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DISSERTATION

Existence and duality results for BV solutions to linear-growth variational problems with measure data

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The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. I mean that profounder beauty which comes from the harmonious order of the parts, and which a pure intelligence can grasp.

Henri Poincaré (1854–1912)

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Summary

Object of this thesis is the study of convex first–order variational functionals $\mathcal{F}$ with linear growth in the gradient variable, coupled with a non–linear integral term with respect to a (possibly signed) Radon measure on given bounded domains of $\mathbb{R}^n$. After achieving a generalized parametric lower semicontinuity result, we provide necessary and sufficient conditions for the existence of minimizers of $\mathcal{F}$ in the space of functions of bounded variation (BV), discussing (counter)examples and borderline cases for the assumptions. A first step in the existence theory consists of approaching semicontinuity of the (anisotropic) total variation problem with measures in continuation to the parametric outcome of [92], further exploiting lifting of functions to extend the result to a broader class of integrals. The transition from the standard total variation integral to possibly non–even anisotropies with measures is addressed by the choice of a suitable signed isoperimetric condition defined on sets of finite perimeter – also equivalently reformulated on multiple subclasses of BV functions. More importantly, we make use of coarea and layer–cake arguments to move from the parametric formulation with (anisotropic) perimeter to (anisotropic) variations with measures. This method represents the foundation of our global existence theory, and it is subsequently employed to achieve a more general existence result. In parallel, we determine the dual maximization problem corresponding to $\mathcal{F}$ set in the class of divergence–measure vector fields. The outcome of duality enables us to reformulate optimality relations for solutions of the two problems in terms of a refined version of Anzellotti's pairing [7] between maximizing fields of assigned divergence measure and distributional derivatives of minimizers of $\mathcal{F}$. By introducing a suitable notion of solution to the Euler–Lagrange equation associated to $\mathcal{F}$, we then recover the usual meaning of necessary – and, under convexity, also sufficient – condition for minimizers of $\mathcal{F}$, demonstrating that our BV formulation provides a natural extension of the Sobolev model in the sense of L^1–relaxation.

Zusammenfassung

Thema dieser Dissertation ist die Untersuchung konvexer Variationsfunktionale erster Ordnung $\mathcal{F}$ mit linearem Wachstum in der Gradientenvariable, gekoppelt mit einem nichtlinearen Integralterm bezüglich eines (möglicherweise signierten) Radon–Maßes auf einem gegebenen beschränkten Gebiet in $\mathbb{R}^n$. Nach dem Erreichen eines verallgemeinerten parametrischen Resultats zur Unterhalbstetigkeit geben wir notwendige und hinreichende Bedingungen für die Existenz von Minimierern von $\mathcal{F}$ im Raum der Funktionen von beschränkter Variation (BV) an und diskutieren (Gegen–)Beispiele und Grenzfälle für die Annahmen. Ein erster Schritt in der Existenztheorie besteht darin, sich der Unterhalbstetigkeit des (anisotropen) Variationsproblems mit Maßen anzunähern, in Anlehnung an das parametrische Resultat von [92], und dann ein "Lifting" von Funktionen zu nutzen, um das Ergebnis auf eine weitere Klasse von Integralen auszuweiten. Der Übergang vom Variationsintegral zu möglicherweise nicht–geraden Anisotropien mit Maßen wird durch die Wahl einer geeigneten signierten isoperimetrischen Bedingung behandelt, die auf Mengen von endlichem Perimeter definiert ist und äquivalent für mehrere Klassen von BV–Funktionen umformuliert wird. Wir verwenden die Koflächen– und die Layer–Cake–Formel, um von der parametrischen Formulierung mit (anisotropem) Perimeter zu (anisotropen) Variationen mit Maßen überzugehen. Diese Methode bildet die Grundlage unserer globalen Existenztheorie und wird anschließend verwendet, um ein allgemeineres Existenzresultat zu erzielen. Parallel dazu bestimmen wir das duale Maximierungsproblem, das $\mathcal{F}$ in der Klasse der Divergenz–Maß–Vektorfelder entspricht. Das Dualitätsresultat ermöglicht es uns, die Optimalitätsbeziehungen für Lösungen der beiden Probleme mit Hilfe einer verfeinerten Version von Anzellottis Paarung [7] zwischen maximierenden Feldern mit gegebenem Maß als Divergenz und Maßableitungen von Minimierern von $\mathcal{F}$ neu zu formulieren. Durch die Einführung eines geeigneten Lösungsbegriffs für die mit $\mathcal{F}$ verbundene Euler–Lagrange–Gleichung gewinnen wir dann die übliche Form der notwendigen – und unter Konvexität auch hinreichenden – Bedingung für Minimierer von $\mathcal{F}$ zurück und zeigen, dass unsere BV–Formulierung eine sinnvolle Erweiterung des Sobolev–Modells im Sinn einer L^1–Relaxierung darstellt.

Declaration of contributions

Concerning the contents of this dissertation, the analysis of isoperimetric conditions in Chapter 3, the existence theory discussed in Chapters 4 and 5, together with the first half of Chapter 6, are mostly based on the articles [51, 52] elaborated in joint work with Thomas Schmidt. The first research paper [51] includes (re)formulations of anisotropic isoperimetric conditions, the theory of parametric and non–parametric lower semicontinuity (Theorem 4.5 and Theorem 4.8 in the present thesis), together with the corresponding existence and consistency results for homogeneous functionals. Statements and proofs of Results 1–3 in full generality – that is, for possibly inhomogeneous integrands – are reprised from our work [52] instead. The findings above have been achieved through a close collaboration with T. Schmidt, and the contribution is shared by both authors in equal amounts.

The discussion on the two types of Γ–convergence in Chapter 6 and the refinement of the Anzellotti pairing expressed in Chapter 7 are mostly the author's original achievements. Regarding the dual formulation in Chapter 8 – including the statements of Result 4 and Result 5 – the credit should be attributed equally to both T. Schmidt and myself. Notably, the contents of the remaining Sections 6.3 and 6.4 of *Part I*, as well as the theory in *Part II*, are being presented for publication here for the first time.

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Chapter 1

Introduction

In this thesis, we treat minimization problems for first–order variational functionals of the kind

$$\mathrm{F}_{u_0}[w] := \int_\Omega f(x, \nabla w(x))\,\mathrm{d}x + \int_\Omega w^*(x)\,\mathrm{d}\mu(x) \tag{1.0.1}$$

among functions w in the Sobolev space $\mathrm{W}^{1,1}(\Omega)$ under the Dirichlet condition on traces $w = u_0$ a.e. on $\partial\Omega$, for some prescribed datum $u_0 \in \mathrm{L}^1(\partial\Omega; \mathcal{H}^{n-1})$. The boundary condition in question motivates our choice of working with bounded Lipschitz domains Ω in $\mathbb{R}^n$, $n \in \mathbb{N}$. In (1.0.1), we consider an integrand $f = f(x,\xi)\colon \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ convex with linear growth in the gradient variable – that is, $\alpha|\xi| \le f(x,\xi) \le \beta(|\xi|+1)$ for positive constants α, β – whereas μ is a finite Radon measure on Ω vanishing on $\mathcal{H}^{n-1}$–negligible sets. Our primary scope is establishing necessary and sufficient conditions for the existence of variational minima for a suitable reformulation of the functional F_{u_0}, defined now in the space of functions of bounded variation BV in Ω. In fact, we observe that equation (1.0.1) results well–posed for functions w e.g. in $\mathrm{C}^1(\Omega)$, as well as for integrable functions with weak gradient $\nabla w \in \mathrm{L}^1(\Omega, \mathbb{R}^n)$ – this latter case being meaningful provided we take into account the precise representative w^* in the measure integral. In a further step, the established theory of Giaquinta–Modica–Soucek [57] for functionals on measures, in combination with some more subtle selection of function representatives in the measure term, allows for a meaningful extension of (1.0.1) from $\mathrm{W}^{1,1}_{u_0}(\Omega)$ to $\mathrm{BV}(\Omega)$, which in view of its (weak–$*$) compactness represents the natural functional space to achieve minima. As a matter of fact, we will primarily work with the functional

$$\mathcal{F}[w] := f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega}) - \int_\Omega w^+\,\mathrm{d}\mu_- + \int_\Omega w^-\,\mathrm{d}\mu_+\,,$$

suitably extending F_{u_0} to all $w \in \mathrm{BV}(\Omega)$ and with notation discussed in the following Section 1.1.

Examples of first-order convex integrals. Concerning the first term in (1.0.1), the most prominent case is the (non–parametric) *area functional*

$$\mathcal{A}(w,\Omega) := \int_\Omega \sqrt{1 + |\nabla w(x)|^2}\,\mathrm{d}x\,,$$

obtained from the area integrand $\mathbb{R}^n \ni \xi \mapsto f(\xi) := \sqrt{1+|\xi|^2}$. Such a denomination arises from the fact that, at least for continuously differentiable functions w, the value $\mathcal{A}(w,\Omega)$ is the surface area of the graph of w on Ω. Minimizers of such functional alone (or, more precisely, the surface determined by their graph) are called *minimal surfaces*, and they represent hyper–surfaces in $\mathbb{R}^{n+1}$ with zero mean–curvature at every point. It is then straightforward to bring up simple variants of the area–functional such as

$$\int_\Omega \sqrt{c^2 + |\nabla w(x)|^2}\,\mathrm{d}x$$

with some constant $c \in (0,\infty)$, as well as more elaborate ones as the following – originating respectively the Finsler and the Riemannian area integral – by setting

$$f(x,\xi) = \sqrt{c^2 + \varphi(x,\xi)^2} \qquad \text{and} \qquad f(x,\xi) = \sqrt{c^2 + g_x(\xi,\xi)}$$

defined in function of a continuous Finsler metric $\varphi \in \mathrm{C}\big(\overline{\Omega} \times \mathbb{R}^n\big)$ or a Riemannian metric g on $\overline{\Omega}$, respectively. The limit case $c = 0$ reduces in the examples above to the corresponding *homogeneous* version

$$\mathrm{TV}(w,\Omega) := \int_\Omega |\nabla w(x)|\,\mathrm{d}x\,, \quad \mathrm{TV}_\varphi(w,\Omega) := \int_\Omega \varphi(x,\nabla w(x))\,\mathrm{d}x\,, \quad \text{and} \quad \int_\Omega \sqrt{g_x(\nabla w(x),\nabla w(x))}\,\mathrm{d}x$$

for $w \in \mathrm{W}^{1,1}(\Omega)$. The first in the last series is the linear–growth functional is obtained for $f(\xi) = |\xi|$ Euclidean norm on $\mathbb{R}^n$ and takes the name of *total variation* (TV) integral, since it computes the total variation of the w on Ω. The TV can be regarded as limit case of the p–Dirichlet energy functional

$$\mathcal{E}_p[w] := \int_\Omega |\nabla w(x)|^p\,\mathrm{d}x\,, \qquad \text{for } p \in (1,\infty)$$

defined for $w \in \mathrm{W}^{1,p}(\Omega)$. Integrals of the form of $\mathcal{E}_p$ will *not* be included in our treatment, being the integrand $f_p(\xi) := |\xi|^p$ only of p–growth in $\mathbb{R}^n$. However, it is worth mentioning in our context, since a standard approach to the degenerate case consists of treating first the p–energy functional in the reflexive, separable space $\mathrm{W}^{1,p}(\Omega)$, and then sending $p \searrow 1$ in the hope of preserving certain properties for TV minimizers in $\mathrm{W}^{1,1}(\Omega)$. In our work, the generalization of TV to the *anisotropic total variation* integral TV_φ plays a central role, together with its parametric counterpart represented by the anisotropic perimeter of a set. In fact, on the one hand, the presence of φ allows to cover multiple natural phenomena not direction–invariant in the gradient variable, as for instance in image restoration and edge detection [47, 72, 30]; on the other hand, the positive homogeneity of $\varphi(x,.)$ comes with the advantage of preserving coarea–type results, thereby ensuring a direct connection to the parametric problem. We shall see later on that the possibly inhomogeneous functional F_{u_0} associated to f can be expressed in terms of the anisotropic variation of the corresponding *recession function* $f^\infty\colon \overline{\Omega} \times \mathbb{R}^n \to [0,\infty)$, and that our minimization strategy for (1.0.1) essentially relies on the reduction to a problem defined on an appropriate derived function of f^∞.

Literature overview and motivation: From semicontinuity to existence results. We first focus on the existence of BV–minima for convex functionals with linear–growth satisfying a (weak) boundary condition – temporarily neglecting the presence of a measure. In the work [57] from the late 1970s, the authors proved the existence of minima under continuity and coercivity assumptions for f. The reasoning proceeds via direct method of calculus of variations, after verifying that BV–coercivity is preserved for the extension to $\mathrm{BV}(\Omega)$ of the functional $w \mapsto \int_\Omega f(.,\nabla w)$ determined by its $\mathrm{L}^1(\Omega)$–*relaxation*, whereas the missing ingredient of lower semicontinuity (in short, LSC) follows from the crucial result of Reshetnyak for functionals on measures; see the later Sections 2.6 and 2.7. At this point, the authors themselves point out how the existence statement can be easily extended to the variant

$$\inf_w \int_\Omega \big[f(x,\nabla w(x)) + g(x,w(x))\big]\,\mathrm{d}x \tag{1.0.2}$$

defined among functions with trace $w = u_0$ on $\partial\Omega$, whenever g is Carathéodory and it satisfies a bound of the kind

$$\left|\int_\Omega g(x,w(x))\,\mathrm{d}x\right| \le \Gamma + \alpha C \int_\Omega |\nabla w(x)|\,\mathrm{d}x \qquad \text{for all } w \in \mathrm{W}^{1,1}_0(\Omega) \tag{1.0.3}$$

with constants $C \in [0,1)$, $\Gamma \in \mathbb{R}$ and α as in the growth condition on f. If the term $w \mapsto \int_\Omega g(.,w)$ is $\mathrm{L}^1(\Omega)$–lower semicontinuous, then a minimizer of (1.0.2) always exists since (1.0.3) guarantees coercivity, whereas semicontinuity of the joint integral follows from the LSC of the two terms separately. The generality of the paper [57] is remarkable, especially considered that at that time the vast majority of the mathematical community was rather invested in the specific study of the *prescribed mean curvature problem* (PMC), with the generalization from minimal surfaces to *surfaces of prescribed mean curvature* being marked by the additional presence of a function $H\colon \Omega \to \mathbb{R}$. Notice that this latter corresponds in our notation to a specific choice in (1.0.1), namely where f is the area integrand and selecting only absolutely continuous measures $\mu := H\mathcal{L}^n$. In the range of a couple of years, the works of Bombieri–Giusti [20], Giaquinta [56], and Giusti [58] dealt with solvability conditions for the PMC problem for surfaces of prescribed mean curvature expressed by an L^n–integrable function H on Ω:

$$\min_{w} \int_\Omega \left(\sqrt{1+|\nabla w(x)|^2} + H(x)w(x)\right) \mathrm{d}x\,, \tag{1.0.4}$$

possibly imposing a (weak) boundary condition $w = u_0$ on $\partial\Omega$. In the framework of [77] and [59], this is further extended by allowing $H = H(x,t)$ for $t \in \mathbb{R}$, which consequently requires a suitable reformulation of the measure term as a double integral. In order to approach the standard version (1.0.4), one needs to introduce an extra bound for H in Ω in terms of $\mathrm{P}(A)$ *perimeter* of $A \subseteq \mathbb{R}^n$ in the sense of De Giorgi:

$$\left|\int_A H(x)\,\mathrm{d}x\,\right| \le C\mathrm{P}(A) \qquad \text{for all measurable } A \subseteq \Omega\,. \tag{IC}$$

In the years, it has become praxis to name the expression above *isoperimetric condition* (IC) (or *linear isoperimetric inequality*) for H with constant $C \in [0,\infty)$, because it follows from the isoperimetric inequality at least in the original case $H \in \mathrm{L}^n(\Omega)$. The bound in (IC) is well–known to be necessary and sufficient to existence of minima of (1.0.4) in the space $\mathrm{BV}(\Omega)$, under the determinant assumption $C < 1$. Interestingly, in the present work, we show that this is preserved for a large class of integrands with linear growth under the IC for very general measures μ (Result 2). To achieve this, we first need to reformulate the functional on BV and in terms of more general Radon measures on Ω – that is, measures possibly admitting a singular part for μ with respect to $\mathcal{L}^n$. Specifically, in the measure term in (1.0.1) we integrate specific representatives of functions which are actually well–defined μ–a.e. in Ω; on the other hand, the representative w^* of $w \in \mathrm{W}^{1,1}(\Omega)$ (as well as w^+, w^- in case $w \in \mathrm{BV}(\Omega)$) cannot capture the behaviour of w on sets of codimension higher than one (see Section 2.2), hence if the measure μ is too fine the integral term is ill–posed. For this reason, we will always assume for our measures the vanishing condition:

$$\text{For Borel } Z \subseteq \Omega\,, \quad \mathcal{H}^{n-1}(Z) = 0 \implies |\mu|(Z) = 0\,.$$

We also record that an L^p–lower semicontinuity theory for functionals including measures in such generality was developed by Carriero–Leaci–Pascali [27, 28, 29] and Pallara [82] under p–growth conditions. However, in our setting with $p = 1$ this is not sufficient to prove the existence of minima because of a lack of compactness in $\mathrm{W}^{1,1}(\Omega)$. Subsequent examinations on existence in the measure case and for linear–growth integrals were carried out in [97] for the area functional under a stronger IC for non–negative measures, as well as in the works [35, 36], these latter however rather focusing on weak solutions to the mean curvature equation in the viscosity sense. In recent times, the independent works of [92] and [67] adopted a similar set of assumptions to approach L^1–semicontinuity for the area functional. Nevertheless, whereas in the paper of Schmidt [92] the analysis is fully parametric along the lines of Massari's results [70], in [67] Leonardi and Comi deal with the reformulation of the prescribed mean curvature problem originally in the form of

$$\int_\Omega \sqrt{1+|\nabla w(x)|^2}\,\mathrm{d}x + \int_\Omega w^*(x)\,\mathrm{d}\mu \tag{1.0.5}$$

and suitably extended to BV. It was precisely the promising results mentioned above driving our motivation in the study of such variational problems with measures, and indeed the present dissertation aims to establish an existence theory which incorporates the findings of both papers in a broader framework. In doing this, we will partially rely on either quoting or rearranging proofs contained in our papers [51, 52].

Isoperimetric conditions. We now aim to a generalization of the condition in (IC) to measures μ in Ω, thereby achieving the new IC

$$|\mu(A)| \leq C\mathrm{P}(A) \quad \text{for all measurable } A \subseteq \Omega \tag{1.0.6}$$

for some $C \in [0, \infty)$. A more rigorous expression of (1.0.6) will appear later on, and we preliminary record that it admits multiple interesting reformulations, which will be object of our study as well. In particular, the *parametric* version of the IC as introduced above – that is, the IC on sets of the kind of (1.0.6) – is equivalent to certain *non–parametric* formulations on suitable spaces of functions on Ω; compare with the results in Chapter 3. Equivalent conditions for (IC) in the non–negative measure framework were already studied by Meyers and Ziemer [76, 98] with the scope of achieving Poincaré's inequality in BV. However, in contrast to our result, the authors there allowed for adjustments of the constants C in the reformulations, whereas we are able to maintain the same bound in all of the statements. Though our objective is minimizing the non–parametric functional (1.0.1), the equivalent versions of the IC turn particularly usual insofar as a frequent shift to the parametric setting will be employed at convenience in the treatment of our problem. Measures satisfying (IC) are sometimes referred to as Guy–David measures, and – in view of their reformulation in terms of BV functions – they identify elements of the dual of $\mathrm{BV}(\Omega)$. Conditions of the form (IC) have been vastly employed in the literature in relation to the area functional (1.0.5), compare with [48, 95, 58]. The justification behind this usage becomes clear when reasoning as follows: If restricting to regular – for instance, twice–continuously differentiable – solutions u to the associated Euler–Lagrange equation

$$\operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right) = \mu \ \text{ in } \Omega\,, \tag{PMC}$$

known as *prescribed mean curvature measure* (PMC) equation, we can integrate by parts on smooth domains $A \Subset \Omega$ of finite perimeter and inward normal vector ν_A to ∂A to get the necessary condition

$$|\mu(A)| = \left|\int_{\partial A} \frac{\nabla u \cdot \nu_A}{\sqrt{1+|\nabla u|^2}}\,\mathrm{d}\mathcal{H}^{n-1}\right| \leq \int_{\partial A} \mathrm{d}\mathcal{H}^{n-1} = |\mathrm{D}\mathbb{1}_A|(\Omega) = \mathrm{P}(A)$$

under the assumption $|\mu|(\partial A) = 0$; see, for instance, the result of [31, Theorem 2.2] for divergence–measure vector fields. Hence, it is *necessary* that (1.0.6) holds for $C \leq 1$. Furthermore, we shall see that condition (1.0.6) is even *sufficient* to the existence of minimizers for (PMC), this latter provided the constant C is *strictly smaller* than one. The equation in (PMC) can be physically interpreted as modelling capillarity, so that a solution function u identifies the position of a membrane (represented by the graph of u) at equilibrium while subjected to a force determined by μ. In the interesting case of measures concentrated on sets of codimension one, such a model admits applications in the study of water striders or thin insects resting on the water surface.

When approaching the broader case of (1.0.1) in the class of functionals for $f\colon \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ with linear growth, we can still hope to determine necessary conditions introducing an upper bound for the measure – this time in terms of a possibly *anisotropic perimeter* defined according to the recession function f^∞. Indeed, in a completely analogous fashion to our computation for the area functional, it is possible to deduce an anisotropic variant of the IC (1.0.6) from the Euler–Lagrange equation

$$\operatorname{div}\big[\nabla_\xi f(\,\cdot\,, \nabla w)\big] = \mu \ \text{ in } \Omega\,, \tag{EL}$$

to which we add the boundary condition $w = u_0$ for the traces on $\partial\Omega$. We will later show that these two equations together preserve the role of necessary and sufficient condition to the existence of minimizers of $\mathcal{F}$.

A complementary PDE perspective. In parallel to the minimization problem for the BV–extension of F_{u_0}, we also concentrate on solving the elliptic PDE with measure (EL) under boundary condition u_0, where f, μ, and u_0 are defined as above. Our interest is driven by the fact that (EL) represents – at least for differentiable $\xi \mapsto f(x,\xi)$ and (weakly) differentiable w – the Euler–Lagrange equation corresponding to (1.0.1). Nevertheless, we shall see that our assumptions are not limited, for instance, to differentiable integrands f. In fact, for the precise expression of (EL) one should replace $\nabla_\xi f(\,\cdot\,,\nabla w)$ with the set $\partial_\xi f(\,\cdot\,,\nabla w) \subseteq \mathbb{R}^n$, denoting by $\partial_\xi f$ the subdifferential of f with respect to the second variable ξ. Yet another more prominent generalization of (EL) concerns its correct formulation in terms of arbitrary functions with derivative measure, i.e., passing from $w \in \mathrm{W}^{1,1}_{u_0}(\Omega)$ to $w \in \mathrm{BV}(\Omega)$. To this aim, we introduce a Borel field $\sigma\colon \Omega \to \mathbb{R}^n$ and impose the condition $\operatorname{div}(\sigma) = \mu$ on the distributional divergence of σ in Ω. Then, we identify σ with an element of the subdifferential $\partial_\xi f(\,\cdot\,,\nabla w)$ pointwise almost everywhere. As we shall see, this will lead to the reformulation of $\sigma \in \partial_\xi f(\,\cdot\,,\nabla w)$ given by

$$\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n) \qquad \text{and} \qquad f(.,\nabla w) + f^*(.,\sigma) = \sigma \cdot \nabla w \quad \text{a.e. in } \Omega\,.$$

The last equality is defined in terms of the (possibly infinite) convex conjugate function $f^*(x,\xi^*) := \sup_{\xi\in\mathbb{R}^n}(\xi^*\cdot\xi - f(x,\xi))$ of f with respect to the gradient variable, defined for all $x \in \Omega$ and $\xi^* \in \mathbb{R}^n$. For instance, one may verify that in the case of the total variation integrand $f(\xi) := |\xi|$, solving the 1–Laplace equation

$$\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right) = H \;\text{ in } \Omega \quad \text{ with } \nabla w \neq 0 \text{ a.e. in } \Omega$$

corresponds to requiring the existence of a divergence–measure vector field σ such that

$$\sigma\cdot\nabla w = |\nabla w| \;\text{ a.e. in } \Omega\,, \qquad ||\sigma||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \leq 1 \qquad \text{and} \qquad \operatorname{div}(\sigma) = \mu\,.$$

Specifically, we notice that the first bound determines $\sigma = \nabla w/|\nabla w|$ wherever $\nabla w \neq 0$, and a fortiori $|\sigma| = 1$ almost everywhere on $\{\nabla w \neq 0\}$. Alternatively, starting from the prescribed mean curvature equation (PMC) and computing the conjugate of f, we obtain

$$\sigma\cdot\nabla w = \sqrt{1+|\nabla w|^2} - \sqrt{1-|\sigma|^2} \;\text{ a.e. in } \Omega\,, \qquad ||\sigma||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \leq 1 \qquad \text{and} \qquad \operatorname{div}(\sigma) = \mu\,.$$

To extend this approach to any function $w \in \mathrm{BV}(\Omega)$, we consider an operator between σ and now the full derivative $\mathrm{D}w$, so that we can extend the scalar product in a suitable way. Such a pairing $[\![\sigma,\mathrm{D}w]\!]$ was introduced by Anzellotti in [7] – see also [10]– to generalize the intuition of Kohn–Temam [64] of a measure pairing between bounded admissible stresses σ and displacement fields w, where the two functions are extremals of a pair of dual convex variational problems arising in Hencky's elasto–plasticity theory. Anzellotti proved that such a pairing represents a Radon measure and that it verifies an extended Gauss–Green formula in BV, and the pairing was related in the follow–up work [12] to minimization problems for linear–growth integrals with additional terms in $H \in \mathrm{L}^n(\Omega)$. Starting from its physical original setting, the theory of Anzellotti has then found various applications in multiple aspects of mathematical analysis. In our context and concerning the minimal gradient problem, we record the contributions of Mercaldo–Segura de León–Trombetti [73, 74], who came up with the notion of *renormalized solution* u_p of the p–Laplace equation

$$\operatorname{div}(|\nabla w|^{p-2}\nabla w) = H \quad \text{ in } \Omega$$

for $H \in \mathrm{L}^1(\Omega)$ and defined for $w \in \mathrm{W}^{1,p}_0(\Omega)$, $p \in (1,\infty)$. Moreover, the authors have showed that the pointwise limit u of u_p computed for $p \searrow 1$ determines a specific BV solution of the 1–Laplace equation

$$\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right) = H \quad \text{in } \Omega \qquad \text{for all } w \in \mathrm{W}_0^{1,1}(\Omega) \text{ with } \nabla w \neq 0 \text{ a.e. in } \Omega\,.$$

Within this framework, solutions are defined in terms of the pairing of Anzellotti computed for truncated vector fields. More recent applications of the pairing in connection to variational functionals with linear growth include, among others, the contributions of [15, 87, 88, 71, 79, 80, 89, 38, 55, 63, 67]. Even solutions to the parabolic problems of mean curvature flow and (anisotropic) total variation flow can be expressed as functions of Anzellotti's pairing, as done for instance in [4, 5, 16, 6, 78, 1].

Furthermore, it is well–known that (EL) in the distributional sense determines a necessary condition to minimization of the functional (1.0.1) *within the Sobolev space* $\mathrm{W}^{1,1}_{u_0}(\Omega)$, and we record that by convexity of $f(x,.)$ it is even sufficient. However, such a procedure cannot be straightforwardly extended to the BV case, and we want to do so employing the latter pairing. A possible approach in this direction was presented by Anzellotti himself in [11], and it involves explicit computations of first variation formulas – for linear–growth integrands but no measure – in a suitable class of admissible test functions. Nonetheless, we believe that an explicit verification of such a first variation condition is rather challenging, since the equality should be tested for *all* admissible functions. Our method proceeds instead along the lines of [67], relying on the introduction of a notion of BV–*weak solution* to (EL) on the basis of certain *extremality relations* between functional minima and bounded vector fields with divergence equal to the given measure μ. Such results are achieved by rephrasing the minimization problem for (1.0.1) via Fenchel's duality into a maximization problem among suitable divergence–measure fields. To this aim, we shall work with suitable generalizations of Anzellotti's pairing which are a better fit for our framework, and which enable a full recovery of the original link between variational and differential formulation.

In the remainder of Chapter 1, we provide an overview of the mathematical objects considered in the dissertation, introducing the precise set of assumptions and highlighting the main original results of each section.

1.1 Linear-growth functionals with no measure

We work in an open bounded set $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary, and we assign a function u_0 in $\mathrm{W}^{1,1}(\mathbb{R}^n)$ whose trace on $\partial\Omega$ represents our (weak) Dirichlet boundary datum. Next, we state the core set of assumptions on the integrand f defining the leading term of the functional. Notice that the following Assumption 1.1 is set on the full domain $\mathbb{R}^n \times \mathbb{R}^n$, whereas for our minimization purposes it would be enough to have the properties listed below just for $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^n$. However, in the sequel to this work, we will often make use of extended integrals defined on some larger set containing the closure of Ω, so that we find it convenient to introduce our hypotheses directly on the whole space $\mathbb{R}^n$. Alternatively, one might require Assumption 1.1 on $\overline{\Omega} \times \mathbb{R}^n$ only, and at need extend first to $f_{x_0} \colon (\overline{\Omega} \cup U_{x_0}) \times \mathbb{R}^n \to [0,\infty)$ on U_{x_0} neighborhood of $x_0 \in \partial\Omega$, to finally achieve a global extension of f via partitions of unity. A more detailed explanation is postponed to Remark 2.64 in the next chapter. Similarly, the choice of introducing the Dirichlet datum u_0 directly as an element of $\mathrm{W}^{1,1}(\mathbb{R}^n)$ – instead of working, for instance, with boundary traces on $\partial\Omega$ – enables us to avoid the passage to Sobolev extensions on neighborhoods of the boundary.

Assumption 1.1 (admissible integrands)**.** We consider lower semicontinuous Borel functions $f \colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ with $f = f(x,\xi)$ such that:

(i) The restriction $\xi \mapsto f(x,\xi)$ is convex for every $x \in \mathbb{R}^n$;

(ii) There exists $\alpha \in (0,\infty)$ such that $f(x,\xi) \geq \alpha|\xi|$ for all $x,\, \xi \in \mathbb{R}^n$;

(iii) There exists $\beta \in [\alpha, \infty)$ such that $f(x,\xi) \leq \beta(|\xi|+1)$ for all x, $\xi \in \mathbb{R}^n$.

Whenever needed, we will also impose the continuity assumption on f with respect to *both* variables, and/or full continuity of the *recession function* f^∞ of f, defined as $f^\infty(x,\xi) := \lim_{t\to 0^+} tf(x,\xi/t)$ for $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$; a more rigorous explanation is postponed to Section 2.6.1.

We point out that condition (ii) is set to achieve *coercivity* of the corresponding integral in $\mathrm{BV}(\Omega)$, whereas (iii) establishes a linear control from above on the growth in the gradient variable. Altogether, assumptions (ii)–(iii) guarantee well–posedness of the first–order functional with null measure

$$\mathrm{F}^0_{u_0}[w] := \int_\Omega f(x,\nabla w(x))\,\mathrm{d}x \qquad \text{for all } w \in \mathrm{W}^{1,1}_{u_0}(\Omega)\,. \tag{1.1.1}$$

For lower semicontinuous, convex, p–coercive functionals of order 1 defined in the space $\mathrm{W}^{1,p}_{u_0}(\Omega)$ with $p \in (1,\infty)$, the existence of minimizers follows from (weak) semicontinuity by the direct method. However, as already mentioned, a lack of compactness prevents a straightforward application to the limit case $p = 1$. For this reason, we suitably extend the functional (1.1.1) to the (weakly–$*$ compact) space $\mathrm{BV}(\Omega)$ achieving

$$f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega}) := \int_\Omega f(.,\nabla w)\,\mathrm{d}x + \int_\Omega f^\infty\left(.,\frac{\mathrm{d}\mathrm{D}^s w}{\mathrm{d}|\mathrm{D}^s w|}\right)\mathrm{d}|\mathrm{D}^s w| + \int_{\partial\Omega} f^\infty\left(.,(w-u_0)\nu_\Omega\right)\mathrm{d}\mathcal{H}^{n-1} \tag{1.1.2}$$

for all $w \in \mathrm{BV}(\Omega)$, where $\overline{w}^{u_0}$ is the extension by u_0 on the complement of Ω. The introduction of the positively 1–homogeneous recession function f^∞ enables us to generalize (1.1.1) to functions w with non–null singular part of the derivative measure $\mathrm{D}^s w$. It is evident that the expression above reduces to $\mathrm{F}^0_{u_0}$ on the Dirichlet class $\mathrm{W}^{1,1}_{u_0}(\Omega)$. In regard to the minimization problem for (1.1.2), we record that the previous (strong) boundary condition $w = u_0$ on $\partial\Omega$ is here replaced by the presence of a non–negative boundary term $\int_{\partial\Omega} f^\infty\left(.,(w-u_0)\nu_\Omega\right)\mathrm{d}\mathcal{H}^{n-1}$, which introduces an anisotropic penalization linear in the distance of w from u_0. Moreover, from [57] we know that the expression (1.1.2) represents the $\mathrm{L}^1(\Omega)$–relaxation of $\mathrm{F}^0_{u_0}$, that is the functional defined as

$$\mathrm{F}^0_{\mathrm{rel}}[w] := \inf\left\{\liminf_{k\to\infty}\int_\Omega f(x,\nabla w_k(x))\,\mathrm{d}x :\ \mathrm{W}^{1,1}_{u_0}(\Omega) \ni w_k \to w \text{ in } \mathrm{L}^1(\Omega)\right\} \qquad \text{for } w \in \mathrm{BV}(\Omega)\,.$$

Then, $\mathrm{F}^0_{\mathrm{rel}}$ is the largest L^1–lower semicontinuous functional below $\mathrm{F}^0_{u_0}$ on the Dirichlet class $\mathrm{W}^{1,1}_{u_0}(\Omega)$ – that is, the LSC *envelope* of $\mathrm{F}^0_{u_0}$. Consequently, if $\mathrm{F}^0_{u_0}$ itself is LSC on its space of definition, we achieve even equality – that means, $\mathrm{F}^0_{\mathrm{rel}} = \mathrm{F}^0_{u_0}$ on $\mathrm{W}^{1,1}_{u_0}(\Omega)$. Under our specific assumptions on f, the fundamental contribute of Reshetnyak's Theorem 2.67 determines semicontinuity of (1.1.2) with respect to the $\mathrm{L}^1(\Omega)$–topology, hence $\mathrm{F}^0_{\mathrm{rel}}[w]$ is larger or equal than the value of the functional (1.1.2) evaluated on each $w \in \mathrm{BV}(\Omega)$, and existence of BV–minima for $w \mapsto f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega})$ follows from the assumption (ii) and the direct method. Furthermore, in case the integrands are even continuous, an approximation result via sequences with fixed boundary datum u_0 together with a version of Reshetnyak's continuity theorem yields exactly equality, whence a full characterization of relaxation for functionals with linear growth. We refer directly to [57] for further details.

1.2 Linear-growth functionals with measures. Assumptions and main results

Clearly, the existence results mentioned above are still in place if we consider a functional defined on $\mathrm{BV}(\Omega)$ given by the sum of our previous $w \mapsto f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega})$ as in (1.1.2) and the measure term

$$\mathrm{BV}(\Omega) \ni w \mapsto \int_\Omega H(x)w(x)\,\mathrm{d}x\,, \tag{1.2.1}$$

which from the embedding $\mathrm{BV}(\Omega) \hookrightarrow \mathrm{L}^{1^*}(\Omega) = \mathrm{L}^{n'}(\Omega)$ is well–defined and weak–$*$ continuous in BV whenever $H \in \mathrm{L}^n(\Omega)$. Therefore, as observed in [57], the sum functional preserves our required semicontinuity; further examples of measure terms which are LSC *per se* were extensively investigated in the literature. In this regard, the innovative aspect of our treatment – as well as in the works [92, 67] – consists of considering as measure–dependent term the functional

$$\mathrm{BV}(\Omega) \ni w \mapsto \int_\Omega w^- \,\mathrm{d}\mu_+ - \int_\Omega w^+ \,\mathrm{d}\mu_- \tag{1.2.2}$$

with representatives $w^\pm$ taking the upper/lower value of w on the jump points, and where $\mu_\pm$ are non–negative finite Radon measures on Ω vanishing on sets of Hausdorff dimension smaller or equal than $n-1$ and satisfying $\mu = \mu_+ - \mu_-$ in Ω. Clearly, when restricting to $\mathrm{W}^{1,1}(\Omega)$, we can replace $w^\pm$ with the precise representative w^* in both integrals. Notice that, whenever $\mu_\pm$ has a non–null density $H \in \mathrm{L}^1(\Omega)$ with respect to $\mathcal{L}^n$, we recover precisely the integral (1.2.1). To ensure that both terms above are well–posed, we need to impose the following assumption.

Assumption 1.2 (admissible measures)**.** For non–negative Radon measures μ_+ and μ_- on Ω, we suppose $\mu_\pm(Z) = 0$ for every $\mathcal{H}^{n-1}$–negligible Borel set $Z \subseteq \Omega$, and that

$$\int_\Omega w^+ \,\mathrm{d}\mu_\pm < \infty \quad \text{for every non–negative } w \in \mathrm{BV}(\Omega)\,.$$

With Assumption 1.2 in place, our complete functional on BV is well–defined and composed of a LSC term in f together with a (possibly non–semicontinuous) measure integrand term (1.2.2). Therefore, we cannot reason as in the examples mentioned above, but we rather aim to establish semicontinuity of the full functional via a *compensation* effect between the two terms arising from a suitable IC depending on the recession function and its mirrored version $\widetilde{f^\infty}(x,\xi) = f^\infty(x,-\xi)$. Explicitly, we shall set

$$-C\mathrm{P}_{\widetilde{f^\infty}}(A) \le \mu_-(A^1) - \mu_+(A^+) \le \mu_-(A^+) - \mu_+(A^1) \le C\mathrm{P}_{f^\infty}(A) \quad \text{for measurable } A \Subset \Omega\,, \tag{1.2.3}$$

with some constant $C \in [0,\infty)$ – and we record that in the following C will be at most equal to one. Here, A^1 and A^+ denote respectively the measure–theoretic interior and closure of A, corresponding in the parametric setting to level sets of the function representatives w^-, w^+. We shall see in Chapter 3 that requiring (1.2.3) corresponds to the validity of a joint so–called f^∞–IC for (μ_-,μ_+) and a $\widetilde{f^\infty}$–IC on Ω with constant C.

Below we state our main result on semicontinuity.

Result 1 (lower semicontinuity)**.** *We let $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, an open bounded Lipschitz Ω in $\mathbb{R}^n$, and $\mu_\pm$ admissible measures on Ω. We consider a* <u>*continuous*</u> *function $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ admissible in the sense of Assumption 1.1 and furthermore satisfying:*

(H1) *The recession function f^∞ is continuous on $\mathbb{R}^n \times \mathbb{R}^n$;*

(H2) *There exists $M \in \mathbb{R}$ such that $f(x,\xi) \ge f^\infty(x,\xi) - M$ holds for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$.*

Then, if (1.2.3) holds with constant $C = 1$, the functional $\mathcal{F}\colon \mathrm{BV}(\Omega) \to \mathbb{R}$ defined as

$$\mathcal{F}[w] := f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega}) - \int_\Omega w^+ \,\mathrm{d}\mu_- + \int_\Omega w^- \,\mathrm{d}\mu_+ \quad \textit{for all } w \in \mathrm{BV}(\Omega) \tag{1.2.4}$$

is lower semicontinuous in $\mathrm{BV}(\Omega)$ with respect to convergence in $\mathrm{L}^1(\Omega)$.

In reference to the assumptions of Result 1, we point out that the choice of (H1) is quite standard, whereas the additional condition (H2) is rather a technical assumption motivated by our proof strategy. Indeed, the presence of (H2) allows us to achieve convexity of some auxiliary functional defined in dimension $n+1$, as further elaborated in the next Section 1.3 and then in more detail in Chapter 5; compare also with [52, Remark 4.9].

From Result 1 and coercivity, we deduce by direct method the claimed existence result.

Result 2 (existence)**.** *Under the hypotheses of Result 1 and assuming that (1.2.3) holds for some $C \in [0, 1)$, the functional $\mathcal{F}$ admits at least a minimizer in $\mathrm{BV}(\Omega)$.*

Here, the strict upper bound for the constant C in (1.2.3) is fundamental to guarantee coercivity and therefore to ensure existence. Nevertheless, theoretically it could be possible to look for functional minimizers in the borderline case $C = 1$ without resorting to the direct method. This will be investigated after the proof of Result 1 in the last section of Chapter 5, and we shall see that under the IC with $C = 1$ the answer to existence is positive only for homogeneous integrals with bounded u_0.

In a wholly analogous manner to the measure–null case (1.1.2), the LSC Result 1 yields the trivial inequality between the $\mathrm{L}^1(\Omega)$–relaxation $\mathrm{F}_{\mathrm{rel}}$ of F_{u_0} and the functional $\mathcal{F}$:

$$\mathcal{F}[w] \leq \mathrm{F}_{\mathrm{rel}}[w] := \inf\left\{\liminf_{k\to\infty}\int_\Omega f(.,\nabla w_k)\,\mathrm{d}x + \int_\Omega w_k^*\,\mathrm{d}\mu \colon\ \mathrm{W}^{1,1}_{u_0}(\Omega) \ni w_k \to w \text{ in } \mathrm{L}^1(\Omega)\right\}$$

for every w in $\mathrm{BV}(\Omega)$. Actually, we are even able to identify $\mathcal{F}$ with the $\mathrm{L}^1(\Omega)$–relaxed functional of the combined integral

$$\mathrm{F}_{u_0}[w] = \int_\Omega f(.,\nabla w)\,\mathrm{d}x + \int_\Omega w^*\,\mathrm{d}\mu \qquad \text{for } w \in \mathrm{W}^{1,1}_{u_0}(\Omega)\,,$$

where the remaining equality follows from a suitable area–strict approximating result. We shall see that such an approximating sequence can only be expected provided that the measures $\mu_\pm$ not only satisfy (1.2.3), but they are even *mutually singular* on Ω – meaning, their supports are disjoint in Ω.

Result 3 (recovery sequences with prescribed boundary values and consistency)**.** *We set a datum $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, an open bounded Lipschitz $\Omega \subseteq \mathbb{R}^n$ and a* <u>*continuous*</u> *integrand f as in Assumption 1.1 such that (H1) holds. We suppose that $\mu_\pm$ are mutually singular admissible measures on Ω. Then, for any $w \in \mathrm{BV}(\Omega)$, there is a sequence $(w_k)_k$ in $\mathrm{W}^{1,1}_{u_0}(\Omega)$ achieving*

$$\mathcal{F}[w] = \lim_{k\to\infty} \mathrm{F}_{u_0}[w_k]\,, \tag{1.2.5}$$

and with extensions $\overline{w}^{u_0} := w\mathbb{1}_\Omega + u_0\mathbb{1}_{U\setminus\overline{\Omega}}$, $\overline{w_k}^{u_0} := w_k\mathbb{1}_\Omega + u_0\mathbb{1}_{U\setminus\overline{\Omega}}$, respectively of w, w_k to some open bounded domain $U \Supset \Omega$, such that

$$\overline{w_k}^{u_0} \xrightarrow[k\to\infty]{} \overline{w}^{u_0} \text{ area–strictly in } \mathrm{BV}(U)\,.$$

Furthermore, recalling that $\mathcal{F}|_{\mathrm{W}^{1,1}_{u_0}(\Omega)} = \mathrm{F}_{u_0}$, from (1.2.5) it is $\inf\limits_{\mathrm{BV}(\Omega)} \mathcal{F} \leq \inf\limits_{\mathrm{W}^{1,1}_{u_0}(\Omega)} \mathrm{F}_{u_0} \leq \lim\limits_{k\to\infty} \mathrm{F}_{u_0}[w_k] = \mathcal{F}[w]$, whence passing to the infimum for w in $\mathrm{BV}(\Omega)$ we deduce the equality

$$\inf_{\mathrm{BV}(\Omega)} \mathcal{F} = \inf_{\mathrm{W}^{1,1}_{u_0}(\Omega)} \mathrm{F}_{u_0}. \tag{1.2.6}$$

If additionally (1.2.3) is in place with $C = 1$, the functional definition of (1.2.4) is ***consistent*** *with the extension by lower semicontinuity of the Sobolev functional F_{u_0}, that is $\mathcal{F} = \mathrm{F}_{\mathrm{rel}}$ on $\mathrm{BV}(\Omega)$.*

The proof of Result 3 is postponed to Section 6.2.1.

1.3 Lower semicontinuity from homogeneous to general functionals

We now briefly mention some intermediate steps in the proof of the semicontinuity Result 1. Our starting point is achieving LSC for the subcase of (1.2.4) where the leading term is given by the *anisotropic total variation functional*, which, taking into account the measure integrand, is

$$\widehat{\Phi}[w] := |\mathrm{D}w|_\varphi(\Omega) + \int_{\partial\Omega} \varphi(\,\cdot\,,(w-u_0)\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Omega w^-\,\mathrm{d}\mu_+ - \int_\Omega w^+\,\mathrm{d}\mu_- \quad \text{for } w \in \mathrm{BV}(\Omega) \tag{1.3.1}$$

under admissibility assumptions for $\mu_\pm$ and defined for some anisotropy $\varphi\colon \overline{\Omega}\times\mathbb{R}^n \to [0,\infty)$ – namely, a Borel function φ positively 1–homogeneous in its second argument. To prove that the functional (1.3.1) is semicontinuous under a suitable IC with constant $C \in [0,1]$, we first obtain lower semicontinuity for the (parametric) *anisotropic perimeter* functional

$$\mathrm{P}_\varphi(A) + \mu_+(A^1) - \mu_-(A^+) \qquad \text{for measurable } A \subseteq \mathbb{R}^n$$

on sequences of Borel sets (locally) convergent in measure. This is achieved by an adaptation of the good exterior approximation result in [92, Lemma 4.4] to our measures $\mu_\pm$ and to anisotropic perimeters. The crucial point lies then in transferring the result to the non–parametric problem via successive applications of the anisotropic coarea result

$$|\mathrm{D}w|_\varphi(U) = \int_{-\infty}^{\infty} \mathrm{P}_\varphi(\{w>t\}\,,U)\,\mathrm{d}t \qquad \text{for all } w \in \mathrm{BV}(U)$$

for $U \subseteq \mathbb{R}^n$ open, combined with layer–cake formulas for the representatives $w^\pm$ of functions w in $\mathrm{BV}(U)$:

$$\int_U w^-\,\mathrm{d}\nu = \int_0^\infty \nu\big(\{w>t\}^1 \cap U\big)\,\mathrm{d}t \qquad \text{and} \qquad \int_U w^+\,\mathrm{d}\nu = \int_0^\infty \nu\big(\{w>t\}^+ \cap U\big)\,\mathrm{d}t\,.$$

We refer respectively to the semicontinuity Theorems 4.5 and 4.8 in Chapter 4 for rigorous statements and their corresponding proofs. Then, once the homogeneous case is sorted out, we investigate LSC of the functional $\mathcal{F}$ in its most general form. The reasoning behind this is twofold.

- On the one hand, we rely on the simple realization that the area of the graph of a differentiable function w over a domain $\Omega \subseteq \mathbb{R}^n$ coincides with the integral of the norm of the normal vector $(\nabla w(x), -1)$ at every point $(x, w(x))$, which in turn is the total variation of the function $x_0+w(x)$ on the cylinder–type domain $(x_0, x) \in (0,1)\times\Omega =: \Omega_\Diamond$. In formulas, we write

  $$\mathcal{A}(w,\Omega) = \int_\Omega \sqrt{1+|\nabla w(x)|^2}\,\mathrm{d}x = |(1,\nabla w)|(\Omega_\Diamond)\,.$$

  Keeping this basic case in mind, we can associate to any integrand $f\colon \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ under Assumption 1.1 a specific choice of anisotropy φ in $\mathbb{R}^{n+1}$ that is now 1–homogeneous in the joint variable (ξ_0,ξ) in $\mathbb{R}\times\mathbb{R}^n$. Here, we additionally insert an extra variable $x_0 \in \mathbb{R}$ to justify integration on the product domain $\Omega_\Diamond$. We briefly mention that the real variables x_0, ξ_0 are introduced to enable the lifting argument mentioned, therefore constructing a homogeneous integrand $\varphi = \varphi((x_0,x),(\xi_0,\xi))$ on $\Omega_\Diamond \times \mathbb{R}^{n+1}$ starting from our original function $f = f(x,\xi)$ defined on $\Omega\times\mathbb{R}^n$. The additional assumptions (H1) and (H2) play here a determinant role, as they enable for such φ to satisfy all hypotheses of the semicontinuity Theorem 2.78. Simultaneously, we introduce the extended measure $\mu_\Diamond := \mathcal{L}^1 \otimes \mu$ defined on $\Omega_\Diamond$, and then work with a restricted class of BV functions on $\Omega_\Diamond$ composed of elements of the form $w_\Diamond(x_0,x) := x_0 + w(x)$ for $w \in \mathrm{BV}(\Omega)$. A more extensive analysis is postponed to Chapter 5.

- Besides, we want to identify the correct isoperimetric conditions to guarantee solutions of the (possibly inhomogeneous) functional $\mathcal{F}$. To fix the ideas, we consider once again the area integrand $f(\xi) = \sqrt{1+|\xi|^2}$ in $\mathbb{R}^n$. As already mentioned, integrating by parts the corresponding Euler–Lagrange equation (PMC) we get the bound $|\mu(A)| \leq \mathrm{P}(A)$ for all $A \Subset \Omega$ regular of finite perimeter. Nonetheless, one may check that the same procedure applied to the 1–Laplace equation
$$\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right) = \mu \ \text{ in } \Omega \qquad \text{for } \nabla w \neq 0 \text{ a.e.}$$
yields exactly the same condition, being $\nabla w/|\nabla w|$ a unit vector field. This suggests that the IC takes into account only the *homogenized* part of f, namely the recession function f^∞. Obviously, in case the integrand is already homogeneous, the function involved in the IC will be f itself. By doing so, we can link our semicontinuity problem in f to our previous LSC result for anisotropic functionals (1.3.1), with a given anisotropy φ on $\mathbb{R}^{n+1}$.

Whereas the LSC and coercivity results remain unaltered in passing from homogeneous to general linear–growth integrands, concerning the existence of minima in the limit case of the IC, we observe a different behaviour. In fact, when (1.2.3) holds for $C = 1$ (meaning, no smaller constant does satisfy the condition), we distinguish the following cases:

- If $u_0 \notin \mathrm{L}^\infty(\mathbb{R}^n)$, in general existence of minima fails. The counterexample 5.15 shows that the infimum might not be achieved already for homogeneous functionals.
- For boundary data $u_0 \in \mathrm{L}^\infty(\mathbb{R}^n)$, the functional $\mathcal{F}$ admits BV–minima only for $f = f^\infty + c$, for some $c \in \mathbb{R}$ (where clearly the presence of a constant does not affect the behaviour of $\mathcal{F}$). Essentially, this only happens when we reduce to the homogeneous case (1.3.1). Such a conclusion arises from a truncation argument, and Example 5.16 shows that this is not expected to hold for arbitrary functionals.

This covers up the totality of admissible cases for existence, since an IC with (minimal) constant strictly larger than one determines not only loss of coercivity, but even unboundedness from below of the functional $\mathcal{F}$.

1.4 Duality theory for linear-growth functionals with measures

Given the minimization problem (1.0.1), we then focus on determining its *dual reformulation* following the theory of convex duality (or *Fenchel's duality*) as extensively discussed in the monograph [44]. The procedure is standard in convex analysis in finite–dimensional spaces, it involves mathematical objects such as the convex conjugate f^* of a function f and the relation between the two functions as expressed in Fenchel's inequality. The scope of such a theory is to obtain a maximization problem in a class of divergence–measure vector fields, which admits solutions and is equivalent to the original one – namely, so that the values of extremals are preserved. This way, we can relate solutions of the two variational problems in order to determine some *optimality conditions*.

We anticipate that, under the usual admissibility assumptions on mutually singular $\mu_\pm$ and letting $\mu := \mu_+ - \mu_-$, the primal variational problem
$$\inf_{\mathrm{BV}(\Omega)} \mathcal{F} = \inf_{w \in \mathrm{BV}(\Omega)} \left(f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega}) - \int_\Omega w^+ \,\mathrm{d}\mu_- + \int_\Omega w^- \,\mathrm{d}\mu_+ \right) \tag{P}$$
admits as a corresponding dual problem
$$\sup_{\substack{\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n) \\ \operatorname{div}(\sigma)=\mu}} \left(\int_\Omega [\sigma \cdot \nabla u_0 - f^*(.,\sigma)] \,\mathrm{d}x + \int_\Omega u_0^* \,\mathrm{d}\mu \right) \tag{P*}$$

for f^* convex conjugate function of f with respect to the gradient variable. A major advantage is that the dual problem always admits maxima, regardless of the existence of minima for $\mathcal{F}$ – considering any element to be a maximizer in case the supremum is $-\infty$. This happens because the term $\sigma \mapsto \int_\Omega f^*(x,\sigma(x))\,\mathrm{d}x$ is always LSC, whereas the remaining integral term depending on ∇u_0 is continuous, hence $\sigma \mapsto \int_\Omega \sigma\cdot\nabla u_0 - f^*(.,\sigma)$ results upper semicontinuous. Moreover, we anticipate that the linear growth assumption on f determines $f^*(.,\sigma)=\infty$ almost everywhere on Ω whenever $||\sigma||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} > \beta$, so the dual problem (P*) is reduced to maximization on a bounded domain in $\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$.

One way to achieve the equivalence of the two problems (P), (P*) is making use of the vast abstract duality theory developed by Ekeland–Temam [44, Chapter III, IV and V] and thus working on the dual space of finite Radon measures on Ω. Nevertheless, this approach is rather involved, as there is no explicit expression for the involved space $[\mathrm{RM}(\Omega,\mathbb{R}^n)]^*$. And at the same time, abstract duality relies on an articulated measurable selection argument to explicitly compute the conjugate functional of the term in f; compare with [44, Chapter IX, Section 2.1]. For such reasons, we rather follow a different approach, determining first the dual of the minimization problem in $\mathrm{W}^{1,1}_{u_0}$, and then justifying the passage to BV via the equality of infima in Result 3. Even concerning duality in the space $\mathrm{W}^{1,1}_{u_0}$, we prefer to employ a step–wise regularization technique instead of the abstract theory. In detail, assuming in any case convexity of the integrand $f = f(x,\xi)$ in ξ, the procedure is the following.

STEP 1. We first prove a duality formula in $\mathrm{W}^{1,2}_{u_0}$ for $\xi \mapsto f(x,\xi)$ differentiable with quadratic growth and $\mathrm{W}^{1,2}$–coercive (Theorem 8.25);

STEP 2. Then (Theorem 8.28) we pass to the dual formulation in $\mathrm{W}^{1,1}_{u_0}$ for $\xi \mapsto f(x,\xi)$ differentiable under linear–growth exploiting coercive approximations of f;

STEP 3. Finally, we achieve the $\mathrm{W}^{1,1}_{u_0}$ duality for any convex $\xi \mapsto f(x,\xi)$ with linear growth (Theorem 8.29) via appropriate sequences of differentiable approximants.

Our main accomplishment in this context combines the existence Result 2 with the consistency Result 3 to achieve the subsequent statement.

Result 4 (duality formula in BV). *We fix $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, $\Omega \subseteq \mathbb{R}^n$ bounded with Lipschitz boundary, and $\mu_\pm$ admissible measures on Ω which identify the Jordan decomposition of the measure μ on Ω. Then, for any <u>continuous</u> integrand $f\colon \mathbb{R}^n\times\mathbb{R}^n \to [0,\infty)$ admissible as in Assumption 1.1(H1), we have the coincidence*

$$(\mathrm{P}) = \inf_{\mathrm{W}^{1,1}_{u_0}(\Omega)} \mathrm{F}_{u_0} = (\mathrm{P}^*) \ \in [-\infty,\infty)$$

and the dual problem (P) always admits maximizers (possibly with value $-\infty$). Furthermore, if the condition (1.2.3) holds for $C \in [0,1)$ in Ω and the additional assumption (H2) is verified, then both infimum and supremum of the primal and dual problem are attained.*

As a corollary of the duality formula, we will see that maximizing vector fields σ of problem (P*) coincide with subgradients of f evaluated on gradients of some minimizers of (P), provided these latter exist.

Result 5 (optimality conditions in BV). *Under the assumptions of Result 4, we let $u \in \mathrm{BV}(\Omega)$ and $\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ with $\operatorname{div}(\sigma)=\mu$. Then, u is a solution of (P) and σ is a solution of (P*) if and only if both of the following hold:*

(a) $\sigma(x) \in \partial_\xi f(x,\nabla u(x))$ *for $\mathcal{L}^n$-a.e. $x\in\Omega$; and*

(b) $[\![\sigma,\mathrm{D}u]\!]^s_{u_0} = f^\infty(.,\mathrm{D}^s u) + f^\infty(.,(u-u_0)\nu_\Omega)\,\mathcal{H}^{n-1} \llcorner \partial\Omega$ *as measures on $\overline{\Omega}$.*

Moreover, under differentiability of $f(x,.)$ on $\mathbb{R}^n$ for a.e. $x \in \Omega$ and assuming the existence of a minimum of (P), the solution σ of the dual problem (P) is unique up to $\mathcal{L}^n$–negligible sets.*

We believe that a brief comment on Result 5 is in order. For $u \in \mathrm{BV}(\Omega)$, we recall that the writing $\mathrm{D}u = \nabla u \mathcal{L}^n + \mathrm{D}^s u$ identifies the Radon–Nikodým decomposition of $\mathrm{D}u$. The symbol $[\![\sigma, \mathrm{D}u]\!]_{u_0}$ in (b) denotes an up–to–the–boundary variant of the Anzellotti pairing between functions and measures, which takes into account the weak Dirichlet boundary condition and is a finite Radon measure on $\mathbb{R}^n$ with singular part $[\![\sigma, \mathrm{D}u]\!]^s_{u_0}$ with respect to $\mathcal{L}^n$; see Chapter 7. We additionally remark that the relation in (a) can be expressed in terms of the convex conjugate function of the integrand f and inverting the roles of u and σ by an application of the conjugate operator, that is

(a′) $\nabla u \in \partial_{\xi^*} f^*(x, \sigma(x))$ for $\mathcal{L}^n$–a.e. $x \in \Omega$.

Applying the properties of the pairing, we shall see that yet another reformulation of (a) in terms of the absolutely continuous part of $[\![\sigma, \mathrm{D}u]\!]_{u_0}$ is

(a″) $(\sigma \cdot \nabla u)\mathcal{L}^n = [\![\sigma, \mathrm{D}u]\!]^a_{u_0} = (f(.,\nabla u) + f^*(.,\sigma))\mathcal{L}^n$ on Ω.

We stress that no joint isoperimetric condition is assumed for Result 5; nevertheless, we shall see (Proposition 5.1) that, in absence of (1.2.3) for $C \in [0, 1]$, the functional $\mathcal{F}$ becomes unbounded from below, hence no minimizers u exist – and even the class of weak solutions of (EL) introduced next would be empty.

Taking the last result 5 into account, we can finally present a meaningful notion of solution to the formal Euler–Lagrange equation (EL), as expressed in the next Definition 1.

Definition 1 (weak solution of Euler–Lagrange for linear–growth integrals with measures)**.** We consider a continuous function $f \colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ admissible as in Assumption 1.1(H1). Given an open set Ω bounded and Lipschitz in $\mathbb{R}^n$ and measures $\mu_\pm$ admissible on Ω representing the Jordan decomposition of μ on Ω, we say that a function $u \in \mathrm{BV}(\Omega)$ is a *weak solution* of

$$\operatorname{div}\big[\nabla_\xi f(\,\cdot\,, \nabla w)\big] = \mu \ \text{ in } \Omega \tag{EL}$$

under the Dirichlet boundary condition $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ if there exists a vector field $\sigma \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n)$ satisfying $\operatorname{div}(\sigma) = \mu$ as distributions on Ω and

$$[\![\sigma, \mathrm{D}u]\!]_{u_0} = f(., \mathrm{D}\overline{u}^{u_0}) \llcorner \overline{\Omega} + f^*(., \sigma)\mathcal{L}^n \llcorner \Omega \quad \text{ as finite measures on } \overline{\Omega}\,.$$

With such a notion at hand, we will prove that (EL) maintains its original meaning of both necessary and sufficient condition to determine functional minimizers in $\mathrm{W}^{1,1}_{u_0}(\Omega)$, that would be our later Proposition 8.32.

1.5 Thesis structure

We have so far presented the general purpose and enunciated the most prominent results of this dissertation. Below we provide a comprehensive and more detailed list of contents and the thesis plan.

- In **Chapter 2** we fix our notation and recap some essential results in geometric measure theory and convex analysis. Specifically, we discuss functions of bounded variation (BV), sets of finite perimeter, and useful slicing properties of rectifiable sets. Then, we introduce functionals of measures and corresponding (semi)continuity properties, as well as the essential assumptions on anisotropies, which pave the way to our semicontinuity theory in the homogeneous case. We also recall properties of truncations of BV functions, in addition to some approximation results for later usage. At the same time, a strong emphasis is given to the notion of representatives of functions and to sets of assigned density. These will play a determinant role in the correct formulation of our minimization problem in BV.

We then dive into the original contributions, which are exposed separately and ordered in two large *Parts*, with focus respectively on the primal minimization problem and on its dual reformulation.

- A common ground for both *Part I* and *Part II* is the presence of admissible measures as introduced in Definition 3.3. In **Chapter 3**, we partially reprise notions from [92] and [67] to provide a rigorous definition of *isoperimetric conditions* (ICs) for pairs of non–negative Radon measures admissible in the previous sense, and we illustrate our assumptions with the help of examples. Then we prove an equivalence result (Theorem 3.11) between parametric and non–parametric formulations of the ICs. Moreover, since the intuitive generalization of (1.0.1) into (1.2.4) is obtained when considering our measure components $\mu_\pm$ identifying the decomposition of μ, in Theorem 3.16 we provide equivalent conditions to ICs in the specific case of mutually singular pairs of measures.

PART I

- **Chapter 4** deals with semicontinuity and existence theory for minima of anisotropic functionals (whose prototype is the TV functional) depending on measures under the IC (1.2.3) with constant $C < 1$ and as formally introduced in Chapter 3. We will first address semicontinuity for the parametric reformulation in (1.3.1). In detail, Lemma 4.4 of Section 4.2 determines a set which nicely approximates sequences of sets of finite perimeter converging in measure; we then employ the latter Lemma and a weaker form of IC to achieve lower semicontinuity of anisotropic perimeters with measures. Finally, Section 4.3 exploits the statement of Theorem 4.5 to prove semicontinuity of the (non–parametric) functional $\widehat{\Phi}$ (Theorem 4.8) via the passage to auxiliary functionals defined on the full space $\mathrm{BV}(\mathbb{R}^n)$.

- The lower semicontinuity results for the homogeneous case are then generalized in **Chapter 5** to the wide class of integrals with linear growth convex in the gradient as introduced in Assumption 1.1. This is achieved by interpreting inhomogeneous functionals as a particular case of a homogeneous one in dimension $n+1$, up to summing some terms which do not affect semicontinuity. Employing the statements of the previous chapter rephrased in terms of an extra lifting variable and similarly extending admissibility of measures as well as the ICs, we are now in the conditions of proving Result 1. Moreover, in Section 5.1 we demonstrate that the value $C = 1$ determines a threshold for coercivity and boundedness from below of the functional, and joining these considerations together we conclude the validity of Result 2 for $C < 1$. The last Section 5.3 takes into account the limit case of the IC, where we compare existence results between the homogeneous and the most general case.

- The scope of **Chapter 6** it twofold. On the one hand, assuming mutual singularity of $\mu_\pm$, we prove the existence of recovery sequences in $\mathrm{W}^{1,1}$ for the full BV–functional in (1.2.4). If one allows for functions of arbitrary trace on the boundary, Theorem 6.2 follows easily from a truncation argument – and we even find smooth approximants. The analysis differs when prescribing a boundary trace u_0 for the recovery sequence, and some more machinery (including multiple applications of area–strict approximations) is required to obtain the convergence (1.2.5) in Result 3. In such a case, via the above–mentioned semicontinuity Result 1, we conclude that even the equality between infima (1.2.6) holds. In the last Section 6.4, we adapt the theory of semicontinuity to achieve the liminf–inequality for the Γ–limit $\mathcal{F}$ of sequences of functionals with varying measures. An analogous Γ–convergence result (Proposition 6.6) for sequences of functionals extended via different boundary values is a straightforward consequence of the strict convergence of traces.

PART II

- The second half of the work encloses the theory of convex duality stated for our minimization problem in *Part I*. To express the minima–maxima optimality relations in BV for our problem, we rely on a specific Radon measure depending on a given L^∞–vector field and (the derivative measure of) a BV function. Such a measure was introduced by Gabriele Anzellotti in [7] for continuous BV functions and divergence vector fields, later on reprised and expressed in broader generality by [32, 33], and finally extended to an up–to–the–boundary variant in [15, 87, 89]. In **Chapter 7**, we provide an adaptation to our framework of the up–to–the boundary pairing by selecting appropriate BV representatives. This latter version of Anzellotti's pairing turns out to be extremely useful when treating boundary problems of the kind considered in *Part I*. Finally, we extend the set of known properties to the new pairing; in detail, we record a recovery sequences result consistent with our choice of representatives (Proposition 7.10), the boundary trace result of Proposition 7.13 which extends the statements in [7, 87], and lastly a control on the measure in terms of conjugate and polars of arbitrary functions.

- The explicit formulation of the dual maximization problem (P*) of (P) is achieved in **Chapter 8** (Section 8.2) via a step–wise approximation procedure. To do so, first of all in Section 8.1 we recap some preliminaries on convex conjugate functions, images of subdifferentials, and infimal convolution. Then, specializing to our case $f = f(x, \xi)$, we determine two kinds of approximants for f (via Moreau regularization or $\mathrm{W}^{1,2}$–coercive sequences) and their related features. We finally combine the duality results with properties of the pairing treated in Chapter 7 to link the solutions of the primal and dual problem, expressing extremality conditions in terms of the pairing itself; with these latter, we achieve Results 4 and 5. Furthermore, going back to the notion of weak solution introduced in Definition 1 to the equation (EL), in Proposition 8.32 this is demonstrated to characterize BV minimizers for the corresponding functional $\mathcal{F}$. In the final sections of the dissertation, we focus on specific conditions to guarantee uniqueness and enunciate some supplementary regularity results for solutions of (P*) and (P).

Chapter 2

Preliminaries

In the n–dimensional Euclidean space $\mathbb{R}^n$, $n \in \mathbb{N}$, we consider Lebesgue–measurable subsets A and we denote with $|A|$ their n–dimensional volume. To express the coincidence of sets or functions up to Lebesgue–negligible sets, we say that equality holds almost everywhere (in short, *a.e.*). We employ the writing $\mathrm{B}_r(x)$ to indicate the open ball centered at point $x \in \mathbb{R}^n$ of radius $r \in (0, \infty)$; for $x = 0$, we just refer to B_r. We recall that $\omega_n := |\mathrm{B}_1|$ is the volume of the unit ball in $\mathbb{R}^n$, whose topological boundary $\partial \mathrm{B}_1$ is the $(n-1)$–sphere $\mathbb{S}^{n-1}$. For any $k \in [0, n]$, we denote with $\mathcal{H}^k$ the k–dimensional Hausdorff measure on $\mathbb{R}^n$. With the usual notation $\mathrm{Int}(A)$, $\overline{A}$ (or, whenever convenient, $\mathrm{cl}(A)$) we refer respectively to the topological interior and closure of the set $A \subseteq \mathbb{R}^n$ with respect to the norm topology, and $\mathbb{1}_A$ is the characteristic function of A. For $p \in [1, \infty]$ and $X \subseteq \mathbb{R}^n$, we indicate with $\mathrm{L}^p(X)$, $\mathrm{W}^{k,p}(X)$, and $\mathrm{C}^k(X)$ respectively the (real valued) Lebesgue space of p–integrable functions, the Sobolev space of k–times weakly differentiable functions in $\mathrm{L}^p(X)$, and the space of k–times continuously differentiable functions on X – each with its corresponding local version obtained adding the subscript *loc*. For functions with compact support, we use the subscript c, whereas for the closure of functions with compact support in a given space we add the subscript 0.

For a locally compact subset X of $\mathbb{R}^n$, $\mathcal{B}(X)$ is the Borel σ–algebra of X – that is, the smallest σ–algebra generated by the open sets of X. A non–negative (Borel) measure μ on X is a σ–additive set function $\mu\colon \mathcal{B}(X) \to [0, \infty]$ such that $\mu(\emptyset) = 0$, and we implicitly understand all measures μ to be completed to the larger σ–algebra of μ–measurable sets in the spirit of [3, Definition 1.40]. A non–negative measure μ is finite provided $\mu(X) < \infty$, and it is Radon in case μ is just locally finite – i.e. $\mu(B) < \infty$ for all $B \in \mathcal{B}(X)$ relatively compact in X. The space of equivalence classes of functions f on X such that $|f|^p$ is μ–integrable is denoted by $\mathrm{L}^p(X; \mu)$, omitting the second argument in case $\mu = \mathcal{L}^n$. Two non–negative measures μ_1, μ_2 on X are *mutually singular* (written $\mu_1 \perp \mu_2$) if there exists $A \in \mathcal{B}(X)$ such that $\mu_1(A) = \mu_2(X \setminus A) = 0$. For $m \in \mathbb{N}$, an $\mathbb{R}^m$–valued (Borel) measure is a σ–additive mapping $\nu\colon \mathcal{B}(X) \to \mathbb{R}^m$ such that $\nu(\emptyset) = 0$. We say that ν is *vector–valued* whenever $m > 1$, whereas ν is a *signed* measure if $m = 1$. For a ν–measurable set $A \subseteq X$, the restriction of ν to A is $\nu \llcorner A(B) := \nu(A \cap B)$. A set function ν defined on the relatively compact Borel sets of X with values in $\mathbb{R}^m$ is a *Radon measure* on X, written $\nu \in \mathrm{RM}_{\mathrm{loc}}(X, \mathbb{R}^m)$, if $\nu(\emptyset) = 0$ and ν is σ–additive. For any $\mathbb{R}^m$–valued measure ν on X, we define its *total variation* $|\nu|$ as the (non–negative) measure

$$|\nu|(B) := \sup \left\{ \sum_{i \in \mathbb{N}} |\nu(B_i)| : \ B = \bigcup_{i \in \mathbb{N}} B_i \ \ B_i \in \mathcal{B}(X) \text{ pairwise disjoint} \right\} \text{ for every } B \in \mathcal{B}(X)\,,$$

which clearly enjoys $|\nu|(B) \geq |\nu(B)|$ for any B Borel. If ν Radon measure is defined on the whole $\mathcal{B}(X)$ and satisfies $|\nu|(X) < \infty$, we say that ν is a *finite Radon measure* on X. Assigned $m \in \mathbb{N}$, the space of all $\mathbb{R}^m$–valued finite Radon measures on X is $\mathrm{RM}(X, \mathbb{R}^m) := \{\nu \in \mathrm{RM}_{\mathrm{loc}}(X, \mathbb{R}^m) : \ |\nu|(X) < \infty\}$, endowed with the norm $||\nu||_{\mathrm{RM}(X, \mathbb{R}^m)} := |\nu|(X)$. From now on, we omit the target space when working with signed Radon measures, and we refer to $\mathrm{RM}_{(\mathrm{loc})}(X)$ in place of $\mathrm{RM}_{(\mathrm{loc})}(X,\mathbb{R})$. The *support* of an $\mathbb{R}^m$–valued measure ν is defined as $\mathrm{supp}(\nu) := \mathrm{cl}\left(\{x \in X : |\nu|(U) > 0 \text{ for all } U \text{ neighborhood of } x \text{ in } X\}\right)$. For a signed measure ν on X, we introduce its *positive* (respectively, *negative*) *part* $\nu_+ := (|\nu| + \nu)/2$ (or resp. $\nu_- := (|\nu| - \nu)/2$). We refer to the writing $\nu = \nu_+ - \nu_-$ as the *Jordan decomposition* of ν. Mutual singularity between measures is extended to vector–valued measures ν_1, ν_2, and we shall say that ν_1 is singular to ν_2 if the corresponding total variation measures are mutually singular, and in such a case we record $|\nu_1 + \nu_2| = |\nu_1| + |\nu_2|$. A measure ν is *concentrated* on the set $B \in \mathcal{B}(X)$ if $|\nu|(X \setminus B) = 0$.

Given a non–negative, σ–finite measure μ and any $\mathbb{R}^m$–valued measure ν, we say that ν is *absolutely continuous* with respect to μ (and write $\nu \ll \mu$) if for every B Borel such that $\mu(B) = 0$, then $|\nu|(B) = 0$. In the definition of absolute continuity, we set the σ–finiteness assumption on μ to render the definition consistent with the statement of Radon–Nikodým's Theorem 2.1 and with the Lebesgue decomposition Theorem 2.2. For instance, regardless of the fact that $\mathcal{H}^k(A) = 0$ for some $k < n$ and measurable $A \subseteq \mathbb{R}^n$ implies $\mathcal{L}^n(A) = 0$, yet we do not formally say that the measure $\mathcal{L}^n$ is absolutely continuous with respect to $\mathcal{H}^k$ – being this latter measure *not* σ–finite on $\mathbb{R}^n$. Measures such as in the last example will play a prominent role in the present work.

2.1 Fundamentals of measure theory

We begin with the following fundamental measure–theoretic result in the formulations of [90, Satz 2.45], [46, Theorem 1.30], and [3, Theorem 1.28].

Theorem 2.1 (Radon–Nikodým). *Given a non–negative, σ–finite measure μ on X and an $\mathbb{R}^m$–valued measure ν on X absolutely continuous with respect to μ, there exists a μ–measurable function $f \colon X \to \mathbb{R}^m$ uniquely defined μ–a.e. in X and such that*

$$\nu(B) = \int_B f \,\mathrm{d}\mu \qquad \text{for all } B \in \mathcal{B}(X)\,,$$

written $\nu = f\mu$. If $\mu \in \mathrm{RM}_{(\mathrm{loc})}(X,\mathbb{R}^m)$, then $f \in \mathrm{L}^1_{(\mathrm{loc})}(X,\mathbb{R}^m;\mu)$.

We refer to μ as *basis measure* and to f as the **Radon-Nikodým density** of ν with respect to μ, and we will denote f via the formal ratio $\mathrm{d}\nu/\mathrm{d}\mu$. Closely related to Theorem 2.1 is the following measure decomposition result (see e.g. [90, Satz 2.46], [46, Theorem 1.31], [3, Theorem 1.28]).

Theorem 2.2 (Lebesgue decomposition). *Let μ be a non–negative, σ–finite measure on X and ν an $\mathbb{R}^m$–valued measure on X. Then there exists a unique pair of $\mathbb{R}^m$–valued measures (ν^a, ν^s) on X such that $\nu = \nu^a + \nu^s$, $\nu^a \ll \mu$ and $\nu^s \perp \mu$.*

The measure ν^a is called the **absolutely continuous part** of ν with respect to μ, whereas ν^s is its **singular part**. Applying Theorem 2.1 to the absolutely continuous part of ν, so that $\nu^a \ll \mu$, we can write $\nu^a = f\mu$ for the Radon–Nikodým density $f = \mathrm{d}\nu^a/\mathrm{d}\mu$. Overall, we obtain the unique writing $\nu = f\mu + \mu^s$.

As a corollary of the two theorems above and recalling that any measure is absolutely continuous with respect to its total variation, we obtain the following.

Corollary 2.3 (polar decomposition). *If μ is an $\mathbb{R}^m$–valued measure on X, there exists a unique function $f \colon X \to \mathbb{R}^m$ such that:*

- *$|f| = 1$ holds $|\mu|$–almost everywhere on X; and*
- *$\mu = f|\mu|$.*

We record that the space $\mathrm{RM}(X, \mathbb{R}^m)$ endowed with the total variation measure as norm is a Banach space; nevertheless, when working with Radon measures, the norm (or strong) convergence is usually too restrictive. For this reason, we rather employ two kinds of weaker convergences – namely, weak–$*$ and (different kinds of) strict convergence – in the space of finite Radon measures, which heavily rely on the characterization of $\mathrm{RM}(X, \mathbb{R}^m)$ as dual space. In fact, Riesz' representation theorem states that $\mathrm{RM}(X, \mathbb{R}^m)$ is isometrically isomorph to the dual space of $\mathrm{C}_0(X, \mathbb{R}^m)$ (the latter being defined as the completion of $\mathrm{C_c}(X, \mathbb{R}^m)$ with respect to the supremum norm) via the identification of any measure $\mu \in \mathrm{RM}(X, \mathbb{R}^m)$ with the functional $\int_X \varphi \cdot \mathrm{d}\mu$ defined on all $\varphi \colon X \to \mathbb{R}^m$ continuous and vanishing at infinity; see, for instance, [3, Theorem 1.54].

From now on, we will replace a generic locally compact $X \subseteq \mathbb{R}^n$ with U open in $\mathbb{R}^n$.

Definition 2.4 (weak–$*$ convergence in RM)**.** For U open in $\mathbb{R}^n$, we say that a sequence $(\mu_k)_k$ in $\mathrm{RM}(U, \mathbb{R}^m)$ converges weakly–$*$ to $\mu \in \mathrm{RM}(U, \mathbb{R}^m)$ for $m \in \mathbb{N}$, and write $\mu_k \overset{*}{\rightharpoonup} \mu$, if $(\mu_k)_k$ weak–$*$ converges to μ as a linear functional on $\mathrm{C}_0(U, \mathbb{R}^m)$, namely if

$$\lim_{k\to\infty} \int_U \varphi \cdot \mathrm{d}\mu_k = \int_U \varphi \cdot \mathrm{d}\mu \qquad \text{for all } \varphi \in \mathrm{C}_0(U, \mathbb{R}^m).$$

Similarly, for $(\mu_k)_k$, μ in $\mathrm{RM}_{\mathrm{loc}}(U, \mathbb{R}^m)$ we define local weak–$*$ convergence employing the Riesz' duality with the space $\mathrm{C_c}(U, \mathbb{R}^m)$. Moreover, we intend convergence of $(\mu_k)_k$ to μ weakly–$*$ in $\mathrm{RM}\left(\overline{U}, \mathbb{R}^m\right)$ as the convergence $\int_{\overline{U}} \varphi \cdot \mathrm{d}\mu_k \xrightarrow[k\to\infty]{} \int_{\overline{U}} \varphi \cdot \mathrm{d}\mu$ for every $\varphi \in \mathrm{C}_0(\overline{U}, \mathbb{R}^m)$ – this latter replaced by $\mathrm{C_c}(\overline{U}, \mathbb{R}^m)$ in case of weak–$*$ convergence in $\mathrm{RM}_{\mathrm{loc}}\left(\overline{U}, \mathbb{R}^m\right)$.

Notice that the identification of $\mathrm{RM}(U, \mathbb{R}^m)$ with the dual space $[\mathrm{C}_0(U, \mathbb{R}^m)]^*$ yields by the Theorem of Banach–Alaoglu the following crucial compactness result.

Proposition 2.5 (weak–$*$ compactness in RM)**.** *Any sequence $(\mu_k)_k$ in $\mathrm{RM}(U, \mathbb{R}^m)$ bounded in norm (i.e. such that $\sup_{k\in\mathbb{N}} |\mu_k|(U) < \infty$) admits a weak–$*$ convergent subsequence to some $\mu \in \mathrm{RM}(U, \mathbb{R}^m)$.*

Moreover, the following useful properties hold.

Proposition 2.6 (semicontinuity of total variation respect to weak–$*$ convergence in RM)**.** *If $(\mu_k)_k$ is a sequence of $\mathbb{R}^m$–valued Radon measures on the open $U \subseteq \mathbb{R}^n$ with μ_k converging weakly–$*$ to some μ in $\mathrm{RM}(U, \mathbb{R}^m)$ as $k \to \infty$, then we have:*

(i) (*lower semicontinuity*). *Any open set $A \subseteq U$ is such that* $\liminf_{k\to\infty} |\mu_k|(A) \geq |\mu|(A)$;

(ii) (*upper semicontinuity*). *If $m = 1$ and all μ_k are non–negative, any compact set K in U satisfies* $\limsup_{k\to\infty} \mu_k(K) \leq \mu(K)$.

Another type of convergence for Radon measures is the stronger notion of Ψ–strict convergence.

Definition 2.7 (Ψ–strict convergence for measures)**.** Let U be locally compact in $\mathbb{R}^n$, $\Psi \in \mathrm{L}^1(U)$ non–negative. We say that the sequence of measures $(\mu_k)_k$ in $\mathrm{RM}(U, \mathbb{R}^m)$ converges **Ψ-strictly** to some μ in $\mathrm{RM}(U, \mathbb{R}^m)$ provided $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathrm{RM}(U, \mathbb{R}^m)$ as $k \to \infty$, and assumed that for the sequence of $(m+1)$–valued measures $(\Psi\mathcal{L}^n, \mu_k)$ it holds $|(\Psi\mathcal{L}^n, \mu_k)|(U) \to |(\Psi\mathcal{L}^n, \mu)|(U)$ as $k \to \infty$. Similarly, Ψ–strict convergence of $(\mu_k)_k$ to μ in $\mathrm{RM}\left(\overline{U}, \mathbb{R}^m\right)$ indicates a weak–$*$ convergence in $\mathrm{RM}\left(\overline{U}, \mathbb{R}^m\right)$ combined with $|(\Psi\mathcal{L}^n, \mu_k)|\left(\overline{U}\right) \to |(\Psi\mathcal{L}^n, \mu)|\left(\overline{U}\right)$ as $k \to \infty$.

We record that reasoning via subsequences one may argue that any sequence $(\mu_k)_k$ converging Ψ–strictly to μ in $\mathrm{RM}(U, \mathbb{R}^m)$ is such that the total variation measure is such that $|(\Psi\mathcal{L}^n, \mu_k)| \overset{*}{\rightharpoonup} |(\Psi\mathcal{L}^n, \mu)|$ weakly–$*$ in $\mathrm{RM}(U)$ for $k \to \infty$. Specifically, in such a case Proposition 2.6 holds for the non–negative measures $|(\Psi\mathcal{L}^n, \mu_k)|$, $|(\Psi\mathcal{L}^n, \mu)|$ in place of μ_k and μ, respectively.

The cases $\Psi \equiv 0$ and $\Psi \equiv 1$ in U (the latter only meaningful for U of finite measure) in Definition 2.7 are of remarkable importance, and the corresponding convergences take the name of **strict** and **area-strict** convergence of Radon measures, respectively. We notice that, under strict convergence, the second requirement in Definition 2.7 reduces to $|\mu_k|(U) \to |\mu|(U)$ for $k \to \infty$. We shall see from a later result by Reshetnyak (Corollary 2.69) that area–strict convergence induces not only weak–$*$, but also strict convergence in RM, whereas the reverse implication in general is not true. In turn, strict convergence is stronger than weak-$*$ convergence, but the inverse implication fails already in dimension one, as presented next.

Example 2.8 (weak–$*$ convergence does not imply strict convergence in RM)**.** We assume $U := (a, b)$, for $-\infty < a < 0 < b < \infty$ and we consider the measures $\mu :\equiv 0$ in U, $\mu_k := \delta_{1/k} - \delta_0$ for any $k \in \mathbb{N}$, observing that $\mu_k \in \mathrm{RM}(U)$ for k large enough. Then, the sequence $(\mu_k)_k$ converges to μ weakly–$*$ in $\mathrm{RM}(U)$, since fixed any $\varphi \in \mathrm{C}_0(U)$ the continuity of φ in 0 returns

$$\int_\Omega \varphi \,\mathrm{d}\mu_k = \int_\Omega \varphi \,\mathrm{d}(\delta_{1/k} - \delta_0) = \varphi(1/k) - \varphi(0) \xrightarrow[k\to\infty]{} 0.$$

Nevertheless, we cannot expect strict convergence in $\mathrm{RM}(U)$ because the mutual singularity of the Dirac measures concentrated on two distinct points yields $|\mu_k| = \delta_{1/k} + \delta_0$ for every k, hence $|\mu_k|(U) \to 2$ for $k \to \infty$, whereas clearly $|\mu|(U) = 0$.

The following statement tells us that, under Ψ–strict convergence, the limsup inequality in Proposition 2.6 can be extended to any closed (not necessarily compact) set.

Proposition 2.9. *If $(\mu_k)_k$ converges Ψ–strictly to μ in $\mathrm{RM}(U, \mathbb{R}^m)$ as $k \to \infty$, then:*

(i) (*upper semicontinuity*). *For any closed set $C \subseteq U$, it is* $\limsup\limits_{k\to\infty} |(\Psi\mathcal{L}^n, \mu_k)|(C) \le |(\Psi\mathcal{L}^n, \mu)|(C)$.

Proof. Recalling that the Ψ–strict convergence induces weak–$*$ convergence $|(\Psi\mathcal{L}^n, \mu_k)| \overset{*}{\rightharpoonup} |(\Psi\mathcal{L}^n, \mu)|$ in $\mathrm{RM}(U)$ for $k \to \infty$, the thesis follows straightforwardly from Proposition 2.6(i) applied to the open set $U \cap C^{\mathrm{c}}$ and by $|(\Psi\mathcal{L}^n, \mu_k)|(U) \to |(\Psi\mathcal{L}^n, \mu)|(U)$ as for $k \to \infty$ by hypothesis. ☐

The following result provides an equivalent condition to Ψ–strict convergence of Radon measures.

Proposition 2.10 (characterization of Ψ–strict convergence)**.** *Let $(\mu_k)_k$, μ be $\mathbb{R}^m$–valued finite Radon measures on the open set $U \subseteq \mathbb{R}^n$ and a non–negative $\Psi \in \mathrm{L}^1(U)$. Then $(\mu_k)_k$ converges Ψ–strictly to μ in $\mathrm{RM}(U, \mathbb{R}^m)$ if and only if:*

(i) *The sequence $(\mu_k)_k$ converges weakly–$*$ in $\mathrm{RM}(U, \mathbb{R}^m)$ to μ as $k \to \infty$; and*

(ii) *It holds* $\lim\limits_{k\to\infty} \int_U \varphi \,\mathrm{d}|(\Psi\mathcal{L}^n, \mu_k)| = \int_U \varphi \,\mathrm{d}|(\Psi\mathcal{L}^n, \mu)|$ *for all bounded functions $\varphi \in \mathrm{C}(U)$.*

We refer the reader to [3, Proposition 1.80] for the proof of the necessary condition in Proposition 2.10; the sufficient one comes straightforward by taking $\varphi \equiv 1$ on U, hence $|(\Psi\mathcal{L}^n, \mu_k)|(U) \to |(\Psi\mathcal{L}^n, \mu)|(U)$ for $k \to \infty$, which together with (i) induces Ψ–strict convergence.

Specifically, if the measures μ_k, μ are non–negative on U, condition (ii) applied to $\Psi \equiv 0$ reads:

$$\int_U \varphi \,\mathrm{d}\mu_k \xrightarrow[k\to\infty]{} \int_U \varphi \,\mathrm{d}\mu \qquad \text{for all bounded } \varphi \in \mathrm{C}(U)\,,$$

valid in particular for all $\varphi \in \mathrm{C}_0(U)$, therefore that in this case (ii) alone induces weak–$*$ convergence $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathrm{RM}(U)$ as $k \to \infty$. We deduce the following characterization of strict convergence.

Corollary 2.11. *For a sequence $(\mu_k)_k$ of non–negative Radon measures on open $U \subseteq \mathbb{R}^n$, and given some $\mu \in \mathrm{RM}(U)$ non–negative, we have*

$$\mu_k \xrightarrow[k\to\infty]{} \mu \text{ strictly in } U \quad \Longleftrightarrow \quad \lim_{k\to\infty} \int_U \varphi \, \mathrm{d}\mu_k = \int_U \varphi \, \mathrm{d}\mu \quad \text{for all bounded } \varphi \in \mathrm{C}(U)\,.$$

For further usage, we also record that Corollary 2.11 determines an extension property for sequences of strictly–converging, non–negative measures on open sets.

Corollary 2.12 (strict convergence is preserved by trivial extensions)**.** *We assume that $(\mu_k)_k$ and μ are finite non–negative Radon measures on $U \subseteq \mathbb{R}^n$ and we introduce the trivially extension measures $\overline{\mu_k}$, $\overline{\mu}$ on $\mathbb{R}^n$ defined as $\overline{\mu_{(k)}}(S) := \mu_{(k)}(S \cap \Omega)$ for all Borel sets S in $\mathbb{R}^n$. Then, if $\mu_k \rightharpoonup \mu$ strictly in $\mathrm{RM}(U)$ as $k \to \infty$, the extensions satisfy $\overline{\mu_k} \rightharpoonup \overline{\mu}$ strictly in $\mathrm{RM}(\mathbb{R}^n)$ for $k \to \infty$.*

Proof. From Corollary 2.11 it suffices to show that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\overline{\mu_k} = \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\overline{\mu} \quad \text{for all } \varphi \in \mathrm{C}(\mathbb{R}^n) \text{ bounded}\,.$$

This is easily obtained, since any function φ in the class above is such that its restriction $\varphi|_U$ is continuous and bounded in U, hence by $\overline{\mu}(U^{\mathrm{c}}) = \overline{\mu_k}(U^{\mathrm{c}}) = 0$ for all k and the characterization of strict convergence $\mu_k \rightharpoonup \mu$ in $\mathrm{RM}(U)$ we read

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\overline{\mu_k} = \lim_{k\to\infty} \int_U \varphi \, \mathrm{d}\mu_k = \int_U \varphi \, \mathrm{d}\mu = \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\overline{\mu}\,,$$

as required, whence the strict convergence of the extensions is proved. □

As a side note, we emphasize that the extension result of Corollary 2.12 does *not* apply to weakly–$*$ convergence only, as exemplified next.

Example 2.13 (weak–$*$ convergence is not preserved)**.** Fix an open interval $U := (a, b)$ for $-\infty < a < b \leq \infty$ and the sequence of Dirac measures $\mu_k := \delta_{a+1/k}$ on U. Choosing any test function $\varphi \in \mathrm{C}_0(U)$, it is

$$\lim_{k\to\infty} \int_U \varphi \, \mathrm{d}\mu_k = \lim_{k\to\infty} \varphi(a + 1/k) = 0 = \int_U \varphi \, \mathrm{d}\mu\,,$$

for the null measure $\mu \equiv 0$ on U, hence $\mu_k \rightharpoonup \mu$ weakly–$*$ in $\mathrm{RM}(U)$ as $k \to \infty$. Nevertheless, supposing that the extended sequence converges weakly–$*$ in the space of finite Radon measures in $\mathbb{R}$, assigned the smooth cut–off function $\widehat{\varphi}$ with $\mathbb{1}_U \leq \widehat{\varphi} \leq \mathbb{1}_{U'}$ for $U' := (a-1, b+1)$, weak–$*$ convergence of extensions would induce

$$0 = \int_{\mathbb{R}} \widehat{\varphi} \, \mathrm{d}\overline{\mu} = \lim_{k\to\infty} \int_{\mathbb{R}} \widehat{\varphi} \, \mathrm{d}\overline{\mu_k} = \lim_{k\to\infty} \widehat{\varphi}(a + 1/k) = 1\,,$$

which is absurd. The issue lies in the fact that the sequence of Dirac measures concentrates closer and closer to the boundary of U, but since $\{a\} \notin U$, the limit measure will be constantly null on U. Actually, one easily verifies that the sequence of extensions $(\overline{\mu_k})_k$ converges (weakly–$*$ and strictly) to the measure δ_a in $\mathrm{RM}(\mathbb{R})$.

We now recall the elementary relation between the measure of superlevel sets of integrable functions and their integral, known as the measure–theoretic statement of Chebyschev's inequality; see, for instance, [85, Section 4.3].

Lemma 2.14 (Chebyshev's inequality)**.** *Given a non–negative measure ν on an open set U in $\mathbb{R}^n$ and a function $u \in \mathrm{L}^1(U;\nu)$, for any $t > 0$ it holds*

$$\nu\big(\{x \in U : \ |u(x)| \geq t\}\big) \leq \frac{1}{t} \int_U |u(x)| \, \mathrm{d}\nu(x)\,.$$

Next, we quote the standard layer–cake formula enabling the rewriting of measure integrals as integration on the real line of measure of superlevel sets; we refer, for instance, to [86, Theorem 8.16].

Theorem 2.15 (layer–cake formula)**.** *We assume U is an open set in $\mathbb{R}^n$. If ν is a non–negative measure on U that is σ–finite and $u\colon U \to [0,\infty)$ is ν–measurable, then for any Borel set $B \subseteq U$ we have*

$$\int_B (\theta \circ u)\,\mathrm{d}\nu = \int_0^\infty \theta'(t)\,\nu(\{u > t\} \cap B)\,\mathrm{d}t$$

for any non–decreasing function $\theta\colon [0,\infty) \to [0,\infty)$ absolutely continuous on $[0,T]$ for every $T < \infty$ and such that $\theta(0) = 0$. In particular, letting $\theta(t) := t$ we deduce

$$\int_B u\,\mathrm{d}\nu = \int_0^\infty \nu\big(\{u > t\} \cap B\big)\,\mathrm{d}t\,. \tag{2.1.1}$$

2.2 Functions of bounded variation

From now on, U will always denote an open subset of $\mathbb{R}^n$; when distinctly discussing the bounded Lipschitz case, we will write Ω instead. At this point, we need to introduce the space of functions of bounded variation on U in terms of distributional gradients, then we will list some fundamental properties following mostly [3, Chapter 3]. For our scopes, it suffices to restrict to real–valued BV functions.

Definition 2.16 (BV functions)**.** A function $u \in \mathrm{L}^1_{(\mathrm{loc})}(U)$ is (locally) of bounded variation in U – written $u \in \mathrm{BV}_{(\mathrm{loc})}(U)$ – if its distributional derivative is a $\mathbb{R}^n$–valued (resp. locally finite) Radon measure on U, i.e. if there exists $\mathrm{D}u \in \mathrm{RM}_{(\mathrm{loc})}(U,\mathbb{R}^n)$ such that

$$\int_U u\,\operatorname{div}(\varphi)\,\mathrm{d}x = -\int_U \varphi \cdot \mathrm{d}\mathrm{D}u \qquad \text{for all } \varphi \in \mathrm{C}^\infty_\mathrm{c}(U,\mathbb{R}^n)\,.$$

Clearly, all functions in $\mathrm{W}^{1,1}(U)$ belong to the space $\mathrm{BV}(U)$, and any Sobolev function u has distributional derivative $\mathrm{D}u$ which reduces to $\nabla u \mathcal{L}^n$; for the absolutely continuous part ∇u of $\mathrm{D}u$ we will omit the symbol $\mathcal{L}^n$ unless required by the context. A mollification argument proves the validity of the constancy theorem even for functions in BV, namely every $u \in \mathrm{BV}_\mathrm{loc}(U)$ with total derivative $\mathrm{D}u \equiv 0$ is almost everywhere constant in each connected component of U.

Definition 2.17 (variation)**.** For a function $u \in \mathrm{L}^1_\mathrm{loc}(U)$ the **variation** of u in $W \Subset U$ open is

$$\mathrm{TV}(u,W) := \sup\left\{\int_W u\,\operatorname{div}(\varphi)\,\mathrm{d}x\colon\ \varphi \in \mathrm{C}^\infty_\mathrm{c}(W,\mathbb{R}^n)\,,\ |\varphi| \le 1 \text{ a.e. in } W\right\}.$$

A function $u \in \mathrm{L}^1(U)$ is in $\mathrm{BV}(U)$ if and only if its total variation $\mathrm{TV}(u,U)$ in U is finite. In such a case, we have $|\mathrm{D}u|(.) = \mathrm{TV}(u,.)$ as variation measure associated to $\mathrm{D}u$, and – with some abuse of notation – we refer to $|\mathrm{D}u|(B)$ as the **total variation** of u on Borel $B \subseteq U$. Another equivalent writing for $\mathrm{TV}(u,B)$ is $\int_B |\mathrm{D}u|$.

A crucial property of the variation is its lower semicontinuity (LSC).

Proposition 2.18 (lower semicontinuity of TV)**.** *For $(u_k)_k$, $u \in \mathrm{L}^1_\mathrm{loc}(U)$ such that $u_k\mathcal{L}^n \overset{*}{\rightharpoonup} u\mathcal{L}^n$ weakly–$*$ in $\mathrm{RM}_\mathrm{loc}(U)$ for $k \to \infty$, it is*

$$\mathrm{TV}(u,U) \le \liminf_{k\to\infty} \mathrm{TV}(u_k,U).$$

Introducing the norm $||u||_{\mathrm{BV}(U)} := ||u||_{\mathrm{L}^1(U)} + |\mathrm{D}u|(U)$ on $\mathrm{BV}(U)$, this latter is a Banach space with respect to the norm (or strong) convergence; moreover, it can be proved that $\mathrm{BV}(\Omega)$ for Ω Lipschitz is continuously embedded into the Lebesgue space on Ω with Sobolev exponent $1^* := n/(n-1) = n'$ (under the convention $1^* = \infty$ whenever $n = 1$), whereas for $p \in [1, 1^*)$ the embedding $\mathrm{BV}(\Omega) \hookrightarrow \mathrm{L}^p(\Omega)$ is even compact.

Although it is necessary to introduce a norm topology on BV, in analogy to the behaviour of the space RM the notion of norm–convergence in BV is too strong in the majority of cases. For this reason, we introduce some weaker types of convergence based on analogous definitions of weak convergence of measures – recalling that the distributional derivative of a function of bounded variation is a finite measure.

Definition 2.19 (weak convergences in BV)**.** Given $(u_k)_k$ and u in $\mathrm{BV}(U)$, we say that $(u_k)_k$ converges:

- weakly–$*$ to u in $\mathrm{BV}(U)$, and we write $u_k \overset{*}{\rightharpoonup} u$, if $u_k\mathcal{L}^n \overset{*}{\rightharpoonup} u\mathcal{L}^n$ in $\mathrm{RM}(U)$ and $\mathrm{D}u_k \overset{*}{\rightharpoonup} \mathrm{D}u$ in $\mathrm{RM}(U, \mathbb{R}^n)$ for $k \to \infty$;
- Ψ–strictly to u in $\mathrm{BV}(U)$ for some $\Psi \in \mathrm{L}^1(U)$ non–negative, if $u_k \to u$ strongly in $\mathrm{L}^1(U)$ and the sequence of derivative measures converges Ψ–strictly in $\mathrm{RM}(U, \mathbb{R}^n)$, this latter meaning $\mathrm{D}u_k \overset{*}{\rightharpoonup} \mathrm{D}u$ weakly–$*$ in $\mathrm{RM}(U, \mathbb{R}^n)$ and $|(\Psi\mathcal{L}^n, \mathrm{D}u_k)|(U) \to |(\Psi\mathcal{L}^n, \mathrm{D}u)|(U)$ for $k \to \infty$. Just like in the measure case, we speak of *strict* convergence of $(u_k)_k$ to u if $\Psi \equiv 0$ and of *area–strict* convergence for $\Psi \equiv 1$ in U with finite measure.

Applying the definition of weak–$*$ convergence in BV, from Proposition 2.18 we read the following semicontinuity result.

Corollary 2.20 (lower semicontinuity of TV with respect to weak–$*$ convergence in BV)**.** *For $u \in \mathrm{BV}(U)$ and a sequence $(u_k)_k$ in $\mathrm{BV}(U)$ such that u_k converges to u weakly–$*$ in $\mathrm{BV}(U)$ as $k \to \infty$, we have*

$$|\mathrm{D}u|(U) \leq \liminf_{k\to\infty} |\mathrm{D}u_k|(U).$$

Proposition 2.21. *Any weak–$*$ converging sequence in $\mathrm{BV}(U)$ is bounded.*

Proof. If $(u_k)_k$, $u \in \mathrm{BV}(U)$ with $u_k \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(U)$, we know that both $(u_k\mathcal{L}^n)_k$ and the corresponding sequence of derivatives converge as finite Radon measures on U. By the characterization of $\mathrm{RM}(U)$ as dual space, from the weak–$*$ convergence $u_k\mathcal{L}^n \overset{*}{\rightharpoonup} u\mathcal{L}^n$ as $k \to \infty$ we deduce that the linear functionals $F_k \colon \mathrm{C}_0(U) \to \mathbb{R}$, $F_k[\varphi] := \int_U u_k\varphi \,\mathrm{d}x$, are such that for any φ the sequence $(F_k[\varphi])_k$ is bounded. Applying the theorem of Banach–Steinhaus, we achieve even boundedness of $(F_k)_k$ in $[\mathrm{C}_0(U)]^*$, meaning $\sup_k ||u_k\mathcal{L}^n||_{[\mathrm{C}_0(U)]^*} = \sup_k ||u_k\mathcal{L}^n||_{\mathrm{RM}(U)} < \infty$. In an analogous way, exploiting $\mathrm{D}u_k \overset{*}{\rightharpoonup} \mathrm{D}u$ in $\mathrm{RM}(U, \mathbb{R}^n)$ for $k \to \infty$, we may also conclude that $(\mathrm{D}u_k)_k$ is bounded in $\mathrm{RM}(U, \mathbb{R}^n)$. Finally, we compute

$$\sup_{k\in\mathbb{N}} ||u_k||_{\mathrm{BV}(U)} \leq \sup_{k\in\mathbb{N}} ||u_k||_{\mathrm{L}^1(U)} + \sup_{k\in\mathbb{N}} ||\mathrm{D}u_k||_{\mathrm{RM}(U,\mathbb{R}^n)} = \sup_{k\in\mathbb{N}} ||u_k\mathcal{L}^n||_{\mathrm{RM}(U)} + \sup_{k\in\mathbb{N}} ||\mathrm{D}u_k||_{\mathrm{RM}(U,\mathbb{R}^n)} < \infty\,.$$

This implies our claimed boundedness in $\mathrm{BV}(U)$. □

As mentioned in Chapter 1, the main reason for extending variational functionals with linear growth is that their natural space of definition $\mathrm{W}^{1,1}$ is *not* a relatively compact space, namely, there exist sequences bounded in norm with no converging subsequence. Instead, we retrieve such a property on the larger space BV.

Theorem 2.22 (compactness in BV)**.** *Assume $U \subseteq \mathbb{R}^n$ is an open set. Then every bounded sequence in $\mathrm{BV}(U)$ admits a weakly–$*$ converging subsequence in $\mathrm{BV}(U)$. If we are in an open, bounded, and Lipschitz $\Omega \subseteq \mathbb{R}^n$, then the weak–$*$ converging subsequence achieves even strong convergence in $\mathrm{L}^1(\Omega)$.*

Notice that the last theorem provides roughly an inverse statement to Proposition 2.21, and its validity sets the basis to the so–called *direct method* in the calculus of variations (see later Theorem 2.47). This latter is employed to show the existence of minimizers for variational problems in BV, and in our framework it will be largely applied; see, for instance, Chapter 5. The second half of the statement in Theorem 2.22 can be achieved as a combination of Banach–Alaoglu and Rellich compactness theorem.

Next we record the following result on BV extensions of functions outside Lipschitz domains, joining the statements in [3, Theorems 3.87 and 3.88] and [3, Corollary 3.89].

Theorem 2.23 (boundary traces and extension of BV functions)**.** *Let* Ω *be an open bounded subset of* $\mathbb{R}^n$ *with Lipschitz boundary and* $u \in \mathrm{BV}(\Omega)$*. Then, for* $\mathcal{H}^{n-1}$*–a.e.* $x \in \partial\Omega$ *there exists* $\mathrm{T}_{\partial\Omega}u(x) \in \mathbb{R}$ *such that*

$$\lim_{r\to 0} r^{-n} \int_{\Omega\cap \mathrm{B}_r(x)} |u(y) - \mathrm{T}_{\partial\Omega}u(x)| \,\mathrm{d}y = 0\,.$$

The boundary trace function $\mathrm{T}_{\partial\Omega}u$ *is in* $\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})$ *and the boundary trace operator* $u \mapsto \mathrm{T}_{\partial\Omega}u$ *is bounded and continuous with respect to strict convergence in* $\mathrm{BV}(\Omega)$*, explicitly*

$$\lim_{k\to\infty} \int_{\partial\Omega} |\mathrm{T}_{\partial\Omega}u_k - \mathrm{T}_{\partial\Omega}u| \,\mathrm{d}\mathcal{H}^{n-1} = 0 \qquad \textit{for all } (u_k)_k,\ u \in \mathrm{BV}(\Omega) \textit{ s.t. } u_k \rightharpoonup u \textit{ strictly in } \mathrm{BV}(\Omega)\,.$$

Moreover, for any given $v \in \mathrm{BV}\big(U \setminus \overline{\Omega}\big)$ *for open* $U \Supset \Omega$*, we introduce the function* w *defined as*

$$w := \begin{cases} u, & \textit{in } \Omega;\\ v, & \textit{in } U\setminus\overline{\Omega}\,.\end{cases}$$

Then, $w \in \mathrm{BV}(U)$ *with derivative measure*

$$\mathrm{D}w = \mathrm{D}u + \big(\mathrm{T}^{\mathrm{int}}_{\partial\Omega}u - \mathrm{T}^{\mathrm{ext}}_{\partial\Omega}v\big)\nu_\Omega \mathcal{H}^{n-1} \llcorner \partial\Omega + \mathrm{D}v \qquad \textit{on } U \tag{2.2.1}$$

for ν_Ω *generalized inner normal vector to* $\partial\Omega$*. Here we denote with* $\mathrm{T}^{\mathrm{int}}_{\partial\Omega}u$ *(resp.* $\mathrm{T}^{\mathrm{ext}}_{\partial\Omega}v$*) the boundary trace function of* u *(resp.* v*) on* $\partial\Omega$*, each calculated as average limit of integrals on half–balls with orientation determined by* ν_Ω *(resp. by its opposite).*

We refer to $\mathrm{T}^{\mathrm{int}}_{\partial\Omega}u$ (resp. $\mathrm{T}^{\mathrm{ext}}_{\partial\Omega}v$) as the inner (resp. outer) boundary trace on $\partial\Omega$ of the function w with respect to Ω. Whenever clear from the context, the trace symbol will be omitted. We additionally record that according to [54, Theorem 1.II] the boundary trace operator is surjective on $\mathrm{W}^{1,1}(\Omega)$, meaning that for any given $\mathcal{H}^{n-1}$–integrable function ψ on $\partial\Omega$ there exists a function in $\mathrm{W}^{1,1}(\Omega)$ with trace ψ.

