coordinates of $\lambda$.

(i) If for $k_j$ rows of length $m_j$ are marked, we call $\mathbf{k} = (k_1,\ldots,k_d)$ the *type* of this row marking. A row marking is called *distinct* if the lowest row for each length $m_j$ is marked.

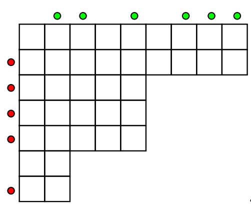
(ii) A distinct column marking of $\lambda$ is a $d$-tupel $\mathbf{z} = (z_1+1,\ldots,z_d+1)$, such that $(z_d+1,\ldots,z_1+1)$ is a distinct row marking of the conjugate partition of $\lambda$.

(iii) We identify a pair $(\mathbf{k};\mathbf{z})$ of such distinct row and column markings with the word $\mathtt{W} = u_{k_1}u_0^{z_1}\cdots u_{k_d}u_0^{z_d} \in \mathcal{U}^{*,\circ}$ and call the pair $(\mathbf{k};\mathbf{z})$ of such distinct markings for short a $\mathtt{W}$-marking of $\lambda$.

(iv) For $\mathtt{W} \in \mathcal{U}^{*,\circ}$, let be $\mathcal{MP}_{\mathtt{W}}$ the set of marked partitions with marking $\mathtt{W}$. The set of all marked partitions is $\mathcal{MP} = \bigcup_{\mathtt{W}\in\mathcal{U}^{*,\circ}} \mathcal{MP}_{\mathtt{W}}$.

We visualize row markings with a coloured dot to the left of the row. Analogously, a column marking is visualized with a coloured dot on top of the column that got marked.

**Example 1.75.** A marked partition of type $u_1u_0u_0u_3u_0u_1$ is



.

We can describe the combinatorial interpretation of Schlesinger–Zudilin $q$MZVs as follows.

**Lemma 1.76** ([14, Proposition 17])**.** *Given a word* $\mathtt{W} \in \mathcal{U}^{*,\circ}$*. Then,* $\zeta_q^{\mathrm{SZ}}(\mathtt{W})$ *is the generating series of marked partitions of type* $\mathtt{W}$*, i.e.,*

$$\zeta_q^{\mathrm{SZ}}(\mathtt{W}) = \sum_{\hat{\lambda} \in \mathcal{MP}_{\mathtt{W}}} q^{|\hat{\lambda}|}.$$

In [14], one obtains a remarkable fact by considering a marked partition and its conjugate (when row markings become column markings and vice versa).

**Lemma 1.77.** *Given a word* $\mathtt{W} \in \mathcal{U}^{*,\circ}$*. The marked partitions of type* $\mathtt{W}$ *are in one-to-one correspondence with marked partitions of type* $\tau(\mathtt{W})$*. An explicit bijection is given by conjugating the Young Tableaus together with the markings.*

Since conjugation does not change the partitioned number, Lemma 1.77 gives directly that the corresponding generating series are the same.

**Corollary 1.78** ([14])**.** *Duality already follows from the one-to-one correspondence of marked partitions of type* $\mathtt{W}$ *and marked partitions of type* $\tau(\mathtt{W})$ *for any* $\mathtt{W} \in \mathcal{U}^{*,\circ}$ *as described in Lemma 1.77.*

**Remark 1.79.** In this way, we have a combinatorial proof of duality, Theorem 1.29. Remarkable about this proof is that it is not only based on the fact that the (integer) coefficients in $\zeta_q^{\mathrm{SZ}}(\mathtt{W})$ and $\zeta_q^{\mathrm{SZ}}(\tau(\mathtt{W}))$, for any $\mathtt{W} \in \mathcal{U}^{*,\circ}$, are the same, but also gives them a combinatorial interpretation as the number of specifically marked partitions from which we constructed a one-to-one correspondence. This way, this proof is more profound than the standard ones using, e.g., a rearrangement of sums.

### 1.3.3 Stuffle product described with marked partitions

In the following, we provide the main result of [15] which is Chapter 3. It consists of a combinatorial description of the stuffle product (for Schlesinger–Zudilin $q$MZVs) using marked partitions. The idea for this is slicing two marked partitions into their *horizontal blocks* (a horizontal block of a (marked) partition is the union of all rows of the corresponding Young Tableau having a given length) and "glueing" them together to a new marked partition, in some way we will present now.

**Definition 1.80** (Definition 3.5)**.** The map $\Phi\colon \mathcal{MP} \times \mathcal{MP} \to \mathcal{MP}$ is defined as follows: Given marked partitions $\widehat{p_1}$ of $N_1$ and $\widehat{p_2}$ of $N_2$, then $\widehat{p} = \Phi(\widehat{p_1}, \widehat{p_2})$ is the marked partition of $N_1 + N_2$ obtained by the following rules:

(i) We set $\Phi(\emptyset, \widehat{p_2}) := \widehat{p_2}$ and $\Phi(\widehat{p_1}, \emptyset) := \widehat{p_1}$.

(ii) The Young Tableau of $\widehat{p}$ is obtained by cutting the Young Tableau of $\widehat{p_1}$ and $\widehat{p_2}$ horizontally below the rows containing corners into their horizontal blocks and glueing them (horizontally again) together to a new Young Tableau. If both, $\widehat{p_1}$ and $\widehat{p_2}$, have horizontal blocks of same length, the ones of $\widehat{p_1}$ will occur above the ones of $\widehat{p_2}$ in the new partition.

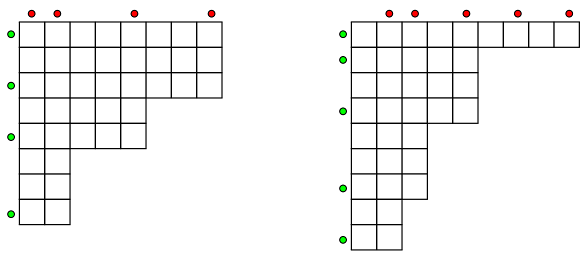
(iii) Keep the markings of the rows.

(iv) If there was a marking in the $j$-th leftmost column of $\widehat{p_1}$ or $\widehat{p_2}$, the $j$-th leftmost column of $\widehat{p}$ will be marked as well.

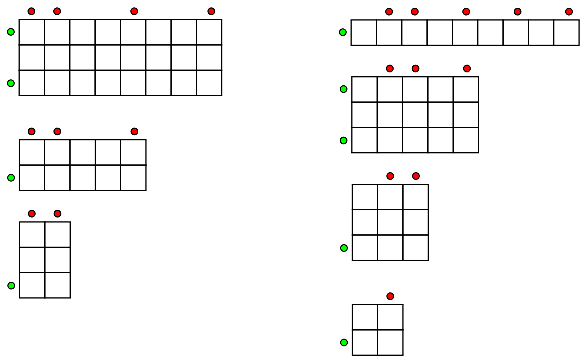
**Remark 1.81.** Note that the map $\Phi$ is associative but not commutative. The underlying Young Tableau of $\Phi(\widehat{p_1}, \widehat{p_2})$ is the same as the one of $\Phi(\widehat{p_2}, \widehat{p_1})$ and also the column markings match. However, the row markings, in general, do not if $\widehat{p_1}$ and $\widehat{p_2}$ have horizontal blocks of the same length.

**Example 1.82** (Example 3.7)**.** Consider the following pair of marked partitions.

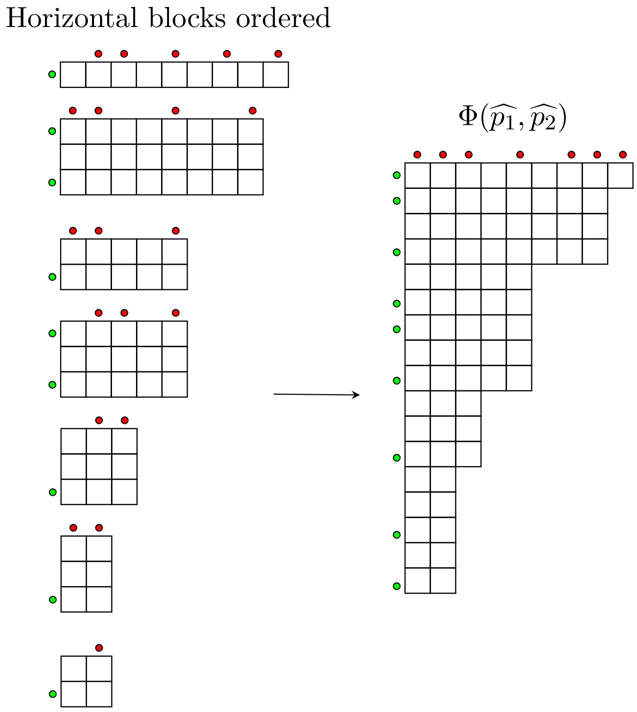
$\widehat{p_1}$ $\widehat{p_2}$



First, we slice them into their horizontal blocks.



Following the definition of $\Phi$, we obtain $\Phi(\widehat{p_1}, \widehat{p_2})$ after sorting the horizontal blocks as the following marked partition:




**Definition 1.83** (Definition 3.8). (i) For $\mathtt{W}_1, \mathtt{W}_2, \mathtt{W} \in \mathcal{U}^{*,\circ}$, we set $m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}} \in \mathbb{Z}_{\geq 0}$ to be the multiplicity of $\mathtt{W}$ in $\mathtt{W}_1 * \mathtt{W}_2$, i.e., to be the unique integer satisfying

$$\mathtt{W}_1 * \mathtt{W}_2 = \sum_{\mathtt{W} \in \mathcal{U}^{*,\circ}} m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}} \mathtt{W}.$$

(ii) For $\mathtt{W}_1, \mathtt{W}_2, \mathtt{W} \in \mathcal{U}^{*,\circ}$ and $\widehat{p} \in \mathcal{MP}_{\mathtt{W}}$, we define

$$m_{\mathtt{W}_1,\mathtt{W}_2;\widehat{p}} := \# \left\{ (\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2} \mid \Phi(\widehat{p_1}, \widehat{p_2}) = \widehat{p} \right\}.$$

Note that, for fixed $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^{*,\circ}$, almost all $m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}}$ are zero. The main result now is how the stuffle product can be interpreted combinatorially using marked partitions.

**Theorem 1.84** (Theorem 3.9). *Consider words $\mathtt{W}_1, \mathtt{W}_2, \mathtt{W} \in \mathcal{U}^{*,\circ}$. For all marked partitions $\widehat{p} \in \mathcal{MP}_{\mathtt{W}}$, we have*

$$m_{\mathtt{W}_1,\mathtt{W}_2;\widehat{p}} = m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}}.$$

*In particular, given $\mathtt{W}_1, \mathtt{W}_2$, $m_{\mathtt{W}_1,\mathtt{W}_2;\widehat{p}}$ only depends on the word $\mathtt{W}$ but not on the marked partition $\widehat{p} \in \mathcal{MP}_{\mathtt{W}}$.*

The proof of Theorem 1.84 is provided in Chapter 3. It uses mainly a combinatorial argument for obtaining a recursion of the numbers $m_{\mathtt{W}_1,\mathtt{W}_2;\widehat{p}}$ that will be a similar recursion as one can obtain for the numbers $m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}}$ using the stuffle product (which can be found in Lemma 3.12). The new aspect of Theorem 1.84 is that it provides a combinatorial and deeper understanding of the stuffle product than it was known so far.

### 1.3.4 Outlook

After this short introduction to marked partitions, we will briefly examine how marked partitions can be used for other aspects of the algebraic structure of ($q$)MZVs.

Both duality and the stuffle product are now understood on marked partitions. Referring to Conjecture 1.43, conjecturally, every linear relation among (Schlesinger–Zudilin) $q$MZVs now can be described using marked partitions.

Nevertheless, the structure of marked partitions has yet to be understood entirely regarding the linear relations among $q$MZVs. For future study, it will be interesting if one can make progress in proving Conjecture 1.49, e.g., using marked partitions. The main idea is that, for fixed $\mathtt{W} \in \mathcal{U}^{*,\circ}$ with $\mathrm{zero}(\mathtt{W}) \geq 1$, one can find for every integer $N$ a bijection from the set of marked partitions of $N$ of type $\mathtt{W}$ to a particular (union of) set(s) of marked partitions of $N$ of type $\mathtt{W}'$ satisfying $\mathtt{W}' \in \mathcal{U}^{*,\circ}$ and $\mathrm{zero}(\mathtt{W}') < \mathrm{zero}(\mathtt{W})$.

## 1.4 The analytic side of Multiple $q$-Zeta Values considered in Paper III

This section introduces the paper [12] that builds Chapter 4, which contains an analytic perspective on Multiple $q$-Zeta Values. As mentioned at the beginning of this chapter, quasi-modular forms are particular $q$MZVs via their Fourier expansion. Similarly, one can consider every Multiple $q$-Zeta Value as Fourier expansion of a function on the upper half plane $\mathbb{H} := \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$ via setting $q = e^{2\pi i\tau}$ for $\tau \in \mathbb{H}$. A common strategy for investigating $q$-series, also for their algebraic structure, is considering their asymptotic expansion. We will do this in Section 1.4.1 for several $q$MZVs and give ideas for an approach for general $q$MZVs. Such asymptotic expansions are of particular interest since the coefficients often are linear combinations of MZVs, i.e., every $\mathbb{Q}$-linear relation among $q$MZVs will give several $\mathbb{Q}$-linear relations among MZVs. The results of Section 1.4.1 were obtained jointly with H. Bachmann, J.–W. van Ittersum, and N. Sato. Furthermore, when considering particular sums of $q$MZVs such as

$$\sum_{d\geq 0} \zeta_q^{\mathrm{SZ}}(u_1^d),$$

one obtains the well-known generating series of partition numbers that has an expression as Euler product. W. Bridges, B. Brindle, K. Bringmann, and J. Franke studied the asymptotic expansion in detail for a large class of them using Wright's Circle Method which we briefly introduce in Section 1.4.2. We will present in Section 1.4.3 the main results of this work, which is Chapter 4.

### 1.4.1 Asymptotics of $q$MZVs

First, recall from Section 1.1.4 that for every bi-bracket $G = g\binom{k_1,\dots,k_d}{z_1,\dots,z_d}$ multiplied with $(1-q)^w$, where $w := k_1 + \cdots + k_d + z_1 + \cdots + z_d$, the limit $q \to 1$ exists (after possible regularization), due to [8, Theorem 1.2]. Then, for $q = e^{-t}$, we want to describe the asymptotic expansion

$$G \sim a_{-w}\frac{1}{t^w} + \sum_{n>-w} a_n t^n \quad (t \to 0),$$

where $a_{-w}$ is the limit mentioned above. A naive guess is that the coefficients $a_n$ are linear combinations of Multiple Zeta Values of lower depth and mixed weight. That $a_{-w} \in \mathcal{Z}$ is already known from [8, Theorem 1.2]. We refer to Lemma 1 and Proposition 1 of [48] for partial results on the asymptotic behaviour of bi-brackets. Another approach can be deduced from Zagier's work [45]. The following statement ([45, Eq. (48)]) is the basis for Lemma 1.86.

**Proposition 1.85.** *Setting $q = e^{-t}$, for all $z \in \mathbb{Z}_{>0}$, we have*

$$g\binom{1}{z} = g\binom{z+1}{0} \sim \frac{\zeta(z+1)}{t^{z+1}} - \frac{1}{z!}\sum_{r=0}^{\infty}(-1)^{r+z}\frac{B_r}{r!}\frac{B_{r+z}}{r+z}t^{r-1} \qquad (t \to 0),$$

*where $B_m$ denotes the mth Bernoulli number.*

By noting that $q\frac{d}{dq} = -\frac{d}{dt}$ for $q = e^{-t}$, we obtain the asymptotic expansion of most bi-brackets in depth 1.

**Lemma 1.86.** *Let be $k, z \in \mathbb{Z}_{>0}$ satisfying $z > k-1$. Setting $q = e^{-t}$, we have*

$$\begin{aligned} g\binom{k}{z} = g\binom{z+1}{k-1} &\sim \frac{\zeta(z-k+2)}{(k-1)!}\frac{1}{t^{z+1}} + (-1)^{z-k}\frac{B_{z-k+1}}{z!(z-k+1)}\frac{1}{t^k} \\ &+ \frac{1}{(k-1)!z!}\sum_{r=0}^{\infty}(-1)^{r+z-k+1}\frac{1}{r!}\frac{B_{r+k}B_{r+z+1}}{(r+k)(r+z+1)}t^r \qquad (t \to 0). \end{aligned}$$

For an investigation of the asymptotic behaviour of bi-brackets, in depth 1, the case $z = k-1$ remains to be considered. For this case, we recall the definition of bi-brackets in depth 1. When setting $q = e^{-t}$, we have

$$\begin{aligned} g\binom{k}{z} &= \frac{1}{z!}\sum_{m>0} m^z \frac{P_k(q^m)}{(1-q^m)^k} \\ &= \frac{1}{z!}\frac{1}{t^z}\sum_{m>0}(mt)^z\frac{P_k(e^{-mt})}{(1-e^{-mt})^k} \end{aligned} \tag{1.86.1}$$

Now, since $P_k(X)$ has no zero in $X = 1$ for all $k \in \mathbb{Z}_{>0}$, we have a pole of order $k-1$ if $z = k-1$. Defining

$$f_{(k,z)}(t) := t^z \frac{P_k(e^{-t})}{(1-e^{-t})^k},$$

we are interested in the asymptotic expansion of

$$g_{(k,z)}(t) := \sum_{m>0} f_{(k,z)}(mt).$$

For $z = k-1$, note that $f_{(k,k-1)}(t)$ has a single pole in $t = 0$, i.e., it has, near $t = 0$, asymptotic development

$$f(t) \sim \sum_{n=-1}^{\infty} b_n t^n \tag{1.86.2}$$

for appropriate $b_n \in \mathbb{R}$. We make use of the following result due to Zagier.

**Lemma 1.87** ([45, Proposition 3])**.** *If*

$$f(t) \sim \sum_{\lambda \geq -1} b_\lambda t^\lambda \quad (t \to 0),$$

*then we have*

$$g(t) = \sum_{m>0} f(mt) \sim \frac{1}{t}\left(b_{-1}\log\frac{1}{t} + I_f^*\right) + \sum_{\lambda > -1}^{\infty} b_\lambda \zeta(-\lambda) t^\lambda \quad (t \to 0).$$

*Here, $I_f^*$ denotes the integral*

$$I_f^* := \int_0^\infty f(t) - b_{-1}\frac{e^{-t}}{t}\, dt.$$

Therefore, Lemma 1.87 gives, together with (1.86.2), the desired asymptotic development for $t \to 0$ of $g\binom{k}{k-1}$ (when setting $q = e^{-t}$) already when knowing the coefficients $b_n$ in (1.86.2).

**Lemma 1.88.** *For every $k \in \mathbb{Z}_{>0}$ and $t \in \mathbb{R}\backslash\{0\}$, we have*

$$f_{k,k-1}(t) = t^{-1} + \frac{(-1)^{k-1}}{(k-1)!}\sum_{n=0}^{\infty}\frac{B_{n+k}}{(n+k)n!}t^{n+k-1}.$$

*In particular, we have this expression as asymptotic behaviour as $t \to 0$.*

*Proof.* Using one of the fundamental properties of Eulerian polynomials, we find

$$\begin{aligned}\frac{P_k(e^{-t})}{(1-e^{-t})^k} &= \frac{1}{(k-1)!}\left(-\frac{d}{dt}\right)^{k-1}\frac{1}{e^t-1}\\ &= \frac{1}{(k-1)!}\left(-\frac{d}{dt}\right)^{k-1}\sum_{n=0}^{\infty}\frac{B_n}{n!}t^{n-1},\end{aligned}$$

implying the lemma after multiplication with $t^{k-1}$. □

For the asymptotic behaviour of $g_{(k,k-1)}(t)$, it remains to compute $I^*_{f_{(k,k-1)}}$.

**Lemma 1.89.** *We have*

$$I^*_{f_{(k,k-1)}} = \int_0^\infty f_{(k,k-1)}(t) - \frac{e^{-t}}{t}\, dt = \delta_{k>1}H_{k-1} + \gamma,$$

*where $H_{k-1}$ denotes the $(k-1)$st harmonic number and $\gamma$ the Euler constant.*

*Sketch of the proof.* We have

$$t^{k-1}\left(-\frac{d}{dt}\right)^{k-1}\frac{e^{-t}}{t} = (k-1)!\frac{e^{-t}}{t} + \sum_{j=0}^{k-2}\binom{k-1}{j}j!e^{-t}t^{k-j-2}.$$

Hence, we obtain

$$f_{(k,k-1)}(t) - \frac{e^{-t}}{t} = \sum_{j=0}^{k-2}\frac{e^{-t}t^{k-j-2}}{(k-j-2)!} + \frac{t^{k-1}}{(k-1)!}\left(-\frac{d}{dt}\right)^{k-1}\left[\frac{1}{e^t-1} - \frac{e^{-t}}{t}\right].$$

This leads to

$$I^*_{f_{(k,k-1)}} = \delta_{k>1}H_{k-1} + \frac{1}{(k-1)!}\int_0^\infty t^{k-1}\left(-\frac{d}{dt}\right)^{k-1}\left[\frac{1}{e^t-1} - \frac{e^{-t}}{t}\right] dt,$$

where $H_{k-1}$ denotes the $(k-1)$st harmonic number. Integrating by parts leads to the recursion

$$\int_0^\infty t^{k-1}\left(-\frac{d}{dt}\right)^{k-1}\left[\frac{1}{e^t-1} - \frac{e^{-t}}{t}\right] dt = (k-1)\int_0^\infty t^{k-2}\left(-\frac{d}{dt}\right)^{k-2}\left[\frac{1}{e^t-1} - \frac{e^{-t}}{t}\right] dt,$$

yielding

$$\int_0^\infty t^{k-1}\left(-\frac{d}{dt}\right)^{k-1}\left[\frac{1}{e^t-1}-\frac{e^{-t}}{t}\right]dt=(k-1)!\gamma.$$

In particular, we conclude

$$I^*_{f_{(k,k-1)}}=\delta_{k>1}H_{k-1}+\gamma,$$

proving the lemma. □

Using Lemmas 1.87, 1.88, 1.89, and $\zeta(-n)=(-1)^n\frac{B_{n+1}}{n+1}$ for nonnegative integers $n$, we obtain now the asymptotic behaviour of $g_{(k,k-1)}(t)$ as $t\to 0$.

**Corollary 1.90.** *For all $k\in\mathbb{Z}_{>0}$, we have, as $t\to 0$,*

$$g_{(k,k-1)}(t)\sim\frac{\log\frac{1}{t}}{t}+\frac{\delta_{k>1}H_{k-1}+\gamma}{t}+\frac{1}{(k-1)!}\sum_{n=0}^{\infty}\frac{B_{n+k}}{(n+k)}\frac{B_{n+1}}{(n+1)!}(-t)^{n+k-1}.$$

With (1.86.1), we obtain the asymptotic expansion of $\mathrm{g}\binom{k}{k-1}$.

**Corollary 1.91.** *For all $k\in\mathbb{Z}_{>0}$, when setting $q=e^{-t}$, we have, as $t\to 0$,*

$$\mathrm{g}\binom{k}{k-1}\sim\frac{\log\frac{1}{t}}{(k-1)!t^k}+\frac{\delta_{k>1}H_{k-1}+\gamma}{(k-1)!t^k}+\frac{1}{(k-1)!^2}\sum_{n=0}^{\infty}\frac{B_{n+k}}{(n+k)}\frac{B_{n+1}}{(n+1)!}(-t)^n.$$

Therefore, the depth 1 case of bi-brackets is done. For higher depth, one approach is to use [17, Theorem 1.4]. The challenge there is computing integrals such as $I^*_f$ in a generalized way. This is current research and is left as an open problem.

### 1.4.2 The Circle Method

Another way to consider $q$MZVs is in viewing them as $q$-series and investigating the asymptotic of the coefficients as the exponent tends to $\infty$. One will do this using Wright's Circle Method. We briefly introduce here the Circle Method which is a tool from complex analysis used in analytic number theory and combinatorics to better understand properties of sequences. We find asymptotic formulas for a general class of partition functions, see Sections 1.4.3 and 4.1.6. This is also for the study of $q$MZVs of interest since every $q$MZV is the generating function of a class of marked partitions, as presented in Section 1.3.

Suppose that a sequence $(c(n))_{n\in\mathbb{N}_0}$ has moderate growth and the *generating function*

$$F(q):=\sum_{n\geq 0}c(n)q^n,$$

is holomorphic in the unit disk with radius of convergence 1. Via Cauchy's integral formula one can then recover the coefficients from the generating function

$$c(n)=\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{F(q)}{q^{n+1}}dq, \tag{1.91.1}$$

for any simple closed curve $\mathcal{C}$ contained in the unit disk orientated counterclockwise. The so-called Circle Method uses the analytic behavior of $F(q)$ near the boundary of the unit circle to obtain asymptotic information about $c(n)$. In fact for "nice" examples this method is automatic and there is a long history for example with the Prime Number Theorem. For instance, if the $c(n)$ are positive and monotonically increasing, it is expected that the part close to $q = 1$ provides the dominant contribution to (1.91.1) (Tauberian Theorems then show this). This part of the curve is the *major arc* and the complement is the *minor arc*. To obtain an asymptotic expansion for $c(n)$, one then evaluates the major arc to some degree of accuracy and bounds the minor arc. Depending on the function $F(q)$, both of these tasks present a variety of difficulties.

In Chapter 4, we are interested in infinite product generating functions of the form

$$F(q) = \prod_{n\geq 1} \frac{1}{(1-q^n)^{f(n)}}.$$

Such generating functions are important in the theory of partitions, but also arise, for example, in representation theory. If the Dirichlet series for $f(n)$ has a single simple pole on the positive real axis and $F$ is "bounded" away from $q = 1$, then Meinardus [36] proved an asymptotic expression for $c(n)$. Debruyne and Tenenbaum [24] eliminated the technical growth conditions on $F$ by adding a few more assumptions on the $f(n)$, which made their result more applicable. The main results of Chapter 4, Theorems 1.93 and 1.94, yield asymptotic expansions given mild assumptions on $f(n)$ and have a variety of new applications.

### 1.4.3 Analytic behaviour of $q$-series studied with the circle method

A particular connection of $q$MZVs to the partition functions is given in the following lemma, which can be found, e.g., in [3] or [14].

**Lemma 1.92.** *Denoting by $p(n)$ the number of partitions of $n$, one has*

$$\sum_{d\geq 0} \zeta_q^{\mathrm{SZ}}(u_1^d) = \sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} \frac{1}{1-q^n}. \tag{1.92.1}$$

In [30], Hardy and Ramanujan used (1.92.1) to show the asymptotic formula

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}, \qquad n \to \infty,$$

which gave birth of the Circle Method. Using modular transformations, one can describe with high precision the analytic behaviour of the product if $q$ is near a root of unity. One further sees directly from the infinite product that dominant singularities occur at such roots of unity with small denominator. These ideas culminate in Rademacher's exact formula for $p(n)$ [39].

With Theorem 1.93 we find, for certain constants $B_j$ and arbitrary $N \in \mathbb{N}$,

$$p(n) = \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n} \left(1 + \sum_{j=1}^{N} \frac{B_j}{n^{\frac{j}{2}}} + O_N\left(n^{-\frac{N+1}{2}}\right)\right).$$

Similarly, one can treat the cases for $k$-th powers (in arithmetic progressions), see [24].

The main goal of Chapter 4 (which is [12]) was to prove asymptotic formulas for a general class of partition functions. To state it, let $f : \mathbb{N} \to \mathbb{N}_0$, set $\Lambda := \mathbb{N} \setminus f^{-1}(\{0\})$, and for $q = e^{-z}$ ($z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$), define

$$G_f(z) := \sum_{n\geq 0} p_f(n)q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)^{f(n)}}, \qquad L_f(s) := \sum_{n\geq 1} \frac{f(n)}{n^s}.$$

We require the following key properties of these objects:

(P1) All poles of $L_f$ are real. Let $\alpha > 0$ be the largest pole of $L_f$. There exists $L \in \mathbb{N}$, such that for all primes $p$, we have $|\Lambda \setminus (p\mathbb{N} \cap \Lambda)| \geq L > \frac{\alpha}{2}$.

(P2) Condition (P2) is attached to $R \in \mathbb{R}^+$. The series $L_f(s)$ converges for some $s \in \mathbb{C}$, has a meromorphic continuation to $\{s \in \mathbb{C} : \mathrm{Re}(s) \geq -R\}$, and is holomorphic on the line $\{s \in \mathbb{C} : \mathrm{Re}(s) = -R\}$. The function $L_f^*(s) := \Gamma(s)\zeta(s+1)L_f(s)$ has only real poles $0 < \alpha := \gamma_1 > \gamma_2 > \cdots$ that are all simple, except the possible pole at $s = 0$, that may be double.

(P3) For some $a < \frac{\pi}{2}$, in every strip $\sigma_1 \leq \sigma \leq \sigma_2$ in the domain of holomorphicity, we uniformly have, for $s = \sigma + it$,

$$L_f(s) = O_{\sigma_1,\sigma_2}\left(e^{a|t|}\right), \qquad |t| \to \infty.$$

Note that (P1) implies that $|\Lambda \setminus (b\mathbb{N} \cap \Lambda)| \geq L > \frac{\alpha}{2}$ for all $b \geq 2$. The analytic properties of $L_f$ are a major ingredient needed to prove the following theorem, as analytic continuation in (P2) gives rise to asymptotic expansions of[2] $\mathrm{Log}(G_f(z))$ and (P3) assists with vertical integration.

**Theorem 1.93** (Theorem 4.5). *Assume (P1) for $L \in \mathbb{N}$, (P2) for $R > 0$, and (P3). Then, for some $M, N \in \mathbb{N}$,*

$$p_f(n) = \frac{C}{n^b} \exp\left(A_1 n^{\frac{\alpha}{\alpha+1}} + \sum_{j=2}^{M} A_j n^{\alpha_j}\right)\left(1 + \sum_{j=2}^{N} \frac{B_j}{n^{\beta_j}} + O_{L,R}\left(n^{-\min\left\{\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\right\}}\right)\right),$$

*where $0 \leq \alpha_M < \alpha_{M-1} < \cdots \alpha_2 < \alpha_1 = \frac{\alpha}{\alpha+1}$ are given by[3] $\mathcal{L}$ (defined in (1.93.1)), and $0 < \beta_2 < \beta_3 < \cdots$ are given by $\mathcal{M} + \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are defined in (1.93.2) and (1.93.3), respectively. The coefficients $A_j$ and $B_j$ can be calculated explicitly; the constants $A_1$, $C$, and $b$ are provided in (1.93.4) and (1.93.5). Moreover, if $\alpha$ is the only positive pole of $L_f$, then we have $M = 1$.*

With the notation of Theorem 1.93, we define

$$\mathcal{L} := \frac{1}{\alpha+1}\mathcal{P}_R + \sum_{\mu \in \mathcal{P}_R}\left(\frac{\mu+1}{\alpha+1} - 1\right)\mathbb{N}_0, \tag{1.93.1}$$

$$\mathcal{M} := \frac{\alpha}{\alpha+1}\mathbb{N}_0 + \left(-\sum_{\mu \in \mathcal{P}_R}\left(\frac{\mu+1}{\alpha+1} - 1\right)\mathbb{N}_0\right) \cap \left[0, \frac{R+\alpha}{\alpha+1}\right), \tag{1.93.2}$$

$$\mathcal{N} := \left\{\sum_{j=1}^{K} b_j\theta_j : b_j, K \in \mathbb{N}_0, \theta_j \in (-\mathcal{L}) \cap \left(0, \frac{R}{\alpha+1}\right)\right\}. \tag{1.93.3}$$

We set, with $\omega_\alpha := \mathrm{Res}_{s=\alpha} L_f(s)$,

$$\begin{aligned} A_1 &:= \left(1 + \frac{1}{\alpha}\right)(\omega_\alpha\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}}, \\ C &:= \frac{e^{L_f'(0)}(\omega_\alpha\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{\frac{1}{2}-L_f(0)}{\alpha+1}}}{\sqrt{2\pi(\alpha+1)}}, \end{aligned} \tag{1.93.4}$$

[2]Throughout we use the principal branch of the logarithm.

[3]We can enlarge the discrete exponent sets at will, since we can always add trivial powers with vanishing coefficients to an expansion. Therefore, from now on we always use this expression, even if the set increases tacitly.

$$b := \frac{1 - L_f(0) + \frac{\alpha}{2}}{\alpha + 1}. \tag{1.93.5}$$

We will provide the proof of the Theorem 1.93 and several examples in Chapter 4. The second main result of Chapter 4 is the following theorem giving the asymptotic expansion in the case that $L_f$ has exactly two positive poles.

**Theorem 1.94** (Theorem 4.29)**.** *Assume that $f : \mathbb{N} \to \mathbb{N}_0$ satisfies the conditions of Theorem 1.93 and that $L_f$ has exactly two positive poles $\alpha > \beta$, such that $\frac{\ell+1}{\ell}\beta < \alpha \leq \frac{\ell}{\ell-1}\beta$ for some $\ell \in \mathbb{N}$. Then we have*

$$p_f(n) = \frac{C}{n^b} \exp\left( A_1 n^{\frac{\alpha}{\alpha+1}} + A_2 n^{\frac{\beta}{\alpha+1}} + \sum_{k=3}^{\ell+1} A_k n^{\frac{(k-1)\beta}{\alpha+1} + \frac{k-2}{\alpha+1} + 2 - k} \right) \\ \times \left( 1 + \sum_{j=2}^{M_1} \frac{B_j}{n^{\nu_j}} + O_{L,R}\left( n^{-\min\left\{ \frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1} \right\}} \right) \right), \qquad (n \to \infty),$$

*with*

$$A_1 := (\omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}} \left(1 + \frac{1}{\alpha}\right), \qquad A_2 := \frac{\omega_\beta \Gamma(\beta)\zeta(\beta+1)}{(\omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{\beta}{\alpha+1}}},$$

*and for all $k \geq 3$*

$$A_k := K_k + \frac{c_1^{\frac{1}{\alpha+1}}}{\alpha} \sum_{m=1}^{\ell} \binom{-\alpha}{m} \sum_{\substack{0 \leq j_1, \ldots, j_\ell \leq m \\ j_1 + \ldots + j_\ell = m \\ j_1 + 2j_2 + \ldots + \ell j_\ell = k-1}} \binom{m}{j_1, j_2, \ldots, j_\ell} \frac{K_2^{j_1} \cdots K_{\ell+1}^{j_\ell}}{c_1^{\frac{m}{\alpha+1}}} \\ + \frac{c_2}{\beta c_1^{\frac{\beta}{\alpha+1}}} \sum_{m=1}^{\ell} \binom{-\beta}{m} \sum_{\substack{0 \leq j_1, \ldots, j_\ell \leq m \\ j_1 + \ldots + j_\ell = m \\ j_1 + 2j_2 + \ldots + \ell j_\ell = k-2}} \binom{m}{j_1, j_2, \ldots, j_\ell} \frac{K_2^{j_1} \cdots K_{\ell+1}^{j_\ell}}{c_1^{\frac{m}{\alpha+1}}}.$$

*Here, $C$ and $b$ are defined in* (1.93.4) *and* (1.93.5)*, the $\nu_j$ run through $\mathcal{M} + \mathcal{N}$, the $K_j$ are given in Lemma 4.28, and $c_1$, $c_2$, and $c_3$ run through* (4.27.4)*.*

### 1.4.4 Outlook

(i) An example of Theorem 1.93 is given in Theorem 4.4 where we studied the asymptotic behaviour of the finite-dimensional representation of $\mathfrak{so}(5)$ which is closely connected with the corresponding Witten zeta function $\zeta_{\mathfrak{so}(5)}$. A more detailed study of $\zeta_{\mathfrak{so}(5)}$ and the proof for Theorem 4.4 is provided in Section 4.5.

(ii) Furthermore, since $q$MZVs, in general, do not have an Euler product representation, Theorem 1.93 gives the asymptotic expansion of particular (infinite) sums of $q$MZVs only, such as in (1.92.1). Therefore, obtaining the asymptotic expansion for general $q$MZVs is an open problem. Nevertheless, the Circle Method has turned out to be a powerful tool in answering such questions over the last decades. This is why studying the asymptotic of $q$MZVs using the Circle Method gives rise to a research project for the future.

(iii) Solving the open problem of finding the asymptotic expansion for all $q$MZVs can help answer the question of which ($\mathbb{Q}$-linear combinations of) $q$MZVs are (quasi-) modular forms.

## 1.5 Conclusion

This section briefly considers how our perspectives to $q$MZVs are connected.

**i)** Our combinatorial approach described in Section 1.3 is directly connected with the algebraic one from Section 1.2. Both duality and the stuffle product are now understood on marked partitions. Referring to Conjecture 1.43, conjecturally, every linear relation among (Schlesinger–Zudilin) $q$MZVs now can be described using marked partitions. This opens the door to various new research projects; it would be interesting to translate several (folklore) conjectures about the algebraic structure of $q$MZVs into terms of marked partitions. For example, Bachmann's Conjecture 1.49 would be of interest to consider on the level of marked partitions. It states - roughly speaking - that every (Schlesinger–Zudilin) $q$MZVis linear combination of generating series of marked partitions as considered where only row markings (resp. only column markings when using duality) are allowed. Hence, for proving Conjecture 1.49, one has to deal with particular bijections (that has to be discovered yet) among several sets of marked partitions, similar to the problem of describing the stuffle product. One approach could be to translate the refined Bachmann Conjecture 1.55 into the "language" of marked partitions.

**ii)** In general, the analytic study of Multiple $q$-Zeta Values is of interest since the coefficients occurring in such asymptotic expansions often (maybe always; this is current research) are $\mathbb{Q}$-linear combinations of Multiple Zeta Values. Hence, by comparing coefficients, a relation among Multiple $q$-Zeta Values gives a set of $\mathbb{Q}$-linear relations among Multiple Zeta Values. In this way, the analytic study of Multiple $q$-Zeta Values has an impact on the algebraic study of Multiple Zeta Values.

**iii)** Another aspect of the analytic study of Multiple $q$-Zeta Values is the connection to quasi-modular forms since every quasi-modular form is a (linear combination of) Multiple $q$-Zeta Values via their Fourier expansion. To study this connection, it is often helpful to consider $q$MZVs as multiple Eisenstein series, introduced in [2]. For more details on multiple Eisenstein series, we refer to the works [2, 3, 4, 5, 6, 7, 22]. Note that there is also another connection of quasi-modular forms and MZVs, conjectured by Broadhurst and Kreimer, see [18] for details.

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# Chapter 2

# Paper I: On the relations satisfied by Multiple *q*-Zeta Values

# On the relations satisfied by Multiple $q$-Zeta Values

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**Abstract.** In 2015, Bachmann [2] conjectured that the $\mathbb{Q}$-vector space $\mathcal{Z}_q^f$ of (formal) $q$-analogues of Multiple Zeta Values ($q$MZVs) is spanned by a very particular set compared to known spanning sets. This work proves that this conjecture is true for a subspace of $\mathcal{Z}_q^f$ spanned by words satisfying some condition on their number of zeros and depth. According to this partial result, we give an explicit approach to the whole conjecture based on particular $\mathbb{Q}$-linear relations among formal Multiple $q$-Zeta Values, which are implied by duality.

## 2.1 Introduction

Given a field $F$ and a countable set $\mathcal{A}$, we call $\mathcal{A}$ also an *alphabet* and elements of $\mathcal{A}$ are referred to as *letters*. Denote by $\operatorname{span}_F \mathcal{A}$ the $F$-vector space spanned by elements of $\mathcal{A}$. Furthermore, monomials of elements in $\mathcal{A}$ (with respect to concatenation) are called *words*. Usually, the neutral element with respect to concatenation is denoted by $\mathbf{1}$ and called the *empty word*. Let $\mathcal{A}^*$ denote the set of words with letters in $\mathcal{A}$, then we write $F\langle\mathcal{A}\rangle$ for the $F$-vector space $\operatorname{span}_F \mathcal{A}^*$, equipped with the non-commutative, but associative multiplication, given by concatenation.

Choosing $F = \mathbb{Q}$ and $\mathcal{A} = \mathcal{U} := \{u_j \mid j \in \mathbb{Z}_{\geq 0}\}$, we define the *stuffle product* to be the $\mathbb{Q}$-bilinear map $*\colon \mathbb{Q}\langle\mathcal{U}\rangle \times \mathbb{Q}\langle\mathcal{U}\rangle \to \mathbb{Q}\langle\mathcal{U}\rangle$ recursively via

$$u_{j_1}\mathtt{W}_1 * u_{j_2}\mathtt{W}_2 = u_{j_1}(\mathtt{W}_1 * u_{j_2}\mathtt{W}_2) + u_{j_2}(u_{j_1}\mathtt{W}_1 * \mathtt{W}_2) + u_{j_1+j_2}(\mathtt{W}_1 * \mathtt{W}_2)$$

for all $j_1, j_2 \in \mathbb{Z}_{\geq 0}$ and $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^*$ with initial condition $\mathbf{1} * \mathtt{W} = \mathtt{W} * \mathbf{1} = \mathtt{W}$ for $\mathtt{W} \in \mathcal{U}^*$. By Hoffman's work [7], $(\mathbb{Q}\langle\mathcal{U}\rangle, *)$ is an associative and commutative $\mathbb{Q}$-algebra. For a word $\mathtt{W} = u_{k_1} \cdots u_{k_r} \in \mathcal{U}^*$, we often write $u_{\mathbf{k}}$ ($u_\emptyset := \mathbf{1}$), where $\mathbf{k} = (k_1, \ldots, k_r)$, and associate the notion of

$$\begin{aligned}
\textit{length}, &\quad \operatorname{len}(\mathtt{W}) := \operatorname{len}(\mathbf{k}) := r,\\
\textit{depth}, &\quad \operatorname{depth}(\mathtt{W}) := \operatorname{depth}(\mathbf{k}) := \#\{k_j \neq 0 \mid 1 \leq j \leq r\},\\
\textit{number of zeros}, &\quad \operatorname{zero}(\mathtt{W}) := \operatorname{zero}(\mathbf{k}) := \#\{k_j = 0 \mid 1 \leq j \leq r\},\\
\textit{weight}, &\quad \operatorname{wt}(\mathtt{W}) := \operatorname{wt}(\mathbf{k}) := |\mathbf{k}| + \operatorname{zero}(\mathtt{W}),
\end{aligned}$$

where $|\mathbf{k}| := k_1 + \cdots + k_r$. Furthermore, we denote $\mathcal{U}^{*,\circ} := \mathcal{U}^* \backslash u_0\mathcal{U}^*$ to be the set of words not starting with $u_0$ and we define the corresponding $\mathbb{Q}$-vector space $\mathbb{Q}\langle\mathcal{U}\rangle^\circ \subset \mathbb{Q}\langle\mathcal{U}\rangle$

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spanned by the words from $\mathcal{U}^{*,\circ}$. Note that $\mathbb{Q}\langle\mathcal{U}\rangle^\circ$ is closed under $*$ which gives rise to a commutative $\mathbb{Q}$-algebra $(\mathbb{Q}\langle\mathcal{U}\rangle^\circ, *)$ (see [7]). The map $\zeta_q^{\mathrm{SZ}}\colon (\mathbb{Q}\langle\mathcal{U}\rangle^\circ, *) \to (\mathbb{Q}[\![q]\!], \cdot)$ is the $\mathbb{Q}$-algebra homomorphism (see [8]) defined via $\zeta_q^{\mathrm{SZ}}(\mathbf{1}) = 1$, $\mathbb{Q}$-linearity, and, with $m_{d+1} := 0$,

$$\zeta_q^{\mathrm{SZ}}\left(u_{k_1}u_0^{z_1}\cdots u_{k_d}u_0^{z_d}\right) := \sum_{m_1>\cdots>m_d>0}\prod_{j=1}^{d}\binom{m_j-m_{j+1}-1}{z_j}\frac{q^{m_jk_j}}{(1-q^{m_j})^{k_j}}, \quad (2.0.1)$$

for any $k_1,\ldots,k_d \in \mathbb{Z}_{>0}$ and $z_1,\ldots,z_d \in \mathbb{Z}_{\geq 0}$ where $d \in \mathbb{Z}_{>0}$ (note that this definition is not the usual one, like in [11], but equivalent to it; this statement can be deduced, e.g., from [5, Theorem 2.18]). We denote by $\mathcal{Z}_q$ the image of $\zeta_q^{\mathrm{SZ}}$ and call elements in $\mathcal{Z}_q$ *(Schlesinger–Zudilin) qMZVs* ((SZ-)$q$MZVs for short). Note that these $q$-series are $q$-analogues of Multiple Zeta Values since in the case $k_1 \geq 2$ and $z_1 = \cdots = z_d = 0$, we have

$$\lim_{q\to 1}(1-q)^{k_1+\cdots+k_d}\zeta_q^{\mathrm{SZ}}(u_{k_1}\cdots u_{k_d}) = \zeta(u_{k_1}\cdots u_{k_d}) := \sum_{m_1>\cdots>m_d>0}\frac{1}{m_1^{k_1}\cdots m_d^{k_d}}.$$

But in this work, we focus purely on the algebraic structure of (SZ-)$q$MZVs and do not consider its implication for classical Multiple Zeta Values. Over the years, several versions of $q$MZVs were introduced (see, e.g., [3, 4, 9, 10, 14]); for an overview, see [5]. Because of Conjecture 2.1 and since the $q$-series on the right of (2.0.1) is invariant under the $\mathbb{Q}$-linear involution $\tau : \mathbb{Q}\langle\mathcal{U}\rangle^\circ \to \mathbb{Q}\langle\mathcal{U}\rangle^\circ$, defined by $\tau(\mathbf{1}) := \mathbf{1}$ and

$$\tau\left(u_{k_1}u_0^{z_1}\cdots u_{k_d}u_0^{z_d}\right) := u_{z_d+1}u_0^{k_d-1}\cdots u_{z_1+1}u_0^{k_1-1}$$

for all $d \in \mathbb{Z}_{>0}$, $k_1,\ldots,k_d \geq 1$, and $z_1,\ldots,z_d \geq 0$ (see [13, Theorem 8.3]), we will consider the algebra of *formal qMZVs*,

$$\mathcal{Z}_q^f := {(\mathbb{Q}\langle\mathcal{U}\rangle^\circ, *)}\big/{T},$$

where $T$ is the $*$-ideal in $\mathbb{Q}\langle\mathcal{U}\rangle^\circ$ generated by $\{\tau(\mathtt{W}) - \mathtt{W} \mid \mathtt{W} \in \mathbb{Q}\langle\mathcal{U}\rangle^\circ\}$. For $\mathtt{W} \in \mathbb{Q}\langle\mathcal{U}\rangle^\circ$, we set $\zeta_q^{\mathrm{f}}(\mathtt{W})$ to be the congruence class of $\mathtt{W}$ in $\mathcal{Z}_q^f$. Note that depth and weight are invariant under $\tau$ while the number of zeros generally is not. Furthermore, playing with $\tau$ and the stuffle product $*$, one obtains non-trivial $\mathbb{Q}$-linear relations among formal $q$MZVs. The following folklore conjecture (see [1]; a published version can be found in [14, Conjecture 1]) states the expectation of how the $\mathbb{Q}$-linear relations among SZ-$q$MZVs look like.

**Conjecture 2.1** (Bachmann)**.** *All $\mathbb{Q}$-linear relations among elements in $\mathcal{Z}_q$ are obtained by the stuffle product $*$ and duality $\tau$.*

I.e., one expects $\mathcal{Z}_q \simeq \mathcal{Z}_q^f$. We will consider in this paper only $\mathbb{Q}$-linear relations in $\mathcal{Z}_q^f$ which are implied by

$$\zeta_q^{\mathrm{f}}\left(\mathtt{W}_1 * (\mathtt{W}_2 - \tau(\mathtt{W}_2))\right) = 0 \quad (2.1.1)$$

for any words $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^{*,\circ}$. For investigating $\mathcal{Z}_q^f$ in more detail, we need the following notion of filtrations.

**Notation 2.2.** (i) For every $(\mathrm{N}, \mathrm{op}) \in \{(\mathrm{Z}, \mathrm{zero}), (\mathrm{D}, \mathrm{depth}), (\mathrm{W}, \mathrm{wt})\}$, $n \in \mathbb{Z}$, and sets $\mathcal{S} \subset \mathbb{Q}\langle\mathcal{U}\rangle^\circ$, $\mathcal{S}' \subset \mathcal{Z}_q^f$, write

$$\mathrm{Fil}_n^{\mathrm{N}}\,\mathcal{S} := \mathrm{span}_{\mathbb{Q}}\left\{\mathtt{W} \in \mathcal{U}^{*,\circ} \mid \mathrm{op}(\mathtt{W}) \leq n\right\} \cap \mathcal{S},$$

$$\mathrm{Fil}_n^{\mathrm{N}}\,\mathcal{S}' := \mathrm{span}_{\mathbb{Q}}\left\{\zeta_q^{\mathrm{f}}\left(\mathtt{W}\right) \in \mathcal{Z}_q^f \mid \mathtt{W} \in \mathcal{U}^{*,\circ},\ \mathrm{op}(\mathtt{W}) \leq n\right\} \cap \mathcal{S}'$$

for the filtration by number of zeros (N = Z), depth (N = D), and weight (N = W) respectively on $\mathcal{S}$ and $\mathcal{S}'$ respectively.

(ii) For $\mathcal{S} \subset \mathbb{Q}\langle\mathcal{U}\rangle^\circ$ or $\mathcal{S} \subset \mathcal{Z}_q^f$, $\mathrm{N}_1, \ldots, \mathrm{N}_m \in \{\mathrm{Z}, \mathrm{D}, \mathrm{W}\}$, where $m \in \mathbb{Z}_{>0}$, and for integers $n_1, \ldots, n_m \in \mathbb{Z}$, we abbreviate

$$\mathrm{Fil}_{n_1,\ldots,n_m}^{\mathrm{N}_1,\ldots,\mathrm{N}_m}\,\mathcal{S} := \bigcap_{j=1}^{m} \mathrm{Fil}_{n_j}^{\mathrm{N}_j}\,\mathcal{S}.$$

The following particular filtration will play a main role in this paper.

**Definition 2.3.** We define

$$\mathcal{Z}_q^{f,\circ} := \mathrm{Fil}_0^{\mathrm{Z}}\,\mathcal{Z}_q^f.$$

At this point, note that

$$\mathrm{Fil}_{z,d,w}^{\mathrm{Z,D,W}}\,\mathbb{Q}\langle\mathcal{U}\rangle^\circ * \mathrm{Fil}_{z',d',w'}^{\mathrm{Z,D,W}}\,\mathbb{Q}\langle\mathcal{U}\rangle^\circ \subset \mathrm{Fil}_{z+z',d+d',w+w'}^{\mathrm{Z,D,W}}\,\mathbb{Q}\langle\mathcal{U}\rangle^\circ \tag{2.3.1}$$

and

$$\tau\left(\mathrm{Fil}_{z,d,w}^{\mathrm{Z,D,W}}\,\mathbb{Q}\langle\mathcal{U}\rangle^\circ\right) = \mathrm{Fil}_{w-z-d,d,w}^{\mathrm{Z,D,W}}\,\mathbb{Q}\langle\mathcal{U}\rangle^\circ \tag{2.3.2}$$

for all $z, z', d, d', w, w' \in \mathbb{Z}$. Hence, considering (2.1.1), $\mathtt{W}_1 * \mathtt{W}_2$ and $\mathtt{W}_1 * \tau(\mathtt{W}_2)$ are, in general, in different filtrations of $\mathbb{Q}\langle\mathcal{U}\rangle^\circ$ regarding the number of zeros since we have, in general $z \neq w - z - d$. Therefore, for given $\mathtt{W} \in \mathcal{U}^{*,\circ}$, it is difficult to find the minimal $z \in \mathbb{Z}_{\geq 0}$ such that $\zeta_q^{\mathrm{f}}\left(\mathtt{W}\right) \in \mathrm{Fil}_z^{\mathrm{Z}}\,\mathcal{Z}_q^f$.

Let us consider a small example of how we use $\mathbb{Q}$-linear relations of shape (2.1.1) to obtain that, e.g., $\zeta_q^{\mathrm{f}}\left(\mathtt{W}\right) \in \mathcal{Z}_q^{f,\circ}$ for $\mathtt{W} = u_2u_0 \in \mathcal{U}^{*,\circ}$. First, we note that

$$u_2u_0 = u_1 * u_1u_0 - 2u_1u_1u_0 - u_1u_1 - u_1u_0u_1.$$

Now,

$$\begin{aligned}
0 = &\,\zeta_q^{\mathrm{f}}\left(u_1 * (u_1u_0 - \tau(u_1u_0))\right) - 2\zeta_q^{\mathrm{f}}\left(\mathbf{1} * (u_1u_1u_0 - \tau(u_1u_1u_0))\right)\\
&- \zeta_q^{\mathrm{f}}\left(\mathbf{1} * (u_1u_0u_1 - \tau(u_1u_0u_1))\right)\\
= &\,\zeta_q^{\mathrm{f}}\left(u_1 * u_1u_0\right) - \zeta_q^{\mathrm{f}}\left(u_1 * u_2\right) - 2\zeta_q^{\mathrm{f}}\left(u_1u_1u_0\right) + 2\zeta_q^{\mathrm{f}}\left(u_2u_1\right)\\
&- \zeta_q^{\mathrm{f}}\left(u_1u_0u_1\right) + \zeta_q^{\mathrm{f}}\left(u_1u_2\right),
\end{aligned}$$

and so,

$$\begin{aligned}
\zeta_q^{\mathrm{f}}\left(u_2u_0\right) = &\,\zeta_q^{\mathrm{f}}\left(u_1 * u_2\right) - 2\zeta_q^{\mathrm{f}}\left(u_2u_1\right) - \zeta_q^{\mathrm{f}}\left(u_1u_1\right) - \zeta_q^{\mathrm{f}}\left(u_1u_2\right)\\
= &\,\zeta_q^{\mathrm{f}}\left(u_1u_2\right) + \zeta_q^{\mathrm{f}}\left(u_3\right) - \zeta_q^{\mathrm{f}}\left(u_2u_1\right) - \zeta_q^{\mathrm{f}}\left(u_1u_1\right) - \zeta_q^{\mathrm{f}}\left(u_1u_2\right) \in \mathcal{Z}_q^{f,\circ}.
\end{aligned} \tag{2.3.3}$$

That formal $q$MZVs are in $\mathcal{Z}_q^{f,\circ}$ already is not just a coincidence, as the following conjecture shows.

**Conjecture 2.4** (Bachmann, [3, Conjecture 3.9])**.** *For all $z, d, w \in \mathbb{Z}_{>0}$, we have*

$$\mathrm{Fil}_{z,d,w}^{\mathrm{Z,D,W}}\,\mathcal{Z}_q^f \subset \mathrm{Fil}_{z+d,w}^{\mathrm{D,W}}\,\mathcal{Z}_q^{f,\circ}. \tag{2.4.1}$$

*In particular, we have $\mathcal{Z}_q^f = \mathcal{Z}_q^{f,\circ}$.*

We say that Bachmann's Conjecture 2.4 *is true for* $(z_0, d_0, w_0) \in \mathbb{Z}^3_{>0}$ if (2.4.1) is true for $(z, d, w) = (z_0, d_0, w_0)$.

Partial results already exist; we will collect them in the following.

**Theorem 2.5.** *(i) By Bachmann ([3, Proposition 4.4]), Bachmann's Conjecture 2.4 is true for all* $(z, 1, w) \in \mathbb{Z}^3_{>0}$.

*(ii) also by Bachmann ([3, Proposition 4.4]), Bachmann's Conjecture 2.4 is true for all* $(1, 2, w) \in \mathbb{Z}^3_{>0}$.

*(iii) by Vleeshouwers ([12, Theorem 5.3]), Bachmann's Conjecture 2.4 is true for all triples* $(z, 2, w) \in \mathbb{Z}^3_{>0}$ *with some parity condition on* $w$,

*(iv) and by Burmester ([6, Theorem 6.4]), Bachmann's Conjecture 2.4 is true for all* $(1, d, w) \in \mathbb{Z}^3_{>0}$.

While the proofs of (i)–(iii) are mainly based on generating series of the corresponding $q$-series, the proof of (iv) uses the stuffle product and duality relations. Using relations among formal Multiple Zeta Values of shape (2.1.1) only suffices to prove the following theorem.

**Theorem 2.6** (Theorem 2.26)**.** *Let be* $z, d \in \mathbb{Z}_{>0}$, $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d_{>0}$, *and consider integers* $1 \leq j_1 \leq j_2 \leq d$. *Deconcatenate* $\mathbf{k}$ *as*

$$\mathbf{k}_{(1;j_1)} = (k_1, \ldots, k_{j_1}),\ \mathbf{k}_{(j_1+1;j_2)} = (k_{j_1+1}, \ldots, k_{j_2}),\ \mathbf{k}_{(j_2+1;d)} = (k_{j_2+1}, \ldots, k_d).$$

*We have*

$$\zeta_q^{\mathrm{f}}\left(u_{\mathbf{k}_{(1;j_1)}}\left(u_{\mathbf{k}_{(j_1+1;j_2)}} * u_{\mathbf{k}_{(j_2+1;d)}} u_0^z\right)\right) \in \sum_{s=1}^{z} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w} \mathcal{Z}_q^f,$$

*where* $w = |\mathbf{k}| + z$.

**Remark 2.7.** (i) Theorem 2.6 is a generalization of Bachmann's Theorem [3, Proposition 4.4] via the case $d = 1$. We have already seen the proof for an example of this theorem using our methods in (2.3.3). We will generalize this approach in Proposition 2.21 to generalize Bachmann's Theorem 2.5(i).

(ii) Note that Theorem 2.6 also generalizes Burmester's Theorem [6, Theorem 6.4] via considering the special cases $z = 1$. For details, we refer to Corollary 2.28.

Extending our methods of playing with relations of shape (2.1.1), we observe the following theorem.

**Theorem 2.8** (Theorem 2.75)**.** *Bachmann's Conjecture 2.4 is true for all* $(z, d, w) \in \mathbb{Z}^3_{>0}$ *with* $z + d \leq 6$.

In this paper, we will use duality and the stuffle product only for an approach to write $\zeta_q^{\mathrm{f}}(\mathtt{W})$ for every $\mathtt{W} \in \mathcal{U}^{*,\circ}$ satisfying $\mathrm{zero}(\mathtt{W}) \geq 1$ as linear combination of $\zeta_q^{\mathrm{f}}(\mathtt{W}')$'s with $\mathrm{zero}(\mathtt{W}') < \mathrm{zero}(\mathtt{W})$ and $\mathtt{W}' \in \mathcal{U}^{*,\circ}$. We need the following notion of $\mathrm{F}_{z,d,w}$ for this.

**Definition 2.9.** For $z, d, w \in \mathbb{Z}_{>0}$, we define

$$\mathrm{F}_{z,d,w} := \mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w-1} \mathcal{Z}_q^f + \sum_{\substack{z'+d'=z+d-1 \\ 0 \leq z' \leq z}} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z',d',w} \mathcal{Z}_q^f.$$

In this paper, our main approach towards Bachmann's Conjecture 2.4 is to strengthen the conjecture as follows and then to investigate the strengthened version for obtaining results like Theorem 2.8.

**Conjecture 2.10** (Refined Bachmann Conjecture)**.** *For all $z, d, w \in \mathbb{Z}_{>0}$, we have*

$$\mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w}\, \mathcal{Z}^f_q \subset \mathrm{F}_{z,d,w}\,. \tag{2.10.1}$$

We say that the refined Bachmann Conjecture 2.10 *is true for* $(z_0, d_0, w_0) \in \mathbb{Z}^3_{>0}$, if (2.10.1) is true for $(z, d, w) = (z_0, d_0, w_0)$.

**Lemma 2.11** (Lemma 2.68)**.** *Fix $z, d, w \in \mathbb{Z}_{>0}$. If the refined Bachmann Conjecture 2.10 is true for $(z, d, w)$ and if Bachmann's Conjecture 2.4 is true for all $(z', d', w') \in \mathbb{Z}^3_{>0}$ with $z' + d' + w' < z + d + w$, then Bachmann's Conjecture 2.4 is true for $(z, d, w)$. In particular, the refined Bachmann Conjecture 2.10 implies Bachmann's Conjecture 2.4.*

To study the refined Bachmann Conjecture 2.10, we will introduce the box product (see Definition 2.15) that provides a connection to the stuffle product (see Lemma 2.56) and allows us to refine the refined Bachmann Conjecture 2.10 for $z \geq d$ again (see Conjecture 2.39). In this way, we obtain another particular result towards the refined Bachmann Conjecture 2.10.

**Theorem 2.12** (Theorem 2.76)**.** *The refined Bachmann Conjecture 2.10 is true for all triples of positive integers $(z, d, w) \in \mathbb{Z}^3_{>0}$ with $1 \leq d \leq 4$.*

Theorem 2.12 will follow mainly using Theorem 2.8 and the investigation of the box product from Section 2.4. Furthermore, Theorem 2.12 is a strong statement since - together with some more results of this paper - now, Bachmann's Conjecture 2.4 is almost proven for $z + d \leq 7$ as well: Namely, following Lemma 2.11, it remains to prove the refined Bachmann Conjecture 2.10 for triples of shape $(2, 5, w) \in \mathbb{Z}^3_{>0}$.

All our main results (and those implied by the box product) are based on $\mathbb{Q}$-linear relations of shape (2.1.1) only. Following our approach to a general proof of the refined Bachmann Conjecture 2.10 (and so of Bachmann's Conjecture 2.4 too), described in Section 2.5, it is conjecturally possible to prove the refined Bachmann Conjecture 2.10 using $\mathbb{Q}$-linear relations of shape (2.1.1) only. Based on our results, it seems that this approach works. Furthermore, our explicit approach has the advantage that it is (compared to other approaches) easy to obtain explicit formulas for $\zeta^{\mathrm{f}}_q(\mathtt{W})$ (with $\mathtt{W} \in \mathcal{U}^{*,\circ}$) as element of $\mathcal{Z}^{f,\circ}_q$. Proposition 2.21, for example, contains such an explicit formula. Nevertheless, the explicitness limits this method in the sense that the larger $z + d$ is in the refined Bachmann Conjecture 2.10, the more confusing the $\mathbb{Q}$-linear relations (2.1.1), one needs to consider following our approach, become.

**Organization of the paper.** Section 2.2 contains the introduction of the box product mentioned. Section 2.3 contains generalizations of theorems by Bachmann and Burmester concerning the refined Bachmann Conjecture 2.10, like Theorem 2.6. In Section 2.4, we will investigate the *box product* and consider its connection to the stuffle product. Furthermore, Section 2.5 contains the rough description of our approach to the refined Bachmann Conjecture 2.10. Using the approach from Section 2.5, in Section 2.6, we prove new partial results towards Bachmann's Conjecture 2.4. Particularly, there, we will provide proofs for Theorems 2.8 and 2.12. Last, Section 2.7 ends the paper with some open questions and a rough generalization of our calculations from Section 2.6.

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## 2.2 Introduction of the box product

In this section, we introduce the box product and consider elementary properties. First, we briefly remark on a property of the stuffle product in the following proposition.

