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The Initial Value Problem for the Generalised Einstein Equations

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Abstract

This thesis discusses (in the setting of exact Courant algebroids (CAs)) three aspects of curvature in Hitchin's generalised geometry: It proves the existence of a unique geometrically preferred canonical Levi-Civita (LC) generalised connection with specified divergence, it introduces tools for generalised submanifold geometry, and it establishes that the generalised Einstein equations admit (in the setting of a closed divergence operator) a maximal globally hyperbolic development for arbitrary initial data.

It is well known that, on an exact CA $E \to M$, $\dim M > 1$, equipped with a generalised metric $\mathcal{G}$ and a divergence operator div, there exist multiple LC generalised connections D with divergence $\operatorname{div}_D = \operatorname{div}$. The available choice complicates the geometric interpretation of quantities which depend on it. This work provides a resolution to this problem in the form of a geometric criterion which uniquely selects a well-known canonical LC generalised connection $D^{\mathcal{G},\operatorname{div}}$ with divergence div. An explicit expression for the generalised Riemann tensor $\mathcal{R}m^D$ of any LC generalised connection D is calculated and then specialised for the canonical connection $D = D^{\mathcal{G},\operatorname{div}}$. The condition of generalised Riemann flatness $\mathcal{R}m^{D^{\mathcal{G},\operatorname{div}}} = 0$ for pairs $(\mathcal{G}, \operatorname{div})$ is investigated. In Riemannian and Lorentzian signature, generalised Riemann flatness is shown to imply complete triviality in the sense that the preferred representative of the Ševera class H must vanish, the semi-Riemannian metric g induced by the generalised metric $\mathcal{G}$ must be flat, and the divergence operator div must agree with the metric-induced divergence $\operatorname{div}^{\mathcal{G}}$. In neutral signature, a non-trivial example of a flat pair $(\mathcal{G}, \operatorname{div})$ is constructed.

Given an immersed manifold $\iota\colon N \to M$, where M is the base of an exact CA $E \to M$, there is a well-known construction of an exact CA $\iota^! E$ over N, the *pullback CA*. Assuming the ambient space E to be equipped with data $(\mathcal{G}, \operatorname{div}, D)$ consisting of a generalised metric $\mathcal{G}$, a divergence operator div, and a generalised connection D, this work describes the data $(\mathcal{H}, \operatorname{div}_N, D^N)$ naturally inherited by $\iota^! E$, and introduces the notion of extrinsic curvature of $\iota^! E$ in the case that $N = \Sigma \subset M$ is a hypersurface. To describe the inherited data, it is shown that the metric $\mathcal{G}$ on the ambient space induces a natural realisation of $\iota^! E$ as a subbundle $E_N \subset \iota^* E$. This allows for a definition of the inherited data $(\mathcal{H}, \operatorname{div}_N, D^N)$ via restriction and projection, e.g. $\mathcal{H} = \mathcal{G}|_{E_N \times E_N}$. To describe the extrinsic curvature of a hypersurface Σ, the *generalised second fundamental form*, the *generalised mean curvature*, and the *conormal extrinsic curvature* are introduced. The latter has no analogue in ordinary differential geometry. Generalised Gauß-Weingarten and Gauß-Codazzi equations are obtained. As applications, the constraint equations for the initial value formulation of the generalised Einstein equations are derived from the generalised Gauß-Codazzi equations, it is shown that generalised Kähler and hyper-Kähler structures are heritable, and a generalised version of the fundamental theorem for hypersurfaces is established. The latter rests on the notion of flatness $\mathcal{R}m^{D^{\mathcal{G},\operatorname{div}}} = 0$ defined via the canonical connection $D^{\mathcal{G},\operatorname{div}}$ associated to the pair $(\mathcal{G}, \operatorname{div})$.

The generalised Einstein equations (GEE) are equations for a pair $(\mathcal{G}, \operatorname{div})$ over an $n+1$-dimensional manifold and consist of the conditions of generalised Ricci and scalar flatness, $\mathcal{R}c(\mathcal{G}, \operatorname{div}) = 0$ and $\mathcal{S}c(\mathcal{G}, \operatorname{div}) = 0$. In the case of exact divergence, they are equivalent to the equations of motion of the bosonic NS-NS sector in type II ten-dimensional supergravity, which features as dynamical objects a Lorentzian metric g, a two-form B, and a scalar field ϕ (the dilaton potential). This work establishes for the case of closed divergence that the GEE have a well-posed initial value problem. In a preliminary section, techniques developed by Ringström in the context of an Einstein-scalar system are disentangled from the specific

choice of matter model to achieve broader applicability. Then, the *generalised Lorenz gauge* is introduced, which is a new gauge condition for the B-field such that, combined with a suitable gauge condition for the metric (e.g. the wave gauge), the GEE become a hyperbolic system of PDEs with principal symbol given by the (dynamical) metric. Applying the techniques from the aforementioned preliminary section, a globally hyperbolic development satisfying these gauge conditions is constructed for an arbitrary set of initial data, and geometric uniqueness is established by comparison of any given development with this gauge-fixed development. Employing a famous result by Choquet-Bruhat and Geroch, it follows that the GEE admit a maximal globally hyperbolic development for any set of initial data.

Zusammenfassung

Diese Dissertation befasst sich im Rahmen exakter Courantalgebroide (CA) mit drei Aspekten von Krümmung in Hitchin's verallgemeinerter Geometrie: Sie beweist die Existenz eines eindeutigen geometrisch ausgezeichneten kanonischen Levi-Civita (LC) verallgemeinerten Zusammenhangs mit vorgegebener Divergenz, sie führt die Werkzeuge verallgemeinerter Untermannigfaltigkeitsgeometrie ein, und sie etabliert, dass die verallgemeinerten Einsteingleichungen im Falle eines geschlossenen Divergenzoperators eine maximale global hyperbolische Entwicklung beliebiger Anfangsdaten zulassen.

Es ist bekannt, dass auf einem exakten CA $E \to M$, $\dim M > 1$, ausgestattet mit einer verallgemeinerten Metrik $\mathcal{G}$ und einem Divergenzoperator div mehrere LC verallgemeinerte Zusammenhänge D mit Divergenz $\mathrm{div}_D = \mathrm{div}$ existieren. Die verfügbare Auswahl erschwert die geometrische Interpretation der von ihr abhängigen Größen. Diese Arbeit präsentiert eine Lösung dieses Problems in Form eines geometrischen Kriteriums, welches eindeutig einen wohlbekannten kanonischen LC verallgemeinerten Zusammenhang $D^{\mathcal{G},\mathrm{div}}$ mit Divergenz div auszeichnet. Der verallgemeinerte Riemanntensor $\mathcal{R}m^D$ eines beliebigen LC verallgemeinerten Zusammenhangs D wird berechnet, und die gefundenen Formeln werden für den kanonischen Zusammenhang $D = D^{\mathcal{G},\mathrm{div}}$ spezialisiert. Die Bedingung verallgemeinerter Riemann-Flachheit $\mathcal{R}m^{D^{\mathcal{G},\mathrm{div}}} = 0$ an Paare $(\mathcal{G}, \mathrm{div})$ wird untersucht. In Riemannscher und Lorentzscher Signatur wird gezeigt, dass verallgemeinerte Riemann-Flachheit in dem Sinne vollständige Trivialität impliziert, dass der bevorzugte Vertreter der Ševeraklasse H verschwindet, die semi-Riemannsche Metrik g, die durch die verallgemeinerte Metrik $\mathcal{G}$ induziert wird, flach ist, und der Divergenzoperator div mit der metrisch induzierten Divergenz $\mathrm{div}^{\mathcal{G}}$ übereinstimmt. In neutraler Signatur wird ein nichttrivales Beispiel eines flachen Paares $(\mathcal{G}, \mathrm{div})$ konstruiert.

Gegeben eine immersierte Mannigfaltigkeit $\iota\colon N \to M$, wobei M die Basis eines exakten CAs $E \to M$ ist, gibt es eine bekannte Konstruktion eines exakten CAs $\iota^! E$ über N, dem *Rückzug CA*. Unter der Annahme, dass E mit einer verallgemeinerten Metrik $\mathcal{G}$, einem Divergenzoperator div, und einem verallgemeinerten Zusammenhang D ausgestattet ist, beschreibt diese Arbeit die Größen $(\mathcal{H}, \mathrm{div}_N, D^N)$, die $\iota^! E$ auf natürliche Weise erbt, und führt im Falle, dass $N = \Sigma \subset M$ eine Hyperfläche ist, den Begriff der äußeren Krümmung von $\iota^! E$ ein. Um die geerbten Größen zu beschreiben wird gezeigt, dass die Metrik $\mathcal{G}$ auf dem umgebenen Raum eine natürliche Realisierung von $\iota^! E$ als Unterbündel $E_N \subset \iota^* E$ induziert. Dies erlaubt die Definition der geerbten Größen $(\mathcal{H}, \mathrm{div}_N, D^N)$ über Restriktion und Projektion, z.B. $\mathcal{H} = \mathcal{G}|_{E_N \times E_N}$. Um die äußere Krümmung einer Hyperfläche Σ zu beschreiben, werden die *verallgemeinerte zweite Fundamentalform*, die *verallgemeinerte mittlere Krümmung*, und die *konormale äußere Krümmung* eingeführt. Letztere hat kein

Analogon in der gewöhnlichen Differentialgeometrie. Verallgemeinerte Gauß-Weingarten und Gauß-Codazzi Gleichungen werden berechnet. Als Anwendungen werden die Zwangsbedingungen der Anfangswertformulierung der verallgemeinerten Einsteingleichungen aus den verallgemeinerten Gauß-Codazzi Gleichungen hergeleitet, es wird die Erblichkeit verallgemeinerter Kähler- und Hyperkählerstrukturen gezeigt, und es wird eine verallgemeinerte Version des Fundamentalsatzes der Hyperflächentheorie etabliert. Letzterer beruht auf dem Begriff der Flachheit $\mathcal{R}m^{D^{\mathcal{G},\mathrm{div}}} = 0$, der über den kanonisch zum Paar $(\mathcal{G}, \mathrm{div})$ assoziierten Zusammenhang $D^{\mathcal{G},\mathrm{div}}$ definiert ist.

Die verallgemeinerten Einsteingleichungen (VEG) sind Gleichungen für ein Paar $(\mathcal{G}, \mathrm{div})$ auf einer $n+1$-dimensionalen Mannigfaltigkeit und bestehen aus den Gleichungen verallgemeinerter Ricci- und Skalarkrümmungs-Flachheit, $\mathcal{R}c(\mathcal{G}, \mathrm{div}) = 0$ und $\mathcal{S}c(\mathcal{G}, \mathrm{div}) = 0$. Im Falle exakter Divergenz sind sie äquivalent zu den Bewegungsgleichungen des bosonischen NS-NS Sektors in Typ II zehndimensionaler Supergravitation, deren dynamische Größen eine Lorentzmetrik g, eine Zweiform B, und ein skalares Feld ϕ (das Dilaton Potential) sind. Diese Arbeit etabliert für den Fall geschlossener Divergenz, dass die Anfangswertaufgabe für die VEG wohlgestellt ist. In einem vorbereitenden Kapitel werden von Ringström im Kontext eines Einstein-Skalar-Systems entwickelte Techniken von den spezifischen Annahmen an das Materiemodell befreit um breitere Anwendbarkeit zu erreichen. Danach wird die *verallgemeinerte Lorenzeichung* eingeführt, welche eine neue Eichbedingung für das B-Feld ist sodass, kombiniert mit einer geeigneten Eichbedingung an die Metrik (z.B. die Welleneichung), die VEG ein hyperbolisches PDG-System mit Hauptsymbol gegeben durch die (dynamische) Metrik werden. Unter Anwendung der im vorbereitenden Kapitel entwickelten Techniken wird eine diese Eichbedingung erfüllende global hyperbolische Entwicklung beliebiger Anfangsdaten konstruiert, und geometrische Eindeutigkeit wird per Vergleich mit dieser eichfixierten Entwicklung gezeigt. Es folgt aus einem berühmten Resultat von Choquet-Bruhat und Geroch, dass die VEG eine maximale global hyperbolische Entwicklung beliebiger Anfangsdaten zulassen.

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Contents

1 Introduction

Hitchin's generalised geometry is a natural modification of differential geometry, in which the Lie algebroid $(TM, [\cdot,\cdot])$ is replaced by a Courant algebroid. It features "generalised" analogues of several elemental geometric quantities such as metrics, connections, and curvature tensors. The beauty of generalised geometry is that it preserves enough of the main structural relationships between these geometric quantities for many important notions, ideas, and results from ordinary differential geometry to remain relevant, while differing in the particulars from the familiar framework in ways which are significant, mathematically interesting, and often physically meaningful [1].

Consider the generalised Einstein equations (GEE) on exact Courant algebroids. The GEE are equations for the fundamental geometric object of generalised geometry, which is a pair $(\mathcal{G}, \mathrm{div})$ consisting of a generalised metric $\mathcal{G}$ and a divergence operator div. The GEE are the combined equations[1] of generalised Ricci flatness $\mathcal{Rc}(\mathcal{G}, \mathrm{div}) = 0$ and generalised scalar flatness $\mathcal{Sc}(\mathcal{G}, \mathrm{div}) = 0$. They are analogous to the ordinary Einstein equations: The definitions of $\mathcal{Rc}$ and $\mathcal{Sc}$ closely resemble that of their ordinary counterparts, as they are the appropriate traces of a generalised Riemann tensor which measures the commutativity of second order generalised covariant derivatives. Finally, the GEE are physically meaningful: Decomposing the pair $(\mathcal{G}, \mathrm{div})$ into a semi-Riemannian metric g, a two-form B, and (assuming exact divergence) a real-valued function ϕ, the GEE are the equations of motion for (g, B, ϕ) viewed as the dynamical fields of the bosonic NS-NS sector in type II ten-dimensional supergravity [2]. Note that, for this reason, generalised geometry is sometimes referred to as a *geometrisation* of said supergravity sector.

This thesis studies the GEE and their initial value problem (IVP). The study of the latter, in particular, is motivated by the relationship of the GEE both to the ordinary Einstein equations and to supergravity.

The text consists of two parts. The first offers a contextualisation of the GEE within the framework of generalised geometry. In particular, it develops the theory of *extrinsic generalised geometry* and obtains a geometric characterisation of a canonical generalised connection $D^{\mathcal{G},\mathrm{div}}$ appearing in [3, 4, 5, 6]. The second part establishes the IVP for the GEE to be well-posed.

The geometric characterisation of the canonical connection $D^{\mathcal{G},\mathrm{div}}$. The fundamental theorem of Riemannian geometry states that, given a Riemannian manifold (M, g), there exists a unique torsion-free connection compatible with the metric. It is well-known that generalised geometry lacks a similar result: Given an exact Courant algebroid $E \to M$, $\dim M > 1$, equipped with a generalised metric $\mathcal{G}$ and a divergence operator div, the space $\mathcal{D}^0(\mathcal{G}, \mathrm{div})$ of torsion-free generalised connections compatible with the pair $(\mathcal{G}, \mathrm{div})$ has dimension at least one (cf. Corollary 2.40). Thus, every quantity whose definition depends on a choice of generalised connection $D \in \mathcal{D}^0(\mathcal{G}, \mathrm{div})$ fails to be an invariant of $(\mathcal{G}, \mathrm{div})$. In particular, this applies to the generalised Riemann tensor (cf. Theorem 3.5). However, it would be desirable for a pair $(\mathcal{G}, \mathrm{div})$ to have a notion of generalised Riemann flatness with which to contrast the notions of generalised Ricci and scalar flatness. Therefore, this thesis concerns itself with the characterisation of the canonical connection $D^{\mathcal{G},\mathrm{div}} \in \mathcal{D}^0(\mathcal{G}, \mathrm{div})$ as the unique geometrically preferred choice, yielding the preferred flatness condition $\mathcal{Rm}^{D^{\mathcal{G},\mathrm{div}}} = 0$.

[1] For reasons that will become apparent in Section 2.6, generalised Ricci flatness does not imply generalised scalar flatness.

Extrinsic generalised geometry. Consider initial data (Σ, h, k) coming from a spacelike hypersurface Σ in a Lorentzian manifold (M, g) with induced metric h and second fundamental form k. Denoting by N the unit normal on Σ, by $G = \mathrm{Rc} - \frac{\mathrm{Sc}}{2} g$ the Einstein tensor of (M, g), and by Sc_Σ the scalar curvature of (Σ, h), it is a well-known application of the Gauß-Codazzi equations that on Σ

$$G(N, N) = \frac{1}{2}\left[\mathrm{Sc}_\Sigma - |k|_h^2 + (\mathrm{tr}_h k)^2\right], \qquad G(N, X) = (\mathrm{div}_\Sigma k)(X) - X(\mathrm{tr}_h k),$$

where $X \in \Gamma(T\Sigma)$ is arbitrary, and in index notation $|k|_h^2 = h^{im} h^{jn} k_{ij} k_{mn}$ as well as $(\mathrm{div}_\Sigma k)_i = h^{mn} \nabla^\Sigma_m k_{ni}$ with ∇^Σ the Levi-Civita connection on Σ. In particular, assuming $G = 0$, one obtains two equations involving h and k. In general relativity, these are respectively referred to as the *energy constraint* and the *momentum constraint*. Clearly, a proper understanding of these constraints demands the theory of extrinsic (i.e. submanifold) geometry.

This thesis develops the theory of extrinsic generalised geometry for semi-Riemannian hypersurfaces. By virtue of the new framework, the thesis offers an elegant derivation and geometric description of the constraint equations for the GEE. This extends the geometrisation of supergravity to the notion of initial data.

The initial value problem for the generalised Einstein equations. Any viable physical theory must make unique predictions given appropriate initial data. This property, more technically referred to as "having a well-posed IVP", took decades to establish for the Einstein vacuum equations: local well-posedness[2] was established in 1952 by Choquet-Bruhat [7], and global well-posedness (i.e. existence of a maximal globally hyperbolic development (MGHD)) was obtained in 1969 by Choquet-Bruhat and Geroch [8].

The main problem was the geometric nature of the Einstein equations, as it caused profound conceptual complications. Due to the diffeomorphism invariance of the theory, fundamental notions such as initial data, uniqueness, and maximality all needed new definitions. The subtleties were sorted out over a number of individual contributions [9]. The modern framework is outlined[3] as follows [10].

In $n+1$ dimensions, a *solution* to the Einstein vacuum equations is a Ricci-flat Lorentz manifold (M, g) of dimension $n+1$. *Initial data* is a tuple (Σ, h, k) consisting of an n-dimensional Riemannian manifold (Σ, h) and a symmetric two-tensor k satisfying certain constraint equations. A *development* of initial data (Σ, h, k) is a solution (M, g) together with a semi-Riemannian embedding $\iota\colon \Sigma \hookrightarrow M$ such that k is realised as the second fundamental form. Developments are *geometrically unique* in the sense that any two developments of the same initial data share a common development to which they, perhaps after restriction to a smaller development, are related by a diffeomorphism. A development is called *globally hyperbolic* if every inextendible timelike curve intersects the initial hypersurface exactly once (i.e. the initial hypersurface is Cauchy). Finally, the MGHD is the (up to diffeomorphism) unique globally hyperbolic development which is an extension of every globally hyperbolic development.

This thesis adapts these notions for the initial value formulation of the GEE (in the setting of a closed divergence operator) and proves that the GEE admit an MGHD for arbitrary initial data. In this, it mostly operates in terms of the decomposed data (g, B, ϕ).

[2] Choquet-Bruhat obtained both local existence and uniqueness, though the latter was already known at the time.

[3] For the sake of brevity, all details relating to time orientations are omitted.

1.1 Outline

The main achievements of this thesis are

1. to characterise, given a generalised metric $\mathcal{G}$ and a divergence operator div, the canonical LC generalised connection $D^{\mathcal{G},\mathrm{div}}$ compatible with div appearing in [3, 4, 5, 6] as the unique geometrically preferred choice (cf. Theorem 3.2),

2. to develop the tools of extrinsic generalised geometry and demonstrate their utility with

   a) the fundamental theorem for generalised hypersurfaces (Theorem 4.46),

   b) a derivation of the constraint equations for the GEE in the language of generalised geometry (cf. Corollaries 4.36 and 4.38), and

3. to establish the well-posedness of the initial value problem (IVP) for the GEE (cf. Theorem 6.33).

Each result rests on its predecessor. The first gives rise to a notion of flatness for a pair $(\mathcal{G}, \mathrm{div})$, and the second rests on this notion in its discussion of the extrinsic geometry induced by a flat ambient space. In turn, the second result contains the constraint equations for the GEE, which are of course relevant to their initial value formulation.

Note that Choquet-Bruhat [11] obtained well-posedness of the Cauchy problem for 11-dimensional $N = 1$ supergravity, which is a system inequivalent but related to the GEE, cf. also Remark 6.3. Note furthermore that, in the analytic category and for Riemannian signature, Bunk, Pino, and Shahbazi [12] have obtained a well-posedness result for gradient generalised Ricci solitons (a system closely related to the GEE) in the language of bundle gerbes.

The thesis is structured as follows.

Chapter 2 is an introduction to Hitchin's generalised geometry and largely based on [13, 6, 14, 5, 15]. It contains no original results.

Section 2.1 is purely pedagogical. The section investigates the *generalised tangent bundle* $\mathbb{T}M = TM \oplus T^*M$ equipped with the natural inner product $\langle \cdot, \cdot \rangle$ given by contraction,

$$\langle X + \xi, X + \xi \rangle = \xi(X) \quad \text{for all } X + \xi \in \mathbb{T}M.$$

It explains that a promotion of a particular symmetry of $(\mathbb{T}M, \langle \cdot, \cdot \rangle)$, the symmetry under B-field transformations, to a fundamental symmetry on par with diffeomorphism invariance naturally leads to the *H-twisted bracket* $[\cdot, \cdot]_H$ on $\mathbb{T}M$ (the twist H being a closed three-form), and then to the notion of *exact Courant algebroids* (CAs). In fact, diffeomorphisms and B-field transformations are united in the notion of *generalised diffeomorphisms*.

A formal approach to exact CAs is presented in Section 2.2, giving their definition as a vector bundle $E \to M$ equipped with a bracket $[\cdot, \cdot]$ on sections of E, an inner product $\langle \cdot, \cdot \rangle$ on E, and an *anchor homomorphism* $\pi \colon E \to TM$, all such that several properties hold; in particular

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E^* \overset{\langle \cdot,\cdot \rangle}{\cong} E \xrightarrow{\pi} TM \longrightarrow 0 \tag{1.1}$$

is required to be a short exact sequence. Basic results on exact CAs are obtained. For example, it is shown (Lemma 2.9) that there exists an *isotropic splitting* $\sigma \colon TM \to E$ of the short exact sequence (1.1), and that such a splitting induces (Theorem 2.10) a Courant isomorphism between E and an H-twisted generalised tangent bundle for some choice of twist H.

Sections 2.3-2.6 introduce geometric objects on exact CAs.

Section 2.3 discusses *generalised metrics*, which are non-degenerate symmetric bilinear forms $\mathcal{G} \in \Gamma(\mathrm{Sym}^2(E))$ satisfying compatibility conditions with the CA structure. The section gives a few equivalent characterisations of generalised metrics. In particular, they are described in terms of their eigenbundles $E_\pm$ (Lemma 2.15) and in terms of pairs (g, σ) consisting of a semi-Riemannian metric g and an isotropic splitting σ (Proposition 2.17). *The preferred representative of the Ševera class* is defined as the twist H of the generalised tangent bundle to which E is related under the Courant isomorphism defined from the splitting σ.

Section 2.4 explains *divergence operators*, which are maps $\Gamma(E) \to C^\infty(M)$ satisfying a Leibniz rule. It is shown that every generalised metric $\mathcal{G}$ induces a divergence operator $\mathrm{div}^{\mathcal{G}}$. The *dilaton* associated to a pair $(\mathcal{G}, \mathrm{div})$ is introduced as the section $e \in \Gamma(E)$ which describes the difference between the metric divergence $\mathrm{div}^{\mathcal{G}}$ and the given divergence operator div,

$$\langle e, \cdot \rangle = \mathrm{div}^{\mathcal{G}} - \mathrm{div}\,.$$

Closed (resp. *exact*) divergence operators are defined as those for which for any generalised metric, the induced dilaton is given as $\langle e, \cdot \rangle = \pi^*\xi$ for some closed (resp. exact) one-form ξ.

Section 2.5 discusses *generalised connections*, which are maps $\Gamma(E) \times \Gamma(E) \to \Gamma(E)$ that are C^∞-linear in the first entry, satisfy a Leibniz rule in the second, and are in some sense compatible with the CA structure. The section introduces the *torsion* of a generalised connection. It recalls that a generalised connection is $\mathcal{G}$-metric if $D\mathcal{G} = 0$, and that it is compatible with a given operator div if the induced divergence div_D,

$$\mathrm{div}_D(a) \coloneqq \mathrm{tr}(Da) \quad \text{for all } a \in \Gamma(E),$$

agrees with div. Given a generalised metric $\mathcal{G}$, the construction of a canonical Levi-Civita generalised connection $D^{\mathcal{G},\mathrm{div}^{\mathcal{G}}}$ with metric divergence is recalled (Corollary 2.36).

Section 2.6 describes the *generalised Riemann*, *Ricci*, and *scalar curvature*. The section shows (Lemmas 2.43 and 2.44) that the generalised Riemann tensor $\mathcal{R}m^D$ is an algebraic curvature tensor, meaning that it is a section of $\mathrm{Sym}^2\Lambda^2 E^*$ which satisfies the Bianchi identity. In fact, denoting by $E_\pm \subset E$ the eigenbundles of the generalised metric, it is observed (Corollary 2.45) that the generalised Riemann tensor is a section of the direct sum of the *pure type subbundle* and the *mixed type subbundle*

$$\Lambda^2 \mathrm{Sym}^2 E_+^* \oplus \Lambda^2 \mathrm{Sym}^2 E_-^* \qquad \text{and} \qquad (\mathrm{Sym}^2 E_+ \oplus \mathrm{Sym}^2 E_-) \wedge (E_+ \vee E_-).$$

Furthermore, it is shown (Propositions 2.47 and 2.49) that both the generalised scalar curvature $\mathcal{S}c^D$ and the restrictions $\mathcal{R}c^\pm \coloneqq \mathcal{R}c^D|_{E_\mp \times E_\pm}$ of the generalised Ricci curvature $\mathcal{R}c^D$ are invariants of the pair $(\mathcal{G}, \mathrm{div})$, i.e. it is shown that they are independent of the choice of Levi-Civita generalised connection D compatible with a given divergence operator.

Chapter 3 presents original results related to the geometric characterisation of the canonical generalised connection $D^{\mathcal{G},\mathrm{div}}$ induced by a pair $(\mathcal{G}, \mathrm{div})$ consisting of a generalised metric $\mathcal{G}$ and a divergence operator div. Sections 3.1-3.3 are based on joint work with Vicente Cortés, Matas Mackevicius, and Thomas Mohaupt [16]. Section 3.4 is based on joint work with Vicente Cortés [17].

Section 3.1 recalls the canonical Levi-Civita (LC) generalised connection $D^{\mathcal{G},\mathrm{div}}$ with divergence div from [3, 4, 5, 6]. It goes on to state the first main result of this thesis, informally summarised as follows.

Theorem 1.1 (Theorem 3.2)**.** *Let $\mathcal{G}$ be a generalised metric and* div *a divergence operator on an exact Courant algebroid E. Then, there is a geometric characterisation of the canonical Levi-Civita generalised connection $D^{\mathcal{G},\mathrm{div}}$ with divergence* div*. In particular, $D^{\mathcal{G},\mathrm{div}}$ is the unique geometrically preferred choice of Levi-Civita generalised connection with specified divergence.*

Idea. The geometric condition selecting $D^{\mathcal{G},\mathrm{div}}$ is the following. The canonical connection $D^{\mathcal{G},\mathrm{div}^{\mathcal{G}}}$ with metric divergence is uniquely constructed from the LC connection of the metric g induced by $\mathcal{G}$. To generalise to arbitrary divergence, we define the map

$$\mathrm{tr}\colon E^*\otimes E^*\otimes E\to E, \qquad (\mathrm{tr}\,\chi)(a)=\mathrm{tr}(\chi(\cdot,a)), \qquad \chi\in E^*\otimes E^*\otimes E,\ a\in E.$$

Note that the difference of any two generalised connections is a section of $E^*\otimes E^*\otimes E$. In fact, if D and D' are both Levi-Civita, their difference is a section of a certain subbundle F. The idea is to show that the structure on E induces a non-degenerate inner product on F for which $\ker\mathrm{tr}|_F$, the kernel of the restricted trace map, is a non-degenerate subspace. Because then, we can consider the orthogonal decomposition $F=\ker\mathrm{tr}|_F\oplus(\ker\mathrm{tr}|_F)^\perp$. And since

$$\mathrm{tr}(D^{\mathcal{G},\mathrm{div}^{\mathcal{G}}}-D^{\mathcal{G},\mathrm{div}})=\mathrm{div}^{\mathcal{G}}-\mathrm{div}=\langle e,\cdot\rangle\in\Gamma(E^*),$$

where $e\in\Gamma(E)$ is the dilaton which describes div, we know that $\mathrm{tr}|_F$ is surjective, which allows us to conclude the existence of a unique element $\chi\in\Gamma((\ker\mathrm{tr}|_F)^\perp)$ such that $\mathrm{tr}\,\chi=\langle e,\cdot\rangle$. In other words, the geometric condition

$$D^{\mathcal{G},\mathrm{div}^{\mathcal{G}}}-D^{\mathcal{G},\mathrm{div}}\in\Gamma((\ker\,\mathrm{tr}|_F)^\perp)$$

uniquely selects $D^{\mathcal{G},\mathrm{div}}$. (□)

Section 3.2 computes the generalised Riemann tensor of the canonical connection $D^{\mathcal{G},\mathrm{div}^{\mathcal{G}}}$ with metric divergence. Section 3.3 generalises the calculation to arbitrary LC generalised connections D, and derives formulas for the generalised Ricci and scalar curvature. Specialised formulas for the generalised Riemann tensor of the canonical connection $D^{\mathcal{G},\mathrm{div}}$ are presented.

Section 3.4 gives a characterisation of pairs $(\mathcal{G},\mathrm{div})$ which satisfy the flatness condition $\mathcal{R}m^{D^{\mathcal{G},\mathrm{div}}}=0$. In Riemannian and Lorentzian signature, it is found (Corollaries 3.17 and 3.19) that for such pairs the generalised metric $\mathcal{G}$ must induce a flat metric g on the base, the preferred representative H of the Ševera class must vanish, and the divergence must be metric, $\mathrm{div}=\mathrm{div}^{\mathcal{G}}$. In neutral signature, a non-trivial flat pair $(\mathcal{G},\mathrm{div})$ is constructed (Example 3.22). Note that [18] provides a characterisation of flat metric connections on transitive Courant algebroids for a weaker notion of flatness.

Chapter 4 presents original results which develop the formalism of extrinsic generalised geometry and demonstrate its utility in applications. The chapter is based on joint work with Vicente Cortés [17].

Let $E\to M$ be an exact CA, and consider an immersion $\iota\colon N\to M$. Section 4.1 recalls the *pullback CA* $\iota^!E\to N$, which is a naturally induced exact CA over N introduced in [19, Lemma 3.7] and [20, Proposition 7.1.]. The section explains (Proposition 4.8) that a generalised metric $\mathcal{G}$ induces a canonical realisation of $\iota^! E$ as a subbundle $E_N \subset \iota^* E$ of the usual pullback bundle $\iota^* E$. Section 4.2 utilises this realisation to describe the pullback of generalised metrics, divergence operators, and generalised connections.

Assume that E is equipped with a generalised metric $\mathcal{G}$, a divergence operator $\operatorname{div} = \operatorname{div}^{\mathcal{G}} - \langle e, \cdot \rangle$, and a generalised connection D. Section 4.2.1 defines the *inherited generalised metric* $\mathcal{H}$ on E_N by restriction,

$$\mathcal{H} := \mathcal{G}|_{E_N \times E_N}.$$

With this, Section 4.2.2 defines the *inherited divergence operator* div_N on E_N as

$$\operatorname{div}_N = \operatorname{div}^{\mathcal{H}} - \langle e^{\|}, \cdot \rangle,$$

where $e^{\|} \in \Gamma(E_N)$ is the orthogonal projection of $\iota^* e \in \Gamma(\iota^* E)$ into E_N. It is shown that the inherited divergence operator div_N is closed (resp. exact) if div is closed (resp. exact).

Section 4.2.3 defines the *inherited generalised connection* D^N on E_N,

$$D^N_u v = \pi^{\|}(D_{\tilde{u}} \tilde{v}) \quad \text{for all } u, v \in \Gamma_{\text{loc}}(E_N) \text{ and extensions } \tilde{u}, \tilde{v} \in \Gamma_{\text{loc}}(E).$$

The following result on the compatibility of the inherited structure is obtained.

Proposition 1.2 (Lemma 4.13 and Proposition 4.16)**.** *Let $(\mathcal{H}, \operatorname{div}_N, D^N)$ be the geometric data on E_N inherited from the data $(\mathcal{G}, \operatorname{div}, D)$ on the ambient Courant algebroid E. Then,*

(i) D^N is torsion-free if D is torsion-free,

(ii) D^N is $\mathcal{H}$-metric if D is $\mathcal{G}$-metric,

(iii) in general, the divergence operator div_{D^N} induced by D^N disagrees with the inherited divergence operator div_N.

In particular, the canonical connection $D^{\mathcal{H}, \operatorname{div}_N}$ of the inherited pair $(\mathcal{H}, \operatorname{div}_N)$ generally disagrees with the inherited connection $(D^{\mathcal{G}, \operatorname{div}})^N$.

As a remedy to (iii), it is shown (Lemma 4.19) that for every LC generalised connection D^N on E_N with divergence $\operatorname{div}_{D^N} = \operatorname{div}_N$, there exists a LC generalised connection D on E with divergence $\operatorname{div}_D = \operatorname{div}$ which induces D^N.

Section 4.3 focuses on the case that $N = \Sigma$ is a hypersurface in M and introduces the *generalised second fundamental forms* $\mathcal{K}^{n_\pm}$, the *conormal extrinsic curvatures* $\mathcal{L}^{\pm}$, and the *generalised mean curvatures* $\mathcal{T}^{\pm}$.

Idea. To explain the definition of the extrinsic curvature quantities in generalised geometry, we assume to work on the H-twisted generalised tangent bundle $\mathbb{T}M$, where H is the preferred representative of the Ševera class. Denote by g the metric induced by the generalised metric $\mathcal{G}$, and by $n \in \Gamma(TM|_\Sigma)$ the unit normal on Σ. Then naturally $E_\Sigma \cong \mathbb{T}\Sigma$, and one obtains the orthogonal decomposition

$$\mathbb{T}M = \mathbb{T}\Sigma \oplus \operatorname{span}\{n, gn\}.$$

In other words, there are two normal directions in $\mathbb{T}M$ and they are spanned by n and $gn = g(n, \cdot)$. It is natural to consider the linear combinations $n_\pm = n \pm gn$ which lie in the metric eigenspaces $E_\pm$. We define for each of these two unit normals a generalised second fundamental form $\mathcal{K}^{n_\pm}$,

$$\mathcal{K}^{n_\pm}(a, b) := \mathcal{G}(D_a n_\pm, b), \qquad a, b \in \Gamma(\mathbb{T}\Sigma).$$

Observe that $n_+ - n_- = 2gn$ has no vector component. By the Leibniz rule for D, this implies tensoriality of $D_{n_+-n_-}$, giving rise to the *conormal extrinsic curvature*

$$\mathcal{L}^{\pm}(a) \coloneqq \mathcal{G}(D_{n_\pm - n_\mp} n_\pm, a), \qquad a \in \Gamma(\mathbb{T}\Sigma).$$

The generalised mean curvatures are obtained by tracing, $\mathcal{T}^{\pm} \coloneqq \operatorname{tr}_{\mathcal{H}} \mathcal{K}^{n_\pm}$. (□)

Formulas decomposing $\mathcal{K}^{n_\pm}, \mathcal{L}^{\pm}$, and $\mathcal{T}^{\pm}$ in terms of appropriate ordinary quantities are obtained (Lemma 4.23). In particular, it is established that the restrictions $\mathcal{K}^{\pm} \coloneqq \mathcal{K}^{n_\pm}|_{E_\mp \times E_\pm}$ to the mixed-type spaces $E_\mp \times E_\pm$ and the generalised mean curvatures $\mathcal{T}^{\pm}$ do not depend on the choice of LC generalised connection with given divergence and hence are invariants of the pair $(\mathcal{G}, \operatorname{div})$. In fact, employing canonical identifications $E_\pm \cong TM$, it is found that in terms of the metric h on Σ, the second fundamental form k of Σ, the twist H, and the dilaton e, one has

$$\mathcal{K}^{\pm}(\mathcal{G}, \operatorname{div}) = k \mp \frac{i_n H}{2}, \qquad \mathcal{T}^{\pm}(\mathcal{G}, \operatorname{div}) = \operatorname{tr}_h k \mp \mathcal{G}(e, n_\pm).$$

Section 4.4 establishes the generalised Gauß-Weingarten (Lemma 4.29) and the generalised Gauß-Codazzi equations (Theorems 4.32 and 4.37). The generalised Gauß-Weingarten equations are direct analogues of their ordinary counterparts:

$$(D_a b)^{\parallel} = D^{\Sigma}_a b, \qquad (D_a b)^{\perp} = -\varepsilon[\mathcal{K}^{n_+}(a,b)n_+ + \mathcal{K}^{n_-}(a,b)n_-], \qquad D_a n_\pm = \mathcal{G}^{-1}\mathcal{K}^{n_\pm}(a),$$

where $a, b \in \Gamma(E_\Sigma)$ are arbitrary and $\varepsilon = g(n,n) = \mathcal{G}(n_\pm, n_\pm)$.

While the generalised Gauß-Codazzi equations have structural similarities to their ordinary counterparts, there are three features of generalised geometry which lead to significant differences between these equations. One, the generalised second fundamental forms $\mathcal{K}^{n_\pm}$ (in contrast to the second fundamental form k) have anti-symmetric parts, which enter the equations. Two, the generalised Riemann tensor splits into two parts – pure and mixed type – of differing behaviour. These types each have their own Gauß equations. Three, the conormal derivative $D_{n_\pm - n_\mp}$ doesn't act as a derivation but as a tensor. This gives rise to additional Codazzi equations.

The second main result of this thesis is a corollary of the generalised Gauß-Codazzi equations, obtained by tracing over them. To state it, recall that $\mathcal{R}c^{\pm}$ denote the mixed-type parts of the generalised Ricci curvature on E, that $\mathcal{S}c$ denotes the generalised scalar curvature on E, that $\mathcal{K}^{\pm}$ denote the mixed-type parts of the generalised second fundamental forms and $\mathcal{A}^{\pm} = \mathcal{H}^{-1}\mathcal{K}^{\pm} \in \Gamma(\operatorname{Hom}(E^{\mp}_\Sigma, E^{\pm}_\Sigma))$ the associated homomorphisms, that $\mathcal{T}^{\pm}$ denote the generalised mean curvatures, and that $n \in \Gamma(TM|_\Sigma)$ and $n_\pm \in \Gamma(E_\pm|_\Sigma)$ denote the unit normals. Furthermore, denote by $\mathcal{S}c_\Sigma$ the generalised scalar curvature on E_Σ.

Theorem 1.3 (Corollaries 4.36 and 4.38)**.** *Let $E \to M$ be an exact Courant algebroid equipped with a generalised metric $\mathcal{G}$ and a divergence operator* div*. Assume that Σ is a hypersurface in M, and consider the inherited structure $(E_\Sigma, \mathcal{H}, \operatorname{div}_\Sigma)$.*

Then, for all $a_\mp \in \Gamma(E^{\mp}_\Sigma)$

$$\begin{aligned} 2\mathcal{R}c^{\pm}(n_\mp, n_\pm) - \varepsilon\mathcal{S}c &= -\varepsilon\mathcal{S}c_\Sigma - |\mathcal{K}^{\pm}|^2 + \frac{(\mathcal{T}^+)^2 + (\mathcal{T}^-)^2}{2}, \\ \mathcal{R}c^{\pm}(a_\mp, n_\pm) &= (\operatorname{div}_\Sigma \mathcal{A}^{\pm})(a_\mp) - \pi a_\mp(\mathcal{T}^{\pm}). \end{aligned} \tag{1.2}$$

Herein, $\varepsilon = g(n,n) = \mathcal{G}(n_\pm, n_\pm)$, and all quantities involved in these equations are invariants of either $(\mathcal{G}, \operatorname{div})$ or the extrinsic geometry $(\mathcal{H}, \operatorname{div}_\Sigma, \mathcal{K}^{\pm}, \mathcal{T}^{\pm})$.

Assuming the ambient space to be generalised Einstein, $\mathcal{R}c^{\pm} = 0$ and $\mathcal{S}c = 0$, equations (1.2) are the constraint equations for the generalised Einstein equations. Taking the dilaton to be closed, Section 4.4 obtains a decomposition of the constraint equations in terms of ordinary objects. The result agrees with the formulas derived in [21, Proposition 5.11] using a description of the relevant supergravity sector in the language of gerbes.

Section 4.5 establishes the third main result of this thesis: the fundamental theorem for generalised hypersurfaces. Consider the ambient Courant algebroid E to be equipped with a pair $(\mathcal{G}, \mathrm{div})$, where $\mathcal{G}$ is a Riemannian generalised metric. Denote by D the canonical connection from Theorem 1.1. Then, E_Σ inherits data $(\mathcal{H}, \mathrm{div}_\Sigma, D^\Sigma, \mathcal{K}, \mathcal{L})$ consisting of a Riemannian generalised metric $\mathcal{H}$, a divergence operator div_Σ, a generalised connection D^Σ, generalised second fundamental forms $\mathcal{K}^{n\pm}$ and conormal extrinsic curvatures $\mathcal{L}^{\pm}$. Because D is assumed to be the canonical connection, this data satisfies a specific set of relations. For the purposes of this outline, data $(\mathcal{H}, \mathrm{div}_\Sigma, D^\Sigma, \mathcal{K}, \mathcal{L})$ consisting of the appropriate type of tensors satisfying these relations is referred to as an *admissible Riemannian extrinsic generalised geometry*. Furthermore, the set of equations which the generalised Gauß-Codazzi equations impose in case of a flat pair $(\mathcal{G}, \mathrm{div})$ on the data $(\mathcal{H}, \mathrm{div}_\Sigma, D^\Sigma, \mathcal{K}, \mathcal{L})$ is referred to as *the flat generalised Gauß-Codazzi equations*.

Theorem 1.4 (Fundamental Theorem of Generalised Hypersurfaces, Theorem 4.46)**.** *Let $E_\Sigma \to \Sigma$ be an exact Courant algebroid equipped with an admissible Riemannian extrinsic generalised geometry $(\mathcal{H}, \mathrm{div}_\Sigma, D^\Sigma, \mathcal{K}, \mathcal{L})$ which satisfies the flat generalised Gauß-Codazzi equations.*

Then, for every simply connected open set $U \subset \Sigma$ there exists a Riemannian (hypersurface) immersion $U \to \mathbb{R}^d$ such that the canonical structure on the untwisted generalised tangent bundle $\mathbb{T}\mathbb{R}^d$ induces $(\mathcal{H}, \mathrm{div}_\Sigma, D^\Sigma, \mathcal{K}, \mathcal{L})$.

In particular, since the geometry of the ambient space is trivial, all parts of the extrinsic generalised geometry of Σ must be trivial, except for the ordinary metric h and second fundamental form k, which have to satisfy the well-known flat Gauß and Codazzi equations from Riemannian geometry.

Idea. Recall that, in the Riemannian case, a flat pair $(\mathcal{G}, \mathrm{div})$ is trivial, i.e. for a flat pair the induced metric g is flat, the preferred representative of the Ševera class H vanishes, and the divergence is metric, $\mathrm{div} = \mathrm{div}^{\mathcal{G}}$. The idea is that these conditions should be encoded in the flat generalised Gauß-Codazzi equations. If we manage to show this, we can conclude from the formulas for $\mathcal{K}$ and $\mathcal{L}$ that all parts of the extrinsic generalised geometry must be trivial, except for h and k, for which the generalised Gauß-Codazzi equations reduce to the ordinary ones. Thus, we are allowed to apply the ordinary fundamental theorem for hypersurfaces (since U is simply connected), and obtain the result.

In proving the claim that the flat generalised Gauß-Codazzi equations force the extrinsic generalised geometry to reduce to an ordinary flat extrinsic geometry, we choose to avoid a repetition of the lengthy calculations which went into the result that generalised Riemann flatness implies complete triviality. Instead, we construct a synthetic generalised Riemann tensor to which that result applies directly. This is done as follows.

First, we define a synthetic conormal bundle $L \to \Sigma$ and a synthetic generalised normal bundle $\mathcal{N} \to \Sigma$. Then, we show that the extrinsic curvature data $(D^\Sigma, \mathcal{K}, \mathcal{L})$ defines an $E_\Sigma \oplus L$-connection $\tilde{D}$ on $E_\Sigma \oplus \mathcal{N}$. We show that $\tilde{D}$ satisfies analogues of the defining properties of the preferred LC generalised connection. In particular, we prove that one can associate a curvature tensor $\mathcal{R}m^{\tilde{D}}$ to $\tilde{D}$, and that this tensor has a decomposition which is analogous to that of the generalised Riemann tensor of the preferred LC generalised connection. In particular, $\mathcal{R}m^{\tilde{D}}$ vanishes as a consequence of the generalised Gauß-Codazzi equations. Trivially extending $\mathcal{R}m^{\tilde{D}}$ to a tensor on $E_\Sigma \oplus \mathcal{N}$, we obtain the synthetic generalised Riemann tensor to which the aforementioned trivialisation result applies. The claim follows. (☐)

Chapter 5 introduces those techniques developed by Ringström [10] which are relevant to the study of any Einstein-matter system as an initial value problem (IVP). The chapter is preliminary and contains no original results. It is based on the preliminary section of the independent work [22] of the author of this thesis.

An Einstein-matter system is a system of geometric PDEs for a tuple (M, g, Φ) consisting of a Lorentzian manifold (M, g) and matter fields Φ (which are a section of a natural vector bundle over the manifold M). The geometric PDEs consist of the Einstein equations

$$\mathrm{Rc} - \frac{\mathrm{Sc}}{2} = T[g, \Phi]$$

and equations of motions for the matter Φ. The latter have to imply energy-momentum conservation, $\operatorname{div} T = 0$, without imposing the Einstein equations.

A useful tool for establishing well-posedness of the IVP for a given Einstein-matter system is [10, Corollary 9.16.], included in this work as Theorem A.4. It states that a quasi-linear hyperbolic system of PDEs for a vector valued function $u\colon \mathbb{R}^{n+1} \to \mathbb{R}^N$, $N \in \mathbb{N}$, of the form

$$g^{\mu\nu}[u]\partial_\mu\partial_\nu u = f[u] \tag{1.3}$$

has a unique maximal solution given compactly supported initial data and minor well-behavedness assumptions on the "metric" g and the non-linearity f. Note that f and g may depend on the point $(t, x) \in \mathbb{R}^{n+1}$ and the values at the point (t, x) of the solution u and its first derivatives $\partial_\mu u$.

Section 5.1 uses Theorem A.4 to prove local (in space and time) existence and uniqueness of solutions to a more geometric version of the system (1.3), for which solutions $u = (g, u')$ consist of a Lorentz metric g and a tensor u' representing other matter fields. We refer to such a system as a *hyperbolic PDE with metric principal symbol*.

In the study of their IVP, there is a well-known problem with the Einstein equations: The Ricci (as well as the Einstein) tensor, understood as a second order differential operator acting on the metric, is not hyperbolic. Diffeomorphism invariance causes the operator to degenerate. One solution to this problem is to employ a gauge condition which breaks diffeomorphism invariance. This work relies on *DeTurck's gauge condition* [23]. Note that this gauge is a special case of Friedrich's "wave gauge with source functions" [24, 25].

Section 5.2 explains that the DeTurck gauge propagates well under the Einstein equations if the equations of motion for the matter imply $\operatorname{div} T = 0$. That is, in such a system, if the gauge condition is implemented on a subset $\Omega \subset \Sigma$ of a spacelike hypersurface, it is implemented on the entire Cauchy development $D(\Omega)$ of that subset.

Section 5.3 considers families of globally hyperbolic subsets of a globally hyperbolic ambient space with the property that there exists a Cauchy hypersurface in the ambient space for which the Cauchy property is preserved under restriction to any member of the family. The section proves global hyperbolicity of unions of arbitrary such families and intersections of finite such families. This is relevant to the patching together of local solutions and gauge transformations.

Chapter 6 presents original results establishing the well-posedness of the initial value problem (IVP) for the generalised Einstein equations (GEE). It is based on the independent work [22] of the author of this thesis.

The GEE are the combined equations of generalised Ricci and scalar flatness, $\mathcal{R}c = 0$ and $\mathcal{S}c = 0$. They are equations for a pair $(\mathcal{G}, \mathrm{div})$ consisting of a generalised metric $\mathcal{G}$ and a divergence operator div. The equation $\mathcal{S}c = 0$ can be understood as the equation of motion for the dilaton, and it is required for energy-momentum conservation to hold (cf. proof of Proposition 6.15).

Section 6.1 gives justification for the focus on the case of closed divergence in this work. In the case of closed divergence, the dilaton e is given by a closed one-form ξ, $e = \pi^*\xi$. Thus, the GEE are equations for a tuple (g, H, ξ) consisting of a Lorentz metric g, a closed three-form H, and a closed one-form ξ (the dilaton), and can be stated as

$$\mathrm{d}^*H = -i_\xi H, \qquad \mathrm{Rc} = \frac{H^2}{4} - \nabla\xi, \qquad \frac{|H|^2}{6} = \mathrm{d}^*\xi + |\xi|^2 . \tag{1.4}$$

Note that the GEE are also referred to as the *string frame* GEE. Note furthermore that, often, a two-form potential B for the three-form H and a scalar potential ϕ for the dilaton ξ are employed.

Section 6.2 adapts for the GEE the initial value formulation of the Einstein-scalar field system developed in [10]. In particular, the section defines (using the constraint equations for the GEE calculated in Section 4.4) *initial data* and the notion of a *development* of such data.

Idea. The first goal is to show that the GEE admit a development of arbitrary initial data. Following in spirit [10], we prove this by construction. The idea is the following. If, locally, the GEE were to form a hyperbolic PDE with metric principal symbol, then we would know from the preliminary Section 5.1 that there exists a solution to the GEE in a coordinate neighbourhood of every point of the initial hypersurface. Furthermore, because these solutions satisfy a uniqueness property, we could patch these solutions together to obtain the desired development.

However, there is a problem with this idea: Viewed as equations for a tuple (g, B, ϕ), the GEE are not locally of the form (1.3). To see this, assume $H = \mathrm{d}B$ and $\xi = \mathrm{d}\phi$. Then, the following three issues are immediate.