With the aim of addressing variational problems under some generalized boundary conditions, we will often deal with the following extension case. Assume that the function $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ is assigned and consider the extension of $u \in \mathrm{BV}(\Omega)$ via u_0 outside Ω as

$$\overline{u}^{u_0} := u\mathbb{1}_\Omega + u_0\mathbb{1}_{\mathbb{R}^n\setminus\overline{\Omega}} \in \mathrm{BV}(\mathbb{R}^n)$$

with boundary traces u, $u_0 \in \mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})$ with respect to Ω. Exploiting the decomposition (2.2.1) of the derivative of $\overline{u}^{u_0}$ and passing to the total variation, it follows

$$|\mathrm{D}\overline{u}^{u_0}| = |\mathrm{D}u| \llcorner \Omega + |u - u_0|\mathcal{H}^{n-1} \llcorner \partial\Omega + |\mathrm{D}u_0| \llcorner \big(\mathbb{R}^n \setminus \overline{\Omega}\big) \qquad \text{as measures on } \mathbb{R}^n\,.$$

Observe that, for these purposes, it would be enough to require u_0 to be defined on an arbitrary open set (bounded, to enable area–strict convergence) contained in the complement of $\overline{\Omega}$. Nevertheless, our necessity of evaluating the datum u_0 on Ω will become evident in Chapter 6.

To simplify notation, we find it convenient to introduce the following notion of generalized Dirichlet class.

Definition 2.24 (class $\mathrm{BV}_{u_0}(\overline{U})$ and convergences)**.** We assume U is an open set in $\mathbb{R}^n$ and we let $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$. We define the class

$$\mathrm{BV}_{u_0}(\overline{U}) := \left\{u \in \mathrm{BV}(U) \colon\ \overline{u}^{u_0} := u\mathbb{1}_U + u_0\mathbb{1}_{\mathbb{R}^n\setminus U} \in \mathrm{BV}(\mathbb{R}^n)\right\}$$

and consider a non–negative function $\Psi \in \mathrm{L}^1(U)$. For $(u_k)_k$ and u in $\mathrm{BV}_{u_0}(\overline{U})$, we say that the sequence $(u_k)_k$ converges weakly–$*$ to u in $\mathrm{BV}_{u_0}(\overline{U})$ provided the extensions satisfy $\overline{u_k}^{u_0} \overset{*}{\rightharpoonup} \overline{u}^{u_0}$ weakly–$*$ on $\mathrm{BV}(\mathbb{R}^n)$ as $k \to \infty$. The sequence converges Ψ–strictly in $\mathrm{BV}_{u_0}(\overline{U})$ if the convergence $\overline{u_k}^{u_0} \rightharpoonup \overline{u}^{u_0}$ is $\overline{\Psi}^0$–strict in $\mathrm{BV}(\mathbb{R}^n)$, where $\overline{\Psi}^0$ extends Ψ to zero outside Ω.

Notice that in the definition of Ψ–convergence in $\mathrm{BV}_{u_0}(\overline{U})$ one may equivalently require $\overline{u_k}^{u_0} \rightharpoonup \overline{u}^{u_0}$ strictly with respect to any non–negative extension of Ψ integrable on the whole $\mathbb{R}^n$ (in place of the trivial extension by 0). From Theorem 2.23, it is clearly $\mathrm{BV}_{u_0}(\overline{\Omega}) = \mathrm{BV}(\Omega)$ whenever $U = \Omega$ is bounded with Lipschitz boundary. Moreover, we highlight that from the a.e.–coincidence $\overline{u_k}^{u_0} = \overline{u}^{u_0} = u_0$ outside $\overline{\Omega}$ for all $k \in \mathbb{N}$, the interesting requirement of convergences in $\mathrm{BV}_{u_0}(\overline{\Omega})$ is the sequence behaviour precisely up to the boundary of Ω. Since all our results are stated on Lipschitz domains, we see how the notion of $\mathrm{BV}_{u_0}(\overline{\Omega})$ is here essentially introduced to address convergences of sequences avoiding repetitive references to extensions in the statements. In fact, the next proposition enables us to achieve weak–$*$ convergence of the extended sequences starting from a convergence in Lipschitz Ω.

Proposition 2.25. *We fix $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, Ω bounded Lipschitz in $\mathbb{R}^n$ and consider $(u_k)_k$, u in $\mathrm{BV}(\Omega)$.*

(i) *If $(u_k)_k$ converges to u weakly–$*$ in $\mathrm{BV}(\Omega)$ as $k \to \infty$, then it also converges weakly–$*$ in $\mathrm{BV}_{u_0}(\overline{\Omega})$.*

(ii) *Consider a function $\Psi \in \mathrm{L}^\infty(\Omega)$ such that $\operatorname{ess\,inf}_\Omega \Psi > 0$. If $(u_k)_k$ converges Ψ–strictly to u in $\mathrm{BV}(\Omega)$ as $k \to \infty$, then the convergence holds even Ψ–strictly in $\mathrm{BV}_{u_0}(\overline{\Omega})$.*

Proof. We first prove the statement (i). By Proposition 2.21 and the assumed convergence, we find that $s := \sup ||u_k||_{\mathrm{BV}(\Omega)} < \infty$. Then, applying the decomposition of derivative measures and the boundedness of the trace operator in Theorem 2.23, we write

$$\begin{aligned}\sup_{k\in\mathbb{N}} |\mathrm{D}\overline{u_k}^{u_0}|(\mathbb{R}^n) &\le \sup_{k\in\mathbb{N}} |\mathrm{D}u_k|(\Omega) + \sup_{k\in\mathbb{N}} ||u_k - u_0||_{\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})} + |\mathrm{D}u_0|\left(\overline{\Omega}^{\mathrm{c}}\right)\\ &\le (C+1)s + C||u_0||_{\mathrm{W}^{1,1}(\Omega)} + |\mathrm{D}u_0|\left(\overline{\Omega}^{\mathrm{c}}\right),\end{aligned}$$

for some $C \in (0,\infty)$. Hence, the sequence $(\overline{u_k}^{u_0})_k$ is bounded in $\mathrm{BV}(\mathbb{R}^n)$, and via Theorem 2.22 plus uniqueness of the weak limit we infer that a subsequence must converge to some $w \in \mathrm{BV}(\mathbb{R}^n)$ weakly–$*$ in $\mathrm{BV}(\mathbb{R}^n)$, and in particular the restriction on Ω yields (in case, passing to another non–relabelled subsequence) the strong convergence $u_k \to w$ in $\mathrm{L}^1(\Omega)$ as $k \to \infty$. At the same time, $u_0\mathcal{L}^n \llcorner \overline{\Omega}^{\mathrm{c}} = \overline{u_k}^{u_0}\mathcal{L}^n \llcorner \overline{\Omega}^{\mathrm{c}} \overset{*}{\rightharpoonup} w\mathcal{L}^n \llcorner \overline{\Omega}^{\mathrm{c}}$, implying $w = u_0$ a.e. on the complement of Ω, and altogether $\overline{u_k}^{u_0} \overset{*}{\rightharpoonup} \overline{u}^{u_0}$ weakly–$*$ in $\mathrm{BV}(\mathbb{R}^n)$, up to subsequences. Moreover, the weak–$*$ convergence $\overline{u_k}^{u_0} \overset{*}{\rightharpoonup} \overline{u}^{u_0}$ in $\mathrm{RM}(\mathbb{R}^n)$ holds for the whole sequence, otherwise we can find $(u_{k_\ell})_\ell$ subsequence and $\widehat{\varphi} \in \mathrm{C}_0(\mathbb{R}^n)$ such that $L_1 := \lim_{\ell\to\infty} \int_{\mathbb{R}^n} \widehat{\varphi}\overline{u_{k_\ell}}^{u_0}\,\mathrm{d}x \neq \int_{\mathbb{R}^n} \widehat{\varphi}\overline{u}^{u_0}\,\mathrm{d}x$. Nevertheless, by assumption the sequence $(u_k\mathcal{L}^n)_k$ is bounded in $\mathrm{RM}(\Omega)$ – and therefore also is its subsequence $(u_{k_\ell})_\ell$. Then, boundedness of $(u_{k_\ell})_\ell$ induces (by weak–$*$ compactness in RM) weak–$*$ convergence of yet another subsequence to $\overline{u}^{u_0}$, which is a contradiction; we then proved $\overline{u_k}^{u_0}\mathcal{L}^n \overset{*}{\rightharpoonup} \overline{u}^{u_0}\mathcal{L}^n$ weakly–$*$ in $\mathrm{RM}(\mathbb{R}^n)$ as $k \to \infty$. We now claim that the whole sequence $(\overline{u_k}^{u_0})_k$ shall converge to $\overline{u}^{u_0}$ in $\mathrm{BV}(\mathbb{R}^n)$. Indeed, if this is not the case, it shall hold $\mathrm{D}\overline{u_k}^{u_0} \overset{*}{\not\rightharpoonup} \mathrm{D}\overline{u}^{u_0}$ weakly–$*$ in $\mathrm{RM}(\mathbb{R}^n,\mathbb{R}^n)$. Therefore, there must be some subsequence $(u_{k_\ell})_\ell$ and some $\widehat{\varphi} \in \mathrm{C}_0(\mathbb{R}^n,\mathbb{R}^n)$ with $L_2 := \lim_{\ell\to\infty} \int_{\mathbb{R}^n} \widehat{\varphi}\cdot\mathrm{d}\mathrm{D}\overline{u_{k_\ell}}^{u_0} \neq \int_{\mathbb{R}^n} \widehat{\varphi}\cdot\mathrm{d}\mathrm{D}\overline{u}^{u_0}$. Still, the sequence $(\overline{u_{k_\ell}}^{u_0})_\ell$ is bounded, and it determines as above yet another subsequence of $(u_{k_\ell})_\ell$ converging to $\overline{u}^{u_0}$ weakly–$*$ in $\mathrm{BV}(\mathbb{R}^n)$, in particular $L_2 = \int_{\mathbb{R}^n} \widehat{\varphi}\cdot\mathrm{d}\mathrm{D}\overline{u}^{u_0}$, which is absurd. Then, according to Definition 2.24 we have proved weak–$*$ convergence of $(u_k)_k$ to u in $\mathrm{BV}_{u_0}(\overline{\Omega})$.

To achieve (ii) instead, we recall that Ψ–strict convergence in some $\mathrm{BV}(U)$ includes strong convergence in $\mathrm{L}^1(U)$ and weak–$*$ convergence of the derivatives $\mathrm{D}u_k \overset{*}{\rightharpoonup} \mathrm{D}u$ in $\mathrm{RM}(U,\mathbb{R}^n)$, thus taking into account (ii) it is left to prove that $\left|\left(\overline{\Psi}^0\mathcal{L}^n,\mathrm{D}\overline{u_k}^{u_0}\right)\right|(\mathbb{R}^n)\to\left|\left(\overline{\Psi}^0\mathcal{L}^n,\mathrm{D}\overline{u}^{u_0}\right)\right|(\mathbb{R}^n)$ as $k\to\infty$, for $\overline{\Psi}^0$ extension of Ψ to zero outside Ω. We employ (2.2.1) to write

$$\begin{aligned}\left(\overline{\Psi}^0\mathcal{L}^n,\mathrm{D}\overline{u_k}^{u_0}\right)&=(\Psi\mathcal{L}^n,\mathrm{D}u_k)\llcorner\Omega+\left(\overline{\Psi}^0\mathcal{L}^n,\mathrm{D}u_k{}^{u_0}\right)\llcorner\partial\Omega+(0,\nabla u_0)\mathcal{L}^n\llcorner\overline{\Omega}^{\mathrm{c}}\\&=(\Psi\mathcal{L}^n,\mathrm{D}u_k)\llcorner\Omega+(u_k-u_0)\mathcal{H}^{n-1}\llcorner\partial\Omega+\nabla u_0\mathcal{L}^n\llcorner\Omega^{\mathrm{c}}\quad\text{ as measures on }\mathbb{R}^n\,.\end{aligned}$$

Besides, Ψ–strict convergence in $\mathrm{BV}(\Omega)$ yields via the Ψ–variant of Reshetnyak's continuity theorem (expressed in [15, Theorem 3.10] and applied to $(\mathrm{D}u_k)_k$ and $\mathrm{D}u$) the strict convergence of $(u_k)_k$ to u in $\mathrm{BV}(\Omega)$; see also the comments subsequent to Corollary 2.69. Then, the strict convergence of the trace operator determines

$$\begin{aligned}\lim_{k\to\infty}\left|\left(\overline{\Psi}^0\mathcal{L}^n,\mathrm{D}\overline{u_k}^{u_0}\right)\right|(\mathbb{R}^n)&=\lim_{k\to\infty}|(\Psi\mathcal{L}^n,\mathrm{D}u_k)|(\Omega)+\lim_{k\to\infty}\int_{\partial\Omega}|u_k-u_0|\,\mathrm{d}\mathcal{H}^{n-1}+||\nabla u_0||_{\mathrm{L}^1(\Omega^{\mathrm{c}})}\\&=|(\Psi\mathcal{L}^n,\mathrm{D}u)|(\Omega)+\int_{\partial\Omega}|u-u_0|\,\mathrm{d}\mathcal{H}^{n-1}+||\nabla u_0||_{\mathrm{L}^1(\Omega^{\mathrm{c}})}\\&=|(\Psi\mathcal{L}^n,\mathrm{D}\overline{u}^{u_0})|(\mathbb{R}^n)\,.\end{aligned}$$

We read out that the sequence $(\overline{u_k}^{u_0})_k$ converges Ψ–strictly to the extension of u by u_0 in $\mathrm{BV}(\mathbb{R}^n)$, and this verifies the claimed Ψ–strict convergence in $\mathrm{BV}_{u_0}\left(\overline{\Omega}\right)$. □

We now consider measurable sets A in $\mathbb{R}^n$ and introduce measure–theoretic counterparts of the topological interior, exterior, and boundary of A. Fixed any $\theta\in[0,1]$, we say that $x\in\mathbb{R}^n$ is a point of **density** θ for A if the following limit exists:

$$\lim_{r\to0}\frac{|\mathrm{B}_r(x)\cap A|}{|\mathrm{B}_r|}=\theta,$$

and we denote with A^θ the set of points of density equals to θ for A. For the limit case $\theta=1$ we talk about **measure-theoretic interior** A^1 of A, whereas A^0 is the **measure-theoretic exterior** set of A, and the **measure-theoretic** (or *essential*) **boundary** ∂_*A of A is the set of points in $\mathbb{R}^n$ of density neither 0 nor 1 – and we record $\partial_*A\subseteq\partial A$. The complement of A^0 is called **measure-theoretic closure** A^+ of A. Notice that measure–theoretic exterior and interior exchange when passing to the complement, that is $(A^{\mathrm{c}})^+=(A^1)^{\mathrm{c}}$ and $(A^+)^{\mathrm{c}}=(A^{\mathrm{c}})^1$.

Remark 2.26. For a measurable set $A\subseteq\mathbb{R}^n$ and open $U\subseteq\mathbb{R}^n$, we have $A^+\cap U\subseteq(A\cap U)^+$ and $A^\theta\cap U\subseteq(A\cap U)^\theta$ for any density $\theta\in[0,1]$.

Relying on the concept of density, we can introduce special representatives of a measurable function $u\colon U\to\mathbb{R}$ defined on an open set U in $\mathbb{R}^n$. In fact, for any $t\in\mathbb{R}$ we introduce the notation $E_t:=\{u>t\}$ for the t–superlevel set of u in U, and one may verify that the family $(E_t^+)_t$ is decreasing with respect to t, thus for any $x\in U$ it exists the **approximate upper limit**

$$u^+(x):=\sup\left\{t\in\mathbb{R}\colon\ x\in E_t^+\right\}$$

of u at point x. Similarly, being $(E_t^1)_t$ decreasing in t, we can set the **approximate lower limit** of our function at point x as

$$u^-(x):=\sup\left\{t\in\mathbb{R}\colon\ x\in E_t^1\right\}\,.$$

Heuristically, if u jumps, we can identify u^+ as the function attaining the upper of the two values, while u^- attains the lower one. It also holds $(-u)^\pm=-u^\mp$, and the estimate $u^-+v^-\leq(u+v)^-\leq$

$(u+v)^+ \leq u^+ + v^+$ is valid in U for any pair of measurable u, v. The average of upper and lower approximate limit is the **precise representative** $u^*(x) := (u^+(x) + u^-(x))/2$ of u at point $x \in U$; with such a representative we even achieve equality in the estimates above, meaning $u^* + v^* = (u+v)^*$. For a measurable $A \subseteq \mathbb{R}^n$, it holds $(\mathbb{1}_A)^+ = \mathbb{1}_{A^+}$ and $(\mathbb{1}_A)^- = \mathbb{1}_{A^1}$.

Following now [3, Section 3.6], we consider a function $u \in \mathrm{L}^1_{\mathrm{loc}}(U)$ and say that $x \in U$ is a **Lebesgue point** of u if there exists some $\widetilde{u}(x) \in \mathbb{R}$ such that

$$\lim_{r \to 0} \fint_{\mathrm{B}_r(x)} |u(y) - \widetilde{u}(x)| \,\mathrm{d}y = 0\,,$$

whereas the points x in U for which such property does not hold are the **approximate discontinuity** points of u, and for these latter we write $x \in S_u$. It can be proved that almost every $x \in U$ is a Lebesgue point (that is, $\mathcal{L}^n(S_u) = 0$), and that $\widetilde{u}$ is a Borel function on $U \setminus S_u$ a.e.–coincident with u. An approximate discontinuity point $x \in S_u$ is an **approximate jump point** for u (written $x \in \mathrm{J}_u$) if there exists a unit vector $\nu_u(x) \in \mathbb{S}^{n-1}$ and two distinct real values a_x, b_x such that

$$\lim_{r \to 0} \fint_{\mathrm{B}^+_r(x)} |u(y) - a_x| \,\mathrm{d}y = \lim_{r \to 0} \fint_{\mathrm{B}^-_r(x)} |u(y) - b_x| \,\mathrm{d}y = 0\,,$$

with average integrals computed on the half–balls

$$\mathrm{B}^\pm_r(x) := \{y \in \mathrm{B}_r(x) \colon \ (y-x) \cdot \nu_u(x) \gtrless 0\}$$

The triplet $(a_x, b_x, \nu_u(x))$ is uniquely determined up to a permutation of the first two elements and up to a change of sign in $\nu_u(x)$. Furthermore, for our $u \in \mathrm{L}^1_{\mathrm{loc}}(U)$ all functions $u^+ = u^- = u^* = \widetilde{u}$ agree on Lebesgue points, whereas for $x \in \mathrm{J}_u$ we record $u^+(x) = \max\{a_x, b_x\}$ and $u^-(x) = \min\{a_x, b_x\}$. Further regularity assumptions on u determine coincidence of all representatives on progressively larger sets in the domain, as presented in the following theorem. For a proof of the full statement, we refer to [3, Theorem 3.78]; the treatment of rectifiable sets mentioned in Theorem 2.27 is postponed to the later Section 2.4.

Theorem 2.27 (Federer–Vol'pert)**.** *For open $U \subseteq \mathbb{R}^n$ and $u \in \mathrm{BV}(U)$, the approximate discontinuity set S_u is countably $\mathcal{H}^{n-1}$–rectifiable, $\mathcal{H}^{n-1}(S_u \setminus \mathrm{J}_u) = 0$, and it holds $\mathrm{D}u \llcorner \mathrm{J}_u = (u^+ - u^-)\nu_u \, \mathcal{H}^{n-1} \llcorner \mathrm{J}_u$ for the orientation $\nu_u \in \mathbb{S}^{n-1}$ determined on the set of approximate jump points as above.*

Consequently, for any BV function u all representatives u^*, u^-, u^+ coincide $\mathcal{H}^{n-1}$–a.e. outside J_u. In case $u \in \mathrm{W}^{1,1}(U)$, then $\mathcal{H}^{n-1}(S_u) = 0$ and thus the representatives are all equal up to $\mathcal{H}^{n-1}$–negligible subsets of U.

We highlight that an alternative definition of approximate upper and lower limit consists of directly selecting the larger among a_x and b_x as upper limit $u^+(x)$, and the remaining value as lower one $u^-(x)$ (compare with [3, Definition 3.67]). However, our choice of introducing $u^\pm$ as supremum level in density points of superlevel sets as done in [48, Section 2.9] stems from the necessity of having *everywhere* defined representatives on the domain of u (even regardless of the function integrability assumption), which turns out to be useful for our later results.

Exploiting the $\mathcal{H}^{n-1}$–characterization of upper and lower approximate limits as jump values of BV functions, and decomposing $\mathrm{J}_u = (\mathrm{J}_{u_+} \cap \mathrm{J}_{u_-}) \cup (\mathrm{J}_{u_+} \setminus \mathrm{J}_{u_-}) \cup (\mathrm{J}_{u_-} \setminus \mathrm{J}_{u_+})$ for $u_\pm$ positive/negative part of $u \in \mathrm{BV}$ up to Hausdorff–negligible sets, we obtain the decomposition lemma.

Lemma 2.28. *For any $u \in \mathrm{BV}(U)$ on open $U \subseteq \mathbb{R}^n$, we have:*

(i) $u^+ = (u_+)^+ - (u_-)^-$ *holding $\mathcal{H}^{n-1}$–a.e. in U;*

(ii) $u^- = (u_+)^- - (u_-)^+$ *holding $\mathcal{H}^{n-1}$–a.e. in U.*

Lemma 2.29. *For a non–negative, σ–finite measure μ on $\mathbb{R}^n$ vanishing on $\mathcal{H}^{n-1}$–negligible sets, assigned $u \in \mathrm{BV}(\mathbb{R}^n)$, it is*

$$\left\{u^+ > t\right\} = \{u > t\}^+ \quad \text{and} \quad \left\{u^- > t\right\} = \{u > t\}^1 \quad \mu\text{–a.e. on } \mathbb{R}^n,\ \text{for a.e. } t \in \mathbb{R}\,.$$

Proof. From the definition of u^+, we find that the chain of inclusions $\{u^+ > t\} \subseteq \{u > t\}^+ \subseteq \{u^+ \geq t\}$ holds $\mathcal{H}^{n-1}$–a.e. in $\mathbb{R}^n$ and for all $t \in \mathbb{R}$. Moreover,

$$\mu\left(\left\{u^+ \geq t\right\}\right) - \mu\left(\left\{u^+ > t\right\}\right) = \mu(\{u^+ \geq t\} \setminus \{u^+ > t\}) = \mu(\{u^+ = t\})\,.$$

At the same time, σ–finiteness of μ determines $\mu\left(\{u^+ = t\}\right) = 0$ for all but countably many t. Altogether, we deduce that $\{u^+ > t\}$ and $\{u > t\}^+$ agree μ–a.e. on $\mathbb{R}^n$. An analogous reasoning applied to $\{u^- > t\} \subseteq \{u > t\}^1 \subseteq \{u^- \geq t\}$ yields the second half of the statement. □

Recalling that BV functions admit finite Radon measures as derivatives, we can apply Theorem 2.2 to the derivative $\mathrm{D}u$ of any $u \in \mathrm{BV}(U)$ with basis measure $\mathcal{L}^n$ to derive the decomposition

$$\mathrm{D}u = \mathrm{D}^a u + \mathrm{D}^s u = \nabla u \mathcal{L}^n + \mathrm{D}^s u \qquad \text{for all } u \in \mathrm{BV}(U)\,,$$

where $\nabla u = \mathrm{d}\mathrm{D}^a u / \mathrm{d}\mathcal{L}^n$ is the Radon–Nikodým density of the absolutely continuous part of the derivative respect to $\mathcal{L}^n$. The singular part of $\mathrm{D}u$, in turn, admits a decomposition into a **jump part** $\mathrm{D}^j u := \mathrm{D}^s u \llcorner \mathrm{J}_u$ and a diffuse part of $\mathrm{D}^s u$ with respect to $\mathcal{L}^n$, that is the **Cantor part** of u defined as $\mathrm{D}^c u := \mathrm{D}^s u \llcorner (U \setminus S_u)$. We record that $\mathrm{D}^c u$ and $\mathrm{D}^j u$ determine mutually singular measures on U. Moreover, the sum $\mathrm{D}^d u := \mathrm{D}^a u + \mathrm{D}^c u$ is known as **diffuse part** of the derivative. The space $\mathrm{SBV}(U)$ of special functions of bounded variation includes all $u \in \mathrm{BV}(U)$ with $\mathrm{D}^c u \equiv 0$. On the contrary, any function $u \in \mathrm{BV}(U)$ such that $\mathrm{D}u = \mathrm{D}^c u$ is called a Cantor function.

We now quote the **Poincaré's inequality** on Sobolev spaces with trace zero on the boundary, see for instance [68, Theorem 13.19]: If $\Omega \subseteq \mathbb{R}^n$ is an open and bounded set and $p \in [1, \infty)$, then there exists a constant $C_p \in (0, \infty)$ depending on Ω, n and p only such that

$$||w||_{\mathrm{L}^p(\Omega)} \leq C_p ||\nabla w||_{\mathrm{L}^p(\Omega,\mathbb{R}^n)} \qquad \text{for all } w \in \mathrm{W}^{1,p}_0(\Omega)\,.$$

Consequently, for any datum $u_0 \in \mathrm{W}^{1,p}(\Omega)$ one computes

$$||\nabla w||^p_{\mathrm{L}^p(\Omega,\mathbb{R}^n)} \geq \widetilde{C_p} ||w||^p_{\mathrm{W}^{1,p}(\Omega)} - \lambda_p ||u_0||^p_{\mathrm{W}^{1,p}(\Omega)} \qquad \text{for all } w \in \mathrm{W}^{1,p}_{u_0}(\Omega) \tag{2.2.2}$$

for suitable constants $\widetilde{C_p}$, $\lambda_p \in (0, \infty)$.

Similar to this, even for BV functions defined on nice enough domains, we find a uniform upper bound by the L^1–norm of the function in terms of its total variation. There exist various formulations of the following Poincaré's inequality, here we state the sharp result obtained by Miranda in [77, Proposition 2] for functions of arbitrary traces. For an equivalent and straightforward proof, we refer to [51, Theorem 2.10].

Theorem 2.30 (Poincaré's inequality in BV). *For an open, bounded Lipschitz $\Omega \subseteq \mathbb{R}^n$ and any $w \in \mathrm{BV}(\Omega)$, it holds*

$$||w||_{\mathrm{L}^1(\Omega)} \leq \gamma_n \left(|\mathrm{D}w|(\Omega) + \int_{\partial\Omega} |w| \,\mathrm{d}\mathcal{H}^{n-1} \right) \tag{2.2.3}$$

where $\gamma_n := \frac{1}{n} \left(|\Omega|/\omega_n\right)^{1/n}$.

Henceforth, we will mainly apply (2.2.3) in the form:

$$||w||_{\mathrm{BV}(\Omega)} \le \widetilde{\gamma_n}|\mathrm{D}\overline{w}^0|(\overline{\Omega}) \quad \text{for all } w \in \mathrm{BV}(\Omega) \tag{2.2.4}$$

with constant $\widetilde{\gamma_n} := \gamma_n + 1$ and for $\overline{w}^0$ extension of w to zero outside Ω.

2.3 Sets of finite perimeter

Next, we focus on a special class of functions in BV, namely on characteristic functions of the so–called *sets of finite perimeter* – also named *Caccioppoli sets*. Such a notion was introduced by the Neapolitan mathematician Renato Caccioppoli already in 1927 for the bidimensional case [24], later improved and extended by the same author to higher dimensions as discussed in the talk [25] at the UMI Congress of 1951. The topic sparked the interest of the mathematical community since the 1950s, starting from the foundational works of Ennio de Giorgi [39, 40] and Herbert Federer with Wendell Fleming [49].

Definition 2.31 (sets of finite perimeter)**.** A measurable set $E \subseteq \mathbb{R}^n$ is of locally finite perimeter in U open if $\mathbb{1}_E \in \mathrm{BV}_{\mathrm{loc}}(U)$. If additionally $|\mathrm{D}\mathbb{1}_E|(U) < \infty$, we say that E is of **finite perimeter** in U, and we write $\mathrm{P}(E,U) := |\mathrm{D}\mathbb{1}_E|(U)$ for the perimeter of E in U.

For $|E \cap U| < \infty$, the characteristic function $\mathbb{1}_E$ is in $\mathrm{L}^1(U)$, hence E is of finite perimeter in U if and only if $\mathbb{1}_E \in \mathrm{BV}(U)$. It is obviously $\mathrm{P}(E^{\mathrm{c}},U) = \mathrm{P}(E,U)$ and $\mathrm{P}(E,U) \le \mathrm{P}(E,U')$ for open $U' \supseteq U$, however the function $\mathrm{P}(.,U)$ is *not* increasing with respect to set inclusion. For sets E of finite perimeter in $U = \mathbb{R}^n$, we will shortly write $\mathrm{P}(E)$ in place of $\mathrm{P}(E,\mathbb{R}^n)$. A basilar property of perimeter inherited from the lower semicontinuity of TV with respect to the $\mathrm{L}^1_{\mathrm{loc}}$–topology is expressed in the following.

Proposition 2.32 (lower semicontinuity of perimeter)**.** *The perimeter in U is lower semicontinuous with respect to local convergence in measure, i.e.*

$$\mathrm{P}(E,U) \le \liminf_{k\to\infty} \mathrm{P}(E_k,U) \quad \textit{whenever} \quad \mathbb{1}_{E_k} \xrightarrow[k\to\infty]{} \mathbb{1}_E \ \textit{in}\ \mathrm{L}^1_{\mathrm{loc}}(U)\,.$$

Yet another fundamental result in the theory of sets of finite perimeter is the (Euclidean) isoperimetric inequality, which establishes a relation between perimeter and volume of measurable sets in $\mathbb{R}^n$. In particular, the estimate (2.3.1) achieved by De Giorgi [41] confirms the intuitive conjecture that n–dimensional balls are the unique (up to Lebesgue–negligible sets) geometric objects in the space $\mathbb{R}^n$ achieving maximal volume under prescribed perimeter value.

Theorem 2.33 (isoperimetric inequality)**.** *For a measurable set E in $\mathbb{R}^n$ with $0 < |E| < \infty$, it is*

$$n\omega_n^{1/n}|E|^{\frac{n-1}{n}} \le \mathrm{P}(E)\,, \tag{2.3.1}$$

with equality holding if and only if $|E \triangle \mathrm{B}_r(x)| = 0$ for some $x \in \mathbb{R}^n$ and $r = (|E|/\omega_n)^{1/n}$.

For a proof, we refer to [69, Section 14.2] or directly to [41]. If E of locally finite perimeter in the open set $U \subseteq \mathbb{R}^n$, following [3, Definition 3.54] we define the **reduced boundary** $\partial^* E$ of E as the collection of $x \in \mathbb{R}^n$ such that:

1. The point x is in the support of $\mathrm{D}\mathbb{1}_E$, i.e. $|\mathrm{D}\mathbb{1}_E|(\mathrm{B}_r(x)) = \mathrm{P}(E,\mathrm{B}_r(x)) > 0$ for all $r > 0$;
2. There exists $\nu_E(x) := \lim_{r\to 0} \frac{\mathrm{D}\mathbb{1}_E(\mathrm{B}_r(x))}{|\mathrm{D}\mathbb{1}_E|(\mathrm{B}_r(x))}$ in $\mathbb{R}^n$; and
3. It holds $|\nu_E(x)| = 1$.

If so, we say that $\nu_E(x) \in \mathbb{S}^{n-1}$ is the **generalized inner normal** to E at point $x \in \partial^* E$. Then, the Lebesgue–Besicovitch derivation theorem [3, Theorem 2.22] yields the existence of a generalized normal ν_E for $|\mathrm{D}\mathbb{1}_E|$–a.e. points of the support of $\mathrm{D}\mathbb{1}_E$ in U; moreover, it holds $\mathrm{D}\mathbb{1}_E = \nu_E|\mathrm{D}\mathbb{1}_E|$ as measures in U with the $|\mathrm{D}\mathbb{1}_E|$–a.e. defined density $\mathrm{d}\mathrm{D}\mathbb{1}_E/\mathrm{d}|\mathrm{D}\mathbb{1}_E| = \nu_E$.

We now state a partial statement for the fundamental structure theorem by De Giorgi, claiming that reduced boundaries have the geometrical structure of generalized hypersurfaces; check out, for instance, [69, Theorem 15.9] or [3, Theorem 3.59].

Theorem 2.34 (De Giorgi, partial statement)**.** *If E is a set of locally finite perimeter in an open set $U \subseteq \mathbb{R}^n$, then $\mathrm{D}\mathbb{1}_E \llcorner U = \nu_E \mathcal{H}^{n-1} \llcorner (U \cap \partial^* E)$ as measures, and thus $|\mathrm{D}\mathbb{1}_E| \llcorner U = \mathcal{H}^{n-1} \llcorner (U \cap \partial^* E)$.*

It follows that for $\mathbb{1}_E \in \mathrm{BV}_{\mathrm{loc}}(U)$ it holds $\mathrm{P}(E, B) = \mathcal{H}^{n-1}(B \cap \partial^* E)$ for any B Borel subset of U. Furthermore, Federer's theorem (see [3, Theorem 3.61] or [69, Theorem 16.2]) asserts that sets of locally finite perimeter have density equal to either 0, 1/2 or 1 on $\mathcal{H}^{n-1}$–a.e. point, and that essential and reduced boundary differ at most on an $\mathcal{H}^{n-1}$–negligible set. More precisely, every E of finite perimeter in U is such that $\partial^* E \cap U \subseteq E^{1/2} \cap U \subseteq \partial_* E \cap U \subseteq \partial E \cap U$ and $\mathcal{H}^{n-1}(\partial_* E \cap U) = \mathcal{H}^{n-1}(\partial^* E \cap U)$.

We recall that by Rademacher's theorem every Lipschitz function $u\colon U \to \mathbb{R}$ on open $U \subseteq \mathbb{R}^n$ is differentiable almost everywhere on U; we refer for example to [69, Theorem 7.8]. In such cases, the fundamental coarea theorem holds.

Lemma 2.35. *For any function $u\colon U \to \mathbb{R}$ defined on open $U \subseteq \mathbb{R}^n$, we introduce the superlevel sets $E_t := \{u > t\}$ for $t \in \mathbb{R}$. Then, if u is continuous in U, we have $\partial E_t = \{u = t\}$ for a.e. $t \in \mathbb{R}$.*

Proof. Let us fix $t \in \mathbb{R}$ and consider first an element $x \in \partial E_t$. If $u(x) > t$, then by continuity there exists some $\bar\varepsilon > 0$ such that $\mathrm{B}_{\bar\varepsilon}(x) \subseteq \{u > t\}$, so $\mathrm{B}_{\bar\varepsilon}(x) \cap \{u \le t\} = \emptyset$, against the definition of topological boundary of $\{u > t\}$. Flipping all inequalities, we would achieve another contradiction by assuming $u(x) < t$. It must then be $x \in \{u = t\}$.

We now take $x \in U$ such that $u(x) = t$ for some t, whereas $x \notin \partial E_t$. Then, it is $\mathrm{B}_{\bar\varepsilon}(x) \cap \{u > t\} = \emptyset$ for some value $\bar\varepsilon > 0$, hence $u(y) \le u(x) = t$ for all $y \in U$ such that $|x - y| < \bar\varepsilon$, meaning x is a point of local maximum for u with value t. However, any real–valued function can only attain (local) maxima in countably many points – see below – thus $x \in \{u = t\} \setminus \partial E_t$ can happen only for countably many t's. This proves that the remaining inclusion $\{u = t\} \subseteq \partial E_t$ holds for almost every real t, as required.

To check the claim, for every $x \in U$ we set $\delta_x := \sup\{\delta > 0 : \ u(y) \le u(x) \text{ for all } y \in \mathrm{B}_\delta(x) \subseteq U\}$, that is the radius of the largest ball in U where x attains the maximum of u. We record that $\delta_x = 0$ if and only if x is not a point of local maximum. Then, if x_α and x_β are two points of maximum such that $f(x_\alpha) \ne f(x_\beta)$, one has $|x_\alpha - x_\beta| \ge \delta_{\alpha,\beta} := \min\left\{\delta_{x_\alpha}, \delta_{x_\beta}\right\} > 0$, which yields $\mathrm{B}_{\delta_{\alpha,\beta}/2}(x_\alpha) \cap \mathrm{B}_{\delta_{\alpha,\beta}/2}(x_\beta) = \emptyset$. However, this is only feasible if the ball centres are countably many in U, and we conclude that there are at most countably many values of local maxima for u. $\square$

We now present the main coarea result as stated in [69, Theorem 18.1].

Theorem 2.36 (coarea formula for Lipschitz functions)**.** *For a Lipschitz function $u\colon U \to \mathbb{R}$ defined on the open set U in $\mathbb{R}^n$, the sets E_t have locally finite perimeter in U for a.e. $t \in \mathbb{R}$, their topological boundary satisfies $\partial E_t \supseteq \partial^* E_t$ for a.e. $t \in \mathbb{R}$,*

$$\mathcal{H}^{n-1}\big(\partial E_t \setminus \partial^* E_t\big) = 0 \quad \text{for a.e. } t \in \mathbb{R},$$

and it holds

$$\int_B |\nabla u| \, \mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{P}(E_t, B) \, \mathrm{d}t = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(B \cap \{u = t\}) \, \mathrm{d}t \quad \text{for all } B \subseteq U \text{ Borel}. \tag{2.3.2}$$

For further usage, we additionally mention an extended version of coarea, see [46, Theorem 3.11] or [3, Remark 2.94].

Corollary 2.37. *For a Lipschitz function $u\colon U \to \mathbb{R}$ and a bounded Borel $g\colon U \to \mathbb{R}$, the coarea formula (2.3.2) can be generalized to*

$$\int_U g|\nabla u|\,\mathrm{d}x = \int_{-\infty}^{\infty}\left(\int_U g(y)\,\mathrm{d}|\mathrm{D}\mathbb{1}_{E_t}|(y)\right)\mathrm{d}t = \int_{-\infty}^{\infty}\left(\int_{U\cap\{u=t\}} g(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\right)\mathrm{d}t\,.$$

An analogous result holds even for arbitrary BV functions, compare with [3, Theorem 3.40] and [46, Theorem 5.9].

Theorem 2.38 (Fleming–Rishel coarea formula)**.** *For $u \in \mathrm{BV}(U)$, the superlevel sets E_t are of finite perimeter in U for a.e. $t \in \mathbb{R}$, and for any $B \subseteq U$ Borel it is*

$$\mathrm{D}u(B) = \int_{-\infty}^{\infty} \mathrm{D}\mathbb{1}_{E_t}(B)\,\mathrm{d}t\,, \qquad |\mathrm{D}u|(B) = \int_{-\infty}^{\infty} |\mathrm{D}\mathbb{1}_{E_t}|(B)\,\mathrm{d}t = \int_{-\infty}^{\infty} \mathrm{P}(E_t, B)\,\mathrm{d}t\,.$$

Proposition 2.39. *For open $U \subseteq \mathbb{R}^n$, $u \in \mathrm{BV}(U)$, and any $E \subseteq \mathbb{R}$ with $\mathcal{L}^1(E) = 0$, we have:*

(i) *The absolutely continuous part ∇u of $\mathrm{D}u$ vanishes a.e. on $u^{-1}(E)$;*

(ii) *The Cantor part $\mathrm{D}^{\mathrm{c}}u$ of $\mathrm{D}u$ vanishes on $\widetilde{u}^{-1}(E)$.*

We now quote the chain rule for functions of bounded variation in [3, Theorem 3.99], incidentally observing that such a result is *not* preserved for vector–valued BV functions.

Theorem 2.40 (chain rule in BV)**.** *We assume $u \in \mathrm{BV}(U)$ for open $U \subseteq \mathbb{R}^n$, and we let $f\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Furthermore, in case $|U| = \infty$ we additionally suppose $f(0) = 0$. Then the composition $f \circ u$ is in $\mathrm{BV}(U)$ and one has*

$$\mathrm{D}(f\circ u) = f'(u)\nabla u\mathcal{L}^n + \big(f(u^+) - f(u^-)\big)\nu_u\mathcal{H}^{n-1}\llcorner \mathrm{J}_u + f'(\widetilde{u})\mathrm{D}^{\mathrm{c}}u\,.$$

2.4 Rectifiability and slicing properties of Hausdorff measures

For the notation in this section we follow [3, Section 2.9] and set $n \in \mathbb{N}$, $k \in [0,n] \cap \mathbb{N}$.

Definition 2.41 (rectifiable sets)**.** Consider an $\mathcal{H}^k$–measurable $E \subseteq \mathbb{R}^n$. We say that:

- E is *countably k–rectifiable*, if $E \subseteq \bigcup_{i\in\mathbb{N}} f_i(\mathbb{R}^k)$ for Lipschitz mappings $f_i : \mathbb{R}^k \to \mathbb{R}^n$;
- E is *countably $\mathcal{H}^k$–rectifiable*, if $\mathcal{H}^k\left(E \setminus \bigcup_{i\in\mathbb{N}} f_i(\mathbb{R}^k)\right) = 0$ for again countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^n$;
- E is *$\mathcal{H}^k$–rectifiable*, if E is countably $\mathcal{H}^k$–rectifiable and $\mathcal{H}^k(E) < \infty$.

Conversely, we shall say that a Borel set $E \subseteq \mathbb{R}^n$ is *purely $\mathcal{H}^k$–unrectifiable* if $\mathcal{H}^k(E \cap F) = 0$ for any countably $\mathcal{H}^k$–rectifiable set F. Moreover, any Borel set E such that $\mathcal{H}^k(E) < \infty$ admits an $\mathcal{H}^k$–a.e. unique decomposition $E = E^{\mathrm{u}} \cup E^{\mathrm{r}}$ where E^{u} is purely $\mathcal{H}^k$–unrectifiable and E^{r} is countably $\mathcal{H}^k$–rectifiable.

Remark 2.42. We recall that the theorem of Federer–Vol'pert 2.27 guarantees countable $\mathcal{H}^{n-1}$–rectifiability for the discontinuity set S_u whenever $u \in \mathrm{BV}(U)$ for open $U \subseteq \mathbb{R}^n$. At the same time, taking into account $\mathcal{H}^{n-1}(S_u \setminus \mathrm{J}_u) = 0$, we read out that even the jump set J_u of u is countably $\mathcal{H}^{n-1}$–rectifiable.

We enunciate a useful slicing property for Hausdorff rectifiable sets, which was originally proved in [48, Theorem 3.2.23].

Theorem 2.43 (product structure on products of rectifiable sets)**.** *We assume m, $n \in \mathbb{N}$. If W is a countably h–rectifiable Borel set in $\mathbb{R}^m$ (with $h \in [0,m] \cap \mathbb{N}$) and Z is a countably $\mathcal{H}^k$–rectifiable Borel subset of $\mathbb{R}^n$ (with $k \in [0,n] \cap \mathbb{N}$), then $W \times Z$ is a countably $\mathcal{H}^{h+k}$–rectifiable subset of $\mathbb{R}^m \times \mathbb{R}^n$ and it holds*

$$\mathcal{H}^{h+k} \llcorner (W \times Z) = (\mathcal{H}^h \llcorner W) \otimes (\mathcal{H}^k \llcorner Z) \qquad \textit{as measures on } \mathbb{R}^m \times \mathbb{R}^n \,.$$

We point out that the assumption on Z cannot be weakened to just *countable* $\mathcal{H}^k$–rectifiability, otherwise the product structure of Theorem 2.43 would be violated; for a counterexample see [48, 2.10.29]. We are especially interested in the setting $h = m = 1$, $k = n - 1$, and $W = \mathbb{R}$. In such a case, Theorem 2.43 and Fubini's theorem yield:

Corollary 2.44 (product structure on products of rectifiable sets in codimension 1)**.** *Any countably $\mathcal{H}^{n-1}$–rectifiable Borel set $Z \subseteq \mathbb{R}^n$ is such that $\mathcal{H}^n \llcorner (\mathbb{R} \times Z) = \mathcal{L}^1 \otimes (\mathcal{H}^{n-1} \llcorner Z)$ as measures on $\mathbb{R}^{n+1}$, and for $h \colon \mathbb{R} \times Z \to [0,\infty)$ Borel it holds*

$$\int_{B \times Z} h(x_0, x) \, \mathrm{d}\mathcal{H}^n(x_0, x) = \int_B \int_Z h(x_0, x) \, \mathrm{d}\mathcal{H}^{n-1}(x) \, \mathrm{d}x_0 \qquad \textit{for any Borel set } B \subseteq \mathbb{R} \,.$$

We refer to [52, Lemma 2.16] for an alternative proof of Corollary 2.44 which does not rely on the general structure result. Another useful lemma asserting that almost every slice of an n–negligible set in $\mathbb{R}^{n+1}$ is negligible with respect to the Hausdorff measure of dimension $n-1$ is stated below, and it is reprised from our work [52, Lemma 2.17].

Lemma 2.45 (slicing negligible sets)**.** *Any $\mathcal{H}^n$–negligible set $Z \subseteq \mathbb{R}^{n+1}$ is such that the slice*

$${}_{x_0}Z := \{x \in \mathbb{R}^n : (x_0, x) \in Z\}$$

is $\mathcal{H}^{n-1}$–negligible for a.e. $x_0 \in \mathbb{R}$.

2.5 Γ-convergence of functionals on BV

We introduce the following notions according to the treatment of Γ–convergence in [37] and [21]. The results are here presented already targeted to our framework, meaning that our formulation refers directly to the space BV.

Definition 2.46 ((equi–)coercivity)**.** We consider functionals $(F_k)_k$, $F \colon \mathrm{BV}(U) \to \overline{\mathbb{R}}$.

- The functional F is coercive (or, more appropriately, *sequentially coercive*) on $\mathrm{BV}(U)$ if for every $t \in \mathbb{R}$ the weak–$*$ closure of $\{F \le t\}$ is sequentially compact in $\mathrm{BV}(U)$ with respect to the weak–$*$ convergence; that is, for every t and every $(u_k)_k \subseteq \overline{\{F \le t\}}$, there is a (non relabelled) subsequence and a limit $u \in \mathrm{BV}(U)$ such that $u_k \overset{*}{\rightharpoonup} u$ weakly–$*$ in $\mathrm{BV}(U)$ as $k \to \infty$.

- The sequence $(F_k)_k$ is equi–coercive on $\mathrm{BV}(U)$ if for every $t \in \mathbb{R}$ there exists a sequentially compact set K_t in $\mathrm{BV}(U)$ with respect to the weak–$*$ topology such that $\{F_k \le t\} \subseteq K_t$ for every $k \in \mathbb{N}$.

Observe that if $F \colon \mathrm{BV}(U) \to \overline{\mathbb{R}}$ is weak–$*$ lower semicontinuous in $\mathrm{BV}(U)$, then every sublevel set is weakly–$*$ closed, and thus F is coercive if and only if $\{F \le t\}$ is sequentially weakly–$*$ compact for every $t \in \mathbb{R}$. Next, we recall the fundamental direct method for proving existence results in BV.

Theorem 2.47 (BV direct method in the calculus of variations)**.** *If $F\colon \mathrm{BV}(U) \to \overline{\mathbb{R}}$ is coercive and weakly–$*$ lower semicontinuous in* $\mathrm{BV}(U)$*, then F has a minimum point. If $U = \Omega$ is an open, bounded Lipschitz domain, the same conclusion is attained for F coercive and $\mathrm{L}^1(\Omega)$–lower semicontinuous.*

Proof. In case $F \equiv \infty$, every point attains the minimum; so we assume $F \not\equiv \infty$ and thus we find a sequence $(u_k)_k$ in $\mathrm{BV}(U)$ such that

$$\lim_{k\to\infty} F[u_k] = \inf_{\mathrm{BV}(U)} F =: m < \infty\,,$$

hence up to subsequences $(u_k)_k \subseteq \{F \leq t_0\}$ for some $t_0 \in \mathbb{R}$. Applying coercivity, $u_k \overset{*}{\rightharpoonup} u$ weakly–$*$ to some $u \in \mathrm{BV}(U)$ as $k \to \infty$, hence by weak–$*$ LSC of F we conclude:

$$m = \liminf_{k\to\infty} F[u_k] \geq F[u]\,,$$

that is u minimizes F in $\mathrm{BV}(U)$.

In the case $U = \Omega$, coercivity yields boundedness of $(u_k)_k$ in $\mathrm{BV}(\Omega)$, and thus arguing via Theorem 2.22 we obtain even strong convergence in $\mathrm{L}^1(\Omega)$ to some $u \in \mathrm{BV}(\Omega)$. The existence result follows then in the same way employing L^1–semicontinuity of F. □

The following useful characterization of equi–coercivity reprises [37, Proposition 7.7].

Proposition 2.48 (characterization of equi–coercivity)**.** *The sequence $(F_k)_k$ is equi–coercive on* $\mathrm{BV}(U)$ *if and only if there exists a functional $\Psi\colon \mathrm{BV}(U) \to \overline{\mathbb{R}}$ such that $F_k \geq \Psi$ on $\mathrm{BV}(U)$ for all k, with Ψ coercive and weakly–$*$ lower semicontinuous in* $\mathrm{BV}(U)$*.*

Proof (only sufficient condition to equi–coercivity). If such a Ψ exists, then for every $k \in \mathbb{N}$ and every $t \in \mathbb{R}$ it is $\{F_k \leq t\} \subseteq \{\Psi \leq t\} =: K_t$, with all K_t weakly–$*$ closed in $\mathrm{BV}(U)$ by characterization of lower semicontinuity, and K_t sequentially compact in $\mathrm{BV}(U)$ with respect to weak–$*$ convergence from the coercivity assumption on Ψ. Hence the sequence $(F_k)_k$ is equi–coercive. □

We now introduce the definition of **Γ-convergence** for a sequence $(F_k)_k$ defined on $\mathrm{BV}(\Omega)$ on a bounded and Lipschitz set $\Omega \subseteq \mathbb{R}^n$. We restrict the analysis to such domains since our final scope is determining properties of Γ–limits for functionals defined on $\mathrm{BV}(\Omega)$.

Definition 2.49 (Γ–convergence of functionals on BV)**.** We say that a sequence $F_k\colon \mathrm{BV}(\Omega) \to \overline{\mathbb{R}}$ of functionals Γ–converges to the functional $F\colon \mathrm{BV}(\Omega) \to \overline{\mathbb{R}}$ in $\mathrm{BV}(\Omega)$ if for all $w \in \mathrm{BV}(\Omega)$ the following two conditions hold.

(i) (liminf inequality). For every sequence $(w_k)_k$ in $\mathrm{BV}(\Omega)$ converging to w in $\mathrm{L}^1(\Omega)$, it is

$$\liminf_{k\to\infty} F_k[w_k] \geq F[w]\,;$$

(ii) (existence of a recovery sequence). There exists some $(w_k)_k$ in $\mathrm{BV}(\Omega)$ converging to w in $\mathrm{L}^1(\Omega)$ such that

$$\lim_{k\to\infty} F_k[w_k] = F[w]\,.$$

If both (i) and (ii) are achieved, we call F the **Γ-limit** of the sequence $(F_k)_k$, and write $F = \Gamma\!-\!\lim_k F_k$.

Notice that another equivalent definition of Γ–convergence is obtained imposing the requirement (i) together with:

(ii′) (limsup inequality). There exists a sequence $(w_k)_k$ in $\mathrm{BV}(\Omega)$ converging to w in $\mathrm{L}^1(\Omega)$ satisfying

$$\limsup_{k\to\infty} F_k[w_k] \le F[w] .$$

Remark 2.50. We observe that if the limsup inequality (ii′) holds for some $(F_k)_k$, F, then for every $w \in \mathrm{BV}(\Omega)$ it is $\limsup_{k\to\infty} \left(\inf_{\mathrm{BV}(\Omega)} F_k\right) \le \limsup_{k\to\infty} F_k[w_k] \le F[w]$, hence passing to the infimum

$$\limsup_{k\to\infty} \left(\inf_{\mathrm{BV}(\Omega)} F_k \right) \le \inf_{\mathrm{BV}(\Omega)} F . \tag{2.5.1}$$

Definition 2.51. For a sequence of functionals $(F_k)_k$ defined on $\mathrm{BV}(\Omega)$, we say that $(u_k)_k$ in $\mathrm{BV}(\Omega)$ is a minimizing sequence if $\lim_{k\to\infty} F_k[u_k] = \lim_{k\to\infty} \left(\inf_{\mathrm{BV}(\Omega)} F_k\right)$.

The following result deals with the convergence of minimal values for sequences of equi–coercive functionals. We refer to [21, Theorem 1.21], [22, Theorem 2.10], or [37, Theorem 7.8] for a proof in general topological or metric spaces.

Theorem 2.52 (fundamental theorem of Γ–convergence)**.** *If $(F_k)_k$ is an equi–coercive sequence of functionals on $\mathrm{BV}(\Omega)$ and $F := \Gamma\text{−}\lim_k F_k$, then F admits minimum in $\mathrm{BV}(\Omega)$ and*

$$\min_{\mathrm{BV}(\Omega)} F = \lim_{k\to\infty} \left(\inf_{\mathrm{BV}(\Omega)} F_k \right) .$$

Furthermore, BV–minimizing sequences for F_k are such that their $\mathrm{L}^1(\Omega)$–limit of subsequences attains the minimum of F.

Proof. As usual, we assume $F \not\equiv \infty$ – otherwise the result is trivial. Let $(u_k)_k$ be a minimizing sequence for $(F_k)_k$ in $\mathrm{BV}(\Omega)$. Bringing in Proposition 2.48, we determine a coercive, weakly–∗ LSC Ψ such that $F_k \ge \Psi$ for every k, hence

$$\limsup_{k\to\infty} \Psi[u_k] \le \lim_{k\to\infty} F_k[u_k] = \lim_{k\to\infty} \left(\inf_{\mathrm{BV}(\Omega)} F_k \right) =: m < \infty ,$$

where finiteness of m follows from (2.5.1) since F is not identically infinity. Then, up to a relabelling of the indices we can assume the existence of some real value t_0 such that $(u_k)_k$ is contained in the weakly–∗ sequentially compact set $\{\Psi \le t_0\} =: K_{t_0}$. Via the compactness Theorem 2.22, we conclude that $(u_k)_k$ converges (again up to subsequences) to some $u \in \mathrm{BV}(\Omega)$ with respect to the $\mathrm{L}^1(\Omega)$–topology. The liminf inequality (i) then yields $\inf_{\mathrm{BV}(\Omega)} F \le F[u] \le m$, whereas from (ii) and applying (2.5.1) it also holds $m \le \inf_{\mathrm{BV}(\Omega)} F$. Altogether, we deduce the minimization property for u and the convergence of the infima $m_k := \inf_{\mathrm{BV}(\Omega)} F_k$ to the minimum m. □

2.6 Functionals on measures

The following notion of functionals depending on measures was introduced by Goffman–Serrin [62] to generalize the definition of the total variation functional $\mathrm{G}[\nu] := |\nu|(.)$ for a Radon measure ν. We first illustrate the abstract theory of functionals G on measures, and later we specialize our treatment to derivative measures of a BV function.

For $m \in \mathbb{N}$, we say that the function $g \colon \mathbb{R}^m \to \mathbb{R}$ is **positively 1-homogeneous** if $g(t\xi) = tg(\xi)$ for all $t \in [0, \infty)$ and all $\xi \in \mathbb{R}^m$. For future usage, we also recall that every $g \colon \mathbb{R}^n \to \mathbb{R}$ positively 1–homogeneous and convex satisfies the subadditivity property

$$g(\xi + \tau) \le g(\xi) + g(\tau) \qquad \text{for all } \xi, \tau \in \mathbb{R}^n , \tag{2.6.1}$$

since $g(\xi+\tau)/2 = g((\xi+\tau)/2) \leq g(\xi/2)+g(\tau/2)$. A straightforward consequence of (2.6.1) is the reverse triangle inequality estimate

$$g(\xi)-g(\tau) = g((\xi-\tau)+\tau)-g(\tau) \leq g(\xi-\tau) \quad \text{for all } \xi,\tau \in \mathbb{R}^n\,. \tag{2.6.2}$$

If U is an open set in $\mathbb{R}^n$ and $g\colon U\times\mathbb{R}^n \to [0,\infty)$ is a Borel function positively 1–homogeneous in the second variable, then we introduce the following functional on measure:

$$\mathrm{G}[\nu](U) := \int_U g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\mathrm{d}\mu \quad \text{for } \nu \in \mathrm{RM}_{\mathrm{loc}}(U,\mathbb{R}^n)\,, \tag{2.6.3}$$

for any choice of non–negative measure μ in $\mathrm{RM}_{\mathrm{loc}}(U)$ such that $\nu \ll \mu$. Observe that if $\widehat{\mu}$ is any other non–negative Radon measure on U such that $\nu \ll \widehat{\mu}$, then $\mu,\widehat{\mu},\nu \ll \mu+\widehat{\mu}$ and thus the Theorem of Radon–Nikodým determines

$$\frac{\mathrm{d}\nu}{\mathrm{d}(\mu+\widehat{\mu})} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\cdot\frac{\mathrm{d}\mu}{\mathrm{d}(\mu+\widehat{\mu})} = \frac{\mathrm{d}\nu}{\mathrm{d}\widehat{\mu}}\cdot\frac{\mathrm{d}\widehat{\mu}}{\mathrm{d}(\mu+\widehat{\mu})} \quad (\mu+\widehat{\mu})\text{–a.e. on } U\,, \tag{2.6.4}$$

where the densities $\mathrm{d}\nu/\mathrm{d}\mu$, $\mathrm{d}\nu/\mathrm{d}\widehat{\mu}$ are defined respectively μ, $\widehat{\mu}$–a.e. on U. Then, by positive homogeneity of g and via the rewriting in (2.6.4) we compute

$$\begin{aligned}
\int_U g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}(\mu+\widehat{\mu})}\right)\mathrm{d}(\mu+\widehat{\mu}) &= \int_U g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\cdot\frac{\mathrm{d}\mu}{\mathrm{d}(\mu+\widehat{\mu})}\right)\mathrm{d}\mu + \int_U g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\cdot\frac{\mathrm{d}\mu}{\mathrm{d}(\mu+\widehat{\mu})}\right)\mathrm{d}\widehat{\mu}\\
&= \int_U g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\frac{\mathrm{d}\mu}{\mathrm{d}(\mu+\widehat{\mu})}\mathrm{d}\mu + \int_U g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\frac{\mathrm{d}\mu}{\mathrm{d}(\mu+\widehat{\mu})}\mathrm{d}\widehat{\mu}\\
&= \int_U g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\frac{\mathrm{d}\mu}{\mathrm{d}(\mu+\widehat{\mu})}\mathrm{d}(\mu+\widehat{\mu})\\
&= \int_U g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\mathrm{d}\mu\,,
\end{aligned}$$

since the density function $\mathrm{d}\mu/\mathrm{d}(\mu+\widehat{\mu})$ is non–negative $(\mu+\widehat{\mu})$–a.e. on U. Symmetrically, exploiting the second equality in (2.6.4) and that $\mathrm{d}\widehat{\mu}/\mathrm{d}(\mu+\widehat{\mu})$ is non–negative $(\mu+\widehat{\mu})$–a.e., it is also:

$$\begin{aligned}
\int_U g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}(\mu+\widehat{\mu})}\right)\mathrm{d}(\mu+\widehat{\mu}) &= \int_U g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}\widehat{\mu}}\cdot\frac{\mathrm{d}\widehat{\mu}}{\mathrm{d}(\mu+\widehat{\mu})}\right)\mathrm{d}\mu + \int_U g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}\widehat{\mu}}\cdot\frac{\mathrm{d}\widehat{\mu}}{\mathrm{d}(\mu+\widehat{\mu})}\right)\mathrm{d}\widehat{\mu}\\
&= \int_U g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}\widehat{\mu}}\right)\frac{\mathrm{d}\widehat{\mu}}{\mathrm{d}(\mu+\widehat{\mu})}\mathrm{d}\mu + \int_U g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}\widehat{\mu}}\right)\frac{\mathrm{d}\widehat{\mu}}{\mathrm{d}(\mu+\widehat{\mu})}\mathrm{d}\widehat{\mu}\\
&= \int_U g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}\widehat{\mu}}\right)\frac{\mathrm{d}\widehat{\mu}}{\mathrm{d}(\mu+\widehat{\mu})}\mathrm{d}(\mu+\widehat{\mu})\\
&= \int_U g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}\widehat{\mu}}\right)\mathrm{d}\widehat{\mu}\,.
\end{aligned}$$

We conclude that all three integrals agree, and thus the definition of $G[\nu]$ is well–posed.

Remark 2.53. We observe that for any pair of measure ν_1, $\nu_2 \in \mathrm{RM}_{\mathrm{loc}}(U,\mathbb{R}^n)$ mutually singular on U and assigned $\nu := \nu_1+\nu_2$, the functional $\mathrm{G}[\nu]$ can be decomposed into the sum of $\mathrm{G}[\nu_1]$ and $\mathrm{G}[\nu_2]$ as measures on U, that means

$$\mathrm{G}[\nu](B) = \mathrm{G}[\nu_1](B)+\mathrm{G}[\nu_2](B) \quad \text{for all Borel sets } B\subseteq U\,.$$

In fact, taking as basis measure the total variation $|\nu| = |\nu_1|+|\nu_2|$, it certainly is ν, ν_1, $\nu_2 \ll |\nu|$ on any B, thus we write

$$\mathrm{G}[\nu](B) = \int_B g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}\right)\mathrm{d}|\nu| = \int_B g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}\right)\mathrm{d}|\nu_1| + \int_B g\left(\cdot,\frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}\right)\mathrm{d}|\nu_2|$$

and we conclude by noticing that $\frac{\mathrm{d}\nu}{\mathrm{d}|\nu|} = \frac{\mathrm{d}\nu_i}{\mathrm{d}|\nu_i|}$ holds $|\nu_i|$–a.e. on B for $i=1,2$.

Actually, for specific convex functionals on measures G we can even achieve a characterization of the measures allowing a further decomposition of G. To this aim, with some abuse of notation we introduce the notion of a strictly convex norm g on the vector space X – and we shall say that X is a *strictly convex space* when equipped with such a norm.

Definition 2.54 (strictly convex norm)**.** If X is a vector space over $\mathbb{R}$, we say that the positively 1–homogeneous mapping $F\colon X \to [0,\infty)$ is a strictly convex norm on X if F is convex, $F(x) > 0$ for all $x \neq 0$, and F satisfies the strict convexity condition:

(SC) For distinct x, $y \in X$ such that $F(x) = F(y) = 1$, we have $F(\lambda x + (1-\lambda)y) < 1$ for every $\lambda \in (0,1)$.

The requirement in (SC) determines that balls with respect to F – that is, all sets of the kind $F^{-1}([0,r]) = \{x \in X : F(x) \leq r\}$ – are strictly convex in X. Notice that strictly convex norms are *not* strictly convex functions, since by positive homogeneity they are affine on half–lines. The easiest examples of strictly convex norms on the Euclidean space are the p–norms $g(x) := |x|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $x \in \mathbb{R}^n$ $(n > 1)$ whenever $p \in (1,\infty)$, whereas the limit cases $p = 1$ and $p = \infty$ – where $|x|_\infty := \max_{i=1,\dots,n} |x_i|$ – lose strict convexity.

Alternatively, one might introduce strictly convex norms employing the equivalent reformulation of (SC) given by:

(SC′) For x, $y \in X$ with $x \neq 0$ such that $F(x) + F(y) = F(x+y)$, it is $y = \gamma x$ for some $\gamma \in [0,\infty)$.

We point out that the reverse implication in (SC′) is always verified, since if $y = \gamma x$ with γ non–negative, the positive homogeneity of F yields $F(x+y) = F((1+\gamma)x) = (1+\gamma)F(x) = F(x) + F(y)$. Proving the equivalence (SC) $\iff$ (SC′) is standard, and we stress that such a proof does not employ the usual assumption for norms of (full) homogeneity of F.

Proposition 2.55. *We consider an open U in $\mathbb{R}^n$ and a mapping $g\colon U \times \mathbb{R}^n \to [0,\infty)$ such that $\xi \mapsto g(x,\xi)$ is a strictly convex norm on $\mathbb{R}^n$ for all $x \in U$. Then, for arbitrary measures ν_1, ν_2 in $\mathrm{RM}_{\mathrm{loc}}(U,\mathbb{R}^n)$, we have*

$$\mathrm{G}[\nu_1+\nu_2] = \mathrm{G}[\nu_1] + \mathrm{G}[\nu_2] \text{ as measures on } U \qquad \iff \qquad \nu_2 = \gamma\nu_1 + \nu\,,$$

where $\gamma\colon U \to [0,\infty)$ and $\nu \in \mathrm{RM}_{\mathrm{loc}}(U,\mathbb{R}^n)$ with $\nu \perp \nu_1$ on U.

Proof. Suppose first that $\mathrm{G}[\nu_1+\nu_2](B) = \mathrm{G}[\nu_1](B) + \mathrm{G}[\nu_2](B)$ for every $B \subseteq U$ Borel. We compute the functionals on measures according to (2.6.3), choosing as basis measure $\mu := |\nu_1| + |\nu_2|$:

$$\mathrm{G}[\nu_1](B) + \mathrm{G}[\nu_2](B) = \int_B \left(g\left(.,\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu}\right) + g\left(.,\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu}\right)\right)\mathrm{d}\mu = \int_B g\left(.,\frac{\mathrm{d}(\nu_1+\nu_2)}{\mathrm{d}\mu}\right)\mathrm{d}\mu = \mathrm{G}[\nu_1+\nu_2](B)\,.$$

By arbitrariness of B, the two quantities above are equal if and only if the integrands agree μ–a.e. on U, that is

$$g\left(x,\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu}(x)\right) + g\left(x,\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu}(x)\right) = g\left(x,\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu}(x) + \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu}(x)\right) \qquad \text{for } \mu\text{–a.e. } x \in U\,. \tag{2.6.5}$$

We now distinguish two cases. On the points $x \in U$ where $\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu}(x) = 0$, the equality in (2.6.5) is always verified since $g(x,0) = 0$. Otherwise, i.e. in case $\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu}(x) \neq 0$, we can apply the strict convexity norm assumption on g restricted to the second variable in the form of (SC′) with vectors $\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu}(x)$, $\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu}(x) \in \mathbb{R}^n$ to obtain $\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu}(x) = \gamma(x)\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu}(x)$ for μ–a.e. x and for $\gamma(x) \geq 0$. Since ν_1 and ν_2 are absolutely continuous with respect to μ, we derive $\nu_2 = \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu}\mu = \gamma\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu}\mu = \gamma\nu_1$ on $\mathrm{supp}(\nu_1)$. Outside the support of ν_1 we don't have any restriction on ν_2, so we can summarize the two cases by arguing the existence of some measure $\nu \in \mathrm{RM}_{\mathrm{loc}}(U, \mathbb{R}^n)$ whose support is contained in the complement of $\mathrm{supp}(\nu_1)$ – hence ν and ν_1 are mutually singular on U – and such that we have $\nu_2 = \gamma\nu_1 + \nu$ on U, as claimed.

If vice versa $\nu_1 \in \mathrm{RM}_{\mathrm{loc}}(U, \mathbb{R}^n)$ and $\nu_2 = \gamma\nu_1 + \nu$ with non–negative γ varying on U and measures $\nu \perp \nu_1$, we easily check that for $B \subseteq U$ Borel it holds

$$\begin{aligned} \mathrm{G}[\nu_2](B) = \mathrm{G}[\gamma\nu_1 + \nu](B) &= \mathrm{G}[\nu](B) + \int_B g\left(\cdot, \frac{\mathrm{d}(\gamma\nu_1)}{\mathrm{d}(|\gamma||\nu_1|)}\right) \mathrm{d}(|\gamma||\nu_1|) \\ &= \mathrm{G}[\nu](B) + \int_B \gamma\, g\left(\cdot, \frac{\mathrm{d}\nu_1}{\mathrm{d}|\nu_1|}\right) \mathrm{d}|\nu_1| \end{aligned}$$

from an application of Remark 2.53, whence we deduce

$$\begin{aligned} \mathrm{G}[\nu_1](B) + \mathrm{G}[\nu_2](B) &= \mathrm{G}[\nu](B) + \int_B (1+\gamma)\, g\left(\cdot, \frac{\mathrm{d}\nu_1}{\mathrm{d}|\nu_1|}\right) \mathrm{d}|\nu_1| \\ &= \mathrm{G}[\nu](B) + \int_B g\left(\cdot, \frac{\mathrm{d}((\gamma+1)\nu_1)}{\mathrm{d}(|\gamma+1||\nu_1|)}\right) \mathrm{d}(|\gamma+1||\nu_1|) \\ &= \mathrm{G}[(1+\gamma)\nu_1](B) + \mathrm{G}[\nu](B) = \mathrm{G}[\nu_1 + \nu_2](B)\,. \end{aligned}$$

This completes the proof of the reverse implication and thus of the equivalence of our two conditions. □

Through the procedure introduced in (2.6.3), we have seen how to define functionals for homogeneous integrands in the second variable. However, we would like to work with functionals on measures – also including inhomogeneous integrands. The following generalised definition serves this purpose well.

Definition 2.56 (functionals from measures)**.** For open $U \subseteq \mathbb{R}^n$, we assume that $g\colon U \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ is Borel and positively 1–homogeneous in $\mathbb{R} \times \mathbb{R}^n$, and we set the functional

$$\mathrm{G}[(\mathcal{L}^n, \nu)](U) := \int_U g\left(\cdot, \frac{\mathrm{d}\mathcal{L}^n}{\mathrm{d}\mu}, \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right) \mathrm{d}\mu \quad \text{for all } \nu \in \mathrm{RM}_{\mathrm{loc}}(U, \mathbb{R}^n)\,.$$

Consistently with the results above, here μ is any non–negative measure in $\mathrm{RM}_{\mathrm{loc}}(U)$ with $(\mathcal{L}^n, \nu) \ll \mu$; for example, we could assign $\mu := \mathcal{L}^n + |\nu|$.

Once again, Theorem 2.1 applied to the $(n+1)$–valued measure $(\mathcal{L}^n, \nu)$ and homogeneity of g in the joint variable yield well–posedness of $\mathrm{G}[(\mathcal{L}^n, \nu)]$ for any ν. Notice that for integrands g independent of the second variable, we reduce to the case of (2.6.3). We are particularly interested in applying Definition 2.56 to the case where ν is the derivative measures of some BV function, i.e. considering the functional $\mathrm{G}[\mathrm{D}w](U) := \mathrm{G}[(\mathcal{L}^n, \mathrm{D}w)](U)$ for $w \in \mathrm{BV}_{\mathrm{loc}}(U)$. Explicitly, we achieve

$$\mathrm{G}[\mathrm{D}w](U) = \int_U g\left(\cdot, \frac{\mathrm{d}\mathcal{L}^n}{\mathrm{d}\mu}, \frac{\mathrm{d}\mathrm{D}w}{\mathrm{d}\mu}\right) \mathrm{d}\mu \quad \text{for all } w \in \mathrm{BV}_{\mathrm{loc}}(U) \tag{2.6.6}$$

defined for any non–negative $\mu \in \mathrm{RM}_{\mathrm{loc}}(U)$ such that $(\mathcal{L}^n, \mathrm{D}w) \ll \mu$.

2.6.1 Functionals with linear growth

We have mentioned in Chapter 1 that our interest lies in first–order functionals of integrands f with linear growth. In such generality, Definition 2.56 cannot be applied straightforwardly to our f, since this latter might *not* be homogeneous in the gradient variable. To overcome such an issue, in [57] the authors presented an adjusted definition of functionals on measures for linear growth integrands, which now depend on two auxiliary functions, the *perspective* and *recession function*.

Unless otherwise specified, we consider arbitrary open domains $U \subseteq \mathbb{R}^n$ and functions $f\colon \overline{U} \times \mathbb{R}^n \to \mathbb{R}$ of linear growth convex in the second entry.

Definition 2.57 (linear growth)**.** We say that a function $f\colon U \times \mathbb{R}^n \to \mathbb{R}$ has **linear growth** in U (meant in the second variable) if there exist constants $\alpha, \beta \in [0, \infty)$ such that

$$\alpha|\xi| \le f(x,\xi) \le \beta(|\xi|+1) \quad \text{for all } (x,\xi) \in U \times \mathbb{R}^n\,. \tag{2.6.7}$$

Definition 2.58 (perspective and recession function)**.** We assume $f\colon U \times \mathbb{R}^n \to \mathbb{R}$ to be convex in the second variable and with linear growth for some constants α, β. The **recession function** of f is $f^\infty\colon U \times \mathbb{R}^n \to [0,\infty)$ defined as

$$f^\infty(x,\xi) := \lim_{t\to 0^+} tf\left(x, \frac{\xi}{t}\right) \quad \text{for all } (x,\xi) \in U \times \mathbb{R}^n\,,$$

whereas the **perspective** (or *homogenized*) **function** of f is $\overline{f}\colon U \times [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\overline{f}(x,t,\xi) := tf\left(x, \frac{\xi}{t}\right) \text{ for } t \in (0,\infty) \quad \text{and} \quad \overline{f}(x,0,\xi) := f^\infty(x,\xi)\,.$$

For any given $x \in U$, the perspective function $\overline{f}(x,.)$ represents an extension of $f(x,.)$ to the positive half–space $[0,\infty) \times \mathbb{R}^n$ in dimension $n+1$. Specifically, it is the extended function of $f(x,.)$ which is positively 1–homogeneous on the half–space and satisfying both conditions

$$\overline{f}(.,1,.) = f \quad \text{and} \quad \overline{f}(.,0,.) = f^\infty\,.$$

We remark that it is the convexity of $\xi \mapsto f(x,\xi)$ to guarantee the existence of the limit in the definition of f^∞. For more general functions f (specifically, such that $f(x,.)$ is continuous but not convex) with linear growth, a double limit notion is required in order to define the recession function; namely, one may set

$$f^\infty(x,\xi) := \liminf_{\substack{(x',\xi')\to(x,\xi)\\ t\to 0^+}} tf\left(x', \frac{\xi'}{t}\right) \quad \text{for all } (x,\xi) \in U \times \left(\mathbb{R}^n \setminus \{0\}\right). \tag{2.6.8}$$

In our framework, the single limit for $t \searrow 0$ for fixed x, ξ suffices, since our necessary assumptions for the minimization problem with functional (1.2.4) always include convexity of the integrand in the gradient variable. This will become clearer while proving the next proposition, where some basic properties of the recession and perspective function are listed. A detailed proof can be found in [52, Lemma 2.8].

Proposition 2.59 (properties of $\overline{f}$ and f^∞)**.** *For f as in Definition 2.58, all of the following hold:*

(i) *The function $\xi \mapsto f^\infty(x,\xi)$ is convex and positively 1–homogeneous in $\mathbb{R}^n$, with*

$$\alpha|\xi| \le f^\infty(x,\xi) \le \beta|\xi| \quad \textit{for all } (x,\xi) \in U \times \mathbb{R}^n. \tag{2.6.9}$$

(ii) *The function $(t,\xi) \mapsto \overline{f}(x,t,\xi)$ is convex and positively 1–homogeneous in $[0,\infty) \times \mathbb{R}^n$, and*

$$\alpha|\xi| \le \overline{f}(x,t,\xi) \le \beta(t+|\xi|) \quad \textit{for all } (x,t,\xi) \in U \times [0,\infty) \times \mathbb{R}^n\,.$$

(iii) *If f is lower semicontinuous, then f^∞ and $\overline{f}$ are lower semicontinuous as well. If both f and f^∞ are continuous, then $\overline{f}$ is continuous.*

Remark 2.60. We record that the lower bound in (2.6.7) – that is, the coercivity assumption (ii) for f in Assumption 1.1 – is not strictly necessary to determine existence of (convex) f^∞ and $\overline{f}$; in fact, by replacing this condition only with boundedness from below

$$f(x,\xi) \geq c_1 \qquad \text{for all } (x,\xi) \in U \times \mathbb{R}^n$$

for a non–negative constant c_1, the result of Proposition 2.59 is still valid with $0 \leq f^\infty(x,\xi) \leq \beta|\xi|$ holding in place of (2.6.9). Such a weaker growth condition will be employed in the duality Section 8.2.

From the convexity assumption, we easily obtain the following variant of the triangular inequality for f and f^∞.

Proposition 2.61. *For f as in Definition 2.58, we have*

$$f(x,\xi) - f(x,\tau) \leq f^\infty(x,\xi-\tau) \qquad \textit{for all} \;\; x \in U, \;\; \textit{all}\; \xi,\tau \in \mathbb{R}^n\,, \tag{2.6.10}$$

and consequently $f(x,\xi) \leq f^\infty(x,\xi) + f(x,0)$ for all $(x,\xi) \in U \times \mathbb{R}^n$.

Proof. For every $x \in U$ and ξ,τ in $\mathbb{R}^n$, convexity of f implies

$$f(x,\xi) = f(x,(\xi-\tau)+\tau) \leq tf\left(x,\frac{\xi-\tau}{t}\right) + (1-t)f\left(x,\frac{\tau}{1-t}\right) \qquad \text{for all } t \in (0,1)\,,$$

and sending t to 0 we obtain just as claimed

$$f(x,\xi) \leq \lim_{t\to 0^+} tf\left(x,\frac{\xi-\tau}{t}\right) + \lim_{t\to 0^+}(1-t)f\left(x,\frac{\tau}{1-t}\right) = f^\infty(x,\xi-\tau) + f(x,\tau)\,.$$

□

Moreover, the convexity of f allows us to determine even a generalized supporting hyperplane inequality for the recession function.

Lemma 2.62. *Any f as in Definition 2.58 achieves*

$$\pm\nabla_\xi f(x,\xi)\cdot\nu \leq f^\infty(x,\pm\nu) \quad \textit{for all } x \in U \;\; \textit{all } \nu \in \mathbb{R}^n \textit{ and a.e. } \xi \in \mathbb{R}^n\,.$$

Proof. We recall that the convexity of f in the second entry yields, via Rademacher's theorem, the existence of the standard gradient $\xi \mapsto \nabla_\xi f(x,\xi)$ for every fixed $x \in U$. We exploit then the supporting hyperplane inequality $f(x,z) \geq f(x,\xi) + \nabla_\xi f(x,\xi)\cdot(z-\xi)$ for $z := \xi \pm \nu$ together with the estimate $f(x,\xi\pm\nu) \leq f(x,\xi) + f^\infty(x,\pm\nu)$ of the previous Proposition 2.61 to achieve the claim. □

As mentioned at the beginning of the section, homogeneity of the perspective function in the joint variable $(t,\xi) \in [0,\infty) \times \mathbb{R}^n$ enables us to extend the definition of functionals on derivative measures (2.6.6) to $\overline{f}$ whenever f has linear growth.

Proposition 2.63. *We assume that the Borel function $f\colon U\times\mathbb{R}^n \to \mathbb{R}$ is convex in the second variable and has linear growth. Then the functional*

$$f(.,\nu)(U) := \int_U \overline{f}\left(.,\frac{\mathrm{d}\mathcal{L}^n}{\mathrm{d}\mu},\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\mathrm{d}\mu \qquad \textit{for } \nu \in \mathrm{RM}_{\mathrm{loc}}(U,\mathbb{R}^n)$$

is well–posed according to Definition 2.56, for any choice of $\mu \in \mathrm{RM}_{\mathrm{loc}}(U)$ non–negative with $(\mathcal{L}^n,\nu) \ll \mu$ on U. Furthermore, it holds

$$f(.,\nu)(U) = \int_U f\left(.,\frac{\mathrm{d}\nu^a}{\mathrm{d}\mathcal{L}^n}\right)\mathrm{d}\mathcal{L}^n + \int_U f^\infty\left(.,\frac{\mathrm{d}\nu^s}{\mathrm{d}|\nu^s|}\right)\mathrm{d}|\nu^s|\,, \tag{2.6.11}$$

where ν^a, ν^s denote respectively the absolutely continuous and singular part of ν with respect to $\mathcal{L}^n$.

Proof. The integral $f(.,\nu)(U)$ is well–posed in view of positive 1–homogeneity of the perspective function $(t,\xi)\mapsto \overline{f}(x,t,\xi)$ from Proposition 2.59(ii). To prove (2.6.11), we consider the measure $\mu := |(\mathcal{L}^n,\nu)|$ and we exploit Remark 2.53 via the Lebesgue decomposition $\nu=\nu^a+\nu^s$, so that we can write $(\mathcal{L}^n,\nu)=(\mathcal{L}^n,\nu^a)+(0,\nu^s)$ with mutually singular addends. From the definition of f^∞ and $\overline{f}$, we then compute

$$\begin{aligned} f(.,\nu)(U) &= \int_U \overline{f}\left(.,\frac{\mathrm{d}\mathcal{L}^n}{\mathrm{d}\mu_1},\frac{\mathrm{d}\nu^a}{\mathrm{d}\mu_1}\right)\mathrm{d}\mu_1 + \int_U \overline{f}\left(.,0,\frac{\mathrm{d}\nu^s}{\mathrm{d}\mu_2}\right)\mathrm{d}\mu_2 \\ &= \int_U \overline{f}\left(.,1,\frac{\mathrm{d}\nu^a}{\mathrm{d}\mathcal{L}^n}\right)\mathrm{d}\mathcal{L}^n + \int_U \overline{f}\left(.,0,\frac{\mathrm{d}\nu^s}{\mathrm{d}|\nu^s|}\right)\mathrm{d}|\nu^s| \\ &= \int_U f\left(.,\frac{\mathrm{d}\nu^a}{\mathrm{d}\mathcal{L}^n}\right)\mathrm{d}\mathcal{L}^n + \int_U f^\infty\left(.,\frac{\mathrm{d}\nu^s}{\mathrm{d}|\nu^s|}\right)\mathrm{d}|\nu^s| \end{aligned}$$

setting $\mu_1 := \mathcal{L}^n$ and $\mu_2 := |\nu^s|$, therefore the decomposition formula (2.6.11) is verified. □

More specifically, for any integrand as in Proposition 2.63 and any $w\in \mathrm{BV}_{\mathrm{loc}}(U)$, the formula above applied to Lebesgue decomposition of $\mathrm{D}w$ becomes

$$f(.,\mathrm{D}w)(U) := \int_U \overline{f}\left(.,\frac{\mathrm{d}\mathcal{L}^n}{\mathrm{d}\mathrm{D}w},\frac{\mathrm{d}\mathrm{D}w}{\mathrm{d}\mu}\right)\mathrm{d}\mu = \int_U f(.,\nabla w)\,\mathrm{d}x + \int_U f^\infty\left(.,\frac{\mathrm{d}\mathrm{D}^s w}{\mathrm{d}|\mathrm{D}^s w|}\right)\mathrm{d}|\mathrm{D}^s w|\,.$$

It is left to provide a generalized meaning to our Dirichlet condition u_0 in BV, and, to do so, we make strong use of the boundary trace and extension Theorem 2.23. We then focus on Borel functions $f\colon U\times\mathbb{R}^n\to\mathbb{R}$ convex in the second variable and with linear growth in U, where $U\Supset\Omega$ and Ω is a bounded Lipschitz domain. Following the pivotal work [57], for any $w\in\mathrm{BV}(\Omega)$ and any datum $u_0\in \mathrm{W}^{1,1}(\mathbb{R}^n)$, we work with the BV extension $\overline{w}^{u_0} := w\mathbb{1}_\Omega + u_0\mathbb{1}_{\mathbb{R}^n\setminus\overline{\Omega}}$ (such that the restriction $\overline{w}^{u_0}|_U$ is in $\mathrm{BV}(U)$) and with the functional

$$\begin{aligned} &f(.,\mathrm{D}\overline{w}^{u_0})(U) \\ &= f(.,\mathrm{D}w)(\Omega) + \int_{\partial\Omega} f^\infty\left(.,(w-u_0)\nu_\Omega\right)\mathrm{d}\mathcal{H}^{n-1} + \int_{U\setminus\Omega} f(.,\nabla u_0)\,\mathrm{d}x \qquad \text{for all } w\in\mathrm{BV}(\Omega)\,. \end{aligned}$$

Remark 2.64 (on an alternative extension procedure)**.** In our later applications of the theory so far presented, we will usually set the domain $U:=\mathbb{R}^n$ and thus consider extensions of any $w\in\mathrm{BV}(\Omega)$ via the given function u_0 on the whole complement of Ω. As a consequence, to write the functional $f(.,\mathrm{D}\overline{w}^{u_0})(\mathbb{R}^n)$ it is required for the integrand to be defined on all $\mathbb{R}^n\times\mathbb{R}^n$. As already mentioned at the beginning of Section 1.1, to do so it suffices to start with some lower semicontinuous f defined on $\overline{\Omega}\times\mathbb{R}^n$ and there satisfying Assumptions (i)–(iii). Then, by Lipschitzianity of the domain, for any $x_0\in\partial\Omega$ we can find a radius r_0 and a bi–Lipschitz invertible mapping $\Phi_{x_0}\colon \mathrm{B}_{r_0}(x_0)\to \mathrm{B}_1$, $\Phi_{x_0}=(\Phi^1_{x_0},\dots,\Phi^n_{x_0})$, such that $\Phi_{x_0}(\mathrm{B}_{r_0}(x_0)\cap\partial\Omega)=\mathrm{B}_1\cap\{x_n=0\}$ and $\Phi_{x_0}(\mathrm{B}_{r_0}(x_0)\cap\Omega)=\mathrm{B}_1\cap\{x_n>0\}$ with B_1 unit ball in $\mathbb{R}^n$ centered at the origin. We then set the function $f_{x_0}(x,\xi):=f(\Phi^{-1}_{x_0}(\Phi_{x_0}(x)-\Phi^n_{x_0}(x)e_n),\xi) = f(\Phi^{-1}_{x_0}(\Phi^1_{x_0}(x),\dots,\Phi^{n-1}_{x_0}(x),0),\xi)$ for all $x\in \mathrm{B}_{r_0}(x_0)\cap\overline{\Omega}^{\mathrm{c}}$ and all $\xi\in\mathbb{R}^n$, in order to extend f constantly along the direction given by the image of $e_n=(0,\dots,0,1)\in\mathbb{R}^n$ via $\Phi^{-1}_{x_0}$. Then, since the function behaviour on the ξ variable remains unaltered, f_{x_0} still satisfies properties (i)–(iii). Furthermore, applying continuity of Φ_{x_0}, $\Phi^{-1}_{x_0}$, and the assumed lower semicontinuity of f on $\partial\Omega\times\mathbb{R}^n$, we deduce that f_{x_0} is even lower semicontinuous on $\left(\mathrm{B}_{r_0}(x_0)\cap\overline{\Omega}^{\mathrm{c}}\right)\times\mathbb{R}^n$. Considering now a partition of unity $\{\rho_i\}_i$ via continuous functions $\rho_i\colon \mathrm{B}_{r_i}(x_i)\to[0,1]$ for suitable $x_i\in\partial\Omega$ such that $\partial\Omega\cap\mathcal{U}=\partial\Omega$ setting $\mathcal{U}:=\bigcup_i \mathrm{B}_{r_i}(x_i)$, we let

$$\widehat{f}(x,\xi) := \begin{cases} f(x,\xi), & \text{if } x\in\overline{\Omega} \\ \sum_i f_{x_i}(x,\xi)\cdot\rho_i(x), & \text{if } x\in\overline{\Omega}^{\mathrm{c}}\cap\mathcal{U}\,. \end{cases}$$

for all $(x,\xi) \in \left(\overline{\Omega} \cup \mathcal{U}\right) \times \mathbb{R}^n$. Then, $\xi \mapsto \widehat{f}(x,\xi)$ is still convex in view of the convexity of each $f_{x_i}(x,.)$, the fact that ρ_i are non–negative, and since for any x it is $\rho_j(x) = 0$ for all but a finite number of indices j. The linear growth conditions (ii)–(iii) for $\widehat{f}$ on $\left(\overline{\Omega} \cup \mathcal{U}\right) \times \mathbb{R}^n$ follow from the analogous properties of f, f_{x_i}, and from boundedness of all ρ_i's. In an analogous way, one may verify that $\widehat{f}$ inherits lower semicontinuity from the component functions f and f_{x_i}'s – also exploiting continuity of ρ_i. Proceeding with further extensions of $\widehat{f}(x,\xi)$ for x outside of the (again Lipschitz) domain $\overline{\Omega} \cup \mathcal{U}$, by subsequent iterations of the construction above we will lastly achieve an extension of f defined on the whole $\mathbb{R}^n$ and which is admissible in the sense of Assumption 3.3.

2.7 Reshetnyak's theorems

In his seminal paper [83], Reshetnyak proved a (semi)continuity result (Theorem 2.65) for positively 1–homogeneous functionals on measures for (semi)continuous integrands under assumptions of weak–$*$ convergence of Radon measures (respectively, strict convergence, for the continuity case); see [83, Theorems 2 and 3]. Such statements can be preserved when considering convex (semi)continuous functionals defined on integrals with linear growth, see Corollary 2.66. In our specific case of the Dirichlet boundary problem in BV, we would also like to extend semicontinuity of the functional up to the boundary of a Lipschitz domain $\Omega \subseteq \mathbb{R}^n$. In such a framework, it is common to assume $\mathcal{L}^n(\partial\Omega) = 0$, and thus the occurrence of weak–$*$ convergence in $\mathrm{RM}_{\mathrm{loc}}\left(\overline{\Omega}, \mathbb{R}^n\right)$ will satisfy our purpose.

For the proofs of the results below we refer, for instance, to [3, Theorems 2.38 and 2.39].

Theorem 2.65 (Reshetnyak semicontinuity; homogeneous version)**.** *We consider a lower semicontinuous function $h\colon U \times \mathbb{R}^n \to [0,\infty)$ which is positively 1–homogeneous and convex in the second variable, and measures $(\nu_k)_k$, ν in $\mathrm{RM}_{\mathrm{loc}}(U,\mathbb{R}^n)$. If $(\nu_k)_k$ converges to ν locally weakly–$*$ in $\mathrm{RM}_{\mathrm{loc}}(U,\mathbb{R}^n)$, then*

$$\liminf_{k\to\infty} \mathrm{H}[\nu_k](U) := \liminf_{k\to\infty} \int_U h\left(.,\frac{\mathrm{d}\nu_k}{\mathrm{d}\mu}\right)\mathrm{d}\mu \geq \int_U h\left(.,\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\mathrm{d}\mu =: \mathrm{H}[\nu](U).$$

Theorem 2.65 can be straightforwardly generalized to possibly non–homogeneous functions, with an application to the (convex, positively 1–homogeneous and LSC) perspective function of $(t,\xi) \mapsto h(x,t,\xi)$ presented in Section 2.6.1, and exploiting the convergence of measures $(\mathcal{L}^n,\nu_k)$, $(\mathcal{L}^n,\nu)$ in $\mathrm{RM}_{\mathrm{loc}}(U,\mathbb{R}^{n+1})$ in place of just ν_k, ν.

Corollary 2.66 (Reshetnyak semicontinuity; non–homogeneous version)**.** *For any lower semicontinuous function $h\colon U \times \mathbb{R}^n \to [0,\infty)$ convex in the second variable, and any $(\nu_k)_k$ in $\mathrm{RM}_{\mathrm{loc}}(U,\mathbb{R}^n)$ converging to ν locally weakly–$*$ in $\mathrm{RM}_{\mathrm{loc}}(U,\mathbb{R}^n)$, we have*

$$\liminf_{k\to\infty} \int_U \overline{h}\left(.,\frac{\mathrm{d}\mathcal{L}^n}{\mathrm{d}\mu},\frac{\mathrm{d}\nu_k}{\mathrm{d}\mu}\right)\mathrm{d}\mu \geq \int_U \overline{h}\left(.,\frac{\mathrm{d}\mathcal{L}^n}{\mathrm{d}\mu},\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\mathrm{d}\mu\,,$$

where $\overline{h}\colon U \times [0,\infty) \times \mathbb{R}^n \to [0,\infty)$ is the perspective function of h as introduced in Definition 2.58. If additionally $\mathcal{L}^n(\partial U) = 0$ and the measures converge weakly–$*$ in $\mathrm{RM}_{\mathrm{loc}}\left(\overline{U},\mathbb{R}^n\right)$, the result can be extended up to the boundary of U, that is $\liminf_{k\to\infty} \mathrm{H}[\nu_k]\left(\overline{U}\right) \geq \mathrm{H}[\nu]\left(\overline{U}\right)$.

For a sketch of proof of the last part of the statement, we refer the reader to [15, Appendix B]. Restricting now to derivative measures of BV functions and assuming $U = \Omega$, we can reformulate the two results above as follows.

Theorem 2.67 (Reshetnyak semicontinuity for admissible functionals)**.** *For $\Omega \subseteq \mathbb{R}^n$ open bounded Lipschitz, $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, and assigned a function $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as in Assumption 1.1, it holds*

$$\liminf_{k\to\infty} f(.,\mathrm{D}\overline{w_k}^{u_0})\left(\overline{\Omega}\right) = \liminf_{k\to\infty} \int_{\overline{\Omega}} \overline{f}\left(.,\frac{\mathrm{d}\mathcal{L}^n}{\mathrm{d}\mu},\frac{\mathrm{dD}w_k}{\mathrm{d}\mu}\right)\mathrm{d}\mu \geq \int_{\overline{\Omega}} \overline{f}\left(.,\frac{\mathrm{d}\mathcal{L}^n}{\mathrm{d}\mu},\frac{\mathrm{dD}w}{\mathrm{d}\mu}\right)\mathrm{d}\mu \;= f(.,\mathrm{D}\overline{w}^{u_0})\left(\overline{\Omega}\right)$$

on sequences $(w_k)_k$ in $\mathrm{BV}(\Omega)$ converging to w in $\mathrm{L}^1(\Omega)$. In other words, by summing the common term in $\overline{\Omega}^{\mathrm{c}}$ we achieve:

$$\liminf_{k\to\infty} f(.,\mathrm{D}\overline{w_k}^{u_0})(U) \geq f(.,\mathrm{D}\overline{w}^{u_0})(U)\,,$$

for any open and bounded set $U \Supset \Omega$ and for $(w_k)_k$, w in $\mathrm{BV}(\Omega)$ converging in $\mathrm{L}^1(\Omega)$ to w.

Proof. We suppose $\liminf_{k\to\infty} f(.,\mathrm{D}\overline{w_k}^{u_0})(U) < \infty$, thus making use of the lower bound in the growth condition it is $\sup_{k\in\mathbb{N}} |\mathrm{D}\overline{w_k}^{u_0}|(U) < \infty$. Since clearly $\overline{w_k}^{u_0} \to \overline{w}^{u_0}$ in $\mathrm{L}^1(U)$ as $k\to\infty$, Theorem 2.22 determines a subsequence[1](not relabelled) and $v \in \mathrm{BV}(U)$ such that $\overline{w_k}^{u_0} \overset{*}{\rightharpoonup} v$ weakly–$*$ in $\mathrm{BV}(U)$ as $k\to\infty$, but the previous convergence in $\mathrm{L}^1(U)$ yields by uniqueness of the limit $v = \overline{w}^{u_0}$ a.e. in U. Specifically, we proved $\mathrm{D}\overline{w_k}^{u_0} \overset{*}{\rightharpoonup} \mathrm{D}\overline{w}^{u_0}$ weakly–$*$ in $\mathrm{RM}(U,\mathbb{R}^n)$ for $k\to\infty$, and an application of Corollary 2.66 yields the thesis. □

We now pass to *continuity* theorems for integral functionals, where in particular the semicontinuity assumption of h in the LSC theorems above is replaced by full continuity, whereas convexity in the second variable is no longer required. However, the continuity result comes with the prize of a stronger assumption – that is, (area–)strict convergence of measures.

Theorem 2.68 (Reshetnyak continuity; homogeneous version)**.** *We consider a continuous mapping $h\colon U\times\mathbb{R}^n \to [0,\infty)$ which is positively 1–homogeneous in the second variable and such that $h(x,\xi) \leq \beta|\xi|$ for some $\beta\in[0,\infty)$. If for the measures $(\nu_k)_k, \nu$ in $\mathrm{RM}(U,\mathbb{R}^n)$ it is $\nu_k \rightharpoonup \nu$ strictly in $\mathrm{RM}(U,\mathbb{R}^n)$ as $k\to\infty$, then*

$$\lim_{k\to\infty} \mathrm{H}[\nu_k](U) = \lim_{k\to\infty}\int_U h\left(.,\frac{\mathrm{d}\nu_k}{\mathrm{d}\mu}\right)\mathrm{d}\mu = \int_U h\left(.,\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\mathrm{d}\mu = \mathrm{H}[\nu](U).$$

Even in the continuity case, the passage to a formulation for non–homogeneous integrands requires positive 1–homogeneity in the joint variable $(t,\xi)\in[0,\infty)\times\mathbb{R}^n$ and the extension to $(n+1)$–dimensional measures. The additional finiteness assumption on the domain is necessary to consider area–strict convergence of BV functions.

Corollary 2.69 (Reshetnyak continuity; non–homogeneous version)**.** *For $U\subseteq\mathbb{R}^n$ open with finite measure, we assume $h\colon U\times\mathbb{R}^n\to[0,\infty)$ to be continuous with $h(x,\xi)\leq\beta(|\xi|+1)$, and such that (2.6.8) exists as a limit. Then, if $(\nu_k)_k$, ν in $\mathrm{RM}(U,\mathbb{R}^n)$ are such that $\nu_k\rightharpoonup\nu$ area–strictly in $\mathrm{RM}(U,\mathbb{R}^n)$ as $k\to\infty$, we have*

$$\lim_{k\to\infty}\int_U \overline{h}\left(.,\frac{\mathrm{d}\mathcal{L}^n}{\mathrm{d}\mu},\frac{\mathrm{d}\nu_k}{\mathrm{d}\mu}\right)\mathrm{d}\mu = \int_U \overline{h}\left(.,\frac{\mathrm{d}\mathcal{L}^n}{\mathrm{d}\mu},\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\mathrm{d}\mu\,.$$

If additionally $\mathcal{L}^n(\partial U)=0$ and the measures converge area–strictly in $\mathrm{RM}\left(\overline{U},\mathbb{R}^n\right)$, the result can be extended up to the boundary of U, yielding $\lim_{k\to\infty}\mathrm{H}[\nu_k]\big(\overline{U}\big) = \mathrm{H}[\nu]\big(\overline{U}\big)$.

An application of Corollary 2.69 to the isotropy $\varphi_0(\xi) := |\xi|$ yields $|\nu_k|(U)\to|\nu|(U)$, hence we deduce strict convergence of $(\nu_k)_k$ to ν from the area–strict convergence of measures. Actually, one could formulate variants of Reshetnyak's continuity theorem by replacing the assumption of area–strict convergence on domains with finite measure with any Ψ–strict convergence in open $U\subseteq\mathbb{R}^n$, provided for $\Psi\in\mathrm{L}^1(U)$ is locally bounded and such that $\operatorname{ess\,inf}_U\Psi>0$, and imposing the refined growth condition $|h(x,\xi)|\leq\Psi(x)+\beta|\xi|$ on $h\colon U\times\mathbb{R}^n\to\mathbb{R}$. The latter continuity result was proved in [15, Theorem 3.10].

We are finally able to state continuity for our integrand f as a straightforward result of Corollary 2.69. Observe that to apply this latter to the perspective function $\overline{f}$, continuity of f^∞ is required

[1]Actually, the procedure described reaches at first semicontinuity only on the selected subsequence of $(w_k)_k$. Then, arguing as usual by contradiction and yet again passing to converging subsequences, one may check that the result of Theorem 2.70 is achieved for the full sequence $(w_k)_k$.

as well – compare with Proposition 2.59(iii). Therefore, the continuity assumption on f only is *not* enough to guarantee the validity of Theorem 2.70, and this justifies the further assumption (H1).

Theorem 2.70 (Reshetnyak continuity for admissible functionals). *For $\Omega \subseteq \mathbb{R}^n$ open bounded and Lipschitz, $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, and the function $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as in Assumption 1.1 such that f and f^∞ are continuous, it holds*

$$\lim_{k\to\infty} f(.,\mathrm{D}\overline{w_k}^{u_0})(\overline{\Omega}) = \lim_{k\to\infty} \int_{\overline{\Omega}} \bar{f}\left(.,\frac{\mathrm{d}\mathcal{L}^n}{\mathrm{d}\mu},\frac{\mathrm{d}\mathrm{D}w_k}{\mathrm{d}\mu}\right)\mathrm{d}\mu = \int_{\overline{\Omega}} \bar{f}\left(.,\frac{\mathrm{d}\mathcal{L}^n}{\mathrm{d}\mu},\frac{\mathrm{d}\mathrm{D}w}{\mathrm{d}\mu}\right)\mathrm{d}\mu \ = f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega})$$

for sequences $(w_k)_k$ in $\mathrm{BV}(\Omega)$ converging to w area–strictly in $\mathrm{BV}(\Omega)$. Consequently, it is even

$$\lim_{k\to\infty} f(.,\mathrm{D}\overline{w_k}^{u_0})(U) = f(.,\mathrm{D}\overline{w}^{u_0})(U)\,,$$

for any open and bounded set $U \Supset \Omega$. In case f is positively 1–homogeneous in the second variable, the continuity result holds even under <u>strict</u> convergence $w_k \rightharpoonup w$ in $\mathrm{BV}(\Omega)$.

Moreover, from the same reasoning following Corollary 2.69 applied now to derivative measures, Theorem 2.70 employed for $f(\xi) = f^\infty(\xi) := |\xi|$ yields strict convergence in $\mathrm{BV}(U)$ from area–strict convergence in $\mathrm{BV}(U)$. Then, the result of Theorem 2.70 achieved up to the boundary of Ω is justified by Proposition 2.25. In fact, this latter applied to $\Psi \equiv 1$ reads out that area–strict convergence of $(w_k)_k$ to w in $\mathrm{BV}(\Omega)$ determines area–strict convergence in $\mathrm{BV}_{u_0}(\overline{\Omega})$, and specifically $\mathrm{D}\overline{w_k}^{u_0} \rightharpoonup \mathrm{D}\overline{w}^{u_0}$ area–strictly in $\mathrm{RM}\left(\overline{\Omega},\mathbb{R}^n\right)$. Finally, the last part of the statement of Corollary 2.69 enables us to reach $\overline{\Omega}$ in the functional convergence.

We additionally record that the full continuity of the integrand f in Theorem 2.70 could be weakened to continuity of $f(x,.)$ for Lebesgue a.e. $x \in \Omega$, provided that the generalized definition of recession function as in (2.6.8) is considered. In any case, one would still need to impose continuity of the perspective function $\bar{f}$ on $\Omega \times \{0\} \times \mathbb{R}^n$. For a more detailed treatment, we refer to [65, Theorem 4] and its variant [15, Theorem 3.10] in possibly unbounded domains.

2.8 Anisotropies and anisotropic variations

In our formulation, an anisotropy on $\mathbb{R}^n$ is any non–negative Borel function $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ such that $\xi \mapsto \varphi(x,\xi)$ is positively 1–homogeneous on $\mathbb{R}^n$. Any φ admits the *mirrored anisotropy* $\widetilde{\varphi}$ defined as

$$\widetilde{\varphi}(x,\xi) := \varphi(x,-\xi) \qquad \text{for all } x,\xi \in \mathbb{R}^n\,.$$

Clearly, $\widetilde{\varphi}$ is still an anisotropy in $\mathbb{R}^n$ and we have $\widetilde{\varphi} \equiv \varphi$ provided φ is even in the second variable, whereas the relation $\widetilde{\widetilde{\varphi}} = \varphi$ always holds. In analogy with the total variation of functions, one introduces the notion of variation in terms of a given anisotropy. In the same way, anisotropic variations of characteristic functions provide a generalization to perimeter measures.

Definition 2.71 (anisotropic TV). Assume φ is an anisotropy on $\mathbb{R}^n$. Given an open $U \subseteq \mathbb{R}^n$ and a Borel set $B \subseteq U$, the **φ-anisotropic total variation** (TV_φ) on B of the function $w \in \mathrm{BV}_{\mathrm{loc}}(U)$ is given by

$$\mathrm{TV}_\varphi(w,B) := |\mathrm{D}w|_\varphi(B) := \int_B \varphi\left(.,\nu_w\right)\mathrm{d}|\mathrm{D}w| \tag{2.8.1}$$

for the Radon–Nikodým density $\nu_w := \mathrm{d}\mathrm{D}w/\mathrm{d}|\mathrm{D}w|$.

Note that (2.8.1) is consistent with our definition of functionals on measures as introduced in (2.6.3) for homogeneous integrands. The TV_φ provides a natural extension of the standard total variation functional, which is retrieved for the isotropy $\varphi_0(\xi) := |\xi|$. For functions $w \in \mathrm{W}^{1,1}_{\mathrm{loc}}(U)$, we can equivalently write $\int_B \varphi(.,\nabla w)\,\mathrm{d}x$ when referring to $\mathrm{TV}_\varphi(w,B)$. Applying the decomposition $\mathrm{D}w = \nabla w\mathcal{L}^n + \mathrm{D}^j w + \mathrm{D}^c w$ for the derivative of any $w \in \mathrm{BV}_{\mathrm{loc}}(U)$, Remark 2.53 yields the rewriting

$$\begin{aligned}\mathrm{TV}_\varphi(w,B) &= \int_B \varphi\left(.,\nabla w\right)\,\mathrm{d}x + \int_B \varphi\left(.,\frac{\mathrm{d}\mathrm{D}^j w}{\mathrm{d}|\mathrm{D}^j w|}\right)\mathrm{d}|\mathrm{D}^j w| + \int_B \varphi\left(.,\frac{\mathrm{d}\mathrm{D}^c w}{\mathrm{d}|\mathrm{D}^c w|}\right)\mathrm{d}|\mathrm{D}^c w| \\ &= |\mathrm{D}^a w|_\varphi(B) + |\mathrm{D}^j w|_\varphi(B) + |\mathrm{D}^c w|_\varphi(B)\,.\end{aligned}$$

We point out that the mirrored anisotropy $\widetilde{\varphi}$ of φ determines the mirrored total variation $\mathrm{TV}_{\widetilde{\varphi}}$ as

$$\mathrm{TV}_{\widetilde{\varphi}}(w,B) = \int_B \varphi\left(.,-\nu_w\right)\mathrm{d}|\mathrm{D}w| = |\mathrm{D}(-w)|_\varphi(B) \qquad \text{for all } w \in \mathrm{BV}_{\mathrm{loc}}(U)\,.$$

Definition 2.72 (anisotropic perimeter)**.** For an anisotropy φ and a set $E \subseteq \mathbb{R}^n$ of locally finite perimeter in open $U \subseteq \mathbb{R}^n$, the **φ-anisotropic perimeter** of E in $B \subseteq U$ is

$$\mathrm{P}_\varphi(E,B) := |\mathrm{D}\mathbb{1}_E|_\varphi(B) = \int_B \varphi\left(.,\nu_E\right)\mathrm{d}|\mathrm{D}\mathbb{1}_E|\,. \tag{2.8.2}$$

Here, ν_E is the $|\mathrm{D}\mathbb{1}_E|$–a.e. defined generalized inner normal vector on the reduced boundary $\partial^* E$ of E as introduced in Section 2.3. We remark that for sets of just *local* finite perimeter, the integrand in (2.8.2) might be infinite. In case $U = \mathbb{R}^n$, consistently with the isotropic case, we neglect the last entry when computing $\mathrm{P}_\varphi(E,\mathbb{R}^n)$.