**Proposition 2.13.** *Let be* $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^{*,\circ}$ *and write*

$$z = \operatorname{zero}(\tau(\mathtt{W}_1)) + \operatorname{zero}(\tau(\mathtt{W}_2)),\ d_1 = \operatorname{depth}(\mathtt{W}_1),\ d_2 = \operatorname{depth}(\mathtt{W}_2),\ w = \operatorname{wt}(\mathtt{W}_1) + \operatorname{wt}(\mathtt{W}_2).$$

*Then, for* $0 \leq s \leq \min\{d_1, d_2\}$*, there are uniquely determined*

$$\mathcal{L}_{\max\{d_1,d_2\}+s} \in \operatorname{span}_{\mathbb{Q}} \{\mathtt{W} \in \mathcal{U}^{*,\circ} \mid \operatorname{depth}(\mathtt{W}) = \max\{d_1, d_2\} + s\}$$

*such that*

$$\mathtt{W}_1 * \mathtt{W}_2 = \sum_{s=0}^{\min\{d_1,d_2\}} \mathcal{L}_{\max\{d_1,d_2\}+s}.$$

*Furthermore, for all* $0 \leq s \leq \min\{d_1, d_2\}$*, we have*

$$\tau\left(\mathcal{L}_{\max\{d_1,d_2\}+s}\right) \in \operatorname{Fil}^{\mathrm{Z,D,W}}_{z-s,\max\{d_1,d_2\}+s,w} \mathbb{Q}\langle\mathcal{U}\rangle^{\circ}.$$

*In particular,* $\tau\left(\mathcal{L}_{\max\{d_1,d_2\}}\right)$ *is the part of* $\tau(\mathtt{W}_1 * \mathtt{W}_2)$ *having the maximum number of zeros and we have*

$$\tau(\mathtt{W}_1 * \mathtt{W}_2) \in \sum_{s=0}^{\min\{d_1,d_2\}} \operatorname{Fil}^{\mathrm{Z,D,W}}_{z-s,\max\{d_1,d_2\}+s,w} \mathbb{Q}\langle\mathcal{U}\rangle^{\circ}.$$

*Proof.* This is a direct consequence of Equations (2.3.1) and (2.3.2). $\square$

Let us consider an example to point out the statement of Proposition 2.13.

**Example 2.14.** Choose $\mathtt{W}_1 = u_2$, $\mathtt{W}_2 = u_1u_2$, i.e., $d_1 = 1$, $d_2 = 2$ in the notion of Proposition 2.13. We have

$$\mathtt{W}_1 * \mathtt{W}_2 = \underbrace{u_3u_2 + u_1u_4}_{=\mathcal{L}_2} + \underbrace{u_2u_1u_2 + 2u_1u_2u_2}_{=\mathcal{L}_3}.$$

Observe

$$\tau(\mathcal{L}_2) = u_1u_0u_1u_0u_0 + u_1u_0u_0u_0u_1, \quad \tau(\mathcal{L}_3) = u_1u_0u_1u_1u_0 + 2u_1u_0u_1u_0u_1.$$

We see that $\tau(\mathcal{L}_2)$ indeed has the maximum number of zeros in the expression $\tau(u_2*u_1u_2)$.

Since we want to reduce the number of zeros, we often will be interested in the part of the stuffle product only that has the maximum number of zeros. Therefore, Proposition 2.13 motivates the definition of the *box product* that basically extracts this part of the stuffle product after one applies $\tau$.

**Definition 2.15** (Box product)**.** The $\mathbb{Q}$-bilinear *box product* $\circledast\colon \mathbb{Q}\langle\mathcal{U}\rangle^\circ \times \mathbb{Q}\langle\mathcal{U}\rangle^\circ \to \mathbb{Q}\langle\mathcal{U}\rangle^\circ$ is defined as follows: For $\mathtt{W}_j \in \mathcal{U}^{*,\circ}$ with $\operatorname{depth}(\mathtt{W}_j) = d_j$, where $j \in \{1,2\}$, we set

$$\mathtt{W}_1 \circledast \mathtt{W}_2 := \mathcal{L}_{\max\{d_1,d_2\}}$$

in the notion of Proposition 2.13.

For illustration, we continue Example 2.14.

**Example 2.16.** We have

$$u_2 \circledast u_1u_2 = u_3u_2 + u_1u_4,$$

which is exactly $\mathcal{L}_2$ of Example 2.14, i.e., after applying $\tau$, one obtains the part of the stuffle product $u_2 * u_1u_2$ having maximum number of zeros. We state and prove the generalization of this observation in Lemma 2.56.

**Corollary 2.17.** *Let be* $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^{*,\circ}$ *and write*

$$z = \operatorname{zero}(\tau(\mathtt{W}_1)) + \operatorname{zero}(\tau(\mathtt{W}_2)),\ d_1 = \operatorname{depth}(\mathtt{W}_1),\ d_2 = \operatorname{depth}(\mathtt{W}_2),\ w = \operatorname{wt}(\mathtt{W}_1) + \operatorname{wt}(\mathtt{W}_2).$$

*Then,*

$$\tau(\mathtt{W}_1 * \mathtt{W}_2) - \tau(\mathtt{W}_1 \circledast \mathtt{W}_2) \in \sum_{s=1}^{\min\{d_1,d_2\}} \operatorname{Fil}^{\mathrm{Z,D,W}}_{z-s,\max\{d_1,d_2\}+s,w} \mathbb{Q}\langle\mathcal{U}\rangle^\circ.$$

*Proof.* This is an immediate consequence of Proposition 2.13 and the definition of the box product. $\square$

**Lemma 2.18.** *Consider the alphabet* $\mathcal{U}\backslash\{u_0\} = \{u_j \mid j \in \mathbb{Z}_{>0}\}$*. The restriction of the box product* $\circledast\colon \mathbb{Q}\langle\mathcal{U}\backslash\{u_0\}\rangle \times \mathbb{Q}\langle\mathcal{U}\backslash\{u_0\}\rangle \to \mathbb{Q}\langle\mathcal{U}\backslash\{u_0\}\rangle$ *can be described as follows. For any two words* $\mathtt{W}_1 = u_{n_1}\cdots u_{n_s}$, $\mathtt{W}_2 = u_{\ell_1}\cdots u_{\ell_r} \in (\mathcal{U}\backslash\{u_0\})^*$*, we set recursively*

$$\mathtt{W}_1 \tilde{\circledast} \mathtt{W}_2 := \begin{cases} 0, & \text{if } s > r,\\ \mathtt{W}_2, & \text{if } \mathtt{W}_1 = \mathbf{1},\\ u_{\ell_1}\left(\mathtt{W}_1 \tilde{\circledast} u_{\ell_2}\cdots u_{\ell_r}\right) + u_{n_1+\ell_1}\left(u_{n_2}\cdots u_{n_s} \tilde{\circledast} u_{\ell_2}\cdots u_{\ell_r}\right), & \text{if } s \le r. \end{cases}$$

*Then,* $\mathtt{W}_1 \tilde{\circledast} \mathtt{W}_2 = \mathtt{W}_1 \circledast \mathtt{W}_2$ *whenever* $\operatorname{len}(\mathtt{W}_1) \le \operatorname{len}(\mathtt{W}_2)$.

Note that the box product satisfies the following connection to the stuffle product.

**Lemma 2.19.** *For all indices of positive integers* $\boldsymbol{n_1}, \boldsymbol{n_2}, \boldsymbol{\ell}$*, we have*

$$u_{\boldsymbol{n_1}} \circledast (u_{\boldsymbol{n_2}} \circledast u_{\boldsymbol{\ell}}) = (u_{\boldsymbol{n_1}} * u_{\boldsymbol{n_2}}) \circledast u_{\boldsymbol{\ell}} = u_{\boldsymbol{n_2}} \circledast (u_{\boldsymbol{n_1}} \circledast u_{\boldsymbol{\ell}}).$$

*Proof.* The proof of the first equality follows by induction on $\operatorname{len}(\boldsymbol{n_1}) + \operatorname{len}(\boldsymbol{n_2})$ and the definition of stuffle and box product. The second equality then follows from the commutativity of the stuffle product and the first equality. $\square$

Next, we make an easy but instrumental observation. For this, we denote for an given index $\mathbf{k} = (k_1, \dots, k_r)$ its *reversed index* by $\operatorname{rev}(\mathbf{k}) := (k_r, \dots, k_1)$.

**Proposition 2.20.** *Given* $\mathbf{n} \in \mathbb{Z}^s_{>0}$, $\boldsymbol{\ell} \in \mathbb{Z}^d_{>0}$ *with* $1 \le s \le d$*. Writing*

$$u_{\mathbf{n}} \circledast u_{\boldsymbol{\ell}} = \sum_{\boldsymbol{\mu} \in \mathbb{Z}^d_{>0}} a_{\boldsymbol{\mu}} u_{\boldsymbol{\mu}}$$

*with $a_{\boldsymbol{\mu}} \in \mathbb{Z}$ appropriate, we have*

$$u_{\mathrm{rev}(\mathbf{n})} \circledast u_{\mathrm{rev}(\boldsymbol{\ell})} = \sum_{\boldsymbol{\mu} \in \mathbb{Z}_{>0}^d} a_{\boldsymbol{\mu}} u_{\mathrm{rev}(\boldsymbol{\mu})}.$$

*Proof.* Using Lemma 2.18 and induction on $\mathrm{len}(\mathbf{n}) + \mathrm{len}(\boldsymbol{\ell})$, the claim follows immediately. □

## 2.3 A common approach to theorems by Bachmann and Burmester

In this section, we consider the cases of $d = 1$ (and $z \in \mathbb{Z}_{>0}$ arbitrary), and $z = 1$ (and $d \in \mathbb{Z}_{>0}$ arbitrary), respectively, of Bachmann's Conjecture 2.4. The first case mainly is a result originally due to Bachmann ([3, Proposition 4.4]), which we will reprove in a way giving explicit formulas for every element of $\mathrm{Fil}^{\mathrm{Z,D,W}}_{z,1,w}\, \mathcal{Z}_q^f$ as linear combination of elements in $\mathrm{Fil}^{\mathrm{D,W}}_{z+1,w}\, \mathcal{Z}_q^f$. The second case is done by Burmester's thesis ([6, Theorem 6.4]), which we will extend in Section 2.3.2.

### 2.3.1 Bachmann's Conjecture 2.4 for $(z, 1, w)$

By [3, Proposition 4.4] (see also Theorem 2.5(i)), it is known that Bachmann's Conjecture 2.4 is true for all triples $(z, 1, w)$. Here, we give an alternative proof which gives an explicit expression in terms of elements in $\mathcal{Z}_q^{f,\circ}$.

**Proposition 2.21.** *For all $k \in \mathbb{Z}_{>0}$ and $z \in \mathbb{Z}_{\geq 0}$, we have that $\zeta_q^{\mathrm{f}}(u_k u_0^z)$ equals*

$$\begin{aligned}
&(-1)^z \sum_{\substack{j_1, j_2 \geq 0 \\ j_1 + j_2 = z}} \sum_{\substack{n_0, \ldots, n_{j_2} \geq 0 \\ n_0 + \cdots + n_{j_2} = k-1}} \sum_{\substack{1 \leq p \leq j_2 \\ 0 \leq \varepsilon_p \leq \min\{1, n_p\}}} \zeta_q^{\mathrm{f}}\left(u_{n_{j_2} - \varepsilon_{j_2} + 1} \cdots u_{n_1 - \varepsilon_1 + 1} u_{n_0 + 1} u_1^{j_1}\right) \\
&+ \sum_{1 \leq j \leq z} \sum_{\substack{\ell_1, \ldots, \ell_j \geq 1 \\ \ell_1 + \cdots + \ell_j \leq z}} \sum_{\substack{j_1, j_2 \geq 0 \\ j_1 + j_2 = z - \ell_1 - \cdots - \ell_j}} \\
&\sum_{\substack{n_0, \ldots, n_{j_2} \geq 0 \\ n_0 + \cdots + n_{j_2} = k-1}} \sum_{\substack{1 \leq p \leq j_2 \\ 0 \leq \varepsilon_p \leq \min\{1, n_p\}}} (-1)^{z-j} \zeta_q^{\mathrm{f}}\left(u_1^{\ell_1} * \cdots * u_1^{\ell_j} * u_{n_{j_2} - \varepsilon_{j_2} + 1} \cdots u_{n_1 - \varepsilon_1 + 1} u_{n_0 + 1} u_1^{j_1}\right).
\end{aligned}$$

*In particular, we have $\zeta_q^{\mathrm{f}}(u_k u_0^z) \in \mathrm{Fil}^{\mathrm{D,W}}_{z+1,k+z}\, \mathcal{Z}_q^{f,\circ}$, yielding Bachmann's Conjecture 2.4 for all triples $(z, 1, w)$.*

*Proof.* First note that a calculation, using the definition of the stuffle product, shows for all $a \in \mathbb{Z}_{>0}$, $b \in \mathbb{Z}_{\geq 0}$ the identity

$$u_a u_0^b = \sum_{\ell=1}^{a-1} (-1)^{\ell-1} u_1^{\ell} * u_{a-\ell} u_0^b + (-1)^{a-1} h(a, b), \tag{2.21.1}$$

where $h(a, b) := \sum_{\substack{j_1, j_2 \geq 0 \\ j_1 + j_2 = a-1}} u_1^{j_1+1}\left(u_1^{j_2} * u_0^b\right)$. Choosing $a = z + 1$ and $b = k - 1$, we obtain

$$u_{z+1} u_0^{k-1} = \sum_{\ell=1}^{z} (-1)^{\ell-1} u_1^{\ell} * u_{z+1-\ell} u_0^{k-1} + (-1)^z h(z+1, k-1).$$

Using the latter formula and (2.21.1) repeatedly, we obtain

$$u_{z+1}u_0^{k-1} = \sum_{1\leq j\leq z} \sum_{\substack{\ell_1,\dots,\ell_j\geq 1\\ \ell_1+\dots+\ell_j\leq z}} (-1)^{z-j} u_1^{\ell_1} * \dots * u_1^{\ell_j} * h(z+1-\ell_1-\dots-\ell_j, k-1) \qquad (2.21.2)$$
$$+ (-1)^z h(z+1,k-1).$$

Now, note that for all $a \in \mathbb{Z}_{>0}$ and $b \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned} h(a,b) &= \sum_{\substack{j_1,j_2\geq 0\\ j_1+j_2=a-1}} u_1^{j_1+1}\left(u_1^{j_2} * u_0^b\right) \\ &= \sum_{\substack{j_1,j_2\geq 0\\ j_1+j_2=a-1}} \sum_{\substack{n_0,\dots,n_{j_2}\geq 0\\ n_0+\dots+n_{j_2}=b}} \sum_{\substack{1\leq p\leq j_2\\ 0\leq\varepsilon_p\leq\min\{1,n_p\}}} u_1^{j_1+1}u_0^{n_0}u_1u_0^{n_1-\varepsilon_1}\cdots u_1u_0^{n_{j_2}-\varepsilon_{j_2}}. \end{aligned}$$

Hence, by $\tau$-invariance of formal $q$MZVs,

$$\zeta_q^{\mathrm{f}}(h(a,b)) = \sum_{\substack{j_1,j_2\geq 0\\ j_1+j_2=a-1}} \sum_{\substack{n_0,\dots,n_{j_2}\geq 0\\ n_0+\dots+n_{j_2}=b}} \sum_{\substack{1\leq p\leq j_2\\ 0\leq\varepsilon_p\leq\min\{1,n_p\}}} \zeta_q^{\mathrm{f}}\left(u_{n_{j_2}-\varepsilon_{j_2}+1}\cdots u_{n_1-\varepsilon_1+1}u_{n_0+1}u_1^{j_1}\right),$$

implying the claim when using (2.21.2) and $\zeta_q^{\mathrm{f}}(u_k u_0^z) = \zeta_q^{\mathrm{f}}(\tau(u_k u_0^z)) = \zeta_q^{\mathrm{f}}\left(u_{z+1}u_0^{k-1}\right)$. From the obtained representation of $\zeta_q^{\mathrm{f}}(u_k u_0^z)$, we get directly $\zeta_q^{\mathrm{f}}(u_k u_0^z) \in \mathrm{Fil}^{\mathrm{D,W}}_{z+1,k+z}\,\mathcal{Z}_q^{f,\circ}$ due to (2.3.1). □

Let us consider an example regarding Proposition 2.21.

**Example 2.22.** For $k = z = 2$, Proposition 2.21 yields

$$\begin{aligned} &\zeta_q^{\mathrm{f}}\left(u_2u_0^2\right) \\ &= \zeta_q^{\mathrm{f}}(u_1 * u_1 * u_2) - 2\zeta_q^{\mathrm{f}}(u_1 * u_2u_1) - \zeta_q^{\mathrm{f}}(u_1 * u_1u_2) - \zeta_q^{\mathrm{f}}(u_1 * u_1u_1) - \zeta_q^{\mathrm{f}}(u_1u_1 * u_2) \\ &\quad + 3\zeta_q^{\mathrm{f}}(u_2u_1u_1) + 2\zeta_q^{\mathrm{f}}(u_1u_2u_1) + \zeta_q^{\mathrm{f}}(u_1u_1u_2) + 3\zeta_q^{\mathrm{f}}(u_1u_1u_1) \\ &= \zeta_q^{\mathrm{f}}(u_4) - \zeta_q^{\mathrm{f}}(u_3u_1) - \zeta_q^{\mathrm{f}}(u_2u_2) - \zeta_q^{\mathrm{f}}(u_2u_1) - \zeta_q^{\mathrm{f}}(u_1u_2) \in \mathrm{Fil}^{\mathrm{D,W}}_{2,4}\,\mathcal{Z}_q^{f,\circ} \subset \mathrm{Fil}^{\mathrm{D,W}}_{3,4}\,\mathcal{Z}_q^{f,\circ}. \end{aligned}$$

### 2.3.2 Bachmann's Conjecture 2.4 for $(1, d, w)$

Given an index $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}_{>0}^d$, we introduce the following notation of subindices

$$\mathbf{k}_{(j_1;j_2)} := \begin{cases} (k_{j_1}, \dots, k_{j_2}), & \text{if } 1 \leq j_1 \leq j_2 \leq d, \\ \emptyset, & \text{else.} \end{cases}$$

**Lemma 2.23.** *Fix $z, d \in \mathbb{Z}_{>0}$ and $\mathbf{k} \in \mathbb{Z}_{>0}^d$. For $1 \leq j \leq d$, we have*

$$\zeta_q^{\mathrm{f}}\left(u_{k_1}\left(u_{\mathbf{k}_{(2;j)}} * u_{\mathbf{k}_{(j+1;d)}}u_0^z\right)\right) \in \sum_{s=1}^{z} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w}\,\mathcal{Z}_q^f,$$

*where $w = |\mathbf{k}| + z$.*

*Proof.* We prove by induction on $d$. The base case $d = 1$ corresponds to Proposition 2.21 since then $j = 1$ and so $\mathbf{k}_{(2;j)} = \mathbf{k}_{(j+1;d)} = \emptyset$. Hence, we may assume $d > 1$ and that Lemma 2.23 is proven already for all smaller values of $d$. First, note that the case $j = d$

follows from $\mathbf{k}_{(j+1;d)} = \emptyset$ in this case and from

$$\sum_{\substack{n_1,\dots,n_{s'}\geq 1\\ n_1+\dots+n_{s'}=z\\ 1\leq s'\leq d}} \zeta_q^{\mathrm{f}}\left(u_{n_1}\cdots u_{n_{s'}} * u_{\mathbf{k}}\right) \in \sum_{s=1}^{z} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w}\,\mathcal{Z}_q^f$$

since

$$\begin{aligned}
&\sum_{\substack{n_1,\dots,n_{s'}\geq 1\\ n_1+\dots+n_{s'}=z\\ 1\leq s'\leq d}} \zeta_q^{\mathrm{f}}\left(u_{n_1}\cdots u_{n_{s'}} * \tau(u_{\mathbf{k}})\right)\\
=&\sum_{\substack{n_1,\dots,n_{s'}\geq 1\\ n_1+\dots+n_{s'}=z\\ 1\leq s'\leq d}} \zeta_q^{\mathrm{f}}\left(u_{n_1}\cdots u_{n_{s'}} * u_1u_0^{k_d-1}\cdots u_1u_0^{k_1-1}\right)\\
=&\sum_{\substack{n_1,\dots,n_{s'}\geq 1\\ n_1+\dots+n_{s'}=z\\ 1\leq s'\leq d}} \zeta_q^{\mathrm{f}}\left(\tau\left(u_{n_1}\cdots u_{n_{s'}} * u_1u_0^{k_d-1}\cdots u_1u_0^{k_1-1}\right)\right)\\
\equiv&\sum_{\substack{n_1,\dots,n_{s'}\geq 1\\ n_1+\dots+n_{s'}=z\\ 1\leq s'\leq d}} \zeta_q^{\mathrm{f}}\left(\tau\left(u_{n_1}\cdots u_{n_{s'}} \circledast u_1u_0^{k_d-1}\cdots u_1u_0^{k_1-1}\right)\right) \mod \sum_{s=1}^{z} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w}\,\mathcal{Z}_q^f.
\end{aligned}$$

The last identity is a consequence of Proposition 2.13 and the definition of the box product. Furthermore, the remaining expression is

$$\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}\left(u_{\mathbf{k}_{(2;d)}} * u_0^z\right)\right) \mod \sum_{s=1}^{z} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w}\,\mathcal{Z}_q^f,$$

which can be verified via induction on $s'+d$ and the definition of the stuffle product. Hence, let be $1 \leq j \leq d-1$ and assume that the claim holds for all larger values of $j$. The induction hypothesis on $d$ implies, since $\mathrm{len}(\emptyset)+\mathrm{len}\left(\mathbf{k}_{(j+2;d)}\right) = d-j-1 < d-1$,

$$\zeta_q^{\mathrm{f}}\left(u_{\mathbf{k}_{(j+1;d)}}u_0^z\right) = \zeta_q^{\mathrm{f}}\left(u_{k_{j+1}}(u_\emptyset * u_{\mathbf{k}_{(j+2;d))}}u_0^z)\right) \in \sum_{s=1}^{z} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d-j+s,w'},$$

where $w' = |\mathbf{k}_{(j+1;d)}| + z$. Hence, by (2.3.1), we obtain

$$\zeta_q^{\mathrm{f}}\left(u_{\mathbf{k}_{(1;j)}} * u_{\mathbf{k}_{(j+1;d)}}u_0^z\right) \in \sum_{s=1}^{z} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w}\,\mathcal{Z}_q^f. \tag{2.23.1}$$

Now, using the definition of the stuffle product, we obtain

$$\begin{aligned}
u_{\mathbf{k}_{(1;j)}} * u_{\mathbf{k}_{(j+1;d)}}u_0^z =& u_{k_1}\left(u_{\mathbf{k}_{(2;j)}} * u_{\mathbf{k}_{(j+1;d)}}u_0^z\right) + u_{k_{j+1}}\left(u_{\mathbf{k}_{(1;j)}} * u_{\mathbf{k}_{(j+2;d)}}u_0^z\right)\\
&+ u_{k_1+k_{j+1}}\left(u_{\mathbf{k}_{(2;j)}} * u_{\mathbf{k}_{(j+2;d)}}u_0^z\right).
\end{aligned}$$

Note that the formal $q$MZVof the second summand on the right-hand side is an element of $\sum_{s=1}^{z} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w}\,\mathcal{Z}_q^f$ due to the assumption on $j$, while the formal $q$MZV of the third one is by induction hypothesis on $d$. Hence, because of (2.23.1), we obtain

$$\zeta_q^{\mathrm{f}}\left(u_{k_1}\left(u_{\mathbf{k}_{(2;j)}} * u_{\mathbf{k}_{(j+1;d)}} u_0^z\right)\right) \in \sum_{s=1}^{z} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w} \mathcal{Z}_q^f,$$

completing the induction step. Therefore, the lemma is proven. □

**Corollary 2.24.** *Fix $z, d \in \mathbb{Z}_{>0}$. For all $\mathbf{k} \in \mathbb{Z}_{>0}^d$, we have*

$$\zeta_q^{\mathrm{f}}\left(u_{\mathbf{k}} u_0^z\right) \in \sum_{s=1}^{z} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w} \mathcal{Z}_q^f,$$

*where $w = |\mathbf{k}| + z$.*

*Proof.* This is the special case $j = 1$ of Lemma 2.23. □

**Corollary 2.25.** *Fix $d \in \mathbb{Z}_{>0}$. For all $\mathbf{k} \in \mathbb{Z}_{>0}^d$, we have*

$$\zeta_q^{\mathrm{f}}\left(u_{\mathbf{k}} u_0 u_0\right) \in \mathrm{Fil}^{\mathrm{D,W}}_{d+2,w} \mathcal{Z}_q^{f,\circ},$$

*where $w = |\mathbf{k}| + 2$.*

*Proof.* The special case $z = 2$ of Corollary 2.24 and $\mathrm{Fil}^{\mathrm{Z,D,W}}_{1,d+1,w} \mathcal{Z}_q^f \subset \mathrm{Fil}^{\mathrm{D,W}}_{d+2,w} \mathcal{Z}_q^{f,\circ}$ by Burmester's Theorem 2.5(iv) yield the claim. □

Lemma 2.23 is a special case of the following theorem.

**Theorem 2.26** (Theorem 2.6)**.** *Let be $z, d \in \mathbb{Z}_{>0}$, $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}_{>0}^d$, and consider integers $1 \leq j_1 \leq j_2 \leq d$. We have*

$$\zeta_q^{\mathrm{f}}\left(u_{\mathbf{k}_{(1;j_1)}}\left(u_{\mathbf{k}_{(j_1+1;j_2)}} * u_{\mathbf{k}_{(j_2+1;d)}} u_0^z\right)\right) \in \sum_{s=1}^{z} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w} \mathcal{Z}_q^f, \tag{2.26.1}$$

*where $w = |\mathbf{k}| + z$.*

*Proof.* We prove by induction on $d$. Note that the base case $d = 1$ follows from Proposition 2.21 since then $j_1 = j_2 = 1$ and so $\mathbf{k}_{(j_1+1;j_2)} = \mathbf{k}_{(j_2+1;d)} = \emptyset$. Hence, choose $d > 1$ and assume the theorem is proven for all smaller values of $d$. Furthermore, note that the case $j_1 = 1$ is nothing else than Lemma 2.23. Hence, let $2 \leq j_1 \leq d$ arbitrary. The claim for $j_2 = j_1$ corresponds to Corollary 2.24 since then $\mathbf{k}_{(j_1+1;j_2)} = \emptyset$. Therefore, assume $j_2 > j_1 > 1$ in the following and that the claim is proven for all possible smaller values of $j_1$, $j_2$ and $\mathrm{len}(\mathbf{k}_{(j_1+1;j_2)}) = j_2 - j_1$, respectively. Using the recursive definition of the stuffle product gives

$$\begin{aligned}
&u_{\mathbf{k}_{(1;j_1)}}\left(u_{\mathbf{k}_{(j_1+1;j_2)}} * u_{\mathbf{k}_{(j_2+1;d)}} u_0^z\right) \\
&= u_{\mathbf{k}_{(1;j_1-1)}}\left(u_{\mathbf{k}_{(j_1+1;j_2)}} * u_{k_{j_1}} u_{\mathbf{k}_{(j_2+1;d)}} u_0^z\right) \\
&\quad - u_{\mathbf{k}_{(1;j_1-1)}} u_{k_{j_1+1}}\left(u_{\mathbf{k}_{(j_1+2;j_2)}} * u_{k_{j_1}} u_{\mathbf{k}_{(j_2+1;d)}} u_0^z\right) \\
&\quad - u_{\mathbf{k}_{(1;j_1-1)}} u_{k_{j_1}+k_{j_1+1}}\left(u_{\mathbf{k}_{(j_1+2;j_2)}} * u_{\mathbf{k}_{(j_2+1;d)}} u_0^z\right)
\end{aligned}$$

Now, the formal $q$MZVof the first summand on the right-hand side is in $\sum_{s=1}^{z} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w} \mathcal{Z}_q^f$ due to the assumption on $j_1$ (since $\mathrm{len}\left(\mathbf{k}_{(1;j_1-1)}\right) = \mathrm{len}\left(\mathbf{k}_{(1;j_1)}\right) - 1$), while the second one is as well due to the assumption on $j_2-j_1$ (since $\operatorname{len}\left(\mathbf{k}_{(j_1+2;j_2)}\right)=\operatorname{len}\left(\mathbf{k}_{(j_1+1;j_2)}\right)-1$), and the third one is due to the induction hypothesis on $d$. In particular, we have

$$\zeta_q^{\mathrm{f}}\left(u_{\mathbf{k}_{(1;j_1)}}\left(u_{\mathbf{k}_{(j_1+1;j_2)}} * u_{\mathbf{k}_{(j_2+1;d)}} u_0^z\right)\right) \in \sum_{s=1}^{z} \operatorname{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w} \mathcal{Z}_q^f,$$

completing the induction step. Hence, the theorem follows. $\square$

**Corollary 2.27.** *Let be $z,d \in \mathbb{Z}_{>0}$, $\mathbf{k}=(k_1,\dots,k_d)\in\mathbb{Z}^d_{>0}$. For all $1\le j\le d$, we have*

$$\sum_{\substack{\ell_j,\dots,\ell_d\ge 0\\ \ell_j+\cdots+\ell_d=z}} \zeta_q^{\mathrm{f}}\left(u_{\mathbf{k}_{(1;j-1)}} u_{k_j} u_0^{\ell_j}\cdots u_{k_d} u_0^{\ell_d}\right) \in \sum_{s=1}^{z} \operatorname{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w} \mathcal{Z}_q^f, \tag{2.27.1}$$

*where $w=|\mathbf{k}|+z$.*

*Proof.* Let be $1\le j\le d$. The corollary is obtained from the special case $j_1=j$, $j_2=d$ of Theorem 2.26 and multiplying out the corresponding stuffle product occurring in (2.26.1) since then $\mathbf{k}_{(j_2+1;d)}=\emptyset$. $\square$

As a corollary of Corollary 2.27, we obtain Burmester's Theorem 2.5(iv).

**Corollary 2.28** (Burmester, [6, Theorem 6.4])**.** *Bachmann's Conjecture 2.4 is true for all $(1,d,w)\in\mathbb{Z}^3_{>0}$.*

*Proof.* Let be $d\in\mathbb{Z}_{>0}$, $\mathbf{k}=(k_1,\dots,k_d)\in\mathbb{Z}^d_{>0}$ and denote $w=|\mathbf{k}|+1$ in the following. Considering Corollary 2.27 with $z=1$ and $j=d$, we obtain $\zeta_q^{\mathrm{f}}(u_{\mathbf{k}}u_0)\in \operatorname{Fil}^{\mathrm{D,W}}_{d+1,w}\mathcal{Z}_q^{f,\circ}$. Now, let be $1\le j'\le d-1$. Considering the difference of (2.27.1) with $z=1$, $j=j'$ and (2.27.1) with $z=1$, $j=j'+1$, we obtain

$$\zeta_q^{\mathrm{f}}\left(u_{\mathbf{k}_{(1;j')}} u_0 u_{\mathbf{k}_{(j'+1;d)}}\right) \in \operatorname{Fil}^{\mathrm{D,W}}_{d+1,w}\mathcal{Z}_q^{f,\circ}.$$

In particular, for every $\mathtt{W}\in\mathcal{U}^{*,\circ}\cap\operatorname{Fil}^{\mathrm{Z,D,W}}_{1,d,w}\mathbb{Q}\langle\mathcal{U}\rangle^\circ$, we have shown $\zeta_q^{\mathrm{f}}(\mathtt{W})\in\operatorname{Fil}^{\mathrm{D,W}}_{d+1,w}\mathcal{Z}_q^{f,\circ}$, i.e., we have $\operatorname{Fil}^{\mathrm{Z,D,W}}_{1,d,w}\mathcal{Z}_q^f\subset\operatorname{Fil}^{\mathrm{D,W}}_{d+1,w}\mathcal{Z}_q^{f,\circ}$, completing the claim. $\square$

**Corollary 2.29.** *Let be $d\in\mathbb{Z}_{\ge 2}$ and $\mathbf{k}=(k_1,\dots,k_d)\in\mathbb{Z}^d_{>0}$. We have*

$$\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_{\mathbf{k}_{(3;d)}}\right)\in\operatorname{Fil}^{\mathrm{D,W}}_{d+2,w}\mathcal{Z}_q^{f,\circ},$$

*where $w=|\mathbf{k}|+2$.*

*Proof.* Consider the difference of (2.27.1) for $z=2$, $j=2$, and (2.27.1) for $z=2$, $j=3$ to obtain, all congruences modulo $\operatorname{Fil}^{\mathrm{Z,D,W}}_{1,d+1,w}\mathcal{Z}_q^f$,

$$\begin{aligned}
0 &\equiv \sum_{\substack{\ell_3,\dots,\ell_d\ge 0\\ \ell_3+\cdots+\ell_d=2}} \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0^{\ell_3}\cdots u_{k_d}u_0^{\ell_d}\right) - \sum_{\substack{\ell_2,\dots,\ell_d\ge 0\\ \ell_2+\cdots+\ell_d=2}} \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0^{\ell_2}\cdots u_{k_d}u_0^{\ell_d}\right)\\
&\equiv -\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{\mathbf{k}_{(3;d)}}\right) - \sum_{\substack{\ell_3,\dots,\ell_d\ge 0\\ \ell_3+\cdots+\ell_d=1}} \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_0^{\ell_3}\cdots u_{k_d}u_0^{\ell_d}\right)\\
&\equiv -\zeta_q^{\mathrm{f}}\left(u_1u_0^{k_d-1}\cdots u_1u_0^{k_3-1}u_3u_0^{k_2-1}u_1u_0^{k_1-1}\right)\\
&\quad - \sum_{\substack{\ell_3,\dots,\ell_d\ge 0\\ \ell_3+\cdots+\ell_d=1}} \zeta_q^{\mathrm{f}}\left(u_{\ell_d+1}u_0^{k_d-1}\cdots u_{\ell_3+1}u_0^{k_3-1}u_2u_0^{k_2-1}u_1u_0^{k_1-1}\right)
\end{aligned}$$

$$\equiv \zeta_q^{\mathrm{f}}\left(u_1u_0^{k_d-1}\cdots u_1u_0^{k_3-1}u_2u_0^{k_2-1}u_2u_0^{k_1-1}\right)-\zeta_q^{\mathrm{f}}\left(u_1*\tau\left(u_{k_1}u_{k_2}u_0u_{\mathbf{k}_{(3;d)}}\right)\right)$$
$$\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_{\mathbf{k}_{(3;d)}}\right).$$

Since $\mathrm{Fil}^{\mathrm{Z,D,W}}_{1,d+1,w}\,\mathcal{Z}_q^f\subset \mathrm{Fil}^{\mathrm{D,W}}_{d+2,w}\,\mathcal{Z}_q^{f,\circ}$ by Corollary 2.28, the claim follows. ☐

## 2.4 Investigation of the box product

First, in Section 2.4.1, we show that several monomials can already be written as a $\mathbb{Q}$-linear combination of non-trivial box products. In Section 2.4.2, we investigate a conjecture (Conjecture 2.39) regarding the structure of box products and give partial results for it. Furthermore, in Section 2.4.3, we study the main connection between the box product and the stuffle product that we will need to prove our main results. Last, in Section 2.4.4, we give some further results about the box product that are interesting for itself but not necessary for the remaining paper.

### 2.4.1 Monomials as linear combination of box products

In the following, we characterize some particular monomials in $\mathbb{Q}\langle\mathcal{U}\backslash\{u_0\}\rangle$ as a linear combination of box products. The results will be important for proving Theorem 2.76.

We will need the $\mathbb{Q}$-vector space spanned by (non-trivial) box products in the following.

**Definition 2.30.** We define

$$\mathcal{P}:=\mathrm{span}_{\mathbb{Q}}\left\{\mathtt{W}_1\circledast\mathtt{W}_2 \mid \mathtt{W}_1,\,\mathtt{W}_2\in(\mathcal{U}\backslash\{u_0\})^*,\,\mathtt{W}_1,\mathtt{W}_2\neq\mathbf{1}\right\}\subset\mathbb{Q}\langle\mathcal{U}\backslash\{u_0\}\rangle. \tag{2.30.1}$$

**Corollary 2.31.** *Given $\boldsymbol{\mu}\in\mathbb{Z}^d_{>0}$ with $d\in\mathbb{Z}_{>0}$. Then $u_{\boldsymbol{\mu}}\in\mathcal{P}$ if and only if $u_{\mathrm{rev}(\boldsymbol{\mu})}\in\mathcal{P}$.*

*Proof.* This is an immediate consequence of Proposition 2.20. ☐

**Lemma 2.32.** *For all $d\in\mathbb{Z}_{>0}$ and $0\le j\le d-1$, we have*

$$u_2^d,\ u_1^ju_{1+d}u_1^{d-j-1}\in\mathcal{P}.$$

*Proof.* A direct calculation shows $u_2^d=u_1^d\circledast u_1^d$, giving the first part of the lemma. Furthermore, for all $0\le j\le d-1$, we have

$$u_1^ju_{1+d}u_1^{d-j-1}=\sum_{a=1}^{d}(-1)^{a-1}\,u_1^a\circledast u_1^ju_{1+d-a}u_1^{d-j-1},$$

giving the second claim of the lemma. ☐

**Lemma 2.33.** *For arbitrary $d\in\mathbb{Z}_{>0}$ and $0\le j\le d-2$, we have*

$$u_1u_2^ju_3u_2^{d-j-2}\in\mathcal{P}.$$

*Proof.* For any $0\le j\le d-2$, one verifies

$$\begin{aligned}&u_1u_2^ju_3u_2^{d-j-2}\\&=\sum_{a=1}^{j+1}(-1)^{a+1}\,u_1^{j-a+1}u_2u_1^{d-j-2}\circledast u_au_1^{d-1}+\sum_{a=1}^{j+2}(-1)^{a+1}\,u_1^{d-a+1}\circledast u_au_1^{d-1}.\end{aligned}$$ ☐

We first need an auxiliary lemma to prove the statements in Corollary 2.35 and Lemma 2.36.

**Lemma 2.34.** *For all $d, \mu_1, \mu_2 \in \mathbb{Z}_{>0}$ with $\mu_1 + \mu_2 \leq d + 2$, we have*

$$u_{\mu_1} u_{\mu_2} (u_1^{d-\mu_1-\mu_2+2} \circledast u_1^{d-2}) \in \mathcal{P}.$$

*Proof.* We prove by induction on $\mu_1$. First, consider $\mu_1 = 1$. Similarly to the proof of Lemma 2.33, we obtain by direct calculation that

$$\begin{aligned} &u_1 u_{\mu_2} (u_1^{d-\mu_2+1} \circledast u_1^{d-2}) \\ =& - u_1 u_{\mu_2-1} (u_1^{d-\mu_2+2} \circledast u_1^{d-2}) - \sum_{\substack{0 \leq a \leq \mu_2 - 3 \\ 0 \leq b \leq 1+a}} (-1)^b \, u_1^{d-\mu_2+b+2} \circledast u_{2+a-b} u_{\mu_2-2-a} u_1^{d-2} \\ &+ \sum_{\substack{0 \leq a \leq \mu_2 - 3 \\ 0 \leq b \leq a}} (-1)^{a+b} \, u_1^{a-b} u_2 u_1^{d-\mu_2+1} \circledast u_{1+b} u_{\mu_2-2-a} u_1^{d-2}. \end{aligned}$$

Hence, we have for all $\mu_2 \in \mathbb{Z}_{>0}$ that $u_1 u_{\mu_2} (u_1^{d-\mu_2+1} \circledast u_1^{d-2}) \in \mathcal{P}$ if and only if we have $u_1 u_{\mu_2-1} (u_1^{d-\mu_2+2} \circledast u_1^{d-2}) \in \mathcal{P}$, giving recursively that $u_1 u_{\mu_2} (u_1^{d-\mu_2+1} \circledast u_1^{d-2}) \in \mathcal{P}$ if and only if

$$u_1 u_3 (u_1^{d-2} \circledast u_1^{d-2}) \in \mathcal{P},$$

which is true since this is the $j = 0$ case of Lemma 2.33.

Now, for $\mu_1 > 1$, assume that the lemma is proven for $\mu_1 - 1$ already. We calculate

$$\begin{aligned} u_{\mu_1} u_{\mu_2} (u_1^{d-\mu_1-\mu_2+2} \circledast u_1^{d-2}) = & \sum_{a=\mu_2}^{d-\mu_1+2} (-1)^{\mu_2+a} \, u_1^{d-\ell_1-a+3} \circledast u_{\mu_1-1} u_a u_1^{d-2} \\ & - u_{\mu_1-1} u_{\ell_2} (u_1^{d-\mu_1-\mu_2+3} \circledast u_1^{d-2}) \\ & + (-1)^{d-\mu_1+1-\mu_2} \, u_{\mu_1-1} u_{d-\mu_1+3} u_1^{d-2}. \end{aligned}$$

I.e., we have $u_{\mu_1} u_{\mu_2} (u_1^{d-\mu_1-\mu_2+2} \circledast u_1^{d-2}) \in \mathcal{P}$ by the assumption that the lemma is proven for $\mu_1 - 1$. □

**Corollary 2.35.** *For all $d \in \mathbb{Z}_{>0}$ and $0 \leq j \leq d$, we have*

$$u_{1+j} u_{d-j+1} u_1^{d-2} \in \mathcal{P}.$$

*Proof.* Setting $\mu_1 = 1 + j$ and $\mu_2 = d - j + 1$ in Lemma 2.34, we obtain the claim. □

Furthermore, Lemma 2.34 is used to prove the following observation.

**Lemma 2.36.** *For arbitrary $d \in \mathbb{Z}_{>0}$ and all $0 \leq j \leq d - 3$, we have*

$$u_2 u_1 u_2^j u_3 u_2^{d-j-3} \in \mathcal{P}.$$

*Proof.* First, a direct calculation gives for all $0 \leq j \leq d - 3$ that

$$\begin{aligned} &u_2 u_1 u_2^j u_3 u_2^{d-j-3} \\ =& \sum_{a=2}^{j+2} (-1)^a \, u_1^{j-a+2} u_2 u_1^{d-j-3} \circledast u_a u_1^{d-1} + \sum_{a=1}^{j+3} (-1)^a \, u_1^{d-a+1} \circledast u_a u_1^{d-1} \end{aligned}$$

$$- \sum_{\substack{a,b\geq 2\\ a+b\leq j+3}} (-1)^{a+b}\, u_1^{j-(a+b)+3} u_2 u_1^{d-j-3} \boxast u_a u_b u_1^{d-2}$$
$$+ (-1)^{j+3}\, u_2 u_{j+3} (u_1^{d-j-3} \boxast u_1^{d-2})$$
$$+ (-1)^{j+3} \sum_{\substack{a,b\geq 2\\ a+b=j+3}} u_{a+2} u_{b+1} (u_1^{d-j-4} \boxast u_1^{d-2}) + u_{a+2} u_b (u_1^{d-j-3} \boxast u_1^{d-2}).$$

Using Lemma 2.34 now yields the claim. □

Collecting the results of this subsection, we have proven the following theorem.

**Theorem 2.37.** *Let be $d \in \mathbb{Z}_{>0}$.*

*(i) For all $0 \leq j \leq d-2$, we have*

$$u_1 u_2^j u_3 u_2^{d-j-2},\ u_2^j u_3 u_2^{d-j-2} u_1 \in \mathcal{P}.$$

*(ii) For all $0 \leq j \leq d$, we have*

$$u_{1+j} u_{d-j+1} u_1^{d-2},\ u_1^{d-2} u_{d-j+1} u_{1+j} \in \mathcal{P}.$$

*(iii) For all $0 \leq j \leq d-3$, we have*

$$u_2 u_1 u_2^j u_3 u_2^{d-j-3},\ u_2^j u_3 u_2^{d-j-3} u_1 u_2 \in \mathcal{P}.$$

*Proof.* Using Corollary 2.31 each, the proof for (i) follows from Lemma 2.33, the proof of (ii) follows from Corollary 2.35, and the proof of (iii) follows from Lemma 2.36. □

### 2.4.2 A conjecture about particular box products and implications

We consider in this section the structure of all box products $u_{\mathbf{n}} \boxast u_{\boldsymbol{\ell}}$ such that $\operatorname{len}(\boldsymbol{\ell})$ and $|\mathbf{n}| + |\boldsymbol{\ell}|$ are fixed. For this, we will need the spaces $\mathcal{S}_{z,d}$ and $\mathcal{T}_{z,d}$ in the following.

**Definition 2.38.** (i) For all $z, d \in \mathbb{Z}_{>0}$, we define

$$\mathcal{T}_{z,d} := \operatorname{span}_{\mathbb{Q}} \left\{ u_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathbb{Z}_{>0}^d,\ |\boldsymbol{\mu}| = z + d \right\},$$
$$\mathfrak{t}_{z,d} := \dim_{\mathbb{Q}} \mathcal{T}_{z,d}.$$

(ii) Furthermore, for all $z, d \in \mathbb{Z}_{>0}$, we define

$$\mathcal{J}_{z,d} := \left\{ (\mathbf{n}, \boldsymbol{\ell}) \,\middle|\, \mathbf{n} \in \mathbb{Z}_{>0}^s,\ \boldsymbol{\ell} \in \mathbb{Z}_{>0}^d,\ 1 \leq s \leq d,\ |\mathbf{n}| + |\boldsymbol{\ell}| = z + d \right\},$$
$$\mathfrak{j}_{z,d,:} = \#\mathcal{J}_{z,d}.$$

and

$$\mathcal{S}_{z,d} := \operatorname{span}_{\mathbb{Q}} \{ u_{\mathbf{n}} \boxast u_{\boldsymbol{\ell}} \mid (\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{z,d} \} = \mathcal{T}_{z,d} \cap \mathcal{P},$$
$$\mathfrak{s}_{z,d} := \dim_{\mathbb{Q}} \mathcal{S}_{z,d}.$$

Based on numerical calculations (see Lemma 2.42), we conjecture the following for the dimension of $\mathcal{S}_{z,d}$.

**Conjecture 2.39.** *For all $z, d \in \mathbb{Z}_{>0}$, we have*

$$\mathfrak{s}_{z,d} = \binom{z+d-1}{\min\{z,d\}-1}. \tag{2.39.1}$$

Given $(z_0, d_0) \in \mathbb{Z}_{>0}$, we say that Conjecture 2.39 *is true for* $(z_0, d_0)$ if (2.39.1) is true for $(z, d) = (z_0, d_0)$. Note the following equivalent formulation for $z \geq d$.

**Corollary 2.40.** *Given $(z, d) \in \mathbb{Z}^2_{>0}$ with $z \geq d$. Conjecture 2.39 is true for $(z, d)$ if and only if $\mathcal{S}_{z,d} = \mathcal{T}_{z,d}$.*

*Proof.* Clearly, for all $z, d \in \mathbb{Z}_{>0}$, one has

$$\mathfrak{t}_{z,d} = \binom{z+d-1}{d-1}$$

since $\mathfrak{t}_{z,d}$ is the number of compositions of $z+d$ into exactly $d$ positive integers. Hence, for $(z, d) \in \mathbb{Z}^2_{>0}$ with $z \geq d$, Conjecture 2.39 is equivalent to $\mathfrak{s}_{z,d} = \mathfrak{t}_{z,d}$ which is equivalent to $\mathcal{S}_{z,d} = \mathcal{T}_{z,d}$ since $\mathcal{S}_{z,d} \subset \mathcal{T}_{z,d}$ and both $\mathcal{S}_{z,d}$ and $\mathcal{T}_{z,d}$ are finite-dimensional $\mathbb{Q}$-vector spaces. □

**Theorem 2.41.** *Fix $d \in \mathbb{Z}_{>0}$. If Conjecture 2.39 is true for $(d, d)$, then it is also true for all $(z, d) \in \mathbb{Z}^2_{>0}$ with $z > d$.*

*Proof.* Fix $d \in \mathbb{Z}_{>0}$ and assume that Conjecture 2.39 is true for $(d, d)$. I.e., by Corollary 2.40, we assume $\mathcal{S}_{d,d} = \mathcal{T}_{d,d}$. This is equivalent to

$$u_{\mathbf{z}} = \sum_{(\mathbf{n},\boldsymbol{\ell}) \in \mathcal{J}_{d,d}} a_{\mathbf{n},\boldsymbol{\ell}}(\mathbf{z})\, u_{\mathbf{n}} \circledast u_{\boldsymbol{\ell}}$$

for all $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{Z}^d_{>0}$ with $|\mathbf{z}| = 2d$ and with $a_{\mathbf{n},\boldsymbol{\ell}}(\mathbf{z}) \in \mathbb{Q}$ appropriate.

Now, assume $z > d$ and let be $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{Z}^d_{>0}$ with $|\mathbf{z}| = z + d$ arbitrary. We can write

$$(z_1, \ldots, z_d) = (z'_1 + \delta_1, \ldots, z'_d + \delta_d)$$

with $\delta_1, \ldots, \delta_d \in \mathbb{Z}_{\geq 0}$ and $\mathbf{z}' = (z'_1, \ldots, z'_d) \in \mathbb{Z}^d_{>0}$ with $|\mathbf{z}'| = 2d$. Hence,

$$u_{\mathbf{z}} = \sum_{(\mathbf{n},\boldsymbol{\ell}) \in \mathcal{J}_{d,d}} a_{\mathbf{n},\boldsymbol{\ell}}(\mathbf{z}')\, u_{\mathbf{n}} \circledast u_{\ell_1+\delta_1} \cdots u_{\ell_d+\delta_d}.$$

Since $\mathbf{z}$ was chosen arbitrary, we obtain $\mathcal{S}_{z,d} = \mathcal{T}_{z,d}$, proving the theorem. □

**Lemma 2.42.** *Conjecture 2.39 is true for all $(z, d) \in \mathbb{Z}^2_{>0}$ with $1 \leq d \leq 8$.*

*Proof.* The proof for $1 \leq z \leq d \leq 8$ is obtained by computer algebra; for details, see Remark 2.60 and the appendix. By Theorem 2.41, Conjecture 2.39 is also true for $z \geq d$ when $1 \leq d \leq 8$, proving the lemma. □

Note that $\mathfrak{s}_{z,d}$ is the dimension of the image of the $\mathbb{Q}$-linear map

$$\begin{aligned} \mathcal{B}_{z,d} \colon \operatorname{span}_{\mathbb{Q}} \mathcal{J}_{z,d} &\longrightarrow \mathcal{T}_{z,d}, \\ (\mathbf{n}, \boldsymbol{\ell}) &\longmapsto u_{\mathbf{n}} \circledast u_{\boldsymbol{\ell}} \end{aligned}$$

that we continue $\mathbb{Q}$-bilinearly. By the rank-nullity theorem, we know that

$$\mathfrak{s}_{z,d} + \dim_{\mathbb{Q}} \ker \mathcal{B}_{z,d} = \dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}} \mathcal{J}_{z,d}. \tag{2.42.1}$$

The right-hand side is given by $\mathfrak{j}_{z,d}$, which is the number of writing $z+d$ as ordered sum of at least $d+1$ and at most $d+\min\{z,d\}$ positive integers, i.e.,

$$\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}} \mathcal{J}_{z,d} = \mathfrak{j}_{z,d} = \sum_{j=1}^{\min\{z,d\}} \binom{z+d-1}{d+j-1}. \tag{2.42.2}$$

Hence, determining $\mathfrak{s}_{z,d}$ now is equivalent to determining $\dim_{\mathbb{Q}} \ker \mathcal{B}_{z,d}$. While it seems to be difficult to obtain a (conjectured) basis of $\mathcal{S}_{z,d}$, we can give a conjectured basis of $\ker \mathcal{B}_{z,d}$ explicitly. To do so, we need the notion of stuffle product and box product on index level. I.e., we set $\mathbf{n} * \emptyset := \emptyset * \mathbf{n} := \mathbf{n}$, $\mathbf{n} \boxast \emptyset := \emptyset \boxast \mathbf{n} := \mathbf{n}$ for every index $\mathbf{n}$. Furthermore, for given indices $\mathbf{n} = (n_1, \ldots, n_s) \in \mathbb{Z}_{>0}^s$, $\mathbf{m} = (m_1, \ldots, m_t) \in \mathbb{Z}_{>0}^t$ with $s, t \geq 1$, we set recursively

$$\begin{aligned} \mathbf{n} * \mathbf{m} := {} & (n_1).((n_2, \ldots, n_s) * \mathbf{m}) + (m_1).(\mathbf{n} * (m_2, \ldots, m_t)) \\ & + (n_1 + m_1).((n_2, \ldots, n_s) * (m_2, \ldots, m_t)) \end{aligned}$$

as formal sum of indices, where $().()$ means the concatenation of indices. Similarly, we define the box product $\mathbf{n} \boxast \mathbf{m}$ to be the part of $\mathbf{n} * \mathbf{m}$ of smallest length.

**Example 2.43.** To illustrate the definition of stuffle product and box product of indices, we consider $\mathbf{n} = (1,2)$ and $\mathbf{m} = (3,2)$. We have

$$\begin{aligned} \mathbf{n} * \mathbf{m} &= (1,2) * (3,2) \\ &= (4,4) + (1,5,2) + (1,3,4) + 2(4,2,2) + (3,3,2) \\ &\quad + (1,2,3,2) + 2(1,3,2,2) + 2(3,1,2,2) + (3,2,1,2) \end{aligned}$$

and

$$\mathbf{n} \boxast \mathbf{m} = (1,2) \boxast (3,2) = (4,4).$$

In the following, for $z, d \in \mathbb{Z}_{>0}$, we consider the set

$$\mathcal{K}_{z,d} := \left\{ (\mathbf{n_1}, \mathbf{n_2} \boxast \boldsymbol{\ell}) - (\mathbf{n_1} * \mathbf{n_2}, \boldsymbol{\ell}) \;\middle|\; \substack{\mathbf{n_1} \in \mathbb{Z}_{>0}^{s_1},\, \mathbf{n_2} \in \mathbb{Z}_{>0}^{s_2},\, \boldsymbol{\ell} \in \mathbb{Z}_{>0}^{d}, \\ 1 \leq s_1, s_2 \leq d,\, |\mathbf{n_1}| + |\mathbf{n_2}| + \boldsymbol{\ell} = \boldsymbol{z} + \boldsymbol{d}} \right\} \subset \operatorname{span}_{\mathbb{Q}} \mathcal{J}_{z,d},$$

where $(\cdot,\cdot)$ is $\mathbb{Q}$-bilinearly continued.

**Lemma 2.44.** *For all $z, d \in \mathbb{Z}_{>0}$, we have* $\operatorname{span}_{\mathbb{Q}} \mathcal{K}_{z,d} \subset \ker \mathcal{B}_{z,d}$.

*Proof.* This is an immediate consequence of Lemma 2.19. □

By numerical calculations (see the appendix), we conjecture that the converse inclusion is also true if $z \leq d$.

**Conjecture 2.45.** *Let be $z, d \in \mathbb{Z}_{>0}$ with $z \leq d$. Then,*

$$\operatorname{span}_{\mathbb{Q}} \mathcal{K}_{z,d} = \ker \mathcal{B}_{z,d}. \tag{2.45.1}$$

We say that Conjecture 2.45 is true for $(z_0, d_0)$ if (2.45.1) is true for $(z,d) = (z_0, d_0)$. Note the following consequence.

**Lemma 2.46.** *Let be $z, d \in \mathbb{Z}_{>0}$ with $z \leq d$. If Conjecture 2.45 is true for $(z, d)$, we have*

$$\mathfrak{d}_{z,d} \geq \binom{z+d-1}{d}.$$

*In particular, if $z = d$ additionally, then Conjecture 2.39 is true for $(d, d)$.*

*Proof.* Let be $z, d \in \mathbb{Z}_{>0}$ with $z \leq d$. We begin by noting that we have

$$\#\mathcal{K}_{z,d} = \sum_{j=2}^{z} \binom{z+d-1}{d+j-1}$$

since $\#\mathcal{K}_{z,d}$ is the number of ways one can write $z+d$ as ordered sum of at least $d+2$ and at most $d + \min\{z, d\}$ $(= d + z$ in case $z \leq d)$ positive integers. Now, if Conjecture 2.45 is true for $(z, d)$, we obtain by (2.42.1) and (2.42.2), that

$$\mathfrak{d}_{z,d} = \mathfrak{j}_{z,d} - \dim_{\mathbb{Q}} \ker \mathcal{B}_{z,d} \geq \sum_{j=1}^{z} \binom{z+d-1}{d+j-1} - \sum_{j=2}^{z} \binom{z+d-1}{d+j-1} = \binom{z+d-1}{d}.$$

In case $z = d$, the right hand side is $\mathfrak{t}_{d,d}$, i.e., we must have equality and so, Conjecture 2.39 is true for $(d, d)$. This completes the proof of the lemma. □

The set $\mathcal{K}_{z,d}$ seems to be of special interest regarding determining a basis of $\ker \mathcal{B}_{z,d}$ as the following refinement of Conjecture 2.45 shows.

**Conjecture 2.47.** *Let be $z, d \in \mathbb{Z}_{>0}$ with $z \leq d$. Then $\mathcal{K}_{z,d}$ is a basis of $\ker \mathcal{B}_{z,d}$.*

As usual, we say that Conjecture 2.47 is true for $(z_0, d_0)$ if $\mathcal{K}_{z_0,d_0}$ is a basis of $\ker \mathcal{B}_{z_0,d_0}$. We give evidence for Conjecture 2.47.

**Lemma 2.48.** *Conjecture 2.47 is true for all $(z, d) \in \mathbb{Z}_{>0}^2$ satisfying $1 \leq z \leq d \leq 8$.*

*Proof.* For $z = 1$ and $d \in \mathbb{Z}_{>0}$, we have $\mathcal{K}_{1,d} = \emptyset$ and $\mathfrak{j}_{z,d} = d = \mathfrak{d}_{1,d}$ as we will show in Lemma 2.52, i.e., $\ker \mathcal{B}_{1,d}$ is the trivial vector space. Hence, Conjecture 2.47 is true for all $(1, d) \in \mathbb{Z}_{>0}^2$. For $z \geq 2$, the claim is obtained by numerical calculations, see the appendix. □

Note the following consequence that Conjecture 2.47 is a refinement of Conjecture 2.39.

**Lemma 2.49.** *Let be $z, d \in \mathbb{Z}_{>0}$ with $z \leq d$. If Conjecture 2.47 is true for $(z, d)$, then also Conjecture 2.39 is true for $(z, d)$.*

*Proof.* Let be $z, d \in \mathbb{Z}_{>0}$ with $z \leq d$ and assume that Conjecture 2.47 is true for $(z, d)$. By (2.42.1) and (2.42.2), then we obtain

$$\mathfrak{d}_{z,d} = \mathfrak{j}_{z,d} - \dim_{\mathbb{Q}} \ker \mathcal{B}_{z,d} = \sum_{j=1}^{z} \binom{z+d-1}{d+j-1} - \sum_{j=2}^{z} \binom{z+d-1}{d+j-1} = \binom{z+d-1}{d},$$

i.e., Conjecture 2.39 is true for $(z, d)$. □

We investigate $\mathfrak{d}_{z,d}$ in the following in more detail.

**Lemma 2.50.** *For all $z, d \in \mathbb{Z}_{>0}$, we have*

$$\mathfrak{d}_{z,d+1} + \mathfrak{d}_{z+1,d} \leq \mathfrak{d}_{z+1,d+1}.$$

*Proof.* Fix $z, d \in \mathbb{Z}_{>0}$. By definition of $\mathfrak{d}_{z,d+1}$, there are $\mathfrak{d}_{z,d+1}$ linearly independent linear combinations

$$\sum_{(\mathbf{n},\boldsymbol{\ell})\in\mathcal{J}_{z,d+1}} a^{(j)}_{\mathbf{n},\boldsymbol{\ell}}(\mathbf{z})\, u_{\mathbf{n}} \boxast u_{\boldsymbol{\ell}} \qquad (1 \le j \le \mathfrak{d}_{z,d+1}).$$

Then, the $\mathfrak{d}_{z,d+1}$ linear combinations ($1 \le j \le \mathfrak{d}_{z,d+1}$ in the following) of case $(z+1, d+1)$,

$$\sum_{(\mathbf{n},\boldsymbol{\ell})\in\mathcal{J}_{z,d+1}} a^{(j)}_{\mathbf{n},\boldsymbol{\ell}}(\mathbf{z})\, u_{\mathbf{n}} \boxast u_{(\ell_1,\dots,\ell_d,\ell_{d+1}+1)}, \tag{2.50.1}$$

are linearly independent as well. Note that all occurring words $u_{\mu_1}\cdots u_{\mu_{d+1}}$ in this linear combinations satisfy $\mu_{d+1} \ge 2$.

Now, by definition of $\mathfrak{d}_{z+1,d}$, there are $\mathfrak{d}_{z+1,d}$ linear independent linear combinations

$$\sum_{(\mathbf{n},\boldsymbol{\ell})\in\mathcal{J}_{z+1,d}} b^{(j)}_{\mathbf{n},\boldsymbol{\ell}}(\mathbf{z})\, u_{\mathbf{n}} \boxast u_{\boldsymbol{\ell}} \qquad (1 \le j \le \mathfrak{d}_{z+1,d}). \tag{2.50.2}$$

Considering for $1 \le j \le \mathfrak{d}_{z+1,d}$ the following linear combinations in case $(z+1, d+1)$

$$\begin{aligned} &\sum_{(\mathbf{n},\boldsymbol{\ell})\in\mathcal{J}_{z+1,d}} b^{(j)}_{\mathbf{n},\boldsymbol{\ell}}(\mathbf{z})\, u_{\mathbf{n}} \boxast u_{\boldsymbol{\ell}} u_1 \\ &= \left( \sum_{(\mathbf{n},\boldsymbol{\ell})\in\mathcal{J}_{z+1,d}} b^{(j)}_{\mathbf{n},\boldsymbol{\ell}}(\mathbf{z})\, u_{\mathbf{n}} \boxast u_{\boldsymbol{\ell}} \right) u_1 + \sum_{(\mathbf{n},\boldsymbol{\ell})\in\mathcal{J}_{z+1,d}} b^{(j)}_{\mathbf{n},\boldsymbol{\ell}}(\mathbf{z}) \left( u_{(n_1,\dots,n_{s-1})} \boxast u_{\boldsymbol{\ell}} \right) u_{1+n_s} \end{aligned} \tag{2.50.3}$$

are linearly independent again because of (2.50.2). Furthermore, they and the ones from (2.50.1) are linearly independent since the latter ones contain words ending in $u_{\mu_{d+1}}$ with $\mu_{d+1} \ge 2$ while the linear independence of (2.50.3) already comes from words ending all in $u_1$.

Summarized, we have proven $\mathfrak{d}_{z,d+1} + \mathfrak{d}_{z+1,d} \le \mathfrak{d}_{z+1,d+1}$. □

**Remark 2.51.** Assuming Conjecture 2.39, the inequality in Lemma 2.50 is an equality if and only if $z \ne d$.

With Lemma 2.50, we can now prove the following partial result towards Conjecture 2.39.