(i) The expression $\mathrm{d}^*\mathrm{d}B$ is not to leading order of the form $g^{\alpha\beta}\partial_\alpha\partial_\beta B_{\mu\nu}$.

(ii) The Ricci tensor is not to leading order of the form $g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu}$.

(iii) There are out-of-place second order derivatives of ϕ in the equation $\mathrm{Rc} = \frac{H^2}{4} - \nabla\xi$.

We can solve issue (ii) with the DeTurck gauge. Similarly, we can solve issue (i) by introducing the generalised Lorenz gauge. To solve issue (iii), we adopt the "Einstein frame". (□)

Section 6.3 introduces the *Einstein frame*. The Einstein frame GEE are the system of PDEs for which every solution $(\tilde{g}, H, \phi)$ is related to a solution (g, H, ϕ) of the string frame GEE (and vice versa) by the conformal transformation

$$\tilde{g} = e^{-2\kappa\phi} g, \qquad \kappa = \frac{1}{\dim M - 2} .$$

Note that defining the Einstein frame requires a potential ϕ for the dilaton, $\xi = \mathrm{d}\phi$, which is only defined locally and up to addition of a constant. The name "Einstein frame" is due to the supergravity literature, in which this conformal transformation is well-known. (Recall [2] that the GEE are equivalent to the equations of motion for (g, B, ϕ) in supergravity.)

Idea. The Einstein frame solves the issue (iii) remarked upon above by absorbing the out-of-place second derivatives of ϕ in the Ricci tensor of the conformally transformed metric $\tilde{g}$. One may view it as the defining property of κ that this is achieved. ($\square$)

Section 6.3 develops the initial value formulation for the Einstein frame GEE, relates (Lemma 6.14) Einstein and string frame initial data, and shows (Proposition 6.15) that in the Einstein frame, energy-momentum conservation is satisfied.

Section 6.4 recalls the DeTurck gauge and introduces the *generalised Lorenz gauge*. The section establishes (Proposition 6.18) that, with both gauge conditions implemented, the Einstein frame GEE form locally a hyperbolic system of PDEs with metric principal symbol (as discussed in Section 5.1). It is shown (Proposition 6.21) that the generalised Lorenz gauge propagates well.

Idea. The two gauge conditions solve issues (i) and (ii) remarked upon above.

For the DeTurck gauge, one introduces a background metric $\bar{g}$ (thus breaking diffeomorphism invariance) and constructs from it a covector $\mathcal{D} = \mathcal{D}[g, \bar{g}]$ and a modified Ricci tensor $\hat{\mathrm{Rc}}_{\mu\nu} = \mathrm{Rc}_{\mu\nu} + \nabla_{(\mu}\mathcal{D}_{\nu)}$. The idea is the following. If the gauge condition $\mathcal{D} = 0$ is implemented, one can consider instead of the Einstein equations the modified equations

$$\hat{\mathrm{Rc}} - \frac{g}{2}\operatorname{tr}_g \hat{\mathrm{Rc}} = T.$$

Then, defining $\mathcal{D}$ carefully, unwanted second derivatives of the metric g are replaced in the modified system by second derivatives of the background metric $\bar{g}$. In this way, issue (ii) is solved.

Issue (i) is due to invariance under B-field transformations. Inspired by the DeTurck gauge, we introduce the modification $\hat{\square}_{\mathrm{Hd}} = -(\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*_{g,\bar{g}})$ of the operator $-\mathrm{d}\mathrm{d}^*$, which is such that the correspondingly modified equations do not suffer from (i) and are, if the generalised Lorenz gauge $0 = \mathrm{d}\mathrm{d}^*_{g,\bar{g}}B$ is implemented, equivalent to the original equations. ($\square$)

Let $\mathcal{I}$ be initial data to the Einstein frame GEE over a manifold Σ. Consider the IVP on the background manifold $M = \mathbb{R} \times \Sigma$ equipped with a background metric $\bar{g}$. Denote by t the canonical coordinate induced by the $\mathbb{R}$-factor.

Note that, because of the gauge invariance under generalised diffeomorphisms (i.e. combinations of diffeomorphisms and B-field transformations), the geometric initial data does not determine all components of the initial values $g_{\mu\nu}|_{t=0}$ and $B_{\mu\nu}|_{t=0}$ on the initial hypersurface $\{t = 0\}$ uniquely. However, to employ the local results developed in the preliminary Section 5.1, these values are required.

Section 6.5 introduces (utilising the canonical transversal vector field ∂_t) geometric conditions on the initial values of the fields which determine them uniquely from the initial data. In particular, these conditions guarantee (Lemma 6.24) that the DeTurck gauge and the generalised Lorenz gauge are initially implemented. Furthermore, the conditions are designed and demonstrated (Lemma 6.25) to be invariant under a change of Einstein frame. (Recall that the Einstein frame only exists locally and is not unique.) In particular, it is established that the DeTurck gauge and the generalised Lorenz gauge are invariant under a change of Einstein frame.

Section 6.6 applies the previously developed theory to the Einstein frame GEE to obtain (Lemmas 6.26 and 6.27) existence and uniqueness of local (in space and time) developments. Theorem 6.28 finally shows that, indeed, the local Einstein frame developments can be patched together to a globally hyperbolic string frame development on a tubular neighbourhood $D \subset M$ of the $\{t = 0\}$-hypersurface. Note that, in this construction, the string frame development is locally related to Einstein frame developments which satisfy the DeTurck gauge and the Lorenz gauge.

In Section 6.7, it is shown (Lemma 6.29) that the generalised Lorenz gauge can be implemented by virtue of a closed B-field transformation. Given an arbitrary development of initial data, it is concluded (Proposition 6.30) that one can locally find a generalised diffeomorphism which relates it to the development constructed in Section 6.6. Patching these local gauge transformations together to obtain a global one, Theorem 6.31 obtains geometric uniqueness of string frame developments.

Section 6.8 employs the famous result by Choquet-Bruhat and Geroch [8] to conclude from the existence of developments and their geometric uniqueness the fourth and final main result of this thesis.

Theorem 1.5 (Theorem 6.33)**.** *Let $\mathcal{I}$ be initial data for the string frame GEE. Then there exists a maximal globally hyperbolic development (MGHD) of $\mathcal{I}$ in the string frame. It is unique up to diffeomorphism.*

Appendix. Appendix A presents results on wave equations employed in chapters 5 and 6. Section A.1 summarises results from [10] on non-linear wave equations over $\mathbb{R}^{n+1}$. Section A.2 recalls a result on tensor wave equations on manifolds from [10], and establishes the related notion of *form wave equations* which encompasses the B-field equation in the GEE (1.4). Section A.3 shows that the B-field equations is a very particular type of form wave equation, and establishes well-posedness results for this type of equation.

Appendix B presents an auxiliary result on the decomposition of the exterior differential and co-differential of a form over a hypersurface.

Appendix C investigates the divergence of tensors in generalised geometry. It establishes that the divergence of mixed-type tensors can be invariantly defined given a pair $(\mathcal{G}, \mathrm{div})$. This is relevant to the interpretation of the constraint equations for the GEE (1.2).

1.2 Notation and Conventions

This section is intended to provide an overview of notation and conventions employed in this thesis.

We assume smoothness throughout.

M always denotes a $d = n + 1$-dimensional manifold. If there is another manifold immersed, we name this N. If N is assumed to be of co-dimension one, we denote it as Σ. For Part II, we assume Σ to be embedded.

Generally, we denote vector fields by capital Latin letters such as $V, W, X, Y, ... \in \Gamma(TM)$. However, while in Part II we denote the unit normal on a hypersurface Σ by N, we change this to a lower case n in Chapter 4 of Part I to avoid confusion with the more general immersed manifold N.

Notation for Part I. We mostly work on an exact Courant algebroid $\pi\colon E \to M$ over M. We generally denote sections of an exact Courant algebroid, i.e. generalised vector fields, by lower case latin letters $a, b, v, w, \ldots$. We try to match letters between corresponding generalised vector fields and usual vector fields, so that we have e.g. $V = \pi v$. Given a generalised metric on E, we denote the associated sections in the ± 1 eigenbundle of E by $n_\pm$, e.g. $\pi n_\pm = n$.

Working with a representative $H \in \Omega^3(M)_{\mathrm{cl}}$ of the Ševera class of E, and given a metric g on M, we make frequent use of the following two contractions of $H^{\otimes 2} = H \otimes H$ with g: $H^{(2)}$, the $(0,4)$ tensor such that $H^{(2)}(X, Y, V, W) = \operatorname{tr}_g H(X, Y, \cdot)H(V, W, \cdot)$, and H^2, the $(0,2)$ tensor such that $H^2(X, Y) = \operatorname{tr}_g H^{(2)}(X, \cdot, Y, \cdot)$. Up to sign, these are the only possible non-trivial contractions of type $(0,4)$ and $(0,2)$, respectively. Note that, with H_X the g-skew-symmetric endomorphism such that $g(H_X Y, Z) = H(X, Y, Z)$ for all $X, Y, Z \in \Gamma(TM)$, we have $H^2(X, Y) = -\operatorname{tr}_g H_X H_Y$. Our convention for the scalar product of alternating forms induced by the semi-Riemannian metric is that we consider forms as special tensors and use the usual scalar product on tensors. In particular,

$$|H|_g^2 := H_{ijk}H^{ijk},$$

where indices were raised with the metric.

Provided a generalised metric $\mathcal{G}$ and a divergence operator div on E, we often work with the generalised vector field $e \in \Gamma(E)$ defined via $\operatorname{div} = \operatorname{div}^{\mathcal{G}} - \langle e, \cdot\rangle$. Due to the splitting induced by $\mathcal{G}$, we can then write $e = 2(X + \xi) \in \Gamma(\mathbb{T}M)$. Of particular importance will be the special case that $X = 0$ and $\xi = \mathrm{d}\phi$ for some function $\phi \in C^\infty(M)$, which we sometimes refer to as "the dilaton".

For the extrinsic curvature, our sign conventions are such that the second fundamental form is given by (recall that n denotes the unit normal) $k(X, Y) = g(\nabla^g_X n, Y)$, and the shape tensor by $AX = \nabla^g_X n$, $X, Y \in \Gamma(T\Sigma)$, in terms of the Levi-Civita connection ∇^g. Our sign convention for the Riemann tensor is such that $\mathrm{Rm}(X, Y)Z = (\nabla^2_{X,Y} - \nabla^2_{Y,X})Z$ and $\mathrm{Rm}(W, Z, X, Y) = g(\mathrm{Rm}(X, Y)Z, W)$. Then $\mathrm{Rc}(X, Y) = \operatorname{tr}_g \mathrm{Rm}(\cdot, X, \cdot, Y)$. We employ similar conventions in the "generalised" case.

Notation in Part II. In our discussion of the initial value problem in Chapters 5 and 6, (M, g) denotes a $d = n+1$-dimensional Lorentzian manifold. We set $\kappa = \frac{1}{d-2}$. We call a time-oriented Lorentzian manifold a *spacetime*. We call a tuple (M, g, H, ϕ) consisting of a spacetime (M, g), a closed three-form $H \in \Omega^3(M)$ and a function $\phi \in C^\infty(M)$ a *SuGra spacetime*. We denote the Beltrami wave operator as $\Box = \nabla^\mu\nabla_\mu$ and the Hodge wave operator as $\Box_{\mathrm{Hd}} = -\mathrm{d}^*\mathrm{d} - \mathrm{d}\mathrm{d}^*$. We denote the interior product with a vector field X by i_X, so that for a one-form ξ we have $i_X(\xi) = \xi(X)$. For convenience, we also write i_ξ for the interior product with the vector field $\xi^\sharp$ obtained from raising an index on a given one-form ξ.

Working in indices, we employ the Einstein summation convention throughout. Greek indices $\alpha, \beta, \mu, \nu, \ldots$ take values $0, \ldots, n$, where 0 is assumed to correspond to a timelike direction. Latin indices $a, b, i, j, \ldots$ take values $1, \ldots, n$ and are assumed to correspond to spacelike directions.

We make use of round brackets to denote normalised symmetrisation, and square brackets to denote normalised antisymmetrisation. Thus, for example, the Lie derivative of the metric can be written as $L_X g_{\mu\nu} = 2\nabla_{(\mu}X_{\nu)}$. Working without indices, we denote the same operations by a superscript "sym" and "antisym", respectively. Thus, $L_X g = 2[\nabla X^\flat]^{\mathrm{sym}}$ with $\flat\colon T^*M \to TM$ the musical isomorphism coming from g.

Given a semi-Riemannian embedded hypersurface Σ in (M, g) with unit normal N, we decompose the restriction of a p-form $A \in \Omega^p(M)$ in terms of its tangent and normal part. The tangent part $A^{\|} \in \Omega^p(\Sigma)$ is the restriction of A to $T\Sigma$, and the normal part $A^{\perp} \in \Omega^{p-1}(\Sigma)$ is the restriction of $i_N A$ to $T\Sigma$. In particular, $A = A^{\|} + \varepsilon\, N^{\flat} \wedge A^{\perp}$, where $N^{\flat} = g(N, \cdot)$ and $\varepsilon = g(N, N)$.

In this setting, it is sometimes of interest to decompose the restricted exterior derivative $\mathrm{d}A|_{\Sigma}$ in terms of contributions from the parallel and normal parts of A and $\nabla_N A$. Denoting by k the second fundamental form on Σ, one finds the following contribution (cf. Lemma B.2)

$$\left(k \cdot A^{\|}\right)_{i_1 \dots i_p} \coloneqq (-1)^p p\, k^{j}_{[i_1} A^{\|}_{i_2 \dots i_p]j}$$

Herein, $\cdot$ is the natural Lie algebra action of endomorphisms on the space of p-forms (for convenience, we use the letter k also for the Weingarten map, i.e. the endomorphism obtained from the second fundamental form by raising an index).

– Part I –

The Einstein Equations in Generalised Geometry

2 Hitchin's Generalised Geometry

This chapter provides an introduction to Hitchin's Generalised Geometry and does not contain any original results. Section 2.1 focuses on communicating ideas from [13, 6] that lead to the notion of an exact Courant algebroid (CA). A formal approach to exact CAs based on [14, 6] is presented in Section 2.2. Sections 2.3-2.5, which introduce geometrical objects on exact CAs, follow in broad strokes [6]. Finally, Section 2.6 gives a description of the generalised Riemann, Ricci and scalar curvature based on [5, 15].

2.1 A Pedagogical Introduction

This pedagogical introduction argues that generalised geometry is a well-motivated geometric theory. It focuses on communicating ideas and providing intuitions.

Given a smooth d-dimensional manifold M, a basic object of study in generalised geometry is the *generalised tangent bundle* $\mathbb{T}M = TM \oplus T^*M$. It admits a natural class of transformations unknown to the (usual) tangent bundle: B-field transformations. This section explains, drawing on ideas from [13, 6], how a notion of isomorphy between generalised tangent bundles which combines diffeomorphisms with B-field transformations leads to the definition of exact Courant algebroids (CAs).

The first thing to note about the generalised tangent bundle is that contraction defines a natural inner product of neutral signature on $\mathbb{T}M$. Explicitly,

$$\langle \cdot,\cdot\rangle : \mathbb{T}M \otimes \mathbb{T}M \longrightarrow C^\infty(M); \qquad \langle X+\xi, Y+\eta\rangle := \frac{1}{2}(\xi(Y)+\eta(X)). \tag{2.1}$$

It should be stressed that this scalar product is part of the manifold structure and should (conceptually) not be understood as a *generalised* version of a semi-Riemannian metric (though formally of course it is a semi-Riemannian metric on $\mathbb{T}M$).

Consider the inner product $\langle\cdot,\cdot\rangle$ as an isomorphism $\flat\colon \mathbb{T}M \to \mathbb{T}^*M$. Writing elements of $\mathbb{T}M = TM \oplus T^*M$ as column vectors and elements of $\mathbb{T}^*M \cong TM \oplus T^*M$ as line vectors, transposition of vectors defines an isomorphism:

$$(\cdot)^t\colon \mathbb{T}M \longrightarrow \mathbb{T}^*M; \qquad \begin{pmatrix} X \\ \xi \end{pmatrix}^t \longmapsto \begin{pmatrix} X, & \xi \end{pmatrix}.$$

With this, we can write $\flat$ in block-diagonal matrix form as

$$\flat = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We follow Hitchin's introduction to generalised geometry [13] in discussing the symmetries of the inner product. The symmetry group of $\langle\cdot,\cdot\rangle$ over an arbitrary point $p \in M$ is $O(d,d)$. Any given element of the corresponding Lie algebra $\mathfrak{so}(d,d)$ can be written in block-diagonal form as

$$\begin{pmatrix} A & \beta \\ B & -A^t \end{pmatrix} \quad \text{for some } A \in \mathrm{End}(T_pM),\ B \in \Lambda^2 T_p^*M, \text{ and } \beta \in \Lambda^2 T_pM,$$

where B is viewed as acting on T_pM by $X \mapsto i_X B$, and similarly for β. Indeed,

$$\flat \begin{pmatrix} A & \beta \\ B & -A^t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} B & -A^t \\ A & \beta \end{pmatrix} = \begin{pmatrix} -A^t & B \\ \beta & A \end{pmatrix}\flat = -\begin{pmatrix} A & \beta \\ B & -A^t \end{pmatrix}^t \flat.$$

The decomposition of an arbitrary element of $\mathfrak{so}(d,d)$ in A, B, and β reveals three types of infinitesimal symmetries of the inner product. Exponentiating an element with only A non-trivial, we obtain the symmetry

$$\exp\begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} = \begin{pmatrix} \exp(A) & 0 \\ 0 & \exp(-A^t) \end{pmatrix}$$

Note that $\exp(A) \in \mathrm{GL}(d)$, and thus invariance of $\langle\cdot,\cdot\rangle$ under these types of elements is implied by diffeomorphism-invariance. Of more interest to us is that exponentiating an element with only B non-trivial yields a *B-field transformation*:

$$\exp(B) = 1 + B, \qquad \text{where we write} \quad B = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}.$$

In the following, we see the notion of exact CAs arise from the promotion of B-field invariance to a fundamental symmetry on par with diffeomorphism invariance. In fact, B-field transformations and diffeomorphisms are united in the notion of generalised diffeomorphisms. Note that, while there is a similar notion of β-field invariance, it is not of explicit importance to us. In a sense, this is a breaking of symmetry.

Definition 2.1 (Generalised Diffeomorphism)**.** The group of *generalised diffeomorphisms on M* is the semi-direct product $\mathrm{Diff}(M) \ltimes \Omega^2(M)$ with composition given by

$$(f_1, B_1) \circ (f_2, B_2) = (f_1 \circ f_2, f_2^* B_1 + B_2).$$

Remark 2.2. In standard terminology, a generalised diffeomorphism relates two generalised tangent bundles $(\mathbb{T}M, [\cdot,\cdot]_{H_1})$ and $(\mathbb{T}N, [\cdot,\cdot]_{H_2})$ with fixed "twisted brackets" $[\cdot,\cdot]_{H_1}$ and $[\cdot,\cdot]_{H_2}$ (we introduce these brackets soon). In particular, a generalised diffeomorphism is required to relate the twists (closed three-forms) H_1 and H_2, i.e. $H_1 = f^*(H_2 + \mathrm{d}B)$. This work refers to such generalised diffeomorphisms as "Courant isomorphisms".

The term "generalised diffeomorphism on M", on the other hand, is reserved for the situation that we want to relate the twisted bundle $(\mathbb{T}M, [\cdot,\cdot]_{H_1})$ to $(\mathbb{T}M, [\cdot,\cdot]_{H_2})$ for (a priori) any choice of twist H_2. For this reason, we arrive at Definition 2.1. Note that this meaning of the term is natural in the context of the generalised Einstein equations, cf. Remark 6.7.

Note that the inverse of an element (f, B) is given by $(f^{-1}, -f_* B)$. The natural action of a generalised diffeomorphism $F = (f, B)$ on a generalised vector field is given by

$$X + \xi \longmapsto f_*(e^B(X + \xi)) = f_*(X + \xi + i_X B).$$

It will become important to note that the *anchor map*

$$\pi\colon \mathbb{T}M \longrightarrow TM, \qquad X + \xi \longmapsto X \tag{2.2}$$

is equivariant under generalised diffeomorphisms in that $\pi(F(X + \xi)) = f_* \pi(X)$.[4]

We now define a bracket on $\mathbb{T}M$ which is a suitable generalisation of the Lie bracket. In this, we follow [6, § 2.2]. Remember that the Lie bracket can be obtained as follows. Given a vector field X on M, there exists a smooth one-parameter family of diffeomorphisms $f(s)$

[4]In fact, from the proof of Proposition 2.11, it is clear that the space of generalised diffeomorphisms on $\mathbb{T}M$ is the space of vector bundle automorphisms of $\mathbb{T}M$ (covering a diffeomorphism of M) which respect $\langle\cdot,\cdot\rangle$ and π.

of M such that $f(0) = \mathrm{id}_M$ and $f'(0) = X$ (e.g. the flow of X, if X is complete). Given any such family, the Lie derivative of a vector field Y is given by

$$L_X Y = \frac{d}{ds}\bigg|_{s=0} f_*^{-1}(s) Y.$$

Similarly, given any generalised vector field $X + \xi$, there exists a smooth one-parameter family of generalised diffeomorphisms $F(s) = (f(s), B(s))$ such that $F(0) = (\mathrm{id}_M, 0)$ and $F'(0) = (X, \mathrm{d}\xi)$. The infinitesimal action of such a family defines the *untwisted Dorfman bracket*:

$$\begin{aligned}[X+\xi, Y+\eta] &\coloneqq \frac{d}{ds}\bigg|_{s=0} F^{-1}(s)(Y+\eta) \\ &= \frac{d}{ds}\bigg|_{s=0} f_*^{-1}(s)[Y + \eta - i_Y f_*(s) B(s)] \\ &= L_X(Y+\eta) - i_Y \mathrm{d}\xi\end{aligned}$$

Note that not all one-parameter families of generalised diffeomorphisms starting at the identity are infinitesimally described by a generalised vector field; This is only true if $B'(0)$ is exact, which is of course locally equivalent to being closed. For this reason, closed B-field transformations are special in that they leave the Dorfman bracket invariant. In fact, focussing on B-field transformations, one can check that

$$[e^B(X+\xi), e^B(Y+\eta)] = e^B[X+\xi, Y+\eta] + i_Y i_X \mathrm{d}B. \tag{2.3}$$

We are lead to introducing the *twist* H, which is a closed three-form on M. Defining the *H-twisted Dorfman bracket*

$$[X+\xi, Y+\eta]_H = L_X(Y+\eta) - i_Y \mathrm{d}\xi + i_Y i_X H \tag{2.4}$$

one finds that a generalised diffeomorphism $F = (f, B)$ with non-closed B-field simply causes a change of twist,

$$F^{-1}[F(\cdot), F(\cdot)]_H = [\cdot,\cdot]_{f^*H + \mathrm{d}B}. \tag{2.5}$$

Definition 2.3 (Twisted Generalised Tangent Bundle)**.** The tuple $(\mathbb{T}M, \pi, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H)$, where π denotes the anchor map (2.2), $\langle\cdot,\cdot\rangle$ the inner product (2.1), and $[\cdot,\cdot]_H$ the H-twisted Dorfman bracket (2.4), is called the *H-twisted generalised tangent bundle.*

Note that (while one may discuss B-field invariance on it) the twisted generalised tangent bundle fundamentally violates even closed B-field invariance. The reason is the direct sum structure $TM \oplus T^*M$, which allows for an explicit and canonical breaking of B-field invariance.[5] In light of the status to which B-field invariance is elevated in the present considerations, this should be viewed as undesirable. In the following, we see how an effort to overcome this issue leads to the notion of exact CAs.

Two aspects of the direct sum structure $TM \oplus T^*M$ are B-field invariant. The first is the vector part of generalised vector fields, as encoded in the B-field invariance of the anchor, $\pi e^B = \pi$. The second is dual. Consider the anchor's dual $\pi^*\colon T^*M \to \mathbb{T}^*M, \xi \mapsto \langle 2\xi, \cdot\rangle$.

[5]This is similar to how a choice of representative M of the diffeomorphism class $[M]$ breaks diffeomorphism invariance. However, in the abstract setting, this choice does not naturally come with *explicit* non-invariant structure. Thus, (as far as the author is aware) there is no insight to be gained from working with the diffeomorphism class $[M]$ instead of the manifold M.

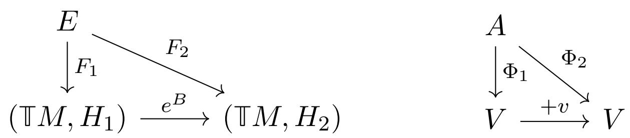
Employing the isomorphism $\mathbb{T}M \cong \mathbb{T}^*M$ coming from $\langle \cdot, \cdot \rangle$, one can write $\pi^* : \mathbb{T}M \to \mathbb{T}M, \xi \mapsto 2\xi$. The relation $e^B\pi^* = \pi^*$ encodes the second B-field invariant aspect of the direct sum structure of $\mathbb{T}M$: the inclusion of the space of one-forms in $\mathbb{T}M$. We have obtained the B-field invariant short exact sequence

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} \mathbb{T}M \xrightarrow{\pi} TM \longrightarrow 0.$$

A splitting of this sequence respecting the inner product is a vector bundle isomorphism $\sigma : TM \to \tilde{T}$ onto an isotropic subbundle $\tilde{T} \subset \mathbb{T}M$ such that $\pi \circ \sigma = \mathrm{id}$. We call such a map an *isotropic splitting*. One can see that the space of isotropic splittings is an affine space over the space of two-forms; Given any isotropic splitting σ_1, any other isotropic splitting σ_2 can be written as $e^B\sigma_1 = \sigma_1 + B$ for a unique two-form B (cf. Lemma 2.9). Thus, the canonical splitting $TM \to TM \oplus T^*M$ is the explicit structure breaking B-field invariance. The notion of an exact CA is obtained by replacing the generalised tangent bundle $\mathbb{T}M$ with an abstract vector bundle $E \to M$ and thus forgetting about the canonical splitting.

Definition 2.4 (Exact Courant Algebroid)**.** An exact Courant algebroid is a tuple $(E, \pi_E, \langle \cdot, \cdot \rangle_E, [\cdot, \cdot]_E)$ consisting of a vector bundle $E \to M$, a vector bundle homomorphism $\pi_E : E \to TM$, and an inner product $\langle \cdot, \cdot \rangle_E$ on E, and a bracket $[\cdot, \cdot]_E$ on sections of E such that there exists a three-form H on M and a vector bundle isomorphism $F : E \to \mathbb{T}M$ covering the identity which relates $(E, \pi_E, \langle \cdot, \cdot \rangle_E, [\cdot, \cdot]_E)$ with the H-twisted generalised tangent bundle $(\mathbb{T}M, \pi, \langle \cdot, \cdot \rangle, [\cdot, \cdot]_H)$.

To repeat the idea, an exact Courant algebroid is a structure which can be identified with an H-twisted generalised tangent bundle for some twist H, but without a canonical choice of identification. Any two choices of isomorphism F_1 and F_2 are related by a unique B-field transformation. Below, we summarise this in a commutative diagram, and provide for comparison a similar commutative diagram for an affine space A over a vector space V. Note that any two isomorphisms Φ_1 and Φ_2 between A and V are related by addition of a unique vector $v \in V$.



In the next section, we learn how to define an exact Courant algebroid axiomatically.

2.2 Exact Courant Algebroids

In this section, we provide the axiomatic definition of exact Courant algebroids (CAs) and prove equivalence to the Definition 2.4 given in the preceding chapter.

We follow [14] in the definition of an exact CA.

Definition 2.5 (Courant Algebroid)**.** A *Courant algebroid* (CA) is a tuple $(E, \pi, \langle \cdot, \cdot \rangle, [\cdot, \cdot])$ consisting of a vector bundle $E \to M$, a vector bundle homomorphism $\pi : E \to TM$ called *anchor*, a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on E, and a bracket $[\cdot, \cdot]$ on sections of E, all such that one has

(i) the Jacobi identity

$$[a, [b, c]] = [b, [a, c]] + [[a, b], c],$$

(ii) compatibility of the bracket with the inner product

$$\pi(c)\langle a, b\rangle = \langle [c, a], b\rangle + \langle a, [c, b]\rangle\,,$$

(iii) the constraint on the symmetric part of the bracket

$$[a, b] + [b, a] = \pi^* \mathrm{d}\langle a, b\rangle\,,$$

(iv) the Leibniz rule

$$[a, fb] = \pi(a)(f)b + f[a, b],$$

(v) and the bracket morphism property

$$\pi[a, b] = L_{\pi a}\pi b.$$

Herein, $a, b, c \in \Gamma(E)$ and $f \in C^\infty(M)$. Note that we make use of the natural isomorphism $E^* \cong E$ provided by the inner product $\langle\cdot,\cdot\rangle$ to view $\pi^*\colon T^*M \to E^*$ as a map $T^*M \to E$.

Remark 2.6. By [14, Proposition 1.1.3], the first three properties imply the last two.

Definition 2.7 (Courant Isomorphism)**.** A *Courant isomorphism* between two Courant algebroids $E_1 \to M_1$ and $E_2 \to M_2$ is an orthogonal vector bundle isomorphism $F\colon E_1 \to E_2$ covering a diffeomorphism $f\colon M_1 \to M_2$ relating the brackets and the anchor maps, i.e.

$$[Fa, Fb]_2 = F[a, b]_1, \qquad \pi_2 Fa = f_*\pi_1 a, \qquad \text{for all } a, b \in \Gamma(E_1).$$

A *Courant automorphism* is a Courant isomorphism between a given Courant algebroid and itself.

Remark 2.8. Note that [14] refers to maps as in Definition 2.7 as Courant autoequivalences, while Courant automorphisms are required to cover the identity. Note furthermore that [6] does not explicitly require Courant isomorphisms to relate the anchor maps, but concludes this instead from the other properties.

As a consequence of these axioms, $\pi \circ \pi^* = 0$ (equivalently, $\operatorname{im}\pi^*$ is isotropic with respect to the symmetric product):

$$\pi(\pi^*\mathrm{d}\langle a, b\rangle) = \pi([a, b] + [b, a]) = L_{\pi a}\pi b + L_{\pi b}\pi a = 0$$

A Courant algebroid is called *exact* if it fits into the short exact sequence (SES)

$$0 \longrightarrow T^* \xrightarrow{\pi^*} E \xrightarrow{\pi} T \longrightarrow 0 \tag{2.6}$$

In this case, $\ker\pi = \operatorname{im}\pi^*$ is isotropic. If a splitting $\sigma\colon TM \to E$ of the sequence respects this isotropy, i.e. if $\operatorname{im}\sigma \subset E$ is isotropic, it is called an *isotropic splitting*.

Note that there always exists an isotropic splitting of the SES (2.6). To see this, let $\sigma_0\colon TM \to E$ be a (potentially non-isotropic) splitting. Then, setting

$$\sigma(X) \coloneqq \sigma_0(X) - \pi^*\langle\sigma_0(X), \sigma_0(\cdot)\rangle\,, \qquad X \in \Gamma(TM)$$

one obtains the isotropic splitting σ:

$$\begin{aligned}\langle\sigma X, \sigma Y\rangle &= \langle\sigma_0 X, \sigma_0 Y\rangle - \langle\pi^*\langle\sigma_0(X), \sigma_0(\cdot)\rangle\,, \sigma_0 Y\rangle - \langle\pi^*\langle\sigma_0(Y), \sigma_0(\cdot)\rangle\,, \sigma_0 X\rangle \\ &= \langle\sigma_0 X, \sigma_0 Y\rangle - \frac{1}{2}\langle\sigma_0 X, \sigma_0 Y\rangle - \frac{1}{2}\langle\sigma_0 X, \sigma_0 Y\rangle = 0\end{aligned}$$

Lemma 2.9. *Let $E \to M$ be an exact CA. The associated space of isotropic splittings is an affine space over the space of two-forms $\Omega^2(M)$.*

Proof. Let $\sigma_{1,2} \colon M \to E$ be isotropic splittings. Then by $\pi \circ \sigma_i = \mathrm{id}_{TM}$, $i = 1, 2$, implies that the image of $\sigma_2 - \sigma_1$ is contained in $\ker \pi = \operatorname{im} \pi^*$. Since π^* is an isomorphism onto its image, we can identify $\sigma_2 - \sigma_1$ with a homomorphism $TM \to T^*M$, or equivalently a tensor $B \in T^*M \otimes T^*M$. From isotropy of both splittings we get

$$0 = |\sigma_2(X)|_E^2 = |\sigma_1(X) + \pi^* B(X)|_E^2 = B(X, X)$$

where $|a|_E^2 = \langle a, a \rangle$ for all $a \in E$. Thus, B is antisymmetric.

Conversely, it is immediate that for any isotropic splitting σ_1 and any two-form $B \in \Omega^2(M)$ the map $\sigma_2 = \sigma_1 + \pi^* B$ defines another isotropic splitting. □

We have seen in the previous chapter an example for an exact Courant algebroid, namely the H-twisted generalised tangent bundle (cf. Definition 2.3). In fact, as we prove below following [6, §2.1], it is the only example up to isomorphy.

We now prove that an isotropic splitting of an exact CA $E \to M$ defines a Courant isomorphism between E and the H-twisted generalised tangent bundle for a particular choice of H. This establishes equivalence between the definition of an exact CA as provided in the preceding section (cf. Definition 2.4) and in this one (cf. Definition 2.5 and the SES (2.6)).

Theorem 2.10. *Let $(E, \pi_E, \langle \cdot, \cdot \rangle_E, [\cdot, \cdot]_E)$ be an exact CA over M, and let $\sigma \colon TM \to E$ be an isotropic splitting. Define a three-form H via*

$$H(X, Y, Z) := 2 \langle [\sigma X, \sigma Y], \sigma Z \rangle, \qquad X, Y, Z \in \Gamma(TM).$$

Then, E is isomorphic to the H-twisted generalised tangent bundle with isomorphism $F : E \to TM \bigoplus T^*M$ given by

$$F^{-1}(X + \xi) = \sigma X + \frac{1}{2} \pi_E^* \xi, \qquad X + \xi \in \Gamma(TM \oplus T^*M)$$

Proof. We follow the proof of the equivalent [6, Proposition 2.10.]. First, we note that the map F indeed is an isomorphism of vector bundles because $\operatorname{rk} E = \operatorname{rk} TM + \operatorname{rk} T^*M$ by the SES (2.6) and the complementarity of the subbundles $\operatorname{im} \sigma$ and $\operatorname{im} \pi^*$ of E. It is also trivial to see that the bracket is preserved:

$$\pi_E F^{-1}(X + \xi) = \pi_E \sigma X = X = \pi(X + \xi), \qquad X + \xi \in \Gamma(\mathbb{T}M).$$

We check with $X + \xi \in \Gamma(\mathbb{T}M)$ that F preserves the inner product:

$$\begin{aligned} \left\langle F^{-1}(X + \xi), F(X + \xi) \right\rangle_E &= \left\langle \sigma X + \frac{1}{2} \pi^* \xi, \sigma X + \frac{1}{2} \pi^* \xi \right\rangle_E = 2 \left\langle \sigma X, \frac{1}{2} \pi^* \xi \right\rangle_E \\ &= \xi(X) = \langle X + \xi, X + \xi \rangle . \end{aligned}$$

Finally, we check that the Dorfman bracket is preserved, i.e. that for $X + \xi, Y + \eta \in \Gamma(\mathbb{T}M)$

$$F[F^{-1}(X + \xi), F^{-1}(Y + \eta)]_E = [X + \xi, Y + \eta]_H = L_X(Y + \eta) - i_Y \mathrm{d}\xi + i_Y i_X H.$$

We use linearity to split the computation in components. We first show $[F^{-1}\xi, F^{-1}\eta]_E = 0$:

$$\begin{aligned}\pi_E[\pi_E^*\xi, \pi_E^*\eta]_E &\overset{(v)}{=} L_{\pi_E\pi_E^*\xi}\pi_E\pi_E^*\eta = 0,\\ \langle[\pi_E^*\xi, \pi_E^*\eta]_E, \sigma Z\rangle_E &\overset{(ii)}{=} \pi_E\pi_E^*\xi\,\langle\pi_E^*\eta, \sigma Z\rangle_E - \langle\pi_E^*\eta, [\pi_E^*\xi, \sigma Z]\rangle_E\\ &= \eta(\pi_E[\pi_E^*\xi, \sigma Z]) = 0.\end{aligned}$$

We introduced $Z \in \Gamma(TM)$, indicated with Roman numerals the definitional CA property employed, and used frequently that $\pi_E \circ \pi_E^* = 0$ and $\operatorname{im}\pi_E^*$ isotropic.

We show next that $F[F^{-1}X, F^{-1}\eta]_E = L_X\eta$. Similarly to before, $\pi_E[\sigma X, \pi_E^*\eta] = 0$. Furthermore with $Z \in \Gamma(TM)$

$$\begin{aligned}\langle[\sigma X, \pi_E^*\eta]_E, \sigma Z\rangle_E &\overset{(v)}{=} X\,\langle\pi_E^*\eta, \sigma Z\rangle_E - \langle\pi_E^*\eta, [\sigma X, \sigma Z]_E\rangle\\ &= X(\eta(Z)) - \eta(L_X Z) = (L_X\eta)(Z) = 2\,\langle L_X\eta, Z\rangle.\end{aligned}$$

We proceed to show that $F[F^{-1}\xi, F^{-1}Y]_E = -i_Y\mathrm{d}\xi$. With the constraint on the symmetric part of the bracket, we find

$$\begin{aligned}[\pi_E^*\xi, \sigma Y]_E &= \pi^*\mathrm{d}\,\langle\pi_E^*\xi, \sigma Y\rangle_E - [\sigma Y, \pi_E^*\xi]_E = \pi^*(\mathrm{d}(\xi(Y))) - F^{-1}(L_Y\xi)\\ &= -F^{-1}(i_Y\mathrm{d}\xi).\end{aligned}$$

Finally, we show that $F[F^{-1}X, F^{-1}Y]_E = L_XY + i_Yi_XH$. The bracket morphism property guarantees $\pi_E[F^{-1}X, F^{-1}Y]_E = L_XY$. Furthermore, by definition of H, for all $Z \in \Gamma(TM)$

$$\langle[\sigma X, \sigma Y], \sigma Z\rangle = \langle H(X,Y), Z\rangle.$$

Note that H is a well-defined three-form: It is antisymmetric in X and Y by the constraint on the symmetric part of the bracket, antisymmetric in Y and Z by the compatibility of the bracket and the inner product, hence totally antisymmetric, and thus the obvious tensoriality in Z implies overall tensoriality. □

Let us give the following characterisation of the automorphism group of the H-twisted generalised tangent bundle $\mathbb{T}M$.

Proposition 2.11. *The automorphism group of the H-twisted generalised tangent bundle $\mathbb{T}M$ consists of those generalised diffeomorphisms $F = (f, B)$ (in the sense of Definition 2.1) which preserve the twist H, i.e. for which $H = f_*(H - \mathrm{d}B)$.*

Proof. Let F be a Courant automorphism of $(\mathbb{T}M, \pi, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H)$. By definition F covers a diffeomorphism $f\colon M \to M$. Consider the vector bundle isomorphism $\Phi := f^* \circ F$, which covers the identity. Φ is orthogonal, as it is a composition of orthogonal vector bundle isomorphisms. By the Courant automorphism property $\pi F = f_*\pi$, we have $\pi\Phi = \pi$. In particular, $\pi\Phi(\xi) = 0$ for $\xi \in T^*M$ and the map $TM \ni X \mapsto \Phi(X) \in \mathbb{T}M$ defines an isotropic splitting of $\mathbb{T}M$ and is thus by Lemma 2.9 a B-field transformation, i.e. $\Phi(X) = e^B(X)$ for some two-form $B \in \Omega^2(M)$. Orthogonality yields that

$$\frac{1}{2}\xi(X) = \langle\Phi(\xi), \Phi(X)\rangle = \left\langle\Phi(\xi), e^BX\right\rangle = \frac{1}{2}\Phi(\xi)(X).$$

In the last equality, we used that $\Phi(\xi) \in \ker\pi = \operatorname{im}\pi^*$. We conclude that $\Phi(\xi) = \xi$. Hence, $\Phi = e^B$ on arbitrary generalised vectors. Thus, $F = f_* \circ e^B$. We finally observe with (2.5) that

$$[X+\xi, Y+\eta]_H = F^{-1}[F(X+\xi), F(Y+\eta)]_H = [X+\xi, Y+\eta]_{f^*H+\mathrm{d}B}.$$

The claim follows. □

To conclude our brief introduction to exact CAs, we cite essentially verbatim the famous classification result [6, Corollary 2.20] for exact CA originally due to Ševera [26].

Theorem 2.12. *Given M a smooth manifold, denote by* $\mathrm{Diff}_0(M)$ *the component of the identity in* $\mathrm{Diff}(M)$ *and set* $\Gamma_M = \mathrm{Diff}(M)/\mathrm{Diff}_0(M)$ *the corresponding mapping class group. Then, the isomorphism classes of exact Courant algebroids on M are in one-to-one correspondence with the quotient* $H^3_{\mathrm{dR}}(M,\mathbb{R})/\Gamma_M$.

Ševera's theorem implies in particular that one can naturally associate to any exact CA a cohomology class $[H] \in H^3_{\mathrm{dR}}(M)$ called the *Ševera class.*

2.3 Generalised Metrics

Generalised metrics provide a conceptual analogue to semi-Riemannian metrics in generalised geometry. After giving the formal definition, we provide a few basic results.

From now on, $E \to M$ denotes an exact CA. The following definition can for example be found in [27, Definition 2.16.].

Definition 2.13 (Generalised Metric)**.** A *generalised metric* on E is a non-degenerate symmetric bilinear form $\mathcal{G} \in \Gamma(\mathrm{Sym}^2 E)$ such that

(i) the restriction $\mathcal{G}|_{\mathrm{Sym}^2 \pi^* T^* M}$ is non-degenerate, and

(ii) the associated endomorphism $\mathcal{G}^{\mathrm{End}} \in \Gamma(\mathrm{End} E)$, $\langle \mathcal{G}^{\mathrm{End}} \cdot, \cdot \rangle = \mathcal{G}$, squares to the identity, $(\mathcal{G}^{\mathrm{End}})^2 = \mathrm{id}_E$.

Example 2.14. A semi-Riemannian metric g on M naturally defines the generalised metric $\mathcal{G}_g = \frac{1}{2}[g \oplus g^{-1}]$ on the (twisted) generalised tangent bundle $\mathbb{T}M$. Explicitly,

$$\mathcal{G}_g(X+\xi, Y+\eta) = \frac{1}{2}[g(X,Y) + g^{-1}(\xi,\eta)] \qquad \forall X+\xi, Y+\eta \in \Gamma(\mathbb{T}M).$$

It is easy to check that the associated endomorphism $\mathcal{G}_g^{\mathrm{End}}$ is given by

$$\mathcal{G}_g^{\mathrm{End}}(X+\xi) = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix}, \qquad X+\xi \in \Gamma(\mathbb{T}M).$$

The following equivalent description of generalised metrics is ubiquitously employed when working with them. It was developed in [28, Definition 2.5.][6].

Lemma 2.15. *There is a natural one-to-one correspondence between generalised metrics and subbundles* $E_+ \subset E$ *such that*

(i) $\langle \cdot, \cdot \rangle|_{\mathrm{Sym}^2 E_+}$ *is non-degenerate and*

(ii) the restriction of the anchor $\pi|_{E_+} : E_+ \to TM$ *is an isomorphism.*

It is given by associating a generalised metric $\mathcal{G}$ with its $+1$ eigenbundle E_+.

[6] There is a slight difference in terminology between the present work and [28]. In [28], a generalised metric need not satisfy the property that $\pi|_{E_+} : E_+ \to TM$ is an isomorphism. A generalised metric which does is called *admissible*.

Remark 2.16. We make a few observations whose extends beyond the proof of Lemma 2.15. We note that $\pi|_{E_+}$ being an isomorphism is equivalent to E_+ and $\pi^* T^* M$ intersecting trivially. We also note that by non-degeneracy of $\langle\cdot,\cdot\rangle|_{E_+}$, the orthogonal subbundle E_- is complementary, and we obtain the orthogonal decomposition $E = E_+ \oplus E_-$. Then also $\langle\cdot,\cdot\rangle|_{E_-}$ is non-degenerate. Furthermore, if there exists a non-zero covector ξ such that $\pi^*\xi \in E_-$, then there exists $a \in E_-$ such that $\xi(\pi a) = \langle \pi^*\xi, a\rangle \neq 0$. However, this implies

$$\left\langle \pi^*\xi, (\pi|_{E_+})^{-1}\pi a \right\rangle = \xi(\pi a) \neq 0,$$

violating the orthogonality of E_+ and E_-. It follows that also E_- and $\pi^* T^* M$ intersect trivially. We denote the inverses of the isomorphisms $\pi|_{E_\pm}$ by $\sigma_\pm = (\pi|_{E_\pm})^{-1}$. Note that $\pi \circ (\sigma_+ - \sigma_-) = 0$, hence $\operatorname{im}(\sigma_+ - \sigma_-) \subset \pi^* T^* M$. In fact, since $\sigma_+ - \sigma_-$ is injective, it is an isomorphism between TM and T^*M. In particular, for any $X \in \Gamma(TM)$ there exists $\xi \in \Gamma(T^*M)$ such that $\sigma_+(X) - \sigma_-(X) = \pi^*\xi$, and thus

$$\begin{aligned} \langle \sigma_+ X, \sigma_+ X\rangle &= \langle \sigma_- X + \pi^*\xi, \sigma_+ X\rangle = \xi(X) \\ &= -\langle \sigma_+ X - \pi^*\xi, \sigma_- X\rangle = -\langle \sigma_- X, \sigma_- X\rangle . \end{aligned} \tag{2.7}$$

One obtains the canonical semi-Riemannian metric $g = \langle \sigma_+(\cdot), \sigma_+(\cdot)\rangle$ on M. One also obtains the canonical isotropic splitting $\sigma = \frac{1}{2}(\sigma_+ + \sigma_-)$, as clearly $\pi \circ \sigma = \mathrm{id}_{TM}$ and by (2.7) also

$$4\,|\sigma X|_E^2 = |\sigma_+ X + \sigma_- X|_E^2 = |\sigma_+ X|_E^2 + |\sigma_- X|_E^2 = 0$$

for all $X \in \Gamma(TM)$. To conclude this remark, we prove that $\sigma_+ - \sigma_- = \pi^*$. Take $\xi \in \Gamma(T^*M)$ arbitrary, and let $X \in \Gamma(TM)$ be such that $\sigma_+(X) - \sigma_-(X) = \pi^*\xi$. Then for all $Y \in \Gamma(TM)$

$$\xi(Y) = \langle \pi^*\xi, \sigma(Y)\rangle = \frac{1}{2}\langle \sigma_+ X - \sigma_- X, \sigma_+ Y + \sigma_- Y\rangle = g(X, Y).$$

Thus, as claimed $X = g^{-1}\xi$.

Proof. **Constructing the generalised metric from the subbundle $E_+ \subset E$.** Take a subbundle $E_+ \subset E$ as in the statement. We obtain a decomposition $E = E_+ \oplus E_-$ as established in Remark 2.16. Define an endomorphism $\mathcal{G}^{\mathrm{End}} \in \Gamma(\operatorname{End} E)$ by demanding $\mathcal{G}^{\mathrm{End}}\big|_{E_\pm} = \pm 1$ and extending linearly. Then clearly $(\mathcal{G}^{\mathrm{End}})^2 = \mathrm{id}_E$ and $\mathcal{G} = \left\langle \mathcal{G}^{\mathrm{End}}\cdot, \cdot \right\rangle$ is a non-degenerate symmetric bilinear form.

It remains to check that $\mathcal{G}|_{\mathrm{Sym}^2 \pi^* T^* M}$ is non-degenerate. Let $\xi \in T_p^* M$ be non-zero, $p \in M$. Then, by Remark 2.16 there exists a non-zero vector $X \in T_p M$ such that $\pi^*\xi = \sigma_+ X - \sigma_- X$. Thus for all $Z \in T_p M$

$$\begin{aligned} \mathcal{G}(\pi^*\xi, \sigma_+ Z - \sigma_- Z) &= \langle \sigma_+ X + \sigma_- X, \sigma_+ Z - \sigma_- Z\rangle = \langle \sigma_+ X, \sigma_+ Z\rangle - \langle \sigma_- X, \sigma_- Z\rangle \\ &= 2\,\langle \sigma_+ X, \sigma_+ Z\rangle , \end{aligned}$$

where in the last line we used (2.7). By non-degeneracy of the inner product on E_+, there exists a vector $Z \in T_p M$ such that this scalar product is non-zero, hence $\mathcal{G}|_{\mathrm{Sym}^2 \pi^* T^* M}$ is non-degenerate.

Constructing the subbundle from the generalised metric. Let $\mathcal{G}$ be a generalised metric. The symmetry of $\mathcal{G}$ implies that the endomorphism $\mathcal{G}^{\mathrm{End}}$ can be diagonalised with respect to an ONB of $\langle\cdot,\cdot\rangle$. Since $(\mathcal{G}^{\mathrm{End}})^2 = \mathrm{id}$, the eigenvalues are $+1$ and -1, and we obtain the orthogonal eigenbundle decomposition $E = E_+ \oplus E_-$. In particular, $\langle \cdot, \cdot \rangle|_{\mathrm{Sym}^2 E_+}$ is non-degenerate. Finally, note that for $\xi \in \pi^* T^* M \cap E_+$, one has

$$\mathcal{G}(\pi^*\xi, \pi^*\eta) = \langle \pi^*\xi, \pi^*\eta \rangle = 0$$

and thus by non-degeneracy of $\mathcal{G}$ on T^*M that $\xi = 0$. Therefore, E_+ and π^*T^*M intersect trivially or, equivalently, $\pi|_{E_+}$ is an isomorphism. □

We have developed the tools for a more intuitive characterisation of generalised metrics, cf. equation (2.19) in [28].

Proposition 2.17. *There is a natural one-to-one correspondence between generalised metrics $\mathcal{G}$ and pairs (g, σ) consisting of a semi-Riemannian metric g and an isotropic splitting σ, obtained by associating the pair (g, σ) with the generalised metric $\mathcal{G}_{g,\sigma} = F_\sigma^* \mathcal{G}_g$, where $\mathcal{G}_g \in \mathrm{Sym}^2(\mathbb{T}M)$ is as in Example 2.14 and $F_\sigma \colon E \cong \mathbb{T}M$ is as in Theorem 2.10. Furthermore, under this correspondence*

$$F_\sigma(E_\pm) = \{X \pm gX \mid X \in TM\}. \tag{2.8}$$

Remark 2.18. If g has signature (p, q), then $\mathcal{G}_{g,\sigma}$ has signature $(2p, 2q)$. If g is Riemannian (resp. Lorentzian), we also call $\mathcal{G}_{g,\sigma}$ Riemannian (resp. Lorentzian).