Observe that, in view of De Giorgi's Theorem 2.34, the φ–perimeter can be equivalently expressed as $\mathrm{P}_\varphi(E,B) = \int_{B\cap\partial^* E} \varphi(.,\nu_E)\,\mathrm{d}\mathcal{H}^{n-1}$. Moreover, being the generalized normal of the complement of E such that

$$\nu_{E^{\mathrm{c}}} := \frac{\mathrm{d}\mathrm{D}\mathbb{1}_{E^{\mathrm{c}}}}{\mathrm{d}|\mathrm{D}\mathbb{1}_{E^{\mathrm{c}}}|} = -\frac{\mathrm{d}\mathrm{D}\mathbb{1}_E}{\mathrm{d}|\mathrm{D}\mathbb{1}_E|} = -\nu_E \quad |\mathrm{D}\mathbb{1}_E| = |\mathrm{D}\mathbb{1}_{E^{\mathrm{c}}}|\text{–a.e.}\,,$$

the mirrored anisotropic perimeter satisfies $\mathrm{P}_{\widetilde{\varphi}}(E,B) = \mathrm{P}_\varphi(E^{\mathrm{c}},B)$.

The following statement is proved in [51, Lemma 2.20] and it extends to anisotropic perimeters the (isotropic) estimate in [92, Lemma 2.9], which takes into account representatives of intersection sets. We underline that its validity will be crucial in order to verify semicontinuity for the anisotropic perimeter coupled with a measure, as – for instance – in the latter Theorem 4.5.

Lemma 2.73. *We consider an open set $U \subseteq \mathbb{R}^n$ and an anisotropy φ such that $\varphi(x,\xi) \le \beta|\xi|$ for all $x,\xi \in \mathbb{R}^n$ and some $\beta \in [0,\infty)$. Then, for sets $A,R,S \subseteq \mathbb{R}^n$ with $\mathrm{P}(A,U)+\mathrm{P}(R,U)+\mathrm{P}(S,U) < \infty$, there hold*

$$\mathrm{P}_\varphi(A\cap R,U) \le \mathrm{P}_\varphi(A,R^1\cap U)+\mathrm{P}_\varphi(R,A^+\cap U) \quad \textit{and} \quad \mathrm{P}_\varphi(A\setminus S,U) \le \mathrm{P}_\varphi(A,S^0\cap U)+\mathrm{P}_{\widetilde{\varphi}}(S,A^+\cap U)\,,$$

where the conditions $\mathcal{H}^{n-1}(\partial^* A\cap\partial^* R\cap U) = 0$ and $\mathcal{H}^{n-1}(\partial^* A\cap\partial^* S\cap U) = 0$ respectively determine equality.

If there exists some $\Gamma \in [0,\infty)$ such that $\varphi \le \Gamma\varphi_0$ everywhere, for any Borel set $B \subseteq U$ we can estimate from above $\mathrm{TV}(w,B)_\varphi \le \Gamma\int_B |\nu_w|\,\mathrm{d}|\mathrm{D}w| = \Gamma|\mathrm{D}w|(B)$ recalling that $\mathrm{Im}(\nu_w) \subseteq \mathbb{S}^{n-1}$, and similarly in the case of a lower bound. Therefore, we obtained:

Proposition 2.74. *If the anisotropy φ satisfies the linear–growth condition*

$$\alpha|\xi| \le \varphi(x,\xi) \le \beta|\xi| \quad \textit{for all } x,\xi \in \mathbb{R}^n \tag{2.8.3}$$

for some $0 < \alpha \le \beta < \infty$, then for any open set $U \subseteq \mathbb{R}^n$ and any $w \in \mathrm{BV}_{\mathrm{loc}}(U)$ we have

$$\alpha|\mathrm{D}w| \le |\mathrm{D}w|_\varphi \le \beta|\mathrm{D}w| \qquad \textit{as measures on } U\,.$$

Any φ satisfying (2.8.3) is said to be **comparable to the Euclidean norm**. In particular, from Proposition 2.74 we deduce that in such a case any set of (locally) finite perimeter in U is a set of (locally) finite φ–perimeter in U, and vice versa.

Remark 2.75. If φ is comparable to the Euclidean norm, then for any given function $w \in \mathrm{BV}_{(\mathrm{loc})}(U)$ the mapping $\mathrm{TV}(w,.)_\varphi$ is a non–negative (locally) finite Radon measure on U. The same applies to the φ–perimeter measure $\mathrm{P}_\varphi(E,.) := |\mathrm{D}\mathbb{1}_E|_\varphi(.)$ on U for any set E of (locally) finite perimeter in U.

Proposition 2.76 (subadditivity of TV_φ with respect to functions)**.** *For any anisotropy φ in $\mathbb{R}^n$ convex in the last n variables, and for any u, $v \in \mathrm{BV}_{\mathrm{loc}}(U)$ on open $U \subseteq \mathbb{R}^n$, it holds*

$$|\mathrm{D}(u+v)|_\varphi \leq |\mathrm{D}u|_\varphi + |\mathrm{D}v|_\varphi \text{ as measures on } U\,. \tag{2.8.4}$$

Proof. Assigned any Borel set $B \subseteq U$, we may take $\mu := |\mathrm{D}u| + |\mathrm{D}v|$ as basis measure and directly compute

$$|\mathrm{D}(u+v)|_\varphi(B) = \int_B \varphi\left(.,\frac{\mathrm{dD}u}{\mathrm{d}\mu} + \frac{\mathrm{dD}v}{\mathrm{d}\mu}\right)\mathrm{d}\mu \leq \int_B \varphi\left(.,\frac{\mathrm{dD}u}{\mathrm{d}\mu}\right)\mathrm{d}\mu + \int_B \varphi\left(.,\frac{\mathrm{dD}v}{\mathrm{d}\mu}\right)\mathrm{d}\mu$$

exploiting convexity of φ. Since both $\mathrm{D}u$, $\mathrm{D}v \ll \mu$, the last two integrals yield the right–hand side of (2.8.4). □

Lemma 2.77. *If φ is an anisotropy on the open set $U \subseteq \mathbb{R}^n$ and $w \in \mathrm{BV}(U)$, then the φ–variation enjoys the following decomposition*

$$|\mathrm{D}w|_\varphi = |\mathrm{D}w_+|_\varphi + |\mathrm{D}w_-|_{\widetilde{\varphi}} \quad \text{as measures on } U\,.$$

We observe that the chain rule Theorem 2.40 guarantees for any function w of (locally) bounded variation that the positive part $w_+ := \max\{w,0\}$ and negative part $w_- := -\min\{w,0\}$ are still of (locally) bounded variation. In case the two functions in Proposition 2.76 are the positive and (the opposite of the) negative part of some BV function w, the subadditivity property (2.8.4) holds as equality – even without convexity of $\xi \mapsto \varphi(x,\xi)$. This can be heuristically justified considering that the derivative measures $\mathrm{D}w_\pm$ are mutually singular on some large part of U (that is, outside the set of jump points J_w where a change of sign of w occurs), and hence here the decomposition of Remark 2.53 applies. For a complete proof of Lemma 2.77 we refer to [51, Lemma 2.15]. An alternative, convexity–based proof can be achieved by strict approximation whenever φ is everywhere continuous and convex in ξ, and such assumptions would be compatible with our admissible class of integrands. Nevertheless, we prefer stating the decomposition result of Lemma 2.77 in the most general set of hypotheses.

A fundamental property of the anisotropic variation for convex anisotropies is its lower semicontinuity with respect to weak–$*$ convergence in BV, as expressed in the following result in consequence of Reshetnyak's Theorem 2.65.

Theorem 2.78 (Reshetnyak semicontinuity for anisotropies)**.** *We consider a lower semicontinuous anisotropy $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ with $\xi \mapsto \varphi(x,\xi)$ convex for every $x \in \mathbb{R}^n$. Given $(u_k)_k$, u in $\mathrm{BV}(U)$ for open $U \subseteq \mathbb{R}^n$, we assume that either:*

(a) *$(u_k)_k$ converges to u weakly–$*$ in $\mathrm{BV}(U)$ as $k \to \infty$; or*

(b) *$(u_k)_k$ converges to u in $\mathrm{L}^1(U)$ with $\varphi(x,\xi) \geq \alpha|\xi|$ for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and some $\alpha \in [0,\infty)$.*

Then we have

$$\liminf_{k\to\infty} |\mathrm{D}u_k|_\varphi(U) \geq |\mathrm{D}u|_\varphi(U)\,. \tag{2.8.5}$$

Proof. Under assumption (a), we can directly apply Theorem 2.65 to the sequence of derivative measures $(\mathrm{D}u_k)_k$ converging to $\mathrm{D}u$ weakly–$*$ in $\mathrm{RM}(U,\mathbb{R}^n)$ to achieve (2.8.5). Suppose now (b) holds. If the minimum limit in (2.8.5) is infinite, the result follows trivially. Otherwise, the lower bound on φ yields

$$\liminf_{k\to\infty} |\mathrm{D}u_k|(U) \le \liminf_{k\to\infty} |\mathrm{D}u_k|_\varphi(U) < \infty\,,$$

hence $\sup_{k\in\mathbb{N}} |\mathrm{D}u_k|(U) < \infty$ up to passage to subsequences, and at the same time the $\mathrm{L}^1(U)$–convergence implies $\sup_{k\in\mathbb{N}} ||u_k||_{\mathrm{L}^1(U)} < \infty$ up to relabelling. An application of Theorem 2.22 yields a (non–relabelled) subsequence and a weak–$*$ limit $v \in \mathrm{BV}(U)$ for the sequence $(u_k)_k$. By uniqueness of the $\mathrm{L}^1(U)$–limit we deduce $v = u$ a.e. on U, as well as

$$\int_U \psi \cdot \mathrm{dD}v = \lim_{k\to\infty} \int_U \psi \cdot \mathrm{dD}u_k = -\lim_{k\to\infty} \int_U \mathrm{div}(\psi) u_k \,\mathrm{d}x = -\int_U \mathrm{div}(\psi) u \,\mathrm{d}x = \int_U \psi \cdot \mathrm{dD}u,$$

holding for any test function $\psi \in \mathrm{C}^\infty_c(U,\mathbb{R}^n)$. Hence $\mathrm{D}v = \mathrm{D}u$ as Radon measures on U, and $u_k \overset{*}{\rightharpoonup} u$ weakly–$*$ in $\mathrm{BV}(U)$. The claimed lower semicontinuity follows now by reduction to the previous condition (a). □

Expectedly, a continuity result for anisotropic variations can be achieved provided we assume full continuity of the integrand and a stronger convergence in BV. Analogously to the corresponding Reshetnyak's result, Theorem 2.79 applies regardless of the convexity of φ.

Theorem 2.79 (Reshetnyak continuity for anisotropies)**.** *We take an open set $U \subseteq \mathbb{R}^n$ and a continuous anisotropy $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ such that $\varphi(x,\xi) \le \beta|\xi|$ for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ for some $\beta \in [0,\infty)$. Then, for any $(u_k)_k$ and u in $\mathrm{BV}(U)$ such that $(u_k)_k$ converges to u strictly in $\mathrm{BV}(U)$ as $k \to \infty$, we have*

$$\lim_{k\to\infty} |\mathrm{D}u_k|_\varphi(U) = |\mathrm{D}u|_\varphi(U)\,. \tag{2.8.6}$$

Moreover, if $U = \Omega$ bounded Lipschitz, the extensions $(\overline{u_k}^{u_0})_k$, $\overline{u}^{u_0}$ via any $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ satisfy

$$\lim_{k\to\infty} |\mathrm{D}\overline{u_k}^{u_0}|_\varphi(\overline{\Omega}) = |\mathrm{D}\overline{u}^{u_0}|_\varphi(\overline{\Omega})\,. \tag{2.8.7}$$

Proof. The first part is a direct consequence of Theorem 2.68 applied to the measures $(\mathrm{D}u_k)_k$ converging strictly to $\mathrm{D}u$ in $\mathrm{RM}(U,\mathbb{R}^n)$. In case $U = \Omega$, we recall that by Theorem 2.23 it is $(\overline{u_k}^{u_0})_k, \overline{u}^{u_0} \in \mathrm{BV}(\mathbb{R}^n)$, and the second limit follows from (2.8.6) in $\mathbb{R}^n$ by cancellation of the common (finite) term on the complement of $\overline{\Omega}$. □

For simplicity, we assign the following name to sequences satisfying condition (2.8.6) or (2.8.7).

Definition 2.80 (φ–strict convergence)**.** Assume φ is an anisotropy on $\mathbb{R}^n$ and U is open therein. We say that:

(i) The sequence $(u_k)_k$ in $\mathrm{BV}(U)$ converges **φ-strictly** in $\mathrm{BV}(U)$ to u, if $u_k \to u$ in $\mathrm{L}^1(U)$ and $|\mathrm{D}u_k|_\varphi(U) \to |\mathrm{D}u|_\varphi(U)$ as $k \to \infty$;

(ii) For $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ and $(u_k)_k$, $u \in \mathrm{BV}_{u_0}(\overline{U})$, the sequence $(u_k)_k$ converges **φ-strictly** in $\mathrm{BV}_{u_0}(\overline{U})$ to u, if the sequence of extensions $\overline{u_k}^{u_0}$ converges φ–strictly on $\mathrm{BV}(\mathbb{R}^n)$ to $\overline{u}^{u_0}$.

Clearly, for continuous anisotropies φ under the growth bound $\varphi(.,\xi) \le \beta|\xi|$, Theorem 2.79 yields

$$u_k \xrightarrow[k\to\infty]{} u \quad \text{strictly in } \mathrm{BV}(\Omega) \implies u_k \xrightarrow[k\to\infty]{} u \quad \varphi\text{–strictly in } \mathrm{BV}_{u_0}(\overline{\Omega})\,.$$

Another essential property of the total variation measure of a BV function is its reformulation as the integral in $\mathbb{R}$ of the perimeter of superlevel sets – meaning, the Fleming–Rishel Coarea Theorem 2.38. We now show that a generalized version of the coarea formula holds for the TV_φ as well. We first prove a preliminary result on inner normals in the reduced boundary for superlevel sets of BV functions.

Lemma 2.81. *Given $u \in \mathrm{BV}(U)$, we consider the superlevel sets $E_t := \{x \in U : u(x) > t\}$ for a.e. $t \in \mathbb{R}$. Then the generalized normal vector to $\partial^* E_t$ coincides with the Radon–Nikodým density of $\mathrm{D}u$ with respect to its total variation up to $\mathcal{H}^{n-1}$–negligible sets, that is*

$$\nu_u = \nu_{E_t} \quad \mathcal{H}^{n-1}\text{–a.e. on } U \cap \partial^* E_t\,, \text{ for a.e. } t \in \mathbb{R}\,. \tag{2.8.8}$$

Proof. Exploiting the result

$$\int_U g \,\mathrm{d}|\mathrm{D}u| = \int_{-\infty}^{\infty} \left(\int_U g(y) \,\mathrm{d}|\mathrm{D}\mathbb{1}_{E_t}|(y) \right) \mathrm{d}t = \int_{-\infty}^{\infty} \left(\int_{U\cap\partial^* E_t} g(y) \,\mathrm{d}\mathcal{H}^{n-1}(y) \right) \mathrm{d}t \tag{2.8.9}$$

of Corollary 2.37 for a Lipschitz function $u\colon U \to \mathbb{R}$ and Borel $g\colon U \to \mathbb{R}$ bounded, we then generalize (2.8.9) to arbitrary $u \in \mathrm{BV}(U)$ – for instance, employing the approximation Theorem 2.103. In a similar way, by Theorem 2.34 we compute

$$\int_U G \cdot \mathrm{d}\mathrm{D}u = \int_{-\infty}^{\infty} \left(\int_U G(y) \cdot \mathrm{d}\mathrm{D}\mathbb{1}_{E_t}(y) \right) \mathrm{d}t = \int_{-\infty}^{\infty} \left(\int_{U\cap\partial^* E_t} G(y) \cdot \nu_{E_t}(y) \,\mathrm{d}\mathcal{H}^{n-1}(y) \right) \mathrm{d}t$$

for any bounded Borel vector field $G\colon U \to \mathbb{R}^n$ and $u \in \mathrm{BV}(U)$. In particular, letting $g := G \cdot \nu_u$ bounded, a combination of the last two formulas yields

$$\int_{-\infty}^{\infty} \left(\int_{U\cap\partial^* E_t} G(y) \cdot \nu_u(y) \,\mathrm{d}\mathcal{H}^{n-1}(y) \right) \mathrm{d}t = \int_{-\infty}^{\infty} \left(\int_{U\cap\partial^* E_t} G(y) \cdot \nu_{E_t}(y) \,\mathrm{d}\mathcal{H}^{n-1}(y) \right) \mathrm{d}t\,.$$

Assuming now $G := (\eta \circ u)\psi$ for some $\eta \in \mathrm{C_c}(\mathbb{R})$ and ψ in a countable and dense set $\mathcal{C}$ in $\mathrm{C_c}(U, \mathbb{R}^n)$, for a Borel representative of G we write

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} \left(\int_{U\cap\partial^* E_t} (\eta \circ u)(y)\psi(y) \cdot (\nu_u(y) - \nu_{E_t}(y)) \,\mathrm{d}\mathcal{H}^{n-1}(y) \right) \mathrm{d}t \\ &= \int_{-\infty}^{\infty} \eta(t) \left(\int_{U\cap\partial^* E_t} \psi(y) \cdot (\nu_u(y) - \nu_{E_t}(y)) \,\mathrm{d}\mathcal{H}^{n-1}(y) \right) \mathrm{d}t\,, \end{aligned}$$

where we applied

$$u(y) = t \quad \text{ for } \mathcal{H}^{n-1}\text{–a.e. } y \in U \cap \partial^* E_t\,, \text{ for a.e. } t \in \mathbb{R}\,.$$

By the fundamental lemma of the calculus of variations, we obtain

$$\int_{U\cap\partial^* E_t} \psi \cdot (\nu_u - \nu_{E_t}) \,\mathrm{d}\mathcal{H}^{n-1} = 0 \quad \text{ for all } \psi \in \mathcal{C}\,, \text{ for a.e. } t \in \mathbb{R}\,.$$

The density of $\mathcal{C}$ in $\mathrm{C_c}(U, \mathbb{R}^n)$ and another application of the fundamental lemma yield (2.8.8). □

We are finally ready to state our anisotropic coarea theorem.

Theorem 2.82 (anisotropic Fleming–Rishel coarea formula)**.** *For an anisotropy φ on $\mathbb{R}^n$ such that $\varphi(x, \xi) \le \beta|\xi|$ for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ for some $\beta \in [0, \infty)$, $u \in \mathrm{BV}(U)$ and any Borel set $B \subseteq U$, it is*

$$|\mathrm{D}u|_\varphi(B) = \int_{-\infty}^{\infty} |\mathrm{D}\mathbb{1}_{E_t}|_\varphi(B) \,\mathrm{d}t = \int_{-\infty}^{\infty} \mathrm{P}_\varphi(E_t, B) \,\mathrm{d}t\,. \tag{2.8.10}$$

Proof. From the a.e.–equality (2.8.8) and applying (2.8.9) to the measurable function $g := \varphi(.,\nu_u) \leq \beta|\nu_u| = \beta$ and thus bounded, we directly employ the definition of φ–variation, φ–perimeter and De Giorgi's Theorem 2.34 to compute

$$\begin{aligned}|\mathrm{D}u|_\varphi(B) = \int_B \varphi(\,\cdot\,,\nu_u)\,\mathrm{d}|\mathrm{D}u| = \int_{-\infty}^{\infty}\int_{B\cap\partial^*E_t} \varphi(\,\cdot\,,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t &= \int_{-\infty}^{\infty}\int_{B\cap\partial^*E_t} \varphi(\,\cdot\,,\nu_{E_t})\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \mathrm{P}_\varphi(E_t,B)\,\mathrm{d}t\,,\end{aligned}$$

proving the statement. □

2.8.1 Polar functions

We introduce the notion of polar function, representing a sort of dual function to the original one.

Definition 2.83 (polar function)**.** For any $g\colon \mathbb{R}^n \to [0,\infty)$ such that $g(\xi) > 0 = g(0)$ for all $\xi \neq 0$, the **polar function** $g^\circ\colon \mathbb{R}^n \to [0,\infty]$ of g is defined as

$$g^\circ(\xi^*) := \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{\xi^*\cdot\xi}{g(\xi)}\,,$$

and the polar inequality follows

$$\xi^*\cdot\xi \leq g^\circ(\xi^*)g(\xi) \qquad \text{for all } \xi,\xi^*\in\mathbb{R}^n\,. \tag{2.8.11}$$

The analysis of equality cases in (2.8.11) is of special importance, being related to the notion of subdifferential of g and thus to minimization properties, see the following Section 2.9.2. We observe that the Cauchy–Schwarz' inequality in $\mathbb{R}^n$ yields for the isotropic variation $\varphi_0 := |.|$ the coincidence $\varphi_0^\circ = \varphi_0$. If g is also positively 1–homogeneous on $\mathbb{R}^n$, we refer to (2.8.11) as **anisotropic Cauchy-Schwarz' inequality** for g.

Any polar g° is a non–negative, positively 1–homogeneous function by definition, and lower semi-continuity follows from being the supremum of affine functions. Observe that $g^\circ(\xi^*) > 0$ for all $\xi^*\neq 0$ with $g^\circ(0) = 0$, moreover g° is always convex since for any $t\in(0,1)$ and any ξ^*, $\xi^{**}\in\mathbb{R}^n$ we have

$$g^\circ(t\xi^*+(1-t)\xi^{**}) \leq t\left(\sup_{\xi\in\mathbb{R}^n\setminus\{0\}}\frac{\xi^*\cdot\xi}{g(\xi)}\right) + (1-t)\left(\sup_{\xi\in\mathbb{R}^n\setminus\{0\}}\frac{\xi^{**}\cdot\xi}{g(\xi)}\right) = tg^\circ(\xi^*) + (1-t)g^\circ(\xi^{**})\,.$$

Accordingly, g° is everywhere continuous, and Rademacher's Theorem states that g° is differentiable a.e. with non–zero gradient in $\mathbb{R}^n$. In the following, we list some properties of polars holding according to the original function.

Proposition 2.84. *We consider $g\colon\mathbb{R}^n\to[0,\infty)$ such that $g(\xi) > 0 = g(0)$ for all $\xi\neq 0$.*

(i) *If g is positively 1–homogeneous, then*

$$g^\circ(\xi^*) = \sup_{\xi\in\mathbb{K}_g} \xi^*\cdot\xi \quad \text{for all } \xi^*\in\mathbb{R}^n,\ \text{ with } \mathbb{K}_g := \{\xi\in\mathbb{R}^n\colon\ g(\xi)\leq 1\} = g^{-1}\big([0,1]\big);$$

(ii) *If there is $\alpha\in[0,\infty)$ such that $g(\xi)\geq\alpha|\xi|$ for all $\xi\in\mathbb{R}^n$, then $g^\circ(\xi^*)\leq\alpha^{-1}|\xi^*|$ for all $\xi^*\in\mathbb{R}^n$ and $\mathbb{K}_g\subseteq\overline{\mathrm{B}_{1/\alpha}}$;*

(iii) *If there is $\beta\in[0,\infty)$ such that $g(\xi)\leq\beta|\xi|$ for all $\xi\in\mathbb{R}^n$, then $g^\circ(\xi^*)\geq\beta^{-1}|\xi^*|$ for all $\xi^*\in\mathbb{R}^n$;*

(iv) *If g is continuous, then the set $\mathbb{K}_g$ is closed in $\mathbb{R}^n$. Assuming additionally $g(\xi) \geq \alpha|\xi|$ for all ξ, then $\mathbb{K}_g$ is even compact, and thus if g is positively 1–homogeneous we have:*

$$\text{for all } \ \xi^* \in \mathbb{R}^n \text{ there exists } \ \overline{\xi} \in \mathbb{K}_g \text{ s.t. } \ g^\circ(\xi^*) = \xi^* \cdot \overline{\xi}\,.$$

Proof. The supremum reformulation in (i) follows directly from homogeneity. To check the upper bound in (ii), we apply the Cauchy–Schwarz' inequality to $\xi^* \in \mathbb{R}^n$: $g^\circ(\xi^*) \leq \sup_{\xi \neq 0} \frac{\xi^* \cdot \xi}{\alpha|\xi|} \leq |\xi^*|/\alpha$ – with inverted inequalities for (iii). Moreover, if $\xi \in \mathbb{K}_g$ the coercivity bound straightforwardly determines $|\xi| \leq g(\xi)/\alpha \leq 1/\alpha$. Finally, to verify (iv) we exploit (i) and (ii), noticing that for any ξ^* assigned the continuous function $\xi \mapsto \xi \cdot \xi^*$ admits maximum in $\mathbb{K}_g$. □

Consider now a mapping $g \colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ with $g = g(x, \xi)$, and its polar function respect to ξ. The following result guarantees continuity not only of $\xi \mapsto g^\circ(x, \xi)$ but of g° in the joint variables (x, ξ).

Proposition 2.85. *Given a continuous function $g \colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ with $\xi \mapsto g(x, \xi)$ positively 1–homogeneous and satisfying $g(x, \xi) \geq \alpha|\xi|$ in $\mathbb{R}^n \times \mathbb{R}^n$, then the polar function $g^\circ \colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ defined with respect to the second variable is still continuous on $\mathbb{R}^n \times \mathbb{R}^n$.*

Proof. We consider a sequence (x_k, ξ_k^*) converging to (x, ξ^*) in $\mathbb{R}^n \times \mathbb{R}^n$. By (iv), for every $k \in \mathbb{N}$ there is $\overline{\xi_k} \in \mathbb{R}^n$ such that $g\big(x_k, \overline{\xi_k}\big) \leq 1$ and $g^\circ(x_k, \xi_k^*) = \xi_k^* \cdot \overline{\xi_k}$. Moreover, $\alpha|\overline{\xi_k}| \leq g\big(x_k, \overline{\xi_k}\big) \leq 1$ determines $\sup_{k\in\mathbb{N}} \big|\overline{\xi_k}\big| \leq 1/\alpha$, hence up to subsequences $(\overline{\xi_k})_k$ converges to some $\overline{\xi} \in \mathbb{R}^n$, and thanks to the continuity of g we have $g\big(x, \overline{\xi}\big) \leq 1$. From this, we conclude

$$\lim_{k\to\infty} g^\circ(x_k, \xi_k^*) = \lim_{k\to\infty} \xi_k^* \cdot \overline{\xi_k} = \xi^* \cdot \overline{\xi} \leq g^\circ(x, \xi^*)\,.$$

Assume now ξ is any vector in $\mathbb{R}^n$ such that $g(x, \xi) \leq 1$. By defining the sequence of elements $\xi_k := \xi/\max\{g(x_k, \xi), 1\}$, we have $g(x_k, \xi_k) \leq 1$ for all k and $\xi_k \to \xi$ in $\mathbb{R}^n$ as $k \to \infty$. Then

$$\xi^* \cdot \overline{\xi} = \lim_{k\to\infty} \xi_k^* \cdot \overline{\xi_k} = \lim_{k\to\infty} g^\circ(x_k, \xi_k^*) \geq \lim_{k\to\infty} \xi_k^* \cdot \xi_k = \xi^* \cdot \xi$$

hence $\xi^* \cdot \overline{\xi} = g^\circ(x, \xi^*)$, therefore we determined continuity of g°. □

Notice that from Proposition 2.85 we infer that polars of continuous, coercive anisotropies on $\mathbb{R}^n$ are still continuous anisotropies on $\mathbb{R}^n$.

2.9 Fundamentals of convex analysis

In this section, we recap some basic notions of convex analysis which will play a fundamental role in our dual formulation of the problem in Chapter 8. We start from the definition of the conjugate function as in [84] or [44]. To fix the notation, we consider X to be a (real) topological vector space and X^* its topological dual space. Endowing X (resp. X^*) with the topology of weak convergence over X^* (resp. X), we obtain a locally convex Hausdorff space. We employ the notation $x^* \cdot x$ to indicate the duality pairing between $x \in X$ and the element $x^* \in X^*$.

2.9.1 Convex conjugate functions

For a generalized function $F \colon X \to \overline{\mathbb{R}}$, the **effective domain** is the set $\mathrm{dom}(F) := \{x \in X \colon F(x) < \infty\}$ and $\mathrm{dom}(F)$ is convex in X provided F is a convex function.

Definition 2.86. Assigned $F\colon X \to \overline{\mathbb{R}}$, the **convex conjugate** function of F is the generalized function $F^*\colon X^* \to \overline{\mathbb{R}}$ defined as

$$F^*(x^*) := \sup_{x\in X} \big(x^* \cdot x - F(x)\big) \quad \text{for all } x^* \in X^* .$$

We observe that F^* is the pointwise supremum on the family of affine functions $x^* \mapsto x^* \cdot x - F(x)$ of X^*, and therefore F^* is convex and lower semicontinuous on X^*. Additionally, the conjugate dual function enjoys $-F(0) \le F(x^*) \le \infty$ for all $x^* \in X^*$. Moreover, if $F \equiv \pm\infty$, we have $F^* \equiv \mp\infty$, and we say that the generalized functions $\pm\infty$ are conjugate to each other.

Definition 2.87. A function $F : X \to (-\infty, \infty]$ is said to be **proper** if $F \not\equiv \infty$.

In particular, proper functions F on X admit conjugate functions satisfying $F^*\colon X^* \to (-\infty, \infty]$. If F is proper, the immediate estimate

$$x \cdot x^* \le F^*(x^*) + F(x) \quad \text{for all } x \in X,\ x^* \in X^* \tag{2.9.1}$$

is called **Fenchel's inequality** for F.

The operation of conjugate can be iterated arbitrarily many times, and we are primarily interested in the *bi–conjugate function* $F^{**} := (F^*)^*$ of F, which on reflexive spaces X is computed as

$$F^{**}(x) = \sup_{x^*\in X^*} (x^* \cdot x - F^*(x^*)) \quad \text{for all } x \in X$$

exploiting the canonical isometric isomorphism $X \simeq X^{**}$. If F^* is proper, the bi–conjugate function of F represents the *lower–semicontinuous convex envelope* (or *LSC convex hull*) of F, namely F^{**} is the largest lower semicontinuous convex function below or equal to F; see, for instance, [44, Chapter I, Proposition 4.1]. Clearly, $F^{**} \equiv F$ on X if and only if F is convex and LSC. In case the space X is also finite–dimensional, convexity of F yields continuity, and the bi–duality operator is the identity precisely on the class of convex functions on X. We now list some immediate properties of conjugate functions.

Proposition 2.88. *For any pair of functions F, $G\colon X \to \overline{\mathbb{R}}$ and a normed space X, we record:*

- *If $F \le G$ in X, then $F^* \ge G^*$ in X^*.*
- *If $G = \lambda F$ for some $\lambda \in (0, \infty)$, then $G^* = \lambda F^*(./\lambda)$.*
- *For any $p \in (0, \infty)$ with Hölder conjugate exponent p', the convex conjugate function of $F := |.|_X^p/p$ is $F^* = |.|_X^{p'}/p'$.*

In a normed space X, the conjugate function of the norm $|.|_X$ can easily be calculated from the inequality of Cauchy–Schwarz as follows. Denoted with $F\colon X \to \mathbb{R}$ the norm function $F(x) := |x|_X$ in X, then the conjugate function F^* of F is

$$F^*(x^*) = \begin{cases} 0, & \text{if } |x^*|_{X^*} \le 1; \\ \infty, & \text{otherwise.} \end{cases}$$

In fact, given $x^* \in X^*$ with $|x^*|_{X^*} \le 1$, by Cauchy–Schwarz it is

$$0 = -\inf_{x\in X} |x|\big(|x^*| + 1\big) \le F^*(x^*) \le \sup_{x\in X} |x|\big(|x^*| - 1\big) = 0\,.$$

If $|x^*| > 1$ instead, we know there exists some $\overline{x} \in X$ with $|\overline{x}| \leq 1$ and satisfying $|x^*| = \overline{x} \cdot x^*$. Then, the estimate (2.9.1) applied to the sequence $(k\overline{x})_k$ yields

$$F^*(x^*) \geq (k\overline{x}) \cdot x^* - F(k\overline{x}) = k\big(|x^*| - |\overline{x}|\big) \geq k\big(|x^*| - 1\big) \quad \text{for all } k \in \mathbb{N}\,,$$

and taking the supremum in k we conclude $F^*(x^*) = \infty$.

Our scope is now generalizing the last consideration achieved for the isotropy $\varphi_0 = |.|$ to the class of positively 1–homogeneous functions. For further employment, we will directly work in the Euclidean setting, specifically $X = \mathbb{R}^n$ for given $n \in \mathbb{N}$, and we will consider as integrands $f = f(x,\xi)$ Borel functions on $\mathbb{R}^n \times \mathbb{R}^n$ convex in the second variable and with linear growth in $\mathbb{R}^n$. In such a case, we record that the convex conjugate function of $\xi \mapsto f(x,\xi)$ is defined as

$$f^*(x,\xi^*) := \sup_{\xi \in \mathbb{R}^n} \big(\xi^* \cdot \xi - f(x,\xi)\big) \quad \text{for all } x,\, \xi^* \in \mathbb{R}^n\,.$$

From the results of Section 2.6.1, we know that f induces a recession function $f^\infty \colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ which is an anisotropy with respect to the second variable, and in particular for all $x \in \mathbb{R}^n$ it holds

$$(f^\infty)^*(x,\xi^*) = \begin{cases} 0, & \text{if } (f^\infty)^\circ(x,\xi^*) \leq 1; \\ \infty, & \text{otherwise}\,. \end{cases}$$

Hence, the only feasible values for conjugates of $f^\infty(x,.)$ (actually, for conjugates of any 1–homogeneous function in ξ) are zero and infinity, with this latter value achieved everywhere on the complement of the closed polar unit ball $\mathbb{K}_{x,(f^\infty)^\circ} := \{\xi \in \mathbb{R}^n \colon\; (f^\infty)^\circ(x,\xi) \leq 1\}$ in $\mathbb{R}^n$ centered in the origin.

Proposition 2.89. *For open $U \subseteq \mathbb{R}^n$, we assume that $f \colon U \times \mathbb{R}^n \to \mathbb{R}$ is convex in the second variable and with linear growth. Fixed $x \in U$, for the convex conjugate function of $\xi \mapsto f(x,\xi)$ evaluated at point $\xi^* \in \mathbb{R}^n$ we have:*

(i) *If $f^*(x,\xi^*) < \infty$, then $(f^\infty)^\circ(x,\xi^*) \leq 1$, where the latter is the polar function computed for the restriction $\xi \mapsto f^\infty(x,\xi)$;*

(ii) *Supposed $(f^\infty)^\circ(x,\xi^*) \leq 1$, then $(f^\infty)^*(x,\xi^*) = 0$;*

(iii) *If (H2) holds and $(f^\infty)^\circ(x,\xi^*) \leq 1$, then the conjugate function evaluated at (x,ξ^*) is finite and satisfies $-f(x,0) \leq f^*(x,\xi^*) \leq M$.*

Proof. To verify (i), we suppose that $(f^\infty)^\circ(x,\xi^*) > 1$ and we claim that $f^*(x,\xi^*) = \infty$. In fact, convexity of $f^\infty(x.,)$ yields continuity in ξ of $f^\infty(x,\xi)$, and the bound in (2.6.9) induces compactness of the set $\{\xi \in \mathbb{R}^n \colon\; f^\infty(x,\xi) \leq 1\}$. Hence, by definition of polar in the second variable, there is $\overline{\xi}$ such that $f^\infty\big(x,\overline{\xi}\big) \leq 1$ fulfilling $\xi^* \cdot \overline{\xi} = (f^\infty)^\circ(x,\xi^*) > 1$. We can thus apply Proposition 2.61 and estimate from below

$$f^*(x,\xi^*) \geq \sup_{\xi \in \mathbb{R}^n} \big(\xi^* \cdot \xi - f^\infty(x,\xi)\big) - c_x \geq k\big(\xi^* \cdot \overline{\xi} - f^\infty\big(x,\overline{\xi}\big)\big) - c_x \geq k\big(\xi^* \cdot \overline{\xi} - 1\big) - c_x$$

for all $k \in \mathbb{N}$ and setting $c_x := f(x,0) \in \mathbb{R}$. A passage to the supremum in k yields the claimed $f^*(x,\xi^*) = \infty$.

The assertion in (ii) is immediately achieved since the assumption $(f^\infty)^\circ(x,\xi^*) \leq 1$ determines $\xi^* \cdot \xi \leq f^\infty(x,\xi)$ for all $\xi \in \mathbb{R}^n$, and via homogeneity of f^∞ we find $0 = -f^\infty(x,0) \leq (f^\infty)^*(x,\xi^*) = \sup_{\xi \in \mathbb{R}^n} \big(\xi^* \cdot \xi - f^\infty(x,\xi)\big) \leq 0$, hence $(f^\infty)^*(x,\xi^*) = 0$ as claimed.

We take now x, ξ^* such that $(f^\infty)^\circ(x,\xi^*) \le 1$ and suppose $f(x,.) \ge f^\infty(x,.) - M$ for some $M \in \mathbb{R}$. Via (ii), we write

$$f^*(x,\xi^*) \le \sup_{\xi\in\mathbb{R}^n} \big(\xi^*\cdot\xi - f^\infty(x,\xi)\big) + M = (f^\infty)^*(x,\xi^*) + M = M\,,$$

which implies our statement (iii) and completes the proof. □

We record that Proposition 2.89 yields for every x the inclusion

$$\mathrm{dom}(f^*(x,.)) \subseteq \mathbb{K}_{x,(f^\infty)^\circ}\,.$$

Furthermore, from (iii), the two sets agree if f also satisfies $\sup_{\xi\in\mathbb{R}^n}[f^\infty(x,\xi) - f(x,\xi)] < \infty$ – recalling that supremum of $f(x,\xi) - f^\infty(x,\xi)$ is in any case finite (and bounded by $f(x,0)$) in consequence of Proposition 2.61.

Remark 2.90. For the prototypical case of the area integrand $f(\xi) = \sqrt{1+|\xi|^2}$ with corresponding recession function $f^\infty(\xi) = (f^\infty)^\circ(\xi) = |\xi|$, one may verify as in [44, Chapter V, Lemma 1.1] that

$$f^*(\xi^*) = \begin{cases} -\sqrt{1-|\xi^*|^2}, & \text{if } \xi^* \in \overline{\mathrm{B}_1};\\ \infty, & \text{otherwise}\,. \end{cases}$$

2.9.2 Subdifferentiability

The notion of subdifferential (respectively subgradient) is useful in order to generalize the notion of differentiability (resp. gradient) to mappings which pointwise do not admit two–sided directional derivatives – for instance, convex functions have one–sided directional derivatives at any point, even when differentiability fails. A subgradient then identifies all possible directions of supporting hyperplanes to the supergraph of the assigned function at a given point. We shall see that such a general tool admits multiple applications in connection with function extremals.

Definition 2.91. Given a generalized function $F\colon X \to \overline{\mathbb{R}}$ and a value $x \in X$, we say that an element x^* in the dual space X^* is a **subgradient** of F at point x – and consequently, that F is **subdifferentiable** at point x – if

$$F(x) + x^*\cdot(z-x) \le F(z) \qquad \text{for all } z \in X\,.$$

The set of all subgradients x^* of F at point x is the **subdifferential** $\partial F(x) \subseteq X^*$. We denote with $\mathrm{Im}(F)$ the collection of all subgradients of F, that is

$$\mathrm{Im}(F) := \bigcup_{x\in X} \partial F(x)\,.$$

A direct consequence of Definition 2.91 is the identification of minimal points of F in terms of the subdifferential:

$$0 \in \partial F(x) \iff F(x) = \min_{y\in X} F(y)\,.$$

When considering convex functions, the definition of subdifferentiability extends the notion of Gateau differentiability (in the following shortly referred to as *differentiability*). Clearly, in our case of interest $X = \mathbb{R}^n$ with the Euclidean norm, functions F which are differentiable at $x \in \mathbb{R}$ admit standard gradient $\nabla F(x)$ as *unique* subgradient of F at x, and thus the subdifferential of F at x is the singleton $\partial F(x) = \{\nabla F(x)\}$. The result is expressed in the next Proposition 2.92 and is quoted from [44, Chapter I, Proposition 5.3].

Proposition 2.92. *Let $F\colon X \to \overline{\mathbb{R}}$ be a convex function and $x \in X$. If F is differentiable in x with differential $F'(x)$, then $\partial F(x) = \{F'(x)\}$. Conversely, for $x \in X$ such that F is finite and continuous in x with subdifferential of F at x being a singleton, then F differentiable in x with $\partial F(x) = \{F'(x)\}$.*

Notice that in general the subdifferential $\partial F(x)$ at any point $x \in X$ is always a closed and convex set (since by definition x^* is a subgradient if and only if x^* satisfies an infinite system of linear inequalities), nevertheless $\partial F(x)$ might be empty for some x. For instance, if on $X = \mathbb{R}^n$ we consider the convex conjugate function f^* of the area integral on $\mathbb{R}^n$ as in Remark 2.90; then clearly f^* is (sub)differentiable on the unit ball, but $\partial f^*(\xi^*) = \emptyset$ if $|\xi^*| \geq 1$ – regardless of the finiteness of f^* on $\partial \mathrm{B}_1$. Nevertheless, in the case of a function F everywhere finite, convexity of F determines $\partial F(x) \neq \emptyset$ for all x; compare with the subsequent Proposition 8.4.

In analogy to what happens in the hyperplane inequality for differentiable functions, the subdifferentiability condition holds with a strict inequality in the case of strictly convex functions.

Remark 2.93. Assume C is a convex domain in X. Then, if for every $x \in C$ it is $\partial F(x) \neq \emptyset$ and for every $x^* \in \partial F(x)$ it holds

$$F(x) + x^* \cdot (z - x) < F(z) \qquad \text{for all } z \in C \setminus \{x\} \tag{2.9.2}$$

with some $F\colon X \to \mathbb{R}$, then F is strictly convex in C. Vice versa, every function $F\colon X \to \mathbb{R}$ strictly convex in C is such that F satisfies the subgradient inequality (2.9.2) in the strict sense for any $x^* \in \partial F(x)$ and any $x \in C$.

Proof. Assume first $x \in C$, and that (2.9.2) is valid for all $z \in C$ with $z \neq x$. Then, for $t \in (0,1)$ we set $u_t := tx + (1-t)z \in C$. We apply the strict inequality in (2.9.2) twice to any element $u_t^* \in \partial F(u_t)$ to compute

$$\begin{aligned}(1-t)F(z) + tF(x) &> (1-t)F(u_t) + (1-t)u_t^* \cdot (z - u_t) + tF(u_u) + tu_t^* \cdot (x - u_t) \\ &= F(u_t) + (1-t)t(z-x) + t(1-t)(x-z) = F(u_t)\,,\end{aligned}$$

which by arbitrariness of x and z yields strict convexity of F in C.

We assume now F to be strictly convex in C and some $x \in C$, $x^* \in \partial F(x)$. For any $z \neq x$ in C, we have $u := (x+z)/2 \in C$ with $u \neq x$, thus exploiting strict convexity and the definition of subgradient, it follows

$$F(x) + F(z) > 2F(u) \geq 2F(x) + 2x^* \cdot (u - x) = 2F(x) + x^* \cdot (z - x)\,,$$

hence (2.9.2) is verified. □

We now want to investigate the equality cases of Fenchel's inequality (2.9.1), and we shall establish a relation between points of the dual space achieving the equality and elements in the subdifferential.

Proposition 2.94 (characterization of subdifferential)**.** *Let $F\colon X \to \overline{\mathbb{R}}$ be proper and fix $x \in X$, $x^* \in X^*$. Then we have*

$$F(x) + F^*(x^*) = x^* \cdot x \qquad \Longleftrightarrow \qquad x^* \in \partial F(x)\,, \tag{2.9.3}$$

meaning that Fenchel's inequality (2.9.1) holds as an equality precisely on subgradients x^* of F at x. Specifically, from the equality above we read $\mathrm{Im}(\partial F) \subseteq \mathrm{dom}(F^*)$.

Proof. The result follows by straightforward computations via definition of the convex conjugate function of F at x^*, namely

$$\begin{aligned}x^* \cdot x - F(x) = F^*(x^*) &\iff x^* \cdot x - F(x) \geq x^* \cdot z - F(z) \quad \text{ for all } z \in X \\ &\iff F(x) + x^* \cdot (z - x) \leq F(z) \quad \text{ for all } z \in X \iff x^* \in \partial F(x)\,.\end{aligned}$$

□

Observe that the assumption $F \not\equiv \infty$ in Proposition 2.94 prevents us from having $F^*(y^*) = -\infty$ for some $y^* \in X^*$, avoiding indeterminate forms of the kind $\infty - \infty$ when computing $F(x) + F^*(x^*)$. Another useful property of strictly convex functions arising from (2.9.3) is that in such a case subdifferentials are pairwise disjoint.

Corollary 2.95. *Any strictly convex function $F\colon X \to \mathbb{R}$ is such that $\partial F(x_1) \cap \partial F(x_2) = \emptyset$ for all $x_1 \neq x_2$ in X.*

Proof. Suppose by contradiction that there exists distinct values $x_1,\ x_2 \in X$ and an element x^* in $\partial F(x_1) \cap \partial F(x_2)$. An application of the equality (2.9.3) yields then $x^* \cdot x_1 - F(x_1) = F^*(x^*) = x^* \cdot x_2 - F(x_2)$, hence rearranging $F(x_1) = F(x_2) + x^* \cdot (x_1 - x_2)$, but this contradicts strict convexity of F according to Remark 2.93. □

Remark 2.96. For a proper, lower semicontinuous and convex function $F\colon X \to (-\infty, \infty]$ on X reflexive, we have

$$x^* \in \partial F(x) \iff x \in \partial F^*(x^*)\,.$$

The equivalence above follows from (2.9.3) applied to F and F^*, while observing that under our assumptions it is $F^{**} = F$. Moreover, the condition $F \not\equiv -\infty$ implies properness of F^*. In fact, if by contradiction $F^* \equiv \infty$, then from definition of bi–conjugate we would have

$$F(x) = (F^*)^*(x) = \sup_{x^* \in X^*} (x^* \cdot x - F^*(x^*)) = -\infty \qquad \text{for all } x \in X\,,$$

which is absurd.

Coming back shortly to our theory of polar functions, we record that elements reaching equality in the anisotropic Cauchy–Schwarz' inequality (2.8.11) are in relation to points in the subdifferential of the polar function, as expressed in the next lemma.

Lemma 2.97 (equality cases in anisotropic Cauchy-Schwarz)**.** *Suppose $g\colon \mathbb{R}^n \to [0, \infty)$ is positively 1–homogeneous and such that $g(\xi) > 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. For every given $\xi^* \in \mathbb{R}^n \setminus \{0\}$, the points $\xi \in \mathbb{R}^n$ achieving the equality in the anisotropic Cauchy–Schwarz' inequality (2.8.11) are characterized by $\xi \in g(\xi)\partial g^\circ(\xi^*)$. In particular, if $\xi \neq 0$ we have*

$$1 \in g(\partial g^\circ(\xi^*))\,.$$

In case g is differentiable at point ξ^*, we deduce that $g(\nabla g^\circ(\xi^*)) = 1$ holds for every differentiability point $\xi^* \in \mathbb{R}^n$ of g° and thus for a.e. $\xi^* \in \mathbb{R}^n$.

Proof. We fix $\xi^* \in \mathbb{R}^n \setminus \{0\}$ and notice that $\xi \in \mathbb{R}^n$ achieves equality in (2.8.11) if and only if ξ^* minimizes the function $\tau^* \mapsto G[\tau^*] := g(\xi)g^\circ(\tau^*) - \tau^* \cdot \xi$ on $\mathbb{R}^n$. This is in turn, equivalent to requiring that

$$0 \in \partial G[\xi^*] = g(\xi)\partial g^\circ(\xi^*) - \xi\,,$$

which holds if and only if $\xi \in g(\xi)g^\circ(\xi^*)$. Assuming then $\xi \in \mathbb{R}^n \setminus \{0\}$, even $g(\xi) \neq 0$ and thus from homogeneity we conclude $1 = g(\xi/g(\xi)) \in g(\partial g^\circ(\xi^*))$. □

2.9.3 Approximate subdifferentials

We already know that the polar function F^* of $F\colon X \to \overline{\mathbb{R}}$ satisfies the Fenchel's inequality $F^*(x^*) + F(x) - x \cdot x^* \geq 0$, with equality attained if and only if $x^* \in X^*$ is a subgradient for F at point $x \in X$. Nevertheless, such a minimum is not always achieved; therefore, it is useful to work with the concept of approximate subdifferentials, which weakens the requirement for standard subdifferentials to the extent that only an upper bound for the above expression is assumed.

Definition 2.98 (approximate subdifferential)**.** For some $\varepsilon \in (0,\infty)$, the collection of elements x^* in X^* such that

$$0 \leq F^*(x^*) + F(x) - x \cdot x^* \leq \varepsilon$$

for some mapping $F\colon X \to \overline{\mathbb{R}}$ and some element $x \in X$ is called the ε–subdifferential of F at x, and x^* is an ε–subgradient of F at x, written $x^* \in \partial_\varepsilon F(x)$.

As for the usual subdifferential, for every x and any ε, the approximate subdifferential $\partial_\varepsilon F(x)$ is a closed and convex subset of X^*. We observe that the sequence $(\partial_\varepsilon F(x))_\varepsilon$ is non–increasing in X^* with respect to the parameter ε, and furthermore $\partial F(x) = \bigcap_{\varepsilon>0} \partial_\varepsilon F(x)$. In case X is reflexive and $F\colon X \to (-\infty,\infty]$ is proper convex and lower semicontinuous, then $F^{**} = F$, and for every $\varepsilon \in (0,\infty)$, $x \in X$, and $x^* \in X^*$, we have

$$x^* \in \partial F_\varepsilon(x) \iff x \in \partial_\varepsilon F^*(x^*)\,, \tag{2.9.4}$$

which extends the relation in Remark 2.96 to approximate subdifferentials.

We now state the major result on approximate subdifferentials as in [44, Chapter I, Theorem 6.2]. This expresses one of the forms of Ekeland's variational principle [43, Theorem 1.1].

Theorem 2.99. *In a Banach space X, we consider a mapping $F\colon X \to \overline{\mathbb{R}}$ that is convex, lower semicontinuous, and not constantly $\pm\infty$ on X. Then, in correspondence of $x \in X$ and $x^* \in \partial_\varepsilon F(x)$ for some $\varepsilon \in (0,\infty)$, there exist $x_\varepsilon \in X$ and $x^*_\varepsilon \in X^*$ such that*

$$|x - x_\varepsilon|_X \leq \sqrt{\varepsilon}\,, \qquad |x^* - x^*_\varepsilon|_{X^*} \leq \sqrt{\varepsilon}\,, \qquad \text{and} \qquad x^*_\varepsilon \in \partial F(x_\varepsilon)\,.$$

2.10 Truncations

Given an open set $U \subseteq \mathbb{R}^n$ and a positive constant M, for every $w\colon U \to \overline{\mathbb{R}}$ we introduce the function

$$w^M := \max\{\min\{w, M\}, -M\}\,. \tag{2.10.1}$$

We call w^M the **truncation** (or *truncated function*) of w at level M. The most relevant properties of truncations of BV functions are listed below.

Lemma 2.100 (properties of truncations)**.** *Let $w \in \mathrm{BV}(U)$ for open $U \subseteq \mathbb{R}^n$, $M \in (0,\infty)$. Then:*

(i) *It is $w^M \in \mathrm{BV}(U) \cap \mathrm{L}^\infty(U)$.*

(ii) *If $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ is an anisotropy, the TV_φ is additive on truncations in the sense*

$$|\mathrm{D}w|_\varphi = |\mathrm{D}w^M|_\varphi + |\mathrm{D}(w - w^M)|_\varphi \qquad \text{as measures on } U\,.$$

In particular, $|\mathrm{D}w^M| \leq |\mathrm{D}w|$ as measures on U.

(iii) *The sequence $(w^M)_M$ converges to w (as well as $(w^M)_\pm \to w_\pm$) in $\mathrm{BV}(U)$ as $M \to \infty$.*

(iv) *If φ is an anisotropy on $\mathbb{R}^n$ such that $\varphi(x,\xi) \leq \beta|\xi|$ for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ with $\beta \in [0,\infty)$, then $w^M \rightharpoonup w$ φ–strictly in $\mathrm{BV}(U)$ for $M \to \infty$.*

(v) *We have the $\mathcal{H}^{n-1}$–a.e. convergences of the representatives $(w^M)^\pm \to w^\pm$ on U for $M \to \infty$.*

Proof. (i) We start by noticing that the functions $t \mapsto \min\{t, M\}$ and $t \mapsto \max\{t, -M\}$ are Lipschitz in $\mathbb{R}$ and vanishing in zero. Hence, the chain rule Theorem 2.40 guarantees that the composition with any BV function w on U is still of bounded variation, namely $w^M \in \mathrm{BV}(U)$. Moreover, from (2.10.1) it is evidently $|w^M| \leq M$ almost everywhere, and the result follows.

(ii) We assume first that $w \in \mathrm{BV}(U)$ is non–negative; then for any $M > 0$ there hold $w - w^M = (w-M)_+$ and $w^M = M - (w-M)_-$ a.e. in U. From the additivity of the φ–variation in Lemma 2.77, we write

$$\begin{aligned}|\mathrm{D}(w-w^M)|_\varphi + |\mathrm{D}w^M|_\varphi &= |\mathrm{D}(w-M)_+|_\varphi + |\mathrm{D}(M-(w-M)_-)|_\varphi \\ &= |\mathrm{D}(w-M)_+|_\varphi + |\mathrm{D}(w-M)_-|_{\widetilde{\varphi}} = |\mathrm{D}(w-M)|_\varphi = |\mathrm{D}w|_\varphi,\end{aligned}$$

as claimed. If $w \in \mathrm{BV}(U)$ is now of arbitrary sign, we exploit the result above for $w_\pm$ and employ $(w-w^M)_\pm = w_\pm - (w_\pm)^M$ and $(w_\pm)^M = (w^M)_\pm$ a.e. on U. Again by additivity, we conclude

$$\begin{aligned}|\mathrm{D}w|_\varphi = |\mathrm{D}w_+|_\varphi + |\mathrm{D}w_-|_{\widetilde{\varphi}} &= |\mathrm{D}(w_+ - (w_+)^M)|_\varphi + |\mathrm{D}(w_+)^M|_\varphi + |\mathrm{D}(w_- - (w_-)^M)|_{\widetilde{\varphi}} + |\mathrm{D}(w_-)^M|_{\widetilde{\varphi}} \\ &= |\mathrm{D}((w-w^M)_+))|_\varphi + |\mathrm{D}(w^M)_+|_\varphi + |\mathrm{D}((w-w^M)_-))|_{\widetilde{\varphi}} + |\mathrm{D}(w^M)_-|_{\widetilde{\varphi}} \\ &= |\mathrm{D}(w-w^M)|_\varphi + |\mathrm{D}w^M|_\varphi\end{aligned}$$

as required. Exploiting now the decomposition above for the isotropy $\varphi_0 := |.|$ in $\mathbb{R}^n$, it is easily computed that the derivative measures satisfy $|\mathrm{D}w^M| \le |\mathrm{D}w|$ in U.

(iii) It is straightforward to verify the $\mathrm{L}^1(U)$–convergence of $(w^M)_M$ to w as $M \to \infty$, thus lower semicontinuity of the total variation and the result of (ii) yield

$$|\mathrm{D}w|(U) \le \liminf_{M\to\infty} |\mathrm{D}w^M|(U) \le |\mathrm{D}w|(U)\,,$$

whence the strict convergence is verified. Finally, by (ii) applied to φ_0, we deduce strong convergence from strict convergence. The same applies to the truncation of the positive and negative parts $(w_\pm)^M$.

(iv) From the growth condition and (iii), it is $0 \le |\mathrm{D}(w-w^M)|_\varphi(U) \le \beta|\mathrm{D}(w-w^M)|(U) \to 0$ as $M \to \infty$, thus φ–convergence follows applying (ii).

(v) For each $x \in U$ such that $w^\pm(x)$ is finite, the conclusion is immediate from $(w^M)^\pm(x) = w^\pm(x)$ for all $M > |w^\pm(x)|$. □

Furthermore, approximate upper and lower limits of truncations induce the following decomposition result.

Lemma 2.101. *If $w\colon U \to [0,\infty)$ is a non–negative measurable function on open $U \subseteq \mathbb{R}^n$ and M is any positive constant, the following decomposition formulas hold*:

(i) *It is $w^+ = (w-w^M)^+ + (w^M)^+$ in U; and*

(ii) *It is $w^- = (w-w^M)^- + (w^M)^-$ in U.*

Proposition 2.102. *Fix a function $w \in \mathrm{BV}(U)$ and a non–negative Borel measure ν on U vanishing on $\mathcal{H}^{n-1}$–negligible subsets. If $w^\pm \in \mathrm{L}^1(U;\nu)$ and $M > 0$, then it is also $(w^M)^\pm \in \mathrm{L}^1(U;\nu)$.*

Proof. We prove it first for the upper approximate limit of w^M. If w is non–negative in U, we bring in the decomposition in Lemma 2.101(i) and that $w^M \le w$ everywhere to write

$$0 \le \int_U (w^M)^+ \,\mathrm{d}\nu = \int_U w^+ \,\mathrm{d}\nu - \int_U (w-w^M)^+ \,\mathrm{d}\nu \le \int_U w^+ \,\mathrm{d}\nu < \infty\,,$$

so $(w^M)^+$ is integrable in U with respect to ν. For functions w of arbitrary sign, it is $(w^M)^+ = ((w^M)_+)^+ - ((w^M)_-)^-$ $\mathcal{H}^{n-1}$–a.e. on U via Lemma 2.28(i), hence

$$\begin{aligned}\int_U |(w^M)^+| \,\mathrm{d}\nu = \int_U |((w^M)_+)^+ - ((w^M)_-)^-| \,\mathrm{d}\nu &\le \int_U \left[((w^M)_+)^+ + ((w^M)_-)^-\right] \mathrm{d}\nu \\ &\le \int_U ((w^M)_+)^+ \,\mathrm{d}\nu + \int_U ((w^M)_-)^+ \,\mathrm{d}\nu \\ &= \int_U ((w_+)^M)^+ \,\mathrm{d}\nu + \int_U ((w_-)^M)^+ \,\mathrm{d}\nu\,,\end{aligned}$$

employing the ν–a.e. equality $((w_\pm)^M)^+ = ((w^M)_\pm)^+$. Applying the previous step to the non–negative functions $w_\pm$, we have obtained that $(w^M)^+ \in \mathrm{L}^1(U;\nu)$. To prove the remaining claim for the lower limit, it suffices to repeat the computations above – this time employing Lemmas 2.101(ii) and 2.28(ii), respectively. □

2.11 Some useful approximation and convergence results

We now collect some convergence theorems holding for BV functions. The first approximation result quoted represents a BV counterpart of the theorem of Meyers–Serrin in the Sobolev space (see [75]), and was originally proved by Anzellotti and Giaquinta in [13, Theorem 1]. An alternative proof can be found in [60, Theorem 1.17].

Theorem 2.103 (Anzellotti–Giaquinta approximation). *For an open set $U \subseteq \mathbb{R}^n$ and any function $u \in \mathrm{BV}(U)$, there exists a sequence $(u_k)_k$ in $\mathrm{C}^\infty(U) \cap \mathrm{W}^{1,1}(U)$ converging strictly in $\mathrm{BV}(U)$ to u.*

We observe that the sequence in Theorem 2.103 does not necessarily converge *strongly* (i.e. in norm) in BV, differently from the result of Meyers–Serrin's theorem in $\mathrm{W}^{1,1}(U)$.

In the following we analyze $\mathcal{H}^{n-1}$–a.e. properties of strictly convergent sequences in BV. We refer to [66, Theorem 3.2] for a proof of Theorem 2.104; analogous results for strongly converging sequences in $\mathrm{W}^{1,1}$ are largely discussed in the classical work of Federer and Ziemer [50, Sections 4 and 10].

Theorem 2.104 (pointwise convergence of BV functions). *Let $U \subseteq \mathbb{R}^n$ be an open set. Consider a sequence $(u_k)_k$ in $\mathrm{BV}(U)$ and a function $u \in \mathrm{BV}(U)$ such that $(u_k)_k$ converges strictly in $\mathrm{BV}(U)$ to u. Then, there exists a subsequence $(u_{k_\ell})_\ell$ such that*

$$u^-(x) \leq \liminf_{\ell\to\infty} u^-_{k_\ell}(x) \leq \limsup_{\ell\to\infty} u^+_{k_\ell}(x) \leq u^+(x) \qquad \text{for } \mathcal{H}^{n-1}\text{–a.e. } x \in U\,.$$

In particular, if $u \in \mathrm{W}^{1,1}(U)$ this yields that the pointwise convergence of the precise representatives $u^*_{k_\ell} \to u^*$ holds $\mathcal{H}^{n-1}$–a.e. in U for $\ell \to \infty$.

We are now interested in (area-)strict approximations of BV functions which are extended by a given boundary datum $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ – namely in the convergence formally introduced in Definition 2.24 for the class BV_{u_0}. To do so, some boundary regularity is needed. Specifically, for the sake of our following treatment, it would be enough to rely on the classical result [18, Lemma B.2], since our extension domains will ultimately be Lipschitz. Nevertheless, we here state the area–strict approximation result in the broader framework of [91, Theorem 1.2].

Theorem 2.105 (area–strict approximation with prescribed boundary datum). *We suppose $\Omega \subseteq \mathbb{R}^n$ is an open bounded set with $\mathrm{P}(\Omega) = \mathcal{H}^{n-1}(\partial\Omega) < \infty$. Then, for every $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ assigned and every $u \in \mathrm{BV}_{u_0}(\overline{\Omega})$, there exists a sequence $(u_k)_k$ in $u_0|_\Omega + \mathrm{C}^\infty_c(\Omega)$ such that $(u_k)_k$ converges to u area–strictly in $\mathrm{BV}_{u_0}(\overline{\Omega})$.*

In turn, Theorem 2.105 relies on the following approximation on sets with finite perimeter, compare with [91, Proposition 4.1].

Proposition 2.106 (interior approximation). *For an open bounded set Ω in $\mathbb{R}^n$ with $\mathrm{P}(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)$ finite and $u \in \mathrm{BV}(\mathbb{R}^n) \cap \mathrm{L}^\infty(\mathbb{R}^n)$, there exists a sequence of open sets $(\Omega_k)_k$ such that $\Omega_k \Subset \Omega$, $\mathrm{P}(\Omega_k) < \infty$ for all $k \in \mathbb{N}$, $|\Omega \setminus \Omega_k| \leq 1/k$ for all k, and the inner traces of u on the boundary satisfy*

$$\int_{\partial^*\Omega_k} \left|\mathrm{T}^{\mathrm{int}}_{\partial^*\Omega_k}(u)\right| \,\mathrm{d}\mathcal{H}^{n-1} \leq \frac{1}{k} + \int_{\partial\Omega} \left|\mathrm{T}^{\mathrm{int}}_{\partial\Omega}(u)\right| \,\mathrm{d}\mathcal{H}^{n-1} \qquad \text{for all } k \in \mathbb{N}\,.$$

It is sometimes useful to produce one–sided (monotone) approximating sequences. For instance, in [26, Theorem 3.3] the authors considered bounded domains Ω in $\mathbb{R}^n$ and functions $u \in \mathrm{BV}(\Omega)$, and determined Sobolev area–strict approximations from above (or, by a change of sign, from below) to u. The aim of the next result is to generalize such a procedure to possibly unbounded domains U with respect to *strict* convergence, and we shall do so employing localizations of U in appropriate balls. For the case of functions with compact support, we follow the cut–off procedure of [92, Lemma 2.21].

Proposition 2.107 (one–sided strict and area–strict approximation)**.** *We consider an open set $U \subseteq \mathbb{R}^n$ and a function $u \in \mathrm{BV}(U)$. Then there exists a sequence $(u_k)_k$ in $\mathrm{W}^{1,1}(U)$ with $u_1 \geq u_k \geq u$ a.e. in U for all k and such that $(u_k)_k$ converges to u strictly in $\mathrm{BV}(U)$. Analogously, there exists $(v_k)_k$ in $\mathrm{W}^{1,1}(U)$ converging strictly to u in $\mathrm{BV}(U)$ and such that $v_1 \leq v_k \leq u$ a.e. in U for all k. In case U is bounded, the approximating sequences $(u_k)_k$, $(v_k)_k$ converge to u even area–strictly in $\mathrm{BV}(U)$. For functions u with compact support in U, we may even take $u_k \in \mathrm{W}^{1,1}_0(U)$ and $v_k \in \mathrm{W}^{1,1}_0(U)$, respectively.*

Proof. For U bounded, the claimed (area–)strict approximating sequence is precisely the one in [26, Theorem 3.3]. We consider then an arbitrary open set U, and we want to achieve one–sided strict convergence. We fix $u_1 \in \mathrm{W}^{1,1}(U)$ such that $u_1 \geq u$ a.e. in U and such that $u_1 - u$ is bounded away from zero a.e. in $U \cap \mathrm{B}_r$ for each $r > 0$. Moreover, for each k it is possible to select a radius $R_k > 0$ achieving $\|u_1\|_{\mathrm{L}^1(U\setminus \mathrm{B}_{R_k})} + \|u\|_{\mathrm{L}^1(U\setminus \mathrm{B}_{R_k})} < \frac{1}{k}$, $|\mathrm{D}u_1|(U\setminus \mathrm{B}_{R_k}) < \frac{1}{k}$, as well as $\|\mathrm{T}^{\mathrm{ext}}_{\partial \mathrm{B}_{R_k}} u_1\|_{\mathrm{L}^1(U\cap \partial \mathrm{B}_{R_k};\mathcal{H}^{n-1})} + \|\mathrm{T}^{\mathrm{int}}_{\partial \mathrm{B}_{R_k}} u\|_{\mathrm{L}^1(U\cap \partial \mathrm{B}_{R_k};\mathcal{H}^{n-1})} < \frac{1}{k}$, and we fix $\varepsilon_k > 0$ such that $u_1 - u \geq \varepsilon_k$ on $U \cap \mathrm{B}_{R_k}$. Then we apply [26, Theorem 3.3] to each $\widetilde{u_k} := u \mathbb{1}_{U\cap \mathrm{B}_{R_k}} + u_1 \mathbb{1}_{U\setminus \mathrm{B}_{R_k}} \in \mathrm{BV}(U)$ restricted to $U \cap \mathrm{B}_{R_k+1}$ and find a sequence $(w_k^\ell)_\ell$ in $\mathrm{W}^{1,1}(U \cap \mathrm{B}_{R_k+1})$ so that $(w_k^\ell)_\ell$ converges to the just introduced function $\widetilde{u_k}$ strictly in $\mathrm{BV}(U \cap \mathrm{B}_{R_k+1})$ as $\ell \to \infty$, and that fulfils $w_k^\ell \geq \widetilde{u_k} \geq u$ a.e. in $U \cap \mathrm{B}_{R_k+1}$ for all ℓ. We extend w_k^ℓ to u_1 on $U \setminus \mathrm{B}_{R_k+1}$ and we consider $\min\{w_k^\ell, u_1\} \in \mathrm{W}^{1,1}(U)$, which agrees with u_1 on $U \setminus B_{R_k}$. Then, our choice of sequences determines

$$\lim_{\ell\to\infty} \|\min\{w_k^\ell, u_1\} - u\|_{\mathrm{L}^1(U)} = \|\widetilde{u_k} - u\|_{\mathrm{L}^1(U)} = \|u_1 - u\|_{\mathrm{L}^1(U\setminus \mathrm{B}_{R_k})} \leq \|u_1\|_{\mathrm{L}^1(U\setminus \mathrm{B}_{R_k})} + \|u\|_{\mathrm{L}^1(U\setminus \mathrm{B}_{R_k})} < \frac{1}{k}\,,$$

and $|\{w_k^\ell \geq u_1\} \cap \mathrm{B}_{R_k}| \leq |\{w_k^\ell - \widetilde{u_k} \geq \varepsilon_k\}| \leq \frac{1}{\varepsilon_k}\|w_k^\ell - \widetilde{u_k}\|_{\mathrm{L}^1(U)} \to 0$ by Lemma 2.14. It follows that for the finite measure $|\mathrm{D}u_1|$ we must have $|\mathrm{D}u_1|(\{w_k^\ell \geq u_1\} \cap \mathrm{B}_{R_k}) \to 0$ for $\ell \to \infty$. All in all, recalling that $|\mathrm{D}w_k^\ell|(U \cap \mathrm{B}_{R_k}) \leq |\mathrm{D}w_k^\ell|(U \cap \mathrm{B}_{R_k+1}) \xrightarrow[\ell\to\infty]{} |\mathrm{D}\widetilde{u_k}|(U \cap \mathrm{B}_{R_k+1}) \leq |\mathrm{D}\widetilde{u_k}|(U)$, we infer

$$\begin{aligned}
&\limsup_{\ell\to\infty} \big|\mathrm{D}\min\{w_k^\ell, u_1\}\big|(U) \\
&\qquad \leq \limsup_{\ell\to\infty} \big[|\mathrm{D}w_k^\ell|(U \cap \mathrm{B}_{R_k}) + |\mathrm{D}u_1|(\{w_k^\ell \geq u_1\} \cap \mathrm{B}_{R_k})\big] + |\mathrm{D}u_1|(U \setminus \mathrm{B}_{R_k}) \\
&\qquad \leq |\mathrm{D}\widetilde{u_k}|(U) + |\mathrm{D}u_1|(U \setminus \mathrm{B}_{R_k}) \\
&\qquad = |\mathrm{D}u|(U \cap \mathrm{B}_{R_k}) + \|\mathrm{T}^{\mathrm{ext}}_{\partial \mathrm{B}_{R_k}} u_1 - \mathrm{T}^{\mathrm{int}}_{\partial \mathrm{B}_{R_k}} u\|_{\mathrm{L}^1(U\cap \partial \mathrm{B}_{R_k};\mathcal{H}^{n-1})} + 2|\mathrm{D}u_1|(U \setminus \mathrm{B}_{R_k}) \\
&\qquad < |\mathrm{D}u|(U) + \frac{3}{k}\,.
\end{aligned}$$

Setting $u_k := \min\{w_{k,\ell_k}, u_1\} \in \mathrm{W}^{1,1}(U)$ with suitably large ℓ_k and for $k \geq 2$, the last computation guarantees that $(u_k)_k$ still converges to u strictly in $\mathrm{BV}(U)$, with clearly $u_1 \geq u_k \geq u$ a.e. in U. This completes the proof for the main $(u_k)_k$ case, whereas the $(v_k)_k$ case follows from the previous one by a change of sign.

If now u has compact support in U, we modify the sequences above by multiplying with a function $\eta \in \mathrm{C}^\infty_{\mathrm{c}}(U)$ such that $\mathbb{1}_{\mathrm{supp}(u)} \leq \eta \leq 1$. Then the resulting sequence $(\eta u_k)_k$ (respectively, $(\eta v_k)_k$) is such that every function has zero trace on the boundary of U – actually, they are even in $\mathrm{W}^{1,1}_{\mathrm{c}}(U)$ –

and from $||u_k||_{\mathrm{L}^1(U\setminus\mathrm{supp}(u))} < 1/k$ for k large enough we deduce $\eta u_k \to \eta u = u$ in $\mathrm{L}^1(U)$ as well as

$$\limsup_{k\to\infty} |\mathrm{D}(\eta u_k)|(U) \leq ||\nabla\eta||_{\mathrm{L}^\infty(U)} \limsup_{k\to\infty} ||u_k||_{\mathrm{L}^1(U\setminus\mathrm{supp}(u))} + \limsup_{k\to\infty} \big(\eta|\mathrm{D}u_k|(U)\big) \leq |\mathrm{D}u|(U)\,.$$

We have therefore proved strict convergence for the sequence $(\eta u_k)_k$ to u, and the same for $(\eta v_k)_k$, which completes the proof of our statement. □

Chapter 3

Isoperimetric conditions

Consider the minimization problem

$$\inf_{w\in \mathrm{BV}(\Omega)} \left(f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega}) - \int_\Omega w^+ \,\mathrm{d}\mu_- + \int_\Omega w^- \,\mathrm{d}\mu_+ \right) \tag{P}$$

for $f\colon \overline{\Omega}\times\mathbb{R}^n \to \mathbb{R}$ with linear growth. In order to justify the appropriate measure assumptions on $\mu := \mu_+ - \mu_-$ on the measure term, we consider the Euler–Lagrange equation corresponding to the full functional, which under suitable differentiability assumptions on $w\colon \Omega\to\mathbb{R}$ and f reads

$$\operatorname{div}\big[\nabla_\xi f(\,\cdot\,,\nabla w)\big] = \mu \quad \text{on } \Omega \tag{EL}$$

for all $w \in \mathrm{W}^{1,1}_{u_0}(\Omega)$ such that the left–hand side is well–posed. Notice that the writing in (EL) is a simplification, since our admissible integrand $f = f(x,\xi)$ might not be everywhere differentiable in $\mathbb{R}^n$, but only a.e. in light of convexity of the restriction $\xi \mapsto f(x,\xi)$ and Rademacher's theorem. Expressively, in the specific case $f = f^\infty$, positive 1–homogeneity yields that the subset $\overline{\Omega}\times\{0\}$ of $\overline{\Omega}\times\mathbb{R}^n$ is composed of singular points for f. In general, assuming that (EL) holds for some function u, we employ integration by parts for divergence–measure vector fields and the convexity estimate $\pm\nabla_\xi f(.,\xi)\cdot\nu \le f^\infty(.,\pm\nu)$ for all ν, $z\in\mathbb{R}^n$ from Lemma 2.62 to write

$$\mp\int_\Omega \psi\,\mathrm{d}\mu = \pm\int_\Omega \nabla_\xi f(.,\nabla u)\cdot\nabla\psi\;\mathrm{d}x \le \int_\Omega f^\infty(.,\pm\nabla\psi)\;\mathrm{d}x \quad \text{for all } \psi\in \mathrm{C}^\infty_{\mathrm{c}}(\Omega)\,. \tag{3.0.1}$$

We observe that, if in place of test functions we integrate by parts (EL) on characteristic functions of a C^1 domain $A \Subset \Omega$, we obtain the analogous *parametric* version

$$\begin{cases} -\mu(A) \le \mathrm{P}_{f^\infty}(A) \\ \mu(A) \le \mathrm{P}_{\widetilde{f^\infty}}(A) \end{cases} \tag{3.0.2}$$

of (3.0.1). The latter bounds provide a necessary condition on the measure μ to the existence of minimizers of (P). We shall say that (3.0.2) represents the *limit* (or *borderline*) case (i.e. with unit constant) of the fundamental *isoperimetric condition* assumption for the pair (μ_-,μ_+) with anisotropy f^∞ and the pair (μ_+,μ_-) with the mirrored anisotropy $\widetilde{f^\infty}$; a rigorous formulation is expressed in the next Definition 3.1. The denomination of limit IC for (3.0.2) – or, equivalently, for (3.0.1) – is motivated by the fact that the value $C = 1$ in the inequalities $-C\mathrm{P}_{f^\infty}(A) \le \mu(A) \le C\mathrm{P}_{\widetilde{f^\infty}}(A)$ results to be an upper bound to ensure both coercivity and semicontinuity.

Throughout the rest of this chapter, we will consider anisotropies φ on $\mathbb{R}^n$ as introduced in Section 2.8. The following definitions and results are stated for quite a general class of non–negative Radon measures ν_1, ν_2, however the reader shall bear in mind that our final aim is applying such theory to the recession function f^∞ in place of φ, as well as to $\mu_\mp$ instead of $\nu_{1/2}$.

Definition 3.1 (anisotropic IC)**.** Assume $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ is an anisotropy. We say that a pair (ν_1, ν_2) of finite, non-negative Radon measures on open $U \subseteq \mathbb{R}^n$ satisfies the **φ-anisotropic isoperimetric condition** (shortly, the φ*–IC*) in U with constant $C \in [0,\infty)$ if it holds

$$\nu_1(A^+) - \nu_2(A^1) \leq C\mathrm{P}_\varphi(A) \quad \text{for all measurable } A \Subset U\,. \tag{3.0.3}$$

For a single non–negative measure ν, the φ–IC is satisfied whenever (3.0.3) holds for $(\nu, 0)$.

In case $\varphi = \varphi_0$ is the standard Euclidean norm in $\mathbb{R}^n$, we speak of *isotropic* (or *standard*) IC. We quote the following result proved in [92, Theorem 7.5 and Lemma 7.2] and establishing equivalent conditions to the isotropic IC for a single measure.

Lemma 3.2 (characterization of isotropic IC for $(\nu,0)$)**.** *For a finite, non–negative Radon measure ν on the open set $U \subseteq \mathbb{R}^n$, the following are equivalent*:

(i) *ν satisfies the isotropic IC with constant C, that is $\nu(A^+) \leq C\mathrm{P}(A)$ for all measurable $A \Subset U$;*

(ii) *It is $\nu(A^1) \leq C\mathrm{P}(A)$ for all measurable $A \Subset U$;*

(iii) *It holds $\nu(Z) = 0$ on every $\mathcal{H}^{n-1}$–negligible Borel set $Z \subseteq U$ and $\int_U |w^*|\,\mathrm{d}\nu \leq C||\nabla w||_{\mathrm{L}^1(U,\mathbb{R}^n)}$ for all $w \in \mathrm{W}^{1,1}_0(U)$.*

Definition 3.3 (admissible measures)**.** We say that a finite, non–negative Radon measure ν on open $U \subseteq \mathbb{R}^n$ is **admissible** if ν satisfies the isotropic IC for some constant $C \in [0,\infty)$.

Notice that the admissibility condition for measures might be alternatively posed assuming the φ–IC holds for some anisotropy φ comparable to the Euclidean norm on $\mathbb{R}^n$ – since the value of the constant C in Definition 3.3 is irrelevant. Furthermore, we record that on suitable domains $U = \Omega$, the admissibility condition in Definition 3.3 is equivalent to requiring that the measure ν is an element of the dual space of $\mathrm{BV}(\Omega)$. For instance, this is the case of bounded Lipschitz domains Ω, where it follows from Poincaré's Theorem 2.30 taking into account [98, Lemma 5.10.4], and our functional reformulation of the IC expressed in the later Theorem 3.11.

The two results below express some immediate consequences of Lemma 3.2.

Corollary 3.4 (continuity properties of measure integrals in $\mathrm{W}^{1,1}_{u_0}$)**.** *Let ν be an admissible measure on an open set $U \subseteq \mathbb{R}^n$ and assume $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$. For any $u \in \mathrm{W}^{1,1}_{u_0}(U)$ there exists a sequence of functions $(u_k)_k$ such that $u_k \in u_0|_U + \mathrm{C}^\infty_\mathrm{c}(U)$ for all k, u_k converging to u in $\mathrm{W}^{1,1}(U)$ as $k \to \infty$, and*

$$\lim_{k\to\infty} \int_U u_k^*\,\mathrm{d}\nu = \int_U u^*\,\mathrm{d}\nu\,.$$

Proof. For $u \in \mathrm{W}^{1,1}_{u_0}(U)$, by density we find a sequence $(v_k)_k$ in $\mathrm{C}^\infty_\mathrm{c}(U)$ converging to $u - u_0$ in $\mathrm{W}^{1,1}(U)$. Then the sequence of functions $u_k := u_0|_U + v_k$ is in the right class and $u_k \to u$ in $\mathrm{W}^{1,1}(U)$ as $k \to \infty$ by definition. Moreover, by admissibility of ν, Lemma 3.2(iii) determines

$$\left|\int_U (u_k - u)^*\,\mathrm{d}\nu\right| \leq \int_U |(u_k-u)^*|\,\mathrm{d}\nu \leq C||\nabla u_k - \nabla u||_{\mathrm{L}^1(U,\mathbb{R}^n)} \quad \text{for every } k \in \mathbb{N}$$

for some non–negative constant C. The thesis is then reached by applying strong convergence in $\mathrm{W}^{1,1}$. □

The notion of admissibility in our framework is motivated by the following result, which, however, requires some regularity assumption for the domain as it relies on the strict approximation of Theorem 2.105.

Proposition 3.5 (characterization of admissible measures). *Consider a non–negative finite Radon measure ν on a bounded Lipschitz set Ω in $\mathbb{R}^n$. Then, ν is admissible if and only if they hold:*

(C1) $\nu(Z) = 0$ *for every $\mathcal{H}^{n-1}$–negligible Borel set $Z \subseteq \Omega$; and*

(C2) *The mapping* $\mathrm{BV}(\Omega) \ni w \mapsto \int_\Omega w^+ \,\mathrm{d}\nu$ *is well–defined in $\overline{\mathbb{R}}$ and it holds $\left|\int_\Omega w^+ \,\mathrm{d}\nu\right| < \infty$ for every $w \in \mathrm{BV}(\Omega)$.*

Recalling that representatives of BV functions are not affected by a change of values on negligible sets for $\mathcal{H}^{n-1}$, the hypothesis (C1) guarantees that our integrals $\int_\Omega w^\pm \,\mathrm{d}\nu$ are at least well–posed. Moreover, we observe that the choice of representatives in (C2) is irrelevant, with passage to the lower approximate limit w^- justified by $w^- = -(-w)^+$ for $w \in \mathrm{BV}(\Omega)$. We infer that (C1)–(C2) represent the weakest assumptions on $\mu_\pm$ such that the full functional $\mathcal{F}$ is well–defined.

Provided (C1) holds, yet another reformulation of (C2) is:

(C3) $w^+ \in \mathrm{L}^1(\Omega;\nu)$ for every $w \in \mathrm{BV}(\Omega)$,

where as above w^+ could be replaced by any representative of w. The equivalence of (C2) and (C3) is motivated by the ν–a.e. estimate $|w^+| = |(w_+)^+ - (w_-)^-| \leq (w_+)^+ + (w_-)^- \leq (w_+)^+ + (w_-)^+$ from Lemma 2.28, hence applying (C2) in the form $\int_\Omega (w_\pm)^+ \mathrm{d}\nu < \infty$ we deduce $\int_\Omega |w^+| \,\mathrm{d}\nu < \infty$.

Remark 3.6. Comparing Definition 3.3 and the Assumption 1.2 in Chapter 1 for an admissible measure, it follows by Lemma 3.2 that the two notions coincide in our case of interest $U = \Omega$.

Proof of Proposition 3.5. We assume that ν satisfies (C1)–(C2) and we want to prove that ν is admissible. Suppose by contradiction that ν does not satisfy the IC in Ω for any $C \in [0,\infty)$, and apply the equivalent condition in Lemma 3.2(ii) to ν. Then, there is a sequence of sets $(A_k)_k$ with $A_k \Subset \Omega$, $\mathrm{P}(A_k) > 0$, and such that

$$\nu(A_k^1) > k\mathrm{P}(A_k) \qquad \text{for all } k \in \mathbb{N}\,. \tag{3.0.4}$$

From (3.0.4) and finiteness of ν, we find some index $\overline{k} \in \mathbb{N}$ such that $\mathrm{P}(A_k) < 1$ for all $k \geq \overline{k}$. If $n = 1$, we already reached a contradiction since any $A \Subset \mathbb{R}$ of positive perimeter satisfies $\mathrm{P}(A) \geq 2$. Suppose now $n \geq 2$. For any k, we let $\alpha_k := 1/(k^2\mathrm{P}(A_k))$ and consider $w := \sum_{k=1}^\infty \alpha_k \mathbb{1}_{A_k}$. The isoperimetric inequality (2.3.1) implies

$$\int_\Omega \alpha_k \mathbb{1}_{A_k} \,\mathrm{d}x = \frac{|A_k|}{k^2\mathrm{P}(A_k)} \leq c_n \frac{\mathrm{P}(A_k)^{1/(n-1)}}{k^2} \leq c_n \frac{1}{k^2} \quad \text{for all } k \geq \overline{k}\,,$$

with $c_n := (n^n\omega_n)^{-1/(n-1)}$. Thus, w is well–defined with $\|w\|_{\mathrm{L}^1(\Omega)} = \sum_{k=1}^\infty \int_\Omega \alpha_k \mathbb{1}_{A_k} \,\mathrm{d}x < \infty$. Moreover, for its derivative measure, we compute

$$|\mathrm{D}w|(\Omega) \leq \sum_{k=1}^\infty \alpha_k |\mathrm{D}\mathbb{1}_{A_k}|(\Omega) = \sum_{k=1}^\infty \alpha_k \mathrm{P}(A_k) = \sum_{k=1}^\infty \frac{1}{k^2} < \infty\,,$$

hence $w \in \mathrm{BV}(\Omega)$. Finally, taking into account that $w^+ \geq w^- \geq \sum_{k=1}^\infty \alpha_k (\mathbb{1}_{A_k})^-$, by Lemma 2.28 and applying (3.0.4) we get

$$\int_\Omega w^+ \,\mathrm{d}\nu \geq \sum_{k=1}^\infty \alpha_k \int_\Omega (\mathbb{1}_{A_k})^- \,\mathrm{d}\nu = \sum_{k=1}^\infty \alpha_k \nu(A_k^1) \geq \sum_{k=1}^\infty \frac{k\mathrm{P}(A_k)}{k^2\mathrm{P}(A_k)} = \sum_{k=1}^\infty \frac{1}{k} = \infty\,,$$

which yields a contradiction with (C2). We conclude that ν is an admissible measure on Ω.

Suppose now ν that is admissible in the sense of Definition 3.3. From Lemma 3.2(iii) we read that the condition (C1) is already in place, together with the estimate

$$\int_\Omega |w^*| \,\mathrm{d}\nu \leq C\|\nabla w\|_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} \qquad \text{for all } w \in \mathrm{W}^{1,1}_0(\Omega)\,.$$

We first pass to the inequality for functions of arbitrary boundary trace by strict approximation. In fact, given $w \in \mathrm{W}^{1,1}(\Omega)$, thanks to Theorem 2.105 we find a sequence $(w_k)_k$ in $\mathrm{W}^{1,1}_0(\Omega)$ converging to w (area–strictly and thus) strictly in $\mathrm{BV}_0(\overline{\Omega})$. Then, Theorem 2.104 yields pointwise $\mathcal{H}^{n-1}$–a.e. convergence of a subsequence $w^*_{k_\ell} \to w^*$ as $\ell \to \infty$, and Fatou's lemma implies

$$\left|\int_\Omega w^* \,\mathrm{d}\nu\right| \leq \int_\Omega |w^*| \,\mathrm{d}\nu \leq \liminf_{\ell\to\infty} \int_\Omega |w^*_{k_\ell}| \,\mathrm{d}\nu \leq C \limsup_{\ell\to\infty} \|\nabla w_{k_\ell}\|_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} < \infty\,, \tag{3.0.5}$$

which is precisely our claim for arbitrary Sobolev functions. To achieve the generalization to BV, we exploit (3.0.5) and the fact that any $w \in \mathrm{BV}(\Omega)$ admits Sobolev functions $w_1, w_2 \in \mathrm{W}^{1,1}(\Omega)$ such that $w_1^* \leq w^- \leq w^+ \leq w_2^*$ holds $\mathcal{H}^{n-1}$–a.e. on Ω. Therefore

$$-\infty < \int_\Omega w_1^* \,\mathrm{d}\nu \leq \int_\Omega w^+ \,\mathrm{d}\nu \leq \int_\Omega w_2^* \,\mathrm{d}\nu < \infty\,,$$

and even the remaining condition (C2) is proved. □

Proposition 3.7 (one–sided continuity for measure integrals). *Consider a non–negative Radon measure ν on open $U \subseteq \mathbb{R}^n$, $\nu(Z) = 0$ for every $\mathcal{H}^{n-1}$–negligible Borel set $Z \subseteq U$ and $w^+ \in \mathrm{L}^1(U;\nu)$ for every $w \in \mathrm{BV}(U)$.*

- *If a sequence $(u_k)_k$ in $\mathrm{W}^{1,1}(U)$ converges to $u \in \mathrm{BV}(U)$ strictly in $\mathrm{BV}(U)$ with $u_1 \geq u_k \geq u$ a.e. in U for all k, there exists a subsequence $(u_{k_\ell})_\ell$ satisfying*

$$\lim_{\ell\to\infty} \int_U u^*_{k_\ell} \,\mathrm{d}\nu = \int_U u^+ \,\mathrm{d}\nu\,. \tag{3.0.6}$$

- *Symmetrically, if $(v_k)_k$ in $\mathrm{W}^{1,1}(U)$ converges to $u \in \mathrm{BV}(U)$ strictly in $\mathrm{BV}(U)$ with $v_1 \leq v_k \leq u$ a.e. in U for all k, then there exists a subsequence $(v_{k_\ell})_\ell$ such that*

$$\lim_{\ell\to\infty} \int_U v^*_{k_\ell} \,\mathrm{d}\nu = \int_U u^- \,\mathrm{d}\nu\,.$$

Proof. The second point can be deduced from the first one simply by applying (3.0.6) to $-v_k$ in place of u_k, $-u$ in place of u and employing the equality $-(-u)^+ = u^-$. In order to prove (3.0.6), we exploit that the chain of inequalities $u_1 \geq u_k \geq u$ holding $\mathcal{L}^n$–a.e. is still valid for the representatives, meaning $u_1^* \geq u_k^* \geq u^+$ $\mathcal{H}^{n-1}$–a.e. Additionally, by Theorem 2.104 there exists a subsequence such that $\lim_{\ell\to\infty} u^*_{k_\ell} = u^+$ $\mathcal{H}^{n-1}$–a.e. in U, and this by hypothesis is preserved ν–a.e. in U. Furthermore (as mentioned in Proposition 3.5), our assumption on ν guarantees also $u_1^*, u^+ \in \mathrm{L}^1(U;\nu)$. We conclude applying dominated convergence theorem to the sequence $(u^*_{k_\ell})_\ell$ to obtain (3.0.6). □

Lemma 3.8 (continuity of ν–integrals along truncations). *Suppose ν is a non–negative Radon measure on an open set $U \subseteq \mathbb{R}^n$, $\nu(Z) = 0$ for every $\mathcal{H}^{n-1}$–negligible Borel set $Z \subseteq U$ and $w^+ \in \mathrm{L}^1(U;\nu)$ for every $w \in \mathrm{BV}(U)$. Then, for every $w \in \mathrm{BV}(U)$ it holds*

$$\lim_{M\to\infty} \int_U \left(w^M\right)^+ \mathrm{d}\nu = \int_U w^+ \,\mathrm{d}\nu \qquad \text{and} \qquad \lim_{M\to\infty} \int_U \left(w^M\right)^- \mathrm{d}\nu = \int_U w^- \,\mathrm{d}\nu\,. \tag{3.0.7}$$

Proof. We recall that by Lemma 2.100(v) the pointwise convergence $(w^M)^\pm \to w^\pm$ for $M \to \infty$ holds first $\mathcal{H}^{n-1}$–a.e., and by assumption on ν even ν–a.e. on U. If w is non–negative, truncations satisfy $0 \le w^{M_1} \le w^{M_2}$ for $M_1 \le M_2$, thus (3.0.7) is achieved by monotone convergence theorem. For w of any sign, we reason with the $\mathcal{H}^{n-1}$-a.e. (and ν–a.e.) decompositions $w^\pm = (w_+)^\pm - (w_-)^\mp$, $(w^M)^\pm = ((w^M)_+)^\pm - ((w^M)_-)^\mp = ((w_+)^M)^\pm - ((w_-)^M)^\mp$ of Lemma 2.28, and finally applying (3.0.7) to $w_\pm$ separately to get the general statement. □

3.1 Characterization of anisotropic ICs for arbitrary pairs of measures

In the last section, we have seen that the admissibility conditions (C1)–(C2) on the single measures ν_1, ν_2 suffice to guarantee

$$w \mapsto \int_\Omega w^- \,\mathrm{d}\nu_2 - \int_\Omega w^+ \,\mathrm{d}\nu_1 \quad \text{well–posed and finite for all } w \in \mathrm{BV}(\Omega)\,.$$

However, bearing in mind the minimization problem (P-hom), we additionally want to impose the necessary condition of the φ–IC in the form of (3.0.3) for a constant small enough. We preliminary observe the following: If surely admissibility of ν_1 only – that means, on $(\nu_1, 0)$ according to Definition 3.3 – determines in light of $\nu_2 \ge 0$ the validity of the (φ–)IC for the pair (ν_1, ν_2) and same constant, the reverse implication does not hold in general, i.e. imposing the (φ–)IC to the separate component measures is in general *stronger*. Actually, the following counterexample illustrates that the two notions of IC (the fully signed and the one for a single measure) may differ due to the subtraction of the terms in the joint IC.

Example 3.9 (a non–trivial signed IC)**.** For $\theta \in (0, 1]$, consider the measures $\nu_1 := (1+\theta/2)\mathcal{H}^1 \llcorner \partial \mathrm{B}_2$ and $\nu_2 := \theta \mathcal{H}^1 \llcorner \partial \mathrm{B}_1$ on $\mathbb{R}^2$. Then both (ν_1, ν_2) and (ν_2, ν_1) satisfy the isotropic IC with constant 1 in $\mathbb{R}^2$, whereas ν_1 alone does not, since for instance $\nu_1\big({\mathrm{B}_2}^+\big) = 4\pi + 2\theta\pi > \mathrm{P}(\mathrm{B}_2)$.

Another difference with the condition of admissibility for measure ν_1 only lies in the fact that for the IC for pairs, we cannot expect reformulations of the kind of Proposition 3.5. Specifically, regardless of the validity of the general IC (3.0.3) for the pair of finite Radon measures (ν_1, ν_2), the single integrals in ν_1, ν_2 might diverge – determining loss of the conditions $w^+ \in \mathrm{L}^1(\Omega; \nu_1) \cap \mathrm{L}^1(\Omega; \nu_2)$ for some $w \in \mathrm{BV}(\Omega)$. Example 3.10 illustrates this pathological case, underlining the necessity of setting (even under joint IC) the admissibility condition for μ_+ and μ_- to guarantee that both measure terms in $\mathcal{F}$ are well–defined.

Example 3.10 (a signed IC without integrability on $\mathrm{W}^{1,1}$)**.** For $i \in \mathbb{N}$ and $n = 2$, consider the circles $S_i := \partial \mathrm{B}_{1/i^2}$ in $\mathbb{R}^2$, and the measures $\nu_1 := \mathcal{H}^1 \llcorner \big(\bigcup_{k=1}^\infty S_{2k-1}\big)$ and $\nu_2 := \mathcal{H}^1 \llcorner \big(\bigcup_{k=1}^\infty S_{2k}\big)$. Then both (ν_1, ν_2) and (ν_2, ν_1) satisfy the isotropic IC in $\mathbb{R}^2$ with constant 1, but the (continuous) function $v \in \mathrm{W}^{1,1}(\mathbb{R}^2)$ defined as $v(x) := (|x|^{-1/2} - 1)_+$ for $x \in \mathbb{R}^2$ satisfies $\int_{\mathbb{R}^2} v^* \,\mathrm{d}\nu_{1/2} = \infty$, since we compute

$$\int_{\mathbb{R}^2} v^* \,\mathrm{d}\nu_1 = \sum_{k=1}^\infty \int_{S_{2k-1}} \big(|x|^{-1/2} - 1\big) \,\mathrm{d}\mathcal{H}^1(x) = \sum_{k=2}^\infty (2k-2) \cdot \mathcal{H}^1\big(S_{2k-1}\big) = 4\pi \sum_{k=2}^\infty \frac{k-1}{(2k-1)^2} = \infty\,,$$
$$\int_{\mathbb{R}^2} v^* \,\mathrm{d}\nu_2 = \sum_{k=1}^\infty \int_{S_{2k}} \big(|x|^{-1/2} - 1\big) \,\mathrm{d}\mathcal{H}^1(x) = \sum_{k=1}^\infty (2k-1) \cdot \mathcal{H}^1\big(S_{2k}\big) = \pi \sum_{k=1}^\infty \frac{2k-1}{2k^2} = \infty\,.$$

The verification of the ICs for the measures in Examples 3.9 and 3.10 can be done exploiting the useful reformulation of the (signed) IC expressed in the subsequent Proposition 3.14, which specifically relies on computing distributional divergences of suitable associated vector fields. For this reason, we postpone the full verification of the isoperimetric condition mentioned in Example 3.9 to our later Section 3.2, whereas for the analogous and yet more involved computations concerning Example 3.10 we directly refer to [51, Section 5]. It is worth observing that the method above (for a continuous, convex in the second entry anisotropy φ) can only be employed in case the component measures $\nu_{1/2}$ are *mutually singular*, otherwise the condition on the divergence is only *necessary* to the IC; indeed, we anticipate that the equivalence stated in Theorem 3.16 only applies to Jordan decompositions of the difference measure $\nu_2 - \nu_1$.

We now want to pass from the functional (i.e. *non–parametric*) definition of the φ–IC to a *parametric* rewriting – namely, bounding measure via the anisotropic perimeter computed on roughly the same set.

Theorem 3.11 (characterizations of anisotropic ICs)**.** *Assume* $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ *is a* <u>*continuous*</u> *anisotropy comparable to the Euclidean norm and fix a constant* $C \in [0,\infty)$*. For a pair of admissible measures* ν_1, ν_2 *on open bounded Lipschitz* Ω *in* $\mathbb{R}^n$*, the following are equivalent*:

(1a) *The pair* (ν_1,ν_2) *satisfies the* φ*–IC in* Ω *with constant* C *– that is,*

$$\nu_1(A^+) - \nu_2(A^1) \le C\mathrm{P}_\varphi(A) \qquad \textit{for all measurable } A \Subset \Omega\,.$$

(1b) *The pair* (ν_1,ν_2) *satisfies*

$$\nu_1(A^+ \cap \Omega) - \nu_2(A^1 \cap \Omega) \le C\mathrm{P}_\varphi(A) \qquad \textit{for all measurable } A \subseteq \Omega\,.$$

(2a) *We have*

$$\int_\Omega v^+ \,\mathrm{d}\nu_1 - \int_\Omega v^- \,\mathrm{d}\nu_2 \le C\left(|\mathrm{D}v|_\varphi(\Omega) + \int_{\partial\Omega} \varphi(\,\cdot\,, v\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1}\right)$$

for all <u>*non–negative*</u> $v \in \mathrm{BV}(\Omega)$*.*

In addition, each of the properties above <u>*implies*</u> *the subsequent condition*:

(2b) *It holds*

$$\int_\Omega v^* \,\mathrm{d}(\nu_1 - \nu_2) \le C \int_\Omega \varphi(\,\cdot\,, \nabla v)\,\mathrm{d}x \tag{3.1.1}$$

for all <u>*non–negative*</u> $v \in \mathrm{W}^{1,1}_0(\Omega)$*.*

We point out that from (2b) we cannot get back to any of the conditions above, since (2b) is expressed for Sobolev functions, and as such it only takes into account the difference measure $\nu_1 - \nu_2$ in the integral. In order to get the backwards implication and recover the specific choice of representatives in either (1a), (1b), or (2a), the additional mutual singularity assumption $\nu_1 \perp \nu_2$ is essential. This case will be considered separately in Section 3.2. Furthermore, we observe that the continuity assumption on φ is necessary to apply our version of Reshetnyak's Theorem 2.79 and reach the boundary of Ω in the proof (1a) $\implies$ (1b) below.

Proof. We first verify that (1a) implies (1b). We consider a measurable $A \subseteq \Omega$, thus (1a) and the lower bound on φ yield that, without loss of generality, we can restrict to sets A of finite perimeter. We apply Proposition 2.106 to the function $1_A \in \mathrm{BV}(\Omega)$ to find an sequence $(\Omega_k)_k$ of open sets such that $\Omega_k \Subset \Omega$, $\mathrm{P}(\Omega_k) < \infty$ for all k, $\Omega_k \to \Omega$ in measure as $k \to \infty$, and

$$\int_{\partial^*\Omega_k} \left|\mathrm{T}^{\mathrm{int}}_{\partial^*\Omega_k}(\mathbb{1}_A)\right| \,\mathrm{d}\mathcal{H}^{n-1} \le \int_{\partial\Omega} \left|\mathrm{T}^{\mathrm{int}}_{\partial\Omega}(\mathbb{1}_A)\right| \,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{k} \qquad \text{for all } k \in \mathbb{N}\,.$$

Since it holds $\bigcup_{k\in\mathbb{N}} (A\cap\Omega_k)^+ = A^+\cap\Omega$ and $\bigcup_{k\in\mathbb{N}} (A\cap\Omega_k)^1 = A^1\cap\Omega$, for the sets $A_k := A\cap\Omega_k \Subset \Omega$ we have

$$\nu_1(A_k^+) \to \nu_1(A^+\cap\Omega) \qquad \text{and} \qquad \nu_2(A_k^1) \to \nu_2(A^1\cap\Omega) \qquad \text{as } k\to\infty\,. \tag{3.1.2}$$

Moreover, we can apply the BV decomposition formula (2.2.1), $\Omega^1=\Omega$ and the result of Federer $\mathcal{H}^{n-1}(\partial\Omega\setminus\partial^*\Omega)=0$ to get

$$\mathrm{D}\mathbb{1}_{A_k} = \mathrm{D}\mathbb{1}_A \llcorner (\Omega_k)^1 + \mathrm{T}^{\mathrm{int}}_{\partial^*\Omega_k}(\mathbb{1}_A)\nu_{\Omega_k}\mathcal{H}^{n-1}\llcorner\partial^*\Omega_k\,, \qquad \mathrm{D}\mathbb{1}_A = \mathrm{D}\mathbb{1}_A\llcorner\Omega + \mathrm{T}^{\mathrm{int}}_{\partial\Omega}(\mathbb{1}_A)\nu_\Omega\mathcal{H}^{n-1}\llcorner\partial\Omega\,.$$

For every k, we then deduce

$$\begin{aligned}\mathrm{P}(A_k) = |\mathrm{D}\mathbb{1}_{A_k}|(\mathbb{R}^n) &= |\mathrm{D}\mathbb{1}_A|\big((\Omega_k)^1\big) + \int_{\partial^*\Omega_k} \big|\mathrm{T}^{\mathrm{int}}_{\partial^*\Omega_k}\mathbb{1}_A\big|\,\mathrm{d}\mathcal{H}^{n-1}\\ &\le\ |\mathrm{D}\mathbb{1}_A|(\Omega) + \int_{\partial\Omega}\big|\mathrm{T}^{\mathrm{int}}_{\partial\Omega}\mathbb{1}_A\big|\,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{k} = |\mathrm{D}\mathbb{1}_A|(\mathbb{R}^n) + \frac1k = \mathrm{P}(A)+\frac1k\,.\end{aligned}$$

This inequality, together with the lower semicontinuity of the perimeter, implies $\mathrm{P}(A_k)\to\mathrm{P}(A)$, and from $A_k\to A$ in measure even $\mathbb{1}_{A_k}\rightharpoonup\mathbb{1}_A$ strictly in $\mathrm{BV}(\mathbb{R}^n)$. With the help of Theorem 2.79, it is also $\mathrm{P}_\varphi(A\cap\Omega_k)\to\mathrm{P}_\varphi(A)$ for $k\to\infty$. The conclusion (1b) follows then from a combination of the assumption (1a) applied to each set A_k, the perimeter convergence, and the measure convergence of (3.1.2).

We prove now that (1b) implies (2a). For $0\le v\in\mathrm{BV}(\Omega)$, we consider the extension $\overline{v}^0\in\mathrm{BV}(\mathbb{R}^n)$ of v by zero, and the trivial measure extensions of ν_1, ν_2 by zero outside Ω. Then the layer–cake formula (2.1.1) in combination with Lemma 2.29 yields

$$\int_\Omega v^+\mathrm{d}\nu_1 - \int_\Omega v^-\mathrm{d}\nu_2 = \int_0^\infty \Big[\nu_1\left(\{\overline{v}^0>t\}^+\cap\Omega\right) - \nu_2\left(\{\overline{v}^0>t\}^1\cap\Omega\right)\Big]\,\mathrm{d}t\,.$$

From the assumption (1b) to each set $\{\overline{v}^0>t\}\subseteq\Omega$ for a.e. $t\ge0$ and by the φ–coarea formula (2.8.10) for $\overline{v}^0$ it is then

$$\begin{aligned}\int_\Omega v^+\mathrm{d}\nu_1 - \int_\Omega v^-\mathrm{d}\nu_2 \le C\int_0^\infty \mathrm{P}_\varphi\left(\{\overline{v}^0\ge t\}\right)\,\mathrm{d}t &= C\big|\mathrm{D}\overline{v}^0\big|_\varphi(\mathbb{R}^n)\\ &= C\left(|\mathrm{D}v|_\varphi(\Omega) + \int_{\partial\Omega}\varphi(\,\cdot\,,v\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1}\right),\end{aligned}$$

which establishes (2a).

The deduction of (1a) from (2a) is easily obtained by reduction to characteristic functions, that is, to $\mathbb{1}_A\in\mathrm{BV}(\Omega)$ for measurable $A\Subset\Omega$, recalling that $(\mathbb{1}_A)^+=\mathbb{1}_{A^+}$ and $(\mathbb{1}_A)^-=\mathbb{1}_{A^1}$.

The first three conditions result in being equivalent, whereas (2b) follows straightforwardly from (2a) by restricting to functions in $\mathrm{W}^{1,1}(\Omega)$ with zero boundary trace and employing the homogeneity condition $\varphi(x,0)=0$ for all $x\in\mathbb{R}^n$. This completes our proof. □

Clearly, any anisotropy φ as in Theorem 3.11 is such that its mirrored function $\widetilde{\varphi}$ still satisfies the same assumptions, and we denote with $(\widetilde{1\mathrm{a}})$, $(\widetilde{1\mathrm{b}})$, $(\widetilde{2\mathrm{a}})$, and $(\widetilde{2\mathrm{b}})$ respectively the statements of Theorem 3.11 applied to $\widetilde{\varphi}$ in place of φ.

With the aim of establishing necessary conditions to minimizers of (P-hom), we now record some equivalences holding specifically for the joint pair of ICs – namely the φ–IC on (μ_1,μ_2) together with the mirrored $\widetilde{\varphi}$–IC on the inverted pair (μ_2,μ_1). Obviously, if φ is even then $\widetilde{\varphi}=\varphi$, and we can directly insert the absolute value in each of the estimates below.