**Lemma 2.52.** *Conjecture 2.39 is true for all pairs $(z, d) \in \mathbb{Z}^2_{>0}$ with $1 \le z \le 3$.*

*Proof.* Note that the proof for $1 \le z \le 2$ is contained in Remark 2.61. Therefore, assume $z = 3$ in the following. For $(z, d) \in \{(3,1), (3,2), (3,3)\}$, the claim follows from Remark 2.61. Hence, consider $d \ge 4$ and prove by induction (with already proven base case $d = 3$) on $d$. By Lemma 2.50, the induction hypothesis, and the case $z = 2$ of the lemma that is proven in Remark 2.61, we know that

$$\mathfrak{d}_{3,d} \ge \mathfrak{d}_{3,d-1} + \mathfrak{d}_{2,d} = \binom{d+1}{2} + \binom{d+1}{1} = \binom{d+2}{2}.$$

Therefore, it suffices to prove $\mathfrak{d}_{3,d} \le \binom{d+2}{2}$. Note that for $(z, d) = (3, d)$ the number of box products spanning $\mathfrak{S}_{3,d}$ is $\binom{d+2}{0} + \binom{d+2}{1} + \binom{d+2}{2}$. I.e., if we can show that $\binom{d+2}{2}$ of those are such that the other $\binom{d+2}{0} + \binom{d+2}{1}$ ones are in their $\mathbb{Q}$-span, we are done. We consider the set of $\binom{d+2}{2}$ box products

$$\mathcal{R}_{3,d} := \left\{ \begin{matrix} u_2u_1 \boxast u_1^d,\ u_1u_2 \boxast u_1^d, \\ u_2 \boxast u_1^{j_1}u_2u_1^{d-j_1-1}, u_1 \boxast u_1^{j_2}u_3u_1^{d-j_2-1}, \\ u_1 \boxast u_1^{j_3}u_2u_1^{j_4}u_2u_1^{d-j_3-j_4-2} \end{matrix} \middle| \begin{matrix} 0 \le j_1 \le d-2, 0 \le j_2 \le d-2, \\ 0 \le j_3, j_4 \le d-2, j_3+j_4 \le d-2 \end{matrix} \right\}.$$

In the following, we show that the other box products in case $(z,d) = (3,d)$ are in the $\mathbb{Q}$-span of $\mathcal{R}_{3,d}$. For $0 \le j_1 \le d-2$, we obtain

$$u_1u_1 \boxast u_1^{j_1}u_2u_1^{d-j_1-1} = \frac{1}{2}\left((u_1 * u_1 - u_2) \boxast u_1^{j_1}u_2u_1^{d-j_1-1}\right) \in \operatorname{span}_{\mathbb{Q}} \mathcal{R}_{3,d} \quad (2.52.1)$$

due to Lemma 2.19 and the definition of $\mathcal{R}_{3,d}$. Furthermore, we have that

$$u_3 \boxast u_1^d = \sum_{j_2=0}^{d-1} u_1 \boxast u_1^{j_2}u_3u_1^{d-j_2-1} - (u_2u_1 + u_1u_2) \boxast u_1^d \quad (2.52.2)$$

is in the $\mathbb{Q}$-span of $\mathcal{R}_{3,d}$. This implies, due to $u_2 * u_1 = u_2u_1 + u_1u_2 + u_3$ and Lemma 2.19, that

$$\begin{aligned} u_2 \boxast u_1^{d-1}u_2 &= u_2 \boxast \left(u_1 \boxast u_1^d - \sum_{j_1=0}^{d-2} u_1^{j_1}u_2u_1^{d-j_1-1}\right) \\ &= (u_2u_1 + u_1u_2 + u_3) \boxast u_1^d - \sum_{j_1=0}^{d-2} u_2 \boxast u_1^{j_1}u_2u_1^{d-j_1-1} \in \operatorname{span}_{\mathbb{Q}} \mathcal{R}_{3,d}. \end{aligned}$$

Similar to (2.52.1), one obtains now

$$u_2 \boxast u_1^{d-1}u_2 \in \operatorname{span}_{\mathbb{Q}} \mathcal{R}_{3,d}.$$

Using (2.52.2), Lemma 2.19 and the definition of $\mathcal{R}_{3,d}$, we get

$$u_1u_1u_1 \boxast u_1^d = \frac{1}{3}\left((u_1 * u_1u_1 - u_2u_1 - u_1u_2) \boxast u_1^d\right) \in \operatorname{span}_{\mathbb{Q}} \mathcal{R}_{3,d},$$

completing the claim. In particular, the lemma is proven for $z = 3$. □

**Proposition 2.53.** *Conjecture 2.39 is true for* $(4,4)$ *and therefore, by Theorem 2.41, for all pairs* $(z,4)$ *with* $z \ge 4$.

*Proof.* Using Corollary 2.40, we have to show $\mathcal{S}_{4,4} = \mathcal{T}_{4,4}$. From Theorem 2.37 and Lemma 2.32, we already have

$$\begin{aligned} &u_2u_2u_2u_2,\ u_5u_1u_1u_1,\ u_1u_5u_1u_1,\ u_1u_1u_5u_1,\ u_1u_1u_1u_5,\ u_1u_3u_2u_2,\ u_1u_2u_3u_2, \\ &u_1u_2u_2u_3,\ u_3u_2u_2u_1,\ u_2u_3u_2u_1,\ u_2u_2u_3u_1,\ u_2u_4u_1u_1,\ u_3u_3u_1u_1,\ u_4u_2u_1u_1, \\ &u_1u_1u_2u_4,\ u_1u_1u_3u_3,\ u_1u_1u_4u_2,\ u_2u_1u_3u_2,\ u_2u_1u_2u_3,\ u_2u_3u_1u_2,\ u_3u_2u_1u_2 \in \mathcal{S}_{4,4}. \end{aligned}$$

Hence, considering $u_1 \boxast u_2u_1u_2u_2$, we obtain $u_3u_1u_2u_2 \in \mathcal{S}_{4,4}$, and so, by Corollary 2.31,we also have $u_2u_2u_1u_3 \in \mathcal{S}_{4,4}$. Now, considering $u_1u_1 \boxast u_1j_1u_2u_1^{j_2}u_2u_1^{j_3}$ for $j_1, j_2, j_3 \in \mathbb{Z}_{\ge 0}$ with $j_1 + j_2 + j_3 = 2$, yields $u_3u_1u_3u_1$, $u_3u_1u_1u_3$, $u_1u_3u_3u_1$, $u_1u_3u_1u_3 \in \mathcal{S}_{4,4}$. Last, consider $u_1 \boxast u_1^{j_1}u_2u_1^{j_2}u_3u_1^{j_3}$ for $j_1, j_2, j_3 \in \mathbb{Z}_{\ge 0}$ with $j_1 + j_2 + j_3 = 1$ immediately gives $u_2u_1u_4u_1$, $u_2u_1u_1u_4$, $u_1u_2u_4u_1$, $u_1u_2u_1u_4 \in \mathcal{S}_{4,4}$, yielding, by Corollary 2.31 again, that $u_4u_1u_2u_1$, $u_4u_1u_1u_2$, $u_1u_4u_2u_1$, $u_1u_4u_1u_2 \in \mathcal{S}_{4,4}$. Therefore, $\mathcal{S}_{4,4} = \mathcal{T}_{4,4}$ follows, completing the proof. □

### 2.4.3 Connection between the box product and the stuffle product

First, to connect the box product with the stuffle product, we introduce the maps $\Psi_{\mathbf{k}}$.

**Definition 2.54.** Fix $d \in \mathbb{Z}_{>0}$ and $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}_{>0}^d$. We define the $\mathbb{Q}$-linear map $\Psi_{\mathbf{k}}\colon \operatorname{span}_{\mathbb{Q}} \{\mathtt{W} \in \mathcal{U}^{*,\circ} \mid \operatorname{len}(\mathtt{W}) = d\} \to \mathbb{Q}\langle\mathcal{U}\rangle^\circ$, given on generators by

$$u_{\mu_1} \cdots u_{\mu_d} \longmapsto u_{\mu_1} u_0^{k_d - 1} \cdots u_{\mu_d} u_0^{k_1 - 1}.$$

Note the following connection of maps $\Psi_{\mathbf{k}}$ with the box product.

**Lemma 2.55.** *Let be $z, d, w \in \mathbb{Z}_{>0}$ and $(\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{z,d}$. Furthermore, let be $\mathbf{k} \in \mathbb{Z}_{>0}^d$ satisfying $|\mathbf{k}| = w - z$. Then,*

$$u_{\mathbf{n}} \boxast \Psi_{\mathbf{k}}(u_{\boldsymbol{\ell}}) = \Psi_{\mathbf{k}}(u_{\mathbf{n}} \boxast u_{\boldsymbol{\ell}}).$$

*Proof.* Using the notation as in the lemma, we note that particularly $\operatorname{depth}(u_{\boldsymbol{\ell}}) = d$. The claim immediately follows by the definition of the box product and the definition of the map $\Psi_{\mathbf{k}}$. □

The following Lemma 2.56 now connects the stuffle product with the box product. It will be the key for proving Theorem 2.69 below and one of the main observations for our approach to the refined Bachmann Conjecture 2.10.

**Lemma 2.56.** *Let be $z, d, w \in \mathbb{Z}_{>0}$ and $(\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{z,d}$. Furthermore, let be $\mathbf{k} \in \mathbb{Z}_{>0}^d$ satisfying $|\mathbf{k}| = w - z$. Then,*

$$\zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}(u_{\mathbf{n}} \boxast u_{\boldsymbol{\ell}})\right) \in \sum_{1 \le s \le \min\{z,d\}} \operatorname{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w} \mathcal{Z}_q^f.$$

*Proof.* Let be $z, d, w \in \mathbb{Z}_{>0}$, $(\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{z,d}$, $\mathbf{k} \in \mathbb{Z}_{>0}^d$ such that $|\mathbf{k}| = w - z$ and write $s'$ for the length of $\mathbf{n}$. I.e., we have, $u_{\mathbf{n}} \in \operatorname{Fil}^{\mathrm{Z,D,W}}_{0,s',|\mathbf{n}|} \mathbb{Q}\langle\mathcal{U}\rangle^\circ$ and $\Psi_{\mathbf{k}}(u_{\boldsymbol{\ell}}) \in \operatorname{Fil}^{\mathrm{Z,D,W}}_{|\mathbf{k}|-d,d,|\mathbf{k}|+|\boldsymbol{\ell}|-d} \mathbb{Q}\langle\mathcal{U}\rangle^\circ$. Since $(\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{z,d}$, we have $|\mathbf{n}| + |\boldsymbol{\ell}| = z + d$. Therefore, (2.3.1) implies, together with the assumption $|\mathbf{k}| = w - z$, that

$$u_{\mathbf{n}} * \Psi_{\mathbf{k}}(u_{\boldsymbol{\ell}}) \in \operatorname{Fil}^{\mathrm{Z,D,W}}_{w-d-z,d+s',w} \mathbb{Q}\langle\mathcal{U}\rangle^\circ.$$

By (2.3.2), this implies now

$$\tau(u_{\mathbf{n}} * \Psi_{\mathbf{k}}(u_{\boldsymbol{\ell}})) \in \operatorname{Fil}^{\mathrm{Z,D,W}}_{z-s',d+s',w} \mathbb{Q}\langle\mathcal{U}\rangle^\circ,$$

yielding, since $1 \le s' \le \min\{z, d\}$,

$$\zeta_q^{\mathrm{f}}\left(u_{\mathbf{n}} * \Psi_{\mathbf{k}}(u_{\boldsymbol{\ell}})\right) = \zeta_q^{\mathrm{f}}\left(\tau(u_{\mathbf{n}} * \Psi_{\mathbf{k}}(u_{\boldsymbol{\ell}}))\right) \in \sum_{1 \le s \le \min\{z,d\}} \operatorname{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w} \mathcal{Z}_q^f.$$

Furthermore, due to Corollary 2.17, we also have

$$\zeta_q^{\mathrm{f}}\left(u_{\mathbf{n}} \boxast \Psi_{\mathbf{k}}(u_{\boldsymbol{\ell}})\right) \in \sum_{1 \le s \le \min\{z,d\}} \operatorname{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w} \mathcal{Z}_q^f.$$

Hence, the lemma follows now from Lemma 2.55. □

**Corollary 2.57.** *Let be $z, d, w \in \mathbb{Z}_{>0}$ and $\boldsymbol{\mu} \in \mathbb{Z}_{>0}^d$ satisfying $|\boldsymbol{\mu}| = z + d$. If $u_{\boldsymbol{\mu}} \in \mathcal{P}$ with $\mathcal{P}$ from* (2.30.1)*, then*

$$\zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}(u_{\boldsymbol{\mu}})\right) \in \sum_{1 \le s \le \min\{z,d\}} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w} \mathcal{Z}_q^f \subset \mathrm{F}_{z,d,w}$$

*for all $\mathbf{k} \in \mathbb{Z}_{>0}^d$ satisfying $|\mathbf{k}| = w - z$.*

*Proof.* Let be $z, d, w \in \mathbb{Z}_{>0}$ and $\boldsymbol{\mu} \in \mathbb{Z}_{>0}^d$ satisfying $|\boldsymbol{\mu}| = z + d$. Furthermore, choose an index $\mathbf{k} \in \mathbb{Z}_{>0}^d$ arbitrary with the property $|\mathbf{k}| = w - z$. Assume $u_{\boldsymbol{\mu}} \in \mathcal{P}$, i.e., we have

$$u_{\boldsymbol{\mu}} = \sum_{(\mathbf{n},\boldsymbol{\ell}) \in \mathcal{J}_{z,d}} a_{\mathbf{n},\boldsymbol{\ell}}\, u_{\mathbf{n}} \circledast u_{\boldsymbol{\ell}}$$

with $a_{\mathbf{n},\boldsymbol{\ell}} \in \mathbb{Q}$ appropriate. Now, for all $(\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{z,d}$, by Lemma 2.56, we have

$$\zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}(u_{\mathbf{n}} \circledast u_{\boldsymbol{\ell}})\right) \in \sum_{1 \le s \le \min\{z,d\}} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w} \mathcal{Z}_q^f.$$

I.e., by $\mathbb{Q}$-linearity of $\zeta_q^{\mathrm{f}}$ and $\Psi_{\mathbf{k}}$, hence we obtain

$$\zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}(u_{\boldsymbol{\mu}})\right) = \sum_{(\mathbf{n},\boldsymbol{\ell}) \in \mathcal{J}_{z,d}} a_{\mathbf{n},\boldsymbol{\ell}}\, \zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}(u_{\mathbf{n}} \circledast u_{\boldsymbol{\ell}})\right) \in \sum_{1 \le s \le \min\{z,d\}} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s,d+s,w} \mathcal{Z}_q^f,$$

completing the claim. □

### 2.4.4 Supplementary results and calculations regarding the box product

We collect in this subsection further results towards the box product that are connected to Conjecture 2.39 but not needed in the following. First, we refine Conjecture 2.39. For this, we define for all $z, d, s_{\min} \in \mathbb{Z}_{>0}$ with $1 \le z \le d$,

$$\begin{aligned} \mathcal{S}_{z,d,s_{\min}} &:= \operatorname{span}_{\mathbb{Q}} \{u_{\mathbf{n}} \circledast u_{\boldsymbol{\ell}} \mid (\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{z,d},\ \operatorname{len}(\mathbf{n}) \ge s_{\min}\} \subset \mathcal{S}_{z,d}, \\ \mathfrak{s}_{z,d,s_{\min}} &:= \dim_{\mathbb{Q}} \mathcal{S}_{z,d,s_{\min}}. \end{aligned}$$

**Conjecture 2.58.** *For all $z, d, s_{\min} \in \mathbb{Z}_{>0}$ with $1 \le z \le d$, we have*

$$\mathfrak{s}_{z,d,s_{\min}} = \binom{z+d-1}{z-s_{\min}}. \tag{2.58.1}$$

Given $(z_0, d_0, s_{\min,0}) \in \mathbb{Z}_{>0}$ with $1 \le z \le d$, we say that Conjecture 2.58 *is true for* $(z_0, d_0, s_{\min,0})$ if (2.58.1) is true for $(z, d, s_{\min}) = (z_0, d_0, s_{\min,0})$.

**Remark 2.59.** With Theorem 2.41, we see that if Conjecture 2.39 is true for $z = d$, then the statement for $z > d$ follows as well. Hence, we can view Conjecture 2.58 (via $s_{\min} = 1$) indeed as a refinement of Conjecture 2.39, despite it is a refinement for $z \le d$ only.

**Remark 2.60.** Conjecture 2.58 is true for all triples $(z, d, s_{\min}) \in \mathbb{Z}_{>0}^3$ with $1 \le z \le d \le 8$ and $1 \le s_{\min} \le 8$. The proof is obtained by computer algebra; for details, see the appendix. One could use the code in the appendix for verifying Conjecture 2.58 also for larger values of $z$ and $d$. The only limit is the computing capacity and time since the code is based on computing ranks of matrices that grow exponentially in $z$ and $d$.

In the next remark, we give an elementary proof, not based on numerical calculations, for the part of Lemma 2.42 that is needed for proving our main results of this paper.

**Remark 2.61.** We could verify Conjecture 2.39 for all pairs $(z,d) \in \mathbb{Z}_{>0}^2$ with $1 \leq d \leq 3$ also without numerical calculations. For this, first, assume $d = 1$ and fix $z \in \mathbb{Z}_{>0}$. Note that $\mathfrak{T}_{z,1} = \operatorname{span}_{\mathbb{Q}} \{u_{z+1}\}$, yielding $\mathfrak{s}_{z,1} \leq \dim_{\mathbb{Q}} \mathfrak{T}_{z,1} = 1$. Furthermore,

$$u_{z+1} = u_1 \boxast u_z \in \mathcal{S}_{z,1},$$

giving $\mathfrak{s}_{z,1} \geq 1$. Hence, Conjecture 2.39 is true for all pairs $(z,1) \in \mathbb{Z}_{>0}^2$ since

$$\mathfrak{s}_{z,1} = 1 = \binom{1+z-1}{\min\{z,1\}-1}.$$

Now, assume $z = 1$ and fix $d \in \mathbb{Z}_{>0}$. In this case, $\mathcal{S}_{1,d} = \operatorname{span}_{\mathbb{Q}} \left\{u_1 \boxast u_1^d\right\}$, i.e., $\mathfrak{s}_{1,d} = 1$. In particular, we have proven Conjecture 2.39 for $z = 1$ since

$$\mathfrak{s}_{1,d} = 1 = \binom{d+1-1}{\min\{1,d\}-1}.$$

Next, assume $d = 2$ and fix $z \in \mathbb{Z}_{\geq 2}$. Note that the case $(z,d) = (1,2)$ follows from the $z = 1$-case we have proven. Note that $\mathfrak{T}_{z,2} = \operatorname{span}_{\mathbb{Q}} \{u_a u_{z+2-a} \mid 1 \leq a \leq z+1\}$. A direct calculation shows

$$u_a u_{z+2-a} = \begin{cases} u_1 u_1 \boxast u_{a-1} u_{z+1-a}, & \text{if } 2 \leq a \leq z, \\ u_1 \boxast u_1 u_z - u_1 u_1 \boxast u_1 u_{z-1}, & \text{if } a = 1, \\ u_1 \boxast u_z u_1 - u_1 u_1 \boxast u_{z-1} u_1, & \text{if } a = z+1. \end{cases}$$

Hence, $u_a u_{z+2-a} \in \mathcal{S}_{z,2}$ for all $1 \leq a \leq z+1$, i.e., $\mathcal{S}_{z,2} = \mathfrak{T}_{z,2}$, giving

$$\mathfrak{s}_{z,2} = \dim_{\mathbb{Q}} \mathfrak{T}_{z,2} = \binom{2+z-1}{\min\{z,2\}-1}$$

since we assumed $z \geq 2 = d$. Hence, Conjecture 2.39 is true for all pairs $(z,2) \in \mathbb{Z}_{>0}^2$.

Now, assume $z = 2$ and fix $d \geq 2$ (since the $(z,d) = (2,1)$-case follows from the $d = 1$-case of the theorem). In this case, $\mathcal{S}_{2,d}$ is spanned by the $d+2$ box products

$$u_1 \boxast u_1^j u_2 u_1^{d-j-1} \quad (0 \leq j \leq d-1), \quad u_1 u_1 \boxast u_1^d, \quad u_2 \boxast u_1^d.$$

Note that all but the last box product are linear independent since $u_1 u_1 \boxast u_1^d$ does not contain any word with letter $u_3$ while $u_1 \boxast u_1^j u_2 u_1^{d-j-1}$ does contain exactly one such one which is unique for fixed $j$. Furthermore, we have

$$u_2 \boxast u_1^d = \sum_{j=0}^{d-1} u_1 \boxast u_1^j u_2 u_1^{d-j-1} - 2 u_1 u_1 \boxast u_1^d,$$

i.e., $u_2 \boxast u_1^d$ is not linearly independent of the box products. Therefore,

$$\mathfrak{s}_{2,d} = d+1 = \binom{d+2-1}{\min\{2,d\}-1}$$

since we assumed $d \geq 2$. This proves Conjecture 2.39 for $z = 2$.

Now, assume $d = 3$. Since the cases $(z,d) \in \{(1,3),(2,3)\}$ follow from the case $z = 1$, respectively $z = 2$, that we have proven already, we may fix $z \in \mathbb{Z}_{\geq 3}$. For $z = 3$, from Lemmas 2.32, 2.33, and 2.36, we obtain $\mathcal{S}_{3,3} = \mathfrak{T}_{3,3}$, yielding, by Corollary 2.40, the claim. For $z > 3$, we apply Theorem 2.41 to obtain the remaining part for the proof that Conjecture 2.39 is true for all pairs $(z,3) \in \mathbb{Z}^2_{>0}$ from the case $z = 3$.

Noting Corollary 2.40, Conjecture 2.39 is equivalent to $\mathcal{S}_{z,d} = \mathcal{T}_{z,d}$ for all $z \geq d$. I.e., in these cases, every $u_{\boldsymbol{\mu}}$ with $\boldsymbol{\mu} \in \mathbb{Z}^d_{>0}$ and $|\boldsymbol{\mu}| = z + d$ conjecturally can be written as $\mathbb{Q}$-linear combination of box products $u_{\mathbf{n}} \boxast u_{\boldsymbol{\ell}}$ with $(\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{z,d}$. With the following lemmas, we reduce the number of such $\boldsymbol{\mu}$'s. For that, we have to show this, which can be seen as progress towards Conjecture 2.39. For this, given $\mathtt{W}_1, \mathtt{W}_2 \in (\mathcal{U} \backslash \{u_0\})^*$, we call the box product $\mathtt{W}_1 \boxast \mathtt{W}_2$ *non-trivial* if $1 \leq \operatorname{len}(\mathtt{W}_1) \leq \operatorname{len}(\mathtt{W}_2)$.

**Lemma 2.62.** *Let be $\boldsymbol{\mu} \in \mathbb{Z}^d_{>0}$ for some $d \geq 1$. Then, $u_{\boldsymbol{\mu}}$ can be written as a linear combination of words ending in $u_1$ and non-trivial box products.*

*Proof.* Choose $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d) \in \mathbb{Z}^d_{>0}$ with $\mu_d > 1$ (for $\mu_d = 1$ there is nothing to prove). Then,

$$u_{\mu_d - 1} \boxast u_{\mu_1} \cdots u_{\mu_{d-1}} u_1 = u_{\boldsymbol{\mu}} + \left(u_{\mu_d - 1} \boxast u_{\mu_1} \cdots u_{\mu_{d-1}}\right) u_1,$$

i.e., after rearranging, one obtains the claim. □

**Lemma 2.63.** *Fix $z, d \in \mathbb{Z}_{>1}$ with $z \geq d \geq 2$. If Conjecture 2.39 is true for $(z, d-1)$, then every $u_{\boldsymbol{\mu}}$ with $\boldsymbol{\mu} \in \mathbb{Z}^d_{>0}$ and $|\boldsymbol{\mu}| = z + d$ can be written as linear combination of words ending in $u_2$ and non-trivial box products.*

*Proof.* Assume $d$ and $z$ as in the lemma. Let be $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d) \in \mathbb{Z}^d_{>0}$ with $|\boldsymbol{\mu}| = z + d$. If $\mu_d = 2$, there is nothing to prove. If $\mu_d > 2$, we proceed as in the proof of Lemma 2.62. If $\mu_d = 1$, by assumption and Theorem 2.41, we have

$$u_{\mu_1} \cdots u_{\mu_{d-1}} = \sum_{(\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{z,d-1}} a_{\mathbf{n},\boldsymbol{\ell}}(\boldsymbol{\mu})\, u_{\mathbf{n}} \boxast u_{\boldsymbol{\ell}}$$

for appropriate $a_{\mathbf{n},\boldsymbol{\ell}}(\boldsymbol{\mu}) \in \mathbb{Q}$. Then,

$$\sum_{(\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{z,d-1}} a_{\mathbf{n},\boldsymbol{\ell}}(\boldsymbol{\mu})\, u_{\mathbf{n}} \boxast u_{\boldsymbol{\ell}} u_1 = u_{\boldsymbol{\mu}} + \sum_{(\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{z,d-1}} a_{\mathbf{n},\boldsymbol{\ell}}(\boldsymbol{\mu}) \left(u_{(n_1, \ldots, n_{s-1})} \boxast u_{\boldsymbol{\ell}}\right) u_{1+n_s}.$$

The latter sum consists of words ending in some $u_{\mu'_d}$ with $\mu'_d \geq 2$. However, such words can be written as linear combinations of words ending in $u_2$ and box products, similar to the proof of Lemma 2.62, completing the proof. □

**Lemma 2.64.** *Fix $z, d \in \mathbb{Z}_{>1}$ with $z \geq d \geq 2$. If Conjecture 2.39 is true for $(z-1, d-1)$, then every $u_{\boldsymbol{\mu}}$ with $\boldsymbol{\mu} \in \mathbb{Z}^d_{>0}$ and $|\boldsymbol{\mu}| = z + d$ can be written as linear combination of words ending in $u_3$ and non-trivial box products.*

*Proof.* Assume $d$ and $z$ as in the lemma. Using Lemma 2.63, we only have to show that a word ending in $u_2$ can be written as a linear combination of words ending in $u_3$ and box products. Choose such a word $u_{\mu_1} \cdots u_{\mu_{d-1}} u_2$, i.e., $2 + \sum_{j=1}^{d-1} \mu_j = z + d$. Then, by assumption, one has

$$u_{\mu_1} \cdots u_{\mu_{d-1}} = \sum_{(\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{z-1,d-1}} a_{\mathbf{n},\boldsymbol{\ell}}(\boldsymbol{\mu})\, u_{\mathbf{n}} \boxast u_{\boldsymbol{\ell}}$$

for appropriate $a_{\mathbf{n},\boldsymbol{\ell}}(\boldsymbol{\mu}) \in \mathbb{Q}$. Hence,

$$\sum_{(\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{z-1,d-1}} a_{\mathbf{n},\boldsymbol{\ell}}(\boldsymbol{\mu})\, u_{\mathbf{n}} \boxast u_{\boldsymbol{\ell}} u_2 = u_{\boldsymbol{\mu}} + \sum_{(\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{z-1,d-1}} a_{\mathbf{n},\boldsymbol{\ell}}(\boldsymbol{\mu}) \left(u_{(n_1, \ldots, n_{s-1})} \boxast u_{\boldsymbol{\ell}}\right) u_{2+n_s}.$$

The latter sum consists of words ending in some $u_{\mu'_d}$ with $\mu'_d \geq 3$. However, such words can be written as linear combinations of words ending in $u_3$ and box products, similar to the proof of Lemma 2.62, completing the proof. $\square$

**Lemma 2.65.** *Let be $\boldsymbol{n} \in \mathbb{Z}^s_{>0}, \boldsymbol{\ell} \in \mathbb{Z}^d_{>0}$ with $1 \leq s \leq d$. Then, $u_{\boldsymbol{n}} \circledast u_{\boldsymbol{\ell}}$ can be written as linear combination of non-trivial box products $u_{\boldsymbol{n}'} \circledast u_{\boldsymbol{\ell}'}$ where $\boldsymbol{\ell}'$ ends in 1.*

*Proof.* Writing $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_d)$, we may assume $\ell_d > 1$ since for $\ell_d = 1$ there is nothing to prove. Then,

$$\begin{aligned} u_{\mathbf{n}} \circledast u_{\boldsymbol{\ell}} &= u_{\mathbf{n}} \circledast \left( u_{\ell_d - 1} \circledast u_{(\ell_1, \ldots, \ell_{d-1}, 1)} - \sum_{j=1}^{d-1} u_{(\ell_1, \ldots, \ell_j + \ell_d - 1, \ldots, \ell_{d-1}, 1)} \right) \\ &= (u_{\mathbf{n}} * u_{\ell_d - 1}) \circledast u_{(\ell_1, \ldots, \ell_{d-1}, 1)} - \sum_{j=1}^{d-1} u_{\mathbf{n}} \circledast u_{(\ell_1, \ldots, \ell_j + \ell_d - 1, \ldots, \ell_{d-1}, 1)}, \end{aligned}$$

where we used Lemma 2.19 in the last step. $\square$

A further result about the numbers $\mathfrak{s}_{z,d}$ is the following lemma that gives a lower bound.

**Lemma 2.66.** *For all $z, d \in \mathbb{Z}_{>0}$, we have $\mathfrak{s}_{z,d} \geq \binom{z+d-2}{d-1}$.*

*Proof.* We prove by induction on $z+d$. For $z = 1$, the claim is clear, since for all $d \in \mathbb{Z}_{>0}$, we have $0 \neq u_1 \circledast u_1^d \in \mathcal{S}_{1,d}$, i.e.,

$$\mathfrak{s}_{1,d} \geq 1 = \binom{1+d-2}{d-1}.$$

For $d = 1$, we have for all $z \in \mathbb{Z}_{>0}$ equality by Lemma 2.42. In particular, the base case $z + d = 2$ is proven. Now, let be $z, d \in \mathbb{Z}_{>1}$ and assume that the lemma is proven for all smaller values of $z + d$. By Lemma 2.50 and the induction hypothesis, we obtain

$$\mathfrak{s}_{z,d} \geq \mathfrak{s}_{z,d-1} + \mathfrak{s}_{z-1,d} \geq \binom{z+d-3}{d-2} + \binom{z+d-3}{d-1} = \binom{z+d-2}{d-1}. \qquad \square$$

We end this subsection with some remark on Conjecture 2.39 that is independent of the rest of the paper.

**Remark 2.67.** Using basic linear algebra, we obtain the following equivalent formulation of Conjecture 2.39 in the cases $z \geq d$. Fix positive integers $d$ and $z$ with $z \geq d$. Conjecture 2.39 is true for the pair $(z, d)$ if and only if the $\binom{z+d-1}{d-1}$ expressions

$$\left\{ \sum_{(\boldsymbol{n},\boldsymbol{\ell}) \in \mathcal{J}_{z,d}} \epsilon^{\boldsymbol{\mu}}_{\boldsymbol{n},\boldsymbol{\ell}} \, u_{\boldsymbol{n}} \circledast u_{\boldsymbol{\ell}} \;\middle|\; \boldsymbol{\mu} \in \mathbb{Z}^d_{>0}, \, |\boldsymbol{\mu}| = z + d \right\}$$

are $\mathbb{Q}$-linearly independent. Here, $\epsilon^{\boldsymbol{\mu}}_{\boldsymbol{n},\boldsymbol{\ell}}$ denotes the multiplicity of $u_{\boldsymbol{\mu}}$ in $u_{\boldsymbol{n}} \circledast u_{\boldsymbol{\ell}}$.

## 2.5 Our approach to the refined Bachmann Conjecture 2.10

In the following, we present the approach with which one is trying to make progress in proving the refined Bachmann Conjecture 2.10. The general idea is to prove by induction on zero($\mathtt{W}$) for $\mathtt{W} \in \mathcal{U}^{*,\circ}$ that $\zeta_q^{\mathrm{f}}(\mathtt{W}) \in \mathcal{Z}_q^{f,\circ}$. This is trivial for the base case zero($\mathtt{W}$) $= 0$.

Thus, we assume $\mathrm{zero}(\mathtt{W}) > 0$. Particularly - for proving the induction step - one has to write $\zeta_q^{\mathrm{f}}(\mathtt{W})$ as a linear combination of $\zeta_q^{\mathrm{f}}(\mathtt{W}')$'s with $\mathtt{W}' \in \mathcal{U}^{*,\circ}$ and $\mathrm{zero}(\mathtt{W}') < \mathrm{zero}(\mathtt{W})$. In our approach, we refine the induction step by showing that for every word $\mathtt{W} \in \mathcal{U}^{*,\circ}$ we can write $\zeta_q^{\mathrm{f}}(\mathtt{W})$ as a linear combination of $\zeta_q^{\mathrm{f}}(\mathtt{W}')$'s with $\mathtt{W}' \in \mathcal{U}^{*,\circ}$ and $\mathrm{zero}(\mathtt{W}') < \mathrm{zero}(\mathtt{W})$, or

$$\mathrm{zero}(\mathtt{W}') = \mathrm{zero}(\mathtt{W}) \quad \text{and} \quad \mathrm{depth}(\mathtt{W}') + \mathrm{wt}(\mathtt{W}') < \mathrm{depth}(\mathtt{W}) + \mathrm{wt}(\mathtt{W})$$

(see the refined Bachmann Conjecture 2.10). The general observation of why the refined Bachmann Conjecture 2.10 is of interest when studying Bachmann's Conjecture 2.4 is given in the following lemma.

**Lemma 2.68** (Lemma 2.11)**.** *Fix $z, d, w \in \mathbb{Z}_{>0}$. If the refined Bachmann Conjecture 2.10 is true for $(z, d, w)$ and if Bachmann's Conjecture 2.4 is true for all $(z', d', w') \in \mathbb{Z}_{>0}^3$ with $z' + d' + w' < z + d + w$, then Bachmann's Conjecture 2.4 is true for $(z, d, w)$. In particular, the refined Bachmann Conjecture 2.10 implies Bachmann's Conjecture 2.4.*

*Proof.* Fix $z, d, w \in \mathbb{Z}_{>0}$ and assume that the refined Bachmann Conjecture 2.10 is true for $(z, d, w)$ and that Bachmann's Conjecture 2.4 is true for all triples $(z', d', w') \in \mathbb{Z}_{>0}^3$ satisfying $z' + d' + w' < z + d + w$. By definition of $\mathrm{F}_{z,d,w}$ and the second part of our assumption, it follows

$$\begin{aligned}\mathrm{F}_{z,d,w} &= \mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w-1} \mathcal{Z}_q^f + \sum_{\substack{z'+d'=z+d-1 \\ 0 \leq z' \leq z}} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z',d',w} \mathcal{Z}_q^f \\ &\subset \mathrm{Fil}^{\mathrm{D,W}}_{z+d,w-1} \mathcal{Z}_q^{f,\circ} + \mathrm{Fil}^{\mathrm{D,W}}_{z+d-1,w} \mathcal{Z}_q^{f,\circ} \subset \mathrm{Fil}^{\mathrm{D,W}}_{z+d,w} \mathcal{Z}_q^{f,\circ}.\end{aligned}$$

Using the assumption $\mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w} \mathcal{Z}_q^f \subset \mathrm{F}_{z,d,w}$, we obtain $\mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w} \mathcal{Z}_q^f \subset \mathrm{Fil}^{\mathrm{D,W}}_{z+d,w} \mathcal{Z}_q^{f,\circ}$, i.e., Bachmann's Conjecture 2.4 for $(z, d, w)$. ☐

For given $z \geq d$, our approach to the refined Bachmann Conjecture 2.10 restricts - independent of the weight $w$ - to prove Conjecture 2.39 for the pair $(z, d)$ as the following theorem shows.

**Theorem 2.69.** *Fix $z, d \in \mathbb{Z}_{>0}$ with $z \geq d$. If Conjecture 2.39 is true for the pair $(z, d)$, then for all $w \in \mathbb{Z}_{>0}$, we have*

$$\mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w} \mathcal{Z}_q^f \subset \sum_{\substack{z'+d'=z+d-1 \\ 0 \leq z' \leq z-1}} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z',d',w} \mathcal{Z}_q^f \subset \mathrm{F}_{z,d,w}\,.$$

*In particular, the refined Bachmann Conjecture 2.10 is true for the triples $(z, d, w) \in \mathbb{Z}_{>0}^3$ with $w$ arbitrary.*

*Proof.* Fix $z, d \in \mathbb{Z}_{>0}$ with $z \geq d$ and assume that Conjecture 2.39 is true for $(z, d)$. This means $u_{\mathbf{z}} \in \mathcal{P}$ for all $\mathbf{z} \in \mathbb{Z}_{>0}^d$ with $|\mathbf{z}| = z + d$. Hence, the claim follows immediately from Corollary 2.57. ☐

**Remark 2.70.** Immediately from Theorems 2.41 and 2.69 the following statement is obtained: If Conjecture 2.39 is true for all $z = d$, then we have

$$\mathrm{Fil}^{\mathrm{Z}}_{z} \mathcal{Z}_q^f \subset \mathrm{Fil}^{\mathrm{Z}}_{d-1} \mathcal{Z}_q^f$$

for all $(z,d) \in \mathbb{Z}^2_{>0}$ with $z \geq d$. More precise, then we have

$$\mathcal{Z}_q^f = \mathcal{Z}_q^{f,\circ} + \sum_{\substack{0 \leq z \leq d-1 \\ d \geq 1}} \operatorname{Fil}^{\mathrm{Z,D,W}}_{z,d,2z+d-1} \mathcal{Z}_q^f.$$

**Remark 2.71.** For $z \geq d$, our approach to Bachmann's Conjecture 2.4, and the refined Bachmann Conjecture 2.10, is to study Conjecture 2.39 in more detail. We will explain this in Section 2.6. For $z < d$, this approach will not suffice since in this case, we have $\mathcal{S}_{z,d} \subsetneq \mathcal{T}_{z,d}$ by Conjecture 2.39 which is numerically explicit verified for small values of $z$ and $d$ (see Lemma 2.42). Hence, we need to extend our approach. We make do with few explicit calculations to prove our main results in Section 2.6. In the outlook, Section 2.7, we abstract our calculations and leave it as an open question whether this generalization is sufficient.

## 2.6 Proof of our main results towards the refined Bachmann Conjecture 2.10

In this section, we first provide the proof of our main results, namely, Theorems 2.8 and 2.12, where some particular statements are black-boxed. We deliver their proofs in Sections 2.6.1, 2.6.2, and 2.6.3.

**Proposition 2.72.** *The refined Bachmann Conjecture 2.10 is true for all $(z,2,w) \in \mathbb{Z}^3_{>0}$.*

*Proof.* Due to case $d = 2$ of Lemma 2.42, Conjecture 2.39 is true for all $(z,2) \in \mathbb{Z}^2_{>0}$ with $z \geq 2$. Theorem 2.69 then implies $\operatorname{Fil}^{\mathrm{Z,D,W}}_{z,2,w} \mathcal{Z}_q^f \subset \mathrm{F}_{z,2,w}$ for all $z, w \in \mathbb{Z}_{>0}$ with $z \geq 2$. Hence, it remains to prove case $z = 1$. However, this follows immediately from the special case $d = 2$ of Corollary 2.28. □

**Proposition 2.73.** *The refined Bachmann Conjecture 2.10 is true for all $(z,3,w) \in \mathbb{Z}^3_{>0}$.*

*Proof.* The case $z = 1$ is proven by Corollary 2.28, the case $z = 2$ will follow from Theorem 2.77 below, and the cases $z \geq 3$ are proven by the $z = 3$ case of Lemma 2.52, Theorem 2.41, and Theorem 2.69. □

**Proposition 2.74.** *The refined Bachmann Conjecture 2.10 is true for all $(z,4,w) \in \mathbb{Z}^3_{>0}$.*

*Proof.* While the case $z = 1$ is proven by Corollary 2.28, the case $z = 2$ will be obtained from Theorem 2.82 below, and the case $z = 3$ will be obtained from Theorem 2.97 below. Furthermore, the cases $z \geq 4$ are proven by Proposition 2.53 and Theorem 2.69, completing the claim. □

We are now able to prove one of our main theorems.

**Theorem 2.75** (Theorem 2.8)**.** *Bachmann's Conjecture 2.4 is true for all $(z,d,w) \in \mathbb{Z}^3_{>0}$ with $z + d \leq 6$.*

*Proof.* For $z + d \leq 3$, the claim is an immediate consequence of Proposition 2.21 and Corollary 2.28. For $z = d = 2$, the claim follows by induction on $w$, using the proven claim for $z + d \leq 3$, Lemma 2.68, and Proposition 2.72 in the induction step. Together with Proposition 2.21 and Corollary 2.28, the claim holds now for $z + d \leq 4$. Furthermore, inductively on $w$, the claim for $(z,d) \in \{(3,2),(2,3)\}$ follows from the already proven claim for $z + d \leq 4$, Lemma 2.68, and Proposition 2.72 (for $(z,d) = (3,2)$), and Proposition 2.73 (for $(z,d) = (2,3)$). Now, using Proposition 2.21 and Corollary 2.28, the claim follows for $z+d \leq 5$. Analogously, for $(z,d) \in \{(4,2),(3,3),(2,4)\}$, the claim follows in each case inductively on $w$, where we use in the induction step the already proven claim for $z+d \leq 5$, Lemma 2.68, and Proposition 2.72 (for $(z,d)=(4,2)$), 2.73 (for $(z,d)=(3,3)$), and 2.74 (for $(z,d)=(2,4)$), respectively. Now, using Proposition 2.21 and Corollary 2.28, the theorem is proven for $z+d \leq 6$ as well, completing the proof. □

Theorem 2.75 is the main ingredient in the proof of the next main theorem.

**Theorem 2.76** (Theorem 2.12)**.** *The refined Bachmann Conjecture 2.10 is true for all triples of positive integers $(z,d,w) \in \mathbb{Z}^3_{>0}$ with $1 \leq d \leq 4$.*

*Proof.* For $1 \leq z < d \leq 4$ and $w \in \mathbb{Z}_{>0}$ arbitrary, we obtain the claim from Theorem 2.75. Furthermore, for $1 \leq d \leq 3$, $z \geq d$ and $w \in \mathbb{Z}_{>0}$ arbitrary, we obtain the claim from Corollary 2.57, Lemma 2.42, and Theorem 2.69. For $d=z=4$ and $w \in \mathbb{Z}_{>0}$ arbitrary, the claim follows from Proposition 2.53 and Corollary 2.57. Hence, for $z \geq d = 4$ and $w \in \mathbb{Z}_{>0}$ arbitrary, the claim is a direct consequence of Corollary 2.57 and Theorem 2.69, proving the theorem finally. □

### 2.6.1 The refined Bachmann Conjecture 2.10 for $(z,d,w)=(2,3,w)$

**Theorem 2.77.** *The refined Bachmann Conjecture 2.10 is true for all $(2,3,w) \in \mathbb{Z}^3_{>0}$, i.e.,*

$$\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0^{z_1}u_{k_2}u_0^{z_2}u_{k_3}u_0^{z_3}\right) \in \mathrm{F}_{2,3,w} \tag{2.77.1}$$

*for all integers $k_j \in \mathbb{Z}_{>0}$, $z_j \in \mathbb{Z}_{\geq 0}$, where $1 \leq j \leq 3$, satisfying $z_1+z_2+z_3=2$ and $w=k_1+k_2+k_3+2$.*

*Proof.* For $k_1=k_2=k_3=1$ and for all $z_1,z_2,z_3 \geq 0$ satisfying $z_1+z_2+z_3=2$, (2.77.1) is true since, after using $\tau$-invariance of $\zeta_q^{\mathrm{f}}$, we have

$$\zeta_q^{\mathrm{f}}\left(u_1u_0^{z_1}u_1u_0^{z_2}u_1u_0^{z_3}\right) = \zeta_q^{\mathrm{f}}\left(u_{z_3+1}u_{z_2+1}u_{z_3+1}\right) \in \mathrm{F}_{2,3,w}\,.$$

Furthermore, for $k_2>1$, (2.77.1) will follow from Lemma 2.79, for $k_3>1$, (2.77.1) will follow from Lemma 2.80, and for $k_1>1$, (2.77.1) will follow from Lemma 2.81, completing the proof of the theorem. □

**Lemma 2.78.** *Let be $k_1,k_2,k_3 \in \mathbb{Z}_{>0}$ and write $w=k_1+k_2+k_3+2$. We have*

$$\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_0\right),\ \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_{k_3}\right) \in \mathrm{F}_{2,3,w}, \tag{2.78.1}$$

$$\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3}\right) \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0\right) \mod \mathrm{F}_{2,3,w} \tag{2.78.2}$$

$$\equiv -\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}\right) \equiv -\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_0\right) \mod \mathrm{F}_{2,3,w}\,. \tag{2.78.3}$$

*In particular, for fixed $k_1,k_2,k_3$, if one of the latter four formal Multiple Zeta Values is in $\mathrm{F}_{2,3,w}$, (2.77.1) is true for the corresponding choice of $k_1,k_2,k_3$.*

*Proof.* First note that (2.78.1) is a consequence of Corollaries 2.25 and 2.29. Furthermore, after using Lemma 2.56 and $\tau$-invariance of formal $q$MZVs, with (2.78.1), we obtain

$$\begin{aligned}
0 &\equiv \zeta_q^{\mathrm{f}}\left(\Psi_{(k_1,k_2,k_3)}(u_1u_1 \circledast u_1u_1u_1)\right) && \mod \mathrm{F}_{2,3,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_2u_0^{k_3-1}u_2u_0^{k_2-1}u_1u_0^{k_1-1} + u_2u_0^{k_3-1}u_1u_0^{k_2-1}u_2u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,3,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0\right) && \mod \mathrm{F}_{2,3,w},
\end{aligned} \tag{2.78.4}$$

$$\begin{aligned}
0 &\equiv \zeta_q^{\mathrm{f}}\left(\Psi_{(k_1,k_2,k_3)}(u_1 \circledast u_1u_2u_1)\right) && \bmod \mathrm{F}_{2,3,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_2u_0^{k_3-1}u_2u_0^{k_2-1}u_1u_0^{k_1-1} + u_1u_0^{k_3-1}u_3u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \bmod \mathrm{F}_{2,3,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3}\right) && \bmod \mathrm{F}_{2,3,w},\\
0 &\equiv \zeta_q^{\mathrm{f}}\left(\Psi_{(k_1,k_2,k_3)}(u_1 \circledast u_1u_1u_2)\right) && \bmod \mathrm{F}_{2,3,w} \quad (2.78.5)\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_2u_0^{k_3-1}u_1u_0^{k_2-1}u_2u_0^{k_1-1} + u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_3u_0^{k_1-1}\right) && \bmod \mathrm{F}_{2,3,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}\right) && \bmod \mathrm{F}_{2,3,w}\,. (2.78.6)
\end{aligned}$$

We obtain (2.78.2) and (2.78.3), by comparing (2.78.4), (2.78.5), and (2.78.6). □

**Lemma 2.79.** *Equation* (2.77.1) *is true for* $k_2 > 1$.

*Proof.* Let be $k_1, k_2, k_3 \in \mathbb{Z}_{>0}$ and write $w = k_1 + k_2 + k_3 + 3$. By (2.3.1), we have

$$u_2u_1 * u_{k_1}u_{k_2}u_{k_3} \in \mathrm{Fil}^{\mathrm{Z,D,W}}_{0,5,w}\, \mathbb{Q}\langle\mathcal{U}\rangle^\circ.$$

Hence, and due to $\tau$-invariance of formal $q$MZVs, we have

$$\begin{aligned}
0 &\equiv \frac{1}{k_2}\zeta_q^{\mathrm{f}}\left(\tau(u_2u_1) * \tau\left(u_{k_1}u_{k_2}u_{k_3}\right)\right) && \bmod \mathrm{F}_{2,3,w}\\
&\equiv \frac{1}{k_2}\zeta_q^{\mathrm{f}}\left(u_1u_1u_0 * u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \bmod \mathrm{F}_{2,3,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_2u_0^{k_3-1}u_2u_0^{k_2}u_1u_0^{k_1-1}\right) + \frac{k_1}{k_2}\zeta_q^{\mathrm{f}}\left(u_1u_1 * u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_1u_0^{k_1}\right) && \bmod \mathrm{F}_{2,3,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_0u_{k_3}u_0\right) + \zeta_q^{\mathrm{f}}\left(\Psi_{(k_1+1,k_2,k_3)}(u_1u_1 \circledast u_1u_1u_1)\right) && \bmod \mathrm{F}_{2,3,w},\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_0u_{k_3}u_0\right) && \bmod \mathrm{F}_{2,3,w},
\end{aligned}$$

where the last step is a consequence of Lemma 2.56. Now, with Lemma 2.78, (2.77.1) indeed is true for $k_2 > 1$. □

**Lemma 2.80.** *Equation* (2.77.1) *is true for* $k_3 > 1$.

*Proof.* Let be $k_1, k_2, k_3 \in \mathbb{Z}_{>0}$ and write $w = k_1 + k_2 + k_3 + 3$. By (2.3.1), we have

$$u_2 * u_{k_1}u_0u_{k_2}u_{k_3} \in \mathrm{Fil}^{\mathrm{Z,D,W}}_{1,4,w}\, \mathbb{Q}\langle\mathcal{U}\rangle^\circ.$$

Hence, and due to $\tau$-invariance of formal $q$MZVs, we have

$$\begin{aligned}
0 &\equiv \frac{1}{k_3}\zeta_q^{\mathrm{f}}\left(\tau(u_2) * \tau\left(u_{k_1}u_0u_{k_2}u_{k_3}\right)\right) && \bmod \mathrm{F}_{2,3,w}\\
&\equiv \frac{1}{k_3}\zeta_q^{\mathrm{f}}\left(u_1u_0 * u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_2u_0^{k_1-1}\right) && \bmod \mathrm{F}_{2,3,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_2u_0^{k_3}u_1u_0^{k_2-1}u_2u_0^{k_1-1}\right) + \frac{k_2}{k_3}\zeta_q^{\mathrm{f}}\left(u_2u_0^{k_3-1}u_1u_0^{k_2}u_2u_0^{k_1-1}\right)\\
&\quad + \frac{k_2}{k_3}\zeta_q^{\mathrm{f}}\left(u_1u_0^{k_3-1}u_2u_0^{k_2}u_2u_0^{k_1-1}\right) + \frac{k_1}{k_3}\zeta_q^{\mathrm{f}}\left(\Psi_{(k_1+1,k_2,k_3)}(u_1 \circledast u_1u_1u_2\right) && \bmod \mathrm{F}_{2,3,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3+1}u_0\right) && \bmod \mathrm{F}_{2,3,w}\,.
\end{aligned}$$

The last step is obtained by Lemmas 2.56 and 2.79. Hence, the lemma is proven by Lemma 2.78. □

Hence, for proving Proposition 2.73, the remaining case is $k_2 = k_3 = 1$.

**Lemma 2.81.** *Equation* (2.77.1) *is true for* $k_1 > 1$.

*Proof.* Let be $k_1, k_2, k_3 \in \mathbb{Z}_{>0}$ with $k_1 > 1$ and write $w = k_1 + k_2 + k_3 + 2$. Due to Lemmas 2.79 and 2.80, we may assume $k_1 > 1$ and $k_2 = k_3 = 1$, i.e., $w = k_1 + 4$ then. By Proposition 2.72, we have $\zeta_q^{\mathrm{f}}(u_{k_1}u_0u_1u_0) \in \mathrm{F}_{2,2,w-1}$ and thus $\zeta_q^{\mathrm{f}}(u_1 * u_{k_1}u_0u_1u_0) \in \mathrm{F}_{2,3,w}$. Multiplying out the latter product and using Proposition 2.72, (2.78.1), and Lemma 2.79, we see that

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}(u_1 * u_{k_1}u_0u_1u_0) \equiv 2\zeta_q^{\mathrm{f}}(u_{k_1}u_0u_1u_1u_0 + u_{k_1}u_1u_0u_1u_0) && \bmod \mathrm{F}_{2,3,w} \\ &\equiv \zeta_q^{\mathrm{f}}(u_{k_1}u_0u_1u_1u_0) && \bmod \mathrm{F}_{2,3,w}, \end{aligned}$$

where the last congruence is obtained from (2.78.4). Thus, the proof of the lemma follows from Lemma 2.78. □

### 2.6.2 The refined Bachmann Conjecture 2.10 for $(z,d,w) = (2,4,w)$

**Theorem 2.82.** *The refined Bachmann Conjecture 2.10 is true for all* $(2,4,w) \in \mathbb{Z}^3_{>0}$, *i.e.,*

$$\zeta_q^{\mathrm{f}}(u_{k_1}u_0^{z_1}u_{k_2}u_0^{z_2}u_{k_3}u_0^{z_3}u_{k_4}u_0^{z_4}) \in \mathrm{F}_{2,4,w} \tag{2.82.1}$$

*for all integers* $k_j \in \mathbb{Z}_{>0}$, $z_j \in \mathbb{Z}_{\geq 0}$, *for* $1 \leq j \leq 4$, *satisfying* $z_1 + z_2 + z_3 + z_4 = 2$ *and* $w = k_1 + k_2 + k_3 + k_4 + 2$.

*Proof.* In the case $k_1 = k_2 = k_3 = k_4 = 1$, (2.82.1) is true since for all $z_1, \ldots, z_4 \geq 0$, we have by $\tau$-invariance of $\zeta_q^{\mathrm{f}}$ that

$$\zeta_q^{\mathrm{f}}(u_1u_0^{z_1}u_1u_0^{z_2}u_1u_0^{z_3}u_1u_0^{z_4}) = \zeta_q^{\mathrm{f}}(u_{z_4+1}u_{z_3+1}u_{z_2+1}u_{z_1+1}) \in \mathrm{Fil}^{\mathrm{D,W}}_{4,w}\,\mathcal{Z}_q^{f,\circ}.$$

In the four cases $k_{i_1},\ k_{i_2},\ k_{i_3} > 1$ with pairwise distinct $i_1, i_2, i_3 \in \{1,2,3,4\}$, (2.82.1) will follow from Lemma 2.87, Proposition 2.88, and Proposition 2.89. Furthermore, the six cases $k_{i_1}, k_{i_2} > 1$ for distinct $i_1, i_2 \in \{1,2,3,4\}$ (and the two other $k_j$'s equal 1) then follow from Lemmas 2.87, 2.90, 2.91, and 2.92. Next, the four cases of $k_i > 1$ ($i \in \{1,2,3,4\}$) (and the three other $k_j$'s equal 1), will follow from Lemmas 2.93, 2.94, 2.95, and 2.96. This completes the proof of the theorem. □

In the following three lemmas, we state some congruences that are true independently of the several cases we might consider.

**Lemma 2.83.** *Let be* $k_1, \ldots, k_4 \in \mathbb{Z}_{>0}$ *and write* $w = k_1 + \cdots + k_4 + 2$. *We have*

$$0 \equiv \zeta_q^{\mathrm{f}}(u_{k_1}u_{k_2}u_{k_3}u_{k_4}u_0u_0) \quad \bmod \mathrm{F}_{2,4,w}, \tag{2.83.1}$$

$$0 \equiv \zeta_q^{\mathrm{f}}(u_{k_1}u_0u_{k_2}u_0u_{k_3}u_{k_4}) \quad \bmod \mathrm{F}_{2,4,w}, \tag{2.83.2}$$

$$0 \equiv \zeta_q^{\mathrm{f}}(u_{k_1}u_{k_2}u_{k_3}u_0u_{k_4}u_0) + \zeta_q^{\mathrm{f}}(u_{k_1}u_{k_2}u_{k_3}u_0u_0u_{k_4}) \quad \bmod \mathrm{F}_{2,4,w}\,. \tag{2.83.3}$$

*Proof.* Note that (2.83.1) is a direct consequence of Corollary 2.25, while (2.83.2) follows from Corollary 2.29. Last, (2.83.3) follows from (2.83.1) and Corollary 2.27 used with the special case $d = 4$, $z = 2$, $j = 3$. □

**Lemma 2.84.** *Let be* $k_1, \ldots, k_4 \in \mathbb{Z}_{>0}$ *and write* $w = k_1 + \cdots + k_4 + 2$. *We have*

$$\begin{aligned} 0 \equiv{}& \zeta_q^{\mathrm{f}}(u_{k_1}u_{k_2}u_{k_3}u_0u_{k_4}u_0) + \zeta_q^{\mathrm{f}}(u_{k_1}u_{k_2}u_0u_{k_3}u_{k_4}u_0) \\ &+ \zeta_q^{\mathrm{f}}(u_{k_1}u_0u_{k_2}u_{k_3}u_{k_4}u_0) \qquad \bmod \mathrm{F}_{2,4,w}, \end{aligned} \tag{2.84.1}$$

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_0u_{k_4}\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0u_{k_4}\right) \mod \mathrm{F}_{2,4,w}, \tag{2.84.2}$$

$$\begin{aligned} 0 \equiv\ & \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_0u_{k_4}\right) \\ & + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3}u_{k_4}\right) \mod \mathrm{F}_{2,4,w}, \end{aligned} \tag{2.84.3}$$

$$\begin{aligned} 0 \equiv\ & \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0u_{k_4}\right) \\ & + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}u_{k_4}\right) \mod \mathrm{F}_{2,4,w}, \end{aligned} \tag{2.84.4}$$

$$\begin{aligned} 0 \equiv\ & \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_0u_{k_4}\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3}u_{k_4}\right) \\ & + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}u_{k_4}\right) \mod \mathrm{F}_{2,4,w}, \end{aligned} \tag{2.84.5}$$

$$\begin{aligned} 0 \equiv\ & \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_{k_4}u_0\right) \\ & + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_0u_{k_4}\right) \\ & + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0u_{k_4}\right) \mod \mathrm{F}_{2,4,w}. \end{aligned} \tag{2.84.6}$$

*Proof.* All relations are, by Lemma 2.56, a consequence of

$$0 \equiv \zeta_q^{\mathrm{f}}\left(\tau(\Psi_{\mathbf{k}}(u_{\mathbf{n}} \circledast u_{\boldsymbol{\ell}}))\right) \mod \mathrm{F}_{2,4,w}$$

with $\mathbf{k} = (k_1, \ldots, k_4)$ each and $(\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{2,4}$, where Lemma 2.83 was applied. Precisely, for (2.84.1), we used $(\mathbf{n}, \boldsymbol{\ell}) = ((1), (2, 1, 1, 1))$, while for obtaining (2.84.2), we used $(\mathbf{n}, \boldsymbol{\ell}) = ((1), (1, 2, 1, 1))$, for (2.84.3), we used $(\mathbf{n}, \boldsymbol{\ell}) = ((1), (1, 1, 2, 1))$, for (2.84.4), we used $(\mathbf{n}, \boldsymbol{\ell}) = ((1), (1, 1, 1, 2)$, for (2.84.5), we used $(\mathbf{n}, \boldsymbol{\ell}) = ((2), (1, 1, 1, 1))$. Furthermore, for (2.84.6), we used $(\mathbf{n}, \boldsymbol{\ell}) = ((1, 1), (1, 1, 1, 1))$. □

**Lemma 2.85.** *Let be $k_1, \ldots, k_4 \in \mathbb{Z}_{>0}$ and write $w = k_1 + \cdots + k_4 + 3$. We have*

$$\begin{aligned} 0 \equiv\ & k_4\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_{k_4+1}u_0\right) - k_3\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3+1}u_{k_4}\right) \\ & - k_2\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2+1}u_{k_3}u_{k_4}\right) \mod \mathrm{F}_{2,4,w}, \end{aligned} \tag{2.85.1}$$

$$0 \equiv k_4\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_{k_4+1}u_0\right) - k_3\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3+1}u_{k_4}\right) \mod \mathrm{F}_{2,4,w}, \tag{2.85.2}$$

$$0 \equiv k_4\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_{k_4+1}u_0\right) + k_2\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_0u_{k_3}u_0u_{k_4}\right) \mod \mathrm{F}_{2,4,w}, \tag{2.85.3}$$

$$0 \equiv k_3\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3+1}u_0u_{k_4}u_0\right) - k_2\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2+1}u_{k_3}u_{k_4}u_0\right) \mod \mathrm{F}_{2,4,w}, \tag{2.85.4}$$

$$0 \equiv k_3\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3+1}u_0u_0u_{k_4}\right) - k_2\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2+1}u_{k_3}u_{k_4}\right) \mod \mathrm{F}_{2,4,w}. \tag{2.85.5}$$

*Proof.* We use $\tau$-invariance of formal $q$MZVs and Corollary 2.28 to see in the following calculations that each of the formal $q$MZVs of stuffle products in the first line indeed is an element of $\mathrm{F}_{2,4,w}$ in the following.