Proof. First of all, note that the description of $\mathcal{G}_g^{\mathrm{End}}$ provided in Example 2.14 makes it obvious that the associated eigenbundles are given by $E_\pm[\mathcal{G}_g] = \{X \pm gX, X \in \Gamma(TM)\}$. Note then that the identity $\pi \circ \sigma_\pm = \mathrm{id}_{TM}$ determines that $\sigma_\pm(X) = X \pm gX$.

It is clear that $\mathcal{G}_{g,\sigma}$ is a generalised metric, as it is the pullback of a generalised metric by a Courant isomorphism. Conversely, recall from Remark 2.16 that a generalised metric $\mathcal{G}$ defines a semi-Riemannian metric g and an isotropic splitting σ on M. We have to check that the constructions are inverse.

Consider the generalised metric $\mathcal{G}_{g,\sigma}$. Denote the induced pair from Remark 2.16 by (g', σ') and the induced isomorphisms $TM \to E_\pm$ by $\sigma'_\pm$. Since $\mathcal{G}_{g,\sigma} = F_\sigma^* \mathcal{G}_g$, we have that

$$F_\sigma(E_\pm[\mathcal{G}_{g,\sigma}]) = E_\pm[\mathcal{G}_g] = \{X \pm gX, X \in \Gamma(TM)\}. \tag{2.9}$$

Thus $F_\sigma \circ \sigma'_\pm(X) = X \pm gX$, implying

$$|X|_{g'}^2 = |\sigma'_+ X|_E^2 = |X + gX|_{\mathbb{T}M}^2 = |X|_g^2$$

and

$$\sigma'(X) = \frac{1}{2}(\sigma'_+(X) + \sigma'_-(X)) = F_\sigma^{-1}(X) = \sigma(X).$$

Thus, $(g', \sigma') = (g, \sigma)$.

Conversely, consider a generalised metric $\mathcal{G}$ inducing a pair (g, σ). Then

$$\begin{aligned} \left|F_\sigma^{-1}(X \pm gX)\right|_{\mathcal{G}}^2 &= \left|\sigma X \pm \frac{1}{2}\pi^* gX\right|_{\mathcal{G}}^2 = \frac{1}{4}\left|\sigma_+ X + \sigma_- X \pm (\sigma_+ X - \sigma_- X)\right|_{\mathcal{G}}^2 \\ &= |\sigma_\pm X|_{\mathcal{G}}^2 = \pm |\sigma_\pm X|_E^2 = |X|_g^2 = \left|F_\sigma^{-1}(X \pm gX)\right|_{\mathcal{G}_{g,\sigma}}^2 \end{aligned}$$

for all $X \in \Gamma(TM)$. Thus, $\mathcal{G} = \mathcal{G}_{g,\sigma}$. Therefore, the correspondence is indeed one-to-one.

Finally, since every generalised metric is of the form $\mathcal{G}_{g,\sigma}$, (2.9) is equivalent to (2.8). □

Let us give a third and last characterisation of generalised metrics. It highlights the way in which a generalised metric is analogous to a (usual) metric, especially in the Riemannian case. We remind ourselves that a semi-Riemannian metric of signature (p, q) is equivalent to a reduction of the structure group from $\mathrm{GL}(d)$ to $O(p, q)$. That is, a Riemannian metric selects with the bundle of orthonormal frames a principal $O(p, q)$-subbundle of the frame bundle, which is a principal $\mathrm{GL}(d)$-bundle. Note furthermore that the eigenbundle decomposition $E = E_+ \oplus E_-$ equivalent to a generalised metric of signature $(2p, 2q)$ reduces the structure group to $O(p, q) \times O(q, p)$. In fact, as has been observed in [28], it is straightforward to see that if one does not require generalised metrics to satisfy property (i) in Definition 2.13, there is a natural one-to-one correspondence between generalised metrics of signature (p, q) and subgroups of $O(d, d)$ isomorphic to $O(p, q) \times O(q, p)$. Property (i) is necessary to achieve the characterisation of generalised metrics in terms of splittings and usual semi-Riemannian metrics given in Proposition 2.17, and is thus needed to establish a connection to supergravity. However, for a Riemannian generalised metric requirement (i) is implied from its positive definiteness. Thus, we still get the characterisation in the Riemannian case.

Remark 2.19. It is also possible [29] to define from a reduction of the structure group arbitrary signature generalised metrics which satisfy property (i). To achieve this, one first has to use that the anchor map reduces the structure group to $\mathrm{GL}(d) \ltimes \Lambda^2\mathbb{R}^d$. Then a generalised metric (with property (i)) is equivalent to a choice of subgroup of $\mathrm{GL}(d) \ltimes \Lambda^2\mathbb{R}^d$ isomorphic to $O(p, q)$. We do not emphasise this viewpoint here, because this reduction of the structure group is not natural in the context of our discussion of generalised connections, which we won't ask to be compatible with the above reduction to $O(p, q)$.

Proposition 2.20. *There is a natural one-to-one correspondence between Riemannian generalised metrics and subgroups of $O(d, d)$ isomorphic to $O(d) \times O(d)$, obtained by associating a Riemannian generalised metric to the subgroup of $O(d) \times O(d)$ respecting the eigenbundle decomposition $E = E_+ \oplus E_-$.*

Note that $O(d)$ and $O(d) \times O(d)$ are respectively the maximal compact subgroups of $\mathrm{GL}(d)$ and $O(d, d)$.

We briefly remark that a generalised metric allows one to associate with the "Polyakov action with Wess-Zumino term" an energy to strings (remember that a semi-Riemannian metric provides a way of associating an energy to a curve), cf. [6, section 2.3.2, i.p. equation (2.25)].

Finally, loosely following [6, Proposition 2.41.], we characterise the space of Courant automorphisms which preserve a given generalised metric $\mathcal{G} = \mathcal{G}(g, \sigma)$. Recall that the isotropic splitting σ enables us to identify E with the H-twisted tangent bundle $\mathbb{T}M$ for some twist H (called the preferred representative of the Ševera class). Recall also that for the twisted generalised tangent bundle we have characterised the space of Courant automorphisms in Proposition 2.11.

Proposition 2.21. *Let $E \to M$ be an exact CA equipped with a generalised metric $\mathcal{G} = \mathcal{G}(g, \sigma)$. Then, the space of Courant automorphisms F of E which preserve the generalised metric, $\mathcal{G} = F^*\mathcal{G} = \mathcal{G}(F(\cdot), F(\cdot))$, is given by the space of diffeomorphisms $f\colon M \to M$ which preserve the metric g and the preferred representative of the Ševera class H, i.e. $g = f^*g$ and $H = f^*H$.*

Proof. Let H be the preferred representative of the Ševera class, and employ the identification of E with the H-twisted generalised tangent bundle induced by σ. Let $F = (f, B)$ be a Courant automorphism of $(\mathbb{T}M, [\cdot,\cdot]_H)$. Then

$$|X|_g^2 + |\xi|_g^2 = 2\,|X+\xi|_{\mathcal{G}}^2 = 2\,|F(X+\xi)|_{\mathcal{G}}^2 = |f_*X|_g^2 + |f_*\xi + f_*B(X)|_g^2\,.$$

Considering the case $\xi = 0$, we find that $B = 0$, and then $f^*g = g$. By Proposition 2.11, $H = f_*(H - \mathrm{d}B) = f_*H$. □

2.4 Divergence Operators

In generalised geometry, the fundamental geometric object is a pair $(\mathcal{G}, \mathrm{div})$ consisting of a generalised metric $\mathcal{G}$ and a divergence operator div. In this section, we introduce divergence operators on exact CAs.

As in (usual) semi-Riemannian geometry, a volume form (more generally, a density; in particular, a generalised metric) induces a divergence operator. However, not every divergence operator in generalised geometry comes from a volume form. Moreover, in addition to a choice of generalised metric, a choice of divergence operator is necessary to obtain a well-defined notion of generalised Ricci curvature. That one can independently choose a generalised metric and a divergence operator enables the encoding of the so-called dilaton in the geometric data, and is therefore crucial to the connection between generalised geometry and the bosonic NS-NS sector in type II ten-dimensional supergravity.

Throughout this section $E \to M$ denotes an exact CA. Recall the notion of a divergence operator [4].

Definition 2.22 (Divergence Operator)**.** A *divergence operator* on E is a map $\mathrm{div}\colon \Gamma(E) \to C^\infty(M)$ satisfying the π-Leibniz rule

$$\mathrm{div}(fa) = \pi(a)(f) + f\,\mathrm{div}(a) \tag{2.10}$$

for all $a \in \Gamma(E)$ and $f \in C^\infty(M)$.

Example 2.23. Let $\mu > 0$ be a nowhere vanishing density on M. Then $\Gamma(E) \ni a \mapsto (L_{\pi a}\mu)/\mu \in C^\infty(M)$ defines the divergence operator div^μ on E. With $\mathcal{G} = \mathcal{G}(g,\sigma)$ a generalised metric on E and μ_g the semi-Riemannian density of g, we denote $\mathrm{div}^{\mathcal{G}} = \mathrm{div}^{\mu_g}$ and refer to this as the *metric divergence*. Note that $\mathrm{div}^{\mathcal{G}}(a) = \mathrm{tr}(\nabla\pi a)$ for all $a \in \Gamma(E)$, where ∇ is the LC connection of g.

Proposition 2.24. *The space of divergence operators is an affine space over* $\Gamma(E^*) \cong \Gamma(E)$.

Proof. This is a trivial consequence of the π-Leibniz rule. □

As a consequence of Propostion 2.24, given a divergence operator div and a generalised metric $\mathcal{G}$, there exists a section $e \in \Gamma(E)$ such that $\mathrm{div} = \mathrm{div}^{\mathcal{G}} - \langle e, \cdot\rangle$. We refer to this generalised vector field as the *generalised dilaton*.

We want to introduce a notion of compatibility between a generalised metric and a divergence operator, which is important for relating different notions of curvature in the literature, cf. Remark 6.6. It builds on the following notion of an infinitesimal isometry, cf. [6, Definitions 2.49. and 2.52.].

Definition 2.25 (Infinitesimal Isometry)**.** Let $\mathcal{G}$ be a generalised metric. A generalised vector field $e \in \Gamma(E)$ is called an *infinitesimal isometry* of $\mathcal{G}$ if $[e,\mathcal{G}] = 0$, where $[e,\mathcal{G}] \in \Gamma(\mathrm{Sym}^2 E)$ is defined by the equation

$$[e,\mathcal{G}](a,b) = \pi e(\mathcal{G}(a,b)) - \mathcal{G}([e,a],b) - \mathcal{G}(a,[e,b]).$$

Definition 2.26 (Compatibilty between Generalised Metrics and Divergence Operators)**.** A pair $(\mathcal{G}, \mathrm{div})$ consisting of a generalised metric $\mathcal{G}$ and a divergence operator div is called *compatible*, if the associated generalised dilaton $e \in \Gamma(E)$ is an infinitesimal isometry.

We give the following well-known characterisation of infinitesimal isometries.

Proposition 2.27. *Let $\mathcal{G}$ be a generalised metric on E, and employ the associated splitting to identify $E \cong \mathbb{T}M$. Then, $e = 2(X + \xi) \in \Gamma(E)$ is an infinitesimal isometry if and only if*

$$L_X g = 0, \qquad d\xi = i_X H.$$

Proof. Let $y = Y + \eta, z = Z + \zeta \in \Gamma(E)$. Then

$$\begin{aligned}[e, \mathcal{G}](y, z) &= \pi e(\mathcal{G}(y, z)) - \mathcal{G}([e, y], z) - \mathcal{G}(y, [e, z]) \\ &= X(g(Y, Z) + g^{-1}(\eta, \zeta)) - g([X, Y], Z) - g^{-1}(L_X \eta - i_Y \mathrm{d}\xi + i_Y i_X H, \zeta) \\ &\quad - g(Y, [X, Z]) - g^{-1}(L_X \zeta - i_Z \mathrm{d}\xi + i_Z i_X H, \eta) \\ &= (L_X g)(Y, Z) + (L_X g^{-1})(\eta, \zeta) + \mathrm{d}\xi(Y, g^{-1}\zeta) - H(X, Y, g^{-1}\zeta) \\ &\quad + \mathrm{d}\xi(Z, g^{-1}\eta) - H(X, Z, g^{-1}\eta).\end{aligned}$$

The claim follows. □

The special case for a compatible pair $(\mathcal{G}, \mathrm{div})$ where $X = 0$ and hence $\mathrm{d}\xi = 0$ is of particular importance, because it is the most relevant to the physics literature. Notably, the correspondence between generalised geometry and the bosonic NS-NS sector of type II ten-dimensional supergravity requires the assumption that ξ is exact [2].

Note that, since any two densities $\mu > 0$ and $\nu > 0$ are related by a positive function, $\mu = e^f \nu$, the associated divergence operators differ always by an exact one-form:

$$\mathrm{div}^\mu(a) - \mathrm{div}^\nu(a) = \frac{L_{\pi a}\mu}{\mu} - \frac{L_{\pi a}\nu}{\nu} = \pi a(f) = \langle \pi^* \mathrm{d}f, a \rangle .$$

In fact, the space of divergence operators coming from a nowhere vanishing density is an affine space over the space of exact one-forms. In particular, the condition that for a pair $(\mathcal{G}, \mathrm{div})$ the generalised dilaton satisfies $X = 0$ and ξ closed (or exact) is independent of the choice of generalised metric $\mathcal{G}$. We have justified the following definition, which is based on [6, Definitions 3.40. and 3.48.].

Definition 2.28 (Closed Divergence and Exact Divergence)**.** A divergence operator div is called *closed* (resp. *exact*) if the difference to a divergence operator div^μ coming from a density $\mu > 0$ is given by a closed (resp. exact) one-form ξ, $\mathrm{div} = \mathrm{div}^\mu - \langle \pi^* \xi, \cdot \rangle$.

Note that every exact divergence operator div is realised by some density $\mu > 0$, $\mathrm{div} = \mathrm{div}^\mu$.

2.5 Generalised Connections

We finally introduce generalised connections. As in usual semi-Riemannian geometry, one can introduce the notion of a torsion-free and metric compatible generalised connection. These connections are called generalised Levi-Civita (LC), and they always exist. However, they are not unique. This non-uniqueness can be used to achieve compatibility with a given divergence operator. A residual non-uniqueness remains (we always assume $\dim M > 1$). A possible resolution for this is the existence of a geometrically preferred canonical generalised LC connection, as we present in 3.1. However, in some settings another resolution presents itself, as often relevant geometrical (or physical) quantities do no not depend on the choice of generalised LC connection. Then, the non-uniqueness may be viewed as acceptable or even preferable, as the demand for independence of choice of generalised LC connection may serve as a selection criterion for relevant quantities.

Let $E \to M$ be an exact CA. Recall the definition of a generalised connection.

Definition 2.29 (Generalised Connection)**.** A *generalised connection* on E is a map $D\colon \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ which

(i) is C^∞ linear in the first entry, $D_{fa} = fD_a$,

(ii) satisfies the π-Leibniz rule, $D_a(fb) = \pi a(f)b + fD_a b$, and

(iii) is compatible with the inner product, $D_a \langle b, c\rangle = \langle D_a b, c\rangle + \langle b, D_a c\rangle$.

Herein, $f \in C^\infty(M)$ and $a, b, c \in \Gamma(E)$ are arbitrary. We denote the space of generalised connections by $\mathcal{D}$.

Due to the compatibility with the inner product [6, Lemma 3.2.], the space of generalised connections is an affine space over $\Gamma(E^* \otimes \mathfrak{so}(E))$, where $\mathfrak{so}(E)$ denotes the bundle of antisymmetric endomorphisms on E.

Example 2.30. Consider the H-twisted generalised tangent bundle $\mathbb{T}M$ with generalised metric $\mathcal{G}_g$ from Example 2.14. Let ∇ be a connection on M. Then one can define the generalised connection D^∇ as

$$D^\nabla_{X+\xi}(Y + \eta) = \nabla_X(Y + \eta), \qquad X + \xi, Y + \eta \in \Gamma(\mathbb{T}M).$$

Definition 2.31 (Torsion Tensor)**.** The *torsion tensor* $T^D \in \Gamma(\Lambda^3 E^*)$ associated to a generalised connection is given by

$$T^D(a, b, c) = \langle D_a b - D_b a - [a, b], c\rangle + \langle D_c a, b\rangle, \qquad a, b, c \in \Gamma(E).$$

We denote the space of torsion-free generalised connections by $\mathcal{D}^0$.

The last term is needed for the expression to define a tensor.[7] It is straightforward to check that indeed $T^D \in \Gamma(\Lambda^3 E^*)$. In fact, fixing a connection ∇ on M and a splitting $E \cong \mathbb{T}M$, one can represent an arbitrary generalised connection on $\mathbb{T}M$ as $D = D^\nabla + \chi$ for $\chi \in \Gamma(E^* \otimes \mathfrak{so}(E))$, and compute its torsion as follows, cf. also [6, Lemma 3.5.].

[7] More specifically, the last term is needed to make the torsion C^∞-linear in in the first entry; In the computation verifying $T(fa, b, c) = fT(a, b, c)$ it is needed to cancel the the last term in $[fa, b] = f[a, b] - \pi(b)(f)a + \pi^*(df)\langle a, b\rangle$, a term to which no analogue appears in the case of a Lie bracket.

Lemma 2.32. *The torsion T^D of $D = D^\nabla + \chi$ is given by*

$$T^D(x,y,z) = \sum_{\sigma(x,y,z)} \left[\left\langle T^\nabla(\pi x, \pi y), z \right\rangle + \chi(x,y,z) - \frac{1}{6} H(\pi x, \pi y, \pi z) \right],$$

where $x, y, z \in \Gamma(E)$, $T^\nabla \in \Omega^2(M) \otimes \Gamma(TM)$ is the torsion of ∇, and we denoted $\chi(x,y,z) = \langle \chi(x,y), z\rangle$.

Proof. Assume first $T^\nabla = 0$. We calculate, writing $x = X + \xi$, $y = Y + \eta$, $z = Z + \zeta$,

$$\begin{aligned}
&T^D(x,y,z)\\
&= \langle D_x y - D_y x - [x,y], z\rangle + \langle D_z x, y\rangle\\
&= \langle \nabla_X(Y+\eta) - \nabla_Y(X+\xi) - [X+\xi, Y+\eta], Z+\zeta\rangle + \langle \nabla_Z(X+\xi), Y+\eta\rangle\\
&\quad + \chi(x,y,z) - \chi(y,x,z) + \chi(z,x,y)\\
&= \sum_{\sigma(x,y,z)} \chi(x,y,z) + \langle \nabla_X \eta - \nabla_Y \xi - L_X \eta + i_Y \mathrm{d}\xi - i_Y i_X H, Z\rangle + \langle \nabla_Z(X+\xi), Y+\eta\rangle\\
&= \sum_{\sigma(x,y,z)} \chi(x,y,z) + H(X,Y,Z).
\end{aligned}$$

In the last line, we used that

$$\begin{aligned}
\langle \nabla_X \eta - L_X \eta, Z\rangle + \langle \eta, \nabla_Z X\rangle &= -\frac{1}{2}\eta(\nabla_X Z - L_X Z) + \langle \eta, \nabla_Z X\rangle = 0,\\
\langle \nabla_Y \xi - i_Y \mathrm{d}\xi, Z\rangle - \langle \nabla_Z \xi, Y\rangle &= i_Z i_Y \mathrm{d}\xi - i_Y i_Z \mathrm{d}\xi = 0.
\end{aligned}$$

Introducing a connection with torsion $\nabla' = \nabla + T^{\nabla'}$, the result follows from noticing that $D^{\nabla'} = D^\nabla + \chi$ with $\chi(a,b,c) = \left\langle T^{\nabla'}(\pi a, \pi b), c\right\rangle$. □

To obtain an elegant description of the space of torsion-free connections, we follow [30] in introducing the projection

$$\partial\colon E^* \otimes \mathfrak{so}(E) \longrightarrow \Lambda^3 E^*, \qquad (\partial\chi)(x,y,z) := \sum_{\sigma(x,y,z)} \langle \chi(x,y), z\rangle\,.$$

Given a Lie algebra subbundle[8] $\mathfrak{g}(E) \subset \mathfrak{so}(E)$, we define its *generalised first prolongation* $(\mathfrak{g}(E))^{\langle 1\rangle}$ as

$$(\mathfrak{g}(E))^{\langle 1\rangle} = \{\chi \in E^* \otimes \mathfrak{g}(E) \mid \partial\chi = 0\}.$$

We see with Lemma 2.32 that the space of torsion-free generalised connections $\mathcal{D}^0$ is an affine space over the space of sections of $(\mathfrak{so}(E))^{\langle 1\rangle}$. More generally, one can observe [30, Proposition 17] that the space of torsion-free generalised connections compatible with a reduction of the structure group from $O(d,d)$ to a subgroup G with Lie algebra $\mathfrak{g} \subset \mathfrak{so}(n,n)$ is (if it is non-empty) an affine space over the space of sections of $(\mathfrak{g}(E))^{\langle 1\rangle}$.

Definition 2.33 (Metric Generalised Connection)**.** Given a generalised metric $\mathcal{G}$, a generalised connection D is called $\mathcal{G}$*-metric* if $D\mathcal{G} = 0$. We denote the space of metric generalised connections by $\mathcal{D}(\mathcal{G})$.

[8] A *Lie algebra bundle* is a vector bundle $E \to M$ with a bracket on each fibre such that locally over a subset $U \subset M$, the structure trivialises as $U \times L$ where L is a Lie algebra.

Note that the connection D^∇ from Example 2.30 is $\mathcal{G}_g$-metric if the connection ∇ is g-metric, where $\mathcal{G}_g$ is the generalised metric from Example (2.14). Since a generalised metric $\mathcal{G}(g,\sigma) \cong \mathcal{G}_g$ reduces the structure group to $O(p,q) \times O(q,p)$ with Lie algebra $\mathfrak{so}(p,q) \oplus \mathfrak{so}(q,p)$, we conclude that, denoting by $\mathfrak{so}(E_+) \oplus \mathfrak{so}(E_-)$ the Lie algebra subbundle of $\mathfrak{so}(E)$, the space of $\mathcal{G}(g,\sigma)$-metric generalised connections is an affine space over the sections of

$$E^* \otimes (\mathfrak{so}(E_+) \oplus \mathfrak{so}(E_-)).$$

This can also be obtained from the observation that a generalised connection D is metric if and only if it preserves the metric eigenbundles $E_\pm$, $D\Gamma(E_\pm) \subset \Gamma(E_\pm)$.

Definition 2.34 (Generalised Levi-Civita Connection)**.** A generalised connection is called *Levi-Civita* (LC), if it is metric and torsion-free. We denote the space of generalised Levi-Civita connections by $\mathcal{D}^0(\mathcal{G})$.

From the discussion of torsion-free generalised connections, we know that $\mathcal{D}^0(\mathcal{G})$ (presuming non-emptiness) is an affine space over the sections of the generalised first prolongation $(\mathfrak{so}(E_+) \oplus \mathfrak{so}(E_-))^{\langle 1 \rangle}$. Note that naturally

$$[\mathfrak{so}(E_+) \oplus \mathfrak{so}(E_-)]^{\langle 1 \rangle} \cong (\mathfrak{so}(E_+))^{\langle 1 \rangle} \oplus (\mathfrak{so}(E_-))^{\langle 1 \rangle}$$

because if $\chi \in (\mathfrak{so}(E_+) \oplus \mathfrak{so}(E_-))^{\langle 1 \rangle}$ then by $\partial\chi = 0$ we have for all $a_\pm, b_\pm, c_\pm \in E_\pm$

$$\chi(a_\mp, b_\pm, c_\pm) = -\chi(c_\pm, a_\mp, b_\pm) - \chi(b_\pm, c_\pm, a_\mp) = 0.$$

Note that the torsion of D^∇ lies by Lemma 2.32 in $\mathfrak{so}(E_+) \oplus \mathfrak{so}(E_-)$ if and only if $T^\nabla = 0$ and $H = 0$. Thus, we can in general not subtract the torsion from D^∇ and expect D^∇ to be generalised LC, even if ∇ is LC. Instead, we have the following Lemma, cf. also [6, Lemma 3.11.].

Lemma 2.35. *The torsion T^D of a metric generalised connection D is a section of*

$$\Lambda^3 E^* \cap [E^* \otimes (\mathfrak{so}(E_+) \oplus \mathfrak{so}(E_-))] = \Lambda^3 E_+^* \oplus \Lambda^3 E_-^* \tag{2.11}$$

(where we use the inner product to identify $\mathfrak{so}(E_\pm) \cong \Lambda^2 E_\pm^*$) if and only if the mixed-type parts of D are given by the Dorfman bracket as

$$D_{a_-} b_+ = \pi_+[a_-, b_+], \qquad D_{a_+} b_- = \pi_-[a_+, b_-], \qquad a_\pm, b_\pm \in \Gamma(E_\pm),$$

where $\pi_\pm = \frac{1}{2}(\mathrm{id}_E \pm \mathcal{G}^{\mathrm{End}})\colon E \to E_\pm$ is the orthogonal projection onto the metric eigenbundles.

Proof. The condition that T^D takes values in (2.11) can equivalently be stated as for all $a_\pm, b_\pm, c_\pm \in \Gamma(E_\pm)$

$$\begin{aligned} 0 &= T^D(a_\pm, b_\mp, c_\mp) \\ &= \langle D_{a_\pm} b_\mp - D_{b_\mp} a_\pm - [a_\pm, b_\mp], c_\mp \rangle + \langle D_{c_\mp} a_\pm, b_\mp \rangle \\ &= \langle D_{a_\mp} b_\pm - [a_\pm, b_\mp], c_\pm \rangle . \end{aligned}$$

In the last line, we used that the metric eigenbundles are $\langle \cdot, \cdot \rangle$-orthogonal and preserved by the metric generalised connection D. ☐

We can now establish the existence of a generalised LC connection, cf. (also in the more general setting of transitive CAs) [3, 4, 5, 6].

Corollary 2.36. *Let $\mathcal{G}(g,\sigma)$ be a generalised metric and ∇ the LC connection of g. Then, the generalised connection D defined for $X, Y \in \Gamma(TM)$, $x_\pm = \sigma_\pm X$, and $y_\pm = \sigma_\pm Y$ as*

$$D_{x_\pm} y_\pm = \sigma_\pm(\nabla_X Y), \qquad D_{x_\mp} y_\pm = \pi_\pm[x_\mp, y_\pm], \tag{2.12}$$

is metric and its torsion T^D takes values in (2.11). In particular, $D^0 \coloneqq D - \frac{1}{3}T^D$ is generalised LC, and the space of generalised LC connections is an affine space over the sections of $(\mathfrak{so}(E_+))^{\langle 1\rangle} \oplus (\mathfrak{so}(E_-))^{\langle 1\rangle}$.

A straightforward calculation employing the metric induced identification $E \cong \mathbb{T}M$ reveals that for $X, Y \in \Gamma(TM)$, $x_\pm = \sigma_\pm X$, and $y_\pm = \sigma_\pm Y$,

$$\begin{aligned}
&2\mathcal{G}([x_\mp, y_\pm], z_\pm)\\
&= 2\mathcal{G}(L_X(Y \pm gY) \pm i_Y \mathrm{d}(gX) + H(X,Y), Z \pm gZ)\\
&= 2g(L_X Y, Z) + (L_X g)(Y,Z) + L_Y(gX)(Z) - Z(g(X,Y)) \pm H(X,Y,Z)\\
&= (L_X g)(Y,Z) + (L_Y g)(X,Z) + g(L_X Y, Z) - Z(g(X,Y)) \pm H(X,Y,Z)\\
&= g(\nabla_Y X, Z) + g(\nabla_Z X, Y) + g(\nabla_X Y, Z) + g(\nabla_Z Y, X) + g(L_X Y, Z)\\
&\quad - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) \pm H(X,Y,Z)\\
&= 2g(\nabla_X Y, Z) \pm H(X,Y,Z).
\end{aligned}$$

where H is the preferred representative of the Ševera class. Similarly, one can compute the torsion T^D for $X, Y, Z \in \Gamma(TM)$ and $x_\pm = \sigma_\pm X$ as

$$\begin{aligned}
&T^D(x_\pm, y_\pm, z_\pm)\\
&= \langle D_{x_\pm} y_\pm - D_{y_\pm} x_\pm - [x_\pm, y_\pm], z_\pm\rangle + \langle D_{z_\pm} x_\pm, y_\mp\rangle\\
&= \pm g\left(\nabla_X Y - \nabla_Y X - \frac{1}{2}[L_X Y + g^{-1}L_X(gY) - g^{-1}i_Y \mathrm{d}(gX) \pm g^{-1}H(X,Y)], Z\right)\\
&\quad \pm g(\nabla_Z X, Y)\\
&= \mp\frac{1}{2}\{(L_X g)(Y,Z) - i_Z i_Y \mathrm{d}(gX) - 2g(\nabla_Z X, Y) \pm H(X,Y,Z)\}\\
&= \mp\frac{1}{2}\{(L_X g)(Y,Z) - L_Y(gX)(Z) + Z(g(X,Y)) - 2g(\nabla_Z X, Y) \pm H(X,Y,Z)\}\\
&= \mp\frac{1}{2}\{g(\nabla_Y X, Z) + g(\nabla_Z X, Y) - g(\nabla_X Y, Z) - g(\nabla_Z Y, X) - g(L_Y X, Z)\\
&\quad - g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \pm H(X,Y,Z)\}\\
&= -\frac{1}{2}H(X,Y,Z).
\end{aligned}$$

Therefore,

$$\begin{aligned}
D^0_{x_\mp} y_\pm &= \sigma_\pm\left(\nabla_X Y \pm \frac{1}{2}g^{-1}H(X,Y)\right) \equiv \sigma_\pm(\nabla^\pm_X Y),\\
D^0_{x_\pm} y_\pm &= \sigma_\pm\left(\nabla_X Y \pm \frac{1}{6}g^{-1}H(X,Y)\right) \equiv \sigma_\pm(\nabla^{\pm 1/3}_X Y).
\end{aligned} \tag{2.13}$$

Note that the sign in front of the $\frac{1}{6}H$-term appears due to the identification $\mathfrak{so}(E_\pm) \cong \Lambda^2 E^*_\pm$ being provided by the inner product.

Remark 2.37. A close inspection of the calculation of T^D reveals a certain uniqueness property of D^0. That is, defining as in Corollary 2.36 $D^0 = D - \frac{1}{3}T^D$ from a generalised connection D as in (2.12), any choice of metric connection ∇ with antisymmetric[9] torsion tensor T^∇ yields the same generalised LC connection D^0. In particular, employing the canonical Bismut connections $\nabla^\pm$ instead of the LC connection ∇ (as is often done in the literature) yields the same connection D^0.

Definition 2.38 (Divergence Compatible Generalised Connection)**.** A generalised connection D induces a divergence operator

$$\mathrm{div}_D\colon \Gamma(E) \longrightarrow C^\infty(M), \qquad a \longmapsto \mathrm{tr}\, Da \in C^\infty(M).$$

A pair (D, div) consisting of a generalised metric and a divergence operator is called *compatible* if $\mathrm{div}_D = \mathrm{div}$. We denote the space of generalised Levi-Civita connections with divergence div by $D^0(\mathcal{G}, \mathrm{div})$.

Compatibility of two generalised connections D and $D' = D + \chi$ with a given divergence operator implies tracelessness of their difference tensor,

$$(\mathrm{tr}\,\chi)(a) \equiv \mathrm{tr}(\chi(\cdot, a)) = 0 \qquad \text{for all } a \in \Gamma(E).$$

In particular, (presuming non-emptiness) $\mathcal{D}^0(\mathcal{G}, \mathrm{div})$ is an affine space over the sections of the kernel of the trace map

$$\mathrm{tr}\colon (\mathfrak{so}(E_+))^{\langle 1\rangle} \oplus (\mathfrak{so}(E_-))^{\langle 1\rangle} \longrightarrow E^*, \qquad \chi \longmapsto \mathrm{tr}\,\chi. \tag{2.14}$$

We prove now that the generalised connection D^0 from Corollary 2.36 provides an example for a generalised LC connection with metric divergence, cf. [3, 4, 5, 6]. Note that D^0 is sometimes also called the *minimal* generalised LC connection [5].

Lemma 2.39. *The generalised LC connection from Corollary 2.36 has metric divergence,* $\mathrm{div}_{D^0} = \mathrm{div}^{\mathcal{G}}$.

Proof. Let $x_\pm = \sigma_\pm(X) \in \Gamma(E_\pm)$. Then, employing local ON frames $\{e_i^\pm = \sigma_\pm(e_i)\}$ of $(E_\pm, \langle\cdot,\cdot\rangle)$ with $\varepsilon_i = \pm\left\langle e_i^\pm, e_i^\pm\right\rangle = g(e_i, e_i)$, we have

$$\begin{aligned}\mathrm{div}_{D^0}(x_\pm) &= \mathrm{tr}\, D^0 x_\pm = \sum_i \varepsilon_i \left\langle D^0_{e_i^+} x_\pm, e_i^+\right\rangle - \sum_i \varepsilon_i \left\langle D^0_{e_i^-} x_\pm, e_i^-\right\rangle \\ &\overset{*}{=} \sum_i \varepsilon_i \left\langle D^0_{e_i^\pm} x_\pm, e_i^\pm\right\rangle = \sum_i \varepsilon_i g(\nabla^{\pm 1/3}_{e_i} X, e_i) = \sum_i g(\nabla_{e_i} X, e_i) = \mathrm{div}_g(X) = \mathrm{div}^{\mathcal{G}}(x_\pm),\end{aligned}$$

where in $*$ we employed metricity of D, and then in the next step formulas (2.13). □

The following result is well-known (also in the more general case of transitive CAs) [3, 4, 5, 6]. We view it as a corollary of Theorem 3.2, which establishes that for every generalised metric $\mathcal{G}$ and divergence operator div, there exists a canonical generalised LC connection $D^{\mathcal{G},\mathrm{div}}$ with divergence div.

Corollary 2.40. $\mathcal{D}^0(\mathcal{G}, \mathrm{div})$ *is an affine space over* $\Gamma(\ker \mathrm{tr})$*, the space of sections of the kernel of the trace map (2.14).*

We conclude our discussion of generalised connections by remarking that Chapter 3 obtains with Theorem 3.2 a geometric characterisation of $D^{\mathcal{G},\mathrm{div}}$. This provides justification for calling $D^{\mathcal{G},\mathrm{div}}$ *the* canonical generalised LC connection.

[9]We call T^∇ antisymmetric if the expression $g(T^\nabla(X,Y),Z)$ is antisymmetric in $X, Y, Z \in TM$.

2.6 The Generalised Riemann Tensor and Other Curvature Invariants

To every[10] generalised connection D, one can associate a generalised Riemann tensor $\mathcal{R}m^D$. Similarly to usual semi-Riemannian geometry, taking a canonical trace yields the "full" generalised Ricci curvature $\overline{\mathcal{R}c}^D$, and given a generalised metric, one can trace once more to obtain the generalised scalar curvature $\mathcal{S}c = \operatorname{tr}_{\mathcal{G}} \overline{\mathcal{R}c}$. With these generalised curvature quantities, one can study in generalised geometry analogues to conditions and systems involving the usual semi-Rimannian curvature quantities. To name a few, this includes generalised Ricci flatness, the generalised Einstein equations[11], and the generalised Ricci flow.

As laid out in Section 2.5, given a pair $(\mathcal{G}, \operatorname{div})$ there is no unique generalised LC connection with given divergence (assuming, as we do, $\dim M > 1$). It turns out that, in general, the generalised Riemann tensor depends on the choice of generalised LC connection with given divergence. Thus, there is no unique generalised Riemann tensor associated to a pair $(\mathcal{G}, \operatorname{div})$. However, there are two ways to remedy this. The first is to refer to the existence of a preferred element $D^{\mathcal{G},\operatorname{div}} \in \mathcal{D}^0(\mathcal{G}, \operatorname{div})$, which renders $\mathcal{R}m^{D^{\mathcal{G},\operatorname{div}}}$ the preferred generalised Riemann tensor associated to the pair $(\mathcal{G}, \operatorname{div})$. We compute this tensor in Sections 3.2 and 3.3. The second way is to only work with invariant quantities: A particular restriction of the full generalised Ricci tensor (referred to as the generalised Ricci tensor) and the generalised scalar curvature are both independent of the choice of connection $D \in \mathcal{D}^0(\mathcal{G}, \operatorname{div})$, and these are the relevant quantities for establishing a connection to supergravity [2].

The following definition is as in [5, 15]. While [5, 15] introduced the definition to the mathematics literature, it was developed in the physics literature in the context of supergravity [31] and double field theory [32].

Definition 2.41 (Generalised Riemann Tensor)**.** Let $D \in \mathcal{D}^0$ be a torsion-free generalised connection on an exact CA E. Then the associated *generalised Riemann tensor* $\mathcal{R}m^D \in \Gamma(\operatorname{Sym}^2 \Lambda^2 E^*)$ is defined by the formula

$$\begin{aligned}\mathcal{R}m^D(a, b, v, w) = \frac{1}{2}\Big\{ \Big\langle D^2_{v,w} b - D^2_{w,v} b, a \Big\rangle + \Big\langle D^2_{b,a} v - D^2_{a,b} v, w \Big\rangle \\ - \operatorname{tr}_E(\langle Dv, w\rangle \langle Db, a\rangle) \Big\}\end{aligned}$$

Herein, $a, b, v, w \in \Gamma(E)$ and $D^2_{v,w} b = D_v D_w b - D_{D_v w} b$.

Definition 2.42 (Algebraic Curvature Tensor)**.** An *algebraic curvature tensor* on E is a section $R \in \Gamma(\operatorname{Sym}^2 \Lambda^2 E^*)$ satisfying the generalised first Bianchi identity

$$\sum_{\sigma(u,v,w)} \mathcal{R}m^D(a, u, v, w) = 0 \qquad \text{for all } a, u, v, w \in \Gamma(E). \tag{2.15}$$

Lemma 2.43. *The formula for the generalised Riemann tensor* $\mathcal{R}m^D$ *indeed defines a section of* $\operatorname{Sym}^2 \Lambda^2 E^*$.

[10] We only discuss the case where D is torsion-free. The general case is discussed for example in [5].

[11] It is common in the literature for generalised Einstein to mean generalised Ricci flat. However, while there is good reason for this nomenclature in Riemannian signature, in general the author of this work views it as preferable for generalised Einstein to mean generalised Ricci flat and generalised scalar flat.

Proof. It is obvious that the defining formula ensures the symmetry $\mathcal{R}m^D(a,b,v,w) = \mathcal{R}m^D(w,v,b,a)$. Note that, employing $T^D = 0$,

$$\begin{aligned}
&\left\langle D^2_{v,w} - D^2_{w,v} b, a\right\rangle = \langle D_v D_w b - D_{D_v w} b - D_w D_v b + D_{D_w v} b, a\rangle \\
&= \pi v \langle D_w b, a\rangle - \langle D_w b, D_v a\rangle - \pi w \langle D_v b, a\rangle + \langle D_v b, D_w a\rangle - \left\langle D_{[v,w]-\langle Dv,w\rangle} b, a\right\rangle \\
&= [\pi v, \pi w] \langle b, a\rangle - \pi v \langle b, D_w a\rangle + \pi w \langle b, D_v a\rangle - \langle D_w b, D_v a\rangle + \langle D_v b, D_w a\rangle \\
&\quad - \left\langle D_{[v,w]} b, a\right\rangle + \operatorname{tr}_E \langle Dv, w\rangle \langle Db, a\rangle \\
&= \left\langle b, D_{[v,w]} a\right\rangle - \pi v \langle b, D_w a\rangle + \pi w \langle b, D_v a\rangle - \langle D_w b, D_v a\rangle + \langle D_v b, D_w a\rangle \\
&\quad + \operatorname{tr}_E \langle Dv, w\rangle \langle Db, a\rangle
\end{aligned}$$

Note that we identify sections of E^* such as $\langle Dv, w\rangle$ with sections of E via the inner product $\langle \cdot, \cdot\rangle$. Thus, the symmetrisation in a and b vanishes:

$$\begin{aligned}
&2\{\mathcal{R}m^D(a,b,v,w) + \mathcal{R}m^D(b,a,v,w)\} \\
&= \left\langle D^2_{v,w} b - D^2_{w,v} b, a\right\rangle + \left\langle D^2_{v,w} a - D^2_{w,v} a, b\right\rangle - \operatorname{tr}_E \langle Dv, w\rangle D \langle a, b\rangle \\
&= \pi[v,w] \langle b, a\rangle - \pi v(\pi w \langle b, a\rangle) + \pi w(\pi v \langle b, a\rangle) \\
&\quad + \operatorname{tr}_E \langle Dv, w\rangle D \langle b, a\rangle - \operatorname{tr}_E \langle Dv, w\rangle D \langle a, b\rangle \\
&= 0
\end{aligned}$$

Furthermore, tensoriality in a is obvious. It follows that $\mathcal{R}m^D \in \Gamma(\mathrm{Sym}^2 \Lambda^2 E^*)$. ☐

We now establish that the generalised Riemann tensor is an algebraic curvature tensor. A proof for the case of arbitrary torsion is given in [5, Theorem 4.13.].

Lemma 2.44. *The generalised Riemann tensor satisfies the first Bianchi identity (2.15).*

Proof. Note first that by $T^D = 0$

$$\begin{aligned}
&\left\langle D^2_{v,w} u - D^2_{v,u} w, a\right\rangle + \left\langle D^2_{v,a} w, u\right\rangle \\
&= \langle D_v D_w u - D_{D_v w} u - D_v D_u w + D_{D_v u} w, a\rangle + \left\langle D^2_{v,a} w, u\right\rangle \\
&= \langle D_v([w,u] - \langle Dw, u\rangle) - D_{D_v w} u + D_{D_v u} w, a\rangle + \left\langle D^2_{v,a} w, u\right\rangle \\
&= \langle D_v [w,u] - D_{D_v w} u + D_{D_v u} w, a\rangle - \langle D_a w, D_v u\rangle ,
\end{aligned}$$

where we understand $\langle Dw, u\rangle$ as a section of E by virtue of $\langle \cdot, \cdot\rangle$. Calculate, employing the above formula and twice more $T^D = 0$,

$$\begin{aligned}
&2 \sum_{\sigma(u,v,w)} \mathcal{R}m^D(a,u,v,w) \\
&= \sum_{\sigma(u,v,w)} \{ \langle D_v[w,u] - D_{D_w u}v + D_{D_u w}v, a\rangle - \langle D_a w, D_v u\rangle \\
&\qquad - \left\langle D^2_{a,u}v, w\right\rangle - \operatorname{tr}\langle Dv, w\rangle \langle Du, a\rangle \} \\
&= \sum_{\sigma(u,v,w)} \{ \left\langle D_v[w,u] - D_{[w,u]-\langle Dw,u\rangle}v, a\right\rangle - \langle D_a w, D_v u\rangle \\
&\qquad - \left\langle D^2_{a,u}v, w\right\rangle - \operatorname{tr}\langle Dv, w\rangle \langle Du, a\rangle \} \\
&= \sum_{\sigma(u,v,w)} \{ \left\langle D_v[w,u] - D_{[w,u]}v, a\right\rangle - \langle D_a w, D_v u\rangle - \left\langle D^2_{a,u}v, w\right\rangle \} \\
&= \sum_{\sigma(u,v,w)} \{ \langle [v,[w,u]], a\rangle - \langle D_a v, [w,u]\rangle - \langle D_a w, D_v u\rangle - \left\langle D^2_{a,u}v, w\right\rangle \} \\
&= \sum_{\sigma(u,v,w)} \{ \langle [u,[v,w]], a\rangle - \langle D_a w, D_v u + [u,v]\rangle - \left\langle D^2_{a,v}w, u\right\rangle \}.
\end{aligned} \tag{2.16}$$

We want to use the Jacobi identity for the Dorfman bracket to show that this expression vanishes. We check that

$$\begin{aligned}
&\langle [u,[v,w]] + [w,[u,v]] + [v,[w,u]], a\rangle \\
&= \langle [u,[v,w]] - [[u,v],w] - [v,[u,w]] + \pi^* \mathrm{d}\langle w,[u,v]\rangle + [v, \pi^*\mathrm{d}\langle w,u\rangle], a\rangle \\
&= \langle \pi^* \mathrm{d}\langle w,[u,v]\rangle + [v, \pi^*\mathrm{d}\langle w,u\rangle], a\rangle \\
&= \pi a \langle w,[u,v]\rangle + \pi v \pi a \langle w,u\rangle - \pi[v,a]\langle w,u\rangle \\
&= \pi a \langle w,[u,v]\rangle + \pi a \pi v \langle w,u\rangle \\
&= \frac{1}{3} \sum_{\sigma(u,v,w)} \{ \pi a \langle w,[u,v]\rangle + \pi a \pi v \langle w,u\rangle \} \\
&= \frac{1}{3} \sum_{\sigma(u,v,w)} \{ \langle D_a w, [u,v]\rangle + \langle w, D_a[u,v]\rangle + \left\langle D^2_{a,v}w + D_{D_a v}w, u\right\rangle \\
&\qquad + \left\langle w, D^2_{a,v}u + D_{D_a v}u\right\rangle + \langle D_a w, D_v u\rangle + \langle D_v w, D_a u\rangle \}.
\end{aligned} \tag{2.17}$$

Note that we expanded the expression over the last two lines in an effort to match the expression to (2.16). To continue this effort, note that, employing $T^D = 0$,

$$\begin{aligned}
&\sum_{\sigma(u,v,w)} \{ \langle D_a w, [u,v]\rangle + \langle D_{D_a v}w, u\rangle + \langle D_{D_a v}u, w\rangle + \langle D_a w, D_v u\rangle + \langle D_v w, D_a u\rangle \} \\
&= \sum_{\sigma(u,v,w)} \{ \langle D_a w, D_u v + D_v u + [u,v]\rangle + \pi(D_a v)\langle w,u\rangle \} \\
&= \sum_{\sigma(u,v,w)} \{ \langle D_a w, 2D_v u + 2[u,v]\rangle - \langle D_{D_a w}u, v\rangle + \langle D_a v, [w,u] + [u,w]\rangle \} \\
&= \sum_{\sigma(u,v,w)} \{ \langle D_a w, 2D_v u + 3[u,v] + [v,u]\rangle - \langle D_{D_a w}u, v\rangle \} \\
&= \sum_{\sigma(u,v,w)} \{ \langle D_a w, 3D_v u + 3[u,v] - D_u v\rangle + \langle D_{D_a w}v, u\rangle - \langle D_{D_a w}u, v\rangle \},
\end{aligned}$$

and then, once more employing $T^D = 0$,

$$\begin{aligned}\sum_{\sigma(u,v,w)} \langle w, D_a[u,v]\rangle &= \sum_{\sigma(u,v,w)} \langle w, D_a(D_u v - D_v u + \langle Du, v\rangle)\rangle \\ &= \sum_{\sigma(u,v,w)} \{ \left\langle w, D^2_{a,u} v + D_{D_a u} v - D^2_{a,v} u - D_{D_a v} u\right\rangle + \left\langle D^2_{a,w} u, v\right\rangle + \langle D_w u, D_a v\rangle \} \\ &= \sum_{\sigma(u,v,w)} \{ \left\langle u, 2D^2_{a,v} w - D^2_{a,v} u\right\rangle + \langle D_w u, D_a v\rangle + \langle D_{D_a w} u, v\rangle - \langle D_{D_a w} v, u\rangle \}.\end{aligned}$$

Inserting into (2.17), we obtain

$$\begin{aligned}&\langle [u,[v,w]] + [w,[u,v]] + [v,[w,u]], a\rangle \\ &= \sum_{\sigma(u,v,w)} \{ \langle D_a w, D_v u + [u,v]\rangle + \left\langle u, D^2_{a,v} w\right\rangle \},\end{aligned}$$

and a comparison to (2.16) yields the first Bianchi identity. □

Corollary 2.45. *For a generalised LC connection $D \in \mathcal{D}^0(\mathcal{G})$, the generalised Riemann tensor $\mathcal{R}m^D$ takes values in the direct sum of the* pure-type subbundle

$$(\mathrm{Sym}^2 \Lambda^2 E_+^*) \oplus (\mathrm{Sym}^2 \Lambda^2 E_-^*)$$

and the mixed-type subbundle

$$(\Lambda^2 E_+^* \oplus \Lambda^2 E_-^*) \vee (E_+^* \wedge E_-^*).$$

Proof. The claim is that $\mathcal{R}m^D(a,b,c,d)$ vanishes if two of the four sections a,b,c,d are in E_+ and two in E_-. If both pairs (a,b) and (c,d) are mixed-type, this is an obvious consequence of D being metric. The remaining cases follow via the first Bianchi identity. □

Definition 2.46 ((Full) Generalised Ricci Tensor)**.** Let D be a torsion-free generalised connection on an exact CA E. The *full generalised Ricci tensor* $\overline{\mathcal{R}c}^D \in \Gamma(\mathrm{Sym}^2 E^*)$ is defined from the generalised Riemann tensor $\mathcal{R}m^D$ by the following trace taken with the inner product $\langle \cdot, \cdot \rangle$:

$$\overline{\mathcal{R}c}^D(a,b) = \mathrm{tr}_E\, \mathcal{R}m^D(\cdot, a, \cdot, b), \qquad a, b \in \Gamma(E).$$

Assuming that $D \in \mathcal{D}^0(\mathcal{G}, \mathrm{div})$, the *generalised Ricci tensor* $\mathcal{R}c^\pm \in \Gamma(E_\mp^* \otimes E_\pm^*)$ is the restriction of the full generalised Ricci tensor,

$$\mathcal{R}c^\pm(\mathcal{G}, \mathrm{div}) = \overline{\mathcal{R}c}^D\Big|_{E_\mp \times E_\pm}.$$

We refer to [33] for a comparison of different notions of generalised Ricci curvature in the literature.

As indicated by the notation, the (restricted) generalised Ricci tensor is independent of the choice of generalised LC connection with given divergence, cf. for example [15, Proposition 4.5.].

Proposition 2.47. *Let $D, D' \in \mathcal{D}^0(\mathcal{G}, \mathrm{div})$. Then $\mathcal{R}c^\pm[D] = \mathcal{R}c^\pm[D']$.*

Proof. By Corollary 2.40, we have $\chi_+ + \chi_- \coloneqq D' - D \in \Gamma(\ker \mathrm{tr})$, where $\mathrm{tr}\colon (\mathfrak{so}(E_+))^{\langle 1\rangle} \oplus (\mathfrak{so}(E_-))^{\langle 1\rangle} \to E^*$ denotes the trace map from equation (2.14). The claim follows from the observation that only the mixed-type components of the generalised Riemann tensor contribute to $\mathcal{R}c^\pm$ (cf. Corollary 2.45), and the formula for these components from Theorem 3.6. $\square$

Definition 2.48 (Generalised Scalar Curvature)**.** Given a torsion-free generalised connection D, its *generalised scalar curvature* $\mathcal{S}c \in C^\infty(M)$ is defined as

$$\mathcal{S}c = \frac{1}{2}\operatorname{tr}_{\mathcal{G}} \overline{\mathcal{R}c}^D.$$

A pair $(\mathcal{G}, \mathrm{div})$ has well-defined associated generalised scalar curvature, cf. for example [15, Lemma 4.11.].