Proposition 3.12 (characterization of joint anisotropic ICs)**.** *Under the same set of assumptions of Theorem 3.11, they are equivalent*:

(1a+1$\widetilde{\text{a}}$) *The pair* (ν_1, ν_2) *satisfies the* φ*–IC with constant* C *and the pair* (ν_2, ν_1) *satisfies the* $\widetilde{\varphi}$*–IC with constant* C*; that is,*

$$-C\mathrm{P}_{\widetilde{\varphi}}(A) \leq \nu_1(A^1) - \nu_2(A^+) \leq \nu_1(A^+) - \nu_2(A^1) \leq C\mathrm{P}_{\varphi}(A) \quad \textit{for all measurable } A \Subset \Omega\,.$$

(2a+2$\widetilde{\text{a}}$) *We have*

$$\int_\Omega v^+ \,\mathrm{d}\nu_1 - \int_\Omega v^- \,\mathrm{d}\nu_2 \leq C\left(|\mathrm{D}v|_\varphi(\Omega) + \int_{\partial\Omega} \varphi(\,\cdot\,, v\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1}\right) \quad \textit{for all } v \in \mathrm{BV}(\Omega)\,. \tag{3.1.3}$$

Remark 3.13. Before approaching the proof of Proposition 3.12, it is worth noticing that any admissible measure ν on Ω is such that not only $(\nu, 0)$ satisfies the isotropic IC with some $C \in [0, \infty)$, but trivially even $(0, \nu)$ satisfies the isotropic IC (for every constant). Then, the reformulation of (2a+2$\widetilde{\text{a}}$) for $\varphi = \varphi_0$ guarantees the validity of

$$\int_\Omega v^+ \,\mathrm{d}\nu \leq C\left(|\mathrm{D}v|(\Omega) + \int_{\partial\Omega} |v|\,\mathrm{d}\mathcal{H}^{n-1}\right) \quad \text{for all } v \in \mathrm{BV}(\Omega)$$

for any ν admissible on Ω.

Proof of Proposition 3.12. From a double application of Theorem 3.11, we already know that (1a+1$\widetilde{\text{a}}$) is equivalent to both

$$\begin{aligned} \int_\Omega v^+ \,\mathrm{d}\nu_1 - \int_\Omega v^- \,\mathrm{d}\nu_2 &\leq C\left(|\mathrm{D}v|_\varphi(\Omega) + \int_{\partial\Omega} \varphi(\,\cdot\,, v\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1}\right) \quad \text{and} \\ -\int_\Omega v^- \,\mathrm{d}\nu_1 + \int_\Omega v^+ \,\mathrm{d}\nu_2 &\leq C\left(|\mathrm{D}v|_{\widetilde{\varphi}}(\Omega) + \int_{\partial\Omega} \widetilde{\varphi}(\,\cdot\,, v\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1}\right) \end{aligned}$$

for all $0 \leq v \in \mathrm{BV}(\Omega)$. Given now $v \in \mathrm{BV}(\Omega)$ of arbitrary sign, we employ the first inequality in the latter estimate to its positive part v_+ and the second one to v_-. Summing them up, we deduce

$$\begin{aligned} \int_\Omega v^+ \,\mathrm{d}\nu_1 - \int_\Omega v^- \,\mathrm{d}\nu_2 &= \int_\Omega ((v_+)^+ - (v_-)^-)\,\mathrm{d}\nu_1 - \int_\Omega ((v_+)^- - (v_-)^+)\,\mathrm{d}\nu_2 \\ &\leq |\mathrm{D}\overline{v_+}^0|_\varphi(\mathbb{R}^n) + |\mathrm{D}\overline{v_-}^0|_{\widetilde{\varphi}}(\mathbb{R}^n) = |\mathrm{D}\overline{v}^0|_\varphi(\mathbb{R}^n)\,, \end{aligned}$$

for the extensions by zero and taking into account the decompositions in Lemmas 2.28 and 2.77. Hence, (3.1.3) applies to BV functions on Ω of any sign.

The remaining implication (2a+2$\widetilde{\text{a}}$) $\implies$ (1a+1$\widetilde{\text{a}}$) follows as in Theorem 3.11 by restriction to characteristic functions, namely for measurable $A \Subset \Omega$ assume $\mathrm{P}(A) < \infty$ and by applying the function estimate first to χ_A first, and then to $-\chi_A$. □

With the previous results at hand, we next express the link between a φ–IC with constant C for two pairs of admissible measures and existence of a suitable divergence–measure field whose divergence is the measure difference and such that $\mathrm{Im}(\varphi^\circ(.,\sigma))$ is almost everywhere contained in the ball centered in the origin and of radius C. The rewriting in (3) turns out to be particularly useful in validating isoperimetric conditions for mutually singular measures; compare with the computations in Section 3.2 for Examples 3.9 and 3.17.

Proposition 3.14. *For the anisotropy* $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ *in Theorem* 3.11 *we further assume convexity of* $\xi \mapsto \varphi(x, \xi)$ *for all* $x \in \mathbb{R}^n$*. Then, for a pair of non–negative, admissible measures* (ν_1, ν_2) *on the Lipschitz set* $\Omega \subseteq \mathbb{R}^n$ *and any constant* $C \in [0, \infty)$*, the following are equivalent:*

(2b+$2\widetilde{\mathrm{b}}$) *We have*

$$-C\int_\Omega \widetilde{\varphi}(\,\cdot\,,\nabla v)\,\mathrm{d}x \le \int_\Omega v^*\,\mathrm{d}(\nu_1-\nu_2) \le C\int_\Omega \varphi(\,\cdot\,,\nabla v)\,\mathrm{d}x$$

for all $v \in \mathrm{W}^{1,1}_0(\Omega)$.

(3) *There exists a vector field* $\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ *with* $\varphi^\circ(x,\sigma(x)) \le C$ *for* $\mathcal{L}^n$*–a.e.* $x \in \Omega$ *and satisfying*

$$\operatorname{div}(\sigma) = \nu_2 - \nu_1 \quad \textit{as distributions on } \Omega\,.$$

The validity of (2b+$2\widetilde{\mathrm{b}}$) is equivalent to verifying that (2b+$2\widetilde{\mathrm{b}}$) itself holds on *non–negative* functions in $\mathrm{W}^{1,1}_0(\Omega)$ only (this is easily achieved passing to positive and negative parts of an arbitrary function in $\mathrm{W}^{1,1}_0(\Omega)$). Altogether, from Theorem 3.11, Proposition 3.12, and this latter Proposition 3.14, we infer that for φ convex the double isoperimetric condition (1a+$1\widetilde{\mathrm{a}}$) – or its equivalent (2a+$2\widetilde{\mathrm{a}}$) – implies (3).

Proof. Suppose first that (3) holds. It suffices to prove (2b+$2\widetilde{\mathrm{b}}$) for non–negative functions in $\mathrm{W}^{1,1}_0(\Omega)$. To do so, we consider a vector field $\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ such that $\varphi^\circ(.,\sigma) \le C$ a.e. in Ω and with $\operatorname{div}(\sigma) = \nu_2 - \nu_1$ on Ω. From definition of polar function φ°, we have

$$\sigma(x)\cdot\xi \le \varphi^\circ(x,\sigma(x))\cdot\varphi(x,\xi) \le C\,\varphi(x,\xi) \qquad \text{for all } x\in\Omega\,,\ \xi\in\mathbb{R}^n\,,$$

and by change of sign in ξ

$$-C\,\widetilde{\varphi}(x,\xi) \le \sigma(x)\cdot\xi \le C\,\varphi(x,\xi) \qquad \text{for all } x\in\Omega\,,\ \xi\in\mathbb{R}^n\,. \tag{3.1.4}$$

Integrating by parts on Ω and exploiting (3.1.4), we arrive at

$$-C\int_\Omega \widetilde{\varphi}(\,\cdot\,,\nabla v)\,\mathrm{d}x \le \int_\Omega v\,\mathrm{d}(\nu_1-\nu_2) \le C\int_\Omega \varphi(\,\cdot\,,\nabla v)\,\mathrm{d}x \quad \text{for all non–negative } v\in\mathrm{C}^\infty_{\mathrm{c}}(\Omega)\,.$$

We extend the estimate above to non–negative Sobolev functions with zero trace on $\partial\Omega$ with the help of Corollary 3.4, which can be applied to both the admissible measures ν_1, ν_2 taken singularly. In fact, concerning the convergence of the φ–variational terms in (3.1.1), we recall that the sequence coming from Corollary 3.4 converges strongly in $\mathrm{W}^{1,1}(\Omega)$, hence even φ–strictly. The proof of (2b+$2\widetilde{\mathrm{b}}$) is then complete.

Vice versa, we assume the validity of (2b+$2\widetilde{\mathrm{b}}$) and introduce the class

$$\mathcal{W} := \big\{\nabla v \,:\, v\in\mathrm{W}^{1,1}_0(\Omega)\big\} \subseteq \mathrm{L}^1\big(\Omega,\mathbb{R}^n\big)\,.$$

We consider the linear functional F on $\mathcal{W}$ as

$$\mathrm{F}[W] = \mathrm{F}[\nabla v] := -\int_\Omega v^*\mathrm{d}(\nu_2-\nu_1) = \int_\Omega v^*\,\mathrm{d}\nu_1 - \int_\Omega v^*\,\mathrm{d}\nu_2 \qquad \text{for all } W=\nabla v\in\mathcal{W}\,.$$

Let now $\mathrm{I}_\varphi\colon \mathrm{L}^1(\Omega,\mathbb{R}^n)\to\overline{\mathbb{R}}$ be defined through

$$\mathrm{I}_\varphi[W] := C\int_\Omega \varphi(\,\cdot\,,W)\,\mathrm{d}x \qquad \text{for all } W\in\mathrm{L}^1(\Omega,\mathbb{R}^n)\,.$$

The role of convexity of $\xi\mapsto\varphi(x,\xi)$ is then decisive to guarantee via (2.6.1) the sublinearity of I_φ in $\mathrm{L}^1(\Omega,\mathbb{R}^n)$. The assumption (3.1.1) also ensures $\mathrm{F}\le\mathrm{I}_\varphi$ on $\mathcal{W}$. Additionally, F is bounded on $\mathcal{W}$ from

$$|\mathrm{F}[W]| \le C\max\left\{\int_\Omega \varphi(\,\cdot\,,\nabla v)\,\mathrm{d}x\,,\int_\Omega \widetilde{\varphi}(\,\cdot\,,\nabla v)\,\mathrm{d}x\right\} \le C\beta\|W\|_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} \qquad \text{for all } W=\nabla v\in\mathcal{W}\,,$$

where $\beta > 0$ is the upper constant in the linear–growth assumption for φ. Thanks to the Hahn–Banach theorem, we can then extend F to a functional $\overline{\mathrm{F}}$ defined on the whole space $\mathrm{L}^1(\Omega, \mathbb{R}^n)$ with same dual norm and such that $\overline{\mathrm{F}} \le \mathrm{I}_\varphi$ is preserved on all $\mathrm{L}^1(\Omega, \mathbb{R}^n)$. From Riesz' representation theorem, we find a vector field $\sigma \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n)$ with $\|\sigma\|_{\mathrm{L}^\infty} = \|\overline{\mathrm{F}}\|_{(\mathrm{L}^1)^*} = \|\mathrm{F}\|_{\mathcal{W}^*} \le C\beta$ and $\overline{\mathrm{F}}[W] = \int_\Omega \sigma \cdot W \,\mathrm{d}x$ whenever $W \in \mathrm{L}^1(\Omega, \mathbb{R}^n)$. In particular, restricting to $\mathcal{W}$ we get

$$-\int_\Omega v^* \,\mathrm{d}(\nu_2 - \nu_1) = \mathrm{F}[\nabla v] = \int_\Omega \sigma \cdot \nabla v \,\mathrm{d}x \qquad \text{for all } v \in \mathrm{W}^{1,1}_0(\Omega)\,.$$

Thus, we have verified $\operatorname{div}(\sigma) = \nu_2 - \nu_1$ as distributions. It is left to check that $\varphi^\circ(x, \sigma(x)) \le C$ holds for a.e. $x \in \Omega$. Observe that we have

$$\int_\Omega \sigma \cdot W \,\mathrm{d}x = \overline{\mathrm{F}}[W] \le \mathrm{I}_\varphi[W] = C \int_\Omega \varphi(\,\cdot\,, W) \,\mathrm{d}x \qquad \text{for all } W \in \mathrm{L}^1(\Omega, \mathbb{R}^n)\,.$$

Specifically, setting $W = \psi\xi$ with arbitrary $0 \le \psi \in \mathrm{C}^\infty_\mathrm{c}(\Omega)$ and $\xi \in \mathbb{R}^n$, the homogeneity of φ determines

$$\int_\Omega \big(\sigma \cdot \xi - C\,\varphi(\,\cdot\,, \xi)\big)\psi \,\mathrm{d}x \le 0 \qquad \text{for all } 0 \le \psi \in \mathrm{C}^\infty_\mathrm{c}(\Omega)\,.$$

Hence, the fundamental lemma of the calculus of variations gives for every $\xi \in \mathbb{R}^n$ a negligible set $N_\xi \subseteq \Omega$ with $\sigma(x) \cdot \xi \le C\,\varphi(x, \xi)$ for $x \in \Omega \setminus N_\xi$, from which

$$\sup_{\xi \in \mathbb{Q}^n \setminus \{0\}} \frac{\sigma(x) \cdot \xi}{\varphi(x, \xi)} \le C \qquad \text{for } x \in \Omega \setminus \bigcup_{\xi \in \mathbb{Q}^n} N_\xi\,.$$

Then, by density of $\mathbb{Q}^n$ in $\mathbb{R}^n$ and continuity of φ in the second variable we arrive at

$$\varphi^\circ(x, \sigma(x)) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{\sigma(x) \cdot \xi}{\varphi(x, \xi)} \le C \qquad \text{for } x \in \Omega \setminus \bigcup_{\xi \in \mathbb{Q}^n} N_\xi\,.$$

Since $\bigcup_{\xi \in \mathbb{Q}^n} N_\xi$ is still $\mathcal{L}^n$–negligible, the claimed estimate for φ° holds, and consequently (3) is verified. □

3.2 Characterization of anisotropic ICs for pairs of mutually singular measures

We complete the analysis of our isoperimetric condition by considering IC for measures with disjoint supports.

Proposition 3.15 (approximation of measure integrals under mutual singularity)**.** *Assume ν_1, ν_2 are non–negative Radon measures on U open in $\mathbb{R}^n$, $\nu_{1/2}(Z) = 0$ for every $\mathcal{H}^{n-1}$–negligible Borel set $Z \subseteq U$ and $w^+ \in \mathrm{L}^1(U; \nu_1) \cap \mathrm{L}^1(U; \nu_2)$ for every $w \in \mathrm{BV}(U)$. Suppose additionally $\nu_1 \perp \nu_2$. Then, for every $u \in \mathrm{BV}(U)$ there is a sequence $(w_k)_k$ in $\mathrm{W}^{1,1}(U)$ such that $(w_k)_k$ converges to u strictly in $\mathrm{BV}(U)$ with*

$$\lim_{k\to\infty} \int_U w_k^* \,\mathrm{d}\nu_1 = \int_U u^+ \,\mathrm{d}\nu_1 \quad \text{and} \quad \lim_{k\to\infty} \int_U w_k^* \,\mathrm{d}\nu_2 = \int_U u^- \,\mathrm{d}\nu_2\,. \tag{3.2.1}$$

If u has even compact support in U, we may additionally obtain $(w_k)_k$ in $\mathrm{W}^{1,1}_0(U)$. In case U is bounded, the convergence of $(w_k)_k$ to u holds even strictly in area.

Additionally, introducing the measure $\nu := \nu_2 - \nu_1$ and the Borel partition of Ω into disjoint U_+, U_- such that $\nu_2 = \nu \llcorner U_+$ and $\nu_1 = -\nu \llcorner U_-$, we have

$$\int_U w_k^* \,\mathrm{d}\nu \xrightarrow[k\to\infty]{} \int_U (u^- \mathbb{1}_{U_+} + u^+ \mathbb{1}_{U_-}) \,\mathrm{d}\nu\,, \tag{3.2.2}$$

as well as

$$w_k^*(x) \xrightarrow[k\to\infty]{} u^-(x)\mathbb{1}_{U_+}(x) + u^+(x)\mathbb{1}_{U_-}(x) \qquad \textit{for } |\nu|\textit{–a.e. } x \in U\,. \tag{3.2.3}$$

Proof. For $u \in \mathrm{BV}(U)$, we employ Proposition 2.107 followed by Proposition 3.7 with measures ν_1, ν_2 and passing to suitable subsequences. This yields a sequence $(u_\ell)_\ell$ in $\mathrm{W}^{1,1}(U)$ which converges to u strictly in $\mathrm{BV}(U)$ (with area–strict convergence in case U is bounded) with $u_\ell \geq u$ a.e. in U for all $\ell \in \mathbb{N}$ and with

$$\lim_{\ell\to\infty} \int_U u_\ell^* \,\mathrm{d}\nu_{1/2} = \int_U u^+ \,\mathrm{d}\nu_{1/2}\,, \tag{3.2.4}$$

as well as a sequence $(v_\ell)_\ell$ in $\mathrm{W}^{1,1}(U)$ (area–)strictly converging to u in $\mathrm{BV}(U)$, $v_\ell \leq u$ a.e. in U for all $\ell \in \mathbb{N}$ and

$$\lim_{\ell\to\infty} \int_U v_\ell^* \,\mathrm{d}\nu_{1/2} = \int_U u^- \,\mathrm{d}\nu_{1/2}\,. \tag{3.2.5}$$

We want to build the sequence $(w_k)_k$ as a sequence of interpolated functions between u_{ℓ_k} and v_{ℓ_k} with a suitable cut–off functions η_k. The decisive point lies in the assumption $\nu_2 \perp \nu_1$, which yields $U = P \cup M$ for some Borel sets P, M such that $\nu_2(M) = \nu_1(P) = 0$. We notice that here P represents our U_+ in the statement (and $M = U_-$), where we here prefer the shorter notation P, M. Then, the function $u^+ - u^-$ is such that $|u^+ - u^-| \leq |u|^+$, hence by assumption $u^+ - u^- \in \mathrm{L}^1(U;\nu_1) \cap \mathrm{L}^1(U;\nu_2)$. The absolute continuity of the Lebesgue integral implies that for every $k \in \mathbb{N}$ there is some $\delta_k > 0$ such that for any Borel set $E \subseteq U$ with $\nu_{1/2}(E) < \delta_k$, then $\int_E (u^+ - u^-)\,\mathrm{d}\nu_{1/2} \leq \frac{1}{k}$. Moreover, finiteness of the measures and the decomposition of U guarantee that for some compact set $K_{k;P} \subseteq P$ it is $\nu_2(U \setminus K_{k;P}) < \delta_k$, and analogously the compact $K_{k;M} \subseteq M$ can be chosen large enough such that $\nu_1(U \setminus K_{k;M}) < \delta_k$. Therefore it follows

$$\int_{U\setminus K_{k;P}} (u^+ - u^-)\,\mathrm{d}\nu_2 \leq \frac{1}{k} \qquad \text{and} \qquad \int_{U\setminus K_{k;M}} (u^+ - u^-)\,\mathrm{d}\nu_1 \leq \frac{1}{k}\,. \tag{3.2.6}$$

From $\mathrm{dist}(K_{k;P}, K_{k;M}) > 0$, in correspondence to each k we can consider a cut–off functions $\eta_k \in \mathrm{C}^1_\mathrm{c}(U)$ with $\eta_k \equiv 1$ on $K_{k;M}$ and $\eta_k \equiv 0$ on $K_{k;P}$ and $0 \leq \eta_k \leq 1$ in U. From the convergences of $(u_\ell)_\ell$ and $(v_\ell)_\ell$ to u it is even $u_\ell - v_\ell \to u$ in $\mathrm{L}^1(U)$ for $\ell \to \infty$, and by Proposition 2.10 the (area–)strict convergence is invariant by multiplication with bounded functions $\eta \in \mathrm{C}(U)$, in the sense that

$$\lim_{\ell\to\infty} \int_U \eta \,\mathrm{d}\big|\big(\mathcal{L}^n, \mathrm{D}u_\ell\big)\big| = \int_U \eta \,\mathrm{d}\big|\big(\mathcal{L}^n, \mathrm{D}u\big)\big|\,,$$

and analogously for the complement cut–off function $1 - \eta$ with sequence v_{ℓ_k} (reducing to strict convergence in case $|U| = \infty$). At this point, we pass to an increasing subsequence $(\ell_k)_k$ such that

$$\|u_{\ell_k} - v_{\ell_k}\|_{\mathrm{L}^1(U)}\, \|\nabla\eta_k\|_{\mathrm{L}^\infty(U,\mathbb{R}^N)} \leq \frac{1}{k}\,,$$

$$\eta_k\big|\big(\mathcal{L}^n, \mathrm{D}u_{\ell_k}\big)\big|(U) \leq \eta_k\big|\big(\mathcal{L}^n, \mathrm{D}u\big)\big|(U) + \frac{1}{k}\,,\quad (1-\eta_k)\big|\big(\mathcal{L}^n, \mathrm{D}v_{\ell_k}\big)\big|(U) \leq (1-\eta_k)\big|\big(\mathcal{L}^n, \mathrm{D}u\big)\big|(U) + \frac{1}{k}\,,$$

$$\int_U (u_{\ell_k}^* - u^+)\,\mathrm{d}\nu_{1/2} \leq \frac{1}{k} \qquad \text{and} \qquad \int_U (u^- - v_{\ell_k}^*)\,\mathrm{d}\nu_{1/2} \leq \frac{1}{k} \tag{3.2.7}$$

for all $k \in \mathbb{N}$. Setting the function $w_k := \eta_k u_{\ell_k} + (1-\eta_k)v_{\ell_k} \in \mathrm{W}^{1,1}(U)$ for every $k \in \mathbb{N}$, we first rewrite

$$\begin{aligned} \int_U w_k^* \,\mathrm{d}\nu_1 &= \int_U u_{\ell_k}^* \,\mathrm{d}\nu_1 - \int_U (1-\eta_k)(u_{\ell_k}^* - v_{\ell_k}^*)\,\mathrm{d}\nu_1\,, \\ \int_U w_k^* \,\mathrm{d}\nu_2 &= \int_U v_{\ell_k}^* \,\mathrm{d}\nu_2 + \int_U \eta_k(u_{\ell_k}^* - v_{\ell_k}^*)\,\mathrm{d}\nu_2\,. \end{aligned} \tag{3.2.8}$$

Applying the ν_1–a.e. and ν_2–a.e. inequalities $v^*_\ell \le u^\pm \le u^*_\ell$, $\eta_k \equiv 1$ on $K_{k;M}$ and $\eta_k \equiv 0$ on $K_{k;P}$, with the help of (3.2.6) and (3.2.7) we estimate

$$0 \le \int_U (1-\eta_k)(u^*_{\ell_k} - v^*_{\ell_k})\,\mathrm{d}\nu_1 \le \int_{U\setminus K_{k;M}} (u^+ - u^-)\,\mathrm{d}\nu_1 + \int_U (u^*_{\ell_k} - u^+)\,\mathrm{d}\nu_1 + \int_U (u^- - v^*_{\ell_k})\,\mathrm{d}\nu_1 \le \frac{3}{k}\,,$$
$$0 \le \int_U \eta_k(u^*_{\ell_k} - v^*_{\ell_k})\,\mathrm{d}\nu_2 \le \int_{U\setminus K_{k;P}} (u^+ - u^-)\,\mathrm{d}\nu_2 + \int_U (u^*_{\ell_k} - u^+)\,\mathrm{d}\nu_2 + \int_U (u^- - v^*_{\ell_k})\,\mathrm{d}\nu_2 \le \frac{3}{k}\,.$$

A passage to the limit in (3.2.8) via the preceding (3.2.4), (3.2.5) yields precisely (3.2.1). Rearranging the terms, we obtain that even (3.2.2) is in place.

We now check that $(w_k)_k$ converges to u (area–)strictly in $\mathrm{BV}(U)$. The convergence in $\mathrm{L}^1(U)$ is straightforward, and by semicontinuity this implies even

$$\liminf_{k\to\infty} |(\mathcal{L}^n, \mathrm{D}w_k)|(U) \ge |(\mathcal{L}^n, \mathrm{D}u)|(U)\,.$$

To obtain the reverse inequality, we decompose the gradient of each w_k as $\nabla w_k = (u_{\ell_k} - v_{\ell_k})\nabla\eta_k + \eta_k\nabla u_{\ell_k} + (1-\eta_k)\nabla v_{\ell_k}$ and from the estimates on the variation we obtain

$$\begin{aligned}
|(\mathcal{L}^n, \mathrm{D}w_k)|(U) &\le \eta_k|(\mathcal{L}^n, \mathrm{D}u_{\ell_k})|(U) + (1-\eta_k)|(\mathcal{L}^n, \mathrm{D}v_{\ell_k})|(U) + \|u_{\ell_k} - v_{\ell_k}\|_{\mathrm{L}^1(U)}\,\|\nabla\eta_k\|_{\mathrm{L}^\infty(U,\mathbb{R}^n)}\\
&= \eta_k|(\mathcal{L}^n, \mathrm{D}u)|(U) + (1-\eta_k)|(\mathcal{L}^n, \mathrm{D}u)|(U) + \frac{3}{k}\\
&= |(\mathcal{L}^n, \mathrm{D}u)|(U) + \frac{3}{k} \qquad \text{for all } k\in\mathbb{N}\,.
\end{aligned}$$

Passing to the maximum limit as $k\to\infty$, we have verified the remaining inequality, whence the (area–)strict convergence.

The claim for u with compact support in U follows from selecting the intermediate approximations u_ℓ and v_ℓ in $\mathrm{W}^{1,1}_0(U)$ as in Proposition 2.107, and by noticing that the sequence $(w_k)_k$ preserves the zero boundary values of its components.

To prove the remaining pointwise convergence, we observe that the concentration of the measure $|\nu|$ yields

$$\int_U |w^*_k - (u^-\mathbb{1}_{U_+} + u^+\mathbb{1}_{U_-})|\,\mathrm{d}|\nu| = \int_U |w^*_k - u^+|\,\mathrm{d}\nu_1 + \int_U |w^*_k - u^-|\,\mathrm{d}\nu_2 \qquad \text{for all } k\in\mathbb{N}\,. \tag{3.2.9}$$

With the help of the estimates above, we compute separately

$$\begin{aligned}
\int_U |w^*_k - u^+|\,\mathrm{d}\nu_1 &\le \int_U |w^*_k - u^*_{\ell_k}|\,\mathrm{d}\nu_1 + \int_U |u^*_{\ell_k} - u^+|\,\mathrm{d}\nu_1\\
&= \int_U (1-\eta_k)(u^*_{\ell_k} - v^*_{\ell_k})\,\mathrm{d}\nu_1 + \int_U (u^*_{\ell_k} - u^+)\,\mathrm{d}\nu_1 \le \frac{3}{k} + \frac{1}{k} = \frac{4}{k}
\end{aligned}$$

as well as

$$\begin{aligned}
\int_U |w^*_k - u^-|\,\mathrm{d}\nu_2 &\le \int_U |w^*_k - v^*_{\ell_k}|\,\mathrm{d}\nu_2 + \int_U |v^*_{\ell_k} - u^-|\,\mathrm{d}\nu_2\\
&= \int_U \eta_k(u^*_{\ell_k} - v^*_{\ell_k})\,\mathrm{d}\nu_2 + \int_U (u^- - v^*_{\ell_k})\,\mathrm{d}\nu_2 \le \frac{4}{k}
\end{aligned}$$

A substitution into (3.2.9) and a passage to the limit yields $w^*_k \to u^-\mathbb{1}_{U_+} + u^+\mathbb{1}_{U_-}$ in $\mathrm{L}^1(U;|\nu|)$ for $k\to\infty$. Possibly passing to a subsequence, we read exactly the $|\nu|$–a.e. pointwise convergence in equation (3.2.3) and therefore we complete the proof. □

Theorem 3.16 (characterizations of anisotropic IC; singular version). *Assume $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ is an anisotropy comparable to the Euclidean norm. We have the following:*

1. *For two non–negative Radon measures ν_1, ν_2 on Ω such that $w^+ \in \mathrm{L}^1(\Omega;\nu_1) \cap \mathrm{L}^1(\Omega;\nu_2)$ for every $w \in \mathrm{BV}(\Omega)$, if ν_1 and ν_2 are mutually singular, then (1a+1ã) implies the vanishing condition (C1) for both ν_1, ν_2 – that is, ν_1 and ν_2 are in fact admissible measures on Ω.*

2. *Assume that φ is also continuous. If ν_1 and ν_2 are two admissible measures on Ω with $\nu_1 \perp \nu_2$, then (1a), (1b), (2a), (2b) are all equivalent (and holding with same constant). Furthermore, if φ is convex in the second variable, any of the joint conditions (1a+1ã), (2a+2ã), (2b+$\widetilde{\text{2b}}$) is equivalent to (3).*

Before proving the Theorem, we record that under the assumption (1) one can actually check that *each one* of the conditions (1a+1ã), (2a+2ã), (2b+$\widetilde{\text{2b}}$), (3) yields that ν_1, ν_2 vanish on Hausdorff–negligible sets; compare with [51, Theorem 4.6] and [31, Proposition 3.1]. Therefore, the mutual singularity assumption on the measures allows us to heuristically split up the conditions on the signed measure $\nu := \nu_2 - \nu_1$ into two ICs (with arbitrary constants) for ν_1, ν_2 separately, thus falling into the case of Lemma 3.2.

Proof of (1). Assume $Z \subseteq \Omega$ is an $\mathcal{H}^{n-1}$–negligible Borel set. The mutual singularity of ν_1, ν_2 yields the existence of a decomposition $\Omega = P \uplus M$ into Borel sets with $\nu_2(M) = \nu_1(P) = 0$. We check first that (1a) – i.e. the upper inequality in (1a+1ã) – implies (C1) for ν_1. Our claim is

$$\nu_1(K) = 0 \qquad \text{for every compact } K \subseteq Z \cap M\,. \tag{3.2.10}$$

In fact, for any $\varepsilon > 0$ there is an open set O with $K \subseteq O \Subset \Omega$ and $\nu_2(O) < \varepsilon$. From a covering argument (see e.g. [92, Lemma 2.7]) we deduce from $\mathcal{H}^{n-1}(K) = 0$ the existence of an open set A with $K \subseteq A \Subset O$ and $\mathrm{P}(A) < \varepsilon$. We exploit (1a) for such A and the upper bound for φ to write

$$\nu_1(A^+) - \nu_2(A^1) \le C\mathrm{P}_\varphi(A) \le C\beta\mathrm{P}(A) < C\beta\varepsilon\,.$$

Moreover, from $\nu_2(A^1) \le \nu_2(O) < \varepsilon$ we deduce even $\nu_1(K) \le \nu_1(A^+) < \varepsilon(C\beta + 1)$, which by arbitrariness of ε yields (3.2.10). Inner regularity of ν_1 determines then $\nu_1(Z) = 0$ as required.
Symmetrically, an analogous argument based on the left–hand side inequality in (1a+1ã) yields $\nu_2(K) = 0$ for every compact $K \subseteq Z \cap P$, and again (C1) for ν_2 follows from the inner regularity.
Proof of (2). Consider first the set of equivalences. With the contribution of Theorem 3.11, it is enough to show that – for instance – (2b) implies (1a). Consider then a measurable set $A \Subset \Omega$, for which we suppose without loss of generality $\mathrm{P}(A) < \infty$ – otherwise even $\mathrm{P}_\varphi(A) = \infty$, and the thesis is trivially verified. We employ Proposition 3.15 for the function $\mathbb{1}_A \in \mathrm{BV}(\Omega)$ compactly supported in Ω to determine a sequence $(w_k)_k$ in $\mathrm{W}^{1,1}_0(\Omega)$, $w_k \rightharpoonup \mathbb{1}_A$ strictly in $\mathrm{BV}(\Omega)$ with

$$\lim_{k\to\infty} \int_\Omega w_k^* \,\mathrm{d}\nu_1 = \int_\Omega (\mathbb{1}_A)^+ \,\mathrm{d}\nu_1 = \nu_1(A^+) \qquad \text{and} \qquad \int_\Omega w_k^* \,\mathrm{d}\nu_2 = \int_\Omega (\mathbb{1}_A)^- \,\mathrm{d}\nu_2 = \nu_2(A^1)\,.$$

Moreover, by Theorem 2.79 and continuity of the integrand φ, the convergence is also φ–strict and thus in particular

$$\lim_{k\to\infty} \int_\Omega \varphi(.,\nabla w_k) = |\mathrm{D}\mathbb{1}_A|_\varphi(\Omega)\,.$$

Bringing in the assumption (2b) spelled out for each $w_k \in \mathrm{W}^{1,1}_0(\Omega)$ and letting $k \to \infty$, from the convergences above we obtain

$$\nu_1(A^+) - \nu_2(A^1) \le C\,|\mathrm{D}\mathbb{1}_A|_\varphi(\Omega) = C\,\mathrm{P}_\varphi(A)\,,$$

which is exactly (1a).

Concerning the last part of the statement, we record that the equivalences of ($1a+1\widetilde{a}$), ($2a+2\widetilde{a}$), ($2b+2\widetilde{b}$) follow from the previous step with the passage to the $\widetilde{\varphi}$–perimeter (resp. $\widetilde{\varphi}$–variation) and the exchange of the roles of ν_1, ν_2. Assuming now the convexity of $\xi \mapsto \varphi(x,\xi)$ for all $x \in \mathbb{R}^n$, Proposition 3.14 guarantees that (3) holds if and only if ($2b+2\widetilde{b}$) is fulfilled, thus our proof is complete. □

As mentioned in Section 3.1, we now illustrate the power of Theorem 3.16 in the verification of isoperimetric conditions for mutually singular, admissible component measures. We start from a very simple case of a non–negative measure satisfying the IC with $C = 1$.

Example 3.17. In $\mathbb{R}^2$, we consider the function $H(x) := 1/|x|$ for $x \in \mathbb{R}^2 \setminus \{(0,0)\}$ and the measure $\nu := H\mathcal{L}^2$. Then, ν satisfies exactly the limit isotropic IC in $\mathbb{R}^2$. In fact, we can work with the associated vector field $\sigma(x) := x/|x|$ for $x \neq 0$ and compute

$$\operatorname{div}\big(x/|x|\big) = (2|x|^2 - x_1^2 - x_2^2)/|x|^3 = 1/|x| = H(x) \quad \text{ for all } x \in \mathbb{R}^2 \setminus \{(0,0)\}\,,$$

hence $\operatorname{div}(\sigma) = \nu$ as measures in $\mathbb{R}^2$, and clearly $||\sigma||_{\mathrm{L}^\infty(\mathbb{R}^2,\mathbb{R}^2)} = 1$. Hence, via Theorem 3.16(2) we deduce that ν satisfies the φ_0–IC with constant 1 (and no smaller constant is allowed). Alternatively, one could check via definition that $\nu(A^+) \leq \mathrm{P}(A)$ for all $A \Subset \mathbb{R}^2$ measurable. Actually, we claim that:

$$\int_A \frac{1}{|x|}\,\mathrm{d}x \leq \mathrm{P}(A) \quad \text{ for all meas. } A \Subset \mathbb{R}^2\,, \text{ with equality iff } |A\Delta \mathrm{B}_R| = 0 \text{ for some ball } \mathrm{B}_R\,. \tag{3.2.11}$$

In fact, for any bounded set A we can find some $R > 0$ such that $|A| = |\mathrm{B}_R|$. Then $|A \setminus \mathrm{B}_R| = |A| - |A \cap \mathrm{B}_R| = |\mathrm{B}_R| - |A \cap \mathrm{B}_R| = |\mathrm{B}_R \setminus A|$ and therefore

$$\begin{aligned}\int_A \frac{1}{|x|}\,\mathrm{d}x = \int_{A\cap \mathrm{B}_R} \frac{1}{|x|}\,\mathrm{d}x + \int_{A\setminus \mathrm{B}_R} \frac{1}{|x|}\,\mathrm{d}x &\leq \int_{A\cap \mathrm{B}_R} \frac{1}{|x|}\,\mathrm{d}x + \int_{A\setminus \mathrm{B}_R} \frac{1}{R}\,\mathrm{d}x \\ &= \int_{A\cap \mathrm{B}_R} \frac{1}{|x|}\,\mathrm{d}x + \int_{\mathrm{B}_R\setminus A} \frac{1}{R}\,\mathrm{d}x = \int_{\mathrm{B}_R} \frac{1}{|x|}\,\mathrm{d}x\,,\end{aligned}$$

which verifies our statement (3.2.11).

Similarly, one can check the validity of the isoperimetric conditions mentioned in Example 3.9.

Proof of Example 3.9. To verify the isotropic IC, we define

$$\sigma(x) := \begin{cases} 0 & \text{for } x \in \mathrm{B}_1 \\ -\theta\frac{x}{|x|^2} & \text{for } x \in \mathrm{B}_2 \setminus \overline{\mathrm{B}_1} \\ 2\frac{x}{|x|^2} & \text{for } x \notin \overline{\mathrm{B}_2} \end{cases}$$

and compute $||\sigma||_{\mathrm{L}^\infty(\mathbb{R}^2,\mathbb{R}^2)} = \max\left\{0,\, \theta \operatorname*{ess\,sup}_{1\leq|x|\leq 2} \frac{1}{|x|},\, 2\operatorname*{ess\,sup}_{|x|\geq 2} \frac{1}{|x|}\right\} = \max\{0,\theta,1\} = 1$, thus σ is a unit vector field. We claim that $\operatorname{div}(\sigma) = \nu_2 - \nu_1$ as distributions on $\mathbb{R}^2$. In fact, the field is continuously differentiable in $\mathbb{R}^2 \setminus (\partial\mathrm{B}_1 \cup \partial\mathrm{B}_2)$, with

$$\operatorname{div}\big(x/|x|^2\big) = (|x|^2 - 2x_1^2 - 2x_2^2)/|x|^4 = 0 \quad \text{ for all } x \in \mathbb{R}^2 \setminus \{(0,0)\}\,. \tag{3.2.12}$$

We now consider a test function $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^2)$; exploiting (3.2.12) in combination with the standard integration by parts formula – e.g., (3.30) in [3]– we obtain

$$
\begin{aligned}
&\operatorname{div}(\sigma)(\psi)\\
&= -\int_{\mathbb{R}^2} \sigma \cdot \nabla\psi \,\mathrm{d}x\\
&= \theta \int_{\mathrm{B}_2\setminus\mathrm{B}_1} \frac{x}{|x|^2}\cdot\nabla\psi(x)\,\mathrm{d}x - 2\int_{\mathrm{B}_2^c}\frac{x}{|x|^2}\cdot\nabla\psi(x)\,\mathrm{d}x\\
&= \theta \int_{\mathrm{B}_2\setminus\mathrm{B}_1} \operatorname{div}\left(\psi(x)\frac{x}{|x|^2}\right)\mathrm{d}x - 2\int_{\mathrm{B}_2^c}\operatorname{div}\left(\psi(x)\frac{x}{|x|^2}\right)\mathrm{d}x\\
&- \theta\int_{\partial(\mathrm{B}_2\setminus\mathrm{B}_1)}\psi(x)\frac{x}{|x|^2}\cdot\nu_{\mathrm{B}_2\setminus\mathrm{B}_1}(x)\,\mathrm{d}\mathcal{H}^1(x) + 2\int_{\partial\mathrm{B}_2}\psi(x)\frac{x}{|x|^2}\cdot\nu_{\mathrm{B}_2^c}(x)\,\mathrm{d}\mathcal{H}^1(x)\\
&= -\theta\int_{\partial\mathrm{B}_2}\psi(x)\frac{x}{|x|^2}\cdot\left(\frac{-x}{|x|}\right)\mathrm{d}\mathcal{H}^1(x) - \theta\int_{\partial\mathrm{B}_1}\psi(x)\frac{x}{|x|^2}\cdot\frac{x}{|x|}\,\mathrm{d}\mathcal{H}^1(x) + 2\int_{\partial\mathrm{B}_2}\psi(x)\frac{x}{|x|^2}\cdot\frac{x}{|x|}\,\mathrm{d}\mathcal{H}^1(x)\\
&= \theta\int_{\partial\mathrm{B}_2}\frac{\psi(x)}{|x|}\,\mathrm{d}\mathcal{H}^1(x) - \theta\int_{\partial\mathrm{B}_1}\frac{\psi(x)}{|x|}\,\mathrm{d}\mathcal{H}^1(x) + 2\int_{\partial\mathrm{B}_2}\frac{\psi(x)}{|x|}\,\mathrm{d}\mathcal{H}^1(x)\\
&= \left(1+\frac{\theta}{2}\right)\psi\mathcal{H}^1\llcorner\partial\mathrm{B}_2 - \theta\psi\mathcal{H}^1\llcorner\partial\mathrm{B}_1 = \nu_1(\psi)-\nu_2(\psi)\,.
\end{aligned}
$$

Therefore, the field σ introduced satisfies $|(\varphi_0)^\circ(\sigma)| = |\sigma| \le 1$ a.e. in $\mathbb{R}^2$ for our integrand $\varphi_0 := |.|$, and the distributional equality $\operatorname{div}(\sigma) = \nu_2 - \nu_1$ on $\mathbb{R}^2$. Since $\nu_1 \perp \nu_2$, an application of Theorem 3.16(2) yields the claimed isotropic ICs for both (ν_1,ν_2) and (ν_2,ν_1) with constant 1. □

Another apparently surprising property of the isoperimetric condition for the pair (ν_1,ν_2) in the case of mutual singularity is the invariance with respect to the choice of representatives.

Proposition 3.18 (anisotropic IC with any representatives). *In case $\nu_1 \perp \nu_2$ and φ is a continuous anisotropy in $\mathbb{R}^n$, then (1a) (or any of its reformulations from Theorem 3.16(2)) is equivalent to*

$$\nu_1(A^\triangle) - \nu_2(A^\square) \le C\mathrm{P}_\varphi(A) \quad \textit{for all measurable } A \Subset \Omega\,, \tag{3.2.13}$$

for any choice of representatives $\triangle, \square \in \{1,+\}$.

Proof. Since $A^1 \subseteq A^+$ for any $A \Subset \Omega$ measurable, we always have

$$\nu_1(A^\triangle) - \nu_2(A^\square) \le \nu_1(A^+) - \nu_2(A^1) \quad \text{for all } \triangle,\square \in \{1,+\}\,.$$

Thus, from (1a) we deduce (3.2.13). Vice versa, we assume (3.2.13) and claim that the function formulation (2b) holds. Consider first a non–negative test function $v \in \mathrm{C}_c^\infty(\Omega)$ and extend it to $\overline{v}^0$ with zero values on the complement of Ω. We can reprise the reasoning already in Theorem 3.11, arguing via layer–cake formula and Lemma 2.29, to obtain

$$\int_\Omega v\,\mathrm{d}(\nu_1-\nu_2) = \int_\Omega v^\triangle \mathrm{d}\nu_1 - \int_\Omega v^\square \mathrm{d}\nu_2 = \int_0^\infty \left(\nu_1\left(\{\overline{v}^0 > t\}^\triangle \cap \Omega\right) - \nu_2\Big(\{\overline{v}^0 > t\}^\square \cap \Omega\Big)\right)\mathrm{d}t\,.$$

The hypothesis (3.2.13) applied to superlevel sets $\{\overline{v}^0 > t\} \Subset \Omega$ for a.e. $t \ge 0$ with the anisotropic coarea formula then yields

$$\int_\Omega v\,\mathrm{d}(\nu_1-\nu_2) \le C\int_0^\infty \mathrm{P}_\varphi\left(\{\overline{v}^0 \ge t\}\right)\mathrm{d}t = C\int_\Omega \varphi(.,\nabla v)\,\mathrm{d}x \quad \text{for all non–negative } v \in \mathrm{C}_c^\infty(\Omega)\,.$$

To obtain the Sobolev estimate, we reason by density of $\mathrm{C}_\mathrm{c}^\infty(\Omega)$ in $\mathrm{W}_0^{1,1}(\Omega)$. To treat the convergence of right–hand side, we employ Reshetnyak's continuity Theorem 2.79, whereas for the measure integrals we observe that

$$\left|\int_\Omega v^* \,\mathrm{d}(\nu_1-\nu_2) - \int_\Omega v_k \,\mathrm{d}(\nu_1-\nu_2)\right| \leq \int_\Omega |(v-v_k)^*| \,\mathrm{d}(\nu_1+\nu_2) \leq (C_1+C_2)||\nabla v_k - \nabla v||_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} \xrightarrow[k\to\infty]{} 0$$

whenever $v_k \to v$ in $\mathrm{W}_0^{1,1}(\Omega)$, by admissibility of measures ν_1, ν_2 (and thus, by the IC with respectively constants C_1, C_2) and exploiting Lemma 3.2(iii). We have then proved that (2b) holds for any non–negative $v \in \mathrm{W}_0^{1,1}(\Omega)$, and finally we get back to the set formulation in (1a) applying Theorem 3.16(2). □

Part I

Primal problem

Chapter 4

Lower semicontinuity of anisotropic total variation problems with measure

In this first chapter of *Part I*, we focus on the issue of minimizing anisotropic versions of the total variation problem with measure. Specifically, assigned an anisotropy $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ and non–negative measures $\mu_\pm$ on open $\Omega \subseteq \mathbb{R}^n$, we want to determine suitable conditions so that the problem

$$\inf_{w\in \mathrm{BV}(\Omega)} \left(|\mathrm{D}\overline{w}^{u_0}|_\varphi(\overline{\Omega}) - \int_\Omega w^+ \,\mathrm{d}\mu_- + \int_\Omega w^- \,\mathrm{d}\mu_+ \right) \tag{P-hom}$$

admits minima for given u_0. We refer to this latter as (P-hom), since it can be regarded as the minimization problem (P) restricted to 1–homogeneous integrands $f = f^\infty = \varphi$. We recall that as usual Ω is an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary, $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, and $\overline{w}^{u_0}$ is the BV–extension of w to u_0 outside Ω. Whenever convenient, we will make use of the functional rewriting

$$\begin{aligned}\widehat{\Phi}[w] &:= |\mathrm{D}w|_\varphi(\Omega) + \int_{\partial\Omega} \varphi(\,\cdot\,,(w-u_0)\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Omega w^- \,\mathrm{d}\mu_+ - \int_\Omega w^+ \,\mathrm{d}\mu_- \\ &= |\mathrm{D}\overline{w}^{u_0}|_\varphi(\overline{\Omega}) + \int_\Omega w^- \,\mathrm{d}\mu_+ - \int_\Omega w^+ \,\mathrm{d}\mu_- \quad \text{for all } w \in \mathrm{BV}(\Omega)\,,\end{aligned} \tag{4.0.1}$$

so that (P-hom) can be reformulated as $\inf_{\mathrm{BV}(\Omega)} \widehat{\Phi}$. In our setting, $\mu_\pm$ are non–negative Radon measures on Ω which heuristically do not read what happens on too small sets in $\mathbb{R}^n$. More rigorously, we shall impose that $\mu_\pm$ vanish on $A \subseteq \Omega$ with Hausdorff dimension strictly smaller than $n-1$ (the same condition already encountered in Chapter 3), meaning that sets of codimension higher than 1 are negligible for $\mu_\pm$. This includes, for instance, prototypical measures of the kind:

- $h\mathcal{L}^n \llcorner \Omega$, for non–negative $h \in \mathrm{L}^1(\Omega)$;
- $f\mathcal{H}^{n-1} \llcorner S$, for a (non–negative) function $f \in \mathrm{L}^1(S;\mathcal{H}^{n-1})$ and $S \subseteq \Omega$ is $\mathcal{H}^{n-1}$–measurable with $0 < \mathcal{H}^{n-1}(S) < \infty$. We point out that S needs not to be countably $\mathcal{H}^{n-1}$–rectifiable, and in principle even purely $\mathcal{H}^{n-1}$–unrectifiable sets S are admissible;
- Any non–negative finite Radon measure concentrated on subsets of Ω of (fractal) Hausdorff dimension strictly between $n-1$ and n.
- Any finite combination of the previous ones.

Moreover, in order to consider (P-hom) we want to impose the additional integrability assumptions $\int_\Omega |w^+|\mathrm{d}\mu_+ < \infty$, $\int_\Omega |w^+|\mathrm{d}\mu_- < \infty$ for all $w \in \mathrm{BV}(\Omega)$, and the admissibility condition expressed in Definition 3.3 will serve exactly to this scope. Clearly, in case both μ_+ and μ_- have no singular part with respect to $\mathcal{L}^n$ in the Radon–Nikodým decomposition, the measure terms in (P-hom) reduce to $\int_\Omega w^\mp \, \mathrm{d}\mu_\pm = \int_\Omega wh_\pm \, \mathrm{d}x$ for some non–negative $h_\pm \in \mathrm{L}^1(\Omega)$, and here the choice of the BV representative of w is irrelevant since already for $w \in \mathrm{L}^1(\Omega)$ it is $w^+ = w^- = w^*$ outside of the $\mathcal{L}^n$–negligible discontinuity set S_w.

Specifically, the aim of this chapter is to achieve semicontinuity of the functional $\widehat{\Phi}$ under suitable isoperimetric conditions. This is obtained reasoning first via the parametric semicontinuity result of Theorem 4.5, and then exploiting a nice extension property for measures in Lemma 4.6. We postpone the treatment of the existence of minimizers for $\widehat{\Phi}$, which will be directly addressed in the enlarged framework of Chapter 5.

4.1 Choice of representatives in relation to semicontinuity

To show by the direct method that $\widehat{\Phi}$ admits minima, we need both coercivity and L^1–lower semicontinuity (LSC) of the full functional in BV. In order to consider the appropriate assumptions on φ, we momentarily discard the measure term – that is, we assume $\mu_+ \equiv \mu_- \equiv 0$. For the remaining functional $\widehat{\Phi}^0[w] := \mathrm{TV}_\varphi[\overline{w}^{u_0}](\overline{\Omega})$ we may employ Reshetnyak's semicontinuity Theorem 2.78 to some open $U \supseteq \overline{\Omega}$ and sequences $\overline{w_{(k)}}^{u_0} := w_{(k)}\mathbb{1}_\Omega + u_0\mathbb{1}_{\mathbb{R}^n\setminus\overline{\Omega}}$ to claim

$$\liminf_{k\to\infty} \widehat{\Phi}^0[w_k] \geq \widehat{\Phi}^0[w] \quad \text{ for all } (w_k)_k,\ w \in \mathrm{BV}(\Omega) \text{ s.t. } \ w_k \to w \text{ in } \mathrm{L}^1(\Omega)\,,$$

provided φ is lower semicontinuous, $\varphi(x,.)$ convex and coercive in $\mathbb{R}^n$. Such a class of anisotropies makes our functional $\widehat{\Phi}$ semicontinuous at least in the basilar null–measure case, and thus the assumptions above represent our minimal set of hypotheses on φ.

Concerning the measure integral in (P-hom) instead, we now want to motivate the choice of approximate lower limit w^- for the integral term with positive sign, and approximate upper limit w^+ for the opposite sign integral, with regard to lower semicontinuity. Indeed, we consider one–sided, monotone non–negative approximations $(w_k)_k \subseteq \mathrm{W}^{1,1}(\Omega)$ of a non–negative $w \in \mathrm{BV}(\Omega)$ and claim:

(i) If $(w_k)_k$ converges φ–strictly to w in $\mathrm{BV}(\Omega)$ from below (implying $w_k^* \leq w^\triangle$ holds $\mu_\pm$–a.e. in Ω for $\triangle \in \{-,+\}$ and all $k \in \mathbb{N}$), it is

$$\lim_{k\to\infty} \widehat{\Phi}^0[w_k] = \widehat{\Phi}^0[w] \quad \text{ and } \quad \liminf_{k\to\infty} \left(\int_\Omega w_k^* \, \mathrm{d}\mu_+\right) \leq \int_\Omega w^\triangle \, \mathrm{d}\mu_+\,.$$

In view of this, assuming that $\mu_- \equiv 0$, the best representative to hope for lower semicontinuity of the full functional $\widehat{\Phi}$ is obtained when $\triangle = -$. Actually, we record that a combination of Theorems 2.79 and 2.104 induces (for some subsequence) the $\mathcal{H}^{n-1}$–a.e. pointwise convergence $w_{k_\ell}^* \to w^-$ in Ω as $\ell \to \infty$, thus arguing via monotone convergence theorem we achieve even

$$\lim_{k\to\infty} \left(\int_\Omega w_k^* \, \mathrm{d}\mu_+\right) = \int_\Omega w^- \, \mathrm{d}\mu_+\,.$$

(ii) Vice versa, for sequences $(w_k)_k$ converging to w φ–strictly in $\mathrm{BV}(\Omega)$ from above – thus in particular $w_k^* \geq w^\triangle$ $\mu_\mp$–a.e. in Ω for all k – we find

$$\lim_{k\to\infty} \widehat{\Phi}^0[w_k] = \widehat{\Phi}^0[w] \quad \text{ and } \quad \liminf_{k\to\infty} \left(-\int_\Omega w_k^* \, \mathrm{d}\mu_-\right) \leq -\int_\Omega w^\triangle \, \mathrm{d}\mu_-\,,$$

hence this time for $\mu_+ \equiv 0$ the minimal difference in the measure term is achieved when $\triangle = +$. Yet again, the φ–strict convergence assumption determines (via Theorems 2.79 and 2.104 and monotone convergence) existence of the limit

$$\lim_{k\to\infty}\left(-\int_\Omega w_k^* \,\mathrm{d}\mu_-\right) = -\int_\Omega w^+ \,\mathrm{d}\mu_- \,.$$

From (i) and (ii), we read that $\widehat{\Phi}$ achieves continuity on suitable (φ–)strictly converging sequences; compare with Result 3. Nevertheless, we point out that we are interested in the *joint* semicontinuity of the functional $\widehat{\Phi}$ – meaning that the measure term does not need to be LSC on its own – and we prove the global LSC result by imposing assumptions which induce a balancing effect between the lower semicontinuous anisotropic TV term and the integrals in μ. To this scope, we shall consider arbitrary sequences in $\mathrm{BV}(\Omega)$ converging with respect to the strong topology in L^1.

The reasoning above intuitively explains the selection of BV representatives in the definition of $\widehat{\Phi}$ – which is also valid for the possibly inhomogeneous functional $\mathcal{F}$, up to passing to area–strictly converging sequences. Next, we provide a rigorous justification of the necessity of $w^\pm$ in the measure term; we do so by first considering functionals $\mathcal{F}_\lambda$ defined taking into account Borel combinations of the extremal values $w^\pm$ in the measure term, and then excluding any other combination when looking for semicontinuity.

Consider the original functional in the Dirichlet class defined by $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$:

$$\mathrm{F}_{u_0}[w] := \int_\Omega f(.,\nabla w)\,\mathrm{d}x + \int_\Omega w^* \,\mathrm{d}\mu \qquad \text{for } w \in \mathrm{W}^{1,1}_{u_0}(\Omega)\,.$$

We are interested in a BV–reformulation of F_{u_0} which extends it in a lower semicontinuous way with respect to convergences in L^1– namely, we look for its $\mathrm{L}^1(\Omega)$–relaxation. In principle, any convex combination of the representatives w^+, w^- of an arbitrary function $w \in \mathrm{BV}(\Omega)$ guarantees at least the extension condition. This motivates the introduction of the λ*–representative* of w, defined as

$$w^\lambda := \lambda w^+ + (1-\lambda)w^- \tag{4.1.1}$$

for a given Borel function $\lambda\colon \Omega \to [0,1]$, and we observe that the λ–representative always reduces to the precise one in the case of Sobolev functions – that is, $w^\lambda = w^*$ $\mathcal{H}^{n-1}$–a.e. on Ω whenever $w \in \mathrm{W}^{1,1}(\Omega)$. The formulation above was elaborated in [67] to investigate the prescribed mean curvature case, and it allows us to consider the family of functionals

$$\mathcal{F}_\lambda[w] := f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega}) + \int_\Omega w^\lambda \,\mathrm{d}\mu \qquad \text{for } w \in \mathrm{BV}(\Omega)\,,$$

each satisfying the fundamental prerequisite $\mathcal{F}_\lambda \equiv \mathrm{F}_{u_0}$ on $\mathrm{W}^{1,1}_{u_0}(\Omega)$. Nevertheless, we shall now see that there is a unique function λ associated to μ for which lower semicontinuity of the full functional $\mathcal{F}_\lambda$ is admissible, and such mapping coincides with the choice $\lambda = \lambda_\mu := \mathbb{1}_{\Omega_-}$ for the Jordan decomposition of the measure μ into its positive part μ_+ and negative part μ_- with $\Omega = \Omega_+ \cup \Omega_-$ and $\mu_+ \llcorner \Omega_- = \mu_- \llcorner \Omega_+ \equiv 0$. Then, the measure integral in $\mathcal{F}_\lambda$ reduces to our usual term

$$\int_\Omega w^{\lambda_\mu}\,\mathrm{d}\mu = \int_\Omega (\mathbb{1}_{\Omega_-} w^+ + \mathbb{1}_{\Omega_+} w^-)\,\mathrm{d}\mu = \int_\Omega w^- \,\mathrm{d}\mu_+ - \int_\Omega w^+ \,\mathrm{d}\mu_- \,,$$

and consequently $\mathcal{F}_\lambda = \mathcal{F}$. We infer that the only suitable candidate for semicontinuity among all $\mathcal{F}_{\lambda_\mu}$ is precisely the functional $\mathcal{F}$ introduced in (1.2.4), which justifies our intuitive choice of representatives expressed in (i)–(ii).

Lemma 4.1. *For a pair of admissible measures $\mu_\pm$ on open Ω, we set $\mu := \mu_+ - \mu_-$. If $\mu_+ \perp \mu_-$, for any $w \in \mathrm{BV}(\Omega)$ we have*

$$\int_\Omega w^\lambda \,\mathrm{d}\mu \geq \int_\Omega w^{\lambda_\mu} \,\mathrm{d}\mu\,, \tag{4.1.2}$$

with equality holding if and only if $\lambda(x) = \lambda_\mu(x)$ for $|\mu|$–a.e. $x \in \mathrm{J}_w$.

For the proof of Lemma 4.1, we refer to [67, Lemma 4.2]. From this result applied for *every* BV function, we deduce the necessity of setting the representative $\lambda = \lambda_\mu$ in the relaxation of F_{u_0}.

Proposition 4.2 (failure of LSC for general $\mathcal{F}_\lambda$)**.** *We fix the open bounded and Lipschitz set $\Omega \subseteq \mathbb{R}^n$, some function $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ and $\mu_\pm$ admissible measures on Ω. We consider $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ admissible as in Assumption 1.1 and such that f and f^∞ are continuous. Moreover, we assume that $|\mu| \geq f\mathcal{H}^{n-1} \llcorner S$ for some strictly positive function $f \in \mathrm{L}^1(S;\mathcal{H}^{n-1})$ with $S \subseteq \Omega$ such that $0 < \mathcal{H}^{n-1}(S) < \infty$ and S is $\mathcal{H}^{n-1}$–rectifiable. Then, for any choice of Borel λ such that $\lambda \neq \lambda_\mu$ up to $\mathcal{H}^{n-1}$–negligible sets in S, the functional $\mathcal{F}_\lambda$ is <u>not</u> lower semicontinuous in $\mathrm{BV}(\Omega)$ with respect to $\mathrm{L}^1(\Omega)$ (or weak–$*$) convergence in $\mathrm{BV}(\Omega)$.*

Proof. Assigned $w \in \mathrm{BV}(\Omega)$, we let $(w_k)_k$ be the sequence in $\mathrm{W}^{1,1}(\Omega)$ obtained by Proposition 3.15 with measures $\mu_\pm$, that means $(w_k)_k$ converges to w area–strictly in $\mathrm{BV}(\Omega)$ with

$$\lim_{k\to\infty} \int_U w_k^* \,\mathrm{d}\mu_- = \int_U w^+ \,\mathrm{d}\mu_- \qquad \text{and} \qquad \int_U w_k^* \,\mathrm{d}\mu_+ = \int_U w^- \,\mathrm{d}\mu_+\,.$$

From the area–strict convergence and Theorem 2.70, we find $f(.,\mathrm{D}\overline{w_k}^{u_0})\big(\overline{\Omega}\big) \to f(.,\mathrm{D}\overline{w}^{u_0})\big(\overline{\Omega}\big)$ for $k \to \infty$. Altogether, $w_k \to w$ in $\mathrm{L}^1(\Omega)$ (and weakly–$*$ in $\mathrm{BV}(\Omega)$) as $k \to \infty$, whereas exploiting the bound in (4.1.2) and $\mathcal{F}_\lambda \equiv \mathcal{F}_{(\lambda_\mu)}$ on $\mathrm{W}^{1,1}(\Omega)$, we also obtain

$$\lim_{k\to\infty} \mathcal{F}_\lambda[w_k] = \mathcal{F}_{\lambda_\mu}[w] < \mathcal{F}_\lambda[w] \quad \text{ for all Borel } \lambda \text{ s.t. } |\mu|\big(\mathrm{J}_w \cap \{\lambda \neq \lambda_\mu\}\big) > 0\,. \tag{4.1.3}$$

This contradicts lower semicontinuity of the functional $\mathcal{F}_\lambda$, unless $\mathrm{J}_w \cap \{\lambda \neq \lambda_\mu\}$ is $|\mu|$–negligible for all $w \in \mathrm{BV}(\Omega)$. Selecting then a BV function $\widehat{w}$ such that $\mathrm{J}_{\widehat{w}} \supseteq S$, by $\mathcal{H}^{n-1}$–rectifiability of $\mathrm{J}_{\widehat{w}}$ it is $\mathcal{H}^{n-1} \llcorner (\mathrm{J}_{\widehat{w}} \cap S) = 0$, and consequently

$$|\mu|\big(\mathrm{J}_{\widehat{w}} \cap \{\lambda \neq \lambda_\mu\}\big) \geq f\mathcal{H}^{n-1}\big(\mathrm{J}_{\widehat{w}} \cap \{\lambda \neq \lambda_\mu\} \cap S\big) = f\mathcal{H}^{n-1}\big(\{\lambda \neq \lambda_\mu\}\big) = \int_{S\cap\{\lambda\neq\lambda_\mu\}} f \,\mathrm{d}\mathcal{H}^{n-1} \geq 0\,,$$

where the last inequality holds as equality if and only if $\mathcal{H}^{n-1}\big(\{\lambda \neq \lambda_\mu\} \cap S\big) = 0$. Therefore, if $\{\lambda \neq \lambda_\mu\}$ is *not* $\mathcal{H}^{n-1}$–negligible in S, we conclude that there exist functions $w \in \mathrm{BV}(\Omega)$ such that $|\mu|\big(\mathrm{J}_w \cap \{\lambda \neq \lambda_\mu\}\big) > 0$, hence (4.1.3) determines the claimed loss of semicontinuity for $\mathcal{F}$. $\square$

Observe that for measures $\mu_\pm$ such that $|\mu|$ is of dimension *strictly larger* than $n-1$ (that means, $|\mu|(A) = 0$ for all measurable $A \subseteq \mathbb{R}^n$ such that $0 \leq \mathcal{H}^{n-1}(A) < \infty$), all representatives agree $|\mu|$–a.e. as a consequence of Theorem 2.27. Then, all functionals $\mathcal{F}_\lambda$, $\mathcal{F}_{\lambda_\mu}$, and the original functional with precise representative coincide, so that all semicontinuity and existence results apply to $\mathcal{F}_\lambda$ as well and for any choice of λ – and we notice that, in this case, the measure term $\int_\Omega u^\lambda \,\mathrm{d}\mu$ is even continuous. For this reason, to violate LSC of some $\mathcal{F}_\lambda$ we need to restrict to our interesting case of measures on $\mathbb{R}^n$ of dimension exactly $n-1$.

4.2 Parametric semicontinuity

We first introduce a weaker notion of isoperimetric condition for measures, which will turn out to be useful in the treatment of semicontinuity for the parametric problem – that is, LSC of the φ–perimeter coupled with measure terms. Notice that the isotropic versions of Definition 4.3, Lemma 4.4, and Theorem 4.5 were originally established in [92]. Concerning our good approximation Lemma 4.4, we emphasize that the statement in [92, Lemma 4.4] already established an analogous result (even though limited to the isotropic case) for non–negative measures ν on $\mathbb{R}^n$ under the vanishing condition (C1). The main difference between [92, Lemma 4.4] and our case lies in the presence there of a slightly weaker version of the technical assumption (4.2.3) – which is nevertheless only required for sets with volume less than some small threshold value $\delta = \delta(\varepsilon)$. Anyway, in Lemma 4.4 below, the validity of (4.2.3) follows by finiteness of ν, therefore once this is verified, we can directly rely on the previous good approximation result.

Definition 4.3 (small–volume φ–anisotropic IC)**.** Assume $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ is an anisotropy. A pair (ν_1, ν_2) of finite non–negative Radon measures on open $U \subseteq \mathbb{R}^n$ satisfies the **small-volume φ-anisotropic isoperimetric condition** in U with constant $C \in [0,\infty)$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\nu_1(A^+) - \nu_2(A^1) \leq C\mathrm{P}_\varphi(A) + \varepsilon \quad \text{for all measurable } A \Subset U \text{ with } |A| < \delta\,. \tag{4.2.1}$$

We say that a single measure ν satisfies the small–volume φ–IC in Ω if (4.2.1) holds for $(\nu, 0)$. As for the IC, we call *isotropic small–volume IC* the corresponding condition holding for $\varphi = \varphi_0$.

The scope of this section is to achieve LSC for the functional $A \mapsto \mathrm{P}_\varphi(A) + \mu_+(A^1) - \mu_-(A^+)$ on the σ–algebra of Borel sets A in $\mathbb{R}^n$, and where we implicitly consider the trivial extension by zero of the measures $\mu_\pm$ to all $\mathbb{R}^n$. Such a functional represents the parametric counterpart of $\widehat{\Phi}$, which can be verified by employing layer–cake and anisotropic coarea formulas. The first step towards semicontinuity is the construction of a Borel set of measure arbitrarily close to the measure of the limit of a converging sequence of sets, while having perimeter arbitrarily small in each approximating set for large indices.

Lemma 4.4 (good exterior approximation)**.** *Let ν be a finite non–negative Radon measure on $\mathbb{R}^n$ with $\nu(Z) = 0$ for every $\mathcal{H}^{n-1}$–negligible $Z \subseteq \mathbb{R}^n$, φ an anisotropy on $\mathbb{R}^n$ such that $\varphi(x,\xi) \leq \beta|\xi|$ for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ with $\beta \in [0,\infty)$, fix $\varepsilon > 0$. Then, for any sequence $(\mathbb{1}_{A_k})_k$ in $\mathrm{BV}(\mathbb{R}^n)$ converging in $\mathrm{L}^1(\mathbb{R}^n)$ to $\mathbb{1}_A \in \mathrm{BV}(\mathbb{R}^n)$ there exists a Borel set $S \subseteq \mathbb{R}^n$ with $\mathbb{1}_S \in \mathrm{BV}(\mathbb{R}^n)$ satisfying*

$$A^+ \subseteq \mathrm{Int}(S)\,, \quad \nu(\overline{S}) < \nu(A^+) + 3\varepsilon\,, \quad \textit{and} \quad \liminf_{k\to\infty} \mathrm{P}_\varphi(S, A_k^+) < \beta\varepsilon\,. \tag{4.2.2}$$

Moreover, if A is bounded, the set S can be chosen bounded too.

Proof. Assigned $\varepsilon > 0$, we can build–up on the result of [92, Lemma 4.4], provided that we prove

$$\nu(A^+) \leq M_\varepsilon \mathrm{P}(A) + \varepsilon \quad \text{for all measurable } A \Subset \mathbb{R}^n \tag{4.2.3}$$

for some $M_\varepsilon \in [0,\infty)$. Observe that (4.2.3) does not represent a small–volume IC for ν on $\mathbb{R}^n$ with $\delta = \infty$, since our constant M_ε is dependent on the choice of ε. Assuming by contradiction that (4.2.3) is false, for each $\ell \in \mathbb{N}$ there exists a measurable set $A_\ell \Subset \mathbb{R}^n$ with $\nu(A_\ell^+) > \ell^2\mathrm{P}(A_\ell) + \varepsilon$. In particular

$$\mathrm{P}(A_\ell) < \frac{\nu(\mathbb{R}^n)}{\ell^2} \qquad \text{and} \qquad \nu(A_\ell^+) > \varepsilon$$

for all $\ell \in \mathbb{N}$. We employ the definition of BV–capacity Cap_1 of any $E \subseteq \mathbb{R}^n$ as

$$\mathrm{Cap}_1(E) := \inf\left\{\int_{\mathbb{R}^n} |\nabla u| \,\mathrm{d}x \,:\, u \in \mathrm{W}^{1,1}(\mathbb{R}^n),\ u \geq 1 \text{ a.e. on } U\,,\ U \text{ open neighborhood of } E\right\}$$

as in [26, Definition 2.1]. It is well–known that Cap_1 is an outer measure on $\mathbb{R}^n$ and satisfies

$$\mathrm{Cap}_1(E) = \inf\left\{\mathrm{P}(H) \,:\, H \subseteq \mathbb{R}^n \text{ measurable}, |H| < \infty, E \subseteq H^+\right\}.$$

Then, for the set

$$E := \bigcap_{k=1}^{\infty} \bigcup_{\ell=k}^{\infty} A_\ell^+$$

we have

$$\nu(E) = \lim_{k\to\infty} \nu\left(\bigcup_{\ell=k}^{\infty} A_\ell^+\right) \geq \liminf_{k\to\infty} \nu(A_k^+) \geq \varepsilon\,.$$

At the same time, we compute

$$\mathrm{Cap}_1(E) \leq \mathrm{Cap}_1\left(\bigcup_{\ell=k}^{\infty} A_\ell^+\right) \leq \sum_{\ell=k}^{\infty} \mathrm{Cap}_1(A_\ell^+) \leq \sum_{\ell=k}^{\infty} \mathrm{P}(A_\ell) \leq \nu(\mathbb{R}^n) \sum_{\ell=k}^{\infty} \ell^{-2}$$

for all $k \in \mathbb{N}$, and by finiteness of ν we deduce $\mathrm{Cap}_1(E) = 0$. Since Cap_1 has the same negligible sets of $\mathcal{H}^{n-1}$ (see for instance [46, Theorem 5.12]), we deduce that even $\nu(E) = 0$. This is in contradiction with the preceding estimate, thus (4.2.3) is valid. We apply then [92, Lemma 4.4] to find the set S with $\mathbb{1}_S \in \mathrm{BV}(\mathbb{R}^n)$ satisfying the first two conditions in (4.2.2), as well as

$$\liminf_{k\to\infty} \mathrm{P}(S, A_k^+) < \varepsilon\,.$$

Finally, the passage to the anisotropic perimeter $\mathrm{P}_\varphi(S, A_k^+)$ in the last estimate is justified by the upper bound on φ. This completes the verification of (4.2.2).

In case $A \Subset \mathbb{R}^n$, the construction of [92, Lemma 4.4] allows us to assume that even S is bounded in $\mathbb{R}^n$, which concludes our proof. □

We are now ready to prove the claimed parametric semicontinuity.

Theorem 4.5 (LSC of anisotropic parametric functional with measures)**.** *We assume that $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ is a lower semicontinuous anisotropy comparable to the Euclidean norm, with $\xi \mapsto \varphi(x,\xi)$ convex for every $x \in \mathbb{R}^n$. Consider finite non–negative Radon measures ν_1, ν_2 on $\mathbb{R}^n$ vanishing on $\mathcal{H}^{n-1}$–negligible sets. If the pair (ν_1,ν_2) satisfies the small–volume φ–IC in $\mathbb{R}^n$ with constant 1 and (ν_2,ν_1) satisfies the small–volume $\widetilde{\varphi}$–IC in $\mathbb{R}^n$ with constant 1, then*

$$\liminf_{k\to\infty} \left[\mathrm{P}_\varphi(A_k) + \nu_2(A_k^1) - \nu_1(A_k^+)\right] \geq \mathrm{P}_\varphi(A) + \nu_2(A^1) - \nu_1(A^+) \tag{4.2.4}$$

whenever A_k and A are measurable in $\mathbb{R}^n$ and A_k converges locally in measure to A.

Proof. Case 1 (bounded domain). Assume first that there is a ball B_r such that $\bigcup_{k=1}^\infty A_k \Subset \mathrm{B}_r$ (and then also $A \Subset \mathrm{B}_r$). Up to passage to subsequences, we suppose it exists $\lim_{k\to\infty}\left[\mathrm{P}_\varphi(A_k)+\nu_2(A_k^1)-\nu_1(A_k^+)\right] < \infty$, and from finiteness of ν_1 and the upper bound on φ we deduce $\mathrm{P}(A) \leq \limsup_{k\to\infty} \mathrm{P}(A_k) < \infty$. In particular, we have $\mathbb{1}_{A_k}, \mathbb{1}_A \in \mathrm{BV}(\mathbb{R}^n)$ for $k \gg 1$ with $\mathbb{1}_{A_k}$ converging in $\mathrm{L}^1(\mathbb{R}^n)$ to $\mathbb{1}_A$. We fix an arbitrary $\varepsilon > 0$ and apply Lemma 4.4 to measure ν_1 with mirrored anisotropy $\widetilde{\varphi}$ to find a Borel, bounded set S with $\mathbb{1}_S \in \mathrm{BV}(\mathbb{R}^n)$ and a subsequence of $(A_k)_k$ such that

$$A^+ \subseteq \mathrm{Int}(S)\,, \qquad \nu_1\big(\overline{S}\big) < \nu_1(A^+) + 3\varepsilon\,, \qquad \lim_{k\to\infty} \mathrm{P}_{\widetilde{\varphi}}(S, A_k^+) < \beta\varepsilon\,,$$

assuming β is the upper constant in the growth bound for φ. Furthermore, another application of Lemma 4.4 to ν_2 for φ on the complements $\mathrm{B}_r \setminus A_k$, $\mathrm{B}_r \setminus A$ determines a second bounded Borel set $S' \subseteq \mathbb{R}^n$ with $\mathbb{1}_{S'} \in \mathrm{BV}(\mathbb{R}^n)$ and yet another subsequence such that

$$(\mathrm{B}_r \setminus A)^+ \subseteq \mathrm{Int}(S')\,, \qquad \nu_2\big(\overline{S'}\big) < \nu_2((\mathrm{B}_r \setminus A)^+) + 3\varepsilon\,, \qquad \lim_{k\to\infty} \mathrm{P}_\varphi(S', (\mathrm{B}_r \setminus A_k)^+) < \beta\varepsilon\,.$$

We define the complement $R \coloneqq \mathrm{B}_r \setminus S'$ of S', which in view of $A_k \cup A \Subset \mathrm{B}_r$, $\partial\mathrm{B}_r \subseteq \mathrm{Int}(S')$, and $R \Subset \mathrm{B}_r$ satisfies $\mathbb{1}_R \in \mathrm{BV}(\mathbb{R}^n)$,

$$\overline{R} \subseteq A^1\,, \qquad \nu_2\big(\mathrm{Int}(R)\big) > \nu_2(A^1) - 3\varepsilon\,, \qquad \lim_{k\to\infty} \mathrm{P}_{\widetilde{\varphi}}(R,(A_k^{\mathrm{c}})^+) < \beta\varepsilon\,.$$

Then, the additivity of the measures $\nu_{1/2}$, $|\mathrm{D}\mathbb{1}_{A_k}|_\varphi$ and the inclusion $\overline{R} \subseteq \mathrm{Int}(S)$ yield

$$\begin{aligned}
\mathrm{P}_\varphi(A_k) &= \mathrm{P}_\varphi\big(A_k, \mathrm{Int}(S) \setminus \overline{R}\big) + \mathrm{P}_\varphi\big(A_k, \mathrm{Int}(S)^{\mathrm{c}}\big) + \mathrm{P}_{\widetilde{\varphi}}\big(A_k^{\mathrm{c}}, \overline{R}\big)\,,\\
\nu_2(A_k^1) &\geq \nu_2\big(A_k^1 \setminus \mathrm{Int}(S)\big) - \nu_2\big((A_k^{\mathrm{c}})^+ \cap \mathrm{Int}(R)\big) + \nu_2\big(\mathrm{Int}(R)\big)\,,\\
\nu_1(A_k^+) &\leq \nu_1\big(A_k^+ \setminus \overline{S}\big) - \nu_1\big((A_k^{\mathrm{c}})^1 \cap \overline{R}\big) + \nu_1\big(\overline{S}\big)\,.
\end{aligned}$$

We combine the inequalities above by computing

$$\begin{aligned}
\lim_{k\to\infty} \big[\mathrm{P}_\varphi(A_k) + \nu_2(A_k^1) - \nu_1(A_k^+)\big] &\geq \liminf_{k\to\infty} \mathrm{P}_\varphi\big(A_k, \mathrm{Int}(S) \setminus \overline{R}\big)\\
&\quad + \liminf_{k\to\infty} \Big[\mathrm{P}_\varphi\big(A_k, \mathrm{Int}(S)^{\mathrm{c}}\big) + \nu_2\big(A_k^1 \setminus \mathrm{Int}(S)\big) - \nu_1\big(A_k^+ \setminus \overline{S}\big)\Big]\\
&\quad + \liminf_{k\to\infty} \Big[\mathrm{P}_{\widetilde{\varphi}}\big(A_k^{\mathrm{c}}, \overline{R}\big) - \nu_2\big((A_k^{\mathrm{c}})^+ \cap \mathrm{Int}(R)\big) + \nu_1\big((A_k^{\mathrm{c}})^1 \cap \overline{R}\big)\Big]\\
&\quad + \nu_2\big(\mathrm{Int}(R)\big) - \nu_1\big(\overline{S}\big)\,. \qquad (4.2.5)
\end{aligned}$$

We now estimate each term in (4.2.5) separately. For the first one, we apply Reshetnyak's Theorem 2.78(b) to the open set $\mathrm{Int}(S) \setminus \overline{R}$ and the inclusions $\overline{R} \subseteq A^1 \subseteq A^+ \subseteq \mathrm{Int}(S)$ to obtain

$$\liminf_{k\to\infty} \mathrm{P}_\varphi(A_k, \mathrm{Int}(S) \setminus \overline{R}) \geq \mathrm{P}_\varphi(A, \mathrm{Int}(S) \setminus \overline{R}) \geq \mathrm{P}_\varphi(A, A^+ \setminus A^1) = \mathrm{P}_\varphi(A)\,. \qquad (4.2.6)$$

For the second term, we first record that since $A^+ \subseteq S$ we get $|A_k \backslash S| \leq |A_k \backslash A| \leq |A_k \triangle A|$, hence the convergence in measure implies $\lim_{k\to\infty} |A_k \backslash S| = 0$. We exploit then the small–volume φ–IC for (ν_1, ν_2) and $A_k \setminus S$ for $k \gg 1$, together with $A_k^+ \setminus \overline{S} \subseteq (A_k \setminus S)^+$, $(A_k \setminus S)^1 \subseteq A_k^1 \setminus \mathrm{Int}(S)$, $S^0 \subseteq \mathrm{Int}(S)^{\mathrm{c}}$ and Lemma 2.73 to deduce

$$\begin{aligned}
\nu_1\big(A_k^+ \setminus \overline{S}\big) - \nu_2\big(A_k^1 \setminus \mathrm{Int}(S)\big) &\leq \nu_1((A_k \setminus S)^+) - \nu_2((A_k \setminus S)^1) \leq \mathrm{P}_\varphi(A_k \setminus S) + \varepsilon\\
&\leq \mathrm{P}_\varphi(A_k, S^0) + \mathrm{P}_{\widetilde{\varphi}}(S, A_k^+) + \varepsilon\\
&\leq \mathrm{P}_\varphi(A_k, \mathrm{Int}(S)^{\mathrm{c}}) + \mathrm{P}_{\widetilde{\varphi}}(S, A_k^+) + \varepsilon
\end{aligned}$$

for $k \gg 1$. Rearranging and passing to the limit, it is then

$$\liminf_{k\to\infty} \big[\mathrm{P}_\varphi(A_k, \mathrm{Int}(S)^{\mathrm{c}}) + \nu_2\big(A_k^1 \setminus \mathrm{Int}(S)\big) - \nu_1\big(A_k^+ \setminus \overline{S}\big)\big] \geq -\lim_{k\to\infty} \mathrm{P}_{\widetilde{\varphi}}(S, A_k^+) - \varepsilon > -(\beta{+}1)\varepsilon\,. \qquad (4.2.7)$$

Analogously, we estimate the third addendum in (4.2.5) via the small–volume $\widetilde{\varphi}$–IC for (ν_2, ν_1) applied to $A_k^{\mathrm{c}} \cap R$ again for $k \gg 1$:

$$\begin{aligned}
\nu_2\big((A_k^{\mathrm{c}})^+ \cap \mathrm{Int}(R)\big) - \nu_1\big((A_k^{\mathrm{c}})^1 \cap \overline{R}\big) &\leq \nu_2((A_k^{\mathrm{c}} \cap R)^+) - \nu_1((A_k^{\mathrm{c}} \cap R)^1) \leq \mathrm{P}_{\widetilde{\varphi}}(A_k^{\mathrm{c}} \cap R) + \varepsilon\\
&\leq \mathrm{P}_{\widetilde{\varphi}}(A_k^{\mathrm{c}}, R^1) + \mathrm{P}_{\widetilde{\varphi}}(R, (A_k^{\mathrm{c}})^+) + \varepsilon\\
&\leq \mathrm{P}_{\widetilde{\varphi}}(A_k^{\mathrm{c}}, \overline{R}) + \mathrm{P}_{\widetilde{\varphi}}(R, (A_k^{\mathrm{c}})^+) + \varepsilon
\end{aligned}$$

for $k \gg 1$, and consequently

$$\liminf_{k\to\infty} \big[\mathrm{P}_{\widetilde{\varphi}}(A_k^{\mathrm{c}}, \overline{R}) - \nu_2\big((A_k^{\mathrm{c}})^+ \cap \mathrm{Int}(R)\big) + \nu_1\big((A_k^{\mathrm{c}})^1 \cap \overline{R}\big)\big] \geq -\lim_{k\to\infty} \mathrm{P}_{\widetilde{\varphi}}(R, (A_k^{\mathrm{c}})^+) - \varepsilon > -(\beta{+}1)\varepsilon\,. \qquad (4.2.8)$$

For the last terms in (4.2.5), we recall that by choice of R and S it is

$$\nu_2\big(\mathrm{Int}(R)\big) > \nu_2(A^1) - 3\varepsilon \qquad \text{and} \qquad \nu_1\big(\overline{S}\big) < \nu_1(A^+) + 3\varepsilon\,. \tag{4.2.9}$$

From the estimates in (4.2.5)-(4.2.9), we conclude that

$$\lim_{k\to\infty} \big[\mathrm{P}_\varphi(A_k) + \nu_2(A^1_k) - \nu_1(A^+_k)\big] \geq \mathrm{P}_\varphi(A) + \nu_2(A^1) - \nu_1(A^+) - 2(\beta{+}4)\varepsilon\,,$$

which by arbitrariness of ε proves the claimed semicontinuity (4.2.4).

Case 2 (general unbounded domain). Without loss of generality, we suppose that it exists finite $\lim_{k\to\infty}\big[\mathrm{P}_\varphi(A_k){+}\nu_2(A^1_k){-}\nu_1(A^+_k)\big]$, and therefore even $\mathrm{P}(A_k) + \mathrm{P}(A) < \infty$ for $k \gg 1$. Then, by the isoperimetric inequality, it is either $|A| < \infty$ or $|A^{\mathrm{c}}| < \infty$.

- If $|A| < \infty$, given $\varepsilon > 0$ there exists a ball large enough such that $|A \setminus \mathrm{B}_r| < \varepsilon$, $\mathrm{P}(A, (\mathrm{B}_r)^{\mathrm{c}}) < \varepsilon$ and $\nu_1((\mathrm{B}_r)^{\mathrm{c}}) < \varepsilon$, $\nu_2((\mathrm{B}_r)^{\mathrm{c}}) < \varepsilon$ because of the finiteness assumption. Then, local convergence of A_k to A implies $|A_k \cap (\mathrm{B}_{r+1} \setminus \mathrm{B}_r)| < \varepsilon$ for $k \gg 1$, and according to

  $$\int_r^{r+1} \mathcal{H}^{n-1}(A^1_k \cap \partial\mathrm{B}_\varrho)\,\mathrm{d}\varrho \leq |A_k \cap (\mathrm{B}_{r+1} \setminus \mathrm{B}_r)| < \varepsilon$$

  we can select a radius $\varrho_k \in [r, r+1]$ such that $\mathcal{H}^{n-1}(A^1_k{\cap}\partial\mathrm{B}_{\varrho_k}) < \varepsilon$ for k large enough. Employing Lemma 2.73, this implies

  $$\mathrm{P}_\varphi(A_k \cap \mathrm{B}_{\varrho_k}) \leq \mathrm{P}_\varphi(A_k, (\mathrm{B}_{\varrho_k})^+) + \mathrm{P}_\varphi(\mathrm{B}_{\varrho_k}, A^1_k) \leq \mathrm{P}_\varphi(A_k) + \beta\mathcal{H}^{n-1}(A^1_k \cap \partial\mathrm{B}_{\varrho_k}) < \mathrm{P}_\varphi(A_k) + \beta\varepsilon$$

  for $k \gg 1$, and – passing to a subsequence – we can assume that the sequence of radii $(\varrho_k)_k$ converges to $\varrho \in [r, r+1]$. At this point, $A_k \cap \mathrm{B}_{\varrho_k} \subseteq \mathrm{B}_{r+1}$ converges to $A \cap \mathrm{B}_\varrho$ in measure, and we apply the result (4.2.4) obtained in the bounded case to such a sequence and, after some minor adjustments, write

  $$\liminf_{k\to\infty} \big[\mathrm{P}_\varphi(A_k \cap \mathrm{B}_{\varrho_k}) + \nu_2(A^1_k) - \nu_1(A^+_k \cap \mathrm{B}_r)\big] \geq \mathrm{P}_\varphi(A, \mathrm{B}_r) + \nu_2(A^1 \cap \mathrm{B}_r) - \nu_1(A^+)\,.$$

  By choice of r and ϱ_k, gathering the estimates above we compute

  $$\begin{aligned}
  &\nu_1(A^+_k \cap \mathrm{B}_r) \geq \nu_1(A^+_k) - \nu_1((\mathrm{B}_r)^{\mathrm{c}}) \geq \nu_1(A^+_k) - \varepsilon \qquad \text{for } k \gg 1\,,\\
  &\nu_2(A^1 \cap \mathrm{B}_r) \geq \nu_2(A^1) - \nu_2((\mathrm{B}_r)^{\mathrm{c}}) \geq \nu_2(A^1) - \varepsilon\,,\\
  &\mathrm{P}_\varphi(A, \mathrm{B}_r) = \mathrm{P}_\varphi(A) - \mathrm{P}_\varphi(A, (\mathrm{B}_r)^{\mathrm{c}}) \geq \mathrm{P}_\varphi(A) - \beta\mathrm{P}(A, (\mathrm{B}_r)^{\mathrm{c}}) \geq \mathrm{P}_\varphi(A) - \beta\varepsilon\,.
  \end{aligned}$$

  Thus, we can pass to

  $$\liminf_{k\to\infty} \big[\mathrm{P}_\varphi(A_k) + \nu_2(A^1_k) - \nu_1(A^+_k)\big] + (\beta+1)\varepsilon \geq \mathrm{P}_\varphi(A) + \nu_2(A^1) - \nu_1(A^+) - (\beta+1)\varepsilon$$

  and by arbitrariness of ε this latter yields (4.2.4).

- If instead $|A^{\mathrm{c}}| < \infty$, we exploit the equalities $(A^{\mathrm{c}})^+ = (A^1)^{\mathrm{c}}$, $(A^+)^{\mathrm{c}} = (A^{\mathrm{c}})^1$ to rewrite

  $$\mathrm{P}_\varphi(A) + \nu_2(A^1) - \nu_1(A^+) = \mathrm{P}_{\widetilde{\varphi}}(A^{\mathrm{c}}) + \nu_1((A^{\mathrm{c}})^1) - \nu_2((A^{\mathrm{c}})^+) + c\,,$$

  where the finite value $c := \nu_2(\mathbb{R}^n) - \nu_1(\mathbb{R}^n)$ does not affect semicontinuity. Reprising the previous step with A^{c}_k and A^{c} in place of A_k and A, with $\widetilde{\varphi}$ in place of φ, and with the roles of ν_2 and ν_1 just exchanged (notice that the recombination matches with our ICs), we complete the deduction of (4.2.4) even in this last setting. □

We conclude this section by adapting the reasoning in [92, Lemma 7.3] to determine an equivalent formulation for the small–volume IC where we extend the measures to the full space by zero. This useful consideration will enable us in Section 4.3 to use the parametric result of Theorem 4.5 for the extended measures $\overline{\mu_\pm}$ on all $\mathbb{R}^n$ and thus obtain lower semicontinuity for the non–parametric functional.

Lemma 4.6 (small–volume IC for extensions)**.** *We consider an anisotropy φ on $\mathbb{R}^n$ comparable with the Euclidean norm with constants $\alpha,\ \beta > 0$, and we assume that ν_1, ν_2 are finite, non–negative Radon measures on the open set $U \subseteq \mathbb{R}^n$ with extensions $\overline{\nu_1}$, $\overline{\nu_2}$ defined as $\overline{\nu_{1/2}}(S) := \nu_{1/2}(S \cap U)$ for Borel $S \subseteq \mathbb{R}^n$. Fix $C \in [0, \infty)$. Then they are equivalent:*

(i) *The pair (ν_1, ν_2) satisfies the small–volume φ–IC in U with constant C;*

(ii) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\overline{\nu_1}(A^+) - \overline{\nu_2}(A^1) \le C\mathrm{P}_\varphi(A, U) + \varepsilon \qquad \textit{for all measurable } A \Subset \mathbb{R}^n \textit{ with } |A| < \delta\,. \tag{4.2.10}$$

In other words, Lemma 4.6 tells us the following: If (ν_1, ν_2) satisfies the small–volume φ–IC in U for some $C \in [0, \infty)$, then $(\overline{\nu_1}, \overline{\nu_2})$ satisfies the small–volume φ–IC in $\mathbb{R}^n$ with same constant C.

Proof. By definition of extended measures, considering sets $A \Subset U$ only, we immediately read (ii) $\Longrightarrow$ (i). We assume then the small–volume φ–IC for (ν_1, ν_2) in U and we claim that (4.2.10) holds. If $d\colon \mathbb{R}^n \to [0, \infty]$ is the function $d(x) := \mathrm{dist}(x, U^{\mathrm{c}})$ for $x \in \mathbb{R}^n$, we know that d is Lipschitz continuous in $\mathbb{R}^n$ with Lipschitz constant 1 and thus by Rademacher's theorem it is $|\nabla d| \le 1$ a.e. on $\mathbb{R}^n$. Clearly, the open sets $\{d < t\}$ are non–increasing as t decreases to 0 with $\bigcup_{t>0}\{d < t\} = U^{\mathrm{c}}$, hence by finiteness of $\overline{\nu_1}$ we obtain $\overline{\nu_1}\big(\{d < t\}\big) \xrightarrow[t \to 0]{} \overline{\nu_1}\big(U^{\mathrm{c}}\big) = 0$. Therefore, given arbitrary $\varepsilon > 0$ we find a $t_0 > 0$ satisfying

$$\overline{\nu_1}\big(\{d < t_0\}\big) < \frac{\varepsilon}{3}\,.$$

We let now $\delta' > 0$ be such the small–volume φ–IC for (ν_1, ν_2) with constant C holds in U with $\varepsilon/3$ and δ', and we set $\delta := \min\big\{\delta', \frac{t_0\varepsilon}{3\beta C}\big\}$. Considering then a measurable $A \Subset \mathbb{R}^n$ with $|A| < \delta$ and $\mathrm{P}_\varphi(A, U) < \infty$ (otherwise, the estimate is trivial), from the lower bound on φ we deduce that A is of finite perimeter in U. Applying Theorem 2.36 we infer

$$\int_0^{t_0} \mathcal{H}^{n-1}\left(A^+ \cap \{d = t\}\right)\,\mathrm{d}t \le \int_0^{\infty} \mathcal{H}^{n-1}\left(A^+ \cap \{d = t\}\right)\,\mathrm{d}t = \int_A |\nabla d|\,\mathrm{d}x \le |A| < \frac{t_0\varepsilon}{3\beta C}\,.$$

Thus, there exists some $\bar{t} \in (0, t_0)$ such that

$$\mathcal{H}^{n-1}\left(A^+ \cap \big\{d = \bar{t}\big\}\right) \le \frac{1}{t_0}\int_0^{t_0} \mathcal{H}^{n-1}\left(A^+ \cap \{d = t\}\right)\,\mathrm{d}t < \frac{\varepsilon}{3\beta C}\,, \tag{4.2.11}$$

and we compute

$$\overline{\nu_1}\big(A^+\big) \le \overline{\nu_1}\big(A^+ \cap \big\{d > \bar{t}\big\}\big) + \overline{\nu_1}\big(\big\{d \le \bar{t}\big\}\big) \le \overline{\nu_1}\big(A^+ \cap \big\{d > \bar{t}\big\}\big) + \overline{\nu_1}\big(\big\{d < t_0\big\}\big)\,. \tag{4.2.12}$$

We cut A away from the boundary of U by letting $E := A \cap \big\{d > \bar{t}\big\} \Subset U$ and from Remark 2.26 we have $A^+ \cap \big\{d > \bar{t}\big\} \subseteq E^+ \Subset U$. At this point we make use of the small-volume φ-IC for (ν_1, ν_2) in U for the set $|E| \le |A| < \delta'$ and of (4.2.12) to get

$$\begin{aligned}\overline{\nu_1}\big(A^+\big) - \overline{\nu_2}\big(A^1\big) &\le \overline{\nu_1}\big(A^+ \cap \big\{d > \bar{t}\big\}\big) + \overline{\nu_1}\big(\big\{d < t_0\big\}\big) - \overline{\nu_2}\big(E^1\big)\\ &\le \nu_1\big(E^+\big) - \nu_2\big(E^1\big) + \frac{\varepsilon}{3} \le C\mathrm{P}_\varphi(E) + \frac{2\varepsilon}{3}\,.\end{aligned} \tag{4.2.13}$$

To estimate the anisotropic perimeter we exploit Lemma 2.73, the upper bound for φ and (4.2.11), and find

$$\begin{aligned}\mathrm{P}_\varphi(E) = \mathrm{P}_\varphi(E,U) &\le \mathrm{P}_\varphi\big(A,\{d>\bar{t}\}^1\cap U\big) + \mathrm{P}_\varphi\big(\{d>\bar{t}\},A^+\cap U\big)\\ &\le \mathrm{P}_\varphi(A,U) + \beta\,\mathrm{P}\big(\{d>\bar{t}\},A^+\big)\\ &= \mathrm{P}_\varphi(A,U) + \beta\,\mathcal{H}^{n-1}\big(A^+\cap\{d=\bar{t}\}\big)\\ &\le \mathrm{P}_\varphi(A,U) + \frac{\varepsilon}{3C}\,. \end{aligned} \tag{4.2.14}$$

The combination of (4.2.13) with (4.2.14) yields the required (4.2.10) and completes the proof. □

4.3 Non-parametric semicontinuity

From the result of Theorem 4.5 in the parametric setting, one may wonder if a weak assumption as the small–volume IC for the pairs (μ_-,μ_+) and (μ_+,μ_-) suffices to obtain semicontinuity for the non–parametric functional $\mathcal{F}$ as well. The answer is nevertheless negative, as illustrated in the following counterexample.

Example 4.7 (failure of non–parametric LSC without limit IC)**.** For $n \ge 2$, let $(A_k)_k$ be any sequence of measurable sets such that $A_k \Subset \Omega$, with pairwise–disjoint closures and $\mathrm{P}(A_k) > 0$ such that $\sum_{k=1}^\infty \mathrm{P}(A_k) < \infty$ (for instance, A_k balls with $\mathrm{P}(A_k) = \delta/k^2$, where $\delta > 0$ small enough). Then, the measure $\mu_- := 2\mathcal{H}^{n-1}\llcorner\big(\bigcup_{k=1}^\infty \partial^* A_k\big)$ does *not* satisfy the (isotropic) IC in Ω with constant 1, as $\mu_-(A_k^+) = 2\mathcal{H}^{n-1}(\partial^* A_k) = 2\mathrm{P}(A_k) > \mathrm{P}(A_k)$ for all $k \in \mathbb{N}$. Additionally, introducing the sequence $u_k := \mathbb{1}_{A_k}/\mathrm{P}(A_k)$ in $\mathrm{BV}(\Omega)$, and bringing in the isoperimetric inequality (2.3.1), we compute $\|u_k\|_{\mathrm{L}^1(\Omega)} = |A_k|\mathrm{P}(A_k)^{-1} \le \mathrm{const}(n)\mathrm{P}(A_k)^{1/(n-1)}$, hence u_k converges in $\mathrm{L}^1(\Omega)$ to zero as $k \to \infty$. Setting then $\mu_+ :\equiv 0$, $u_0 :\equiv 0$, and the isotropy $\varphi = \varphi_0 := |.|$, lower semicontinuity of $\widehat{\Phi}$ fails along such sequence since $\widehat{\Phi}[u_k] = |\mathrm{D}u_k|(\Omega) - \int_\Omega u_k^+\,\mathrm{d}\mu_- = \big[\mathrm{P}(A_k) - 2\mathcal{H}^{n-1}(\partial^* A_k)\big]/\mathrm{P}(A_k) = -1$ for all $k \in \mathbb{N}$, whereas $\widehat{\Phi}[0] = 0$. Nevertheless, we record that by [92, Theorem 8.2] the double perimeter measure μ_- satisfies the *small–volume* (isotropic) IC in $\mathbb{R}^n$ – and thus, on Ω – with constant precisely 1.

In the next step, we want to prove Result 1 restricted to homogeneous integrals; that is, achieving the following Theorem 4.8. To do so, Example 4.7 indicates the need for stronger assumptions compared to parametric semicontinuity. In any case, we record that Theorem 4.5 represents a decisive ingredient in the proof of the non–parametric LSC.

Theorem 4.8 (LSC of anisotropic TV functional with measures)**.** *We fix an open bounded Lipschitz set $\Omega \subseteq \mathbb{R}^n$, admissible measures $\mu_\pm$ on Ω, and $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$. For an anisotropy $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ comparable to the Euclidean norm we assume:*

(a) *φ is continuous; and*

(b) *the restriction $\xi \mapsto \varphi(x,\xi)$ is convex for all $x \in \mathbb{R}^n$.*

Then, supposed that the pair (μ_-,μ_+) satisfies the φ–IC in Ω with constant 1 and (μ_+,μ_-) the $\widetilde{\varphi}$–IC in Ω with constant 1, the anisotropic total variation functional $\widehat{\Phi}$ introduced in (4.0.1) is lower semicontinuous in $\mathrm{BV}(\Omega)$ with respect to convergence in $\mathrm{L}^1(\Omega)$.

We postpone the proof of Theorem 4.8 to the very end of this section. Preliminary to this is the introduction of an auxiliary functional defined extending $\widehat{\Phi}$ on $\mathbb{R}^n$, and the same for its mirrored version with inverted roles of the measure components.

Definition 4.9 (auxiliary functionals)**.** For an anisotropy φ on $\mathbb{R}^n$ and given admissible measures $\mu_\pm$ on open bounded Lipschitz $\Omega \subseteq \mathbb{R}^n$, we define

$$\begin{aligned}\overline{\Phi}[w] &:= |\mathrm{D}w|_\varphi(\mathbb{R}^n) + \int_\Omega w^-\,\mathrm{d}\mu_+ - \int_\Omega w^+\,\mathrm{d}\mu_- \quad \text{and}\\ \widetilde{\Phi}[w] &:= |\mathrm{D}w|_{\widetilde{\varphi}}(\mathbb{R}^n) + \int_\Omega w^-\,\mathrm{d}\mu_- - \int_\Omega w^+\mathrm{d}\mu_+\,,\end{aligned}$$

for $w \in \mathrm{BV}(\mathbb{R}^n)$.

We now observe that our main functional $\widehat{\Phi}$ enjoys the following decomposition.

Remark 4.10. We consider $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, an anisotropy φ on $\mathbb{R}^n$, and $\mu_\pm$ admissible measures on Ω. We have

$$\widehat{\Phi}[w] = \overline{\Phi}\left[(\overline{w}^{u_0})_+\right] + \widetilde{\Phi}\left[(\overline{w}^{u_0})_-\right] - |\mathrm{D}u_0|_\varphi\big(\mathbb{R}^n \setminus \overline{\Omega}\big) \qquad \text{for all } w \in \mathrm{BV}(\Omega)\,,$$

with $\overline{w}^{u_0} = w\mathbb{1}_\Omega + u_0\mathbb{1}_{\mathbb{R}^n\setminus\overline{\Omega}} \in \mathrm{BV}(\mathbb{R}^n)$. In fact, we apply the decomposition for TV_φ in Lemma 2.77 to write

$$\begin{aligned}|\mathrm{D}\overline{w}^{u_0}|_\varphi\big(\overline{\Omega}\big) &= |\mathrm{D}\overline{w}^{u_0}|_\varphi\big(\mathbb{R}^n\big) - |\mathrm{D}u_0|_\varphi\big(\mathbb{R}^n \setminus \overline{\Omega}\big)\\ &= |\mathrm{D}(\overline{w}^{u_0})_+|_\varphi\big(\mathbb{R}^n\big) + |\mathrm{D}(\overline{w}^{u_0})_-|_{\widetilde{\varphi}}\big(\mathbb{R}^n\big) - |\mathrm{D}u_0|_\varphi\big(\mathbb{R}^n \setminus \overline{\Omega}\big).\end{aligned}$$

We can additionally exploit the $\mathcal{H}^{n-1}$–a.e. (and thus, $\mu_\pm$–a.e.) equalities $w^+ = (w_+)^+ - (w_-)^-$ and $w^- = (w_+)^- - (w_-)^+$ of Lemma 2.28 together with $w_\pm = \overline{w}_\pm$ on Ω to rewrite

$$\int_\Omega w^-\,\mathrm{d}\mu_+ - \int_\Omega w^+\,\mathrm{d}\mu_- = \int_\Omega [(\overline{w}^{u_0})_+]^-\,\mathrm{d}\mu_+ - \int_\Omega [(\overline{w}^{u_0})_+]^+\,\mathrm{d}\mu_- + \int_\Omega [(\overline{w}^{u_0})_-]^-\,\mathrm{d}\mu_- - \int_\Omega [(\overline{w}^{u_0})_-]^+\,\mathrm{d}\mu_+\,.$$

The claim follows just by rearranging the terms according to the definitions of $\widehat{\Phi}$, $\overline{\Phi}$, and $\widetilde{\Phi}$.

Working now with truncated functions $w^M := \max\{\min\{w, M\}, -M\}$ of some $w \in \mathrm{BV}(\mathbb{R}^n)$, we will initially prove L^1–semicontinuity for $\overline{\Phi}$, $\widetilde{\Phi}$ on suitable sequences in $\mathrm{BV}(\mathbb{R}^n)$, and lastly we will merge the partial results to show LSC of the functional $\widehat{\Phi}$.

Lemma 4.11. *If the anisotropy φ satisfies $\varphi(x,\xi) \le \beta|\xi|$ for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and some $\beta \in [0,\infty)$, then the functionals $\overline{\Phi}$, $\widetilde{\Phi}$ are continuous on truncations, that is: For any $w \in \mathrm{BV}(\mathbb{R}^n)$ and any positive constant M, it results*

$$\lim_{M\to\infty} \overline{\Phi}[w^M] = \overline{\Phi}[w] \qquad \textit{and} \qquad \lim_{M\to\infty} \widetilde{\Phi}[w^M] = \widetilde{\Phi}[w]\,.$$

Proof. The statement follows from the convergence of each term separately. In fact, for the anisotropic TV we employ Lemma 2.100(iv) to get $|\mathrm{D}w^M|_\varphi(\mathbb{R}^n) \to |\mathrm{D}w|_\varphi(\mathbb{R}^n)$ for $M \to \infty$ (and analogous for the mirrored anisotropy $\widetilde{\varphi}$, satisfying the same bound $\widetilde{\varphi}(x,.) \le \beta|.|$), whereas the convergence of the measure terms is justified by Lemma 3.8. □

Lemma 4.12. *We suppose $\mu_\pm$ are admissible measures on Ω and φ is an anisotropy in $\mathbb{R}^n$ such that*

$$\alpha|\xi| \le \varphi(x,\xi) \le \beta|\xi| \quad \textit{for all } (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n\,, \tag{4.3.1}$$

with constants $0 < \alpha \le \beta < \infty$, moreover for φ we assume (a)–(b). We let (μ_-,μ_+) satisfy the φ–IC in Ω with constant 1, and fix a non–negative function $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$. Then, for all constant $M > 0$ and all non–negative $w \in \mathrm{BV}(\mathbb{R}^n)$ such that $w = u_0$ $\mathcal{L}^n$–a.e. in Ω^c, it holds

$$\overline{\Phi}[w] \ge \overline{\Phi}[w^M] + c_M \qquad \textit{with } c_M := \alpha||\nabla u_0 - \nabla {u_0}^M||_{\mathrm{L}^1(\Omega^c,\mathbb{R}^n)} - \beta||u_0 - {u_0}^M||_{\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})}\,. \tag{4.3.2}$$

As a consequence of the convergence of ${u_0}^M$ to u_0 in $\mathrm{W}^{1,1}(\mathbb{R}^n)$ in Lemma 2.100(iii), and of strict continuity of the trace Theorem 2.23, the estimate (4.3.2) yields $c_M \to 0$ as $M \to \infty$.

The continuity assumption (a) for φ is necessary to pass from the formulation of φ–IC for (μ_-, μ_+) on sets (according to Definition 3.1) to φ–IC on functions exploiting Theorem 3.11. Nevertheless, if an alternative definition of anisotropic IC according to (2a) would be given instead, the sole lower semicontinuity and convexity of $\varphi(x, .)$ would suffice to achieve Lemma 4.12 and its corollaries – including the semicontinuity result of Theorem 4.8.