Now, by (2.83.2) and (2.84.4), we have

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(\tau(u_2) * \tau\left(u_{k_1}u_0u_{k_2}u_{k_3}u_{k_4}\right)\right) && \mod \mathrm{F}_{2,4,w} \\ &\equiv \zeta_q^{\mathrm{f}}\left(u_1u_0 * u_1u_0^{k_4-1}u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_2u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w} \\ &\equiv k_4\zeta_q^{\mathrm{f}}\left(u_2u_0^{k_4}u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_2u_0^{k_1-1}\right) - k_3\zeta_q^{\mathrm{f}}\left(u_1u_0^{k_4-1}u_1u_0^{k_3}u_1u_0^{k_2-1}u_3u_0^{k_1-1}\right) \\ &\quad - k_2\zeta_q^{\mathrm{f}}\left(u_1u_0^{k_4-1}u_1u_0^{k_3-1}u_1u_0^{k_2}u_3u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w} \\ &\equiv k_4\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_{k_4+1}u_0\right) - k_3\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3+1}u_{k_4}\right) \\ &\quad - k_2\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2+1}u_{k_3}u_{k_4}\right) && \mod \mathrm{F}_{2,4,w}, \end{aligned}$$

proving (2.85.1). Furthermore, using (2.83.2), we have

$$0 \equiv \zeta_q^{\mathrm{f}}\left(\tau(u_2) * \tau\left(u_{k_1}u_{k_2}u_0u_{k_3}u_{k_4}\right)\right) \mod \mathrm{F}_{2,4,w}$$

$$\begin{aligned}
&\equiv \zeta_q^{\mathrm{f}}\left(u_1u_0 * u_1u_0^{k_4-1}u_1u_0^{k_3-1}u_2u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w}\\
&\equiv k_4\zeta_q^{\mathrm{f}}\left(u_2u_0^{k_4}u_1u_0^{k_3-1}u_2u_0^{k_2-1}u_1u_0^{k_1-1}\right)\\
&\quad - k_3\zeta_q^{\mathrm{f}}\left(u_1u_0^{k_4-1}u_1u_0^{k_3}u_3u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w}\\
&\equiv k_4\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_{k_4+1}u_0\right) - k_3\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3+1}u_{k_4}\right) && \mod \mathrm{F}_{2,4,w},
\end{aligned}$$

proving (2.85.2). Now, applying (2.83.3) yields

$$\begin{aligned}
0 &\equiv \zeta_q^{\mathrm{f}}\left(\tau(u_2) * \tau\left(u_{k_1}u_{k_2}u_{k_3}u_0u_{k_4}\right)\right) && \mod \mathrm{F}_{2,4,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_1u_0 * u_1u_0^{k_4-1}u_2u_0^{k_3-1}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w}\\
&\equiv k_4\zeta_q^{\mathrm{f}}\left(u_2u_0^{k_4}u_2u_0^{k_3-1}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right)\\
&\quad + k_2\zeta_q^{\mathrm{f}}\left(u_1u_0^{k_4-1}u_2u_0^{k_3-1}u_2u_0^{k_2}u_1u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w}\\
&\equiv k_4\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_{k_4+1}u_0\right) + k_2\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_0u_{k_3}u_0u_{k_4}\right) && \mod \mathrm{F}_{2,4,w},
\end{aligned}$$

proving (2.85.3). Next, use (2.83.1) and (2.84.1) to obtain

$$\begin{aligned}
0 &\equiv \zeta_q^{\mathrm{f}}\left(\tau(u_2) * \tau\left(u_{k_1}u_{k_2}u_{k_3}u_{k_4}u_0\right)\right) && \mod \mathrm{F}_{2,4,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_1u_0 * u_2u_0^{k_4-1}u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w}\\
&\equiv k_3\zeta_q^{\mathrm{f}}\left(u_2u_0^{k_4-1}u_2u_0^{k_3}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right)\\
&\quad - k_2\zeta_q^{\mathrm{f}}\left(u_2u_0^{k_4-1}u_1u_0^{k_3-1}u_1u_0^{k_2}u_2u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w}\\
&\equiv k_3\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3+1}u_0u_{k_4}u_0\right) - k_2\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2+1}u_{k_3}u_{k_4}u_0\right) && \mod \mathrm{F}_{2,4,w},
\end{aligned}$$

proving (2.85.4). Now, (2.83.1) and (2.84.5) imply

$$\begin{aligned}
0 &\equiv \zeta_q^{\mathrm{f}}\left(\tau(u_2u_0) * \tau\left(u_{k_1}u_{k_2}u_{k_3}u_{k_4}\right)\right) && \mod \mathrm{F}_{2,4,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_2u_0 * u_1u_0^{k_4-1}u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w}\\
&\equiv k_3\zeta_q^{\mathrm{f}}\left(u_1u_0^{k_4-1}u_3u_0^{k_3}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right)\\
&\quad - k_2\zeta_q^{\mathrm{f}}\left(u_1u_0^{k_4-1}u_1u_0^{k_3-1}u_1u_0^{k_2}u_3u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w}\\
&\equiv k_3\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3+1}u_0u_0u_{k_4}\right) - k_2\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2+1}u_{k_3}u_{k_4}\right) && \mod \mathrm{F}_{2,4,w},
\end{aligned}$$

proving (2.85.5). This completes the proof of the lemma. □

**Corollary 2.86.** *Let be $k_1, \dots, k_4 \in \mathbb{Z}_{>0}$ and write $w = k_1 + \cdots + k_4 + 3$. We have*

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2+1}u_{k_3}u_0u_{k_4}\right) \mod \mathrm{F}_{2,4,w}, \tag{2.86.1}$$
$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_0u_{k_3}u_0u_{k_4}\right) \mod \mathrm{F}_{2,4,w}, \tag{2.86.2}$$
$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_{k_4+1}u_0\right) \mod \mathrm{F}_{2,4,w}, \tag{2.86.3}$$
$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_0u_{k_4+1}\right) \mod \mathrm{F}_{2,4,w}\,. \tag{2.86.4}$$

*Proof.* Adding (2.85.4) and (2.85.5), yields, applying (2.83.3),

$$\begin{aligned}
0 &\equiv -k_2\left(\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2+1}u_{k_3}u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2+1}u_{k_3}u_{k_4}\right)\right) && \mod \mathrm{F}_{2,4,w}\\
&\equiv k_2\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2+1}u_{k_3}u_0u_{k_4}\right) && \mod \mathrm{F}_{2,4,w},
\end{aligned}$$

where the last step follows from (2.84.4). Hence, (2.86.1) is proven. Furthermore, (2.86.2) is deducted from (2.84.2) and (2.86.1). Now, (2.86.3) follows from (2.85.3) and (2.86.2). Since (2.86.4) is a consequence of (2.86.3) and (2.83.3), the corollary is proven. ☐

**Lemma 2.87.** *Equation* (2.82.1) *is true for* $k_2, k_4 > 1$.

*Proof.* Let be $k_1, k_2, k_3, k_4 \in \mathbb{Z}_{>0}$ and write $w = k_1 + k_2 + k_3 + k_4 + 4$. By Equations (2.83.1), (2.84.1), and (2.86.3), we have

$$\begin{aligned}
0 &\equiv \zeta_q^{\mathrm{f}}\left(\tau(u_1u_3) * \tau\left(u_{k_1}u_{k_2}u_{k_3}u_{k_4}\right)\right) && \mod \mathrm{F}_{2,4,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_1u_0u_0u_1 * u_1u_0^{k_4-1}u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w}\\
&\equiv k_4k_2\zeta_q^{\mathrm{f}}\left(u_2u_0^{k_4}u_1u_0^{k_3-1}u_1u_0^{k_2}u_2u_0^{k_1-1}\right)\\
&\quad + \zeta_q^{\mathrm{f}}\left(u_1u_0^{k_4-1}\left(u_1u_0u_0u_1 * u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right)\right)\\
&\quad + \zeta_q^{\mathrm{f}}\left(u_2u_0^{k_4-1}u_1\left(u_0u_0u_1 * u_0^{k_3-1}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right)\right) && \mod \mathrm{F}_{2,4,w}\,.
\end{aligned}$$

Now, by (2.84.2), (2.84.4), (2.84.6), and (2.86.1), the latter is, modulo $\mathrm{F}_{2,4,w}$, congruent

$$\begin{aligned}
&k_4k_2\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2+1}u_{k_3}u_{k_4+1}u_0\right) - \binom{k_3+1}{2}\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3+2}u_0u_{k_4}u_0\right)\\
&+ k_3k_2\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2+1}u_{k_3+1}u_{k_4}u_0\right) - \binom{k_2+1}{2}\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2+2}u_{k_3}u_{k_4}\right).
\end{aligned}$$

Using (2.85.2), (2.85.3), (2.85.4), and (2.85.5), the latter is, modulo $\mathrm{F}_{2,4,w}$, congruent

$$\begin{aligned}
&- k_2k_3\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_0u_0u_{k_3+1}u_{k_4}\right) - \frac{1}{2}k_2k_3\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2+1}u_{k_3+1}u_{k_4}u_0\right)\\
&+ k_3k_2\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2+1}u_{k_3+1}u_{k_4}u_0\right) - \frac{1}{2}k_2k_3\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_{k_3+1}u_0u_0u_{k_4}\right)\\
\equiv{}& k_2k_3\Bigg(-\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_0u_0u_{k_3+1}u_{k_4}\right) - \frac{1}{2}\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_{k_3+1}u_0u_0u_{k_4}\right)\\
&+\frac{1}{2}\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2+1}u_{k_3+1}u_{k_4}u_0\right)\Bigg) \qquad \mod \mathrm{F}_{2,4,w}\,.
\end{aligned}$$

With (2.84.2), (2.84.3), (2.84.6), and (2.86.2), one obtains so

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_0u_{k_3+1}u_{k_4}u_0\right) \mod \mathrm{F}_{2,4,w}\,. \tag{2.87.1}$$

Now, this, together with (2.84.3) and (2.86.2) imply

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_0u_0u_{k_3+1}u_{k_4}\right) \mod \mathrm{F}_{2,4,w}\,. \tag{2.87.2}$$

Furthermore, (2.85.2) and (2.87.2) imply

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_0u_{k_3}u_{k_4+1}u_0\right) \mod \mathrm{F}_{2,4,w}\,.$$

Note that by Lemma 2.83, Corollary 2.86, and the congruences in Lemma 2.84, the claim follows. ☐

**Proposition 2.88.** *Equation* (2.82.1) *is true for* $k_1, k_3, k_4 > 1$.

*Proof.* Let be $k_1, k_2, k_3, k_4 \in \mathbb{Z}_{>0}$ with $k_1, k_3, k_4 > 1$ and write $w = k_1 + k_2 + k_3 + k_4 + 2$. For all $z_2, z_3, z_4 \geq 0$ with $z_2 + z_3 + z_4 = 2$, using Theorem 2.77 in the first step and

Lemma 2.87 additionally in the second step, we have

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1} * u_{k_2} u_0^{z_2} u_{k_3} u_0^{z_3} u_{k_4} u_0^{z_4}\right) \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1} u_{k_2} u_0^{z_2} u_{k_3} u_0^{z_3} u_{k_4} u_0^{z_4}\right) \mod \mathrm{F}_{2,4,w}\,.$$

Using this observation, for $z_1 \geq 1$, $z_2, z_3, z_4 \geq 0$ with $z_1 + \cdots + z_4 = 2$, we have, using Corollary 2.28 in the first step due to $z_2 + z_3 + z_4 \leq 1$,

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{z_1} * \tau\left(u_{k_1} u_{k_2} u_0^{z_2} u_{k_3} u_0^{z_3} u_{k_4} u_0^{z_4}\right)\right) && \mod \mathrm{F}_{2,4,w} \\ &\equiv \zeta_q^{\mathrm{f}}\left(u_{z_1} * u_{z_4+1} u_0^{k_4-1} u_{z_3+1} u_0^{k_3-1} u_{z_2+1} u_0^{k_2-1} u_1 u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w} \\ &\equiv \zeta_q^{\mathrm{f}}\left(u_{z_4+1} u_0^{k_4-1} u_{z_3+1} u_0^{k_3-1} u_{z_2+1} u_0^{k_2-1} u_{z_1+1} u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w} \\ &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1} u_0^{z_1} u_{k_2} u_0^{z_2} u_{k_3} u_0^{z_3} u_{k_4} u_0^{z_4}\right) && \mod \mathrm{F}_{2,4,w}\,. \end{aligned}$$

This completes the proof of the proposition. □

**Proposition 2.89.** *Equation* (2.82.1) *is true for* $k_1, k_2, k_3 > 1$.

*Proof.* Let be $k_1, k_2, k_3, k_4 \in \mathbb{Z}_{>0}$ with $k_1, k_2, k_3 > 1$ and write $w = k_1 + k_2 + k_3 + k_4 + 2$. Using Lemma 2.87 and Proposition 2.88, we obtain for $z_1, z_2, z_4 \geq 0$ with $z_1 + z_2 + z_4 = 2$ that

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_4} u_0^{z_4} * u_{k_1} u_0^{z_1} u_{k_2} u_0^{z_2} u_{k_3}\right) \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1} u_0^{z_1} u_{k_2} u_0^{z_2} u_{k_3} u_{k_4} u_0^{z_4}\right) \mod \mathrm{F}_{2,4,w},$$

where we used Proposition 2.21 and Proposition 2.73 for the first congruence. Now, for all $z_1, \ldots, z_4 \geq 0$ with $z_1 + \cdots + z_4 = 2$ and $z_3 > 0$, we have

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{z_3} * \tau\left(u_{k_1} u_0^{z_1} u_{k_2} u_0^{z_2} u_{k_3} u_{k_4} u_0^{z_4}\right)\right) && \mod \mathrm{F}_{2,4,w} \\ &\equiv \zeta_q^{\mathrm{f}}\left(u_{z_3} * u_{z_4+1} u_0^{k_4-1} u_1 u_0^{k_3-1} u_{z_2+1} u_0^{k_2-1} u_{z_1+1} u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w} \\ &\equiv \zeta_q^{\mathrm{f}}\left(u_{z_4+1} u_0^{k_4-1} u_{z_3+1} u_0^{k_3-1} u_{z_2+1} u_0^{k_2-1} u_{z_1+1} u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w} \\ &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1} u_0^{z_1} u_{k_2} u_0^{z_2} u_{k_3} u_0^{z_3} u_{k_4} u_0^{z_4}\right) && \mod \mathrm{F}_{2,4,w}\,. \end{aligned}$$

This completes the proof of the proposition. □

Lemma 2.87 and Propositions 2.88 and 2.89, show that Theorem 2.82 is true when three of the $k_j$ are larger than 1. Hence, in the following, we will prove the remaining cases that two of the $k_j$'s are larger 1.

**Lemma 2.90.** *Equation* (2.82.1) *is true for* $k_3, k_4 > 1$.

*Proof.* Let be $k_1, k_2, k_3, k_4 \in \mathbb{Z}_{>0}$ with $k_3, k_4 > 1$ and write $w = k_1 + k_2 + k_3 + k_4 + 2$. According to Lemma 2.87 and Proposition 2.88, we may assume $k_1 = k_2 = 1$. Using Proposition 2.73 for the first two steps in the following calculation, while using Equations (2.83.2), (2.84.2), and (2.86.1) for the last step, we have

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_1 * u_1 u_0 u_{k_3} u_0 u_{k_4}\right) && \mod \mathrm{F}_{2,4,w} \\ &\equiv 2\zeta_q^{\mathrm{f}}\left(u_1 u_1 u_0 u_{k_3} u_0 u_{k_4}\right) + \zeta_q^{\mathrm{f}}\left(u_1 u_0 u_1 u_{k_3} u_0 u_{k_4}\right) \\ &\quad + \zeta_q^{\mathrm{f}}\left(u_1 u_0 u_{k_3} u_1 u_0 u_{k_4}\right) + \zeta_q^{\mathrm{f}}\left(u_1 u_0 u_{k_3} u_0 u_1 u_{k_4}\right) \\ &\quad + \zeta_q^{\mathrm{f}}\left(u_1 u_0 u_{k_3} u_0 u_{k_4} u_1\right) && \mod \mathrm{F}_{2,4,w} \\ &\equiv \zeta_q^{\mathrm{f}}\left(u_1 u_1 u_0 u_{k_3} u_0 u_{k_4}\right) && \mod \mathrm{F}_{2,4,w}\,. \end{aligned} \tag{2.90.1}$$

This implies, with (2.84.2) again,

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_1 u_0 u_1 u_{k_3} u_0 u_{k_4}\right) \mod \mathrm{F}_{2,4,w}\,. \tag{2.90.2}$$

Now, using Proposition 2.73 for the first step, then using (2.83.2), (2.86.1), and Lemma 2.87 for the second step, then applying (2.90.1) and (2.90.2), we obtain

$$\begin{aligned}
0 &\equiv \frac{1}{4}\Big(\zeta_q^{\mathrm{f}}\left(u_1 u_0 * u_1 u_0 u_{k_3} u_{k_4}\right) - \zeta_q^{\mathrm{f}}\left(u_1 * u_1 u_0 u_{k_3} u_{k_4} u_0\right)\Big) && \mod \mathrm{F}_{2,4,w} \\
&\equiv \zeta_q^{\mathrm{f}}\left(u_1 u_1 u_0 u_0 u_{k_3} u_{k_4}\right) + \frac{1}{2}\zeta_q^{\mathrm{f}}\left(u_1 u_1 u_0 u_{k_3} u_0 u_{k_4}\right) \\
&\quad + \frac{1}{4}\zeta_q^{\mathrm{f}}\left(u_1 u_0 u_1 u_{k_3} u_0 u_{k_4}\right) && \mod \mathrm{F}_{2,4,w} \\
&\equiv \zeta_q^{\mathrm{f}}\left(u_1 u_1 u_0 u_0 u_{k_3} u_{k_4}\right) && \mod \mathrm{F}_{2,4,w}\,.
\end{aligned}$$

The lemma follows using the relations in Lemma 2.84. □

**Lemma 2.91.** *Equation* (2.82.1) *is true for* $k_2, k_3 > 1$.

*Proof.* Let be $k_1, k_2, k_3, k_4 \in \mathbb{Z}_{>0}$ with $k_2, k_3 > 1$ and write $w = k_1 + k_2 + k_3 + k_4 + 2$. According to Proposition 2.89 and Lemma 2.87, we may assume $k_1 = k_4 = 1$. Using Proposition 2.73 for the first step and Lemmas 2.87 and 2.90 for the second one, we obtain

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_1 * u_1 u_0 u_0 u_{k_2} u_{k_3}\right) \equiv \zeta_q^{\mathrm{f}}\left(u_1 u_0 u_0 u_{k_2} u_{k_3} u_1\right) \mod \mathrm{F}_{2,4,w},$$

giving by (2.84.5), respectively by (2.84.4) and (2.86.1),

$$\begin{aligned}
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_1 u_{k_2} u_{k_3} u_0 u_0 u_1\right) \mod \mathrm{F}_{2,4,w}, \\
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_1 u_0 u_{k_2} u_{k_3} u_1 u_0\right) \mod \mathrm{F}_{2,4,w}\,.
\end{aligned} \tag{2.91.1}$$

Note that (2.91.1) implies by (2.83.3)

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_1 u_{k_2} u_{k_3} u_0 u_1 u_0\right) \mod \mathrm{F}_{2,4,w},$$

completing, together with (2.83.1), (2.83.2), (2.86.1), (2.86.2), (2.87.1), and (2.87.2), the proof of the lemma. □

**Lemma 2.92.** *Equation* (2.82.1) *is true for* $k_1 > 1$ *and one of* $k_2, k_3, k_4$ *larger* 1.

*Proof.* Let be $k_1, k_2, k_3, k_4 \in \mathbb{Z}_{>0}$ with $k_1 > 1$ and write $w = k_1 + k_2 + k_3 + k_4 + 2$. First, assume that one of $k_3, k_4$ larger 1 as well. For $z_2, z_3, z_4 \geq 0$ with $z_2 + z_3 + z_4 = 2$, we have, using Proposition 2.73 in the first step and Lemmas 2.87, 2.90, and 2.91 for the second one,

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1} * u_{k_2} u_0^{z_2} u_{k_3} u_0^{z_3} u_{k_4} u_0^{z_4}\right) \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1} u_{k_2} u_0^{z_2} u_{k_3} u_0^{z_3} u_{k_4} u_0^{z_4}\right) \mod \mathrm{F}_{2,4,w}\,.$$

Now, for all $z_1 > 0, z_2, z_3, z_4 \geq 0$ with $z_1 + \cdots + z_4 = 2$, using Corollary 2.28, we obtain

$$\begin{aligned}
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{z_1} * \tau\left(u_{k_1} u_{k_2} u_0^{z_2} u_{k_3} u_0^{z_3} u_{k_4} u_0^{z_4}\right)\right) && \mod \mathrm{F}_{2,4,w} \\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{z_1} * u_{z_4+1} u_0^{k_4-1} u_{z_3+1} u_0^{k_3-1} u_{z_2+1} u_0^{k_2-1} u_1 u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w} \\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{z_4+1} u_0^{k_4-1} u_{z_3+1} u_0^{k_3-1} u_{z_2+1} u_0^{k_2-1} u_{z_1+1} u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w} \\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1} u_0^{z_1} u_{k_2} u_0^{z_2} u_{k_3} u_0^{z_3} u_{k_4} u_0^{z_4}\right) && \mod \mathrm{F}_{2,4,w},
\end{aligned}$$

showing that (2.82.1) holds for $k_1, k_3 > 1$, and for $k_1, k_4 > 1$ as well.

It remains considering the case of $k_1, k_2 > 1$ with $k_3, k_4 \in \mathbb{Z}_{>0}$ arbitrary. Note that for $z_3, z_4 \geq 0$ with $z_3 + z_4 = 2$, we have by the previous results of this proof and Lemmas 2.87, 2.90, and 2.91,

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_3} u_0^{z_3} u_{k_4} u_0^{z_4} * u_{k_1} u_{k_2}\right) \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1} u_{k_2} u_{k_3} u_0^{z_3} u_{k_4} u_0^{z_4}\right) \mod \mathrm{F}_{2,4,w} .$$

By Corollary 2.28 for the first congruence and for the second, again, by the previous results of this proof and Lemmas 2.87, 2.90, and 2.91, we have

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_3} u_{k_4} u_0 * u_{k_1} u_0 u_{k_2}\right) \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1} u_0 u_{k_2} u_{k_3} u_{k_4} u_0\right) \mod \mathrm{F}_{2,4,w} .$$

Using the previous results of this proof and (2.83.1), (2.83.2), (2.86.2), (2.86.1), and Lemma 2.84, we obtain that (2.82.1) also holds true for $k_1, k_2 > 1$, completing the proof. □

As in the proof of Theorem 2.82 mentioned, for completing the proof of Theorem 2.82, it remains to consider the cases where one of the $k_j$'s is larger 1 while the other three equal 1.

**Lemma 2.93.** *Equation* (2.82.1) *is true for* $k_3 > 1$.

*Proof.* Let be $k_1, k_2, k_3, k_4 \in \mathbb{Z}_{>0}$ with $k_3 > 1$ and write $w = k_1 + k_2 + k_3 + k_4 + 2$. According to Lemmas 2.90, 2.91, 2.92, we may assume $k_1 = k_2 = k_4 = 1$, i.e., $w = k_3 + 5$. Using Proposition 2.73 for the first congruence, Corollary 2.28 and Proposition 2.73 for the second one, and (2.83.2), (2.84.5), and (2.86.4) for the third one, we have

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_1 * u_1 u_0 u_0 u_1 u_{k_3}\right) && \mod \mathrm{F}_{2,4,w} \\ &\equiv 2\zeta_q^{\mathrm{f}}\left(u_1 u_1 u_0 u_0 u_1 u_{k_3}\right) + \zeta_q^{\mathrm{f}}\left(u_1 u_0 u_1 u_0 u_1 u_{k_3}\right) \\ &\quad + 2\zeta_q^{\mathrm{f}}\left(u_1 u_0 u_0 u_1 u_1 u_{k_3}\right) + \zeta_q^{\mathrm{f}}\left(u_1 u_0 u_0 u_1 u_{k_3} u_1\right) && \mod \mathrm{F}_{2,4,w} \\ &\equiv \zeta_q^{\mathrm{f}}\left(u_1 u_0 u_0 u_1 u_{k_3} u_1\right) && \mod \mathrm{F}_{2,4,w} . \end{aligned} \tag{2.93.1}$$

Furthermore, using Proposition 2.73 for the first congruence, Corollary 2.28, Proposition 2.73 and Equations (2.83.2), (2.84.2), and (2.86.1) for the second congruence, gives

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_1 * u_1 u_0 u_{k_3} u_0 u_1\right) \equiv \zeta_q^{\mathrm{f}}\left(u_1 u_1 u_0 u_{k_3} u_0 u_1\right) \mod \mathrm{F}_{2,4,w}, \tag{2.93.2}$$

and so, by (2.84.2) again,

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_1 u_0 u_1 u_{k_3} u_0 u_1\right) \mod \mathrm{F}_{2,4,w} . \tag{2.93.3}$$

Now, (2.84.4) in combination with (2.93.1) and (2.93.3) implies

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_1 u_0 u_1 u_{k_3} u_1 u_0\right) \mod \mathrm{F}_{2,4,w} . \tag{2.93.4}$$

Using Corollary 2.28 for the first congruence and, for the second one, Corollary 2.28, Propositions 2.72 and 2.73 and Equations (2.84.1), (2.102.11), (2.83.2), (2.93.2), (2.93.3), we obtain

$$\begin{aligned} 0 &\equiv \frac{1}{4}\Big(\zeta_q^{\mathrm{f}}\left(u_1 u_0 u_1 * u_1 u_0 u_{k_3}\right) - \zeta_q^{\mathrm{f}}\left(u_1 u_0 * u_1 u_0 u_1 u_{k_3}\right)\Big) && \mod \mathrm{F}_{2,4,w} \\ &\equiv \zeta_q^{\mathrm{f}}\left(u_1 u_1 u_0 u_0 u_{k_3} u_1\right) && \mod \mathrm{F}_{2,4,w} . \end{aligned} \tag{2.93.5}$$

The remaining proof follows directly from Lemma 2.84. □

**Lemma 2.94.** *Equation* (2.82.1) *is true for* $k_4 > 1$.

*Proof.* Let be $k_1, k_2, k_3, k_4 \in \mathbb{Z}_{>0}$ with $k_4 > 1$ and write $w = k_1 + k_2 + k_3 + k_4 + 2$. According to Lemmas 2.90, 2.87, 2.92, we may assume $k_1 = k_2 = k_3 = 1$, i.e., $w = k_4 + 5$. Using Corollary 2.28 for the first congruence, and for the second one, Corollary 2.28, Proposition 2.73, and Equations (2.84.6), (2.86.3), and (2.93.4), we have

$$0 \equiv \frac{1}{4}\left(\zeta_q^{\mathrm{f}}(u_1u_0 * u_1u_0u_1u_{k_4})\right) \equiv \zeta_q^{\mathrm{f}}(u_1u_1u_0u_0u_1u_{k_4}) \mod \mathrm{F}_{2,4,w}\,.$$

This, (2.86.4), and (2.93.5), gives, together with Proposition 2.73,

$$0 \equiv \zeta_q^{\mathrm{f}}(u_1 * u_1u_1u_0u_0u_{k_4}) \equiv \zeta_q^{\mathrm{f}}(u_1u_1u_0u_1u_0u_{k_4}) \mod \mathrm{F}_{2,4,w}\,.$$

The remaining part of the proof follows from (2.83.1), (2.83.2), (2.86.3), (2.86.4), and Lemma 2.84. □

**Lemma 2.95.** *Equation* (2.82.1) *is true for* $k_2 > 1$.

*Proof.* Let be $k_1, k_2, k_3, k_4 \in \mathbb{Z}_{>0}$ with $k_2 > 1$ and write $w = k_1 + k_2 + k_3 + k_4 + 2$. According to Lemmas 2.87, 2.91, 2.92, we may assume $k_1 = k_3 = k_4 = 1$, i.e., $w = k_2 + 5$. First note that, by Proposition 2.73 and Lemma 2.93, one has

$$0 \equiv \frac{1}{2}\zeta_q^{\mathrm{f}}(u_1 * u_1u_0u_0u_{k_2}u_1) \equiv \zeta_q^{\mathrm{f}}(u_1u_0u_0u_{k_2}u_1u_1) \mod \mathrm{F}_{2,4,w},$$

giving, with (2.84.4) and (2.86.1),

$$0 \equiv \zeta_q^{\mathrm{f}}(u_1u_0u_{k_2}u_1u_1u_0) \mod \mathrm{F}_{2,4,w}, \tag{2.95.1}$$

Furthermore, by Proposition 2.73 for the first congruence, Corollary 2.28, Proposition 2.73 and Lemma 2.93, Equations (2.84.1), (2.86.2), and (2.95.1) for the second congruence, we obtain

$$0 \equiv \zeta_q^{\mathrm{f}}(u_1 * u_1u_{k_2}u_0u_1u_0) \equiv \zeta_q^{\mathrm{f}}(u_1u_{k_2}u_0u_1u_1u_0) \mod \mathrm{F}_{2,4,w}\,.$$

The remaining part of the proof follows from (2.83.1), (2.83.2), (2.86.2), (2.86.1), and Lemma 2.84, immediately. □

**Lemma 2.96.** *Equation* (2.82.1) *is true for* $k_1 > 1$.

*Proof.* Let be $k_1, k_2, k_3, k_4 \in \mathbb{Z}_{>0}$ with $k_1 > 1$ and write $w = k_1 + k_2 + k_3 + k_4 + 2$. According to Lemma 2.92, we may assume $k_2 = k_3 = k_4 = 1$, i.e., $w = k_1 + 5$. For any $z_2, z_3, z_4 \geq 0$ with $z_2 + z_3 + z_4 = 2$, we have, using Corollary 2.28, Proposition 2.73, and Lemmas 2.93, 2.94, and 2.95 for the third congruence,

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}(u_{k_1} * \tau(u_{z_4+1}u_{z_3+1}u_{z_2+1})) && \mod \mathrm{F}_{2,4,w} \\ &\equiv \zeta_q^{\mathrm{f}}(u_{k_1} * u_1u_0^{z_2}u_1u_0^{z_3}u_1u_0^{z_4}) && \mod \mathrm{F}_{2,4,w} \\ &\equiv \zeta_q^{\mathrm{f}}(u_{k_1}u_1u_0^{z_2}u_1u_0^{z_3}u_1u_0^{z_4}) && \mod \mathrm{F}_{2,4,w}\,. \end{aligned}$$

This implies, for any $z_1 > 0, z_2, z_3, z_4 \geq 0$ with $z_1 + \cdots + z_4 = 2$, using Proposition 2.73 for the first congruence and, additionally, Corollary 2.28 for the third congruence,

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}(u_{z_1} * \tau(u_{k_1}u_1u_0^{z_2}u_1u_0^{z_3}u_1u_0^{z_4})) && \mod \mathrm{F}_{2,4,w} \\ &\equiv \zeta_q^{\mathrm{f}}\left(u_{z_1} * u_{z_4+1}u_{z_3+1}u_{z_2+1}u_1u_0^{k_1-1}\right) && \mod \mathrm{F}_{2,4,w} \end{aligned}$$

$$\equiv \zeta_q^{\mathrm{f}}\left(u_{z_4+1}u_{z_3+1}u_{z_2+1}u_{z_1+1}u_0^{k_1-1}\right) \qquad \bmod \ \mathrm{F}_{2,4,w}$$
$$\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0^{z_1}u_1u_0^{z_2}u_1u_0^{z_3}u_1u_0^{z_4}\right) \qquad \bmod \ \mathrm{F}_{2,4,w},$$

completing the proof of the lemma. □

### 2.6.3 The refined Bachmann Conjecture 2.10 for $(z, d, w) = (3, 4, w)$

**Theorem 2.97.** *The refined Bachmann Conjecture 2.10 is true for all* $(3, 4, w) \in \mathbb{Z}^3_{>0}$*, i.e.,*

$$\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0^{z_1}u_{k_2}u_0^{z_2}u_{k_3}u_0^{z_3}u_{k_4}u_0^{z_4}\right) \in \mathrm{F}_{3,4,w} \tag{2.97.1}$$

*for all integers* $k_j \in \mathbb{Z}_{>0}$*,* $z_j \in \mathbb{Z}_{\geq 0}$*, for* $1 \leq j \leq 4$*, satisfying* $z_1 + z_2 + z_3 + z_4 = 3$ *and* $w = k_1 + k_2 + k_3 + k_4 + 3$*.*

*Proof.* In the case $k_1 = k_2 = k_3 = k_4 = 1$, (2.97.1) is true since, by $\tau$-invariance of $\zeta_q^{\mathrm{f}}$, for any $z_1, \ldots, z_4 \geq 0$, we have

$$\zeta_q^{\mathrm{f}}\left(u_1u_0^{z_1}u_1u_0^{z_2}u_1u_0^{z_3}u_1u_0^{z_4}\right) \equiv \zeta_q^{\mathrm{f}}\left(u_{z_4+1}u_{z_3+1}u_{z_2+1}u_{z_1+1}\right) \in \mathcal{Z}_q^{f,\circ}.$$

For $k_3 > 1$, (2.97.1) will follow from Lemma 2.101, for $k_4 > 1$, (2.97.1) will follow from Lemma 2.102, for $k_2 > 1$, (2.97.1) will follow from Lemma 2.103, and for $k_1 > 1$, (2.97.1) will follow from Lemma 2.104. This completes the proof of the theorem. □

First, we will consider some relations we need more than once.

**Lemma 2.98.** *Let be* $k_1, \ldots, k_4 \in \mathbb{Z}_{>0}$ *and write* $w = k_1 + \cdots + k_4 + 3$*. We have*

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_{k_4}u_0u_0u_0\right) \quad \bmod \ \mathrm{F}_{3,4,w}, \tag{2.98.1}$$
$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_{k_3}u_0u_{k_4}\right) \quad \bmod \ \mathrm{F}_{3,4,w}\,. \tag{2.98.2}$$

*Proof.* Congruence (2.98.1) is a special case of Corollary 2.24. Setting $\mathbf{k} := (k_1, \ldots, k_4)$, Equation (2.98.2) follows from Lemma 2.56 and (2.98.1) via

$$\zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}\left(u_1u_2^3\right)\right) \equiv \sum_{j=0}^{3}(-1)^{j-1}\zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}\left(u_1^j \circledast u_{4-j}u_1^3\right)\right) \equiv 0 \quad \bmod \ \mathrm{F}_{3,4,w}\,.$$ □

Next, we consider relations coming from products with no $u_0$ in one of the factors.

**Lemma 2.99.** *Let be* $k_1, \ldots, k_4 \in \mathbb{Z}_{>0}$ *and write* $w = k_1 + \cdots + k_4 + 3$*. We have*

$$\begin{aligned} 0 \equiv\ & \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_{k_4}u_0u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_{k_4}u_0u_0\right) \\ & + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_{k_4}u_0u_0\right) \qquad \bmod \ \mathrm{F}_{3,4,w}, \end{aligned} \tag{2.99.1}$$

$$\begin{aligned} 0 \equiv\ & \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}u_0u_{k_4}\right) \\ & + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_0u_{k_3}u_{k_4}\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_0u_{k_2}u_{k_3}u_{k_4}\right) \quad \bmod \ \mathrm{F}_{3,4,w}, \end{aligned} \tag{2.99.2}$$

$$\begin{aligned} 0 \equiv\ & \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_{k_4}u_0u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_0u_{k_4}u_0\right) \\ & + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_0u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0u_{k_4}u_0\right) \quad \bmod \ \mathrm{F}_{3,4,w}, \end{aligned} \tag{2.99.3}$$

$$\begin{aligned} 0 \equiv\ & \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_{k_4}u_0u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0u_{k_4}u_0\right) \\ & + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_{k_3}u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}u_{k_4}u_0\right) \quad \bmod \ \mathrm{F}_{3,4,w}, \end{aligned} \tag{2.99.4}$$

$$\begin{aligned} 0 \equiv\ & \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0u_0u_{k_4}\right) \\ & + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_{k_3}u_0u_{k_4}\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}u_0u_{k_4}\right) \quad \bmod \ \mathrm{F}_{3,4,w}, \end{aligned} \tag{2.99.5}$$

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_{k_3}u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_{k_3}u_0u_{k_4}\right) \\ &\quad + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_0u_{k_3}u_{k_4}\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_0u_{k_3}u_{k_4}\right) \mod \mathrm{F}_{3,4,w}, \end{aligned} \tag{2.99.6}$$

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_{k_4}u_0u_0u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_0u_{k_4}u_0\right) \\ &\quad + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3}u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}u_{k_4}u_0\right) \mod \mathrm{F}_{3,4,w}, \end{aligned} \tag{2.99.7}$$

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_{k_4}u_0u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_0u_0u_{k_4}\right) \\ &\quad + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3}u_0u_{k_4}\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}u_0u_{k_4}\right) \mod \mathrm{F}_{3,4,w}, \end{aligned} \tag{2.99.8}$$

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_{k_4}u_0u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_0u_0u_{k_4}\right) \\ &\quad + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_0u_{k_3}u_{k_4}\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_0u_{k_3}u_{k_4}\right) \mod \mathrm{F}_{3,4,w}, \end{aligned} \tag{2.99.9}$$

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_{k_4}u_0u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0u_0u_{k_4}\right) \\ &\quad + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_0u_{k_3}u_{k_4}\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_0u_{k_2}u_{k_3}u_{k_4}\right) \mod \mathrm{F}_{3,4,w}, \end{aligned} \tag{2.99.10}$$

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_{k_3}u_{k_4}u_0\right) \\ &\quad + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_0u_{k_3}u_{k_4}\right) \\ &\quad + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_0u_{k_2}u_{k_3}u_{k_4}\right) \mod \mathrm{F}_{3,4,w}, \end{aligned} \tag{2.99.11}$$

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_0u_0u_{k_4}\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_0u_{k_3}u_{k_4}\right) \\ &\quad + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_0u_{k_2}u_{k_3}u_{k_4}\right) \mod \mathrm{F}_{3,4,w}, \end{aligned} \tag{2.99.12}$$

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_{k_4}u_0u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_{k_4}u_0u_0\right) \\ &\quad + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_{k_4}u_0u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_0u_0u_{k_4}\right) \\ &\quad + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0u_0u_{k_4}\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_0u_{k_3}u_{k_4}\right) \mod \mathrm{F}_{3,4,w}, \end{aligned} \tag{2.99.13}$$

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_0u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3}u_{k_4}u_0\right) \\ &\quad + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3}u_0u_{k_4}\right) \\ &\quad + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}u_0u_{k_4}\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_0u_{k_3}u_{k_4}\right) \mod \mathrm{F}_{3,4,w}, \end{aligned} \tag{2.99.14}$$

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_0u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0u_{k_4}u_0\right) \\ &\quad + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_{k_3}u_{k_4}u_0\right) \mod \mathrm{F}_{3,4,w}. \end{aligned} \tag{2.99.15}$$

*Proof.* All relations are a consequence of Lemma 2.98 and, by Lemma 2.56,

$$0 \equiv \zeta_q^{\mathrm{f}}\left(\tau(\Psi_{\mathbf{k}}(u_{\mathbf{n}} \circledast u_{\boldsymbol{\ell}}))\right) \mod \mathrm{F}_{3,4,w}$$

with $\mathbf{k} = (k_1, \ldots, k_4)$ and $(\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{3,4}$ each. Precisely, we used $(\mathbf{n}, \boldsymbol{\ell}) = ((1), (3,1,1,1))$ for (2.99.1), $(\mathbf{n}, \boldsymbol{\ell}) = ((1), (1,1,1,3))$ for (2.99.2), $(\mathbf{n}, \boldsymbol{\ell}) = ((1), (2,2,1,1))$ for (2.99.3). Furthermore, we used $(\mathbf{n}, \boldsymbol{\ell}) = ((1), (2,1,1,2))$ for (2.99.4), $(\mathbf{n}, \boldsymbol{\ell}) = ((1), (1,2,1,2))$ for (2.99.5), $(\mathbf{n}, \boldsymbol{\ell}) = ((1), (1,1,2,2))$ for (2.99.6), $(\mathbf{n}, \boldsymbol{\ell}) = ((2), (2,1,1,1))$ for (2.99.7). Furthermore, we used $(\mathbf{n}, \boldsymbol{\ell}) = ((2), (1,2,1,1))$ for (2.99.8), $(\mathbf{n}, \boldsymbol{\ell}) = ((2), (1,1,2,1))$ for (2.99.9), $(\mathbf{n}, \boldsymbol{\ell}) = ((2), (1,1,1,2))$ for (2.99.10), $(\mathbf{n}, \boldsymbol{\ell}) = ((1,1), (1,1,1,2))$ for (2.99.11). Furthermore, we used $(\mathbf{n}, \boldsymbol{\ell}) = ((3), (1,1,1,1))$ for (2.99.12), $(\mathbf{n}, \boldsymbol{\ell}) = ((2,1), (1,1,1,1))$ for (2.99.13), $(\mathbf{n}, \boldsymbol{\ell}) = ((1,2), (1,1,1,1))$ for (2.99.14), and $(\mathbf{n}, \boldsymbol{\ell}) = ((1,1,1), (1,1,1,1))$ for (2.99.15). □

Note that we have the following conclusions.

**Lemma 2.100.** *Let be $\mathbf{k} = (k_1, \ldots, k_4) \in \mathbb{Z}_{>0}^4$ and write $w = |\mathbf{k}| + 3$. For all $1 \le j \le 4$, we have*

$$0 \equiv \zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}\left(u_1^{j-1}u_4u_1^{4-j} + u_2^{j-1}u_1u_2^{4-j}\right)\right) \mod \mathrm{F}_{3,4,w}, \tag{2.100.1}$$

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_0u_{k_4}u_0\right) + \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3}u_{k_4}u_0\right) \tag{2.100.2}$$

$$\begin{aligned}
&+\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3}u_0u_{k_4}\right) && \bmod \mathrm{F}_{3,4,w}, \\
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_{k_4}u_0u_0\right)+\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_{k_4}u_0u_0\right) && && (2.100.3)\\
&+\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_0u_0u_{k_4}\right) && \bmod \mathrm{F}_{3,4,w}, \\
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3}u_0u_{k_4}\right)+\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_0u_{k_3}u_{k_4}\right) && \bmod \mathrm{F}_{3,4,w}, && (2.100.4)\\
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_{k_4}u_0u_0\right)+\zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_0u_{k_3}u_{k_4}\right) && \bmod \mathrm{F}_{3,4,w}, && (2.100.5)\\
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_{k_4}u_0u_0\right)+\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3}u_0u_{k_4}\right) && \bmod \mathrm{F}_{3,4,w}, && (2.100.6)\\
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}u_{k_4}u_0\right)+\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_0u_0u_{k_4}\right) && \bmod \mathrm{F}_{3,4,w}\,. && (2.100.7)
\end{aligned}$$

*Proof.* The proof of (2.100.1) is obtained from Lemma 2.56 and the direct calculation

$$\begin{aligned}
0 &\equiv \sum_{p=1}^{3}(-1)^p\zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}\left(u_1^p \circledast u_1^{j-1}u_{4-p}u_1^{4-j}\right)\right) && \bmod \mathrm{F}_{3,4,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}\left(u_1^{j-1}u_4u_1^{4-j}+u_2^{j-1}u_1u_2^{4-j}\right)\right) && \bmod \mathrm{F}_{3,4,w}\,.
\end{aligned}$$

Note that (2.100.4) is a consequence of (2.98.1), (2.99.3), (2.99.8), (2.99.7), when using Equation (2.99.14). Analogously, (2.100.5) is a consequence of (2.98.1), (2.99.6), (2.99.9), (2.99.10), using (2.99.13). Furthermore, we obtain (2.100.6) with (2.99.13), (2.99.15), and with the case $j=3$ of (2.100.1), in a similar way, using Lemma 2.56,

$$\begin{aligned}
0 &\equiv \zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}(u_1 \circledast u_1u_2u_1u_2)\right)-\zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}(u_2 \circledast u_1u_2u_1u_1)\right)\\
&\quad -\zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}(u_2 \circledast u_1u_1u_1u_2)\right)-\zeta_q^{\mathrm{f}}\left(u_1u_0^{k_4-1}u_2u_0^{k_3-1}u_3u_0^{k_2-1}u_1u_0^{k_1-1}\right) \quad \bmod \mathrm{F}_{3,4,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_{k_4}u_0u_0\right)+\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3}u_0u_{k_4}\right) \quad \bmod \mathrm{F}_{3,4,w}
\end{aligned}$$

and we obtain (2.100.7) with (2.99.14), (2.99.15), and with case $j=3$ of (2.100.1),

$$\begin{aligned}
0 &\equiv \zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}(u_1 \circledast u_2u_1u_1u_1)\right)-\zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}(u_2 \circledast u_1u_1u_2u_1)\right)\\
&\quad -\zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}(u_2 \circledast u_2u_1u_1u_1)\right) \quad \bmod \mathrm{F}_{3,4,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}u_{k_4}u_0\right)+\zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_0u_0u_{k_4}\right) \ \bmod \mathrm{F}_{3,4,w},
\end{aligned}$$

completing the proof of the lemma. □

For the proof of Theorem 2.97, it remains to consider the cases where we have for one $j \in \{1,2,3,4\}$ that $k_j > 1$.

**Lemma 2.101.** *Equation* (2.97.1) *is true for* $k_3 > 1$.

*Proof.* Let be $k_1, k_2, k_3, k_4 \in \mathbb{Z}_{>0}$ and write $w = k_1+k_2+k_3+k_4+4$. By (2.98.2) and (2.99.15), we obtain

$$\begin{aligned}
0 &\equiv \frac{1}{k_3}\zeta_q^{\mathrm{f}}\left(\tau(u_1u_1u_2) * \tau\left(u_{k_1}u_{k_2}u_{k_3}u_{k_4}\right)\right) && \bmod \mathrm{F}_{3,4,w}\\
&\equiv \frac{1}{k_3}\zeta_q^{\mathrm{f}}\left(u_1u_0u_1u_1 * u_1u_0^{k_4-1}u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \bmod \mathrm{F}_{3,4,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_2u_0^{k_4-1}u_1u_0^{k_3}u_2u_0^{k_2-1}u_2u_0^{k_1-1}\right) && \bmod \mathrm{F}_{3,4,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_{k_3+1}u_{k_4}u_0\right) && \bmod \mathrm{F}_{3,4,w}\,. && (2.101.1)
\end{aligned}$$

Similar, using (2.98.2), (2.99.15), (2.101.1), we have

$$\begin{aligned}
0 \equiv & -\frac{1}{k_2}\zeta_q^{\mathrm{f}}\left(\tau(u_1u_2u_1) * \tau\left(u_{k_1}u_{k_2}u_{k_3}u_{k_4}\right)\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & -\frac{1}{k_2}\zeta_q^{\mathrm{f}}\left(u_1u_1u_0u_1 * u_1u_0^{k_4-1}u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & \zeta_q^{\mathrm{f}}\left(u_2u_0^{k_4-1}u_2u_0^{k_3-1}u_2u_0^{k_2}u_1u_0^{k_1-1}\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_0u_{k_3}u_0u_{k_4}u_0\right) && \bmod \mathrm{F}_{3,4,w}\,. \qquad (2.101.2)
\end{aligned}$$

Furthermore, using (2.99.15), (2.101.2), (2.99.11), we have

$$\begin{aligned}
0 \equiv & \frac{1}{k_3}\zeta_q^{\mathrm{f}}\left(\tau(u_2u_1) * \tau\left(u_{k_1}u_0u_{k_2}u_{k_3}u_{k_4}\right)\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & \frac{1}{k_3}\zeta_q^{\mathrm{f}}\left(u_1u_1u_0 * u_1u_0^{k_4-1}u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_2u_0^{k_1-1}\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & \zeta_q^{\mathrm{f}}\left(u_2u_0^{k_4-1}u_2u_0^{k_3}u_1u_0^{k_2-1}u_2u_0^{k_1-1}\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3+1}u_0u_{k_4}u_0\right) && \bmod \mathrm{F}_{3,4,w}\,. \qquad (2.101.3)
\end{aligned}$$

This implies by (2.99.15) and (2.101.1)

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3+1}u_0u_{k_4}u_0\right) \quad \bmod \mathrm{F}_{3,4,w}\,. \tag{2.101.4}$$

Now, by equations (2.99.15), (2.100.2), (2.100.3), (2.101.4), (2.101.2), we have

$$\begin{aligned}
0 \equiv & -\frac{1}{k_2}\zeta_q^{\mathrm{f}}\left(\tau(u_2u_1) * \tau\left(u_{k_1}u_{k_2}u_0u_{k_3}u_{k_4}\right)\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & -\frac{1}{k_2}\zeta_q^{\mathrm{f}}\left(u_1u_1u_0 * u_1u_0^{k_4-1}u_1u_0^{k_3-1}u_2u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & \zeta_q^{\mathrm{f}}\left(u_2u_0^{k_4-1}u_3u_0^{k_3-1}u_1u_0^{k_2}u_1u_0^{k_1-1}\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_{k_3}u_0u_0u_{k_4}u_0\right) && \bmod \mathrm{F}_{3,4,w}\,. \qquad (2.101.5)
\end{aligned}$$

Considering (2.100.4), this implies

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2+1}u_0u_{k_3}u_{k_4}\right) \quad \bmod \mathrm{F}_{3,4,w}\,. \tag{2.101.6}$$

Next, we consider, using Corollary 2.28, (2.99.14), (2.100.2), (2.100.3),

$$\begin{aligned}
0 \equiv & \frac{1}{k_3}\zeta_q^{\mathrm{f}}\left(\tau(u_2u_0u_1) * \tau\left(u_{k_1}u_{k_2}u_{k_3}u_{k_4}\right)\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & \frac{1}{k_3}\zeta_q^{\mathrm{f}}\left(u_1u_2u_0 * u_1u_0^{k_4-1}u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & \zeta_q^{\mathrm{f}}\left(u_2u_0^{k_4-1}u_3u_0^{k_3}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3+1}u_0u_0u_{k_4}u_0\right) && \bmod \mathrm{F}_{3,4,w}\,. \qquad (2.101.7)
\end{aligned}$$

Now, a consequence of (2.100.4) is

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_0u_{k_3+1}u_{k_4}\right) \quad \bmod \mathrm{F}_{3,4,w}\,.$$

In a similar way, we obtain by Corollary 2.28, (2.99.13), (2.100.2), (2.100.3),

$$\begin{aligned}
0 &\equiv \frac{1}{k_3}\zeta_q^{\mathrm{f}}\left(\tau(u_2u_1u_0) * \tau\left(u_{k_1}u_{k_2}u_{k_3}u_{k_4}\right)\right) && \bmod \ \mathrm{F}_{3,4,w} \\
&\equiv \frac{1}{k_3}\zeta_q^{\mathrm{f}}\left(u_2u_1u_0 * u_1u_0^{k_4-1}u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \bmod \ \mathrm{F}_{3,4,w} \\
&\equiv \zeta_q^{\mathrm{f}}\left(u_3u_0^{k_4-1}u_2u_0^{k_3}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \bmod \ \mathrm{F}_{3,4,w} \\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3+1}u_0u_{k_4}u_0u_0\right) && \bmod \ \mathrm{F}_{3,4,w}\,. \qquad (2.101.8)
\end{aligned}$$

By (2.100.5), one obtains

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_0u_{k_3+1}u_{k_4}\right) \quad \bmod \ \mathrm{F}_{3,4,w}\,. \qquad (2.101.9)$$

From Corollary 2.28, (2.101.7), (2.101.5), and (2.101.2) we immediately get

$$\begin{aligned}
0 &\equiv \frac{1}{k_2}\zeta_q^{\mathrm{f}}\left(\tau(u_2u_1) * \tau\left(u_{k_1}u_{k_2}u_{k_3}u_0u_{k_4}\right)\right) && \bmod \ \mathrm{F}_{3,4,w} \\
&\equiv \frac{1}{k_2}\zeta_q^{\mathrm{f}}\left(u_1u_1u_0 * u_1u_0^{k_4-1}u_2u_0^{k_3-1}u_1u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \bmod \ \mathrm{F}_{3,4,w} \\
&\equiv \zeta_q^{\mathrm{f}}\left(u_1u_0^{k_4-1}u_3u_0^{k_3-1}u_2u_0^{k_2}u_1u_0^{k_1-1}\right) && \bmod \ \mathrm{F}_{3,4,w} \\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_0u_{k_3}u_0u_0u_{k_4}\right) && \bmod \ \mathrm{F}_{3,4,w}, \qquad (2.101.10)
\end{aligned}$$

and so, by (2.100.7),

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2+1}u_{k_3}u_{k_4}u_0\right) \quad \bmod \ \mathrm{F}_{3,4,w}\,. \qquad (2.101.11)$$

This implies, using (2.99.2), (2.101.6), (2.101.2),

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2+1}u_{k_3}u_0u_{k_4}\right) \quad \bmod \ \mathrm{F}_{3,4,w}\,. \qquad (2.101.12)$$

Also, from (2.99.7), using (2.101.5), (2.101.11), and (2.98.1), we obtain

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_0u_0u_{k_3}u_{k_4}u_0\right) \quad \bmod \ \mathrm{F}_{3,4,w}\,. \qquad (2.101.13)$$

This leads to, using (2.99.14), (2.101.5), (2.101.6), (2.101.11), (2.101.12),

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_0u_0u_{k_3}u_0u_{k_4}\right) \quad \bmod \ \mathrm{F}_{3,4,w}\,. \qquad (2.101.14)$$

A consequence of (2.100.6) then is

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2+1}u_{k_3}u_{k_4}u_0u_0\right) \quad \bmod \ \mathrm{F}_{3,4,w}\,. \qquad (2.101.15)$$

By Corollary 2.28, (2.99.4), (2.99.2), and (2.101.1), we have

$$\begin{aligned}
0 &\equiv -\frac{1}{k_4}\zeta_q^{\mathrm{f}}\left(\tau(u_1u_2) * \tau\left(u_{k_1}u_0u_{k_2}u_{k_3}u_{k_4}\right)\right) && \bmod \ \mathrm{F}_{3,4,w} \\
&\equiv -\frac{1}{k_4}\zeta_q^{\mathrm{f}}\left(u_1u_0u_1 * u_1u_0^{k_4-1}u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_2u_0^{k_1-1}\right) && \bmod \ \mathrm{F}_{3,4,w} \\
&\equiv \zeta_q^{\mathrm{f}}\left(u_3u_0^{k_4}u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_2u_0^{k_1-1}\right) && \bmod \ \mathrm{F}_{3,4,w} \\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_{k_4+1}u_0u_0\right) && \bmod \ \mathrm{F}_{3,4,w}\,. \qquad (2.101.16)
\end{aligned}$$

Hence, by Theorem 2.82 for the first congruence and by applying (2.99.4), (2.99.2), (2.101.3) afterwards, we see that

$$
\begin{aligned}
0 \equiv & -\frac{1}{k_3}\zeta_q^{\mathrm{f}}\left(\tau(u_2) * \tau\left(u_{k_1}u_0u_{k_2}u_{k_3}u_{k_4}u_0\right)\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & -\frac{1}{k_3}\zeta_q^{\mathrm{f}}\left(u_1u_0 * u_2u_0^{k_4-1}u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_2u_0^{k_1-1}\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & \, \zeta_q^{\mathrm{f}}\left(u_3u_0^{k_4-1}u_1u_0^{k_3}u_1u_0^{k_2-1}u_2u_0^{k_1-1}\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & \, \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3+1}u_{k_4}u_0u_0\right) && \bmod \mathrm{F}_{3,4,w}\,. \quad (2.101.17)
\end{aligned}
$$

Now, (2.100.6) yields

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3+1}u_0u_{k_4}\right) \quad \bmod \mathrm{F}_{3,4,w}\,.$$

Furthermore, (2.101.17) implies with (2.99.10) and (2.101.9), respectively (2.99.1) and (2.101.8),

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3+1}u_0u_0u_{k_4}\right) \quad \bmod \mathrm{F}_{3,4,w},$$

respectively,

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3+1}u_{k_4}u_0u_0\right) \quad \bmod \mathrm{F}_{3,4,w}\,.$$

The latter implies by using (2.99.13) for the first congruence, then (2.100.7) for the second one, (2.99.7) for the third one, and (2.99.14) for the last one,

$$
\begin{aligned}
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3+1}u_0u_0u_{k_4}\right) && \bmod \mathrm{F}_{3,4,w}, \\
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3+1}u_{k_4}u_0\right) && \bmod \mathrm{F}_{3,4,w}, \\
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3+1}u_{k_4}u_0\right) && \bmod \mathrm{F}_{3,4,w}, \\
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3+1}u_0u_{k_4}\right) && \bmod \mathrm{F}_{3,4,w}\,.
\end{aligned}
$$

This completes the proof of the lemma. □

**Lemma 2.102.** *Equation* (2.97.1) *is true for* $k_4 > 1$.

*Proof.* Let be $k_1, k_2, k_3, k_4 \in \mathbb{Z}_{>0}$ and write $w = k_1 + k_2 + k_3 + k_4 + 4$. From (2.101.16), we obtain by (2.100.6)

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3}u_0u_{k_4+1}\right) \quad \bmod \mathrm{F}_{3,4,w}\,. \qquad (2.102.1)$$

From Corollary 2.28, Lemma 2.101, (2.99.6), (2.101.6), one sees

$$
\begin{aligned}
0 \equiv & -\frac{1}{k_4}\zeta_q^{\mathrm{f}}\left(\tau(u_1u_2) * \tau\left(u_{k_1}u_{k_2}u_0u_{k_3}u_{k_4}\right)\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & -\frac{1}{k_4}\zeta_q^{\mathrm{f}}\left(u_1u_0u_1 * u_1u_0^{k_4-1}u_1u_0^{k_3-1}u_2u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & \, \zeta_q^{\mathrm{f}}\left(u_3u_0^{k_4}u_1u_0^{k_3-1}u_2u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \bmod \mathrm{F}_{3,4,w} \\
\equiv & \, \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_{k_4+1}u_0u_0\right) && \bmod \mathrm{F}_{3,4,w}, \quad (2.102.2)
\end{aligned}
$$

With (2.99.1), this implies

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_{k_4+1}u_0u_0\right) \quad \bmod \mathrm{F}_{3,4,w}, \qquad (2.102.3)$$

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_0u_{k_3}u_{k_4+1}\right) \quad \bmod \ \mathrm{F}_{3,4,w}\,.$$

The second congruence is a consequence of the first one and (2.100.5).

Furthermore, Theorem 2.82 for the first congruence, Lemma 2.101 and (2.99.5) for the third one, give

$$\begin{aligned}
0 &\equiv \frac{1}{k_4}\zeta_q^{\mathrm{f}}\left(\tau(u_2) * \tau\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0u_{k_4}\right)\right) && \bmod \ \mathrm{F}_{3,4,w} \\
&\equiv \frac{1}{k_4}\zeta_q^{\mathrm{f}}\left(u_1u_0 * u_1u_0^{k_4-1}u_2u_0^{k_3-1}u_1u_0^{k_2-1}u_2u_0^{k_1-1}\right) && \bmod \ \mathrm{F}_{3,4,w} \\
&\equiv \zeta_q^{\mathrm{f}}\left(u_2u_0^{k_4}u_2u_0^{k_3-1}u_1u_0^{k_2-1}u_2u_0^{k_1-1}\right) && \bmod \ \mathrm{F}_{3,4,w} \\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0u_{k_4+1}u_0\right) && \bmod \ \mathrm{F}_{3,4,w}, \quad (2.102.4)
\end{aligned}$$

and so, applying case $j = 3$ of (2.100.1),

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_0u_{k_3}u_{k_4+1}\right) \quad \bmod \ \mathrm{F}_{3,4,w}\,. \tag{2.102.5}$$

Furthermore, by Theorem 2.82 for the first congruence and by Lemma 2.101 and (2.99.2) for the third one, we observe

$$\begin{aligned}
0 &\equiv \frac{1}{k_4}\zeta_q^{\mathrm{f}}\left(\tau(u_2) * \tau\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}u_{k_4}\right)\right) && \bmod \ \mathrm{F}_{3,4,w} \\
&\equiv \frac{1}{k_4}\zeta_q^{\mathrm{f}}\left(u_1u_0 * u_1u_0^{k_4-1}u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_3u_0^{k_1-1}\right) && \bmod \ \mathrm{F}_{3,4,w} \\
&\equiv \zeta_q^{\mathrm{f}}\left(u_2u_0^{k_4}u_1u_0^{k_3-1}u_1u_0^{k_2-1}u_3u_0^{k_1-1}\right) && \bmod \ \mathrm{F}_{3,4,w} \\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}u_{k_4+1}u_0\right) && \bmod \ \mathrm{F}_{3,4,w}\,. \quad (2.102.6)
\end{aligned}$$

This implies, using (2.100.7), and (2.99.13) for the second congruence additionally,

$$\begin{aligned}
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_0u_0u_{k_4+1}\right) \quad \bmod \ \mathrm{F}_{3,4,w}, && (2.102.7) \\
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_{k_3}u_0u_0u_{k_4+1}\right) \quad \bmod \ \mathrm{F}_{3,4,w}\,.
\end{aligned}$$

Now, (2.99.9), (2.102.4), (2.102.2), (2.102.7) yield

$$\begin{aligned}
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_0u_{k_3}u_{k_4+1}\right) \quad \bmod \ \mathrm{F}_{3,4,w}, \\
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_0u_{k_4+1}u_0\right) \quad \bmod \ \mathrm{F}_{3,4,w}\,. && (2.102.8)
\end{aligned}$$

The second congruence is a consequence of the first one and (2.100.4). Using (2.102.8) and equations (2.99.7) and (2.102.6), we see that

$$\begin{aligned}
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_0u_{k_3}u_{k_4+1}u_0\right) \quad \bmod \ \mathrm{F}_{3,4,w}, \\
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2}u_{k_3}u_0u_{k_4+1}\right) \quad \bmod \ \mathrm{F}_{3,4,w}, && (2.102.9)
\end{aligned}$$

where the second congruence is implied by the first one and (2.99.14).

Combining (2.99.8), (2.102.3), (2.102.1), (2.102.9), we have

$$\begin{aligned}
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_{k_3}u_0u_0u_0u_{k_4+1}\right) \quad \bmod \ \mathrm{F}_{3,4,w}, && (2.102.10) \\
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2}u_0u_{k_3}u_{k_4+1}u_0\right) \quad \bmod \ \mathrm{F}_{3,4,w}\,.
\end{aligned}$$

The second congruence is a consequence of the first one and case $j = 2$ of (2.100.1).

Now, (2.99.12), (2.102.10), (2.102.5), (2.98.1) give

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_0u_{k_2}u_{k_3}u_{k_4+1}\right) \mod \mathrm{F}_{3,4,w}, \\ 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0u_{k_3}u_0u_{k_4+1}u_0\right) \mod \mathrm{F}_{3,4,w}. \end{aligned} \tag{2.102.11}$$

The second congruence is a consequence of the first one and case $j = 4$ of (2.100.1) additionally. This completes the proof of the Lemma. □

**Lemma 2.103.** *Equation* (2.97.1) *is true for* $k_2 > 1$.

*Proof.* Let be $k_1, k_2, k_3, k_4 \in \mathbb{Z}_{>0}$ and write $w = k_1 + k_2 + k_3 + k_4 + 4$. Note that by Theorem 2.82 for the first congruence and by Lemmas 2.101 and 2.102, and Equations (2.101.13) and (2.101.2) for the third congruence, we have

$$\begin{aligned} 0 &\equiv \frac{1}{k_2}\zeta_q^{\mathrm{f}}\left(\tau(u_2) * \tau\left(u_{k_1}u_{k_2}u_0u_{k_3}u_{k_4}u_0\right)\right) && \mod \mathrm{F}_{3,4,w} \\ &\equiv \frac{1}{k_2}\zeta_q^{\mathrm{f}}\left(u_1u_0 * u_2u_0^{k_4-1}u_1u_0^{k_3-1}u_2u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \mod \mathrm{F}_{3,4,w} \\ &\equiv \zeta_q^{\mathrm{f}}\left(u_3u_0^{k_4-1}u_1u_0^{k_3-1}u_2u_0^{k_2}u_1u_0^{k_1-1}\right) && \mod \mathrm{F}_{3,4,w} \\ &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_0u_{k_2+1}u_{k_3}u_0u_{k_4}\right) && \mod \mathrm{F}_{3,4,w}. \end{aligned} \tag{2.103.1}$$

By (2.99.1) and (2.101.15), this yields

$$0 \equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_{k_3}u_0u_{k_4}u_0u_0\right) \mod \mathrm{F}_{3,4,w}, \tag{2.103.2}$$

leading to, by using (2.100.5) and then (2.99.13),

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2+1}u_0u_0u_{k_3}u_{k_4}\right) \mod \mathrm{F}_{3,4,w}, \\ 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2+1}u_{k_3}u_0u_0u_{k_4}\right) \mod \mathrm{F}_{3,4,w}. \end{aligned}$$

Combining (2.99.8), (2.103.2), (2.101.14), and (2.101.12), we obtain

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_{k_3}u_0u_0u_0u_{k_4}\right) \mod \mathrm{F}_{3,4,w}, \\ 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2+1}u_0u_{k_3}u_{k_4}u_0\right) \mod \mathrm{F}_{3,4,w}. \end{aligned}$$

The second congruence is a consequence of the first one and case $j = 2$ of (2.100.1).