Proposition 2.49. *Let $D, D' \in \mathcal{D}^0(\mathcal{G}, \mathrm{div})$. Then $\mathcal{S}c[D] = \mathcal{S}c[D']$.*

Proof. As in the proof of Proposition 2.47, we have $D' - D = \chi_+ + \chi_- \in \Gamma(\ker \mathrm{tr})$. Taking local ON frames $\{e_i^\pm = \sigma_\pm e_i\}$ for $E_\pm$ and writing $\varepsilon_i = \pm\left\langle e_i^\pm, e_i^\pm\right\rangle$, we have

$$\begin{aligned}
2\mathcal{S}c[D'] = \operatorname{tr}_{\mathcal{G}} \overline{\mathcal{R}c}^{D'} &= \sum_i \varepsilon_i \left\{\overline{\mathcal{R}c}^{D'}(e_i^+, e_i^+) - \overline{\mathcal{R}c}^{D'}(e_i^-, e_i^-)\right\} \\
&\overset{*}{=} \sum_{i,j} \varepsilon_i\varepsilon_j \left\{\mathcal{R}m^{D'}(e_j^+, e_i^+, e_j^+, e_i^+) - \mathcal{R}m^{D'}(e_j^-, e_i^-, e_j^-, e_i^-)\right\} \\
&\overset{\#}{=} \sum_{i,j} \varepsilon_i\varepsilon_j \left\{\mathcal{R}m^{D}(e_j^+, e_i^+, e_j^+, e_i^+) - \mathcal{R}m^{D}(e_j^-, e_i^-, e_j^-, e_i^-)\right\}, \\
&= 2\mathcal{S}c[D].
\end{aligned}$$

In $*$, we employed Corollary 2.45. In $\#$, we employed the formulas from Theorem 3.5 and that due to $\partial\chi_\pm = 0$ and the antisymmetry of $\chi_\pm$ in the last two entries

$$\begin{aligned}
&\sum_{i,j} \varepsilon_i\varepsilon_j \Big\{\mathcal{G}(\chi_\pm(e_j^\pm, e_i^\pm), \chi_\pm(e_i^\pm, e_j^\pm)) + \mathcal{G}(\chi_\pm(e_i^\pm, e_j^\pm), \chi_\pm(e_j^\pm, e_i^\pm)) \\
&\qquad\qquad + \operatorname{tr}_{\mathcal{G}} \chi_\pm(\cdot, e_i^\pm, e_j^\pm)\chi_\pm(\cdot, e_j^\pm, e_i^\pm)\Big\} \\
&= \sum_{i,j,k} \varepsilon_i\varepsilon_j\varepsilon_k \Big\{\chi_\pm(e_i^\pm, e_j^\pm, e_k^\pm)\chi_\pm(e_j^\pm, e_i^\pm, e_k^\pm) + \chi_\pm(e_i^\pm, e_k^\pm, e_j^\pm)\chi_\pm(e_k^\pm, e_i^\pm, e_j^\pm) \\
&\qquad\qquad + \chi_\pm(e_i^\pm, e_j^\pm, e_k^\pm)\chi_\pm(e_i^\pm, e_k^\pm, e_j^\pm)\Big\} \\
&= \sum_{i,j,k} \varepsilon_i\varepsilon_j\varepsilon_k \chi_\pm(e_i^\pm, e_j^\pm, e_k^\pm) \Big\{\chi_\pm(e_j^\pm, e_i^\pm, e_k^\pm) + \chi_\pm(e_k^\pm, e_j^\pm, e_i^\pm) + \chi_\pm(e_i^\pm, e_k^\pm, e_j^\pm)\Big\} \\
&= 0.
\end{aligned}$$

$\square$

3 The Canonical Generalised Levi-Civita Connection and its Curvature

This chapter establishes a new canonical and geometric criterion which uniquely selects a generalised Levi-Civita connection $D^{\mathcal{G},\mathrm{div}}$ with specified divergence.

Sections 3.1 - 3.3 present the results of [16]. In Section 3.1, we give the definition of $D^{\mathcal{G},\mathrm{div}}$ as found in [3, 4, 5, 6], explain the new criterion, and show as the first main result of this thesis that indeed $D^{\mathcal{G},\mathrm{div}}$ is uniquely selected. As the preferred nature of $D^{\mathcal{G},\mathrm{div}}$ passes on to its generalised Riemann, Ricci, and scalar curvature, we present the computation of these curvature invariants for the case of metric divergence in Section 3.2 and then for arbitrary divergence operators in Section 3.3.

Finally, in Section 3.4, we present the results obtained in [17] characterising pairs $(\mathcal{G}, \mathrm{div})$ which are generalised Riemann flat, $\mathcal{R}m^{D^{\mathcal{G},\mathrm{div}}} = 0$.

3.1 The Canonical Generalised Levi-Civita Connection

This section contains the results of [16, chapter 2].

In Section 2.5 and specifically in Corollary 2.36, we explained that for any generalised metric $\mathcal{G}$ there exists a canonical generalised LC connection D^0 with metric divergence. Note also the uniqueness property of D^0 mentioned in Remark 2.37. In this section, we explain for an arbitrary pair $(\mathcal{G}, \mathrm{div})$ the construction of a canonical generalised LC connection $D^{\mathcal{G},\mathrm{div}}$ with divergence div from the generalised LC connection $D^0 = D^{\mathcal{G},\mathrm{div}^{\mathcal{G}}}$ with metric divergence $\mathrm{div}^{\mathcal{G}}$, as appears in [3, 4, 5, 6]. We then introduce the canonical and geometric criterion developed in [16] which uniquely selects $D^{\mathcal{G},\mathrm{div}}$.

Throughout this section, we consider an exact CA $E \to M$ with generalised metric $\mathcal{G}$ and divergence operator div. Denote by D^0 the generalised LC connection from Corollary 2.36, and recall from there that the space of generalised LC connections $\mathcal{D}^0(\mathcal{G})$ is an affine space over the sections of the generalised first prolongation $(\mathfrak{so}(E_+) \oplus \mathfrak{so}(E_-))^{\langle 1 \rangle}$. Recall also from Lemma 2.39 that D^0 has metric divergence.

Employing the ideas from [6, 4], we inspect the generalised first prolongation

$$[\mathfrak{so}(E_+) \oplus \mathfrak{so}(E_-)]^{\langle 1 \rangle} = (\mathfrak{so}(E_+))^{\langle 1 \rangle} \oplus (\mathfrak{so}(E_-))^{\langle 1 \rangle}.$$

Note that generalised vectors $e_\pm \in E_\pm$ induce the canonical elements $\chi_\pm^{e_\pm} \in (\mathfrak{so}(E_\pm))^{\langle 1 \rangle}$,

$$\chi_\pm^{e_\pm}(a_\pm, b_\pm) = \langle a_\pm, b_\pm \rangle \, e_\pm - \langle b_\pm, e_\pm \rangle \, a_\pm. \tag{3.1}$$

Recall that $(\mathrm{tr}\, \chi)(a) \equiv \mathrm{tr}\, \chi(\cdot, a)$ for $\chi \in (\mathfrak{so}(E))^{\langle 1 \rangle}$ and $a \in E$. Denoting by

$$\mathrm{tr}_\pm \colon (\mathfrak{so}(E_\pm))^{\langle 1 \rangle} \longrightarrow E_\pm^*, \qquad \chi \longmapsto \mathrm{tr}\, \chi \tag{3.2}$$

the restrictions of the trace map from (2.14) to $(\mathfrak{so}(E_\pm))^{\langle 1 \rangle}$, one finds that for all $a_\pm \in E_\pm$

$$(\mathrm{tr}_\pm \chi_\pm^{e_\pm})(a_\pm) = \sum_i \varepsilon_i^\pm \left[\left\langle f_i^\pm, a_\pm \right\rangle \left\langle e_\pm, f_i^\pm \right\rangle - \left\langle f_i^\pm, f_i^\pm \right\rangle \langle e_\pm, a^\pm \rangle \right] = (1-d) \, \langle e_\pm, a_\pm \rangle \,,$$

where $f_i^\pm$ is a local ON frame for $(E_\pm, \langle \cdot, \cdot \rangle)$ and $\varepsilon_i^\pm = \left\langle f_i^\pm, f_i^\pm \right\rangle$. Thus, the canonical maps $e_\pm \mapsto \chi_\pm^{e_\pm}$ induce the natural decompositions

$$(\mathfrak{so}(E_\pm))^{\langle 1 \rangle} \cong E_\pm \oplus \ker \mathrm{tr}_\pm \,. \tag{3.3}$$

In particular, denoting by $e = e_+ + e_- \in \Gamma(E)$ the generalised dilaton, $\mathrm{div} = \mathrm{div}^{\mathcal{G}} - \langle e, \cdot\rangle$, we obtain a canonical generalised LC connection $D^{\mathcal{G},\mathrm{div}}$ with divergence div (cf. for example [6, Lemma 3.17.]):

$$D^{\mathcal{G},\mathrm{div}} = D^0 + \frac{1}{d-1}(\chi_+^{e_+} + \chi_-^{e_-}). \tag{3.4}$$

One might worry that there are other canonical ways of splitting (3.3), and thus other canonical generalised LC connections with given divergence. In fact, in some situations where additional data (such as a semi-Riemannian hypersurface foliation of the base, $M = (\Sigma_t)_t$) is provided, this is the case (compare Lemma 4.18). We present in the following the solution developed in [16] to this issue of non-uniqueness.

Note that the decomposition (3.3) is orthogonal with respect to the scalar product induced by the inner product $\langle\cdot,\cdot\rangle$ (or the up to sign identical generalised metric $\mathcal{G}$), as for $\chi \in \ker \mathrm{tr}_\pm$ and local ON frames $f_i^\pm$ of $(E_\pm, \langle\cdot,\cdot\rangle)$ with $\varepsilon_i^\pm = \left\langle f_i^\pm, f_i^\pm\right\rangle$

$$\begin{aligned}\langle \chi, \chi_\pm^{e_\pm}\rangle &= \sum_{i,j,k} \varepsilon_i^\pm \varepsilon_j^\pm \varepsilon_k^\pm \chi(f_i^\pm, f_j^\pm, f_k^\pm)\left[\left\langle f_i^\pm, f_j^\pm\right\rangle \left\langle e_\pm, f_k^\pm\right\rangle - \left\langle f_i^\pm, f_k^\pm\right\rangle \left\langle e_\pm, f_j^\pm\right\rangle\right]\\ &= \sum_{i,k} \varepsilon_i^\pm \varepsilon_k^\pm \chi(f_i^\pm, f_i^\pm, f_k^\pm)\left\langle e_\pm, f_k^\pm\right\rangle - \sum_{i,j} \varepsilon_i^\pm \varepsilon_j^\pm \chi(f_i^\pm, f_j^\pm, f_i^\pm)\left\langle e_\pm, f_j^\pm\right\rangle\\ &= 0\end{aligned}$$

The last equality follows from antisymmetry of $\chi(\cdot,\cdot,\cdot)$ in the last two entries and $\mathrm{tr}\,\chi = 0$.

The following result resolves the issue of non-uniqueness of the decomposition (3.3) by establishing the decomposition (3.3) as the unique decomposition orthogonal with respect to $\langle\cdot,\cdot\rangle$ (and equivalently $\mathcal{G}$). Thus,

$$D^{\mathcal{G},\mathrm{div}} - D^0 \in \Gamma\left([\ker \mathrm{tr}_+]^\perp \oplus [\ker \mathrm{tr}_-]^\perp\right) \cong \Gamma(E_+ \oplus E_-) = \Gamma(E)$$

provides the previously advertised canonical and geometrical criterion selecting the element $D^{\mathcal{G},\mathrm{div}} \in \mathcal{D}^0(\mathcal{G}, \mathrm{div})$ uniquely.

Lemma 3.1. *The kernels of the trace-maps* tr_+ *and* tr_- *as in (3.2) are nondegenerate with respect to the metrics induced by* $\langle\cdot,\cdot\rangle$ *(and equivalently* $\mathcal{G}$*) on* $(\mathfrak{so}(E_+))^{\langle 1\rangle}$ *and* $(\mathfrak{so}(E_-))^{\langle 1\rangle}$*.*

Proof. The proof is analogous for both tr_+ and tr_- and reduces to the following algebraic statement. Consider a pseudo-Euclidean vector space V. Then we claim that the kernel of

$$\mathrm{tr} : \mathfrak{so}(V)^{\langle 1\rangle} \longrightarrow V^* \tag{3.5}$$

is nondegenerate with respect to the induced scalar product. To prove this we decompose $V = L \oplus P$ as an orthogonal sum of a negative definite subspace L and a positive definite subspace P. We decompose

$$V^* \otimes \wedge^2 V^* = (L^* \otimes \wedge^2 L^*) \oplus (L^* \otimes \wedge^2 P^*) \oplus (P^* \otimes L^* \wedge P^*) \oplus \tag{3.6}$$

$$(L^* \otimes L^* \wedge P^*) \oplus (P^* \otimes \wedge^2 L^*) \oplus (P^* \otimes \wedge^2 P^*), \tag{3.7}$$

where the induced scalar product is negative definite on terms in the first line and positive definite on terms in the second line. Next we show that

$$\mathfrak{so}(V)^{\langle 1\rangle} = \{\chi \in V^* \otimes \wedge^2 V^* \mid \partial\chi = 0\} \subset V^* \otimes \wedge^2 V^*$$

is a nondegenerate subspace, where $\partial : V^* \otimes \wedge^2 V^* \to \wedge^3 V^*$ is given by

$$(\partial\chi)(u,v,w) = \sum_{\sigma(u,v,w)} \chi(u,v,w).$$

It clear that the equation $\partial\chi = 0$ decouples into four independent equations corresponding to the tensor power of L in the decomposition (3.6)-(3.7). Since (3.6) collects the terms of odd degree in L and (3.7) those of even degree, the equation $\partial\chi = 0$ defines a subspace which is a direct sum of a subspace $\mathfrak{so}(V)^{\langle 1\rangle}_{\mathrm{odd}}$ of (3.6) and a subspace $\mathfrak{so}(V)^{\langle 1\rangle}_{\mathrm{even}}$ of (3.7). The latter subspaces are definite and therefore their sum is nondegenerate. Finally, the kernel of (3.5) splits as

$$\ker\left(\mathrm{tr} : \mathfrak{so}(V)^{\langle 1\rangle}_{\mathrm{odd}} \to L^*\right) \oplus \ker\left(\mathrm{tr} : \mathfrak{so}(V)^{\langle 1\rangle}_{\mathrm{even}} \to P^*\right)$$

and is therefore as well a sum of two definite subspaces. □

Summarising, we have established the following

Theorem 3.2. *Let $\mathcal{G}$ be a generalised metric and* div *a divergence operator on an exact CA E. Denote by $H \in \Omega^3_{\mathrm{cl}}(M)$ the preferred representative of the Ševera class, by g the semi-Riemannian metric induced by $\mathcal{G}$, and by ∇ its LC connection.*

Then, there is a unique canonical LC generalised connection $D^{\mathcal{G},\mathrm{div}} = D^0 + \chi$ with divergence div *characterised as follows:*

(i) $D^0 = D^1 - \frac{1}{3}T^{D^1} \in \mathcal{D}^0(\mathcal{G}, \mathrm{div}^{\mathcal{G}})$, where D^1 is the unique metric compatible generalised connection with pure-type torsion and pure-type operators given by ∇, i.e.

$$D^1_{\sigma_\pm X}\sigma_\pm Y = \sigma_\pm(\nabla_X Y) \qquad \text{for all } X, Y \in \Gamma(TM),$$

(ii) $\chi \in \Gamma((\ker\mathrm{tr})^\perp) \subset \mathfrak{so}(E_+) \oplus \mathfrak{so}(E_-)$ such that $\mathrm{tr}\,\chi = \mathrm{div} - \mathrm{div}^{\mathcal{G}}$.

Furthermore, recalling the connections $\nabla^\pm$ and $\nabla^{\pm 1/3}$ from (2.13),

(a) if D^0 is as in (i), then for all $X, Y \in \Gamma(TM)$

$$D^0_{\sigma_\mp X}\sigma_\pm Y = \sigma_\pm(\nabla^\pm_X Y) \quad \text{and} \quad D^0_{\sigma_\pm X}\sigma_\pm Y = \sigma_\pm(\nabla^{\pm 1/3}_X Y),$$

(b) if χ is as in (ii), then $\chi = \frac{1}{d-1}(\chi^{e_+}_+ + \chi^{e_-}_-)$, where $e = e_+ + e_- \in \Gamma(E)$ is such that $\mathrm{div} = \mathrm{div}^{\mathcal{G}} - \langle e, \cdot\rangle$ and $\chi^{e_\pm}_\pm$ are as in (3.1).

Proof. By Lemma 2.35, D^1 is determined uniquely. Therefore, D^0 is determined uniquely. We established (a) in (2.13), and $\mathrm{div}_{D^0} = \mathrm{div}^{\mathcal{G}}$ in Lemma 2.39. Finally, by the discussion presented in this chapter, there is a unique $\chi \in (\ker\mathrm{tr})^\perp$ satisfying (iii), and it is given as in (b). □

3.2 Curvature Computations for Metric Divergence

This section reproduces [16, chapter 3].

Recall from Corollary 2.45 that the generalised Riemann tensor is a sum of its pure-type part in $\mathrm{Sym}^2 \Lambda^2 E^*_+ \oplus \mathrm{Sym}^2 \Lambda^2 E^*_-$ and its mixed-type part in $(E^*_+ \wedge E^*_-) \vee \Lambda^2 E^*_+ \oplus (E^*_- \wedge E^*_+) \vee \Lambda^2 E^*_-$. These two components are determined separately in the following two theorems.

Theorem 3.3. *The pure-type part of the Riemann tensor of the canonical generalised Levi-Civita connection D with metric divergence is given by the following formulas. For all $a, b, v, w \in \Gamma(E_+)$ we have*

$$\begin{aligned}\mathcal{R}m^D(a,b,v,w) &= g(\mathrm{Rm}(v,w)b,a) + \frac{1}{36}H^{(2)}(a,v,w,b)\\ &\quad + \frac{1}{36}H^{(2)}(v,b,w,a) + \frac{1}{18}H^{(2)}(v,w,b,a),\end{aligned}$$

where $H^{(2)}$ denotes the contraction of $H^{\otimes 2} \in \Gamma(\mathrm{Sym}^2 \bigwedge^3 T^*M)$ in the first and fourth argument using the metric g and we use the notation $F(a,b,v,w) := F(\pi a, \pi b, \pi v, \pi w)$ for any tensor F on M. Similarly, for all $a, b, v, w \in \Gamma(E_-)$ we have

$$\begin{aligned}-\mathcal{R}m^D(a,b,v,w) &= g(\mathrm{Rm}(v,w)b,a) + \frac{1}{36}H^{(2)}(a,v,w,b)\\ &\quad + \frac{1}{36}H^{(2)}(v,b,w,a) + \frac{1}{18}H^{(2)}(v,w,b,a).\end{aligned}$$

More compactly, we can write both cases in one formula as

$$\begin{aligned}\pm\mathcal{R}m^D(a,b,v,w) &= g(\mathrm{Rm}(v,w)b,a) + \frac{1}{36}H^{(2)}(a,v,w,b)\\ &\quad + \frac{1}{36}H^{(2)}(v,b,w,a) + \frac{1}{18}H^{(2)}(v,w,b,a),\end{aligned}$$

where now the pure-type insertions are $a, b, v, w \in \Gamma(E_\pm)$.

Proof. We consider only the case $a, b, v, w \in \Gamma(E_+)$, since the other case is similar. We will compute each term in the following formula:

$$\begin{aligned}2\mathcal{R}m^D(a,b,v,w) &= \mathfrak{S}\langle D_v D_w b, a\rangle - \mathfrak{S}\langle D_{D_v w} b, a\rangle - \langle (Dv)^* w, (Db)^* a\rangle\\ &= \mathfrak{S}\pi(v)\langle D_w b, a\rangle - \mathfrak{S}\langle D_w b, D_v a\rangle - \mathfrak{S}\langle D_{D_v w} b, a\rangle\\ &\quad - \langle (Dv)^* w, (Db)^* a\rangle,\end{aligned}$$

where we use the notation

$$\mathfrak{S}L(v,w,b,a) := L(v,w,b,a) - L(w,v,b,a) + L(b,a,v,w) - L(a,b,v,w)$$

for any differential operator $L : \Gamma(E^{\otimes 4}) \to C^\infty(M)$. Note in the second line $\mathfrak{S}\langle D_w b, D_v a\rangle$ stands for $\mathfrak{S}L(v,w,b,a)$ with $L(v,w,b,a) = \langle D_w b, D_v a\rangle$.

For the calculations we can assume that the sections a, b, v, w have constant scalar products and vanishing D-derivatives at the point p at which we compute the Riemann tensor. We begin with the case $a, b, v, w \in \Gamma(E_+)$ and compute term by term. From (2.13), we have

$$\pi(v)\langle D_w b, a\rangle = vg(\nabla_w b, a) + \frac{1}{6}vH(w,b,a).$$

At p we obtain

$$\begin{aligned}\mathfrak{S}\pi(v)\langle D_w b, a\rangle|_p &= 2g(\mathrm{Rm}(v,w)b,a)|_p + \frac{1}{6}(dH)(v,w,b,a)|_p\\ &= 2g(\mathrm{Rm}(v,w)b,a)|_p.\end{aligned}$$

We compute the next term using an arbitrary local frame $(e_I) = (e_i, e_{\hat{j}})$ of E adapted to the decomposition $E = E_+ \oplus E_-$ and the notation $(e^I) = (e^i, e^{\hat{j}})$ for the frame such that $\langle e^I, e_J \rangle = \delta^I_J$:

$$\langle D_w b, D_v a \rangle = \sum_{I=1}^{2n} \langle D_w b, e_I \rangle \langle D_v a, e^I \rangle = \sum_{i=1}^{n} \langle D_w b, e_i \rangle \langle D_v a, e^i \rangle.$$

At p this evaluates to

$$\langle D_w b, D_v a \rangle|_p = \frac{1}{36} H^{(2)}(w, b, v, a)|_p.$$

Hence

$$\begin{aligned} -\mathfrak{S} \langle D_w b, D_v a \rangle|_p &= -\frac{1}{36} H^{(2)}(w, b, v, a)|_p + \frac{1}{36} H^{(2)}(v, b, w, a)|_p \\ &\quad - \frac{1}{36} H^{(2)}(a, v, b, w)|_p + \frac{1}{36} H^{(2)}(b, v, a, w)|_p \\ &= -\frac{1}{18} H^{(2)}(w, b, v, a)|_p + \frac{1}{18} H^{(2)}(v, b, w, a)|_p. \end{aligned}$$

Next we compute

$$\langle D_{D_v w} b, a \rangle = \sum_{I=1}^{2n} \langle D_v w, e_I \rangle \langle D_{e^I} b, a \rangle = \sum_{i=1}^{n} \langle D_v w, e_i \rangle \langle D_{e^i} b, a \rangle,$$

which evaluates at p to

$$\langle D_{D_v w} b, a \rangle|_p = \frac{1}{36} H^{(2)}(v, w, b, a)|_p.$$

Thus

$$-\mathfrak{S} \langle D_{D_v w} b, a \rangle|_p = -\frac{1}{9} H^{(2)}(v, w, b, a)|_p.$$

Finally we compute

$$\begin{aligned} \langle (Dv)^* w, (Db)^* a \rangle &= \sum_{I=1}^{2n} \langle D_{e_I} v, w \rangle \langle D_{e^I} b, a \rangle \\ &= \sum_{i=1}^{n} \langle D_{e_i} v, w \rangle \langle D_{e^i} b, a \rangle + \sum_{j=1}^{n} \langle D_{e_{\hat{j}}} v, w \rangle \langle D_{e^{\hat{j}}} b, a \rangle. \end{aligned}$$

Using (2.13) and evaluating at p we get

$$\begin{aligned} \langle (Dv)^* w, (Db)^* a \rangle|_p &= \frac{1}{36} H^{(2)}(v, w, b, a)|_p - \frac{1}{4} H^{(2)}(v, w, b, a)|_p \\ &= -\frac{2}{9} H^{(2)}(v, w, b, a)|_p, \end{aligned}$$

where the minus sign is due to the fact that $\pi_- : E_- \to (TM, g)$ is an isometry with respect to $\mathcal{G}$ and therefore an anti-isometry with respect to $\langle \cdot, \cdot \rangle$. This finishes the proof in the case $a, b, v, w \in \Gamma(E_+)$. □

The following theorem can be extracted from [6, § 3.4]. For completeness, we give an independent proof using the conventions and results of our paper.

Theorem 3.4. *The mixed-type components of the generalised Riemann curvature are given by*

$$\begin{aligned}
&\pm 2\mathcal{R}m^D(a,\bar{b},v,w)\\
&= \mathrm{Rm}(a,\bar{b},v,w) \mp \frac{1}{2}[\nabla_a H](\bar{b},v,w) \pm \frac{1}{6}[\nabla_{\bar{b}} H](a,v,w)\\
&\quad - \frac{1}{12}H^{(2)}(\bar{b},w,a,v) - \frac{1}{12}H^{(2)}(w,a,\bar{b},v) - \frac{1}{6}H^{(2)}(a,\bar{b},v,w),
\end{aligned}$$

where $a,v,w \in \Gamma(E_\pm)$ and $\bar{b} \in \Gamma(E_\mp)$ arbitrary.

Proof. We have to compute

$$\pm 2\mathcal{R}m^D(a,\bar{b},v,w) = \pm \left\langle D^2_{\bar{b},a}v - D^2_{a,\bar{b}}v, w \right\rangle .$$

We start with the first term.

$$\begin{aligned}
&\pm \left\langle D^2_{\bar{b},a}v, w\right\rangle\\
&= \pm \left\langle D_{\bar{b}}D_a v - D_{D_{\bar{b}}a}v, w\right\rangle\\
&= g(\nabla_{\bar{b}}D_a v, w) \pm \frac{1}{2}H(\bar{b}, D_a v, w) - g(\nabla_{D_{\bar{b}}a}v, w) \mp \frac{1}{6}H(D_{\bar{b}}a, v, w)\\
&= g(\nabla^2_{\bar{b},a}v, w) \pm \frac{1}{6}[\nabla_{\bar{b}}(H(a,v))](w) \pm \frac{1}{2}H(\bar{b},\nabla_a v, w) - \frac{1}{12}H^{(2)}(\bar{b},w,a,v)\\
&\quad \mp \frac{1}{2}g(\nabla_{H(\bar{b},a)}v, w) \mp \frac{1}{6}H(\nabla_{\bar{b}}a, v, w) - \frac{1}{12}H^{(2)}(\bar{b},a,v,w)\\
&= g(\nabla^2_{\bar{b},a}v, w) \mp \frac{1}{6}[\nabla_{\bar{b}}(H(v))](a,w) \pm \frac{1}{2}H(\bar{b},\nabla_a v, w) - \frac{1}{12}H^{(2)}(\bar{b},w,a,v)\\
&\quad \mp \frac{1}{2}g(\nabla_{H(\bar{b},a)}v, w) - \frac{1}{12}H^{(2)}(\bar{b},a,v,w)
\end{aligned}$$

Similarly

$$\begin{aligned}
&\pm \left\langle D^2_{a,\bar{b}}v, w\right\rangle\\
&= \pm \left\langle D_a D_{\bar{b}} v - D_{D_a\bar{b}}v, w\right\rangle\\
&= g(\nabla_a D_{\bar{b}} v, w) \pm \frac{1}{6}H(a, D_{\bar{b}} v, w) - g(\nabla_{D_a\bar{b}}v, w) \mp \frac{1}{2}H(D_a\bar{b}, v, w)\\
&= g(\nabla^2_{a,\bar{b}}v, w) \pm \frac{1}{2}[\nabla_a(H(\bar{b},v))](w) \pm \frac{1}{6}H(a,\nabla_{\bar{b}} v, w) - \frac{1}{12}H^{(2)}(a,w,\bar{b},v)\\
&\quad \pm \frac{1}{2}g(\nabla_{H(a,\bar{b})}v, w) \mp \frac{1}{2}H(\nabla_a\bar{b}, v, w) + \frac{1}{4}H^{(2)}(a,\bar{b},v,w)\\
&= g(\nabla^2_{a,\bar{b}}v, w) \mp \frac{1}{2}[\nabla_a(H(v))](\bar{b},w) \pm \frac{1}{6}H(a,\nabla_{\bar{b}} v, w) - \frac{1}{12}H^{(2)}(a,w,\bar{b},v)\\
&\quad \pm \frac{1}{2}g(\nabla_{H(a,\bar{b})}v, w) + \frac{1}{4}H^{(2)}(a,\bar{b},v,w)
\end{aligned}$$

The result follows. ☐

3.3 Curvature Computations for Arbitrary Divergence

This section reproduces [16, chapter 3].

Let $E \to M$ be an exact Courant algebroid with semi-Riemannian generalised metric $\mathcal{G}$ and divergence operator $\operatorname{div} = \operatorname{div}^{\mathcal{G}} - \langle e, \cdot \rangle$. In this section, we compute the components of the generalised Riemann tensor for any generalised Levi-Civita connection D with divergence div. We specialise these formulas for the case $D = D^{\mathcal{G},\operatorname{div}}$.

Employing Corollary 2.36, we write $D = D^0 + \chi_+ + \chi_-$, where D^0 is the canonical generalised Levi-Civita connection with metric divergence and $\chi_\pm \in \Gamma((\mathfrak{so}(E_\pm))^{\langle 1 \rangle})$. When specialising to $D = D^{\mathcal{G},\operatorname{div}}$, we have $(d-1)\chi_\pm = \chi_\pm^{e_\pm}$, cf. Theorem 3.2.

From the decomposition $D = D^0 + \chi$, it follows that $\mathcal{R}m^D$ is given by $\mathcal{R}m^{D^0}$ plus some extra terms that involve $\chi = \chi_+ + \chi_-$. We already computed $\mathcal{R}m^{D^0}$ in Section 3.2. Thus, to obtain formulas for $\mathcal{R}m^D$, it remains to calculate the extra terms. There are exactly two types of extra terms: terms involving a covariant derivative of χ and terms which are quadratic and algebraic in χ.

Theorem 3.5. *The pure-type components of the generalised Riemann curvature are given by*

$$
\begin{aligned}
&\pm \mathcal{R}m^D(a,b,v,w)\\
&= \operatorname{Rm}(a,b,v,w) - \frac{1}{36}H^{(2)}(a,v,b,w) - \frac{1}{36}H^{(2)}(b,v,w,a) - \frac{1}{18}H^{(2)}(v,w,a,b)\\
&\quad \pm \frac{1}{2}\Big\{[D^0_v\chi_\pm](w,b,a) - [D^0_w\chi_\pm](v,b,a) + [D^0_b\chi_\pm](a,v,w) - [D^0_a\chi_\pm](b,v,w)\Big\}\\
&\quad - \mathcal{G}(\chi_\pm(v,a),\chi_\pm(w,b)) + \mathcal{G}(\chi_\pm(w,a),\chi_\pm(v,b))\\
&\quad - \mathcal{G}(\chi_\pm(b,w),\chi_\pm(a,v)) + \mathcal{G}(\chi_\pm(a,w),\chi_\pm(b,v))\\
&\quad + \operatorname{tr}_{\mathcal{G}} \chi_\pm(\cdot,v,w)\chi_\pm(\cdot,b,a).
\end{aligned}
$$

In particular, if $D = D^{\mathcal{G},\operatorname{div}}$,

$$
\begin{aligned}
&\pm \mathcal{R}m^D(a,b,v,w)\\
&= \operatorname{Rm}(a,b,v,w) - \frac{1}{36}H^{(2)}(a,v,b,w) - \frac{1}{36}H^{(2)}(b,v,w,a) - \frac{1}{18}H^{(2)}(v,w,a,b)\\
&\quad \pm \frac{1}{2(d-1)}\Big\{[D^0_v\chi_\pm^{e_\pm}](w,b,a) - [D^0_w\chi_\pm^{e_\pm}](v,b,a)\\
&\qquad\qquad + [D^0_b\chi_\pm^{e_\pm}](a,v,w) - [D^0_a\chi_\pm^{e_\pm}](b,v,w)\Big\}\\
&\quad + \frac{1}{2(d-1)^2}\left\{2\mathcal{G}(e_\pm,e_\pm)[\mathcal{G}(w,a)\mathcal{G}(v,b) - \mathcal{G}(v,a)\mathcal{G}(w,b)]\right.\\
&\qquad\qquad + \mathcal{G}(a,e_\pm)[\mathcal{G}(w,b)\mathcal{G}(v,e_\pm) - \mathcal{G}(v,b)\mathcal{G}(w,e_\pm)]\\
&\qquad\qquad \left. + \mathcal{G}(b,e_\pm)[\mathcal{G}(v,a)\mathcal{G}(w,e_\pm) - \mathcal{G}(w,a)\mathcal{G}(v,e_\pm)]\right\}.
\end{aligned}
$$

Herein, $a,b,v,w \in \Gamma(E_\pm)$ arbitrary, we denoted $d \coloneqq \dim M$, and from now on we use the notation $\operatorname{Rm}(a,b,v,w) \coloneqq \operatorname{Rm}(\pi a, \pi b, \pi v, \pi w)$.

Proof. We calculate

$$
\begin{aligned}
D_v D_w b &= \left(D^0 + \chi_\pm\right)_v \left(D^0 + \chi_\pm\right)_w b\\
&= D^0_v D^0_w b + \chi_\pm(v, D^0_w b) + D^0_v[\chi_\pm(w,b)] + \chi_\pm(v,\chi_\pm(w,b)).
\end{aligned}
$$

Similarly,

$$
D_{D_v w} b - D^0_{D^0_v w} b = \chi_\pm(D^0_v w, b) + D^0_{\chi_\pm(v,w)} b + \chi_\pm(\chi_\pm(v,w),b)
$$

and

$$\begin{aligned}
&\mathrm{tr}_E(\langle Dv, w\rangle \langle Db, a\rangle) - \mathrm{tr}_E(\left\langle D^0 v, w\right\rangle \left\langle D^0 b, a\right\rangle) \\
&= \mathrm{tr}_E\, \chi_\pm(\cdot, v, w) \left\langle D^0 b, a\right\rangle + \mathrm{tr}_E \left\langle D^0 v, w\right\rangle \chi_\pm(\cdot, b, a) + \mathrm{tr}_E\, \chi_\pm(\cdot, v, w)\chi_\pm(\cdot, b, a).
\end{aligned}$$

Therefore, employing that $\sum_{\sigma(a,b,c)} \chi_\pm(a, b, c) = 0$,

$$\begin{aligned}
&2\mathcal{R}m^D(a, b, v, w) - 2\mathcal{R}m^{D^0}(a, b, v, w) \\
&= [D^0_v\chi_\pm](w, b, a) - [D^0_w\chi_\pm](v, b, a) + [D^0_b\chi_\pm](a, v, w) - [D^0_a\chi_\pm](b, v, w) \\
&\quad + \mathrm{tr}_E \Big\{\left\langle D^0 b, a\right\rangle [\chi_\pm(w, v, \cdot) - \chi_\pm(v, w, \cdot) - \chi_\pm(\cdot, v, w)] \\
&\qquad\qquad + \left\langle D^0 v, w\right\rangle [\chi_\pm(a, b, \cdot) - \chi_\pm(b, a, \cdot) - \chi_\pm(\cdot, b, a)]\Big\} \\
&\quad + \chi_\pm(v, \chi_\pm(w, b), a) - \chi_\pm(w, \chi_\pm(v, b), a) + \chi_\pm(b, \chi_\pm(a, v), w) - \chi_\pm(a, \chi_\pm(b, v), w) \\
&\quad - \chi_\pm(\chi_\pm(v, w), b, a) + \chi_\pm(\chi_\pm(w, v), b, a) - \chi_\pm(\chi_\pm(b, a), v, w) + \chi_\pm(\chi_\pm(a, b), v, w) \\
&\quad - \mathrm{tr}_E\, \chi_\pm(\cdot, v, w)\chi_\pm(\cdot, b, a) \\
&= [D^0_v\chi_\pm](w, b, a) - [D^0_w\chi_\pm](v, b, a) + [D^0_b\chi_\pm](a, v, w) - [D^0_a\chi_\pm](b, v, w) \\
&\quad - \langle\chi_\pm(v, a), \chi_\pm(w, b)\rangle + \langle\chi_\pm(w, a), \chi_\pm(v, b)\rangle \\
&\quad - \langle\chi_\pm(b, w), \chi_\pm(a, v)\rangle + \langle\chi_\pm(a, w), \chi_\pm(b, v)\rangle \\
&\quad + \mathrm{tr}_E\, \chi_\pm(\cdot, v, w)\chi_\pm(\cdot, b, a).
\end{aligned}$$

Finally, we simplify the last three lines for the case $\chi_\pm = \chi^{e\pm}_\pm$ as

$$\begin{aligned}
&- \langle\chi^{e\pm}_\pm(v, a), \chi^{e\pm}_\pm(w, b)\rangle + \langle\chi^{e\pm}_\pm(w, a), \chi^{e\pm}_\pm(v, b)\rangle \\
&- \langle\chi^{e\pm}_\pm(b, w), \chi^{e\pm}_\pm(a, v)\rangle + \langle\chi^{e\pm}_\pm(a, w), \chi^{e\pm}_\pm(b, v)\rangle \\
&+ \mathrm{tr}_E\, \chi^{e\pm}_\pm(\cdot, v, w)\chi^{e\pm}_\pm(\cdot, b, a) \\
&= \{2\langle e_\pm, e_\pm\rangle [\langle w, a\rangle \langle v, b\rangle - \langle v, a\rangle \langle w, b\rangle] \\
&\qquad + \langle a, e_\pm\rangle [\langle w, b\rangle \langle v, e_\pm\rangle - \langle v, b\rangle \langle w, e_\pm\rangle] \\
&\qquad + \langle b, e_\pm\rangle [\langle v, a\rangle \langle w, e_\pm\rangle - \langle w, a\rangle \langle v, e_\pm\rangle]\}\,.
\end{aligned}$$

Employing Theorem 3.3, which asserts that

$$\begin{aligned}
\pm\mathcal{R}m^{D^0}(a, b, v, w) = \mathrm{Rm}(a, b, v, w) &- \frac{1}{36}H^{(2)}(a, v, b, w) \\
&- \frac{1}{36}H^{(2)}(b, v, w, a) - \frac{1}{18}H^{(2)}(v, w, a, b),
\end{aligned}$$

the result follows. □

Theorem 3.6. *The mixed-type components of the generalised Riemann curvature are given by*

$$\begin{aligned}
&\pm 2\mathcal{R}m^D(a, \bar b, v, w) \\
&= \mathrm{Rm}(a, \bar b, v, w) \mp \frac{1}{2}[\nabla_a H](\bar b, v, w) \pm \frac{1}{6}[\nabla_{\bar b} H](a, v, w) \\
&\quad - \frac{1}{12}H^{(2)}(\bar b, w, a, v) - \frac{1}{12}H^{(2)}(w, a, \bar b, v) - \frac{1}{6}H^{(2)}(a, \bar b, v, w) \\
&\quad \pm [D^0_{\bar b}\chi_\pm](a, v, w).
\end{aligned}$$

Herein, $a, v, w \in \Gamma(E_\pm)$ and $\bar b \in \Gamma(E_\mp)$ arbitrary. In particular, if $D = D^{\mathcal{G},\mathrm{div}}$ and denoting $d := \dim M$, the formula applies with $\chi_\pm = \frac{1}{d-1}\chi^{e\pm}_\pm$.

Proof. We calculate

$$\begin{aligned} D_{\bar{b}} D_a v &= D^0_{\bar{b}} D^0_a v + D^0_{\bar{b}}[\chi_\pm(a, v)], \\ D_a D_{\bar{b}} v &= D^0_a D^0_{\bar{b}} v + \chi_\pm(a, D^0_{\bar{b}} v). \end{aligned}$$

Similarly,

$$\begin{aligned} D_{D_{\bar{b}} a} v &= D^0_{D^0_{\bar{b}} a} v + \chi_\pm(D^0_{\bar{b}} a, v), \\ D_{D_a \bar{b}} v &= D^0_{D^0_a \bar{b}} v. \end{aligned}$$

Put together, we obtain

$$2\mathcal{R}m^D(a, \bar{b}, v, w) - 2\mathcal{R}m^{D^0}(a, \bar{b}, v, w) = [D^0_{\bar{b}} \chi_\pm](a, v, w).$$

Employing Theorem 3.4, the result follows. □

As corollaries, we obtain the pure-type components of the full generalised Ricci tensor and then the generalised scalar curvature.

Corollary 3.7. *The pure-type part of the generalised Ricci curvature is given by*

$$\begin{aligned} \overline{\mathcal{R}c}^D(a, b)\Big|_{E_\pm \times E_\pm} = \mathrm{Rc} - \frac{1}{12} H^2 \pm \frac{\mathrm{div}_g(e_\pm)}{d-1}\, g \pm \frac{d-2}{d-1}[g\nabla e_\pm]^{\mathrm{sym}} \\ + \frac{3-2d}{2(d-1)^2}\, |e_\pm|^2_{\mathcal{G}}\, g + \frac{d-2}{2(d-1)^2} g\pi e_\pm \otimes g\pi e_\pm. \end{aligned}$$

Proof. This is a straightforward computation. Let $a, b \in \Gamma(E_\pm)$.

$$
\begin{aligned}
&\overline{\mathcal{R}c}^D(a,b)\\
&= \operatorname{tr}_E \mathcal{R}m^D(\cdot,a,\cdot,b)\\
&= \operatorname{tr}_{E_\pm} \mathcal{R}m^D(\cdot,a,\cdot,b)\\
&= \operatorname{tr}_g \Big\{ \mathrm{Rm}(\cdot,a,\cdot,b) - \frac{1}{36}H^{(2)}(\cdot,\cdot,a,b) - \frac{1}{36}H^{(2)}(a,\cdot,b,\cdot) - \frac{1}{18}H^{(2)}(\cdot,b,\cdot,a)\\
&\quad \pm \frac{1}{2(d-1)} \Big\{ [D^0_{(\cdot)}\chi^{e_\pm}_\pm](b,a,\cdot) - [D^0_b\chi^{e_\pm}_\pm](\cdot,a,\cdot)\\
&\qquad\qquad + [D^0_a\chi^{e_\pm}_\pm](\cdot,\cdot,b) - [D^0_{(\cdot)}\chi^{e_\pm}_\pm](a,\cdot,b)\Big\}\\
&\quad + \frac{1}{2(d-1)^2}\{2\mathcal{G}(e_\pm,e_\pm)[\mathcal{G}(b,\cdot)\mathcal{G}(\cdot,a) - \mathcal{G}(\cdot,\cdot)\mathcal{G}(b,a)]\\
&\qquad\qquad + \mathcal{G}(\cdot,e_\pm)[\mathcal{G}(b,a)\mathcal{G}(\cdot,e_\pm) - \mathcal{G}(\cdot,a)\mathcal{G}(b,e_\pm)]\\
&\qquad\qquad + \mathcal{G}(a,e_\pm)[\mathcal{G}(\cdot,\cdot)\mathcal{G}(b,e_\pm) - \mathcal{G}(b,\cdot)\mathcal{G}(\cdot,e_\pm)]\}\}\\
&\overset{*}{=} \mathrm{Rc}(a,b) - \frac{1}{12}H^2(a,b)\\
&\quad \pm \frac{1}{2(d-1)}\Big\{ g(b,a)\operatorname{div}_g(e_\pm) - g(D^0_b e_\pm, a) + (d-1)g(D^0_b e_\pm, a)\\
&\qquad\qquad + (d-1)g(D^0_a e_\pm, b) + g(a,b)\operatorname{div}_g(e_\pm) - g(D^0_a e_\pm, b)\Big\}\\
&\quad + \frac{1}{2(d-1)^2}\Big\{ 2\,|e_\pm|^2_{\mathcal{G}}\,[\mathcal{G}(b,a) - d\,\mathcal{G}(b,a)] + \mathcal{G}(b,a)\,|e_\pm|^2_{\mathcal{G}} - \mathcal{G}(e_\pm,a)\mathcal{G}(b,e_\pm)]\\
&\qquad\qquad + \mathcal{G}(a,e_\pm)[d\,\mathcal{G}(b,e_\pm) - \mathcal{G}(b,e_\pm)]\}\}\\
&\overset{\#}{=} \mathrm{Rc}(a,b) - \frac{1}{12}H^2(a,b) \pm \frac{g(a,b)}{d-1}\operatorname{div}_g(e_\pm) \pm \frac{d-2}{d-1}[g\nabla e_\pm]^{\mathrm{sym}}(a,b)\\
&\quad + \frac{1}{2(d-1)^2}\Big\{(3-2d)\,|e_\pm|^2_{\mathcal{G}}\, g(a,b) + (d-2)g(e_\pm,a)g(b,e_\pm)\Big\}.
\end{aligned}
$$

In $*$, we used that

$$[D^0_u\chi^{e_\pm}_\pm](v,w,x) = g(v,w)g(D^0_u e_\pm, x) - g(v,x)g(D^0_u e_\pm, w),$$

for $u, v, w, x \in \Gamma(E_\pm)$. In $\#$, we used that

$$g(D^0_a e_\pm, b) + g(D^0_a e_\pm, b) = 2g(\nabla e_\pm)^{\mathrm{sym}}(a,b),$$

see (2.13). □

Corollary 3.8. *The generalised scalar curvature is given by*

$$\mathcal{S}c(\mathcal{G},\operatorname{div}) = \mathrm{Sc} - \frac{|H|^2}{12} + \operatorname{div}^{\mathcal{G}}(e_+ - e_-) - \frac{1}{2}\,|e|^2_{\mathcal{G}}. \tag{3.8}$$

Herein, $\operatorname{div} = \operatorname{div}^{\mathcal{G}} - \langle e, \cdot\rangle$.

Proof. We compute with Corollary 3.7

$$
\begin{aligned}
\operatorname{tr}_{\mathcal{G}} \overline{\mathcal{R}c}^D\Big|_{E_\pm\times E_\pm} &= \mathrm{Sc} - \frac{1}{12}\,|H|^2 \pm \frac{d}{d-1}\operatorname{div}_g(e_\pm) \pm \frac{d-2}{d-1}\operatorname{div}_g(e_\pm)\\
&\quad + \frac{1}{2(d-1)^2}\Big\{(3-2d)\,|e_\pm|^2_{\mathcal{G}}\, d + (d-2)\,|e_\pm|^2_{\mathcal{G}}\Big\}\\
&= \mathrm{Sc} - \frac{1}{12}\,|H|^2 \pm 2\operatorname{div}_g(e_\pm) - |e_\pm|^2_{\mathcal{G}}.
\end{aligned}
$$

Thus,

$$\begin{aligned}\mathcal{S}c &= \frac{1}{2}\left\{\operatorname{tr}_{\mathcal{G}} \overline{\mathcal{R}c}^D\Big|_{E_+\times E_+} + \operatorname{tr}_{\mathcal{G}} \overline{\mathcal{R}c}^D\Big|_{E_-\times E_-}\right\}\\ &= \mathrm{Sc} - \frac{1}{12}|H|^2 + \operatorname{div}_g(e_+ - e_-) - \frac{1}{2}|e|^2_{\mathcal{G}},\end{aligned}$$

as claimed. □

Corollary 3.9. *The trace of the full generalised Ricci tensor with respect to the scalar product* $\langle\cdot,\cdot\rangle$ *is given by:*

$$\operatorname{tr}_E \overline{\mathcal{R}c}^D = 2\operatorname{div}_g(e) - (|e_+|^2_{\mathcal{G}} - |e_-|^2_{\mathcal{G}}).$$

Proof. This follows from the formulas in the previous proof taking into account that $\mathcal{G}$ and $\langle\cdot,\cdot\rangle$ differ by a minus sign on E_-. □

Proposition 3.10. *The following generalised Kretschmann scalar is a (non-trivial) geometric invariant of the pair* $(\mathcal{G}, \operatorname{div})$*:*

$$|\mathcal{R}m^D|^2_{\mathcal{G}} = \mathcal{G}^{II'}\mathcal{G}^{JJ'}\mathcal{G}^{KK'}\mathcal{G}^{LL'}\mathcal{R}m^D_{IJKL}\mathcal{R}m^D_{I'J'K'L'},$$

where the sum is over all components.

Proof. The invariance follows from the fact that the generalised connection $D = D^{\mathcal{G},\operatorname{div}}$ is canonically associated with the pair $(\mathcal{G}, \operatorname{div})$. Inspection of the formulas for the generalised Riemann tensor in the case $H = 0$, $\operatorname{div} = \operatorname{div}^{\mathcal{G}}$ show that

$$\left(|\mathcal{R}m^D|^2_{\mathcal{G}}\right)\Big|_{H=0,e=0} = 4|\mathrm{Rm}|^2_g,$$

and therefore the non-triviality of the invariant. □

Corollary 3.11. *The mixed-type components of the generalised Ricci curvature are given by*

$$4\mathcal{R}c(\mathcal{G},\operatorname{div})|_{E_\mp\times E_\pm} = 4\mathrm{Rc} - H^2 \mp 2\mathrm{d}^*H + 4[\nabla\xi]^{\mathrm{sym}} \pm 4[\nabla g X]^{\mathrm{antisym}} \mp 2H(\xi). \tag{3.9}$$

Herein, $\operatorname{div} = \operatorname{div}^{\mathcal{G}} - 2\langle X+\xi,\cdot\rangle$ *and the* $(0,2)$ *tensor* H^2 *is defined by* $H^2(X,Y) = \operatorname{tr}_g H^{(2)}(X,\cdot,Y,\cdot)$.