Proof. We fix a level $M > 0$ and w as in the statement. The φ–IC in the form of (2a) applied to $w - w^M \geq 0$ with $C = 1$ reads

$$-\int_\Omega (w - w^M)^+ \,\mathrm{d}\mu_- + \int_\Omega (w - w^M)^- \,\mathrm{d}\mu_+ \geq -|\mathrm{D}\left(w - w^M\right)|_\varphi(\Omega) - \int_{\partial\Omega} \varphi\left(\,\cdot\,, (w - w^M)\nu_\Omega\right) \mathrm{d}\mathcal{H}^{n-1} .$$

Then, with the help of Lemmas 2.100(ii) and 2.101 we estimate the difference

$$\begin{aligned}
\overline{\Phi}[w] - \overline{\Phi}[w^M] &= |\mathrm{D}(w - w^M)|_\varphi(\mathbb{R}^n) + \int_\Omega (w - w^M)^- \,\mathrm{d}\mu_+ - \int_\Omega (w - w^M)^+ \,\mathrm{d}\mu_- \\
&\geq |\mathrm{D}(w - w^M)|_\varphi(\mathbb{R}^n \setminus \Omega) - \int_{\partial\Omega} \varphi\left(\,\cdot\,, (w - w^M)\,\nu_\Omega\right) \mathrm{d}\mathcal{H}^{n-1} \\
&\geq |\mathrm{D}(w - w^M)|_\varphi(\mathbb{R}^n \setminus \overline{\Omega}) - |\mathrm{D}(w - w^M)|_\varphi(\partial\Omega) - \int_{\partial\Omega} \varphi\left(\,\cdot\,, (w - w^M)\,\nu_\Omega\right) \,\mathrm{d}\mathcal{H}^{n-1} .
\end{aligned}$$

Moreover, the convexity of φ in the second variable yields

$$|\mathrm{D}(w - w^M)|_\varphi(\partial\Omega) \geq \int_{\partial\Omega} \varphi\left(\,\cdot\,, (w - w^M)\,\nu_\Omega\right) \mathrm{d}\mathcal{H}^{n-1} - \int_{\partial\Omega} \varphi\left(\,\cdot\,, (u_0 - {u_0}^M)\,\nu_\Omega\right) \mathrm{d}\mathcal{H}^{n-1}$$

by an application of (2.6.2). Recalling that $w = u_0$ outside $\overline{\Omega}$, we obtain

$$\overline{\Phi}[w] - \overline{\Phi}[w^M] \geq |\mathrm{D}(u_0 - (u_0)^M)|_\varphi(\mathbb{R}^n \setminus \overline{\Omega}) - \int_{\partial\Omega} \varphi\left(\,\cdot\,, (u_0 - {u_0}^M)\,\nu_\Omega\right) \,\mathrm{d}\mathcal{H}^{n-1} ,$$

and thus employing (4.3.1) we have verified the validity of (4.3.2) with appropriate constant c_M. $\square$

At this point, we can prove the claimed lower semicontinuity for the auxiliary functionals $\overline{\Phi}$ and $\widetilde{\Phi}$. In this regard, we mention that the double (strong) IC could be replaced in the following Proposition 4.14 by the weaker condition:

(μ_-, μ_+) satisfy the φ–IC in Ω with constant 1; and
(μ_+, μ_-) satisfy the *small–volume* $\widetilde{\varphi}$–IC in Ω with constant 1,

with inverted assumptions on φ, $\mu_\pm$ for the subsequent Proposition 4.15. In fact, we shall see in the proof that the first condition is necessary to apply the truncation argument in Lemma 4.12, whereas the parametric result 4.5 only requires the small–volume IC with constant 1 on $\mathbb{R}^n$ and for both pairs of measures, thus arguing via Lemma 4.6 this is achieved under the small–volume IC on Ω only. Nevertheless, since ultimately the double (strong) IC is required to obtain LSC of the full functional $\widehat{\Phi}$, we prefer not to distinguish the cases and use directly the formulation in Proposition 4.14.

We now demonstrate via Lemma 4.12 that truncations for sufficiently high levels – namely, above the L^∞ norm of the outer datum – achieve small values of the functional $\widehat{\Phi}$.

Corollary 4.13. *For an anisotropy φ as in Lemma 4.12 and admissible measures $\mu_\pm$ on Ω, we assume that (μ_-,μ_+) satisfies the φ–IC in Ω with constant 1, and that (μ_-,μ_+) satisfies the $\widetilde{\varphi}$–IC in Ω with constant 1. If $u_0 \in \mathrm{L}^\infty(\mathbb{R}^n)\cap \mathrm{W}^{1,1}(\mathbb{R}^n)$, then we obtain*

$$\widehat{\Phi}[w] \geq \widehat{\Phi}[w^M] \quad \textit{for all } w \in \mathrm{BV}(\Omega) \textit{ and all } M \geq \|u_0\|_{\mathrm{L}^\infty(\mathbb{R}^n)}\,.$$

Proof. We fix an arbitrary $w \in \mathrm{BV}(\Omega)$. Choosing a level $M \geq \|u_0\|_{\mathrm{L}^\infty(\mathbb{R}^n)} \geq \|(u_0)_\pm\|_{\mathrm{L}^\infty(\mathbb{R}^n)}$, we record ${u_0}^M = u_0$, as well as $((\overline{w}^{u_0})_\pm)^M = \big((\overline{w}^{u_0})^M\big)_\pm = \big(\overline{w^M}^{u_0}\big)_\pm$ everywhere. From Lemma 4.12 applied to $(\overline{w}^{u_0})_+$, we read $\overline{\Phi}[(\overline{w}^{u_0})_+] \geq \overline{\Phi}\big[\big(\overline{w^M}^{u_0}\big)_+\big]$ because of $(u_0)_+ = ({u_0}_+)^M \geq 0$. Another application of Lemma 4.12 to $(\overline{w}^{u_0})_-$ and $(\widetilde{\varphi},\mu_+,\mu_-)$ in place of (φ,μ_-,μ_+) yields the symmetrical result $\overline{\Phi}[(\overline{w}^{u_0})_-] \geq \overline{\Phi}\big[\big(\overline{w^M}^{u_0}\big)_-\big]$. We complete the reasoning summing up the previous estimates with the help of Remark 4.10. $\square$

Proposition 4.14 (LSC of $\overline{\Phi}$ on non–negative functions with given value outside Ω)**.** *We assume that φ is an anisotropy as in Lemma 4.12. Let $\mu_\pm$ be admissible measures on Ω such that (μ_-,μ_+) satisfies the φ–IC in Ω with constant 1 and (μ_+,μ_-) the $\widetilde{\varphi}$–IC in Ω with constant 1, moreover fix a <u>non–negative</u> $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$. Then, it holds*

$$\liminf_{k\to\infty} \overline{\Phi}[w_k] \geq \overline{\Phi}[w]$$

for every $(w_k)_k$, w in $\mathrm{BV}(\mathbb{R}^n)$ such that $w_k \to w$ in $\mathrm{L}^1(\mathbb{R}^n)$, supposed w_k, $w \geq 0$ a.e. in Ω and $w_k = w = u_0$ a.e. in Ω^{c} for all k.

Proof. Let $(w_k)_k$, w as in the statement and $M > 0$, so that up to subsequences it is $w_k \to w$ pointwise a.e. in $\mathbb{R}^n$. Employing the usual notation $\overline{\mu_\pm}$ for the extended measures on $\mathbb{R}^n$ and Lemma 2.29, we write

$$\big\{(w_k^M)^+ > t\big\} = \big\{w_k^M > t\big\}^+ \quad \text{and} \quad \big\{(w_k^M)^- > t\big\} = \big\{w_k^M > t\big\}^1 \quad \overline{\mu_\pm}\text{–a.e. in } \mathbb{R}^n, \quad \text{for } \mathcal{L}^1\text{–a.e. } t > 0\,.$$

By Lemma 4.12, the anisotropic coarea formula of Theorem 2.82, and a layer–cake argument for $w_k^M \geq 0$ we find

$$\begin{aligned}\overline{\Phi}[w_k] \geq \overline{\Phi}[w_k^M] + c_M &= c_M + |\mathrm{D}w_k^M|_\varphi(\mathbb{R}^n) + \int_{\mathbb{R}^n} \big(w_k^M\big)^- \,\mathrm{d}\overline{\mu_+} - \int_{\mathbb{R}^n} \big(w_k^M\big)^+ \,\mathrm{d}\overline{\mu_-}\\ &= c_M + \int_0^\infty \Big[\mathrm{P}_\varphi\left(\{w_k^M > t\}\right) + \overline{\mu_+}\left(\{w_k^M > t\}^1\right) - \overline{\mu_-}\left(\{w_k^M > t\}^+\right)\Big]\,\mathrm{d}t\,.\end{aligned}$$

From the control $w_k^M \leq M$ it is $\overline{\mu_-}\big(\{w_k^M > t\}^+\big) \leq \mathbb{1}_{(0,M)}(t)\mu_-(\Omega)$ for all k, with the function $\mathbb{1}_{(0,M)}\mu_-(\Omega) \in \mathrm{L}^1((0,\infty))$, thus all integrands in the last term are uniformly bounded from below by a summable function. This justifies the application of Fatou's lemma, hence

$$\liminf_{k\to\infty} \overline{\Phi}[w_k] \geq c_M + \int_0^\infty \liminf_{k\to\infty} \Big[\mathrm{P}_\varphi\left(\{w_k^M > t\}\right) + \overline{\mu_+}\left(\{w_k^M > t\}^1\right) - \overline{\mu_-}\left(\{w_k^M > t\}^+\right)\Big]\,\mathrm{d}t\,.$$

We notice now that given $w_k^M = w^M$ outside Ω, the pointwise convergence $w_k^M \to w^M$ for $k \to \infty$ determines convergence in measure of $A_{k,M}^t := \{w_k^M > t\}$ to $A_M^t := \{w^M > t\}$ for $k \to \infty$, whenever $t \geq 0$ is such that $|\{w^M = t\}| = 0$. Since this latter is valid for $\mathcal{L}^1$–a.e. $t \geq 0$, we are in the conditions of applying our preceding parametric semicontinuity. Via Lemma 4.6, we deduce validity of the corresponding small–volume ICs for the extended measures $\overline{\mu_\pm}$ on $\mathbb{R}^n$, therefore bringing in Theorem 4.5 we find

$$\liminf_{k\to\infty} \overline{\Phi}[w_k] \geq c_M + \int_0^\infty \Big[\mathrm{P}_\varphi\left(\{w^M > t\}\right) + \overline{\mu_+}\left(\{w^M > t\}^1\right) - \overline{\mu_-}\left(\{w^M > t\}^+\right)\Big]\,\mathrm{d}t = \overline{\Phi}[w^M] + c_M\,.$$

At last, we send $M \to \infty$ employing Lemma 4.11 and recalling that the sequence $(c_M)_M$ in Lemma 4.12 decreases to zero as M diverges. This verifies our claim. $\square$

Proposition 4.15 (semicontinuity of $\widetilde{\Phi}$ on non–negative functions with given value outside Ω)**.** *Under the same assumptions of Proposition 4.14, it also holds*

$$\liminf_{k\to\infty} \widetilde{\Phi}[w_k] \geq \widetilde{\Phi}[w]\,,$$

on sequences $(w_k)_k$ in $\mathrm{BV}(\mathbb{R}^n)$ such that $w_k \to w$ in $\mathrm{L}^1(\mathbb{R}^n)$ with w_k, $w \geq 0$ a.e. in Ω and $w_k = w = u_0$ a.e. in Ω^{c} for all k.

Proof. The thesis follows from Proposition 4.14 by exchanging the role of φ with $\widetilde{\varphi}$ and switching $\mu_\pm$ – note that the φ– and $\widetilde{\varphi}$–IC vary accordingly. □

We are finally in the position of proving semicontinuity of $\widehat{\Phi}$ by joining the partial results on $\overline{\Phi}$, $\widetilde{\Phi}$.

Proof of Theorem 4.8. Consider a sequence $(w_k)_k$ in $\mathrm{BV}(\Omega)$ such that $w_k \to w$ in $\mathrm{L}^1(\Omega)$ with limit function $w \in \mathrm{BV}(\Omega)$. Then the sequence of extended functions $\overline{w_k}^{u_0}$ is in $\mathrm{BV}(\mathbb{R}^n)$ and such that $\big(\overline{w_k}^{u_0}\big)_\pm \to \big(\overline{w}^{u_0}\big)_\pm$ in $\mathrm{L}^1(\mathbb{R}^n)$. In view of $\big(\overline{w_k}^{u_0}\big)_\pm$, $\big(\overline{w}^{u_0}\big)_\pm \geq 0$ and $\big(\overline{w_k}^{u_0}\big)_\pm = \big(\overline{w}^{u_0}\big)_\pm = (u_0)_\pm$ outside Ω, Propositions 4.14 and 4.15 yield

$$\liminf_{k\to\infty} \overline{\Phi}\big[\big(\overline{w_k}^{u_0}\big)_+\big] \geq \overline{\Phi}\big[\big(\overline{w}^{u_0}\big)_+\big] \qquad \text{and} \qquad \liminf_{k\to\infty} \widetilde{\Phi}\big[\big(\overline{w_k}^{u_0}\big)_-\big] \geq \widetilde{\Phi}\big[\big(\overline{w}^{u_0}\big)_-\big].$$

Rearranging the terms according to Remark 4.10, we arrive at

$$\liminf_{k\to\infty} \widehat{\Phi}[w_k] \geq \widehat{\Phi}[w]\,,$$

which is the claimed lower semicontinuity of $\widehat{\Phi}$. □

Chapter 5

Minimization of linear-growth functionals with measure terms

We now want to prove Result 2, that means investigating the existence of minimizers for the linear–growth functional with measures

$$\mathcal{F}[w] := f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega}) - \int_\Omega w^+ \,\mathrm{d}\mu_- + \int_\Omega w^- \,\mathrm{d}\mu_+ \qquad \text{for all } w \in \mathrm{BV}(\Omega)\,,$$

defined under Assumptions 1.1 for the integrand f and for suitable admissible measures $\mu_\pm$ on Lipschitz Ω. We recall that the leading term $w \mapsto f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega})$ is set as functional on the measure $\mathrm{D}\overline{w}^{u_0}$ according to (1.1.2) and in terms of the recession function f^∞ of f and according to Section 2.6. As already mentioned in Chapter 1, our main goal is to prove coercivity and lower semicontinuity of $\mathcal{F}$ (that is, Result 1) to apply the direct method.

5.1 Coercivity

Next we shall see that an isoperimetric condition of the type introduced in Chapter 3 for the pairs of measures μ_+, μ_- in Ω is necessary and sufficient to gain coercivity of the non–parametric functional $\mathcal{F}$, provided the constant C strictly smaller than one (with necessary condition valid up to the limit case $C = 1$). This outcome is in line with our expectations, as it matches the classical results for linear–growth functionals with a measure term of the type $H\mathcal{L}^n$ for H integrable in Ω.

We begin by showing that, in the absence of the joint f^∞–IC and $\widetilde{f^\infty}$–IC with constant 1, one loses not only coercivity, but even boundedness from below of the functional. Consequently, the existence of BV–minimizers for $\mathcal{F}$ is completely ruled out.

Proposition 5.1 (necessity of limit IC for coercivity)**.** *We consider $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, $\mu_\pm$ admissible measures on Ω and a Borel integrand $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ with linear growth and convex restriction $\xi \mapsto f(x,\xi)$ for all $x \in \overline{\Omega}$. If $\mathcal{F}$ is bounded from below on $\mathrm{BV}(\Omega)$, then (μ_-,μ_+) satisfies the f^∞–IC in Ω with constant 1 and (μ_+,μ_-) satisfies the $\widetilde{f^\infty}$–IC in Ω with constant 1.*

Proof. We prove the thesis by passing to its contra–positive formulation. Suppose first that $\mathcal{F}$ is bounded from below whereas the f^∞–IC in Ω for (μ_-,μ_+) is violated. We can thus find some measurable set $A \Subset \Omega$ such that

$$\mathrm{P}_{f^\infty}(A) < \mu_-(A^+) - \mu_+(A^1)\,.$$

Then, the sequence of elements $u_k := k\mathbb{1}_A \in \mathrm{BV}(\Omega)$ for $k \in \mathbb{N}$ is such that each u_k has zero boundary trace on $\partial\Omega$, and it admits as derivative the decomposition $\mathrm{D}u_k = \nabla u_k + \mathrm{D}^s u_k = k\mathrm{D}\mathbb{1}_A$. For any given

k, we estimate

$$\begin{aligned}\mathcal{F}[u_k] &= \int_\Omega f\,(.,0)\;\mathrm{d}x + k\int_\Omega f^\infty\left(.,\frac{\mathrm{dD}\mathbb{1}_A}{\mathrm{d}|\mathrm{D}\mathbb{1}_A|}\right)\mathrm{d}|\mathrm{D}\mathbb{1}_A| + \int_{\partial\Omega} f^\infty(\,\cdot\,,-u_0\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\quad - \int_\Omega k(\mathbb{1}_A)^+\,\mathrm{d}\mu_- + \int_\Omega k(\mathbb{1}_A)^-\,\mathrm{d}\mu_+\\ &= \int_\Omega f(\,\cdot\,,0)\;\mathrm{d}x + k\mathrm{P}_{f^\infty}(A) + \int_{\partial\Omega}\widetilde{f^\infty}(\,\cdot\,,u_0\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1} - k\mu_-(A^+) + k\mu_+(A^1)\\ &\le k\left(\mathrm{P}_{f^\infty}(A) - \mu_-(A^+) + \mu_+(A^1)\right) + \beta|\Omega| + \beta||u_0||_{\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})}\,,\end{aligned}$$

with β being the upper bound in the linear growth of f. Therefore, sending $k\to\infty$ we obtain $\lim_{k\to\infty}\mathcal{F}[u_k]=-\infty$, against boundedness from below.

If instead it is the $\widetilde{f^\infty}$–IC in Ω for (μ_+,μ_-) to be violated for some (possibly different) $A\Subset\Omega$, it is the sequence $(-k\mathbb{1}_A)_k$ to determine a contradiction. □

We now check that the ICs above with constant *strictly smaller* than 1 guarantee coercivity.

Proposition 5.2 (sufficiency of ICs with $C<1$ for coercivity). *We take $u_0\in\mathrm{W}^{1,1}(\mathbb{R}^n)$, $\mu_\pm$ admissible measures on Ω, $f\colon\mathbb{R}^n\times\mathbb{R}^n\to[0,\infty)$ Borel with linear growth, $f(x,.)$ convex in $\mathbb{R}^n$ for all $x\in\overline{\Omega}$ and f^∞ continuous. We additionally assume that at least one of the following conditions is satisfied:*

(1) *There is a constant $M\in\mathbb{R}$ such that $f(x,\xi)\ge f^\infty(x,\xi)-M$ holds for all $(x,\xi)\in\Omega\times\mathbb{R}^n$;*

(2) *The integrand f is continuous on $\mathbb{R}^n\times\mathbb{R}^n$.*

Fix a constant $C\in[0,1)$. If (μ_-,μ_+) satisfies the f^∞–IC in Ω with constant C and (μ_+,μ_-) satisfies the $\widetilde{f^\infty}$–IC on Ω with constant C, then the functional $\mathcal{F}$ is coercive on $\mathrm{BV}(\Omega)$. If instead the ICs hold for C exactly 1 and the assumption (1) is in place, then $\mathcal{F}$ is bounded from below – but not necessarily BV–coercive.

Proof. For $C<1$, we claim coercivity in the sense

$$\mathcal{F}[w]\ge\Gamma\|w\|_{\mathrm{BV}(\Omega)}-L\qquad\text{for all }w\in\mathrm{BV}(\Omega)\,,\ \text{for some }\Gamma\in\mathbb{R}^+,\ L\in\mathbb{R}\,.$$

For any $w\in\mathrm{BV}(\Omega)$, Proposition 3.12 with the continuous anisotropy f^∞ in place of φ and the assumed isoperimetric conditions yield

$$\int_\Omega w^+\;\mathrm{d}\mu_- - \int_\Omega w^-\;\mathrm{d}\mu_+ \le C\left(|\mathrm{D}w|_{f^\infty}(\Omega)+\int_{\partial\Omega} f^\infty(\,\cdot\,,w\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1}\right). \tag{5.1.1}$$

Supposed first that (1) is in place, we employ (5.1.1) together with the and convexity of $f^\infty(x.,)$ from Proposition 2.59(i) to obtain

$$\begin{aligned}\mathcal{F}[w] &\ge |\mathrm{D}w|_{f^\infty}(\Omega) - M|\Omega| + \int_{\partial\Omega} f^\infty\left(.,(w-u_0)\nu_\Omega\right)\;\mathrm{d}\mathcal{H}^{n-1} - \int_\Omega w^+\,\mathrm{d}\mu_- + \int_\Omega w^-\;\mathrm{d}\mu_+\\ &\ge (1-C)\left(|\mathrm{D}w|_{f^\infty}(\Omega)+\int_{\partial\Omega} f^\infty\left(.,w\nu_\Omega\right)\mathrm{d}\mathcal{H}^{n-1}\right) - M|\Omega| - \int_{\partial\Omega} f^\infty\left(.,u_0\nu_\Omega\right)\;\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

Making use of the growth condition with positive bounds α, β, via (2.6.9) a fortiori it holds

$$\begin{aligned}\mathcal{F}[w] &\ge \alpha(1-C)\left(|\mathrm{D}w|(\Omega)+\int_{\partial\Omega}|w|\,\mathrm{d}\mathcal{H}^{n-1}\right) - M|\Omega| - \beta||u_0||_{\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})}\\ &\ge \alpha(1-C)\|w\|_{\mathrm{BV}(\Omega)}/\widetilde{\gamma}_n - M|\Omega| - \beta||u_0||_{\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})}\end{aligned}$$

by an application of Poincaré's inequality (2.2.4). This completes our coercivity claim whenever C is smaller than one. Inserting $C = 1$ in the last estimate, we verify the existence of a lower bound for $\mathcal{F}$.

Alternatively, if (2) applies, we claim that

$$f(x,\xi) \geq C_* f^\infty(x,\xi) - M \quad \text{for all } (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \tag{5.1.2}$$

for every $C_* \in (C,1)$ and some corresponding $M \in \mathbb{R}$, and then coercivity would follow from the same reasoning as above. In fact, continuity of both f and f^∞ is inherited by the perspective function $\overline{f}$ (see Lemma 2.59(ii)), and specifically for any given $x \in \mathbb{R}^n$ the function $(t,\xi) \mapsto \overline{f}(x,t,\xi)$ is uniformly continuous in the compact set $K := \{(t,\xi) \in [0,\infty) \times \mathbb{R}^n : t + |\xi| \leq 1\}$. We can thus determine an increasing function $\omega\colon [0,\infty) \to [0,\infty)$ with $\omega(0) = 0 = \lim_{t\to 0^+} \omega(t)$ such that

$$\left| f\left(x, \frac{\xi}{t}\right) - f^\infty\left(x, \frac{\xi}{t}\right)\right| = \frac{\left|\overline{f}(x,t,\xi) - \overline{f}(x,0,\xi)\right|}{t} \leq \frac{\omega(t)}{t} \quad \text{for all } x \in \mathbb{R}^n\,, (t,\xi) \in K \text{ with } t \neq 0\,.$$

For any $z \in \mathbb{R}^n$, we define the vector $\tau_z := (1,z)/(1+|z|)$ and observe $\tau_z \in K$. We infer

$$|f(x,z) - f^\infty(x,z)| \leq \widetilde{\omega}(|z|)(1+|z|) \quad \text{for all } (x,z) \in \mathbb{R}^n \times \mathbb{R}^n \tag{5.1.3}$$

for a radial function $\widetilde{\omega}$ decreasing to zero as $|z| \to \infty$. Let now $R \gg 0$ be such that for some $c \in (0,(1-C)\alpha)$ it is $\widetilde{\omega}(|z|) \leq c$ for all $|z| > R$. From (5.1.3) and the linear growth assumption, we read out

$$f^\infty(x,z) - f(x,z) \leq c + c|z| \leq c + cf^\infty(x,z)/\alpha \quad \text{for all } (x,z) \in \mathbb{R}^n \times \overline{\mathrm{B}_R}^{\mathrm{c}}\,,$$

so that rearranging the terms and setting $C_* := 1 - c/\alpha \in (C,1)$ we obtain $f(x,z) \geq C_* f^\infty(x,z) - c$ for all $(x,z) \in \mathbb{R}^n \times \overline{\mathrm{B}_R}^{\mathrm{c}}$. If instead $|z| \leq R$, we directly compute $f(x,z) - C_* f^\infty(x,z) \geq -\beta C_*|z| \geq -\beta C_* R$. Altogether, letting for instance $M := \max\{c, \beta C_* R\}$, we have verified the validity of (5.1.2). Analogously to the case (1), we can exploit the ICs and (5.1.2) to achieve

$$\mathcal{F}[w] \geq (C_* - C)\left(|\mathrm{D}w|_{f^\infty}(\Omega) + \int_{\partial\Omega} f^\infty\left(.,w\nu_\Omega\right) \mathrm{d}\mathcal{H}^{n-1}\right) - M|\Omega| - \int_{\partial\Omega} f^\infty\left(.,u_0\nu_\Omega\right) \mathrm{d}\mathcal{H}^{n-1}$$

for all $w \in \mathrm{BV}(\Omega)$, and coercivity in BV is once again a direct consequence of Poincaré's inequality. □

Observe that to enable boundedness of $\mathcal{F}$ from below in the limit case of the ICs we cannot be dispensed from condition (1), as illustrated in the following counterexample.

Example 5.3. In the Euclidean plane, we set $\Omega := \mathrm{B}_2 \subseteq \mathbb{R}^2$, the integrand $f(\xi) := |\xi| + 1 - \sqrt{|\xi|+1}$ and measures $\mu_+ :\equiv 0$, $\mu_- := H\mathcal{L}^2 \llcorner \Omega$ with $H(x) = |x|^{-1}$. Then we have $f^\infty(\xi) = |\xi|$, and clearly $f(\xi) - |\xi| = 1 - \sqrt{|\xi|+1} \to -\infty$ as $|\xi| \to \infty$. We record that in [51, Section 5] we proved that μ_- satisfies (exactly) the limit IC. Setting the sequence of stretched truncations $v_k := k^2 \max\{\min\{w_k,1\},0\}$ of the functions $w_k(x) := 1 - k(|x|-1)$, it is $v_k \in \mathrm{BV}(\Omega)$ with $\mathrm{supp}(v_k) = \overline{\mathrm{B}_{1+1/k}}$, and one may compute $|\mathrm{D}v_k|(\Omega) = \pi(2k^2+k) = \int_\Omega v_k \,\mathrm{d}\mu_-$. Therefore, selecting the zero boundary datum $u_0 :\equiv 0$, the corresponding functional achieves $\mathcal{F}[v_k] = \int_\Omega \left[1 - \sqrt{|\nabla v_k|+1}\right] \mathrm{d}x = \pi\left[1 - \sqrt{k^3+1}\right]\left(\frac{2}{k} + \frac{1}{k^2}\right)$ for all k. It is then $\lim_{k\to\infty} \mathcal{F}[v_k] = -\infty$, and since f, $\mu_\pm$ fulfil all the remaining assumptions in Proposition 5.2 but (1) – included the alternative (2) – this confirms the necessity of (1) to boundedness of $\mathcal{F}$ from below.

5.2 Lower semicontinuity

5.2.1 Functional reformulation with extra variable

The reasoning to prove the LSC Result 1 in the general case of integrands f *not* 1–homogeneous is based on Theorem 4.8 and on a suitable reformulation of the problem. To follow the reasoning behind this latter, consider for instance the area functional $f(z) := \sqrt{1+|z|^2}$ for $z \in \mathbb{R}^n$, with the recession function given by the norm $f^\infty(z) = |z|$ and perspective function $\overline{f}(t,z) = |(t,z)|$ for $(t,z) \in [0,\infty) \times \mathbb{R}^n$. We observe that the (possibly even infinite) area of the surface graph determined by the differentiable $w\colon U \to \mathbb{R}$ on open $U \subseteq \mathbb{R}^n$ corresponds to the total variation of the function $w_\Diamond$ defined as $w_\Diamond(x_0,x) := x_0 + w(x)$ for $x_0 \in (0,1)$ and $x \in U$, that is

$$\mathcal{A}(w,U) = \int_U \sqrt{1+|\nabla w|^2}\,\mathrm{d}x = \int_{(0,1)\times U} |\nabla w_\Diamond| = \mathrm{TV}(w_\Diamond,(0,1)\times U)\,.$$

The same principle applies to arbitrary functionals with linear growth and passing to BV functions: Denoting by $U_\Diamond$ the generalized cylinder in $\mathbb{R}\times\mathbb{R}^n$ given by $(0,1)\times U$, we claim that the following holds

$$f(.,\mathrm{D}w)\big(U\big) = |\mathrm{D}w_\Diamond|_{\overline{f}}\big(U_\Diamond\big) \qquad \text{for all } w \in \mathrm{BV}(U)\,,$$

where the latter integral is well–posed on $w_\Diamond$, since $\overline{f}$ is positively 1–homogeneous on $(t,z) \in [0,\infty)\times\mathbb{R}^n$. Actually, the right–hand side does not involve exactly $\overline{f}$ but the auxiliary anisotropy φ generated from f as in Definition 5.4. The rigorous statement of the formula above will be made explicit in the later Proposition 5.9 directly for Lipschitz domains.

In a similar way, for admissible measures $\mu_\pm$ on U we extend the measure integrals $\int_U w^\pm\,\mathrm{d}\mu_\mp$ to appropriate integrals on $\mathbb{R}^{n+1}$ of the lifted function $w_\Diamond$ and corresponding upper/lower approximate limits, computed with respect to the product measures $\mu_{\Diamond\pm} := \mathcal{L}^1 \llcorner (0,1) \otimes \mu_\pm$. Nevertheless, to apply the definition of anisotropic variation on arbitrary open sets $\mathcal{U}$ of $\mathbb{R}^{n+1}$ and arbitrary BV functions on $\mathcal{U}$, we need our anisotropy to be defined on the full space $\mathbb{R}^{n+1}$. To overcome the issue, we introduce a function φ which essentially corresponds to the even extension of $\overline{f}$ to the half–space $(-\infty,0)\times\mathbb{R}^n$; see Definition 5.4 below. With this at hand, we are finally able to transfer properties holding for the homogeneous functional $\widehat{\Phi}$ (this time, on $\mathbb{R}^{n+1}$) associated to the general case of $\mathcal{F}$ – including, for instance, semicontinuity.

Definition 5.4. Assume $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ is Borel with linear growth, and that $f(x,.)$ is convex in $\mathbb{R}^n$ for all $x \in \mathbb{R}^n$. We introduce the auxiliary function $\varphi\colon (\mathbb{R}\times\mathbb{R}^n)\times(\mathbb{R}\times\mathbb{R}^n) \to [0,\infty)$ defined as

$$\varphi(x_0,x,\xi_0,\xi) := \overline{f}(x,|\xi_0|,\xi) \qquad \text{for all } (x_0,x),(\xi_0,\xi) \in \mathbb{R}\times\mathbb{R}^n\,.$$

Notice that φ is in fact independent of the value of $x_0 \in \mathbb{R}$, and such an extra variable is set as a placeholder to ensure that φ is well–defined on the whole $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ – in accordance with the theory of anisotropies in dimension $n+1$. We now list some useful properties:

- It is $\varphi(x_0,x,\pm 1,\xi) = f(x,\xi)$ for all $x_0 \in \mathbb{R}$ and $x,\xi \in \mathbb{R}^n$, specifically $\varphi(x_0,x,\pm1,0) = f(x,0)$;
- $\varphi(x_0,x,\pm1,-\xi) = f(x,-\xi)$;
- $\varphi(x_0,x,0,\xi) = f^\infty(x,\xi)$ and $\varphi(x_0,x,0,-\xi) = \widetilde{f^\infty}(x,\xi)$;
- The mapping φ is constant in the variable ξ_0 whenever f is positively 1–homogeneous in ξ, and in such a case $\varphi(x_0,x,\xi_0,\xi) = f(x,\xi) = f^\infty(x,\xi)$;
- If f and f^∞ are continuous, from Proposition 2.59 we deduce continuity of φ on the full space.

Remark 5.5. It is plain to see that the conclusions of Results 1 and 2 are not affected by adding constants to the integrand f in the functional $\mathcal{F}$. For this reason, in this section, we find it convenient to work with the following adjusted assumptions:

(LIN′) There exist $0 \leq \alpha \leq \beta < \infty$ such that $\alpha\sqrt{1+|\xi|^2} \leq f(x,\xi) \leq \beta\sqrt{1+|\xi|^2}$ for all (x,ξ) in $\mathbb{R}^n \times \mathbb{R}^n$.

(H2′) It is $f(x,\xi) \geq f^\infty(x,\xi)$ for all x, $\xi \in \mathbb{R}^n$.

Under the new assumptions (LIN′)–(H2′), it is easily verified that the function φ represents an anisotropy in $\mathbb{R}^{n+1}$.

Proposition 5.6. *Let f, φ be as in Definition 5.4 and such that (LIN′) applies. Then φ is an anisotropy on $\mathbb{R}^{n+1}$ in the joint variable (ξ_0,ξ), φ comparable to the norm in $\mathbb{R}^{n+1}$ with*

$$\alpha|(\xi_0,\xi)| \leq \varphi(x_0,x,\xi_0,\xi) \leq \sqrt{2}\beta|(\xi_0,\xi)| \quad \textit{for all } (x_0,x),(\xi_0,\xi) \in \mathbb{R}\times\mathbb{R}^n\,. \tag{5.2.1}$$

Moreover, if f also satisfies (H2′), then $(\xi_0,\xi) \mapsto \varphi(x_0,x,\xi_0,\xi)$ is convex with

$$\varphi(x_0,x,\xi_0,\xi) \geq f^\infty(x,\xi) \quad \textit{for all } (x_0,x),(\xi_0,\xi) \in \mathbb{R}\times\mathbb{R}^n\,.$$

Proof. Positive homogeneity of φ in (ξ_0,ξ) is a direct consequence of Proposition 2.59(ii). We want to show that (5.2.1) holds. In fact, for $\xi_0 = 0$ it is evidently $\varphi(x_0,x,0,\xi) = f^\infty(x,\xi) \leq \beta|\xi| \leq \sqrt{2}\beta|(0,\xi)|$, and same for the lower bound $\varphi(x_0,x,0,\xi) = f^\infty(x,\xi) \geq \alpha|\xi| = \alpha|(0,\xi)|$. For $\xi_0 \neq 0$, again the linear–growth assumption on f yields $\varphi(x_0,x,\xi_0,\xi) = |\xi_0|\cdot f\left(x,\frac{\xi}{|\xi_0|}\right) \leq \beta(|\xi_0|+|\xi|) \leq \sqrt{2}\beta|(\xi_0,\xi)|$, and at the same time, $\varphi(x_0,x,\xi_0,\xi) \geq \alpha(|\xi_0|+|\xi|) \geq \alpha|(\xi_0,\xi)|$, therefore (5.2.1) is verified.

We assume now (H2′) and claim convexity of φ with respect to the last two entries. Trivially, if $f = f^\infty$ everywhere, then $\varphi(x_0,x,\xi_0,\xi) = f(x,\xi)$ and convexity follows. We then assume that f is not 1–homogeneous. Proposition 2.59(ii) guarantees that $\varphi(x_0,x,.,.) = \overline{f}(x,.,.)$ is convex on the half–space $[0,\infty)\times\mathbb{R}^n$, and by evenness of φ in ξ_0, $\varphi(x_0,x,.,.)$ is convex on $(-\infty,0)\times\mathbb{R}^n$ as well. We fix now $(x_0,x) \in \mathbb{R}\times\mathbb{R}^n$ and we assume that $\xi_0, \widehat{\xi_0} \in \mathbb{R}\setminus\{0\}$ are such that $\operatorname{sgn}(\xi_0) = -\operatorname{sgn}(\widehat{\xi_0})$. We suppose by contradiction the existence of some $\overline{\lambda} \in (0,1)$ and some vectors ξ, $\widehat{\xi} \in \mathbb{R}^n$ such that

$$\varphi(x_0,x,\overline{\xi_0},\overline{\xi}) > \overline{\lambda}\varphi(x_0,x,\xi_0,\xi) + (1-\overline{\lambda})\varphi(x_0,x,\widehat{\xi_0},\widehat{\xi})\,, \tag{5.2.2}$$

where we set the convex combinations $\overline{\xi_0} := \overline{\lambda}\xi_0 + (1-\overline{\lambda})\widehat{\xi_0}$ and $\overline{\xi} := \overline{\lambda}\xi + (1-\overline{\lambda})\widehat{\xi}$, and without loss of generality we may assume $\overline{\xi_0} = 0$. We compute employing the hypothesis (H2′) and convexity of $f^\infty(x,.)$:

$$\begin{aligned} f^\infty(x,\overline{\xi}) = \varphi(x_0,x,0,\overline{\xi}) &> \overline{\lambda}\,\overline{f}(x,|\xi_0|,\xi) + (1-\overline{\lambda})\overline{f}(x,|\widehat{\xi_0}|,\widehat{\xi}) \\ &= \overline{\lambda}|\xi_0| f\left(x,\frac{\xi}{|\xi_0|}\right) + (1-\overline{\lambda})|\widehat{\xi_0}| f\left(x,\frac{\widehat{\xi}}{|\widehat{\xi_0}|}\right) \\ &\geq \overline{\lambda} f^\infty(x,\xi) + (1-\overline{\lambda}) f^\infty(x,\widehat{\xi}) \\ &\geq f^\infty(x,\overline{\xi})\,, \end{aligned}$$

which is absurd. We conclude that there is no convex combination such that (5.2.2) holds true, namely $(\xi_0,\xi) \mapsto \varphi(x_0,x,\xi_0,\xi)$ is convex on $\mathbb{R}\times\mathbb{R}^n$. Finally, convexity of φ in the last two variables induces

$$\begin{aligned} f^\infty(x,\xi) = \varphi(x_0,x,0,\xi) = \varphi\left(x_0,x,\frac{(\xi_0,\xi)}{2}+\frac{(-\xi_0,\xi)}{2}\right) &\leq \frac{\varphi(x_0,x,\xi_0,\xi)+\varphi(x_0,x,-\xi_0,\xi)}{2} \\ &= \varphi(x_0,x,\xi_0,\xi) \end{aligned}$$

exploiting the evenness of $\varphi(x_0,x,.,\xi)$, and the claimed inequality is demonstrated. $\square$

We fix the following notation.

Definition 5.7 (extended functions and measures)**.** We consider an open bounded Lipschitz set Ω in $\mathbb{R}^n$ and the generalized cylinder $\Omega_\Diamond := (0,1)\times\Omega$ in $\mathbb{R}^{n+1}$. We associate to any $w\colon \Omega \to \overline{\mathbb{R}}$ the function $w_\Diamond\colon \Omega_\Diamond \to \overline{\mathbb{R}}$ defined as

$$w_\Diamond(x_0,x) := x_0 + w(x) \qquad \text{for all } x_0 \in (0,1),\ x\in\Omega\,.$$

Moreover, in correspondence to any measure ν on Ω, we introduce the following

$$\nu_\Diamond := \mathcal{L}^1 \llcorner (0,1)\otimes \nu\,.$$

Assuming now that $w \in \mathrm{BV}(\Omega)$, it is easy to check that $w_\Diamond \in \mathrm{BV}(\Omega_\Diamond)$ with $(n+1)$–valued derivative measure $\mathrm{D}w_\Diamond = \mathcal{L}^1 \otimes (\mathcal{L}^n, \mathrm{D}w)$ on $\Omega_\Diamond$. We record that since ν is a finite measure vanishing on $\mathcal{H}^{n-1}$–negligible sets, an application of Fubini's theorem yields

$$\int_{\Omega_\Diamond} (w_\Diamond)^\pm \,\mathrm{d}\nu_\Diamond = \int_\Omega \left(\int_0^1 x_0\,\mathrm{d}x_0\right)\mathrm{d}\nu + \int_\Omega w^\pm\,\mathrm{d}\nu = \frac{\nu(\Omega)}{2} + \int_\Omega w^\pm\,\mathrm{d}\nu \qquad \text{for all } w\in\mathrm{BV}(\Omega)\,. \tag{5.2.3}$$

Notice that if $\nu\in\mathrm{RM}(\Omega)$, then $\nu_\Diamond$ belongs to $\mathrm{RM}(\Omega_\Diamond)$.

Remark 5.8. Assume that $(\nu_k)_k$ is a sequence of of finite non–negative Radon measures on Ω such that $\nu_k \overset{*}{\rightharpoonup} \nu$ weakly–$*$ in $\mathrm{RM}(\Omega)$ as $k\to\infty$ to some $\nu\in\mathrm{RM}(\Omega)$. Then the sequence $(\nu_{k\Diamond})_k$ converges to $\nu_\Diamond$ weakly–$*$ in $\mathrm{RM}(\Omega_\Diamond)$ for $k\to\infty$. If the convergence of $(\nu_k)_k$ is strict, then the extensions $(\nu_{k\Diamond})_k$ converge even strictly in $\mathrm{RM}(\Omega_\Diamond)$.

Proof. We let $\psi\in \mathrm{C}_0(\Omega_\Diamond)$ and employ the theorem of Fubini to decompose

$$\int_{\Omega_\Diamond} \psi\,\mathrm{d}\nu_\Diamond = \int_0^1 \left(\int_\Omega \psi(x_0,x)\,\mathrm{d}\nu(x)\right)\mathrm{d}x_0\,, \tag{5.2.4}$$

observing that the restriction $\Omega\ni x\mapsto \psi(x_0,x)$ is in $\mathrm{C}_0(\Omega)$ for all fixed $x_0\in(0,1)$ – and clearly an analogous decomposition holds for each measure $\nu_{k\Diamond}$. Then, weak–$*$ convergence of the original measures induces by duality the bound $\sup_k ||\nu_k||_{[\mathrm{C}_0(\Omega)]^*} = \sup_k ||\nu_k||_{\mathrm{RM}(\Omega)} = \sup_{k\in\mathbb{N}} |\nu_k|(\Omega) =: c < \infty$ (compare with the proof of Proposition 2.21), and at the same time we also find the pointwise convergence

$$f_k(x_0) := \int_\Omega \psi(x_0,.)\,\mathrm{d}\nu_k \xrightarrow[k\to\infty]{} \int_\Omega \psi(x_0,.)\,\mathrm{d}\nu =: f(x_0) \qquad \text{for all } x_0\in(0,1)\,.$$

Moreover, we observe that the control $|f_k(x_0)| \le ||\psi(x_0,.)||_{\mathrm{L}^\infty(\Omega)}\cdot \nu_k(\Omega) \le c||\psi(x_0,.)||_{\mathrm{L}^\infty(\Omega)} =: g(x_0)$ holds uniformly in k, with $\int_0^1 g(x_0)\,\mathrm{d}x_0 \le c||\psi||_{\mathrm{L}^\infty(\Omega_\Diamond)} < \infty$. We are thus in the conditions of applying dominated convergence theorem, so that taking into account (5.2.4) the claimed weak–$*$ convergence $\nu_{k\Diamond} \overset{*}{\rightharpoonup} \nu_\Diamond$ follows.

If we further assume that $\nu_k \rightharpoonup \nu$ strictly, by definition of extensions it is precisely

$$\nu_{k\Diamond}(\Omega_\Diamond) = \mathcal{L}^1((0,1))\cdot\nu_k(\Omega) \xrightarrow[k\to\infty]{} \nu(\Omega) = \nu_\Diamond(\Omega_\Diamond)\,,$$

which yields strict convergence in $\mathrm{RM}(\Omega_\Diamond)$. ☐

We turn now to the claimed rewriting of the functional $f\big(\mathrm{D}\overline{w}^{u_0}\big)$ in terms of our anisotropy φ.

Proposition 5.9. *For a Borel integrand $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ convex in the second entry, we assume (LIN′) and (H2′). Then, for any $w \in \mathrm{BV}(\Omega)$ and any $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ we have*

$$|\mathrm{D}w_\Diamond|_\varphi(\Omega_\Diamond) = f(.,\mathrm{D}w)(\Omega) \quad \textit{and} \tag{5.2.5}$$

$$\int_{\partial\Omega_\Diamond} \varphi\left(.,(w_\Diamond - u_{0\Diamond})\nu_{\Omega_\Diamond}\right) \mathrm{d}\mathcal{H}^n = \int_{\partial\Omega} f^\infty(.,(w-u_0)\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1} + 2\int_\Omega f(.,0)|w-u_0|\,\mathrm{d}x\,, \tag{5.2.6}$$

where the notation $w_\Diamond - u_{0\Diamond}$ refers to the boundary traces on $\partial\Omega_\Diamond$.

Proof. To verify (5.2.5), we consider the decomposition of the derivative measure $\mathrm{D}w_\Diamond = \nabla w_\Diamond \mathcal{L}^{n+1} + \mathrm{D}^s w_\Diamond = (\mathcal{L}^1 \otimes \mathcal{L}^n, \mathcal{L}^1 \otimes \nabla w \mathcal{L}^n) + (0, \mathcal{L}^1 \otimes \mathrm{D}^s w)$ of $w_\Diamond$ on $\Omega_\Diamond$. Recalling that $\varphi(x_0,x,0,\xi) = f^\infty(x,\xi)$ for all x, $\xi \in \mathbb{R}^n$ and all $x_0 \in (0,1)$, we write

$$\begin{aligned}
\int_\Omega f^\infty\left(.,\frac{\mathrm{d}\mathrm{D}^s w}{\mathrm{d}|\mathrm{D}^s w|}\right)\mathrm{d}|\mathrm{D}^s w| &= \mathcal{L}^1\left((0,1)\right)\cdot\int_\Omega \varphi\left(1,x,0,\frac{\mathrm{d}\mathrm{D}^s w}{\mathrm{d}|\mathrm{D}^s w|}(x)\right)\mathrm{d}|\mathrm{D}^s w|(x)\\
&= \int_{(0,1)\times\Omega} \varphi\left((x_0,x),\frac{\mathrm{d}\mathrm{D}^s w_\Diamond}{\mathrm{d}|\mathrm{D}^s w_\Diamond|}(x_0,x)\right)\mathrm{d}|\mathrm{D}^s w_\Diamond|(x_0,x)\\
&= \int_{\Omega_\Diamond} \varphi\left(\cdot,\frac{\mathrm{d}\mathrm{D}^s w_\Diamond}{\mathrm{d}|\mathrm{D}^s w_\Diamond|}\right)\mathrm{d}|\mathrm{D}^s w_\Diamond|\,.
\end{aligned} \tag{5.2.7}$$

For the absolutely continuous term instead, we apply $\varphi(1,x,1,\xi) = f(x,\xi)$ for all x, $\xi \in \mathbb{R}^n$ and Fubini's theorem to get

$$\begin{aligned}
\int_\Omega f\left(.,\nabla w\right)\,\mathrm{d}x &= \mathcal{L}^1\left((0,1)\right)\cdot\int_\Omega \varphi\left(1,x,1,\nabla w(x)\right)\mathrm{d}x\\
&= \int_{(0,1)\times\Omega} \varphi\left((x_0,x),\nabla w_\Diamond(x_0,x)\right)\,\mathrm{d}(\mathcal{L}^1\otimes\mathcal{L}^n)(x_0,x)\\
&= \int_{\Omega_\Diamond} \varphi\left(\cdot,\nabla w_\Diamond\right)\,\mathrm{d}\mathcal{L}^{n+1}\,.
\end{aligned} \tag{5.2.8}$$

Then, (5.2.5) is obtained summing up (5.2.7) and (5.2.8) as follows:

$$\begin{aligned}
|\mathrm{D}w_\Diamond|_\varphi(\Omega_\Diamond) = \int_{\Omega_\Diamond} \varphi\left(\cdot,\frac{\mathrm{d}\mathrm{D}w_\Diamond}{\mathrm{d}|\mathrm{D}w_\Diamond|}\right)\mathrm{d}|\mathrm{D}w_\Diamond| &= \int_{\Omega_\Diamond} \varphi\left(\cdot,\nabla w_\Diamond\right)\,\mathrm{d}\mathcal{L}^{n+1} + \int_{\Omega_\Diamond} \varphi\left(\cdot,\frac{\mathrm{d}\mathrm{D}^s w_\Diamond}{\mathrm{d}|\mathrm{D}^s w_\Diamond|}\right)\mathrm{d}|\mathrm{D}^s w_\Diamond|\\
&= \int_\Omega f\left(.,\nabla w\right)\,\mathrm{d}\mathcal{L}^n + \int_\Omega f^\infty\left(.,\frac{\mathrm{d}\mathrm{D}^s w}{\mathrm{d}|\mathrm{D}^s w|}\right)\mathrm{d}|\mathrm{D}^s w|\\
&= f(.,\mathrm{D}w)(\Omega)\,.
\end{aligned}$$

Consider now the boundary term and decompose $\partial\Omega_\Diamond = ([0,1]\times\partial\Omega) \cup (\{0,1\}\times\Omega)$ in $\mathbb{R}\times\mathbb{R}^n$, with inward normal vector $\nu_{\Omega_\Diamond}$ to $\Omega_\Diamond$ satisfying

$$\nu_{\Omega_\Diamond}(x_0,x) = (0,\nu_\Omega(x))\mathbb{1}_{(0,1)\times\partial\Omega} + (1,0)\mathbb{1}_{\{0\}\times\Omega} + (-1,0)\mathbb{1}_{\{1\}\times\Omega}\,, \tag{5.2.9}$$

where for each vector the first component is real–valued and the second entry is $\mathbb{R}^n$–valued. Moreover, the difference of traces clearly attains $w_\Diamond(x_0,x) - u_{0\Diamond}(x_0,x) = w(x) - u_0(x)$ for $\mathcal{H}^n$–a.e. $(x_0,x) \in \partial\Omega_\Diamond$.

Altogether, being φ independent of the value of its first entry, we compute

$$\begin{aligned}
&\int_{\partial\Omega_\Diamond} \varphi(\,\cdot\,,(w_\Diamond - u_{0\Diamond})\nu_{\Omega_\Diamond})\,\mathrm{d}\mathcal{H}^n \\
&= \int_{[0,1]\times\partial\Omega} \varphi(x_0,x,(w(x)-u_0(x))\cdot(0,\nu_\Omega(x)))\,\mathrm{d}\mathcal{H}^n(x_0,x) \\
&\quad + \int_{\{0\}\times\Omega} \varphi(x_0,x,(w(x)-u_0(x))\cdot(1,0))\,\mathrm{d}\mathcal{H}^n(x_0,x) \\
&\qquad + \int_{\{1\}\times\Omega} \varphi(x_0,x,(w(x)-u_0(x))\cdot(-1,0))\,\mathrm{d}\mathcal{H}^n(x_0,x) \\
&= \int_{[0,1]\times\partial\Omega} \varphi(x_0,x,0,(w(x)-u_0(x))\nu_\Omega(x))\,\mathrm{d}\mathcal{H}^n(x_0,x) \\
&\quad + \int_\Omega \varphi(0,x,w(x)-u_0(x),0)\,\mathrm{d}x + \int_\Omega \varphi(1,x,u_0(x)-w(x),0)\,\mathrm{d}x \\
&= \int_{[0,1]\times\partial\Omega} f^\infty(x,(w(x)-u_0(x))\nu_\Omega(x))\,\mathrm{d}\mathcal{H}^n(x_0,x) + 2\int_\Omega f(.,0)|w-u_0|\,\mathrm{d}x\,.
\end{aligned}$$

Bringing in Corollary 2.44 applied to the countably $\mathcal{H}^{n-1}$–rectifiable set $S=\partial\Omega$, it is

$$\int_{\partial\Omega_\Diamond} \varphi(\,\cdot\,,(w_\Diamond - u_{0\Diamond})\nu_{\Omega_\Diamond})\,\mathrm{d}\mathcal{H}^n = \mathcal{L}^1((0,1))\int_{\partial\Omega} f^\infty(.,(w-u_0)\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1} + 2\int_\Omega f(.,0)|w-u_0|\,\mathrm{d}x\,.$$

This returns precisely our claim (5.2.6) and completes the proof of the statement. □

5.2.2 Isoperimetric conditions with extra variable

Lemma 5.10. *For $\Omega\subseteq\mathbb{R}^n$ and $\Omega_\Diamond$ defined as above and given any $w\in\mathrm{BV}(\Omega_\Diamond)$, the restriction $w(x_0,.)$ is in $\mathrm{BV}(\Omega)$ for a.e. $x_0\in(0,1)$ and it holds*

$$w(x_0,.)^\pm(x) = w^\pm(x_0,x) \qquad \textit{for } \mathcal{H}^{n-1}\textit{–a.e. } x\in\Omega\,.$$

If f and φ are as Definition 5.4 with f, f^∞ continuous under (LIN′)–(H2′), we also have

$$\int_0^1 |\mathrm{D}_x w(x_0,.)|_{f^\infty}(\Omega)\,\mathrm{d}x_0 \le |\mathrm{D}w|_\varphi(\Omega_\Diamond) \qquad \textit{and} \tag{5.2.10}$$

$$\int_0^1 \left(\int_{\partial\Omega} f^\infty(.,w(x_0,.)\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1}\right)\mathrm{d}x_0 \le \int_{\partial\Omega_\Diamond} \varphi(.,w\nu_{\Omega_\Diamond})\,\mathrm{d}\mathcal{H}^n\,. \tag{5.2.11}$$

For a complete proof, we refer to [52, Lemma A.2]. In the following we just record that continuity of the integrand f is necessary to pass from the result of (5.2.10) and (5.2.11) on Sobolev functions to BV, exploiting the strict approximation result of Theorem 2.103 and Reshetnyak's continuity Theorem 2.79 for our auxiliary φ. We now verify that the admissibility of a measure ν on Ω is inherited by the extended measure $\nu_\Diamond$ on the domain $\Omega_\Diamond$.

Lemma 5.11. *If ν is an admissible measure on Ω, then $\nu_\Diamond$ is still an admissible measure on $\Omega_\Diamond$ and vice versa.*

Proof. We assume first that ν is admissible on Ω. Exploiting the characterization in Proposition 3.5, we claim both

$$\nu_\Diamond(Z)=0 \qquad \text{for every } \mathcal{H}^n\text{–negligible Borel set } Z\subseteq\Omega_\Diamond \qquad \text{and} \tag{5.2.12}$$

$$\left| \int_{\Omega_\Diamond} w^+ \,\mathrm{d}\nu_\Diamond \right| < \infty \qquad \text{for all } w \in \mathrm{BV}(\Omega_\Diamond)\,. \tag{5.2.13}$$

Then, considering a Borel $Z \subseteq \Omega_\Diamond$ with $\mathcal{H}^n(Z) = 0$ and employing Lemma 2.45, we assert that almost every slice ${}_{x_0}Z := \{x \in \Omega : (x_0, x) \in Z\}$ is Borel in Ω and satisfies $\mathcal{H}^{n-1}({}_{x_0}Z) = 0$. The admissibility of ν on Ω implies $\nu({}_{x_0}Z) = 0$ for a.e. $x_0 \in (0,1)$. From Fubini's theorem for $\nu_\Diamond$, we find

$$\nu_\Diamond(Z) = (\mathcal{L}^1 \llcorner (0,1) \otimes \nu)(Z) = \int_0^1 \nu({}_{x_0}Z)\,\mathrm{d}x_0 = 0\,,$$

from which we read (5.2.12).

To achieve (5.2.13), we observe that it suffices proving that for non–negative $w \in \mathrm{BV}(\Omega_\Diamond)$, and then extending to functions of any sign exploiting the ν–a.e. writing $w^+ = (w_+)^+ - (w_-)^-$ from Lemma 2.28 and via the previous (5.2.12). We recall now that from admissibility of ν, any $w \in \mathrm{BV}(\Omega)$ is such that $w^+ \in \mathrm{L}^1(\Omega;\nu)$. Therefore, Lemma 5.10 and the admissibility of ν guarantees that the integrals $\int_\Omega w(x_0,.)^+(x)\,\mathrm{d}\nu(x)$, $\int_\Omega w^+(x_0,x)\,\mathrm{d}\nu(x)$ agree, and that $w(x_0,.) \in \mathrm{BV}(\Omega)$ for a.e. $x_0 \in (0,1)$. Therefore, Theorem 3.11(2a) applied to $(\nu, 0)$ guarantees the existence of some $C \in [0,\infty)$ satisfying

$$\int_\Omega w^+(x_0,x)\,\mathrm{d}\nu(x) \le C \left(|\mathrm{D}_x w(x_0,.)|(\Omega) + \int_{\partial\Omega} |w(x_0,.)|\,\mathrm{d}\mathcal{H}^{n-1} \right) \qquad \text{for a.e. } \ x_0 \in (0,1)\,. \tag{5.2.14}$$

We now integrate (5.2.14) for $x_0 \in (0,1)$ using Fubini's theorem, (5.2.10) and (5.2.11) for the total variation to conclude

$$\int_{\Omega_\Diamond} w^+ \,\mathrm{d}\nu_\Diamond = \int_0^1 \left(\int_\Omega w^+(x_0,.)\,\mathrm{d}\nu \right) \mathrm{d}x_0 \le C \left(|\mathrm{D}w_\Diamond|(\Omega_\Diamond) + \int_{\partial\Omega_\Diamond} |w_\Diamond|\,\mathrm{d}\mathcal{H}^n \right) < \infty\,,$$

where we recall that $w_\Diamond \in \mathrm{BV}(\Omega_\Diamond)$, thus its trace is $\mathcal{H}^n$–integrable on the boundary of the Lipschitz domain $\Omega_\Diamond$.

Conversely, we assume the product measure $\nu_\Diamond = \mathcal{L}^1 \llcorner (0,1) \otimes \nu$ to be admissible on $\Omega_\Diamond$, and we take any Borel $Z \subseteq \Omega$ such that $\mathcal{H}^{n-1}(Z) = 0$. Then, the set $(0,1) \times Z \subseteq \Omega_\Diamond$ is Borel and Corollary 2.44 yields $\mathcal{H}^n((0,1) \times Z) = \mathcal{L}^1((0,1)) \cdot \mathcal{H}^{n-1}(Z) = 0$, hence the admissibility condition on $\nu_\Diamond$ guarantees $0 = \nu_\Diamond((0,1) \times Z) = \mathcal{L}^1((0,1)) \cdot \nu(Z) = \nu(Z)$ as required. We let now $w \in \mathrm{BV}(\Omega)$ be non–negative and construct the function $\widehat{w}(x_0,x) := w(x)$ defined for $x_0 \in (0,1)$ and $x \in \Omega$. Then, $\widehat{w}$ is in $\mathrm{BV}(\Omega_\Diamond)$ non–negative, $\widehat{w}^+(x_0,.) = \widehat{w}(x_0,.)^+ = w^+$ holds $\mathcal{H}^{n-1}$–a.e. for a.e. $x_0 \in (0,1)$ by Lemma 5.10, and therefore from Proposition 3.5 for $\nu_\Diamond$ in $\Omega_\Diamond$ we know

$$\int_\Omega w^+ \,\mathrm{d}\nu = \mathcal{L}^1((0,1)) \cdot \int_\Omega w^+(x)\,\mathrm{d}\nu(x) = \int_0^1 \left(\int_\Omega \widehat{w}^+(x_0,x)\,\mathrm{d}\nu(x) \right) \mathrm{d}x_0 = \int_{\Omega_\Diamond} \widehat{w}^+ \,\mathrm{d}\nu_\Diamond < \infty\,, \tag{5.2.15}$$

hence again the characterization in Proposition 3.5 determines admissibility of ν on Ω. □

Keeping in mind our ultimate goal of applying the semicontinuity Theorem 4.8 in dimension $n+1$ to approach LSC of the full functional $\mathcal{F}$ on $\mathbb{R}^n$, we would like to preserve not just admissibility of the single component measures $\mu_{\Diamond\pm}$, but even the ICs for (μ_-, μ_+) on Ω when passing to $\mu_{\Diamond\pm}$ on $\Omega_\Diamond$. This is actually true, as demonstrated in the following proposition.

Proposition 5.12 (anisotropic ICs with extra variable)**.** *We consider f, its corresponding mapping φ as in Definition 5.4, and admissible measures ν_1, ν_2 on Ω. Moreover, we assume continuity of f and f^∞, together with (LIN′)–(H2′). Assigned $C \in [0,\infty)$, the following hold:*

(i) *If the pair (ν_1, ν_2) satisfies the f^∞–IC in Ω with constant C, then also $(\nu_{1\Diamond}, \nu_{2\Diamond})$ satisfies the φ–IC in $\Omega_\Diamond$ with constant C.*

(ii) *If the pair* (ν_2, ν_1) *satisfies the* $\widetilde{f^\infty}$*–IC in* Ω *with constant* C*, then also* $(\nu_{2\lozenge}, \nu_{1\lozenge})$ *satisfies the* $\widetilde{\varphi}$*–IC in* $\Omega_\lozenge$ *with same* C*.*

Proof. It suffices to prove the first point, since (ii) can be obtained by (i) replacing $(\nu_1, \nu_2, f^\infty, \varphi)$ with $\big(\nu_2, \nu_1, \widetilde{f^\infty}, \widetilde{\varphi}\big)$. Observe that Lemma 5.11 guarantees admissibility of both $\nu_{1\lozenge}$, $\nu_{2\lozenge}$ separately on $\Omega_\lozenge$. It is then possible to employ the ICs characterization in Theorem 3.11 for (ν_1, ν_2) and $(\nu_{1\lozenge}, \nu_{2\lozenge})$, respectively. Therefore, (i) results once we verify for instance that

$$\int_{\Omega_\lozenge} w^+ \,\mathrm{d}\nu_{1\lozenge} - \int_{\Omega_\lozenge} w^- \,\mathrm{d}\nu_{2\lozenge} \le C \left(|\mathrm{D}w|_\varphi(\Omega_\lozenge) + \int_{\partial\Omega_\lozenge} \varphi(\cdot, w\nu_{\Omega_\lozenge}) \,\mathrm{d}\mathcal{H}^n \right) \text{ for all } w \in \mathrm{BV}(\Omega_\lozenge),\ w \ge 0\,. \tag{5.2.16}$$

Assume that w is a non–negative function in $\mathrm{BV}(\Omega_\lozenge)$. Arguing via Lemma 5.10, almost every restriction $\Omega \ni x \mapsto w(x_0, x)$ is non–negative and belongs to in $\mathrm{BV}(\Omega)$, hence Theorem 3.11 for (ν_1, ν_2) in Ω and integrand f^∞ yields

$$\int_\Omega w^+(x_0, .) \,\mathrm{d}\nu_1 - \int_\Omega w^-(x_0, .) \,\mathrm{d}\nu_2 \le C \left(|\mathrm{D}w(x_0, .)|_{f^\infty}(\Omega) + \int_{\partial\Omega} f^\infty(\cdot, w(x_0, .)\nu_\Omega) \,\mathrm{d}\mathcal{H}^{n-1} \right)$$

for a.e. $x_0 \in (0,1)$. With the aid of the estimates (5.2.10)–(5.2.11) and once again via Fubini, we reach

$$\begin{aligned} \int_{\Omega_\lozenge} w^+ \,\mathrm{d}\nu_{1\lozenge} - \int_{\Omega_\lozenge} w^- \,\mathrm{d}\nu_{2\lozenge} &= \int_0^1 \left(\int_\Omega w^+(x_0, \cdot) \,\mathrm{d}\nu_1 - \int_\Omega w^-(x_0, \cdot) \,\mathrm{d}\nu_2 \right) \mathrm{d}x_0 \\ &\le C \int_0^1 \left(|\mathrm{D}w(x_0, .)|_{f^\infty}(\Omega) + \int_{\partial\Omega} f^\infty(\cdot, w(x_0, .)\nu_\Omega) \,\mathrm{d}\mathcal{H}^{n-1} \right) \mathrm{d}x_0 \\ &\le C \left(|\mathrm{D}w|_\varphi(\Omega_\lozenge) + \int_{\partial\Omega_\lozenge} \varphi(., w\nu_{\Omega_\lozenge}) \,\mathrm{d}\mathcal{H}^n \right). \end{aligned}$$

This achieves the claim (5.2.16), and the φ–IC in (i) follows. □

We observe that in general the implication of Proposition 5.12 cannot be inverted – that is, φ–ICs on extensions do not imply f^∞–ICs for the original pairs of measures. In fact, the equivalence holds for specific cases of integrands f only – for example, the translated area $f(\xi) := \sqrt{1 + |\xi|^2} - 1$ or any homogeneous integrand.

Proposition 5.13. *Each of the implications in* (i), (ii) *in Proposition* 5.12 *can be inverted provided* $f(x, 0) = 0$ *for a.e.* $x \in \Omega$*.*

Proof. We assume that the φ–IC holds for $(\nu_{1\lozenge},\ \nu_{2\lozenge})$ in $\Omega_\lozenge$ with constant C, and fix a non–negative w in $\mathrm{BV}(\Omega)$. We introduce the function $\widehat{w}(x_0, x) := w(x)$ for $(x_0, x) \in (0,1) \times \Omega$, observing that $\widehat{w} \in \mathrm{BV}(\Omega_\lozenge)$ with $\mathrm{d}\mathrm{D}\widehat{w}(x_0, x)/\mathrm{d}|\mathrm{D}\widehat{w}|(x_0, x) = (0, \mathrm{d}\mathrm{D}w(x)/\mathrm{d}|\mathrm{D}w|(x))$ for $|\mathrm{D}\widehat{w}|$–a.e. $(x_0, x) \in \Omega_\lozenge$. We can then exploit Fubini as in (5.2.15) in combination with Theorem 3.11(2a) on $\Omega_\lozenge$ to compute

$$\begin{aligned} \int_\Omega w^+ \,\mathrm{d}\nu_1 - \int_\Omega w^- \,\mathrm{d}\nu_2 &= \int_{\Omega_\lozenge} \widehat{w}^+ \,\mathrm{d}\nu_{1\lozenge} - \int_{\Omega_\lozenge} \widehat{w}^- \,\mathrm{d}\nu_{2\lozenge} \\ &\le C \left(|\mathrm{D}\widehat{w}|_\varphi(\Omega_\lozenge) + \int_{\partial\Omega_\lozenge} \varphi(\cdot, \widehat{w}\nu_{\Omega_\lozenge}) \,\mathrm{d}\mathcal{H}^n \right). \end{aligned} \tag{5.2.17}$$

Explicitly, by definition of φ we have

$$|\mathrm{D}\widehat{w}|_\varphi(\Omega_\lozenge) = \int_{(0,1)\times\Omega} \varphi\left(1, x, 0, \frac{\mathrm{d}\mathrm{D}w(x)}{\mathrm{d}|\mathrm{D}w|(x)}\right) \mathrm{d}|\mathrm{D}\widehat{w}|(x_0, x) = \int_\Omega f^\infty\left(x, \frac{\mathrm{d}\mathrm{D}w(x)}{\mathrm{d}|\mathrm{D}w|(x)}\right) \mathrm{d}|\mathrm{D}w|(x) = |\mathrm{D}w|_{f^\infty}(\Omega),$$

whereas for the boundary term we observe that $\partial\Omega_\Diamond = ([0,1]\times\partial\Omega) \cup (\{0\}\times\Omega) \cup (\{1\}\times\Omega)$ with inner normal $\nu_{\Omega_\Diamond}$ to $\partial\Omega_\Diamond$ decomposed as in (5.2.9). It is then

$$\int_{\{0\}\times\Omega} \varphi(\,.\,,\widehat{w}\nu_{\Omega_\Diamond})\,\mathrm{d}\mathcal{H}^n = \int_\Omega \varphi(0,x,w(x),0)\,\mathrm{d}\mathcal{L}^n(x) = \int_\Omega f(\,.\,,0)|w|\,\mathrm{d}x\,,$$
$$\int_{\{1\}\times\Omega} \varphi(\,.\,,\widehat{w}\nu_{\Omega_\Diamond})\,\mathrm{d}\mathcal{H}^n = \int_\Omega \varphi(1,x,-w(x),0)\,\mathrm{d}\mathcal{L}^n(x) = \int_\Omega f(\,.\,,0)|w|\,\mathrm{d}x$$

as well as

$$\begin{aligned}\int_{[0,1]\times\partial\Omega} \varphi(\,.\,,\widehat{w}\nu_{\Omega_\Diamond})\,\mathrm{d}\mathcal{H}^n &= \int_{[0,1]\times\partial\Omega} \varphi(1,x,0,\widehat{w}\nu_\Omega(x))\,\mathrm{d}\mathcal{H}^n(x_0,x)\\ &= \int_{[0,1]\times\partial\Omega} f^\infty(x,w(x)\nu_\Omega(x))\,\mathrm{d}\mathcal{H}^n(x_0,x)\\ &= \int_{\partial\Omega} f^\infty(\,.\,,w\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1}\end{aligned}$$

bringing in Corollary 2.44. Overall, the expression in (5.2.17) becomes

$$\int_\Omega w^+\,\mathrm{d}\nu_1 - \int_\Omega w^-\,\mathrm{d}\nu_2 \le C\left(|\mathrm{D}w|_{f^\infty}(\Omega) + \int_{\partial\Omega} f^\infty(\,.\,,w\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1} + 2\int_\Omega f(\,.\,,0)|w|\,\mathrm{d}x\right),$$

and this latter implies the f^∞–IC for (ν_1,ν_2) on Ω again by Theorem 3.11(2a), supposed $f(.,0)=0$ holds on Ω. □

5.2.3 Proof of the lower semicontinuity Result 1

Exploiting the latter reformulations, we can finally approach the semicontinuity theorem in our most general setting. We record that by collecting (5.2.3)–(5.2.6) we can rewrite our target functional as

$$\mathcal{F}[w] = \widehat{\Phi}[w_\Diamond] - 2\int_\Omega f(.,0)|w-u_0|\,\mathrm{d}x - \frac{\mu(\Omega)}{2} \qquad \text{for all } w\in\mathrm{BV}(\Omega)\,, \tag{5.2.18}$$
$$\text{where } \widehat{\Phi}[w_\Diamond] = |\mathrm{D}w_\Diamond|_\varphi(\Omega_\Diamond) + \int_{\partial\Omega_\Diamond} \varphi\left(.,(w_\Diamond - u_{0\Diamond})\nu_{\Omega_\Diamond}\right)\mathrm{d}\mathcal{H}^n - \int_{\Omega_\Diamond} (w_\Diamond)^+\,\mathrm{d}\mu_{-\Diamond} + \int_{\Omega_\Diamond} (w_\Diamond)^-\,\mathrm{d}\mu_{+\Diamond}\,.$$

Here we consider the measure $\mu := \mu_+ - \mu_-$ and φ, $\Omega_\Diamond$, $w_\Diamond$ and $\mu_{\pm\Diamond}$ as introduced in Section 5.2.1.

Proof of Result 1. Assume that $(u_k)_k$ is a sequence in $\mathrm{BV}(\Omega)$ converging in $\mathrm{L}^1(\Omega)$ to $u\in\mathrm{BV}(\Omega)$ as $k\to\infty$. We observe first that the linear–growth condition on f determines $0\le f(x,0)\le\beta$ for all $x\in\Omega$, hence in particular

$$\left|\int_\Omega f(x,0)|u_k-u_0|\,\mathrm{d}x - \int_\Omega f(x,0)|u-u_0|\,\mathrm{d}x\right| \le \int_\Omega f(x,0)|u_k-u|\,\mathrm{d}x \le \beta\|u_k-u_0\|_{\mathrm{L}^1(\Omega)} \xrightarrow[k\to 0]{} 0\,.$$

Moreover, the lifted functions

$$u_\Diamond(x_0,x) := x_0 + u(x)\,,\quad u_{k\Diamond}(x_0,x) := x_0 + u_k(x) \qquad \text{for } (x_0,x)\in(0,1)\times\Omega =: \Omega_\Diamond$$

are in $\mathrm{BV}(\Omega_\Diamond)$ and satisfy $u_{k\Diamond}\to u_\Diamond$ in $\mathrm{L}^1(\Omega_\Diamond)$ for $k\to\infty$. We make use of the rewriting of $\mathcal{F}$ in (5.2.18) via the functional $\widehat{\Phi}\colon \mathrm{BV}(\Omega_\Diamond)\to\mathbb{R}$ restricted to the class of functions $w_\Diamond$ and with continuous anisotropic integrand $\varphi\colon\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\to[0,\infty)$, noticing that φ is convex in (ξ_0,ξ) in view of Proposition 5.6. We recall that the f^∞–IC with constant 1 for (μ_-,μ_+) on Ω and the $\widetilde{f^\infty}$–IC with constant 1 for (μ_+,μ_-) on Ω (which jointly take the form of (1.2.3)) yield via Proposition 5.12 the

φ–IC for $(\mu_{-\lozenge}, \mu_{+\lozenge})$ as well as the $\widetilde{\varphi}$-IC for $(\mu_{+\lozenge}, \mu_{-\lozenge})$ in $\Omega_\lozenge$, both again with constant 1. We are thus in the conditions of applying Theorem 4.8 to the functional $\widehat{\Phi}$ on our bounded Lipschitz set $\Omega_\lozenge$ in $\mathbb{R}^{n+1}$ – where we assign any boundary datum in $\mathrm{W}^{1,1}(\mathbb{R}^{n+1})$ with trace $u_{0\lozenge}$ on $\partial\Omega_\lozenge$ – to conclude

$$\begin{aligned}\liminf_{k\to\infty} \mathcal{F}[u_k] &= \liminf_{k\to\infty} \widehat{\Phi}[u_{k\lozenge}] - 2\lim_{k\to\infty}\int_\Omega f(.,0)|u_k - u_0|\,\mathrm{d}x - \frac{\mu(\Omega)}{2}\\ &\geq \widehat{\Phi}[u_\lozenge] - 2\int_\Omega f(.,0)|u-u_0|\,\mathrm{d}x - \frac{\mu(\Omega)}{2} = \mathcal{F}[u]\,,\end{aligned}$$

and the claimed lower semicontinuity follows. □

5.3 Existence of minimizers and analysis of the limit case for the IC

As announced, semicontinuity and coercivity build up to the existence of BV minima, provided the ICs are fulfilled for constants *strictly smaller* than 1.

Proof of Result 2. We consider a minimizing sequence $(u_k)_k$ for $\mathcal{F}$ in $\mathrm{BV}(\Omega)$. The assumption $C < 1$ for the ICs guarantees via Proposition 5.2 the BV–coercivity of the functional $\mathcal{F}$, and from this we deduce boundedness of a subsequence of $(u_k)_k$ in $\mathrm{BV}(\Omega)$. We employ the compactness Theorem 2.22 to determine a function $u \in \mathrm{BV}(\Omega)$ such that (up to relabelling) $u_k \to u$ in $\mathrm{L}^1(\Omega)$ as $k \to \infty$. Ultimately, the semicontinuity Result 1 yields

$$\inf_{w\in\mathrm{BV}(\Omega)} \mathcal{F}[w] = \liminf_{k\to\infty} \mathcal{F}[u_k] \geq \mathcal{F}[u]\,,$$

and from this we read minimality of u for $\mathcal{F}$. □

We stress that if the joint f^∞, $\widetilde{f^\infty}$-IC (1.2.3) holds for some constant *strictly larger* than 1 only, there is no hope to minimize $\mathcal{F}$ – being the functional unbounded from below according to Proposition 5.1. Moreover, the loss of coercivity under ICs with constant exactly 1 prevents us from achieving existence via the direct method. Nevertheless, in the borderline case we could still hope to minimize $\mathcal{F}$ bypassing coercivity – but still relying on the LSC Result 1, valid even for $C = 1$. To answer this open question, in what follows we analyze existence in the so–called *limit* (or *borderline*) case of the IC, that is, assuming as smallest constant C *precisely* 1. In doing so, we shall distinguish between homogeneous integrands $f = f^\infty$ and general cases. Namely, for anisotropic integrands we will use the findings in Chapter 4 to demonstrate that truncations of minimizing sequences are still minimizing when the datum u_0 is a.e. bounded, and consequently the compactness machinery enables us to achieve minimizers. Interestingly, the presence of minima is not guaranteed in case $u_0 \notin \mathrm{L}^\infty$, as well as for general non–homogeneous functionals (now regardless of the boundedness of u_0). This is illustrated by Example 5.15 and Example 5.16, respectively.

Theorem 5.14 (existence in the limit IC; homogeneous case). *We assume* $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n) \cap \mathrm{L}^\infty(\mathbb{R}^n)$ *and that* $f : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ *is an anisotropy comparable to the Euclidean norm and such that*:

(a) f *is continuous*; *and*

(b) *The restriction* $\xi \mapsto f(x,\xi)$ *is convex for all* $x \in \mathbb{R}^n$.

If admissible measures $\mu_\pm$ *on* Ω *are such that* (μ_-, μ_+) *satisfies the* f*–IC in* Ω *with* (*precisely*) *constant* $C = 1$ *and* (μ_+, μ_-) *satisfies the* $\widetilde{f}$*–IC in* Ω *with constant* 1, *then the functional* $\mathcal{F}$ *admits at least a minimizer in* $\mathrm{BV}(\Omega)$.

Observe that as an assumption we require positive 1–homogeneity of $f(x,.)$ on $\mathbb{R}^n$ for every x, thus in particular f agrees with its recession function.

Proof. To begin with, we fix a level $M \geq \|u_0\|_{L^\infty(\mathbb{R}^n)}$. Arguing via Corollary 4.13 exploiting the convexity assumption and replacing $\widehat{\Phi}$ with $\mathcal{F}$, we get

$$\mathcal{F}[w^M] \leq \mathcal{F}[w] \quad \text{for all } w \in \mathrm{BV}(\Omega)\,.$$

Once again we assume that $(u_k)_k$ is a minimizing sequence for $\mathcal{F}$ in $\mathrm{BV}(\Omega)$. Then, $\left(u_k^M\right)_k$ is still a minimizing sequence for $\mathcal{F}$, and from $\|u_k^M\|_{L^\infty(\Omega)} \leq M$ we have the uniform bound $|\int_\Omega \left(u_k^M\right)^{\pm} \mathrm{d}\mu_\mp| \leq M\mu_\pm(\Omega)$. Then – possibly passing to a subsequence – from finiteness of $\mu_\pm$ we infer

$$\sup_{k\in\mathbb{N}} f\Big(\mathrm{D}\overline{u_k^M}^{u_0}\Big)(\overline{\Omega}) - M|\mu|(\Omega) \leq \sup_{k\in\mathbb{N}} \mathcal{F}\left[u_k^M\right] < \infty\,,$$

setting $|\mu| := \mu_+ + \mu_-$ on Ω. Employing the coercivity bound $f(x,\xi) \geq \alpha|\xi|$ for some $\alpha > 0$, we further estimate from below the variation:

$$\begin{aligned}\infty > \sup_{k\in\mathbb{N}} f\Big(\mathrm{D}\overline{u_k^M}^{u_0}\Big)(\overline{\Omega}) &\geq \alpha\bigg(\left|\mathrm{D}u_k^M\right|(\Omega) + \int_{\partial\Omega} \left|u_k^M\right| \mathrm{d}\mathcal{H}^{n-1} - \int_{\partial\Omega} |u_0| \,\mathrm{d}\mathcal{H}^{n-1}\bigg)\\ &= \alpha\left|\mathrm{D}\overline{u_k^M}^0\right|(\overline{\Omega}) - \alpha\|u_0\|_{L^1(\partial\Omega;\mathcal{H}^{n-1})}\,.\end{aligned}$$

Poincaré's inequality (2.2.4) implies then boundedness of $\left(u_k^M\right)_k$ in $\mathrm{BV}(\Omega)$, and by exactly the same reasoning as in Result 2 the $L^1(\Omega)$–limit $u_M \in \mathrm{BV}(\Omega)$ of a suitable subsequence of (u_k^M) is a minimizer of $\mathcal{F}$. $\square$

Nevertheless, some issues arise when taking a Dirichlet datum whose trace is not bounded on $\partial\Omega$. For a proof of Example 5.15 we refer to our work [51, Section 5.3].

Example 5.15 (failure of existence in the limit IC and unbounded datum; homogeneous case)**.** For $n = 2$, we consider the domain $\Omega := \{x \in B_2 : x_2 > -1\}$ and any $u_0 \in W^{1,1}(\mathbb{R}^2)$ extending $u_0(x) := (|x|-1)^{-\alpha}$ with $x \in B_3 \setminus \overline{\Omega}$, $\alpha \in (0, 1/2)$. Assume $\mu_+ :\equiv 0$ and $\mu_- :\equiv H\mathcal{L}^2$ on Ω with $H(x) := |x|^{-1}$ for $x \in \Omega \setminus \{(0,0)\}$. Then μ_- satisfies the isotropic IC in $\mathbb{R}^2$ – and thus in Ω – exactly with constant 1 (compare with Example 3.17), yet the functional $\mathcal{F}$ with isotropic integrand $f(\xi) := |\xi|$ does not admit minimizers in $\mathrm{BV}(\Omega)$.

Proof of Example 5.15. We claim that the infimum of the functional $\mathcal{F}$ is never achieved. Preliminary to computations, we observe that the radially–symmetric function u_0 has a pole in $(0,-1) \in \partial\Omega$, and that from definition of H a candidate minimizer $u \in \mathrm{BV}(\Omega)$ for $\mathcal{F}$ would be radially symmetric and forced to increase on circles with smaller and smaller radius $r \searrow 1$, until reaching $u \equiv \infty$ on B_1, which however contradicts the requirement $u \in L^1(\Omega)$. In detail, our assumption on α guarantees that $u_0 \in W^{1,1}(B_3 \setminus \overline{\Omega})$, and replacing u_0 with any radially symmetric extension that is in $W^{1,1}(\mathbb{R}^2)$, we may rewrite

$$\begin{aligned}\mathcal{F}[w] &= |\mathrm{D}\overline{w}^{u_0}|\big(\mathbb{R}^2\big) - |\mathrm{D}u_0|\big(\overline{\Omega}^{\mathrm{c}}\big) - \int_{\mathbb{R}^2} \frac{\overline{w(x)}^{u_0}}{|x|}\,\mathrm{d}x + \int_{\Omega^{\mathrm{c}}} \frac{u_0(x)}{|x|}\,\mathrm{d}x\\ &= |\mathrm{D}\overline{w}^{u_0}|\big(\mathbb{R}^2\big) - \int_{\mathbb{R}^2} \frac{\overline{w(x)}^{u_0}}{|x|}\,\mathrm{d}x + \mathrm{const}(u_0,\Omega)\,.\end{aligned}$$

We then rather work with the equivalent functional $\overline{\mathcal{F}}$ defined on $\mathrm{BV}(\mathbb{R}^2)$ restricted to the class of functions which agree with (an extension of) u_0 outside Ω, that is, computing

$$\bar{\mathrm{I}} := \inf_{\substack{w\in\mathrm{BV}(\mathbb{R}^2)\\ w=u_0 \text{ on } \Omega^{\mathrm{c}}}} \overline{\mathcal{F}}[w]\,.$$

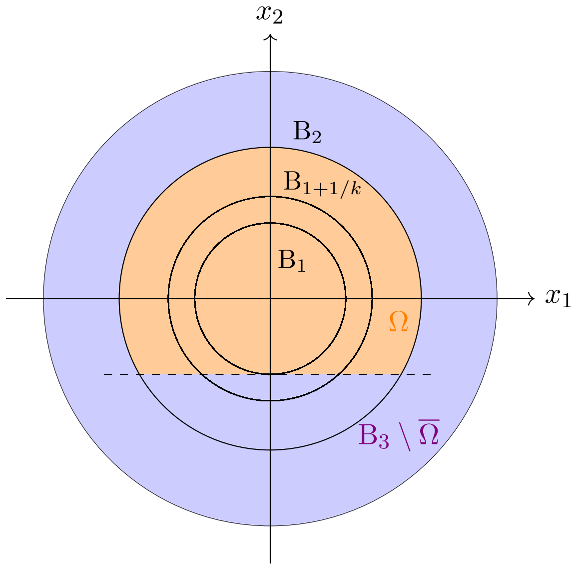

Figure: The domain Ω of Example 5.15 (in orange). When considering the functional $\overline{\mathcal{F}}$ extended to all $\mathbb{R}^2$, we choose a minimizing sequence $(u_k)_k$ composed of truncations of u_0 at level k^α on the portion of the ball of radius $1 + 1/k$ contained in Ω. Outside the violet domain $\mathrm{B}_3 \setminus \overline{\Omega}$, u_k coincides with any Sobolev extension of u_0.

Define the sequence of truncated functions

$$u_k(x) := \begin{cases} k^\alpha & \text{for } x \in \mathrm{B}_{1+1/k} \cap \Omega \\ u_0(x) & \text{for } x \in \mathbb{R}^2 \setminus (\mathrm{B}_{1+1/k} \cap \Omega)\,, \end{cases}$$

for which by symmetry of u_0 the superlevel sets $\{u_k > t\}$ are balls centered in the origin and some radius $r(t)$ whenever $t \in (0, k^\alpha)$ – and in detail $r(t) = 1 + t^{-1/\alpha}$ when $t \in (2^{-\alpha}, k^\alpha)$. Then, each u_k belongs to $\mathrm{BV}(\mathbb{R}^2)$ and agrees with u_0 outside Ω. We then evaluate the modified functional applying the coarea and layer–cake formulas:

$$\begin{aligned}
\overline{\mathcal{F}}[u_k] &= \int_0^{k^\alpha} \left(\mathrm{P}(\mathrm{B}_{r(t)}) - \int_{\mathrm{B}_{r(t)}} \frac{1}{|x|}\,\mathrm{d}x \right) \mathrm{d}t + \int_{k^\alpha}^\infty \left(\mathrm{P}(\{u_k \geq t\}) - \int_{\{u_k \geq t\}} \frac{1}{|x|}\,\mathrm{d}x \right) \mathrm{d}t \\
&= \int_{k^\alpha}^\infty \left(\mathrm{P}(\{u_k \geq t\}) - \int_{\{u_k \geq t\}} \frac{1}{|x|}\,\mathrm{d}x \right) \mathrm{d}t \\
&\leq \int_{k^\alpha}^\infty \mathrm{P}(\{u_k \geq t\})\,\mathrm{d}t \\
&= |\mathrm{D}u_k|(\mathrm{B}_{1+1/k}) \\
&= |\mathrm{D}u_0|(\mathrm{B}_{1+1/k} \setminus \Omega) + |\mathrm{D}u_k|(\mathrm{B}_{1+1/k} \cap \partial\Omega) \qquad \text{for all } k \in \mathbb{N}\,,
\end{aligned}$$

where the equality $\mathrm{P}(\mathrm{B}_{r(t)}) = \int_{\mathrm{B}_{r(t)}} \frac{1}{|x|}$ for a.e. t is justified by limit IC in the form of (3.2.11). Therefore, by integrability of u_0 it is $\limsup_{k\to\infty} \overline{\mathcal{F}}[u_k] \leq 0$, and hence $\bar{\mathrm{I}} \leq 0$. At the same time, for an arbitrary

w in the class introduced above, we have

$$\begin{aligned}\overline{\mathcal{F}}[w] &= |\mathrm{D}w|\left(\{w\geq 0\}\right)+|\mathrm{D}w|\left(\{w<0\}\right)-\int_{\{w\geq 0\}}\frac{w(x)}{|x|}\,\mathrm{d}x-\int_{\{w<0\}}\frac{w(x)}{|x|}\,\mathrm{d}x\\ &\geq |\mathrm{D}w|\left(\{w\geq 0\}\right)-\int_{\{w\geq 0\}}\frac{w(x)}{|x|}\,\mathrm{d}x\\ &= \int_0^\infty\left(\mathrm{P}(\{w\geq t\})-\int_{\{w\geq t\}}\frac{1}{|x|}\,\mathrm{d}x\right)\mathrm{d}t\geq 0\,,\end{aligned}$$

(thus $\bar{\mathrm{I}}=0$) and both inequalities are equalities if and only if for a.e. $t>0$ there exists some $R_t>0$ such that $|\{w\geq t\}\,\Delta\,\mathrm{B}_{R_t}|=0$. Specifically, being $w=u_0$ on $\overline{\Omega}^{\mathrm{c}}$ by assumption, it is $\{w\geq t\}\cap\Omega^{\mathrm{c}}=\{u_0\geq t\}\cap\Omega^{\mathrm{c}}=\mathrm{B}_{1+t^{-1/\alpha}}\cap\Omega^{\mathrm{c}}$ for a.e. $t>2^{-\alpha}$. This yields that the only possibility is $R_t=r(t)=1+t^{-1/\alpha}$ for $t>2^{-\alpha}$, thus letting $t\to\infty$ we would have $\{w=\infty\}=\mathrm{B}_1$ up to negligible sets, which contradicts $w\in\mathrm{BV}(\mathbb{R}^2)$. By this latter, we have verified that $\mathcal{F}$ does not admit minima in $\mathrm{BV}(\Omega)$. □

Concerning the case $C=1$ of the IC again, an adaptation to the area integral of the counterexample above enables us to rule out the existence of minimizers even for bounded data u_0, provided the integrand f is *not* homogeneous.

Example 5.16 (failure of existence in the limit IC; inhomogeneous case)**.** In the bidimensional space, we take $\Omega:=\mathrm{B}_1\subseteq\mathbb{R}^2$, $\mu_+:\equiv 0$, $\mu_-:\equiv H\mathcal{L}^2$ with $H(x):=|x|^{-1}$ for $x\in\Omega\setminus\{(0,0)\}$. Then, for the area integrand $f(\xi):=\sqrt{1+|\xi|^2}$ and datum $u_0:\equiv 0$, the corresponding functional $\mathcal{F}$ has no minimum in $\mathrm{BV}(\Omega)$.

We point out that the counterexample to existence of BV minima with the IC with constant 1 can be achieved for any anisotropic area integral of the kind $f(\xi):=\sqrt{1+\varphi(\xi)^2}$, where φ is a convex anisotropy such that $\varphi(\xi)>0$ for all $\xi\in\mathbb{R}^2\setminus\{(0,0)\}$. In such a case, the domain Ω to be considered for Example 5.16 would be the $\widetilde{\varphi}^\circ$–unit ball $\{\widetilde{\varphi}^\circ<1\}$ in $\mathbb{R}^2$, and the measure $\mu_-:\equiv(\widetilde{\varphi}^\circ)^{-1}\mathcal{L}^2$. Further details on this can be found in [52, Example 3.7].

Proof of Example 5.16. In view of Example 3.17, we know that μ_- satisfies the isotropic IC in $\mathbb{R}^2$ with constant exactly 1. We then exploit the IC–reformulation for functions $w\in\mathrm{BV}(\Omega)$ in Remark 3.13 to achieve

$$\mathcal{F}[w]:=\int_\Omega\sqrt{1+|\mathrm{D}w|^2}+\int_{\partial\Omega}|w|\,\mathrm{d}\mathcal{H}^1-\int_\Omega\frac{w(x)}{|x|}\,\mathrm{d}x>|\mathrm{D}w|(\Omega)+\int_{\partial\Omega}|w|\,\mathrm{d}\mathcal{H}^1-\int_\Omega\frac{w(x)}{|x|}\,\mathrm{d}x\geq 0\,.$$

The sequence of radially decreasing functions $u_k(x):=k(1-|x|)$, $x\in\Omega$, determines a minimizing sequence for the functional. In fact, $u_k\in\mathrm{W}^{1,1}_0(\Omega)$ with $|\nabla u_k|=k$ for every k, and one computes

$$\begin{aligned}\mathcal{F}[u_k]=\int_\Omega\sqrt{1+k^2}\,\mathrm{d}x-k\int_\Omega\frac{1-|x|}{|x|}\,\mathrm{d}x=\pi\big(\sqrt{1+k^2}+k\big)-k\int_\Omega\frac{1}{|x|}\,\mathrm{d}x&=\pi\big(\sqrt{1+k^2}-k\big)\\ &=\frac{\pi}{k+\sqrt{1+k^2}}\xrightarrow[k\to\infty]{}0\,.\end{aligned}$$

Overall, we have $\inf_{\mathrm{BV}(\Omega)}\mathcal{F}=0<\mathcal{F}[w]$ for all $w\in\mathrm{BV}(\Omega)$, and thus our counterexample is verified. □

Chapter 6

Recovery sequences and Γ-convergence

The first half of this chapter is devoted to showing that there exist $\mathrm{W}^{1,1}_{u_0}$–recovery sequences for the functional $\mathcal{F}$ – that is, to prove Result 3. We observe that, under suitable ICs for the pairs of admissible measures $(\mu_\mp, \mu_\pm)$ mutually singular on Ω, our Result 3 determines consistency of our BV–reformulation

$$\mathrm{BV}(\Omega) \ni w \mapsto \mathcal{F}_{(u_0)}[w] := f(.,\mathrm{D}\overline{w}^{u_0})\big(\overline{\Omega}\big) - \int_\Omega w^+ \,\mathrm{d}\mu_- + \int_\Omega w^- \,\mathrm{d}\mu_+$$

in relation to the original functional

$$\mathrm{W}^{1,1}_{u_0}(\Omega) \ni w \mapsto \mathrm{F}_{u_0}[w] := \int_\Omega f(.,\nabla w)\,\mathrm{d}x + \int_\Omega w^* \,\mathrm{d}(\mu_+ - \mu_-)\,,$$

in the sense that $\mathcal{F}$ expresses the L^1–relaxation on $\mathrm{BV}(\Omega)$ of F_{u_0}. We want to stress that a major difference of Result 3 in comparison to the semicontinuity Result 1 lies in the necessary singularity assumption $\mu_+ \perp \mu_-$. This hypothesis is not entirely unexpected, in fact if the supports of $\mu_\pm$ intersect on some portion of Ω, then the measure term $\int_\Omega w^* \,\mathrm{d}(\mu_+ - \mu_-)$ for $w \in \mathrm{W}^{1,1}(\Omega)$ will be subjected to a partial cancellation effect because Sobolev functions enjoy the $\mathcal{H}^{n-1}$–a.e. coincidence of all representatives – which by admissibility holds also $\mu_\pm$–a.e. Such a phenomenon does not apply for the term $\int_\Omega w^- \,\mathrm{d}\mu_+ - \int_\Omega w^+ \,\mathrm{d}\mu_-$ whenever $w \in \mathrm{BV}(\Omega) \setminus \mathrm{W}^{1,1}(\Omega)$. The following example illustrates the necessity of the mutual singularity assumption for the component measures to enable consistency.

Example 6.1 (failure of recovery sequence with prescribed boundary values for $\mu_+ \not\perp \mu_-$)**.** We consider the open unit ball $\Omega := \mathrm{B}_1$ in $\mathbb{R}^2$ and the isotropy $f(\xi) = f^\infty(\xi) := |\xi|$. We fix a Dirichlet boundary datum $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^2)$ with trace $u_0(x_1,x_2) := \mathrm{sgn}(x_1)$ for all $(x_1,x_2) \in \partial\Omega$, and admissible measures $\mu_\pm := \mathcal{H}^1 \llcorner \mathrm{H}$ in Ω, for $\mathrm{H} := \{0\} \times (-1,1)$. Then, both pairs (μ_-,μ_+) and (μ_+,μ_-) satisfy the (isotropic) IC in Ω with constant $C = 1/2$, since from [92, Proposition A.2] given any measurable $A \Subset \Omega$ we write

$$\mu_-(A^+) - \mu_+(A^1) \le \mu_-(A^+) \le \mathrm{P}(A)/2 \quad \text{and} \quad \mu_+(A^+) - \mu_-(A^1) \le \mu_+(A^+) \le \mathrm{P}(A)/2\,,$$

confirming our ICs. At the same time, however, Result 3 fails since it is

$$\min_{w \in \mathrm{BV}(\Omega)} \mathcal{F}[w] = 0 < 4 \le \inf_{w \in \mathrm{W}^{1,1}_{u_0}(\Omega)} \mathrm{F}_{u_0}[w]\,.$$

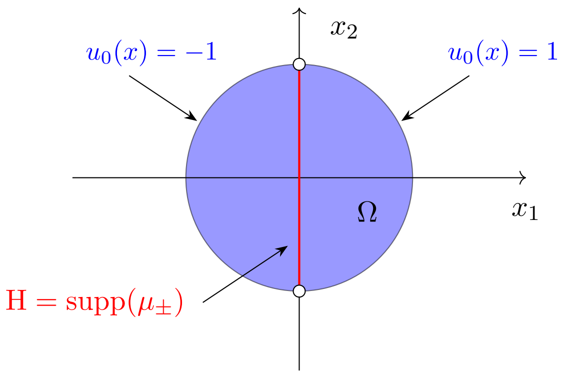

Figure: The domain of Example 6.1.

In fact, we estimate the functional F_{u_0} on $\mathrm{W}^{1,1}_{u_0}$ from below via

$$\begin{aligned}\mathrm{F}_{u_0}[w] = \mathrm{TV}_{u_0}[w](\Omega) \geq \int_\Omega \frac{\partial w}{\partial x_1}\,\mathrm{d}x &= \int_{-1}^{1}\left(\int_{-\sqrt{1-x_2^2}}^{\sqrt{1-x_2^2}} \frac{\partial w}{\partial x_1}(x_1,x_2)\,\mathrm{d}x_1\right)\mathrm{d}x_2\\
&= \int_{-1}^{1}\left[w\left(\sqrt{1-x_2^2},x_2\right) - w\left(-\sqrt{1-x_2^2},x_2\right)\right]\mathrm{d}x_2\\
&= \int_{-1}^{1}\left[\mathrm{sgn}\left(\sqrt{1-x_2^2}\right) - \mathrm{sgn}\left(-\sqrt{1-x_2^2}\right)\right]\mathrm{d}x_2\\
&= \int_{-1}^{1} 2\,\mathrm{d}x_2 = 4 \quad \text{for all } w \in \mathrm{W}^{1,1}_{u_0}(\Omega)\,.\end{aligned}$$

Allowing for jumps on the measure support instead, we would have

$$\begin{aligned}\mathrm{F}_{u_0}[w] = |\mathrm{D}\overline{w}^{u_0}|(\overline{\Omega}) - \int_\Omega (w^+ - w^-)\,\mathrm{d}\mu_- &= |\mathrm{D}\overline{w}^{u_0}|(\overline{\Omega}) - \int_{\{0\}\times(-1,1)} (w^+ - w^-)\,\mathrm{d}\mathcal{H}^1\\
&= |\mathrm{D}\overline{w}^{u_0}|(\overline{\Omega}) - |\mathrm{D}w|(\mathrm{H}) \geq 0 \quad \text{for all } w \in \mathrm{BV}(\Omega)\,,\end{aligned}$$

with equality $\mathrm{F}_{w_0}[\overline{u_0}] = 0$ achieved for the function $\overline{w_0} \in \mathrm{BV}(\Omega) \setminus \mathrm{W}^{1,1}(\Omega)$ defined as $\overline{w_0}(x_1,x_2) := \mathrm{sgn}(x_1)$ for all $(x_1,x_2) \in \Omega$.

6.1 Recovery sequences with no boundary condition

To approach our recovery sequence result, we first present an approximation theorem via smooth functions with *arbitrary* boundary trace – which again strongly relies on the mutual singularity of the component measures and on the results of Chapter 3. Before diving into the proof of Theorem 6.2, we record that the assumption $\mu_+ \perp \mu_-$ is here unavoidable. In contrast, the coercivity condition (ii) of Assumption 1.1 for f in the statement of Theorem 6.2 is not essential, being Reshetnyak's continuity Theorem 2.70 still valid when replacing (ii) with just a lower bound $f(x,\xi) \geq c_1$ for some $c_1 \in \mathbb{R}$.

Theorem 6.2 (smooth recovery sequence with no boundary condition)**.** *We assume that a Borel function $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ is convex in the second entry, has linear growth, and is such that f, f^∞ are continuous. We let $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ and fix admissible measures $\mu_\pm$ on bounded Lipschitz $\Omega \subseteq \mathbb{R}^n$ such that $\mu_+ \perp \mu_-$ and $\mu := \mu_+ - \mu_-$. Then, for any $w \in \mathrm{BV}(\Omega)$ there is some $(w_k)_k$ in $\mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ such that all of the following hold.*

(i) *The sequence $(w_k)_k$ converges strictly in area to w in $\mathrm{BV}(\Omega)$ as $k \to \infty$. As a consequence, for any open bounded $U \Supset \Omega$, the extensions $\overline{w}^{u_0}$, $(\overline{w_k}^{u_0})_k$ are such that $\overline{w_k}^{u_0} \rightharpoonup \overline{w}^{u_0}$ area–strictly in $\mathrm{BV}(U)$ as $k \to \infty$;*

(ii) *We have* $\lim\limits_{k\to\infty} f(.,\mathrm{D}\overline{w_k}^{u_0})(\overline{\Omega}) = f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega})$;

(iii) *The measure integrals satisfy* $\lim\limits_{k\to\infty}\int_\Omega w_k\,\mathrm{d}\mu = \lim\limits_{k\to\infty}\int_\Omega w_k^*\,\mathrm{d}\mu = \int_\Omega w^-\,\mathrm{d}\mu_+ - \int_\Omega w^+\,\mathrm{d}\mu_-$.

Furthermore, from (i)–(iii) *we read the convergence* $\lim\limits_{k\to\infty}\mathcal{F}[w_k] = \mathcal{F}[w]$, *and thus*

$$\inf_{\mathrm{BV}(\Omega)}\mathcal{F} = \inf_{\mathrm{W}^{1,1}(\Omega)}\mathcal{F}. \tag{6.1.1}$$

Proof. Assigned the function $w\in\mathrm{BV}(\Omega)$, we fix a positive level M. Exploiting the mutual singularity of $\mu_\pm$, Proposition 3.15 with measures μ_- instead of ν_1 and μ_+ in place of ν_2 determines a sequence $(w_{j,M})_j$ in $\mathrm{W}^{1,1}(\Omega)$ converging to w^M area–strictly in $\mathrm{BV}(\Omega)$ and with

$$\lim_{j\to\infty}\int_\Omega (w_{j,M})^*\,\mathrm{d}\mu = \int_\Omega (w^M)^-\,\mathrm{d}\mu_+ - \int_\Omega (w^M)^+\,\mathrm{d}\mu_-\,. \tag{6.1.2}$$

Notice that since $|w^M|\le M$ almost everywhere, without loss of generality, we may also assume that $w_{j,M}\in\mathrm{L}^\infty(\Omega)$ for every $j\in\mathbb{N}$. We then apply Reshetnyak's result in the form of Theorem 2.70 to the extension by u_0 of $(w_{j,M})_j$ in Ω, and then again to extensions of the sequence $(w^M)_M$ (recalling that truncations enjoy $w^M\to w$ strongly in $\mathrm{BV}(\Omega)$ as $M\to\infty$ from Lemma 2.100(iii)) to write

$$\lim_{M\to\infty}\left(\lim_{j\to\infty} f\big(.,\mathrm{D}\overline{w_{j,M}}^{u_0}\big)\big(\overline{\Omega}\big)\right) = \lim_{M\to\infty} f\big(.,\mathrm{D}\overline{w^M}^{u_0}\big)\big(\overline{\Omega}\big) = f\big(.,\mathrm{D}\overline{w}^{u_0}\big)\big(\overline{\Omega}\big)\,.$$

Employing (6.1.2) and the continuity result of Lemma 3.8, parallelly we get

$$\lim_{M\to\infty}\left(\lim_{j\to\infty}\int_\Omega (w_{j,M})^*\,\mathrm{d}\mu\right) = \int_\Omega w^-\,\mathrm{d}\mu_+ - \int_\Omega w^+\,\mathrm{d}\mu_-\,.$$

Then, the sequence $(\widehat{w_k})_k$ of functions $\widehat{w_k} := w_{j_k,k}$ for j_k large enough is such that $\widehat{w_k}\in\mathrm{W}^{1,1}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ for every k, $\widehat{w_k}\rightharpoonup w$ strictly in area in $\mathrm{BV}(\Omega)$ for $k\to\infty$, and all items (i), (ii), and (iii) are verified for the sequence $(\widehat{w_k})_k$.

We now extend the thesis to smooth functions with the help of Meyers–Serrin's theorem in [75]. In fact, we approximate each $\widehat{w_k}$ via a sequence $(v_{j,k})_j$ in $\mathrm{C}^\infty(\Omega)\cap\mathrm{W}^{1,1}(\Omega)$ converging strongly as $j\to\infty$ – where again boundedness of every $\widehat{w_k}$ guarantees that we can assume $(v_{j,k})_j\in\mathrm{L}^\infty(\Omega)$. Then, the convergence of the main terms $\lim\limits_{j\to\infty} f(.,\mathrm{D}v_{j,k})(\Omega) = f(.,\mathrm{D}\widehat{w_k})(\Omega)$ follows from Theorem 2.70, and by convergence of traces also $\lim\limits_{j\to\infty}\int_{\partial\Omega} f^\infty\left(.,(v_{j,k}-u_0)\nu_\Omega\right)\mathrm{d}\mathcal{H}^{n-1} = \int_{\partial\Omega} f^\infty\left(.,(\widehat{w_k}-u_0)\nu_\Omega\right)\mathrm{d}\mathcal{H}^{n-1}$ for any fixed k. Additionally, we record that the admissibility assumption for $\mu_\pm$ together with Remark 3.13 for both $(\mu_\pm,0)$ determines

$$\begin{aligned}\left|\int_\Omega v_{j,k}\,\mathrm{d}\mu - \int_\Omega \widehat{w_k}^*\,\mathrm{d}\mu\right| &\le \left|\int_\Omega (v_{j,k}-\widehat{w_k})^*\,\mathrm{d}\mu_+\right| + \left|\int_\Omega (v_{j,k}-\widehat{w_k})^*\,\mathrm{d}\mu_-\right|\\ &\le (C_+ + C_-)\left(|\mathrm{D}(v_{j,k}-\widehat{w_k})|(\Omega) + \int_{\partial\Omega}|v_{j,k}-\widehat{w_k}|\,\mathrm{d}\mathcal{H}^{n-1}\right)\xrightarrow[j\to\infty]{} 0\end{aligned}$$

for some constants $C_\pm\in[0,\infty)$ and exploiting the essential assumption of strong convergence. Finally, we once again pass to a suitable subsequence $(j_k)_k$ such that the sequence $w_k := v_{j_k,k}$ is precisely the one in the statement.

Lastly, the existence of such a recovery sequence $(w_k)_k$ for every $w\in\mathrm{BV}(\Omega)$ implies $\inf\limits_{\mathrm{BV}(\Omega)}\mathcal{F}\le\inf\limits_{\mathrm{W}^{1,1}(\Omega)}\mathcal{F}\le\lim\limits_{k\to\infty}\mathcal{F}[w_k] = \mathcal{F}[w]$, and the passage to the infimum in $u\in\mathrm{BV}(\Omega)$ yields exactly (6.1.1). □

6.2 Recovery sequences with prescribed boundary values

To prove Result 3, we first reduce to approximations of Sobolev functions attaining boundary value u_0 in $\mathrm{L}^\infty(\mathbb{R}^n)$, and then we extend by density to arbitrary u_0 (i.e. not necessarily bounded) via strict approximation (Proposition 3.15) to reach arbitrary BV functions. Whenever useful, we will here use the notation $\mathcal{F}_{u_0}$ in place of $\mathcal{F}$ to stress the dependence on the boundary datum u_0.