Furthermore, combining (2.99.9), (2.103.1), (2.101.10), and (2.101.6), we obtain

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2+1}u_0u_0u_0u_{k_3}u_{k_4}\right) \mod \mathrm{F}_{3,4,w}. \\ 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0u_{k_2+1}u_{k_3}u_0u_{k_4}u_0\right) \mod \mathrm{F}_{3,4,w}. \end{aligned}$$

The second congruence is a consequence of the first one and case $j = 3$ of (2.100.1). This completes the proof of the lemma. □

**Lemma 2.104.** *Equation* (2.97.1) *is true for* $k_1 > 1$.

*Proof.* Let be $k_1, k_2, k_3, k_4 \in \mathbb{Z}_{>0}$ with $k_1 > 1$ and write $w = k_1 + k_2 + k_3 + k_4 + 3$. Using Proposition 2.73 for the first congruence and Lemmas 2.101, 2.102, 2.103 afterwards, for all $z_2, z_3, z_4 \geq 0$ with $z_2 + z_3 + z_4 = 3$, we obtain

$$\begin{aligned} 0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1} * u_{k_2}u_0^{z_2}u_{k_3}u_0^{z_3}u_{k_4}u_0^{z_4}\right) && \mod \mathrm{F}_{3,4,w} \\ &\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_{k_2}u_0^{z_2}u_{k_3}u_0^{z_3}u_{k_4}u_0^{z_4}\right) && \mod \mathrm{F}_{3,4,w}. \end{aligned} \tag{2.104.1}$$

Now, choose $z_1 \geq 1, z_2, z_3, z_4 \geq 0$ with $z_1+\cdots+z_4 = 3$. Then, we obtain by Theorem 2.82 (in case $z_1 = 1$), Corollary 2.28 (in case $z_1 = 2$), and (2.104.1),

$$\begin{aligned}
0 &\equiv \zeta_q^{\mathrm{f}}\left(u_{z_1} * \tau\left(u_{k_1}u_{k_2}u_0^{z_2}u_{k_3}u_0^{z_3}u_{k_4}u_0^{z_4}\right)\right) && \bmod \mathrm{F}_{3,4,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{z_1} * u_{z_4+1}u_0^{k_4-1}u_{z_3+1}u_0^{k_3-1}u_{z_2+1}u_0^{k_2-1}u_1u_0^{k_1-1}\right) && \bmod \mathrm{F}_{3,4,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{z_4+1}u_0^{k_4-1}u_{z_3+1}u_0^{k_3-1}u_{z_2+1}u_0^{k_2-1}u_{z_1+1}u_0^{k_1-1}\right) && \bmod \mathrm{F}_{3,4,w}\\
&\equiv \zeta_q^{\mathrm{f}}\left(u_{k_1}u_0^{z_1}u_{k_2}u_0^{z_2}u_{k_3}u_0^{z_3}u_{k_4}u_0^{z_4}\right) && \bmod \mathrm{F}_{3,4,w}\,.
\end{aligned}$$

This completes the proof of the lemma. □

## 2.7 Conclusion and outlook

With $\mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w}\,\mathcal{Z}_q^f \subset \mathrm{F}_{z,d,w}$ for all $(z,d,w) \in \mathbb{Z}^3_{>0}$ (the refined Bachmann Conjecture 2.10), we gave a refinement of Bachmann's Conjecture 2.4 and proved several cases. For $z \geq d$, we gave a strategy for a general proof. Furthermore, for $z < d$, we were also able to prove the cases $1 \leq d \leq 4$. One can generalize our approach as described in the following paragraph.

**Approach to the refined Bachmann Conjecture 2.10 in case $z < d$.** We fix positive integers $z, d, w \in \mathbb{Z}_{>0}$ with $z < d$ in the following and assume throughout the whole paragraph that

$$\mathrm{Fil}^{\mathrm{Z,D,W}}_{\tilde{z},\tilde{d},\tilde{w}}\,\mathcal{Z}_q^f \subset \mathrm{F}_{\tilde{z},\tilde{d},\tilde{w}}$$

for $\tilde{z} \leq z$, $\tilde{d} < d$, $\tilde{w} < w$ is proven already. Note that the approach from case $z \geq d$ will not suffice for the case $z < d$ since $\mathcal{S}_{z,d} \subsetneq \mathcal{T}_{z,d}$ in this case by Conjecture 2.39. Therefore, we extend this approach as follows. Fix throughout this paragraph an index $\mathbf{k} = (k_1,\ldots,k_d) \in \mathbb{Z}^d_{>0}$ with $|\mathbf{k}| = w - z$. Besides

$$S^{(1)}_{z,d,\mathbf{k}} := \left\{\zeta_q^{\mathrm{f}}\left(\Psi_{\mathbf{k}}(u_{\mathbf{n}} \circledast u_{\boldsymbol{\ell}})\right) \mid (\mathbf{n},\boldsymbol{\ell}) \in \mathcal{J}_{z,d}\right\} \subset \mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w}\,\mathcal{Z}_q^f$$

(the inclusion follows from Lemma 2.56), we consider

$$S^{(2)}_{z,d,\mathbf{k}} := \left\{\zeta_q^{\mathrm{f}}\left(\tau(\tau(\mathtt{W}_{\mathbf{n},\mathbf{m}}) * \tau(\mathtt{W}_{\boldsymbol{\ell},\mathbf{k}'}))\right) \,\middle|\, \begin{smallmatrix} (\mathbf{n},\boldsymbol{\ell})\in\mathcal{J}_{z,d},\, \mathbf{m}\in\mathbb{Z}^{\mathrm{len}(\mathbf{n})}_{\geq 0},\, |\mathbf{m}|\leq \mathrm{len}(\mathbf{n})+d-z,\\ \mathbf{k}'\in\mathbb{Z}^d_{>0},\, k_j\geq k'_j\geq 1\ (1\leq j\leq d),\\ |\mathbf{m}|+|\mathbf{k}'|=s+|\mathbf{k}|,\, \mathrm{wt}(\mathtt{W}_{\mathbf{n},\mathbf{m}})+\mathrm{wt}(\mathtt{W}_{\boldsymbol{\ell},\mathbf{k}'})=w \end{smallmatrix}\right\},$$

where

$$\mathtt{W}_{\mathbf{n},\mathbf{m}} := u_{m_1}u_0^{n_s-1}\cdots u_{m_s}u_0^{n_1-1}, \quad \mathtt{W}_{\boldsymbol{\ell},\mathbf{k}'} = u_{k'_1}u_0^{\ell_d-1}\cdots u_{k'_d}u_0^{\ell_1-1}.$$

**Remark 2.105.** Note that we have $S^{(1)}_{z,d,\mathbf{k}} \subset S^{(2)}_{z,d,\mathbf{k}}$ for all $z, d \in \mathbb{Z}_{>0}$ with $z < d$ and for all $\mathbf{k} \in \mathbb{Z}^d_{>0}$.

Furthermore, we consider

$$S^{(3)}_{z,d,\mathbf{k}} := \left\{\zeta_q^{\mathrm{f}}\left(u_{\sigma(k_1)}u_0^{e_1}\cdots u_{\sigma(k_{s'})}u_0^{e_{s'}} * u_{\sigma(k_{s'+1})}u_0^{e_{s'+1}}\cdots u_{\sigma(k_d)}u_0^{e_d}\right) \,\middle|\, \begin{smallmatrix} \sigma\in\mathcal{S}_{\mathbf{k}},\, 1\leq s'\leq d-1,\\ \mathbf{e}=(e_1,\ldots,e_d)\in\mathbb{Z}^d_{\geq 0},\, |\mathbf{e}|=z \end{smallmatrix}\right\},$$

where $\mathcal{S}_{\mathbf{k}}$ is the set of permutations on $\{k_j \mid 1 \leq j \leq d\}$.

Similarly to the proof of Lemma 2.56, we can show the following.

**Lemma 2.106.** *Fix $z, d, w \in \mathbb{Z}_{>0}$. For all $(\mathbf{n}, \boldsymbol{\ell}) \in \mathcal{J}_{z,d}$, $\mathbf{k}, \mathbf{k}' \in \mathbb{Z}_{>0}^d$, and $\mathbf{m} \in \mathbb{Z}_{\geq 0}^s$, where $s = \mathrm{len}(\mathbf{n})$, satisfying $|\mathbf{k}| = w - z$, $|\mathbf{m}| \leq \mathrm{len}(\mathbf{n}) + d - z$ and $k_j \geq k_j' \geq 1$ for all $1 \leq j \leq d$, $|\mathbf{m}| + |\mathbf{k}'| = s + |\mathbf{k}|$, $\mathrm{wt}(\mathtt{W}_{\mathbf{n},\mathbf{m}}) + \mathrm{wt}(\mathtt{W}_{\boldsymbol{\ell},\mathbf{k}'}) = w$, we have*

$$\zeta_q^{\mathrm{f}}\left(\tau(\tau(\mathtt{W}_{\mathbf{n},\mathbf{m}}) * \tau(\mathtt{W}_{\boldsymbol{\ell},\mathbf{k}'}))\right) \in \sum_{1 \leq s' \leq s} \mathrm{Fil}^{\mathrm{Z,D,W}}_{z-s',d+s',w} \mathcal{Z}_q^f.$$

*In particular, we have $S^{(2)}_{z,d,\mathbf{k}} \subset \mathrm{F}_{z,d,w}$.*

Let us consider an example for illustration of Lemma 2.106.

**Example 2.107.** Denote $w = k_1' + k_2' + k_3' + 2$ in the following and choose

$$\mathbf{n} = (1), \quad \mathbf{m} = (2), \quad \boldsymbol{\ell} = (1,1,1), \quad \mathbf{k}' = (k_1', k_2', k_3') \in \mathbb{Z}_{>0}^3$$

in the notation of Lemma 2.106. First, we see that $\mathtt{W}_{\mathbf{n},\mathbf{m}} * \mathtt{W}_{\boldsymbol{\ell},\mathbf{k}'} = u_2 * u_{k_1'} u_{k_2'} u_{k_3'} \in \mathcal{F}$, where $\mathcal{F} = \mathrm{Fil}^{\mathrm{Z,D,W}}_{0,4,w} \mathbb{Q}\langle\mathcal{U}\rangle^\circ + \mathrm{Fil}^{\mathrm{Z,D,W}}_{1,3,w-1} \mathbb{Q}\langle\mathcal{U}\rangle^\circ$. Furthermore, we have

$$\begin{aligned}
&\tau(\tau(u_2) * \tau(u_{k_1'} u_{k_2'} u_{k_3'})) \\
={}& \tau\left(u_1 u_0 * u_1 u_0^{k_3'-1} u_1 u_0^{k_2'-1} u_1 u_0^{k_1'-1}\right) \\
\equiv{}& \tau\Big(k_3' u_2 u_0^{k_3'} u_1 u_0^{k_2'-1} u_1 u_0^{k_1'-1} + k_2' u_2 u_0^{k_3'-1} u_1 u_0^{k_2'} u_1 u_0^{k_1'-1} \\
&+ k_2' u_1 u_0^{k_3'-1} u_2 u_0^{k_2'} u_1 u_0^{k_1'-1} + k_1' u_2 u_0^{k_3'-1} u_1 u_0^{k_2'-1} u_1 \\
&+ k_1' u_1 u_0^{k_3'-1} u_2 u_0^{k_2'-1} u_1 + k_1' u_1 u_0^{k_3'-1} u_1 u_0^{k_2'-1} u_2 u_0^{k_1'}\Big) && \bmod \mathcal{F} \\
\equiv{}& k_3' u_{k_1'} u_{k_2'} u_{k_3'+1} u_0 + k_2' u_{k_1'} u_{k_2'+1} u_{k_3'} u_0 + k_2' u_{k_1'} u_{k_2'+1} u_0 u_{k_3'} \\
&+ k_1' u_{k_1'+1} u_{k_2'} u_{k_3'} u_0 + k_1' u_{k_1'+1} u_{k_2'} u_0 u_{k_3'} + k_1' u_{k_1'+1} u_0 u_{k_2'} u_{k_3'} && \bmod \mathcal{F}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&k_3' \zeta_q^{\mathrm{f}}\left(u_{k_1'} u_{k_2'} u_{k_3'+1} u_0\right) + k_2' \zeta_q^{\mathrm{f}}\left(u_{k_1'} u_{k_2'+1} u_{k_3'} u_0\right) + k_2' \zeta_q^{\mathrm{f}}\left(u_{k_1'} u_{k_2'+1} u_0 u_{k_3'}\right) \\
&+ k_1' \zeta_q^{\mathrm{f}}\left(u_{k_1'+1} u_{k_2'} u_{k_3'} u_0\right) + k_1' \zeta_q^{\mathrm{f}}\left(u_{k_1'+1} u_{k_2'} u_0 u_{k_3'}\right) + k_1' \zeta_q^{\mathrm{f}}\left(u_{k_1'+1} u_0 u_{k_2'} u_{k_3'}\right) \in \mathrm{F}_{2,3,w}.
\end{aligned}$$

Compared to the linear combinations in $S^{(1)}_{z,d,\mathbf{k}}$, it stands out that the latter linear combination is not a linear combination of words with the same multiplicity and the same non-$u_0$ letters in the same order. Nevertheless, all occurring words $u_{k_1} u_0^{z_1} u_{k_2} u_0^{z_2} u_{k_3} u_0^{z_3}$ satisfy $k_j \geq k_j'$ and $\sum_{j=1}^{3} (k_j - k_j') = 1 = |\mathbf{m}| - s = d - z$.

Furthermore, we have the following.

**Lemma 2.108.** *Fix $z, d, w \in \mathbb{Z}_{>0}$ with $z < d$ and assume that $\mathrm{Fil}^{\mathrm{Z,D,W}}_{z',d',w'} \mathcal{Z}_q^f \subset \mathrm{F}_{z',d',w'}$ is proven already for $z' \leq z$, $d' < d$, $w' < w$. Then, for every index $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}_{>0}^d$ and for all permutations $\sigma$ on $\{k_1, \ldots, k_d\}$, $1 \leq s' \leq d-1$, and $\mathbf{e} = (e_1, \ldots, e_d) \in \mathbb{Z}_{\geq 0}^d$ satisfying $|\mathbf{e}| = z$, we have*

$$\zeta_q^{\mathrm{f}}\left(u_{\sigma(k_1)} u_0^{e_1} \cdots u_{\sigma(k_{s'})} u_0^{e_{s'}} * u_{\sigma(k_{s'+1})} u_0^{e_{s'+1}} \cdots u_{\sigma(k_d)} u_0^{e_d}\right) \in \mathrm{F}_{z,d,w}.$$

*In particular, we have $S^{(3)}_{z,d,\mathbf{k}} \subset \mathrm{F}_{z,d,w}$.*

With the proofs of Theorems 2.8 and 2.12, we gave evidence for the following conjecture for $d \leq 4$.

**Conjecture 2.109.** *Fix $z, d, w \in \mathbb{Z}_{>0}$ with $z < d$ and assume that* $\mathrm{Fil}^{\mathrm{Z,D,W}}_{z',d',w'} \mathcal{Z}_q^f \subset \mathrm{F}_{z',d',w'}$ *is proven already for all $z' \leq z$, $d' < d$, $w' < w$. Then, for every $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d_{>0}$ and for every word $\mathtt{W} = u_{k_1} u_0^{z_1} \cdots u_{k_d} u_0^{z_d} \in \mathcal{U}^{*,\circ}$ satisfying* $\mathrm{zero}(\mathtt{W}) = z$, $\mathrm{depth}(\mathtt{W}) = d$, *and* $\mathrm{wt}(\mathtt{W}) = w$, *we have*

$$\zeta_q^{\mathrm{f}}(\mathtt{W}) \in \mathrm{span}_{\mathbb{Q}}\left(S^{(2)}_{z,d,\mathbf{k}} \cup S^{(3)}_{z,d,\mathbf{k}}\right) + \mathrm{F}_{z,d,w} \subset \mathrm{F}_{z,d,w} \,. \tag{2.109.1}$$

*In particular, then we have* $\mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w} \mathcal{Z}_q^f \subset \mathrm{F}_{z,d,w}$.

**Remark 2.110.** Note that the inclusion in (2.109.1) follows from Lemmas 2.106 and 2.108.

**Remark 2.111.** We can refine our approach to Conjecture 2.109 as follows. First, we will use for $\mathbf{k} \in \mathbb{Z}^d_{>0}$ satisfying $\#\{k_j > 1\} \geq d - z$ the linear combinations from $S^{(2)}_{z,d,\mathbf{k}}$ only to show (2.109.1). For the remaining cases, we then may assume without loss of generality that $\#\{k_j = 1\} \geq z$ and use both, $S^{(2)}_{z,d,\mathbf{k}}$ and $S^{(3)}_{z,d,\mathbf{k}}$ to prove (2.109.1). More precise, we consider the cases of $j_0 := \#\{k_j = 1\}$ with increasing $j_0 \geq z$. The intuitive reason for this is that, for given $j_0$, on the one hand we may assume that the cases for smaller values of $j_0$ are proven, making the linear combinations from $S^{(2)}_{z,d,\mathbf{k}}$ easier to handle since parts of them are in $\mathrm{F}_{z,d,w}$ already. On the other hand, the more entries of $\mathbf{k}$ are the same (for our purposes: one), the less formal Multiple Zeta Values of different words occur in the linear combinations from $S^{(3)}_{z,d,\mathbf{k}}$.

**Conclusion.** For $z < d$, our strategy also works in the small cases $1 \leq d \leq 4$ as shown, but there is still much to do for the general proof. More concretely, we conclude with the following open questions:

(i) How can one prove Conjecture 2.39 in general?

(ii) Conjecturally, Conjecture 2.39 can be proven via induction on $z$, $d$, or $z + d$.

(iii) Regarding Conjecture 2.39, we conjecturally have $\mathfrak{z}_{z,d} = \mathfrak{z}_{d,z}$ for all $z, d \in \mathbb{Z}_{>0}$. Can one prove this equality?

(iv) How to prove Conjecture 2.47 in general?

(v) How can one prove $\mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w} \mathcal{Z}_q^f \subset \mathrm{F}_{z,d,w}$ for $z < d$ in general?

(vi) Similar to Proposition 2.21, our approach for showing $\mathrm{Fil}^{\mathrm{Z,D,W}}_{z,d,w} \mathcal{Z}_q^f \subset \mathrm{F}_{z,d,w}$ is suitable to obtain for all words $\mathtt{W} \in \mathcal{U}^{*,\circ}$ an explicit formula $\zeta_q^{\mathrm{f}}(\mathtt{W}) = \zeta_q^{\mathrm{f}}(\mathcal{L})$, where $\mathcal{L}$ is a linear combination of products of elements in $\mathcal{Z}_q^{f,\circ}$. With some engagement following our calculations, this already can be done now for all words $\mathtt{W} \in \mathcal{U}^{*,\circ}$ satisfying $\mathrm{zero}(\mathtt{W}) + \mathrm{depth}(\mathtt{W}) \leq 6$. What do they look like? Can one find some systematics such that one can derive such formulas also for $\mathrm{zero}(\mathtt{W}) + \mathrm{depth}(\mathtt{W}) > 6$ (which would prove Bachmann's Conjecture 2.4 in particular)?

## 2.8 Code of the calculations in Chapter 2

The numerical calculations of Chapter 2 were done using Python. In this appendix, the original source code is presented.

### 2.8.1 Computations regarding Lemma 2.42

#### Setup and basic functions

We begin with the required packages.

```
import numpy as np
import itertools
import math
from ast import literal_eval
```

The first definitions were elementary for the main calculations.

**Function 2.112.** The function d(z,d,s) returns $\binom{z+d-1}{z-s}$ for integers $z, d, s \in \mathbb{Z}_{>0}$ with $s \leq z \leq d$, which is conjecturally $\mathfrak{d}_{z,d,s}$ (see Conjecture 2.58).

```
def d(z,d,s):
    if (z <= d) and (s <= z):
        return(math.comb(z+d-1,z-s))
    elif (z <= d) and (s > z):
        return(0)
```

**Function 2.113.** The function part(r,s) returns the list of all ordered partitions of $r$ into exactly $s$ non-negative integers.

```
def part(r,s):
    if s<=0:
        return([[]])
    else:
        P = []
        for S in set(itertools.combinations(range(r+s-1), s-1)):
            p = []
            I = [-1] + list(S) + [r+s-1]
            for i in range(len(I)):
                if i > 0:
                    p.append(I[i]-I[i-1]-1)
            P.append(p)
        return(P)
```

**Function 2.114.** The function ppart(r,s) returns the list all ordered partitions into exactly $s$ positive integers.

```
def ppart(r,s):
    if s<=0 or r<s:
        return([[]])
    else:
        P = []
        for p in part(r-s,s):
            q = p
            for j in range(len(p)):
                q[j] += 1
            P.append(q)
        P.sort()
        return(P)
```

**Function 2.115.** The function Indices(z,d) returns the list of all indices $\boldsymbol{\mu} \in \mathbb{Z}^d_{>0}$ with $|\boldsymbol{\mu}| = z + d$.

```
def Indices(z,d):
    if z==0:
        return([d*[1]])
    else:
        I = []
```

```
        for index in Indices(z-1,d):
            for k in range(d):
                indi = index[:k] + [index[k]+1] + index[k+1:]
                if indi not in I:
                    I.append(indi)
        I.sort()
        return(I)
```

### The box product

In this section, we implement the box product as linear combination of $u_{\mu} \in (\mathcal{U}\backslash\{u_0\})^*$. Furthermore, for a set of box products, we implement the adjacency matrix whichs rows will correspond to the linear combinations and the columns to the words $u_{\mu}$, i.e., the entries are the coefficient of a word in a linear combination of box products.

We begin with the box product.

**Function 2.116.** The function `box(index1,index2)` returns $u_{\texttt{index1}} \boxast u_{\texttt{index2}}$ as follows. It returns a dictionary `D` containing as keys the indices `ind` satisfying that $u_{\texttt{ind}}$ occurs in the box product $u_{\texttt{index1}} \boxast u_{\texttt{index2}}$ with multiplicity $\neq 0$; the value `D[ind]` then is the multiplicity of $u_{\texttt{ind}}$ in $u_{\texttt{index1}} \boxast u_{\texttt{index2}}$.

```
def box(index1,index2):
    D = {}
    s = len(index1)
    d = len(index2)
    if s>d:
        return(D)
    elif index1 == []:
        D[str(index2)] = 1
    else:
        for S in set(itertools.combinations(range(d), s)):
            L = list(S)
            L.sort()
            ind = []
            for k in range(d):
                if k in L:
                    ind.append(index2[k]+index1[L.index(k)])
                else:
                    ind.append(index2[k])
            D[str(ind)] = 1
        return(D)
```

Based on `box`, we introduce the following function representing $u_{\texttt{index1}} \boxast u_{\texttt{index2}}$ as dictionary `D` with keys $\texttt{ind} \in \mathbb{Z}_{>0}^{\mathrm{len}(\texttt{index2})}$, satisfying

$$|\texttt{ind}| = |\texttt{index1}| + |\texttt{index2}|,$$

and with `D[ind]` being the multiplicity of $u_{\texttt{ind}}$ in the box product $u_{\texttt{index1}} \boxast u_{\texttt{index2}}$.

```
def BOX(index1,index2):
    s = len(index1)
    d = len(index2)
    z = sum(index1)+sum(index2)-d
    I = Indices(z,d)
    D = {}
    for ind in I:
        D[str(ind)] = 0
    if s>d or sum(index1)+sum(index2) != z+d:
        return(D)
    elif index1 == [] and sum(index2) == z+d:
```

```
        D[str(index2)] = 1
    else:
        for ind in box(index1,index2):
            D[ind] = box(index1,index2)[ind]
    return(D)
```

Let us consider an example to see the difference between the functions `box` and `BOX`.

**Example 2.117.** We have

$$u_2 \circledast u_1u_1u_1 = u_3u_1u_1 + u_1u_3u_1 + u_1u_1u_3.$$

Now, `box([2],[1,1,1])` returns

```
{'[3, 1, 1]': 1, '[1, 3, 1]': 1, '[1, 1, 3]': 1}
```

and `BOX([2],[1,1,1])` returns

```
{'[1, 1, 3]': 1,
 '[1, 2, 2]': 0,
 '[1, 3, 1]': 1,
 '[2, 1, 2]': 0,
 '[2, 2, 1]': 0,
 '[3, 1, 1]': 1}.
```

### Dimension of spaces spanned by box products

We considered in the paper the dimension of spaces spanned by several box products (in particular, $\mathcal{S}_{z,d}$). Numerically, we will obtain such dimensions as the rank of the coefficient matrix of the box products that span the space we consider, interpreted as linear combination of words $u_\mu \in (\mathcal{U}\backslash\{u_0\})^*$. For this, we introduce the function `MATR`.

**Function 2.118.** The function `Dim(P)` takes a list `P` of box products, given in shape of `BOX(index1,index2)`, and returns the dimension of the space they span. This is done via computing the rank of the coefficient matrix (as list of lists) of these box products with rows corresponding to the box products, columns corresponding to the coefficient of words $u_\mu \in (\mathcal{U}\backslash\{u_0\})^*$.

```
def Dim(P):
    M = []
    for prod in P:
        I = []
        for index in prod:
            I.append(prod[index])
        M.append(I)
    rk = np.linalg.matrix_rank(M)
    return(rk)
```

### LaTeX-Output

We will consider subspaces of $\mathcal{S}_{z,d}$ for several $z, d \in \mathbb{Z}_{>0}$. Usually, we skip the cases of $z = 1$ or $d = 1$ since we already know the dimension of the corresponding subspace in these cases. The function `MatLatex` produces the LaTeX-code of a table in which we collect our calculations.

**Function 2.119.** The function `MatLatex(M,cap)` gives the LaTeX-code of the table with caption `cap` and three entries in each cell. Here, `M` is a list of lists with four entries each. They are all of shape

$$[z, d, \mathrm{rk}, \dim],$$

where $z$ defines the column, $d$ defines the row, rk is the (numerical) dimension of the subspace of $\mathcal{S}_{z,d}$ we consider, while dim is the corresponding conjectured dimension each. Every cell consists of two numbers, where the first one in black is the (numerically obtained) dimension of the subspace of $\mathcal{S}_{z,d}$ we consider and the second number is in blue the conjectured dimension of the subspace of $\mathcal{S}_{z,d}$ we consider.

```
def MatLatex(M,cap):
    dmin = M[0][0]
    dmax = M[-1][0]
    zmin = M[0][1]
    zmax = M[-1][1]
    B = "\\begin{figure}[h]\n \\centering\n \\caption{"+cap+"}\n \\begin{tabular}{|" + "c|".join("" for j in range(zmin,zmax+2)) + "c|}\n \\hline\n"
    E = "\\end{tabular}\n \\end{figure}"
    newM = (dmax - dmin + 1)*[(zmax - zmin + 1)*["&-"]]
    S = "d$\\backslash$ z&" + "&".join(str(j) for j in range(zmin,zmax+1)) + "\\\\ \\hline\n"
    for result in M:
        helpstr = "&" + str(result[2]) + "\\ \\textcolor{blue}{"+str(result[3])+"}
        dact = result[0] - dmin
        zact = result[1] - zmin
        rowact = newM[dact]
        newM = newM[:(result[0] - dmin)] + [rowact[:zact] + [helpstr] + rowact[zact+1:]] + newM[(result[0] - dmin+1):]
    for j in range(dmax - dmin + 1):
        S = S + str(dmin + j)
        for k in range(zmax-zmin+1):
            S = S + newM[j][k]
        S = S + "\\\\ \\hline\n"
    return(B+S+E)
```

Next, we produce the function giving the desired table for the dimension of $\mathcal{S}_{z,d,s_{\min}}$ for some $s_{\min}$ and $2 \leq z, d$ up to an upper bound we declare in the input.

**Function 2.120.** Choosing $zmax, dmax, smin \in \mathbb{Z}_{>0}$, the following function returns the tabular according to Function 2.119 where in black the computed dimension of the space $\mathcal{S}_{zmax,dmax,smin}$ is displayed, while in blue the conjectured dimension (coming from Conjecture 2.58) appears.

```
def Tabular(zmax,dmax,smin):
    M = []
    for z in range(2,zmax+1):
        for d in range(2,dmax+1):
            P = []
            for k in range(smin,min(d,z)+1):
                S = ppart(d+z,d+k)
                for partition in S:
                    P.append(BOX(partition[:k],partition[k:]))
            rk = Dim(P)
            M.append([d,z,rk,d(z,d,smin)])
    if smin != 1:
        cap = "Dimension of $\\mathcal{S}_{z,d,"+str(smin)+"}$."
    else:
        cap = "Dimension of $\\mathcal{S}_{z,d}$."
    return(MatLatex(M,cap))
```

### Results

In the following, we present several results of our calculations. Recall that every cell of the following tables consists of two numbers, where the first one in black is the (numerically obtained) dimension of the subspace of $\mathcal{S}_{z,d}$ we consider and the second number is in blue the conjectured dimension from Conjecture 2.58.

**Remark 2.121.** (i) Using `Tabular(8,8,1)`, we obtain that Conjecture 2.58 is true for $2 \leq z \leq d \leq 8$ and $s_{\min} = 1$, i.e., Conjecture 2.39 is true for $z, d \leq 8$:

FIGURE 2.1: Dimension of $\mathcal{S}_{z,d}$.

| d\ z | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 2 | 3 3 | - | - | - | - | - | - |
| 3 | 4 4 | 10 10 | - | - | - | - | - |
| 4 | 5 5 | 15 15 | 35 35 | - | - | - | - |
| 5 | 6 6 | 21 21 | 56 56 | 126 126 | - | - | - |
| 6 | 7 7 | 28 28 | 84 84 | 210 210 | 462 462 | - | - |
| 7 | 8 8 | 36 36 | 120 120 | 330 330 | 792 792 | 1716 1716 | - |
| 8 | 9 9 | 45 45 | 165 165 | 495 495 | 1287 1287 | 3003 3003 | 6435 6435 |

(ii) Using `Tabular(8,8,2)`, we obtain that Conjecture 2.58 is true for $2 \leq z \leq d \leq 8$ and $s_{\min} = 2$:

FIGURE 2.2: Dimension of $\mathcal{S}_{z,d,2}$.

| d\ z | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 2 | 1 1 | - | - | - | - | - | - |
| 3 | 1 1 | 5 5 | - | - | - | - | - |
| 4 | 1 1 | 6 6 | 21 21 | - | - | - | - |
| 5 | 1 1 | 7 7 | 28 28 | 84 84 | - | - | - |
| 6 | 1 1 | 8 8 | 36 36 | 120 120 | 330 330 | - | - |
| 7 | 1 1 | 9 9 | 45 45 | 165 165 | 495 495 | 1287 1287 | - |
| 8 | 1 1 | 10 10 | 55 55 | 220 220 | 715 715 | 2002 2002 | 5005 5005 |

(iii) Using `Tabular(8,8,3)`, we obtain that Conjecture 2.58 is true for $2 \leq z \leq d \leq 8$ and $s_{\min} = 3$:

FIGURE 2.3: Dimension of $\mathcal{S}_{z,d,3}$.

| d\ z | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 2 | 0 0 | - | - | - | - | - | - |
| 3 | 0 0 | 1 1 | - | - | - | - | - |
| 4 | 0 0 | 1 1 | 7 7 | - | - | - | - |
| 5 | 0 0 | 1 1 | 8 8 | 36 36 | - | - | - |
| 6 | 0 0 | 1 1 | 9 9 | 45 45 | 165 165 | - | - |
| 7 | 0 0 | 1 1 | 10 10 | 55 55 | 220 220 | 715 715 | - |
| 8 | 0 0 | 1 1 | 11 11 | 66 66 | 286 286 | 1001 1001 | 3003 3003 |

(iv) Using `Tabular(8,8,4)`, we obtain that Conjecture 2.58 is true for $2 \leq z \leq d \leq 8$ and $s_{\min} = 4$:

FIGURE 2.4: Dimension of $\mathcal{S}_{z,d,4}$.

| d\ z | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 2 | 0 0 | - | - | - | - | - | - |
| 3 | 0 0 | 0 0 | - | - | - | - | - |
| 4 | 0 0 | 0 0 | 1 1 | - | - | - | - |
| 5 | 0 0 | 0 0 | 1 1 | 9 9 | - | - | - |
| 6 | 0 0 | 0 0 | 1 1 | 10 10 | 55 55 | - | - |
| 7 | 0 0 | 0 0 | 1 1 | 11 11 | 66 66 | 286 286 | - |
| 8 | 0 0 | 0 0 | 1 1 | 12 12 | 78 78 | 364 364 | 1365 1365 |

(v) Using `Tabular(8,8,5)`, we obtain that Conjecture 2.58 is true for $2 \leq z \leq d \leq 8$ and $s_{\min} = 5$:

FIGURE 2.5: Dimension of $\mathcal{S}_{z,d,5}$.

| d\ z | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 2 | 0 0 | - | - | - | - | - | - |
| 3 | 0 0 | 0 0 | - | - | - | - | - |
| 4 | 0 0 | 0 0 | 0 0 | - | - | - | - |
| 5 | 0 0 | 0 0 | 0 0 | 1 1 | - | - | - |
| 6 | 0 0 | 0 0 | 0 0 | 1 1 | 11 11 | - | - |
| 7 | 0 0 | 0 0 | 0 0 | 1 1 | 12 12 | 78 78 | - |
| 8 | 0 0 | 0 0 | 0 0 | 1 1 | 13 13 | 91 91 | 455 455 |

(vi) Using `Tabular(8,8,6)`, we obtain that Conjecture 2.58 is true for $2 \leq z \leq d \leq 8$ and $s_{\min} = 6$:

FIGURE 2.6: Dimension of $\mathcal{S}_{z,d,6}$.

| d\ z | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 2 | 0 0 | - | - | - | - | - | - |
| 3 | 0 0 | 0 0 | - | - | - | - | - |
| 4 | 0 0 | 0 0 | 0 0 | - | - | - | - |
| 5 | 0 0 | 0 0 | 0 0 | 0 0 | - | - | - |
| 6 | 0 0 | 0 0 | 0 0 | 0 0 | 1 1 | - | - |
| 7 | 0 0 | 0 0 | 0 0 | 0 0 | 1 1 | 13 13 | - |
| 8 | 0 0 | 0 0 | 0 0 | 0 0 | 1 1 | 14 14 | 105 105 |

(vii) Using `Tabular(8,8,7)`, we obtain that Conjecture 2.58 is true for $2 \leq z \leq d \leq 8$ and $s_{\min} = 7$:

FIGURE 2.7: Dimension of $\mathcal{S}_{z,d,7}$.

| d\ z | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 2 | 0 0 | - | - | - | - | - | - |
| 3 | 0 0 | 0 0 | - | - | - | - | - |
| 4 | 0 0 | 0 0 | 0 0 | - | - | - | - |
| 5 | 0 0 | 0 0 | 0 0 | 0 0 | - | - | - |
| 6 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 | - | - |
| 7 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 | 1 1 | - |
| 8 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 | 1 1 | 15 15 |

(viii) Using `Tabular(8,8,8)`, we obtain that Conjecture 2.58 is true for $2 \leq z \leq d \leq 8$ and $s_{\min} = 8$:

FIGURE 2.8: Dimension of $\mathcal{S}_{z,d,8}$.

| d\ z | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 2 | 0 0 | - | - | - | - | - | - |
| 3 | 0 0 | 0 0 | - | - | - | - | - |
| 4 | 0 0 | 0 0 | 0 0 | - | - | - | - |
| 5 | 0 0 | 0 0 | 0 0 | 0 0 | - | - | - |
| 6 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 | - | - |
| 7 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 | - |
| 8 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 | 1 1 |

### 2.8.2 Computations regarding Lemma 2.48

#### Setup and basic functions

We use the same setup as in Section 2.8.1 and the functions `part` and `ppart` from there.

#### Stuffle product and box product

We define the stuffle product on index level and call the function `stuffleprod`.

**Function 2.122.** For indices `L1` and `L2` (input as lists), the function `stuffleprod`(`L1`, `L2`) returns a list of indices (as lists) with the property that their formal sum is exactly the stuffle product `L1` $*$ `L2`.

```
def stuffleprod(L1,L2):
    if len(L1) == 0:
        return([L2])
    elif len(L2) == 0:
        return([L1])
    L = []
    for L3 in stuffleprod(L1[1:],L2):
        L.append([L1[0]]+L3)
    for L3 in stuffleprod(L1,L2[1:]):
        L.append([L2[0]]+L3)
    for L3 in stuffleprod(L1[1:],L2[1:]):
        L.append([L1[0]+L2[0]]+L3)
    return(L)
```

Furthermore, we define the box product on index level and call the function `boxprod`.

**Function 2.123.** For two indices L1 and L2 (input as lists), the function `boxprod(L1, L2)` returns a list of indices (as lists) with the property that their formal sum is exactly the box product L1 $*$ L2.

```
def boxprod(L1,L2):
    s = len(L1)
    d = len(L2)
    if s>d:
        return([])
    elif s==0:
        return([L2])
    L = []
    for L3 in boxprod(L1[1:],L2[1:]):
        L.append([L1[0]+L2[0]]+L3)
    for L3 in boxprod(L1,L2[1:]):
        L.append([L2[0]]+L3)
    return(L)
```

### The numbers $\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}} \mathcal{K}_{z,d}$

First, we implement for given $1 \leq z \leq d$ the conjectured dimension of $\operatorname{span}_{\mathbb{Q}} \mathcal{K}_{z,d}$. Following Conjecture 2.39, (2.42.1), and (2.42.2), this number is

$$\sum_{j=2}^{z} \binom{z+d-1}{d+j-1}. \tag{2.123.1}$$

**Function 2.124.** For $z, d \in \mathbb{Z}_{>0}$ with $z \leq d$, the function `kerneldimconj` returns the conjectured dimension of $\operatorname{span}_{\mathbb{Q}} \mathcal{K}_{z,d}$, which is given by (2.123.1).

```
def kerneldimconj(z,d):
    S = 0
    for j in range(d+1,z+d):
        S = S + math.comb(z+d-1,j)
    return(S)
```

The next function returns for given $1 \leq z \leq d$ the number $\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}} \mathcal{K}_{z,d}$.

**Function 2.125.** Let be $z, d \in \mathbb{Z}_{>0}$ with $z \leq d$. The function `kerneldim(z,d)` returns the number $\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}} \mathcal{K}_{z,d}$ via computing ranks of matrices.

```
def kerneldim(z,d):
    Rel = []
    for s in range(d+2,z+d+1):
        for partition in ppart(z+d,s):
            for t in range(d+1,s):
                Mind = partition[t:]
                Lind = partition[:d]
                Nind = partition[d:t]
                D = {}
                for s in range(d+1,z+d+1):
                    for ppartition in ppart(z+d,s):
                        D[str(ppartition)] = 0
                for P in boxprod(Mind,Lind):
                    D[str(Nind+P)] = D[str(Nind+P)] + 1
                for P in stuffleprod(Nind,Mind):
                    D[str(P+Lind)] = D[str(P+Lind)] - 1
                R = []
                for key in D:
                    R.append(D[key])
                Rel.append(R)
    return(np.linalg.matrix_rank(Rel))
```

### Results

Via

```
for d in range(2,9):
    for z in range(2,d+1):
        print(z,d,(kerneldim(z,d),kerneldimconj(z,d)))
```

we obtain in the following in each row four entries, the first one corresponding to $z$, the second to $d$, the third to the numerical result for $\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}} \mathcal{K}_{z,d}$, and the fourth is the value we expect for $\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}} \mathcal{K}_{z,d}$:

```
2 2 (1, 1)
2 3 (1, 1)
3 3 (6, 6)
2 4 (1, 1)
3 4 (7, 7)
4 4 (29, 29)
2 5 (1, 1)
3 5 (8, 8)
4 5 (37, 37)
5 5 (130, 130)
2 6 (1, 1)
3 6 (9, 9)
4 6 (46, 46)
5 6 (176, 176)
6 6 (562, 562)
2 7 (1, 1)
3 7 (10, 10)
4 7 (56, 56)
5 7 (232, 232)
6 7 (794, 794)
7 7 (2380, 2380)
2 8 (1, 1)
3 8 (11, 11)
4 8 (67, 67)
5 8 (299, 299)
6 8 (1093, 1093)
7 8 (3473, 3473)
8 8 (9949, 9949)
```

**Remark 2.126.** Regarding our results, Lemma 2.48 is proven.

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# Chapter 3

# Paper II: Combinatorial interpretation of the stuffle product

# Combinatorial interpretation of the stuffle product

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**Mathematics Subject Classification:** 05A17, 11M32

**Abstract.** We show how the quasi–shuffle product, Schlesinger–Zudilin Multiple $q$–Zeta Values (SZ-$q$MZVs) satisfy, behaves on the level of partitions. For this, we work with marked partitions, which are partitions in whose Young–Tableau rows and columns are marked in some way. Together with the description of duality using marked partitions (see [4]) and the conjecture by Bachmann [1] that all linear relations among $q$MZVs are implied by duality and the stuffle product, this paper completes conjecturally the description of the structure of $q$MZVs using marked partitions.

## 3.1 Introduction

Multiple $q$-Zeta Values, $q$MZVs for short, can be seen as generalizations of MZVs as well as (quasi-)modular forms or as generating functions of particular types of partitions. They are $q$-series giving back a Multiple Zeta Value (or a $\mathbb{Q}$-linear combination of them) in the limit $q \to 1$, often after modifying the series via multiplicating with some power of $1-q$. In this paper, we focus on $q$MZVs introduced by Schlesinger [7] and Zudilin [10]. For an overview of $q$MZVs, see, e.g., [4].

In the following, we consider $\mathcal{U} := \{u_j \mid j \in \mathbb{Z}_{\geq 0}\}$. We call $\mathcal{U}$ also an *alphabet*, and elements of $\mathcal{U}$ are referred to as *letters*. Furthermore, monomials of elements in $\mathcal{U}$ (with respect to concatenation) are called *words*. Usually, the neutral element with respect to concatenation is denoted by $\mathbf{1}$ and called the *empty word*. Let $\mathcal{U}^*$ denote the set of words with letters in $\mathcal{U}$, then we write $\mathbb{Q}\langle\mathcal{U}\rangle$ for the $\mathbb{Q}$-vector space $\operatorname{span}_{\mathbb{Q}}\mathcal{U}^*$, equipped with the non-commutative, but associative multiplication, given by concatenation. We define the *stuffle product* to be the $\mathbb{Q}$-bilinear map $*\colon \mathbb{Q}\langle\mathcal{U}\rangle \times \mathbb{Q}\langle\mathcal{U}\rangle \to \mathbb{Q}\langle\mathcal{U}\rangle$ recursively via

$$u_{j_1}\mathtt{W}_1 * u_{j_2}\mathtt{W}_2 := u_{j_1}\left(\mathtt{W}_1 * u_{j_2}\mathtt{W}_2\right) + u_{j_2}\left(u_{j_1}\mathtt{W}_1 * \mathtt{W}_2\right) + u_{j_1+j_2}\left(\mathtt{W}_1 * \mathtt{W}_2\right)$$

for all $j_1, j_2 \in \mathbb{Z}_{\geq 0}$ and $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^*$ with initial condition $\mathbf{1} * \mathtt{W} = \mathtt{W} * \mathbf{1} = \mathtt{W}$ for all words $\mathtt{W} \in \mathcal{U}^*$. By Hoffman ([5]), $(\mathbb{Q}\langle\mathcal{U}\rangle, *)$ is an associative and commutative $\mathbb{Q}$-algebra. For a word $\mathtt{W} = u_{k_1}\cdots u_{k_r} \in \mathcal{U}^*$, we associate the *length*, $\operatorname{len}(\mathtt{W}) := r$ and the *depth*, which is $\operatorname{depth}(\mathtt{W}) := \#\{k_j \neq 0 \mid 1 \leq j \leq r\}$. Furthermore, we write $\mathcal{U}^{*,\circ} := \mathcal{U}^* \backslash u_0\mathcal{U}^*$ for the set of words in $\mathcal{U}^*$ not starting with $u_0$ and we denote by $\mathbb{Q}\langle\mathcal{U}\rangle^\circ$ the corresponding subspace of $\mathbb{Q}\langle\mathcal{U}\rangle$, i.e., the $\mathbb{Q}$-vector space generated by words not starting in $u_0$. Note that $\mathbb{Q}\langle\mathcal{U}\rangle^\circ$ is closed under $*$ which gives rise to a commutative $\mathbb{Q}$-algebra $(\mathbb{Q}\langle\mathcal{U}\rangle^\circ, *)$

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(see [5]). The map $\zeta_q^{\mathrm{SZ}}\colon (\mathbb{Q}\langle\mathcal{U}\rangle^\circ, *) \to (\mathbb{Q}[\![q]\!], \cdot)$ is the $\mathbb{Q}$-algebra homomorphism (see [6]) defined via $\zeta_q^{\mathrm{SZ}}(\mathbf{1}) = 1$, $\mathbb{Q}$-linearity, and, with $m_{d+1} := 0$,

$$\zeta_q^{\mathrm{SZ}}\left(u_{k_1}u_0^{z_1}\cdots u_{k_d}u_0^{z_d}\right) := \sum_{m_1>\cdots>m_d>0}\prod_{j=1}^{d}\binom{m_j - m_{j+1} - 1}{z_j}\frac{q^{m_jk_j}}{(1-q^{m_j})^{k_j}},$$

for any $k_1,\ldots,k_d \in \mathbb{Z}_{>0}$ and $z_1,\ldots,z_d \in \mathbb{Z}_{\geq 0}$ where $d \in \mathbb{Z}_{>0}$ (note that this definition is not the usual one, like in [8], but equivalent to it; this statement can be deduced, e.g., from [4, Theorem 2.18]). We denote by $\mathcal{Z}_q$ the image of $\zeta_q^{\mathrm{SZ}}$ and call elements in $\mathcal{Z}_q$ *(Schlesinger–Zudilin)-qMZVs* ((SZ-)$q$MZVs for short). Remarkable is that SZ-$q$MZVs are invariant under the involution $\tau : \mathbb{Q}\langle\mathcal{U}\rangle^\circ \to \mathbb{Q}\langle\mathcal{U}\rangle^\circ$ , defined by $\mathbb{Q}$-linearity, $\tau(\mathbf{1}) := \mathbf{1}$, and

$$\tau\left(u_{k_1}u_0^{z_1}\cdots u_{k_d}u_0^{z_d}\right) := u_{z_d+1}u_0^{k_d-1}\cdots u_{z_1+1}u_0^{k_1-1}$$

for all $d \in \mathbb{Z}_{>0}$, $k_1,\ldots,k_d \geq 1$, and $z_1,\ldots,z_d \geq 0$ (see [9, Theorem 8.3]; $\tau$ is often referred to as *duality*). Note at this point the following folklore conjecture by Bachmann (see [1]; a published version can be found in [11, Conjecture 1]) about the structure of $\mathcal{Z}_q$.

**Conjecture 3.1** (Bachmann)**.** *All $\mathbb{Q}$-linear relations among elements in $\mathcal{Z}_q$ are obtained by the stuffle product $*$ and duality $\tau$.*

Furthermore, the space $\mathcal{Z}_q$ contains all quasi-modular forms via their $q$-expansion

$$\zeta_q^{\mathrm{SZ}}(\mathtt{W}) = \sum_{N\geq 0}\psi_N(\mathtt{W})q^N,$$

where $\psi_N(\mathtt{W})$ denotes the $N$-th Fourier coefficient of $\zeta_q^{\mathrm{SZ}}(\mathtt{W})$ for any $\mathtt{W} \in \mathcal{U}^{*,\circ}$ (see [2]). The Fourier coefficients of modular forms have been a key feature of their study. This paper gives a combinatorial approach to the Fourier coefficients of $q$MZVs, interpreted as finite sums over so-called *marked partitions* (they were introduced in [4]). In particular, we will describe the stuffle product as a pairing on marked partitions.

We will use the following combinatorial interpretation of $\psi_N$ developed in [4]. Let $p$ be a partition of $N$ with $d$ distinct parts $m_j$ with multiplicities $n_j$, meaning that we have

$$m_1 > \cdots > m_d > 0, \quad n_1,\ldots,n_d \in \mathbb{Z}_{>0}, \quad N = m_1n_1 + \cdots + m_dn_d.$$

We shall mark rows with a dot in the Young Tableau of $p$. If for $k_j$ rows of length $m_j$ are marked, we call $\mathbf{k} = (k_1,\ldots,k_d)$ the *type* of this row marking. A row marking is called *distinct* if the lowest row for each length $m_j$ is marked. Furthermore, a distinct column marking of $p$ is an $d$-tupel $\mathbf{z} = (z_1+1,\ldots,z_d+1)$, such that $(z_d+1,\ldots,z_1+1)$ is a distinct row marking of the conjugate partition of $p$. A pair $(\mathbf{k};\mathbf{z})$ of such distinct markings is identified with $\mathtt{W} = u_{k_1}u_0^{z_1}\cdots u_{k_d}u_0^{z_d} \in \mathcal{U}^{*,\circ}$ and called for short a $\mathtt{W}$-marking of $p$.

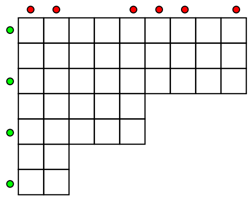
**Definition 3.2.** (i) We interpret $\emptyset$ as the unique marked partition (of $N = 0$) of type $\mathbf{1}$.

(ii) For any $\mathtt{W} \in \mathcal{U}^{*,\circ}$, we define $\mathcal{MP}_{\mathtt{W}}$ as the set of all marked partitions of type $\mathtt{W}$.

(iii) We denote by $\mathcal{MP} := \bigcup_{\mathtt{W}\in\mathcal{U}^{*,\circ}} \mathcal{MP}_{\mathtt{W}}$ the set of all marked partitions.

(iv) Given a (marked) partition, we call the union of all rows in the Young Tableau having a given length a *horizontal block (of the partition/Young Tableau).*

**Example 3.3.** The following is a marked partition of type $\mathtt{W} = u_2u_0u_0u_1u_1u_0$ of the integer $N = 9 \cdot 3 + 5 \cdot 2 + 2 \cdot 2 = 41$.



It consists of three horizontal blocks in the notation of Definition 3.2.

One has the following connection of marked partitions and the Fourier coefficient of SZ-$q$MZVs.

**Proposition 3.4** ([4])**.** *For all $N \in \mathbb{Z}_{\geq 0}$ and $\mathtt{W} \in \mathcal{U}^{*,\circ}$, we have*

$$\psi_N(\mathtt{W}) = \#\mathcal{MP}_{\mathtt{W}}.$$

For the main theorem about the combinatorial interpretation of the product of SZ-$q$MZVs, we need the following pairing $\Phi$ on the set of marked partitions.

**Definition 3.5.** The map $\Phi\colon \mathcal{MP} \times \mathcal{MP} \to \mathcal{MP}$ is defined as follows: Given marked partitions $\widehat{p_1}$ of $N_1$ and $\widehat{p_2}$ of $N_2$, then $\widehat{p} = \Phi(\widehat{p_1}, \widehat{p_2})$ is the marked partition of $N_1 + N_2$ obtained by the following rules:

(i) We set $\Phi(\emptyset, \widehat{p_2}) := \widehat{p_2}$ and $\Phi(\widehat{p_1}, \emptyset) := \widehat{p_1}$.

(ii) The Young Tableau of $\widehat{p}$ is obtained by cutting the Young Tableau of $\widehat{p_1}$ and $\widehat{p_2}$ horizontally below the rows containing corners into their horizontal blocks and glueing them (horizontally again) together to a new Young Tableau. If both, $\widehat{p_1}$ and $\widehat{p_2}$, have horizontal blocks of same length, the ones of $\widehat{p_1}$ will occur above the ones of $\widehat{p_2}$ in the new partition.

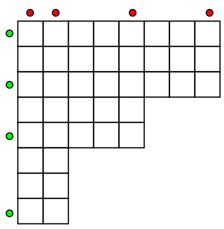
(iii) Keep the markings of the rows.

(iv) If there was a marking in the $j$-th leftmost column of $\widehat{p_1}$ or $\widehat{p_2}$, the $j$-th leftmost column of $\widehat{p}$ will be marked as well.

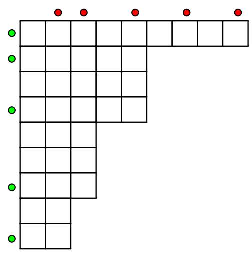
**Remark 3.6.** Note that the map $\Phi$ is associative but not commutative. The underlying Young Tableau of $\Phi(\widehat{p_1}, \widehat{p_2})$ is the same as the one of $\Phi(\widehat{p_2}, \widehat{p_1})$ and also the column markings match but the row markings, in general, do not if $\widehat{p_1}$ and $\widehat{p_2}$ have blocks of same length.

**Example 3.7.** Consider the following pair of marked partitions.

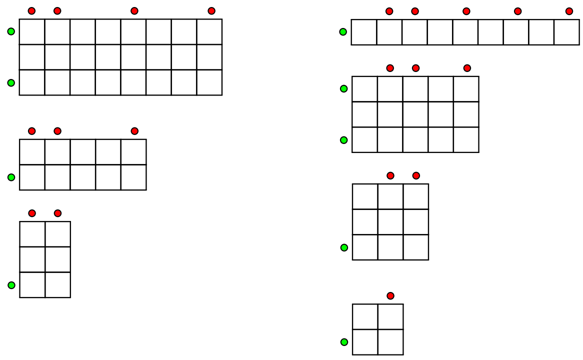
$\widehat{p_1}$



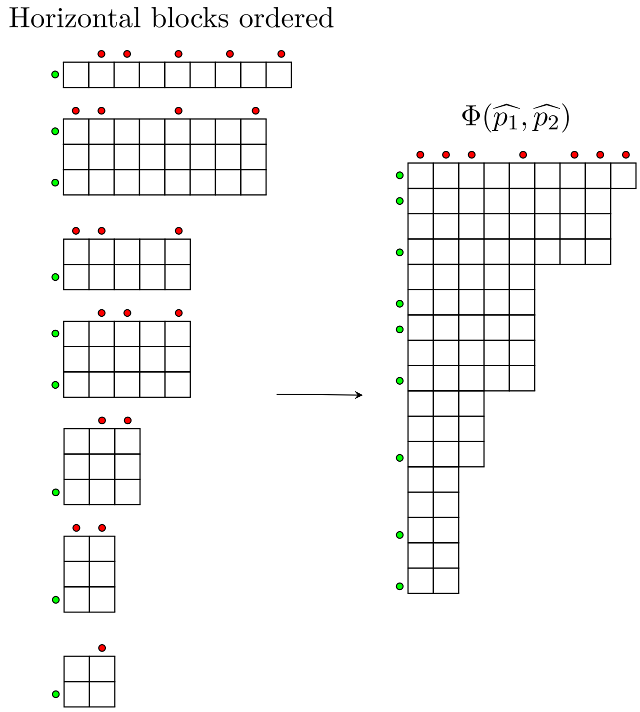
$\widehat{p_2}$



We slice them into their horizontal blocks.



Following the definition of $\Phi$, we obtain $\Phi(\widehat{p_1}, \widehat{p_2})$ after sorting the horizontal blocks as the following marked partition:




**Definition 3.8.** (i) For $\mathtt{W}_1, \mathtt{W}_2, \mathtt{W} \in \mathcal{U}^{*,\circ}$, we set $m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}} \in \mathbb{Z}_{\geq 0}$ to be the multiplicity of $\mathtt{W}$ in $\mathtt{W}_1 * \mathtt{W}_2$, i.e., to be the unique integer satisfying

$$\mathtt{W}_1 * \mathtt{W}_2 = \sum_{\mathtt{W} \in \mathcal{U}^{*,\circ}} m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}} \mathtt{W}.$$

(ii) For $\mathtt{W}_1, \mathtt{W}_2, \mathtt{W} \in \mathcal{U}^{*,\circ}$ and $\widehat{p} \in \mathcal{MP}_{\mathtt{W}}$, we define

$$m_{\mathtt{W}_1,\mathtt{W}_2;\widehat{p}} := \#\left\{(\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2} \mid \Phi(\widehat{p_1}, \widehat{p_2}) = \widehat{p}\right\}.$$

Note that, for fixed $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^{*,\circ}$, almost all $m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}}$ are zero.

**Statement of results.** The main result of this paper states how the stuffle product can be interpreted combinatorially using marked partitions.

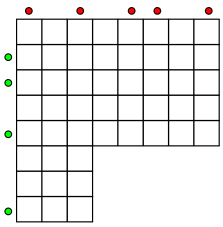
**Theorem 3.9** (Theorem 3.17)**.** *Let be* $\mathtt{W}_1, \mathtt{W}_2, \mathtt{W} \in \mathcal{U}^{*,\circ}$*. For all* $\widehat{p} \in \mathcal{MP}_{\mathtt{W}}$*, we have*

$$m_{\mathtt{W}_1,\mathtt{W}_2;\widehat{p}} = m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}}.$$

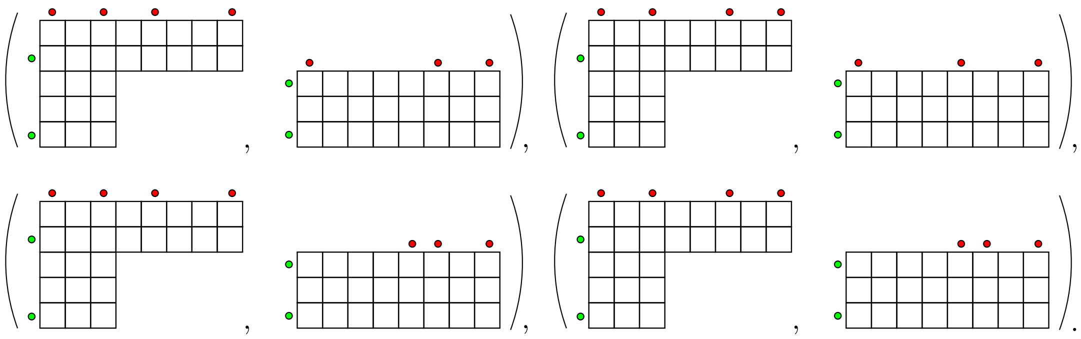
*In particular, given* $\mathtt{W}_1, \mathtt{W}_2$*,* $m_{\mathtt{W}_1,\mathtt{W}_2;\widehat{p}}$ *only depends on the word* $\mathtt{W}$ *but not on the marked partition* $\widehat{p} \in \mathcal{MP}_{\mathtt{W}}$*.*

Remarkable about Theorem 3.9 is that now, conjecturally, all linear relations among Multiple $q$-Zeta Values can be described combinatorially using marked partitions. This is due to Conjecture 3.1 and since duality already can be described using marked partitions, see [4]; Theorem 3.9 now gives the combinatorial interpretation of the stuffle product using marked partitions.

**Example 3.10.** Let be $\mathtt{W}_1 = u_1u_0u_1u_0$, $\mathtt{W}_2 = u_2u_0u_0$, and $\mathtt{W} = u_3u_0u_0u_1u_0$. Note that we have $m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}} = 4$. Furthermore, let be

$\widehat{p} =$  $\in \mathcal{MP}_{\mathtt{W}}$.

The $(\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2}$ satisfying $\Phi(\widehat{p_1}, \widehat{p_2}) = \widehat{p}$ are



In particular, the claim of Theorem 3.9 in this case is true since we have

$$m_{\mathtt{W}_1,\mathtt{W}_2;\widehat{p}} = 4 = m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}}.$$

**Organization of the paper.** In Section 3.2, we consider a recursion of the stuffle product. This will be the key for proving the main theorem in Section 3.3 where we show that the numbers $m_{\mathtt{W}_1,\mathtt{W}_2;\widehat{p}}$ and $m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}}$ satisfy the same recursion.

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## 3.2 About the stuffle product

Note the following characterization of the stuffle product, which is equivalent to the definition of the stuffle product.

**Proposition 3.11.** *Let be* $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^*$ *and* $j_1, j_2 \in \mathbb{Z}_{\geq 0}$. *Then,*

$$\mathtt{W}_1 u_{j_1} * \mathtt{W}_2 u_{j_2} = (\mathtt{W}_1 * \mathtt{W}_2 u_{j_2})\, u_{j_1} + (\mathtt{W}_1 u_{j_1} * \mathtt{W}_2)\, u_{j_2} + (\mathtt{W}_1 * \mathtt{W}_2)\, u_{j_1+j_2}.$$

*Proof.* The proof is obtained by induction on $\mathrm{len}(\mathtt{W}_1) + \mathrm{len}(\mathtt{W}_2)$ where one uses the definition of the stuffle product in the induction step. $\square$

In preparation for proving Theorem 3.9, we need the following recursion the stuffle product satisfies.

**Lemma 3.12.** *Let be* $\mathtt{W}_1'$, $\mathtt{W}_2' \in \mathcal{U}^*$, $j_1, j_2 \in \mathbb{Z}_{>0}$, *and* $n_1, n_2 \in \mathbb{Z}_{\geq 0}$. *Consider*

$$\mathtt{W}_1 = \mathtt{W}_1' u_{j_1} u_0^{n_1}, \quad \mathtt{W}_2 = \mathtt{W}_2' u_{j_2} u_0^{n_2}.$$

*We have*

$$\begin{aligned}
\mathtt{W}_1 * \mathtt{W}_2 = & \sum_{\substack{0 \leq k \leq j \leq n_2 \\ 0 \leq \varepsilon \leq \min\{1, n_2 - j\}}} \binom{n_1 + k}{n_1} \binom{n_1}{j-k} \left(\mathtt{W}_1' * \mathtt{W}_2' u_{j_2} u_0^{n_2 - j - \varepsilon}\right) u_{j_1} u_0^{n_1 + k} \\
& + \sum_{\substack{0 \leq k \leq j \leq n_1 \\ 0 \leq \varepsilon \leq \min\{1, n_1 - j\}}} \binom{n_2 + k}{n_2} \binom{n_2}{j-k} \left(\mathtt{W}_1' u_{j_1} u_0^{n_1 - j - \varepsilon} * \mathtt{W}_2'\right) u_{j_2} u_0^{n_2 + k} \\
& + \sum_{k=0}^{n_2} \binom{n_1 + k}{n_1} \binom{n_1}{n_2 - k} (\mathtt{W}_1' * \mathtt{W}_2') u_{j_1 + j_2} u_0^{n_1 + k}.
\end{aligned}$$

*Proof.* We first prove the statement for $n_1 = 0$ by induction on $n_2$. Note that we have to show for $n_1 = 0$ and $n_2 \in \mathbb{Z}_{\geq 0}$ that

$$\begin{aligned}
\mathtt{W}_1' u_{j_1} * \mathtt{W}_2' u_{j_2} u_0^{n_2} = & \sum_{\substack{0 \leq j \leq n_2 \\ 0 \leq \varepsilon \leq \min\{1, n_2 - j\}}} \left(\mathtt{W}_1' * \mathtt{W}_2' u_{j_2} u_0^{n_2 - j - \varepsilon}\right) u_{j_1} u_0^{j} \\
& + (\mathtt{W}_1' u_{j_1} * \mathtt{W}_2')\, u_{j_2} u_0^{n_2} + (\mathtt{W}_1' * \mathtt{W}_2')\, u_{j_1 + j_2} u_0^{n_2}.
\end{aligned}$$

The statement for the base case $n_2 = 0$ reduces to the equivalent definition of the stuffle product of $\mathtt{W}_1 * \mathtt{W}_2$, which is deduced from Proposition 3.11, and hence, the claim follows in this case. Therefore, let be $n_2 \in \mathbb{Z}_{>0}$ and assume that the claim holds for $n_1 = 0$ and all smaller values of $n_2$. Then, by Proposition 3.11, we have

$$\begin{aligned}
\mathtt{W}_1 * \mathtt{W}_2 = & \mathtt{W}_1' u_{j_1} * \mathtt{W}_2' u_{j_2} u_0^{n_2} \\
= & (\mathtt{W}_1' * \mathtt{W}_2' u_{j_2} u_0^{n_2})\, u_{j_1} + \left(\mathtt{W}_1' u_{j_1} * \mathtt{W}_2' u_{j_2} u_0^{n_2 - 1}\right) u_0 + \left(\mathtt{W}_1' * \mathtt{W}_2' u_{j_2} u_0^{n_2 - 1}\right) u_{j_1}.
\end{aligned}$$

Using the induction hypothesis for the second summand, we obtain

$$\begin{aligned}
\mathtt{W}_1 * \mathtt{W}_2 = & (\mathtt{W}_1' * \mathtt{W}_2' u_{j_2} u_0^{n_2})\, u_{j_1} + \left(\mathtt{W}_1' * \mathtt{W}_2' u_{j_2} u_0^{n_2 - 1}\right) u_{j_1} \\
& + \left( \sum_{\substack{0 \leq j \leq n_2 - 1 \\ 0 \leq \varepsilon \leq \min\{1, n_2 - 1 - j\}}} \left(\mathtt{W}_1' * \mathtt{W}_2' u_{j_2} u_0^{n_2 - j - 1 - \varepsilon}\right) u_{j_1} u_0^{j} \right. \\
& \left. + (\mathtt{W}_1' u_{j_1} * \mathtt{W}_2')\, u_{j_2} u_0^{n_2 - 1} + (\mathtt{W}_1' * \mathtt{W}_2')\, u_{j_1 + j_2} u_0^{n_2 - 1} \right) u_0
\end{aligned}$$

$$= \sum_{\substack{0\le j\le n_2\\0\le\varepsilon\le\min\{1,n_2-j\}}} \left(\mathtt{W}_1' * \mathtt{W}_2' u_{j_2} u_0^{n_2-j-\varepsilon}\right) u_{j_1} u_0^{j} + \left(\mathtt{W}_1' u_{j_1} * \mathtt{W}_2'\right) u_{j_2} u_0^{n_2} + \left(\mathtt{W}_1' * \mathtt{W}_2'\right) u_{j_1+j_2} u_0^{n_2},$$

completing the induction step. Next, for $n_1 \in \mathbb{Z}_{\ge 0}$ and $n_2 = 0$, the proof follows similarly by induction on $n_1$.