Proof. We first compute the traces over the subspaces E_+ and E_-. Let $a \in \Gamma(E_\pm)$ and $\bar{b} \in \Gamma(E_\mp)$. Then

$$\begin{aligned}&2\operatorname{tr}_{E_\pm}\mathcal{R}m^D(\cdot,\bar{b},\cdot,a)\\ &= \operatorname{tr}_g\Big\{\mathrm{Rm}(\cdot,\bar{b},\cdot,a) \mp \frac{1}{2}[\nabla_{(\cdot)}H](\bar{b},\cdot,a) \pm \frac{1}{6}[\nabla_{\bar{b}}H](\cdot,\cdot,a)\\ &\quad - \frac{1}{12}H^{(2)}(\bar{b},a,\cdot,\cdot) - \frac{1}{12}H^{(2)}(a,\cdot,\bar{b},\cdot) - \frac{1}{6}H^{(2)}(\cdot,\bar{b},\cdot,a)\\ &\quad \pm\frac{1}{d-1}[D^0_{\bar{b}}\chi^{e_\pm}_\pm](\cdot,\cdot,a)\Big\}\\ &= \mathrm{Rc}(\bar{b},a) \mp \frac{1}{2}\mathrm{d}^*H(\bar{b},a) - \frac{1}{4}H^2(\bar{b},a) \pm g(D^0_{\bar{b}}e_\pm, a)\\ &= \mathrm{Rc}(\bar{b},a) \mp \frac{1}{2}\mathrm{d}^*H(\bar{b},a) - \frac{1}{4}H^2(\bar{b},a)\\ &\quad \pm g(\nabla_{\bar{b}}X,a) + [\nabla_{\bar{b}}\xi](a) + \frac{1}{2}H(\bar{b},X,a) \pm \frac{1}{2}H(\bar{b},\xi,a),\end{aligned}$$

where in the last line we used that

$$\begin{aligned}\pi D^0_{\bar b} e_\pm &= \nabla^\pm_{\bar b}\pi e_\pm = \nabla^\pm_{\bar b}(X \pm g^{-1}\xi)\\ &= \nabla_{\bar b}X \pm g^{-1}\left[\nabla_{\bar b}\xi \pm \frac{1}{2}\left(H(\bar b, X) \pm H(\bar b, g^{-1}\xi)\right)\right].\end{aligned}$$

Interchanging "+" with "−" and a with $\bar b$, we obtain that

$$\begin{aligned}&2\operatorname{tr}_{E_\mp}\mathcal{R}m^D(\cdot, a, \cdot, \bar b)\\ &= \operatorname{Rc}(a, \bar b) \pm \frac{1}{2}\mathrm{d}^* H(a, \bar b) - \frac{1}{4}H^2(a, \bar b)\\ &\quad \mp g(\nabla_a X, \bar b) + [\nabla_a\xi](\bar b) + \frac{1}{2}H(a, X, \bar b) \mp \frac{1}{2}H(a, \xi, \bar b).\end{aligned}$$

Finally, we compute

$$\begin{aligned}&4\,\mathcal{R}c(\mathcal{G}, \operatorname{div})(\bar b, a) = 4\ \operatorname{tr}_E \mathcal{R}m^D(\cdot, \bar b, \cdot, a)\\ &= 4\ \operatorname{tr}_{E_\pm}\mathcal{R}m^D(\cdot, \bar b, \cdot, a) + 4\ \operatorname{tr}_{E_\mp}\mathcal{R}m^D(\cdot, \bar b, \cdot, a)\\ &= 4\operatorname{Rc}(\bar b, a) \mp 2\,\mathrm{d}^* H(\bar b, a) - H^2(\bar b, a)\\ &\quad \pm 4[\nabla gX]^{\mathrm{antisym}}(\bar b, a) + 4[\nabla\xi]^{\mathrm{sym}}(\bar b, a) \mp 2H(\xi, \bar b, a).\end{aligned}$$

□

Inspecting the proof closely, we recover in the case of a compatible pair $(\mathcal{G}, \operatorname{div})$ a formula for the mixed-type part of the generalised Ricci curvature often used as a definition in the literature (e.g. in [6]). Note that this result was established in [33, Theorem 8].

Corollary 3.12. *It holds $\mathcal{R}c(\mathcal{G}, \operatorname{div})(\bar b, a) = 2\operatorname{tr}_{E_\pm}\mathcal{R}m^D(\cdot, \bar b, \cdot, a)$ if and only if $(\mathcal{G}, \operatorname{div})$ is a compatible pair (cf. Definition 2.27).*

Proof. From the proof of Corollary 3.11, we can see that

$$\operatorname{tr}_{E_\pm}\mathcal{R}m^D(\cdot, \bar b, \cdot, a) - \operatorname{tr}_{E_\mp}\mathcal{R}m^D(\cdot, a, \cdot, \bar b) = \frac{1}{2}[L_X g + \mathrm{d}\xi - i_X H](\bar b, a).$$

The result follows with the characterisation of compatible pairs presented in Proposition 2.27.

□

3.4 Flat Semi-Riemannian Courant Algebroids

This section reproduces [17, chapter 6].

In this section we study exact semi-Riemannian Courant algebroids for which the canonical Levi-Civita generalised connection is flat. We may call them canonically flat. For Riemannian and Lorentzian signature we show that canonical flatness implies complete triviality, by which we mean that the Courant algebroid is untwisted ($H = 0$), the divergence operator coincides with the metric divergence (constant dilaton in physics terminology) and the underlying Riemannian or Lorentzian metric is flat. We begin with two simple consequences of the generalised Einstein equations.

In this section, we always assume the exact Courant algebroid to be equipped with the semi-Riemannian generalised metric $\mathcal{G}$ and a divergence operator div, which we express as $\operatorname{div} = \operatorname{div}^{\mathcal{G}} - \langle e, \cdot\rangle$ for some $e \in \Gamma(E)$. Employing the splitting induced by the generalised metric, we also write $e = 2(X + \xi) \in \Gamma(\mathbb{T}M)$.

Lemma 3.13. *The generalised Einstein equations $\mathcal{R}c = 0$ and $\mathcal{S}c = 0$ imply*

$$0 = \frac{|H|^2}{6} - \mathrm{d}^*\xi - \frac{1}{2}|e|^2_{\mathcal{G}}. \tag{3.10}$$

We will sometimes refer to this as the "dilaton's equation of motion".

Proof. We compute the trace of the expression for the generalised Ricci curvature (3.9) as

$$\mathrm{tr}_g[\mathcal{R}c \circ (\sigma_\pm, \sigma_\mp)] = \mathrm{Sc} - \frac{|H|^2}{4} - \mathrm{d}^*\xi. \tag{3.11}$$

The dilaton's equation of motion is obtained by subtracting the trace (3.11) from the generalised scalar curvature (3.8). □

Corollary 3.14. *Let $E \to M$ be an exact CA over a closed and orientable manifold M. Let $(\mathcal{G}, \mathrm{div})$ be a pair consisting of a Riemannian generalised metric $\mathcal{G}$ and an exact divergence operator* $\mathrm{div} = \mathrm{div}^{\mathcal{G}} - 2\langle \xi, \cdot\rangle$, $\xi \in \Gamma(T^*M)$, $\xi = \mathrm{d}\phi$.

Then, generalised Ricci and scalar flatness, i.e. $\mathcal{R}c = 0$ and $\mathcal{S}c = 0$, imply $\mathrm{Rc} = 0$, $H = 0$ *and* $\xi = 0$. *In particular, generalised Riemann flatness implies complete triviality, i.e.* $\mathrm{Rm} = 0$, $H = 0$, *and* $\xi = 0$.

Proof. We follow the argument as given in [6, Theorem 3.50.]. The dilaton's equation of motion is for $e = 2\xi$ given as

$$0 = \frac{|H|^2}{6} - \mathrm{d}^*\xi - |\xi|^2. \tag{3.12}$$

Integrating (3.12) with the volume form $\mu = e^{-2\phi}\mathrm{vol}_g$, one gets

$$0 = \int_M \left(\frac{|H|^2}{6} - \mathrm{d}^*\mathrm{d}\phi - |\mathrm{d}\phi|^2\right)\mu = \int_M \left(\frac{|H|^2}{6} + |\mathrm{d}\phi|^2\right)\mu.$$

The result follows. □

We want to show that generalised Riemann flatness implies complete triviality without assuming exactness of the divergence operator nor compactness and orientability. The following result shows conformal flatness, independent of signature and topology.

Theorem 3.15. *Let $D \in \mathcal{D}^0(\mathcal{G}, \mathrm{div})$ be the canonical connection, denote* $\mathrm{div} = \mathrm{div}^{\mathcal{G}} - \langle e, \cdot\rangle$. *Then $\mathcal{R}m^D = 0$ implies that the Weyl tensor of (M, g) vanishes. In particular, if $d = \dim M \geq 4$, (M, g) is conformally flat.*

More precisely, the quadratic components of the Riemann tensor are given for perpendicular vector fields $A, B \in \Gamma(TM)$, $g(A, B) = 0$, as

$$\begin{aligned}\mathrm{Rm}(A, B, A, B) &= \frac{\pm\,\mathrm{div}^{\mathcal{G}}(e_\pm)}{2d(d-1)}|A|^2|B|^2 \\ &+ \frac{3}{4(d-1)^2}\left\{2|e_\pm|^2|A|^2|B|^2 - |B|^2 g(A, \pi e_\pm)^2 - |A|^2 g(B, \pi e_\pm)^2\right\}.\end{aligned} \tag{3.13}$$

Furthermore, it holds $\mathrm{div}^{\mathcal{G}}(e_+) = -\,\mathrm{div}^{\mathcal{G}}(e_-)$, $\pi e_+ = \varepsilon\,\pi e_-$ *for some* $\varepsilon \in \{+1, -1\}$, *and*

$$\nabla H = 0, \qquad g\nabla^\pm e_\pm = \frac{\mathrm{div}^{\mathcal{G}}(e_\pm)}{d}g, \qquad 0 = \frac{|H|^2}{6} + \mathrm{div}^{\mathcal{G}}(e_+) - |e_+|^2. \tag{3.14}$$

Remark 3.16. The proof of the above statement does not rely on working on an exact Courant algebroid. Instead, one can consider the following setting.

Let $p \in M$, ∇ a torsion-free connection[12] at p, and $g \in J^1_p(M, \mathrm{Sym}^2(M))$ such that g_p is non-degenerate and $\nabla g|_p = 0$. Let furthermore $\mathrm{Rm} \in \mathrm{Sym}^2(\Lambda^2 T^*_p M)$, $H \in J^1_p(M, \Lambda^3 T^* M)$, and $e = X + \xi \in J^1_p(M, TM \oplus T^*M)$. Assume that at p the equations in Propositions 3.5 and 3.6 hold with the generalised Riemann tensor set to zero.[13] Then, also equations (3.13) and (3.14) hold at p.

Proof. Let us start by considering the following components of the mixed-type part of the generalised Riemann tensor. Take $A, V, W \in \Gamma(TM)$, denote $a = \sigma_\pm A, \bar{a} = \sigma_\mp A, v = \sigma_\pm V$, and $w = \sigma_\pm W$. By Theorem 3.6 we obtain from $\mathcal{R}m^D = 0$

$$\begin{aligned} 0 &= 2\mathcal{R}m^D(a, \bar{a}, v, w) \\ &= -\frac{1}{3}[\nabla_A H](A, V, W) + \frac{1}{d-1}[D^0_{\bar{a}} \chi^{e_\pm}_\pm](a, v, w). \end{aligned} \tag{3.15}$$

We calculate

$$[D^0_{\bar{a}} \chi^{e_\pm}_\pm](a, v, w) = g(A, V) g(\nabla^\pm_A e_\pm, W) - g(A, W) g(\nabla^\pm_A e_\pm, V).$$

Considering (3.15) with $V = A \perp W$, we can conclude $0 = [D^0_{\bar{a}} \chi^{e_\pm}_\pm](a, a, w)$ and thus $g(\nabla^\pm_A e_\pm, W) = 0$. In particular the tensor $g\nabla^\pm e_\pm$ is symmetric and also $0 = [D^0_{\bar{a}} \chi^{e_\pm}_\pm](a, v, w)$ for $V \perp A$. It follows that

$$0 = [D^0_{\bar{a}} \chi^{e_\pm}_\pm](a, v, w) \qquad \text{for all } A, V, W \in \Gamma(TM).$$

Inserting this back into (3.15), we obtain that $[\nabla_A H](A, V, W) = 0$. By a polarisation argument, it follows that $[\nabla_A H](B, V, W) = -[\nabla_B H](A, V, W)$ for all $A, B, V, W \in \Gamma(TM)$. We conclude that for all $A, B, V, W \in \Gamma(TM)$

$$\begin{aligned} 0 &= [\mathrm{d}H](A, B, V, W) \\ &= [\nabla_A H](B, V, W) - [\nabla_B H](A, V, W) + [\nabla_V H](A, B, W) - [\nabla_W H](A, B, V) \\ &= 4[\nabla_A H](B, V, W), \end{aligned}$$

i.e. $\nabla H = 0$. With this, we see that

$$\begin{aligned} 0 &= 2(d-1)\mathcal{R}m^D(a, \bar{b}, v, w) - 2(d-1)\mathcal{R}m^D(w, \bar{v}, b, a) \\ &= [D^0_{\bar{b}} \chi^{e_\pm}_\pm](a, v, w) - [D^0_{\bar{v}} \chi^{e_\pm}_\pm](w, b, a) \end{aligned} \tag{3.16}$$

since $\mathrm{Rm}, H^{(2)} \in \Gamma(\mathrm{Sym}^2 \bigwedge^2 T^* M)$. Taking the trace of this equation over A, V and assuming $B = W$, we obtain $d \cdot g(\nabla_B e_\pm, B) = |B|^2 \operatorname{div}^{\mathcal{G}}(e_\pm)$, implying

$$g\nabla^\pm e_\pm = \frac{\operatorname{div}^{\mathcal{G}}(e_\pm)}{d} g$$

since the tensor $g\nabla^\pm e_\pm$ is symmetric, as shown above. Therefore

$$[D^0_{\bar{b}} \chi^{e_\pm}_\pm](a, v, w) = \frac{\operatorname{div}^{\mathcal{G}}(e_\pm)}{d} \left[g(a, v) g(b, w) - g(a, w) g(b, v) \right].$$

[12] A connection at p is a bilinear map $\nabla : T_p M \times \Gamma(TM) \to T_p M, (v, X) \mapsto \nabla_v X$ satisfying the Leibniz rule: $\nabla_v(fX) = v(f) X_p + f(p) \nabla_v X$, $f \in C^\infty(M)$. Note that $\nabla_v X$ depends only on the 1-jet of X at p. We will write $\nabla_v X|_p$ to emphasise that $\nabla_v X$ is defined at p.

[13] Note that these equations immediately imply that Rm satisfies the Bianchi identity and hence is an algebraic curvature tensor.

To conclude the investigation of the mixed-type generalised Riemann tensor, we compute its quadratic components as

$$\begin{aligned}\pm 2\mathcal{R}m^D(a,\bar{b},a,b) &= \mathrm{Rm}(A,B,A,B) - \frac{1}{4}H^{(2)}(A,B,A,B)\\ &\quad \pm \frac{\mathrm{div}^{\mathcal{G}}(e_\pm)}{d(d-1)}\mathrm{CS}(A,B),\end{aligned} \tag{3.17}$$

where we introduced

$$\mathrm{CS}(A,B) := |A|^2\,|B|^2 - g(A,B)^2.$$

We can conclude from comparing the cases "+" and "−" in (3.17) that $\mathcal{R}m^D = 0$ implies

$$\mathrm{div}^{\mathcal{G}}(e_+) = -\,\mathrm{div}^{\mathcal{G}}(e_-). \tag{3.18}$$

Now, we move our attention to the pure-type tensor. To that end, note that

$$\begin{aligned}&[D^0_b\chi^{e_\pm}_\pm - D^0_{\bar{b}}\chi^{e_\pm}_\pm](a,v,w)\\ &= g(a,v)g([\nabla^{\pm 1/3} - \nabla^\pm]_B e_\pm, w) - g(a,w)g([\nabla^{\pm 1/3} - \nabla^\pm]_B e_\pm, v)\\ &= \pm\frac{1}{3}[g(a,w)H(b,e_\pm,v) - g(a,v)H(b,e_\pm,w)]\end{aligned}$$

and hence

$$\begin{aligned}[D^0_b\chi^{e_\pm}_\pm](a,v,w) &= \frac{\mathrm{div}^{\mathcal{G}}(e_\pm)}{d}\,[g(a,v)g(b,w) - g(a,w)g(b,v)]\\ &\quad \pm \frac{1}{3}[g(a,w)H(b,e_\pm,v) - g(a,v)H(b,e_\pm,w)]\end{aligned}$$

so that

$$\begin{aligned}&[D^0_v\chi^{e_\pm}_\pm](w,b,a) - [D^0_w\chi^{e_\pm}_\pm](v,b,a) + [D^0_b\chi^{e_\pm}_\pm](a,v,w) - [D^0_a\chi^{e_\pm}_\pm](b,v,w)\\ &= \frac{\mathrm{div}^{\mathcal{G}}(e_\pm)}{d}\{g(w,b)g(v,a) - g(w,a)g(v,b) - g(v,b)g(w,a) + g(v,a)g(w,b)\\ &\qquad + g(a,v)g(b,w) - g(a,w)g(b,v) - g(b,v)g(a,w) + g(b,w)g(a,v)\}\\ &\quad \pm \frac{1}{3}\{g(w,a)H(v,e_\pm,b) - g(w,b)H(v,e_\pm,a)\\ &\qquad - g(v,a)H(w,e_\pm,b) + g(v,b)H(w,e_\pm,a)\\ &\qquad + g(a,w)H(b,e_\pm,v) - g(a,v)H(b,e_\pm,w)\\ &\qquad - g(b,w)H(a,e_\pm,v) + g(b,v)H(a,e_\pm,w)\}\\ &= \frac{4\ \mathrm{div}^{\mathcal{G}}(e_\pm)}{d}[g(w,b)g(v,a) - g(w,a)g(v,b)].\end{aligned}$$

Now, with this and Proposition 3.5, we compute the quadratic terms of the pure-type tensor

$$\begin{aligned}\pm\mathcal{R}m^D(a,b,a,b) &= \mathrm{Rm}(A,B,A,B) - \frac{1}{12}H^{(2)}(A,B,A,B)\\ &\quad \pm \frac{2\ \mathrm{div}^{\mathcal{G}}(e_\pm)}{d(d-1)}\mathrm{CS}(A,B) - Q(A,B)\end{aligned} \tag{3.19}$$

where we defined

$$Q(A,B) \coloneqq \frac{1}{2(d-1)^2}\Big\{2\,|e_\pm|^2\,\mathrm{CS}(A,B) - |B|^2\,g(A,e_\pm)^2 - |A|^2\,g(B,e_\pm)^2 \\ +2g(A,B)g(A,e_\pm)g(B,e_\pm)\Big\}\,. \tag{3.20}$$

Note that, by (3.18) and $\mathcal{R}m^D = 0$, Q has to be independent of the choice of sign, and thus $\pi e_+ = \varepsilon\,\pi e_-$ for some $\varepsilon \in \{+1,-1\}$. More precisely,

$$\begin{cases} e = 2X \ \text{ and } \ e_\pm = X \pm gX, & \text{if } \ \varepsilon = +1, \\ e = 2\xi \ \text{ and } \ e_\pm = \pm g^{-1}\xi \pm \xi, & \text{if } \ \varepsilon = -1. \end{cases} \tag{3.21}$$

Comparing the quadratic expressions (3.17) and (3.19), we obtain that

$$\frac{1}{6}H^{(2)}(A,B,A,B) = \mp\frac{\mathrm{div}^{\mathcal{G}}(e_\pm)}{d(d-1)}\mathrm{CS}(A,B) + Q(A,B) \tag{3.22}$$

so that $\mathcal{R}m^D = 0$ implies

$$\begin{aligned} 2\mathrm{Rm}(A,B,A,B) &= \frac{1}{2}H^{(2)}(A,B,A,B) \mp 2\frac{\mathrm{div}^{\mathcal{G}}(e_\pm)}{d(d-1)}\mathrm{CS}(A,B) \\ &= \mp 5\,\frac{\mathrm{div}^{\mathcal{G}}(e_\pm)}{d(d-1)}\mathrm{CS}(A,B) + 3Q(A,B) \end{aligned}$$

From this, we derive the quadratic components of the Ricci tensor as

$$2\mathrm{Rc}(B,B) = \mp 5\,\frac{\mathrm{div}^{\mathcal{G}}(e_\pm)}{d}\,|B|^2 + \frac{3}{2(d-1)^2}\Big\{(2d-3)\,|e_\pm|^2\,|B|^2 - (d-2)g(B,e_\pm)^2\Big\}\,.$$

Therefore,

$$\begin{aligned} 2\mathrm{Rc} &= \mp 5\,\frac{\mathrm{div}^{\mathcal{G}}(e_\pm)}{d}g + \frac{3}{2(d-1)^2}\Big\{(2d-3)\,|e_\pm|^2\,g - (d-2)e_\pm\otimes e_\pm\Big\} \\ &= \frac{3\,|e_\pm|^2 \mp 5\ \mathrm{div}^{\mathcal{G}}(e_\pm)}{d}g + \frac{3(d-2)}{2(d-1)^2}\left\{\frac{|e_\pm|^2}{d}g - e_\pm\otimes e_\pm\right\}. \end{aligned} \tag{3.23}$$

For the Ricci scalar, we obtain

$$2\mathrm{Sc} = 3\,|e_\pm|^2 \mp 5\ \mathrm{div}^{\mathcal{G}}(e_\pm)$$

and for the trace-free part of the Ricci tensor

$$\begin{aligned} 2Z &\coloneqq 2\mathrm{Rc} - \frac{2\mathrm{Sc}}{d} \\ &= \frac{3(d-2)}{2(d-1)^2}\left[\frac{|e_\pm|^2}{d}g - e_\pm\otimes e_\pm\right]. \end{aligned}$$

We obtain, in the decomposition of the Riemann tensor $\mathrm{Rm} = S + E + W$ into scalar curvature part, trace-free Ricci part and Weyl part, assuming $A \perp B$

$$S(A,B,A,B) \coloneqq \frac{\mathrm{Sc}}{d(d-1)}\,|A|^2\,|B|^2 = \frac{3\,|e_\pm|^2 \mp 5\ \mathrm{div}^{\mathcal{G}}(e_\pm)}{2d(d-1)}\,|A|^2\,|B|^2$$

and, again assuming $A \perp B$

$$
\begin{aligned}
&E(A,B,A,B)\\
&:= \frac{1}{d-2}\left\{Z(A,A)\,|B|^2 + Z(B,B)\,|A|^2\right\}\\
&= \frac{|B|^2}{2(d-2)}\left\{\frac{3(d-2)}{2(d-1)^2}\left[\frac{1}{d}\,|e_\pm|^2\,|A|^2 - g(A,e_\pm)^2\right]\right\}\\
&\quad + \frac{|A|^2}{2(d-2)}\left\{\frac{3(d-2)}{2(d-1)^2}\left[\frac{1}{d}\,|e_\pm|^2\,|B|^2 - g(B,e_\pm)^2\right]\right\}\\
&= \frac{3}{4(d-1)^2}\left\{\frac{2}{d}\,|e_\pm|^2\,|A|^2\,|B|^2 - |A|^2\,g(B,e_\pm)^2 - |B|^2\,g(A,e_\pm)^2\right\}.
\end{aligned}
$$

Therefore, with $A \perp B$

$$
\begin{aligned}
&2[E+S](A,B,A,B)\\
&= \frac{\mp 5\ \mathrm{div}^{\mathcal{G}}(e_\pm)}{d(d-1)}\mathrm{CS}(A,B) + 3Q(A,B)\\
&= 2\,\mathrm{Rm}(A,B,A,B),
\end{aligned}
$$

implying that the Weyl tensor vanishes!

Finally, we note that employing $\pi e_+ = \varepsilon \pi e_-$, the dilaton's equation of motion (3.10) becomes

$$
0 = \frac{|H|^2}{6} - \mathrm{d}^*\xi - |e_+|^2 .
$$

In view of (3.21), this coincides with the last equation in (3.14): if $\varepsilon = +1$, then $\xi = 0$, but also $\mathrm{div}^{\mathcal{G}}(e_+) = 0$, since $\mathrm{div}^{\mathcal{G}}(e_+ - e_-) = \mathrm{div}^{\mathcal{G}}(0) = 0$ and $\mathrm{div}^{\mathcal{G}}(e_+ + e_-) = 0$ by (3.18). And if $\varepsilon = -1$, then $\mathrm{div}^{\mathcal{G}}(e_+) = -\mathrm{d}^*\xi$. □

The following corollary shows that, in the Riemannian case, we obtain complete triviality.

Corollary 3.17. *$\mathcal{R}m^D = 0$ and $\dim M > 2$ implies that $|H|_g^2 = |e_\pm|_{\mathcal{G}}^2 = 0$, $\nabla^\pm \pi e_\pm = \nabla \pi e_\pm = 0$ and $\iota_{\pi e_\pm} H = 0$. In particular, if $\mathcal{G}$ is Riemannian, we obtain complete triviality, i.e.* $\mathrm{Rm} = 0, H = 0$, *and* $e = 0$.

Remark 3.18. Note that this Corollary holds under the assumptions described in Remark 3.16.

Proof. Recall from (3.21) that $e = 2(X+\xi) \in \Gamma(T \oplus T^*)$ has $\xi = 0$ if $\varepsilon = +1$ and $X = 0$ if $\varepsilon = -1$. Consider the equation

$$
g\nabla^\pm \pi e_\pm = \frac{\mathrm{div}^{\mathcal{G}}(e_\pm)}{d} g. \tag{3.24}
$$

Its antisymmetric parts

$$
\mathrm{d}(g\pi e_\pm) \mp \iota_{\pi e_\pm} H = 0
$$

decouple in any case and yield, in particular, $\iota_{\pi e_\pm} H = 0$ and, hence, $\iota_{\pi e_\pm} H^{(2)} = 0$. It follows that

$$
H^{(2)}(\pi e_+, B, \pi e_+, B) = 0 \qquad \text{for all } B \in \Gamma(TM) \text{ such that } g(e_+, B) = 0,
$$

which is by (3.22) equivalent to

$$\pm\frac{\operatorname{div}^{\mathcal{G}}(e_\pm)}{d(d-1)} = \frac{|e_\pm|^2}{2(d-1)^2}. \tag{3.25}$$

If $\varepsilon = +1$, the left hand side is only independent of sign if it vanishes, and so $|e_\pm|^2 = |X|^2 = 0$. In that case, from the dilaton's equation of motion in (3.14), we obtain $|H|^2 = 0$ and from (3.24) with $\iota_{\pi e_\pm} H = 0$ that $\nabla^\pm \pi e_\pm = \nabla \pi e_\pm = 0$.

From now on, we can assume $\varepsilon = -1$, equivalently $e = 2\xi \in \Gamma(T^*M)$. Equations (3.24) and (3.25) become

$$\nabla\xi = -\frac{\mathrm{d}^*\xi}{d}g, \qquad \frac{|\xi|^2}{2(d-1)} = -\frac{\mathrm{d}^*\xi}{d}. \tag{3.26}$$

From the dilaton's equation of motion in (3.14), we obtain that

$$0 = \frac{|H|^2}{6} - \frac{d-2}{2(d-1)}|\xi|^2$$

and then with $\nabla H = 0$ also $\nabla|\xi|^2 = 0$ (since $d > 2$). This, however, implies

$$0 = \nabla_\xi |\xi|^2 = 2g(\nabla_\xi \xi, \xi) = \frac{|\xi|^4}{2(d-1)}.$$

Inserting this back into (3.26) and the dilaton's equation of motion, we respectively obtain $\nabla\xi = 0$ and $|H|^2 = 0$. With $\iota_{\pi e_\pm} H = 0$, we obtain that also in this case $\nabla^\pm \pi e_\pm = \nabla \pi e_\pm = 0$. □

Corollary 3.19. *If $n+1 = \dim M > 2$ and $\mathcal{G}$ is Lorentzian, $\mathcal{R}m^D = 0$ implies complete triviality.*

Proof. Let $p \in M$. We know that $\pi e_+|_p$ is a null vector. We can complete $\pi e_+|_p$ to a basis $\{e_0, ..., e_n\}, U = e_0, V = e_1$, such that $\pi e_+|_p = V$, $|U|^2 = 0$, $g(U,V) = -2$ and for $i, j = 2, ..., n$

$$g(e_i, e_j) = \delta_{ij}, \qquad g(e_i, U) = g(e_i, V) = 0.$$

It follows that

$$H^{(2)}(U, e_i, U, e_i) = \sum_{\mu,\nu} g^{\mu\nu} H(e_\mu, U, e_i) H(e_\nu, U, e_i) = \sum_{j=2}^n g^{jj}[H(e_j, U, e_i)]^2 \geq 0. \tag{3.27}$$

At the same time

$$Q(U, e_i) = \frac{-1}{2(d-1)^2}|e_i|^2 g(U,V)^2 \leq 0.$$

Noting that all possibly non-vanishing components of Q are of the form $Q(U, e_i)$, we conclude that $H^{(2)}(e_\mu, e_\nu, e_\mu, e_\nu) = 6Q(e_\mu, e_\nu) = 0$, cf. (3.22), (3.14), and Corollary 3.17. This implies $H = 0$: from (3.27), we already know that $H(U, e_i, e_j) = 0$ for all $i, j \in \{2, ..., n\}$. Furthermore, from Corollary 3.17, we also know that $\iota_V H = 0$. Now $|H|^2 = 0$ implies $H(e_i, e_j, e_k) = 0$ for all $i, j, k \in \{2, ..., n\}$.

Finally, we can use e.g. (3.19) to conclude that $\mathrm{Rm} = 0$. The formula for the Ricci tensor (3.23) then implies that $e_\pm = 0$. □

Proposition 3.20. *Assume that* $\mathcal{R}m^D = 0$*. Then, the factor of conformal flatness is given by*

$$\operatorname{grad}_g \varphi = \frac{\sigma\sqrt{3}}{2(d-1)}\,\pi e_+, \tag{3.28}$$

where $\sigma \in \{+1, -1\}$*. In other words, if* φ *is a (local) solution of (3.28), then* $\tilde{g} = e^{2\varphi} g$ *is flat.*

Proof. For $\varphi \in C^\infty(M)$ to be such that $\tilde{g} = e^{2\varphi} g$ is flat, it has to solve

$$\mathrm{Rc} = (d-2)\,(\nabla \mathrm{d}\varphi - \mathrm{d}\varphi \otimes \mathrm{d}\varphi) + \left(\Delta\varphi + (d-2)\,|\mathrm{d}\varphi|^2\right) g. \tag{3.29}$$

We compare this to the Ricci tensor (3.23)

$$\mathrm{Rc} = -\frac{3(d-2)}{4(d-1)^2} g\pi e_\pm \otimes g\pi e_\pm,$$

where we made use of Corollary 3.17. Since $|e_\pm|^2 = 0$ and $\nabla \pi e_\pm = 0$ by Corollary 3.17, we see that indeed (3.28) provides a solution to (3.29). □

Remark 3.21. That we get one solution $\varphi_\pm$ for the factor of conformal flatness in (3.28) for each of the two values $\sigma = \pm 1$ implies that we have a pair of conformally equivalent flat metrics $e^{2\varphi_\pm} g$.

To see this more explicitly, we focus on $\varphi = \varphi_+$. We denote $\tilde{g} = e^{2\varphi} g$ (which is a flat metric) and $\tilde{\nabla}$ for the Levi-Civita connection of $\tilde{g}$. Then

$$0 = \nabla \mathrm{d}\varphi = \tilde{\nabla}\mathrm{d}\varphi + 2\mathrm{d}\varphi \otimes \mathrm{d}\varphi - |\mathrm{d}\varphi|^2\, g = \tilde{\nabla}\mathrm{d}\varphi + 2\mathrm{d}\varphi \otimes \mathrm{d}\varphi, \tag{3.30}$$

implying that $e^{-2\varphi} g = e^{-4\varphi}\tilde{g}$ is again flat since its Ricci curvature vanishes:

$$\mathrm{Rc}[e^{-4\varphi}\tilde{g}] = -(d-2)\left(-2\tilde{\nabla}\mathrm{d}\varphi - 4\mathrm{d}\varphi \otimes \mathrm{d}\varphi\right) - \left(-2\tilde{\Delta}\varphi + 4(d-2)\,|d\varphi|^2_{\tilde{g}}\right)\tilde{g} = 0.$$

Herein, we used that $\tilde{\Delta}\varphi = -2\,|\mathrm{d}\varphi|^2_{\tilde{g}} = 0$.

Example 3.22. Consider $M = \mathbb{R}^{2m}$, $d = 2m > 2$, endowed with the flat metric

$$\tilde{g} = \sum_{i=1}^{m} \left((dx^i)^2 - (dy^i)^2\right)$$

of signature (m, m) and its LC connection $\tilde{\nabla}$, where $(x^\mu)_{1\le\mu\le 2m} = (x^1, ..., x^m, y^1, ..., y^m)$ are standard coordinates. Replacing (x^1, y^1) by $(u, v) = (x^1 + y^1, x^1 - y^1)$, we obtain a coordinate system for which ∂_u and ∂_v are null. We consider only the region $u > 0$. Set $\varphi = \frac{\log(u)}{2}$. Then $\mathrm{d}\varphi = \frac{\mathrm{d}u}{2u} = \frac{e^{-2\varphi}}{2}\mathrm{d}u$, and hence (compare this to (3.30))

$$\tilde{\nabla}\mathrm{d}\varphi = -2\mathrm{d}\varphi \otimes \mathrm{d}\varphi.$$

It follows that $\nabla d\varphi = 0$, for the LC connection ∇ of $g = e^{-2\varphi}\tilde{g}$. Hence the Ricci curvature of g is given by

$$\mathrm{Rc} = -(d-2)\mathrm{d}\varphi \otimes \mathrm{d}\varphi.$$

To relate this to the dilaton, we set

$$\pi e_+ = \frac{2(d-1)}{\sqrt{3}} \operatorname{grad}_g \varphi = \frac{2(d-1)}{\sqrt{3}} \partial_v,$$

where we used that $g(\partial_v) = d\varphi$. Note that we remain agnostic as to the value of ε in $\pi e_+ = \varepsilon\, \pi e_-$. We make the ansatz

$$H = \sum_{i=2}^{m} f(u)\, g\pi e_+ \wedge \mathrm{d}x^i \wedge \mathrm{d}y^i, \quad f > 0.$$

A small computation reveals that

$$\begin{aligned} H^{(2)} &= \sum_{\mu,\nu} g^{\mu\nu} H(\partial_\mu) \otimes H(\partial_\nu) \\ &= \sum_{i=2}^{m} f^2 e^{2\varphi} \left[(g\pi e_+ \wedge \mathrm{d}y^i)^{\otimes 2} - (g\pi e_+ \wedge \mathrm{d}x^i)^{\otimes 2} \right]. \end{aligned}$$

Therefore, for $A, B \in \Gamma(TM)$

$$\begin{aligned} &H^{(2)}(A,B,A,B) \\ &= f^2 e^{2\varphi} \sum_{i=2}^{m} \left\{ [g(\pi e_+, A) B_{y^i} - g(\pi e_+, B) A_{y^i}]^2 - [g(\pi e_+, A) B_{x^i} - g(\pi e_+, B) A_{x^i}]^2 \right\} \\ &= f^2 e^{4\varphi} \left\{ -g(\pi e_+, A)^2 \left|\rho B\right|_g^2 - g(\pi e_+, B)^2 \left|\rho A\right|_g^2 + 2 g(\pi e_+, A) g(\pi e_+, B) g(\rho A, \rho B) \right\}, \end{aligned}$$

where ρ denotes the orthogonal projection onto the distribution $\{dx^1 = dy^1 = 0\} \subset TM$. Denoting $A_1 = A - \rho A$ and $B_1 = B - \rho B$, we compute Q as (cf. (3.20))

$$\begin{aligned} &2(d-1)^2 Q(A,B) \\ &= -\left|B\right|^2 g(A, e_+)^2 - \left|A\right|^2 g(B, e_+)^2 + 2 g(A,B) g(A, e_+) g(B, e_+) \\ &= -\left|\rho B\right|^2 g(A, e_+)^2 - \left|\rho A\right|^2 g(B, e_+)^2 + 2 g(\rho A, \rho B) g(A, e_+) g(B, e_+) \\ &\quad - \left|B_1\right|^2 g(A, e_+)^2 - \left|A_1\right|^2 g(B, e_+)^2 + 2 g(A_1, B_1) g(A, e_+) g(B, e_+). \end{aligned}$$

Note that the last line vanishes, since up to a sign it is the norm-squared of

$$\begin{aligned} g(A_1, e_+) B_1 - g(B_1, e_+) A_1 &= g(\partial_u, e_+) \{ A_u (B_u \partial_u + B_v \partial_v) - B_u (A_u \partial_u + A_v \partial_v) \} \\ &= g(\partial_u, e_+)(A_u B_v - A_v B_u) \partial_v, \end{aligned}$$

which is a null vector. We conclude that $H^{(2)}(A,B,A,B) = 6Q(A,B)$ if and only if $f = \frac{\sqrt{3}}{d-1} e^{-2\varphi}$. Note that for $\mu \neq 1, m+1$

$$\begin{aligned} \nabla(e^{-\varphi} \mathrm{d}x^\mu) &= \tilde{\nabla}(e^{-\varphi} \mathrm{d}x^\mu) + e^{-\varphi} \mathrm{d}\varphi \otimes \mathrm{d}x^\mu + e^{-\varphi} \mathrm{d}x^\mu \otimes \mathrm{d}\varphi - e^{-\varphi} \tilde{g}(\mathrm{d}\varphi, \mathrm{d}x^\mu) \tilde{g} \\ &= e^{-\varphi} \mathrm{d}x^\mu \otimes \mathrm{d}\varphi = \frac{\sqrt{3}}{2}(d-1) e^{-\varphi} \mathrm{d}x^\mu \otimes g\pi e_+ \end{aligned}$$

so that

$$\begin{aligned} \nabla H &= \sum_{i=2}^{m} \frac{\sqrt{3}}{d-1} \nabla(g\pi e_+ \wedge e^{-\varphi} \mathrm{d}x^i \wedge e^{-\varphi} \mathrm{d}y^i) \\ &= \sum_{i=2}^{m} \frac{3}{2} e^{-2\varphi} \left[\mathrm{d}x^i \otimes g\pi e_+ \wedge g\pi e_+ \wedge \mathrm{d}y^i + \mathrm{d}y^i \otimes g\pi e_+ \wedge \mathrm{d}x^i \wedge g\pi e_+ \right] \\ &= 0. \end{aligned}$$

Because g is conformally flat by construction and $H^{(2)}(A, B, A, B) = 6Q(A, B)$, we find that generalised Riemann flatness is equivalent to (cf. also (3.17, 3.19))

$$\mathrm{Rc} = \frac{1}{4}H^2 = -\frac{f^2(d-2)}{4}e^{4\varphi}g\pi e_+ \otimes e\pi e_+ = -\frac{3(d-2)}{4(d-1)^2}g\pi e_+ \otimes g\pi e_+,$$

which is true by construction.

4 Extrinsic Generalised Geometry

Let $E \to M$ be an exact CA equipped with a generalised metric $\mathcal{G}$, a divergence operator div, and a generalised connection D, and consider a semi-Riemannian immersion $\iota \colon N \to M$.

This chapter presents the theory of extrinsic generalised geometry developed in [17]. In Section 4.1, we recall the induced exact CA $\iota^! E \to N$ (the pullback CA), and show that a choice of Courant transversal bundle induces a canonical realisation of $\iota^! E$ as a subbundle of $\iota^* E$. In particular, since a generalised metric induces with the Courant normal bundle a canonical Courant transversal bundle, one obtains the canonical realisation $E_N \subset \iota^* E$. We see in Section 4.2 that E_N inherits a generalised metric $\mathcal{H}$, a divergence operator div_N, and a generalised connection D^N. We develop the notion of a generalised second fundamental form and generalised mean curvature in Section 4.3, and establish generalised Gauß and Weingarten equations, as well as generalised Gauß and Codazzi equations in Section 4.4. Finally, we use the developed theory to establish in Section 4.5 the the second main result of this thesis: the fundamental theorem for generalised hypersurfaces.

4.1 The Pullback of an Exact Courant Algebroid

This section reproduces [17, chapter 3].

In this section, let M be a smooth manifold, and $(E, \pi, [\cdot,\cdot], \langle\cdot,\cdot\rangle)$ an exact CA over M. Let furthermore $\iota \colon N \to M$ an immersion. We denote by $[H] \in H^3(M, \mathbb{R})$ the Ševera class of E.

We recall the well-known definition of the pullback $\iota^! E$, cf. [19, Lemma 3.7], [20, Proposition 7.1.].

Definition 4.1 (Pullback CA)**.** Denote by $K = \operatorname{Ann}(TN) \subset \iota^*(T^*M)$ the annihilator of TN. View it as a subbundle of $\iota^* E$ (by identifying T^*M with $\pi^* T^* M \subset E$ via $\frac{1}{2}\pi^*$), and denote by $K^\perp \subset \iota^* E$ the orthogonal bundle with respect to the inner product on E. Then the *pullback CA* $\iota^! E$ of E is defined as the vector bundle

$$\iota^! E := K^\perp \big/ K$$

equipped with the anchor $\iota^! \pi$, bracket $\iota^! [\cdot,\cdot]$ and inner product $\iota^! \langle\cdot,\cdot\rangle$ that are naturally inherited from E.

The bracket $\iota^! [a, b]$ of $a, b \in \Gamma_{\mathrm{loc}}(\iota^! E)$ is defined as follows. Take $\tilde{a}, \tilde{b} \in \Gamma_{\mathrm{loc}}(K^\perp)$ arbitrary lifts of a, b, and then $a_*, b_* \in \Gamma_{\mathrm{loc}}(E)$ smooth continuation of $\iota_* \tilde{a}, \iota_* \tilde{b}$. Then $\iota^*[a_*, b_*] \in \Gamma_{\mathrm{loc}}(K^\perp)$ and $\iota^! [a, b]$ is its projection to $\iota^! E$. One can easily show the following Lemma, see [19, Lemma 3.7], [20, Proposition 7.1.].

Lemma 4.2. *The bracket is well-defined on* $\Gamma(\iota^! E)$ *and* $(\iota^! E, \iota^! \pi, \iota^! [\cdot,\cdot], \iota^! \langle\cdot,\cdot\rangle)$ *is an exact CA with Ševera class* $\iota^*[H]$*.*

It will at times be useful to work with a concrete realisation of the pullback CA, from which we will see why the pullback Courant algebroid $\iota^! E$ is exact. To give it, we introduce the following notion.

Definition 4.3 (Courant Transversal Bundle)**.** A *Courant transversal bundle* $\mathcal{T} \subset \iota^* E$ for N is a subbundle such that the sequence

$$0 \longrightarrow K \xrightarrow{\pi^*} \mathcal{T} \xrightarrow{\pi} \nu \longrightarrow 0$$

is exact for some subbundle $\nu \subset \iota^*TM$ transversal to TN. A *splitting* of an exact Courant algebroid E over M is an isomorphism $E \cong \mathbb{T}M$ of Courant algebroids, where the Dorfman bracket on the generalised tangent bundle is in general twisted. We say that a splitting $F : E \cong \mathbb{T}M$ and a Courant transversal bundle are *adapted*, if $F(\mathcal{T}) = \nu \oplus K$.

Remark 4.4. In particular, a Courant transversal bundle is of rank

$$\operatorname{rk} \mathcal{T} = 2 \operatorname{codim} N = 2(\dim M - \dim N),$$

and such that

(i) $\nu := \pi(\mathcal{T})$ is a transversal bundle for N, i.e. $\iota^*TM = TN \oplus \nu$,

(ii) the annihilator $\operatorname{Ann}(TN) \subset \iota^*(T^*M)$ is contained in $\mathcal{T}$.

A generalised metric induces a canonical choice of Courant transversal bundle.

Definition 4.5 (Courant Normal Bundle)**.** Let $\mathcal{G}(g, \sigma)$ be a generalised metric on E such that the immersion $N \to (M, g)$ is semi-Riemannian. Then, the *Courant normal bundle* is the subbundle

$$\mathcal{N} := K \oplus \mathcal{G}^{\text{End}} K = F_\sigma^{-1}(\nu \oplus g\nu) \subset E.$$

Herein, ν denotes the normal bundle induced by g, and F_σ is the Courant isomorphism induced by σ, cf. Theorem 2.10.

Lemma 4.6. *For every exact Courant algebroid E over M and any immersion $\iota : N \to M$ there exists a Courant transversal bundle $\mathcal{T}$.*

(i) For any splitting $F : E \cong \mathbb{T}M$ there is an adapted Courant transversal bundle $\mathcal{T}$.

(ii) For any Courant transversal bundle $\mathcal{T}$ there is an adapted splitting of E.

Proof. Let $\nu \subset \iota^*TM$ be a subbundle transversal to TN and $\tilde{\nu}$ any subbundle of ι^*E which projects isomorphically to ν under the anchor. Then $\mathcal{T} = K \oplus \tilde{\nu}$ is a Courant transversal bundle. This proves the first claim.

Next we prove (i). Given a splitting $F : E \cong \mathbb{T}M$ and a transversal bundle ν, $\mathcal{T} := K \oplus F^{-1}(\nu)$ is a Courant transversal bundle.

To prove (ii) let $F : E \cong \mathbb{T}M$ be a splitting and $\mathcal{T}$ a Courant transversal bundle. Then $F(\mathcal{T}) = K \oplus \operatorname{graph}(\phi)$, where $\phi \in \Gamma(\operatorname{Hom}(\nu, \iota^*T^*M))$. Decomposing $\phi = \phi_K + \psi$, according to $\iota^*T^*M = K \oplus \operatorname{Ann}(\nu)$, we see that we can remove the component $\psi \in \Gamma(\operatorname{Hom}(\nu, \operatorname{Ann}(\nu))) \cong \Gamma(K \wedge \operatorname{Ann}(\nu)) \subset \Gamma(\iota^* \bigwedge^2 T^*M)$ by a not necessarily closed B-field transformation on M. This yields a new splitting $F_1 : E \cong \mathbb{T}M$ such that $F_1(\mathcal{T}) = K \oplus \operatorname{graph}(\phi_K) = K \oplus \nu$. □

Proposition 4.7. *Let $\mathcal{T} \subset \iota^*E$ be a Courant transversal bundle. Then the orthogonal complement $E_N := \mathcal{T}^\perp$ with respect to the inner product $\langle \cdot, \cdot \rangle$ on E is a concrete realisation of the quotient vector bundle $\iota^! E$ as a subbundle of $K^\perp$ complementary to K.*

In particular, we obtain the exact Courant algebroid $(E_N, \iota^*\pi, [\cdot, \cdot]_N, \iota^* \langle \cdot, \cdot \rangle)$, which is canonically isomorphic to $\iota^! E$. Its bracket $[\cdot, \cdot]_N$ is defined by the formula

$$[u, v]_N := \pi^{\|}(\iota^*[\bar{u}, \bar{v}]),$$

where $\bar{u}, \bar{v}$ denote arbitrary extensions of $u, v \in \Gamma_{\text{loc}}(E_N) \subset \Gamma_{\text{loc}}(\iota^*E)$ to sections $\bar{u}, \bar{v} \in \Gamma_{\text{loc}}(E)$ and $\pi^{\|} : \iota^*E = \mathcal{T} \oplus E_N \to E_N$ denotes the projection.

Proof. To see that E_N is a realisation of the quotient vector bundle $\iota^! E$, we choose an adapted splitting $F: E \to \mathbb{T}M$, which exists by Lemma 4.6. Then $F(\mathcal{T}) = K \oplus \nu$ and, hence,

$$\begin{aligned} K^\perp &= (F^{-1}(\operatorname{Ann} TN))^\perp = F^{-1}((\operatorname{Ann} TN)^\perp) = F^{-1}(TN \oplus \operatorname{Ann} \nu \oplus \operatorname{Ann} TN) \\ &= F^{-1}((\nu \oplus \operatorname{Ann} TN)^\perp) \oplus K = \mathcal{T}^\perp \oplus K = E_N \oplus K \end{aligned}$$

Therefore, $E_N \cong \iota^! E$ as vector bundles.

To see that also $E_N \cong \iota^! E$ on the level of CAs, we note that, identifying E_N and $\iota^! E$ as vector bundles, the inner product and the anchor defined on the two bundles are identical. It remains to see that also $[\cdot,\cdot]_N$ and $\iota^![\cdot,\cdot]$ correspond to each other under the identification. This is obvious from the observation that the identification $\iota^! E = K^\perp / K \cong E_N$ is precisely induced by the map $\pi^{\|}|_{K^\perp} : K^\perp \to E_N$. □

4.2 Inherited Geometrical Structures

In this section, which reproduces [17, chapter 4], we consider a semi-Riemannian immersion $\iota\colon N \to M$ into the base manifold of a semi-Riemannian exact CA $E \to M$. To be precise, the immersion is semi-Riemannian with respect to the metric g associated to the generalised metric $\mathcal{G}$ on E, that is $h = \iota^* g$ is nondegenerate. We consider the canonical representation $E_N := \mathcal{N}^\perp$ of the pullback CA $\iota^! E$ coming from the Courant normal bundle $\mathcal{N}$, cf. Definition 4.5. We denote by $F\colon E \cong \mathbb{T}M$ the isomorphism induced by $\mathcal{G} = \mathcal{G}(g, \sigma)$, cf. Theorem 2.10. The goal is to show that a generalised metric, a divergence operator, and a generalised connection induce the same respective structure on E_N.

4.2.1 Generalised Metrics

In this section, we explain the inherited generalised metric $\mathcal{H}$ on E_N.

Proposition 4.8. *The generalised metric $\mathcal{G}$ on E induces a generalised metric $\mathcal{H}$ on E_N. Furthermore, over N, $\mathcal{G}$ is completely determined by $\mathcal{H}$ together with a $\mathcal{G}$-orthonormal set $\{\sqrt{2} n_i\} \subset \Gamma(\mathcal{N})$, $i = 1, \dots, \operatorname{codim} N$, of vectors with $\langle\cdot,\cdot\rangle$-isotropic span.*

Proof. We claim that the symmetric bilinear form $\mathcal{H} := \iota^* \mathcal{G}|_{E_N \times E_N}$ defines a generalised metric. Noting that $\mathcal{N} = \mathcal{N} \cap E_+ \oplus \mathcal{N} \cap E_-$, we immediately conclude that $\mathcal{H}$ is non-degenerate, as the decomposition $E = \mathcal{N} \oplus E_N$ is orthogonal also with respect to $\mathcal{G}$.

We now check that $\mathcal{H}|_{\operatorname{Sym} T^*N}$ is non-degenerate. We remind ourselves that T^*N is identified with the subbundle $\operatorname{Ann}(\nu) \subset \iota^* T^*M$. With this identification, the restriction $\pi^*|_{T^*N} : T^*N \to \iota^* E$ maps to E_N such that the resulting map $T^*N \to E_N$ can be identified with the adjoint of the anchor $E_N \to TN$. Therefore, $\mathcal{H}|_{\operatorname{Sym}^2 T^*N} = \mathcal{G}|_{\operatorname{Sym}^2 (\operatorname{Ann} \nu)}$, and non-degeneracy of this restriction follows again from orthogonality of $E = \mathcal{N} \oplus E_N$, since $\operatorname{Ann} \nu \subset E_N$.

To see that $\mathcal{H}^{\mathrm{End}}$ is involutive, note that the subbundles $\mathcal{N}, E_N = \mathcal{N}^\perp \subset \iota^* E$ are invariant under $\mathcal{G}^{\mathrm{End}}$ and non-degenerate with respect to $\mathcal{G}$. Therefore, $\mathcal{H}^{\mathrm{End}} = \iota^* \mathcal{G}^{\mathrm{End}}|_{E_N} : E_N \to E_N$, implying that $(\mathcal{H}^{\mathrm{End}})^2 = \mathrm{id}$. We have concluded that $\mathcal{H}$ is indeed a generalised metric.