Theorem 6.3 (recovery sequences in $\mathrm{W}^{1,1}$ under bounded datum)**.** *We consider $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n) \cap \mathrm{L}^\infty(\mathbb{R}^n)$ and Ω bounded Lipschitz in $\mathbb{R}^n$. We let $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ be Borel, convex in the second entry, with linear growth (possibly even dropping the coercivity condition (ii) in Assumption 1.1) and such that f, f^∞ are continuous. Then, every $w \in \mathrm{W}^{1,1}(\Omega)$ admits some $(v_k)_k$ in $\mathrm{W}^{1,1}_{u_0}(\Omega)$ satisfying each of the following:*

(i) *The sequence $(v_k)_k$ converges area–strictly in $\mathrm{BV}_{u_0}(\overline{\Omega})$ to w as $k \to \infty$;*

(ii) *It is $\lim\limits_{k\to\infty} f(.,\mathrm{D}v_k)(\Omega) = f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega})$;*

(iii) *For any admissible measure ν on Ω, one has $\lim\limits_{k\to\infty} \int_\Omega v_k^* \,\mathrm{d}\nu = \int_\Omega w^* \,\mathrm{d}\nu$.*

From (i)–(iii) applied to our usual measures $\mu_\pm$, we deduce the convergence $\lim\limits_{k\to\infty} \mathrm{F}_{u_0}[v_k] = \mathcal{F}_{u_0}[w]$.

Proof. We work with truncations w^M of any assigned function w at level M. We first select levels $M \geq \|u_0\|_{\mathrm{L}^\infty(\mathbb{R}^n)}$, and recall that this ensures $\overline{u^M}^{u_0} = (\overline{u}^{u_0})^M$ for any $u \in \mathrm{BV}(\Omega)$. Employing Theorem 2.105, we approximate w^M via a sequence $(u_\ell)_\ell$ in $\mathrm{W}^{1,1}_{u_0}(\Omega)$ such that $(u_\ell)_\ell$ converges to w^M area–strictly in $\mathrm{BV}_{u_0}(\overline{\Omega})$ as $\ell \to \infty$. We now claim that the sequence of truncations $u_\ell^M \in \mathrm{W}^{1,1}_{u_0}(\Omega)$ preserves the convergence, that is $u_\ell^M \rightharpoonup w^M$ area–strictly in $\mathrm{BV}_{u_0}(\overline{\Omega})$ for $\ell \to \infty$. Indeed, assuming $U \Supset \Omega$ is open and bounded in $\mathbb{R}^n$, one inequality in the convergence of the derivative measures follows from the $\mathrm{L}^1(U)$–convergence of the extensions, coupled with the semicontinuity Theorem 2.67 for $f(\xi) = \sqrt{1+|\xi|^2}$. On the other hand, it is $|\nabla u^M|\mathcal{L}^n \leq |\nabla u|\mathcal{L}^n$ and $|\mathrm{D}^s u^M| \leq |\mathrm{D}^s u|$ as measures on Ω for any $u \in \mathrm{BV}(\Omega)$, therefore in particular

$$
\begin{aligned}
\mathcal{A}\left[\overline{u^M}^{u_0}\right](U) \leq \limsup_{\ell\to\infty} \mathcal{A}\left[\overline{u_\ell^M}^{u_0}\right](U) &= \limsup_{\ell\to\infty} \left(\int_U \sqrt{1+|\nabla \overline{u_\ell^M}^{u_0}|^2}\,\mathrm{d}x + \left|\mathrm{D}^s u_\ell^M\right|(\Omega) \right) \\
&\leq \limsup_{\ell\to\infty} \left(\int_U \sqrt{1+|\nabla \overline{u_\ell}^{u_0}|^2}\,\mathrm{d}x + |\mathrm{D}^s u_\ell|(\Omega) \right) \\
&= \lim_{\ell\to\infty} \mathcal{A}[\overline{u_\ell}^{u_0}](U) = \mathcal{A}[\overline{u^M}^{u_0}](U)\,,
\end{aligned}
$$

whence area–strict convergence. We fall thus into the conditions of Theorem 2.70, which yields

$$
\lim_{\ell\to\infty} f\big(.,\mathrm{D}{u_\ell}^M\big)(\Omega) = f\big(.,\mathrm{D}\overline{w^M}^{u_0}\big)\big(\overline{\Omega}\big) \tag{6.2.1}
$$

and as a consequence of the strong convergence in Lemma 2.100(iii) it even applies

$$
\lim_{M\to\infty} f\big(.,\mathrm{D}\overline{w^M}^{u_0}\big)\big(\overline{\Omega}\big) = f\big(.,\mathrm{D}\overline{w}^{u_0}\big)\big(\overline{\Omega}\big)\,. \tag{6.2.2}
$$

Concerning the measure integrals instead, we make use of the $\mathcal{H}^{n-1}$–a.e. convergence $\big(u_\ell^M\big)^* \to \big(w^M\big)^*$ in Ω of Theorem 2.104 as $\ell \to \infty$, of the assumption (C1) for ν and of the uniform bound $||u_\ell^M||_{\mathrm{L}^\infty(\Omega)} \leq M$ in ℓ to infer by dominated convergence that

$$
\lim_{\ell\to\infty} \int_\Omega \big(u_\ell^M\big)^* \,\mathrm{d}\nu = \int_\Omega \big(w^M\big)^* \,\mathrm{d}\nu\,. \tag{6.2.3}
$$

Sending now M to infinity, Lemma 3.8 applied to the measure ν induces

$$\lim_{M\to\infty}\int_\Omega \left(w^M\right)^* \,\mathrm{d}\nu = \int_\Omega w^* \,\mathrm{d}\nu\,. \tag{6.2.4}$$

We finally combine the convergences above by selecting a sequence of levels $(M_k)_k$ with $M_k \geq \|u_0\|_{\mathrm{L}^\infty(\mathbb{R}^n)}$ for all k, $\lim_{k\to\infty} M_k = \infty$, and defining

$$v_k := (u_{\ell_k})^{M_k} \in \mathrm{W}^{1,1}_{u_0}(\Omega)\,,$$

where for each k we set ℓ_k large enough to have area–strict convergence of v_k to w in $\mathrm{BV}_{u_0}\big(\overline{\Omega}\big)$ for $k\to\infty$, that is our (i). Finally, from (6.2.1), (6.2.2) in the specific case of v_k we deduce (ii), whereas (6.2.3) and (6.2.4) together build up (iii). □

We next record a useful approximation when treating unbounded boundary data.

Lemma 6.4 (varying the boundary values)**.** *We take $f\colon \mathbb{R}^n\times\mathbb{R}^n \to [0,\infty)$ Borel, convex in the second entry and with linear growth (alternatively, just imposing the upper bound $f(x,\xi)\leq \beta(|\xi|+1)$). Moreover, we assume that the functions u_0, $(u_{0,k})_k$ are in $\mathrm{W}^{1,1}(\mathbb{R}^n)$ and $u_{0,k}\to u_0$ in $\mathrm{W}^{1,1}(\mathbb{R}^n)$ as $k\to\infty$. For the open, bounded and Lipschitz set Ω in $\mathbb{R}^n$, we fix a sequence of functions $z_k\in \mathrm{W}^{1,1}_{u_{0,k}}(\Omega)$ for $k\in\mathbb{N}$. Then, there exists a sequence $(u_k)_k$ in $\mathrm{W}^{1,1}_{u_0}(\Omega)$ such that $u_k - z_k\to 0$ in $\mathrm{W}^{1,1}(\Omega)$ for $k\to\infty$ – and consequently also $(\overline{u_k}^{u_0} - \overline{z_k}^{u_{0,k}})_k$ converges to 0 in $\mathrm{W}^{1,1}(\mathbb{R}^n)$ – and at the same time satisfying*

$$\lim_{k\to\infty}\left(\int_\Omega f(\,\cdot\,,\nabla u_k)\,\mathrm{d}x - \int_\Omega f(\,\cdot\,,\nabla z_k)\,\mathrm{d}x\right) = 0\,. \tag{6.2.5}$$

Moreover, for any admissible measure ν on Ω we also have

$$\lim_{k\to\infty}\left(\int_\Omega u_k^*\,\mathrm{d}\nu - \int_\Omega z_k^*\,\mathrm{d}\nu\right) = 0\,. \tag{6.2.6}$$

Proof. Introducing the functions $u_k := z_k - u_{0,k} + u_0$ for every k, it is $u_k\in \mathrm{W}^{1,1}_{u_0}(\Omega)$ for all k, and the convergence of $(u_{0,k})_k$ induces strong convergence of $(u_k - z_k)_k$ to 0 in $\mathrm{W}^{1,1}(\Omega)$. Then, the strict continuity of Theorem 2.23 yields the $\mathrm{W}^{1,1}(\mathbb{R}^n)$–convergence of $(\overline{u_k}^{u_0} - \overline{z_k}^{u_{0,k}})_k$ to 0.

To achieve (6.2.5), we make use of the pointwise convexity estimate (2.6.10) and of the linear–growth assumption to deduce $f(.,\nabla u_k) - f(.,,\nabla w_k) \leq f^\infty(.,\nabla u_k - \nabla w_k) \leq \beta|\nabla u_k - \nabla w_k|$ a.e. in Ω for every $k\in\mathbb{N}$, so $u_k - z_k\to 0$ in $\mathrm{W}^{1,1}(\Omega)$ determines

$$\limsup_{k\to\infty}\left(\int_\Omega f(\,\cdot\,,\nabla u_k)\,\mathrm{d}x - \int_\Omega f(\,\cdot\,,\nabla z_k)\,\mathrm{d}x\right) \leq 0\,.$$

An exchange of the roles of u_k and z_k above yields the opposite inequality, and (6.2.5) results.

To verify the last convergence for admissible ν, we exploit Remark 3.13 with both $\pm(u_k - z_k)$ in $\mathrm{W}^{1,1}(\Omega)$ and find some $C\in[0,\infty)$ such that

$$\begin{aligned}\left|\int_\Omega u_k^*\,\mathrm{d}\nu - \int_\Omega z_k^*\,\mathrm{d}\nu\right| &\leq C\bigg(\int_\Omega |\nabla(u_k - z_k)|\,\mathrm{d}x + \int_{\partial\Omega}|u_{0,k} - u_0|\,\mathrm{d}\mathcal{H}^{n-1}\bigg)\\ &= C||\nabla(u_{0,k} - u_0)||_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} + C||u_{0,k} - u_0||_{\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})}\,,\end{aligned}$$

where by Theorem 2.23 the right–hand side decreases to zero as $k\to\infty$. Then, the proof of (6.2.6) and consequently of the full statement is complete. □

6.2.1 Proof of the consistency Result 3

We finally turn to the main recovery sequence result with the prescribed boundary datum u_0.

Proof of Result 3. Assume $w \in \mathrm{BV}(\Omega)$. To prove (1.2.5), we claim that specifically it holds

$$\lim_{k\to\infty} f(.,\mathrm{D}w_k)(\Omega) = f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega}), \tag{6.2.7}$$

$$\lim_{k\to\infty}\int_\Omega w_k^*\,\mathrm{d}\mu_- = \int_\Omega w^+\,\mathrm{d}\mu_- \qquad \text{and} \qquad \lim_{k\to\infty}\int_\Omega w_k^*\,\mathrm{d}\mu_+ = \int_\Omega w^-\,\mathrm{d}\mu_+ \tag{6.2.8}$$

for some $(w_k)_k$ in $\mathrm{W}^{1,1}_{u_0}(\Omega)$ such that $w_k \rightharpoonup w$ area–strictly in $\mathrm{BV}_{u_0}(\overline{\Omega})$ as $k\to\infty$.

Step 1. We consider first $u_0 \in \mathrm{L}^\infty(\mathbb{R}^n)$. We exploit the essential assumption $\mu_+ \perp \mu_-$ to apply Theorem 6.2 and obtain a sequence $(u_k)_k$ in $\mathrm{W}^{1,1}(\Omega)$ converging to w area–strictly in $\mathrm{BV}(\Omega)$ with $\lim_{k\to\infty} f(.,\mathrm{D}\overline{u_k}^{u_0})(\overline{\Omega}) = f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega})$ and

$$\lim_{k\to\infty}\int_\Omega u_k^*\,\mathrm{d}\mu_- = \int_\Omega w^+\,\mathrm{d}\mu_-\,, \qquad \lim_{k\to\infty}\int_\Omega u_k^*\,\mathrm{d}\mu_+ = \int_\Omega w^-\,\mathrm{d}\mu_+\,.$$

Being u_0 bounded, we may apply Theorem 6.3 to each u_k and obtain a sequence $(v_k^\ell)_\ell$ in $\mathrm{W}^{1,1}_{u_0}(\Omega)$ with $v_k^\ell \rightharpoonup u_k$ area–strictly in $\mathrm{BV}_{u_0}(\overline{\Omega})$ for $\ell\to\infty$,

$$\lim_{\ell\to\infty} f(.,\mathrm{D}v_k^\ell)(\Omega) = f(.,\mathrm{D}\overline{u_k}^{u_0})(\overline{\Omega}),$$

$$\lim_{\ell\to\infty}\int_\Omega (v_k^\ell)^*\,\mathrm{d}\mu_- = \int_\Omega u_k^*\,\mathrm{d}\mu_- \qquad \text{and} \qquad \int_\Omega (v_k^\ell)^*\,\mathrm{d}\mu_+ = \int_\Omega u_k^*\,\mathrm{d}\mu_+\,.$$

The claim is therefore reached by passing to a subsequence for $w_k := v_k^{\ell_k} \in \mathrm{W}^{1,1}_{u_0}(\Omega)$ with ℓ_k large enough, observing that $w_k \rightharpoonup w$ area–strictly in $\mathrm{BV}_{u_0}(\overline{\Omega})$ for $k\to\infty$ with the sequence verifying (6.2.7), (6.2.8).

Step 2. We assume now $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ and determine via density a sequence $(u_{0,k})_k$ in $\mathrm{W}^{1,1}(\mathbb{R}^n)\cap \mathrm{L}^\infty(\mathbb{R}^n)$ with $u_{0,k}\to u_0$ in $\mathrm{W}^{1,1}(\mathbb{R}^n)$ as $k\to\infty$. Exploiting the result of *Step 1* for each datum $u_{0,k}$, we find some $(w_k^\ell)_\ell$ in $\mathrm{W}^{1,1}_{u_{0,k}}(\Omega)$ with $w_k^\ell \rightharpoonup w$ area–strictly in $\mathrm{BV}_{u_{0,k}}(\overline{\Omega})$ for $\ell\to\infty$, $\lim\limits_{\ell\to\infty} f(.,\mathrm{D}w_k^\ell)(\Omega) = f(.,\mathrm{D}\overline{w}^{u_{0,k}})(\overline{\Omega})$, and $\lim\limits_{\ell\to\infty}\int_\Omega (w_k^\ell)^*\,\mathrm{d}\mu_\pm = \int_\Omega w^\mp\,\mathrm{d}\mu_\pm$. We pass once again to a sequence $(z_k)_k$ in $\mathrm{W}^{1,1}_{u_{0,k}}(\Omega)$ defined by $z_k := w_k^{\ell_k} \in \mathrm{W}^{1,1}_{u_{0,k}}(\Omega)$ with ℓ_k suitably large, then we record: $\overline{z_k}^{u_{0,k}}$ converges to $\overline{w}^{u_0}$ area–strictly in $\mathrm{BV}(U)$ for $k\to\infty$ for any $U \Supset \Omega$ bounded,

$$\lim_{k\to\infty} f(.,\mathrm{D}z_k)(\Omega) = f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega})\,, \quad \text{and} \tag{6.2.9}$$

$$\lim_{k\to\infty}\int_\Omega z_k^*\,\mathrm{d}\mu_\pm = \int_\Omega w^\mp\,\mathrm{d}\mu_\pm\,. \tag{6.2.10}$$

Finally, an application of Lemma 6.4 for the measures $\mu_\pm$ and $(z_k)_k$ yields a sequence $(w_k)_k$ in $\mathrm{W}^{1,1}_{u_0}(\Omega)$ with $w_k - z_k \to 0$ in $\mathrm{W}^{1,1}(\Omega)$, $\overline{w_k}^{u_0} - \overline{z_k}^{u_{0,k}} \to 0$ (strongly) in $\mathrm{BV}(U)$, and such that

$$\lim_{k\to\infty}\Big(f(.,\mathrm{D}w_k)(\Omega) - f(.,\mathrm{D}z_k)(\Omega)\Big) = 0\,, \tag{6.2.11}$$

$$\lim_{k\to\infty}\left(\int_\Omega w_k^*\,\mathrm{d}\mu_\pm - \int_\Omega z_k^*\,\mathrm{d}\mu_\pm\right) = 0\,. \tag{6.2.12}$$

The combination of (6.2.9) and (6.2.11) implies (6.2.7), while joining (6.2.10) and (6.2.12) together we get (6.2.8).

A rearrangement of the terms determines $\mathcal{F}_{u_0}[w] = \lim_{k\to\infty} \mathrm{F}_{u_0}[w_k]$, thus the claimed equality (1.2.5) follows.

Moreover, if we assume the f^∞–IC in Ω with constant 1 for the pair (μ_-, μ_+) and the symmetric $\widetilde{f^\infty}$–IC in Ω with constant 1 for (μ_+, μ_-), the functional $\mathcal{F}$ results lower semicontinuous by Result 1 respect to the $\mathrm{L}^1(\Omega)$–topology. Then, the definition of relaxed functional of F_{u_0} yields

$$\mathrm{F}_{\mathrm{rel}}[w] := \inf\left\{ \liminf_{k\to\infty} \mathrm{F}_{u_0}[u_k] \colon\ \mathrm{W}^{1,1}_{u_0}(\Omega) \ni u_k \to w \text{ in } \mathrm{L}^1(\Omega) \right\} \geq \mathcal{F}[w] \quad \text{ for all } w \in \mathrm{BV}(\Omega)\,.$$

Conversely, we can approximate any $w \in \mathrm{BV}(\Omega)$ via a recovery sequence $(w_k)_k$ in $\mathrm{W}^{1,1}_{u_0}(\Omega)$, so that bringing into play (1.2.5) we have

$$\mathcal{F}_{u_0}[w] = \lim_{k\to\infty} \mathrm{F}_{u_0}[w_k] \geq \mathrm{F}_{\mathrm{rel}}[w] \quad \text{ for all } w \in \mathrm{BV}(\Omega)\,.$$

In conclusion, $\mathcal{F}_{(u_0)} = \mathrm{F}_{\mathrm{rel}}$ on $\mathrm{BV}(\Omega)$ and the Result 3 is verified. □

As a matter of fact, one may check that the sequence $(w_k)_k$ in the statement of Result 3 also satisfies the pointwise convergence of representatives

$$w_k^* \xrightarrow[k\to\infty]{} w^-\mathbb{1}_{\Omega_+} + w^-\mathbb{1}_{\Omega_+} \quad |\mu|\text{–a.e. in } \Omega\,,$$

where as usual $|\mu| = \mu_+ + \mu_-$ and $\Omega_\pm$ is the corresponding Hahn's decomposition of Ω. The statement follows from the pointwise convergence of all sequences involved in the proofs of Theorem 6.2 and Theorem 6.3, which in turn relies strongly on the vanishing condition for $\mu_\pm$ imposed in Assumption 1.2.

Corollary 6.5 (convexity). *We assume the same set of hypotheses of Result 3 and we additionally suppose that (μ_-, μ_+) satisfies the f^∞–IC in Ω with constant 1 and (μ_+, μ_-) satisfies the $\widetilde{f^\infty}$–IC in Ω with constant 1. Then the functional $\mathcal{F}$ is convex on $\mathrm{BV}(\Omega)$.*

Proof. We record that $\mathcal{F} \equiv \mathrm{F}_{u_0}$ on $\mathrm{W}^{1,1}_{u_0}(\Omega)$, with the functional F_{u_0} evidently convex on its space of definition – being the sum of a convex term and a linear one. Assume now $\lambda \in (0,1)$, u and $v \in \mathrm{BV}(\Omega)$ with corresponding area–strict approximations $(u_k)_k$, $(v_k)_k$ from Result 3. Clearly, the sequence of functions $w_k := \lambda u_k + (1-\lambda)v_k \in \mathrm{W}^{1,1}_{u_0}(\Omega)$ converges to $w := \lambda u + (1-\lambda)v$ in $\mathrm{L}^1(\Omega)$ for $k \to \infty$. From convexity in $\mathrm{W}^{1,1}$ and the semicontinuity Result 1, we directly compute

$$\mathcal{F}[w] \leq \liminf_{k\to\infty} \mathcal{F}[w_k] = \liminf_{k\to\infty} \mathrm{F}_{u_0}[w_k] \leq \lambda \liminf_{k\to\infty} \mathrm{F}_{u_0}[u_k] + (1-\lambda) \liminf_{k\to\infty} \mathrm{F}_{u_0}[v_k] = \lambda\mathcal{F}[u] + (1-\lambda)\mathcal{F}[v]\,,$$

where we applied (1.2.5) to both (u_k) and $(v_k)_k$. The functional $\mathcal{F}$ results then to be convex on $\mathrm{BV}(\Omega)$, as required. □

6.3 Γ-convergence of linear-growth functionals when varying the boundary values

We fix two admissible measures $\mu_\pm$ and let our generalized Dirichlet datum vary in the sequence $(u_{0,k})_k$ in $\mathrm{W}^{1,1}(\mathbb{R}^n)$. In what follows, the functional dependence on $\mu := \mu_+ - \mu_-$ and the boundary datum is stressed out by the writings $\mathcal{F}^\mu_{u_0}$, $\mathcal{F}^\mu_{u_{0,k}} \colon \mathrm{BV}(\Omega) \to \mathbb{R}$. We recall that it is

$$\mathcal{F}^\mu_{u_{0,k}}[w] := f(.,\mathrm{D}\overline{w}^{u_{0,k}})(\overline{\Omega}) - \int_\Omega w^+ \,\mathrm{d}\mu_- + \int_\Omega w^- \,\mathrm{d}\mu_+ \qquad \text{for all } w \in \mathrm{BV}(\Omega)\,.$$

Specifically, scope of this section is extending the lower semicontinuity Result 1 together with Result 3 to prove Γ–convergence of the sequence of functionals $\mathcal{F}^\mu_{u_{0,k}}$ according to Definition 2.49. To this aim, we shall impose for the sequence of boundary data at least strict convergence near $\partial\Omega$. Our primary outcome is stated in the following.

Proposition 6.6 (Γ–convergence of $\mathcal{F}^{\mu}_{u_{0,k}}$)**.** *We consider a continuous Borel integrand $f\colon \mathbb{R}^n\times\mathbb{R}^n \to [0,\infty)$ admissible as in Assumption 1.1 with (H1)–(H2). On the open, bounded and Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, we take mutually singular, admissible measures $\mu_\pm$ such that (μ_-,μ_+) satisfies the f^∞–IC in Ω with constant 1, and (μ_+,μ_-) satisfies the $\widetilde{f^\infty}$–IC in Ω with constant 1. If the sequence $(u_{0,k})_k$ is in $\mathrm{W}^{1,1}(\mathbb{R}^n)$ and the sequence of traces of $u_{0,k}$ converges to the trace of u_0 in $\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})$ as $k\to\infty$, then we have $\mathcal{F}^{\mu}_{u_0} = \Gamma\text{–}\lim_k \mathcal{F}^{\mu}_{u_{0,k}}$ in $\mathrm{L}^1(\Omega)$.*

Proof. We need to verify both the liminf inequality (i) and the recovery sequence statement (ii). To prove (i), we suppose that a sequence $(w_k)_k$ in $\mathrm{BV}(\Omega)$ converges in $\mathrm{L}^1(\Omega)$ to some BV function w. We notice that each element $\mathcal{F}^{\mu}_{u_{0,k}}$ differs from the functional $\mathcal{F}^{\mu}_{u_0}$ on the boundary term solely, that is

$$\mathcal{F}^{\mu}_{u_{0,k}}[w] = \mathcal{F}^{\mu}_{u_0}[w] + \int_{\partial\Omega} f^\infty(\,\cdot\,,(w-u_{0,k})\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1} - \int_{\partial\Omega} f^\infty(\,\cdot\,,(w-u_0)\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1} \qquad \text{for all } k\in\mathbb{N},$$

and we set $\mathcal{I}_k := \mathcal{F}^{\mu}_{u_{0,k}}[w_k] - \mathcal{F}^{\mu}_{u_0}[w_k]$ for assigned k. If $\beta>0$ is the upper bound in the linear-growth of f, via convexity and the estimates in (2.6.1), (2.6.2) applied to f in the second entry, we observe that $|f^\infty(.,\xi) - f^\infty(.,\tau)| \le \beta|\xi-\tau|$ for all ξ, $\tau\in\mathbb{R}^n$. Therefore, the convergence of traces implies

$$\begin{aligned} |\mathcal{I}_k| &:= \left| \int_{\partial\Omega} f^\infty(\,\cdot\,,(w_k-u_{0,k})\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1} - \int_{\partial\Omega} f^\infty(\,\cdot\,,(w_k-u_0)\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1} \right| \\ &\le \beta\|u_{0,k}-u_0\|_{\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})} \xrightarrow[k\to\infty]{} 0\,. \end{aligned} \tag{6.3.1}$$

Then, with the help of semicontinuity Result 1, we conclude

$$\liminf_{k\to\infty} \mathcal{F}^{\mu}_{u_{0,k}}[w_k] = \liminf_{k\to\infty} \mathcal{F}^{\mu}_{u_0}[w_k] + \lim_{k\to\infty} \mathcal{I}_k \ge \mathcal{F}^{\mu}[w]\,,$$

which yields the required estimate in (i).

The verification of the remaining condition (ii) is easily done by fixing any function $w\in\mathrm{BV}(\Omega)$ and the constant sequence $w_k := w$ for all k. Then, employing the notation above and the previous estimate (6.3.1), we have

$$\lim_{k\to\infty} \mathcal{F}^{\mu}_{u_{0,k}}[w_k] = \lim_{k\to\infty} \mathcal{F}^{\mu}_{u_{0,k}}[w] = \mathcal{F}^{\mu}_{u_0}[w] + \lim_{k\to\infty} \mathcal{I}_k = \mathcal{F}^{\mu}_{u_0}[w]\,.$$

We have finally achieved the Γ–convergence result. □

With Γ–convergence at hand, the next natural step is proving the convergence of infima according to Theorem 2.52.

Proposition 6.7. *We take Borel f as in Assumption 1.1(H1)–(H2), $\mu_\pm$ admissible measures on open bounded Lipschitz $\Omega\subseteq\mathbb{R}^n$ such that (μ_-,μ_+) satisfies the f^∞–IC in Ω with constant $C\in[0,1)$ and (μ_+,μ_-) satisfies the $\widetilde{f^\infty}$–IC on Ω with same constant. Moreover, we consider u_0, $(u_{0,k})_k$ in $\mathrm{W}^{1,1}(\mathbb{R}^n)$ with $u_{0,k}\to u_0$ in $\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})$ as $k\to\infty$ in the sense of traces. Then, the sequence of functionals $\mathcal{F}^{\mu}_{u_{0,k}}$ is equi–coercive in $\mathrm{L}^1(\Omega)$.*

Proof. From Proposition 2.48, it suffices to determine an $\mathrm{L}^1(\Omega)$–lower semicontinuous and coercive functional $\Psi\colon \mathrm{BV}(\Omega)\to\overline{\mathbb{R}}$ with $\mathcal{F}^{\mu}_{u_{0,k}}\ge\Psi$ for all k. As in Proposition 5.2(2), we apply the ICs and the linear–growth condition with constants α, β to write

$$\begin{aligned} \mathcal{F}^{\mu}_{u_{0,k}}[w] &\ge \alpha(1-C)\left(|\mathrm{D}w|(\Omega) + \int_{\partial\Omega}|w|\,\mathrm{d}\mathcal{H}^{n-1}\right) - M|\Omega| - \beta\|u_{0,k}\|_{\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})} \\ &\ge \alpha(1-C)\|w\|_{\mathrm{BV}(\Omega)}/\widetilde{\gamma}_n - M|\Omega| - \beta\sup_{k\in\mathbb{N}}\|u_{0,k}\|_{\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})} \\ &=: \Psi[w] \qquad \text{for all } w\in\mathrm{BV}(\Omega)\,, \end{aligned} \tag{6.3.2}$$

via an application of Poincaré's inequality (2.2.4). Here, $\sup_{k\in\mathbb{N}} ||u_{0,k}||_{\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})}$ is finite in view of the L^1–convergence of the traces $(u_{0,k})_k$. Our functional Ψ is visibly coercive in $\mathrm{BV}(\Omega)$; moreover, if $(w_k)_k$ is a sequence in $\mathrm{BV}(\Omega)$ converging to w with respect to the $\mathrm{L}^1(\Omega)$–norm convergence, then via lower semicontinuity of the variation, it is

$$\liminf_{k\to\infty} ||w_k||_{\mathrm{BV}(\Omega)} = \lim_{k\to\infty} |w_k\mathcal{L}^n|(\Omega) + \liminf_{k\to\infty} |\mathrm{D}w_k|(\Omega) \geq |w\mathcal{L}^n|(\Omega) + |\mathrm{D}w|(\Omega) = ||w||_{\mathrm{BV}(\Omega)}\,,$$

so Ψ is even lower semicontinuous, and the claimed equi–coercivity of $\mathcal{F}^{\mu}_{u_{0,k}}$ is verified. □

We continue our analysis by proving that minimizing sequences for $\mathcal{F}^{\mu}_{u_{0,k}}$ satisfy by equi–coercivity an $\mathrm{L}^1(\Omega)$–compactness result.

Corollary 6.8. *We assume the same set of hypotheses as in Proposition 6.7, additionally imposing continuity of f. For any fixed $k \in \mathbb{N}$, we suppose that each $u_k \in \mathrm{BV}(\Omega)$ minimizes the corresponding functional $\mathcal{F}^{\mu}_{u_{0,k}}$ – that is $\mathcal{F}^{\mu}_{u_{0,k}}[u_k] = \min_{\mathrm{BV}(\Omega)} \mathcal{F}^{\mu}_{u_{0,k}}$, where we know a minimum exists in view of Result 2. Then, $(u_k)_k$ admits an $\mathrm{L}^1(\Omega)$–converging subsequence to some $u \in \mathrm{BV}(\Omega)$.*

Proof. First of all, we observe that the strict convergence assumption on $(u_{0,k})_k$ yields

$$\mathcal{F}^{\mu}_{u_{0,k}}[0] = \int_{\Omega} f(.,0)\,\mathrm{d}x + \int_{\partial\Omega} \widetilde{f^{\infty}}(.,u_{0,k})\,\mathrm{d}\mathcal{H}^{n-1} \leq \beta|\Omega| + \beta \sup_{k\in\mathbb{N}} ||u_{0,k}||_{\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})} < \infty \qquad \text{for all } k \in \mathbb{N}\,.$$

Then, for a sequence $(u_k)_k$ as in the statement and making use of (6.3.2), we write

$$\infty > \sup_{k\in\mathbb{N}} \mathcal{F}^{\mu}_{u_{0,k}}[0] \geq \sup_{k\in\mathbb{N}} \left(\min_{\mathrm{BV}(\Omega)} \mathcal{F}^{\mu}_{u_{0,k}} \right) = \sup_{k\in\mathbb{N}} \mathcal{F}^{\mu}_{u_{0,k}}[u_k] \geq \sup_{k\in\mathbb{N}} \Psi[u_k]\,.$$

Ultimately, the coercivity of Ψ and Theorem 2.22 induce the existence of a suitable subsequence converging in $\mathrm{L}^1(\Omega)$ to some limit $u \in \mathrm{BV}(\Omega)$, as claimed. □

At this point, we record that the Γ–convergence of Proposition 6.6 combined with equi–coerciveness of Proposition 6.7 determines via Theorem 2.52 convergence of the infimal values to the minimum value of the Γ–limit $\mathcal{F}^{\mu}_{u_0}$. Note that the constant C in the isoperimetric conditions must be *strictly smaller* than 1 to enforce equi–coerciveness. At the same time, by imposing continuity of the integrand, we argue via Result 2 that $\mathcal{F}^{\mu}_{u_0}$ and all the functionals $\mathcal{F}^{\mu}_{u_{0,k}}$ attain a BV minimum. Altogether, we have obtained the following result.

Corollary 6.9 (convergence of minima of $\mathcal{F}^{\mu}_{u_{0,k}}$)**.** *Under the hypotheses of Proposition 6.6 and additionally assuming that the constant for the ICs is $C \in [0,1)$, we have*

$$\min_{\mathrm{BV}(\Omega)} \mathcal{F}^{\mu}_{u_0} = \lim_{k\to\infty} \left(\min_{\mathrm{BV}(\Omega)} \mathcal{F}^{\mu}_{u_{0,k}} \right) \qquad \textit{for all } u_0, (u_{0,k})_k \in \mathrm{W}^{1,1}(\mathbb{R}^n)\,,\ u_{0,k}|_{\Omega} \xrightarrow[k\to\infty]{} u_0|_{\Omega} \textit{ strictly}\,. \tag{6.3.3}$$

Moreover, any sequence $(u_k)_k$ in $\mathrm{BV}(\Omega)$ of minimizers in the sense $\mathcal{F}^{\mu}_{u_{0,k}}[u_k] = \min_{\mathrm{BV}(\Omega)} \mathcal{F}^{\mu}_{u_{0,k}}$ converges up to subsequences in $\mathrm{L}^1(\Omega)$ to some $u \in \mathrm{BV}(\Omega)$ minimizing $\mathcal{F}^{\mu}_{u_0}$.

We record that the existence of a converging subsequence for $(u_k)_k$ in the statement follows from Corollary 6.8, whereas the minimizing property of the limit u is a consequence of (6.3.3).

6.4 Γ-convergence of linear-growth functionals when varying the measures

As usual, we work on open, bounded, and Lipschitz sets Ω in $\mathbb{R}^n$. Our principal result is Theorem 6.11, stating Γ–convergence of the functionals $\mathcal{F}^{\mu_k}_{u_0}$ to $\mathcal{F}^{\mu}_{u_0}$ for sequences of Radon measures $(\mu_k)_k$ converging *strictly* to some Radon measure μ on Ω. Here, the pairs $((\mu_k)_\mp, (\mu_k)_\pm)$, $(\mu_\mp, \mu_\pm)$ satisfy the f^∞– (or $\widetilde{f^\infty}$–, respectively) isoperimetric condition in Ω with constant $C = 1$. We observe that the assumptions to be set in Theorem 6.11 for the measures μ_k (at least for k large enough) are indeed the same required to show the semicontinuity Result 1 for the functional $\mathcal{F}^{\mu}_{u_0}$. What's more, to prove the liminf inequality of the Γ–convergence we actually rely on a generalized version of the good exterior approximation result in Lemma 4.4, and we then base our analysis in a similar way to our previous proof of (first parametric, then non-parametric) semicontinuity. Whenever possible, we conveniently avoid repetitions of the analogous steps encountered in Chapter 4 in the corresponding proofs, rather highlighting just the steps where the presence of sequences $(\mu_k)_k$ plays a distinctive role. In the proof of the Γ–convergence, the limsup estimate (ii′) – or rather, its variant (ii) – is obtained by a combination of a method for recovery sequences adapted from [67] with the assumed strict convergence of measures.

It is worth noting that, whereas the limsup inequality does not require other assumptions than the admissibility of mutually singular measures, the validity of the isoperimetric condition for μ_k with $k \gg 1$ is unavoidable to obtain the liminf inequality. This is evident from the fact that lower semicontinuity fails already for the single functional $\mathcal{F}^{\mu}_{u_0}$ if μ does *not* satisfy the f^∞–IC with constant 1, as already illustrated in Example 4.7. Furthermore, we claim that our choice of measure convergence is sharp. In fact, Example 6.10 shows that strict convergence of μ_k to μ is necessary to Theorem 6.14, since there exist sequences of measure converging (only) weakly–$*$ in $\mathrm{RM}(\Omega)$ which violate the liminf inequality for the corresponding sequence of functionals.

Example 6.10 (failure of liminf inequality for $\mathcal{F}^{\mu_k}_{u_0}$ under weak–$*$ convergence only)**.** For $n = 1$, let Ω be the open real interval (a, b), for $a < 0 < 1 < b$, $f(\xi) := |\xi|$, $u_0 :\equiv 0$, and consider the sequence of functions $u_k \colon \Omega \to [0, 1]$ with

$$u_k(x) := \begin{cases} 1, & a < x \le 0; \\ 1 - kx, & 0 < x \le \frac{1}{k}; \\ 0, & \frac{1}{k} < x < b. \end{cases}$$

It is easy to check that $u_k \to u := \mathbb{1}_{(a,0)}$ in $\mathrm{L}^1(\Omega)$ for $k \to \infty$, with $u_k \in \mathrm{BV}(\Omega)$ for every k, and that $\mathrm{D}u_k = -k\mathcal{L}^1 \llcorner (0, \frac{1}{k})$, $\mathrm{D}u = \mathrm{D}\mathbb{1}_{(a,0)}$ on Ω – thus $|\mathrm{D}u|(\Omega) = |(1-0)\mathcal{H}^0 \llcorner \{0\}| = 1$. Considering the Dirac delta measures $(\mu_k)_+ := \delta_{1/k}$, $(\mu_k)_- := \delta_0$ for any k, we let $\mu_k := (\mu_k)_+ - (\mu_k)_-$. Then, we already know from Example 2.8 that $(\mu_k)_k$ converges to $\mu :\equiv 0$ weakly–$*$ but not strictly in $\mathrm{RM}(\Omega)$ as $k \to \infty$. We compute now

$$\begin{aligned}
\mathcal{F}^{\mu_k}_{u_0}[u_k] &= |\mathrm{D}u_k|(\Omega) + \int_{\partial\Omega} |u_k| \,\mathrm{d}\mathcal{H}^0 - \int_\Omega u_k \,\mathrm{d}\delta_0 + \int_\Omega u_k \,\mathrm{d}\delta_{1/k} \\
&= \int_{(0,1/k)} |-k| \,\mathrm{d}\mathcal{L}^1 + |u_k(a)| + |u_k(b)| - \int_\Omega u_k \,\mathrm{d}\delta_0 + \int_\Omega u_k \,\mathrm{d}\delta_{1/k} \\
&= 1 + 1 - u_k(0) + u_k(1/k) = 1, \qquad \text{for all } k \in \mathbb{N}.
\end{aligned}$$

Consider the measures $\mu_\pm :\equiv 0$; for the limit functional with $\mu \equiv 0$ it is

$$\mathcal{F}^{\mu}_{u_0}[u] = |\mathrm{D}u|(\Omega) + \int_{\partial\Omega} |u| \,\mathrm{d}\mathcal{H}^1 - \int_\Omega u^+ \,\mathrm{d}\mu_- + \int_\Omega u^- \,\mathrm{d}\mu_+ = 1 + |u(a)| + |u(b)| - 0 = 2\,,$$

whence $\lim_{k\to\infty} \mathcal{F}^{\mu_k}_{u_0}[u_k] = 1 < 2 = \mathcal{F}^{\mu}_{u_0}[u]$, against the liminf requirement of Γ–convergence. We observe that, alternatively, any choice of $\mu_\pm := \delta_{x_0}$ for $x_0 \in \Omega \setminus \{0\}$ would work, since the component measures are still non–negative, finite on Ω with zero difference, and such that the measure terms in $\mathcal{F}^{\mu}_{u_0}[u]$ still cancel out because of continuity of the limit function u in x_0.

Next we state the main Γ–convergence result for the sequence $\mathcal{F}^{\mu_k}_{u_0}$.

Theorem 6.11 (Γ–convergence of $\mathcal{F}^{\mu_k}_{u_0}$)**.** *We consider $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ and a continuous integrand f under Assumption 1.1(H1)–(H2). Furthermore, we assume that $\mu_\pm$ and $(\mu_{k\pm})_k$ are admissible measures on the open, bounded, Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ such that:*

- *The measures μ_+ and μ_- are mutually singular on Ω;*
- *The pairs (μ_{k-}, μ_{k+}) satisfy the f^∞–IC in Ω with constant 1 for $k \gg 1$ and (μ_{k+}, μ_{k-}) satisfy the $\widetilde{f^\infty}$–IC in Ω with constant 1 for $k \gg 1$; and*
- *The sequences $(\mu_k)_\pm$ converge respectively to $\mu_\pm$ strictly in $\mathrm{RM}(\Omega)$ for $k \to \infty$.*

Then, the Γ–convergence $\mathcal{F}^{\mu}_{u_0} = \Gamma{-}\lim_k \mathcal{F}^{\mu_k}_{u_0}$ holds, under the usual convention $\mu := \mu_+ - \mu_-$ and $\mu_k := (\mu_k)_+ - (\mu_k)_-$ for all $k \in \mathbb{N}$.

The statement of Theorem 6.11 will be achieved by combining a recovery sequence argument (Theorem 6.13) with the minimum limit result of Theorem 6.14.

6.4.1 Limsup inequality

Lemma 6.12. *We take $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, Ω bounded Lipschitz set in $\mathbb{R}^n$, and any Borel function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ convex in the second variable with linear growth. Then, for $w \in \mathrm{C}(\Omega) \cap \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and admissible measures $\mu_\pm$, $(\mu_k)_\pm$ on Ω such that $(\mu_k)_\pm$ converges to $\mu_\pm$ strictly in $\mathrm{RM}(\Omega)$ as $k \to \infty$, it is*

$$\lim_{k\to\infty} \mathcal{F}^{\mu_k}_{u_0}[w] = \mathcal{F}^{\mu}_{u_0}[w]\,.$$

Proof. For any fixed $k \in \mathbb{N}$, we note that the integral terms in f cancel out in the difference $\mathcal{F}^{\mu_k}_{u_0}[w] - \mathcal{F}^{\mu}_{u_0}[w]$. Moreover, continuity of w allows us to identify all representatives in the measure integrals, thus we directly compute

$$\begin{aligned}\left|\mathcal{F}^{\mu_k}_{u_0}[w] - \mathcal{F}^{\mu}_{u_0}[w]\right| &= \left| -\int_\Omega w \,\mathrm{d}(\mu_k)_- + \int_\Omega w\,\mathrm{d}(\mu_k)_+ + \int_\Omega w\,\mathrm{d}\mu_- - \int_\Omega w\,\mathrm{d}\mu_+ \right| \\ &\le \left| \int_\Omega w\,\mathrm{d}\mu_- - \int_\Omega w\,\mathrm{d}(\mu_k)_- + \right| + \left| \int_\Omega w\,\mathrm{d}(\mu_k)_+ - \int_\Omega w\,\mathrm{d}\mu_+ \right| \xrightarrow[k\to\infty]{} 0\,,\end{aligned}$$

with the last convergences justified by the characterization of strict convergence in Proposition 2.10 applied with $\psi :\equiv 0$ and measures $(\mu_k)_\pm$, $\mu_\pm$ on Ω. □

We now employ Lemma 6.12 together with the smooth approximation result of Theorem 6.2 with arbitrary u_0 in $\mathrm{W}^{1,1}(\mathbb{R}^n)$ to deduce the recovery sequence property (ii) for $\mathcal{F}^{\mu_k}_{u_0}$. Observe that for the validity of Theorem 6.13 it is not necessary to assume the f^∞–IC for the measures μ_k or μ – differently from the liminf inequality result stated in Theorem 6.14. On the other hand, the mutual singularity assumption for the component measures of μ is required to establish the recovery sequence in Theorem 6.2.

Theorem 6.13 (existence of a recovery sequence for $\mathcal{F}^{\mu_k}_{u_0}$)**.** *With Assumption 1.1(H1) for f continuous, and assigned $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, we consider admissible measures $\mu_\pm$, $(\mu_k)_\pm$ on open bounded Lipschitz Ω in $\mathbb{R}^n$. We further assume $\mu_+ \perp \mu_-$ on Ω. If $(\mu_k)_\pm \rightharpoonup \mu_\pm$ strictly as Radon measures on Ω as $k \to \infty$, then every $w \in \mathrm{BV}(\Omega)$ admits a sequence $(w_k)_k$ in $\mathrm{BV}(\Omega)$ such that $w_k \to w$ in $\mathrm{L}^1(\Omega)$ and satisfying*

$$\lim_{k\to\infty} \mathcal{F}^{\mu_k}_{u_0}[w_k] = \mathcal{F}^{\mu}_{u_0}[w].$$

Proof. We fix $w \in \mathrm{BV}(\Omega)$. From the recovery sequence Theorem 6.2, in correspondence to w there is a sequence $(v_\ell)_\ell$, which in particular belongs to $\mathrm{C}(\Omega) \cap \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, $(v_\ell)_\ell$ converges to w in $\mathrm{L}^1(\Omega)$ as $\ell \to \infty$, and additionally

$$\lim_{\ell \to \infty} \mathcal{F}^{\mu}_{u_0}[v_\ell] = \mathcal{F}^{\mu}_{u_0}[w]. \tag{6.4.1}$$

We then exploit the strict convergence of measures and apply Lemma 6.12 to each v_ℓ to write

$$\lim_{k \to \infty} \mathcal{F}^{\mu_k}_{u_0}[v_\ell] = \mathcal{F}^{\mu}_{u_0}[v_\ell] \quad \text{ for all } \ell \in \mathbb{N}\,,$$

specifically in correspondence to any ℓ there exists some $m_\ell \in \mathbb{N}$ such that

$$\left|\mathcal{F}^{\mu_k}_{u_0}[v_j] - \mathcal{F}^{\mu}_{u_0}[v_j]\right| \le \frac{1}{\ell} \quad \text{ for all } k \ge m_\ell,\ j \in \{1, \dots, \ell\}\,, \tag{6.4.2}$$

and without loss of generality we may take $(m_\ell)_\ell$ increasing in ℓ and such that $m_\ell \to \infty$ for $\ell \to \infty$. We now define the sequence $(w_k)_k$ for $w_k := v_{\ell_k}$ whenever $m_{\ell_k} < k \le m_{\ell_k+1}$ for $\ell_k \in \mathbb{N}$. Then, it is clearly $||w_k - w||_{\mathrm{L}^1(\Omega)} = ||v_{\ell_k} - w||_{\mathrm{L}^1(\Omega)} \to 0$ as $k \to \infty$. Moreover, for any given k and for $\ell_k \in \mathbb{N}$ such that $m_{\ell_k} < k \le m_{\ell_k+1}$, we can apply (6.4.1) and find

$$\left|\mathcal{F}^{\mu_k}_{u_0}[w_k] - \mathcal{F}^{\mu}_{u_0}[u]\right| \le \left|\mathcal{F}^{\mu_k}_{u_0}[v_{\ell k}] - \mathcal{F}^{\mu}_{u_0}[v_{\ell k}]\right| + \left|\mathcal{F}^{\mu}_{u_0}[v_{\ell k}] - \mathcal{F}^{\mu}_{u_0}[w]\right| \le \frac{1}{\ell_k} + \left|\mathcal{F}^{\mu}_{u_0}[v_{\ell k}] - \mathcal{F}^{\mu}_{u_0}[w]\right|.$$

Letting $k \to \infty$, with the help of (6.4.2) we obtain

$$\limsup_{k \to \infty} \left|\mathcal{F}^{\mu_k}_{u_0}[w_k] - \mathcal{F}^{\mu}_{u_0}[w]\right| \le 0\,,$$

and the sequence $(w_k)_k$ realizes (ii) of Definition 2.49, therefore our thesis results. □

6.4.2 Liminf inequality

We turn now to the more involved proof of (i) for the sequence $(\mathcal{F}^{\mu_k}_{u_0})_k$. To achieve the desired estimate

$$\liminf_{k \to \infty} \mathcal{F}^{\mu_k}_{u_0}[u_k] \ge \mathcal{F}^{\mu}_{u_0}[u] \quad \text{ for every } u,\ (u_k)_k \text{ in } \mathrm{BV}(\Omega) \text{ such that } u_k \to u \text{ in } \mathrm{L}^1(\Omega)\,, \tag{6.4.3}$$

we provide an adaptation of the techniques employed to semicontinuity with fixed measure – that would be our preceding Result 1. Our goal is to determine the following.

Theorem 6.14 (liminf inequality for $\mathcal{F}^{\mu_k}_{u_0}$)**.** *Suppose the integrand f satisfies Assumption 1.1(H1)–(H2) with f <u>continuous</u> on $\mathbb{R}^n$, $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, with admissible measures $\mu_\pm$, $(\mu_k)_\pm$ on the bounded Lipschitz $\Omega \subseteq \mathbb{R}^n$ such that:*

- *The pairs (μ_{k-}, μ_{k+}) satisfy the f^∞–IC in Ω with constant 1 for $k \gg 1$;*
- *The pairs (μ_{k+}, μ_{k-}) satisfy the $\widetilde{f^\infty}$–IC in Ω with constant 1 for $k \gg 1$; and*
- *$(\mu_k)_\pm \rightharpoonup \mu_\pm$ strictly in $\mathrm{RM}(\Omega)$ for $k \to \infty$.*

Then, (6.4.3) holds true.

Before reaching the generality of Theorem 6.14, we first need to rephrase our parametric semicontinuity Theorem 4.5 to allow for a joint convergence of both sets and measures. In the following Theorem 6.15, those computations in common with our previous LSC result are omitted, and we rather focus on the steps where the additional presence of $(\nu_{i,k})_k$ is decisive.

Theorem 6.15 (liminf inequality for anisotropic parametric functionals with sequences of measures)**.** *Consider a lower semicontinuous anisotropy $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ comparable to the Euclidean norm with $\xi \mapsto \varphi(x,\xi)$ convex for every $x \in \mathbb{R}^n$, and finite non–negative Radon measures ν_1, ν_2, $(\nu_{1,k})_k$, $(\nu_{2,k})_k$ on $\mathbb{R}^n$ vanishing on $\mathcal{H}^{n-1}$–negligible sets. If the pairs $(\nu_{1,k},\nu_{2,k})$ satisfy the small–volume φ–IC in $\mathbb{R}^n$ with constant 1 and $(\nu_{2,k},\nu_{1,k})$ satisfies the small–volume $\widetilde{\varphi}$–IC in $\mathbb{R}^n$ with constant 1 for $k \gg 1$, then we have*

$$\liminf_{k\to\infty} \left[\mathrm{P}_\varphi(A_k) + \nu_{2,k}(A_k^1) - \nu_{1,k}(A_k^+)\right] \geq \mathrm{P}_\varphi(A) + \nu_2(A^1) - \nu_1(A^+) \tag{6.4.4}$$

whenever A_k and A are measurable in $\mathbb{R}^n$, A_k converging locally in measure to A, and at least one of the following applies:

(a) *The set A is bounded and we have the weak–$*$ convergences $\nu_{1,k} \overset{*}{\rightharpoonup} \nu_1$ and $\nu_{2,k} \overset{*}{\rightharpoonup} \nu_2$ in $\mathbb{R}^n$ as $k \to \infty$; or*

(b) *The convergence of $(\nu_{1,k})_k$ to ν_1 and the convergence of $(\nu_{2,k})_k$ to ν_2 are strict in $\mathbb{R}^n$ for $k \to \infty$.*

Proof. Possibly passing to a subsequence, we suppose the existence of the limit

$$\lim_{k\to\infty} \left[\mathrm{P}_\varphi(A_k){+}\nu_{2,k}(A_k^1){-}\nu_{1,k}(A_k^+)\right] < \infty\,,$$

otherwise (6.4.4) is trivial. Since all measures $\nu_{1,k}$ are finite, from the upper bound on φ we get even $\limsup_{k\to\infty} \mathrm{P}(A_k) < \infty$, thus also $\mathrm{P}(A) < \infty$. We now distinguish two cases according to the different assumptions (a) or (b).

Case (a). From $|A| + \mathrm{P}(A) + \limsup_{k\to\infty} \mathrm{P}(A_k) < \infty$, we deduce that $\mathbb{1}_{A_k}, \mathbb{1}_A \in \mathrm{BV}(\mathbb{R}^n)$ for $k \gg 1$ with $\mathbb{1}_{A_k} \to \mathbb{1}_A$ in $\mathrm{L}^1(\mathbb{R}^n)$. Fixed $\varepsilon > 0$, we repeat the procedure of Theorem 4.5 to find the bounded Borel sets S, S' coming from Lemma 4.4 and $R := \mathrm{B}_r \setminus S'$ satisfying all the same estimates. We can then adjust the reference proof and write

$$\begin{aligned}
\nu_{2,k}(A_k^1) &\geq \nu_{2,k}\big(A_k^1 \setminus \mathrm{Int}(S)\big) - \nu_{2,k}\big((A_k^{\mathrm{c}})^+ \cap \mathrm{Int}(R)\big) + \nu_{2,k}\big(\mathrm{Int}(R)\big)\,,\\
\nu_{1,k}(A_k^+) &\leq \nu_{1,k}\big(A_k^+ \setminus \overline{S}\big) - \nu_{2,k}\big((A_k^{\mathrm{c}})^1 \cap \overline{R}\big) + \nu_{2,k}\big(\overline{S}\big)
\end{aligned}$$

for k large enough. Altogether, we find

$$\begin{aligned}
\lim_{k\to\infty} \left[\mathrm{P}_\varphi(A_k) + \nu_{2,k}(A_k^1) - \nu_{1,k}(A_k^+)\right] &\geq \liminf_{k\to\infty} \mathrm{P}_\varphi\big(A_k, \mathrm{Int}(S) \setminus \overline{R}\big)\\
&\quad + \liminf_{k\to\infty} \Big[\mathrm{P}_\varphi\big(A_k, \mathrm{Int}(S)^{\mathrm{c}}\big) + \nu_{2,k}\big(A_k^1 \setminus \mathrm{Int}(S)\big) - \nu_{1,k}\big(A_k^+ \setminus \overline{S}\big)\Big]\\
&\quad + \liminf_{k\to\infty} \Big[\mathrm{P}_{\widetilde{\varphi}}\big(A_k^{\mathrm{c}}, \overline{R}\big) - \nu_{2,k}\big((A_k^{\mathrm{c}})^+ \cap \mathrm{Int}(R)\big) + \nu_{1,k}\big((A_k^{\mathrm{c}})^1 \cap \overline{R}\big)\Big]\\
&\quad + \liminf_{k\to\infty} \Big[\nu_{2,k}\big(\mathrm{Int}(R)\big) - \nu_{1,k}\big(\overline{S}\big)\Big]\,.
\end{aligned}$$

We now analyze the behavior of the single terms. The first addend is independent of the measure, therefore the same result of Theorem 4.5 applies, so that

$$\liminf_{k\to\infty} \mathrm{P}_\varphi\big(A_k, \mathrm{Int}(S) \setminus \overline{R}\big) \geq \mathrm{P}_\varphi(A)\,. \tag{6.4.5}$$

The second term can be estimated by the convergence in measure $\lim_{k\to\infty} |A_k \setminus S| = 0$ and via the small–volume φ–IC for $(\nu_{1,k},\nu_{2,k})$ to $A_k \setminus S$ for $k \gg 1$, yielding

$$\begin{aligned}
\nu_{1,k}\big(A_k^+ \setminus \overline{S}\big) - \nu_{2,k}\big(A_k^1 \setminus \mathrm{Int}(S)\big) &\leq \nu_{1,k}((A_k \setminus S)^+) - \nu_{2,k}((A_k \setminus S)^1) \leq \mathrm{P}_\varphi(A_k \setminus S) + \varepsilon\\
&\leq \mathrm{P}_\varphi(A_k, S^0) + \mathrm{P}_{\widetilde{\varphi}}(S, A_k^+) + \varepsilon\\
&\leq \mathrm{P}_\varphi(A_k, \mathrm{Int}(S)^{\mathrm{c}}) + \mathrm{P}_{\widetilde{\varphi}}(S, A_k^+) + \varepsilon \qquad \text{for } k \gg 1\,,
\end{aligned}$$

where we employed Lemma 2.73 and $A_k^+ \setminus \overline{S} \subseteq (A_k \setminus S)^+$, $(A_k \setminus S)^1 \subseteq A_k^1 \setminus \mathrm{Int}(S)$, $S^0 \subseteq \mathrm{Int}(S)^{\mathrm{c}}$. Passing then to the limits, it is

$$\liminf_{k\to\infty} \left[\mathrm{P}_\varphi(A_k, \mathrm{Int}(S)^{\mathrm{c}}) + \nu_{2,k}\big(A_k^1 \setminus \mathrm{Int}(S)\big) - \nu_{1,k}\big(A_k^+ \setminus \overline{S}\big)\right] \geq -\lim_{k\to\infty} \mathrm{P}_{\widetilde{\varphi}}(S, A_k^+) - \varepsilon > -(\beta+1)\varepsilon\,. \quad (6.4.6)$$

For the third term, an application of the small–volume $\widetilde{\varphi}$–IC for $(\nu_{2,k}, \nu_{1,k})$ to $A_k^{\mathrm{c}} \cap R$ again for $k \gg 1$ implies

$$\begin{aligned}\nu_{2,k}\big((A_k^{\mathrm{c}})^+ \cap \mathrm{Int}(R)\big) - \nu_{1,k}\big((A_k^{\mathrm{c}})^1 \cap \overline{R}\big) &\leq \nu_{2,k}((A_k^{\mathrm{c}} \cap R)^+) - \nu_{1,k}((A_k^{\mathrm{c}} \cap R)^1) \leq \mathrm{P}_{\widetilde{\varphi}}(A_k^{\mathrm{c}} \cap R) + \varepsilon \\ &\leq \mathrm{P}_{\widetilde{\varphi}}(A_k^{\mathrm{c}}, R^1) + \mathrm{P}_{\widetilde{\varphi}}(R, (A_k^{\mathrm{c}})^+) + \varepsilon \\ &\leq \mathrm{P}_{\widetilde{\varphi}}(A_k^{\mathrm{c}}, \overline{R}) + \mathrm{P}_{\widetilde{\varphi}}(R, (A_k^{\mathrm{c}})^+) + \varepsilon,\end{aligned}$$

and therefore

$$\liminf_{k\to\infty} \left[\mathrm{P}_{\widetilde{\varphi}}(A_k^{\mathrm{c}}, \overline{R}) - \nu_{2,k}\big((A_k^{\mathrm{c}})^+ \cap \mathrm{Int}(R)\big) + \nu_{1,k}\big((A_k^{\mathrm{c}})^1 \cap \overline{R}\big)\right] \geq -\lim_{k\to\infty} \mathrm{P}_{\widetilde{\varphi}}(R, (A_k^{\mathrm{c}})^+) - \varepsilon > -(\beta+1)\varepsilon\,. \quad (6.4.7)$$

Lastly, we recall that our selection of the sets is such that

$$\nu_2\big(\mathrm{Int}(R)\big) > \nu_2(A^1) - 3\varepsilon \qquad \text{and} \qquad \nu_1\big(\overline{S}\big) < \nu_1(A^+) + 3\varepsilon\,. \quad (6.4.8)$$

Employing then lower semicontinuity for the sequence $(\nu_{2,k})$ converging to ν_2 weakly–$*$ in $\mathrm{RM}(\mathbb{R}^n)$, by (6.4.8) it is

$$\liminf_{k\to\infty} \nu_{2,k}\big(\mathrm{Int}(R)\big) \geq \nu_2\big(\mathrm{Int}(R)\big) > \nu_2(A^1) - 3\varepsilon\,.$$

Analogously, the weak–$*$ convergence of $\nu_{1,k}$ to ν_1 on $\mathbb{R}^n$ applied to $\overline{S}$ compact (see Proposition 2.6(ii)) together with (6.4.8) induces

$$\limsup_{k\to\infty} \nu_{1,k}\big(\overline{S}\big) \leq \nu_1\big(\overline{S}\big) < \nu_1(A^1) + 3\varepsilon\,.$$

In sum, we have obtained

$$\liminf_{k\to\infty} \Big[\nu_{2,k}\big(\mathrm{Int}(R)\big) - \nu_{1,k}\big(\overline{S}\big)\Big] \geq \nu_2(A^1) - \nu_1(A^+) - 6\varepsilon\,. \quad (6.4.9)$$

It is left to collect the estimates in (6.4.5)–(6.4.9) to conclude that

$$\lim_{k\to\infty} \left[\mathrm{P}_\varphi(A_k) + \nu_{2,k}(A_k^1) - \nu_{1,k}(A_k^+)\right] \geq \mathrm{P}_\varphi(A) + \nu_2(A^1) - \nu_1(A^+) - 2(\beta+4)\varepsilon\,,$$

and the claim (6.4.4) follows by arbitrariness of ε.

Case (b). Assume now A is any measurable set in $\mathbb{R}^n$. From $\mathrm{P}(A) < \infty$ for and the isoperimetric inequality, it is either $|A| < \infty$ or $|A^{\mathrm{c}}| < \infty$.

(b1) Supposed first $|A| < \infty$ and assigned $\varepsilon > 0$, we let $r \in (0, \infty)$ be large enough for having

$$|A \setminus \mathrm{B}_r| < \varepsilon\,, \qquad \mathrm{P}(A, \mathrm{B}_r^{\mathrm{c}}) < \varepsilon\,, \qquad \nu_1(\mathrm{B}_r^{\mathrm{c}}) < \varepsilon, \qquad \text{and} \qquad \nu_2(\mathrm{B}_r^{\mathrm{c}}) < \varepsilon\,.$$

Repeating the same steps of Theorem 4.5, we find $|A_k \cap (\mathrm{B}_{r+1} \setminus \mathrm{B}_r)| < \varepsilon$ for $k \gg 1$, $\varrho_k, \varrho \in [r, r+1]$ such that $\varrho_k \to \varrho$ as $k \to \infty$, $\mathcal{H}^{n-1}(A_k^1 \cap \partial \mathrm{B}_{\varrho_k}) < \varepsilon$ for all k and thus $\mathrm{P}_\varphi(A_k \cap \mathrm{B}_{\varrho_k}) < \mathrm{P}_\varphi(A_k) + \beta\varepsilon$ for $k \gg 1$. We exploit now the result of Case (a) for the sequence $A_k \cap \mathrm{B}_{\varrho_k} \subseteq \mathrm{B}_{r+1}$ with $A_k \cap \mathrm{B}_{\varrho_k} \to A \cap \mathrm{B}_\varrho$, so that after some manipulations and via Remark 2.26 we get

$$\liminf_{k\to\infty} \left[\mathrm{P}_\varphi(A_k \cap \mathrm{B}_{\varrho_k}) + \nu_{2,k}(A_k^1) - \nu_{1,k}(A_k^+ \cap \mathrm{B}_r)\right] \geq \mathrm{P}_\varphi(A, \mathrm{B}_r) + \nu_2(A^1 \cap \mathrm{B}_r) - \nu_1(A^+)\,.$$

Besides, the (strict and thus) weak–$*$ convergence for $(\nu_{1,k})_k$ guarantees by Proposition 2.6 the validity of $\nu_1(\mathrm{B}_r^{\mathrm{c}}) \geq \limsup_{k\to\infty} \nu_{1,k}(\mathrm{B}_r^{\mathrm{c}})$, hence in correspondence to ε it is

$$\nu_{1,k}(A_k^+ \cap \mathrm{B}_r) \geq \nu_{1,k}(A_k^+) - \nu_{1,k}(\mathrm{B}_r^{\mathrm{c}}) \geq \nu_{1,k}(A_k^+) - \nu_1(\mathrm{B}_r^{\mathrm{c}}) - \varepsilon \geq \nu_{1,k}(A_k^+) - 2\varepsilon \quad \text{for } k \gg 1\,, \tag{6.4.10}$$

and we record $\nu_2(A^1 \cap \mathrm{B}_r) \geq \nu_2(A^1) - \nu_2(\mathrm{B}_r^{\mathrm{c}}) \geq \nu_2(A^1) - \varepsilon$. Therefore, by Lemma 2.73 we achieve

$$\begin{aligned}
&\liminf_{k\to\infty} \left[\mathrm{P}_\varphi(A_k) + \nu_{2,k}(A_k^1) - \nu_{1,k}(A_k^+)\right] + (2\beta+1)\varepsilon \\
&\geq \liminf_{k\to\infty} \left[\mathrm{P}_\varphi(A_k \cap \mathrm{B}_{\varrho_k}) + \nu_{2,k}(A_k^1) - \nu_{1,k}(A_k^+ \cap \mathrm{B}_r)\right] \\
&\geq \mathrm{P}_\varphi(A, \mathrm{B}_r) + \nu_2(A^1 \cap \mathrm{B}_r) - \nu_1(A^+) \\
&\geq \mathrm{P}_\varphi(A) + \nu_2(A^1) - \nu_1(A^+) - (\beta+1)\varepsilon\,,
\end{aligned}$$

and again arbitrariness of ε yields (6.4.4).

(b2) If $|A^{\mathrm{c}}| < \infty$, for any $k \in \mathbb{N}$, we pass to the complements and thus rephrase the functional as

$$\mathrm{P}_\varphi(A_k) + \nu_{2,k}(A_k^1) - \nu_{1,k}(A_k^+) = \mathrm{P}_{\widetilde{\varphi}}(A_k^{\mathrm{c}}) + \nu_{1,k}((A_k^{\mathrm{c}})^1) - \nu_{2,k}((A_k^{\mathrm{c}})^+) + C_k$$

with $C_k := \nu_{2,k}(\mathbb{R}^n) - \nu_{1,k}(\mathbb{R}^n)$ converging to $C := \nu_2(\mathbb{R}^n) - \nu_1(\mathbb{R}^n)$ as $k \to \infty$ in view of strict convergence. Exploiting (6.4.4) for the sequence of sets (A_k^{c}) converging to A^{c} in measure, inserting $\widetilde{\varphi}$ in place of φ and exchanging the roles of $\nu_{1,k}$ and $\nu_{2,k}$, we find

$$\begin{aligned}
\liminf_{k\to\infty} \left[\mathrm{P}_\varphi(A_k) + \nu_{2,k}(A_k^1) - \nu_{1,k}(A_k^+)\right] &\geq \liminf_{k\to\infty} \left[\mathrm{P}_{\widetilde{\varphi}}(A_k^{\mathrm{c}}) + \nu_{1,k}((A_k^{\mathrm{c}})^1) - \nu_{2,k}((A_k^{\mathrm{c}})^+)\right] \\
&\quad + \lim_{k\to\infty} C_k \\
&\geq \mathrm{P}_{\widetilde{\varphi}}(A^{\mathrm{c}}) + \nu_1((A^{\mathrm{c}})^1) - \nu_2((A^{\mathrm{c}})^+) + C \\
&= \mathrm{P}_\varphi(A) + \nu_2(A^1) - \nu_1(A^+)\,.
\end{aligned}$$

With this latter, the statement (6.4.4) is verified even under assumption (b). □

Concerning the assumptions in the latter Theorem 6.15, we want to stress that *strict* convergence of both sequences of measures can be weakened to just weak–$*$ convergence in $\mathbb{R}^n$ *for the only component* $\nu_{2,k}$. In fact – for A of *finite* measure – the proof of (6.4.9) in the bounded case (a), as well as the verification of (6.4.10) in the unbounded case (b1) just rely on the properties of weak–$*$ convergence. If $|A| = \infty$ instead, we observe that the sequence $(C_k)_k$ in the proof of (b2) by superadditivity it suffices to have just the inequality $\liminf_{k\to\infty} C_k = \liminf_{k\to\infty} \nu_{2,k}(\mathbb{R}^n) - \lim_{k\to\infty} \nu_{1,k}(\mathbb{R}^n) \geq \nu_2(\mathbb{R}^n) - \nu_1(\mathbb{R}^n) = C$, which holds if $\nu_{1,k} \rightharpoonup \nu_1$ strictly and $\nu_{2,k} \overset{*}{\rightharpoonup} \nu_2$ weakly–$*$ in $\mathbb{R}^n$. Nevertheless, bearing in mind our successive goal of Lemma 6.16, the liminf–inequality for *both* $\overline{\Phi}^{\mu_k}$ and its symmetrical $\widetilde{\Phi}^{\mu_k}$ is only achieved under strict convergence of both components $\mu_{k\pm}$.

At this stage, to approach the non–parametric result corresponding to (6.4.4), we reprise the decomposition introduced in Section 4.3 for the anisotropic functional $\widehat{\Phi}_{u_0}^\mu$ in $\mathrm{BV}(\Omega)$ via the auxiliary functionals $\overline{\Phi}^\mu$, $\widetilde{\Phi}^\mu$ defined on the full space $\mathrm{BV}(\mathbb{R}^n)$ instead. Then, fixing an anisotropy φ on $\mathbb{R}^n$ and a pair of non–negative admissible measures $\mu_\pm$ on Ω, we recall Remark 4.10:

$$\widehat{\Phi}_{u_0}^\mu[w] = \overline{\Phi}^\mu\big[(\overline{w}^{u_0})_+\big] + \widetilde{\Phi}^\mu\big[(\overline{w}^{u_0})_-\big] - |\mathrm{D}u_0|_\varphi\big(\mathbb{R}^n \setminus \overline{\Omega}\big) \qquad \text{for all } w \in \mathrm{BV}(\Omega)\,,$$

where we set $\mu := \mu_+ - \mu_-$, $\overline{w}^{u_0} = w\mathbb{1}_\Omega + u_0\mathbb{1}_{\mathbb{R}^n\setminus\overline{\Omega}}$ extension of w, and $\overline{\Phi}^\mu$, $\widetilde{\Phi}^\mu \colon \mathrm{BV}(\mathbb{R}^n) \to \mathbb{R}$ with

$$\overline{\Phi}^\mu[w] := |\mathrm{D}w|_\varphi(\mathbb{R}^n) + \int_\Omega w^- \,\mathrm{d}\mu_+ - \int_\Omega w^+ \,\mathrm{d}\mu_-\,, \quad \widetilde{\Phi}^\mu[w] := |\mathrm{D}w|_{\widetilde{\varphi}}(\mathbb{R}^n) + \int_\Omega w^- \,\mathrm{d}\mu_- - \int_\Omega w^+ \mathrm{d}\mu_+$$

for all $w \in \mathrm{BV}(\mathbb{R}^n)$.

Lemma 6.16 (liminf inequality for $\overline{\Phi}^{\mu}$, $\widetilde{\Phi}^{\mu}$ on non–negative functions with given value outside Ω)**.** *If the anisotropy φ on $\mathbb{R}^n$ is comparable to the Euclidean norm and under (a)–(b), we consider sequences of admissible measures $(\mu_{k+})_k$, $(\mu_{k-})_k$ on Ω and μ_-, μ_+ also admissible on Ω such that:*

- *The pairs (μ_{k-}, μ_{k+}) satisfy the φ–IC in Ω with constant 1 for $k \gg 1$;*
- *The pairs (μ_{k+}, μ_{k-}) satisfy the $\widetilde{\varphi}$–IC in Ω with constant 1 for $k \gg 1$; and*
- *The convergence $\mu_{k\pm} \rightharpoonup \mu_{\pm}$ is strict in $\mathrm{RM}(\Omega)$ for $k \to \infty$.*

Then, given a non–negative datum $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, we have

$$\liminf_{k\to\infty} \overline{\Phi}^{\mu_k}[w_k] \geq \overline{\Phi}^{\mu}[w] \quad \text{and} \quad \liminf_{k\to\infty} \widetilde{\Phi}^{\mu_k}[w_k] \geq \widetilde{\Phi}^{\mu}[w]\,, \tag{6.4.11}$$

for every $(w_k)_k$, w in $\mathrm{BV}(\mathbb{R}^n)$ such that $w_k \to w$ in $\mathrm{L}^1(\mathbb{R}^n)$ with $w_k, w \geq 0$ a.e. in Ω and $w_k = w = u_0$ a.e. in Ω^{c} for all k.

Proof. Before approaching the proof, we observe that the estimate for $\widetilde{\Phi}^{\mu}$ can be obtained from the liminf inequality for the functional $\overline{\Phi}^{\mu}$ by an exchange of the roles of $\mu_{k\pm}$, $\mu_{\pm}$, and φ with $\widetilde{\varphi}$. It suffices then to prove the first inequality in (6.4.11), and we shall do so by a slight adaptation of Proposition 4.14. Indeed, for $(w_k)_k$ and w as in the statement and implicitly passing to the usual measure extensions of $\mu_{\pm}$, $\mu_{k\pm}$ to all $\mathbb{R}^n$, for any fixed level $M > 0$, by Lemma 2.29 it is

$$\big\{(w_k^M)^+ > t\big\} = \big\{w_k^M > t\big\}^+ \quad \text{and} \quad \big\{(w_k^M)^- > t\big\} = \big\{w_k^M > t\big\}^1 \qquad \mu_{k\pm}\text{– and } \mu_{\pm}\text{–a.e. in } \mathbb{R}^n$$

and for a.e. $t > 0$. Then, for any $k \in \mathbb{N}$ we employ Lemma 4.12, the anisotropic coarea formula and layer–cake decomposition to write

$$\begin{aligned}\overline{\Phi}^{\mu_k}[w_k] \geq \overline{\Phi}^{\mu_k}[w_k^M] + c_M &= c_M + |\mathrm{D}w_k^M|_{\varphi}(\mathbb{R}^n) + \int_{\mathbb{R}^n} \left(u_k^M\right)^- \,\mathrm{d}\mu_{k+} - \int_{\mathbb{R}^n} \left(u_k^M\right)^+ \,\mathrm{d}\mu_{k-} \\ &= c_M + \int_0^{\infty}\Big[\mathrm{P}_{\varphi}\left(\big\{w_k^M > t\big\}\right) + \mu_{k+}\left(\big\{w_k^M > t\big\}^1\right) - \mu_{k-}\left(\big\{w_k^M > t\big\}^+\right)\Big]\,\mathrm{d}t\,,\end{aligned}$$

and we record that c_M is independent on the choice of measures and infinitesimal for $M \to \infty$. Moreover, from $0 \leq w_k^M \leq M$ and by $\mu_{k-}(\Omega) \to \mu_-(\Omega)$ for $k \to \infty$, it is $\mu_{k-}\big(\big\{w_k^M > t\big\}^+\big) \leq \mathbb{1}_{(0,M)}(t)\mu_{k-}(\Omega) \leq \mathbb{1}_{(0,M)}(t)(\mu_-(\Omega)+1)$ for $k \gg 1$, with $\mathbb{1}_{(0,M)}(\mu_-(\Omega)+1) \in \mathrm{L}^1((0,\infty))$. With the help of Fatou's lemma, we compute

$$\liminf_{k\to\infty} \overline{\Phi}_{\mu_k}[w_k] \geq c_M + \int_0^{\infty} \liminf_{k\to\infty} \Big[\mathrm{P}_{\varphi}\left(\big\{w_k^M > t\big\}\right) + \mu_{k+}\left(\big\{w_k^M > t\big\}^1\right) - \mu_{k-}\left(\big\{w_k^M > t\big\}^+\right)\Big]\,\mathrm{d}t\,.$$

The a.e.–pointwise convergence of (a subsequence of) $w_k^M \to w^M$ as $k \to \infty$ and the condition $w_k^M = w^M$ outside $\overline{\Omega}$ imply that the superlevel sets $A_{k,M}^t := \big\{w_k^M > t\big\}$ converge in measure to $A_M^t := \big\{w^M > t\big\}$ for $k \to \infty$ whenever $t \geq 0$ satisfies $|\{w^M = t\}| = 0$ – and this latter applies for a.e. $t \geq 0$. Now the decisive strict convergence assumption $\mu_{k\pm} \to \mu_{\pm}$ in Ω is inherited by the sequence of extended measures from Corollary 2.12. Therefore, since the extensions of $\mu_{k\pm}$ converge to those of $\mu_{\pm}$ strictly in $\mathbb{R}^n$, we are in the conditions of applying the parametric result of Theorem 6.15(b) to deduce

$$\liminf_{k\to\infty} \overline{\Phi}^{\mu_k}[w_k] \geq c_M + \int_0^{\infty}\Big[\mathrm{P}_{\varphi}\left(\big\{w^M > t\big\}\right) + \mu_+\Big(\big\{w^M > t\big\}^1\Big) - \mu_-\Big(\big\{w^M > t\big\}^+\Big)\Big]\,\mathrm{d}t = c_M + \overline{\Phi}^{\mu}[w^M]\,.$$

To complete the statement, it is just left to $M \to \infty$ recalling that $c_M \to 0$ and applying the convergence in Lemma 4.11. ∎

The results so far obtained are collected in the following statement, which then represents the homogeneous version of our claimed Theorem 6.14.

Theorem 6.17 (liminf inequality for anisotropic total variation functional with measures)**.** *Adopting the same set of assumptions on φ, $\mu_\pm$, $\mu_{k\pm}$ of Lemma 6.16, for any $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ we have*

$$\liminf_{k\to\infty} \widehat{\Phi}^{\mu_k}_{u_0}[w_k] \geq \widehat{\Phi}^{\mu}_{u_0}[w] \qquad \textit{for every } w,\ (w_k)_k \textit{ in } \mathrm{BV}(\Omega) \textit{ such that } w_k \to w \textit{ in } \mathrm{L}^1(\Omega)\,.$$

Proof. We consider a sequence (w_k) converging to w as in the statement. In analogy to the proof of Theorem 4.8, the latter Lemma 6.16 applied to both sequences of positive and negative parts of the functions as in the statement and extended to $\mathbb{R}^n$ yields

$$\liminf_{k\to\infty} \overline{\Phi}^{\mu_k}[(\overline{w_k}^{u_0})_+] \geq \overline{\Phi}_{\mu}[(\overline{w}^{u_0})_+] \qquad \text{and} \qquad \liminf_{k\to\infty} \widetilde{\Phi}^{\mu_k}[(\overline{w_k}^{u_0})_-] \geq \overline{\Phi}_{\mu}[(\overline{w}^{u_0})_-].$$

Our thesis follows by summing up the terms with the help of Remark 4.10. □

Finally, we approach the proof of Theorem 6.14 for general linear–growth integrands f analogously to what was done in the semicontinuity Result 1. Clearly, the latter represents the subcase of Theorem 6.14 occurring when we restrict to $\mu_{k\pm} = \mu_\pm$ for all $k \in \mathbb{N}$.

Proof of Theorem 6.14. We assume that the sequence $(w_k)_k \subseteq \mathrm{BV}(\Omega)$ converges in $\mathrm{L}^1(\Omega)$ to some w in $\mathrm{BV}(\Omega)$, and we introduce the corresponding functions $w_\Diamond$, $w_{k\Diamond} \in \mathrm{BV}(\Omega_\Diamond)$ as well as the measures $\mu_{\Diamond\pm}$, $\mu_{k\pm\Diamond} \in \mathrm{RM}(\Omega_\Diamond)$ as from Definition 5.7. We recall that Lemma 5.11 establishes admissibility on $\Omega_\Diamond$ of the new measures. Consider now the anisotropy φ on $\mathbb{R}^{n+1}$ corresponding to f as in Definition 5.4. Similarly to our considerations in Remark 5.5, the statement of Theorem 6.14 is preserved under vertical translations of the integrand f, thus without loss of generality we may apply Proposition 5.12 and claim that the φ–IC holds in $\Omega_\Diamond$ with constant 1 for $(\mu_{k-\Diamond}, \mu_{k+\Diamond})$ and all $k \gg 1$, as well as the $\widetilde{\varphi}$–IC with constant 1 in $\Omega_\Diamond$ for $(\mu_{k+\Diamond}, \mu_{k-\Diamond})$ for $k \gg 1$. With such a notation, the formulation in (5.2.18) in our framework becomes

$$\begin{aligned}
\mathcal{F}^{\mu}_{u_0}[w] &= \widehat{\Phi}^{\mu_\Diamond}_{u_0\Diamond}[w_\Diamond] - 2\int_\Omega f(.,0)|w-u_0|\,\mathrm{d}x - \frac{\mu(\Omega)}{2}\,,\\
\mathcal{F}^{\mu_k}_{u_0}[w_k] &= \widehat{\Phi}^{\mu_{k\Diamond}}_{u_0\Diamond}[w_{k\Diamond}] - 2\int_\Omega f(.,0)|w_k-u_0|\,\mathrm{d}x - \frac{\mu_k(\Omega)}{2} \qquad \text{for all } k \in \mathbb{N}\,.
\end{aligned}$$

Furthermore, by Remark 5.8 the convergences $\mu_{k+\Diamond} \rightharpoonup \mu_{+\Diamond}$ and $\mu_{k-\Diamond} \rightharpoonup \mu_{-\Diamond}$ are strict in $\mathrm{BV}(\Omega_\Diamond)$ for $k \to \infty$, and this allows an application of Theorem 6.17 to our anisotropic functional $\widehat{\Phi}^{\mu_\Diamond}_{u_0\Diamond}$ in dimension $n+1$. Altogether, we achieved

$$\begin{aligned}
\liminf_{k\to\infty} \mathcal{F}^{\mu_k}_{u_0}[w_k] &= \liminf_{k\to\infty} \widehat{\Phi}^{\mu_{k\Diamond}}_{u_0\Diamond}[w_{k\Diamond}] - 2\lim_{k\to\infty}\int_\Omega f(.,0)|w_k-u_0|\,\mathrm{d}x - \frac{\mu(\Omega)}{2}\\
&\geq \widehat{\Phi}^{\mu_\Diamond}_{u_0\Diamond}[w_\Diamond] - 2\int_\Omega f(.,0)|w-u_0|\,\mathrm{d}x - \frac{\mu(\Omega)}{2} = \mathcal{F}^{\mu}_{u_0}[w]\,,
\end{aligned}$$

which establishes the liminf inequality (i) for $\mathcal{F}^{\mu}_{u_0}$. □

As already done in Section 6.3 for the Γ–convergence of the functionals with respect to a variation of the boundary datum, we would like to apply the general result of Theorem 2.52 to the functionals $\mathcal{F}^{\mu_k}_{u_0}$ and deduce convergence of infima.

Corollary 6.18 (convergence of minima of $\mathcal{F}_{u_0}^{\mu_k}$). *Under the hypotheses of Theorem 6.11 and supposing that the mentioned ICs hold with constant $C \in [0,1)$, we have*

$$\min_{\mathrm{BV}(\Omega)} \mathcal{F}_{u_0}^{\mu} = \lim_{k\to\infty} \left(\min_{\mathrm{BV}(\Omega)} \mathcal{F}_{u_0}^{\mu_k} \right). \tag{6.4.12}$$

Moreover, if $(u_k)_k \subseteq \mathrm{BV}(\Omega)$ is a sequence of minimizers (that is, $\mathcal{F}_{u_0}^{\mu_k}[u_k] = \min_{\mathrm{BV}(\Omega)} \mathcal{F}_{u_0}^{\mu_k}$ for every k), then there exists $u \in \mathrm{BV}(\Omega)$ such that up to subsequences $u_k \to u$ in $\mathrm{L}^1(\Omega)$ as $k \to \infty$, with u achieving the minimum of $\mathcal{F}_{u_0}^{\mu}$. Furthermore, the combination of (6.4.12) with the consistency Result 3 delivers

$$\inf_{\mathrm{W}^{1,1}(\Omega)} \mathcal{F}_{u_0}^{\mu} = \min_{\mathrm{BV}(\Omega)} \mathcal{F}^{\mu} = \lim_{k\to\infty} \left(\min_{\mathrm{BV}(\Omega)} \mathcal{F}^{\mu_k} \right) = \lim_{k\to\infty} \left(\inf_{\mathrm{W}^{1,1}(\Omega)} \mathcal{F}_{u_0}^{\mu_k} \right).$$

Proof. From Theorem 6.11, we know that $\mathcal{F}_{u_0}^{\mu} = \Gamma\text{–}\lim_k \mathcal{F}_{u_0}^{\mu_k}$. Therefore, (6.4.12) follows straightforwardly from the abstract Theorem 2.52, provided the equi–coercivity of $(\mathcal{F}_{u_0}^{\mu_k})_k$ is verified. To do so, we repeat the steps of Proposition 6.7 employing the IC for the pairs $(\mu_{k\mp}, \mu_{k\pm})$ for k large enough and determining the $\mathrm{L}^1(\Omega)$–lower semicontinuous, BV–coercive functional $\Psi[w] := \alpha(1-C)||w||_{\mathrm{BV}(\Omega)}/\widetilde{\gamma}_n - M|\Omega| - \beta \sup_{k\in\mathbb{N}} ||u_{0,k}||_{\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})}$ defined for all $w \in \mathrm{BV}(\Omega)$ and such that $\Psi \le \mathcal{F}_{u_0}^{\mu_k}$ for all $k \gg 1$. From Proposition 2.48, we then read equi–coercivity of the sequence. Moreover, the control $\Psi \le \mathcal{F}_{u_0}^{\mu_k}$ implies that any minimizing sequence $(u_k)_k$ is bounded in $\mathrm{BV}(\Omega)$, so by compactness we argue the existence of an $\mathrm{L}^1(\Omega)$–limit $u \in \mathrm{BV}(\Omega)$, and our claim (6.4.12) follows from Theorem 2.52. □

Part II

Dual problem

Chapter 7

Generalized Anzellotti pairings

For an arbitrary open set U in $\mathbb{R}^n$, we recall that the Riesz–Markov representation theorem states that any finite Radon measure on U is also a distribution via the inclusion $\mathrm{C}_\mathrm{c}^\infty(U) \subseteq \mathrm{C}_0(U)$; see for instance [3, Theorem 1.54], [46, Theorem 1.38]. Moreover, any $\sigma \in \mathrm{L}^1_{\mathrm{loc}}(U,\mathbb{R}^n)$ induces a distributional divergence $\mathrm{div}(\sigma)$ on U defined via

$$\mathrm{div}(\sigma)(\psi) := -\int_U \sigma \cdot \nabla\psi \, \mathrm{d}x \qquad \text{for all } \psi \in \mathrm{C}_\mathrm{c}^\infty(U)\,.$$

In case $\sigma \in \mathrm{L}^\infty_{\mathrm{loc}}(U,\mathbb{R}^n)$, by a density argument we can even set

$$\mathrm{div}(\sigma)(\psi) := \lim_{k\to\infty} \mathrm{div}(\sigma)(\psi_k) = -\lim_{k\to\infty}\int_U \sigma \cdot \nabla\psi_k \, \mathrm{d}x = -\int_U \sigma \cdot \nabla\psi \, \mathrm{d}x$$

for all $\psi \in \mathrm{W}^{1,1}_0(U)$ and for any sequence $(\psi_k)_k$ in $\mathrm{C}_\mathrm{c}^\infty(U)$ converging in $\mathrm{W}^{1,1}(U)$ to ψ. If it is certainly true that every element of $\mathrm{RM}(U)$ represents a distribution on U, the converse does not apply – that means, not all vector fields σ on U admit distributional divergence as a finite Radon measure. To this aim, we introduce the following classes of **bounded divergence-measure fields** on U as

$$\begin{aligned}
\mathcal{DM}_{\mathrm{loc}}(U,\mathbb{R}^n) &:= \{\sigma \in \mathrm{L}^\infty_{\mathrm{loc}}(U,\mathbb{R}^n)\colon\ \mathrm{div}(\sigma) \text{ is a signed Radon measure on } U\} \quad \text{and}\\
\mathcal{DM}(U,\mathbb{R}^n) &:= \{\sigma \in \mathrm{L}^\infty(U,\mathbb{R}^n)\colon\ \mathrm{div}(\sigma) \in \mathrm{RM}(U)\}\,.
\end{aligned}$$

Observe that for $n = 1$ it is $\mathcal{DM}(U) = \mathrm{BV}(U)$. An interesting property of the elements of $\mathcal{DM}_{\mathrm{loc}}$ is that their divergence vanishes on sets negligible with respect to Hausdorff measures of codimension one.

Proposition 7.1 (absolute continuity of divergence–measures)**.** *Any $\sigma \in \mathcal{DM}_{\mathrm{loc}}(U,\mathbb{R}^n)$ is such that*

$$|\mathrm{div}(\sigma)|(A) = 0 \qquad \textit{for all Borel sets } A \subseteq U \textit{ such that } \mathcal{H}^{n-1}(A) = 0\,.$$

For the original statement and proof of Proposition 7.1, we refer the reader to [31, Proposition 3.1]. Clearly, the result implies that the positive and negative parts of $\mathrm{div}(\sigma)$ in the sense of the Radon–Nikodým decomposition still enjoy the same absolute continuity property.

Next, we introduce different types of pairing between L^∞ divergence–measure vector fields and (derivatives of) functions of bounded variation, where the features of the pairings are crucially influenced by the choice of BV representatives.

7.1 Inner pairing

We first present a version of the inner pairing defined according to the precise representative u^* of a BV function u on an open U in $\mathbb{R}^n$. This corresponds to the original pairing introduced by Gabriele Anzellotti in [7], which is in turn inspired by a pairing between admissible stresses and strains adopted in plasticity theory for a given subclass of functions of bounded deformation on the three–dimensional space; see [64, 9, 10]. Furthermore, Anzellotti verified that the pairing is a Radon measure in U absolutely continuous with respect to $|\mathrm{D}u|$ (see later Theorem 7.5, directly stated for our subsequent inner pairing) and proved the normal trace with the integration by parts formula of Theorem 7.3.

Definition 7.2 (Anzellotti pairing)**.** We consider a function $u \in \mathrm{BV}(U)$ and a vector field σ in $\mathcal{DM}(U,\mathbb{R}^n)$ with distributional divergence on U given by $\mathrm{div}(\sigma)$ and such that $u^* \in \mathrm{L}^1(U;|\mathrm{div}(\sigma)|)$. The (standard, inner) **Anzellotti pairing** is the distribution $[\![\sigma,\mathrm{D}u]\!]^*$ on U defined as

$$[\![\sigma,\mathrm{D}u]\!]^* := \mathrm{div}(u\sigma) - u^*\mathrm{div}(\sigma)\,,$$

thus explicitly

$$[\![\sigma,\mathrm{D}u]\!]^*(\psi) = -\int_U u\sigma\cdot\nabla\psi\ \mathrm{d}x - \int_U \psi u^*\,\mathrm{d}(\mathrm{div}(\sigma)) \quad \text{for all } \psi\in \mathrm{C}^\infty_\mathrm{c}(U)\,. \tag{7.1.1}$$

The *bilinearity* of the standard pairing $[\![\sigma,\mathrm{D}u]\!]^*$ is inherited from the linearity of the precise representative u^* of $u\in\mathrm{BV}(U)$ and of distributional divergences. Moreover, if we restrict to Sobolev functions $u\in \mathrm{W}^{1,1}(U)$ – so that $\psi u\in \mathrm{W}^{1,1}_0(U)$ for all test functions ψ – we can compute

$$[\![\sigma,\mathrm{D}u]\!]^*(\psi) = -\int_U u\sigma\cdot\nabla\psi\ \mathrm{d}x + \int_U \sigma\cdot\nabla(\psi u)\ \mathrm{d}x = \int_U \psi\sigma\cdot\nabla u\ \mathrm{d}x \qquad \text{for all } \psi\in\mathrm{C}^\infty_\mathrm{c}(U)\,,$$

namely, the pairing of Anzellotti reduces to the scalar product of σ and the gradient of u in $\mathbb{R}^n$:

$$[\![\sigma,\mathrm{D}u]\!]^* = (\sigma\cdot\nabla u)\mathcal{L}^n\llcorner U \qquad \text{for all } u\in\mathrm{W}^{1,1}(U)\,.$$

We record that in its first appearances of [7, Definition 1.4] the pairing was introduced for the following different combinations of entries:

(a) $u\in\mathrm{BV}(U)\cap\mathrm{L}^{p'}(U)$ and $\sigma\in\mathrm{L}^p(U,\mathbb{R}^n)$ with $\mathrm{div}(\sigma)\in\mathrm{L}^p(U)$, for any choice of $1\le p\le n$; or

(b) $u\in\mathrm{BV}(U)\cap\mathrm{L}^\infty(U)\cap\mathrm{C}(U)$ and $\sigma\in\mathcal{DM}(U,\mathbb{R}^n)$.

The possibilities above guarantee that $[\![\sigma,\mathrm{D}u]\!]^*$ always exists finite, without requiring the integrability condition $u^*\in\mathrm{L}^1(U;|\mathrm{div}(\sigma)|)$ – and actually, in both options (a)–(b) there is no need to consider representatives of u. Later on, Chen–Frid proposed a slightly improved version of the pairing respect getting rid of the continuity assumption in (b) and introducing the precise representative u^* – but still keeping the boundedness assumption on u to enforce integrability. In fact, the authors observed in [31, Proposition 3.1] (our previous Proposition 7.1) that the divergence of a field σ in $\mathcal{DM}(U,\mathbb{R}^n)$ vanishes $\mathcal{H}^{n-1}$–a.e. in U, hence representatives of BV functions can be integrated respect to divergences of such σ in the second term of (7.1.1).

Another interesting result of the pairing is the possibility to recover an integration by parts formula, written by means of a weakly defined trace on the boundary of the normal component of σ as presented in [8]. The following statement follows from slightly adapting and combining [7, Theorems 1.2 and 1.9].

Theorem 7.3 (normal trace for the standard inner pairing)**.** *We assume that Ω is open, bounded, and Lipschitz in $\mathbb{R}^n$. For any $\sigma\in\mathcal{DM}(\Omega)$, there exists a unique normal trace $\sigma^*_n\in\mathrm{L}^\infty(\partial\Omega;\mathcal{H}^{n-1})$ such that:*

- $\sigma_n^*(x) = \sigma(x) \cdot \nu(x)$ *for all* $x \in \partial\Omega$, *provided* $\sigma \in \mathrm{C}^1(\overline{\Omega}, \mathbb{R}^n)$; *and*
- $||\sigma_n^*||_{\mathrm{L}^\infty(\partial\Omega;\mathcal{H}^{n-1})} \leq ||\sigma||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}$.

Moreover, given $u \in \mathrm{BV}(\Omega)$ *such that* $u^* \in \mathrm{L}^1(\Omega; |\mathrm{div}(\sigma)|)$, *one has*

$$[\![\sigma, \mathrm{D}u]\!]^*(\Omega) = \int_{\partial\Omega} \sigma_n^* u \, \mathrm{d}\mathcal{H}^{n-1} - \int_\Omega u^* \, \mathrm{d}(\mathrm{div}(\sigma)) , \tag{7.1.2}$$

where in the boundary term we mean u *to be the* $\mathcal{H}^{n-1}$*–defined boundary trace* $\mathrm{T}_{\partial\Omega} u$ *on* $\partial\Omega$.

In particular, the integration by parts formula (7.1.2) evaluated on Sobolev functions reads:

$$\int_\Omega \sigma \cdot \nabla u \, \mathrm{d}x = \int_{\partial\Omega} \sigma_n^* u \, \mathrm{d}\mathcal{H}^{n-1} - \int_\Omega u^* \, \mathrm{d}(\mathrm{div}(\sigma)) \qquad \text{for all } u \in \mathrm{W}^{1,1}(U) .$$

We now mention a refined version of the pairing where we replace u^* with different BV representatives of u, intending to apply the pairing to pairs of solutions of our problems (P), (P*). Analogous adaptations were already the object of study, for instance, of [87, 88, 89]. Specifically, our representative is defined according to the Jordan decomposition of $\mathrm{div}(\sigma)$ achieved by $\mathrm{div}(\sigma)_+ \llcorner U_- = \mathrm{div}(\sigma)_- \llcorner U_+ \equiv 0$, where $U = U_+ \cup U_-$. We record that in general one could set a pairing $[\![\sigma, \mathrm{D}u]\!]^\lambda$ for any λ – meaning, for an arbitrary Borel representative $u^\lambda := \lambda u^+ + (1-\lambda) u^-$ (see Section 4.1) of $u \in \mathrm{BV}(\Omega)$. This is in fact the framework introduced in [33]. However, the decision of working expressively with the representative $\lambda = \lambda_{\mathrm{div}(\sigma)}$ in our case is motivated by the fact that no other Borel function determines lower semicontinuity of the corresponding functional $\mathcal{F}_\lambda$. The connection with our variational problem will become evident in Section 8.3.

Definition 7.4 (inner pairing)**.** Assume $u \in \mathrm{BV}(U)$, $\sigma \in \mathcal{DM}(U, \mathbb{R}^n)$ with distributional divergence decomposed into $\mathrm{div}(\sigma) = \mathrm{div}(\sigma)_+ - \mathrm{div}(\sigma)_-$ on U. Moreover, we take $u^\pm \in \mathrm{L}^1(U; |\mathrm{div}(\sigma)|)$. Then, the **inner pairing** of σ, $\mathrm{D}u$ with representative $\lambda_{\mathrm{div}(\sigma)}$ is

$$\begin{aligned} [\![\sigma, \mathrm{D}u]\!] \equiv [\![\sigma, \mathrm{D}u]\!]^{\lambda_{\mathrm{div}(\sigma)}} &:= \mathrm{div}(u\sigma) - u^{\lambda_{\mathrm{div}(\sigma)}} \mathrm{div}(\sigma) \\ &= \mathrm{div}(u\sigma) + u^+ \mathrm{div}(\sigma)_- - u^- \mathrm{div}(\sigma)_+ , \end{aligned}$$

with $[\![\sigma, \mathrm{D}u]\!]$ well–defined as a distribution on U. Employing the notation of (4.1.1) in Section 4.1 for u, we recall that $\lambda_{\mathrm{div}(\sigma)} := \mathbb{1}_{U_-}$. Explicitly, we have

$$[\![\sigma, \mathrm{D}u]\!](\psi) = -\int_U u\sigma \cdot \nabla\psi \, \mathrm{d}x + \int_U \psi u^+ \, \mathrm{d}(\mathrm{div}(\sigma)_-) - \int_U \psi u^- \, \mathrm{d}(\mathrm{div}(\sigma)_+) \qquad \text{for all } \psi \in \mathrm{C}_\mathrm{c}^\infty(U) .$$

Observe that Proposition 7.1 combined with the vanishing assumption (C3) for $\mathrm{div}(\sigma)_\pm$ guarantees that the integrals $\int_U \psi u^\pm \, \mathrm{d}(\mathrm{div}(\sigma)_\mp)$ exist finite for any choice of the test function ψ, hence well–posedness of $[\![\sigma, \mathrm{D}u]\!]$ follows. In the future employment of the pairing for $\mathrm{div}(\sigma)$ (with Jordan decomposition given by $\mathrm{div}(\sigma)_\pm$ *admissible* measures on Lipschitz domains Ω as in Chapter 8), the result of Proposition 3.5 ensures that the condition $u^\pm \in \mathrm{L}^1(\Omega; \mathrm{div}(\sigma)_+) \cap \mathrm{L}^1(\Omega; \mathrm{div}(\sigma)_+)$ is automatically satisfied for any choice of $u \in \mathrm{BV}(\Omega)$.

In contrast to the standard pairing of Definition 7.2, the inner pairing just introduced is *not* bilinear. In fact, it only applies the positive 1–homogeneity property $[\![t\sigma, \mathrm{D}u]\!] = t[\![\sigma, \mathrm{D}u]\!] = [\![\sigma, \mathrm{D}(tu)]\!]$ for all $t \in [0, \infty)$. Moreover, if we restrict to functions $u \in \mathrm{W}^{1,1}(U)$, the equality $u^+ = u^- = u^*$ holds in U up to $\mathcal{H}^{n-1}$–negligible sets, and by Proposition 7.1 this is preserved $\mathrm{div}(\sigma)_\pm$–a.e. Therefore, the inner pairing reduces to the Anzellotti pairing of in Definition 7.2, that is

$$[\![\sigma, \mathrm{D}u]\!] = [\![\sigma, \mathrm{D}u]\!]^* = (\sigma \cdot \nabla u) \mathcal{L}^n \llcorner U \qquad \text{for all } u \in \mathrm{W}^{1,1}(U) . \tag{7.1.3}$$

We now list some elementary properties of the inner pairing, starting from Theorem 7.5. An analogous result to (7.1.4) for the standard pairing $[\![\sigma, \mathrm{D}u]\!]^*$ defined for u under assumptions (a) or (b) is already to be found in Anzellotti's seminal work [7, Theorem 1.5 and Corollary 1.6], whereas for a proof in terms of arbitrary Borel representatives u^λ we refer to a synthesis of [33, Propositions 4.4 and 4.7] with [32, Theorems 3.3 and 4.12]. A valuable finding of [32] (not present in the original work [7]) is the characterization of the singular part of the λ–pairing, which we quote in Theorem 7.7. In detail, we will distinguish within the measure $[\![\sigma, \mathrm{D}u]\!]^s$ between a jump part and a Cantor part, and then express their respective properties.

Theorem 7.5 (inner pairing as measure and decomposition)**.** *In the setting of Definition 7.4, the inner pairing $[\![\sigma, \mathrm{D}u]\!]$ represents a finite signed Radon measure on U. Moreover, $[\![\sigma, \mathrm{D}u]\!]$ is absolutely continuous with respect to* Du *and precisely it satisfies*

$$\big|[\![\sigma, \mathrm{D}u]\!]\big| \leq ||\sigma||_{\mathrm{L}^\infty(U,\mathbb{R}^n)}|\mathrm{D}u| \qquad \textit{as measures on } U\,. \tag{7.1.4}$$

Being $[\![\sigma, \mathrm{D}u]\!] \in \mathrm{RM}(U)$, we can consider its Radon–Nikodým decomposition with respect to $\mathcal{L}^n$. Specifically, the absolutely continuous part is explicitly determined by

$$[\![\sigma, \mathrm{D}u]\!]^a = ([\![\sigma, \mathrm{D}u]\!]^*)^a = (\sigma \cdot \nabla u)\mathcal{L}^n \quad \textit{on } U\,. \tag{7.1.5}$$

We now want to further decompose $[\![\sigma, \mathrm{D}u]\!]^s$ in analogy to the decomposition of distributional derivatives of BV functions. In fact, the upper bound achieved in (7.1.4) proves absolute continuity of $[\![\sigma, \mathrm{D}u]\!]$ with respect to $|\mathrm{D}u|$ – which is also inherited by the singular parts $[\![\sigma, \mathrm{D}u]\!]^s \ll |\mathrm{D}^s u|$. This enables us, via Theorem 2.1, to introduce the following definition.

Definition 7.6 (decomposition of singular part of the inner pairing)**.** For u, σ as in Definition 7.4, we define *jump part* of the inner pairing $[\![\sigma, \mathrm{D}u]\!]$ the measure defined by

$$[\![\sigma, \mathrm{D}u]\!]^j := \frac{\mathrm{d}[\![\sigma, \mathrm{D}u]\!]^s}{\mathrm{d}|\mathrm{D}^s u|}|\mathrm{D}^j u|$$

and symmetrically its *Cantor part* will be

$$[\![\sigma, \mathrm{D}u]\!]^c := \frac{\mathrm{d}[\![\sigma, \mathrm{D}u]\!]^s}{\mathrm{d}|\mathrm{D}^s u|}|\mathrm{D}^c u|\,,$$

for $\mathrm{d}[\![\sigma, \mathrm{D}u]\!]^s/\mathrm{d}|\mathrm{D}^s u|$ Radon–Nikodým density of the inner pairing with respect to $\mathrm{D}^s u$.

Following again [33], we can then characterize jump and Cantor part of the pairing in terms of the inner/outer normal traces $\mathrm{Tr}^{\mathrm{e}}(\sigma, \mathrm{J}_u)$, $\mathrm{Tr}^{\mathrm{e}}(\sigma, \mathrm{J}_u) \in \mathrm{L}^\infty(\mathrm{J}_u; \mathcal{H}^{n-1})$ of $\sigma \in \mathcal{DM}(\Omega, \mathbb{R}^n)$ on the jump set J_u of u with respect to the orientation given by the direction of jump ν_u – reprising the definitions introduced in [2, Section 3]. We observe that such a definition is well–posed, since the jump set J_u is countably $\mathcal{H}^{n-1}$–rectifiable in $\mathbb{R}^n$ in view of Theorem 2.27.

Theorem 7.7. *For u and σ as above, the components of the singular part of the pairing satisfy*:

- $[\![\sigma, \mathrm{D}u]\!]^j = ([\![\sigma, \mathrm{D}u]\!]^*)^j + (u^+ - u^-)|\mathrm{div}(\sigma)| \llcorner \mathrm{J}_u /2$
  $= (\mathrm{Tr}^{\mathrm{e}}(\sigma, \mathrm{J}_u)\mathbb{1}_{U_-} + \mathrm{Tr}^{\mathrm{i}}(\sigma, \mathrm{J}_u)\mathbb{1}_{U_+})(u^+ - u^-)\mathcal{H}^{n-1} \llcorner \mathrm{J}_u$
  $= (\mathrm{Tr}^{\mathrm{e}}(\sigma, \mathrm{J}_u)\mathbb{1}_{U_-} + \mathrm{Tr}^{\mathrm{i}}(\sigma, \mathrm{J}_u)\mathbb{1}_{U_+})|\mathrm{D}^j u|$,
- $[\![\sigma, \mathrm{D}u]\!]^c = ([\![\sigma, \mathrm{D}u]\!]^*)^c$*, and by* (7.1.4) *the Cantor part $[\![\sigma, \mathrm{D}u]\!]^c$ is concentrated on the support of* $\mathrm{D}^c u$*.*

- *Moreover, if the approximate discontinuity set S_σ of σ is such that $|\mathrm{D}^c u|(S_\sigma) = 0$ (we record that from $\sigma \in \mathrm{L}^\infty_{\mathrm{loc}}(U, \mathbb{R}^n) \subseteq \mathrm{L}^1_{\mathrm{loc}}(U, \mathbb{R}^n)$ it is $\mathcal{L}^n(S_\sigma) = 0$, so the condition above is equivalent to requiring $|\mathrm{D}^d u|(S_\sigma) = 0$), then the diffuse part of the pairing can be expressed as $[\![\sigma, \mathrm{D}u]\!]^d = \widetilde{\sigma} \cdot \mathrm{D}^d u$, where $\widetilde{\sigma}$ is the approximate limit of σ.*

Note that applying (7.1.3) together with (7.1.5) we obtain

$$[\![\sigma, \mathrm{D}u]\!] = (\sigma \cdot \nabla u)\mathcal{L}^n \llcorner U \quad \text{ and } \quad [\![\sigma, \mathrm{D}u]\!]^s = 0 \qquad \text{for all } u \in \mathrm{W}^{1,1}(U)\,.$$

The next remark states that the pairing reduces to the product measure of field and derivative measure if we restrict to continuous divergence–measure fields, see for instance [33, Remark 4.10].