For the remaining cases $n_1, n_2 \in \mathbb{Z}_{>0}$, we prove by induction on $n_1+n_2$ (with already proven base case $n_1+n_2=1$). Hence, fix $n_1, n_2 \in \mathbb{Z}_{>0}$ and assume that the claim holds for all smaller values of $n_1+n_2$. We have, by Proposition 3.11,

$$\begin{aligned}
&\mathtt{W}_1 * \mathtt{W}_2\\
&=\mathtt{W}_1' u_{j_1} u_0^{n_1} * \mathtt{W}_2' u_{j_2} u_0^{n_2}\\
&= \left(\mathtt{W}_1' u_{j_1} u_0^{n_1-1} * \mathtt{W}_2' u_{j_2} u_0^{n_2} + \mathtt{W}_1' u_{j_1} u_0^{n_1} * \mathtt{W}_2' u_{j_2} u_0^{n_2-1} + \mathtt{W}_1' u_{j_1} u_0^{n_1-1} * \mathtt{W}_2' u_{j_2} u_0^{n_2-1}\right) u_0.
\end{aligned}$$

Applying the induction hypothesis for each of the three summands, we obtain

$$\begin{aligned}
&\left(\mathtt{W}_1' u_{j_1} u_0^{n_1-1} * \mathtt{W}_2' u_{j_2} u_0^{n_2}\right) u_0\\
&= \sum_{\substack{0\le k\le j\le n_2\\0\le\varepsilon\le\min\{1,n_2-j\}}} \binom{n_1-1+k}{n_1-1}\binom{n_1-1}{j-k} \left(\mathtt{W}_1' * \mathtt{W}_2' u_{j_2} u_0^{n_2-j-\varepsilon}\right) u_{j_1} u_0^{n_1+k}\\
&\quad+ \sum_{\substack{0\le k\le j\le n_1-1\\0\le\varepsilon\le\min\{1,n_1-1-j\}}} \binom{n_2+k}{n_2}\binom{n_2}{j-k} \left(\mathtt{W}_1' u_{j_1} u_0^{n_1-1-j-\varepsilon} * \mathtt{W}_2'\right) u_{j_2} u_0^{n_2+k+1}\\
&\quad+ \sum_{k=0}^{n_2} \binom{n_1-1+k}{n_1-1}\binom{n_1-1}{n_2-k} (\mathtt{W}_1' * \mathtt{W}_2') u_{j_1+j_2} u_0^{n_1+k}\\
&= \sum_{\substack{0\le k\le j\le n_2\\0\le\varepsilon\le\min\{1,n_2-j\}}} \binom{n_1-1+k}{n_1-1}\binom{n_1-1}{j-k} \left(\mathtt{W}_1' * \mathtt{W}_2' u_{j_2} u_0^{n_2-j-\varepsilon}\right) u_{j_1} u_0^{n_1+k}\\
&\quad+ \sum_{\substack{1\le k\le j\le n_1\\0\le\varepsilon\le\min\{1,n_1-j\}}} \binom{n_2+k-1}{n_2}\binom{n_2}{j-k} \left(\mathtt{W}_1' u_{j_1} u_0^{n_1-j-\varepsilon} * \mathtt{W}_2'\right) u_{j_2} u_0^{n_2+k}\\
&\quad+ \sum_{k=0}^{n_2} \binom{n_1-1+k}{n_1-1}\binom{n_1-1}{n_2-k} (\mathtt{W}_1' * \mathtt{W}_2') u_{j_1+j_2} u_0^{n_1+k},
\end{aligned}$$

and

$$\begin{aligned}
&\left(\mathtt{W}_1' u_{j_1} u_0^{n_1} * \mathtt{W}_2' u_{j_2} u_0^{n_2-1}\right) u_0\\
&= \sum_{\substack{0\le k\le j\le n_2-1\\0\le\varepsilon\le\min\{1,n_2-1-j\}}} \binom{n_1+k}{n_1}\binom{n_1}{j-k} \left(\mathtt{W}_1' * \mathtt{W}_2' u_{j_2} u_0^{n_2-1-j-\varepsilon}\right) u_{j_1} u_0^{n_1+k+1}\\
&\quad+ \sum_{\substack{0\le k\le j\le n_1\\0\le\varepsilon\le\min\{1,n_1-j\}}} \binom{n_2-1+k}{n_2-1}\binom{n_2-1}{j-k} \left(\mathtt{W}_1' u_{j_1} u_0^{n_1-j-\varepsilon} * \mathtt{W}_2'\right) u_{j_2} u_0^{n_2+k}\\
&\quad+ \sum_{k=0}^{n_2-1} \binom{n_1+k}{n_1}\binom{n_1}{n_2-1-k} (\mathtt{W}_1' * \mathtt{W}_2') u_{j_1+j_2} u_0^{n_1+k+1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{1\le k\le j\le n_2\\ 0\le\varepsilon\le\min\{1,n_2-j\}}} \binom{n_1+k-1}{n_1}\binom{n_1}{j-k}\left(\mathtt{W}'_1 * \mathtt{W}'_2 u_{j_2} u_0^{n_2-j-\varepsilon}\right) u_{j_1} u_0^{n_1+k}\\
&\quad+ \sum_{\substack{0\le k\le j\le n_1\\ 0\le\varepsilon\le\min\{1,n_1-j\}}} \binom{n_2-1+k}{n_2-1}\binom{n_2-1}{j-k}\left(\mathtt{W}'_1 u_{j_1} u_0^{n_1-j-\varepsilon} * \mathtt{W}'_2\right) u_{j_2} u_0^{n_2+k}\\
&\quad+ \sum_{k=1}^{n_2} \binom{n_1+k-1}{n_1}\binom{n_1}{n_2-k}(\mathtt{W}'_1 * \mathtt{W}'_2) u_{j_1+j_2} u_0^{n_1+k},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\mathtt{W}'_1 u_{j_1} u_0^{n_1-1} * \mathtt{W}'_2 u_{j_2} u_0^{n_2-1}\right) u_0\\
&= \sum_{\substack{0\le k\le j\le n_2-1\\ 0\le\varepsilon\le\min\{1,n_2-1-j\}}} \binom{n_1-1+k}{n_1-1}\binom{n_1-1}{j-k}\left(\mathtt{W}'_1 * \mathtt{W}'_2 u_{j_2} u_0^{n_2-1-j-\varepsilon}\right) u_{j_1} u_0^{n_1+k}\\
&\quad+ \sum_{\substack{0\le k\le j\le n_1-1\\ 0\le\varepsilon\le\min\{1,n_1-1-j\}}} \binom{n_2-1+k}{n_2-1}\binom{n_2-1}{j-k}\left(\mathtt{W}'_1 u_{j_1} u_0^{n_1-1-j-\varepsilon} * \mathtt{W}'_2\right) u_{j_2} u_0^{n_2+k}\\
&\quad+ \sum_{k=0}^{n_2-1} \binom{n_1-1+k}{n_1-1}\binom{n_1-1}{n_2-1-k}(\mathtt{W}'_1 * \mathtt{W}'_2) u_{j_1+j_2} u_0^{n_1+k}\\
&= \sum_{\substack{0\le k< j\le n_2\\ 0\le\varepsilon\le\min\{1,n_2-j\}}} \binom{n_1-1+k}{n_1-1}\binom{n_1-1}{j-1-k}\left(\mathtt{W}'_1 * \mathtt{W}'_2 u_{j_2} u_0^{n_2-j-\varepsilon}\right) u_{j_1} u_0^{n_1+k}\\
&\quad+ \sum_{\substack{0\le k< j\le n_1\\ 0\le\varepsilon\le\min\{1,n_1-j\}}} \binom{n_2-1+k}{n_2-1}\binom{n_2-1}{j-1-k}\left(\mathtt{W}'_1 u_{j_1} u_0^{n_1-j-\varepsilon} * \mathtt{W}'_2\right) u_{j_2} u_0^{n_2+k}\\
&\quad+ \sum_{k=0}^{n_2-1} \binom{n_1-1+k}{n_1-1}\binom{n_1-1}{n_2-1-k}(\mathtt{W}'_1 * \mathtt{W}'_2) u_{j_1+j_2} u_0^{n_1+k}.
\end{aligned}
$$

Now, using the identity

$$
\binom{\ell_1-1}{\ell_2-1}\binom{\ell_2-1}{\ell_3-1} + \binom{\ell_1-1}{\ell_2}\binom{\ell_2}{\ell_3} + \binom{\ell_1-1}{\ell_2-1}\binom{\ell_2-1}{\ell_3} = \binom{\ell_1}{\ell_2}\binom{\ell_2}{\ell_3}
$$

for $\ell_1, \ell_2, \ell_3 \in \mathbb{Z}_{\ge 0}$, we obtain

$$
\begin{aligned}
&\sum_{\substack{0\le k\le j\le n_2\\ 0\le\varepsilon\le\min\{1,n_2-j\}}} \binom{n_1-1+k}{n_1-1}\binom{n_1-1}{j-k}\left(\mathtt{W}'_1 * \mathtt{W}'_2 u_{j_2} u_0^{n_2-j-\varepsilon}\right) u_{j_1} u_0^{n_1+k}\\
&+ \sum_{\substack{1\le k\le j\le n_2\\ 0\le\varepsilon\le\min\{1,n_2-j\}}} \binom{n_1+k-1}{n_1}\binom{n_1}{j-k}\left(\mathtt{W}'_1 * \mathtt{W}'_2 u_{j_2} u_0^{n_2-j-\varepsilon}\right) u_{j_1} u_0^{n_1+k}\\
&+ \sum_{\substack{0\le k< j\le n_2\\ 0\le\varepsilon\le\min\{1,n_2-j\}}} \binom{n_1-1+k}{n_1-1}\binom{n_1-1}{j-1-k}\left(\mathtt{W}'_1 * \mathtt{W}'_2 u_{j_2} u_0^{n_2-j-\varepsilon}\right) u_{j_1} u_0^{n_1+k}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\substack{0\le k\le j\le n_2\\ 0\le\varepsilon\le\min\{1,n_2-j\}}} \binom{n_1-1+k}{n_1-1}\binom{n_1-1}{j-k}\left(\mathtt{W}_1'*\mathtt{W}_2'u_{j_2}u_0^{n_2-j-\varepsilon}\right)u_{j_1}u_0^{n_1+k}\\
&\quad+\sum_{\substack{0\le k\le j\le n_2\\ 0\le\varepsilon\le\min\{1,n_2-j\}}} \binom{n_1+k-1}{n_1}\binom{n_1}{j-k}\left(\mathtt{W}_1'*\mathtt{W}_2'u_{j_2}u_0^{n_2-j-\varepsilon}\right)u_{j_1}u_0^{n_1+k}\\
&\quad+\sum_{\substack{0\le k\le j\le n_2\\ 0\le\varepsilon\le\min\{1,n_2-j\}}} \binom{n_1-1+k}{n_1-1}\binom{n_1-1}{j-1-k}\left(\mathtt{W}_1'*\mathtt{W}_2'u_{j_2}u_0^{n_2-j-\varepsilon}\right)u_{j_1}u_0^{n_1+k}\\
&= \sum_{\substack{0\le k\le j\le n_2\\ 0\le\varepsilon\le\min\{1,n_2-j\}}} \binom{n_1+k}{n_1}\binom{n_1}{j-k}\left(\mathtt{W}_1'*\mathtt{W}_2'u_{j_2}u_0^{n_2-j-\varepsilon}\right)u_{j_1}u_0^{n_1+k},
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{\substack{1\le k\le j\le n_1\\ 0\le\varepsilon\le\min\{1,n_1-j\}}} \binom{n_2+k-1}{n_2}\binom{n_2}{j-k}\left(\mathtt{W}_1'u_{j_1}u_0^{n_1-j-\varepsilon}*\mathtt{W}_2'\right)u_{j_2}u_0^{n_2+k}\\
&\quad+\sum_{\substack{0\le k\le j\le n_1\\ 0\le\varepsilon\le\min\{1,n_1-j\}}} \binom{n_2-1+k}{n_2-1}\binom{n_2-1}{j-k}\left(\mathtt{W}_1'u_{j_1}u_0^{n_1-j-\varepsilon}*\mathtt{W}_2'\right)u_{j_2}u_0^{n_2+k}\\
&\quad+\sum_{\substack{0\le k<j\le n_1\\ 0\le\varepsilon\le\min\{1,n_1-j\}}} \binom{n_2-1+k}{n_2-1}\binom{n_2-1}{j-1-k}\left(\mathtt{W}_1'u_{j_1}u_0^{n_1-j-\varepsilon}*\mathtt{W}_2'\right)u_{j_2}u_0^{n_2+k}\\
&= \sum_{\substack{0\le k\le j\le n_1\\ 0\le\varepsilon\le\min\{1,n_1-j\}}} \binom{n_2+k-1}{n_2}\binom{n_2}{j-k}\left(\mathtt{W}_1'u_{j_1}u_0^{n_1-j-\varepsilon}*\mathtt{W}_2'\right)u_{j_2}u_0^{n_2+k}\\
&\quad+\sum_{\substack{0\le k\le j\le n_1\\ 0\le\varepsilon\le\min\{1,n_1-j\}}} \binom{n_2-1+k}{n_2-1}\binom{n_2-1}{j-k}\left(\mathtt{W}_1'u_{j_1}u_0^{n_1-j-\varepsilon}*\mathtt{W}_2'\right)u_{j_2}u_0^{n_2+k}\\
&\quad+\sum_{\substack{0\le k\le j\le n_1\\ 0\le\varepsilon\le\min\{1,n_1-j\}}} \binom{n_2-1+k}{n_2-1}\binom{n_2-1}{j-1-k}\left(\mathtt{W}_1'u_{j_1}u_0^{n_1-j-\varepsilon}*\mathtt{W}_2'\right)u_{j_2}u_0^{n_2+k}\\
&= \sum_{\substack{0\le k\le j\le n_1\\ 0\le\varepsilon\le\min\{1,n_1-j\}}} \binom{n_2+k}{n_2}\binom{n_2}{j-k}\left(\mathtt{W}_1'u_{j_1}u_0^{n_1-j-\varepsilon}*\mathtt{W}_2'\right)u_{j_2}u_0^{n_2+k},
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=0}^{n_2}\binom{n_1-1+k}{n_1-1}\binom{n_1-1}{n_2-k}(\mathtt{W}_1'*\mathtt{W}_2')u_{j_1+j_2}u_0^{n_1+k}\\
&+\sum_{k=1}^{n_2}\binom{n_1+k-1}{n_1}\binom{n_1}{n_2-k}(\mathtt{W}_1'*\mathtt{W}_2')u_{j_1+j_2}u_0^{n_1+k}\\
&+\sum_{k=0}^{n_2-1}\binom{n_1-1+k}{n_1-1}\binom{n_1-1}{n_2-1-k}(\mathtt{W}_1'*\mathtt{W}_2')u_{j_1+j_2}u_0^{n_1+k}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=0}^{n_2} \binom{n_1-1+k}{n_1-1}\binom{n_1-1}{n_2-k} (\mathtt{W}_1' * \mathtt{W}_2') u_{j_1+j_2} u_0^{n_1+k} \\
&\quad + \sum_{k=0}^{n_2} \binom{n_1+k-1}{n_1}\binom{n_1}{n_2-k} (\mathtt{W}_1' * \mathtt{W}_2') u_{j_1+j_2} u_0^{n_1+k} \\
&\quad + \sum_{k=0}^{n_2} \binom{n_1-1+k}{n_1-1}\binom{n_1-1}{n_2-1-k} (\mathtt{W}_1' * \mathtt{W}_2') u_{j_1+j_2} u_0^{n_1+k} \\
&= \sum_{k=0}^{n_2} \binom{n_1+k}{n_1}\binom{n_1}{n_2-k} (\mathtt{W}_1' * \mathtt{W}_2') u_{j_1+j_2} u_0^{n_1+k}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\mathtt{W}_1 * \mathtt{W}_2 \\
&= \left(\mathtt{W}_1' u_{j_1} u_0^{n_1-1} * \mathtt{W}_2' u_{j_2} u_0^{n_2}\right) u_0 + \left(\mathtt{W}_1' u_{j_1} u_0^{n_1} * \mathtt{W}_2' u_{j_2} u_0^{n_2-1}\right) u_0 \\
&\quad + \left(\mathtt{W}_1' u_{j_1} u_0^{n_1-1} * \mathtt{W}_2' u_{j_2} u_0^{n_2-1}\right) u_0 \\
&= \sum_{\substack{0\le k\le j\le n_2 \\ 0\le \varepsilon\le \min\{1,n_2-j\}}} \binom{n_1+k}{n_1}\binom{n_1}{j-k} \left(\mathtt{W}_1' * \mathtt{W}_2' u_{j_2} u_0^{n_2-j-\varepsilon}\right) u_{j_1} u_0^{n_1+k} \\
&\quad + \sum_{\substack{0\le k\le j\le n_1 \\ 0\le \varepsilon\le \min\{1,n_1-j\}}} \binom{n_2+k}{n_2}\binom{n_2}{j-k} \left(\mathtt{W}_1' u_{j_1} u_0^{n_1-j-\varepsilon} * \mathtt{W}_2'\right) u_{j_2} u_0^{n_2+k} \\
&\quad + \sum_{k=0}^{n_2} \binom{n_1+k}{n_1}\binom{n_1}{n_2-k} (\mathtt{W}_1' * \mathtt{W}_2') u_{j_1+j_2} u_0^{n_1+k},
\end{aligned}
$$

completing the induction step and providing proof of the lemma. □

We write in the following

$$
\delta_\bullet := \begin{cases} 1, & \text{if } \bullet \text{ is true,} \\ 0, & \text{if } \bullet \text{ is false} \end{cases}
$$

for the Kronecker Delta, as usual. Using Lemma 3.12, we obtain the following recursion for the numbers $m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}}$.

**Proposition 3.13.** *Let be* $\mathtt{W}_1, \mathtt{W}_2, \mathtt{W} \in \mathcal{U}^{*,\circ}$.

*(i) If* $\mathtt{W}_1 = \mathbf{1}$, *we have* $m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}} = \delta_{\mathtt{W}=\mathtt{W}_2}$.

*(ii) If* $\mathtt{W}_2 = \mathbf{1}$, *we have* $m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}} = \delta_{\mathtt{W}=\mathtt{W}_1}$.

*(iii) If* $\mathtt{W} = \mathbf{1}$, *we have* $m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}} = \delta_{\mathtt{W}_1=\mathtt{W}_2=\mathbf{1}}$.

*(iv) If* $\mathtt{W}_1, \mathtt{W}_2, \mathtt{W} \neq \mathbf{1}$, *write*

$$
\mathtt{W}_1 = \mathtt{W}_1' u_{j_1} u_0^{n_1}, \quad \mathtt{W}_2 = \mathtt{W}_2' u_{j_2} u_0^{n_2}, \quad \mathtt{W} = \mathtt{W}' u_{j_3} u_0^{n_3}
$$

*with unique* $\mathtt{W}_1', \mathtt{W}_2', \mathtt{W}' \in \mathcal{U}^{*,\circ}$, $j_1, j_2, j_3 \in \mathbb{Z}_{>0}$, *and* $n_1, n_2, n_3 \in \mathbb{Z}_{\ge 0}$. *Then,*

$$
m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}} = \sum_{\substack{0\le k\le j\le n_2 \\ 0\le \varepsilon\le \min\{1,n_2-j\}}} \binom{n_1+k}{n_1}\binom{n_1}{j-k} m_{\mathtt{W}_1',\mathtt{W}_2' u_{j_2} u_0^{n_2-j-\varepsilon};\mathtt{W}'} \delta_{\substack{j_1=j_3, \\ n_1+k=n_3}}
$$

$$+ \sum_{\substack{0\le k\le j\le n_1\\ 0\le\varepsilon\le\min\{1,n_1-j\}}} \binom{n_2+k}{n_2}\binom{n_2}{j-k} m_{\mathtt{W}_1'u_{j_1}u_0^{n_1-j-\varepsilon},\mathtt{W}_2';\mathtt{W}'}\,\delta_{\substack{j_2=j_3,\\ n_2+k=n_3}}$$
$$+ \sum_{k=0}^{n_2} \binom{n_1+k}{n_1}\binom{n_1}{n_2-k} m_{\mathtt{W}_1',\mathtt{W}_2';\mathtt{W}'}\,\delta_{\substack{j_1+j_2=j_3,\\ n_1+k=n_3}}.$$

*Proof.* While (i), (ii), and (iii) are evident following the definition of the stuffle product, (iv) is an immediate consequence of Lemma 3.12. □

## 3.3 Proof of our main theorem

After we have shown the recursion for the stuffle product in Lemma 3.12, we can now prove our main theorem. The idea is relatively simple: We show combinatorially that the numbers $m_{\mathtt{W}_1,\mathtt{W}_2;\widehat{p}}$ satisfy the same recursion as the numbers $m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}}$ such that the claim will follow by induction on depth($\mathtt{W}$). First, we need some notion to clarify our combinatorial arguments in the proof of the main theorem.

**Definition 3.14.** Let be $\mathtt{W}\in\mathcal{U}^{*,\circ}$ and $\widehat{p}\in\mathcal{MP}_{\mathtt{W}}$.

(i) We denote by $\mathrm{sm}(\widehat{p})$ the minimal length of the parts of $\widehat{p}$.

(ii) We denote by $\mathrm{C}(\widehat{p})\subset\{1,\dots,\mathrm{sm}(\widehat{p})\}$ the column markings of $\widehat{p}$ of columns that occur in the horizontal block of minimal length of $\widehat{p}$, i.e., $j\in\mathrm{C}(\widehat{p})$ if and only if the $j$-th leftmost column in $\widehat{p}$ has a marking and $j\le\mathrm{sm}(\widehat{p})$.

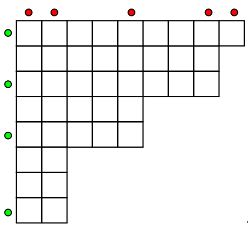
(iii) We denote by $(\widehat{p})_{-1}$ the marked partition arising from $\widehat{p}$ when removing from $\widehat{p}$ the horizontal block of minimal length $\mathrm{sm}(\widehat{p})$ and the corresponding row markings and all column markings from the $j$-th leftmost column if $j\in\mathrm{C}(\widehat{p})$.

(iv) We denote by $(\widehat{p})_1$ the marked partition arising from $\widehat{p}$ as the horizontal block of minimal length $\mathrm{sm}(\widehat{p})$ of $\widehat{p}$ by keeping the row markings and with a column marking in the $j$-th leftmost column if and only if $j\in\mathrm{C}(\widehat{p})$.

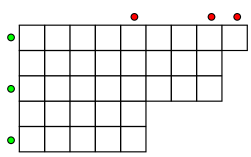
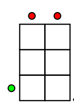
Note that we will use sometimes the phrase that "$m$ is column marking of $\widehat{p}$" when meaning that the $m$-th column of $\widehat{p}$ is marked.

Let us consider an example towards the notation from Definition 3.14.

**Example 3.15.** Consider

$\widehat{p} =$  .

Using the notation from Definition 3.14, we have $\mathrm{sm}(\widehat{p})=2$ and $\mathrm{C}(\widehat{p})=\{1,2\}$. Furthermore, we have

$(\widehat{p})_{-1} =$  and $(\widehat{p})_1 =$ .

**Remark 3.16.** (i) For all $\mathtt{W} \in \mathcal{U}^{*,\circ}$ and $\widehat{p} \in \mathcal{MP}_{\mathtt{W}}$, we have

$$\widehat{p} = \Phi((\widehat{p})_{-1}, (\widehat{p})_1).$$

(ii) For $\mathtt{W} \in \mathcal{U}^{*,\circ}$ with $\operatorname{depth}(\mathtt{W}) \geq 1$, write $\mathtt{W} = \mathtt{W}' u_j u_0^n$ with $\mathtt{W}' \in \mathcal{U}^{*,\circ}$, $j \in \mathbb{Z}_{>0}$, and $n \in \mathbb{Z}_{\geq 0}$ uniquely determined. For all $\widehat{p} \in \mathcal{MP}_{\mathtt{W}}$, we have $(\widehat{p})_{-1} \in \mathcal{MP}_{\mathtt{W}'}$.

With the additional notation from Definition 3.14, we are now ready to prove our main theorem stating that $\Phi$ describes the stuffle product on the level of marked partitions.

**Theorem 3.17** (Theorem 3.9)**.** *Let be $\mathtt{W}_1, \mathtt{W}_2, \mathtt{W} \in \mathcal{U}^{*,\circ}$. For all $\widehat{p} \in \mathcal{MP}_{\mathtt{W}}$, we have*

$$m_{\mathtt{W}_1,\mathtt{W}_2;\widehat{p}} = m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}}.$$

*In particular, given $\mathtt{W}_1, \mathtt{W}_2$, $m_{\mathtt{W}_1,\mathtt{W}_2;\widehat{p}}$ only depends on the word $\mathtt{W}$ but not on the marked partition $\widehat{p} \in \mathcal{MP}_{\mathtt{W}}$.*

*Proof.* We begin with the three special cases $\mathtt{W}_1 = \mathbf{1}$, $\mathtt{W}_2 = \mathbf{1}$, $\mathtt{W} = \mathbf{1}$. First, if $\mathtt{W}_1 = \mathbf{1}$, we note that

$$\mathtt{W}_1 * \mathtt{W}_2 = \mathbf{1} * \mathtt{W}_2 = \mathtt{W}_2,$$

i.e., for all $\mathtt{W} \in \mathcal{U}^{*,\circ}$, by Proposition 3.13(i), we have

$$m_{\mathbf{1},\mathtt{W}_2;\mathtt{W}} = \delta_{\mathtt{W}=\mathtt{W}_2}.$$

Furthermore, we have $\mathcal{MP}_{\mathtt{W}_1} = \mathcal{MP}_{\mathbf{1}} = \{\emptyset\}$. Hence, for all $\mathtt{W} \in \mathcal{U}^{*,\circ}$ and $\widehat{p} \in \mathcal{MP}_{\mathtt{W}}$, we have

$$\begin{aligned}
m_{\mathtt{W}_1,\mathtt{W}_2;\widehat{p}} &= \#\left\{(\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2} \mid \Phi(\widehat{p_1}, \widehat{p_2}) = \widehat{p}\right\} \\
&= \#\left\{(\emptyset, \widehat{p_2}) \in \mathcal{MP}_{\mathbf{1}} \times \mathcal{MP}_{\mathtt{W}_2} \mid \Phi(\emptyset, \widehat{p_2}) = \widehat{p}\right\} \\
&= \#\left\{(\emptyset, \widehat{p_2}) \in \mathcal{MP}_{\mathbf{1}} \times \mathcal{MP}_{\mathtt{W}_2} \mid \widehat{p_2} = \widehat{p}\right\} \\
&= \delta_{\mathtt{W}=\mathtt{W}_2} \\
&= m_{\mathtt{W}_1,\mathtt{W}_2;\mathtt{W}}.
\end{aligned}$$

I.e., if $\mathtt{W}_1 = \mathbf{1}$, the claim follows. Similarly (using Proposition 3.13(ii)), we obtain the claim for $\mathtt{W}_2 = \mathbf{1}$. Next, consider the special case of $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^{*,\circ}$ arbitrary and $\mathtt{W} = \mathbf{1}$. Then,

$$m_{\mathtt{W}_1,\mathtt{W}_2;\mathbf{1}} = \delta_{\mathtt{W}_1=\mathtt{W}_2=\mathbf{1}}.$$

Furthermore, we have $\mathcal{MP}_{\mathtt{W}} = \mathcal{MP}_{\mathbf{1}} = \{\emptyset\}$ and so

$$\begin{aligned}
m_{\mathtt{W}_1,\mathtt{W}_2;\emptyset} &= \#\left\{(\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2} \mid \Phi(\widehat{p_1}, \widehat{p_2}) = \emptyset\right\} \\
&= \#\left\{(\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2} \mid \widehat{p_1} = \widehat{p_2} = \emptyset\right\} \\
&= \delta_{\mathtt{W}_1=\mathtt{W}_2=\mathbf{1}} \\
&= m_{\mathtt{W}_1,\mathtt{W}_2;\mathbf{1}},
\end{aligned}$$

where the last step follows from Proposition 3.13(iii). I.e., the claim follows also for all $\mathtt{W}_1, \mathtt{W}_2 \in \mathcal{U}^{*,\circ}$ when $\mathtt{W} = \mathbf{1}$.

Therefore, we may assume $\mathtt{W}_1, \mathtt{W}_2, \mathtt{W} \neq \mathbf{1}$ in the following. We write

$$\mathtt{W}_1 = \mathtt{W}_1' u_{j_1} u_0^{n_1}, \quad \mathtt{W}_2 = \mathtt{W}_2' u_{j_2} u_0^{n_2}, \quad \mathtt{W} = \mathtt{W}' u_{j_3} u_0^{n_3},$$

where $\mathtt{W}_1', \mathtt{W}_2', \mathtt{W}' \in \mathcal{U}^{*,\circ}$, $j_1, j_2, j_3 \in \mathbb{Z}_{>0}$, and $n_1, n_2, n_3 \in \mathbb{Z}_{\geq 0}$ are uniquely determined. We prove the claim of the theorem by induction on $\operatorname{depth}(\mathtt{W})$ and note that the base case $\operatorname{depth}(\mathtt{W}) = 0$ has already been proven since then $\mathtt{W} = \mathbf{1}$. Therefore, we may assume $\operatorname{depth}(\mathtt{W}) > 0$ and that the claim holds for all smaller values of $\operatorname{depth}(\mathtt{W})$. Let be $\widehat{p} \in \mathcal{MP}_{\mathtt{W}}$ arbitrary. Particularly, in $(\widehat{p})_1$, there are exactly $j_3$ rows marked and $n_3 + 1$ columns, including the row and column, respectively, containing the corner. To obtain $m_{\mathtt{W}_1, \mathtt{W}_2; \widehat{p}}$, we need to count the pairs $(\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2}$ of marked partitions such that $\Phi(\widehat{p_1}, \widehat{p_2}) = \widehat{p}$. In particular, we have

$$(\Phi(\widehat{p_1}, \widehat{p_2}))_1 = (\widehat{p})_1.$$

There are three distinct cases we will study.

(i) $\operatorname{sm}(\widehat{p_1}) < \operatorname{sm}(\widehat{p_2})$: I.e., the horizontal block of minimal length of $\widehat{p_1}$ (neglecting the column markings) builds the whole horizontal block of minimal length of $\widehat{p}$ (neglecting the column markings).

(ii) $\operatorname{sm}(\widehat{p_1}) > \operatorname{sm}(\widehat{p_2})$: I.e., the horizontal block of minimal length of $\widehat{p_2}$ (neglecting the column markings) builds the whole horizontal block of minimal length of $\widehat{p}$ (neglecting the column markings).

(iii) $\operatorname{sm}(\widehat{p_1}) = \operatorname{sm}(\widehat{p_2})$: I.e., the horizontal block of minimal length of $\widehat{p}$ is the horizontal block of minimal length of $\widehat{p_1}$ above horizontal block of minimal length of $\widehat{p_2}$, in particular $(\widehat{p})_1 = \Phi((\widehat{p_1})_1, (\widehat{p_2})_1)$.

**Case (i).** Let us consider (i) first. We want to find the number

$$\#\left\{(\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2} \mid \operatorname{sm}(\widehat{p_1}) < \operatorname{sm}(\widehat{p_2}),\ \Phi(\widehat{p_1}, \widehat{p_2}) = \widehat{p}\right\}.$$

Note that for $(\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2}$ with $\operatorname{sm}(\widehat{p_1}) < \operatorname{sm}(\widehat{p_2})$ and $(\widehat{p_1}, \widehat{p_2}) = \widehat{p}$, we have

$$\widehat{p} = \Phi(\widehat{p_1}, \widehat{p_2}) = \Phi((\Phi(\widehat{p_1}, \widehat{p_2}))_{-1}, (\widehat{p})_1).$$

Furthermore, we have

$$(\Phi(\widehat{p_1}, \widehat{p_2}))_{-1} = \Phi\left((\widehat{p_1})_{-1}, \widetilde{\widehat{p_2}}\right) \in \mathcal{MP}_{\mathtt{W}'},$$

where $\widetilde{\widehat{p_2}}$ is the marked partition $\widehat{p_2}$ without column markings in $\mathrm{C}(\widehat{p}) \setminus \{\operatorname{sm}(\widehat{p})\}$. Note that we have

$$(\widehat{p_1})_{-1} \in \mathcal{MP}_{\mathtt{W}_1'} \quad \text{and} \quad \widetilde{\widehat{p_2}} \in \mathcal{MP}_{\mathtt{W}_2' u_{j_2} u_0^{n_2 - j - \varepsilon}},$$

where $j = \#\left((\mathrm{C}(\widehat{p}) \setminus \{\operatorname{sm}(\widehat{p})\}) \cap \mathrm{C}(\widehat{p_2})\right)$ and

$$\varepsilon = \begin{cases} 0, & \text{if } \operatorname{sm}(\widehat{p}) \notin \mathrm{C}(\widehat{p_2}), \\ 1, & \text{if } \operatorname{sm}(\widehat{p}) \in \mathrm{C}(\widehat{p_2}). \end{cases}$$

Now, fix $0 \leq j \leq n_2$, $0 \leq \varepsilon \leq \min\{1, n_2 - j\}$ and marked partitions

$$\widehat{q_1} \in \mathcal{MP}_{\mathtt{W}_1'} \quad \text{and} \quad \widetilde{\widehat{q_2}} \in \mathcal{MP}_{\mathtt{W}_2' u_{j_2} u_0^{n_2 - j - \varepsilon}}$$

such that $\mathrm{sm}\,(\widehat{p}) < \mathrm{sm}\left(\widetilde{\widehat{q_2}}\right)$ and

$$\widehat{q} := \Phi\left(\widehat{q_1}, \widetilde{\widehat{q_2}}\right) = (\widehat{p})_{-1} \in \mathcal{MP}_{\mathtt{W}'}.$$

Hence, we are interested in the number of pairs $(\widehat{q}', \widehat{q_2}) \in \mathcal{MP}_{u_{j_1}u_0^{n_1}} \times \mathcal{MP}_{\mathtt{W}_2}$ such that $\mathrm{sm}\,(\widehat{q}') < \mathrm{sm}\,(\widehat{q_2})$,

$$\widehat{p} = \Phi(\Phi(\widehat{q_1}, \widehat{q}'), \widehat{q_2}),$$

and such that $\widehat{q_2}$ without column markings in $\mathrm{C}\,(\widehat{p}) \setminus \{\mathrm{sm}\,(\widehat{p})\}$ is $\widetilde{\widehat{q_2}}$ (in accordance with the notation above). Note that the underlying Young Tableau of $\widehat{q}'$ consists of exactly one horizontal block and is uniquely determined by $(\widehat{p})_1$, as well as the row markings of $\widehat{q}'$. In particular, this is possible only if $j_1 = j_3$, and implies $\mathrm{sm}\,(\widehat{q}') = \mathrm{sm}\,(\widehat{p})$. Furthermore, since

$$\#(\mathrm{C}\,(\widehat{q}') \setminus \{\mathrm{sm}\,(\widehat{q}')\}) = n_1, \quad \#(\mathrm{C}\,(\widehat{p}) \setminus \{\mathrm{sm}\,(p)\}) = n_3, \quad \text{and} \quad \mathrm{C}\,(\widehat{q}') \subset \mathrm{C}\,(\widehat{p}),$$

we have $\binom{n_3}{n_1}$ choices to determine $\mathrm{C}\,(\widehat{q}')$ and so to determine $\widehat{q}'$. Now, for fixed $\widehat{q}'$, $\widehat{q_2}$ is determined up to the column markings in $\mathrm{C}\,(\widehat{p})$ by definition. The other $n_3 - n_1$ column markings of $\mathrm{C}\,(\widehat{p}) \setminus \{\mathrm{sm}\,(\widehat{p})\}$ have to be column markings of $\widehat{q_2}$. Moreover, by definition of $\widetilde{\widehat{q_2}}$, therefore $j - (n_3 - n_1)$ column markings of $\widehat{q_2}$ that do not belong to $\widetilde{\widehat{q_2}}$ are column markings of $\widehat{q}'$ as well, which is possible in $\binom{n_1}{j-(n_3-n_1)}$ ways, determining $\widehat{q_2}$ finally. Hence, we have proven that

$$\begin{aligned}
&\#\left\{(\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2} \mid \mathrm{sm}\,(\widehat{p_1}) < \mathrm{sm}\,(\widehat{p_2}),\ \Phi(\widehat{p_1}, \widehat{p_2}) = \widehat{p}\right\} \\
&= \sum_{\substack{0 \le j \le n_2 \\ 0 \le \varepsilon \le \min\{1, n_2 - j\}}} \binom{n_3}{n_1} \binom{n_1}{j - (n_3 - n_1)} m_{\mathtt{W}_1', \mathtt{W}_2' u_{j_2} u_0^{n_2 - j - \varepsilon}; (\widehat{p})_{-1}} \delta_{j_1 = j_3} \\
&= \sum_{\substack{0 \le j \le n_2 \\ 0 \le \varepsilon \le \min\{1, n_2 - j\}}} \binom{n_3}{n_1} \binom{n_1}{j - (n_3 - n_1)} m_{\mathtt{W}_1', \mathtt{W}_2' u_{j_2} u_0^{n_2 - j - \varepsilon}; \mathtt{W}'} \delta_{j_1 = j_3},
\end{aligned} \tag{3.17.1}$$

where the last step follows from the induction step since $\mathrm{depth}(\mathtt{W}') = \mathrm{depth}(\mathtt{W}) - 1$.

**Case (ii).** Now, considering (ii), analogously, we obtain

$$\begin{aligned}
&\#\left\{(\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2} \mid \mathrm{sm}\,(\widehat{p_1}) > \mathrm{sm}\,(\widehat{p_2}),\ \Phi(\widehat{p_1}, \widehat{p_2}) = \widehat{p}\right\} \\
&= \sum_{\substack{0 \le j \le n_1 \\ 0 \le \varepsilon \le \min\{1, n_1 - j\}}} \binom{n_3}{n_2} \binom{n_2}{j - (n_3 - n_2)} m_{\mathtt{W}_1' u_{j_1} u_0^{n_1 - j - \varepsilon}, \mathtt{W}_2'; \mathtt{W}'} \delta_{j_2 = j_3}.
\end{aligned} \tag{3.17.2}$$

**Case (iii).** We want to find the number

$$\#\left\{(\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2} \mid \mathrm{sm}\,(\widehat{p_1}) = \mathrm{sm}\,(\widehat{p_2}),\ \Phi(\widehat{p_1}, \widehat{p_2}) = \widehat{p}\right\}.$$

Note that for $(\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2}$ with $\mathrm{sm}\,(\widehat{p_1}) = \mathrm{sm}\,(\widehat{p_2})$ and $(\widehat{p_1}, \widehat{p_2}) = \widehat{p}$, we have

$$(\widehat{p})_{-1} = \Phi((\widehat{p_1})_{-1}, (\widehat{p_2})_{-1}) \quad \text{and} \quad (\widehat{p})_1 = \Phi((\widehat{p_1})_1, (\widehat{p_2})_1).$$

Furthermore, for $h \in \{1,2\}$, we have

$$(\widehat{p_h})_{-1} \in \mathcal{MP}_{\mathtt{W}'_h}, \quad (\widehat{p_h})_1 \in \mathcal{MP}_{u_{j_h} u_0^{n_h}}, \quad (\widehat{p})_{-1} \in \mathcal{MP}_{\mathtt{W}'}, \quad \text{and} \quad (\widehat{p})_1 \in \mathcal{MP}_{u_{j_3} u_0^{n_3}}.$$

Now, for $h \in \{1,2\}$, fix marked partitions $\widehat{q_h} \in \mathcal{MP}_{\mathtt{W}'_h}$ such that

$$\Phi(\widehat{q_1}, \widehat{q_2}) = (\widehat{p})_{-1}.$$

We are interested in the number of pairs $(\widehat{q_1}', \widehat{q_2}') \in \mathcal{MP}_{u_{j_1} u_0^{n_1}} \times \mathcal{MP}_{u_{j_2} u_0^{n_2}}$ such that

$$\Phi(\widehat{q_1}', \widehat{q_2}') = (\widehat{p})_1.$$

Note that the underlying Young Tableau of both $\widehat{q_1}'$ and $\widehat{q_2}'$ are uniquely determined by $(\widehat{p})_1$, as well as the row markings of $\widehat{q_1}'$ and $\widehat{q_2}'$. In particular, this is possible only if $j_3 = j_1 + j_2$, and implies $\mathrm{sm}\,(\widehat{q_1}') = \mathrm{sm}\,(\widehat{q_2}') = \mathrm{sm}\,(\widehat{p})$.

Now, since

$$\#(\mathrm{C}\,(\widehat{q_1}') \setminus \{\mathrm{sm}\,(\widehat{q_1}')\}) = n_1, \quad \#(\mathrm{C}\,(\widehat{p}) \setminus \{\mathrm{sm}\,(p)\}) = n_3, \quad \text{and} \quad \mathrm{C}\,(\widehat{q_1}') \subset \mathrm{C}\,(\widehat{p}),$$

we have $\binom{n_3}{n_1}$ choices to determine the column markings of $\widehat{q_1}'$ which determines $\widehat{q_1}'$. Furthermore, since $\mathrm{C}\,(\widehat{q_1}') \cup \mathrm{C}\,(\widehat{q_2}') = \mathrm{C}\,((\widehat{p})_1)$, we have that $n_2 - (n_3 - n_1)$ column markings of $\widehat{q_2}'$, different from $\mathrm{sm}\,(\widehat{q_2}')$, belong to $\widehat{q_1}'$ as well, which is possible in $\binom{n_1}{n_2-(n_3-n_1)}$ ways when $\widehat{q_1}'$ is already determined, determining $\widehat{q_2}'$ finally.

Hence, by using $\widehat{p} = \Phi((\widehat{p})_{-1}, (\widehat{p})_1)$, we have proven

$$\begin{aligned}
&\# \{(\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2} \mid \mathrm{sm}\,(\widehat{p_1}) = \mathrm{sm}\,(\widehat{p_2}),\ \Phi(\widehat{p_1}, \widehat{p_2}) = \widehat{p}\} \\
&= \binom{n_3}{n_1} \binom{n_1}{n_2 - (n_3 - n_1)} m_{\mathtt{W}'_1, \mathtt{W}'_2; (\widehat{p})_{-1}} \delta_{j_3 = j_1 + j_2} \\
&= \binom{n_3}{n_1} \binom{n_1}{n_2 - (n_3 - n_1)} m_{\mathtt{W}'_1, \mathtt{W}'_2; \mathtt{W}'} \delta_{j_3 = j_1 + j_2},
\end{aligned} \tag{3.17.3}$$

where the last step follows from the induction hypothesis since $\mathrm{depth}(\mathtt{W}') = \mathrm{depth}(\mathtt{W}) - 1$.

**Conclusion.** Now, (3.17.1), (3.17.2), and (3.17.3) yield by definition of $m_{\mathtt{W}_1, \mathtt{W}_2; \widehat{p}}$ that

$$\begin{aligned}
&m_{\mathtt{W}_1, \mathtt{W}_2; \widehat{p}} \\
&= \# \{(\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2} \mid \mathrm{sm}\,(\widehat{p_1}) < \mathrm{sm}\,(\widehat{p_2}),\ \Phi(\widehat{p_1}, \widehat{p_2}) = \widehat{p}\} \\
&\quad + \# \{(\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2} \mid \mathrm{sm}\,(\widehat{p_1}) > \mathrm{sm}\,(\widehat{p_2}),\ \Phi(\widehat{p_1}, \widehat{p_2}) = \widehat{p}\} \\
&\quad + \# \{(\widehat{p_1}, \widehat{p_2}) \in \mathcal{MP}_{\mathtt{W}_1} \times \mathcal{MP}_{\mathtt{W}_2} \mid \mathrm{sm}\,(\widehat{p_1}) = \mathrm{sm}\,(\widehat{p_2}),\ \Phi(\widehat{p_1}, \widehat{p_2}) = \widehat{p}\} \\
&= \sum_{\substack{0 \le k \le j \le n_2 \\ 0 \le \varepsilon \le \min\{1, n_2 - j\}}} \binom{n_1 + k}{n_1} \binom{n_1}{j - k} m_{\mathtt{W}'_1, \mathtt{W}'_2 u_{j_2} u_0^{n_2 - j - \varepsilon}; \mathtt{W}'} \delta_{\substack{j_1 = j_3, \\ n_1 + k = n_3}} \\
&\quad + \sum_{\substack{0 \le k \le j \le n_1 \\ 0 \le \varepsilon \le \min\{1, n_1 - j\}}} \binom{n_2 + k}{n_2} \binom{n_2}{j - k} m_{\mathtt{W}'_1 u_{j_1} u_0^{n_1 - j}, \mathtt{W}'_2; \mathtt{W}'} \delta_{\substack{j_2 = j_3, \\ n_2 + k = n_3}} \\
&\quad + \sum_{k=0}^{n_2} \binom{n_1 + k}{n_1} \binom{n_1}{n_2 - k} m_{\mathtt{W}'_1, \mathtt{W}'_2; \mathtt{W}'} \delta_{\substack{j_1 + j_2 = j_3, \\ n_1 + k = n_3}} \\
&= m_{\mathtt{W}_1, \mathtt{W}_2; \mathtt{W}},
\end{aligned}$$

where the last step immediately follows from Proposition 3.13(iv). This completes the induction step, so the theorem is proven. □

**Remark 3.18.** Marked partitions seem a powerful tool for studying the coefficients in the $q$-expansion of $q$MZVs. With them, we give the (algebraic) behaviour of $q$MZVs a combinatorial interpretation. In this paper, we did this for the stuffle product. In [4], we already did this for duality in the Schlesinger–Zudilin model of $q$MZVs (and in the Bradley–Zhao model for an involution similar to $\tau$). For future works, it would be interesting, for example, to describe problems like Bachmann's conjecture (bi-brackets and brackets span the same $\mathbb{Q}$-vector space, see [3, Conjecture 4.3]) combinatorially with marked partitions and making progress in proving them.

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# Chapter 4

# Paper III: Asymptotic expansions for partitions generated by infinite products

# Asymptotic expansions for partitions generated by infinite products

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**Abstract.** Recently, Debruyne and Tenenbaum proved asymptotic formulas for the number of partitions with parts in $\Lambda \subset \mathbb{N}$ ($\gcd(\Lambda) = 1$) and good analytic properties of the corresponding zeta function, generalizing work of Meinardus. In this paper, we extend their work to prove asymptotic formulas if $\Lambda$ is a multiset of integers and the zeta function has multiple poles. In particular, our results imply an asymptotic formula for the number of irreducible representations of degree $n$ of $\mathfrak{so}(5)$. We also study the Witten zeta function $\zeta_{\mathfrak{so}(5)}$, which is of independent interest.

## 4.1 Introduction and statement of results

### 4.1.1 The Circle Method

In analytic number theory and combinatorics, one uses complex analysis to better understand properties of sequences. Suppose that a sequence $(c(n))_{n\in\mathbb{N}_0}$ has moderate growth and the *generating function*

$$F(q) := \sum_{n\geq 0} c(n)q^n,$$

is holomorphic in the unit disk with radius of convergence 1. Via Cauchy's integral formula one can then recover the coefficients from the generating function

$$c(n) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{F(q)}{q^{n+1}}dq, \tag{4.0.1}$$

for any simple closed curve $\mathcal{C}$ contained in the unit disk orientated counterclockwise. The so-called Circle Method uses the analytic behavior of $F(q)$ near the boundary of the

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unit circle to obtain asymptotic information about $c(n)$. In fact for "nice" examples this method is automatic and there is a long history for example with the Prime Number Theorem. For instance, if the $c(n)$ are positive and monotonically increasing, it is expected that the part close to $q = 1$ provides the dominant contribution to (4.0.1) (Tauberian Theorems then show this). This part of the curve is the *major arc* and the complement is the *minor arc*. To obtain an asymptotic expansion for $c(n)$, one then evaluates the major arc to some degree of accuracy and bounds the minor arc. Depending on the function $F(q)$, both of these tasks present a variety of difficulties.

In the present paper, we are interested in infinite product generating functions of the form

$$F(q) = \prod_{n\geq 1} \frac{1}{(1-q^n)^{f(n)}}.$$

Such generating functions are important in the theory of partitions, but also arise, for example, in representation theory. If the Dirichlet series for $f(n)$ has a single simple pole on the positive real axis and $F$ is "bounded" away from $q = 1$, then Meinardus [30] proved an asymptotic expression for $c(n)$. Debruyne and Tenenbaum [17] eliminated the technical growth conditions on $F$ by adding a few more assumptions on the $f(n)$, which made their result more applicable. Our main results, Theorems 4.5 and 4.29, yield asymptotic expansions given mild assumptions on $f(n)$ and have a variety of new applications.

### 4.1.2 The classical partition function

Let $n \in \mathbb{N}$. A weakly decreasing sequence of positive integers that sum to $n$ is called a *partition* of $n$. The number of partitions is denoted by $p(n)$. If $\lambda_1 + \ldots + \lambda_r = n$, then the $\lambda_j$ are called the *parts* of the partition. The partition function has no elementary closed formula, nor does it satisfy any finite order recurrence. However, setting $p(0) := 1$, its generating function has the following product expansion

$$\sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} \frac{1}{1-q^n}, \tag{4.0.2}$$

where $|q| < 1$. In [23], Hardy and Ramanujan used (4.0.2) to show the asymptotic formula

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}, \qquad n \to \infty,$$

which gave birth of the Circle Method. Using modular transformations one can describe with high precision the analytic behavior of the product if $q$ is near a root of unity. One further sees directly from the infinite product that dominant singularities occur at such roots of unity with small denominator. These ideas culminate in Rademacher's exact formula for $p(n)$ [33].

With Theorem 4.5 we find, for certain constants $B_j$ and arbitrary $N \in \mathbb{N}$,

$$p(n) = \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n} \left(1 + \sum_{j=1}^{N} \frac{B_j}{n^{\frac{j}{2}}} + O_N\left(n^{-\frac{N+1}{2}}\right)\right). \tag{4.0.3}$$

Similarly, one can treat the cases for $k$-th powers (in arithmetic progressions), see [17].

### 4.1.3 Plane partitions

Another application is an asymptotic formula for plane partitions. A *plane partition of size $n$* is a two-dimensional array of non-negative integers $\pi_{j,k}$ for which $\sum_{j,k} \pi_{j,k} = n$, such that $\pi_{j,k} \geq \pi_{j,k+1}$ and $\pi_{j,k} \geq \pi_{j+1,k}$ for all $j, k \in \mathbb{N}$. We denote the number of plane partitions of $n$ by $\mathrm{pp}(n)$. MacMahon [25] proved that

$$\sum_{n\geq 0} \mathrm{pp}(n) q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)^n}.$$

Using Theorem 4.5, we recover Wright's asymptotic formula[2] [38]

$$\mathrm{pp}(n) = \frac{C}{n^{\frac{25}{36}}} e^{A_1 n^{\frac{2}{3}}} \left(1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{2(j-1)}{3}}} + O_N\left(n^{-\frac{2(N+1)}{3}}\right)\right),$$

where the constants $B_j$ are explicitly computable,

$$C := \frac{\zeta(3)^{\frac{7}{36}} e^{\zeta'(-1)}}{2^{\frac{11}{36}} \sqrt{3\pi}}, \qquad A_1 := \frac{3\zeta(3)^{\frac{1}{3}}}{2^{\frac{2}{3}}}$$

with $\zeta$ the Riemann zeta function.

### 4.1.4 Partitions into polygonal numbers

The $n$-th *$k$-gonal number* is given by[3] ($k \in \mathbb{N}_{\geq 3}$)

$$P_k(n) := \frac{1}{2}\left((k-2)n^2 + (4-k)n\right). \tag{4.0.4}$$

The study of representations of integers as sums of polygonal numbers has a long history. Fermat conjectured in 1638 that every $n \in \mathbb{N}$ may be written as the sum of at most $k$ $k$-gonal numbers which was finally proved by Cauchy. Let $p_k(n)$ denotes the number of partitions of $n$ into $k$-gonal numbers. We have the generating function

$$\sum_{n\geq 0} p_k(n) q^n = \prod_{n\geq 1} \frac{1}{1-q^{P_k(n)}}.$$

The $p_k(n)$ have the following asymptotics.[4]

**Theorem 4.1.** *We have, for all*[5] $N \in \mathbb{N}$,

$$p_k(n) = \frac{C(k) e^{A(k) n^{\frac{1}{3}}}}{n^{\frac{5k-6}{6(k-2)}}} \left(1 + \sum_{j=1}^{N} \frac{B_{j,k}}{n^{\frac{j}{3}}} + O_N\left(n^{-\frac{N+1}{3}}\right)\right),$$

*where the $B_{j,k}$ can be computed explicitly and*

$$C(k) := \frac{(k-2)^{\frac{6-k}{6(k-2)}} \Gamma\left(\frac{2}{k-2}\right) \zeta\left(\frac{3}{2}\right)^{\frac{k}{3(k-2)}}}{2^{\frac{3k-2}{2(k-2)}} \sqrt{3} \pi^{\frac{4k-9}{3(k-2)}}}, \qquad A(k) := \frac{3}{2}\left(\sqrt{\frac{\pi}{k-2}} \zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}}.$$

[2] Note the well-known typographic error in Wrights asymptotic, he is off by a factor $\sqrt{3}$.

[3] Note that these count points in polygons.

[4] Note that asymptotics for polynomial partitions were investigated in a more general setting by Dunn–Robles [19].

[5] Explicit asymptotic formulas for $p_3(n)$, $p_4(n)$, and $p_5(n)$ are given in Corollary 4.33.

**Remark 4.2.** Theorem 4.1 strengthens an asymptotic formula of Brigham for $\log(p_k(n))$ (see page 191 of [7] part D).

### 4.1.5 Numbers of finite-dimensional representations of Lie algebras

The special unitary group $\mathfrak{su}(2)$ has (up to equivalence) one irreducible representation $V_k$ of each dimension $k \in \mathbb{N}$. Each $n$-dimensional representation $\bigoplus_{k=1}^{\infty} r_k V_k$ corresponds to a unique partition

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_r, \qquad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 1 \tag{4.2.1}$$

such that $r_k$ counts the number of $k$ in (4.2.1). As a result, the number of representations equals $p(n)$. It is natural to ask whether this can be generalized. The next case is the unitary group $\mathfrak{su}(3)$, whose irreducible representations $W_{j,k}$ indexed by pairs of positive integers. Note that (see Chapter 5 of [22]) $\dim(W_{j,k}) = \frac{1}{2}jk(j+k)$. Like in the case of $\mathfrak{su}(2)$, a general $n$-dimensional representation decomposes into a sum of these $W_{j,k}$, again each with some multiplicity. So analogous to (4.0.2), the numbers $r_{\mathfrak{su}(3)}(n)$ of $n$-dimensional representations, have the generating function

$$\sum_{n\geq 0} r_{\mathfrak{su}(3)}(n) q^n = \prod_{j,k\geq 1} \frac{1}{1-q^{\frac{jk(j+k)}{2}}},$$

again with $r_{\mathfrak{su}(3)}(0) := 1$. In [34], Romik proved that, as $n \to \infty$,

$$r_{\mathfrak{su}(3)}(n) \sim \frac{C_0}{n^{\frac{3}{5}}} \exp\left(A_1 n^{\frac{2}{5}} + A_2 n^{\frac{3}{10}} + A_3 n^{\frac{1}{5}} + A_4 n^{\frac{1}{10}}\right),$$

with explicit constants[6] $C_0, A_1, \ldots, A_4$ expressible in terms of zeta and gamma values. Two of the authors [8] improved this to an analogue of formula (4.0.3), namely, for any $N \in \mathbb{N}_0$, we have

$$\begin{aligned} r_{\mathfrak{su}(3)}(n) =& \frac{C_0}{n^{\frac{3}{5}}} \exp\left(A_1 n^{\frac{2}{5}} + A_2 n^{\frac{3}{10}} + A_3 n^{\frac{1}{5}} + A_4 n^{\frac{1}{10}}\right) \\ & \times \left(1 + \sum_{j=1}^{N} \frac{C_j}{n^{\frac{j}{10}}} + O_N\left(n^{-\frac{N+1}{10}}\right)\right), \end{aligned} \tag{4.2.2}$$

as $n \to \infty$, where the constants $C_j$ do not depend on $N$ and $n$ and can be calculated explicitly. The expansion (4.2.2) with explicit values for $A_j$ $(1 \leq j \leq 4)$ and $C_0$, can also be obtained using Theorem 4.29.

This framework generalizes to other groups. For example, one can investigate the *Witten zeta function* for $\mathfrak{so}(5)$, which is (for more background to this function, see [27] and [28])

$$\zeta_{\mathfrak{so}(5)}(s) := \sum_{\varphi} \frac{1}{\dim(\varphi)^s} = 6^s \sum_{n,m\geq 1} \frac{1}{m^s n^s (m+n)^s (m+2n)^s}, \tag{4.2.3}$$

where the $\varphi$ are running through the finite-dimensional irreducible representations of $\mathfrak{so}(5)$. We prove the following; for the more precise statement see Theorem 4.43.

**Theorem 4.3.** *The function $\zeta_{\mathfrak{so}(5)}$ has a meromorphic continuation to $\mathbb{C}$ whose positive poles are simple and occur for $s \in \{\frac{1}{2}, \frac{1}{3}\}$.*

[6] Note that Romik used different signs for the constants in the exponential.

It is well-known that the finite-dimensional representations of $\mathfrak{so}(5)$ can be doubly indexed as $(\varphi_{j,k})_{j,k\in\mathbb{N}}$ with $\dim(\varphi_{j,k}) = \frac{1}{6}jk(j+k)(j+2k)$, which explains the last equality in (4.2.3). A general $n$-dimensional representation decomposes as a sum of these $\varphi_{j,k}$, each with some multiplicity. Therefore, as in the case $\mathfrak{su}(3)$, we find that

$$\sum_{n\geq 0} r_{\mathfrak{so}(5)}(n)q^n = \prod_{j,k\geq 1} \frac{1}{1-q^{\frac{jk(j+k)(j+2k)}{6}}}.$$

We prove the following.

**Theorem 4.4.** *As $n\to\infty$, we have, for any $N\in\mathbb{N}$,*

$$r_{\mathfrak{so}(5)}(n) = \frac{C}{n^{\frac{7}{12}}} \exp\left(A_1 n^{\frac{1}{3}} + A_2 n^{\frac{2}{9}} + A_3 n^{\frac{1}{9}} + A_4\right)\left(1+\sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{j-1}{9}}} + O_N\left(n^{-\frac{N+1}{9}}\right)\right),$$

*where $C$, $A_1$, $A_2$, $A_3$, and $A_4$ are given in* (4.45.6)–(4.45.7) *and the $B_j$ can be calculated explicitly.*

### 4.1.6 Statement of results

The main goal of this paper is to prove asymptotic formulas for a general class of partition functions. To state it, let $f:\mathbb{N}\to\mathbb{N}_0$, set $\Lambda := \mathbb{N}\setminus f^{-1}(\{0\})$, and for $q=e^{-z}$ ($z\in\mathbb{C}$ with $\mathrm{Re}(z)>0$), define

$$G_f(z) := \sum_{n\geq 0} p_f(n)q^n = \prod_{n\geq 1}\frac{1}{(1-q^n)^{f(n)}}, \qquad L_f(s) := \sum_{n\geq 1}\frac{f(n)}{n^s}. \tag{4.4.1}$$

We require the following key properties of these objects:

(P1) All poles of $L_f$ are real. Let $\alpha>0$ be the largest pole of $L_f$. There exists $L\in\mathbb{N}$, such that for all primes $p$, we have $|\Lambda\setminus(p\mathbb{N}\cap\Lambda)|\geq L>\frac{\alpha}{2}$.

(P2) Condition (P2) is attached to $R\in\mathbb{R}^+$. The series $L_f(s)$ converges for some $s\in\mathbb{C}$, has a meromorphic continuation to $\{s\in\mathbb{C}:\mathrm{Re}(s)\geq -R\}$, and is holomorphic on the line $\{s\in\mathbb{C}:\mathrm{Re}(s)=-R\}$. The function $L_f^*(s):=\Gamma(s)\zeta(s+1)L_f(s)$ has only real poles $0<\alpha:=\gamma_1>\gamma_2>\dots$ that are all simple, except the possible pole at $s=0$, that may be double.

(P3) For some $a<\frac{\pi}{2}$, in every strip $\sigma_1\leq\sigma\leq\sigma_2$ in the domain of holomorphicity, we uniformly have, for $s=\sigma+it$,

$$L_f(s) = O_{\sigma_1,\sigma_2}\left(e^{a|t|}\right), \qquad |t|\to\infty.$$

Note that (P1) implies that $|\Lambda\setminus(b\mathbb{N}\cap\Lambda)|\geq L>\frac{\alpha}{2}$ for all $b\geq 2$. The analytic properties of $L_f$ are a major ingredient needed to prove the following theorem, as analytic continuation in (P2) gives rise to asymptotic expansions of[7] $\mathrm{Log}(G_f(z))$ and (P3) assists with vertical integration.