Finally, note that g is determined by the induced metric h and a unit normal frame $\{U_i\}$. Thus, the generalised metric $\mathcal{G} \cong (g, F)$ is fully specified by the generalised metric $\mathcal{H} \cong (h, F|_{E_N})$ and the $\mathcal{G}$-orthonormal set $\{\sqrt{2} F^{-1}(U_i)\}$ of $\mathcal{N}$, which spans a maximally isotropic subbundle of $\mathcal{N}$. □

Corollary 4.9. *The generalised metric $\mathcal{G}$ on E induces a generalised metric $\iota^!\mathcal{G}$ on $\iota^!E$.*

Proof. It suffices to define $\iota^!\mathcal{G}$ as the generalised metric on $\iota^!E$ which corresponds to the generalised metric $\mathcal{H}$ of Proposition 4.8 under the canonical isomorphism $\iota^!E \cong E_N$. □

4.2.2 Divergence Operators

In this section, we explain how a divergence operator div on E induces a divergence operator div_N on E_N.

Definition 4.10 (Induced Divergence Operator)**.** Let $\mathrm{div} = \mathrm{div}^{\mathcal{G}} - \langle e, \cdot\rangle$ be a divergence operator on an exact semi-Riemannian CA $(E, \mathcal{G})$ with induced semi-Riemannian metric g on M. Let $\iota\colon N \to (M, g)$ be a semi-Riemannian immersion with Courant normal bundle $\mathcal{N}$. Then we define on $E_N = \mathcal{N}^\perp$ the divergence operator

$$\mathrm{div}_N := \mathrm{div}^{\mathcal{H}} - \langle e^\parallel, \cdot\rangle.$$

Herein, $e^\parallel$ is the orthogonal projection of $\iota^* e$ into E_N, and $\mathcal{H}$ is the generalised metric induced on E_N by $\mathcal{G}$.

We note that, in general, div_N depends on the choice of generalised metric $\mathcal{G}$. One can reduce this to a dependence only on the metric g in the tuple $\mathcal{G} \cong (g, F)$, if one demands the expression $\langle e, \cdot\rangle$ to uniquely define an element in $(\iota^!E)^*$. This amounts to $\iota^* e \in \Gamma(K^\perp)$ or, equivalently, $\pi(\iota^* e) \in \Gamma(TN)$.

Recall from Section 2.4 that a pair $(\mathcal{G}, \mathrm{div})$ consisting of a generalised metric and a divergence operator is called *compatible* if the generalised vector field $e = 2(X + \xi) \in \Gamma(E)$ defined by the equation $\mathrm{div} = \mathrm{div}^{\mathcal{G}} - \langle e, \cdot\rangle$ satisfies

$$0 = L_X g, \qquad \mathrm{d}\xi = H(X). \tag{4.1}$$

Recall furthermore that the div is called *closed* if $X = 0$ and $\mathrm{d}\xi = 0$, and *exact* if $X = 0$ and $\xi = \mathrm{d}\phi$ for some $\phi \in C^\infty(M)$.

Proposition 4.11. *Let $\mathrm{div} = \mathrm{div}^{\mathcal{G}} - \langle e, \cdot\rangle$ be a divergence operator on the exact semi-Riemannian CA $(E, \mathcal{G})$. Assuming $\iota^* e \in \Gamma(K^\perp)$, the induced pair $(\mathcal{H}, \mathrm{div}_N)$ is compatible, closed, and exact if the pair $(\mathcal{G}, \mathrm{div})$ has the respective attribute. (Note that the assumption $\iota^* e \in \Gamma(K^\perp)$ is always satisfied if $(\mathcal{G}, \mathrm{div})$ is closed or, in particular, exact).*

Proof. We work in the splitting provided by the generalised metric, and denote $e = 2(X+\xi)$. The condition $\iota^* e \in \Gamma(K^\perp)$ translates to $\iota^* X = X^\parallel \in \Gamma(TN)$, where $X^\parallel$ denotes the component of $\iota^* X$ tangent to N.

Throughout this proof, we denote $\varphi^\parallel := \iota^*\varphi|_{\bigwedge^k TN} \in \Gamma(\bigwedge^k T^*N)$ for any $\varphi \in \Gamma(\bigwedge^k T^*M)$. In particular, $\xi^\parallel$ is given by $\iota^*\xi|_{TN} \in \Gamma(TN) \cong \Gamma(\mathrm{Ann}(\nu))$.

First we assume that $(\mathcal{G}, \mathrm{div})$ is compatible and deduce that $\mathcal{H}$ is compatible with $\mathrm{div}_N = \mathrm{div}^{\mathcal{H}} - 2\left\langle X^\parallel + \xi^\parallel, \cdot\right\rangle$. In fact, from (4.1) see that

$$L_{X^\parallel} h = L_X g|_{TN} = 0$$

and

$$\mathrm{d}^N \xi^\parallel = (\mathrm{d}\xi)^\parallel = (H(X))^\parallel = H^\parallel(X^\parallel)$$

wherein d^N denotes the exterior derivative on N. This proves compatibility of $(\mathcal{H}, \mathrm{div}_N)$.

Assume next $(\mathcal{G}, \mathrm{div})$ to be closed. Then closedness of the induced pair immediately follows from $\mathrm{d}^N \xi^{\parallel} = (\mathrm{d}\xi)^{\parallel} = 0$.

Finally, exactness of the induced pair is an immediate consequence of exactness of $(\mathcal{G}, \mathrm{div})$, as with $\xi = \mathrm{d}\phi$

$$\xi^{\parallel} = (\mathrm{d}\phi)^{\parallel} = \mathrm{d}^N \iota^* \phi$$

This finishes the proof. □

Remark 4.12. By a slight generalisation of the above proof, we see that Proposition 4.11 holds under the weaker assumption that the normal component $X^{\perp} \in \Gamma(\nu)$ of $X = \pi e/2 \in \Gamma(TM)$ extends to a Killing field $X^{\perp} \in \Gamma(TM)$ satisfying $H(X^{\perp}) = 0$.

4.2.3 Generalised Connections

Here we explain the inheritance of generalised connections. We assume as before that $(E, \mathcal{G})$ is a semi-Riemannian exact CA over M, and that $\iota \colon N \to (M, g)$ is a semi-Riemannian immersion with respect to the semi-Riemannian metric g associated with $\mathcal{G}$. We respectively denote by $\mathcal{H}$ and h the induced generalised metric on E_N and the usual metric on N.

The following lemma establishes the basis of our discussion.

Lemma 4.13. *Denote by $\pi^{\parallel} \colon \iota^* E \to E_N$ the projection along the Courant normal bundle $\mathcal{N}$. Then restriction and projection $D \mapsto D^N = \pi^{\parallel} D|_{E_N}$ of generalised connections on E defines a map to the space of generalised connections on E_N. More precisely, given $u, v \in \Gamma_{\mathrm{loc}}(E_N)$ and extensions $\tilde{u}, \tilde{v} \in \Gamma_{\mathrm{loc}}(E)$ we define*

$$D^N_u v := \pi^{\parallel}(D_{\tilde{u}} \tilde{v}).$$

The map $D \mapsto D^N$ maps $\mathcal{G}$-metric to $\mathcal{H}$-metric generalised connections and torsion-free to torsion-free generalised connections.

Furthermore, it maps the canonical connection $D^0 \in \mathcal{D}^0(\mathcal{G}, \mathrm{div}^{\mathcal{G}})$ to the canonical connection $D^{N0} \in \mathcal{D}^0(\mathcal{H}, \mathrm{div}^{\mathcal{H}})$, thus $D^{N0} = (D^0)^N$.

Proof. Using the fact that $\iota^* \pi u \in \Gamma(TN)$, it is easy to see that $\iota^*(D_{\tilde{u}} \tilde{v})$, and hence $\pi^{\parallel}(D_{\tilde{u}} \tilde{v})$, is independent of the extensions $\tilde{u}, \tilde{v}$ of $u, v \in \Gamma_{\mathrm{loc}}(E_N)$. This shows that the generalised connection $D^N = \pi^{\parallel} D|_{E_N}$ is well-defined for any generalised connection D on E. We recall that $\mathcal{N} = \mathcal{N} \cap E_+ \oplus \mathcal{N} \cap E_-$ and, hence, the decomposition $\iota^* E = \mathcal{N} \oplus E_N$ is orthogonal for both, $\langle \cdot, \cdot \rangle$ and $\mathcal{G}$. Therefore, D^N is always a generalised connection on E_N and is metric if D is:

$$\begin{aligned} \pi(u) \langle v, w \rangle &= \langle D_u v, w \rangle + \langle v, D_u w \rangle = \left\langle D^N_u v, w \right\rangle + \left\langle v, D^N_u w \right\rangle, \\ \pi(u) \mathcal{G}(v, w) &= \mathcal{G}(D_u v, w) + \mathcal{G}(v, D_u w) = \mathcal{G}(D^N_u v, w) + \mathcal{G}(v, D^N_u w), \end{aligned}$$

for all $u, v, w \in \Gamma(E_N)$. A similar calculation shows that the torsion 3-forms of D and D^N are related by $T^{D^N} = (T^D)^{\parallel}$. In particular, $T^D = 0$ implies $T^{D^N} = 0$.

For the last part of the statement, observe that the eigenbundles of $\mathcal{H}^{\mathrm{End}}$ are given by $(E_N)_{\pm} := \pi^{\parallel} E_{\pm}$, and consider the explicit description of the pure-type and mixed-type parts of D^0 and D^{N0} from (2.13). Since the LC connection on (M, g) gets mapped onto the LC connection on (N, h) under restriction and projection, and since the twist on E_N is given by $H^{\parallel} = \iota^* H|_{\bigwedge^3 TN}$, the result follows. □

Going on a brief tangent, we apply this result to show that, under natural conditions on the semi-Riemannian immersion $N \to M$, a generalised Kähler structure over M induces one over N. Recall that a *generalised almost Hermitian structure* is a pair $(\mathcal{G}, \mathcal{J})$ consisting of a generalised metric $\mathcal{G}$ and a generalised almost complex structure $\mathcal{J} \in \Gamma(\mathrm{End} E)$ (i.e. $\mathcal{J}^2 = -1$ and $\mathcal{J}$ is $\langle \cdot, \cdot \rangle$-symmetric) such that $\mathcal{J}$ is $\mathcal{G}$-antisymmetric. Recall furthermore that a generalised almost Hermitian structure is called a *generalised Kähler structure* if both $\mathcal{J}_1 = \mathcal{J}$ and $\mathcal{J}_2 = \mathcal{G}^{\mathrm{End}} \mathcal{J}$ are integrable (i.e. if their respective Nijenhuis tensors vanish).

Corollary 4.14. *Let $(E, \mathcal{G}, \mathcal{J})$ be an exact Courant algebroid over M endowed with a (possibly indefinite) generalised Kähler structure. Furthermore let $N \subset M$ be a semi-Riemannian submanifold and $(E_N, \mathcal{H})$ the induced exact semi-Riemannian Courant algebroid. Assume that $\mathcal{J} E_N \subset E_N$. Then $(E_N, \mathcal{H}, \mathcal{J}|_{E_N})$ is generalised Kähler.*

Proof. According to [30] the generalised Kähler property for a generalised almost Hermitian structure $(\mathcal{G}, \mathcal{J})$ on a Courant algebroid E is equivalent to the existence of a LC generalised connection such that $D\mathcal{J} = 0$. By virtue of Lemma 4.13 we know that the induced generalised connection D^N is a Levi-Civita generalised connection for $\mathcal{H}$. Therefore it suffices to show that $D^N \mathcal{J}_N = 0$ for the generalised almost complex structure $\mathcal{J}_N := \mathcal{J}|_{E_N}$. Let $u, v \in \Gamma(E_N)$. Then

$$(D^N_u \mathcal{J}_N) v = D^N_u (\mathcal{J}_N v) - \mathcal{J}_N D^N_u v = \pi^{\|}(D_u \mathcal{J}) v = 0.$$

□

Similarly, using the characterisation in [30] of generalised hyper-Kähler structures as generalised almost hyper-Hermitian structures $(\mathcal{G}, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$ admitting a LC generalised connection D such that $D\mathcal{J}_\alpha = 0$ for all $\alpha \in \{1, 2, 3\}$ we obtain the following.

Corollary 4.15. *Let $(E, \mathcal{G}, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$ be an exact Courant algebroid over M endowed with a (possibly indefinite) generalised hyper-Kähler structure. Furthermore let $N \subset M$ be a semi-Riemannian submanifold and $(E_N, \mathcal{H})$ the induced exact semi-Riemannian Courant algebroid. Assume that $\mathcal{J}_\alpha E_N \subset E_N$ for all $\alpha \in \{1, 2, 3\}$. Then $(E_N, \mathcal{H}, \mathcal{J}_\alpha|_{E_N}, \alpha = 1, 2, 3)$ is generalised hyper-Kähler.*

The map $D \mapsto D^N$ does not, in general, preserve compatibility with a given divergence operator div.

Proposition 4.16. *Let* div *be any divergence operator on the semi-Riemannian CA E and D the canonical generalised LC connection with divergence operator* $\mathrm{div} = \mathrm{div}^{\mathcal{G}} - \langle e, \cdot \rangle$. *Then the reduced generalised LC connection D^N has the divergence operator (we always assume* $\dim N > 1$ *and hence* $\dim M - 1 \neq 0$*)*

$$\mathrm{div}_{D^N} = \mathrm{div}^{\mathcal{H}} - \frac{\dim N - 1}{\dim M - 1} \langle e^{\|}, \cdot \rangle,$$

while the reduction of div *is*

$$\mathrm{div}_N = \mathrm{div}^{\mathcal{H}} - \langle e^{\|}, \cdot \rangle.$$

In particular,

$$\mathrm{div}_{D^N} = \mathrm{div}_N + \frac{\dim M - \dim N}{\dim M - 1} \langle e^{\|}, \cdot \rangle.$$

More generally, for any generalised LC connection $D = D^0 + \chi$ with divergence operator div, *$D^0 \in \mathcal{D}^0(\mathcal{G}, \mathrm{div}^{\mathcal{G}})$ the canonical generalised connection from Corollary 2.36, we have*

$$\mathrm{div}_{D^N}(v) = \mathrm{div}_N(v) - \mathrm{tr}_{\mathcal{N}}(\chi v), \quad v \in \Gamma(E_N). \tag{4.2}$$

Proof. The reduction of the divergence operator $\operatorname{div} = \operatorname{div}^{\mathcal{G}} - \langle e, \cdot\rangle$ to $\operatorname{div}_N = \operatorname{div}^{\mathcal{H}} - \left\langle e^{\|}, \cdot\right\rangle$ is by definition.

Write $D = D^0 + \chi$, where $\chi \in \Gamma(\mathfrak{so}(E_+)^{\langle 1\rangle} \oplus \mathfrak{so}(E_-)^{\langle 1\rangle})$ is such that $\operatorname{tr}(\chi v) = -\langle e, v\rangle$ for all $v \in E$, cf. Corollary 2.40. Recall that by Lemma 4.13, D^0 reduces to the canonical generalised connection $(D^0)^N = \pi^{\|} D^0 = D^{N0} \in \mathcal{D}^0(\mathcal{H}, \operatorname{div}^{\mathcal{H}})$ with divergence operator $\operatorname{div}_{D^{N0}} = \operatorname{div}^{\mathcal{H}}$. It follows for all $v \in \Gamma(E_N)$ that

$$\operatorname{div}_{D^N}(v) = \operatorname{div}_{D^{N0}}(v) + \operatorname{tr}_{E_N} \pi^{\|}(\chi v)$$

where $\operatorname{tr}_{E_N} \pi^{\|}(\chi v) = \operatorname{tr}_{E_N}(\chi v) = \operatorname{tr}(\chi v) - \operatorname{tr}_{\mathcal{N}}(\chi v) = -\langle e^{\|}, v\rangle - \operatorname{tr}_{\mathcal{N}}(\chi v)$.

Considering now the canonical LC connection D with divergence div from Theorem 3.2, we have $\chi = \frac{1}{\dim M - 1}(\chi_+^{e_+} + \chi_-^{e_-})$. For this choice and $v \in (E_N)_+$ we obtain

$$(\dim\ M - 1)\operatorname{tr}_{\mathcal{N}}(\chi v) = \sum n_i^*(\langle n_i, v\rangle e_+ - \langle e_+, v\rangle n_i) = -(\dim M - \dim N)\langle e_+^{\|}, v\rangle$$

where (n_i) is a (local) orthonormal frame of $\mathcal{N}_+ := \mathcal{N} \cap E_+$ and (n_i^*) is the dual frame of $\mathcal{N}_+^*$. As a consequence,

$$\operatorname{div}_{D^N} = \operatorname{div}_{D^{N0}} - \frac{\dim N - 1}{\dim M - 1}\langle e^{\|}, \cdot\rangle = \operatorname{div}^{\mathcal{H}} - \frac{\dim N - 1}{\dim M - 1}\langle e^{\|}, \cdot\rangle,$$

as claimed. □

We denote by $\mathcal{D}^0(\mathcal{G}, \operatorname{div}; N)$ the subspace of generalised connections in $\mathcal{D}^0(\mathcal{G}, \operatorname{div})$ that project onto connections in $\mathcal{D}^0(\mathcal{H}, \operatorname{div}_N)$.

Recall the maps $\operatorname{tr}_\pm \colon (\mathfrak{so}(E_\pm))^{\langle 1\rangle} \to E_\pm^*, \chi_\pm \mapsto \operatorname{tr}\chi_\pm$ from (3.2).

Proposition 4.17. *$\mathcal{D}^0(\mathcal{G}, \operatorname{div}; N)$ is an affine space over the space of sections $\chi \in \Gamma(\ker\operatorname{tr}_+ \oplus \ker\operatorname{tr}_-)$ such that $\chi^{\|} = \iota^*\chi|_{E_N^3} \in \ker\operatorname{tr}_{E_N}$. By abuse of notation, we will refer to this space as $\Gamma(\ker\operatorname{tr} \cap \ker\operatorname{tr}_N)$.*

Proof. The next lemma establishes that $\mathcal{D}^0(\mathcal{G}, \operatorname{div}; N)$ is non-empty. The claim is a consequence. □

Lemma 4.18. *Let $D^0 \in \mathcal{D}^0(\mathcal{G}, \operatorname{div}^{\mathcal{G}})$ be the canonical generalised LC connection with metric divergence. Define the generalised connection $D = D^0 + \frac{\chi_+^N + \chi_-^N}{\dim M - 1}$, where $\chi_\pm^N \in \Gamma(\mathfrak{so}(E_\pm)^{\langle 1\rangle})$ is such that $\operatorname{tr}_E \chi_\pm^N = -(\dim M - 1)e_\pm$ and on N*

$$\begin{aligned}&\left[\chi_\pm^N - \chi_\pm^{e_\pm}\right](a_\pm, b_\pm, c_\pm)\\ &:= \frac{\dim M - \dim N}{\dim N - 1}\chi_\pm^{e_\pm^{\|}}(a_\pm^{\|}, b_\pm^{\|}, c_\pm^{\|}) - \chi_\pm^{e_\pm^{\|}}(a_\pm^{\perp}, b_\pm^{\perp}, c_\pm^{\|}) - \chi_\pm^{e_\pm^{\|}}(a_\pm^{\perp}, b_\pm^{\|}, c_\pm^{\perp}),\end{aligned} \tag{4.3}$$

where we have abbreviated $\chi(a, b, c) := \langle\chi(a, b), c\rangle$. Then $D \in \mathcal{D}^0(\mathcal{G}, \operatorname{div}; N)$.

Proof. First, we check that the demand (4.3) is compatible with $\operatorname{tr}_E \chi_\pm^N = -(\dim M - 1)e_\pm$, as

$$\begin{aligned}\operatorname{tr}_E\left[\chi_\pm^N - \chi_\pm^{e_\pm}\right] &= \frac{\dim M - \dim N}{\dim N - 1}\operatorname{tr}\chi_{N\pm}^{e_\pm^{\|}} - \sum_i \underbrace{\varepsilon_i^\pm \chi_\pm^{e_\pm^{\|}}(n_i^\pm, \cdot, n_i^\pm)}_{-\langle e_\pm^{\|}, \cdot\rangle}\\ &= (\dim M - \dim N)\langle -e_\pm^{\|} + e_\pm^{\|}, \cdot\rangle = 0,\end{aligned}$$

where $\overset{e^{\parallel}_{\pm}}{\chi}_{N\pm} \in \Gamma(E_N^* \otimes \mathfrak{so}(E_N))$ is the tensor obtained by restriction (and projection) of $\overset{e^{\parallel}_{\pm}}{\chi}_{\pm}$ and $(n_i^{\pm})$ is an orthonormal basis of $\mathcal{N}_{\pm}$ and $\varepsilon_i^{\pm} = \langle n_i^{\pm}, n_i^{\pm} \rangle = -\varepsilon_i^{\mp}$. Thus, $\chi^N_{\pm}$ is well-defined.

It is immediate that $\chi^N_{\pm} - \overset{e_{\pm}}{\chi}_{\pm} \in \Gamma(\ker \mathrm{tr}_{\pm})$, as it is antisymmetric in the last two entries, its total antisymmetrisation vanishes, and by definition $\mathrm{tr}_E[\chi^N_{\pm} - \overset{e_{\pm}}{\chi}_{\pm}] = 0$. Hence $D \in \mathcal{D}^0(\mathcal{G}, \mathrm{div})$. Furthermore, the restriction satisfies

$$\chi^N_{\pm}\Big|_{E_N} = \frac{\dim M - 1}{\dim N - 1} \overset{e^{\parallel}_{\pm}}{\chi}_{N\pm}$$

which proves that $\pi^{\parallel} D \in \mathcal{D}^0(\mathcal{H}, \mathrm{div}_N)$ is the canonical connection. □

Lemma 4.19. *Assume that $N \subset M$ is a submanifold. Then, the restriction of the map $D \mapsto D^N$ to $\mathcal{D}^0(\mathcal{G}, \mathrm{div}; N)$ is a surjection onto $\mathcal{D}^0(\mathcal{H}, \mathrm{div}_N)$.*

Proof. We only have to check that restriction onto E_N gives a surjective map

$$\Gamma(\ker \mathrm{tr} \cap \ker \mathrm{tr}_N) \longrightarrow \Gamma(\ker \mathrm{tr}_{E_N} : \mathfrak{so}(E_N)^{\langle 1 \rangle} \to E_N^*).$$

This, however, is immediate, as every element of $\ker \mathrm{tr}_{E_N}$ can be linearly extended to an element of $\ker \mathrm{tr}_N$ by requiring it to vanish on the normal bundle $\mathcal{N}$. It can then be extended to a section $\chi \in \Gamma(\ker \mathrm{tr})$ such that $\mathrm{tr}_{E_N} \chi^{\parallel} = 0$, thus by definition $\chi \in \Gamma(\ker \mathrm{tr} \cap \ker \mathrm{tr}_N)$. (Recall that the name $\Gamma(\ker \mathrm{tr} \cap \ker \mathrm{tr}_N)$ is an abuse of notation.) □

4.3 Extrinsic Curvature

This section reproduces [17, chapter 5.1].

Let $\pi\colon E \to M$ be an exact CA with generalised semi-Riemannian metric $\mathcal{G} \cong (g, \sigma)$ and compatible divergence operator $\operatorname{div} = \operatorname{div}^{\mathcal{G}} - \langle e, \cdot \rangle$, where $e \in \Gamma(E)$. Let $\Sigma \subset M$ be a hypersurface[14] which is nondegenerate with respect to g. Recall that the generalised metric $\mathcal{G}$ induces a generalised metric $\mathcal{H}$ on the exact CA E_Σ, see Proposition 4.8. Denote by $n \in \Gamma(TM|_\Sigma)$ a unit normal on Σ, and furthermore $n_\pm = \sigma_\pm n$.

Definition 4.20 (Generalised Second Fundamental Form)**.** We define the *generalised extrinsic curvature* or *generalised second fundamental form* of Σ with respect to $D \in \mathcal{D}^0(\mathcal{G}, \operatorname{div})$ to be

$$\mathcal{K}^{n_\pm} \in \Gamma(E_\Sigma^* \otimes E_\Sigma^*); \qquad \mathcal{K}^{n_\pm}(a, b) := \mathcal{G}(D_a n_\pm, b) = \pm \langle D_a n_\pm, b \rangle$$

for $a, b \in \Gamma(E_\Sigma)$. The *shape tensor* $\mathcal{A}^{n_\pm} := Dn_\pm \in \Gamma(\operatorname{End} E_\Sigma)$ is the corresponding endomorphism under the isomorphism $\mathcal{G} : E_\Sigma \to E_\Sigma^*$.

The "mixed-type" components of the generalised second fundamental form are of particular importance, hence we denote them by

$$\mathcal{K}^\pm := \mathcal{K}^{n_\pm}|_{E_\Sigma^\mp \times E_\Sigma^\pm}$$

and by $\mathcal{A}^\pm = \mathcal{G}^{-1}\mathcal{K}^\pm : E_\Sigma^\mp \to E_\Sigma^\pm$ the corresponding components of the shape tensor. We also introduce the *conormal extrinsic curvature* $\mathcal{L}^\pm \in \Gamma(E_\Sigma^*)$, which is defined as

$$\mathcal{L}^\pm(a) := \mathcal{G}(D_{n_\pm - n_\mp} n_\pm, a).$$

Note that $\pi(n_\pm - n_\mp) = 0$, so that the right hand side does not depend on any derivatives of n. The conormal extrinsic curvature can be understood as measuring how close a generalised LC connection with divergence div is to projecting to a generalised LC connection with divergence div_N on the hypersurface Σ:

Lemma 4.21. *Let $D \in \mathcal{D}^0(\mathcal{G}, \operatorname{div})$. Then $D \in \mathcal{D}^0(\mathcal{G}, \operatorname{div}; \Sigma)$ if and only if both conormal extrinsic curvatures $\mathcal{L}^+$ and $\mathcal{L}^-$ vanish.*

Proof. Let $v_\pm$ be a section of $E_\Sigma^\pm$. We can assume without loss of generality that E is a twisted generalised tangent bundle and, thus, $n_\pm = n \pm gn$ and $v_\pm = V \pm gV$, where $V \in \Gamma(TN)$. We extend n and V such that $\nabla_n n = \nabla_n V = 0$. Recalling that $[n_-, n_+] \in \ker \pi$, we see that $\langle v_+, [n_-, n_+] \rangle = \frac{1}{2}[n_-, n_+](\pi v_+) = 0$. This proves that $\langle D_{n_-} v_+, n_+ \rangle = -\langle v_+, D_{n_-} n_+ \rangle = -\langle v_+, [n_-, n_+] \rangle = 0$. As a consequence, the condition $\mathcal{L}^+ = 0$ reduces to $\langle D_{n_+} v_+, n_+ \rangle = 0$. Similarly, $\mathcal{L}^- = 0$ reduces to $\langle D_{n_-} v_-, n_- \rangle = 0$. Therefore $\mathcal{L}^+ = \mathcal{L}^- = 0$ reduces to $0 = \operatorname{tr}_\mathcal{N}(Dv_+) = \operatorname{tr}_\mathcal{N}(Dv_-)$. From $\nabla_n V = 0$ it follows that $\langle D^0_{n_\pm} v_\pm, n_\pm \rangle = 0$, in view of the formulas (2.13) for the canonical connection D^0. We have proven that the tensor $\chi = D - D^0$ has $\operatorname{tr}_\mathcal{N}(\chi v) = 0$ for all $v \in \Gamma(E_\Sigma)$ if and only if the normal conormal curvatures $\mathcal{L}^\pm$ of D are zero. From equation (4.2) we see that this means that $D \in \mathcal{D}^0(\mathcal{G}, \operatorname{div}; \Sigma)$ if and only if $\mathcal{L}^+ = \mathcal{L}^- = 0$. □

Corollary 4.22. *The conormal extrinsic curvature of the canonical generalised LC connection D^0 with metric divergence vanishes.*

[14] All results trivially generalise to $\iota\colon \Sigma \to M$ an immersion of co-dimension one.

Proof. This follows from the fact that the canonical generalised LC connection D^0 with metric divergence projects to the canonical generalised LC connection D^{N0} with metric divergence. □

Recall from Corollary 2.40 that the space of torsion-free, metric and divergence compatible generalised connections is an affine space over the space of sections of $\ker \mathrm{tr}_+ \oplus \ker \mathrm{tr}_-$, where $\mathrm{tr}_\pm \colon \mathfrak{so}(E_\pm)^{\langle 1\rangle} \to E^*_\pm$ are given as in (3.2). Thus, denoting $d = \dim M$ and taking the canonical connection $D = D^0 + \frac{1}{d-1}(\chi_+^{e_+} + \chi_-^{e_-}) \in \mathcal{D}^0(\mathcal{G}, \mathrm{div})$ from Theorem 3.2, we can describe an arbitrary connection in $\mathcal{D}^0(\mathcal{G}, \mathrm{div})$ as

$$D^\chi \coloneqq D + \chi = D + \chi_+ + \chi_- \tag{4.4}$$

where $\chi_\pm \in \Gamma(\ker \mathrm{tr}_\pm)$. We need the following computational result.

Lemma 4.23. *Denote $\chi_\pm^\perp(a_\pm, b_\pm) \coloneqq \chi_\pm(a_\pm, b_\pm, n_\pm)$. The pure-type and mixed-type parts of the generalised second fundamental form induced by D^χ are given by*

$$\begin{aligned} \mathcal{K}^{n_\pm}|_{E_\Sigma^\pm \times E_\Sigma^\pm} &= k \mp \chi_\pm^\perp - \frac{\langle e, n_\pm\rangle}{\dim \Sigma} h \mp \frac{i_n H}{6} \\ \mathcal{K}^{n_\pm}|_{E_\Sigma^\mp \times E_\Sigma^\pm} &= k \mp \frac{i_n H}{2} \end{aligned} \tag{4.5}$$

Herein, we employed the isometries $\sigma_\pm \colon (TM, g) \cong (E_\pm, \mathcal{G})$ to identify TM with $E_\pm$.

Moreover, we have for the conormal extrinsic curvature

$$\mathcal{L}^\pm = \frac{\varepsilon}{\dim \Sigma} \langle e_\pm, \cdot\rangle \mp \chi_\pm^\perp(n_\pm, \cdot) \tag{4.6}$$

where $\varepsilon = g(n, n)$.

Proof. We calculate for $a = a_+ + a_-, b = b_+ + b_- \in \Gamma(E_\Sigma)$, $a_\pm, b_\pm \in \Gamma(E_\Sigma^\pm)$:

$$\begin{aligned}
&\mathcal{G}\left(D^\chi_a n_\pm, b\right) \\
&= \mathcal{G}\left(D^\chi_{a_+ + a_-} n_\pm, b_+ + b_-\right) \\
&= \mathcal{G}\left(D^\chi_{a_\pm} n_\pm, b_\pm\right) + \mathcal{G}\left(D^\chi_{a_\mp} n_\pm, b_\pm\right) \\
&= \mathcal{G}\left(D^0_{a_\pm} n_\pm + \left(\frac{\chi_\pm^{e_\pm}}{\dim M - 1} + \chi_\pm\right)(a_\pm, n_\pm), b_\pm\right) + g(\nabla^\pm_{\pi a_\mp} n, \pi b_\pm) \\
&= g(\nabla^{\pm 1/3}_{\pi a_\pm} n, \pi b_\pm) \pm \frac{1}{\dim \Sigma}\Big[\underbrace{\langle a_\pm, n_\pm\rangle}_{=0} \langle e_\pm, b_\pm\rangle - \langle e_\pm, n_\pm\rangle \langle a_\pm, b_\pm\rangle\Big] \\
&\qquad \pm \chi_\pm(a_\pm, n_\pm, b_\pm) + g(\nabla^\pm_{\pi a_\mp} n, \pi b_\pm) \\
&= g(\nabla^{\pm 1/3}_{\pi a_\pm} n, \pi b_\pm) - \frac{\langle e, n_\pm\rangle}{\dim \Sigma} h(\pi a_\pm, \pi b_\pm) \mp \chi_\pm^\perp(a_\pm, b_\pm) + g(\nabla^\pm_{\pi a_\mp} n, \pi b_\pm) \\
&= g(\nabla_{\pi a} n, \pi b_\pm) \pm \frac{1}{6}[H(\pi a_\pm, n, b_\pm) + 3H(\pi a_\mp, n, \pi b_\pm)] \\
&\qquad \mp \chi_\pm^\perp(a_\pm, b_\pm) - \frac{\langle e, n_\pm\rangle}{\dim \Sigma} h(\pi a_\pm, \pi b_\pm) \\
&= k(\pi a, \pi b_\pm) \mp \chi_\pm^\perp(a_\pm, b_\pm) - \frac{\langle e, n_\pm\rangle}{\dim \Sigma} h(\pi a_\pm, \pi b_\pm) \\
&\qquad \mp \frac{1}{6}[H(n, \pi a_\pm, \pi b_\pm) + 3H(n, \pi a_\mp, \pi b_\pm)].
\end{aligned}$$

Similarly, we calculate

$$\begin{aligned}
\mathcal{L}^{\pm}(a_\pm) &= \mathcal{G}(D_{n_\pm - n_\mp} n_\pm, a) \\
&= \mathcal{G}\left(D^0_{n_\pm} n_\pm + \left(\frac{\chi_\pm^{e_\pm}}{\dim M - 1} + \chi_\pm \right)(n_\pm, n_\pm) - D^0_{n_\mp} n_\pm, a_\pm \right) \\
&= g(\nabla_n^{\pm 1/3} n - \nabla_n^{\pm} n, \pi a_\pm) \pm \left(\frac{\chi_\pm^{e_\pm}}{\dim \Sigma} + \chi_\pm \right)(n_\pm, n_\pm, a_\pm) \\
&= \frac{\pm 1}{\dim \Sigma} [\, \langle n_\pm, n_\pm \rangle \langle e_\pm, a_\pm \rangle - \langle n_\pm, a_\pm \rangle \langle n_\pm, e_\pm \rangle \,] \mp \chi_\pm^{\perp}(n_\pm, a_\pm) \\
&= \frac{\varepsilon}{\dim \Sigma} \langle e_\pm, a_\pm \rangle \mp \chi_\pm^{\perp}(n_\pm, a_\pm).
\end{aligned}$$

This proves the claim. □

Corollary 4.24. *The mixed-type tensors $\mathcal{K}^\pm$ only depend on the choice of generalised metric $\mathcal{G}$, and the sum $\mathcal{K}^+ + \mathcal{K}^-$ of the mixed-type extrinsic curvatures is symmetric:*

$$\mathcal{K}^-(a_+, b_-) = \mathcal{K}^+(b_-, a_+). \tag{4.7}$$

Note that the pure-type tensors depend on the choice of χ. In some settings, one may wish for the generalised metric $\mathcal{G}$ and divergence operator div to determine all relevant geometrical objects. Then, the quantities of interest are obtained by extracting the χ-independent part of the pure-type tensors, i.e. by dividing out the action of $\Gamma(\ker \mathrm{tr})$. The result does not depend on whether we consider $\Gamma(\ker \mathrm{tr})$ acting on $\mathcal{D}^0(\mathcal{G}, \mathrm{div})$ or $\Gamma(\ker \mathrm{tr} \cap \ker \mathrm{tr}_\Sigma)$ acting on $\mathcal{D}^0(\mathcal{G}, \mathrm{div}; \Sigma)$, respectively.

Lemma 4.25. *Denote by $\mathrm{End}^0(E_\Sigma^\pm)$ the space of traceless endomorphisms on $E_\Sigma^\pm$ and by $\mathrm{Bil}^0(E_\Sigma^\pm)$ the space of traceless bilinear forms with respect to $\langle \cdot, \cdot \rangle$. Then the following gives a well-defined surjection:*

$$\ker \mathrm{tr}_\pm|_\Sigma \longrightarrow \mathrm{Bil}^0(E_\Sigma^\pm) \cong \mathrm{End}^0(E_\Sigma^\pm); \qquad \chi_\pm \longrightarrow \chi_\pm^\perp$$

Furthermore, the map remains surjective if we restrict it to $(\ker \mathrm{tr})|_\Sigma \cap \ker \mathrm{tr}_\Sigma$.

In particular, the induced map on the level of sections $\Gamma(\ker \mathrm{tr} \cap \ker \mathrm{tr}_\Sigma) \to \Gamma(\ker \mathrm{tr}_{E_\Sigma})$ is surjective. We denote also this map as $\chi_\pm \mapsto \chi_\pm^\perp$.

Proof. It is clear that the bilinear form $\chi_\pm^\perp$ defines an endomorphism $\chi_\pm^\perp$ by the identification $\chi_\pm^\perp(a, b) = \langle \chi_\pm^\perp a, b \rangle$. Antisymmetry of $\chi_\pm$ in the last two entries implies that $\chi_\pm(\alpha_\pm, n_\pm, n_\pm) = 0$ for all $\alpha_\pm \in E_\pm|_\Sigma$. Tracelessness of $\chi_\pm$ thus implies

$$0 = [\mathrm{tr}\, \chi_\pm](n_\pm) = -\left[\mathrm{tr}_{E_\Sigma} \chi_\pm^\perp \right]$$

so that indeed the map is well-defined. To prove surjectivity, we show, for instance, how to construct a pre-image $\chi_+ \in (\ker \mathrm{tr}_+)|_\Sigma \cap \ker \mathrm{tr}_{\Sigma +}$ of an arbitrary element $T \in \mathrm{Bil}^0(E_\Sigma^+)$. Let χ_+ be the element of the subbundle

$$\left[(E_\Sigma^+)^* \otimes (E_\Sigma^+)^* \wedge \langle n_+, \cdot \rangle \right] \oplus \left[\langle n_+, \cdot \rangle \otimes \bigwedge^2 (E_\Sigma^+)^* \right] \subset \left(E_+^* \otimes \bigwedge^2 E_+^* \right)\Big|_\Sigma$$

which has the following values

$$\chi_+(a, b, n_+) = T(a, b), \quad \chi_+(n_+, a, b) = T(b, a) - T(a, b)$$

where $a, b \in \Gamma(E_\Sigma^+)$. Tracelessness of T implies $\operatorname{tr}\chi_+ = \operatorname{tr}_\Sigma \chi_+ = 0$, and

$$\begin{aligned}(\partial\chi_+)(a,b,n_+) &= \chi_+(a,b,n_+) + \chi_+(b,n_+,a) + \chi_+(n_+,a,b) \\ &= T(a,b) - T(b,a) + (T(b,a) - T(a,b)) = 0.\end{aligned}$$

Therefore χ_+ is an element of $\ker \operatorname{tr}_+|_\Sigma$. Also, $\chi_+^\perp(a,b) = \chi_+(a,b,n_+) = T(a,b)$ for all $a, b \in E_\Sigma^+$. Thus, χ_+ is a pre-image of T. □

It follows that the only component of the pure-type operators independent of the choice of generalised LC connection with specified divergence operator is the trace.

Definition 4.26 (Generalised Mean Curvature)**.** The *generalised mean curvature* is defined as the trace

$$\mathcal{T}^\pm := \operatorname{tr}_\mathcal{H} \mathcal{K}^{n_\pm}.$$

From Lemma 4.23, we immediately get the following result.

Corollary 4.27. *The generalised mean curvature is given by*

$$\mathcal{T}^\pm = \operatorname{tr}_h k - \langle e, n_\pm \rangle .$$

In particular, it is independent of the choice of sign if and only if $g(\pi e, n) = 0$.

We now derive a counterpart of the formula $k = \frac{1}{2} L_n g$ in generalised geometry.

Lemma 4.28. *On E_Σ, it holds*

$$\frac{1}{2}[n_\pm, \mathcal{G}] = \mathcal{K}^+ + \mathcal{K}^-.$$

Proof. This follows from direct computation. We only consider n_+, the case of n_- is completely analogous. Let us consider $a, b \in \Gamma(E)$, and write $a = a_+ + a_-$, $b = b_+ + b_-$. Then

$$\begin{aligned}[n_+, \mathcal{G}](a_\pm, b_\pm) &= \pi(n_+)\mathcal{G}(a_\pm, b_\pm) - \mathcal{G}([n_+, a_\pm], b_\pm) - \mathcal{G}(a_\pm, [n_+, b_\pm]) \\ &= \pm(\pi(n_+)\langle a_\pm, b_\pm\rangle - \langle [n_+, a_\pm], b_\pm\rangle - \langle a_\pm, [n_+, b_\pm]\rangle) \\ &= 0.\end{aligned}$$

Furthermore

$$\begin{aligned}[n_+, \mathcal{G}](a_+, b_-) &= \pi(n_+)\mathcal{G}(a_+, b_-) - \mathcal{G}([n_+, a_+], b_-) - \mathcal{G}(a_+, [n_+, b_-]) \\ &= \langle [n_+, a_+], b_-\rangle - \langle a_+, [n_+, b_-]\rangle \\ &= 2\langle [b_-, n_+], a_+\rangle \\ &= 2\mathcal{G}(D_{b_-} n_+, a_+) \\ &= 2\mathcal{K}^+(b_-, a_+).\end{aligned}$$

Symmetry of $[n_\pm, \mathcal{G}]$ together with (4.7) implies the claim. □

4.4 Generalised Gauß and Codazzi Equations

In this section, which reproduces [17, section 5.1], we determine the counterpart of the Gauß and Codazzi equations in generalised geometry. We still consider a hypersurface $\Sigma \subset M$ which is nondegenerate with respect to g, denote by $\mathcal{H}$ the induced metric on the exact CA E_Σ, by $n \in \Gamma(TM|_\Sigma)$ a unit normal on Σ, and $n_\pm = \sigma_\pm n$. Let $D \in \mathcal{D}^0(\mathcal{G}, \mathrm{div})$ and $D^\Sigma \in \mathcal{D}^0(\mathcal{H})$ the inherited connection. Denote $\varepsilon = g(n,n) = \mathcal{G}(n_\pm, n_\pm)$. Recall that, in general, the inherited divergence operator div_Σ does not agree with the divergence operator div_{D^Σ} coming from the inherited connection D^Σ, i.e. $\mathrm{div}_\Sigma \neq \mathrm{div}_{D^\Sigma}$.

The starting point is the decomposition of the ambient generalised connection in tangent and normal parts.

Lemma 4.29 (Generalised Gauß and Weingarten equations)**.** *Let $a, b \in \Gamma(E_\Sigma)$ and denote by $a_\pm, b_\pm$ their projections to $\Gamma(E_\Sigma^\pm)$. Then*

$$\begin{aligned}(D_a b_\pm)^\parallel &= D_a^\Sigma b_\pm,\\ \mathcal{G}(D_a b_\pm, n_\pm) &= -\mathcal{K}^{n_\pm}(a, b_\pm),\\ D_a n_\pm &= \mathcal{A}^{n_\pm}(a).\end{aligned} \tag{4.8}$$

Herein, $(\cdot)^\parallel \colon \iota^* E \to E_\Sigma$ denotes the orthogonal projection.

Proof. The last equation holds by definition of the generalised shape tensor and the first by definition of D^Σ, see Lemma 4.13. We recall that the values of $D_{\tilde{a}}\tilde{b}$ along Σ do not depend on the chosen extensions $\tilde{a}, \tilde{b}$ of a, b. For that reason the tildes are omitted. Finally,

$$\mathcal{G}(D_a b_\pm, n_\pm) = -\mathcal{G}(b_\pm, D_a n_\pm) = -\mathcal{K}^{n_\pm}(a, b_\pm).$$

□

The following are useful computational identities.

Lemma 4.30. *For all $u_\pm, v_\pm \in \Gamma(E_\Sigma^\pm)$ and $a, b \in \Gamma(E_\Sigma)$, it holds*

(i) $\mathcal{G}([n_\pm - n_\mp, a], b) = 0$,

(ii) $\mathcal{G}\left(D_{n_\pm - n_\mp} u_\pm, v_\pm\right) = \mathcal{K}^{n_\pm}(u_\pm, v_\pm) - \mathcal{K}^{n_\pm}(v_\pm, u_\pm)$, *and*

(iii) $\mathcal{G}\left(D_{n_\pm - n_\mp} u_\pm, n_\pm\right) = -\mathcal{L}^\pm(u_\pm)$.

Furthermore, assuming an extension of n such that $\nabla_n n = 0$, one has $[n_\pm - n_\mp, a] = 0$.

Proof. To prove (i) it suffices to pick a splitting of E adapted to the normal bundle, so that $n_\pm = n \pm n^\flat$, where $n^\flat := gn$. Then

$$\mathcal{G}([n_\pm - n_\mp, a], b) = \pm 2\mathcal{G}([n^\flat, a]_H, b) = \mp 2\mathcal{G}(i_{\pi a} \mathrm{d}n^\flat, b) = \mp \mathrm{d}n^\flat(\pi a, \pi \mathcal{H}^{\mathrm{End}} b) = 0,$$

where the last equality follows from $\Sigma \subset M$ being a submanifold. Note that, if $\nabla_n n = 0$, then $\mathrm{d}n^\flat = 0$ and therefore $[n_\pm - n_\mp, a] = 0$.

Employing (i), we find (ii):

$$\begin{aligned}\mathcal{G}(D_{n_\pm - n_\mp} u_\pm, v_\pm) &= \mathcal{G}(D_{u_\pm}(n_\pm - n_\mp) + [n_\pm - n_\mp, u_\pm], v_\pm) - \mathcal{G}(D_{v_\pm}(n_\pm - n_\mp), u_\pm)\\ &= \mathcal{K}^{n_\pm}(u_\pm, v_\pm) - \mathcal{K}^{n_\pm}(v_\pm, u_\pm).\end{aligned}$$

Employing metricity of D, we find

$$\mathcal{G}(D_{n_\pm - n_\mp} u_\pm, n_\pm) = -\mathcal{G}(u_\pm, D_{n_\pm - n_\mp} n_\pm) = -\mathcal{L}^\pm(u_\pm)$$

and have thus established (iii). □

To establish the generalised Gauß and Codazzi equations, we compute a decomposition of the parallel part of $D^2_{v,w}b$ in terms of $(D^\Sigma)^2_{v,w}b$, the generalised second fundamental form, and $D_{n_\pm}b$.

Lemma 4.31. *The second derivative D^2 is given by*

$$\begin{aligned}\mathcal{G}(a, D^2_{v,w}b_\pm) &= \mathcal{G}(a, (D^\Sigma)^2_{v,w}b_\pm) - \varepsilon\mathcal{K}^{n_\pm}(v,a)\mathcal{K}^{n_\pm}(w,b_\pm)\\ &\quad + \varepsilon \sum_{\sigma\in\{+,-\}} \mathcal{K}^{n_\sigma}(v,w)\mathcal{G}(a, D_{n_\sigma}b_\pm)\end{aligned}$$

where $a, v, w \in \Gamma(E_\Sigma)$ and $b_\pm \in \Gamma(E_\pm)$ such that $b_\pm|_\Sigma \in \Gamma(E^\pm_\Sigma)$.