Remark 7.8 (inner pairing for continuos vector fields)**.** For $\sigma \in \mathcal{DM}(U, \mathbb{R}^n) \cap \mathrm{C}(U, \mathbb{R}^n)$ and any $u \in \mathrm{BV}(U)$ such that $u^\pm \in \mathrm{L}^1(U; |\mathrm{div}(\sigma)|)$, we have

$$[\![\sigma, \mathrm{D}u]\!] = \sigma \cdot \mathrm{D}u \quad \text{ as measures on } U\,.$$

7.2 Up-to-the-boundary pairing

Our next goal is adapting the notions of [87, Definition 3.2], [88, Definition 3.2], and [15, Definition 5.1] to introduce an extended version of the inner pairing $[\![\sigma, \mathrm{D}u]\!]$. Such *up–to–the–boundary pairing* $[\![\sigma, \mathrm{D}u]\!]_{u_0}$ will represent a distribution on the whole space $\mathbb{R}^n$ taking into account the extension of u by a given datum $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, while preserving positive 1–homogeneity of $\sigma \mapsto [\![\sigma, \mathrm{D}u]\!]_{u_0}$. In accordance to the extension result of Theorem 2.23, to guarantee that $\mathrm{D}\overline{u}^{u_0}$ is well–defined and finite we shall impose some additional regularity of the domain, working as in the previous chapters with open bounded Lipschitz subsets Ω of $\mathbb{R}^n$. The rest of Chapter 7 is devoted to listing or proving various properties of the up–to–the–boundary pairing, most of which are adapted from the corresponding results valid for the inner pairing. We point out that the introduction of a new pairing $[\![\sigma, \mathrm{D}u]\!]_{u_0}$ is meant to mirror the Dirichlet boundary condition u_0 in the main functional in (1.2.4), as explicated in the following Chapter 8.

Definition 7.9 (up–to–the–boundary pairing)**.** We assign a datum $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, $u \in \mathrm{BV}(\Omega)$ and $\sigma \in \mathcal{DM}(\Omega, \mathbb{R}^n)$ with $u_0^*|_\Omega$, $u^\pm \in \mathrm{L}^1(\Omega; |\mathrm{div}(\sigma)|)$. We introduce the **up-to-the-boundary pairing** $[\![\sigma, \mathrm{D}u]\!]_{u_0}$ defined through

$$[\![\sigma, \mathrm{D}u]\!]_{u_0} := \mathrm{div}\big(\overline{(u-u_0)\sigma}\big) - (u-u_0)^- \,\overline{\mathrm{div}(\sigma)_+} + (u-u_0)^+ \,\overline{\mathrm{div}(\sigma)_-} + (\sigma \cdot \nabla u_0)\mathcal{L}^n \llcorner \Omega$$

as a distribution on $\mathbb{R}^n$, where we consider the extensions $\overline{(u-u_0)\sigma} := (u-u_0)\sigma \mathbb{1}_\Omega + 0 \cdot \mathbb{1}_{\mathbb{R}^n \setminus \overline{\Omega}}$ and $\overline{\mathrm{div}(\sigma)}(A) := \mathrm{div}(\sigma)(A \cap \Omega)$ for any $A \subseteq \mathbb{R}^n$ Borel. Explicitly, it is

$$\begin{aligned} &[\![\sigma, \mathrm{D}u]\!]_{u_0}(\psi) \qquad (7.2.1)\\ &:= -\int_\Omega (u-u_0)\sigma \cdot \nabla\psi \,\mathrm{d}x - \int_\Omega \psi (u-u_0)^- \,\mathrm{d}(\mathrm{div}(\sigma)_+) + \int_\Omega \psi(u-u_0)^+ \,\mathrm{d}(\mathrm{div}(\sigma)_-) + \int_\Omega \psi\sigma \cdot \nabla u_0 \,\mathrm{d}x \end{aligned}$$

for all $\psi \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^n)$.

Proposition 7.10 (continuity property for the up–to–the–boundary pairing)**.** *Assume $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ and $\sigma \in \mathcal{DM}(\Omega, \mathbb{R}^n)$ such that $w^\pm \in \mathrm{L}^1(\Omega; |\mathrm{div}(\sigma)|)$ for every $w \in \mathrm{BV}(\Omega)$. Then, for $u \in \mathrm{BV}(\Omega)$ and any sequence $(w_k)_k$ in $\mathrm{W}^{1,1}(\Omega)$ satisfying the conditions:*

(i) *$w_k \to u$ in $\mathrm{L}^1(\Omega)$ as $k \to \infty$; and*

(ii) *$w_k^* \to u^\pm$ pointwise $\mathrm{div}(\sigma)_\mp$–a.e. in Ω as $k \to \infty$,* *we have convergence of the up–to–the–boundary pairing in the distributional sense, that is*

$$\lim_{k\to\infty} [\![\sigma, \mathrm{D}w_k]\!]_{u_0}(\psi) = [\![\sigma, \mathrm{D}u]\!]_{u_0}(\psi) \qquad \textit{for all } \psi \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^n)\,. \tag{7.2.2}$$

Remark 7.11. Notice that for instance the approximating sequence $(w_k)_k$ in the statement of Proposition 7.10 can be chosen as the one in Proposition 3.15 (where ν_1, ν_2 represent the mutually singular measures $\mathrm{div}(\sigma)_\mp$), or as the one in Result 3 (now with $\mu_\pm = \mathrm{div}(\sigma)_\mp$, see details in Section 6.2.1). We record that in this latter case the sequence achieves even the boundary value u_0 in the strong sense, meaning the condition $\lim_{k\to\infty} \int_\Omega w_k^* \,\mathrm{d}(\mathrm{div}(\sigma))_\mp = \int_\Omega w^\pm \,\mathrm{d}(\mathrm{div}(\sigma)_\mp$ is satisfied specifically for some $(w_k)_k$ in $\mathrm{W}^{1,1}_{u_0}(\Omega)$ converging to w even area–strictly in $\mathrm{BV}_{u_0}(\overline{\Omega})$.

Proof of Proposition 7.10. Fixed a test function $\psi \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^n)$, we first claim that it holds

$$\lim_{k\to\infty} \int_\Omega \psi w_k^* \,\mathrm{d}(\mathrm{div}(\sigma)) = \int_\Omega \psi u^- \,\mathrm{d}(\mathrm{div}(\sigma)_+) - \int_\Omega \psi u^+ \,\mathrm{d}(\mathrm{div}(\sigma)_-)\,. \tag{7.2.3}$$

In fact, the convergences $w_\ell^* \to u^-$ pointwise $\mathrm{div}(\sigma)_+$–a.e. for $\ell \to \infty$, and $w_\ell^* \to u^-$ pointwise $\mathrm{div}(\sigma)_-$–a.e. in Ω are preserved when multiplying by ψ. Fixing any truncation level $M > 0$, the $\mathrm{div}(\sigma)_\mp$–a.e. convergences $((\psi w_\ell)^M)^* = ((\psi w_\ell)^*)^M = (\psi w_\ell^*)^M \to (\psi u^\pm)^M = ((\psi u)^\pm)^M = ((\psi u)^M)^\pm$ are respectively in place on Ω. In view of the uniform bound $|(\psi w_\ell^*)^M(x)| \le M$ for all $\ell \in \mathbb{N}$ and for $\mathrm{div}(\sigma)_\mp$–a.e. $x \in \Omega$, we may apply the dominated convergence theorem to achieve

$$\lim_{\ell\to\infty} \int_\Omega (\psi w_\ell^*)^M \,\mathrm{d}(\mathrm{div}(\sigma)_-) = \int_\Omega (\psi u^+)^M \,\mathrm{d}(\mathrm{div}(\sigma)_-)\,,$$

$$\lim_{\ell\to\infty} \int_\Omega (\psi w_\ell^*)^M \,\mathrm{d}(\mathrm{div}(\sigma)_+) = \int_\Omega (\psi u^-)^M \,\mathrm{d}(\mathrm{div}(\sigma)_+)\,.$$

We now record that both measures $\mathrm{div}(\sigma)_\pm$ satisfy the assumptions Lemma 3.8 – because of Proposition 7.1 and given that (C3) for $|\mathrm{div}(\sigma)|$ holds by assumption. We can therefore pass to the limit as M diverges, yielding respectively

$$\lim_{M\to\infty} \int_\Omega (\psi u^+)^M \,\mathrm{d}(\mathrm{div}(\sigma)_-) = \int_\Omega \psi u^+ \,\mathrm{d}(\mathrm{div}(\sigma)_-)\,,$$

$$\lim_{M\to\infty} \int_\Omega (\psi u^-)^M \,\mathrm{d}(\mathrm{div}(\sigma)_+) = \int_\Omega \psi u^- \,\mathrm{d}(\mathrm{div}(\sigma)_+)\,.$$

Finally, by selection of a suitable sequence of levels $(M_k)_k$ such that $\lim_{k\to\infty} M_k = \infty$ and choosing for every k the function $w_k := (w_{\ell_k})^{M_k}$ with ℓ_k large enough, by summing up term by term in the estimates above we verify the claimed convergence (7.2.3). Moreover, exploiting assumption (i) it clearly holds $\lim_{k\to\infty} \int_\Omega (w_k - u_0)\sigma \cdot \nabla\psi \,\mathrm{d}x = \int_\Omega (u - u_0)\sigma \cdot \nabla\psi \,\mathrm{d}x$. The thesis (7.2.2) is then promptly achieved via the distributional definition of the pairing in (7.2.1). □

Lemma 7.12 (convergence of up–to–the–boundary pairing on truncations)**.** *Assigned u, u_0, σ as in Definition 7.9, for any level $M \in (0,\infty)$ we find*

$$\lim_{M\to\infty} [\![\sigma, \mathrm{D}u^M]\!]_{u_0^M}(\psi) = [\![\sigma, \mathrm{D}u]\!]_{u_0}(\psi) \qquad \textit{for all } \psi \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^n)\,.$$

Proof. Given M positive and $u_0^*|_\Omega$, $u^\pm \in \mathrm{L}^1(\Omega; |\mathrm{div}(\sigma)|)$, we record that by Proposition 2.102 it is $(u_0^M)^*$, $(u^M)^\pm \in \mathrm{L}^1(\Omega; |\mathrm{div}(\sigma)|)$, hence the pairing $[\![\sigma, \mathrm{D}u^M]\!]_{u_0^M}$ is well–posed. Then, for any ψ in $\mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^n)$ we compute

$$\begin{aligned}[\![\sigma, \mathrm{D}u^M]\!]_{u_0^M}(\psi) = &- \int_\Omega (u^M - u_0^M)\sigma \cdot \nabla\psi \,\mathrm{d}x - \int_\Omega \psi(u^M - u_0^M)^- \,\mathrm{d}(\mathrm{div}(\sigma)_+) \\ &+ \int_\Omega \psi(u^M - u_0^M)^+ \,\mathrm{d}(\mathrm{div}(\sigma)_-) + \int_\Omega \psi\sigma \cdot \nabla u_0^M \,\mathrm{d}x\,,\end{aligned}$$

with the first and last term converging respectively to $-\int_\Omega (u-u_0)\sigma\cdot\nabla\psi$ and $\int_\Omega \psi\sigma\cdot\nabla u_0$ in view of the strong convergences $u^M \to u$, $u_0^M \to u_0$ in $\mathrm{BV}(\Omega)$ as $M\to\infty$ from Lemma 2.100(iii). Furthermore, Lemma 2.100(v) yields the $\mathcal{H}^{n-1}$–a.e. pointwise convergences of the representatives $(w^M)^\pm \to w^\pm$ on Ω for $M\to\infty$, which via Proposition 7.1 are preserved $\mathrm{div}(\sigma)_\pm$–a.e. in Ω. It follows that

$$\psi(u^M-u_0^M)^\pm = \psi(u^M)^\pm - \psi(u_0^M)^* \xrightarrow[M\to\infty]{} \psi u^\pm - \psi u_0^* = \psi(u-u_0)^\pm \qquad |\mathrm{div}(\sigma)|\text{–a.e. in } \Omega\,.$$

In addition, from Lemma 2.100(v) we read the $\mathrm{div}(\sigma)_\pm$–a.e. inequalities $|(w^M)^\pm| \le |w^\pm|$ on Ω for any $w\in\mathrm{BV}(\Omega)$ and any M. Thus, we have $|\psi(u^M-u_0^M)^\pm| \le |\psi|\cdot|(u^M)^\pm| + |\psi|\cdot|(u_0^M)^*| \le ||\psi||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}\,(|u^\pm|+|u_0^*|) =: g^\pm$, with $g^\pm\in\mathrm{L}^1(\Omega;|\mathrm{div}(\sigma)|)$, therefore our assumptions on the distributional divergence holds. Finally, we employ the dominated convergence theorem to conclude that

$$\lim_{M\to\infty}\int_\Omega \psi(u^M-u_0^M)^\pm\,\mathrm{d}(\mathrm{div}(\sigma)_\mp) = \int_\Omega \psi(u-u_0)^\pm\,\mathrm{d}(\mathrm{div}(\sigma)_\mp)\,,$$

and this completes the proof. □

We have seen that the inner pairing reduces to the scalar product on Sobolev functions. We now prove the analogous version for the up–to–the–boundary pairing on Lipschitz domains, at the same time extending the normal trace result of Theorem 7.3.

Proposition 7.13 (up–to–the–boundary pairing on Sobolev functions)**.** *We assume* $u_0\in\mathrm{W}^{1,1}(\mathbb{R}^n)$ *and that* $\sigma\in\mathcal{DM}(\Omega,\mathbb{R}^n)$ *such that* $u_0^*|_\Omega\in\mathrm{L}^1(\Omega;|\mathrm{div}(\sigma)|)$.

(i) (*statement with prescribed boundary values*). *For every function* $u\in\mathrm{W}^{1,1}_{u_0}(\Omega)$ *such that* u^* *belongs to* $\mathrm{L}^1(\Omega;|\mathrm{div}(\sigma)|)$, *it holds* $[\![\sigma,\mathrm{D}u]\!]_{u_0} = (\sigma\cdot\nabla u)\mathcal{L}^n\llcorner\Omega$ *as measures on* $\mathbb{R}^n$ *– or equivalently*

$$[\![\sigma,\mathrm{D}u]\!]_{u_0}(\psi) = \int_\Omega \psi\sigma\cdot\nabla u\,\mathrm{d}x \qquad \textit{for all } \psi\in\mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^n)\,.$$

(ii) (*statement with arbitrary traces*). *In correspondence to* σ, *there exists a unique normal trace* $\sigma_n^*\in\mathrm{L}^\infty(\partial\Omega;\mathcal{H}^{n-1})$ *with*

$$||\sigma_n^*||_{\mathrm{L}^\infty(\partial\Omega;\mathcal{H}^{n-1})} \le ||\sigma||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \tag{7.2.4}$$

and such that for every $u\in\mathrm{W}^{1,1}(\Omega)$ *with* $u^*\in\mathrm{L}^1(\Omega;|\mathrm{div}(\sigma)|)$ *we have*

$$[\![\sigma,\mathrm{D}u]\!]_{u_0} = (\sigma\cdot\nabla u)\mathcal{L}^n\llcorner\Omega + (u-u_0)\sigma_n^*\,\mathcal{H}^{n-1}\llcorner\partial\Omega \qquad \textit{as measures on } \mathbb{R}^n\,,$$

which means (*omitting the boundary trace symbol in the last integral*) *that*

$$[\![\sigma,\mathrm{D}u]\!]_{u_0}(\psi) = \int_\Omega \psi\sigma\cdot\nabla u\,\mathrm{d}x + \int_{\partial\Omega}\psi(u-u_0)\sigma_n^*\,\mathrm{d}\mathcal{H}^{n-1} \qquad \textit{for all } \psi\in\mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^n)\,. \tag{7.2.5}$$

We clarify that the normal trace σ_n^* in (i) is the same function introduced in [7], and satisfying the integration by parts formula (7.1.2) for the Anzellotti pairing.

Proof. The result of (i) follows once we verify (ii). We assume $u\in\mathrm{W}^{1,1}(\Omega)$ so that $u^*\in\mathrm{L}^1(\Omega;|\mathrm{div}(\sigma)|)$, and we prove that (7.2.5) is satisfied. From [87, Lemma 3.3] we know that a normal trace for σ with upper bound (7.2.4) and satisfying (7.2.5) exists whenever u, $u_0|_\Omega\in\mathrm{W}^{1,1}(\Omega)\cap\mathrm{L}^\infty(\Omega)$. We observe that in the Sobolev case, the pairing introduced in Definition 7.9 and the one in [87, Definition 3.2] agree, as a consequence of the $\mathcal{H}^{n-1}$–a.e. equalities $u^+=u^-=u^*$ on Ω for u in $\mathrm{W}^{1,1}$. Assigned a positive level M, we can thus apply the result of [87, Lemma 3.3] to u^M, $u_0^M|_\Omega\in\mathrm{W}^{1,1}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ and write

$$[\![\sigma,\mathrm{D}u^M]\!]_{u_0^M}(\psi) = \int_\Omega \psi\sigma\cdot\nabla u^M\,\mathrm{d}x + \int_{\partial\Omega}\psi(u^M-u_0^M)\sigma_n^*\,\mathrm{d}\mathcal{H}^{n-1} \qquad \text{for all } \psi\in\mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^n)\,.$$

Being $u^M \to u$, $u_0^M \to u_0$ strongly in $\mathrm{W}^{1,1}(\Omega)$ as $M \to \infty$, Theorem 2.23 and $\sigma_n^* \in \mathrm{L}^\infty(\partial\Omega; \mathcal{H}^{n-1})$ determine

$$\lim_{M\to\infty} [\![\sigma, \mathrm{D}u^M]\!]_{u_0^M}(\psi) = \int_\Omega \psi\sigma \cdot \nabla u \, \mathrm{d}x + \int_{\partial\Omega} \psi(u - u_0)\sigma_n^* \, \mathrm{d}\mathcal{H}^{n-1} \quad \text{for all } \psi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n). \tag{7.2.6}$$

Joining the result of Proposition 7.13 with (7.2.6), we obtain the claimed global statement (7.2.5) even for unbounded functions u. □

Proposition 7.14 (up–to–the–boundary pairing as measure). *For any choice of u, u_0 and σ as in Definition 7.9, σ such that $w^\pm \in \mathrm{L}^1(\Omega; |\mathrm{div}(\sigma)|)$ for every $w \in \mathrm{BV}(\Omega)$, the pairing $[\![\sigma, \mathrm{D}u]\!]_{u_0}$ is a finite signed Radon measure on $\mathbb{R}^n$ supported in $\overline{\Omega}$ with*

$$\left|[\![\sigma, \mathrm{D}u]\!]_{u_0} - (u-u_0)\sigma_n^* \, \mathcal{H}^{n-1} \llcorner \partial\Omega\right| \le ||\sigma||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} |\mathrm{D}u| \llcorner \Omega \quad \textit{as measures on } \mathbb{R}^n. \tag{7.2.7}$$

Moreover, $[\![\sigma, \mathrm{D}u]\!]_{u_0}$ is absolutely continuous with respect to $\mathrm{D}\overline{u}^{u_0} \llcorner \overline{\Omega}$, and it holds

$$\left|[\![\sigma, \mathrm{D}u]\!]_{u_0}\right| \le ||\sigma||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} |\mathrm{D}\overline{u}^{u_0}| \llcorner \overline{\Omega} \quad \textit{as measures on } \mathbb{R}^n \tag{7.2.8}$$

for the extension $\overline{u}^{u_0} := u\mathbb{1}_\Omega + u_0 \mathbb{1}_{\mathbb{R}^n \setminus \overline{\Omega}}$.

Proof. We prove first that (7.2.7) holds in the distributional sense, namely

$$\left|[\![\sigma, \mathrm{D}u]\!]_{u_0}(\psi) - \int_{\partial\Omega} \psi(u-u_0)\sigma_n^* \, \mathrm{d}\mathcal{H}^{n-1}\right| \le ||\sigma||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \int_\Omega |\psi| \, \mathrm{d}|\mathrm{D}u| \quad \text{for all } \psi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n). \tag{7.2.9}$$

In fact, from Proposition 3.15 applied to $\mathrm{div}(\sigma)_-$ in place of ν_1 and $\mathrm{div}(\sigma)_+$ in place of ν_2, we can find some sequence $(w_k)_k$ in $\mathrm{W}^{1,1}(\Omega)$ strictly converging to u in $\mathrm{BV}(\Omega)$ and such that, given the Hahn's decomposition of Ω into $\Omega_\pm$, it is $w_k^* \to u^- \mathbb{1}_{\Omega_+} + u^+ \mathbb{1}_{\Omega_-}$ pointwise $|\mathrm{div}(\sigma)|$–a.e. in Ω as $k \to \infty$, and $\int_\Omega w_k^* \, \mathrm{d}(\mathrm{div}(\sigma)) \to \int_\Omega (u^- \mathbb{1}_{\Omega_+} + u^+ \mathbb{1}_{\Omega_-}) \, \mathrm{d}(\mathrm{div}(\sigma))$ as $k \to \infty$. Restricting the pointwise convergence to Ω_+, we find $w_k^* \to u^-$ pointwise $\mathrm{div}(\sigma)_+$–a.e. in Ω_+ (and thus in Ω) for $k \to \infty$, and symmetrically restricting to Ω_- it is $w_k^* \to u^-$ pointwise $\mathrm{div}(\sigma)_-$–a.e. in Ω_- (thus in Ω). We can then rely on the result of Proposition 7.10 to obtain that $[\![\sigma, \mathrm{D}w_k]\!]_{u_0}(\psi) \to [\![\sigma, \mathrm{D}u]\!]_{u_0}(\psi)$ as $k \to \infty$. On the other hand, we exploit the strict convergence assumption and Proposition 2.10 to argue

$$\limsup_{k\to\infty} \int_\Omega \psi\sigma \cdot \nabla w_k \, \mathrm{d}x \le ||\sigma||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \int_\Omega |\psi| \, \mathrm{d}|\mathrm{D}u|.$$

Employing now strict continuity of traces and the rewriting in Proposition 7.13(ii), we compute with the help of the convergences above

$$\begin{aligned} \mathrm{I}_k^{u_0}[\psi] := [\![\sigma, \mathrm{D}u]\!]_{u_0}(\psi) - \int_{\partial\Omega} \psi(u-u_0)\sigma_n^* \, \mathrm{d}\mathcal{H}^{n-1} &= \lim_{k\to\infty} \left([\![\sigma, \mathrm{D}w_k]\!]_{u_0}(\psi) - \int_{\partial\Omega} \psi(w_k - u_0)\sigma_n^* \, \mathrm{d}\mathcal{H}^{n-1}\right) \\ &= \lim_{k\to\infty} \int_\Omega \psi\sigma \cdot \nabla w_k \\ &\le ||\sigma||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \int_\Omega |\psi| \, \mathrm{d}|\mathrm{D}u|. \end{aligned} \tag{7.2.10}$$

By arbitrariness of $\psi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n)$, we can apply (7.2.10) to $-\psi$ to get

$$-\mathrm{I}_k^{u_0}[\psi] = \mathrm{I}_k^{u_0}[-\psi] \le ||\sigma||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \int_\Omega |\psi| \, \mathrm{d}|\mathrm{D}u|.$$

Altogether, we found $\left|\mathrm{I}_k^{u_0}[\psi]\right| \le ||\sigma||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \int_\Omega |\psi| \, \mathrm{d}|\mathrm{D}u|$ for all $\psi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n)$, which completes the proof of (7.2.9). Then, via the theorems of Hahn–Banach and Riesz–Markov, the last estimate determines that $[\![\sigma, \mathrm{D}u]\!]_{u_0}$ is a finite signed Radon measure on $\mathbb{R}^n$ satisfying (7.2.7).

Additionally, Proposition 7.13(ii) states that $||\sigma_n^*||_{\mathrm{L}^\infty(\partial\Omega;\mathcal{H}^{n-1})} \leq ||\sigma||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}$, thus even the bound (7.2.8) follows. To show that $\operatorname{supp}(\llbracket\sigma, \mathrm{D}u\rrbracket_{u_0}) \subseteq \overline{\Omega}$, we take a test function $\widehat{\psi} \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n)$ with support contained in $\overline{\Omega}^\mathrm{c}$; then, $\widehat{\psi} \equiv 0$ and $\nabla\widehat{\psi} \equiv 0$ on Ω, and checking via the explicit writing (7.2.1) we achieve $\llbracket\sigma, \mathrm{D}u\rrbracket_{u_0}(\widehat{\psi}) = 0$. With this latter, the statement is entirely proved. □

Remark 7.15. For any u, u_0 and σ as in Definition 7.9, the up–to–the–boundary measure $\llbracket\sigma, \mathrm{D}u\rrbracket_{u_0}$ evaluated on the whole space $\mathbb{R}^n$ takes the value of

$$\llbracket\sigma, \mathrm{D}u\rrbracket_{u_0}(\mathbb{R}^n) = \llbracket\sigma, \mathrm{D}u\rrbracket_{u_0}(\overline{\Omega}) = \int_\Omega \sigma \cdot \nabla u_0 - \int_\Omega (u - u_0)^- \,\mathrm{d}(\operatorname{div}(\sigma)_+) + \int_\Omega (u - u_0)^+ \,\mathrm{d}(\operatorname{div}(\sigma)_-)\,.$$

Proof. Considering the test function on $\mathbb{R}^n$ given by a cut–off function $\widehat{\psi}$ such that $\mathbb{1}_\Omega \leq \widehat{\psi} \leq \mathbb{1}_U$ for open $U \Supset \Omega$, then clearly $\widehat{\psi} \equiv 1$ on Ω with vanishing gradient on Ω. Applying the distributional definition in (7.2.1), we write

$$\llbracket\sigma, \mathrm{D}u\rrbracket_{u_0}(\widehat{\psi}) = \int_\Omega \sigma \cdot \nabla u_0 - \int_\Omega (u - u_0)^- \,\mathrm{d}(\operatorname{div}(\sigma)_+) + \int_\Omega (u - u_0)^+ \,\mathrm{d}(\operatorname{div}(\sigma)_-)\,.$$

On the other hand, reading $\llbracket\sigma, \mathrm{D}u\rrbracket_{u_0}$ as Radon measure concentrated on $\overline{\Omega}$ as from Proposition 7.14, for the same test function $\widehat{\psi}$ one has

$$\begin{aligned}\llbracket\sigma, \mathrm{D}u\rrbracket_{u_0}(\widehat{\psi}) = \int_{\mathbb{R}^n} \widehat{\psi} \,\mathrm{d}\llbracket\sigma, \mathrm{D}u\rrbracket_{u_0} &= \int_{\overline{\Omega}} \widehat{\psi} \,\mathrm{d}\llbracket\sigma, \mathrm{D}u\rrbracket_{u_0} + \int_{\mathbb{R}^n\setminus\overline{\Omega}} \widehat{\psi} \,\mathrm{d}\llbracket\sigma, \mathrm{D}u\rrbracket_{u_0} \\ &= \int_{\overline{\Omega}} \mathrm{d}\llbracket\sigma, \mathrm{D}u\rrbracket_{u_0} = \llbracket\sigma, \mathrm{D}u\rrbracket_{u_0}(\overline{\Omega})\,.\end{aligned}$$

The final result follows by comparing the two equations. □

We now show that the upper bound in Proposition 7.14 determines that the up–to–the–boundary pairing restricted to $\partial\Omega$ coincides with the trace boundary integral of $u - u_0$.

Corollary 7.16 (up–to–the–boundary–pairing on the boundary)**.** *For u, u_0, σ as in Definition 7.9, we have*

$$\llbracket\sigma, \mathrm{D}u\rrbracket_{u_0} \llcorner \partial\Omega = \llbracket\sigma, \mathrm{D}u\rrbracket_{u_0}^j \llcorner \partial\Omega = (u - u_0)\sigma_n^* \,\mathcal{H}^{n-1} \llcorner \partial\Omega\,. \tag{7.2.11}$$

Proof. We consider the measure bound (7.2.7) restricted to $\partial\Omega$, which yields both

$$\pm\llbracket\sigma, \mathrm{D}u\rrbracket_{u_0} \llcorner \partial\Omega \mp (u - u_0)\sigma_n^* \,\mathcal{H}^{n-1} \llcorner \partial\Omega \leq ||\sigma||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}(|\mathrm{D}u| \llcorner \Omega) \llcorner \partial\Omega = 0\,.$$

Therefore, we deduce that $(u - u_0)\sigma_n^* \,\mathcal{H}^{n-1} \leq \llbracket\sigma, \mathrm{D}u\rrbracket_{u_0} \leq (u - u_0)\sigma_n^* \,\mathcal{H}^{n-1}$ must hold as an equality of measures on $\partial\Omega$, implying (7.2.11). □

Furthermore, when restricting the new pairing to the interior of its support, we recover precisely the previous inner pairing $\llbracket\sigma, \mathrm{D}u\rrbracket$.

Remark 7.17 (inner and up–to–the–boundary pairing agree on Ω)**.** We consider $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, $u \in \mathrm{BV}(\Omega)$ and $\sigma \in \mathcal{DM}(\Omega, \mathbb{R}^n)$ such that $u_0^*|_\Omega$, $u^\pm \in \mathrm{L}^1(\Omega; |\operatorname{div}(\sigma)|)$. Then it holds

$$\llbracket\sigma, \mathrm{D}u\rrbracket = \llbracket\sigma, \mathrm{D}u\rrbracket_{u_0} \llcorner \Omega \quad \text{as measures on } \Omega\,.$$

Proof. It is enough to show that the pairings agree as distributions on Ω. Let then $\psi \in \mathrm{C}_c^\infty(\Omega)$ and consider the trivial extension $\overline{\psi}^0 := \psi \mathbb{1}_\Omega + 0 \cdot \mathbb{1}_{\mathbb{R}^n \setminus \overline{\Omega}} \in \mathrm{C}_c^\infty(\mathbb{R}^n)$. Recalling that $[\![\sigma, \mathrm{D}u]\!]_{u_0}$ is supported on $\overline{\Omega}$ and that $\overline{\psi}^0|_{\overline{\Omega}} = \psi$ on Ω, we employ (7.2.1) for $\overline{\psi}^0$ to compute

$$\begin{aligned}
[\![\sigma, \mathrm{D}u]\!]_{u_0}(\psi) = [\![\sigma, \mathrm{D}u]\!]_{u_0}(\overline{\psi}) &= -\int_\Omega (u-u_0)\sigma\cdot\nabla\psi \,\mathrm{d}x - \int_\Omega \psi u^- \,\mathrm{d}(\mathrm{div}(\sigma)_+) + \int_\Omega \psi u^+ \,\mathrm{d}(\mathrm{div}(\sigma)_-) \\
&\quad + \int_\Omega \psi\sigma\cdot\nabla u_0 \,\mathrm{d}x + \int_\Omega \psi u_0^* \,\mathrm{d}(\mathrm{div}(\sigma)) \\
&= [\![\sigma, \mathrm{D}u]\!](\psi) + \int_\Omega u_0\sigma\cdot\nabla\psi \,\mathrm{d}x + \int_\Omega \psi\sigma\cdot\nabla u_0 \,\mathrm{d}x + \int_\Omega \psi u_0^* \,\mathrm{d}(\mathrm{div}(\sigma))\,.
\end{aligned}$$

We now observe that $\psi u_0 \in \mathrm{W}_0^{1,1}(\Omega)$, hence the definition of distributional divergence of σ yields

$$\int_\Omega \psi u_0^* \,\mathrm{d}(\mathrm{div}(\sigma)) = -\int_\Omega \sigma\cdot\nabla(\psi u_0)\,\mathrm{d}x = -\int_\Omega u_0\sigma\cdot\nabla\psi\,\mathrm{d}x - \int_\Omega \psi\sigma\cdot\nabla u_0\,\mathrm{d}x\,,$$

and by substitution in the previous chain of equalities we obtain $[\![\sigma, \mathrm{D}u]\!]_{u_0}(\psi) = [\![\sigma, \mathrm{D}u]\!](\psi)$, so the thesis follows. □

In the subsequent Propositions 7.18 and 7.20, we state other upper bounds for the new pairing. In detail, the goal of our first Proposition 7.18 is to extend the control (7.2.8) on the pairing to anisotropic bounds.

Proposition 7.18 (upper bound in terms of polar functions)**.** *We consider a* <u>*continuous*</u> *anisotropy φ on $\mathbb{R}^n$ with linear growth, i.e. $\alpha|\xi| \le \varphi(x,\xi) \le \beta|\xi|$ for some $\alpha,\beta \in (0,\infty)$. Moreover, we let $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, $u \in \mathrm{BV}(\Omega)$ and $\sigma \in \mathcal{DM}(\Omega,\mathbb{R}^n)$ such that all $w \in \mathrm{BV}(\Omega)$ satisfy $w^\pm \in \mathrm{L}^1(\Omega;|\mathrm{div}(\sigma)|)$. Then, recalling the writing $|\mathrm{D}\overline{u}^{u_0}|_\varphi \llcorner \overline{\Omega} = |\mathrm{D}u|_\varphi \llcorner \Omega + \varphi(.,(u-u_0)\nu_\Omega)\mathcal{H}^{n-1}\llcorner\partial\Omega$, the following estimate holds*:

$$[\![\sigma, \mathrm{D}u]\!]_{u_0} \le \|\varphi^\circ(.,\sigma)\|_{\mathrm{L}^\infty(\Omega)} |\mathrm{D}\overline{u}^{u_0}|_\varphi \llcorner \overline{\Omega} \qquad \textit{as measures on } \mathbb{R}^n\,. \tag{7.2.12}$$

Proof. We first argue that the polar function $\varphi^\circ(x,.)$ of $\varphi(x,.)$ is well–defined for all $x \in \Omega$ since the linear–growth assumption on φ induces $\varphi(x,\xi) > 0 = \varphi(x,0)$ for all $(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Consider then a non–negative test function $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^n)$. We exploit the recovery sequence argument in the proof of Result 3 to suitably approximate u via a sequence $(w_k)_k$ in $\mathrm{W}_{u_0}^{1,1}(\Omega)$. Then, a combination of Proposition 7.10 with the reduction to the pairing on Sobolev functions $(w_k)_k$ with trace u_0 on $\partial\Omega$ as in Proposition 7.13(i), determines

$$[\![\sigma, \mathrm{D}u]\!]_{u_0}(\psi) = \lim_{k\to\infty} [\![\sigma, \mathrm{D}w_k]\!]_{u_0}(\psi) \;= \lim_{k\to\infty}\int_\Omega \psi\sigma\cdot\nabla w_k\,\mathrm{d}x\,.$$

Via the polar inequality $\xi\cdot\xi^* \le \varphi^\circ(x,\xi)\varphi(x,\xi^*)$, we can thus argue that

$$[\![\sigma, \mathrm{D}u]\!]_{u_0}(\psi) \le \liminf_{k\to\infty}\int_\Omega \psi\varphi^\circ(.,\sigma)\varphi(.,\nabla w_k)\,\mathrm{d}x \le \|\varphi^\circ(.,\sigma)\|_{\mathrm{L}^\infty(\Omega)} \liminf_{k\to\infty}\int_\Omega \psi\varphi(.,\nabla w_k)\,\mathrm{d}x\,. \tag{7.2.13}$$

On the other hand, strict convergence of $(w_k)_k$ to u in $\mathrm{BV}_{u_0}(\overline{\Omega})$ induces by Theorem 2.79 the φ–strict convergence of $(w_k)_k$ to u in $\mathrm{BV}_{u_0}(\overline{\Omega})$. With these latter ingredients at hand, we can employ Proposition 2.10 with strict convergence of the non–negative measures $\nu_k := |\mathrm{D}\overline{w_k}^{u_0}|_\varphi$, $\nu := |\mathrm{D}\overline{u}^{u_0}|_\varphi$ on $\mathbb{R}^n$ and for our (continuous, bounded) ψ. Simplifying the common terms in the integrals, we obtain

$$\lim_{k\to\infty}\int_\Omega \psi\varphi(.,\nabla w_k)\,\mathrm{d}x = \int_\Omega \psi\,\mathrm{d}|\mathrm{D}u|_\varphi + \int_{\partial\Omega} \psi\varphi(.,(u-u_0)\nu_\Omega)\,.$$

Recalling now our previous result (7.2.13), we conclude that

$$[\![\sigma, \mathrm{D}u]\!]_{u_0}(\psi) \leq ||\varphi^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)} \left(\int_\Omega \psi \,\mathrm{d}|\mathrm{D}u|_\varphi + \int_{\partial\Omega} \psi\varphi(.,(u-u_0)\nu_\Omega)\mathrm{d}\mathcal{H}^{n-1} \right) \tag{7.2.14}$$

for all non–negative $\psi \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^n)$, which establishes our thesis (7.2.12). □

We observe that the generalizations from an upper bound in terms of the φ_0–variation (as in Proposition 7.14) to any φ–variation of the type considered in the latter Proposition 7.18 come with a drawback. In fact, in general, the non–evenness of φ in the second entry prevents us from achieving exactly the opposite control from below in the following (7.2.12). We additionally observe that by reprising the proof of (7.2.12) – this time working with non–positive functions – one obtains

$$-[\![\sigma, \mathrm{D}u]\!]_{u_0}(\psi) \leq ||\varphi^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)} \left(\int_\Omega \psi \,\mathrm{d}|\mathrm{D}u|_{\widetilde{\varphi}} + \int_{\partial\Omega} \psi\widetilde{\varphi}(.,(u-u_0)\nu_\Omega)\,\mathrm{d}\mathcal{H}^{n-1} \right)$$

for all $\psi \in \mathrm{C}^\infty_c(\mathbb{R}^n)$, $\psi \geq 0$. Then, for anisotropies φ such that the restriction $\xi \mapsto \varphi(x,\xi)$ is *even* in $\mathbb{R}^n$, summing up with (7.2.14) on positive and negative parts of test functions, we get the φ–anisotropic version

$$\big|[\![\sigma, \mathrm{D}u]\!]_{u_0}\big| \leq ||\varphi^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)}|\mathrm{D}\overline{u}^{u_0}|_\varphi \llcorner \overline{\Omega}$$

of (7.2.8), which however does not hold for arbitrary anisotropies φ.

Remark 7.19. Exploiting the decomposition Theorem 7.7 and Remark 7.17, the upper bound for the pairing expressed in Proposition 7.18 ensures that

$$[\![\sigma, \mathrm{D}u]\!]^j = (\mathrm{Tr}^\mathrm{e}(\sigma, \mathrm{J}_u)\mathbb{1}_{\Omega_-} + \mathrm{Tr}^\mathrm{i}(\sigma, \mathrm{J}_u)\mathbb{1}_{\Omega_+})|\mathrm{D}^j u| \leq ||\varphi^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)}|\mathrm{D}^j u|_\varphi \quad \text{on } \Omega$$

for the jump part of the pairing, whereas the Cantor part enjoys

$$[\![\sigma, \mathrm{D}u]\!]^c \leq ||\varphi^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)}|\mathrm{D}^c u|_\varphi \quad \text{on } \Omega\,.$$

A more general bound for the pairing as in Proposition 7.18 can be achieved also in terms of non–homogeneous functions $f = f(x,\xi)$ with linear growth, and it reads out

$$[\![\sigma, \mathrm{D}u]\!]_{u_0} \leq ||f^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)} f(.,\mathrm{D}\overline{u}^{u_0})\llcorner\overline{\Omega} \qquad \text{as measures on } \mathbb{R}^n\,,$$

holding provided it is $f(x,\xi) > 0 = f(x,0)$ for all $(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ with f, f^∞ continuous everywhere and convex in the second entry. In fact, the polar inequality (2.8.11) still applies to this setting, and the proof of Proposition 7.18 can be adjusted to area–strict converging sequences exploiting the inhomogeneous version of Reshetnyak's theorem, as mentioned in the following.

Proposition 7.20 (upper bound in terms of conjugate functions)**.** *We consider a continuous function $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ admissible as in Assumption 1.1(H1), and $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$. We take $u \in \mathrm{BV}(\Omega)$ and a vector field $\sigma \in \mathcal{DM}(\Omega,\mathbb{R}^n)$ such that $w^\pm \in \mathrm{L}^1(\Omega; |\mathrm{div}(\sigma)|)$ for all $w \in \mathrm{BV}(\Omega)$. Then it holds*

$$[\![\sigma, \mathrm{D}u]\!]_{u_0} \leq f(.,\mathrm{D}\overline{u}^{u_0})\llcorner\overline{\Omega} + f^*(.,\sigma)\mathcal{L}^n\llcorner\Omega \qquad \textit{as measures on } \mathbb{R}^n\,, \tag{7.2.15}$$

with inequality trivially satisfied in case $f^*(.,\sigma) = \infty$ on a set of positive measure in Ω.

Observe that from Proposition 7.20 we can deduce the previous result of Proposition 7.18. In fact, assuming that $\xi \mapsto f(x,\xi)$ is positively 1–homogeneous, by Proposition 2.89 the conjugate $f^*(.,\sigma)$ is a.e.–finite if and only if $||f^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)} \leq 1$ – and in such a case it is even $f^*(.,\sigma) = 0$ almost everywhere. Then, the bound in (7.2.15) is trivial if we take a field σ such that $||f^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)} > 1$. Otherwise, by homogeneity of f, f° and of the pairing in the first entry, without loss of generality we may assume precisely $||f^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)} = 1$, hence (7.2.12) follows.

Proof of Proposition 7.20. We slightly adapt the result of Proposition 7.18 to the possibly inhomogeneous framework. Since the measure $[\![\sigma, \mathrm{D}u]\!]_{u_0}$ is supported in the closure of Ω, to get (7.2.15) it suffices to prove

$$[\![\sigma, \mathrm{D}u]\!]_{u_0}(\psi) \le \int_{\overline{\Omega}} \psi \, \mathrm{d}f(.,\mathrm{D}\overline{u}^{u_0}) + \int_{\Omega} \psi f^*(.,\sigma)\,\mathrm{d}x \qquad \text{for all non–negative } \psi \in \mathrm{C}^\infty_\mathrm{c}(U)\,, \tag{7.2.16}$$

for any open bounded set $U \Supset \Omega$. Then, letting a non–negative $\psi \in \mathrm{C}^\infty_\mathrm{c}(U)$, we employ area–strict approximation of u via $(w_k)_k$ in $\mathrm{W}^{1,1}_{u_0}(\Omega)$ as in Result 3. Arguing as in Proposition 7.18 and via Fenchel's inequality $\xi \cdot \xi^* \le f(x,\xi) + f^*(x,\xi^*)$ for all $x, \xi, \xi^* \in \mathbb{R}^n$, we get

$$\begin{aligned} [\![\sigma, \mathrm{D}u]\!]_{u_0}(\psi) = \lim_{k\to\infty} \int_\Omega \psi\sigma\cdot\nabla w_k \;\mathrm{d}x &\le \liminf_{k\to\infty} \int_\Omega \psi\left(f(.,\nabla w_k) + f^*(.,\sigma)\right)\,\mathrm{d}x \\ &= \liminf_{k\to\infty}\left(\int_\Omega \psi f(.,\nabla w_k)\,\mathrm{d}x\right) + \int_\Omega \psi f^*(.,\sigma)\,\mathrm{d}x\,. \end{aligned} \tag{7.2.17}$$

Recall that the convergence $w_k \to u$ for $k \to \infty$ is strict in area on $\mathrm{BV}_{u_0}\big(\overline{\Omega}\big)$, so by Reshetnyak's Theorem 2.70 it holds $\lim_{k\to\infty} f(.,\mathrm{D}\overline{w_k}^{u_0})(U) = f(.,\mathrm{D}\overline{u}^{u_0})\big(U\big)$, whereas the semicontinuity Theorem 2.67 yields

$$\liminf_{k\to\infty} f(.,\mathrm{D}\overline{w_k}^{u_0})(A) \ge f(.,\mathrm{D}\overline{u}^{u_0})(A) \qquad \text{for all } A \subseteq U \text{ open}\,.$$

We are thus in the conditions of applying [3, Proposition 1.80] to the non–negative Radon measures $f(.,\mathrm{D}\overline{w_k}^{u_0}) \llcorner U$, $f(.,\mathrm{D}\overline{u}^{u_0}) \llcorner U$ and the function ψ, hence

$$\lim_{k\to\infty} \int_\Omega \psi f(.,\nabla w_k)\,\mathrm{d}x = \int_{\overline{\Omega}} \psi\,\mathrm{d}f\big(.,\mathrm{D}\overline{u}^{u_0}\big)\,.$$

In conclusion, using (7.2.17) we derive the claimed (7.2.16), which induces the corresponding inequality between measures. □

Chapter 8

Duality theory for linear-growth functionals with measures

As usual, we set a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ and a datum $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$. For suitable measures $\mu_\pm$ admissible on Ω as in Definition 3.3, the $\mathrm{BV}(\Omega)$–minimization problem for the functional $\mathcal{F}$ of (1.2.4) reads out

$$\inf_{w\in \mathrm{BV}(\Omega)} \left(f(.,\mathrm{D}\overline{w}^{u_0})(\overline{\Omega}) - \int_\Omega w^+ \,\mathrm{d}\mu_- + \int_\Omega w^- \,\mathrm{d}\mu_+ \right). \tag{P}$$

We are now concerned with a dual reformulation of the (primal) problem (P) in the sense of convex analysis; for further details on dual formulations, we refer to the monograph [44]. In general, duals of first–order variational problems in proper integrand $f\colon \Omega\times\mathbb{R}^n \to \overline{\mathbb{R}}$ are formulated in terms of the convex conjugate function of f with respect to the second entry, which according to Section 2.9 is the function $f^*\colon \Omega\times\mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ given by

$$f^*(x,\xi^*) := \sup_{\xi\in\mathbb{R}^n} \big(\xi^*\cdot\xi - f(x,\xi)\big) \qquad \text{for all } (x,\xi^*) \in \Omega\times\mathbb{R}^n\,.$$

Then, the convex dual problem of (P) represents a *maximization* problem in some dual function space, and it is characterized by the condition of attaining the same extremal value of (P). The aim of the present section is to determine the appropriate dual variational problem for the functional $\inf_{\mathrm{BV}(\Omega)} \mathcal{F}$, and therefore obtain Result 4. Specifically, under mutual singularity of $\mu_\pm$, we claim that our dual formulation is expressed by

$$\sup_{\substack{\sigma\in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)\\ \mathrm{div}(\sigma)=\mu}} \left(\int_\Omega [\sigma\cdot\nabla u_0 - f^*(.,\sigma)]\,\mathrm{d}x + \int_\Omega u_0^*\,\mathrm{d}\mu \right). \tag{P*}$$

Here, the supremum is taken among bounded divergence–measure fields $\sigma \in \mathcal{DM}(\Omega,\mathbb{R}^n)$ with divergence equals to the measure $\mu := \mu_+ - \mu_-$; in short, (P*) is a maximization problem in the class

$$\mathcal{DM}_\mu(\Omega,\mathbb{R}^n) := \{\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)\colon\ \mathrm{div}(\sigma) = \mu \ \text{ as measures on } \Omega\}\,.$$

Such a duality formula represents a meaningful extension of previous results, in particular of [15] (in the case of linear–growth integrands without measures) and of [67] (for the area problem with signed admissible measures). We shall see in the following that the relation between the primal and dual problems can be straightforwardly expressed in terms of the pairings introduced in Chapter 7. Similar strategies are employed in [89] to treat dual obstacle problems for TV and area, as well as in the study of (super)solutions to the corresponding 1–Laplace and minimal surface equation in [87].

The standard approach to convex duality relies on the abstract theory enunciated in full generality in [44, Chapter 3]. Although the universality of such a method would enable us to tackle problem (P) as well, here we adopted a step–wise approximation procedure (via coercive integrands with quadratic growth) to determine the dual of the original minimization problem $\inf_{\mathrm{W}^{1,1}_{u_0}} \mathcal{F}_{u_0}$, and then we make use of the Result 3 to achieve that (P*) is also the dual of $\inf_{\mathrm{BV}} \mathcal{F}$. We believe that the advantage of our approach privileges a better understanding of the maximization problem at each step; instead, working with abstract theory in the full space $\mathrm{BV}(\Omega)$ would lead to more involved computations to determine the appropriate dual spaces – and this becomes increasingly complex given the generality of our admissible integrands f. We additionally highlight that to obtain duality formulas in the Sobolev spaces, any pairs $\mu_\pm$ of admissible measures on Ω are allowed, whereas the additional condition of mutual singularity is imposed only at a later point, namely when applying Result 3.

The closest literature results to the statements of our Chapter 8 are to be found in the dual formulation (and consequent optimality conditions) addressed in [15, Theorems 1.1 and 2.2] – where, however, the assumption $\mu \equiv 0$ determines a major difference to our treatment. Indeed, the reference deals with the BV–relaxation of first–order variational problems of the kind

$$\inf_{w \in \mathrm{W}^{1,1}_{u_0}(U,\mathbb{R}^N)} \int_U f(.,\nabla w)\,\mathrm{d}x$$

where U is an open – possibly even unbounded – set in $\mathbb{R}^n$, and the mapping $w\colon U \to \mathbb{R}^N$ is N–valued for $N \in \mathbb{N}$ given. Here, the authors consider integrands $f\colon U \times \mathbb{R}^{Nn} \to \mathbb{R}$ convex in $\mathbb{R}^N$ and under the more general linear–growth assumption

$$|f(x,\xi)| \le \Psi(x) + \beta|\xi| \qquad \text{for all } (x,\xi) \in U \times \mathbb{R}^{Nn},$$

supposing $\Psi \in \mathrm{L}^1(U)$ is non–negative and $\beta \in [0,\infty)$. The technique in [15] has some analogies with ours, insofar as it does not employ the abstract duality theory. Nevertheless, in there the approach to duality is based on the construction of approximative solutions for the dual problem via Ekeland's variational principle [43].

Remark 8.1. It is of common knowledge that one inequality between (P) and (P*) is straightforward. In fact, we record that the Fenchel's inequality for $f(x,.)$ yields $\xi^*\cdot\xi \le f(x,\xi) + f^*(x,\xi^*)$ for all $x \in \Omega$, $\xi,\ \xi^* \in \mathbb{R}^n$. Then, for any $w \in \mathrm{W}^{1,1}_{u_0}(\Omega)$ and any $\sigma \in \mathcal{DM}_\mu(\Omega,\mathbb{R}^n)$, by the definition of distributional divergence applied to $w - u_0 \in \mathrm{W}^{1,1}_0(\Omega)$ it is easy to verify that

$$\begin{aligned}
\int_\Omega f(.,\nabla w)\,\mathrm{d}x + \int_\Omega (w-u_0)^*\,\mathrm{d}\mu &\ge \int_\Omega \nabla w\cdot\sigma\,\mathrm{d}x - \int_\Omega f^*(.,\sigma)\,\mathrm{d}x + \int_\Omega (w-u_0)^*\,\mathrm{d}(\mathrm{div}(\sigma))\\
&= \int_\Omega \nabla w\cdot\sigma\,\mathrm{d}x - \int_\Omega f^*(.,\sigma)\,\mathrm{d}x - \int_\Omega \sigma\cdot\nabla(w-u_0)\,\mathrm{d}x\\
&= \int_\Omega \sigma\cdot\nabla u_0\,\mathrm{d}x - \int_\Omega f^*(.,\sigma)\,\mathrm{d}x\,, \qquad (8.0.1)
\end{aligned}$$

and we notice that $f^*(.,\sigma) = \infty$ somewhere in Ω is also allowed. Then, it is straightforward to estimate

$$(\mathrm{P}) = \inf_{w \in \mathrm{W}^{1,1}_{u_0}(\Omega)} \left(\int_\Omega f\,(.,\nabla w)\,\mathrm{d}x + \int_\Omega w^*\,\mathrm{d}\mu \right) \ge \sup_{\sigma \in \mathcal{DM}_\mu(\Omega,\mathbb{R}^n)} \left(\int_\Omega [\sigma\cdot\nabla u_0 - f^*(.,\sigma)]\,\mathrm{d}x + \int_\Omega u_0^*\,\mathrm{d}\mu \right).$$

It is thus left to check the validity of the remaining inequality to conclude that (P*) is actually in duality with (P).

Our procedure for convex duality consists of the following steps:

- We first approach duality in the class $\mathrm{W}^{1,2}_{u_0}(\Omega)$ under convexity and differentiability assumptions for an integrand f of quadratic growth and $\mathrm{W}^{1,2}$–coercive (Theorem 8.25).
- We then get rid of the coercivity assumption by approximating the integrand via a sequence of $\mathrm{W}^{1,2}$–coercive functions $f_k(x,\xi) := f(x,\xi) + \varepsilon_k|\xi|^2/2$ with $(\varepsilon_k)_k$ decreasing to zero as $k \to \infty$. This way, in Theorem 8.26 we are able to determine the dual problem for differentiable integrands with at most quadratic growth, where we still maximize among vector fields in a suitable subclass of elements in $\mathrm{L}^2(\Omega, \mathbb{R}^n)$.
- Employing the result above, in Theorem 8.28 we then extend to minimization in $\mathrm{W}^{1,1}_{u_0}(\Omega)$ under the same differentiability assumptions, now considering measures in the dual of BV as our usual and integrands with linear growth. The dual problem is obtained by maximizing the same quantity as in Theorems 8.25 and 8.26, this time among vector fields in $\mathcal{DM}(\Omega, \mathbb{R}^n)$ with assigned measure divergence.
- Finally, in Theorem 8.29 we reach the generality of Result 4 for admissible integrand f, reasoning by approximations via Moreau envelopes $f^k(x,\xi) := \inf_{\tau^* \in \mathbb{R}^n} \big(f(x,\tau^*) + |\xi^* - \tau^*|^2/2\varepsilon_k\big)$. Here, the differentiability property of f^k enables us to reduce to the previous step and then, sending $k \to \infty$, we recover the problem for f.

The diagram below summarizes the procedure just described. We point out that the same final result of Theorem 8.29 could be alternatively achieved by removing the assumptions of differentiability first, and coercivity at last; in such a case, the usage of the sequences $(f_k)_k$, $(f^k)_k$ would also be inverted. In conclusion, we record that the techniques employed in each step – namely, the functional approximation via p–growth functionals for $p \searrow 1$, as well as the Moreau–Yosida regularization of integrands – are well–known and employed in a large variety of fields. Nevertheless, this approach is, to our knowledge, novel in the context of duality for convex functionals with linear growth, already in the null–measure case.

f	convex differentiable	convex
coercive quadratic growth	Theorem 8.25 $\downarrow$ f_k coercive, quadratic growth	
linear growth and duality in $\mathrm{W}^{1,2}$	Theorem 8.26 $\downarrow$	
linear growth and duality in $\mathrm{W}^{1,1}$	Theorem 8.28 $\longrightarrow$ f^k differentiable, linear growth	Theorem 8.29

Clearly, the choice of the quadratic growth in the preliminary Theorems 8.25 and 8.26 is not a must, as any dual reformulation in a separable space $\mathrm{W}^{1,p}(\Omega)$ for $p \in (1,\infty)$ would lead to the same outcome when passing to the linear case $p = 1$. However, the advantage of $p = 2$ consists in working in a Hilbert space: This prevents us from introducing yet another dual space in the preparatory dual problem and in the consequent duality of the Moreau approximations f^k in terms of the previously introduced f_k; see the discussion following Lemma 8.20.

8.1 Preliminaries to convex duality

In correspondence to a generalized function $g\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, we consider the greatest lower semicontinuous function majorized by g, that is, the function whose epigraph is the closure in $\mathbb{R}^{n+1}$ of the epigraph of g. We call such function the **closure** (or *lower semicontinuous hull*) $\mathrm{cl}(g)$ of g, and it is

$$\mathrm{cl}(g)(x_0) = \liminf_{x \to x_0} g(x) \qquad \text{for all } x_0 \in \mathbb{R}^n .$$

We say that g is **closed** if $\mathrm{cl}(g) = g$, and this happens if and only if g is lower semicontinuous on $\mathbb{R}^n$. The next result corresponds to [84, Theorem 12.2] and it establishes a connection between closure and convex conjugates of functions. We record that in contrast to the reference, we also assume properness for the sake of consistency with our definition of closure; compare with the argumentation at the beginning of [84, Section 7].

Theorem 8.2. *For a proper convex function $g\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, we have $g^* = \mathrm{cl}(g)^*$ and $g^{**} = \mathrm{cl}(g)$.*

We now introduce the notion of relative interior of a set, which provides a suitable extension of the concept of topological interior and which turns out to be especially useful when dealing with lower–dimensional geometrical objects. To do so, we first recall that the *affine hull* of a set $S \subseteq \mathbb{R}^n$ is the smallest affine set containing S, namely

$$\mathrm{aff}(S) := \left\{ \sum_i \lambda_i x_i \, : x_i \in S \, , \lambda_i \in \mathbb{R} \, , \sum_i \lambda_i = 1 \right\} .$$

Definition 8.3 (relative interior of a set)**.** For any given $S \subseteq \mathbb{R}^n$, the **relative interior** $\mathrm{ri}(S)$ of S is the interior of S relative to its affine hull, that is

$$\mathrm{ri}(S) := \{ x \in \mathbb{R}^n \colon \; \mathrm{B}_\varepsilon(x) \cap \mathrm{aff}(S) \subseteq S \text{ for some } \varepsilon > 0 \} .$$

It is naturally $\mathrm{ri}(S) \supseteq \mathrm{Int}(S)$, whereas for n–dimensional sets S in $\mathbb{R}^n$ we have $\mathrm{aff}(S) = \mathbb{R}^n$ and the two notions of interior coincide.

Proposition 8.4. *Any proper, convex function $g\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is such that the relative interior $\mathrm{ri}(\mathrm{dom}(g))$ is non empty, and $\partial g(x) \neq \emptyset$ for any $x \in \mathrm{ri}(\mathrm{dom}(g))$. Specifically, every convex function $g\colon \mathbb{R}^n \to \mathbb{R}$ is everywhere subdifferentiable.*

Proof. The convexity of g yields in turn convexity of $\mathrm{dom}(g)$ in $\mathbb{R}^n$, and since g is proper, we have $\mathrm{dom}(g) \neq \emptyset$. From [84, Theorem 6.2], even $\mathrm{ri}(\mathrm{dom}(g))$ is convex and non–empty. We can thus consider a point $x \in \mathrm{ri}(\mathrm{dom}(g))$, and the conclusion on $\partial g(x)$ follows from [42, Theorem 2.79]. Specifically, if g is everywhere finite we have $\mathrm{ri}(\mathrm{dom}(g)) = \mathrm{dom}(g) = \mathbb{R}^n$, thus $\partial g(x) \neq \emptyset$ for all $x \in \mathbb{R}^n$, hence g it subdifferentiable at every point. □

Proposition 8.5. *The convex conjugate function g^* of a proper, convex function $g\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is still proper and convex.*

Proof. Since the convexity of g^* follows from the definition, we focus on the proof of properness. From $g \not\equiv \infty$ we know that it is $g^*(\xi^*) \in (-\infty, \infty]$ for all $\xi^* \in \mathbb{R}^n$. Moreover, employing Proposition 8.4 there exists $x \in \mathrm{ri}(\mathrm{dom}(g))$ and $\xi^* \in \partial g(x)$. By the definition of subdifferential, we write

$$\xi^* \cdot \xi - g(\xi) \leq \xi^* \cdot x - g(x) \quad \text{for all } \xi \in \mathbb{R}^n ,$$

and passing to the supremum in ξ we deduce $g^*(\xi^*) = \xi^* \cdot x - g(x)$. Moreover, by $x \in \mathrm{dom}(g)$, it holds $g(x) \neq \pm\infty$ and this yields $g^*(\xi^*) \in \mathbb{R}$, showing that g^* is a proper function as well. □

To achieve the result below for the image of subdifferential of a strictly convex function, we elaborate by adapting the proof of [15, Proposition 3.7].

Proposition 8.6. *For any strictly convex function $g\colon \mathbb{R}^n \to \mathbb{R}$, the set $\mathrm{Im}(\partial g)$ is open and convex in $\mathbb{R}^n$. Moreover, it holds $g^* \in \mathrm{C}^1(\mathrm{Im}(\partial g))$.*

Proof. To prove that $\mathrm{Im}(\partial g)$ is open in $\mathbb{R}^n$, we assume $\xi_0^* \in \partial g(\xi_0)$ for some $\xi_0 \in \mathbb{R}^n$ and determine a neighborhood of ξ_0 entirely enclosed in $\mathrm{Im}(\partial g)$. We claim that there exist positive ε and a real constant M such that

$$g(\xi) - \xi_0^* \cdot \xi \geq \varepsilon|\xi| - M \quad \text{for all } \xi \in \mathbb{R}^n . \tag{8.1.1}$$

If this latter is verified, then for all $\xi^* \in \mathrm{B}_\varepsilon(\xi_0^*)$ the distance is $\mathrm{d}(\xi^*) := |\xi^* - \xi_0^*| < \varepsilon$, and thus we have

$$h_{\xi^*}(\xi) := g(\xi) - \xi^* \cdot \xi \geq (g(\xi) - \xi_0^* \cdot \xi) + \xi \cdot (\xi_0^* - \xi^*) \geq (\varepsilon - \mathrm{d}(\xi^*))|\xi| - M \xrightarrow[|\xi|\to\infty]{} \infty ,$$

therefore the continuous function h_{ξ^*} admits global minimum $\widehat{\xi}$ in some closed ball of radius large enough. Explicitly, we obtain $g(\widehat{\xi}) + \xi^* \cdot (\xi - \widehat{\xi}) \leq g(\xi)$ for all ξ, hence $\xi^* \in \partial g(\widehat{\xi})$ and openness of $\mathrm{Im}(\partial g)$ follows.

To prove now the estimate in (8.1.1), we first assume that g is a non–negative function vanishing in the origin and set $\xi_0 = 0$, $\xi_0^* = 0$. Then, for $\xi \in (\mathrm{B}_1)^{\mathrm{c}}$ we write $\lambda := |\xi|^{-1} \in (0, 1]$ and we apply convexity to infer $g(\lambda\xi) \leq \lambda g(\xi) + (1 - \lambda)g(0) = \lambda g(\xi)$. Specifically, strict convexity of g determines $\varepsilon := \inf_{\partial \mathrm{B}_1} g > 0$, so that $g(\xi) \geq g(\lambda\xi)/\lambda \geq \varepsilon|\xi|$ for all $\xi \in (\mathrm{B}_1)^{\mathrm{c}}$. In the unit ball instead, we consider $\varepsilon|\xi| - \varepsilon = \varepsilon(|\xi| - 1) < 0 \leq g(\xi)$ for $|\xi| < 1$. All in all, we proved $g \geq \varepsilon(|.| - 1)$ in $\mathbb{R}^n$, thus (8.1.1) results holding in this specific case.
To extend the lower bound to any strictly convex g and any ξ_0, $\xi_0^* \in \partial g(\xi_0)$, we exploit that for the auxiliary function $\xi \mapsto \breve{g}(\xi) := g(\xi + \xi_0) - g(\xi_0) - \xi_0^* \cdot \xi$ it is $\breve{g}(0) = 0 \leq \breve{g}(\xi)$ for all ξ (since ξ_0^* is a subdifferential of g at ξ_0), and the minimization property implies $0 \in \partial\breve{g}(0)$. Applying thus (8.1.1) to $\breve{g}$ with $\xi_0^* = 0$, we compute

$$g(\xi) - \xi_0^* \cdot \xi = \breve{g}(\xi - \xi_0) + g(\xi_0) - \xi_0^* \cdot \xi_0 \geq \varepsilon|\xi| - (\varepsilon|\xi_0| - g(\xi_0) + \xi_0^* \cdot \xi_0) \quad \text{for all } \xi \in \mathbb{R}^n .$$

In sum, we have obtained precisely the claimed (8.1.1) even in the most general case.

We now address convexity of $\mathrm{Im}(\partial g)$ in $\mathbb{R}^n$ again relying on (8.1.1). Assigned a pair of vectors ξ_1^*, ξ_2^* in $\mathrm{Im}(\partial g)$ and their linear combination $\xi^* := \lambda\xi_1^* + (1 - \lambda)\xi_2^*$ with $\lambda \in [0, 1]$, for any $\xi \in \mathbb{R}^n$ we distinguish two cases:

- If $\xi \cdot (\xi_1^* - \xi^*) \geq 0$, the inequality in (8.1.1) applied to ξ_1^* (with corresponding constants ε_1, M_1) yields $h_{\xi^*}(\xi) \geq (g(\xi) - \xi_1^* \cdot \xi) + \xi \cdot (\xi_1^* - \xi^*) \geq \varepsilon_1|\xi| - M_1$.
- Otherwise, from the definition of ξ^*, it is $(1 - \lambda)\xi \cdot (\xi_1^* - \xi_2^*) = \xi \cdot (\xi_1^* - \xi^*) < 0$, hence it follows $\xi \cdot (\xi_2^* - \xi^*) = \lambda\xi \cdot (\xi_2^* - \xi_1^*) \geq 0$. As above, exploiting now the estimate in (8.1.1) for ξ_2^*, ε_2, M_2, we deduce $h_{\xi^*}(\xi) \geq \varepsilon_2|\xi| - M_2$.

The conclusion is the validity of $h_{\xi^*}(\xi) \geq \varepsilon|\xi| - M$ for all $\xi \in \mathbb{R}^n$, with some $\varepsilon > 0$, $M \in \mathbb{R}$. Reprising then the same reasoning in the openness proof, we conclude that the linear combination ξ^* is an element of $\mathrm{Im}(\partial g)$ and therefore this latter set is convex.

We now prove the differentiability statement for the conjugate function. By the definition of convex conjugate itself and Proposition 8.5, the function $g^*\colon \mathbb{R}^n \to (-\infty, \infty]$ is convex with $\mathrm{dom}(g^*) \neq \emptyset$, hence g^* is continuous on $\mathrm{Int}(\mathrm{dom}(g^*))$ and consequently on $\mathrm{Im}(\partial g) = \mathrm{Int}(\mathrm{Im}(\partial g)) \subseteq \mathrm{Int}(\mathrm{dom}(g^*))$. We consider any $\xi^* \in \mathrm{Im}(\partial g)$ – thus $\xi^* \in \partial g(\xi)$ for some $\xi \in \mathbb{R}^n$ – and by Remark 2.96 we find $\xi \in \partial g^*(\xi^*)$. Actually, we claim that $\partial g^*(\xi^*) = \{\xi\}$ – otherwise we would have another element $\widetilde{\xi} \in \partial g^*(\xi^*) \setminus \{\xi\}$, and once again Remark 2.96 determines $\xi^* \in \partial g(\xi) \cap \partial g(\widetilde{\xi}) = \emptyset$, with the last equality following from the strict convexity making use of Corollary 2.95. Hence, ξ is the only subgradient of g^* at point ξ^* and exploiting Proposition 2.92 we infer differentiability of g^* in ξ^* with $\xi = \nabla g(\xi^*)$. By arbitrariness of ξ^* in the open set $\mathrm{Im}(\partial g)$ and by $g^* \in \mathrm{C}(\mathrm{Im}(\partial g))$, we find that g^* is continuous and differentiable in $\mathrm{Im}(\partial g)$. Recalling that convex, differentiable functions have continuous gradient on their domain, we may conclude the thesis. □

We now present a result on the strict convexity of the convex conjugate function following from the differentiability of the function. We record that such property can only hold on convex sets contained in the effective domain – being the value of the conjugate function everywhere infinity outside the domain. Notice that no convexity of the primal function is needed for this purpose.

Proposition 8.7. *We assume $g\colon \mathbb{R}^n \to \mathbb{R}$ to be everywhere differentiable. Then the convex conjugate function g^* is strictly convex on each convex set contained in* $\mathrm{Im}(\partial g)$.

Proof. Let C be a convex set in $\mathrm{Im}(\partial g)$ and recall that Proposition 2.94 ensures finiteness of g^* on C. Since conjugates are always convex, we assume by contradiction the existence of a distinct pair of elements ξ_1^*, ξ_2^* in C such that g^* is affine on the line segment connecting ξ_1^* and ξ_2^*. In correspondence to the midpoint $\xi^* := (\xi_1^* + \xi_2^*)/2 \in C$, there is some $\xi \in \mathbb{R}^n$ such that $\xi^* \in \partial g(\xi)$, and the differentiability assumption forces $\xi^* = \nabla g(\xi)$. Proposition 2.94 then yields

$$0 = g(\xi) + g^*(\xi^*) - \xi^* \cdot \xi = \frac{1}{2}\big(g(\xi) + g^*(\xi_1^*) - \xi_1^* \cdot \xi\big) + \frac{1}{2}\big(g(\xi) + g^*(\xi_2^*) - \xi_2^* \cdot \xi\big)\,,$$

where each addend is non–negative by the definition of conjugate. It is then $g(\xi) + g^*(\xi_i^*) = \xi_i^* \cdot \xi$ for $i = 1, 2$, meaning $\{\xi_1^*, \xi_2^*\} \subseteq \partial g(\xi) = \xi^*$, which is impossible for $\xi_1^* \neq \xi_2^*$. We have then proved the strict convexity of g^* in C. □

We observe that for differentiable functions g such that the set $\mathrm{Im}(g)$ is convex in $\mathbb{R}^n$ (e.g. assuming g strictly convex everywhere, compare with Proposition 8.6), strict convexity of g^* in the latter Proposition 8.7 can be extended to the whole $\mathrm{Im}(\partial g)$.

From Proposition 2.94, we already know the inclusion $\mathrm{Im}(\partial g) \subseteq \mathrm{dom}(g^*)$ to hold for any $g\colon \mathbb{R}^n \to \mathbb{R}$. In addition to this, one may prove that – under strict convexity and linear growth – even a stronger statement holds (Proposition 8.9). To achieve the preliminary Lemma 8.8, we will make use of the ε–subdifferentiability approximation result from Section 2.9.3.

Lemma 8.8. *Any convex $g\colon \mathbb{R}^n \to \mathbb{R}$ is such that* $\mathrm{dom}(g^*) \subseteq \overline{\mathrm{Im}(\partial g)}$.

Proof. Assigned an element $\xi^* \in \mathrm{dom}(g^*)$, we have $0 \leq h_{\xi^*}(\xi) := g^*(\xi^*) + g(\xi) - \xi^* \cdot \xi < \infty$ for all $\xi \in \mathbb{R}^n$, with $\inf_{\xi \in \mathbb{R}^n} h_{\xi^*}(\xi) = 0$ by the definition of conjugate function. Then, for every $k \in \mathbb{N}$ there exists some $\xi_k \in \mathbb{R}^n$ such that $0 \leq h_{\xi^*}(\xi_k) \leq 1/k$, hence we have $\xi^* \in \partial_{1/k}\, g(\xi_k)$. We apply (2.9.4) to the finite, convex (and thus in particular LSC) function g to deduce that in correspondence to each $k \in \mathbb{N}$ it holds $\xi_k \in \partial_{1/k}\, g^*(\xi^*)$. Taking into account Theorem 2.99, for each k we find $\widehat{\xi}_k^* \in \mathbb{R}^n$, $\widehat{\xi}_k \in \partial g^*(\widehat{\xi}_k^*)$ with $|\xi^* - \widehat{\xi}_k^*| \leq 1/\sqrt{k}$. It follows that $\xi^* = \lim_{k\to\infty} \widehat{\xi}_k^*$ with each $\widehat{\xi}_k^* \in \partial g(\widehat{\xi}_k)$, thus $\big(\widehat{\xi}_k^*\big)_k \subseteq \mathrm{Im}(\partial g)$. This yields the claim $\xi^* \in \overline{\mathrm{Im}(\partial g)}$. □

Proposition 8.9. *Let $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ satisfy Assumption 1.1 with strict convexity of $f(x,.)$ for all $x \in \mathbb{R}^n$. Then we have $\mathrm{Im}(\partial_\xi f(x,.)) = \mathrm{Int}(\mathrm{dom}(f^*(x.,)))$ for all $x \in \mathbb{R}^n$. If Assumption (H2) is also in place, we obtain $\mathrm{Im}(\partial_\xi f(x,.)) = \{\xi \in \mathbb{R}^n : (f^\infty)^\circ(x,\xi) < 1\}$.*

Proof. We fix $x \in \mathbb{R}^n$. The inclusion $\mathrm{Im}(\partial_\xi f(x,.)) \subseteq \mathrm{dom}(f^*(x.,))$ follows from the equality case in Fenchel's inequality, thus exploiting openness of the image of subdifferentials for strictly convex functions in Proposition 8.6, it applies $\mathrm{Im}(\partial_\xi f(x,.)) = \mathrm{Int}(\mathrm{Im}(\partial_\xi f(x,.))) \subseteq \mathrm{Int}(\mathrm{dom}(f^*(x.,)))$. On the other hand, Lemma 8.8 yields $\mathrm{dom}(f^*(x,.)) \subseteq \overline{\mathrm{Im}(\partial_\xi f(x,.))}$, and the thesis is achieved passing to interiors.

If we furthermore assume (H2), from Proposition 2.89 it is

$$\mathbb{K}_{x,(f^\infty)^\circ} := \{\xi \in \mathbb{R}^n : (f^\infty)^\circ(x,\xi) \le 1\} = \mathrm{dom}(f^*(x,.))$$

and by continuity of the polar function $\xi \mapsto (f^\infty)^\circ(x,\xi)$ one has

$$\mathrm{Int}(\mathbb{K}_{x,(f^\infty)^\circ}) = \{\xi \in \mathbb{R}^n : (f^\infty)^\circ(x,\xi) < 1\} = \mathrm{Int}(\mathrm{dom}(f^*(x,.)))\,,$$

thus, we conclude exploiting equality in the general case. □

Lemma 8.10 (bound on gradients of convex functions with p–growth)**.** *We assume that $g\colon \mathbb{R}^n \to \mathbb{R}$ is a convex mapping such that*

$$|g(\xi)| \le \beta(\gamma + |\xi|)^p \quad \textit{for all } \xi \in \mathbb{R}^n \tag{8.1.2}$$

holds with $p \in [1,\infty)$ and some $\beta,\gamma \in [0,\infty)$. Then, there exists a constant $\beta' = \beta'(\beta,\gamma,p) \in (0,\infty)$ such that

$$|g(\xi) - g(\tau)| \le \beta'(\gamma + |\xi| + |\tau|)^{p-1}|\xi - \tau| \quad \textit{for all } \ \xi,\tau \in \mathbb{R}^n\,.$$

For a proof we refer the reader to [61, Lemma 5.2] or [34, Proposition 2.32].

Remark 8.11. From the last bound we deduce that, if f satisfies the p–growth condition (8.1.2), then for any $\xi \in \mathbb{R}^n$ it is

$$z^* \in \partial g(\xi) \implies |z^*| \le \beta'(\gamma + 2|\xi|)^{p-1}\,. \tag{8.1.3}$$

Indeed, for $z^* = 0$ the inequality is trivially satisfied. Otherwise, by the definition of z^* subgradient for g at ξ, we have

$$z^* \cdot \tau \le \frac{g(\xi + t\tau) - g(\xi)}{t} \le \beta'|\tau|(\gamma + |\xi + t\tau| + |\xi|)^{p-1} \quad \text{for all } \tau \in \mathbb{R}^n \ \text{ all } t \in (0,\infty)\,.$$

Specifically, setting $\tau := z^*/|z^*|$ and then sending t to 0 we argue

$$|z^*| \le \beta' \liminf_{t\to 0^+}\left(\gamma + \left|\xi + t\frac{z^*}{|z^*|}\right| + |\xi|\right)^{p-1} = \beta'(\gamma + 2|\xi|)^{p-1}\,.$$

Remark 8.12. We record that the results above in the case of convex functions g of *linear growth* yield Lipschitz continuity of g with Lipschitz constant β', as well as the boundedness of the image of subdifferentials:

$$\mathrm{Im}(\partial g) := \bigcup_{\xi \in \mathbb{R}^n} \partial g(\xi) \subseteq \overline{\mathrm{B}_{\beta'}}\,.$$

8.1.1 Infimal convolution

We introduce the operator of infimal (or inf–)convolution acting on two – or in general, on a finite number of – functions; interestingly, the inf–convolution applied to convex functions turns out to be in duality with the operation of sum between convex conjugates. The terminology originates from the fact that, when only two component functions are considered, the argument of the infimum in the inf–convolution is analogous to the one involved in the standard integral convolution.

Definition 8.13 (inf–convolution)**.** For any pair of proper functions $f_1, f_2 \colon \mathbb{R}^n \to \overline{\mathbb{R}}$, the **infimal convolution** (or *inf–convolution*) of f_1 and f_2 is the function $f_1 \square f_2 \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ defined on $x \in \mathbb{R}^n$ by

$$(f_1 \square f_2)(x) := \inf_{y \in \mathbb{R}^n} (f_1(y) + f_2(x-y)) \in [-\infty, \infty] .$$

Notice that $f_1 \square f_2 \not\equiv \infty$, with the properness assumption on both components preventing indeterminate forms $\infty - \infty$. If f_1, f_2 are also convex, then even $f_1 \square f_2$ is convex, but it might not be proper as the infimum could be $-\infty$.

The definition of inf–convolution can be easily extended to a finite number $N \in \mathbb{N}$ of proper functionals $f_1, f_2, \dots, f_N$ on $\mathbb{R}^n$ as follows: if $x \in \mathbb{R}^n$, we let

$$(f_1 \square f_2 \square \dots \square f_N)(x) := \inf \{ f_1(x_1) + f_2(x_2) + \dots + f_N(x_N) \colon x_i \in \mathbb{R}^n,\ x_1 + x_2 + \dots + x_N = x \} .$$

The following dual property is very useful when dealing with extrema of convex functions. For a full proof we refer to [84, Theorem 16.4] or [42, Theorem 2.107].

Theorem 8.14 (inf–convolution and conjugate)**.** *We consider proper functions $f_1, \dots, f_N \colon \mathbb{R}^n \to \overline{\mathbb{R}}$.*

(i) *It is $(f_1 \square f_2 \square \dots \square f_N)^* = f_1^* + f_2^* + \dots + f_N^*$ and the resulting function is proper if and only if $\bigcap_{i=1}^n \operatorname{dom}(f_i^*) \neq \emptyset$.*

(ii) *If all $f_1, \dots, f_N$ are convex and $\bigcap_{i=1}^n \operatorname{dom}(f_i) \neq \emptyset$, then we have $(\operatorname{cl}(f_1) + \operatorname{cl}(f_2) + \dots + \operatorname{cl}(f_N))^* = \operatorname{cl}(f_1^* \square f_2^* \square \dots \square f_N^*)$.*

If specifically $\bigcap_{i=1}^n \operatorname{ri}(\operatorname{dom}(f_i)) \neq \emptyset$, we can remove the closure in the last formula and get $(f_1 + f_2 + \dots + f_N)^* = f_1^* \square f_2^* \square \dots \square f_N^*$.

Proof. For simplicity of notation, we only prove the result for $N = 2$, with an identical procedure for any finite number N of component functions. To show (i), we fix $x^* \in \mathbb{R}^n$ and explicitly compute

$$\begin{aligned}
(f_1 \square f_2)^*(x^*) = \sup_{x \in \mathbb{R}^n} (x^* \cdot x - (f_1 \square f_2)(x)) &= \sup_{x \in \mathbb{R}^n} \left(x^* \cdot x - \inf_{\substack{x_1, x_2 \in \mathbb{R}^n \\ x_1 + x_2 = x}} f_1(x_1) + f_2(x_2) \right) \\
&= \sup_{x \in \mathbb{R}^n} \sup_{\substack{x_1, x_2 \in \mathbb{R}^n \\ x_1 + x_2 = x}} (x^* \cdot x - f_1(x_1) - f_2(x_2)) \\
&= \sup_{x_1, x_2 \in \mathbb{R}^n} (x^* \cdot x_1 - f_1(x_1) + x^* \cdot x_2 - f_2(x_2)) \\
&= f_1^*(x^*) + f_2^*(x^*).
\end{aligned}$$

Moreover, since conjugates of proper functions are always bounded from below, $f_1^* + f_2^*$ is proper if and only if $\operatorname{dom}(f_1^* + f_2^*) = \operatorname{dom}(f_1^*) \cap \operatorname{dom}(f_2^*)$ is not empty.

We turn now to the proof of (ii) assuming convexity of f_1 and f_2, which induces via Proposition 8.5 properness of the respective conjugate functions f_1^*, f_2^*. We can thus apply (i) to such conjugates and write

$$(f_1^* \square f_2^*)^* = (f_1^*)^* + (f_2^*)^* = \operatorname{cl}(f_1) + \operatorname{cl}(f_2)$$

where we applied $\mathrm{cl}(f) = f^{**}$ for f proper and convex. Passing again to the conjugates, via Theorem 8.2 we conclude

$$(\mathrm{cl}(f_1) + \mathrm{cl}(f_2))^* = (f_1^* \Box f_2^*)^{**} = \mathrm{cl}(f_1^* \Box f_2^*)\,,$$

where the latter equality holds provided $f_1^* \Box f_2^*$ is proper and convex. This condition is, in fact, verified, since the assumption $\mathrm{dom}(f_1) \cap \mathrm{dom}(f_2) \neq \emptyset$ determines by $\mathrm{cl}(f_i) \leq f_i$ the following

$$\mathrm{dom}(f_1^{**} + f_2^{**}) = \mathrm{dom}(f_1^{**}) \cap \mathrm{dom}(f_2^{**}) = \mathrm{dom}(\mathrm{cl}(f_1)) \cap \mathrm{dom}(\mathrm{cl}(f_2)) \neq \emptyset\,,$$

and from the result of the preceding (i) applied to f_i^* in place of f_i, the inf–convolution $f_1^* \Box f_2^*$ is proper, and we have demonstrated (ii).

If we furthermore assume that the relative interiors of f_1 and f_2 intersect, we can apply [84, Theorem 9.3] to write $\mathrm{cl}(f_1 + f_2) = \mathrm{cl}(f_1) + \mathrm{cl}(f_2)$, and the previous equality becomes

$$(f_1 + f_2)^* = (\mathrm{cl}(f_1 + f_2))^* = \mathrm{cl}(f_1^* \Box f_2^*)$$

in view of $(\mathrm{cl}(f))^* = f^*$ from Theorem 8.2. Moreover, if $\mathrm{ri}(\mathrm{dom}(f_1)) \cap \mathrm{ri}(\mathrm{dom}(f_2)) \neq \emptyset$, one can also verify lower semicontinuity of $f_1^* \Box f_2^*$, hence it coincides with its closure and the equality results. □

8.1.2 Recap on weak lower semicontinuity for 0-order and 1$^{\text{st}}$-order integrals

We now quote the crucial weak semicontinuity result for convex functionals of zero and first order. In the one–dimensional case, the results below date back to Leonida Tonelli's work [96] in the 1920s, later on refined by James Serrin and extended to multiple variables in [93, Theorem 11].

Theorem 8.15 (weak L^p–LSC of convex functionals $g = g(x, y)$)**.** *Let $M, N \in \mathbb{N}$ and $p \in [1, \infty]$. Assume $U \subseteq \mathbb{R}^N$ is open and $g\colon U \times \mathbb{R}^M \to \mathbb{R} \cup \{\infty\}$ is a Borel function such that $g(x, .)$ is lower semicontinuous and convex in $\mathbb{R}^M$ for $\mathcal{L}^N$–a.e. $x \in U$. Moreover, suppose there exist $a \in \mathrm{L}^1(U)$ and $b \in \mathrm{L}^{p'}(U, \mathbb{R}^M)$ such that*

$$g(x, \xi) \geq a(x) + b(x) \cdot \xi \quad \text{for all } \xi \in \mathbb{R}^M \text{ and for a.e. } x \in U\,. \tag{8.1.4}$$

Then the functional

$$\mathrm{L}^p(U, \mathbb{R}^M) \ni w \mapsto \int_U g(., w)\, \mathrm{d}x$$

is weakly lower semicontinuous in $\mathrm{L}^p(U, \mathbb{R}^M)$ (or weak–∗ LSC in case $p = \infty$, under the convention $1/\infty = 0$). If additionally $|U| < \infty$, we attain weak lower semicontinuity in $\mathrm{L}^q(U, \mathbb{R}^M)$ for all $q \geq p$.

For a proof of Theorem 8.15 we refer for instance to [34, Theorem 3.20] or [53, Theorem 6.54]. Furthermore, we point out that any $g(x.,)\colon \mathbb{R}^M \to \mathbb{R} \cup \{\infty\}$ proper and convex satisfies some version of (8.1.4), as the next result shows.

Proposition 8.16. *For any proper, convex function $g\colon \mathbb{R}^M \to \mathbb{R} \cup \{\infty\}$ there exist $a \in \mathbb{R}$ and $b \in \mathbb{R}^M$ such that*

$$g(\xi) \geq a + b \cdot \xi \quad \text{for all } \xi \in \mathbb{R}^M\,.$$

Proof. We employ Proposition 8.4 to find an element $\tau \in \mathrm{ri}(\mathrm{dom}(g)) \subseteq \mathbb{R}^M$ and a subgradient ξ^* in $\partial g(\tau)$. Then, by the definition of subdifferentials

$$g(\xi) \geq (g(\tau) - \xi^* \cdot \tau) + \xi^* \cdot \xi \quad \text{for all } \xi \in \mathbb{R}^M\,,$$

and by finiteness of $g(\tau)$ is finite we conclude the statement. □

More precisely, in the case of integrands with p–growth on bounded domains, the last considerations imply the validity of the weak lower semicontinuity result expressed in Theorem 8.15.

Corollary 8.17. *We consider an open set $U \subseteq \mathbb{R}^N$ with $|U| < \infty$. Let $g\colon U \times \mathbb{R}^M \to \mathbb{R}$ be a Borel function with $g(x,.)$ convex in $\mathbb{R}^M$ for $\mathcal{L}^N$–a.e. $x \in U$, and assume that g has p–growth in ξ in the sense*

$$|g(x,\xi)| \leq \beta(\gamma + |\xi|)^p \quad \textit{for a.e. } x \in U \textit{ and every } \xi \in \mathbb{R}^M$$

for constants $\beta, \gamma \in [0,\infty)$ and $p \in [1,\infty)$. Then, the bound in (8.1.4) is always satisfied even with the functions a, $b \in \mathrm{L}^\infty(U)$.

Proof. From our convexity assumption, we know that the subdifferential $\partial_\xi g(x,\xi)$ is non–empty for a.e. $x \in U$ (Proposition 8.4). In particular, assigned a vector $\tau_x^* \in \partial_\xi g(x,0)$, we have

$$g(x,\xi) \geq g(x,0) + \tau_x^* \cdot \xi \geq -\beta\gamma^p + \tau_x^* \cdot \xi \quad \text{for all } \xi \in \mathbb{R}^M \,,$$

exploiting the p–growth condition. Moreover, from (8.1.3) it applies $|\tau_x^*| \leq \beta'\gamma^{p-1}$, hence τ_x^* is in every Lebesgue space, hence we have verified (8.1.4) for uniformly bounded coefficients. □

An immediate consequence of Theorem 8.15 is the $\mathrm{W}^{1,p}$–semicontinuity of first–order functionals, obtained by setting there $M := n \times N$.

Corollary 8.18 (weak $\mathrm{W}^{1,p}$–LSC of convex functionals $g = g(x,z)$)**.** *Under the assumptions of Theorem 8.15 with $M := n \times N$, $n \in \mathbb{N}$, the functional*

$$\mathrm{W}^{1,p}(U,\mathbb{R}^{n\times N}) \ni w \mapsto \int_U g(.,\nabla w)\,\mathrm{d}x$$

is lower semicontinuous with respect to weak convergence in $\mathrm{W}^{1,p}(U,\mathbb{R}^{n\times N})$ (respectively, weak–∗ LSC for $p = \infty$). On domains of finite measure, the functional is even weakly LSC in $\mathrm{W}^{1,q}\big(U,\mathbb{R}^{n\times N}\big)$ for all $q \geq p$.

8.1.3 Moreau envelope and regularization

Given an arbitrary mapping $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, we fix the notation

$$c_1 \leq f(x,\xi) \leq c_2(1 + |\xi|) \quad \text{for all } x,\xi \in \mathbb{R}^n \tag{8.1.5}$$

with some constant c_1, $c_2 \in [0,\infty)$. Observe that the bound in (8.1.5) is weaker than our usual linear-growth assumption (2.6.7) insofar as the coercivity control on f may fail. For f as above, we introduce a sequence of non–negative values $(\varepsilon_k)_k$ such that $\varepsilon_k \searrow 0$, and for each $k \in \mathbb{N}$ we consider the function

$$f_k(x,\xi) := f(x,\xi) + \varepsilon_k|\xi|^2/2 \quad \text{for all } x,\xi \in \mathbb{R}^n \,.$$

From now on, we will work with the convex conjugate function of f_k with respect to the second variable, that is

$$f_k^*(x,\xi^*) := \sup_{\xi\in\mathbb{R}^n} \big(\xi \cdot \xi^* - f_k(x,\xi)\big) \quad \text{for all } x,\xi^* \in \mathbb{R}^n \,.$$

Moreover, we recall that the operation of conjugates inverts inequalities, thus $f_k^*(x,\xi^*) \leq f^*(x,\xi^*)$ for all $(x,\xi^*) \in \mathbb{R}^n \times \mathbb{R}^n$ and for every $k \in \mathbb{N}$. In the next Proposition 8.19, we verify that f_k^* defines a weakly lower semicontinuous functional in each $\mathrm{L}^p(U)$ for any open, bounded set $U \subseteq \mathbb{R}^n$.

Proposition 8.19 (weak L^p–LSC of functionals on conjugates)**.** *For Borel $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$, we assume (8.1.5). Then the functional $w \mapsto \int_U f^*(.,w)\,\mathrm{d}x$ and each $w \mapsto \int_U f_k^*(.,w)\,\mathrm{d}x$ defined on bounded $U \subseteq \mathbb{R}^n$ are weak lower semicontinuous in $\mathrm{L}^p(U,\mathbb{R}^n)$ for any $p \in [1,\infty)$ (and weak–∗ LSC in $\mathrm{L}^\infty(U,\mathbb{R}^n)$).*

Proof. Fix $k \in \mathbb{N}$. We know that conjugate functions are Borel with $\xi^* \mapsto f^*(x,\xi^*)$, $\xi^* \mapsto f_k^*(x,\xi^*)$ lower semicontinuous and convex for any given $x \in U$. Furthermore, (8.1.4) is valid for both f^* and f_k^*, since

$$f^*(x,\xi^*) \geq f_k^*(x,\xi^*) \geq 0 \cdot \xi^* - f_k(x,0) = -f(x,\xi) \geq -c_2 = -c_2 + 0 \cdot \xi^* \quad \text{for all } x \in U\,.$$

Being $c_2 \in \mathrm{L}^1(U)$, the claimed weak(-$*$) lower semicontinuity follows from Theorem 8.15. $\square$

Lemma 8.20 (pointwise convergence of Moreau envelope)**.** *We suppose that $g\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper, convex function such that $g(0) \in \mathbb{R}$. For $k \in \mathbb{N}$, we let $(\varepsilon_k)_k$ be a decreasing sequence converging to 0 for $k \to \infty$ and set $g_k(\xi) := g(\xi) + \varepsilon_k|\xi|^2/2$ for $\xi \in \mathbb{R}^n$. Then, for any $\xi^* \in \mathbb{R}^n$ we obtain pointwise convergence of the conjugates $g_k^*(\xi^*) \to g^*(\xi^*)$ as $k \to \infty$.*

Proof. We first record that Proposition 8.4 ensures that the relative interior $\mathrm{ri}(\mathrm{dom}(g))$ is not empty. We define the function $p_k := \varepsilon_k|.|^2/2$ on $\mathbb{R}^n$, noticing that by Proposition 2.88 the convex conjugate of p_k is given by $p_k^* = |.|^2/2\varepsilon_k$. Then, exploiting Theorem 8.14(ii) for the set $\mathrm{ri}(\mathrm{dom}(g)) \cap \mathrm{ri}(\mathrm{dom}(p_k)) = \mathrm{ri}(\mathrm{dom}(g))$, we compute

$$g_k^*(\xi^*) = (g + p_k)^*(\xi^*) = (g^* \mathbin{\square} p_k^*)(\xi^*) = \inf_{\tau^* \in \mathbb{R}^n}\left(g^*(\tau^*) + \frac{1}{2\varepsilon_k}|\xi^* - \tau^*|^2\right) \tag{8.1.6}$$

for any given $\xi^* \in \mathbb{R}^n$. Since g is proper with $g(0) \in \mathbb{R}$, we have $g^*(\tau^*) = \sup_{z \in \mathbb{R}^n}(z \cdot \tau^* - g(z)) \geq -g(0)$ for all $\tau^* \in \mathbb{R}^n$, so $\inf g^* > -\infty$. Moreover, Proposition 8.5 yields $g^* \not\equiv \infty$, hence $\inf g^* \in \mathbb{R}$. We also record that $g_k \geq g$ determines via Proposition 2.88 that $g^* \geq g_k^*$ for all $k \in \mathbb{N}$, and therefore $g^*(\xi^*) \geq \limsup_{k\to\infty} g_k^*(\xi^*)$ is trivial for any $\xi^* \in \mathbb{R}^n$. To prove the remaining inequality, we fix $\xi^* \in \mathbb{R}^n$. If $g^*(\xi^*) = \inf g^*$, then by (8.1.6) it holds $g^*(\xi^*) \geq g_k^*(\xi^*) \geq \inf g^*$ for every k, and clearly the equality is preserved passing to the limit. We now assume that $g^*(\xi^*) - \inf g^* > 0$. For any positive M and any $k \in \mathbb{N}$, we consider the constant $C_k := [2\varepsilon_k M]^{1/2} \in (0,\infty)$ and the corresponding ball $\mathrm{B}_k := \mathrm{B}_{C_k}$ in $\mathbb{R}^n$, where $\mathrm{B}_k \to \emptyset$ in measure as $k \to \infty$. Then it is

$$g^*(\tau^*) + \frac{1}{2\varepsilon_k}|\xi^* - \tau^*|^2 \geq g^*(\tau^*) + M \qquad \text{for all } \tau^* \notin \xi^* + \mathrm{B}_k\,,$$

whereas lower semicontinuity of g^* yields

$$\lim_{k\to\infty}\left(\inf_{\tau^* \in \xi^* + \mathrm{B}_k} g^*(\tau^*)\right) = \lim_{\varepsilon_k \to 0}\left(\inf_{\tau^* \in \xi^* + \mathrm{B}_k} g^*(\tau^*)\right) \geq g^*(\xi^*)\,.$$

Taking into account (8.1.6), we obtain

$$\begin{aligned} g_k^*(\xi^*) &\geq \min\left\{M + \inf_{\tau^* \in \xi^* + \mathrm{B}_k} g^*(\tau^*),\ \inf_{\tau^* \in \xi^* + \mathrm{B}_k}\left(g^*(\tau^*) + \frac{1}{2\varepsilon_k}|\xi^* - \tau^*|^2\right)\right\} \\ &\geq \min\left\{M + \inf g^*,\ \inf_{\tau^* \in \xi^* + \mathrm{B}_k} g^*(\tau^*)\right\} \qquad \text{for all } k \in \mathbb{N}\,. \end{aligned}$$

Hence, sending $k \to \infty$ we deduce

$$\begin{aligned} g^*(\xi^*) \geq \limsup_{k\to\infty} g_k^*(\xi^*) &\geq \liminf_{k\to\infty}\left(\min\left\{M + \inf g^*,\ \inf_{\xi^* + \mathrm{B}_k} g^*\right\}\right) \\ &\geq \min\left\{M + \inf g^*,\ g^*(\xi^*)\right\} =: m_{M,\xi^*}\,. \end{aligned} \tag{8.1.7}$$

In case $g^*(\xi^*) = \infty$, we obtain $m_{M,\xi^*} = M + \inf g^* < \infty$, therefore $\infty \geq \limsup_{k\to\infty} g_k^*(\xi^*) \geq M + \inf g^*$, and the pointwise convergence follows by arbitrariness of M. If instead $g^*(\xi^*) < \infty$, we can set $M := g^*(\xi^*) - \inf g^* \in (0,\infty)$, so that $m_{M,\xi^*} = g^*(\xi^*)$, and from (8.1.7) we read our thesis. $\square$

The function obtained in (8.1.6) as convex conjugate of each

$$g_k(\xi) := g(\xi) + \varepsilon_k|\xi|^2/2 \quad \text{for all } \xi \in \mathbb{R}^n$$

approximating g is of distinctive importance in convex analysis, and it is called **Moreau envelope** g^k of g with parameter ε_k. This function represents a specific type of infimal convolution, where one of the components in Definition 8.13 is always fixed and equal to (a positive multiple of) the Euclidean norm squared. The Moreau envelope was introduced in [81], and it can be defined on any Hilbert space H; nevertheless, for our actual purposes, it suffices to consider $\mathrm{H} = \mathbb{R}^n$. We observe that our preceding Lemma 8.20 establishes pointwise convergence of the sequence of Moreau envelopes g^k to the convex conjugate function of g, that is

$$g^k(\xi) = (g_k)^*(\xi) = \inf_{\tau^* \in \mathbb{R}^n} \left(g^*(\tau^*) + \frac{1}{2\varepsilon_k}|\xi^* - \tau^*|^2 \right) \xrightarrow[k\to\infty]{} g^*(\xi)\,.$$

Definition 8.21 (Moreau envelope)**.** We assume that $g\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper function and $\lambda \in (0,\infty)$ is a given parameter. The *Moreau envelope* (or *Moreau–Yosida regularization*) of g with parameter λ is the function $g^\lambda\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ defined for $z \in \mathbb{R}^n$ as

$$g^\lambda(z) := \inf_{y \in \mathbb{R}^n} \left(g(y) + \frac{1}{2\lambda}|z-y|^2 \right).$$

Example 8.22. The Moreau envelope with parameter λ of the norm function $g(z) := |z|$ for $z \in \mathbb{R}^n$ is given by the *Huber function*

$$g^\lambda(z) = \begin{cases} |z|^2/2\lambda; & |z| \le \lambda; \\ |z| - \lambda/2; & |z| > \lambda\,. \end{cases}$$

In fact, given $z \in \mathbb{R}^n$ our function $y \mapsto h_z(y) := g(y) + \frac{1}{2\lambda}|z-y|^2$ is continuous and convex in $\mathbb{R}^n$ with $h_z(y) \to \infty$ for $|y| \to \infty$, hence it admits minima in a ball large enough. We recall that y is a minimum of h_z if and only if $0 \in \partial h_z(y)$. For $y \neq 0$, h_z is even differentiable with $\nabla h_z(y) = y/|y| + (y-z)/\lambda$, thus its critical points $\overline{y} \in \mathbb{R}^n \setminus \{0\}$ satisfy $z = (\lambda + |\overline{y}|)\overline{y}/|\overline{y}|$ (hence $|\overline{y}| = |z| - \lambda \ge 0$), meaning they are all points having same direction of z and at distance λ. Then, we compute $h_z(\overline{y}) = (|z| - \lambda) + \lambda/2 = |z| - \lambda/2$, provided $|z| \ge \lambda$. For $y = 0$ instead, we compute $\partial h_z(0) = \{v + (y-z)/\lambda :\, v \in \overline{\mathrm{B}_1}\}|_{y=0} = \{v - z/\lambda :\, v \in \overline{\mathrm{B}_1}\}$, thus the minimality condition is achieved if $|z|/\lambda \le 1$, and in such a case it would be $h_z(0) = |z|^2/2\lambda$. The claim is then proved.

One of the advantages of working with Moreau's envelopes is that they improve regularity for the minimization problem in g^λ, while at the same time preserving minimality. In fact, it can be easily checked that g and g^λ attain the same infima, as

$$\inf_{z\in\mathbb{R}^n} g^\lambda(z) = \inf_{z\in\mathbb{R}^n} \inf_{y\in\mathbb{R}^n} \left(g(y) + \frac{1}{2\lambda}|z-y|^2 \right) = \inf_{y\in\mathbb{R}^n} \left(g(y) + \inf_{z\in\mathbb{R}^n} \frac{1}{2\lambda}|z-y|^2 \right) = \inf_{z\in\mathbb{R}^n} g(z)$$

for any choice of λ. Further, the Moreau envelope of a convex and lower semicontinuous function g is always continuously differentiable – even when no further regularity of g is assumed. Such property is stated in the next fundamental result due to Moreau [81, Proposition 7d]; compare also with [23, Lemme 2.1] and [94, Chapter IV, Proposition 1.8] for a self–contained proof in the Hilbert setting.

Theorem 8.23 (Moreau)**.** *For any proper, convex and lower semicontinuous function $g\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ and any parameter $\lambda \in (0,\infty)$, the Moreau envelope g^λ is convex, lower semicontinuous and differentiable on $\mathbb{R}^n$. Moreover, for any $z \in \mathbb{R}^n$, the infimum in the definition of $g^\lambda(z)$ is attained on $\overline{y} \in \mathbb{R}^n$ if and only if $z \in \overline{y} + \lambda\partial g(\overline{y})$.*

In the following, we will denote with g^k the Moreau envelope of g with corresponding parameter ε_k, building a decreasing sequence of non–negative real parameters as $k \to \infty$. For convex functions g with linear growth, the convergence of the Moreau envelope g^k to g as $k \to \infty$ holds not just pointwise as shown in Lemma 8.20, but the limit is in fact uniform. In detail, we can determine an upper bound for the difference between the original function g and each regularization g^k.

Proposition 8.24. *If $g\colon \mathbb{R}^n \to \mathbb{R}$ is convex with linear growth $|g(\xi)| \leq \beta(\gamma + |\xi|)$ for all $\xi \in \mathbb{R}^n$, with $\beta, \gamma \in [0, \infty)$, and given a sequence $(\varepsilon_k)_k$ decreasing to zero as $k \to \infty$, we consider the Moreau envelope of g of parameter ε_k, i.e. the function*

$$g^k(\xi) := \inf_{\tau \in \mathbb{R}^n} \left(g(\tau) + \frac{1}{2\varepsilon_k}|\xi - \tau|^2 \right) \qquad \text{for all } \xi \in \mathbb{R}^n\,.$$

Then, there is a constant $c \in (0, \infty)$ achieving the uniform bound

$$0 \leq g(\xi) - g^k(\xi) \leq c \cdot \varepsilon_k \qquad \text{for all } \xi \in \mathbb{R}^n \text{ and for all } k \in \mathbb{N}\,.$$

Proof. The inequality $g^k \leq g$ follows by definition. For the remaining bound, we fix $\xi \in \mathbb{R}^n$ and recall that Lemma 8.10 for $p = 1$ implies Lipschitz continuity of g with Lipschitz constant β'. To be precise, it holds $g(\xi) - g(\xi - \tau) \leq \beta'|\tau|$ for all $\tau \in \mathbb{R}^n$. We obtain

$$g^k(\xi) = \inf_{\tau \in \mathbb{R}^n} \left(g(\xi - \tau) + \frac{1}{2\varepsilon_k}|\tau|^2 \right) \geq g(\xi) + \inf_{\tau \in \mathbb{R}^n} \left(\frac{1}{2\varepsilon_k}|\tau|^2 - \beta'|\tau| \right).$$

It is then straightforward to compute the minimum of the radial function $h(\rho) := \rho^2/2\varepsilon_k - \beta'\rho$ in $[0, \infty)$, which is achieved for $\rho = \beta'\varepsilon_k$, determining $h(\beta'\varepsilon_k) = -(\beta')^2\varepsilon_k/2$. Altogether, by setting $c := (\beta')^2/2$, we have proved the remaining inequality. □

8.2 Convex duality

With the preliminaries in Section 8.1 at hand, we are now able to approach the preparatory duality results. As explained at the beginning of the chapter, we are going to treat first the standard case of duality for convex, coercive, and differentiable integrands with a measure. We implicitly assume that Ω is our usual open, bounded, Lipschitz domain in $\mathbb{R}^n$.

Theorem 8.25 (convex duality formula in $\mathrm{W}^{1,2}$ under coercivity and differentiability)**.** *We consider a datum $u_0 \in \mathrm{W}^{1,2}(\mathbb{R}^n)$ and a (possibly signed) Radon measure μ on Ω vanishing on $\mathcal{H}^{n-1}$–negligible sets, μ satisfying*

$$\left| \int_\Omega w^* \,\mathrm{d}\mu \right| \leq C||\nabla w||_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} \qquad \text{for all } w \in \mathrm{W}^{1,2}_0(\Omega) \tag{8.2.1}$$

for some constant $C \in [0, \infty)$. Moreover, we suppose that $u_0^*|_\Omega \in \mathrm{L}^1(\Omega; |\mu|)$. We fix a Borel integrand $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$, $f = f(x, \xi)$ convex and differentiable in ξ for a.e. given $x \in \mathbb{R}^n$, and we assume the existence of c_1, $c_2 \in [0, \infty)$ such that

$$c_1|\xi|^2 \leq f(x, \xi) \leq c_2(1 + |\xi|)^2 \qquad \text{for all } x, \xi \in \mathbb{R}^n\,.$$

Then, we obtain the duality formula:

$$\min_{w \in \mathrm{W}^{1,2}_{u_0}(\Omega)} \left(\int_\Omega f\,(.,\nabla w)\,\mathrm{d}x + \int_\Omega w^*\,\mathrm{d}\mu \right) = \max_{\substack{\sigma \in \mathrm{L}^2(\Omega,\mathbb{R}^n) \\ \mathrm{div}(\sigma) = \mu}} \left(\int_\Omega [\sigma \cdot \nabla u_0 - f^*(.,\sigma)]\,\mathrm{d}x + \int_\Omega u_0^*\,\mathrm{d}\mu \right). \tag{8.2.2}$$

Moreover, there exists a unique maximizer $\overline{\sigma}$ of the dual problem.

We notice that the assumptions on μ determine well–posedness of the measure term $\int_\Omega w^* \, \mathrm{d}\mu$, moreover for all $w \in \mathrm{W}^{1,2}_{u_0}(\Omega)$ it is

$$\left|\int_\Omega w^* \, \mathrm{d}\mu\right| \le \left|\int_\Omega (w-u_0)^* \, \mathrm{d}\mu\right| + \int_\Omega |u_0^*| \, \mathrm{d}|\mu| \le C||\nabla(w-u_0)||_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} + ||u_0||_{\mathrm{L}^1(\Omega;|\mu|)} < \infty \,.$$

This, combined with the growth assumption on f, determines the finiteness of $\int_\Omega f\,(.,\nabla w)\,\mathrm{d}x + \int_\Omega w^* \, \mathrm{d}\mu$ in $\mathrm{W}^{1,2}_{u_0}(\Omega)$. We can now proceed with proving the first duality result.