**Theorem 4.5.** *Assume* (P1) *for $L\in\mathbb{N}$,* (P2) *for $R>0$, and* (P3)*. Then, for some $M,N\in\mathbb{N}$,*

$$p_f(n) = \frac{C}{n^b}\exp\left(A_1 n^{\frac{\alpha}{\alpha+1}} + \sum_{j=2}^{M} A_j n^{\alpha_j}\right)\left(1+\sum_{j=2}^{N}\frac{B_j}{n^{\beta_j}} + O_{L,R}\left(n^{-\min\left\{\frac{2L-\alpha}{2(\alpha+1)},\frac{R}{\alpha+1}\right\}}\right)\right),$$

[7]Throughout we use the principal branch of the logarithm.

*where* $0 \leq \alpha_M < \alpha_{M-1} < \cdots \alpha_2 < \alpha_1 = \frac{\alpha}{\alpha+1}$ *are given by*[8] $\mathcal{L}$ *(defined in* (4.6.1)*), and* $0 < \beta_2 < \beta_3 < \dots$ *are given by* $\mathcal{M} + \mathcal{N}$*, where* $\mathcal{M}$ *and* $\mathcal{N}$ *are defined in* (4.6.2) *and* (4.6.3)*, respectively. The coefficients* $A_j$ *and* $B_j$ *can be calculated explicitly; the constants* $A_1$*,* $C$*, and* $b$ *are provided in* (4.6.4) *and* (4.6.5)*. Moreover, if* $\alpha$ *is the only positive pole of* $L_f$*, then we have* $M = 1$*.*

**Remark 4.6.**

(1) Debruyne and Tenenbaum [17] proved Theorem 4.5 in the special case that $f$ is the indicator function of a subset $\Lambda$ of $\mathbb{N}$. They also assumed that the associated $L$-function can be analytically continued except for one pole in $0 < \alpha \leq 1$. In (P1), the assumption that $|\Lambda \setminus (p\mathbb{N} \cap \Lambda)| \geq L$ is used in Lemma 4.17 to bound minor arcs, whereas the additional assumption $L > \frac{\alpha}{2}$, that was automatically satisfied in [17], ensures that the bounds for the minor arcs are sufficient.

(2) The complexity of the exponential term depends on the number and positions of the positive poles of $L_f$. Theorem 4.29 is more explicit and covers the case of exactly two positive poles. This case has importance for representation numbers of $\mathfrak{su}(3)$ and $\mathfrak{so}(5)$.

In Section 4.2, we collect some analytic tools, properties of special functions and useful properties of asymptotic expansions that are heavily used throughout the paper. In Section 4.3, we apply the Circle Method and calculate asymptotic expansions for the saddle point $\varrho_n$ and the value of the generating function $G_f(\varrho_n)$. In Section 4.4, we complete the proof of Theorem 4.5, and we also state and prove a more explicit version of Theorem 4.5 in the case that $L_f$ has two positive poles (Theorem 4.29). The proofs of Theorems 4.1, 4.3, and 4.4 are given in Section 4.5; this includes a detailed study of the Witten zeta function $\zeta_{\mathfrak{so}(5)}$ which is of independent interest.

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## Notation

For $\beta \in \mathbb{R}$, we denote by $\{\beta\} := \beta - \lfloor \beta \rfloor$ the *fractional part* of $\beta$. As usual, we set $\mathbb{H} := \{\tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0\}$ and $\mathbb{E} := \{z \in \mathbb{C} : |z| < 1\}$. For $\delta > 0$, we define

$$\mathcal{C}_\delta := \left\{z \in \mathbb{C} \colon |\operatorname{Arg}(z)| \leq \tfrac{\pi}{2} - \delta\right\},$$

where Arg uses the principal branch of the complex argument. For $r > 0$ and $z \in \mathbb{C}$, we set

$$B_r(z) := \{w \in \mathbb{C} : |w - z| < r\}.$$

[8] We can enlarge the discrete exponent sets at will, since we can always add trivial powers with vanishing coefficients to an expansion. Therefore, from now on we always use this expression, even if the set increases tacitly.

For $a, b \in \mathbb{R}$, we let $\mathcal{R}_{a,b;K}$ be the rectangle with vertices $a \pm iK$ and $b \pm iK$, and we let $\partial\mathcal{R}_{a,b;K}$ be the path along the boundary of $\mathcal{R}_{a,b;K}$, surrounded once counterclockwise. For $-\infty \leq a < b \leq \infty$, we denote $S_{a,b} := \{z \in \mathbb{C} : a < \mathrm{Re}(z) < b\}$. We also set, for real $\sigma_1 \leq \sigma_2$ and $\delta > 0$,

$$S_{\sigma_1,\sigma_2,\delta} := \{s \in \mathbb{C} : \sigma_1 \leq \mathrm{Re}(s) \leq \sigma_2\} \Bigg\backslash \left( B_\delta\left(\frac{1}{2}\right) \cup \bigcup_{j=-\infty}^{1} B_\delta\left(\frac{j}{3}\right)\right).$$

For $k \in \mathbb{N}$ and $s \in \mathbb{C}$, the *falling factorial* is $(s)_k := s(s-1)\cdots(s-k+1)$. For $f : \mathbb{N} \to \mathbb{N}_0$, we let $\mathcal{P}$ be the set of poles of $L_f^*$, and for $R > 0$ we denote by $\mathcal{P}_R$ the union of the poles of $L_f^*$ greater than $-R$ with $\{0\}$. We define

$$\mathcal{L} := \frac{1}{\alpha+1}\mathcal{P}_R + \sum_{\mu\in\mathcal{P}_R}\left(\frac{\mu+1}{\alpha+1} - 1\right)\mathbb{N}_0, \tag{4.6.1}$$

$$\mathcal{M} := \frac{\alpha}{\alpha+1}\mathbb{N}_0 + \left(-\sum_{\mu\in\mathcal{P}_R}\left(\frac{\mu+1}{\alpha+1} - 1\right)\mathbb{N}_0\right) \cap \left[0, \frac{R+\alpha}{\alpha+1}\right), \tag{4.6.2}$$

$$\mathcal{N} := \left\{\sum_{j=1}^{K} b_j\theta_j : b_j, K \in \mathbb{N}_0, \theta_j \in (-\mathcal{L}) \cap \left(0, \frac{R}{\alpha+1}\right)\right\}. \tag{4.6.3}$$

We set, with $\omega_\alpha := \mathrm{Res}_{s=\alpha} L_f(s)$,

$$A_1 := \left(1 + \frac{1}{\alpha}\right)(\omega_\alpha\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}}, \qquad C := \frac{e^{L_f'(0)}(\omega_\alpha\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{\frac{1}{2}-L_f(0)}{\alpha+1}}}{\sqrt{2\pi(\alpha+1)}}, \tag{4.6.4}$$

$$b := \frac{1 - L_f(0) + \frac{\alpha}{2}}{\alpha+1}. \tag{4.6.5}$$

## 4.2 Preliminaries

In this section, we collect and prove some tools used in this paper.

### 4.2.1 Tools from complex analysis

We require the following results from complex analysis. The first theorem describes Taylor coefficients of the inverse of a biholomorphic function; for a proof, see Corollary 11.2 on p. 437 of [11].

**Proposition 4.7.** *Let $\phi : B_r(0) \to D$ be a holomorphic function such that $\phi(0) = 0$ and $\phi'(0) \neq 0$, with $\phi(z) =: \sum_{n\geq1} a_n z^n$. Then $\phi$ is locally biholomorphic and its local inverse of $\phi$ has a power series expansion $\phi^{-1}(w) =: \sum_{k\geq1} b_k w^k$, where*

$$b_k = \frac{1}{k a_1^k} \sum_{\substack{\ell_1,\ell_2,\ell_3\ldots\geq0 \\ \ell_1+2\ell_2+3\ell_3+\cdots=k-1}} (-1)^{\ell_1+\ell_2+\ell_3+\cdots} \frac{k\cdots(k-1+\ell_1+\ell_2+\cdots)}{\ell_1!\ell_2!\ell_3!\cdots}\left(\frac{a_2}{a_1}\right)^{\ell_1}\left(\frac{a_3}{a_1}\right)^{\ell_2}\cdots.$$

To deal with certain zeros of holomorphic functions, we require the following result from complex analysis, the proof of which is quickly obtained from Exercise 7.29 (i) in [10].

**Proposition 4.8.** *Let $r > 0$ and let $\phi_n : B_r(0) \to \mathbb{C}$ be a sequence of holomorphic functions that converges uniformly on compact sets to a holomorphic function $\phi : B_r(0) \to \mathbb{C}$, with $\phi'(0) \neq 0$. Then there exist $r > \kappa_1 > 0$ and $\kappa_2 > 0$ such that, for all $n$ sufficiently large, the restrictions $\phi_n|_{B_{\kappa_1}(0)} : B_{\kappa_1}(0) \to \phi_n(B_{\kappa_1}(0))$ are biholomorphic and $B_{\kappa_2}(0) \subset \phi_n(B_{\kappa_1}(0))$. In particular, the restrictions $\phi_n^{-1}|_{B_{\kappa_2}(0)} : B_{\kappa_2}(0) \to \phi_n^{-1}(B_{\kappa_2}(0))$ are biholomorphic functions.*

### 4.2.2 Asymptotic expansions

We require two classes of asymptotic expansions.

**Definition 4.9.** Let $R \in \mathbb{R}$.

(1) Let $g : \mathbb{R}^+ \to \mathbb{C}$ be a function. Then $g \in \mathcal{K}(R)$ if there exist real numbers $\nu_{g,1} < \nu_{g,2} < \nu_{g,3} < \cdots < \nu_{g,N} < R$ and complex numbers $a_{g,j}$ such that

$$g(x) = \sum_{j=1}^{N_g} \frac{a_{g,j}}{x^{\nu_{g,j}}} + O_R\left(x^{-R}\right), \qquad (x \to \infty).$$

(2) Let $\phi$ be holomorphic on the right half-plane. Then $\phi \in \mathcal{H}(R)$ if there are real numbers $\nu_{\phi,1} < \nu_{\phi,2} < \nu_{\phi,3} < \cdots < \nu_{\phi,N} < R$ and $a_{\phi,j} \in \mathbb{C}$ such that, for all $k \in \mathbb{N}_0$ and $0 < \delta < \frac{\pi}{2}$,

$$\phi^{(k)}(z) = \sum_{j=1}^{N_\phi} (\nu_{\phi,j})_k a_{\phi,j} z^{\nu_{\phi,j}-k} + O_{\delta,R,k}\left(|z|^{R-k}\right), \qquad (z \to 0, z \in \mathcal{C}_\delta). \tag{4.9.1}$$

If there is no risk of confusion, then we write $N$, $\nu_j$, and $a_j$ in the above. The $R$-dependence of the error only matters if $R$ varies, for instance, if we can choose it to be arbitrarily large.

Note that any sequence $g(n)$ with

$$g(n) = \sum_{j=1}^{N} \frac{a_j}{n^{\nu_j}} + O_R\left(n^{-R}\right), \qquad (n \to \infty), \tag{4.9.2}$$

can be extended to a function $g$ in $\mathcal{K}(R)$. Conversely, each function in $\mathcal{K}(R)$ can be restricted to a sequence $\{g(n)\}_{n\in\mathbb{N}}$ satisfying (4.9.2). In addition, we include functions in $\mathcal{K}(R)$ that have asymptotic expansion as in (1), but are initially defined only on intervals $(r, \infty)$ for some large $r > 0$. The reason for this is that it does not matter how the function is defined up to $r$, and therefore it can always be continued to $(0, \infty)$. If $g \in \mathcal{K}(R)$ for all $R > 0$, then we write

$$g(x) = \sum_{j\geq 1} \frac{a_j}{x^{\nu_j}}, \qquad (x \to \infty). \tag{4.9.3}$$

We use the same abbreviation if $\phi \in \mathcal{H}(R)$ for all $R > 0$. In this case we write $g \in \mathcal{K}(\infty)$ and $\phi \in \mathcal{H}(\infty)$, respectively. In some situations, we write for $R \in \mathbb{R} \cup \{\infty\}$

$$g(x) = \sum_{j=1}^{N} \frac{a_{g,j}}{x^{\nu_{g,j}}} + O_R\left(x^{-R}\right),$$

where $R$ might depend on the choice of the function $g$. If $R = \infty$, then one may ignore the error $O_R(x^{-R})$ and use the notation (4.9.3) instead. We have the following useful lemmas, that can be obtained by a straightforward calculation.

**Lemma 4.10.** *Let $R_1, R_2 \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $g \in \mathcal{K}(R_1)$, and $h \in \mathcal{K}(R_2)$. Then we have the following:*

(1) *We have $\lambda g \in \mathcal{K}(R_1)$ and $g+h \in \mathcal{K}(\min\{R_1, R_2\})$. The exponents $\nu_{g+h,j}$ run through*

$$(\{\nu_{g,j} \colon 1 \leq j \leq N_g\} \cup \{\nu_{h,j} \colon 1 \leq j \leq N_h\}) \cap (-\infty, \min\{R_1, R_2\}).$$

(2) *We have $gh \in \mathcal{K}(\min\{R_1 + \nu_{h,1}, R_2 + \nu_{g,1}\})$. The exponents $\nu_{gh,j}$ run through*

$$(\{\nu_{g,j} \colon 1 \leq j \leq N_g\} + \{\nu_{h,j} \colon 1 \leq j \leq N_h\}) \cap (-\infty, \min\{R_1 + \nu_{h,1}, R_2 + \nu_{g,1}\}).$$

We next deal with compositions of asymptotic expansions with holomorphic functions.

**Lemma 4.11.** *Let $0 < R \leq \infty$, $g \in \mathcal{K}(R)$ with $\nu_{g,1} = 0$ and $h$ holomorphic at $a_{g,1}$. Then $(h \circ g)(x)$ is defined for all $x > 0$ sufficiently large, and we have $h \circ g \in \mathcal{K}(R)$ with*

$$\{\nu_{h\circ g,j} : 1 \leq j \leq N_{h\circ g}\} = \left( \sum_{j=1}^{N_g} \nu_{g,j} \mathbb{N}_0 \right) \cap [0, R).$$

We need a similar result for general asymptotic expansions.

**Lemma 4.12.** *Let $0 < R_1, R_2 \leq \infty$, $\phi \in \mathcal{H}(R_1)$, $g \in \mathcal{K}(R_2)$, and $R := \min\{R_2 - \nu_{g,1}, \nu_{g,1} R_1\}$. Assume $\nu_{g,1} > 0$ and $g(x) > 0$ for $x$ sufficiently large. Then $\phi \circ g \in \mathcal{K}(R)$, $a_{\phi\circ g,1} = a_{\phi,1} a_{g,1}^{\nu_{\phi,1}}$, and*

$$\{\nu_{\phi\circ g,j} \colon 1 \leq j \leq N_{\phi\circ g}\} = \left( \nu_{g,1}\{\nu_{\phi,1}, ..., \nu_{\phi,N_\phi}\} + \sum_{j=2}^{N_g} (\nu_{g,j} - \nu_{g,1}) \mathbb{N}_0 \right) \cap (-\infty, R).$$

### 4.2.3 Special functions

The following theorem collects some facts about the Gamma function.

**Proposition 4.13** (see [1, 35])**.** *Let $\gamma$ denote the Euler–Mascheroni constant.*

(1) *The gamma function $\Gamma$ is holomorphic on $\mathbb{C} \setminus (-\mathbb{N}_0)$ with simple poles in $-\mathbb{N}_0$. For $n \in \mathbb{N}_0$ we have $\operatorname{Res}_{s=-n} \Gamma(s) = \frac{(-1)^n}{n!}$.*

(2) *For $s = \sigma + it \in \mathbb{C}$ with $\sigma \in I$ for a compact interval $I \subset [\frac{1}{2}, \infty)$, we uniformly have*

$$\max\left\{1, |t|^{\sigma - \frac{1}{2}}\right\} e^{-\frac{\pi|t|}{2}} \ll_I |\Gamma(s)| \ll_I \max\left\{1, |t|^{\sigma - \frac{1}{2}}\right\} e^{-\frac{\pi|t|}{2}}.$$

*The bound also holds for compact intervals $I \subset \mathbb{R}$ if $|t| \geq 1$.*

(3) *Near $s = 0$, we have the Laurent series expansion $\Gamma(s) = \frac{1}{s} - \gamma + O(s)$.*

(4) *For all $s \in \mathbb{C} \setminus \mathbb{Z}$, we have $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$.*

For $s, z \in \mathbb{C}$ with $s \notin -\mathbb{N}$, the *generalized Binomial coefficient* is defined by

$$\binom{s}{z} := \frac{\Gamma(s+1)}{\Gamma(z+1)\Gamma(s-z+1)}.$$

We require the following properties of the Riemann zeta function.

**Proposition 4.14** (see [2, 9, 35])**.**

(1) *The $\zeta$-function has a meromorphic continuation to $\mathbb{C}$ with only a simple pole at $s=1$ with residue 1. For $s \in \mathbb{C}$ we have (as identity between meromorphic functions)*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

(2) *For $I := [\sigma_0, \sigma_1]$ and $s = \sigma + it \in \mathbb{C}$, there exists $m_I \in \mathbb{Z}$, such that for $\sigma \in I$*

$$\zeta(s) \ll (1+|t|)^{m_I}, \qquad (|t| \to \infty).$$

(3) *Near $s=1$, we have the Laurent series expansion $\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$.*

For the Saddle Point Method we need the following estimate.

**Lemma 4.15.** *Let $\mu_n$ be an increasing unbounded sequence of positive real numbers, $B>0$, and $P$ a polynomial of degree $m \in \mathbb{N}_0$. Then we have*

$$\int_{-\mu_n}^{\mu_n} P(x) e^{-Bx^2} dx = \int_{-\infty}^{\infty} P(x) e^{-Bx^2} dx + O_{B,P}\left(\mu_n^{\frac{m-1}{2}} e^{-B\mu_n^2}\right).$$

Finally, we require the following in our study of the Witten zeta function $\zeta_{\mathfrak{so}(5)}$.

**Lemma 4.16.** *Let $n \in \mathbb{N}_0$. The function $g : \mathbb{R} \to \mathbb{R}$ defined as $g(u) := e^{|u|} \int_{-\infty}^{\infty} |v|^n e^{-|v|-|v+u|} dv$ satisfies $g(u) = O_n(u^{n+1})$ as $|u| \to \infty$.*

*Proof.* Let $u \geq 0$. Then we have

$$g(u) = \frac{n!}{2^{n+1}} \sum_{j=0}^{n} \frac{2^j}{j!} u^j + \frac{u^{n+1}}{n+1} + \frac{n!}{2^{n+1}} = O_n\left(u^{n+1}\right).$$

The lemma follows, since $g$ is an even function. □

## 4.3 Minor and major arcs

### 4.3.1 The minor arcs

For $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, we define, with $G_f$ given in (4.4.1),

$$\Phi_f(z) := \mathrm{Log}(G_f(z)).$$

Note that we assume throughout, that the function $f$ grows polynomially, which is implicitly part of (P2). We apply Cauchy's Theorem, writing

$$p_f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(n(\varrho_n + it) + \Phi_f(\varrho_n + it)\right) dt,$$

where $\varrho_n \to 0^+$ is determined in Subsection 4.3.3. We split the integral into two parts, the major and minor arcs, for any $\beta \geq 1$

$$p_f(n) = \frac{e^{\varrho_n n}}{2\pi} \int_{|t| \leq \varrho_n^\beta} \exp\left(int + \Phi_f(\varrho_n + it)\right) dt + \frac{e^{\varrho_n n}}{2\pi} \int_{\varrho^\beta \leq |t| \leq \pi} \exp\left(int + \Phi_f(\varrho_n + it)\right) dt. \tag{4.16.1}$$

The first integral provides the main terms in the asymptotic expansion for $p_f(n)$, the second integral is negligible, as the following lemma shows.

**Lemma 4.17.** *Let $1 < \beta < 1 + \frac{\alpha}{2}$ and assume that $f$ satisfies the conditions of Theorem 4.5. Then*

$$\int_{\frac{\varrho_n^\beta}{2\pi} \leq |t| \leq \frac{1}{2}} e^{2\pi i n t} G_f(\varrho_n + 2\pi i t) dt \ll_L \varrho_n^{L+1} G_f(\varrho_n).$$

*Sketch of proof.* The proof may be adapted from [17, Lemma 3.1]. That is, we estimate the quotient,

$$\frac{|G_f(\varrho_n + 2\pi i t)|}{G_f(\varrho_n)} \leq \prod_{m \geq 1} \left(1 + \frac{16||mt||^2}{e^{m\varrho_n} m^2 \varrho_n^2}\right)^{-\frac{f(m)}{2}},$$

where $||x||$ is the distance from $x$ to the nearest integer. We then throw away $m$-th factors depending on the location of $t \in [\frac{\varrho_n^\beta}{2\pi}, \frac{1}{2}]$. The proof follows [17, Lemma 3.1] *mutatis mutandis*; key facts are hypotheses (P1) and (P3) of Theorem 4.5 and that (which follows from [35, Theorem 7.28 (1)])

$$\sum_{1 \leq m \leq x} f(m) \sim \frac{\operatorname{Res}_{s=\alpha} L_f}{\alpha} x^\alpha. \qquad \square$$

### 4.3.2 Inverse Mellin transforms for generating functions

We start this subsection with a lemma on the asymptotic behavior of the function $\Phi_f$ near $z = 0$.

**Lemma 4.18.** *Let $f : \mathbb{N} \to \mathbb{N}_0$ satisfy (P2) with $R > 0$ and (P3). Fix some $0 < \delta < \frac{\pi}{2} - a$. Then we have, as $z \to 0$ in $C_\delta$,*

$$\Phi_f(z) = \sum_{\nu \in -\mathcal{P}_R \setminus \{0\}} \operatorname{Res}_{s=-\nu} L_f^*(s) z^\nu - L_f(0) \operatorname{Log}(z) + L_f'(0) + O_R\left(|z|^R\right).$$

*For the $k$-th derivative ($k \in \mathbb{N}$), we have*

$$\Phi_f^{(k)}(z) = \sum_{\nu \in -\mathcal{P}_R \setminus \{0\}} (\nu)_k \operatorname{Res}_{s=-\nu} L_f^*(s) z^{\nu-k} + \frac{(-1)^k (k-1)! L_f(0)}{z^k} + O_{R,k}\left(|z|^{R-k}\right).$$

*Proof.* With $J_f(s; z) := L_f^*(s) z^{-s}$, we obtain, for $\kappa \in \mathbb{N}_0$,

$$2\pi i \Phi_f^{(\kappa)}(z) = \frac{d^\kappa}{dz^\kappa}\left(\int_{-R-i\infty}^{-R+i\infty} + \lim_{K \to \infty}\left(\int_{\partial\mathcal{R}_{-R,\alpha+1;K}} + \int_{\alpha+1-iK}^{-R-iK} + \int_{-R+iK}^{\alpha+1+iK}\right)\right) J_f(s; z) ds. \tag{4.18.1}$$

Here we use (P2), giving that there are no poles of $J_f(s; z)$ on the path of integration. By Proposition 4.14 (2), [8, Theorem. 2.1 (3)], and (P3), we find a constant $c(R, \kappa)$ such that, as $|v| \to \infty$,

$$\left|L_f^*(-R + iv)\right| \ll_R (1 + |v|)^{c(R,\kappa)} e^{-\left(\frac{\pi}{2} - a\right)|v|}.$$

This yields, with Leibniz's integral rule and $0<\delta<\frac{\pi}{2}-a$,

$$\left|\frac{d^\kappa}{dz^\kappa}\int_{-R-i\infty}^{-R+i\infty} J_f(s;z)ds\right| \ll_{R,\kappa} |z|^{R-\kappa}.$$

For the second integral in (4.18.1), applying the Residue Theorem gives

$$\frac{d^\kappa}{dz^\kappa}\lim_{K\to\infty}\frac{1}{2\pi i}\int_{\partial\mathcal{R}_{-R,\alpha+1;K}} J_f(s;z)ds$$
$$=\sum_{\nu\in-\mathcal{P}_R\setminus\{0\}}(\nu)_\kappa \operatorname{Res}_{s=-\nu} L_f^*(s)z^{\nu-\kappa}+\frac{d^\kappa}{dz^\kappa}\left(-L_f(0)\operatorname{Log}(z)+L_f'(0)\right),$$

since $s=0$ is a double pole of $J_f(s;z)$. For the last two integrals in (4.18.1) we have, for some $m(I)\in\mathbb{N}_0$, depending on $I:=[-R,\alpha+1]$,

$$\left|\int_{-R\pm iK}^{\alpha+1\pm iK} J_f(s;z)ds\right| \ll_I (1+|K|)^{m(I)}\max\left\{|z|^{\alpha+1},|z|^{-R}\right\}e^{-(\delta-a)|K|},$$

which vanishes as $K\to\infty$ and thus the claim follows by distinguishing $\kappa=0$ and $\kappa\in\mathbb{N}$. $\square$

### 4.3.3 Approximation of saddle points

We now approximately solve the saddle point equations

$$-\Phi_f'(\varrho)=n=-\Phi_f'(\varrho_n). \tag{4.18.2}$$

The following proposition provides an asymptotic formula for certain functions.

**Proposition 4.19.** *Let $\phi\in\mathcal{H}(R)$ with $R>0$, $\nu_{\phi,1}<0$, and $a_{\phi,1}>0$. Assume that $\phi(\mathbb{R}^+)\subset\mathbb{R}$. Then we have the following:*

(1) *There exists a positive sequence $(\varrho_n)_{n\in\mathbb{N}}$, such that for all $n$ sufficiently large, $\phi(\varrho_n)=n$ holds.*

(2) *We have[9] $\varrho\in\mathcal{K}(1-\frac{R+1}{\nu_{\phi,1}})$, $a_{\varrho,1}=a_{\phi,1}^{-\frac{1}{\nu_{\phi,1}}}$, and the corresponding exponent set*

$$\{\nu_{\varrho,j}:1\le j\le N_\varrho\}=\left(-\frac{1}{\nu_{\phi,1}}+\sum_{j=1}^{N_\phi}\left(1-\frac{\nu_{\phi,j}}{\nu_{\phi,1}}\right)\mathbb{N}_0\right)\cap\left(-\infty,1-\frac{R+1}{\nu_{\phi,1}}\right).$$

*In particular, we have $\varrho_n\to 0^+$.*

*Proof.* In the proof we abbreviate $\nu_n:=\nu_{\phi,n}$ and $a_n:=a_{\phi,n}$.
(1) For $n\in\mathbb{N}$, set

$$\psi_n(w):=-1+\frac{1}{n}\phi\left(\left(\frac{n}{a_1}\right)^{\frac{1}{\nu_1}}w\right).$$

As $\phi$ is holomorphic on the right-half plane by assumption, so are the $\psi_n$. Using (4.9.1), write

$$\psi_n(w)=w^{\nu_1}-1+E_n(w), \tag{4.19.1}$$

[9] Recall that we can consider the sequence $\varrho_n$ as a function on $\mathbb{R}^+$.

where the error satisfies

$$E_n(w) = \frac{1}{n}\sum_{j=2}^{N_\phi} a_j \left(\frac{n}{a_1}\right)^{\frac{\nu_j}{\nu_1}} w^{\nu_j} + O_R\left(n^{\frac{R}{\nu_1}-1}|w|^R\right).$$

We next show that, for all $n$ sufficiently large, the $\psi_n$ only have one zero near $w=1$. We argue with Rouché's Theorem. First, we find that, for $n$ sufficiently large, the inequality

$$|E_n(w)| < |1-w^{\nu_1}| + |w^{\nu_1} - 1 + E_n(w)| = |1-w^{\nu_1}| + |\psi_n(w)| \tag{4.19.2}$$

holds on the entire boundary of $B_{\kappa(\nu_1)}(1)$, with $0 < \kappa(\nu_1) < \frac{1}{2}$ sufficiently small such that $w \mapsto 1-w^{\nu_1}$ only has one zero in $B_{\kappa(\nu_1)}(1)$. By Rouché's Theorem and (4.19.2), for $n$ sufficiently large $\psi_n$ also has exactly one zero in $B_{\kappa(\nu_1)}(1)$. We denote this zero of $\psi_n$ by $w_n$. It is real as $\phi$ is real-valued on the positive real line and a holomorphic function. One can show that $\varrho_n = (\frac{n}{a_1})^{\frac{1}{\nu_1}} w_n > 0$ satisfies $\phi(\varrho_n) = n$.
(2) We first give an expansion for $w_n$. By Proposition 4.8, there exists $\kappa > 0$, such that for all $n$ sufficiently large and all $z \in B_\kappa(0)$, the inverse functions $\psi_n^{-1}$ of $\psi_n$ are defined and holomorphic in $B_\kappa(1)$. Using this, we can calculate $w_n$, satisfying $\psi_n(w_n) = 0$. For this, let

$$h_n(w) := \psi_n(w+1) - \psi_n(1).$$

We have $h_n(0) = 0$, and we find, with Theorem 4.7,

$$w_n - 1 = h_n^{-1}(-\psi_n(1)) = \sum_{m\geq 1} (-1)^m b_m(n)\psi_n(1)^m,$$

where the $b_m$ can be explicitly calculated. First, $\psi_n(1)^m$ ($m \in \mathbb{N}_0$) have expansions in $n$ by (4.19.1) and Lemma 4.11. They have exponent set $\sum_{2\leq j\leq N_\phi}(1-\frac{\nu_j}{\nu_1})\mathbb{N}_0 \cap [0, 1-\frac{R}{\nu_1})$. We find, for $k \in \mathbb{N}$,

$$\psi_n^{(k)}(1) = \frac{1}{n}\sum_{j=1}^{N_\phi} (\nu_j)_k a_j \left(\frac{n}{a_1}\right)^{\frac{\nu_j}{\nu_1}} + O_R\left(n^{\frac{R}{\nu_1}-1}\right). \tag{4.19.3}$$

Again by Lemma 4.11, and (4.19.3), $\psi_n^{(k)}(1)$ ($k \in \mathbb{N}_0$) has expansions in $n$, with exponent set $(\sum_{2\leq j\leq N_\phi}(1-\frac{\nu_j}{\nu_1})\mathbb{N}_0) \cap [0, 1-\frac{R}{\nu_1})$. By Lemma 4.11 we have the following expansion in $n$

$$\psi_n'(1)^{-m} = \left(\nu_1 + \frac{1}{n}\sum_{j=2}^{N_\phi} \nu_j a_j \left(\frac{n}{a_1}\right)^{\frac{\nu_j}{\nu_1}} + O_R\left(n^{\frac{R}{\nu_1}-1}\right)\right)^{-m}$$

with exponent set $(\sum_{2\leq j\leq N_\phi}(1-\frac{\nu_j}{\nu_1})\mathbb{N}_0) \cap [0, 1-\frac{R}{\nu_1})$. By the formula in Theorem 4.7, the $b_m(n)$ are essentially sums and products of terms $\psi_n'(1)^{-1}$ and $\psi_n^{(k)}(1)$, where $k \geq 2$. Hence, $b_m(n)$ has an expansion in $n$, with exponent set $(\sum_{2\leq j\leq N_\phi}(1-\frac{\nu_j}{\nu_1})\mathbb{N}_0) \cap [0, 1-\frac{R}{\nu_1})$, and according to Proposition 4.10, the same holds for finite linear combinations $\sum_{1\leq m\leq M}(-1)^m b_m(n)\psi_n(1)^m$. As $\psi_n(1) = O(n^{\frac{\nu_2}{\nu_1}-1})$ for $n \to \infty$, one has, for $M$ sufficiently large and not depending on $n$,

$$\sum_{m\geq M+1} (-1)^m b_m(n)\psi_n(1)^m = O_R\left(n^{\frac{R}{\nu_1}-1}\right).$$

Now, as $w_n \sim 1$, we conclude the theorem recalling that $\varrho_n = (\frac{n}{a_1})^{\frac{1}{\nu_1}} w_n$. □

We next apply Theorem 4.19 to $-\Phi_f'$. For the proof one may use Corollary 4.18 with $k = 1$.

**Corollary 4.20.** *Let $\varrho_n$ solve* (4.18.2)*. Assume that $f\colon \mathbb{N} \to \mathbb{N}_0$ satisfies the conditions of Theorem 4.5. Then $\varrho \in \mathcal{K}(\frac{R}{\alpha+1}+1)$ with $a_{\varrho,1} = a_{-\Phi_f',1}^{\frac{1}{\alpha+1}} = (\omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}}$ and we have*

$$\{\nu_{\varrho,j}\colon 1 \le j \le N_\varrho\} = \left(\frac{1}{\alpha+1} - \sum_{\mu \in \mathcal{P}_R} \left(\frac{\mu+1}{\alpha+1} - 1\right)\mathbb{N}_0\right) \cap \left[\frac{1}{\alpha+1}, \frac{R}{\alpha+1}+1\right).$$

### 4.3.4 The major arcs

In this subsection we approximate, for some $1 + \frac{\alpha}{3} < \beta < 1 + \frac{\alpha}{2}$,

$$I_n := \int_{|t| \le \varrho_n^\beta} \exp(\Phi_f(\varrho_n + it) + int)dt,$$

where $\alpha$ is the largest positive pole of $L_f$. The following lemma can be shown using [17, §4].

**Lemma 4.21.** *Let $f : \mathbb{N} \to \mathbb{N}_0$ satisfy the conditions of Theorem 4.5, $\varrho_n$ solve* (4.18.2)*, and $N \in \mathbb{N}$. Then we have*

$$I_n = \sqrt{2\pi} G_f(\varrho_n) \left( \frac{1}{\sqrt{\Phi_f''(\varrho_n)}} + \sum_{2 \le k \le \frac{3H(N+\alpha)}{2\alpha}} \frac{(2k)!\lambda_{2k}(\varrho_n)}{2^k k! \Phi_f''(\varrho_n)^{k+\frac{1}{2}}} + O_N\left(\varrho_n^N\right)\right),$$

*where $H := \lceil \frac{N}{3(\beta - 1 - \frac{\alpha}{3})} \rceil + 1$ and*

$$\lambda_{2k}(\varrho) := (-1)^k \sum_{h=1}^H \frac{1}{h!} \sum_{\substack{3 \le m_1, \ldots, m_h \le \frac{3(N+\alpha)}{\alpha} \\ m_1 + \cdots + m_h = 2k}} \prod_{j=1}^h \frac{\Phi_f^{(m_j)}(\varrho)}{m_j!}.$$

The following lemma shows that the first term in Lemma 4.21 dominates the others; its proof follows with Lemma 4.12, Corollary 4.18, and Corollary 4.20 by a straightforward calculation.

**Lemma 4.22.** *Let $k \ge 2$ and assume the conditions as in Lemma 4.21. Then we have*

$$\frac{\lambda_{2k}(\varrho_n)}{\Phi_f''(\varrho_n)^{k+\frac{1}{2}}} = \sum_{j=1}^M \frac{b_j}{n^{\eta_j}} + O_R\left(n^{-R+1+\left(k - \lfloor \frac{2k}{3} \rfloor + \frac{3}{2}\right)\frac{\alpha}{\alpha+1}}\right),$$

*where the $\eta_j$ run through*

$$\frac{\alpha+2}{2(\alpha+1)} + \frac{\alpha}{\alpha+1}\mathbb{N}_0 + \left(-\sum_{\mu \in \mathcal{P}_R} \left(\frac{\mu+1}{\alpha+1} - 1\right)\mathbb{N}_0\right) \cap \left[0, \frac{R+\alpha}{\alpha+1}\right).$$

We next use Lemma 4.12 and Corollary 4.20 to give an asymptotic expansion for $G_f(\varrho_n)$.

**Lemma 4.23.** *Assume that $f : \mathbb{N} \to \mathbb{N}_0$ satisfies the conditions of Theorem 4.5. Then, we have*

$$G_f(\varrho_n) = \frac{e^{L_f'(0)} n^{\frac{L_f(0)}{\alpha+1}}}{a_{-\Phi_f',1}^{\frac{L_f(0)}{\alpha+1}}} \exp\left(\frac{1}{\alpha}(\omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}} n^{\frac{\alpha}{\alpha+1}} + \sum_{j=2}^{M} C_j n^{\beta_j}\right)$$
$$\times \left(1 + \sum_{j=1}^{N} \frac{B_j}{n^{\delta_j}} + O_R\left(n^{-\frac{R}{\alpha+1}}\right)\right),$$

*where $0 \le \beta_M < \cdots < \beta_2 < \frac{\alpha}{\alpha+1}$ run through $\mathcal{L}$ and $0 < \delta_1 < \delta_2 < \cdots < \delta_N$ through $\mathcal{M} + \mathcal{N}$.*

*Proof.* Let $\phi(z) := \Phi_f(z) + L_f(0)\mathrm{Log}(z)$ and $F := \phi \circ \varrho$. By Lemma 4.18, Proposition 4.19, and Lemma 4.12 we find that

$$\Phi_f(\varrho_n) + L_f(0)\log(\varrho_n) = L_f'(0) + \sum_{j=1}^{N_F} \frac{a_{F,j}}{n^{\nu_{F,j}}} + O_R\left(n^{-\frac{R}{\alpha+1}}\right), \tag{4.23.1}$$

where $\nu_{F,j}$ run through (the inclusion follows by Corollary 4.20)

$$\left(-\frac{1}{\alpha+1}\mathcal{P}_R + \sum_{j=2}^{N_\varrho}\left(\nu_{\varrho,j} - \frac{1}{\alpha+1}\right)\mathbb{N}_0\right) \cap \left(-\infty, \frac{R}{\alpha+1}\right)$$
$$\subset \left(-\frac{1}{\alpha+1}\mathcal{P}_R - \sum_{\mu \in \mathcal{P}_R}\left(\frac{\mu+1}{\alpha+1} - 1\right)\mathbb{N}_0\right) \cap \left(-\infty, \frac{R}{\alpha+1}\right). \tag{4.23.2}$$

Note that, again by Lemma 4.12 and Corollary 4.18, we obtain

$$a_{F,1} = a_{\phi,1} a_{\varrho,1}^{\nu_{\phi,1}} = \frac{1}{\alpha}(\omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}}.$$

We split the sum in (4.23.1) into two parts: one with nonpositive $\nu_{F,1}, \dots, \nu_{F,M}$, say, and the one with positive $\nu_{F,j} < \frac{R}{\alpha+1}$. Note that $M$ is bounded and independent of $R$. Exponentiating (4.23.1) yields

$$\exp(\Phi_f(\varrho_n)) = \varrho_n^{-L_f(0)} e^{L_f'(0)} \exp\left(\sum_{j=M+1}^{N_F} \frac{a_{F,j}}{n^{\nu_{F,j}}} + O_R\left(n^{-\frac{R}{\alpha+1}}\right)\right) \exp\left(\sum_{j=1}^{M} \frac{a_{F,j}}{n^{\nu_{F,j}}}\right).$$

Note that the positive $\nu_{F,j}$ run through (4.23.2) with $-\infty$ replaced by 0. By Lemma 4.11, we have

$$\exp\left(\sum_{j=M+1}^{N_F} \frac{a_{F,j}}{n^{\nu_{F,j}}} + O_R\left(n^{-\frac{R}{\alpha+1}}\right)\right) = 1 + \sum_{j=1}^{K} \frac{H_j}{n^{\varepsilon_j}} + O_R\left(n^{-\frac{R}{\alpha+1}}\right)$$

for some $K \in \mathbb{N}$ and with exponents $\varepsilon_j$ running through $\mathcal{N}$. Recall that, by Corollary 4.20, we have $\varrho_n \sim a_{\varrho,1} n^{-\frac{1}{\alpha+1}}$. Now set $h(n) := n^{-\frac{L_f(0)}{\alpha+1}} \varrho_n^{-L_f(0)}$.
A straightforward calculation using Corollary 4.20 shows that $h \in \mathcal{K}(\frac{R+\alpha}{\alpha+1})$ with exponent set $(-\sum_{\mu \in \mathcal{P}_R}(\frac{\mu+1}{\alpha+1} - 1)\mathbb{N}_0) \cap [0, \frac{R+\alpha}{\alpha+1}) \subset \mathcal{M}$ and $a_{h,1} = a_{-\Phi_f',1}^{-\frac{L_f(0)}{\alpha+1}}$. By Proposition 4.10 (2),

we obtain, for some $N \in \mathbb{N}$, $B_j \in \mathbb{C}$, and $\delta_j$ running through $\mathcal{M} + \mathcal{N}$,

$$h(n) \left( 1 + \sum_{j=1}^{K} \frac{H_j}{n^{\varepsilon_j}} + O_R\left(n^{-\frac{R}{\alpha+1}}\right) \right) = a_{h,1} \left( 1 + \sum_{j=1}^{N} \frac{B_j}{n^{\delta_j}} + O_R\left(n^{-\frac{R}{\alpha+1}}\right) \right).$$

Setting $C_j := a_{F,j}$ for $1 \leq j \leq M$, the lemma follows. □

Another important step for the proof of our main theorem is the following lemma.

**Lemma 4.24.** *Let $f : \mathbb{N} \to \mathbb{N}_0$ satisfy the conditions of Theorem 4.5. Then we have, as $n \to \infty$,*

$$e^{n\varrho_n} = \exp\left( (\omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}} n^{\frac{\alpha}{\alpha+1}} + \sum_{j=2}^{M} a_{\varrho,j} n^{\eta_j} \right) \left( 1 + \sum_{j=1}^{N} \frac{D_j}{n^{\mu_j}} + O_R\left(n^{-\frac{R}{\alpha+1}}\right) \right)$$

*for some $1 \leq M \leq N_\varrho$, with $\frac{\alpha}{\alpha+1} > \eta_2 > \cdots > \eta_M \geq 0$ running through $\mathcal{L}$ and the $\mu_j$ through $\mathcal{N}$.*

*Proof.* Let $g(n) := n\varrho_n$. By Corollary 4.20 we have $g \in \mathcal{K}(\frac{R}{\alpha+1})$ with exponent set

$$\{\nu_{g,j} : 1 \leq j \leq N_\varrho\} = \left( -1 + \frac{1}{\alpha+1} - \sum_{\mu \in \mathcal{P}_R} \left( \frac{\mu+1}{\alpha+1} - 1 \right) \mathbb{N}_0 \right) \cap \left[ -1 + \frac{1}{\alpha+1}, \frac{R}{\alpha+1} \right).$$

Hence, for some $1 \leq M \leq N_\varrho$, we obtain

$$e^{n\varrho_n} = \exp\left( a_{-\Phi'_f,1}^{\frac{1}{\alpha+1}} n^{\frac{\alpha}{\alpha+1}} + \sum_{j=2}^{M} \frac{a_{\varrho,j}}{n^{\nu_{g,j}}} \right) \exp\left( \sum_{j=M+1}^{N_\varrho} \frac{a_{\varrho,j}}{n^{\nu_{g,j}}} + O_R\left(n^{-\frac{R}{\alpha+1}}\right) \right)$$

with $-\frac{\alpha}{\alpha+1} < \nu_{g,2} < \cdots < \nu_{g,M} \leq 0 < \nu_{g,M+1} < \cdots < \nu_{g,N_\varrho}$. By Lemma 4.18 we obtain $a_{-\Phi'_f,1}^{\frac{1}{\alpha+1}} = (\omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}}$.
Note that the exponents $0 < \nu_{g,M+1} < \cdots < \nu_{g,N_\varrho}$ run through

$$\left( -\frac{\alpha}{\alpha+1} - \sum_{\mu \in \mathcal{P}_R} \left( \frac{\mu+1}{\alpha+1} - 1 \right) \mathbb{N}_0 \right) \cap \left( 0, \frac{R}{\alpha+1} \right).$$

By Lemma 4.11, $\exp(\sum_{j=M+1}^{N_\varrho} \frac{a_{\varrho,j}}{n^{\nu_{g,j}}} + O_R(n^{-\frac{R}{\alpha+1}}))$ is in $\mathcal{K}(\frac{R}{\alpha+1})$, with exponent set

$$\left\{ \sum_{j=1}^{K} b_j \theta_j : K, b_j \in \mathbb{N}_0,\ \theta_j \in \left( -\frac{\alpha}{\alpha+1} - \sum_{\mu \in \mathcal{P}_R} \left( \frac{\mu+1}{\alpha+1} - 1 \right) \mathbb{N}_0 \right) \cap \left( 0, \frac{R}{\alpha+1} \right) \right\}.$$

As $\alpha \in \mathcal{P}_R$, this is a subset of $\mathcal{N}$, so the above exponents are given by $\mathcal{N}$, proving the lemma. □

The following corollary is very helpful to prove our main theorem.

**Corollary 4.25.** *Let $f\colon \mathbb{N} \to \mathbb{N}_0$ satisfy the conditions of Theorem 4.5. Then we have*

$$e^{n\varrho_n} G_f(\varrho_n) = \frac{e^{L'_f(0)} n^{\frac{L_f(0)}{\alpha+1}}}{a_{-\Phi'_f,1}^{\frac{L_f(0)}{\alpha+1}}} \exp\left( A_1 n^{\frac{\alpha}{\alpha+1}} + \sum_{j=2}^{M} A_j n^{\alpha_j} \right) \left( 1 + \sum_{j=1}^{N} \frac{E_j}{n^{\eta_j}} + O_R\left(n^{-\frac{R}{\alpha+1}}\right) \right),$$

*with $A_1$ defined in* (4.6.4), $\frac{\alpha}{\alpha+1} > \alpha_2 > \cdots > \alpha_M \geq 0$ *running through* $\mathcal{L}$, *and* $\eta_j$ *through* $\mathcal{M}+\mathcal{N}$.

## 4.4 Proof of Theorem 4.5

### 4.4.1 The general case

The following lemma follows by a straightforward calculation, using (4.16.1) and Lemmas 4.17, 4.21, and 4.22.

**Lemma 4.26.** *Let $f:\mathbb{N}\to\mathbb{N}_0$ satisfy the conditions of Theorem 4.5. Then we have*

$$p_f(n) = \frac{e^{n\varrho_n} G_f(\varrho_n)}{\sqrt{2\pi}} \left( \sum_{j=1}^{M} \frac{d_j}{n^{\nu_j}} + O_{L,R}\left( n^{-\min\left\{ \frac{L+1}{\alpha+1}, \frac{R+\alpha}{\alpha+1} + \frac{\alpha+2}{2(\alpha+1)} \right\}} \right) \right)$$

*for some $M \in \mathbb{N}$, $d_1 = \frac{1}{\sqrt{\alpha+1}} (\omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{2(\alpha+1)}}$, and the $\nu_j$ run through*

$$\frac{\alpha+2}{2(\alpha+1)} + \frac{\alpha}{\alpha+1}\mathbb{N}_0 + \left( - \sum_{\mu \in \mathcal{P}_R} \left( \frac{\mu+1}{\alpha+1} - 1 \right) \mathbb{N}_0 \right) \cap \left[ 0, \frac{R+\alpha}{\alpha+1} \right).$$

*In particular, we have $\nu_1 = \frac{\alpha+2}{2(\alpha+1)}$.*

We prove the following lemma.

**Lemma 4.27.** *Assume that $f$ satisfies the conditions of Theorem 4.5 and that $L_f$ has only one positive pole $\alpha$. Then we have*

$$n\varrho_n + \Phi_f(\varrho_n) = (\omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}} \left(1+\frac{1}{\alpha}\right) n^{\frac{\alpha}{\alpha+1}} - L_f(0)\log(\varrho_n) + L_f'(0) + o(1).$$

*Proof.* By Lemma 4.18, we have

$$\Phi_f(\varrho_n) = \frac{\omega_\alpha \Gamma(\alpha)\zeta(\alpha+1)}{\varrho_n^\alpha} - L_f(0)\log(\varrho_n) + L_f'(0) + O\left(\varrho_n^{R_0}\right), \tag{4.27.1}$$

where

$$R_0 := \begin{cases} -\max\limits_{\nu \in \mathcal{P}_R \cap (-R,0)} \nu & \text{if } \mathcal{P}_R \cap (-R,0) \neq \emptyset, \\ R & \text{otherwise.} \end{cases} \tag{4.27.2}$$

To show the lemma, we need an expansion for $\varrho_n$. We have, by (4.18.2) and again by Corollary 4.18,

$$-\Phi_f'(\varrho_n) = \frac{\omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1)}{\varrho_n^{\alpha+1}} + \frac{L_f(0)}{\varrho_n} + O\left(\varrho_n^{R_0-1}\right).$$

By Corollary 4.20, we have an expansion for $\varrho_n$ with an error $o(1)$. We iteratively find the first terms. By Corollary 4.20 we have $\varrho_n \sim a_{-\Phi_f',1}^{\frac{1}{\alpha+1}} n^{-\frac{1}{\alpha+1}}$, as $n\to\infty$. We next determine the second order term in $\varrho_n = \frac{a_{-\Phi_f',1}^{\frac{1}{\alpha+1}}}{n^{\frac{1}{\alpha+1}}} + \frac{K_2}{n^{\kappa_2}} + o(n^{-\kappa_2})$ for some $\kappa_2 < \frac{1}{\alpha+1}$

and $K_2 \in \mathbb{C}$. We choose $\kappa$ in

$$n\left(1+\frac{K_2}{a_{-\Phi_f',1}^{\frac{1}{\alpha+1}} n^{\kappa_2-\frac{1}{\alpha+1}}}\right)^{-\alpha-1}+\frac{L_f(0)}{a_{-\Phi_f',1}^{\frac{1}{\alpha+1}}} n^{\frac{1}{\alpha+1}}\left(1+\frac{K_2}{a_{-\Phi_f',1}^{\frac{1}{\alpha+1}} n^{\kappa_2-\frac{1}{\alpha+1}}}\right)^{-1}=n+O(n^\kappa)$$

as small as possible. One finds that

$$\frac{(\alpha+1)K_2}{a_{-\Phi_f',1}^{\frac{1}{\alpha+1}}} n^{1-\kappa_2+\frac{1}{\alpha+1}}=\frac{L_f(0)}{a_{-\Phi_f',1}^{\frac{1}{\alpha+1}}} n^{\frac{1}{\alpha+1}},$$

and hence

$$\varrho_n=\frac{a_{-\Phi_f',1}^{\frac{1}{\alpha+1}}}{n^{\frac{1}{\alpha+1}}}+\frac{L_f(0)}{(\alpha+1)n}+o\left(\frac{1}{n}\right). \tag{4.27.3}$$

Plugging (4.27.3) into $\Phi_f$ leads, by (4.27.1), to

$$\Phi_f\left(\frac{a_{-\Phi_f',1}^{\frac{1}{\alpha+1}}}{n^{\frac{1}{\alpha+1}}}+\frac{L_f(0)}{(\alpha+1)n}+o\left(\frac{1}{n}\right)\right)=\frac{a_{-\Phi_f',1}^{\frac{1}{\alpha+1}}}{\alpha} n^{\frac{\alpha}{\alpha+1}}-\frac{L_f(0)}{\alpha+1}-L_f(0)\log(\varrho_n)+L_f'(0)+o(1).$$

As a result, using (4.27.3), we conclude the claim. □

We are now ready to prove Theorem 4.5.

*Proof of Theorem 4.5.* Corollaries 4.20 and 4.25 with Lemmas 4.26 and 4.27 give the asymptotic for $p_f(n)$. We use Lemma 4.23 to calculate the exponents and (4.6.4) as well as (4.6.5) for the constants. Throughout we use Lemma 4.10 (2) to deal with the expansions of products of functions. □

### 4.4.2 The case of two positive poles of $L_f$

If $\alpha > 0$ is the only positive pole of $L_f$, then we can calculate the single term in the exponential in the asymptotic of $p_f(n)$ explicitly, by Theorem 4.5. In this subsection we assume that $L_f$ has exactly two positive simple poles, $\alpha$ and $\beta$. In this case, Corollary 4.18 with $k = 1$ gives

$$-\Phi_f'(z)=\frac{c_1}{z^{\alpha+1}}+\frac{c_2}{z^{\beta+1}}+\frac{c_3}{z}+O_R\left(|z|^{R_0-1}\right)$$

with $R_0$ from (4.27.2). Above we set $c_j := a_{-\Phi_f',j}$ for $1 \le j \le 3$, i.e., by Lemma 4.18

$$c_1=\omega_\alpha\Gamma(\alpha+1)\zeta(\alpha+1),\quad c_2=\omega_\beta\Gamma(\beta+1)\zeta(\beta+1),\quad c_3=L_f(0). \tag{4.27.4}$$

In the next lemma, we approximate the saddle point in this special situation.

**Lemma 4.28.** *Let $f$ satisfy the conditions of Theorem 4.5. Additionally assume that $L_f$ has exactly two positive poles $\alpha$ and $\beta$ that satisfy $\frac{\ell+1}{\ell}\beta < \alpha \le \frac{\ell}{\ell-1}\beta$ for some $\ell \in \mathbb{N}$, where we treat the case $\ell = 1$ simply as $2\beta < \alpha$. Then there exists $0 < r \le \frac{R}{\alpha+1}$ such that*

$$\varrho_n=\sum_{j=1}^{\ell+1}\frac{K_j}{n^{(j-1)\left(1-\frac{\beta+1}{\alpha+1}\right)+\frac{1}{\alpha+1}}}+\frac{c_3}{(\alpha+1)n}+O_R\left(n^{-r-1}\right) \tag{4.28.1}$$

*for some constants $K_j$ independent of $n$ and $c_3$ as in* (4.27.4)*. In particular, we have*

$$K_1 = c_1^{\frac{1}{\alpha+1}}, \quad K_2 = \frac{c_2}{(\alpha+1)c_1^{\frac{\beta}{\alpha+1}}}, \quad K_3 = \frac{c_2^2(\alpha-2\beta)}{2(\alpha+1)^2 c_1^{\frac{2\beta+1}{\alpha+1}}},$$
$$K_4 = \frac{c_2^3\left(2\alpha^2-9\alpha\beta-2\alpha+9\beta^2+3\beta\right)}{6(\alpha+1)^3 c_1^{\frac{3\beta+2}{\alpha+1}}},$$
$$K_5 = \frac{c_2^4(6\alpha^3-44\alpha^2\beta-15\alpha^2+96\alpha\beta^2+56\alpha\beta+6\alpha-64\beta^3-48\beta^2-8\beta)}{24(\alpha+1)^4 c_1^{\frac{4\beta+3}{\alpha+1}}}.$$

*Proof.* By Corollary 4.20, the exponents of $\varrho_n$ that are at most 1 are given by combinations

$$\frac{1}{\alpha+1}+(j-1)\left(1-\frac{\beta+1}{\alpha+1}\right)+m\left(1-\frac{1}{\alpha+1}\right)\le 1,$$

with $j\in\mathbb{N}$ and $m\in\mathbb{N}_0$. A straightforward calculation shows that $\frac{\ell+1}{\ell}\beta<\alpha\le\frac{\ell}{\ell-1}\beta$ if and only if

$$0<\frac{1}{\alpha+1}+(j-1)\left(1-\frac{\beta+1}{\alpha+1}\right)\le 1$$

for all $1\le j\le \ell+1$ but not for $j>\ell+1$. Together with the error term induced by Corollary 4.20, (4.28.1) follows. Assuming $\ell\ge 5$, $K_1$–$K_5$ and the term $\frac{c_3}{(\alpha+1)n}$ can be determined iteratively. □

We are now ready to prove asymptotic formulas if $L_f$ has exactly two positive poles.

**Theorem 4.29.** *Assume that $f:\mathbb{N}\to\mathbb{N}_0$ satisfies the conditions of Theorem 4.5 and that $L_f$ has exactly two positive poles $\alpha>\beta$, such that $\frac{\ell+1}{\ell}\beta<\alpha\le\frac{\ell}{\ell-1}\beta$ for some $\ell\in\mathbb{N}$. Then we have*

$$p_f(n)=\frac{C}{n^b}\exp\left(A_1 n^{\frac{\alpha}{\alpha+1}}+A_2 n^{\frac{\beta}{\alpha+1}}+\sum_{k=3}^{\ell+1}A_k n^{\frac{(k-1)\beta}{\alpha+1}+\frac{k-2}{\alpha+1}+2-k}\right)$$
$$\times\left(1+\sum_{j=2}^{M_1}\frac{B_j}{n^{\nu_j}}+O_{L,R}\left(n^{-\min\left\{\frac{2L-\alpha}{2(\alpha+1)},\frac{R}{\alpha+1}\right\}}\right)\right),\qquad (n\to\infty),$$

*with*

$$A_1:=(\omega_\alpha\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}}\left(1+\frac{1}{\alpha}\right),\qquad A_2:=\frac{\omega_\beta\Gamma(\beta)\zeta(\beta+1)}{(\omega_\alpha\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{\beta}{\alpha+1}}},\quad (4.29.1)$$

*and for all $k\ge 3$*

$$A_k:=K_k+\frac{c_1^{\frac{1}{\alpha+1}}}{\alpha}\sum_{m=1}^{\ell}\binom{-\alpha}{m}\sum_{\substack{0\le j_1,\dots,j_\ell\le m\\ j_1+\dots+j_\ell=m\\ j_1+2j_2+\dots+\ell j_\ell=k-1}}\binom{m}{j_1,j_2,\dots,j_\ell}\frac{K_2^{j_1}\cdots K_{\ell+1}^{j_\ell}}{c_1^{\frac{m}{\alpha+1}}}$$
$$+\frac{c_2}{\beta c_1^{\frac{\beta}{\alpha+1}}}\sum_{m=1}^{\ell}\binom{-\beta}{m}\sum_{\substack{0\le j_1,\dots,j_\ell\le m\\ j_1+\dots+j_\ell=m\\ j_1+2j_2+\dots+\ell j_\ell=k-2}}\binom{m}{j_1,j_2,\dots,j_\ell}\frac{K_2^{j_1}\cdots K_{\ell+1}^{j_\ell}}{c_1^{\frac{m}{\alpha+1}}}.$$

*Here, $C$ and $b$ are defined in* (4.6.4) *and* (4.6.5)*, the $\nu_j$ run through $\mathcal{M}+\mathcal{N}$, the $K_j$ are given in Lemma 4.28, and $c_1$, $c_2$, and $c_3$ run through* (4.27.4)*.*

*Proof.* Assume that $g : \mathbb{N} \to \mathbb{C}$ has an asymptotic expansion as $n \to \infty$ and denote by $[g(n)]_*$ the part with nonnegative exponents. With Lemmas 4.18 and 4.26 we obtain, using that $L_f$ has exactly two positive poles in $\alpha$ and $\beta$,

$$p_f(n) = \frac{C}{n^b} \exp\left(\left[n\varrho_n + \frac{c_1}{\alpha\varrho_n^\alpha} + \frac{c_2}{\beta\varrho_n^\beta}\right]_*\right)\left(1 + \sum_{j=2}^{M_1} \frac{a_j}{n^{\delta_j}} + O_{L,R}\left(n^{-\min\left\{\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\right\}}\right)\right)$$

with the $\delta_j$ running through $\mathcal{M}$. With the Binomial Theorem and Lemma 4.28, we find

$$\frac{c_1}{\alpha\varrho_n^\alpha} = \frac{c_1^{\frac{1}{\alpha+1}}}{\alpha} n^{\frac{\alpha}{\alpha+1}} \left(1 + \sum_{m\geq 1} \binom{-\alpha}{m} \left(\sum_{j=2}^{\ell+1} \frac{K_j c_1^{-\frac{1}{\alpha+1}}}{n^{(j-1)\left(1-\frac{\beta+1}{\alpha+1}\right)}} + \frac{c_3 c_1^{-\frac{1}{\alpha+1}}}{(\alpha+1)n^{\frac{\alpha}{\alpha+1}}} + o\left(n^{-\frac{\alpha}{\alpha+1}}\right)\right)^m\right). \tag{4.29.2}$$

By definition, $[\frac{c_1}{\alpha\varrho_n^\alpha}]_*$ is the part of the expansion of $\frac{c_1}{\alpha\varrho_n^\alpha}$ involving nonnegative powers of $n$, i.e., for $m \geq 2$ in the sum on the right of (4.29.2) we can ignore the term

$$\frac{c_3}{(\alpha+1)c_1^{\frac{1}{\alpha+1}} n^{\frac{\alpha}{\alpha+1}}} + o\left(n^{-\frac{\alpha}{\alpha+1}}\right).$$

Applying the Multinomial Theorem to (4.29.2) gives

$$\frac{c_1}{\alpha\varrho_n^\alpha} = \frac{c_1^{\frac{1}{\alpha+1}}}{\alpha} n^{\frac{\alpha}{\alpha+1}} - \frac{c_3}{\alpha+1} + \frac{c_1^{\frac{1}{\alpha+1}}}{\alpha} \sum_{m=1}^{\ell} \binom{-\alpha}{m} \sum_{\substack{0\leq j_1,j_2,\dots,j_\ell\leq m\\ j_1+\cdots+j_\ell=m}} \binom{m}{j_1, j_2, \dots, j_\ell} \frac{K_2^{j_1}\cdots K_{\ell+1}^{j_\ell}}{c_1^{\frac{m}{\alpha+1}}}$$
$$\times n^{\frac{(j_1+2j_2+\cdots+\ell j_\ell)\beta}{\alpha+1} + \frac{j_1+2j_2+\cdots+\ell j_\ell - 1}{\alpha+1} - (j_1+2j_2+\cdots+\ell j_\ell - 1)} + o(1). \tag{4.29.3}$$

Similarly, we have

$$\frac{c_2}{\beta\varrho_n^\beta} = \frac{c_2}{\beta c_1^{\frac{\beta}{\alpha+1}}} n^{\frac{\beta}{\alpha+1}} + \frac{c_2}{\beta c_1^{\frac{\beta}{\alpha+1}}} \sum_{m=1}^{\ell} \binom{-\beta}{m} \sum_{\substack{0\leq j_1,j_2,\dots,j_\ell\leq m\\ j_1+\cdots+j_\ell=m}} \binom{m}{j_1, j_2, \dots, j_\ell} \frac{K_2^{j_1}\cdots K_{\ell+1}^{j_\ell}}{c_1^{\frac{m}{\alpha+1}}}$$
$$\times n^{\frac{(j_1+2j_2+\cdots+\ell j_\ell+1)\beta}{\alpha+1} + \frac{j_1+2j_2+\cdots+\ell j_\ell}{\alpha+1} - (j_1+2j_2+\cdots+\ell j_\ell)} + o(1). \tag{4.29.4}$$

Finally, we obtain, with Lemma 4.28,

$$[n\varrho_n]_* = K_1 n^{\frac{\alpha}{\alpha+1}} + \sum_{m=1}^{\ell} K_{m+1} n^{\frac{m\beta}{\alpha+1} + \frac{m-1}{\alpha+1} - (m-1)} + \frac{c_3}{\alpha+1}. \tag{4.29.5}$$

Combining (4.29.3), (4.29.4), and (4.29.5), we find that

$$\left[n\varrho_n + \frac{c_1}{\alpha\varrho_n^\alpha} + \frac{c_2}{\beta\varrho_n^\beta}\right]_* = \left(1 + \frac{1}{\alpha}\right) c_1^{\frac{1}{\alpha+1}} n^{\frac{\alpha}{\alpha+1}} + \frac{c_2}{\beta c_1^{\frac{\beta}{\alpha+1}}} n^{\frac{\beta}{\alpha+1}} + \sum_{k=2}^{\ell} A_{k+1} n^{\frac{k\beta}{\alpha+1} + \frac{k-1}{\alpha+1} - (k-1)},$$

where

$$A_k = K_k + \frac{c_1^{\frac{1}{\alpha+1}}}{\alpha} \sum_{m=1}^{\ell} \binom{-\alpha}{m} \sum_{\substack{0\leq j_1,j_2,\dots,j_\ell\leq m\\ j_1+\cdots+j_\ell=m\\ j_1+2j_2+\cdots+\ell j_\ell=k-1}} \binom{m}{j_1, j_2, \dots, j_\ell} \frac{K_2^{j_1}\cdots K_{\ell+1}^{j_\ell}}{c_1^{\frac{m}{\alpha+1}}}$$

$$+\frac{c_2}{\beta c_1^{\frac{\beta}{\alpha+1}}}\sum_{m=1}^{\ell}\binom{-\beta}{m}\sum_{\substack{0\le j_1,j_2,\ldots,j_\ell\le m\\ j_1+\cdots+j_\ell=m\\ j_1+2j_2+\cdots+\ell j_\ell=k-2}}\binom{m}{j_1,j_2,\ldots,j_\ell}\frac{K_2^{j_1}\cdots K_{\ell+1}^{j_\ell}}{c_1^{\frac{m}{\alpha+1}}}.$$

Note that we have by definition of $c_1$, $c_2$ (see (4.27.4)), $K_1$, and $K_2$ (see Lemma 4.28),

$$A_1=\left(1+\frac{1}{\alpha}\right)c_1^{\frac{1}{\alpha+1}}=\left(1+\frac{1}{\alpha}\right)(\omega_\alpha\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}},$$
$$A_2=\frac{c_2}{\beta c_1^{\frac{\beta}{\alpha+1}}}=\frac{\omega_\beta\Gamma(\beta)\zeta(\beta+1)}{(\omega_\alpha\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{\beta}{\alpha+1}}},$$

which gives (4.29.1). Hence we indeed obtain, as $n\to\infty$, for suitable $M_1\in\mathbb{N}$

$$p_f(n)=\frac{C}{n^b}\exp\left(A_1n^{\frac{\alpha}{\alpha+1}}+A_2n^{\frac{\beta}{\alpha+1}}+\sum_{k=3}^{\ell+1}A_kn^{\frac{(k-1)\beta}{\alpha+1}+\frac{k-2}{\alpha+1}-(k-2)}\right)$$
$$\times\left(1+\sum_{j=2}^{M_1}\frac{B_j}{n^{\nu_j}}+O_{L,R}\left(n^{-\min\left\{\frac{2L-\alpha}{2(\alpha+1)},\frac{R}{\alpha+1}\right\}}\right)\right),$$

where the $\nu_j$ run, as in Theorem 4.5, through $\mathcal{M}+\mathcal{N}$. This proves the theorem. □

## 4.5 Proofs of Theorems 4.1, 4.3, and 4.4

We require the zeta function associated to a polynomial $P$,

$$Z_P(s):=\sum_{n\ge1}\frac{1}{P(n)^s}$$

with $P(n)>0$ for $n\in\mathbb{N}$. In particular, we consider $P=P_k$ (see (4.0.4)). The following lemma ensures that all the $P_k$ satisfy (P1) with $L$ arbitrary large.