Proof. First, a direct computation employing Lemma 4.29 reveals

$$\begin{aligned}\mathcal{G}(a, D_vD_wb_\pm) &= \mathcal{G}\left(a, D_v\left\{D^\Sigma_w b_\pm - \varepsilon n_\pm \mathcal{K}^{n_\pm}(w,b_\pm)\right\}\right)\\ &= \mathcal{G}(a, D^\Sigma_v D^\Sigma_w b_\pm) - \varepsilon\mathcal{K}^{n_\pm}(v,a)\mathcal{K}^{n_\pm}(w,b_\pm).\end{aligned}$$

Also

$$\begin{aligned}\mathcal{G}(a, D_{D_vw}b_\pm) &= \mathcal{G}\left(a, D_{D^\Sigma_v w}b_\pm - \varepsilon\mathcal{K}^{n_+}(v,w)D_{n_+}b_\pm - \varepsilon\mathcal{K}^{n_-}(v,w)D_{n_-}b_\pm\right)\\ &= \mathcal{G}\left(a, D^\Sigma_{D^\Sigma_v w}b_\pm - \sum_{\sigma\in\{+,-\}} \varepsilon\mathcal{K}^{n_\sigma}(v,w)D_{n_\sigma}b_\pm\right).\end{aligned}$$

The result follows. □

Theorem 4.32 (Generalised Gauß equations)**.** *The pure-type part of the generalised Riemann tensors obtained from D and D^Σ are related by the following Gauß equation, where $a, b, v, w \in \Gamma(E^\pm_\Sigma)$:*

$$\begin{aligned}&\pm 2\varepsilon\left\{\mathcal{R}m^D(a,b,v,w) - \mathcal{R}m^{D^\Sigma}(a,b,v,w)\right\}\\ &= \mathcal{K}^{n_\pm}(w,a)\mathcal{K}^{n_\pm}(v,b) - \mathcal{K}^{n_\pm}(v,a)\mathcal{K}^{n_\pm}(w,b)\\ &\quad + \mathcal{K}^{n_\pm}(a,w)\mathcal{K}^{n_\pm}(b,v) - \mathcal{K}^{n_\pm}(b,w)\mathcal{K}^{n_\pm}(a,v)\\ &\quad + [\mathcal{K}^{n_\pm}(v,w) - \mathcal{K}^{n_\pm}(w,v)][\mathcal{K}^{n_\pm}(b,a) - \mathcal{K}^{n_\pm}(a,b)].\end{aligned}$$

The mixed-type part satisfies with $a, v, w \in \Gamma(E^\pm_\Sigma), \bar{b} \in \Gamma(E^\mp_\Sigma)$

$$\begin{aligned}&\pm 2\varepsilon\left\{\mathcal{R}m^D(a,\bar{b},v,w) - \mathcal{R}m^{D^\Sigma}(a,\bar{b},v,w)\right\}\\ &= \mathcal{K}^{n_\pm}(a,w)\mathcal{K}^{\pm}(\bar{b},v) - \mathcal{K}^{\pm}(\bar{b},w)\mathcal{K}^{n_\pm}(a,v)\\ &\quad + [\mathcal{K}^{n_\pm}(v,w) - \mathcal{K}^{n_\pm}(w,v)]\mathcal{K}^{\pm}(\bar{b},a).\end{aligned}$$

Proof. We begin by computing with the help of Lemma 4.31 for general sections $a, b, v, w \in \Gamma(E_\Sigma)$ (technically, we have to work with extensions of b and v to sections of E)

$$\begin{aligned}&2\mathcal{R}m^D(a,b,v,w)\\ &= \left\langle D^2_{v,w}b - D^2_{w,v}b, a\right\rangle + \left\langle D^2_{b,a}v - D^2_{a,b}v, w\right\rangle - \mathrm{tr}_E(\langle Dv, w\rangle\langle Db, a\rangle)\\ &= 2\mathcal{R}m^{D^\Sigma}(a,b,v,w)\\ &\quad + \varepsilon\sum_\sigma\Big\{-\sigma\mathcal{K}^{n_\sigma}(v,a)\mathcal{K}^{n_\sigma}(w,b) + \sigma\mathcal{K}^{n_\sigma}(w,a)\mathcal{K}^{n_\sigma}(v,b)\\ &\quad - \sigma\mathcal{K}^{n_\sigma}(b,w)\mathcal{K}^{n_\sigma}(a,v) + \sigma\mathcal{K}^{n_\sigma}(a,w)\mathcal{K}^{n_\sigma}(b,v) - \sigma\langle D_{n_\sigma}v, w\rangle\langle D_{n_\sigma}b, a\rangle\Big\}\\ &\quad + \sum_\sigma\left\{(\mathcal{K}^{n_\sigma}(v,w) - \mathcal{K}^{n_\sigma}(w,v))\langle D_{n_\sigma}b, a\rangle + (\mathcal{K}^{n_\sigma}(b,a) - \mathcal{K}^{n_\sigma}(a,b))\langle D_{n_\sigma}v, w\rangle\right\}.\end{aligned}$$

We restrict to the pure-type case, and then make use of Lemma 4.30 (ii):

$$
\begin{aligned}
&\pm 2\varepsilon \left\{ \mathcal{R}m^D(a,b,v,w) - \mathcal{R}m^{D^\Sigma}(a,b,v,w) \right\} \\
&= \mathcal{K}^{n_\pm}(w,a)\mathcal{K}^{n_\pm}(v,b) - \mathcal{K}^{n_\pm}(v,a)\mathcal{K}^{n_\pm}(w,b) + \mathcal{G}(D_{n_\pm}b,a)[\mathcal{K}^{n_\pm}(v,w) - \mathcal{K}^{n_\pm}(w,v)] \\
&\quad + \mathcal{K}^{n_\pm}(a,w)\mathcal{K}^{n_\pm}(b,v) - \mathcal{K}^{n_\pm}(b,w)\mathcal{K}^{n_\pm}(a,v) + \mathcal{G}(D_{n_\pm}v,w)[\mathcal{K}^{n_\pm}(b,a) - \mathcal{K}^{n_\pm}(a,b)] \\
&\quad - \mathcal{G}(D_{n_\pm}v,w)\mathcal{G}(D_{n_\pm}b,a) + \mathcal{G}(D_{n_\mp}v,w)\mathcal{G}(D_{n_\mp}b,a) \\
&= \mathcal{K}^{n_\pm}(w,a)\mathcal{K}^{n_\pm}(v,b) - \mathcal{K}^{n_\pm}(v,a)\mathcal{K}^{n_\pm}(w,b) \\
&\quad + \mathcal{K}^{n_\pm}(a,w)\mathcal{K}^{n_\pm}(b,v) - \mathcal{K}^{n_\pm}(b,w)\mathcal{K}^{n_\pm}(a,v) \\
&\quad + \mathcal{G}(D_{n_\pm - n_\mp}v,w)[\mathcal{K}^{n_\pm}(b,a) - \mathcal{K}^{n_\pm}(a,b)] \\
&= \mathcal{K}^{n_\pm}(w,a)\mathcal{K}^{n_\pm}(v,b) - \mathcal{K}^{n_\pm}(v,a)\mathcal{K}^{n_\pm}(w,b) \\
&\quad + \mathcal{K}^{n_\pm}(a,w)\mathcal{K}^{n_\pm}(b,v) - \mathcal{K}^{n_\pm}(b,w)\mathcal{K}^{n_\pm}(a,v) \\
&\quad + [\mathcal{K}^{n_\pm}(v,w) - \mathcal{K}^{n_\pm}(w,v)][\mathcal{K}^{n_\pm}(b,a) - \mathcal{K}^{n_\pm}(a,b)].
\end{aligned}
$$

Finally, restricting to the mixed-type case, we get with $a,v,w \in \Gamma(E^\pm_\Sigma)$ and $\bar{b} \in \Gamma(E^\mp_\Sigma)$

$$
\begin{aligned}
&\pm 2\varepsilon \left\{ \mathcal{R}m^D(a,\bar{b},v,w) - \mathcal{R}m^{D^\Sigma}(a,\bar{b},v,w) \right\} \\
&= \mathcal{K}^{n_\pm}(a,w)\mathcal{K}^{\pm}(\bar{b},v) - \mathcal{K}^{\pm}(\bar{b},w)\mathcal{K}^{n_\pm}(a,v) \\
&\quad \pm \mathcal{K}^{\pm}(\bar{b},a)\left\langle D_{n_\pm}v,w\right\rangle \mp \mathcal{K}^{\mp}(a,\bar{b})\left\langle D_{n_\mp}v,w\right\rangle .
\end{aligned}
$$

The result follows from symmetry of $\mathcal{K}^\pm + \mathcal{K}^\mp$ (see Corollary 4.24) and Lemma 4.30 (ii). □

To present the next result succinctly, we introduce the new operations $\{\cdot\}^{\sigma\text{-sym}}$ and $\{\cdot\}^{\sigma\text{-antisym}}$ of "σ-symmetrisation" and "σ-antisymmetrisation". They are defined for a pair $(T^\sigma)_{\sigma=\pm}$ of mixed-type tensors $T^\pm \in \Gamma(E^*_\mp \otimes E^*_\pm)$, as follows

$$
\begin{aligned}
2\{T^\sigma\}^{\sigma\text{-sym}}(\bar{a},b) &= 2[T^\pm + T^\mp]^{\text{sym}}(\bar{a},b) = T^\pm(\bar{a},b) + T^\mp(b,\bar{a}), \\
2\{T^\sigma\}^{\sigma\text{-antisym}}(\bar{a},b) &= 2[T^\pm + T^\mp]^{\text{antisym}}(\bar{a},b) = T^\pm(\bar{a},b) - T^\mp(b,\bar{a}).
\end{aligned}
$$

We use curly brackets in the context of σ-(anti)symmetrisation, and round or square bracket in the context of usual (anti)symmetrisation. Note that with σ-antisymmetrisation, one can express (4.7) as $\{\mathcal{K}^\sigma\}^{\sigma\text{-antisym}} = 0$.

Corollary 4.33. *The pure-type part of the generalised Ricci tensors on E and E_N satisfy the following Gauß equation, where $a,b \in \Gamma(E^\pm_\Sigma)$*

$$
\begin{aligned}
&\varepsilon \left\{ \overline{\mathcal{R}c}^D(a,b) - \overline{\mathcal{R}c}^{D^\Sigma}(a,b) \right\} \\
&= \pm \mathcal{R}m^D(n_\pm,a,n_\pm,b) + [(\mathcal{K}^{n_\pm})^2 - \mathcal{T}^\pm\mathcal{K}^{n_\pm}]^{\text{sym}}(a,b) + 2[(\mathcal{K}^{n_\pm})^{\text{antisym}}]^2(a,b).
\end{aligned}
$$

The mixed-type parts satisfy with $\bar{a} \in \Gamma(E^\mp_\Sigma), b \in \Gamma(E^\pm_\Sigma)$

$$
\begin{aligned}
&\varepsilon \left\{ \mathcal{R}c^\pm(\bar{a},b) - \mathcal{R}c^\pm_\Sigma(\bar{a},b) \right\} \\
&= \pm 2\{\mathcal{R}m^D(n_\sigma,\cdot,n_\sigma,\cdot)\}^{\sigma\text{-antisym}}(\bar{a},b) + \left\{ (\mathcal{K}^{n_\sigma})^2 - \mathcal{T}^\sigma\mathcal{K}^\sigma \right\}^{\sigma\text{-sym}}(\bar{a},b) \\
&\quad + 2\left\{ (\mathcal{K}^{n_\sigma})^{\text{antisym}}(\mathcal{A}^\sigma\cdot,\cdot) \right\}^{\sigma\text{-sym}}(\bar{a},b),
\end{aligned}
$$

where $(\mathcal{K}^{n_\pm})^2(\bar{a},b) := \mathcal{K}^{n_\pm}(\mathcal{A}^\pm\bar{a},b)$ and $\mathcal{A}^\pm$ is defined by $\mathcal{G}(\mathcal{A}^\pm\bar{a},b) = \mathcal{K}^\pm(\bar{a},b)$. The expression $(\mathcal{K}^{n_\sigma})^{\text{antisym}}$ stands for the antisymmetric part of the tensor $\mathcal{K}^{n_\sigma}$, $\sigma \in \{\pm\}$.

Proof. For the pure-type equation, we calculate from contraction of the Gauß equation for the generalised Riemann tensors, Theorem 4.32,

$$
\begin{aligned}
&2\varepsilon \left\{ \overline{\mathcal{R}c}^{D}(a,b) - \overline{\mathcal{R}c}^{D^{\Sigma}}(a,b) \right\} \mp 2\mathcal{R}m^{D}(n_\pm, a, n_\pm, b) \\
&= \mathcal{K}^{n_\pm}(\mathcal{A}^{n_\pm}b, a) - \mathcal{K}^{n_\pm}(b,a)\mathcal{T}^{\pm} + \mathcal{K}^{n_\pm}(\mathcal{A}^{n_\pm}a, b) - \mathcal{T}^{\pm}\mathcal{K}^{n_\pm}(a,b) \\
&\quad + \sum_i \varepsilon_i [\mathcal{K}^{n_\pm}(b, e_i^\pm) - \mathcal{K}^{n_\pm}(e_i^\pm, b)][\mathcal{K}^{n_\pm}(e_i^\pm, a) - \mathcal{K}^{n_\pm}(a, e_i^\pm)] \\
&= 2[(\mathcal{K}^{n_\pm})^2 - \mathcal{T}^{\pm}\mathcal{K}^{n_\pm}]^{\mathrm{sym}}(a,b) + 4[(\mathcal{K}^{n_\pm})^{\mathrm{antisym}}]^2(a,b)
\end{aligned}
$$

where $\varepsilon_i = \pm\langle e_i^\pm, e_i^\pm \rangle$. The statement regarding the mixed-type parts follows from a similar calculation employing Theorem 4.32:

$$
\begin{aligned}
&2\varepsilon \left\{ \mathcal{R}c^{\pm}(\bar{a},b) - \mathcal{R}c^{\pm}_{\Sigma}(\bar{a},b) \right\} \\
&= \pm 2\varepsilon \left\{ \sum_i \varepsilon_i \mathcal{R}m^{D}(e_i^\pm, \bar{a}, e_i^\pm, b) - \sum_i \varepsilon_i \mathcal{R}m^{D^{\Sigma}}(e_i^\pm, \bar{a}, e_i^\pm, b) + \varepsilon \mathcal{R}m^{D}(n_\pm, \bar{a}, n_\pm, b) \right\} \\
&\quad \mp 2\varepsilon \left\{ \sum_i \varepsilon_i \mathcal{R}m^{D}(e_i^\mp, \bar{a}, e_i^\mp, b) - \sum_i \varepsilon_i \mathcal{R}m^{D^{\Sigma}}(e_i^\mp, \bar{a}, e_i^\mp, b) + \varepsilon \mathcal{R}m^{D}(n_\mp, \bar{a}, n_\mp, b) \right\} \\
&= \pm 2\mathcal{R}m^{D}(n_\pm, \bar{a}, n_\pm, b) + 2(\mathcal{K}^{n_\pm})^2(\bar{a},b) - \mathcal{T}^{\pm}\mathcal{K}^{\pm}(\bar{a},b) - \mathcal{K}^{n_\pm}(b, \mathcal{A}^{\pm}\bar{a}) \\
&\quad \mp 2\mathcal{R}m^{D}(n_\mp, \bar{a}, n_\mp, b) + 2(\mathcal{K}^{n_\mp})^2(b,\bar{a}) - \mathcal{T}^{\mp}\mathcal{K}^{\mp}(b,\bar{a}) - \mathcal{K}^{n_\mp}(\bar{a}, \mathcal{A}^{\mp}b).
\end{aligned} \tag{4.9}
$$

This concludes the proof. □

Remark 4.34. If we assume the pair $(\mathcal{G}, \mathrm{div})$ to be compatible, we can simplify the result of Corollary 4.33 for the mixed-type case. By Corollary 3.12, it holds

$$
\mathcal{R}c^{\pm}(\bar{a},b) = 2\,\mathrm{tr}_{E_\pm}\,\mathcal{R}m^{D}(\cdot, \bar{a}, \cdot, b)
$$

so that

$$
\begin{aligned}
&\varepsilon \left\{ \mathcal{R}c^{\pm}(\bar{a},b) - \mathcal{R}c^{\pm}_{\Sigma}(\bar{a},b) \right\} \\
&\quad = \pm 2\mathcal{R}m^{D}(n_\pm, \bar{a}, n_\pm, b)] + \left[(\mathcal{K}^{n_\pm})^2 - \mathcal{T}^{\pm}\mathcal{K}^{\pm} \right](\bar{a},b) + 2(\mathcal{K}^{n_\pm})^{\mathrm{antisym}}(\mathcal{A}^{\pm}\bar{a}, b).
\end{aligned}
$$

For the next Proposition, we need the following

Lemma 4.35. *Let $D \in \mathcal{D}^0(\mathcal{G}, \mathrm{div}; \Sigma)$. Then*

$$
\begin{aligned}
&\overline{\mathcal{R}c}^{D}(n_+, n_+) + \overline{\mathcal{R}c}^{D}(n_-, n_-) - \overline{\mathcal{R}c}^{D}(n_+, n_-) - \overline{\mathcal{R}c}^{D}(n_-, n_+) \\
&= \sum_{\pm} \left\{ -\mathrm{tr}_{\mathcal{H}}(\mathcal{K}^{n_\pm})^2 + \frac{1}{2} \left| \mathcal{K}^{n_\pm}|_{E_\pm \oplus E_\pm} \right|^2_{\mathcal{H}} + \frac{1}{2} |\mathcal{K}^{\pm}|^2_{\mathcal{H}} \right\}.
\end{aligned}
$$

Proof. For this calculation, we assume $\nabla_n n = 0$. As $D \in \mathcal{D}^0(\mathcal{G}, \mathrm{div}; \Sigma)$, this implies $D_{n_\mp} n_\pm = 0$ by the proof of Lemma 4.21, and then $D_{n_\pm} n_\pm = 0$ by the statement of

Lemma 4.21.

$$
\begin{aligned}
&\overline{\mathcal{R}c}^D(n_+,n_+)+\overline{\mathcal{R}c}^D(n_-,n_-)-\overline{\mathcal{R}c}^D(n_+,n_-)-\overline{\mathcal{R}c}^D(n_-,n_+)\\
&=\sum_{i=1}^n\varepsilon_i\sum_\pm\pm\{\mathcal{R}m^D(e_i^\pm,n_+,e_i^\pm,n_+)+\mathcal{R}m^D(e_i^\pm,n_-,e_i^\pm,n_-)\\
&\qquad-2\mathcal{R}m^D(e_i^\pm,n_+,e_i^\pm,n_-)\}\\
&=\sum_{i=1}^n\varepsilon_i\sum_\pm\pm\frac{1}{2}\left\{2\left\langle D_{e_i^\pm}D_{n_\pm}n_\pm-D_{n_\pm}D_{e_i^\pm}n_\pm-D_{D_{e_i^\pm}n_\pm}n_\pm+D_{D_{n_\pm}e_i^\pm}n_\pm,e_i^\pm\right\rangle\right.\\
&\qquad-\operatorname{tr}_E\left(\left\langle De_i^\pm,n_\pm\right\rangle\left\langle Dn_\pm,e_i^\pm\right\rangle\right)\\
&\qquad-2\left\langle D_{e_i^\pm}D_{n_\mp}n_\pm-D_{n_\mp}D_{e_i^\pm}n_\pm-D_{D_{e_i^\pm}n_\mp}n_\pm+D_{D_{n_\mp}e_i^\pm}n_\pm,e_i^\pm\right\rangle\\
&\qquad\left.+2\operatorname{tr}_E\left(\left\langle De_i^\pm,n_-\right\rangle\left\langle Dn_+,e_i^\pm\right\rangle\right)\right\}\\
&\overset{*}{=}\sum_{i=1}^n\varepsilon_i\sum_\pm\pm\left\{\left\langle -D_{n_\pm}D_{e_i^\pm}n_\pm-D_{D_{e_i^\pm}n_\pm}n_\pm+D_{D_{n_\pm}e_i^\pm}n_\pm,e_i^\pm\right\rangle\right.\\
&\qquad-\frac{1}{2}\operatorname{tr}_E\left(\left\langle De_i^\pm,n_\pm\right\rangle\left\langle Dn_\pm,e_i^\pm\right\rangle\right)\\
&\qquad\left.-\left\langle -D_{n_\mp}D_{e_i^\pm}n_\pm-D_{D_{e_i^\pm}n_\mp}n_\pm+D_{D_{n_\mp}e_i^\pm}n_\pm,e_i^\pm\right\rangle\right\}\\
&=\sum_{i=1}^n\varepsilon_i\sum_\pm\pm\left\{\mp n(\mathcal{K}^{n_\pm}(e_i^\pm,e_i^\pm))\pm\mathcal{K}^{n_\pm}(e_i^\pm,D_{n_\pm}e_i^\pm)\mp\mathcal{K}^{n_\pm}(\mathcal{A}^{n_\pm}(e_i^\pm),e_i^\pm)\right.\\
&\qquad\pm\mathcal{K}^{n_\pm}(D_{n_\pm}e_i^\pm,e_i^\pm)+\frac{1}{2}\operatorname{tr}_E\left(\mathcal{K}^{n_\pm}(\cdot,e_i^\pm)\mathcal{K}^{n_\pm}(\cdot,e_i^\pm)\right)\\
&\qquad\left.\pm n(\mathcal{K}^{n_\pm}(e_i^\pm,e_i^\pm))\mp\mathcal{K}^{n_\pm}(e_i^\pm,D_{n_\mp}e_i^\pm)\pm\mathcal{K}^{n_\pm}(\mathcal{A}^{n_\mp}(e_i^\pm),e_i^\pm)\pm\mathcal{K}^{n_\pm}(D_{n_\mp}e_i^\pm,e_i^\pm)\right\}\\
&\overset{\#}{=}\sum_\pm\left\{-\operatorname{tr}_{\mathcal{H}}(\mathcal{K}^{n_\pm})^2+\frac{1}{2}\left|\mathcal{K}^{n_\pm}\right|_E^2+\operatorname{tr}(\mathcal{A}^{n_\pm}\mathcal{A}^{n_\mp})\right\}\\
&=\sum_\pm\left\{-\operatorname{tr}_{\mathcal{H}}(\mathcal{K}^{n_\pm})^2+\frac{1}{2}\left|\mathcal{K}^{n_\pm}|_{E_\pm\oplus E_\pm}\right|_{\mathcal{H}}^2+\frac{1}{2}\left|\mathcal{K}^\pm\right|_{\mathcal{H}}^2\right\}.
\end{aligned}
$$

In $*$, we used that in the preceding line both terms vanish, and that $D_{n_\pm}n_\pm=D_{n_\mp}n_\pm=0$. In $\#$, we used that the first terms in the first and third line cancel, and that

$$\sum_i\varepsilon_i\left\{\mathcal{K}^{n_\pm}(D_{n_\pm-n_\mp}e_i^\pm,e_i^\pm)+\mathcal{K}^{n_\pm}(e_i^\pm,D_{n_\pm-n_\mp}e_i^\pm)\right\}=\pi(n_\pm-n_\mp)\mathcal{T}^\pm=0.$$

□

Corollary 4.36. *The generalised scalar curvatures on E and E_Σ are related by the Gauß equation*

$$2\mathcal{R}c^\pm(n_\mp,n_\pm)-\varepsilon\mathcal{S}c=-\varepsilon\mathcal{S}c_\Sigma-\left|\mathcal{K}^\pm\right|^2+\frac{(\mathcal{T}^+)^2+(\mathcal{T}^-)^2}{2}.$$

Herein, $\varepsilon=g(n,n)$ where n is the unit normal on Σ and $\mathcal{G}\cong(g,F)$.

Proof. This is obtained by taking the trace of the pure-type generalised Gauß equation, cf.

Corollary 4.33:

$$\begin{aligned}
&4\varepsilon\left\{\mathcal{S}c - \mathcal{S}c_\Sigma\right\}\\
&= \sum_\pm \left\{4\overline{\mathcal{R}c}^D(n_\pm, n_\pm) - 2(\mathcal{T}^\pm)^2 + 4\,\mathrm{tr}_\mathcal{H}(\mathcal{K}^{n_\pm})^2 - 2\left|\mathcal{K}^{n_\pm}|_{E_\pm\oplus E_\pm}\right|^2_\mathcal{H}\right\}\\
&= 8\mathcal{R}c^\pm(n_\mp, n_\pm) - 2(\mathcal{T}^+)^2 - 2(\mathcal{T}^-)^2 + 2\sum_\pm |\mathcal{K}^\pm|^2_\mathcal{H}\,.
\end{aligned}$$

For the last equality, we assumed $D \in \mathcal{D}^0(\mathcal{G}, \mathrm{div}; \Sigma)$ and employed Lemma 4.35. □

The above has, in the spirit of the ADM formulation of general relativity, the interpretation of a generalised energy constraint, if Σ is a spacelike hypersurface in the base manifold of a generalised Lorentzian CA. In order to compute the generalised momentum constraint, we derive the generalised Codazzi equations.

Theorem 4.37 (Generalised Codazzi equations)**.** *The normal components of the generalised Riemann curvature satisfy the Codazzi equations*

$$\pm 2\mathcal{R}m^D(a, \bar b, n_\pm, w) = [D^\Sigma_{\bar b}\mathcal{K}^{n_\pm}](a, w) - [D^\Sigma_a\mathcal{K}^\pm](\bar b, w) + \varepsilon\mathcal{K}^\pm(\bar b, a)\mathcal{L}^\pm(w)$$

and

$$\begin{aligned}
&\pm 2\mathcal{R}m^D(a, b, n_\pm - n_\mp, w)\\
&= [D^\Sigma_w\mathcal{K}^{n_\pm}](a, b) - [D^\Sigma_w\mathcal{K}^{n_\pm}](b, a) + [D^\Sigma_b\mathcal{K}^{n_\pm}](a, w) - [D^\Sigma_a\mathcal{K}^{n_\pm}](b, w)\\
&\quad + \varepsilon\left\{\mathcal{L}^\pm(w)[\mathcal{K}^{n_\pm}(b, a) - \mathcal{K}^{n_\pm}(a, b)] + \mathcal{L}^\pm(b)\mathcal{K}^{n_\pm}(w, a) - \mathcal{L}^\pm(a)\mathcal{K}^{n_\pm}(w, b)\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\pm 2\mathcal{R}m^D(a, n_\pm - n_\mp, n_\pm - n_\mp, b)\\
&= 2[(\mathcal{K}^{n_\pm})^2]^{\mathrm{sym}}(a, b) - 2\mathcal{K}^{n_\pm}(a, \mathcal{A}^{n_\pm}b) - \mathcal{K}^\pm(\mathcal{A}^\mp a, b)\\
&\quad + \mathrm{tr}_\mathcal{H}\left[\mathcal{K}^{n_\pm}(\cdot, a)\mathcal{K}^{n_\pm}(\cdot, b)\right]_{E_\pm\times E_\pm} - [D^\Sigma_a\mathcal{L}^\pm](b) - [D^\Sigma_b\mathcal{L}^\pm](a)
\end{aligned}$$

and

$$\begin{aligned}
&\pm 2\mathcal{R}m^D(a, n_\pm - n_\mp, n_\pm - n_\mp, \bar b)\\
&= 4\{(\mathcal{K}^{n_\sigma})^2\}^{\sigma\text{-antisym}}(\bar b, a) - 2\{\mathcal{K}^{n_\sigma}(\cdot, \mathcal{A}^\sigma\cdot)\}^{\sigma\text{-antisym}} + [D^\Sigma_a\mathcal{L}^\mp](\bar b) - [D^\Sigma_{\bar b}\mathcal{L}^\pm](a).
\end{aligned}$$

Herein, $a, b, w \in \Gamma(E^\pm_\Sigma)$ and $\bar b \in \Gamma(E^\mp_\Sigma)$.

Proof. The result follows from straightforward calculation. We check that

$$\begin{aligned}
&\pm 2\mathcal{R}m^D(a, \bar b, n_\pm, w)\\
&= \mathcal{G}(D_{\bar b}D_a n_\pm - D_a D_{\bar b}n_\pm - D_{D_{\bar b}a}n_\pm + D_{D_a\bar b}n_\pm, w)\\
&= +\mathcal{G}\left(D^\Sigma_{\bar b}[\mathcal{A}^{n_\pm}a] - D^\Sigma_a[\mathcal{A}^\pm\bar b], w\right) - \mathcal{K}^{n_\pm}(D^\Sigma_{\bar b}a, w) + \mathcal{K}^\pm(D^\Sigma_a\bar b, w)\\
&\quad - \varepsilon\mathcal{G}(D_{\bar b}a, n_\pm)\mathcal{G}(D_{n_\pm}n_\pm, w) + \varepsilon\mathcal{G}(D_a\bar b, n_\mp)\mathcal{G}(D_{n_\mp}n_\pm, w)\\
&= [D^\Sigma_{\bar b}\mathcal{K}^{n_\pm}](a, w) - [D^\Sigma_a\mathcal{K}^\pm](\bar b, w) + \varepsilon\mathcal{K}^\pm(\bar b, a)\mathcal{L}^\pm(w),
\end{aligned}$$

where we used the symmetry of $\mathcal{L}^+ + \mathcal{L}^-$, see Corollary 4.24. In the following calculations, we assume for simplicity $\nabla_n n = 0$ and hence $D_{n_\mp}n_\pm = [n_\mp, n_\pm] = 0$. With the help of

Lemma 4.30 we compute the following:

$$\begin{aligned}
&\pm 2\mathcal{R}m^D(a,b,n_\pm - n_\mp, w)\\
&= \mathcal{G}(D_{n_\pm - n_\mp} D_w b - D_w D_{n_\pm - n_\mp} b - D_{D_{n_\pm - n_\mp} w} b + D_{D_w(n_\pm - n_\mp)} b, a)\\
&\quad + \mathcal{G}\left(D_b D_a n_\pm - D_a D_b n_\pm - D_{D_b a} n_\pm + D_{D_a b} n_\pm, w\right)\\
&\quad \mp \operatorname{tr}_E(\mathcal{G}(Dn_\pm, w)\mathcal{G}(Db, a))\\
&= [D^\Sigma_w \mathcal{K}^{n_\pm}](a,b) - [D^\Sigma_w \mathcal{K}^{n_\pm}](b,a) \pm \mathcal{G}(D^\Sigma_{\mathcal{G}(D_{(\cdot)} n_\pm, w)} b, a)\\
&\quad + \varepsilon\left[\mathcal{L}^\pm(b)\mathcal{K}^{n_\pm}(w,a) - \mathcal{L}^\pm(a)\mathcal{K}^{n_\pm}(w,b)\right]\\
&\quad + [D^\Sigma_b \mathcal{K}^{n_\pm}](a,w) - [D^\Sigma_a \mathcal{K}^{n_\pm}](b,w) + \varepsilon\, \mathcal{L}^\pm(w)[\mathcal{K}^{n_\pm}(b,a) - \mathcal{K}^{n_\pm}(a,b)]\\
&\quad \mp \operatorname{tr}_E(\mathcal{G}(Dn_\pm, w)\mathcal{G}(Db, a))\\
&= [D^\Sigma_w \mathcal{K}^{n_\pm}](a,b) - [D^\Sigma_w \mathcal{K}^{n_\pm}](b,a) + [D^\Sigma_b \mathcal{K}^{n_\pm}](a,w) - [D^\Sigma_a \mathcal{K}^{n_\pm}](b,w)\\
&\quad + \varepsilon\left\{\mathcal{L}^\pm(w)[\mathcal{K}^{n_\pm}(b,a) - \mathcal{K}^{n_\pm}(a,b)] + \mathcal{L}^\pm(b)\mathcal{K}^{n_\pm}(w,a) - \mathcal{L}^\pm(a)\mathcal{K}^{n_\pm}(w,b)\right\}.
\end{aligned}$$

Herein, $\mathcal{G}(D_{(\cdot)} n_\pm, w) \in \Gamma(E^*_\Sigma)$ is identified with an element of $\Gamma(E_\Sigma)$ by means of the scalar product $\langle \cdot, \cdot \rangle$. For the next Codazzi equation, one has

$$\begin{aligned}
&\pm 2\mathcal{R}m^D(a, n_\pm - n_\mp, n_\pm - n_\mp, b)\\
&= \mathcal{G}\left(D_{n_\pm - n_\mp} D_b n_\pm - D_b D_{n_\pm - n_\mp} n_\pm - D_{D_{n_\pm - n_\mp} b} n_\pm + D_{D_b(n_\pm - n_\mp)} n_\pm, a\right)\\
&\quad + \mathcal{G}\left(D_{n_\pm - n_\mp} D_a n_\pm - D_a D_{n_\pm - n_\mp} n_\pm - D_{D_{n_\pm - n_\mp} a} n_\pm + D_{D_a(n_\pm - n_\mp)} n_\pm, b\right)\\
&\quad \mp \operatorname{tr}_E(\mathcal{G}(Dn_\pm, b)\mathcal{G}(Dn_\pm, a)).
\end{aligned}$$

We calculate

$$\begin{aligned}
&\mathcal{G}\left(D_{n_\pm - n_\mp} D_b n_\pm - D_b D_{n_\pm - n_\mp} n_\pm - D_{D_{n_\pm - n_\mp} b} n_\pm + D_{D_b(n_\pm - n_\mp)} n_\pm, a\right)\\
&\quad = \mathcal{K}^{n_\pm}(\mathcal{A}^{n_\pm} b, a) - \mathcal{K}^{n_\pm}(a, \mathcal{A}^{n_\pm} b) - [D^\Sigma_b \mathcal{L}](a) \pm \operatorname{tr}_E \mathcal{G}(Dn_\pm, b)\mathcal{G}(Dn_\pm, a).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\pm 2\mathcal{R}m^D(a, n_\pm - n_\mp, n_\pm - n_\mp, b)\\
&= \mathcal{K}^{n_\pm}(\mathcal{A}^{n_\pm} a, b) + \mathcal{K}^{n_\pm}(\mathcal{A}^{n_\pm} b, a) - \mathcal{K}^{n_\pm}(a, \mathcal{A}^{n_\pm} b) - \mathcal{K}^{n_\pm}(b, \mathcal{A}^{n_\pm} a)\\
&\quad - [D^\Sigma_a \mathcal{L}^\pm](b) - [D^\Sigma_b \mathcal{L}^\pm](a) \pm \operatorname{tr}_E \mathcal{K}^{n_\pm}(\cdot, a)\mathcal{K}^{n_\pm}(\cdot, b)\\
&= 2[(\mathcal{K}^{n_\pm})^2]^{\mathrm{sym}}(a,b) - 2\mathcal{K}^{n_\pm}(a, \mathcal{A}^{n_\pm} b)\\
&\quad - [D^\Sigma_a \mathcal{L}^\pm](b) - [D^\Sigma_b \mathcal{L}^\pm](a) \pm \left[\operatorname{tr}_{E_\pm} \mathcal{K}^{n_\pm}(\cdot, a)\mathcal{K}^{n_\pm}(\cdot, b) + \operatorname{tr}_{E_\mp} \mathcal{K}^{n_\pm}(\cdot, a)\mathcal{K}^{n_\pm}(\cdot, b)\right],
\end{aligned}$$

where, using Corollary 4.24,

$$\pm \operatorname{tr}_{E_\mp} \mathcal{K}^{n_\pm}(\cdot, a)\mathcal{K}^{n_\pm}(\cdot, b) = -\operatorname{tr}_{\mathcal{H}} \mathcal{K}^\pm(\cdot, a)\mathcal{K}^\pm(\cdot, b) = -\operatorname{tr}_{\mathcal{H}} \mathcal{K}^\mp(a, \cdot)\mathcal{K}^\pm(\cdot, b).$$

Analogously, we obtain the last Codazzi equation by calculating

$$\begin{aligned}
&\pm 2\mathcal{R}m^D(a, n_\pm - n_\mp, n_\pm - n_\mp, \bar{b})\\
&= \mathcal{G}\left(D_{n_\pm - n_\mp} D_{\bar{b}} n_\pm - D_{\bar{b}} D_{n_\pm - n_\mp} n_\pm - D_{D_{n_\pm - n_\mp} \bar{b}} n_\pm + D_{D_{\bar{b}}(n_\pm - n_\mp)} n_\pm, a\right)\\
&\quad + \mathcal{G}\left(D_{n_\pm - n_\mp} D_a n_\mp - D_a D_{n_\pm - n_\mp} n_\mp - D_{D_{n_\pm - n_\mp} a} n_\mp + D_{D_a(n_\pm - n_\mp)} n_\mp, \bar{b}\right)\\
&\quad \mp \operatorname{tr}_E(\mathcal{G}(Dn_\mp, \bar{b})\mathcal{G}(Dn_\pm, a)).
\end{aligned}$$

We note that

$$\begin{aligned}&\mathcal{G}\left(D_{n_\pm - n_\mp} D_{\bar b} n_\pm - D_{\bar b} D_{n_\pm - n_\mp} n_\pm - D_{D_{n_\pm - n_\mp}\bar b} n_\pm + D_{D_{\bar b}(n_\pm - n_\mp)} n_\pm, a\right)\\ &= \mathcal{K}^{n_\pm}(\mathcal{A}^\pm \bar b, a) - \mathcal{K}^{n_\pm}(a, \mathcal{A}^\pm \bar b) - [D^\Sigma_{\bar b}\mathcal{L}^\pm](a) \pm \operatorname{tr}_E \mathcal{G}(Dn_\mp, \bar b)\mathcal{G}(Dn_\pm, a)\end{aligned}$$

(cf. Lemma 4.30) and obtain

$$\begin{aligned}&\pm 2\mathcal{R}m^D(a, n_\pm - n_\mp, n_\pm - n_\mp, \bar b)\\ &= \mathcal{K}^{n_\pm}(\mathcal{A}^\pm \bar b, a) - \mathcal{K}^{n_\pm}(a, \mathcal{A}^\pm \bar b) - \mathcal{K}^{n_\mp}(\mathcal{A}^\mp a, \bar b) + \mathcal{K}^{n_\mp}(\bar b, \mathcal{A}^\mp a)\\ &\quad + [D^\Sigma_a \mathcal{L}^\mp](\bar b) - [D^\Sigma_{\bar b}\mathcal{L}^\pm](a) \pm \operatorname{tr}_E \mathcal{K}^{n_\pm}(\cdot, a)\mathcal{K}^{n_\mp}(\cdot, \bar b)\\ &= 4\{(\mathcal{K}^{n_\sigma})^2\}^{\sigma\text{-antisym}}(\bar b, a) - 2\{\mathcal{K}^{n_\sigma}(\cdot, \mathcal{A}^\sigma \cdot)\}^{\sigma\text{-antisym}} + [D^\Sigma_a\mathcal{L}^\mp](\bar b) - [D^\Sigma_{\bar b}\mathcal{L}^\pm](a),\end{aligned}$$

where we used that

$$\begin{aligned}&\pm \operatorname{tr}_E \mathcal{K}^{n_\pm}(\cdot, a)\mathcal{K}^{n_\mp}(\cdot, \bar b)\\ &= \pm\left[\operatorname{tr}_{E_\pm}\mathcal{K}^{n_\pm}(\cdot, a)\mathcal{K}^{n_\mp}(\cdot, \bar b) + \operatorname{tr}_{E_\mp}\mathcal{K}^{n_\pm}(\cdot, a)\mathcal{K}^{n_\mp}(\cdot, \bar b)\right]\\ &= \mathcal{K}^{n_\pm}(\mathcal{A}^\pm \bar b, a) - \mathcal{K}^{n_\mp}(\mathcal{A}^\mp a, \bar b).\end{aligned}$$

This proves the claim. □

As a Corollary, we obtain the generalised momentum constraint.

Corollary 4.38. *The mixed components of the generalised Ricci tensor on E are given as follows:*

$$\mathcal{R}c^\pm(a_\mp, n_\pm) = (\operatorname{div}_\Sigma \mathcal{A}^\pm)(a_\mp) - \pi a_\mp(\mathcal{T}^\pm).$$

Herein, $a_\mp \in \Gamma(E^\mp_\Sigma)$ arbitrary, $\operatorname{div}_\Sigma \mathcal{A}^\pm$ is the divergence of the mixed-type tensor $\mathcal{A}^\pm$ with respect to $(\mathcal{G}, \operatorname{div})$, as in Lemma C.2.

Proof. Take the trace over a and w in the mixed-type equation of Theorem 4.37, choosing $D \in \mathcal{D}^0(\mathcal{G}, \operatorname{div}; \Sigma)$ so that $\operatorname{div}_\Sigma = \operatorname{div}_{D^\Sigma}$. Note that for such D the conormal extrinsic curvature $\mathcal{L}^\pm$ vanishes. □

Translating this equation into the non-generalised language, one can employ the following Lemma to re-express the constraint equations for the generalised Einstein tensor $\mathcal{R}c^+ + \mathcal{R}c^- - \frac{1}{2}\mathcal{S}c \cdot \mathcal{G}$ in terms of ordinary (rather than "generalised") geometric objects on Σ.

Lemma 4.39. *It holds*

$$(\operatorname{div}_\Sigma \mathcal{A}^\pm)(a_\mp) = \operatorname{div}^{e^\pm}\left(k \mp \frac{\iota_n H}{2}\right)(\pi a_\mp) - \frac{1}{4}H^2(n, \pi a_\mp)$$

where $a_\mp \in \Gamma(E^\mp_\Sigma)$ is arbitrary, and $\operatorname{div}^{e^\pm} : \Gamma(T\Sigma) \overset{h}{\cong} \Gamma(T^*\Sigma) \to C^\infty(\Sigma)$ is defined by $\operatorname{div}^{e^\pm}(X) = \operatorname{div}_h(X) \mp g(X, \pi e^\pm)$.

Proof. We denote by $D^\Sigma \in \mathcal{D}^0(\mathcal{H}, \operatorname{div}_\Sigma)$ the generalised LC connection from Theorem 3.2. We furthermore denote by $\nabla^{\Sigma\pm}$ the connections on Σ with torsion $\pm\iota^* H$, cf. (2.13). We choose an orthonormal frame e_i for $T\Sigma$ and put $\varepsilon_i = g(e_i, e_i)$. Then, we calculate with Lemma C.2

$$
\begin{aligned}
&(\operatorname{div}_\Sigma \mathcal{A}^\pm)(a_\mp)\\
&= (\operatorname{div}^{\mathcal{H},\pm} \mathcal{A}^\pm)(a_\mp) - \langle e, \mathcal{A}^\pm \rangle (a_\mp)\\
&= \sum_i \varepsilon_i \left(\nabla^{\Sigma\mp}_{e_i} \left[k \mp \frac{\iota_n H}{2} \right] \right) (\pi a_\mp, e_i) \mp \left[k \mp \frac{\iota_n H}{2} \right] (\pi a_\mp, \pi e^\pm)\\
&= \operatorname{div}^{e^\pm} \left[k \mp \frac{\iota_n H}{2} \right] (\pi a_\mp) \pm \frac{1}{2} \sum_i \varepsilon_i \left[k \mp \frac{\iota_n H}{2} \right] (H(e_i, \pi a_\mp), e_i)\\
&= \operatorname{div}^{e^\pm} \left[k \mp \frac{\iota_n H}{2} \right] (\pi a_\mp) - \frac{1}{4} H^2(n, \pi a_\mp).
\end{aligned}
$$

□

Corollary 4.40. *Assume that on the ambient exact CA $E \to M$, the generalised Einstein equations are satisfied, i.e. $\mathcal{R}c = 0$ and $\mathcal{S}c = 0$. If $e = 2\xi$, $\mathrm{d}\xi = 0$, the constraint equations for the generalised Einstein equations are given by*

$$
\begin{aligned}
-\varepsilon \mathrm{Sc}_\Sigma + (\operatorname{tr} k)^2 - |k|^2 &= -\varepsilon \frac{\left|H^\parallel\right|^2}{12} + \frac{\left|H^\perp\right|^2}{4} - 2\varepsilon (\mathrm{d}^\Sigma)^* \xi^\parallel + 2(\operatorname{tr} k) x - \varepsilon \left|\xi^\parallel\right|^2 - x^2,\\
\operatorname{div}_h k - \mathrm{d}^\Sigma \operatorname{tr} k &= \frac{1}{4} \left\langle H^\perp, H^\parallel \right\rangle - \mathrm{d}^\Sigma x + i_{\xi^\parallel} k,\\
0 &= \left((\mathrm{d}^\Sigma)^* + i_{\xi^\parallel} \right) H^\perp.
\end{aligned}
$$

Herein, we decomposed on Σ

$$
H = H^\parallel + \varepsilon\, n^\flat \wedge H^\perp, \qquad \xi = \xi^\parallel + \varepsilon\, x\, n^\flat
$$

and denoted by $\left\langle H^\perp, H^\parallel \right\rangle$ the one-form obtained from contracting $H^\perp$ with $H^\parallel$ (matching corresponding entries) by means of the induced metric h. The first equation is equivalent to the generalised energy constraint, and the other two are equivalent to the generalised momentum constraint. (Note that the case of a space-like hypersurface $(\Sigma, h = g|_\Sigma)$ in a Lorentzian manifold (M, g) corresponds to $\varepsilon = -1$.)

Proof. Recall from Corollary 4.36 that the generalised energy constraint is given by

$$
0 = -\varepsilon \mathcal{S}c_\Sigma - |\mathcal{K}^\pm|^2 + \frac{(\mathcal{T}^+)^2 + (\mathcal{T}^-)^2}{2}.
$$

The generalised scalar curvature for $\mathcal{S}c_\Sigma$ is given by, cf. (3.8),

$$
\mathcal{S}c_\Sigma = \mathrm{Sc}_\Sigma - \frac{\left|H^\parallel\right|^2}{12} - 2(\mathrm{d}^\Sigma)^* \xi^\parallel - \left|\xi^\parallel\right|^2,
$$

where we have used that

$$
\frac{1}{2} \left|2\xi^\parallel\right|^2_{\mathcal{H}} = 2\mathcal{H}(\xi^\parallel, \xi^\parallel) = 2 \left\langle \mathcal{H}\xi^\parallel, \xi^\parallel \right\rangle = h^{-1}(\xi^\parallel, \xi^\parallel) = \left|\xi^\parallel\right|^2_h.
$$

Furthermore, by Lemma 4.23 and Corollary 4.27, we have

$$
|\mathcal{K}^\pm|^2_{\mathcal{H}} = |k|^2 + \frac{\left|H^\perp\right|^2}{4}, \qquad (\mathcal{T}^\pm)^2 = (\operatorname{tr} k - x)^2.
$$

Equivalence of the first equation to the generalised energy constraint follows from substitution.

Now recall the generalised momentum constraint

$$0 = (\operatorname{div}_\Sigma \mathcal{A}^\pm)(a_\mp) - \pi a_\mp(\mathcal{T}^\pm).$$

By Lemma 4.39, we have

$$(\operatorname{div}_\Sigma \mathcal{A}^\pm)(a_\mp) = (\operatorname{div}_h - i_{\xi^\parallel})\left(k \mp \frac{H^\perp}{2}\right)(\pi a_\mp) - \frac{1}{4}\left\langle H^\perp, H^\parallel\right\rangle(\pi a_\mp),$$

where we have used that $\pi(e_\pm) = \pm h^{-1}\xi^\parallel$ and $i_{\xi^\parallel} := i_{h^{-1}\xi^\parallel}$. The result follows from considering the sum and the difference of the cases "+" and "−". □

4.5 The Fundamental Theorem for Generalised Hypersurfaces

This section reproduces [17, chapter 7].

The fundamental theorem for hypersurfaces is a classical result [34, Chapter VII, Theorem 7.1] establishing that every tuple (Σ, h, k) consisting of a Riemannian manifold (Σ, h) and a symmetric two-tensor k satisfying the flat Gauß and Codazzi equations is locally equivalent to a hypersurface in flat space equipped with the induced metric and exterior curvature tensor. In this section, we establish the analogous result for the generalised case.

The idea in our proof is to make use of the result that generalised Riemann flatness implies complete triviality (Corollary 3.17) to reduce to the classical situation, where we already have the theorem. The problem is that we do not have access to the generalised Riemann tensor of an ambient space to which we could apply Corollary 3.17. Our solution to this is inspired by the classical case.

One approach in the classical case (cf., though in slightly different settings, the proofs of [35, Theorem 8.1] or [36, Theorem 2.3]) is to define a synthetical version $\tilde{T}$ of the tangent space to the ambient manifold by defining the "normal bundle" $\nu = \Sigma \times \mathbb{R}$ with generator n and setting $\tilde{T} = T\Sigma \oplus \nu$. One then defines a connection $\tilde{\nabla}$ on $\tilde{T}$ by setting $\tilde{\nabla}_X Y = \nabla^\Sigma_X Y - k(X,Y)n$ and $\tilde{\nabla}_X n = AX$, where $k = hA$. One finds that $\tilde{\nabla}$ is flat and hence that $(\tilde{T}, \tilde{\nabla})$ can be locally identified with $M \times \mathbb{R}^d$, where $d - 1 = \dim \Sigma$. After some further steps,[15] one obtains from this the local immersion into $\mathbb{R}^d$. Note that $\tilde{\nabla}$ is not a $\tilde{T}$-connection, i.e. n is not allowed as a direction entry, owing to the fact that one cannot know the normal derivative of an object only defined on Σ.

Transferring the approach from the classical to the generalised case, one synthetically defines the generalised normal bundle as the span $\mathcal{N} = \langle n, n^\flat \rangle$ and then the generalised tangent bundle $\tilde{E}_\Sigma = E_\Sigma \oplus \mathcal{N}$. Now, one might naively expect that one should define an E_Σ-connection $\tilde{D}$ on $\tilde{E}_\Sigma$, in the hope that it is flat so that one can somehow apply Corollary 3.17 to its curvature tensor, reduce to the classical case and proceed as there. However, the key insight is that such a connection would not encode sufficient information. It turns out that the needed extra input are the covariant derivatives in the conormal direction $n^\flat$. In contrast to those in the normal direction, these can be meaningfully defined, as their computation does not involve taking an actual derivative. In particular, one can compute $\tilde{D}_{n^\flat} T$ for a generalised tensor T only defined over Σ. Thus, introducing $L = \langle n^\flat \rangle$, $\tilde{D}$ should be defined as an $E_\Sigma \oplus L$-connection on $\tilde{E}_\Sigma$. Indeed, we find in Lemma 4.42 that one can associate to such a connection a well-defined torsion tensor $T^{\tilde{D}} \in \Gamma(\Lambda^3(E_\Sigma \oplus L)^*)$, as well as a generalised Riemann tensor $\mathcal{R}m^{\tilde{D}}$ defined on the bundles (4.15 - 4.17). It is to this curvature tensor that we will apply Corollary 3.17 and obtain the fundamental theorem for generalised hypersurfaces.

We begin now by describing the general assumptions of this section and presenting some preparatory constructions.

General assumptions. Let $(E_\Sigma, \mathcal{H}, \mathrm{div}_\Sigma, D^\Sigma, \mathcal{K}, \mathcal{L})$ be a tuple consisting of an exact CA $E_\Sigma \to \Sigma$ over a $(d-1)$-dimensional manifold Σ, equipped with a generalised Riemannian metric $\mathcal{H}$, a divergence operator div_Σ, the canonical generalised LC connection D^Σ on $(E_\Sigma, \mathcal{H})$ such that (cf. Proposition 4.16)

$$\mathrm{div}_{D^\Sigma} = \mathrm{div}_\Sigma + \frac{1}{d-1} \langle e_\Sigma, \cdot \rangle \,, \tag{4.10}$$

[15] We actually go through the needed steps in our proof of Theorem 4.46 as we do not refer to but reprove the classical result.

two sections $\mathcal{L}^{\pm} \in \Gamma((E_\Sigma^{\pm})^*)$, and two bilinear forms $\mathcal{K}^{n_\pm} \in \Gamma((E_\Sigma)^* \otimes (E_\Sigma^{\pm})^*)$. The latter we ask to decompose as in (4.5) and (4.6) with $\chi_\pm^\perp = 0$, i.e.

$$\begin{aligned} \mathcal{K}^{n_\pm}|_{E_\Sigma^\pm \times E_\Sigma^\pm} &= k - \frac{e_\pm^\perp}{d-1} h \mp \frac{H^\perp}{6} \\ \mathcal{K}^{n_\pm}|_{E_\Sigma^\mp \times E_\Sigma^\pm} &= k \mp \frac{H^\perp}{2} \\ \mathcal{L}^\pm &= \frac{\langle e_\Sigma, \cdot \rangle}{d-1} \end{aligned} \tag{4.11}$$

for the Riemannian metric h associated with $\mathcal{H}$, a symmetric two-tensor k, two functions $e_\pm^\perp \in C^\infty(\Sigma)$ and a two-form $H^\perp$.

Remark 4.41. Note that the decomposition of $\mathcal{K}^{n_\pm}$ into k, $e_\pm^\perp$ and $H^\perp$ is unique.

Extending the Courant algebroid and the generalised connection. Define the trivial vector bundles $\nu := \Sigma \times \mathbb{R}$ and $\mathcal{N} := \Sigma \times \mathbb{R}^2$ and denote their trivialising frames by n and $\{n_+, n_-\}$. Define then $\tilde{T} := T\Sigma \oplus \nu$ and $\tilde{E}_\Sigma := E_\Sigma \oplus \mathcal{N}$. We extend the inner product onto $\tilde{E}_\Sigma$ by requiring it to make the direct sum orthogonal and setting $\langle n_\pm, n_\pm \rangle = \pm 1$, $\langle n_+, n_- \rangle = 0$. Similarly, we define a second symmetric tensor $\tilde{\mathcal{G}}$ on $\tilde{E}_\Sigma$ by the conditions that the subbundles $\tilde{E}_\Sigma^\pm := E_\Sigma^\pm \oplus \mathbb{R}n_\pm$ are $\tilde{\mathcal{G}}$-orthogonal and that $\tilde{\mathcal{G}} = \pm\langle \cdot, \cdot \rangle > 0$ on those subbundles. We define a positive definite symmetric tensor $\tilde{g}$ on $\tilde{T}$ such that it agrees with h on $T\Sigma$ and makes n a unit normal to that subspace.