Proof of Theorem 8.25. In a completely analogous manner to (8.0.1), integrating by parts on $\mathrm{W}^{1,2}_0(\Omega)$ we verify that the inequality "$\ge$" in (8.2.2) is trivially satisfied. To address the reverse inequality, we first claim that the primal problem in (8.2.2) admits a minimum in $\mathrm{W}^{1,2}_{u_0}(\Omega)$. We start by noticing that convexity of $f(x,.)$ and the quadratic growth assumption determine via Corollaries 8.17 and 8.18 weak lower semicontinuity of $w \mapsto \int_\Omega f\,(.,\nabla w)\,\mathrm{d}x$ in $\mathrm{W}^{1,2}(\Omega)$. Moreover, the condition (8.2.1) yields that the linear functional $w \mapsto \int_\Omega w^* \, \mathrm{d}\mu$ is bounded in $\mathrm{W}^{1,2}_0(\Omega)$ with μ an element of the dual $\left(\mathrm{W}^{1,2}_0(\Omega)\right)^*$, and the functional $w \mapsto \int_\Omega w^* \, \mathrm{d}\mu$ is weakly continuous on $\mathrm{W}^{1,2}_0(\Omega)$. Then, if $(w_k)_k$, w are in $\mathrm{W}^{1,2}_{u_0}(\Omega)$ and $w_k \rightharpoonup w$ weakly as $k \to \infty$ in $\mathrm{W}^{1,2}(\Omega)$, it suffices to apply the continuity result to $w_k - u_0$, $w - u_0 \in \mathrm{W}^{1,2}_0(\Omega)$ and simplify the common terms in view of $|\int_\Omega u_0^* \, \mathrm{d}\mu| < \infty$. Altogether, the joint functional is weakly LSC on $\mathrm{W}^{1,2}_{u_0}(\Omega)$. Furthermore, we observe that BV coercivity follows from (2.2.2) together with Young's inequality in the variant $ab \le \varepsilon a^2/2 + b^2/2\varepsilon$ for $a,\, b \ge 0$ and all $\varepsilon > 0$, since

$$\begin{aligned}
&\int_\Omega f(.,\nabla w) + \int_\Omega w^* \, \mathrm{d}\mu \\
&= \int_\Omega f(.,\nabla w) + \int_\Omega (w-u_0)^* \, \mathrm{d}\mu + \int_\Omega u_0^* \, \mathrm{d}\mu \\
&\ge c_1 ||\nabla w||^2_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} - C||\nabla(w-u_0)||_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} - ||u_0||_{\mathrm{L}^1(\Omega;|\mu|)} \\
&\ge (c_1 - C\varepsilon/2)||\nabla w||^2_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} - C/2\varepsilon - C||\nabla u_0||_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} - ||u_0||_{\mathrm{L}^1(\Omega;|\mu|)} \\
&\ge \widetilde{C_2}(c_1 - C\varepsilon/2)||w||^2_{\mathrm{W}^{1,2}(\Omega)} - \lambda_2(c_1 - C\varepsilon/2)||u_0||^2_{\mathrm{W}^{1,2}(\Omega)} - C/2\varepsilon - C||\nabla u_0||_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} - ||u_0||_{\mathrm{L}^1(\Omega;|\mu|)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } w \in \mathrm{W}^{1,2}_{u_0}(\Omega)\,.
\end{aligned}$$

Then, letting $\varepsilon \in (0, 2c_1/C)$ we achieve coercivity in $\mathrm{W}^{1,2}_{u_0}$. By direct method we can then argue the existence of some $\overline{w}$ minimizer for $w \mapsto \int_\Omega f(.,\nabla w) + \int_\Omega w^* \, \mathrm{d}\mu$ in the class $\mathrm{W}^{1,2}_{u_0}(\Omega)$, and we set the vector field $\overline{\sigma}\colon \Omega \to \mathbb{R}^n$ such that $\overline{\sigma}(x) := \nabla_\xi f(x, \nabla\overline{w}(x))$ for a.e. $x \in \Omega$. We now show that $\overline{\sigma}$ achieves the maximum in the dual problem. From the minimality of $\overline{w}$, the necessary condition of the Euler–Lagrange equation in $\mathrm{W}^{1,2}_{u_0}$ yields $\mathrm{div}(\overline{\sigma}) = \mu$. To verify that $\overline{\sigma}$ belongs to $\mathrm{L}^2(\Omega,\mathbb{R}^n)$, we simply apply Lemma 8.10 with $p = 2$ to $\xi \mapsto f(x,\xi)$. We thus find a constant $c_2{}' > 0$ such that

$$\begin{aligned}
\int_\Omega |\overline{\sigma}|^2 \, \mathrm{d}x = \int_\Omega |\nabla_\xi f(.,\nabla\overline{w})|^2 \, \mathrm{d}x &\le (c_2{}')^2 \int_\Omega (1 + 2|\nabla\overline{w}|)^2 \, \mathrm{d}x \\
&= (c_2{}')^2 \left[|\Omega| + 4||\nabla\overline{w}||_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} + 4||\nabla\overline{w}||^2_{\mathrm{L}^2(\Omega,\mathbb{R}^n)}\right] < \infty \,.
\end{aligned}$$

We furthermore recall that Proposition 2.94 induces

$$\overline{\sigma}(x) \cdot \nabla\overline{w}(x) = f(x,\nabla\overline{w}(x)) + f^*(x,\overline{\sigma}(x)) \qquad \text{for a.e. } x \in \Omega\,. \tag{8.2.3}$$

Finally, integration by parts on $\overline{w} - u_0 \in \mathrm{W}^{1,2}_0(\Omega)$ yields

$$\begin{aligned}
\int_\Omega f(.,\nabla\overline{w})\,\mathrm{d}x + \int_\Omega (\overline{w}-u_0)^* \, \mathrm{d}\mu &= \int_\Omega \nabla\overline{w} \cdot \overline{\sigma} \, \mathrm{d}x - \int_\Omega f^*(.,\overline{\sigma})\,\mathrm{d}x + \int_\Omega (\overline{w}-u_0)^* \, \mathrm{d}(\mathrm{div}(\overline{\sigma})) \\
&= \int_\Omega \overline{\sigma} \cdot \nabla u_0 \, \mathrm{d}x - \int_\Omega f^*(.,\overline{\sigma})\,\mathrm{d}x\,,
\end{aligned}$$

hence $\overline{\sigma}$ must attain the maximum of the problem on the right–hand side, which concludes the proof of (8.2.4).

We further notice that the necessary and sufficient condition for some maximizer $\sigma \in \mathrm{L}^2(\Omega, \mathbb{R}^n)$ with $\operatorname{div}(\sigma) = \mu$ is that $\sigma(x) \in \partial_\xi f(x, \nabla\overline{w}(x))$ for a.e. $x \in \Omega$, since otherwise (8.2.3) fails and the inequality between primal and dual problem is strict. Moreover, the differentiability assumption on f in ξ determines $\partial_\xi f(., \nabla\overline{w}) = \{\nabla f(., \nabla\overline{w})\}$ almost everywhere in Ω, therefore (8.2.3) is achieved as an equality for the only vector field $\overline{\sigma}$, hence uniqueness of solutions for the dual problem. □

Theorem 8.26 (convex duality formula in $\mathrm{W}^{1,2}$ under differentiability)**.** *Assume that $u_0 \in \mathrm{W}^{1,2}(\mathbb{R}^n)$ and that μ is a (possibly signed) Radon measure on Ω vanishing on $\mathcal{H}^{n-1}$–negligible sets of Ω and satisfying (8.2.1), $u_0^*|_\Omega \in \mathrm{L}^1(\Omega; |\mu|)$. Then, for any Borel integrand $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ convex and differentiable in the second variable ξ for a.e. $x \in \mathbb{R}^n$, f satisfying the linear–growth condition (8.1.5), we have*

$$\inf_{w\in \mathrm{W}^{1,2}_{u_0}(\Omega)} \left(\int_\Omega f(., \nabla w)\, \mathrm{d}x + \int_\Omega w^*\, \mathrm{d}\mu \right) = \max_{\substack{\sigma\in \mathrm{L}^2(\Omega,\mathbb{R}^n)\\ \operatorname{div}(\sigma)=\mu}} \left(\int_\Omega [\sigma\cdot\nabla u_0 - f^*(., \sigma)]\, \mathrm{d}x + \int_\Omega u_0^*\, \mathrm{d}\mu \right). \tag{8.2.4}$$

We preliminarily observe that the linear–growth assumption (8.1.5) is suitable to our purposes, since any $w \in \mathrm{W}^{1,2}_{u_0}(\Omega)$ is such that $||\nabla w||_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} \le |\Omega|^{1/2}||\nabla w||_{\mathrm{L}^2(\Omega,\mathbb{R}^n)}$, hence

$$c_1|\Omega| \le \int_\Omega f(., \nabla w)\, \mathrm{d}x \le c_2|\Omega| + c_2|\Omega|^{1/2}||\nabla w||_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} < \infty \qquad \text{for all } w \in \mathrm{W}^{1,2}_{u_0}(\Omega)\,,$$

therefore, the control on the measure term given by (8.2.1) yields finiteness of the joint functional.

Proof of Theorem 8.26. As usual, the inequality "$\ge$" in (8.2.4) is trivial. To address the other inequality, we introduce the bounded, positive, decreasing sequence $(\varepsilon_k)_k$ such that $\varepsilon_k \searrow 0$ as $k \to \infty$. We generate $\mathrm{W}^{1,2}$–coercive approximations $(f_k)_k$ of f by letting

$$f_k(x, \xi) := f(x, \xi) + \varepsilon_k|\xi|^2/2 \qquad \text{for all } x, \xi \in \mathbb{R}^n$$

for any given $k \in \mathbb{N}$. Clearly, it is $f_k(x, \xi) \ge c_1 + \varepsilon_k|\xi|^2/2$, so the sequence $(f_k)_k$ is $\mathrm{W}^{1,2}(\Omega)$–coercive, while preserving convexity and differentiability in ξ. Then, an application of Theorem 8.25 to each $\xi \mapsto f_k(x, \xi)$ yields

$$\min_{w\in \mathrm{W}^{1,2}_{u_0}(\Omega)} \left(\int_\Omega f_k(., \nabla w)\, \mathrm{d}x + \int_\Omega w^*\, \mathrm{d}\mu \right) = \max_{\substack{\sigma\in \mathrm{L}^2(\Omega,\mathbb{R}^n)\\ \operatorname{div}(\sigma)=\mu}} \left(\int_\Omega [\sigma\cdot\nabla u_0 - f_k^*(., \sigma)]\, \mathrm{d}x + \int_\Omega u_0^*\, \mathrm{d}\mu \right). \tag{8.2.5}$$

If we now verify the following estimates (A) and (B), the remaining inequality "$\le$" in (8.2.4) follows, and the duality formula for the functional in f is proved.

$$\inf_{w\in \mathrm{W}^{1,2}_{u_0}(\Omega)}\left(\int_\Omega f(., \nabla w)\, \mathrm{d}x + \int_\Omega (w-u_0)^*\, \mathrm{d}\mu\right) \le \liminf_{k\to\infty} \min_{w\in \mathrm{W}^{1,2}_{u_0}(\Omega)}\left(\int_\Omega f_k(., \nabla w)\, \mathrm{d}x + \int_\Omega (w-u_0)^*\mathrm{d}\mu\right) \tag{A}$$

$$\sup_{\substack{\sigma\in \mathrm{L}^2(\Omega,\mathbb{R}^n)\\ \operatorname{div}(\sigma)=\mu}} \left(\int_\Omega \sigma\cdot\nabla u_0\, \mathrm{d}x - \int_\Omega f^*(., \sigma)\, \mathrm{d}x \right) \ge \limsup_{k\to\infty} \max_{\substack{\sigma\in \mathrm{L}^2(\Omega,\mathbb{R}^n)\\ \operatorname{div}(\sigma)=\mu}} \left(\int_\Omega [\sigma\cdot\nabla u_0 - f_k^*(., \sigma)]\, \mathrm{d}x \right). \tag{B}$$

Proof of (A). We recall that for any $w \in \mathrm{W}^{1,2}(\Omega)$ it is

$$\int_\Omega f_k(., \nabla w)\, \mathrm{d}x = \int_\Omega f(., \nabla w)\, \mathrm{d}x + \frac{\varepsilon_k}{2}||\nabla w||^2_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} \xrightarrow[k\to\infty]{} \int_\Omega f(., \nabla w)\, \mathrm{d}x\,.$$

If for $k \in \mathbb{N}$ we consider a minimum $w_k \in \mathrm{W}^{1,2}_{u_0}(\Omega)$ of the primal problem with integrand f_k, we directly get the claimed inequality (A) via

$$\begin{aligned}\inf_{w \in \mathrm{W}^{1,2}_{u_0}(\Omega)} \left(\int_\Omega f\,(.,\nabla w) + \int_\Omega (w-u_0)^*\,\mathrm{d}\mu \right) &= \inf_{w \in \mathrm{W}^{1,2}_{u_0}(\Omega)} \lim_{k\to\infty} \left(\int_\Omega f_k\,(.,\nabla w) + \int_\Omega (w-u_0)^*\,\mathrm{d}\mu \right) \\ &\le \liminf_{k\to\infty} \left(\int_\Omega f_k\,(.,\nabla w_k) + \int_\Omega (w_k-u_0)^*\,\mathrm{d}\mu \right) \\ &= \liminf_{k\to\infty} \left(\min_{w \in \mathrm{W}^{1,2}_{u_0}(\Omega)} \int_\Omega f_k\,(.,\nabla w) + \int_\Omega (w-u_0)^*\,\mathrm{d}\mu \right).\end{aligned}$$

Assume now that each σ_k is a maximizer of the dual problem corresponding to f_k. Before approaching the proof of (B), we need to ensure the following.

Claim. The sequence $(\sigma_k)_k$ is bounded in $\mathrm{L}^2(\Omega,\mathbb{R}^n)$. Looking back at the proof of Theorem 8.25, we know that fixed $k \in \mathbb{N}$ it is $\sigma_k := \nabla_\xi f_k(.,\nabla w_k)$ for some w_k minimum of the primal problem in f_k. This yields $\sigma_k \in \mathrm{L}^2(\Omega,\mathbb{R}^n)$ with $\operatorname{div}(\sigma_k) = \mu$ and

$$\int_\Omega \sigma_k \cdot \nabla u_0 \,\mathrm{d}x - \int_\Omega f_k^*(.,\sigma_k)\,\mathrm{d}x = \max_{\substack{\sigma \in \mathrm{L}^2(\Omega,\mathbb{R}^n)\\ \operatorname{div}(\sigma)=\mu}} \left(\int_\Omega \sigma \cdot \nabla u_0 \,\mathrm{d}x - \int_\Omega f_k^*(.,\sigma)\,\mathrm{d}x \right).$$

Explicitly, we have $\sigma_k = \nabla_\xi f(.,\nabla w_k) + \varepsilon_k \nabla w_k$ almost everywhere in Ω, and assumed that c_2' is the positive constant arising from Lemma 8.10 applied to $\xi \mapsto f(x,\xi)$ (thus, c_2' is employed in place of β'), our growth assumption on f determines $|\nabla_\xi f(x,\nabla w_k(x))| \le \max\{c_1, c_2'\} =: c_3$ for a.e. $x \in \Omega$, hence we estimate from above

$$|\sigma_k|^2 \le 2|\nabla_\xi f(.,\nabla w_k)|^2 + 2\varepsilon_k^2 |\nabla w_k|^2 \le 2c_3^2 + 2\varepsilon_k^2|\nabla w_k|^2 \quad \mathcal{L}^n\text{–a.e. on } \Omega\,,$$

from which it follows

$$\sup_{k\in\mathbb{N}} ||\sigma_k||^2_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} \le 2c_3^2 + 2\sup_{k\in\mathbb{N}} \varepsilon_k^2 ||\nabla w_k||^2_{\mathrm{L}^2(\Omega,\mathbb{R}^n)}\,.$$

If we show that the sequence of minimizers $(\varepsilon_k w_k)_k$ is bounded in $\mathrm{L}^2(\Omega,\mathbb{R}^n)$, then our claim is proved. To this aim, we fix $k \in \mathbb{N}$ and compute recalling that w_k minimizes $\mathrm{W}^{1,2}_{u_0}(\Omega) \ni w \mapsto \int_\Omega f\,(.,\nabla w)\,\mathrm{d}x + \int_\Omega (w-u_0)^*\,\mathrm{d}\mu$ and exploiting (8.2.1) for $w_k - u_0 \in \mathrm{W}^{1,2}_0(\Omega)$:

$$\begin{aligned}\frac{\varepsilon_k}{2}||\nabla w_k||^2_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} &\le \int_\Omega f_k(.,\nabla w_k) - c_1|\Omega| \\ &\le \int_\Omega f_k(.,\nabla u_0) + \int_\Omega (u_0-w_k)^*\,\mathrm{d}\mu - c_1|\Omega| \\ &\le \int_\Omega f(.,\nabla u_0) + \frac{\varepsilon_k}{2}||\nabla u_0||^2_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} + C||\nabla u_0 - \nabla w_k||_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} - c_1|\Omega| \\ &\le \int_\Omega f(.,\nabla u_0) + \frac{\varepsilon_k}{2}||\nabla u_0||^2_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} + C||\nabla u_0||_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} + C||\nabla w_k||_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} - c_1|\Omega| \\ &\le (C+c_2)||\nabla u_0||_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} + c_2 + \frac{\varepsilon_k}{2}||\nabla u_0||^2_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} + C||\nabla w_k||_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} - c_1|\Omega|\,.\end{aligned}$$

Setting $\widehat{C} := (C+c_2)||\nabla u_0||_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} + c_2 - c_1|\Omega|$, multiplying by $2\varepsilon_k$, and finally with the help of Young's inequality, we obtain

$$\begin{aligned}\varepsilon_k^2||\nabla w_k||^2_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} &\le 2\varepsilon_k\widehat{C} + \varepsilon_k^2||\nabla u_0||^2_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} + 2C\varepsilon_k||\nabla w_k||_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} \\ &\le 2\varepsilon_k\widehat{C} + \varepsilon_k^2||\nabla u_0||^2_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} + \varepsilon_k^2||\nabla w_k||^2_{\mathrm{L}^2(\Omega,\mathbb{R}^n)}/2 + 4C^2\,,\end{aligned}$$

that is $\varepsilon_k^2\|\nabla w_k\|^2_{\mathrm{L}^2(\Omega,\mathbb{R}^n)}/2 \le 2\varepsilon_k\widehat{C} + \varepsilon_k^2\|\nabla u_0\|^2_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} + 4C^2$, with the upper bound converging to the constant $4C^2$ as $k \to \infty$. Substituting in (8.2), we have verified boundedness of the sequence $(\sigma_k)_k$.

Proof of (B). From the $\mathrm{L}^2(\Omega,\mathbb{R}^n)$–boundedness of $(\sigma_k)_k$ and via the theorem of Banach–Alaoglu, we infer the existence of a (non relabelled) subsequence and of a weak limit $\overline{\sigma} \in \mathrm{L}^2(\Omega,\mathbb{R}^n)$ such that $\sigma_k \rightharpoonup \overline{\sigma}$ weakly in $\mathrm{L}^2(\Omega,\mathbb{R}^n)$ as $k \to \infty$. Then, it still holds $\operatorname{div}(\overline{\sigma}) = \mu$ as distributions (in particular, $\operatorname{div}(\overline{\sigma})$ is a Radon measure) on Ω, since for every $\psi \in \mathrm{C}^\infty_\mathrm{c}(\Omega)$ weak convergence determines

$$\int_\Omega \psi \,\mathrm{d}(\operatorname{div}(\overline{\sigma})) = -\int_\Omega \overline{\sigma}\cdot\nabla\psi \,\mathrm{d}x = -\lim_{k\to\infty}\int_\Omega \sigma_k\cdot\nabla\psi\,\mathrm{d}x = \lim_{k\to\infty}\int_\Omega \psi\,\mathrm{d}(\operatorname{div}(\sigma_k)) = \int_\Omega \psi\,\mathrm{d}\mu\,.$$

We now claim that

$$\liminf_{k\to\infty}\int_\Omega f_k^*(.,\sigma_k)\,\mathrm{d}x \ge \int_\Omega f^*(.,\overline{\sigma})\,\mathrm{d}x\,. \tag{8.2.6}$$

In fact, Proposition 8.19 guarantees weak lower semicontinuity in $\mathrm{L}^2(\Omega,\mathbb{R}^n)$ of each f_j^*, specifically

$$\liminf_{k\to\infty}\int_\Omega f_j^*(.,\sigma_k)\,\mathrm{d}x \ge \int_\Omega f_j^*(.,\overline{\sigma})\,\mathrm{d}x \qquad \text{for all } j\in\mathbb{N}\,.$$

At the same time, being $(\varepsilon_k)_k$ decreasing in k, the sequence of convex conjugates $(f_k^*)_k$ is increasing in k, and Lemma 8.20 applied to $f(x,.)$ proper and convex implies the pointwise convergence $f_k^*(x,\xi^*) \nearrow f^*(x,\xi^*)$ as $k\to\infty$, with the uniform lower bound

$$f_k^*(x,\xi^*) \ge -f_k(x,0) = -f(x,0) \ge -c_1 \in \mathrm{L}^1(\Omega) \qquad \text{for all } k\in\mathbb{N}$$

coming from the assumption (8.1.5). We apply thus the monotone convergence theorem to deduce

$$\lim_{k\to\infty}\int_\Omega f_k^*(.,\tau)\,\mathrm{d}x = \int_\Omega f^*(.,\tau)\,\mathrm{d}x \qquad \text{for every } \tau\in\mathrm{L}^2(\Omega,\mathbb{R}^n)\,. \tag{8.2.7}$$

On the other hand, assuming that the indices $j,k\in\mathbb{N}$ are such that $j\le k$, from semicontinuity we read

$$\liminf_{k\to\infty}\int_\Omega f_k^*(.,\sigma_k)\,\mathrm{d}x \ge \liminf_{k\to\infty}\int_\Omega f_j^*(.,\sigma_k)\,\mathrm{d}x \ge \int_\Omega f_j^*(.,\overline{\sigma})\,\mathrm{d}x\,,$$

and sending $j\to\infty$ we deduce the stated (8.2.6) in view of the limit in (8.2.7). Altogether, collecting the results above, we find

$$\begin{aligned}\int_\Omega[\overline{\sigma}\cdot\nabla u_0 - f^*(.,\overline{\sigma})]\,\mathrm{d}x &\ge \lim_{k\to\infty}\int_\Omega \sigma_k\cdot\nabla u_0\,\mathrm{d}x + \limsup_{k\to\infty}\left(-\int_\Omega f_k^*(.,\sigma_k)\,\mathrm{d}x\right)\\ &= \limsup_{k\to\infty}\int_\Omega[\sigma_k\cdot\nabla u_0 - f_k^*(.,\sigma_k)]\,\mathrm{d}x\\ &= \limsup_{k\to\infty}\max_{\substack{\sigma\in\mathrm{L}^2(\Omega,\mathbb{R}^n)\\ \operatorname{div}(\sigma)=\mu}}\left(\int_\Omega[\sigma\cdot\nabla u_0 - f_k^*(.,\sigma)]\,\mathrm{d}x\right),\end{aligned} \tag{8.2.8}$$

and since $\overline{\sigma}$ is an admissible field passing to the supremum in (8.2.8) on $\sigma\in\mathrm{L}^2(\Omega,\mathbb{R}^n)$ with divergence μ we conclude that (B) holds.

With the help of (A), (B) we can finally then send $k\to\infty$ in (8.2.5) and obtain

$$\int_\Omega[\overline{\sigma}\cdot\nabla u_0 - f^*(.,\overline{\sigma})]\,\mathrm{d}x \ge \inf_{w\in\mathrm{W}^{1,2}_{u_0}(\Omega)}\left(\int_\Omega f\,(.,\nabla w)\,\mathrm{d}x + \int_\Omega (w-u_0)^*\,\mathrm{d}\mu\right),$$

which yields the final statement (8.2.4). Furthermore, from the latter inequality, we even read that $\overline{\sigma}$ achieves the maximum of the dual problem. □

We now want to extend our duality formula from the $\mathrm{W}^{1,2}$–setting of Theorem 8.26 to the required space $\mathrm{W}^{1,1}$. To do so, the next approximation result for the infimum of the functional $\mathcal{F} = \mathcal{F}^{\mu}_{u_0}$ on $\mathrm{W}^{1,1}$ comes in handy.

Lemma 8.27 (convergence of infima for sequences in $\mathrm{W}^{1,1}$ with arbitrary boundary values)**.** *Consider admissible measures $\mu_\pm$ on Ω and a Borel $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ under the linear–growth condition (8.1.5) and such that $\xi \mapsto f(x,\xi)$ is convex for a.e. $x \in \mathbb{R}^n$. Then, the following holds:*

$$\inf_{\mathrm{W}^{1,1}_{u_0}(\Omega)} \mathcal{F}^{\mu}_{u_0} = \lim_{k\to\infty} \left(\inf_{\mathrm{W}^{1,1}_{u_{0,k}}(\Omega)} \mathcal{F}^{\mu}_{u_{0,k}} \right) \qquad \text{for all } u_0, (u_{0,k})_k \in \mathrm{W}^{1,1}(\Omega)\,,\ u_{0,k} \xrightarrow[k\to\infty]{} u_0 \text{ in } \mathrm{W}^{1,1}(\Omega)\,.$$

Proof. Let $u_0 \in \mathrm{W}^{1,1}(\Omega)$ and a sequence $(u_{0,k})$ converging to u_0 in $\mathrm{W}^{1,1}(\Omega)$. We fix an arbitrary $v \in \mathrm{W}^{1,1}_0(\Omega)$ and an element τ^*_x in the (non–empty) set $\partial_\xi f(x, \nabla v(x) + \nabla u_0(x))$ for $x \in \Omega$. Then, the subgradient inequality determines

$$\begin{aligned}\int_\Omega f(.,\nabla(v+u_0))\,\mathrm{d}x &\le \int_\Omega f(.,\nabla(v+u_{0,k}))\,\mathrm{d}x + \int_\Omega \tau^*_x \cdot (\nabla u_0 - \nabla u_{0,k})\,\mathrm{d}x \\ &\le \int_\Omega f(.,\nabla(v+u_{0,k}))\,\mathrm{d}x + \max\left\{c_1, c_2'\right\} \|\nabla u_0 - \nabla u_{0,k}\|_{\mathrm{L}^1(\Omega,\mathbb{R}^n)}\end{aligned}$$

for all $k \in \mathbb{N}$, where we make use of the gradient bound in Lemma 8.10 in terms of the adjusted constant $c_2' > 0$. For the measure term instead, we exploit the admissibility condition in the form of Remark 3.13 for $\mu_\pm$ with respective constants $C_\pm \in [0,\infty)$ and write

$$\begin{aligned}&\int_\Omega (v+u_0)^*\,\mathrm{d}(\mu_+ - \mu_-) \\ &\le \int_\Omega (v+u_{0,k})^*\,\mathrm{d}(\mu_+ - \mu_-) + (C_+ + C_-)\big[\|\nabla u_0 - \nabla u_{0,k}\|_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} + \|u_0 - u_{0,k}\|_{\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})}\big],\end{aligned}$$

again valid for every k. Then, the sequence of elements

$$C_k := \left(\max\left\{c_1, c_2'\right\} + C_+ + C_-\right)\left(\|\nabla u_0 - \nabla u_{0,k}\|_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} + \|u_0 - u_{0,k}\|_{\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})}\right)$$

is such that $C_k \to 0$ as $k \to \infty$ because of strong convergence. Therefore, passing to the infimum for $v \in \mathrm{W}^{1,1}_0(\Omega)$ we conclude

$$\inf_{\mathrm{W}^{1,1}_{u_0}(\Omega)} \mathcal{F}^{\mu}_{u_0} \le \liminf_{k\to\infty} \left(\inf_{\mathrm{W}^{1,1}_{u_{0,k}}(\Omega)} \mathcal{F}^{\mu}_{u_0} + C_k \right) = \liminf_{k\to\infty} \inf_{\mathrm{W}^{1,1}_{u_{0,k}}(\Omega)} \mathcal{F}^{\mu}_{u_0}\,.$$

The reverse inequality is obtained by switching the roles of u_0 and $u_{0,k}$, thus our claim is proved. □

Theorem 8.28 (convex duality formula in $\mathrm{W}^{1,1}$ under differentiability)**.** *We take $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ and $\mu_\pm$ admissible measures on Ω with $\mu := \mu_+ - \mu_-$. Then, the duality formula holds:*

$$\inf_{w\in \mathrm{W}^{1,1}_{u_0}(\Omega)} \left(\int_\Omega f(.,\nabla w)\,\mathrm{d}x + \int_\Omega w^*\,\mathrm{d}\mu \right) = \sup_{\sigma \in \mathcal{DM}_\mu(\Omega,\mathbb{R}^n)} \left(\int_\Omega [\sigma\cdot\nabla u_0 - f^*(.,\sigma)]\,\mathrm{d}x + \int_\Omega u_0^*\,\mathrm{d}\mu \right) \tag{8.2.9}$$

for any Borel $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ convex and differentiable in the second variable ξ for a.e. $x \in \mathbb{R}^n$, with f satisfying the linear–growth condition (8.1.5). Furthermore, if the common value in (8.2.9) is larger than $-\infty$, then the dual problem admits a solution.

We would like to emphasize that under the assumptions above, it is allowed for the infimum

$$\mathrm{I} := \inf_{w \in \mathrm{W}^{1,1}_{u_0}(\Omega)} \left(\int_\Omega f\,(.,\nabla w)\;\mathrm{d}x + \int_\Omega w^*\,\mathrm{d}\mu \right)$$

in (8.2.9) to achieve the value $-\infty$. Nevertheless, when later imposing the f^∞–isoperimetric conditions with constant 1 for the pairs of measures, from Proposition 5.2 we are safe that $\mathrm{I} > -\infty$ (at least in the standard case of assumptions (H1)–(H2)). Then, from the statement of Theorem 8.28 we deduce even the existence of maximizers for the corresponding dual problem.

In comparison to the statements of the $\mathrm{W}^{1,2}$–duality formulas in Theorem 8.25 and Theorem 8.26, it is worth noticing that the assumption $u_0^*|_\Omega \in \mathrm{L}^1(\Omega;|\mu|)$ drops out, since it is automatically implied by the admissibility of the component measures $\mu_\pm$; compare with Proposition 3.5. At the same time, if $\mu_\pm$ are admissible on Ω as in the statement above, then μ fulfils (8.2.1), since exploiting the rewriting of Remark 3.13 with respective constants $C_\pm$ we find

$$\left| \int_\Omega w^*\,\mathrm{d}\mu \right| \le (C_+ + C_-)||\nabla w||_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} \le (C_+ + C_-)|\Omega|^{1/2}||\nabla w||_{\mathrm{L}^2(\Omega,\mathbb{R}^n)} \qquad \text{for all } w \in \mathrm{W}^{1,2}_0(\Omega)\,.$$

Proof of Theorem 8.28. From integration by parts, the infimum I is always larger than or equal to the supremum on the class of admissible vector fields. We focus now on the reverse inequality in (8.2.9), noticing that this is always satisfied in case $\mathrm{I} = -\infty$. We assume then $\mathrm{I} > -\infty$ and we distinguish two cases according to the integrability of the datum u_0.

Step 1. We suppose first $u_0 \in \mathrm{W}^{1,2}(\mathbb{R}^n)$. Exploiting the result of Theorem 8.26 in $\mathrm{W}^{1,2}(\Omega)$ with (strong) Dirichlet condition u_0, we find

$$\mathrm{I} \le \inf_{w \in \mathrm{W}^{1,2}_{u_0}(\Omega)} \left(\int_\Omega f\,(.,\nabla w)\;\mathrm{d}x + \int_\Omega w^*\,\mathrm{d}\mu \right) \le \int_\Omega [\overline{\sigma}\cdot\nabla u_0 - f^*(.,\overline{\sigma})]\;\mathrm{d}x + \int_\Omega u_0^*\,\mathrm{d}\mu\,, \tag{8.2.10}$$

with $\overline{\sigma}$ the weak limit in $\mathrm{L}^2(\Omega,\mathbb{R}^n)$ of the sequence $(\sigma_k)_k$ of maximizers of the dual problem in f_k. Then, our thesis is achieved once we verify that $\overline{\sigma} \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$.

We suppose by contradiction that $\operatorname{ess\,sup}_{x\in\Omega} |\overline{\sigma}(x)| > c_2$. The growth assumption (8.1.5) for the integrand f together with Lemma 2.59(i) and Proposition 2.84(iii) would then imply

$$\operatorname*{ess\,sup}_{x\in\Omega}\,(f^\infty)^\circ(x,\overline{\sigma}(x)) \ge \operatorname*{ess\,sup}_{x\in\Omega} |\overline{\sigma}(x)|/c_2 > 1\,,$$

hence by Proposition 2.89(i) even $\int_\Omega f^*(.,\overline{\sigma})\;\mathrm{d}x = \infty$, and consequently from the result of (8.2.10) we read

$$-\infty < \mathrm{I} \le \int_\Omega [\overline{\sigma}\cdot\nabla u_0 - f^*(.,\overline{\sigma})]\;\mathrm{d}x + \int_\Omega u_0^*\,\mathrm{d}\mu = -\infty\,,$$

which is impossible. Therefore, it is necessary for $\overline{\sigma}$ to be almost everywhere bounded by the constant c_2, and in particular $\overline{\sigma} \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ as claimed. Since we already proved in Theorem 8.26 that $\operatorname{div}(\overline{\sigma}) = \mu$, from (8.2.10) we conclude the validity of (8.2.9) and the existence of a maximum under $\mathrm{W}^{1,2}$–data.

Step 2. We suppose now that u_0 is in $\mathrm{W}^{1,1}(\mathbb{R}^n)$ (but not necessarily in $\mathrm{W}^{1,2}(\mathbb{R}^n)$) and we approximate the restriction of u_0 to Ω via a sequence $(u_{0,k})_k$ in $\mathrm{W}^{1,2}(\Omega)$, with $u_{0,k} \to u_0|_\Omega$ strongly in

$\mathrm{W}^{1,1}(\Omega)$. We are thus in the conditions of applying Lemma 8.27 and the results of *Step 1* to each $u_{0,k}$ and therefore get

$$\begin{aligned} -\infty < \mathrm{I} &= \lim_{k\to\infty}\left(\inf_{\mathrm{W}^{1,1}_{u_{0,k}}(\Omega)}\int_\Omega f\left(.,\nabla w\right)\,\mathrm{d}x + \int_\Omega w^*\,\mathrm{d}\mu\right) \\ &= \lim_{k\to\infty}\left(\int_\Omega [\sigma_k\cdot\nabla u_{0,k} - f^*(.,\sigma_k)]\,\mathrm{d}x + \int_\Omega (u_{0,k})^*\,\mathrm{d}\mu\right) \end{aligned} \tag{8.2.11}$$

with $\sigma_k \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ maximizer of the dual problem in $\mathrm{L}^2(\Omega,\mathbb{R}^n)$ for datum $u_{0,k}$ and k large. We notice that $\mathrm{I} > -\infty$ induces $\inf_{\mathrm{W}^{1,1}_{u_{0,k}}(\Omega)}\int_\Omega f\left(.,\nabla w\right)\,\mathrm{d}x + \int_\Omega w^*\,\mathrm{d}\mu > -\infty$ for k large enough, thus the existence of some maximizer σ_k is guaranteed by *Step 1* whenever $k \gg 1$. Explicitly, we have proved that $\sup_{k\in\mathbb{N}}||\sigma_k||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \le c_2$, so the theorem of Banach–Alaoglu yields the existence of a vector field $\overline{\sigma} \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ which is the weak–$*$ limit in $\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ of a non–relabelled subsequence of $(\sigma_k)_k$, and we record that by lower semicontinuity of the dual norm it is still $||\overline{\sigma}||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \le c_2$.

We now argue that such $\overline{\sigma}$ achieves the equality in (8.2.9). Clearly, since $\mathrm{div}(\sigma_k) = \mu$ for all k, weak–$*$ convergence in L^∞ implies $\mathrm{div}(\overline{\sigma}) = \mu$ as measures on Ω. Furthermore, one may check in a similar flavour to the procedure of Theorem 8.26 (exploiting this time the weak–$*$ semicontinuity in $\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ for each conjugate f_k^* according to Proposition 8.19) the validity of

$$\liminf_{k\to\infty}\int_\Omega f^*(.,\sigma_k)\,\mathrm{d}x \ge \liminf_{k\to\infty}\int_\Omega f_k^*(.,\sigma_k)\,\mathrm{d}x \ge \int_\Omega f^*(.,\overline{\sigma})\,\mathrm{d}x\,, \tag{8.2.12}$$

recalling that $f_j^* \le f^*$ a.e. in Ω and for every $j\in\mathbb{N}$. The convergence

$$\lim_{k\to\infty}\int_\Omega \sigma_k\cdot\nabla u_{0,k}\,\mathrm{d}x = \int_\Omega \overline{\sigma}\cdot\nabla u_0\,\mathrm{d}x \tag{8.2.13}$$

is justified by the $\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ weak–$*$ convergence of σ_k to $\overline{\sigma}$ as a consequence of

$$\begin{aligned} \left|\int_\Omega (\sigma_k\cdot\nabla u_{0,k} - \overline{\sigma}\cdot\nabla u_0)\,\mathrm{d}x\right| &\le ||\sigma_k||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}||\nabla(u_{0,k}-u_0)||_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} + \left|\int_\Omega (\sigma_k-\overline{\sigma})\cdot\nabla u_0\,\mathrm{d}x\right| \\ &\le c_2||\nabla(u_{0,k}-u_0)||_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} + \left|\int_\Omega (\sigma_k-\overline{\sigma})\cdot\nabla u_0\,\mathrm{d}x\right| \xrightarrow[k\to\infty]{} 0\,. \end{aligned}$$

Lastly, the admissibility requirement for μ_+ and μ_- enables via Remark 3.13 the computation of the following limit

$$\limsup_{k\to\infty}\left|\int_\Omega (u_{0,k}-u_0)^*\mathrm{d}\mu\right| \le (C_+ + C_-)\limsup_{k\to\infty}\left(||\nabla(u_0-u_{0,k})||_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} + ||u_0-u_{0,k}||_{\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})}\right) = 0\,.$$

Combining the estimates (8.2.11)–(8.2.13), we complete the statement by writing

$$\mathrm{I} \le \lim_{k\to\infty}\left(\int_\Omega [\sigma_k\cdot\nabla u_{0,k} - f^*(.,\overline{\sigma_k})]\,\mathrm{d}x + \int_\Omega (u_{0,k})^*\,\mathrm{d}\mu\right) \le \int_\Omega [\overline{\sigma}\cdot\nabla u_0 - f^*(.,\overline{\sigma})]\,\mathrm{d}x + \int_\Omega u_0^*\,\mathrm{d}\mu\,,$$

and being $\overline{\sigma}$ in the right class of competitors, this latter estimate closes the proof of (8.2.9) for general data u_0 in $\mathrm{W}^{1,1}(\Omega)$. □

The final step consists of getting rid of the differentiability assumption; this is obtained by exploiting the convex conjugate function of the approximating sequence f_k considered above, namely the Moreau regularization f^k of f with parameter ε_k.

Theorem 8.29 (general convex duality formula in $\mathrm{W}^{1,1}$)**.** *We fix $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, $\mu_\pm$ admissible measures on Ω and we set $\mu := \mu_+ - \mu_-$. Then, the duality formula*

$$\inf_{w\in \mathrm{W}^{1,1}_{u_0}(\Omega)} \left(\int_\Omega f\,(.,\nabla w)\,\mathrm{d}x + \int_\Omega w^*\,\mathrm{d}\mu\right) = \sup_{\sigma\in\mathcal{DM}_\mu(\Omega,\mathbb{R}^n)} \left(\int_\Omega [\sigma\cdot\nabla u_0 - f^*(.,\sigma)]\,\mathrm{d}x + \int_\Omega u_0^*\,\mathrm{d}\mu\right) \quad (8.2.14)$$

holds for any $f\colon \mathbb{R}^n\times\mathbb{R}^n \to [0,\infty)$ Borel, convex in the second variable ξ for a.e. $x\in\mathbb{R}^n$ and under (8.1.5). Furthermore, if the common value is larger than $-\infty$, then the supremum in (8.2.14) is achieved.

We preliminarily observe that the additional assumption $\inf_{\mathrm{W}^{1,1}_{u_0}(\Omega)} \mathcal{F} > -\infty$ is guaranteed, for instance, if (H1) holds and $\mu_\pm$ satisfy the appropriate ICs with constant $C\in(0,1)$ in Ω under continuity of f (or alternatively if C is exactly 1, provided that both (H1), (H2) are in place). In fact, in such cases, we can apply the boundedness from below in Proposition 5.2 and the trivial inequality $\inf_{\mathrm{BV}}\mathcal{F} \le \inf_{\mathrm{W}^{1,1}_{u_0}} \mathrm{F}_{u_0}$.

Proof. We set the value

$$\mathrm{I} := \inf_{w\in \mathrm{W}^{1,1}_{u_0}(\Omega)} \left(\int_\Omega f\,(.,\nabla w)\,\mathrm{d}x + \int_\Omega w^*\,\mathrm{d}\mu\right) \in [-\infty,\infty)$$

and as usual we integrate by parts to have $\mathrm{I} \ge \left(\int_\Omega [\sigma\cdot\nabla u_0 - f^*(.,\sigma)]\,\mathrm{d}x + \int_\Omega u_0^*\,\mathrm{d}\mu\right)$ for all fields $\sigma\in\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ with $\mathrm{div}(\sigma)=\mu$, so we can directly address the non–trivial inequality. We introduce a decreasing sequence $\varepsilon_k \searrow 0$ for $k\to\infty$ and the Moreau envelope f^k of f with constant ε_k with respect to the second variable only, namely

$$f^k(x,\xi) := \inf_{\tau\in\mathbb{R}^n} \left(f(x,\tau) + \frac{1}{2\varepsilon_k}|\xi-\tau|^2\right) \qquad \text{for } (x,\xi)\in\Omega\times\mathbb{R}^n\,.$$

Then, each f^k still satisfies (8.1.5), since $c_1 \le \inf_{z\in\mathbb{R}^n} f(x,z) \le f^k(x,\xi) \le f(x,\xi) \le c_2(1+|\xi|)$, and Theorem 8.23 ensures convexity and differentiability of $f^k(x,.)$ for almost every point x in Ω. We exploit the previous duality result of Theorem 8.28 applied to each f^k while letting

$$\mathrm{I}^k := \inf_{w\in \mathrm{W}^{1,1}_{u_0}(\Omega)} \left(\int_\Omega f^k\,(.,\nabla w)\,\mathrm{d}x + \int_\Omega w^*\,\mathrm{d}\mu\right) \le \mathrm{I}$$

for every $k\in\mathbb{N}$ and recalling the inequality $(f^k)^* \ge f^*$, thus it is

$$\begin{aligned}\limsup_{k\to\infty} \mathrm{I}^k &= \limsup_{k\to\infty} \sup_{\sigma\in\mathcal{DM}_\mu(\Omega,\mathbb{R}^n)} \left(\int_\Omega \sigma\cdot\nabla u_0\,\mathrm{d}x + \int_\Omega u_0^*\,\mathrm{d}\mu - \int_\Omega (f^k)^*(.,\sigma)\,\mathrm{d}x\right)\\ &\le \sup_{\sigma\in\mathcal{DM}_\mu(\Omega,\mathbb{R}^n)} \left(\int_\Omega \sigma\cdot\nabla u_0\,\mathrm{d}x + \int_\Omega u_0^*\,\mathrm{d}\mu - \int_\Omega f^*(.,\sigma)\,\mathrm{d}x\right).\end{aligned}$$

If we prove that $\mathrm{I}^k \to \mathrm{I}$ as $k\to\infty$, we achieve the thesis (8.2.14). Taking into account Proposition 8.24, there exists a uniform constant $c\in(0,\infty)$ such that $f(x,\xi) \le f^k(x,\xi) + c\cdot\varepsilon_k$ for all $k\in\mathbb{N}$ and $(x,\xi)\in\Omega\times\mathbb{R}^n$. Thus, integration on Ω yields $\mathrm{I}^k \le \mathrm{I} \le \mathrm{I}^k + c\cdot\varepsilon_k|\Omega|$, and sending $k\to\infty$ we infer the existence of the limit I.

We claim now that if $\mathrm{I} > -\infty$, there exists a maximizer σ for the problem obtained by convex duality. Since the sequence of infima $(\mathrm{I}_k)_k$ converges to I, it shall be $\mathrm{I}_k > -\infty$ for $k \gg 1$ as well.

Via the result of the preceding Theorem 8.28 applied to f^k for every k large enough, this yields the existence of a dual vector field $\sigma_k \in \mathcal{DM}_\mu(\Omega, \mathbb{R}^n)$ such that

$$-\infty < \mathrm{I}^k = \sup_{\sigma \in \mathcal{DM}_\mu(\Omega,\mathbb{R}^n)} \left(\int_\Omega [\sigma \cdot \nabla u_0 - (f^k)^*(.,\sigma)]\,\mathrm{d}x + \int_\Omega u_0^*\,\mathrm{d}\mu \right)$$
$$= \int_\Omega [\sigma_k \cdot \nabla u_0 - (f^k)^*(.,\sigma_k)]\,\mathrm{d}x + \int_\Omega u_0^*\,\mathrm{d}\mu \tag{8.2.15}$$

for k large enough. We can then repeat the same strategy of Theorem 8.28 for each integrand f^k (with linear growth given by the same constants c_1, c_2 of f) to find $\sup_{k\in\mathbb{N}} ||\sigma_k||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \leq c_2$, hence up to subsequences $\sigma_k \overset{*}{\rightharpoonup} \overline{\sigma}$ weakly–$*$ in $\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ and $\operatorname{div}(\overline{\sigma}) = \mu$ on Ω. Then, the conjugate function f^* induces weak–$*$ lower semicontinuity of the corresponding functional in $\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ as in Proposition 8.19, and from this

$$\liminf_{k\to\infty} \int_\Omega (f^k)^*(.,\sigma_k)\;\mathrm{d}x \geq \liminf_{k\to\infty} \int_\Omega f^*(.,\sigma_k)\;\mathrm{d}x \geq \int_\Omega f^*(.,\overline{\sigma})\;\mathrm{d}x\,.$$

In conclusion, passing to the limit as $k \to \infty$ in (8.2.15) we deduce

$$-\infty < \mathrm{I} = \lim_{k\to\infty} \left(\int_\Omega [\sigma_k \cdot \nabla u_0 - (f^k)^*(.,\sigma_k)]\,\mathrm{d}x \right) + \int_\Omega u_0^*\,\mathrm{d}\mu \leq \int_\Omega [\overline{\sigma} \cdot \nabla u_0 - f^*(.,\overline{\sigma})]\,\mathrm{d}x + \int_\Omega u_0^*\,\mathrm{d}\mu\,.$$

This verifies the duality estimate (8.2.14) and yields simultaneously maximality of $\overline{\sigma}$ for the dual problem. □

Combining the theorems above, we have demonstrated the main duality result in BV.

Proof of Result 4. The equality between (P) and (P*) is a plain consequence of the duality expressed in Theorem 8.29, and we pass to infima in BV via the consistency equality (1.2.6) of Result 3 for mutually singular measures $\mu_\pm$. Lastly, we record that a maximum for (P*) is attained whenever the common value is not $-\infty$.

If additionally (μ_-,μ_+) verifies the f^∞–IC with constant $C \in [0,1)$ and (μ_+,μ_-) the $\widetilde{f^\infty}$–IC with constant C for on Ω, Result 2 guarantees the existence of a minimum for the primal problem, and thus Theorem 8.29 asserts existence of a maximal vector field for the dual problem, as claimed. □

8.3 Weak solutions of generalized Euler-Lagrange equations

From the duality Result 4, we can finally deduce the characterization of optimal pairs of minimizers-maximizers for the dual problems (P), (P*). Such optimality conditions will be expressed in terms of the up–to–the–boundary pairing of Section 7.2 with boundary datum our assigned $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, and with entries respective a maximizer σ of (P*) and the derivative measure $\mathrm{D}u$ of a minimizer u for the primal problem (P).

Corollary 8.30. *We assume all hypotheses of Result 4 for u_0, Ω, $\mu_\pm$, and f. Assigned a pair of functions $(u,\sigma) \in \mathrm{BV}(\Omega) \times \mathcal{DM}_\mu(\Omega,\mathbb{R}^n)$, then they are equivalent:*

(i) *The function u minimizes (P) and the function σ maximizes (P*).*

(ii) *It holds $[\![\sigma, \mathrm{D}u]\!]_{u_0} = f(.,\mathrm{D}\overline{u}^{u_0}) \llcorner \overline{\Omega} + f^*(.,\sigma)\mathcal{L}^n \llcorner \Omega$ as measures on $\mathbb{R}^n$, with the usual notation $\overline{u}^{u_0} := u\mathbb{1}_\Omega + u_0\mathbb{1}_{\mathbb{R}^n\setminus\overline{\Omega}}$.*

In particular, restricting to the interior of Ω with the help of Remark 7.17, we read that a minimizing–maximizing pair (u,σ) for (P)–(P*) satisfies the duality formula for the inner pairing

$$\llbracket\sigma,\mathrm{D}u\rrbracket = f(.,\mathrm{D}u) + f^*(.,\sigma)\mathcal{L}^n \qquad \text{as measures on } \Omega\,.$$

Furthermore, the finiteness of the measure $\llbracket\sigma,\mathrm{D}u\rrbracket_{u_0}$ on $\mathbb{R}^n$ – or equivalently, on its support $\overline{\Omega}$ – induces $f^*(.,\sigma)<\infty$ almost everywhere in Ω for any maximizer σ. In turn, this latter is equivalent via Proposition 2.89(i) to requiring the condition $||(f^\infty)^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)}\le 1$ on any maximizer σ of (P*).

Proof of Corollary 8.30. Rearranging the terms in the duality Result 4, u attains the minimum of (P) and σ the maximum of (P*) if and only if

$$\int_\Omega [\sigma\cdot\nabla u_0 - f^*(.,\sigma)]\,\mathrm{d}x = f(.,\mathrm{D}\overline{u}^{u_0})(\overline{\Omega}) - \int_\Omega (u-u_0)^+\,\mathrm{d}(\mathrm{div}(\sigma)_-) + \int_\Omega (u-u_0)^-\,\mathrm{d}(\mathrm{div}(\sigma)_+)\,,$$

and we notice that this latter implicitly yields $f^*(.,\sigma)<\infty$ Lebesgue a.e. on Ω. We now exploit Remark 7.15 to rewrite the equality as

$$\llbracket\sigma,\mathrm{D}u\rrbracket_{u_0}(\overline{\Omega}) = f(.,\mathrm{D}\overline{u}^{u_0})(\overline{\Omega}) + \int_\Omega f^*(.,\sigma)\,\mathrm{d}x\,.$$

Then, all measure components above are finite on $\overline{\Omega}$, and manipulating via subadditivity we obtain that the condition in (i) holds if and only if

$$0 = \left(\llbracket\sigma,\mathrm{D}u\rrbracket_{u_0} - f(.,\mathrm{D}\overline{u}^{u_0}) - f^*(.,\sigma)\mathcal{L}^n\right)(\overline{\Omega}) \le \sum_{B\in\mathcal{B}} \left(\llbracket\sigma,\mathrm{D}u\rrbracket_{u_0} - f(.,\mathrm{D}\overline{u}^{u_0}) - f^*(.,\sigma)\mathcal{L}^n\right)(B)$$

for every Borel partition $\mathcal{B}$ of $\overline{\Omega}$. Recalling Proposition 7.20, the right–hand side of the estimate above is always less or equal to zero on any B. Thus, equality is satisfied if and only if the measure in the last term is null, that is, whenever $\llbracket\sigma,\mathrm{D}u\rrbracket_{u_0} = f(.,\mathrm{D}\overline{u}^{u_0})\llcorner\overline{\Omega} + f^*(.,\sigma)\mathcal{L}^n\llcorner\Omega$ as Radon measures, and the corollary follows. □

Via the decomposition of our pairing $\llbracket\sigma,\mathrm{D}u\rrbracket_{u_0}$ into absolutely continuous and singular parts with respect to the Lebesgue measure, we obtain the optimality conditions in the required form of Result 5.

Proof of Result 5. Fix $u\in\mathrm{BV}(\Omega)$ and $\sigma\in\mathcal{DM}_\mu(\Omega,\mathbb{R}^n)$. Considering the up–to–the–boundary pairing $\llbracket\sigma,\mathrm{D}u\rrbracket_{u_0}$, we can first reduce to the inner pairing on Ω via Remark 7.17, and then decompose this latter into sum of an absolutely continuous $\llbracket\sigma,\mathrm{D}u\rrbracket^a$ and a singular part $\llbracket\sigma,\mathrm{D}u\rrbracket^s$ with respect to $\mathcal{L}^n$. Then, with the help of (7.1.5) and (7.2.11), we find

$$\llbracket\sigma,\mathrm{D}u\rrbracket^a_{u_0} = (\llbracket\sigma,\mathrm{D}u\rrbracket_{u_0}\llcorner\Omega)^a = (\sigma\cdot\nabla u)\mathcal{L}^n\,.$$

Moreover, taking into account the Radon–Nikodým decomposition of the measure defined by f with respect to $\mathrm{D}\overline{u}^{u_0}$, we have

$$\begin{aligned} f(.,\mathrm{D}\overline{u}^{u_0})\llcorner\overline{\Omega} &= f(.,\mathrm{D}u)^a\llcorner\Omega + f(.,\mathrm{D}\overline{u}^{u_0})^s\llcorner\overline{\Omega} \\ &= f(.,\nabla u)\mathcal{L}^n\llcorner\Omega + f^\infty(.,\mathrm{D}^s u)\llcorner\Omega + f^\infty\left(.,(u-u_0)\nu_\Omega\right)\mathcal{H}^{n-1}\llcorner\partial\Omega\,. \end{aligned}$$

Comparing the absolutely continuous and singular parts in the last two writings, with the help of Corollary 8.30, we deduce that (u,σ) is a minimizing–maximizing pair if and only if they hold

(a″) $(\sigma\cdot\nabla u)\mathcal{L}^n = (f(.,\nabla u)+f^*(.,\sigma))\mathcal{L}^n$ on Ω; and

(b) $\llbracket\sigma,\mathrm{D}u\rrbracket^s_{u_0} = f^\infty(.,\mathrm{D}^s u)\llcorner\Omega + f^\infty\left(.,(u-u_0)\nu_\Omega\right)\mathcal{H}^{n-1}\llcorner\partial\Omega$.

We observe that (a′′) determines the equality case of Fenchel's inequality for the conjugate of f in the second entry, thus by Proposition 2.94 the condition in (a′′) is equivalent to setting $\sigma \in \partial_\xi f(.,\nabla u)$ almost everywhere in Ω, and the optimality conditions are proved.

Assume now $f(x,.)$ is differentiable in $\mathbb{R}^n$ for a.e. $x \in \Omega$, meaning that the subdifferential of $f(x,.)$ is single–valued at each point and it coincides with the standard gradient. Then, if σ_1, σ_2 are two maximizers of (P*) and $u \in \mathrm{BV}(\Omega)$ is a minimum of (P), we must have $\sigma_1(x), \sigma_2(x) \in \partial_\xi f(x,\nabla u(x)) = \{\nabla_\xi f(x,\nabla u(x))\}$ for a.e. x in Ω, meaning $\sigma_1 = \sigma_2$ and the (a.e.–)uniqueness is proved. This concludes the proof of Result 5. □

Remark 8.31. We point out that setting the condition (a) obtained in Result 5 – or its equivalent (a′′) – is exactly the same as requiring $\nabla u \in \partial_{\xi^*} f^*(.,\sigma)$ almost everywhere in Ω. In fact, our admissibility conditions on f allow for an application of Remark 2.96 to the restriction $\xi \mapsto f(x,\xi)$ for almost every $x \in \Omega$.

We recall that in general the pairing $[\![\sigma, \mathrm{D}u]\!]_{u_0}$ is a signed measure for any σ, u fixed. Nevertheless, (b) yields that at least the singular part of the pairing is *non–negative* when evaluated on a pair of solutions of (P) and (P*), respectively. For the absolutely continuous part of the pairing, instead, we cannot exclude a change of sign – as the first term $f(.,\nabla u)\mathcal{L}^n$ is always non–negative by our assumptions on f, but the polar function $f^*(.,\sigma)$ is allowed to be negative on possibly large sets.

To conclude our claims expressed in Chapter 1, we are left to show well–posedness of Definition 1 of weak solutions to the generalized Euler–Lagrange equation for the functional $\mathcal{F}$ – that is, we claim that being solutions of (EL) according to Definition 1 is a necessary and sufficient condition to BV–minimality of the corresponding functional $\mathcal{F}$.

Proposition 8.32 (weak solutions are functional minimizers)**.** *Under all assumptions of Definition 1, $u \in \mathrm{BV}(\Omega)$ minimizes $\mathcal{F}$ if and only if u is a weak solution of* (EL) *with boundary value u_0.*

Proof. We assume first that u attains the minimum in (1.2.4), and argue via Result 4 the existence of some $\overline{\sigma} \in \mathcal{DM}_\mu(\Omega,\mathbb{R}^n)$ achieving the maximum of (P*). Thus, Corollary 8.30 yields the equality between measures $[\![\overline{\sigma}, \mathrm{D}u]\!]_{u_0} = f(.,\mathrm{D}\overline{u}^{u_0}) \llcorner \overline{\Omega} + f^*(.,\overline{\sigma})\mathcal{L}^n$ on $\overline{\Omega}$, so u is a weak solution of (EL) with generalized boundary datum u_0.

Assuming conversely that $u \in \mathrm{BV}(\Omega)$ is a weak solution as in Definition 1, we consider any vector field $\overline{\sigma} \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ with $\mathrm{div}(\overline{\sigma})_\pm = \mu_\pm$. Keeping in mind Remark 7.15, we compute

$$\begin{aligned}
\mathcal{F}[u] &= f(.,\mathrm{D}\overline{u}^{u_0})(\overline{\Omega}) - \int_\Omega u^+ \,\mathrm{d}\mu_- + \int_\Omega u^- \,\mathrm{d}\mu_+ \\
&= [\![\overline{\sigma}, \mathrm{D}u]\!]_{u_0}(\overline{\Omega}) - \int_\Omega f^*(.,\overline{\sigma})\,\mathrm{d}x - \int_\Omega u^+ \,\mathrm{d}\mu_- + \int_\Omega u^- \,\mathrm{d}\mu_+ \\
&= \int_\Omega [\overline{\sigma}\cdot\nabla u_0 - f^*(.,\overline{\sigma})] + \int_\Omega u_0^* \,\mathrm{d}\mu \\
&\leq \sup_{\sigma \in \mathcal{DM}_\mu(\Omega,\mathbb{R}^n)} \int_\Omega [\sigma\cdot\nabla u_0 - f^*(.,\sigma)] + \int_\Omega u_0^* \,\mathrm{d}\mu = \inf_{\mathrm{BV}(\Omega)} \mathcal{F}
\end{aligned}$$

where in the last equality we applied the duality formula of Result 4. Altogether, we deduce minimality of u for the functional $\mathcal{F}$, so that the equivalence follows. □

8.3.1 Extremality relations for area and TV problem with measure

We now consider the prototypical integrands of area $f(\xi) := \sqrt{1+|\xi|^2}$ and total variation $f(\xi) := |\xi|$ to explicit the behaviour of minimizers u of (P) and corresponding vector fields σ maximizing the dual problem (P*) in these two cases. We observe that such extremality relations have been partially established already in [44] for minimal surfaces, being later on generalized in [87, 89, 67].

Example 8.33 (optimality relations for area problem with measure)**.** The area is the standard example of a strictly convex, everywhere differentiable function $f = f(\xi)$ with gradient $\nabla f(\xi) = \xi/\sqrt{1+|\xi|^2}$ and polar function as expressed in Remark 2.90. Given $u \in \mathrm{BV}(\Omega)$, Result 5 ensures that u is a weak solution of the prescribed mean curvature measure equation

$$\operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right) = \mu \text{ in } \Omega$$

with datum $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ if and only if there is a vector field $\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ such that $\operatorname{div}(\sigma) = \mu$ in Ω, with σ uniquely defined Lebesgue almost everywhere on Ω as $\sigma = \nabla u/\sqrt{1+|\nabla u|^2}$, and generalized pairing satisfying

$$[\![\sigma, \mathrm{D}u]\!]^s = |\mathrm{D}^s u| \text{ on } \Omega \quad \text{ and } \quad (u-u_0)\sigma_n^* = |u-u_0| \ \mathcal{H}^{n-1}\text{–a.e. on } \partial\Omega \tag{8.3.1}$$

for the normal trace $\sigma_n^* \in \mathrm{L}^\infty(\partial\Omega;\mathcal{H}^{n-1})$ of σ obtained in Proposition 7.13. Notice that the differentiability of the integrand (and hence, strict convexity of its polar) reflects on the $\mathcal{L}^n$–a.e. uniqueness of the associated vector field σ. Specific to this case is the geometric meaning of σ as projection on the $\mathbb{R}^n$ plane of the normal vector $\nu_u := (\nabla u, -1)/\sqrt{1+|\nabla u|^2} \in \mathbb{R}^n \times \mathbb{R}$ to the surface graph of u on its regular points. Furthermore, we observe that $\sigma \cdot \nabla u \geq 0$ almost everywhere, and $|\sigma| \nearrow 1$ whenever $|\nabla u|$ approaches infinity, whereas $||\sigma||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} < 1$ for solutions u with gradient bounded everywhere (compare with the following Proposition 8.38). In any case, in order not to violate $\nabla u \in \mathrm{L}^1(\Omega,\mathbb{R}^n)$, the set of points on which σ is unitary has to be negligible in Ω – that is, $\mathcal{L}^n(\{x \in \Omega : |\sigma(x)| = 1\}) = 0$. Regarding the normal trace σ_n^*, we recall that by Proposition 7.13(ii) applied to the area functional we also have $||\sigma_n^*||_{\mathrm{L}^\infty(\partial\Omega;\mathcal{H}^{n-1})} \leq ||\sigma||_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \leq 1$, thus, taking into account the optimality condition on the boundary, we achieve $|u-u_0| = (u-u_0)\sigma_n^* \leq |\sigma_n^*| \cdot |u-u_0| \leq |u-u_0| \ \mathcal{H}^{n-1}$–a.e. on $\partial\Omega$. Altogether, on the portion of $\partial\Omega$ where u does not agree with the value u_0, the normal trace σ_n^* of σ takes only the values ± 1 up to $\mathcal{H}^{n-1}$–negligible sets, depending on the respective sign of $u - u_0$, namely:

- $\sigma_n^* \equiv 1 \ \mathcal{H}^{n-1}$–a.e. on $\{x \in \partial\Omega : u(x) > u_0(x)\}$;
- $\sigma_n^* \equiv -1 \ \mathcal{H}^{n-1}$–a.e. on $\{x \in \partial\Omega : u(x) < u_0(x)\}$; and
- $\sigma_n^*(x) \in [-1,1]$ on $\mathcal{H}^{n-1}$–a.e. point $x \in \partial\Omega$ where $u(x) = u_0(x)$.

Then, if $v \in \mathrm{BV}(\Omega)$ is any other solution of (PMC), we shall see in Proposition 8.35 that strict convexity induces $\nabla v = \nabla u$ a.e. in Ω, and the optimality condition on the boundary applied to v yields $|v-u_0|\mathcal{H}^{n-1} \llcorner \partial\Omega = [\![\sigma, \mathrm{D}v]\!]^s_{u_0} \llcorner \partial\Omega = (v-u_0)\sigma_n^*\mathcal{H}^{n-1} \llcorner \partial\Omega = (v-u_0)\mathcal{H}^{n-1} \llcorner (\partial\Omega \cap \{u > u_0\}) + (u_0 - v)\mathcal{H}^{n-1} \llcorner (\partial\Omega \cap \{u < u_0\}) + (v-u)\mathcal{H}^{n-1} \llcorner (\partial\Omega \cap \{u = u_0\})$. From this latter, repeating the same analysis with the roles of u and v inverted, we deduce the $\mathcal{H}^{n-1}$–a.e. coincidence $\{u \geq u_0\} = \{v \geq u_0\}$ on $\partial\Omega$, meaning that for $\mathcal{H}^{n-1}$–almost every $x \in \partial\Omega$ all minimizers of the area integrand have trace lying either entirely above or entirely below the trace $u_0(x)$, with equality potentially admitted.

We also observe that the $\mathcal{L}^n$–a.e. vinculum on its divergence $\operatorname{div}(\nabla u/\sqrt{1+|\nabla u|^2}) = \operatorname{div}(\nu_u) = \mu$ is to be meant as generalized formulation of the prescribed mean curvature of the graph of u, to which we reduce for an absolutely continue measure μ and regular u. Further properties of u in relation to σ are described in the next Section 8.5.

Focusing on the unidimensional case, the condition $\operatorname{div}(\sigma) = \mu$ is satisfied by any $\sigma \in \mathcal{DM}(\Omega) = \mathrm{BV}(\Omega)$ such that $\sigma(x) = \mu((-\infty, x) \cap \Omega) + c$, for some $c \in \mathbb{R}$ and all $x \in \Omega$. From the a.e. vinculum $\sigma(x) = u'(x)/\sqrt{1+u'(x)^2}$, we read out that σ is oriented as the (absolutely continuous part of the) derivative of u – meaning $\sigma \geq 0$ where u is increasing and vice versa – thus if u increases in a neighborhood of a point $x_0 \in \partial\Omega$, the trace fulfils $\sigma_n^*(x_0) \geq 0$, hence $u(x_0) \geq u_0(x_0)$. Notably, the necessary conditions on the trace above yield that on almost every boundary point $x_0 \in \partial\Omega$ such that

$u(x_0) \neq u_0(x_0)$ we necessarily have $|\sigma_n^*(x_0)| = 1$ and $|u'(x_k)| \to \infty$ for $(x_k)_k \in \Omega$ converging to x_0, so the absolutely continuous part of the gradient of solutions blows up in proximity to the boundary – with appropriate sign consistent with $\operatorname{sgn}(u(x_0) - u_0(x_0))$. Another peculiarity of the prescribed mean curvature problem in dimension one is expressed in [67, Theorem 8.1] as a consequence of the explicit (and unique up to translations) solution σ of $\operatorname{div}(\sigma) = \mu$ in $\mathbb{R}$. In fact, if the domain is a bounded interval $\Omega := (a, b)$ and μ is an admissible measure non–negative in some Borel set $U \subseteq \Omega$, then the relation $\sigma(x) = \mu((a, x)) + c$ implies that σ and u' are non–decreasing in U, thus the extremality relations yield $\mathrm{D}^s u \equiv 0$ in U. So, all minimizers u belong to $\mathrm{W}^{1,1}(U)$ and are here convex (therefore, local Lipschitz continuous). Obviously, the opposite assumption $\mu \geq 0$ in U would imply concavity of u instead.

Example 8.34 (optimality relations for TV problem with measure)**.** Minimizers for the total variation with measure μ enjoy, as expected, worse regularity properties in comparison to the area case, in consequence of the loss of differentiability and strict convexity – and clearly the same applies to any anisotropic variant $f(\xi) = \varphi(\xi)$. Indeed, if we consider the standard isotropic integrand $f(\xi) := |\xi|$, the heuristic Euler–Lagrange equation (EL) reads out

$$\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right) = \mu \quad \text{in } \Omega\,,$$

and our duality theory yields that, provided a solution $u \in \mathrm{BV}(\Omega)$ of (P) exists, we cannot expect uniqueness of maximizers, since every vector field $\sigma \in \mathcal{DM}_\mu(\Omega, \mathbb{R}^n)$ such that σ is a.e. defined by $\nabla u / |\nabla u|$ on $\{\nabla u \neq 0\}$ and $|\sigma| \leq 1$ a.e. on Ω (in particular, a.e. on $\{\nabla u = 0\}$) is admissible. Geometrically, we are now considering normal vectors σ to level sets $\{u = t\}$ for any $t \in \mathbb{R}$, wherever these are defined. Notice that for the singular part of $[\![\sigma, \mathrm{D}u]\!]_{u_0}$ the same conditions (8.3.1) of area hold, as the two integrands share the same recession function.

Specifically, in dimension $n = 1$, the equality $\sigma(x)u'(x) = |u'(x)|$ determines $\sigma > 0$ for u strictly increasing and vice versa, while any direction of σ is allowed on the points where the absolutely continuous part of the derivative vanishes. Moreover, even if the measure has a constant sign, we cannot conclude that minimizers of the total variation functional with measure are continuous, nor can we exclude the presence of a Cantor part in their derivative; compare with the area case above. The reason lies in the fact that the set $\{x \in \Omega : |\sigma(x)| = 1\}$ is here always non–negligible, unless u is piecewise constant in Ω.

8.4 An excursus on the uniqueness of dual solutions

We have seen that Result 5 implies uniqueness of the dual vector field σ solution of (P*) for integrands $f = f(x, \xi) \colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ satisfying Assumption 1.1(H1)–(H2) and additionally supposing

(U1) $\xi \mapsto f(x, \xi)$ is differentiable in $\mathbb{R}^n$ for a.e. $x \in \Omega$.

We find such an assumption (U1) convenient to verify, as it only involves the primal function f. Nevertheless, for the purpose of uniqueness of σ, it would suffice to impose the hypothesis

(U2) $\xi \mapsto f^*(x, \xi)$ is strictly convex in $\operatorname{dom}(f^*(x.,))$ for a.e. $x \in \Omega$

in place of (U1). In fact, supposing that (U2) holds and given σ_1, $\sigma_2 \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n)$ with same divergence $\operatorname{div}(\sigma_1) = \mu = \operatorname{div}(\sigma_2)$, with both vector fields achieving the maximum M of (P*), we introduce the set $\Sigma := \{x \in \Omega : \sigma_1(x) \neq \sigma_2(x)\}$ and the combination $\sigma := (\sigma_1 + \sigma_2)/2$, which surely preserves the condition $\sigma \in \mathcal{DM}_\mu(\Omega, \mathbb{R}^n)$. Moreover, since σ_1, σ_2 maximize (P*), it shall be $\sigma_1(x)$, $\sigma_2(x) \in$

$\mathrm{dom}(f^*(x.,))$ for a.e. $x \in \Omega$. Then, recalling $\sigma_1 = \sigma_2 = \sigma$ a.e. on $\Omega \setminus \Sigma$, supposed $\mathcal{L}^n(\Sigma) > 0$ we compute

$$
\begin{aligned}
M - \int_\Omega u_0^* \,\mathrm{d}\mu &\geq \int_\Omega \sigma \cdot \nabla u_0 - \int_{(\Omega\setminus\Sigma)\,\cup\,\Sigma} f^*(.,\sigma) \\
&> \frac{1}{2}\left[\int_\Omega \sigma_1 \cdot \nabla u_0 - \int_\Sigma f^*(.,\sigma_1) - \int_{\Omega\setminus\Sigma} f^*(.,\sigma)\right] \\
&\quad + \frac{1}{2}\left[\int_\Omega \sigma_2 \cdot \nabla u_0 - \int_\Sigma f^*(.,\sigma_2) - \int_{\Omega\setminus\Sigma} f^*(.,\sigma)\right] = M - \int_\Omega u_0^* \,\mathrm{d}\mu
\end{aligned}
$$

by linearity of the first term and strict convexity of the conjugate function in the second entry. The contradiction determines that Σ is Lebesgue negligible, therefore, the uniqueness of the dual vector field is achieved. Observe that this procedure does not impose further assumptions on μ, and indeed (U2) is often assumed in the framework of classical convex duality – that is, for convex variational integrals without measures, see e.g. [44, Theorem 3.1, Chapter V]. Nevertheless, it might not be straightforward to verify an assumption on the dual function, such as (U2), and our preference falls rather on (U1) – as previously illustrated.

We recall that a standard condition leading to uniqueness common in the literature, is the stronger

(U3) $\xi \mapsto f(x,\xi)$ is differentiable and strictly convex in $\mathbb{R}^n$ for a.e. $x \in \Omega$,

where again we want to impose Assumption 1.1(H1) and continuity of f (and clearly (U3) implies our condition (U1)). This is for instance the setting of Bildhauer and Fuchs [19, 17] and [18, Chapter 2], however there only limited to x–independent integrands $f = f(\xi)$. In detail, in [19] the authors assume an additional order of continuous differentiability and a specific bound on the Hessian of f to prove existence of a dual solution as weak limit for $\varepsilon \to 0$ of ε–coercive approximations of the dual problem – as a side note, we record that the strategy to achieve such solutions is similar to our proof of Theorem 8.26. In second place, uniqueness of the weak limit σ is proved in the follow–up paper [17] exploiting that:

- Differentiability of $f(x,.)$ yields strict convexity of $f^*(x,.)$ on every convex set in $\mathrm{Im}(\partial_\xi f(x,.))$, applying Proposition 8.7 to $g_x(\xi) := f(x,\xi)$ for every $x \in \Omega$;
- Strict convexity of $f(x,.)$ determines convexity of the whole $\mathrm{Im}(\partial_\xi f(x,.))$; compare with Proposition 8.6.

Coming back to our assumed (U1), we record that Proposition 8.7 induces strict convexity of $\xi^* \mapsto f^*(x,\xi^*)$ on every convex domain in $\mathrm{Im}(\partial_\xi f(x,.))$. Moreover, it can be proved that for our lower semicontinuous function $\xi \mapsto f(x,\xi)$ the set $E_x := \mathrm{Int}(\mathrm{Im}(\partial_\xi f(x,.)))$ is convex in $\mathbb{R}^n$ for a.e. x in Ω – see e.g. [15, Proposition 3.7] – hence (U1) yields strict convexity of $f^*(x,.)$ in E_x. Thus, if the assigned integrand f is such that all dual solutions $\sigma \in \mathcal{DM}_\mu(\Omega,\mathbb{R}^n)$ satisfy $\sigma(x) \in E_x$ for a.e. $x \in \Omega$, uniqueness of maximizers follows from strict convexity by assuming that σ_1, σ_2 are solutions of (P*) (and from Result 5 both $\sigma_1(x)$, $\sigma_2(x)$ belong to the convex set $\mathrm{Im}(\partial_\xi f(x,.))$ for a.e. $x \in \Omega$). Reprising then the exact same reasoning above for the linear combination $\sigma := (\sigma_1 + \sigma_2)/2$, we can again conclude that $\sigma(x) \in \mathrm{Im}(\partial_\xi f(x,.))$, and thus strict convexity of $f^*(x,\sigma(x))$ forces uniqueness.

8.5 Further properties of weak solutions and extremal divergence-measure vector fields

From the extremality relations obtained in Section 8.3, we are able to extrapolate information on the regularity of BV–minimizers for problem (P) in relation to different assumptions on the associated dual solutions of (P*) and vice versa. Our first observation is a straightforward consequence of the upper bound for the pairing in terms of polar functions, expressing a first connection between regularity of solutions and the L^∞–norm of corresponding vector fields. Explicitly, a notable consequence of the strict convexity of the integrand is that the absolutely continuous part of the derivative of minimizers is uniquely determined.

Proposition 8.35 (uniqueness of gradient for strictly convex integrands)**.** *For bounded open Lipschitz Ω in $\mathbb{R}^n$ and any $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, we assume that $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ is continuous under Assumption 1.1(H1) and that $\mu_\pm$ are mutually singular, admissible measures on Ω. We consider any two minimizers $u,\, v \in \mathrm{BV}(\Omega)$ of (P). Then, if $f(x,.)$ is strictly convex in $\mathbb{R}^n$ for all $x \in \Omega$, it holds $\nabla u = \nabla v$ almost everywhere in Ω.*

Proof. Fixed $x \in \Omega$, we argue via strict convexity and Proposition 8.6 that $f^*(x,.) \in \mathrm{C}^1(\mathrm{Im}(\partial_\xi f(x,.)))$, in particular $\partial_{\xi^*} f^*(x,\xi^*) = \{\nabla_{\xi^*} f^*(x,\xi^*)\}$ for all $\xi^* \in \mathrm{Im}(\partial_\xi f(x,.))$. The presence of minimizers for (P) yields by Results 4 and 5 the existence of $\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$, $\mathrm{div}(\sigma) = \mu_+ - \mu_-$ in Ω and with $\sigma(x) \in \mathrm{Im}(\partial_\xi f(x,.))$ for a.e. $x \in \Omega$. Hence, from the minimality of both u and v, the optimality relation (a′) implies $\nabla u(x), \nabla v(x) \in \partial_{\xi^*} f^*(x,\sigma(x)) = \{\nabla_{\xi^*} f^*(x,\sigma(x))\}$ for almost every $x \in \Omega$, yielding a.e.–uniqueness of the gradients. □

The result of Proposition 8.35 is achieved straightforwardly in the standard case of strictly convex integrands with linear growth and $\mu \equiv 0$ – even without involving the dual formulation, via strict convexity of the integrand $w \mapsto \int_\Omega f(.,\nabla w)\,\mathrm{d}x$ on $\mathrm{BV}(\Omega)$. Nevertheless, once we allow the presence of arbitrary measures $\mu_\pm$ under some IC, the direct reasoning to achieve uniqueness of the gradients fails since the measure term $w \mapsto \int_\Omega w_\mu^\lambda\,\mathrm{d}\mu := \int_\Omega w^-\,\mathrm{d}\mu_+ - \int_\Omega w^+\,\mathrm{d}\mu_-$ is *not* convex in $\mathrm{BV}(\Omega)$ (unless μ is of dimension strictly larger than $n-1$).

We additionally point out that in general – that is, for f with linear growth and $\mu \not\equiv 0$ – we cannot exclude the presence of a singular part of the derivative of minimizers, even under continuity assumptions on σ, as there could be verticalizations of the gradient which do not contradict continuity of the field. Nevertheless, by setting appropriate additional assumptions on the dual field – for example, σ not reaching the boundary of the unit ball with respect to the polar function of f^∞ – we can achieve Sobolev solutions of the primal problem (see Proposition 8.36), or Lipschitz solutions (in Proposition 8.38). Analogous results for divergence–free vector fields can be found in [15], where the authors assume continuity of σ and $\sigma(x) \in \mathrm{Int}(\mathrm{Im}(\partial_\xi f(x,.)))$. In connection to this, we recall that $\mathrm{Int}(\mathrm{Im}(\partial_\xi f(x,.)))$ can be replaced with $\mathrm{Im}(\partial_\xi f(x,.))$ in case f is strictly convex in the second variable; then, joining together Propositions 2.89(i) and 2.94, one has

$$\begin{aligned}\sigma(x) \in \mathrm{Int}(\mathrm{Im}(\partial_\xi f(x,.))) \subseteq \mathrm{Int}(\mathrm{dom}(f^*(x,.))) &\subseteq \mathrm{Int}(\{\xi \in \mathbb{R}^n : (f^\infty)^\circ(x,\xi) \leq 1\}) \\ &= \{\xi \in \mathbb{R}^n : (f^\infty)^\circ(x,\xi) < 1\}\,,\end{aligned}$$

and the connection with our assumption becomes evident. Nevertheless, in presence of a measure term as in our treatment, we don't expect the validity of the (in)equalities in [15, Theorem 5.2, Lemma 5.6] to be preserved, and since these latter play there a fundamental role to get regularity results in [15, Section 2.4], we rather privilege a different approach.

Proposition 8.36 ($\mathrm{W}^{1,1}$ regularity)**.** *Given $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ and a bounded Lipschitz Ω in $\mathbb{R}^n$, we consider the minimization problem (P) with continuous integrand f as in Assumption 1.1(H1) and with mutually singular, admissible measures $\mu_\pm$ on Ω. If $\sigma \in \mathcal{DM}_\mu(\Omega,\mathbb{R}^n)$ solves the dual problem (P*) with $\|(f^\infty)^\circ(.,\sigma)\|_{\mathrm{L}^\infty(\Omega)} < 1$, then any minimizer u of (P) is in $\mathrm{W}^{1,1}_{u_0}(\Omega)$. Furthermore, assuming strict convexity of $f(x.,)$ for all x in connected Ω, the solution of (P) is unique up to additive constants – that is, any minimizer $v \in \mathrm{BV}(\Omega)$ is such that $v = u + c$ $\mathcal{L}^n$–a.e. in Ω for some $c \in \mathbb{R}$.*

Proof. Let $u \in \mathrm{BV}(\Omega)$ be a minimizer of (P). Employing Proposition 7.18 for the recession function, we write

$$[\![\sigma, \mathrm{D}u]\!]_{u_0} \leq ||(f^\infty)^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)} f^\infty(., \mathrm{D}\overline{u}^{u_0}) \llcorner \overline{\Omega}\,,$$

thus, the restriction to the singular part of both measures the extremality relation (b) delivers

$$f^\infty(., \mathrm{D}^s\overline{u}^{u_0}) \llcorner \overline{\Omega} = [\![\sigma, \mathrm{D}u]\!]^s_{u_0} \leq ||(f^\infty)^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)} f^\infty(., \mathrm{D}^s\overline{u}^{u_0}) \llcorner \overline{\Omega} < f^\infty(., \mathrm{D}^s\overline{u}^{u_0}) \llcorner \overline{\Omega}$$

whenever the measure $f^\infty(., \mathrm{D}^s\overline{u}^{u_0})$ is not identically zero on $\overline{\Omega}$. Recalling that the linear–growth assumption determines $\alpha|\mathrm{D}^s\overline{u}^{u_0}| \leq f^\infty(., \mathrm{D}^s\overline{u}^{u_0}) \leq \beta|\mathrm{D}^s\overline{u}^{u_0}|$ as measures, to avoid a contradiction it shall hold $\mathrm{D}^s u \equiv 0$ in Ω, as well as the $\mathcal{H}^{n-1}$–a.e. equality $u - u_0 = 0$ for the traces on $\partial\Omega$, and the claimed regularity follows.

If additionally $f(x, .)$ is everywhere strictly convex, from of Proposition 8.35 we know that any two minimizers u, v have the same absolutely continuous part of the derivative measure, and by u, $v \in \mathrm{W}^{1,1}(\Omega)$ and the constancy theorem we conclude that $v - u$ is almost everywhere constant. □

The last result for functional minimizers u can be further improved to Lipschitz continuity if we have stronger information on the essential bound of the polar of f^∞ evaluated on a solution of the dual problem. In detail, the behaviour of f^* on such a solution determines whether the gradient of u might blow up to infinity or not. We first recall the following standard result for the Sobolev space $\mathrm{W}^{1,\infty}$.

Theorem 8.37 (characterization of $\mathrm{W}^{1,\infty}$)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary and a function $w \colon \Omega \to \mathbb{R}$. Then w is Lipschitz continuous on Ω if and only if $w \in \mathrm{W}^{1,\infty}(\Omega)$.*

The equivalence above is meant for the continuous representative of w. A proof of Theorem 8.37 is contained, for instance, in [45, Theorem 4, Section 5.8.2], where the characterization is first proved for $\Omega = \mathbb{R}^n$ under compact support of w, and then the result is generalized to regular bounded domains via the Sobolev extension operator. Notice that Theorem 4 in [45] is stated for C^1 domains just for consistency with the extension result [45, Theorem 1, Section 5.4], but actually both theorems are preserved under the sole Lipschitz regularity of the boundary.

Proposition 8.38 (sufficient condition to Lipschitzianity of minimizers)**.** *For Ω bounded Lipschitz and connected in $\mathbb{R}^n$, we consider a function $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ and we let $f \colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be a continuous admissible integrand according to Assumption 1.1(H1)–(H2). Furthermore, we consider mutually singular, admissible measures $\mu_\pm$ on Ω, and a divergence–measure field $\sigma \in \mathcal{DM}_\mu(\Omega, \mathbb{R}^n)$ maximizing (P*). If $||(f^\infty)^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)} < 1$, then any minimizer u of $\mathcal{F}$ is Lipschitz continuous in Ω with $\mathcal{H}^{n-1}$–a.e. equality of the traces $u = u_0$ on $\partial\Omega$. Assume that $L \in (0, 1)$ is such that $||(f^\infty)^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)} \leq L$. Then, in case it holds $Lf^*(.,\sigma/L) = f^*(.,\sigma)$ almost everywhere on Ω and u_0 is constant on $\partial\Omega$, all solutions u are even constant up to the boundary of Ω. Otherwise, no minimizer u exists.*

Before turning to the proof, we record that the claimed Lipschitzianity of u implies uniform continuity on Ω, and thus it is possible to extend u to $\overline{\Omega}$ by continuity. Then, if the claimed equality for the traces holds, we conclude that u_0 shall be continuous on $\partial\Omega$.

Moreover, if for the admissible measures $\mu_\pm$ we impose our usual hypotheses of f^∞–IC for (μ_-, μ_+) and $\widetilde{f^\infty}$–IC for (μ_+, μ_-) in Ω, both with constant $L \in [0, 1)$ such that $||(f^\infty)^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)} \leq L$, then Result 2 guarantees the existence of minima for $\mathcal{F}$. At the same time, we record that the presence of the ICs for mutually singular component measures is equivalent via Theorem 3.16 to the statement:

$$\text{it exists } \tau \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n)\,, \quad \mathrm{div}(\tau) = \mu\,, \quad \text{and} \quad (f^\infty)^\circ(x, \tau(x)) \leq L \ \text{ for a.e. } x \in \Omega\,.$$

Nevertheless, we carefully observe that the vector field $\tau \in \mathcal{DM}_\mu(\Omega, \mathbb{R}^n)$ above does not represent in general a maximizer σ of the dual problem – for which we do know $(f^\infty)^\circ(.,\sigma) \leq 1$ to hold almost everywhere, but not necessarily $(f^\infty)^\circ(x, \tau(x)) \leq L$.

Proof of Proposition 8.38. Without loss of generality, we may suppose that the constant M in Assumption (H2) is larger or equal than zero, since if $f \geq f^\infty - M$ holds for some negative M, we can always replace it with 0. Assuming that the pair $(u, \sigma) \in \mathrm{BV}(\Omega) \times \mathcal{DM}_\mu(\Omega, \mathbb{R}^n)$ is minimizing–maximizing for our problems in duality, Corollary 8.30 establishes $[\![\sigma, \mathrm{D}u]\!]_{u_0} = f(., \mathrm{D}\overline{u}^{u_0}) + f^*(., \sigma)\mathcal{L}^n$ as measures on $\overline{\Omega}$, and therefore it must be $f^*(., \sigma) < \infty$ almost everywhere – in detail via Proposition 2.89 even $(f^\infty)^\circ(., \sigma) \leq 1$ a.e. on Ω. We fix $L \in (0, 1)$ such that $||(f^\infty)^\circ(., \sigma)||_{\mathrm{L}^\infty(\Omega)} \leq L$. From the upper bound in Proposition 7.20 and positive homogeneity of the pairing in the first entry, we can write

$$\big(f(., \mathrm{D}\overline{u}^{u_0}) + f^*(., \sigma)\mathcal{L}^n\big)/L = [\![\sigma, \mathrm{D}u]\!]_{u_0}/L \leq f(., \mathrm{D}\overline{u}^{u_0}) + f^*(., \sigma/L)\mathcal{L}^n \quad \text{as measures on } \overline{\Omega}\,,$$

hence rearranging the terms

$$f(., \mathrm{D}\overline{u}^{u_0}) \llcorner \overline{\Omega} \leq \frac{Lf^*(., \sigma/L) - f^*(., \sigma)}{1 - L} \mathcal{L}^n \llcorner \Omega \tag{8.5.1}$$

for any $L \in (0, 1)$. Denoting by ∇u the absolutely continuous part of $\mathrm{D}u$, we apply the estimate $f^*(., \xi^*) \geq \xi^* \cdot \nabla u - f(., \nabla u)$ a.e. in Ω for all $\xi^* \in \mathbb{R}^n$, with equality achieved by $\xi^* = \sigma$ in view of the optimality condition in the form of (a$''$). It follows that

$$\begin{aligned} \mathcal{M}_L := \frac{Lf^*(., \sigma/L) - f^*(., \sigma)}{1 - L} &\geq \frac{1}{1 - L} \left(L\sigma \cdot \nabla u/L - Lf(., \nabla u) - \sigma \cdot \nabla u + f(., \nabla u)\right) \\ &= f(., \nabla u) \geq \alpha |\nabla u|\,, \end{aligned}$$

holding almost everywhere on Ω, hence the measure $\mathcal{M}_L \mathcal{L}^n \llcorner \Omega$ is absolutely continuous and non–negative on Ω. Moreover, under the assumption $||(f^\infty)^\circ(., \sigma)||_{\mathrm{L}^\infty(\Omega)} \leq L$, the measure $\mathcal{M}_L \mathcal{L}^n$ is even Radon on Ω, since from $(f^\infty)^\circ(., \sigma) \leq L < 1$ we deduce via Proposition 2.89(iii) and the 1–homogeneity of the polar function both $f^*(., \sigma/L) \leq M$ and $f^*(., \sigma) \geq -f(., 0) \geq -\beta$. Therefore, we have

$$\mathcal{M}_L \leq \frac{1}{1 - L} \sup_{\substack{\tau \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n) \\ (f^\infty)^\circ(., \tau) \leq L}} \left(Lf^*(., \tau/L) - f^*(., \tau)\right) \leq \frac{LM + \beta}{1 - L} \in \mathrm{L}^1(\Omega)\,.$$

In conclusion, from (8.5.1) we obtain $\alpha |\mathrm{D}\overline{u}^{u_0}| \leq f(., \mathrm{D}\overline{u}^{u_0}) \leq \mathcal{M}_L \mathcal{L}^n$ as inequality between finite measures on $\overline{\Omega}$.

We now distinguish the following cases:

- If $\mathcal{M}_L \mathcal{L}^n \equiv 0$ on Ω, the last estimate implies $\mathrm{D}\overline{u}^{u_0} \equiv 0$ on $\overline{\Omega}$, hence u is constant in Ω with $u = u_0$ as traces on $\partial\Omega$, which is only admissible if the datum u_0 was constant on the whole boundary.

- If instead the measure $\mathcal{M}_L \mathcal{L}^n$ is non–negative but not everywhere zero on Ω, passing to the Lebesgue decomposition the chain of inequalities above yields $\mathrm{D}^s \overline{u}^{u_0} \equiv 0$ on $\overline{\Omega}$ (in particular, $u = u_0$ as traces on $\partial\Omega$) and thus $\mathrm{D}u = \nabla u \mathcal{L}^n$ on Ω. At the same time, for the absolutely continuous part of the gradient, it shall hold

$$|\nabla u| \leq \mathcal{M}_L/\alpha \leq \frac{LM + \beta}{\alpha(1 - L)} < \infty \quad \text{a.e. on } \Omega\,.$$

  Exploiting Theorem 8.37, we conclude that u has a Lipschitz continuous representative on Ω as claimed.

□

We point out that in general we cannot rule out the case $\mathcal{M}_L\mathcal{L}^n \equiv 0$ in Ω, as the values of $\mathcal{M}_L$ vary depending on our choice of f and σ. However, in the *homogeneous* case $f = f^\infty$ it is for sure $\mathcal{M}_L = 0$ almost everywhere on Ω for every $L \in (0,1)$, since Proposition 2.89(ii) applied to the anisotropy f determines $f^*(.,\sigma/L) = f^*(.,\sigma) \equiv 0$ a.e. on Ω for any $\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ satisfying $f^\circ(.,\sigma/L) \le 1$ – and this implies a fortiori even $f^\circ(.,\sigma) < 1$. Altogether, we have reached the following conclusion: If a dual solution $\sigma \in \mathcal{DM}_\mu(\Omega,\mathbb{R}^n)$ under homogeneous integrand satisfies $||(f^\infty)^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)} \le L$ for some $L \in (0,1)$, then only constant solutions of the primal problem are allowed. Conversely, the necessary condition $||(f^\infty)^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)} \le 1$ determines that in presence of a non–constant minimizer of (P) for $f = f^\infty$ we have precisely $||(f^\infty)^\circ(.,\sigma)||_{\mathrm{L}^\infty(\Omega)} = 1$. In fact, one can even prove a local formulation of the last result, namely in the homogeneous case we cannot expect to find maximizers of the dual problem with L^∞ norm of the polar *strictly smaller* than 1 on any subset of Ω with positive measure, unless the associated solution of the primal problem is here constant.

Proposition 8.39. *On open bounded Lipschitz $\Omega \subseteq \mathbb{R}^n$, we take $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, a continuous integrand $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ positively 1–homogeneous and convex in ξ with linear growth for constants $0 < \alpha \le \beta < \infty$. Let $\mu_\pm$ be admissible measures on Ω with $\mu_+ \perp \mu_-$. If (u,σ) is a minimizing–maximizing pair of solutions for (P) and (P*) respectively, then $\sigma \in \mathcal{DM}_\mu(\Omega,\mathbb{R}^n)$ is such that $f^\circ(.,\sigma) = 1$ almost everywhere outside the critical set $K := \{x \in \Omega : \nabla u(x) = 0\}$.*

Proof. For u, σ as in the statement, the optimality condition in Corollary 8.30 and finiteness of the up–to–the–boundary pairing determine $f^*(.,\sigma) < \infty$ a.e. on Ω, so necessarily $f^\circ(.,\sigma) \le 1$ a.e. in Ω (via Proposition 2.89(i)). Then, we assume $|\Omega \setminus K| > 0$, $L \in (0,1]$, and $f^\circ(.,\sigma) = L$ on a measurable set S of positive measure on the complement of K in Ω. Recalling that as usual homogeneity of f implies $f^*(.,\sigma/L) = 0$ a.e. on S and $f^*(.,\sigma) \equiv 0$ even a.e. on Ω, via Proposition 7.20 and again Corollary 8.30 it is

$$f(.,\mathrm{D}\overline{u}^{u_0})/L = [\![\sigma,\mathrm{D}u]\!]_{u_0}/L = [\![\sigma/L,\mathrm{D}u]\!]_{u_0} \le f(.,\mathrm{D}\overline{u}^{u_0}) + f^*(.,\sigma/L)\mathcal{L}^n = f(.,\mathrm{D}\overline{u}^{u_0}) \text{ on } S\,,$$

and since in particular $S \cap \partial\Omega = \emptyset$, the linear–growth assumption determines

$$0 \le \alpha(1-L)|\mathrm{D}u| \le (1-L)f(.,\mathrm{D}u) \le 0 \qquad \text{as measures on } S\,,$$

and restricting to the absolutely continuous part we find $(1-L)\int_S |\nabla u| = 0$. However, since $\nabla u \ne 0$ a.e. on S, to reach equality the only possibility is having $L = 1$, and the arbitrariness of S implies $f^\circ(.,\sigma) = 1$ a.e. on $\Omega \setminus K$ as claimed. □

We now get a uniqueness result for solutions of the primal problem – provided the IC is satisfied with a constant strictly smaller than 1. If such an assumption is weakened to the only limit IC in Ω, we expect a loss of uniqueness, since solutions allow to be shifted vertically on Ω (away from the boundary to satisfy the trace requirement) while still preserving the minimal value of $\mathcal{F}$.

Proposition 8.40 (uniqueness of Sobolev minimizers for strictly convex integrands)**.** *We assume the domain Ω to be connected, $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ and $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ continuous, strictly convex in the second variable and such that Assumption 1.1(H1) is in place. Furthermore, we consider mutually singular, admissible measures $\mu_\pm$ on Ω such that (μ_-,μ_+) satisfies the f^∞–IC in Ω and (μ_+,μ_-) the $\widetilde{f^\infty}$–IC in Ω both with constant $C \in [0,1)$. If $u \in \mathrm{W}^{1,1}(\Omega)$ achieves the minimum of (P) and $v \in \mathrm{BV}(\Omega)$ is another minimizer such that $u = v$ holds $\mathcal{H}^{n-1}$–a.e. on $\partial\Omega$, then $v = u$ holds $\mathcal{L}^n$–a.e. on Ω.*

Proof. By Proposition 8.35 and strict convexity of the integrand, it is $\nabla u = \nabla v$ almost everywhere on Ω, thus $v(x) = u(x) + c(x)$ for a.e. $x \in \Omega$ for some $c \in \mathrm{BV}(\Omega)$ such that $\mathrm{D}^a c \equiv 0$, $\mathrm{D}^s c \equiv \mathrm{D}^s v$

as measures in Ω. Moreover, the equality of traces determines $\mathrm{Tr}(c) = 0$ $\mathcal{H}^{n-1}$–a.e. on $\partial\Omega$. The minimality of u and v, combined with the $\mathcal{H}^{n-1}$–a.e. equality $v^{\pm} = u^* + c^{\pm}$, yields

$$
\begin{aligned}
0 = \mathcal{F}[v] - \mathcal{F}[u] &= f(.,\mathrm{D}v)(\Omega) - \int_\Omega v^+ \,\mathrm{d}\mu_- + \int_\Omega v^- \,\mathrm{d}\mu_+ - f(.,\mathrm{D}u)(\Omega) - \int_\Omega u^* \,\mathrm{d}\mu \\
&= \int_\Omega (f(.,\nabla v) - f(.,\nabla u)) \,\mathrm{d}x + f^\infty(.,\mathrm{D}^s v)(\Omega) - \int_\Omega c^+ \,\mathrm{d}\mu_- + \int_\Omega c^- \,\mathrm{d}\mu_+ \\
&= f^\infty(.,\mathrm{D}^s c)(\Omega) - \int_\Omega c^+ \,\mathrm{d}\mu_- + \int_\Omega c^- \,\mathrm{d}\mu_+ \,,
\end{aligned}
\tag{8.5.2}
$$

where we applied $\mathrm{D}^s u \equiv 0$, $\nabla u = \nabla v$ a.e. and $u - u_0 = v - u_0$ as traces on $\partial\Omega$. We then exploit the characterization of the joint ICs in the form of (3.1.3) with our given constant C to the function c with null trace to get

$$\int_\Omega c^+ \,\mathrm{d}\mu_- - \int_\Omega c^- \,\mathrm{d}\mu_+ \le C f^\infty(.,\mathrm{D}c)(\Omega) = C f^\infty(.,\mathrm{D}^s c)(\Omega)\,.$$

A substitution into (8.5.2) determines

$$(1 - C) f^\infty(.,\mathrm{D}^s c)(\Omega) \le 0\,,$$

which by $C < 1$ proves $\mathrm{D}^s c \equiv 0$ on Ω. Since by the definition of c it is also $\mathrm{D}^a c \equiv 0$ on Ω, from the constancy theorem in BV and the null trace condition we find $c \equiv 0$ on Ω. In sum, we have achieved the expected equality $v = u + c = u$ valid almost everywhere on Ω. □

Observe that, without assuming $\mathrm{D}^s u \equiv 0$ or $u \equiv v$ as traces on $\partial\Omega$ in Proposition 8.40, the equality (8.5.2) in general might not be reached. Our last statement exploits homogeneity – and specifically, strict convexity of sublevel sets of the function $f^\infty(x,.)$ – to determine a relation between the singular part of derivatives of two minimizers of $\mathcal{F}$. We notice that Proposition 8.41 applies in particular to both area and total variation.

Proposition 8.41 (sign invariance of the singular part). *For $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ and a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, we consider two minimizers u, $v \in \mathrm{BV}(\Omega)$ of $\mathcal{F}$ for a continuous integrand f under Assumption 1.1(H1) and with mutually singular, admissible measures $\mu_\pm$ on Ω. We additionally assume that u is such that $\mu_\pm(\mathrm{J}_u) = 0$, and that $\xi \mapsto f^\infty(x,\xi)$ is a strictly convex norm on $\mathbb{R}^n$ for all $x \in \Omega$ according to Definition 2.54. Then, either $u \in \mathrm{W}^{1,1}(\Omega)$ or there exists non–negative $\gamma\colon \Omega \to [0,\infty)$ and a measure $\nu \in \mathrm{RM}(\Omega,\mathbb{R}^n)$ singular to $\mathrm{D}^s u$ and to $\mathcal{L}^n$ such that $\mathrm{D}^s v = \gamma \mathrm{D}^s u + \nu$ on Ω.*

Proof. If we let $\mu := \mu_+ - \mu_-$, from the optimality relation (b) in Result 5 on the interior of Ω applied to any solution σ in $\mathcal{DM}_\mu(\Omega,\mathbb{R}^n)$ of the dual problem (P*), we know $[\![\sigma,\mathrm{D}u]\!]^s = [\![\sigma,\mathrm{D}u]\!]^s_{u_0} \llcorner \Omega = f^\infty(.,\mathrm{D}^s u) \llcorner \Omega$, and similarly for v. We now record that the assumption on u implies $\mu_\pm(\{u^- < u^+\}) \le \mu_\pm(\mathrm{J}_u) = 0$, so the representatives of u agree $\mu_\pm$–almost everywhere on Ω. Hence, summing up the inner pairings recalling that $\mathrm{div}(\sigma) = \mu$, we directly compute

$$
\begin{aligned}
[\![\sigma,\mathrm{D}u]\!] + [\![\sigma,\mathrm{D}v]\!] &= \mathrm{div}(u\sigma) + \mathrm{div}(v\sigma) + (u^* + v^+)(\mathrm{div}(\sigma))_- - (u^* + v^-)(\mathrm{div}(\sigma))_+ \\
&= \mathrm{div}((u+v)\sigma) + (u+v)^+\mu_- - (u+v)^-\mu_+ \\
&= [\![\sigma,\mathrm{D}(u+v)]\!].
\end{aligned}
$$

Exploiting the upper bound on the pairing in Proposition 7.20 and taking into account its singular parts only, we achieve

$$
\begin{aligned}
f^\infty(.,\mathrm{D}^s u) + f^\infty(.,\mathrm{D}^s v) = [\![\sigma,\mathrm{D}u]\!]^s + [\![\sigma,\mathrm{D}v]\!]^s = ([\![\sigma,\mathrm{D}u]\!] + [\![\sigma,\mathrm{D}v]\!])^s &= [\![\sigma,\mathrm{D}(u+v)]\!]^s \\
&\le f^\infty(.,\mathrm{D}^s(u+v)) \\
&\le f^\infty(.,\mathrm{D}^s u) + f^\infty(.,\mathrm{D}^s v)
\end{aligned}
$$

as measures on Ω, where the last step follows from convexity of $f^\infty(x,.)$. We then deduce the equality between measures $f^\infty(.,\mathrm{D}^s u) + f^\infty(.,\mathrm{D}^s v) = f^\infty(.,\mathrm{D}^s(u+v))$ on Ω, therefore by Proposition 2.55 it is either $\mathrm{D}^s u \equiv 0$, or $\mathrm{D}^s v = \gamma \mathrm{D}^s u + \nu$ for a non–negative function γ on Ω and some measure $\nu \in \mathrm{RM}(\Omega,\mathbb{R}^n)$ singular to $\mathrm{D}^s u$ on Ω. $\square$

Proposition 8.41 tells us that the jump direction of solutions of (P) is preserved, namely that any two solutions u, v jump either both upwards or downwards on the set $\mathrm{J}_u \cap \mathrm{J}_v$. The result above applies for instance to the area integrand $f(\xi) := \sqrt{1+|\xi|^2}$, whose recession function $f^\infty(\xi) = |\xi|$ is strictly convex *as norm* in $\mathbb{R}^n$. In such a case, the strict convexity of the function f also yields via Proposition 8.35 coincidence of the absolutely continuous parts of the gradient for solutions. Then, if $u \in \mathrm{BV}(\Omega) \setminus \mathrm{W}^{1,1}(\Omega)$ is a $\mu_\pm$–a.e. continuous minimizer of the functional

$$\mathrm{BV}(\Omega) \ni w \mapsto \int_\Omega \sqrt{1+|\nabla w|^2}\,\mathrm{d}x + \int_{\partial\Omega} |w-u_0|\,\mathrm{d}\mathcal{H}^{n-1} - \int_\Omega w^+\,\mathrm{d}\mu_- + \int_\Omega w^-\,\mathrm{d}\mu_+\,,$$

any other minimizer v is such that its derivative measure can be decomposed into mutually singular parts as $\mathrm{D}v = \nabla u + \gamma \mathrm{D}^s u + \nu$.

We have already established in Proposition 2.59 that convexity in the second entry of f under linear growth yields convexity in the same variable of the corresponding recession function. A follow–up question could be the following: Does strict convexity of $f^\infty(x,.)$ as a norm follow from strict convexity of $f(x,.)$ and/or vice versa? If yes, we could unify the statements of Propositions 8.35 and 8.41 in the hope of extrapolating information on absolutely continuous and singular parts of pairs of minimizers of $\mathcal{F}$ at the same time. Unfortunately, in general, the answer to both questions is negative.

In fact, the following Proposition 8.42 can be applied to norm functions g *not* strictly convex to find counterexamples to the first assertion above, already valid for x–independent functions. The statement below is quoted from [14, Proposition 3.1] and slightly simplified for our purposes.

Proposition 8.42. *For any* 1*–homogeneous, convex function* $g\colon \mathbb{R}^n \to [0,\infty)$*, there exists a smooth, strictly convex function* $f\colon \mathbb{R}^n \to [0,\infty)$ *such that:*

- f *has linear growth, i.e.* $\alpha|\xi| \le f(\xi) \le \beta(1+|\xi|)$ *for all* $\xi \in \mathbb{R}^n$*, with positive constants* α*,* β*;*
- g *is the recession function of* f*, meaning* $g(\xi) = \lim_{t\to 0^+} t f(\xi/t)$ *for* $\xi \in \mathbb{R}^n$*.*

On the other hand, an easy counterexample shows the presence of strictly convex (as norm) recession functions which do *not* stem from a strictly convex integrand – since we cannot exclude the presence of flat traits for f on bounded domains, which will be neglected in the calculation of f^∞.

Example 8.43. We consider any positive L and the map $f_L\colon \mathbb{R}^n \to [0,\infty)$ defined via

$$f_L(\xi) := \begin{cases} 0, & \text{if } |\xi| \le L; \\ |\xi| - L, & \text{otherwise}. \end{cases}$$

Then, f_L is not 1–homogeneous – but f_L is manifestly convex (*not* strictly convex) in $\mathbb{R}^n$. Still, for t small enough we compute

$$t f_L(\xi/t) = |\xi| - tL \xrightarrow[t\to 0]{} |\xi| = f_L^\infty(\xi)\,,$$

hence, its recession function is a strictly convex norm.

Nevertheless, we point out that from the results of Section 2.9 it is possible to relate at least strict convexity of the polar $(f^\infty)^\circ(x,.)$ to strict convexity (on the effective domain) of the conjugate function of f. In fact, we recall that under strict convexity of $f(x,.)$ the set $\mathrm{dom}(f^*(x,.))$ is at least convex in $\mathbb{R}^n$ for every x – this latter follows from combining Propositions 8.6 and 8.9 together. When additionally imposing (H2), Proposition 2.89 guarantees that the equality between sets $\mathbb{K}_{x,(f^\infty)^\circ} := \{\xi \in \mathbb{R}^n : (f^\infty)^\circ(x,\xi) \leq 1\} = \mathrm{dom}(f^*(x,.))$ holds for all x. Getting back now to our Example 8.43, we can compute the conjugate function of f_L as

$$f_L^*(\xi^*) = \max\left\{ \sup_{|\xi|\leq L} \xi^* \cdot \xi;\ L + \sup_{|\xi|>L} (\xi^* \cdot \xi - |\xi|) \right\} = \max\left\{ L|\xi^*|;\ L + \sup_{|\xi|>L} |\xi|(|\xi^*| - 1) \right\},$$

hence for $|\xi^*| \leq 1$ it is $f_L^*(\xi^*) = L$, while for $|\xi^*| > 1$ we have $f_L^*(\xi^*) = \infty$. In conclusion, $\mathrm{dom}(f_L^*) = \overline{\mathrm{B}_1}$ is strictly convex in $\mathbb{R}^n$. At the same time, we observe that $\{(f_L^\infty)^\circ \leq 1\} = \{f_L^\infty \leq 1\} = \overline{\mathrm{B}_1}$, which it is consistent with our f_L^∞ being a strictly convex norm.

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