**Lemma 4.30.** *Let $k\ge3$ be an integer and let*

$$\Lambda^{[k]}:=\{P_k(n):n\in\mathbb{N}\}.$$

*For every prime $p$, we have $|\Lambda^{[k]}\setminus(\Lambda^{[k]}\cap p\mathbb{N})|=\infty$.*

We next show that (P2) and (P3) hold.

**Proposition 4.31.** *Let $k\in\mathbb{N}$ with $k\ge3$.*

(1) *The function $Z_{P_k}$ has a meromorphic continuation to $\mathbb{C}$ with at most simple poles in $\frac12-\mathbb{N}_0$. The positive pole lies in $s=\frac12$.*

(2) *We have $Z_{P_k}(s)\ll Q_k(|\operatorname{Im}(s)|)$ as $|\operatorname{Im}(s)|\to\infty$ for some polynomial $Q_k$.*

*Proof.* (1) The meromorphic continuation of $Z_{P_k}$ to $\mathbb{C}$ follows by [29, Theorem B]. By [29, Theorem A (ii)] the only possible poles (of order at most one) are located at $\frac12-\frac12\mathbb{N}_0$. Holomorphicity in $-\mathbb{N}_0$ is a direct consequence of [29, Theorem C]. Finally, note that $P_k(n)\ll_k n^2$. Thus, as $x\to\infty$,

$$\sum_{1\le n\le x}\frac{1}{P_k(n)^{\frac12}}\gg_k\sum_{1\le n\le x}\frac1n.$$

This proves the existence of a pole in $s=\frac{1}{2}$, completing the proof.
(2) This result follows directly by [29, Proposition 1 (iii)]. $\square$

To apply Theorem 4.5, it remains to compute $Z_{P_k}(0)$ and $Z'_{P_k}(0)$, as well as $\operatorname{Res}_{s=\frac{1}{2}} Z_{P_k}(s)$.

**Proposition 4.32.** *Let $k \in \mathbb{N}$ with $k \geq 3$.*

(1) *We have $Z_{P_k}(0) = \frac{1}{2-k}$ and*

$$Z'_{P_k}(0) = \frac{\log\left(\frac{k-2}{2}\right)}{k-2} + \log\left(\Gamma\left(\frac{2}{k-2}\right)\right) - \log(2\pi).$$

(2) *We have* $\operatorname{Res}_{s=\frac{1}{2}} Z_{P_k}(s) = \sqrt{\frac{1}{2(k-2)}}$.

*Proof.* (1) Since the roots of $P_k$ are not in $\mathbb{R}_{\geq 1}$, we may use [29, Theorem D] to obtain that $Z_{P_k}(0) = \frac{1}{2-k}$. For the derivative, one applies [29, Theorem E] yielding

$$Z'_{P_k}(0) = \frac{\log\left(\frac{k-2}{2}\right)}{k-2} + \log\left(\Gamma\left(\frac{2}{k-2}\right)\right) - \log(2\pi).$$

(2) Since $Z_{P_k}(s) = (\frac{2}{k-2})^s \sum\limits_{n\geq 1} (n - \frac{k-4}{k-2})^{-s} n^{-s}$, the result follows as the sum has residue $\frac{1}{2}$ at $s = \frac{1}{2}$ by equation (16) of [29]. $\square$

The previous three lemmas are used to prove Theorem 4.1.

*Proof of Theorem 4.1.* We may apply Theorem 4.5 as Lemma 4.30 and Lemma 4.31 ensure that conditions (P1)–(P3) are satisfied. Hence, one obtains an asymptotic formula for $p_k(n)$. The constants occurring in Theorem 4.5 are computed using (4.6.4), (4.6.5), and Lemma 4.32. That the exponential consists only of the term $A_1 n^{\frac{1}{3}}$ follows by Theorem 4.5, since $Z_{P_k}(s)$ has exactly one positive pole, lying in $s = \frac{1}{2}$. Note that we are allowed to choose $L$ and $R$ arbitrarily large due to Lemma 4.30 and Lemma 4.31 (1). $\square$

We consider some special cases of Theorem 4.1.

**Corollary 4.33.** *For triangular numbers, squares, and pentagonal numbers, respectively, we have*

$$p_3(n) \sim \frac{\zeta\left(\frac{3}{2}\right)}{2^{\frac{7}{2}}\sqrt{3}\pi n^{\frac{3}{2}}} \exp\left(\frac{3}{2}\pi^{\frac{1}{3}}\zeta\left(\frac{3}{2}\right)^{\frac{2}{3}} n^{\frac{1}{3}}\right), \; p_4(n) \sim \frac{\zeta\left(\frac{3}{2}\right)^{\frac{2}{3}}}{2^{\frac{7}{3}}\sqrt{3}\pi^{\frac{7}{6}} n^{\frac{7}{6}}} \exp\left(\frac{3}{2^{\frac{4}{3}}}\pi^{\frac{1}{3}}\zeta\left(\frac{3}{2}\right)^{\frac{2}{3}} n^{\frac{1}{3}}\right),$$

$$p_5(n) \sim \frac{\Gamma\left(\frac{2}{3}\right)\zeta\left(\frac{3}{2}\right)^{\frac{5}{9}}}{2^{\frac{13}{6}} 3^{\frac{4}{9}} \pi^{\frac{11}{9}} n^{\frac{19}{18}}} \exp\left(\frac{3^{\frac{2}{3}}}{2}\pi^{\frac{1}{3}}\zeta\left(\frac{3}{2}\right)^{\frac{2}{3}} n^{\frac{1}{3}}\right).$$

The next lemma shows that $\prod_{j,k\geq 1}(1-q^{\frac{jk(j+k)(j+2k)}{6}})^{-1}$ satisfies (P1) for $L$ arbitrarily large.

**Lemma 4.34.** *Let $f\colon \mathbb{N} \to \mathbb{N}_0$ be defined by*

$$f(n) := \left|\left\{(j,k) \in \mathbb{N}^2 : \frac{jk(j+k)(j+2k)}{6} = n\right\}\right|.$$

*Then, for all primes $p$, we have $|\Lambda \setminus (\Lambda \cap p\mathbb{N})| = \infty$.*

For investigating the function $\zeta_{\mathfrak{so}(5)}$, we need the *Mordell–Tornheim zeta function*, defined by

$$\zeta_{\mathrm{MT},2}(s_1, s_2, s_3) := \sum_{m,n\geq 1} m^{-s_1} n^{-s_2} (m+n)^{-s_3}.$$

By [27], for $\mathrm{Re}(s) > 1$ and some $-\mathrm{Re}(s) < c < 0$ we get a relation between $\zeta_{\mathrm{MT},2}$ and $\zeta_{\mathfrak{so}(5)}$ via

$$\zeta_{\mathfrak{so}(5)}(s) = \frac{6^s}{2\pi i \Gamma(s)} \int_{c-i\infty}^{c+i\infty} \Gamma(s+z)\Gamma(-z)\zeta_{\mathrm{MT},2}(s, s-z, 2s+z)dz. \tag{4.34.1}$$

We have the following theorem.

**Theorem 4.35** ([26, Theorem 1])**.** *The function $\zeta_{\mathrm{MT},2}$ has a meromorphic continuation to $\mathbb{C}^3$ and its singularities satisfy $s_1 + s_3 = 1 - \ell, s_2 + s_3 = 1 - \ell, s_1 + s_2 + s_3 = 2$, with $\ell \in \mathbb{N}_0$.*

Fix $M \in \mathbb{N}_0$ and $0 < \varepsilon < 1$. Let $\mathrm{Re}(s_1), \mathrm{Re}(s_3) > 1$, $\mathrm{Re}(s_2) > 0$, and $s_2 \notin \mathbb{N}$. Then, for $\mathrm{Re}(s_2) < M + 1 - \varepsilon$, we have (see equation (5.3) in [26])

$$\begin{aligned}\zeta_{\mathrm{MT},2}&(s_1, s_2, s_3)\\ &= \frac{\Gamma(s_2+s_3-1)\Gamma(1-s_2)}{\Gamma(s_3)}\zeta(s_1+s_2+s_3-1)\\ &\quad + \sum_{m=0}^{M-1} \binom{-s_3}{m} \zeta(s_1+s_3+m)\zeta(s_2-m)\\ &\quad + \frac{1}{2\pi i}\int_{M-\varepsilon-i\infty}^{M-\varepsilon+i\infty} \frac{\Gamma(s_3+w)\Gamma(-w)}{\Gamma(s_3)}\zeta(s_1+s_3+w)\zeta(s_2-w)dw. \end{aligned} \tag{4.35.1}$$

The first two summands on the right-hand side of (4.35.1) extend meromorphically to $\mathbb{C}^3$, so to show that (4.34.1) extends meromorphically, we consider (4.35.1). Note that $\mathrm{Re}(w) = M - \varepsilon$. To avoid poles on the line of integration, we assume that

$$\mathrm{Re}(s_3 + w) > 0 \Leftrightarrow \mathrm{Re}(s_3) > \varepsilon - M, \tag{4.35.2}$$

$$\mathrm{Re}(s_1 + s_3 + w) > 1 \Leftrightarrow \mathrm{Re}(s_1) + \mathrm{Re}(s_3) > 1 - M + \varepsilon, \tag{4.35.3}$$

$$\mathrm{Re}(s_2 - w) < 1 \Leftrightarrow \mathrm{Re}(s_2) < 1 + M - \varepsilon. \tag{4.35.4}$$

Note that the final condition is already assumed above.

By Propostition 2.6 (2), the integral converges compactly and the integrands are locally holomorphic. Thus, the integral is a holomorphic function in the region defined by (4.35.2), (4.35.3), and (4.35.4).

Recalling (4.34.1), we are interested in $\zeta_{\mathrm{MT},2}(s, s-z, 2s+z)$. By Theorem 4.35, this function is meromorphic in $\mathbb{C}^2$ and holomorphic outside the hyperplanes defined by $3s + z = 1 - \ell$, $3s = 1 - \ell$, and $4s = 2$, where $\ell \in \mathbb{N}_0$. With (4.35.1), we obtain

$$\begin{aligned}\zeta_{\mathrm{MT},2}(s, s-z, 2s+z) &= \frac{\Gamma(3s-1)\Gamma(z+1-s)}{\Gamma(2s+z)}\zeta(4s-1)\\ &\quad + \sum_{m=0}^{M-1}\binom{-2s-z}{m}\zeta(3s+z+m)\zeta(s-z-m) + I_M(s;z), \end{aligned} \tag{4.35.5}$$

where $s \in \mathbb{C} \setminus \{\frac{1}{2}, \frac{1-\ell}{3}\}$, and

$$I_M(s;z) := \frac{1}{2\pi i} \int_{M-\varepsilon-i\infty}^{M-\varepsilon+i\infty} \frac{\Gamma(2s+z+w)\Gamma(-w)}{\Gamma(2s+z)} \zeta(3s+z+w)\zeta(s-z-w)dw.$$

The following lemma shows that $I_M(s;z)$ is holomorphic in $z$. To state it let

$$\mu = \mu_{M,\sigma} := \max\{-1+\sigma-M+\varepsilon, 1-3\sigma-M+\varepsilon, -2\sigma-M+\varepsilon\}.$$

**Lemma 4.36.** *Let $s = \sigma + it \in \mathbb{C}$, $M \in \mathbb{N}_0$, and $0 < \varepsilon < 1$. Then $z \mapsto I_M(s;z)$ is holomorphic in $S_{\mu,\infty}$.*

*Proof.* If $z \in S_{\mu,\infty}$, then $\mathrm{Re}(2s+z+w) > 0$, $\mathrm{Re}(3s+z+w) > 1$, and $\mathrm{Re}(s-z-w) < 1$ for $w \in \mathbb{C}$ satisfying $\mathrm{Re}(w) = M-\varepsilon$, so $\Gamma(2s+z+w)$, $\zeta(3s+z+w)$, and $\zeta(s-z-w)$ have no poles on the path of integration. As $0 < \varepsilon < 1$, we have $M-\varepsilon \notin \mathbb{N}_0$, so $w \mapsto \Gamma(-w)$ has no pole if $\mathrm{Re}(w) = M-\varepsilon$. As a result, no pole is located on the path of integration, and by Proposition 4.13 (2) and the uniform polynomial growth of the zeta function along vertical strips we find that the integral converges uniformly on compact subsets of $S_{\mu,\infty}$. ☐

The next lemma shows, that $I_M$ is bounded polynomially in certain vertical strips. A proof is obtained using Propositions 4.13 (2) and 4.14 (2).

**Lemma 4.37.** *Let $\sigma_1 < \sigma_2$ and $\sigma_3 < \sigma_4$, such that $S_{\sigma_3,\sigma_4} \subset S_{\mu,\infty}$ for all $s \in S_{\sigma_1,\sigma_2}$ and fix $0 < \varepsilon < 1$ sufficiently small. In $S_{\sigma_1,\sigma_2} \times S_{\sigma_3,\sigma_4}$ the function $(s,z) \mapsto I_M(s;z)$ is holomorphic and satisfies $|I_M(s;z)| \leq P_{\sigma_1,\sigma_2,\sigma_3,\sigma_4,M}(|\mathrm{Im}(s)|, |\mathrm{Im}(z)|)$ for some polynomial $P_{\sigma_1,\sigma_2,\sigma_3,\sigma_4,M}(X,Y) \in \mathbb{R}[X,Y]$.*

Next we investigate $\zeta_{\mathrm{MT},2}(s, s-z, 2s+z)$ for fixed $s$ more in detail.

**Lemma 4.38.** *Let $s \in \mathbb{C} \setminus \{\frac{1}{2}, \frac{1}{3} - \frac{1}{3}\mathbb{N}_0\}$. Then $z \mapsto \zeta_{\mathrm{MT},2}(s, s-z, 2s+z)$ is holomorphic in the entire complex plane except for possibly simple poles in $z = 1-\ell-3s$ with $\ell \in \mathbb{N}_0$.*

*Proof.* As holomorphicity is a local property, it suffices to consider arbitrary right half-planes. By Lemma 4.36, for $M$ sufficiently large, $I_M$ is holomorphic in an arbitrary right half-plane. By (4.35.1), possible poles of $\zeta_{\mathrm{MT},2}(s, s-z, 2s+z)$ therefore lie in $z = s-\ell$ and in $z = -3s-m-\ell$, $\ell \in \mathbb{N}$. A direct calculation shows that the residue at $z = s-\ell$ vanishes if $\ell \leq M-1$. Consequently, for a fixed pole $s-\ell$, we can choose $M$ sufficiently large such that we only have to consider the of (4.35.1). This gives the claim. ☐

We are now ready to prove growth properties of $\zeta_{\mathrm{MT},2}$. As we need to avoid critical singular points, we focus on incomplete half-planes of the type $S_{\sigma_1,\sigma_2,\delta}$ (with $\delta > 0$ arbitrarily small).

**Lemma 4.39.** *Let $\sigma_1 < \sigma_2$, $\sigma_3 < \sigma_4$ with $1-3\sigma_1 < \sigma_3$ and $\delta > 0$ arbitrarily small. For $(s,z) \in S_{\sigma_1,\sigma_2,\delta} \times S_{\sigma_3,\sigma_4}$, we have, for some polynomial $P_{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\delta}$ only depending on $S_{\sigma_1,\sigma_2,\delta}$ and $S_{\sigma_3,\sigma_4}$,*

$$|\zeta_{\mathrm{MT},2}(s, s-z, 2s+z)| \leq P_{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\delta}(|\mathrm{Im}(s)|, |\mathrm{Im}(z)|).$$

*If $\sigma_1 < 0$, for all $s \in U$ with $U \subset S_{\sigma_1,\sigma_2}$, a sufficiently small neighborhood of 0, we have*

$$\left| \frac{\zeta_{\mathrm{MT},2}(s, s-z, 2s+z)}{\Gamma(s)} \right| \leq P_{\sigma_3,\sigma_4,U}(|\mathrm{Im}(z)|),$$

*where the polynomial $P_{\sigma_3,\sigma_4,U}$ only depends on $\sigma_3$, $\sigma_4$, and $U$.*

We need another lemma dealing with the poles of the Mordell–Tornheim zeta function.

**Lemma 4.40.** *Let $k \in \mathbb{N}_0$. Then the meromorphic function $s \mapsto \zeta_{\mathrm{MT},2}(s, s-k, 2s+k)$ is holomorphic for $s = -\ell$ with $\ell \in \mathbb{N}_{\geq \frac{k}{2}}$ and has possible simple poles at $s = \ell \in \mathbb{N}_0$ with $0 \leq \ell < \frac{k}{2}$. In particular, $s \mapsto \Gamma(s+k)\zeta_{\mathrm{MT},2}(s, s-k, 2s+k)\Gamma(s)^{-1}$ is holomorphic at $s = -\ell$ with $\ell \in \mathbb{N}_0$.*

*Proof.* Let $s$ lie in a bounded neighborhood of $-\ell$. We use (4.35.5) with $s = k$. Analogous to the proof of Lemma 4.36, the function $s \mapsto I_M(s;k)$ is holomorphic in a neighborhood of $s = -\ell$. The analysis of the remaining terms is straightforward, and the lemma follows. $\square$

The next lemma states where the integral of (4.34.1) defining $\zeta_{\mathfrak{so}(5)}$ is a meromorphic function.

**Lemma 4.41.** *Let $\varepsilon > 0$ be sufficiently small and let $K \in \mathbb{N}$. Then the function*

$$s \mapsto \frac{1}{2\pi i\Gamma(s)} \int_{K-\varepsilon-i\infty}^{K-\varepsilon+i\infty} \Gamma(s+z)\Gamma(-z)\zeta_{\mathrm{MT},2}(s, s-z, 2s+z)dz \tag{4.41.1}$$

*is meromorphic on the half plane $\{s \in \mathbb{C} : \mathrm{Re}(s) > \frac{1-K+\varepsilon}{3}\}$ with at most simple poles in $\{\frac{1}{2}, \frac{1}{3} - \frac{1}{3}\mathbb{N}_0\} \setminus (-\mathbb{N}_0)$ (with $\mathrm{Re}(s) > \frac{1-K+\varepsilon}{3}$) and grows polynomially on vertical strips with finite width.*

*Proof.* We first show holomorphicity in $S_{\sigma_1,\sigma_2,\delta}$ with $\frac{1-K+\varepsilon}{3} < \sigma_1 < \sigma_2$ and $0 < \delta < 1$. Since $\mathrm{Re}(s) > \frac{1-K+\varepsilon}{3} > -K+\varepsilon$, there are no poles of $\Gamma(s+z)\Gamma(-z)$ on the path of integration $\mathrm{Re}(z) = K - \varepsilon$. By Lemma 4.38, $z \mapsto \zeta_{\mathrm{MT},2}(s, s-z, 2s+z)$ has no poles for $s \in S_{\sigma_1,\sigma_2,\delta}$, as $\mathrm{Re}(z + 3s - 1) = K - \varepsilon + 3\,\mathrm{Re}(s) - 1 > 0$. By Proposition 4.13 (2), Lemma 4.39, and Lemma 4.16, the integral is holomorphic away from singularities and grows polynomially on vertical strips of finite width.

We are left to show that (4.41.1) has at most a simple pole at $s = s_0$, where $s_0 \in \{\frac{1}{2}, \frac{1}{3} - \frac{1}{3}\mathbb{N}_0\} \setminus (-\mathbb{N}_0)$ with $s_0 \geq \frac{1-K+\varepsilon}{3}$. Recall the representation of $\zeta_{\mathrm{MT},2}$ in (4.35.5). By Lemma 4.37

$$\int_{K-\varepsilon-i\infty}^{K-\varepsilon+i\infty} \Gamma(s+z)\Gamma(-z)I_M(s;z)dz$$

converges absolutely and uniformly on any sufficiently small compact subset $C$ containing $s_0$ for $M$ sufficiently large. Similarly, by Propositions 4.14 (2) and 4.13 (2),

$$\int_{K-\varepsilon-i\infty}^{K-\varepsilon+i\infty} \Gamma(s+z)\Gamma(-z) \sum_{m=0}^{M-1} \binom{-2s-z}{m} \zeta(3s+z+m)\zeta(s-z-m)dz$$

converges absolutely and uniformly in $C$. In particular, both integrals continue holomorphically to $s_0$. As $s \mapsto \frac{1}{\Gamma(s)}$ is entire, it is sufficient to study

$$\frac{\Gamma(3s-1)\zeta(4s-1)}{\Gamma(s)} \int_{K-\varepsilon-i\infty}^{K-\varepsilon+i\infty} \frac{\Gamma(s+z)\Gamma(-z)\Gamma(1+z-s)}{\Gamma(2s+z)}dz$$

around $s_0$. Again, by Proposition 4.13 (2), the integral converges absolutely and uniformly in $C$. As $\frac{\Gamma(3s-1)\zeta(4s-1)}{\Gamma(s)}$ has at most a simple pole in $s_0$ and a removable singularity if $s_0 \in -\mathbb{N}_0$, the proof of the lemma is complete. $\square$

The following lemma is a refinement of Lemma 4.41 for the specific case that $z \in \mathbb{Z}$ and follows from Lemma 4.37, by using Propositions 4.13 and 4.14.

**Lemma 4.42.** *Let $k \in \mathbb{N}_0$ with $0 \le k \le K-1$. Then, for all $\sigma_1 < \sigma_2$, there exists a polynomial $P_{K,\sigma_1,\sigma_2}$, such that, uniformly for all $\sigma_1 \le \mathrm{Re}(s) \le \sigma_2$ and $|\mathrm{Im}(s)| \ge 1$,*

$$|\zeta_{\mathrm{MT},2}(s, s-k, 2s+k)| \le P_{K,\sigma_1,\sigma_2}(|\mathrm{Im}(s)|).$$

The following theorem shows that the function $\zeta_{\mathfrak{so}(5)}$ satisfies the conditions of Theorem 4.5 and gives the more precise statement of Theorem 4.3.

**Theorem 4.43.** *The function $\zeta_{\mathfrak{so}(5)}$ extends to a meromorphic function in $\mathbb{C}$ and is holomorphic in $\mathbb{N}_0$. For $K \in \mathbb{N}$ and $0 < \varepsilon < 1$, we have, on $S_{\frac{1-K+\varepsilon}{3},\infty}$,*

$$\begin{aligned}\zeta_{\mathfrak{so}(5)}(s) = \frac{6^s}{\Gamma(s)} \sum_{k=0}^{K-1} \frac{(-1)^k \Gamma(s+k)}{k!} \zeta_{\mathrm{MT},2}(s, s-k, 2s+k) \\ + \frac{6^s}{2\pi i \Gamma(s)} \int_{K-\varepsilon-i\infty}^{K-\varepsilon+i\infty} \Gamma(s+z)\Gamma(-z)\zeta_{\mathrm{MT},2}(s, s-z, 2s+z)dz. \quad (4.43.1)\end{aligned}$$

*All poles of $\zeta_{\mathfrak{so}(5)}$ are simple and contained in $\{\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}, \dots\}$. Furthermore, for all $\sigma_0 \le \sigma \le \sigma_1$ as $|\mathrm{Im}(s)| \to \infty$, for some polynomial depending only on $\sigma_0$ and $\sigma_1$,*

$$|\zeta_{\mathfrak{so}(5)}(s)| \le P_{\sigma_0,\sigma_1}(|\mathrm{Im}(s)|).$$

*Proof.* Assume $\mathrm{Re}(s) > 1$. By Lemma 4.38, the only poles of the integrand in (4.34.1) in $S_{-\mathrm{Re}(s),\infty}$ lie at $z \in \mathbb{N}_0$. By shifting the path to the right of $\mathrm{Re}(z) = M - \varepsilon$, we find, with Lemma 4.39 and the Residue Theorem, that (4.43.1) holds on $S_{1,\infty}$. By Lemma 4.41 the right-hand side is a meromorphic function on $S_{\frac{1-K+\varepsilon}{3},\infty}$. By Theorem 4.35, the functions $s \mapsto \zeta_{\mathrm{MT},2}(s, s-k, 2s+k)$ only have possible (simple) poles for $s_1 + s_3 = 3s + k = 1 - \ell$, $s_2 + s_3 = 3s = 1 - \ell$, $s_1 + s_2 + s_3 = 4s = 2$, with $\ell \in \mathbb{N}_0$, i.e., for $s \in \{\frac{1}{2}, \frac{1}{3}, 0, -\frac{1}{3}, -\frac{2}{3}, -1, \dots\}$. However, by Lemma 4.40 the sum in (4.43.1) continues holomorphically to $-\mathbb{N}_0$, so the sum only contributes possible poles $s \in \mathcal{S} := \{\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}, -\frac{4}{3}, \dots\}$. Note that this argument does not depend on the choice of $K$. On the other hand, if we choose $K$ sufficiently large, then the integral in (4.43.1) is a holomorphic function around $s = -m$ for fixed but arbitrary $m \in \mathbb{N}_0$, and it only contributes poles in $\mathcal{S}$ in $S_{\frac{1-K+\varepsilon}{3},\infty}$ by Lemma 4.41, where $0 < \varepsilon < 1$. So the statement about the poles follows if $K \to \infty$.

We are left to show the polynomial bound. With Lemma 4.42 we obtain the bound for the finite sum, as we chose $K$ in terms of $\sigma_0$ and $\sigma_1$. Lemma 4.41 implies the polynomial bound for the integral. □

To apply Theorem 4.5 we require $\zeta_{\mathfrak{so}(5)}(0)$.

**Proposition 4.44.** *We have $\zeta_{\mathfrak{so}(5)}(0) = \frac{3}{8}$.*

*Proof.* Since $I_M(s; z)$ is holomorphic in $s$ for $z \in S_{\mu,\infty}$ by Lemma 4.37 and $\Gamma(s)$ has a pole in $s = 0$,

$$\lim_{s \to 0} \frac{I_M(s; z)}{\Gamma(s)} = 0. \quad (4.44.1)$$

Let $K \in \mathbb{N}$. For $z \in \mathbb{C}$ with $\mathrm{Re}(z) = K - \frac{1}{2}$ and $m \in \mathbb{N}_0$, we have $\pm(z+m) \neq 1$. Hence, $s \mapsto \binom{-2s-z}{m}\zeta(3s+z+m)\zeta(s-z-m)$ is holomorphic at $s = 0$. This implies that for $z \in \mathbb{C}$ with $\mathrm{Re}(z) = K - \frac{1}{2}$, we have

$$\lim_{s \to 0} \binom{-2s-z}{m} \frac{\zeta(3s+z+m)\zeta(s-z-m)}{\Gamma(s)} = 0.$$

Using this, (4.43.1) with $\varepsilon = \frac{1}{2}$, (4.44.1), Proposition 4.13 (4), and Lebesgue's dominated convergence theorem, we obtain, for integers $K \geq 3$,

$$\lim_{s\to 0} \frac{6^s}{2\pi i\Gamma(s)} \int_{K-\frac{1}{2}-i\infty}^{K-\frac{1}{2}+i\infty} \Gamma(s+z)\Gamma(-z)\zeta_{\mathrm{MT},2}(s,s-z,2s+z)dz = \frac{i}{72}\int_{K-\frac{1}{2}-i\infty}^{K-\frac{1}{2}+i\infty} \frac{1}{\sin(\pi z)}dz.$$

Since $\sin(\pi(z+1)) = -\sin(\pi z)$ and

$$\lim_{L\to\infty} \int_{K-\frac{1}{2}-iL}^{K+\frac{1}{2}-iL} \frac{1}{\sin(\pi z)}dz = \lim_{L\to\infty} \int_{K+\frac{1}{2}+iL}^{K-\frac{1}{2}+iL} \frac{1}{\sin(\pi z)}dz = 0,$$

the Residue Theorem implies that

$$\lim_{s\to 0} \tfrac{6^s}{2\pi i\Gamma(s)} \int_{K-\frac{1}{2}-i\infty}^{K-\frac{1}{2}+i\infty} \Gamma(s+z)\Gamma(-z)\zeta_{\mathrm{MT},2}(s,s-z,2s+z)dz = \tfrac{1}{72}\operatorname{Res}_{z=K} \tfrac{\pi}{\sin(\pi z)} = \tfrac{(-1)^K}{72}. \tag{4.44.2}$$

In the following we use that $\zeta(s)$ does not have a pole in $s = \pm m$ for $m \in \mathbb{N}_{\geq 2}$, implying that $s \mapsto \binom{-2s-1}{m-1}\zeta(3s+m)\zeta(s-m)$ is holomorphic at $s = 0$.
Moreover $s \mapsto \Gamma(s+k)\binom{-2s-k}{m}\zeta(3s+k+m)\zeta(s-k-m)$ is holomorphic at $s = 0$ for $(k,m) \in (\mathbb{N}\times\mathbb{N}_0)\setminus\{(1,0)\}$. Thus, using Propositions 4.13 (3) and 4.14 (3) and the fact that $\zeta(-1) = -\frac{1}{12}$ and[10] $\zeta(0) = -\frac{1}{2}$, we obtain, with (4.35.5),

$$\begin{aligned}\lim_{s\to 0} \frac{6^s}{\Gamma(s)} \sum_{k=0}^{K-1} &\frac{(-1)^k\Gamma(s+k)}{k!}\zeta_{\mathrm{MT},2}(s,s-k,2s+k)\\ =&\frac{3}{8} + \frac{(-1)^{K+1}}{72} + \lim_{s\to 0} I_M(s;0) + \sum_{k=1}^{K-1} \frac{(-1)^k}{k} \lim_{s\to 0} \frac{I_M(s;k)}{\Gamma(s)}.\end{aligned} \tag{4.44.3}$$

Since, by Lemma 4.37, $s \mapsto I_M(s;k)$ is holomorphic at $s = 0$ for every $k \in \mathbb{N}_0$ and $\frac{1}{\Gamma(s)}$ vanishes in $s = 0$, we have

$$\lim_{s\to 0} \frac{I_M(s;k)}{\Gamma(s)} = 0.$$

Applying the Lebesgue dominated convergence theorem gives $\lim\limits_{s\to 0} I_M(s;0) = 0$, yielding the claim with (4.43.1), (4.44.2), and (4.44.3). □

Furthermore, we need certain residues of $\zeta_{\mathfrak{so}(5)}$.

**Proposition 4.45.** *The poles of $\zeta_{\mathfrak{so}(5)}$ are precisely $\{\frac{1}{2}\} \cup \{\frac{d}{3} \notin \mathbb{Z} : d \leq 1$ odd$\}$. We have*

$$\operatorname{Res}_{s=\frac{1}{2}} \zeta_{\mathfrak{so}(5)}(s) = \frac{\sqrt{3}\Gamma\left(\frac{1}{4}\right)^2}{8\sqrt{\pi}}.$$

*Moreover for $d \in \mathbb{Z}_{\leq 1} \setminus (-3\mathbb{N}_0)$,*

$$\operatorname{Res}_{s=\frac{d}{3}} \zeta_{\mathfrak{so}(5)}(s) = \frac{3^{\frac{d}{3}-\frac{3}{2}}\pi\Gamma\left(\frac{d}{6}\right)\zeta\left(\frac{4d}{3}-1\right)}{2^{\frac{d}{3}-1}(1-d)!\Gamma\left(\frac{d}{3}\right)^2\Gamma\left(\frac{d}{2}\right)} \binom{d}{3}\left(1+2^{\frac{2d}{3}-1}\right). \tag{4.45.1}$$

[10]In the published version, $\zeta(0) = \frac{1}{2}$ was written; the sign was corrected here. Note that the (corrected) sign does not affect the calculation since $\zeta(0)$ occurs squared only.

*In particular, we have*

$$\operatorname{Res}_{s=\frac{1}{3}} \zeta_{\mathfrak{so}(5)}(s) = \frac{2^{\frac{1}{3}}+1}{3^{\frac{2}{3}}}\zeta\left(\frac{1}{3}\right).$$

*Proof.* With Lemma 4.41, near $s=\frac{1}{2}$, we can choose $K=1$ in (4.43.1) and obtain

$$\begin{aligned}
&\operatorname{Res}_{s=\frac{1}{2}} \zeta_{\mathfrak{so}(5)}(s)\\
&= \lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)\\
&\quad\times\left(6^s\zeta_{\mathrm{MT},2}(s,s,2s)+\frac{6^s}{2\pi i\Gamma(s)}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s+z)\Gamma(-z)\zeta_{\mathrm{MT},2}(s,s-z,2s+z)dz\right).
\end{aligned}$$

Now, we have

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)6^s\zeta_{\mathrm{MT},2}(s,s,2s)=\frac{\sqrt{3}\pi}{2\sqrt{2}}.$$

On the other hand, we find

$$\begin{aligned}
&\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)\frac{6^s}{2\pi i\Gamma(s)}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s+z)\Gamma(-z)\zeta_{\mathrm{MT},2}(s,s-z,2s+z)dz\\
&\quad=\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)\frac{6^s\Gamma(3s-1)\zeta(4s-1)}{2\pi i\Gamma(s)}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s+z)\Gamma(-z)\Gamma(z+1-s)dz, \quad (4.45.2)
\end{aligned}$$

since $s\mapsto\frac{\Gamma(s+z)\Gamma(-z)\zeta(3s+z)\zeta(s-z)}{\Gamma(s)}$ and $s\mapsto\frac{\Gamma(s+z)\Gamma(-z)I_1(s;z)}{\Gamma(s)}$ are holomorphic if $\operatorname{Re}(z)=\frac{1}{2}$. Shifting the path to the left and using [21, 9.113], Proposition 4.13 (1), 15.4.26 of [31], and Proposition 4.13 (4) we obtain that (4.45.2) equals

$$\frac{\sqrt{3}\pi}{2\sqrt{2}}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;-1\right)-\frac{\sqrt{3}\pi}{2\sqrt{2}}=\frac{\sqrt{3}\Gamma\left(\frac{1}{4}\right)^2}{8\sqrt{\pi}}-\frac{\sqrt{3}\pi}{2\sqrt{2}}.$$

This proves the first part of the proposition.

Now, let $d\in\mathbb{Z}_{\leq 1}\setminus(-3\mathbb{N}_0)$ and choose $0<\varepsilon<\frac{1}{3}$, and also $K,M>1-d$. We have, by (4.43.1),

$$\begin{aligned}
\operatorname{Res}_{s=\frac{d}{3}}\zeta_{\mathfrak{so}(5)}(s)&=\lim_{s\to\frac{d}{3}}\frac{\left(s-\frac{d}{3}\right)6^s}{\Gamma(s)}\sum_{k=0}^{K-1}\frac{(-1)^k\Gamma(s+k)}{k!}\zeta_{\mathrm{MT},2}(s,s-k,2s+k)\\
&\quad+\lim_{s\to\frac{d}{3}}\frac{\left(s-\frac{d}{3}\right)6^s}{2\pi i\Gamma(s)}\int_{K-\varepsilon-i\infty}^{K-\varepsilon+i\infty}\Gamma(s+z)\Gamma(-z)\zeta_{\mathrm{MT},2}(s,s-z,2s+z)dz. \quad (4.45.3)
\end{aligned}$$

Note that $\lim\limits_{s\to\frac{d}{3}}(s-\frac{d}{3})I_M(s;k)=0$ because of holomorphicity of $I_M$ by Lemma 4.37 and

$$\lim_{s\to\frac{d}{3}}\left(s-\frac{d}{3}\right)\zeta(3s+k+m)=\frac{1}{3}\delta_{m=1-d-k}.$$

Thus we obtain, by (4.35.5) and (15.4.26) of [31],

$$\begin{aligned}
&\lim_{s\to\frac{d}{3}} \frac{\left(s-\frac{d}{3}\right)6^s}{\Gamma(s)} \sum_{k=0}^{K-1} \frac{(-1)^k\Gamma(k+s)}{k!}\zeta_{\mathrm{MT},2}(s,s-k,k+2s)\\
&= \frac{6^{\frac{d}{3}}\zeta\left(\frac{4d}{3}-1\right)}{3(1-d)!\Gamma\left(\frac{d}{3}\right)} \left(\sum_{k=0}^{K-1} \frac{(-1)^{k+d+1}\Gamma\left(k+1-\frac{d}{3}\right)\Gamma\left(k+\frac{d}{3}\right)}{k!\Gamma\left(k+\frac{2d}{3}\right)}\right.\\
&\qquad\left.+\sum_{k=0}^{1-d}(-1)^k\binom{1-d}{k}\frac{\Gamma\left(k+\frac{d}{3}\right)\Gamma\left(1-\frac{2d}{3}-k\right)}{\Gamma\left(\frac{d}{3}\right)}\right)\\
&= \frac{6^{\frac{d}{3}}\zeta\left(\frac{4d}{3}-1\right)}{3(1-d)!\Gamma\left(\frac{d}{3}\right)} \left(\sum_{k=0}^{K-1} \frac{(-1)^{k+d+1}\Gamma\left(k+1-\frac{d}{3}\right)\Gamma\left(k+\frac{d}{3}\right)}{k!\Gamma\left(k+\frac{2d}{3}\right)}\right.\\
&\qquad\left.+\Gamma\left(1-\frac{2d}{3}\right){}_2F_1\left(\frac{d}{3},d-1;\frac{2d}{3};-1\right)\right)\\
&= \frac{6^{\frac{d}{3}}\zeta\left(\frac{4d}{3}-1\right)}{3(1-d)!\Gamma\left(\frac{d}{3}\right)} \sum_{k=0}^{K-1} \frac{(-1)^{k+d+1}\Gamma\left(k+1-\frac{d}{3}\right)\Gamma\left(k+\frac{d}{3}\right)}{k!\Gamma\left(k+\frac{2d}{3}\right)}\\
&\qquad+\frac{3^{\frac{d}{3}-1}\zeta\left(\frac{4d}{3}-1\right)\Gamma\left(1-\frac{2d}{3}\right)\Gamma\left(\frac{2d}{3}\right)\Gamma\left(\frac{d}{6}\right)}{2^{\frac{d}{3}}(1-d)!\Gamma\left(\frac{d}{3}\right)^2\Gamma\left(\frac{d}{2}\right)}. \qquad (4.45.4)
\end{aligned}$$

For the integral in (4.45.3), we obtain that

$$\begin{aligned}
&\lim_{s\to\frac{d}{3}} \frac{\left(s-\frac{d}{3}\right)6^s}{2\pi i\Gamma(s)} \int_{K-\varepsilon-i\infty}^{K-\varepsilon+i\infty} \Gamma(s+z)\Gamma(-z)\zeta_{\mathrm{MT},2}(s,s-z,2s+z)dz\\
&= \frac{(-1)^{d+1}6^{\frac{d}{3}}\zeta\left(\frac{4d}{3}-1\right)}{3(1-d)!\Gamma\left(\frac{d}{3}\right)} \frac{1}{2\pi i}\int_{K-\varepsilon-i\infty}^{K-\varepsilon+i\infty} \frac{\Gamma\left(z+\frac{d}{3}\right)\Gamma\left(z+1-\frac{d}{3}\right)\Gamma(-z)}{\Gamma\left(z+\frac{2d}{3}\right)}dz. \qquad (4.45.5)
\end{aligned}$$

By shifting the path of integration to the left such that all poles of $\Gamma(\frac{d}{3}+z)\Gamma(1-\frac{d}{3}+z)\Gamma(-z)$ except the ones in $\mathbb{N}_0$ lie left to the path of integration, we obtain with formula (9.113) of [21]

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{K-\varepsilon-i\infty}^{K-\varepsilon+i\infty} \frac{\Gamma\left(z-\frac{d}{3}\right)\Gamma\left(z+1-\frac{d}{3}\right)\Gamma(-z)}{\Gamma\left(z+\frac{2d}{3}\right)}dz\\
&\qquad= \frac{\Gamma\left(\frac{d}{3}\right)\Gamma\left(1-\frac{d}{3}\right)}{\Gamma\left(\frac{2d}{3}\right)}{}_2F_1\left(\frac{d}{3},1-\frac{d}{3};\frac{2d}{3};-1\right)\\
&\qquad\quad+\sum_{k=0}^{K-1}\frac{(-1)^{k+1}\Gamma\left(k+\frac{d}{3}\right)\Gamma\left(k+1-\frac{d}{3}\right)}{k!\Gamma\left(k+\frac{2d}{3}\right)}\\
&\qquad= \frac{\Gamma\left(1-\frac{d}{3}\right)\Gamma\left(\frac{d}{6}\right)}{2\Gamma\left(\frac{d}{2}\right)} - \sum_{k=0}^{K-1}\frac{(-1)^{k}\Gamma\left(k+\frac{d}{3}\right)\Gamma\left(k+1-\frac{d}{3}\right)}{k!\Gamma\left(k+\frac{2d}{3}\right)},
\end{aligned}$$

where the final equality is due to (15.4.26) of [31]. Equation (4.45.1) follows by this calculation together with (4.45.3), (4.45.4), (4.45.5), and Proposition 4.13 (4). Finally note that (4.45.1) vanishes for even $d \le 1$. □

Now we are ready to prove Theorem 4.4.

*Proof of Theorem 4.4.* Note that by Lemma 4.34 and Theorem 4.43 all conditions of Theorem 4.5 are satisfied (with $L$ and $R \notin \frac{1}{3}\mathbb{N}$ arbitrary large). As $\zeta_{\mathfrak{so}(5)}$ has, by Proposition 4.45, exactly two positive poles $\alpha := \frac{1}{2} > \frac{1}{3} =: \beta$, Theorem 4.29 applies with $\ell = 3$, and we obtain

$$\begin{aligned}&r_{\mathfrak{so}(5)}(n)\\&= \frac{C}{n^b}\exp\left(A_1 n^{\frac{1}{3}} + A_2 n^{\frac{2}{9}} + A_3 n^{\frac{1}{9}} + A_4\right)\left(1 + \sum_{j=2}^{N+1}\frac{B_j}{n^{\frac{j-1}{9}}} + O_N\left(n^{-\frac{N+1}{9}}\right)\right),\ (n\to\infty).\end{aligned}$$

So we are left to calculate $c$, $b$, $A_1$, $A_2$, $A_3$, and $A_4$. By Proposition 4.44, $\zeta_{\mathfrak{so}(5)}(0) = \frac{3}{8}$ and by Proposition 4.45, $\operatorname{Res}_{s=\frac{1}{2}} \zeta_{\mathfrak{so}(5)}(s)$, $\omega_{\frac{1}{2}} = \frac{\sqrt{3}\Gamma(\frac{1}{4})^2}{8\sqrt{\pi}}$ and $\omega_{\frac{1}{3}} = \frac{2^{\frac{1}{3}}+1}{3^{\frac{2}{3}}}\zeta(\frac{1}{3})$. Hence, by (4.27.4), we get

$$c_1 \frac{\sqrt{3}\Gamma\left(\frac{1}{4}\right)^2\zeta\left(\frac{3}{2}\right)}{16}, \qquad c_2 = 3^{-\frac{5}{3}}\left(2^{\frac{1}{3}}+1\right)\Gamma\left(\frac{1}{3}\right)\zeta\left(\frac{1}{3}\right)\zeta\left(\frac{4}{3}\right).$$

Moreover, by Lemma 4.28, we have

$$K_2 = \frac{2c_2}{3c_1^{\frac{2}{9}}}, \quad K_3 = -\frac{c_2^2}{27c_1^{\frac{10}{9}}}.$$

Now, we compute $A_1$, $C$, and $b$ by (4.6.4) and $A_2$, $A_3$, $A_4$ by Theorem 4.29 and obtain

$$b = \frac{7}{12}, \qquad C = \frac{e^{\zeta'_{\mathfrak{so}(5)}(0)}\Gamma\left(\frac{1}{4}\right)^{\frac{1}{6}}\zeta\left(\frac{3}{2}\right)^{\frac{1}{12}}}{2^{\frac{1}{3}}3^{\frac{11}{24}}\sqrt{\pi}}, \qquad A_1 = \frac{3^{\frac{4}{3}}\Gamma\left(\frac{1}{4}\right)^{\frac{4}{3}}\zeta\left(\frac{3}{2}\right)^{\frac{2}{3}}}{2^{\frac{8}{3}}}, \tag{4.45.6}$$

$$A_2 = \frac{2^{\frac{8}{9}}\left(2^{\frac{1}{3}}+1\right)\Gamma\left(\frac{1}{3}\right)\zeta\left(\frac{1}{3}\right)\zeta\left(\frac{4}{3}\right)}{3^{\frac{7}{9}}\Gamma\left(\frac{1}{4}\right)^{\frac{4}{9}}\zeta\left(\frac{3}{2}\right)^{\frac{2}{9}}}, \qquad A_3 = -\frac{2^{\frac{40}{9}}\left(2^{\frac{1}{3}}+1\right)^2\Gamma\left(\frac{1}{3}\right)^2\zeta\left(\frac{1}{3}\right)^2\zeta\left(\frac{4}{3}\right)^2}{3^{\frac{44}{9}}\Gamma\left(\frac{1}{4}\right)^{\frac{20}{9}}\zeta\left(\frac{3}{2}\right)^{\frac{10}{9}}},$$

$$A_4 = \frac{2^8\left(2^{\frac{1}{3}}+1\right)^3\Gamma\left(\frac{1}{3}\right)^3\zeta\left(\frac{1}{3}\right)^3\zeta\left(\frac{4}{3}\right)^3}{3^8\Gamma\left(\frac{1}{4}\right)^4\zeta\left(\frac{3}{2}\right)^2}. \tag{4.45.7}$$

This proves the theorem. □

## 4.6 Open problems

We are led by our work to the following questions:

(i) Is there a simple expression for $\zeta'_{\mathfrak{so}(5)}(0)$?

(ii) Can one weaken the hypothesis that $f(n) \ge 0$ for all $n$ in Theorem 4.5? An important application would be that the $r_f(n)$ are eventually positive. There are many special cases in the literature (see [12, 13, 14, 15, 16]), but to the best of our knowledge no general asymptotic formula has been proved.[11]

(iii) In [20], Erdős proved by elementary means that if $S \subset \mathbb{N}$ has natural density $d$ and $1_S$ is the indicator function of $S$, then $\log(p_{1_S}(n)) \sim \pi\sqrt{\frac{2dn}{3}}$. Referring to Theorem 4.5, can one prove by elementary means that for any $\varepsilon > 0$

$$\log(r_f(n)) = A_1 n^{\frac{\alpha}{\alpha+1}} + \sum_{j=2}^{M} A_j n^{\alpha_j} + O(n^{\varepsilon})?$$

(iv) Can one "twist" the products in Theorem 4.5 by $w \in \mathbb{C}$ and prove asymptotic formulas for the (complex) coefficients of

$$\prod_{n\geq 1} \frac{1}{(1-wq^n)^{f(n)}}?$$

If $f(n) = n$ or $f(n) = 1$, then such asymptotics were shown to determine zero attractors of polynomials (see [3, 4]) and equidistribution of partition statistics see [5, 6]), and the general case of $|w| \neq 1$ was treated by Parry [32]. Nevertheless, all of these results require that $L_f(s)$ has only a single simple pole with positive real part.

(v) In Theorem 4.5, can one write down explicit or recursive expressions for the constants $A_j$ in the exponent, say in the case that $L_f(s)$ has three positive poles?

(vi) Can one prove limit shapes for the partitions generated by (4.4.1) in the sense of [18, 37]?

## Conflicts of interest statement

All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

## Data Availability statement

No datasets were generated or analysed during the current study.

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# Abstract

In this dissertation we investigate objects from number theory, $q$-analogues of Multiple Zeta Values, with respect to their algebraic structure.

Multiple Zeta Values are real numbers that appear in various areas of mathematics and theoretical physics. Apart from this, the simple zeta values, i.e. the values of the Riemann zeta function $\zeta$ at integer digits greater than or equal to 2, are also widely used. Little is known about the algebraic properties of the latter values. For example, only one such value of the form $\zeta(2n+1)$ for $n \in \mathbb{Z}_{>0}$ is known to be irrational - namely for $\zeta(3)$ (for $\zeta(2n)$ more is known). If you are now interested in the algebraic structure of these single zeta values, it is in a way natural to consider the so-called Multiple Zeta Values as a generalization, since the product of two Multiple Zeta Values is in turn an integer linear combination of Multiple Zeta Values. For the investigation of their (algebraic) structure, for example, it is natural to study $\mathbb{Q}$-linear relations between Multiple Zeta Values due to various known representations of the product as a linear combination of Multiple Zeta Values.

An $q$-analogue of a Multiple Zeta Value is an $q$-series, which (after possible modification) results in a Multiple Zeta Value in the limit $q \to 1$. Just like the Multiple Zeta Values, their $q$-analogues also fulfill many linear relations in analogy to the Multiple Zeta Values. It is often practical to consider the algebraic structure of Multiple Zeta Values in order to avoid unwanted effects of real numbers. This paper is devoted to the structure of these $q$-analogues. We present different approaches for this purpose: An algebraic, a combinatorial and an analytic one.

The algebraic approach investigates the $\mathbb{Q}$-algebra of $q$-analogues, $\mathcal{Z}_q$, using well-known methods such as duality and the stuffle product representation of the product of $q$-analogues. Here, a special class of linear relations is systematically exploited to obtain new results, in particular with regard to a conjecture from Bachmann's dissertation. For this purpose, so-called formal multiple $q$-zeta values, as introduced in Burmester's dissertation, are used, which algebraically abstract the considered $q$-analogues with respect to the considered class of relations. Bachmann's conjecture states that the algebra of formal multiple $q$-zeta values $\mathcal{Z}_q^f$ corresponds to the subalgebra $\mathcal{Z}_q^{f,\circ}$, where $\mathcal{Z}_q^{f,\circ}$ initially appears much 'smaller' than $\mathcal{Z}_q$ due to its definition. We will further refine this conjecture and give a more precise approach than previously known to a (hopefully) general proof of the conjecture, which still remains open.

The combinatorial approach, on the other hand, leads via so-called marked partitions, which are partitions in whose Young diagram rows and columns are marked in a certain way. All $q$-analogues of Multiple Zeta Values are the generating series of special marked partitions, as is already known from my master thesis. Having clarified in the latter how duality can be described by labeled partitions, we now give an explicit description of the stuffle product at the level of labeled partitions. This is innovative as it allows us to derive a deeper understanding of the stuffle product. Moreover, any linear relation between Multiple $q$-Zeta Values can now presumably be described by labeled partitions, since duality and stuffle product presumably imply all such linear relations. A seminal question is how labeled partitions can be used to prove algebraic conjectures about Multiple ($q$-)Zeta Values.

Finally, the analytical approach deals with the asymptotic behavior of $q$-analogues. There are two different asymptotic behaviors to investigate: By setting $q = e^{-t}$ ($t > 0$) and

looking at the asymptotic evolution of the $q$-analogue for $t \to 0$. Or by examining the asymptotic behavior of the coefficients of $q^n$ of the corresponding $q$-series for $n \to \infty$. Both turn out to be difficult, so that both approaches in this work provide the asymptotic evolution only of special $q$-analogues and leave the evolution for general $q$-analogues of Multiple Zeta Values open as a further subject of research. However, the chosen approach via the asymptotic development of the Fourier coefficients by means of Wright's circle method provides the asymptotic development of many other $q$-series relevant in number theory and beyond.

Each of the three approaches to the algebraic structure of $q$-analogues of Multiple Zeta Values raises new questions and at the same time shows ways to continue.

# Zusammenfassung

In dieser Dissertation untersuchen wir Objekte aus der Zahlentheorie, $q$-Analoga Multipler Zetawerte, hinsichtlich ihrer algebraischen Struktur.

Multiple Zetawerte sind reelle Zahlen, welche in verschiedenen Gebieten der Mathematik und der theoretischen Physik auftauchen. Davon abgesehen finden die einfachen Zetawerte, also die Werte der Riemann'schen Zeta-Funktion $\zeta$ an ganzzahligen Stellen größer oder gleich 2, ebenfalls breite Anwendung. Von dem letztgenannten Werten ist wenig über ihre algebraischen Eigenschaften bekannt. Zum Beispiel weiß man lediglich von genau einem solchen Wert der Form $\zeta(2n+1)$ für $n \in \mathbb{Z}_{>0}$, dass er irrational ist - nämlich für $\zeta(3)$ (für $\zeta(2n)$ ist mehr bekannt). Interessiert man sich nun für die algebraische Struktur dieser einfachen Zetawerte, ist es in gewisser Weise natürlich, als Verallgemeinerung die sogenannten Multiplen Zetawerte zu betrachten, da das Produkt zweier Multipler Zetawerte wiederum eine ganzzahlige Linearkombination Multipler Zetawerte ist. Für die Untersuchung derer (algebraischer) Struktur ist es zum Beispiel aufgrund verschiedener bekannter Darstellungen des Produkts als Linearkombination Multipler Zetawerte wiederum natürlich, $\mathbb{Q}$-Linearrelationen zwischen Multiplen Zetawerten zu studieren.

Ein $q$-Analogon eines Multiplen Zetawerts ist eine $q$-Reihe, welche (nach eventueller Modifikation) im Grenzwert $q \to 1$ einen Multiplen Zetawert ergibt. Ebenso wie die Multiplen Zetawerte erfüllen auch deren $q$-Analoga viele Linearrelationen in Analogie zu den Multiplen Zetawerten. Oftmals ist es praktisch, für die Untersuchung der algebraischen Struktur Multipler Zetawerte deren $q$-Analoga zu betrachten, um nicht gewollte Effekte reeller Zahlen zu umgehen. Diese Arbeit widmet sich nun der Struktur dieser $q$-Analoga. Wir stellen hierfür unterschiedliche Zugangsmöglichkeiten vor: Einen algebraischen, einen kombinatorischen und einen analytischen.

Der algebraische Zugang untersucht die $\mathbb{Q}$-Algebra der $q$-Analoga, $\mathcal{Z}_q$, mit altbekannten Mitteln wie der Dualität und der stuffle-Produkt-Darstellung des Produkts von $q$-Analoga. Hierbei werden eine spezielle Klasse von Linearrelationen systematisch ausgenutzt, um neue Resultate, insbesondere hinsichtlich einer Vermutung aus Bachmanns Dissertation, zu erlangen. Hierfür werden sogenannte formale Multiple $q$-Zetawerte, wie in Burmesters Dissertation eingeführt, verwendet, welche die betrachteten $q$-Analoga hinsichtlich der betrachteten Klasse von Relationen algebraisch abstrahiert. Die genannte Vermutung von Bachmann sagt aus, dass die Algebra der formalen Multiplen $q$-Zetawerte $\mathcal{Z}_q^f$ mit der Unteralgebra $\mathcal{Z}_q^{f,\circ}$ übereinstimmt, wobei $\mathcal{Z}_q^{f,\circ}$ durch ihre Definition zunächst wesentlich 'kleiner' als $\mathcal{Z}_q$ erscheint. Wir werden diese Vermutung weiter verfeinern und geben einen präziseren Ansatz als bislang bekannt zu einem (hoffentlich) allgemeinen Beweis der Vermutung, welcher nach wie vor offen bleibt.

Der kombinatorische Zugang hingegen führt über sogenannte markierte Partitionen, welche Partitionen sind, in deren Young-Diagramm Zeilen und Spalten auf gewisse Weise markiert sind. Sämtliche $q$-Analoga Multipler Zetawerte sind die Erzeugendenreihe von speziellen markierten Partitionen, wie aus meiner Masterarbeit bereits bekannt ist. Nachdem in letzterer geklärt wurde, wie Dualität durch markierte Partitionen beschrieben werden kann, geben wir nun eine explizite Beschreibung des stuffle-Produkts auf dem Level der markierten Partitionen. Dies ist innovativ, da sich hieraus ein tieferes Verständnis des stuffle-Produkts ableiten lässt. Zudem kann nun vermutungsweise jede Linearrelation zwischen Multiplen $q$-Zetawerten durch markierte Partitionen beschrieben werden, da Dualität und stuffle-Produkt vermutungsweise alle solche Linearrelationen implizieren. Eine zukunftsweisende Frage ist, wie sich markierte Partitionen zum Beweis algebraischer Vermutungen über Multiple ($q$-)Zetawerte einsetzen lassen.

Der analytische Zugang zuletzt beschäftigt sich mit dem asymptotischen Verhalten von $q$-Analoga. Hierbei gibt es zwei unterschiedliche asymptotische Verhalten zu untersuchen: Indem man $q = e^{-t}$ $(t > 0)$ setzt und die asymptotische Entwicklung des $q$-Analogons für $t \to 0$ betrachtet. Oder indem man das asymptotische Verhalten der Koeffizienten von $q^n$ der entsprechenden $q$-Reihe für $n \to \infty$ untersucht. Beides stellt sich als schwierig heraus, sodass beide Ansätze in dieser Arbeit die asymptotische Entwicklung nur von speziellen $q$-Analoga liefern und die Entwicklung für allgemeine $q$-Analoga von Multiplen Zetawerten als weiteren Forschungsgegenstand offen lässt. Jedoch gibt der gewählte Ansatz über die asymptotische Entwicklung der Fourierkoeffizienten mittels der Kreismethode nach Wright die asymptotische Entwicklung sehr vieler weiterer, in der Zahlentheorie und darüber hinaus, relevanter $q$-Reihen.

Durch jede der drei Herangehensweise an die algebraische Struktur von $q$-Analoga Multipler Zetawerte werden neue Fragestellungen aufgeworfen und zugleich Wege zur Fortsetzung aufgezeigt.

# Publications related to this dissertation

**i)** Section 1.3 is based on a revised version of

[15] Benjamin Brindle. "Combinatorial interpretation of the Schlesinger-Zudilin stuffle product". Preprint, arXiv:2409.16966 (2024).

The revised version of this preprint builds Chapter 3.

**ii)** Section 1.4 is based on

[12] Walter Bridges et al. "Asymptotic expansions for partitions generated by infinite products". In: *Math. Ann.* 390.2 (2024), pp. 2593–2632.

The paper builds Chapter 4.

# Eigenanteilserklärung

i) Chapter 1: Chapter 1 was written completely by the candidate. It builds the introduction of this dissertation. The new results presented in Subsection 1.4.1 are jointly due to H. Bachmann, J.-W. v. Ittersum, N. Sato, and the candidate who contributed equally. Furthermore, the new results in Subsection 1.4.1 are printed with permission of H. Bachmann, J.-W. v. Ittersum, and N. Sato. The use of Section 1.4.1 is to motivate the results of Section 1.4.3 and Chapter 4. All results in Chapter 1 besides those from Section 1.4.1 are due to the candidate with no contribution of other people.

ii) Chapter 2 ([16]): Chapter 2 corresponds to the work [16] and was written by the candidate. All results up to Lemma 2.64 are due to the candidate. Lemma 2.64 was obtained in a discussion with A. Burmester; both A. Burmester and the candidate contributed equally to Lemma 2.64. The entire work [16] is reproduced in this dissertation with permission of A. Burmester for Lemma 2.64. Some parts of the text are taken verbatim from that work for Section 1.2. The work [16] is intended to submit to a journal.

iii) Chapter 3 (revised version of [15]): Chapter 3 corresponds to a revised version of the preprint [15]. The candidate wrote it, and all results are due to the candidate. The motivation for the project has its origin in a question by H. Bachmann who asked whether the quasi-shuffle product similar to the stuffle product, that so-called *balanced Multiple q-Zeta Values* (introduced in [21, 23]) satisfy, can be described using marked partitions. Some parts of the text are taken verbatim from that preprint for Section 1.3.

iv) Chapter 4 ([12]): Chapter 4 corresponds to the paper [12], co-authored by W. Bridges, K. Bringmann, and J. Franke. The four authors contributed equally to the work. The entire paper [12] is reproduced in this dissertation with permission of all authors. Some parts of the text are taken verbatim from that paper for Section 1.4.3.

# Eidesstattliche Erklärung

Hiermit versichere ich an Eides statt, die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen benutzt zu haben. Sofern im Zuge der Erstellung der vorliegenden Dissertationsschrift generative Künstliche Intelligenz (gKI) basierte elektronische Hilfsmittel verwendet wurden, versichere ich, dass meine eigene Leistung im Vordergrund stand und dass eine vollständige Dokumentation aller verwendeten Hilfsmittel gemäß der Guten wissenschaftlichen Praxis vorliegt. Ich trage die Verantwortung für eventuell durch die gKI generierte fehlerhafte oder verzerrte Inhalte, fehlerhafte Referenzen, Verstöße gegen das Datenschutz- und Urheberrecht oder Plagiate.

Hamburg, 31. März 2025

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