Now we define an E_Σ-connection $\tilde{D}$ on $\tilde{E}_\Sigma$ by

$$\tilde{D}_a b_\pm := D_a^\Sigma b_\pm - \mathcal{K}^{n_\pm}(a, b_\pm) n_\pm, \quad \tilde{D}_a n_\pm := \mathcal{A}^{n_\pm}(a), \quad a \in E_\Sigma,\ b_\pm \in \Gamma(E_\Sigma^\pm),$$

where $\mathcal{A}^{n_\pm} := \mathcal{H}^{-1}\mathcal{K}^{n_\pm}$. Then one can check that $\tilde{D}\tilde{\mathcal{G}} = \tilde{D}\langle \cdot, \cdot \rangle = 0$. Next we consider the line bundle $L \to \Sigma$ spanned by $n_+ - n_-$ and define an anchor map $\tilde{\pi}$ for $E_\Sigma \oplus L \to \Sigma$ as the composition of the natural projection $E_\Sigma \oplus L \to E_\Sigma$ with the anchor of E_Σ. We also extend the Dorfman bracket of E_Σ trivially to $\Gamma(E_\Sigma \oplus L)$ such that $[\Gamma(L), \Gamma(L)] = 0$ and $[v, f(n_+ - n_-)] = -[f(n_+ - n_-), v] = \pi(v)(f)(n_+ - n_-)$ for all $v \in \Gamma(E_\Sigma)$, $f \in C^\infty(\Sigma)$. We note that $(E_\Sigma \oplus L, \langle \cdot, \cdot \rangle|_{E_\Sigma \oplus L}, [\cdot, \cdot])$ satisfies the axioms of a Courant algebroid with exception of the non-degeneracy of the scalar product.

Lemma 4.42. *There is a unique extension of the E_Σ-connection $\tilde{D}$ to an $(E_\Sigma \oplus L, \tilde{\pi})$-connection on $\tilde{E}_\Sigma = E_\Sigma \oplus \mathcal{N}$ (which we denote by the same symbol $\tilde{D}$) which is still metric for $\tilde{\mathcal{G}}$ and $\langle \cdot, \cdot \rangle$ and satisfies the following formula (inspired by Lemma 4.30 (ii) and the definition of the conormal exterior curvature $\mathcal{L}^\pm$ in the case of an actual hypersurface)*

$$\begin{aligned} \tilde{D}_{n_\pm - n_\mp} n_\pm &= \mathcal{B}^\pm := \mathcal{H}^{-1}\mathcal{L}^\pm, \\ \mathcal{G}(\tilde{D}_{n_\pm - n_\mp} a_\pm, b_\pm) &= \mathcal{K}^{n_\pm}(a_\pm, b_\pm) - \mathcal{K}^{n_\pm}(b_\pm, a_\pm), \\ \mathcal{G}(\tilde{D}_{n_\pm - n_\mp} a_\pm, n_\pm) &= -\mathcal{L}^\pm(a_\pm). \end{aligned} \tag{4.12}$$

Proof. To check that $\tilde{D}\langle \cdot, \cdot \rangle = 0$ one can easily evaluate the covariant derivative $\langle \tilde{D}_u v, w \rangle + \langle v, \tilde{D}_u w \rangle - \pi(u)\langle v, w \rangle$ on elements $u \in E_\Sigma \oplus L$, $v, w \in \Gamma(\tilde{E}_\Sigma)$, where it is sufficient to consider sections of $E_\Sigma^\pm$ and the sections $n_+ - n_-$ and $n_\pm$ of L and $\mathcal{N}$, respectively. Since the decomposition $\tilde{E}_\Sigma = \tilde{E}_\Sigma^+ \oplus \tilde{E}_\Sigma^-$ is invariant under $\tilde{D}$, we see that also $\tilde{D}\mathcal{G} = 0$. □

We now construct an auxiliary ambient exact CA $E_M \to M$ restricting to $\tilde{E}_\Sigma$ over Σ and reproducing the structure on it. On E_M we construct *an* extension D^M of $\tilde{D}$ to a (generally non-flat) canonical Levi-Civita connection realising the given exterior curvature tensors.

Lemma 4.43. *There exists an exact Courant algebroid $\pi_M\colon E_M \to M$ over a d-dimensional manifold M containing Σ as a submanifold and on E_M a generalised metric $\mathcal{G}_M$, a divergence operator* div_M*, and finally a vector bundle isomorphism $\Phi\colon \tilde{E}_\Sigma \cong E_M|_\Sigma$ such that*

(1) $\Phi^*\mathcal{G}_M = \tilde{\mathcal{G}}$ *and* $\Phi^* \langle\cdot,\cdot\rangle_M = \langle\cdot,\cdot\rangle$,

(2) $\Phi(n)$ *becomes, under the $\mathcal{G}_M$-induced identification $E_M \cong \mathbb{T}M$, the unit normal* $\mathbf{n} \in \Gamma(TM)$*, and*

(3) denoting by H_M the preferred representative of the Ševera class and writing $\operatorname{div}_M = \operatorname{div}^{\mathcal{G}_M} - \langle e_M, \cdot\rangle_M$,

$$H_M|_\Sigma = H_\Sigma + \mathbf{n}^\flat \wedge H^\perp, \qquad e_M|_\Sigma = \Phi(e_\Sigma) + e_+^\perp \mathbf{n}_+ - e_-^\perp \mathbf{n}_-.$$

where $\mathbf{n}_\pm = \mathbf{n} \pm \mathbf{n}^\flat$.

In particular, with D^M the canonical generalised Levi Civita connection on E_M with divergence div_M*, it holds for all $a, b, c \in \Gamma(E_\Sigma \oplus L)$ and all $\beta \in \Gamma(\tilde{E}_\Sigma)$*

$$\langle [a^*, b^*]_M, c^*\rangle_M|_\Sigma = \langle [a,b], c\rangle, \qquad D^M_{a^*}\beta^*|_\Sigma = \Phi(\tilde{D}_a \beta). \tag{4.13}$$

Herein, a^*, b^*, c^*, and β^* denote arbitrary extensions of respectively $\Phi(a), \Phi(b), \Phi(c)$, and $\Phi(\beta)$ to sections of E_M.

Proof. Consider $M = \Sigma \times \mathbb{R}$. Identify E_Σ with $\mathbb{T}\Sigma$ via the isomorphism induced by the generalised metric $\mathcal{H}$, denote the corresponding closed three-form by $H_\Sigma \in \Omega^3(\Sigma)$. Denote $\mathbf{n} = \partial_t$ and $\mathbf{n}^\flat = \mathrm{d}t$, and then $\mathbf{n}_\pm = \mathbf{n} \pm \mathbf{n}^\flat$. Arbitrarily extend $H = H_\Sigma + \mathbf{n}^\flat \wedge H^\perp$ to a closed three-form on M and $e = e_\Sigma + e_+^\perp \mathbf{n}_+ - e_-^\perp \mathbf{n}_-$ to a section of $\mathbb{T}M$. Define a generalised metric $\mathcal{G}_M$ on M such that over Σ, it is determined by $\mathcal{H}$ and $\mathbf{n}$ as indicated in Proposition 4.8. We consider now $E_M = \mathbb{T}M$ as the generalised tangent bundle with twist H, standard scalar product $\langle\cdot,\cdot\rangle_M = \langle\cdot,\cdot\rangle$, generalised metric $\mathcal{G}_M$, and divergence operator $\operatorname{div}_M = \operatorname{div}^{\mathcal{G}} - \langle e, \cdot\rangle$. Define $\Phi\colon \tilde{E}_\Sigma \to E_M|_\Sigma$ on $E_\Sigma = \mathbb{T}\Sigma$ as the canonical identification of $\mathbb{T}\Sigma$ with the orthogonal subbundle $\langle \mathbf{n}_+, \mathbf{n}_-\rangle^\perp \subset E_M$, and on $\mathcal{N}$ by demanding $n_\pm \mapsto \mathbf{n}_\pm$. In this way, $\Phi(E_\Sigma)$ is identified with the semi-Riemannian Courant algebroid over Σ induced from E_M.

Now consider the canonical generalised Levi-Civita connection D^M. By the assumption (4.10) made on $\operatorname{div}_{D^\Sigma}$ and Proposition 4.16, D^Σ is the Φ-pullback of the generalised Levi-Civita connection that D^M induces on $\Phi(\mathbb{T}\Sigma) \subset \mathbb{T}M$. The statement in (4.13) regarding the generalised connections is then due to Lemmas 4.29 and 4.30. Regarding the statement for the bracket, note that L is the radical of $(E_\Sigma \oplus L, \langle\cdot,\cdot\rangle)$ and

$$\pi_M([a^*, b^*]_M)|_\Sigma = \mathcal{L}_{\pi_M a^*} \pi_M b^*|_\Sigma = \mathcal{L}_{\pi a} \pi b$$

is tangent to the submanifold $\Sigma \subset M$ ($\mathcal{L}_X Y$ stands for the Lie derivative of vector fields) and thus $[a^*, b^*]_M|_\Sigma$ is a section of $\Phi(E_\Sigma \oplus L)$. Hence, for $a, b, c \in \Gamma(E_\Sigma)$, the statement for the brackets reduces to Proposition 4.7. For more general a, b, and c, it follows from the observation that the bracket of a pair of sections of E_M extending a section of L and a section of $E_\Sigma \oplus L$ satisfies on Σ the defining equations for the bracket on $E_\Sigma \oplus L$ up to terms in L. Using the projection

$$\rho\colon E_M|_\Sigma = \Phi(\tilde{E}_\Sigma) = \Phi(E_\Sigma \oplus L \oplus \langle n\rangle) \to \Phi(E_\Sigma \oplus \langle n\rangle)$$

we can write these relations as follows. For all $\xi, \eta \in \Gamma(E_M)$ such that $\xi|_\Sigma = f\mathbf{n}^\flat$ and $\eta|_\Sigma = g\mathbf{n}^\flat$ for some functions $f, g \in C^\infty(\Sigma)$ and all $a \in \Gamma(E_M)$ such that $a|_\Sigma \in \Gamma(\Phi(E_\Sigma))$ the following equations hold on Σ:

$$[\xi, \eta]_M = 0 \qquad \text{and} \qquad \rho[a, \xi]_M = -\rho[\xi, a]_M = 0$$

□

Corollary 4.44. *We can define a torsion-tensor $T^{\tilde{D}} \in \Gamma(\Lambda^3(E_\Sigma \oplus L)^*)$ for $\tilde{D}$ via the defining formula in Definition 2.31, and this tensor vanishes. Via the defining formula in Definition 2.41, we can associate to $\tilde{D}$ the generalised Riemann tensor $\mathcal{R}m^{\tilde{D}}$, which is a section of the direct sum of the pure-type subbundle*

$$\operatorname{Sym}^2 \Lambda^2 \left(E_\Sigma^+\right)^* \oplus \operatorname{Sym}^2 \Lambda^2 \left(E_\Sigma^-\right)^*, \tag{4.14}$$

the mixed-type subbundle

$$\left(\Lambda^2(\tilde{E}_\Sigma^+)^* \oplus \Lambda^2(\tilde{E}_\Sigma^-)^*\right) \vee \left((E_\Sigma^+)^* \wedge (E_\Sigma^-)^*\right) \tag{4.15}$$

and the conormal subbundles

$$(L^* \wedge (E_\Sigma^+)^* \oplus L^* \wedge (E_\Sigma^-)^*) \vee (\Lambda^2(E_\Sigma^+)^* \oplus \Lambda^2(E_\Sigma^-)^*) \quad \textit{and} \tag{4.16}$$

$$\operatorname{Sym}^2(L^* \wedge (E_\Sigma^+)^* \oplus L^* \wedge (E_\Sigma^-)^*), \tag{4.17}$$

which are respectively linear and quadratic in L^*, where $L := \langle n_+ - n_- \rangle$.

Remark 4.45. Technically, on the mixed-type subbundle, the defining formula in Definition 2.41 cannot be employed, since in the direction entry of $\tilde{D}$ there could feature contributions proportional to $n = (n_+ + n_-)/2$, which would be ill-defined. For instance, $n_+ = \frac{1}{2}(n_+ - n_-) + n$ contains such a contribution. However, we can deal with this by formally assuming that also the derivative $\tilde{D}_n$ respects the decomposition $\tilde{E}_\Sigma = \tilde{E}_\Sigma^+ \oplus \tilde{E}_\Sigma^-$, as then one obtains immediately that the terms featuring such a problematic derivative vanish. Alternatively, if one wishes to avoid such formal evaluations, one can adapt the definition of $\mathcal{R}m^{\tilde{D}}$ on the mixed-type subbundle to simply exclude these undesired terms, which yields a definition more in line with that of the generalised Riemann tensors $\mathcal{R}m^\pm_{\mathrm{GF}}$ found in [6].

Proof. Employing the generalised connection D^M from Lemma 4.43, which extends $\tilde{D}$, the formulas for $T^{\tilde{D}}$ and $\mathcal{R}m^{\tilde{D}}$ respectively equate by (4.13) to restrictions of the formulas for the torsion T^{D^M} and the curvature $\mathcal{R}m^{D^M}$ of D^M, which are well-defined tensors. □

The Generalised Fundamental Theorem for Hypersurfaces. Assume now that the geometric data $(E_\Sigma, \mathcal{H}, \operatorname{div}_\Sigma, D^\Sigma, \mathcal{K}, \mathcal{L})$, specified under the general assumptions of Section 4.5, satisfy the equations of Gauß and Codazzi for a flat ambient Riemannian CA, i.e. the equations from Theorems 4.32 and 4.37 with an ambient space of vanishing generalised Riemann curvature. Then, we obtain the announced theorem.

Theorem 4.46. *Under the above assumptions, for every simply connected open set $U \subset \Sigma$ there exists a Riemannian (hypersurface) immersion $U \to \mathbb{R}^d$ such that the canonical structure on the untwisted generalised tangent bundle $\mathbb{TR}^d$ induces $(\mathcal{H}, \operatorname{div}_\Sigma, D^\Sigma, \mathcal{K}, \mathcal{L})$.*

In particular, $e = 0$, $H = 0$, $\mathcal{L} = 0$, and (Σ, h, k) satisfies the well-known flat Gauß and Codazzi equations from Riemannian geometry.

Remark 4.47. The Riemannian immersion $U \to \mathbb{R}^d$ is unique up to isometries of $\mathbb{R}^d$. This follows from the fundamental theorem for hypersurfaces, which includes this uniqueness statement in the version [34, Chapter VII, Theorem 7.2] and applies since (Σ, h, k) satisfies the classical Gauß and Codazzi equations.

Proof. Denote by $\tilde{D}$ the $(E_\Sigma \oplus L)$-connection from Lemma 4.42, and by $\mathcal{R}m^{\tilde{D}}$ its generalised Riemann curvature. Note that the components of the generalised Riemann tensor involved in the Gauß and Codazzi equations (cf. Theorems 4.32 and 4.37) precisely determine the tensor on the subbundles (4.14), (4.15), (4.16) and (4.17).

Next we relate the $(E_\Sigma \oplus L)$-connection $\tilde{D}$ to a triple $(\tilde{\nabla}, H, e)$ consisting of:

1. the metric connection $\tilde{\nabla}$ on $\tilde{T}$ obtained from setting $\tilde{\nabla}_X Y = \nabla^\Sigma_X Y - k(X,Y)n$, $\tilde{\nabla}_X n = AX$, where $k = hA$ and ∇^Σ is the Levi-Civita connection of Σ,
2. $H = H_\Sigma + n^\flat \wedge H^\perp \in \Gamma(\Lambda^3 \tilde{T}^*)$, and
3. $e = e_\Sigma + e^\perp_+ n_+ - e^\perp_- n_- \in \Gamma(\tilde{E}_\Sigma)$.

To that end, we introduce the related connections $\tilde{\nabla}^\pm$ and $\tilde{\nabla}^{\pm 1/3}$ on $\tilde{T}$ by setting, cf. (2.13),

$$\tilde{\nabla}^\pm_X = \tilde{\nabla}_X \pm \frac{1}{2} H_X, \qquad \tilde{\nabla}^{\pm 1/3}_X = \tilde{\nabla}_X \pm \frac{1}{6} H_X$$

Proposition 4.48. *Employing the obvious identifications $\tilde{E}^\pm_\Sigma \cong \tilde{T}$ mapping $n_\pm$ to n and restricting to the anchor on $E^\pm_\Sigma$, the relationship between $\tilde{D}$ and the triple $(\tilde{\nabla}, H, e)$ is given by, compare Theorem 3.2,*

$$\begin{aligned} \tilde{D}_a \beta &= \tilde{\nabla}^{\pm 1/3}_a \beta + \frac{1}{d-1} \chi^{e\pm}_\pm (a, \beta), \qquad \tilde{D}_{\bar{a}} \beta = \tilde{\nabla}^\pm_{\bar{a}} \beta, \\ \tilde{D}_{n_\pm - n_\mp} \beta &= \mp \frac{1}{3} H^\perp(\beta) + \frac{1}{d-1} \chi^{e\pm}_\pm (n_\pm, \beta) \end{aligned} \tag{4.18}$$

where $\chi^{e\pm}_\pm$ is formally defined as in (3.1), $a \in \Gamma(E^\pm_\Sigma), \bar{a} \in \Gamma(E^\mp_\Sigma)$, and $\beta \in \Gamma(\tilde{E}^\pm_\Sigma)$. In particular, the restrictions of $\tilde{D}$ to pure-type entries $\Gamma(E^\pm_\Sigma) \times \Gamma(\tilde{E}^\pm_\Sigma)$ and mixed-type entries $\Gamma(E^\pm_\Sigma) \times \Gamma(\tilde{E}^\mp_\Sigma)$ respectively define two connections in $\tilde{T}$.

Proof. This follows from applying properties (a) and (b) in Theorem 3.2 to the extension D^M of $\tilde{D}$ to the canonical generalised LC connection on E_M discussed in Lemma 4.43. □

Since $\tilde{\nabla}$ is a metric connection in $\tilde{T} \to \Sigma$, it has a curvature tensor $\mathrm{Rm}^{\tilde{\nabla}}$, defined via the classical formula

$$\mathrm{Rm}^{\tilde{\nabla}}(\tilde{V}, \tilde{W}, X, Y) = \tilde{g}\left(\tilde{V}, \tilde{\nabla}_X \tilde{\nabla}_Y \tilde{W} - \tilde{\nabla}_Y \tilde{\nabla}_X \tilde{W} - \tilde{\nabla}_{[X,Y]} \tilde{W}\right), \tag{4.19}$$

where $X, Y \in \Gamma(T\Sigma)$ and $\tilde{V}, \tilde{W} \in \Gamma(\tilde{T})$. For better comparison with generalised curvatures, we symmetrically extend it to a section $\mathrm{Rm}^{\tilde{\nabla}}$ of

$$(\Lambda^2 \tilde{T}^*) \vee (\Lambda^2 T^* \Sigma).$$

Note that $\mathrm{Rm}^{\tilde{\nabla}}$ is related to $\mathrm{Rm}^{\nabla^\Sigma}$ by the classical Gauß and Codazzi equations. From these, together with the Bianchi identity for $\mathrm{Rm}^{\nabla^\Sigma}$ and the symmetry of k it follows that $\mathrm{Rm}^{\tilde{\nabla}}$ satisfies the Bianchi identities

$$\begin{aligned} 0 &= \sum_{\sigma(X,Y,Z)} \mathrm{Rm}^{\tilde{\nabla}}(X, Y, Z, V) \\ 0 &= \sum_{\sigma(X,Y,Z)} \mathrm{Rm}^{\tilde{\nabla}}(X, Y, Z, n) = \sum_{\sigma(n,X,Y)} \mathrm{Rm}^{\tilde{\nabla}}(Z, n, X, Y) \end{aligned} \tag{4.20}$$

where $X, Y, Z, V \in \Gamma(T\Sigma)$. The second equality in the second line follows from the symmetries of $\mathrm{Rm}^{\tilde{\nabla}}$.

Lemma 4.49. *The generalised Riemann tensor $\mathcal{R}m^{\tilde{D}}$ and the Riemann tensor $\mathrm{Rm}^{\tilde{\nabla}}$ are related via the formulas given in Theorems 3.5 and 3.6, where $\mathrm{Rm}^{\tilde{\nabla}}$ takes the place of Rm, $\mathcal{R}m^{\tilde{D}}$ takes the place of $\mathcal{R}m^{D}$, $H = H_\Sigma + n^\flat \wedge H^\perp$ and $e = e_\Sigma + e_+^\perp n_+ - e_-^\perp n_-$.*

Remark 4.50. We briefly clarify which formulas Lemma 4.49 claims to hold for the conormal components of the generalised Riemann tensor, given that these components are absent from Theorems 3.5 and 3.6, which only feature formulas for pure- and mixed-type components.

To start with, we note that the curvature components specified in Theorems 3.5 and 3.6 fully determine any algebraic generalised Riemann tensor, cf. Definition 2.42 and Lemma 2.44. The assertion in Lemma 4.49 is that, formally employing the multi-linearity of the tensor and then applying the replacement prescription detailed in its statement to the extracted equations, we obtain formulas that hold in our setting. In particular, Lemma 4.49 claims that for every component of $\mathcal{R}m^{\tilde{D}}$ that is well-defined, we obtain in this way an equation for it whose right-hand side is also well-defined.

Proof. Consider the extension D^M of $\tilde{D}$ from Lemma 4.43 and their relationship (4.13). As D^M is a canonical generalised LC connection, Theorems 3.5 and 3.6 apply to it. Since E_M is equipped with structure inducing H and e on Σ, and since $\mathcal{R}m^{D^M} = \mathcal{R}m^{\tilde{D}}$ on the domain of the latter, the result follows. □

Let us consider $\Sigma \subset M = \Sigma \times \mathbb{R}$ via the canonical embedding $p \mapsto (p, 0)$. We denote $n = \partial_t$, $n_\pm = \partial_t \pm \mathrm{d}t$, and define at every point $p \in \Sigma$ the following data. First, we extend $\tilde{\nabla}|_p$ to a connection on M at p by setting $\tilde{\nabla}_n X|_p = \partial_t X|_p + A(X)|_p$ for any vector field X tangent to Σ and $\tilde{\nabla}_n(fn)|_p = (\partial_t f)n|_p$, $f \in C^\infty(M)$. We then extend $\mathrm{Rm}^{\tilde{\nabla}}|_p$ to a tensor[16] $\mathrm{Rm} \in \mathrm{Sym}^2(\Lambda^2 T_p^* M)$ by demanding $\mathrm{Rm}(X, n, n, X) = 0$. Finally, we define $g \in J_p^1(M, \mathrm{Sym}^2(M))$, $H \in J_p^1(M, \Lambda^3 M)$, and $e = 2(X + \xi) \in J_p^1(TM \oplus T^*M)$ by requiring the following:

$$\begin{aligned}
&g_p = \mathrm{d}t_p^2 + h_p, && H_p = H_\Sigma|_p + \mathrm{d}t_p \wedge H_p^\perp, && e_p = [e_\Sigma + e_+^\perp n_+ - e_-^\perp n_-]_p \\
&\tilde{\nabla} g|_p = 0, && \tilde{\nabla}_n H|_p = 0, && \tilde{\nabla}_n e|_p = 0, \\
&\tilde{\nabla}_X H_p = \left[\tilde{\nabla}_X (H_\Sigma + \mathrm{d}t \wedge H^\perp)\right]_p, && && \tilde{\nabla}_X e|_p = \left[\tilde{\nabla}_X (e_\Sigma + e_+^\perp n_+ - e_-^\perp n_-)\right]_p.
\end{aligned}$$

These definitions imply the equations one obtains from Theorems 3.5 and 3.6 with the generalised Riemann tensor assumed to vanish - the non-trivial part of this statement being that it applies to the equations that come from the assumed vanishing of the components

$$\mathcal{R}m^{\tilde{D}}(a, n_\mp, v, w), \qquad \mathcal{R}m^{\tilde{D}}(a, n_\pm, n_\pm, w), \qquad \mathcal{R}m^{\tilde{D}}(n_\mp, n_\pm, n_\pm, w)$$

which lie in neither of the bundles (4.14), (4.15), (4.16) and (4.17). Thus, by Remarks 3.16 and 3.18, we can apply Corollary 3.17 to obtain that $H_p = 0, e_p = 0$, and $\mathrm{Rm}_p = 0$ for all $p \in \Sigma$. In particular, the inital connection $\tilde{\nabla}$ in $\tilde{T} \to \Sigma$ is flat, and we can find a parallel local ON frame $\{e_i\}$ for $\tilde{T}$ on an open set $p \in U \subset \Sigma$. This defines the trivialising isomorphism

$$\phi \colon (\tilde{T}, \tilde{\nabla}) \longrightarrow (\underline{\mathbb{R}}^d, \nabla)$$

[16] In fact, from the Bianchi identities (4.20) for $\mathrm{Rm}^{\tilde{\nabla}}$, it follows that Rm satisfies the Bianchi identity as well and hence is (even a priori) an algebraic curvature tensor.

where $\tilde{\nabla} = \phi^*\nabla$. Then for all $X, Y \in \Gamma(T\Sigma)$,

$$\nabla_X \phi(Y) = \phi(\tilde{\nabla}_X Y) = \phi(\nabla^\Sigma_X Y - k(X,Y)n)$$

so that the one-form $\alpha := \phi|_{T\Sigma} \in \Omega^1(\Sigma, \underline{\mathbb{R}}^d)$ satisfies

$$\begin{aligned}
\mathrm{d}\alpha(X,Y) &= \nabla_X \alpha(Y) - \nabla_Y \alpha(X) - \alpha([X,Y]) \\
&= \alpha\left(\nabla^\Sigma_X Y - k(X,Y)n\right) - \alpha\left(\nabla^\Sigma_Y X - k(Y,X)n\right) - \alpha([X,Y]) \\
&= \alpha(\nabla^\Sigma_X Y - \nabla^\Sigma_Y X - [X,Y]) \\
&= 0
\end{aligned}$$

By assumption, U is simply connected. Hence, we can find $\varphi\colon U \to \mathbb{R}^d$ such that $\alpha = \mathrm{d}\varphi$. Note that φ is an immersion because α has, restricted to the fibre at any point, maximal rank. After possibly shrinking U, φ becomes an embedding. Employing the splitting $F_\Sigma\colon E_\Sigma \cong \mathbb{T}\Sigma$ obtained from $\mathcal{H}$, we find with

$$\bar{\varphi} := \begin{pmatrix} \varphi_* & 0 \\ 0 & (\varphi^*)^{-1} \end{pmatrix}$$

that

$$F := \bar{\varphi} \circ F_\Sigma|_U \colon E_U \longrightarrow \mathbb{T}\mathbb{R}^d$$

is as desired. □

– Part II –

The Initial Value Problem for the Generalised Einstein Equations

5 Techniques Applicable to IVPs for Einstein-Matter Systems

This chapter consists of the preliminary section of the preprint [22] (adding in proofs) and does not contain any original results. It presents those parts of the work [10] by Ringström which are relevant to any Einstein-matter system. While all results presented in this section are heavily based on [10], most of them are only established implicitly in [10] (cf. also Remark 6.2). For completeness, we include proofs to those results here.

For an explanation of the general idea for proving the well-posedness of an IVP, we refer to the beginning of Chapter 6.

5.1 Quasi-Linear Hyperbolic PDEs with Metric Principal Symbol

In this section, we prove the existence and uniqueness of solutions to quasi-linear hyperbolic PDEs with principal symbol given by the inverse of a Lorentzian metric which is part of the dynamics, provided that the non-linearity satisfies a minor well-behavedness assumption. In an appropriate gauge, many important Einstein-matter systems are of this type, cf. Remark 6.1.

Let $M = \mathbb{R} \times \Sigma$ for a smooth manifold Σ. Denote by $L(M) \subset \mathrm{Sym}^2(TM)$ the bundle of non-degenerate symmetric two-tensors with Lorentzian signature, by E' a vector subbundle of the tensor bundle $\mathcal{T}M$, and set $E = L(M) \times E'$. Take an arbitrary point $p \in \Sigma$, and a coordinate neighbourhood (U_Σ, x^m) of p, and define coordinates $x^\mu = (x^0 = t, x^m)$ on $U = \mathbb{R} \times U_\Sigma$. Then, we study on U the PDE

$$g^{\mu\nu}\partial_\mu\partial_\nu u - f[u] = 0, \qquad u = (g, u') \in \Gamma(E|_U) \tag{5.1}$$

where $f \in C^\infty(L_n \times \mathbb{R}^{N'+(n+1)N+n+1}, \mathbb{R}^N)$ with $N' = \mathrm{rank}\, E'$, $N = N' + (n+1)^2$, and L_n the space of $n+1$-dimensional Lorentz matrices. We employed the notation

$$f[u](t,x) \coloneqq f(t, x, g_{\mu\nu}, u'_j, \partial_\mu u_j).$$

We finally make the assumption that

$$f[u](t,x) = F(g_{\mu\nu}, u'_j, \partial_\mu u_j, \partial^\alpha B_l(t,x))$$

for some "background fields" $B \in C^\infty(M, \mathbb{R}^K)$, $K \in \mathbb{N}_0$, and a smooth $\mathbb{R}^N$-valued function F depending on variables as indicated ($\alpha \in \mathbb{N}_0^{n+1}$ represents multi-indices up to degree $k \in \mathbb{N}_0$, $|\alpha| \le k$) such that $F(g_{\mu\nu}, 0, ..., 0) = 0$.

Existence of Local Solutions. The following proposition provides a generalisation of a result implicitly established in [10, proof of Theorem 14.2].

Proposition 5.1. *Let $U_0, U_1 \in C^\infty(U_\Sigma, E)$ be initial data for the equation (5.1) such that, writing $U_0 = (g_0, U'_0)$, $(g_0)_{\mu\nu}$ takes values in the space of canonical Lorentz matrices $\mathcal{C}_n$.*

Then, for every open neighbourhood $O \subset U$ of p, there exists an open neighbourhood $W \subset O$ of p on which the equation (5.1) admits a globally hyperbolic development $u = (g, u')$ of the initial data on $W \cap \Sigma$ given by restriction of U_0 and U_1 (i.e. $u = (g, u')$ solves (5.1), (W, g) is globally hyperbolic, and we have $U_0|_{W\cap\Sigma} = u|_{W\cap\Sigma}$ as well as $U_1|_{W\cap\Sigma} = \partial_\mu u|_{W\cap\Sigma}$). W can be chosen such that the component-matrix $(g_{\mu\nu})_{\mu\nu}$ takes values in the space of canonical Lorentz matrices $\mathcal{C}_n$ and, in particular, such that $\mathrm{grad}\, t$ is timelike on W.

Proof. The idea is to apply some technical modifications to the equation (5.1) such that one can apply Theorem A.4.

Take an open neighbourhood $p \in V \subset M$ with compact closure $\overline{V}$ contained in U. Take furthermore an open subset $\mathcal{U} \subset \mathrm{Sym}_{n\times n}$ containing $(g_0)_{ij}(q)$ for all $q \in \overline{V} \cap \Sigma$ such that the closure $\overline{\mathcal{U}}$ is compact and contains only positive definite matrices. We then take a smooth function $(A_{\mu\nu})\colon \mathbb{R}^{d\times d} \times \mathbb{R} \times \Sigma \to \mathcal{C}_n$ such that

(i) $A_{00} \in [-2, -1/4]$, $A_{0i} \in [-2, 2]$, and (A_{ij}) is positive definite everywhere with positive lower and upper bound, and

(ii) for (t, x) in V, one has $A_{00}(g, t, x) = g_{00}$ if $g_{00} \in [-3/2, -1/2]$, $A_{0i}(g, t, x) = g_{0i}$ if $g_{0i} \in [-1, 1]$, and $A_{ij}(g, t, x) = g_{ij}$ for $g_{ij} \in \mathcal{U}$, and

(iii) $A_{\mu\nu}(g, t, x)$ is independent of its arguments for (t, x) outside of a compact subset of $\mathbb{R}^d$.

We denote by $(A^{\mu\nu})$ the inverse matrix. Consider now the PDE (5.1), replacing the components $g^{\mu\nu}$ with the components $A^{\mu\nu}$. We note that, trivially extending A to be a function in $C^\infty(\mathbb{R}^{N+dN+d}, \mathcal{C}_n)$ results in a C^∞ (N, n)-admissible metric, cf. Definition A.2.

We further modify the non-linearity. Consider a bump function $b_1 \in C_0^\infty[U]$ such that $b_1 = 1$ on $\overline{V}$. Then, consider $\hat{f} \in C^\infty(\mathbb{R}^{N+(n+1)N+(n+1)}, \mathbb{R}^N)$ such that

$$\hat{f}_i[u](t, x) = F_i(A_{\mu\nu}(g, t, x), u'_j, \partial_\mu u_j, b_1(t, x)\partial^\alpha B_l(t, x))$$

We immediately conclude from the assumptions made on the function F that $\hat{f}[0]$ has compact support. We also note that the norm of the continuous functions $A_{\mu\nu}(g)$ and $b_1 \partial^\alpha B_l$ is bounded from above, as they are constant outside of a compact subset. Hence, all derivatives $\partial^\beta \hat{f}_i(t, x, \xi)$, β an arbitrary multi-index, satisfy the estimate $\partial^\beta \hat{f}_i(t, x, |\xi|) \leq h_{i,\beta}(|\xi|))$ for the following continuous and increasing function $h_{i,\beta}$:

$$h_{i,\beta}(M) \coloneqq \max_{(t,x)\in K, |\xi|\leq M} \partial^\beta \hat{f}_i(t, x, \xi) = \max_{(t,x)\in \mathbb{R}^d, |\xi|\leq M} \partial^\beta \hat{f}_i(t, x, \xi)$$

Herein, K is a compact subset of $\mathbb{R}^{n+1}$ containing the support of b_1 and the set on which $A_{\mu\nu}(g, t, x)$ is non-constant. It follows that f is a C^∞ (N, n)-admissible non-linearity, cf. Definition A.3.

To construct initial data compatible with Theorem A.4, take another bump function $b \in C_0^\infty(U_\Sigma)$ such that $b = 1$ on $\overline{V} \cap \Sigma$. Then consider as initial data bU_0 and bU_1. Notice that we can understand these functions naturally as elements of $C^\infty(\Sigma, \mathbb{R}^N)$, as we have coordinates on U_Σ, and can smoothly extend it as the zero-function outside of U_Σ.

Applying the described modifications to the system (5.1), it becomes such that Theorem A.4 can be applied. We get a smooth local solution $u = (g, u')$. The smoothness implies the existence of an open neighbourhood $(0, p) \in W \subset \mathbb{R} \times \Sigma$ such that the components of $g_{\mu\nu}$ take values for which $g_{\mu\nu} = A_{\mu\nu}$. We denote by $\pi\colon \mathbb{R} \times \Sigma \to \Sigma$ the natural projection, and demand W to be such that $b_1|_W = 1$, $W \subset V$, and $\Sigma \cap W$ is a Cauchy hypersurface in (W, g). Then, on W, u is a solution of the original system (5.1). □

Uniqueness of Local Solutions. We want to discuss the uniqueness of local solutions to equations of the type (6.1), as guaranteed to exist by Proposition 5.1.

First, we cite essentially verbatim [10, Lemma 12.8], which is the backbone of our proof of local uniqueness.

Proposition 5.2. *Let (M,g) be an $(n+1)$-dimensional spacetime and let us assume that there is a smooth spacelike Cauchy hypersurface Σ. Let p be a point to the future of Σ and assume that there are geodesic normal coordinates (O,ϕ) centered at p such that the compact*[17] *set $J^-(p)\cap J^+(\Sigma)$ is contained in O. Assume $u\colon O\to\mathbb{R}^l$ solves over $J^-(p)\cap J^+(\Sigma)$ the equation*

$$\Box_g u + Xu + \kappa u = 0,$$

where $X\in C^\infty(O,\mathbb{R}^n\otimes \mathrm{Mat}_{l\times l})$ is an $l\times l$ matrix of vector fields, and $\kappa\in C^\infty(O,\mathrm{Mat}_{l\times l})$ is an $l\times l$ matrix. Assume furthermore that u and $\operatorname{grad} u$ vanish on $\Sigma\cap J^-(p)$. Then u and $\operatorname{grad} u$ vanish in $J^-(p)\cap J^+(\Sigma)$.

We discuss local uniqueness in the following setting. Let

$$f\in C^\infty(L_n\times\mathbb{R}^{N'+(n+1)N+n+1},\mathbb{R}^N),$$

where $N'\in\mathbb{N}_0$ and $N=N'+(n+1)^2$. Denote $\Sigma=\{0\}\times\mathbb{R}^n\subset\mathbb{R}^{n+1}$. For $i=1,2$, let $W_i\subset O_i\subset\mathbb{R}^{n+1}$ be open subsets such that $W_1\cap W_2\cap\Sigma\neq\emptyset$, and denote $\Sigma_i\coloneqq W_i\cap\Sigma$. Let $g_i\colon O_i\to\mathcal{C}_n$ be a Lorentzian metric and $u_i'\colon O_i\to\mathbb{R}^{N'}$, $i=1,2$, be such that over W_i, $u_i=(g_i,u_i')$ solves

$$g_i^{\mu\nu}\partial_\mu\partial_\nu u_i = f[u_i]$$

Assume furthermore that O_i is convex with respect to g_i, and that Σ_i is spacelike Cauchy in (W_i,g_i). Denote by $J_i^\pm$ the causal future/past with respect to g_i in W_i.

Corollary 5.3. *Let $p\in W_1\cap W_2$ be a point to the future of Σ_1 and Σ_2 such that $J_i^-(p)\subset W_1\cap W_2$ for both $i=1,2$. Assume that $u_1=u_2$ and $\operatorname{grad} u_1=\operatorname{grad} u_2$ on $\Sigma_i\cap J_i^-(p)$ for both $i=1,2$. Then $u_1=u_2$ on $J_i^-(p)\cap J_i^+(\Sigma_i)$ for both $i=1,2$.*

Proof. The idea is similar to the one expressed in the proof of Lemma [10, Lemma 9.7.]. It is based on the observation that for a function $h\in C^\infty(U,\mathbb{R}^l)$ with $U\subset\mathbb{R}^k$ convex open and $k,l\in\mathbb{N}_0$, it holds for all $x,y\in U$ that

$$h(x)-h(y)=\underbrace{\int_0^1 Dh|_{tx+(1-t)y}\mathrm{d}t}_{=:L}\,(x-y).$$

Crucially, the right hand side is a linear operator L (depending on x and y) acting on the difference $(x-y)$.

Denote $u=u_1$ and $v=u_2$, $g_u=g_1$ and $g_v=g_2$, and $f_u=f[u_1]$ and $f_v=f[u_2]$. Then

$$g_u^{\mu\nu}\partial_\mu\partial_\nu(u-v)=(g_v^{\mu\nu}-g_u^{\mu\nu})\partial_\mu\partial_\nu v+f_u-f_v \tag{5.2}$$

We observe that $\mathcal{C}_n$ is a convex open subset of $\mathrm{Sym}^2(n+1)\cong\mathbb{R}^{(n+1)(n+2)/2}$. Hence, there exist linear operators $f_1,f_2^\mu\in\mathrm{End}(\mathbb{R}^N)$ and a linear operator $F=(F^{\mu\nu})\in\mathrm{Hom}(\mathbb{R}^N,\mathbb{R}^{(n+1)^2})$ such that

$$\begin{aligned} f_u-f_v &= f_1(u-v)+f_2^\mu\partial_\mu(u-v),\\ g_u^{\mu\nu}-g_v^{\mu\nu} &= [F(u-v)]^{\mu\nu}\end{aligned}$$

Inserting this into (5.2), we obtain that

$$g_u^{\mu\nu}\partial_\mu\partial_\nu(u-v)=[F(u-v)]^{\mu\nu}\partial_\mu\partial_\nu v+f_1(u-v)+f_2^\mu\partial_\mu(u-v)$$

Applying Proposition 5.2, it follows that $u=v$ on $J_1^-(p)\cap J_1^+(\Sigma_1)$. Exchanging u and v, one obtains the result for $i=2$. □

[17] The formulation of [10, Lemma 12.8] is such that compactness of $J^-(p)\cap J^+(\Sigma)$ is assumed instead of claimed. However, this compactness is automatic, cf. for example [37, Theorem 8.3.12.].

To establish local uniqueness in the desired sense, we need the following basic result from Lorentzian geometry. We cite essentially verbatim that part of [10, Lemma 10.10.] which is relevant to us.

Lemma 5.4. *Let (M, g) be a spacetime with smooth spacelike Cauchy hypersurface S. If $U \subset S$ is open, $q \in J^+(S)$ and $J^-(q) \cap J^+(S) \subset U$, then if $q_i \in J^+(S)$ are such that $q_i \to q$, we have $J^-(q_i) \cap J^+(S) \subset U$ for i large enough.*

Note that an equivalent formulation of the conclusion is that there exists an open neighbourhood V of q such that for all $r \in V$ we have $J^-(r) \cap J^+(S) \subset U$.

The following Proposition establishes local uniqueness in the desired sense. It presents an argument found in [10, proof of Theorem 14.2].

Proposition 5.5. *Assume that $\overline{W}_i$ is compact and contained in O_i, $i = 1, 2$. Assume that $u_1 = u_2$ and $\operatorname{grad} u_1 = \operatorname{grad} u_2$ on $\Sigma_1 \cap \Sigma_2$. Then $u_1 = u_2$ on the whole intersection $W_1 \cap W_2$.*

Proof. The difficulty in seeing that the solutions coincide on $W_1 \cap W_2$ is that we do not have global hyperbolicity of the intersection $W_1 \cap W_2$ with respect to $g_1 = g[u_1]$ or $g_2 = g[u_2]$. Define

$$S_t := ([0, t] \times \Sigma) \cap \overline{W}_1 \cap \overline{W}_2$$

and then

$$\begin{aligned} \mathcal{A} := \{t \geq 0 \mid u_1 = u_2 \text{ on } S_t \text{ and} \\ J_1^-(r) \cap J_1^+(\Sigma_1) = J_2^-(r) \cap J_2^+(\Sigma_2) \text{ for all } r \in S_t\}. \end{aligned}$$

By assumption $0 \in \mathcal{A}$, so $\mathcal{A} \neq \emptyset$. $\mathcal{A}$ is also closed, since the solutions u_i are continuous and the sets $J_i^-(r)$ can be written as the closure of the union of sets $J_i^-(r_j)$ for a sequence $r_j \in J_i^-(r)$ converging to r.

We claim that $\mathcal{A}$ is open. Take $t \in \mathcal{A}$. First, note that by Lemma 5.4

$$J_i^-(s) \cap J_i^+(\Sigma_i) \subset W_1 \cap W_2 \qquad \text{for both } i = 1, 2 \tag{5.3}$$

is an open condition on $s \in S_\infty := (\mathbb{R}_{\geq 0} \times \Sigma) \cap \overline{W}_1 \cap \overline{W}_2$. By assumption, it is satisfied for all $s \in S_t$. By compactness of $S_t \subset S_\infty$, there exists $\tau > t$ such that (5.3) holds for all $s \in S_\tau$. Therefore, there exists $\epsilon > 0$ such that $t + \epsilon \in \mathcal{A}$ unless there is a sequence $r_j = (t_j, p_j) \in S_\infty$ such that $t_j \to t$ from above and for all j

$$u_1|_{r_j} \neq u_2|_{r_j} \tag{5.4}$$

Assume that r_j is such a sequence. By compactness of S_∞, there exists $p \in \Sigma$ such that $p_j \to p$ (possibly after passing to a subsequence). However, it holds (5.3) for $t_j < \tau$, i.e. for j large enough. For such j, we conclude with Corollary 5.3 that $u_1 = u_2$ on $J_i^-(r_j) \cap J_i^+(\Sigma_i)$ for both $i = 1, 2$, in direct contradiction with (5.4). Hence, $\mathcal{A}$ is open.

It follows that $\mathcal{A} = [0, \infty)$. In particular, $u_1 = u_2$ on S_∞. By time reversal, the claim holds on $\overline{W}_1 \cap \overline{W}_2$. □

5.2 DeTurck's Gauge Condition

In this section, we show that the DeTurck gauge propagates well, and we show that one can locally implement it by acting with a diffeomorphism. We are interested in DeTurck's gauge condition, because it casts the Einstein vacuum equations a system of the form of quasi-linear hyperbolic PDEs with principal symbol given by the metric, as discussed in section 5.1.

The DeTurck gauge propagates well. Let $\mathcal{D} \in \Gamma(T^*M)$ (later given by (6.14)), and assume that the modified Ricci tensor $\hat{\mathrm{Rc}}_{\mu\nu} = \mathrm{Rc}_{\mu\nu} + \nabla_{(\mu}\mathcal{D}_{\nu)}$ (later as in (6.15)) satisfies the following "Einstein equations":

$$\hat{\mathrm{Rc}} - \frac{\mathrm{tr}_g \hat{\mathrm{Rc}}}{2} g = T$$

Herein, T is any divergence-free two-tensor. We refer to $\mathcal{D} = 0$ as *DeTurck's gauge condition*.

In this setting, we follow [10, chapter 14.1] to show that the initial vanishing of $\mathcal{D}$ and $\nabla\mathcal{D}$ on a subset Ω of an initial hypersurface Σ implies that $\mathcal{D}$ vanishes on the Cauchy development $D(\Omega)$.

Lemma 5.6. *Let (M, g) be a globally hyperbolic Lorentzian manifold with spacelike Cauchy hypersurface Σ. Let $\bar{g}$ be another Lorentzian metric on M, the background metric. Assume that $\hat{\mathrm{Rc}} - \frac{1}{2}\mathrm{tr}_g \hat{\mathrm{Rc}} = T$, where T a $(0, 2)$-tensor such that $\mathrm{div}\, T = 0$. Assume that on $\Omega \subset \Sigma$, $\mathcal{D} = 0$ and $\nabla\mathcal{D} = 0$. Then $\mathcal{D} = 0$ on the entire Cauchy development $D(\Omega)$.*

Proof. We compute the divergence of the "Einstein tensor" obtained from the modified Ricci tensor as

$$\begin{aligned}\nabla^\mu \left(\hat{\mathrm{Rc}}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\ \mathrm{tr}\,\hat{\mathrm{Rc}}\right) &= \nabla^\mu \left(\hat{\mathrm{Rc}}_{\mu\nu} - \mathrm{Rc}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\ \mathrm{tr}(\hat{\mathrm{Rc}} - \mathrm{Rc})\right)\\ &= \nabla^\mu\nabla_{(\mu}\mathcal{D}_{\nu)} - \frac{1}{2}\nabla_\nu\nabla^\mu\mathcal{D}_\mu\\ &= \frac{1}{2}\nabla^\mu\nabla_\mu\mathcal{D}_\nu + R_\mu{}^{\lambda\mu}{}_\nu\mathcal{D}_\lambda\\ &= \frac{1}{2}\nabla^\mu\nabla_\mu\mathcal{D}_\nu + R^\lambda_\nu\mathcal{D}_\lambda.\end{aligned}$$

Since we assume to solve $\hat{\mathrm{Rc}} - \frac{1}{2}g\ \mathrm{tr}\,\hat{\mathrm{Rc}} = T$ and $\mathrm{div}\, T = 0$, we have

$$\frac{1}{2}\nabla^\mu\nabla_\mu\mathcal{D}_\nu + R^\lambda_\nu\mathcal{D}_\lambda = 0.$$

This wave equation has no source term. Thus, we can apply Theorem A.6 to conclude that $\mathcal{D}$ vanishes on all of $D(\Omega)$. □

Implementing the DeTurck gauge. The existence of a diffeomorphism implementing the DeTurck gauge is a classical result [24, section 2]. In our proof, we follow the argument presented in [10, proof of Theorem 14.3].

Given a Lorentzian manifold (M, g) and a background metric $\bar{g}$, DeTurck's gauge condition is

$$0 = \mathcal{D}_\mu[g, \bar{g}] = g_{\mu\nu}g^{\alpha\beta}(\Gamma^\nu_{\alpha\beta} - \bar{\Gamma}^\nu_{\alpha\beta}).$$

Note that the right-hand side is coordinate-independent and thus yields a well-defined covector $\mathcal{D}$. If it is clear which metrics are being referred to, we also write $\mathcal{D} = \mathcal{D}[g] = \mathcal{D}[g, \bar{g}]$. We discuss this gauge condition in Section 6.4 and in particular in Remark 6.17.

We need the following auxiliary result, which is essentially verbatim [10, Lemma 12.5.].

Lemma 5.7. *Let (M, g) be an $(n + 1)$-dimensional spacetime and let Σ be a smooth spacelike n-dimensional submanifold. If $p \in \Sigma$ there is a chart (U, x) with $p \in U$ and $x = (x^0, ..., x^n)$ such that $q \in U_\Sigma := U \cap \Sigma$ if and only if $q \in U$ and $x^0(q) = 0$. Furthermore*

$\partial_{x^0}|_q$ is the future directed unit normal to Σ for $q \in U_\Sigma$. If we fix $\epsilon > 0$ and denote by $g_{\mu\nu}$ the components of g in the coordinates x, then we can assume U to be such that $|g_{0i}| \leq \epsilon$, $i = 1, ..., n$ on U. If we let $a = g_{00}(p)$ and $b > 0$ be such that $g_{ij}(p)$, considered as a positive definite matrix, is bounded from below by b, we can assume that $g_{00} < a/2$ and that g_{ij}, considered as a positive definite matrix, is bounded from below by $b/2$.

Remark 5.8. Consider $p \in \Sigma$. Given any coordinate system $(\hat{U}_\Sigma, \hat{x} = (\hat{x}^1, ..., \hat{x}^n))$ of Σ centered at p, one can choose the coordinate system x in Lemma 5.7 such that $U_\Sigma \subset \hat{U}_\Sigma$ and $(x^1, ..., x^n)|_{U_\Sigma} = \hat{x}|_{U_\Sigma}$. This follows from close inspection of the proof presented in [10].

The following result is implicitly established in [10, proof of Theorem 14.3].

Proposition 5.9. *Let $(M, \bar{g})$ and (M', g) be smooth spacetimes, and Σ a smooth spacelike hypersurface M that is also embedded in M' with embedding $i\colon \Sigma \hookrightarrow M'$. Denote by N and N' the respective future-directed unit normal on Σ and $i(\Sigma)$. Then, for all $p \in i(\Sigma)$ there exist neighbourhoods $p \in W' \subset M'$ and $i^{-1}(p) \in W \subset M$ and a diffeomorphism $f\colon W \to W'$ such that*

(i) f^*g is in DeTurck gauge with respect to $\bar{g}$, i.e $\mathcal{D}[f^*g] = 0$, and

(ii) $f_*N_q = N'_{i(q)}$ for all $q \in W_\Sigma := W \cap \Sigma$, and

(iii) $f_*|_{TW_\Sigma} = i_*$.

Proof. Take $p \in i(\Sigma)$, and coordinates (U', x) of M' centered at p as in Lemma 5.7. Restrict them to coordinates $\hat{x} = (x^1, ..., x^n)|_{U'_\Sigma}$ on $U'_\Sigma = U' \cap i(\Sigma)$. Then define coordinates $\hat{y}^i := \hat{x}^i \circ i$ on $U_\Sigma := i^{-1}(U'_\Sigma)$, and extend these to Σ-adapted coordinates (V, y) of M with $(y^1, ..., y^n) = \hat{y}$ that are as in Lemma 5.7, cf. Remark 5.8. In summary:

$$\begin{array}{ccccc} M & \hookleftarrow & \Sigma & \overset{i}{\hookrightarrow} & M' \\ (V, y) & \longleftarrow & (U_\Sigma, \hat{y}) & \overset{i^*}{\longleftarrow} & (U', x) \end{array}$$

Now take a bump function $\eta \in C_0^\infty(\mathbb{R}^{n+1})$, $0 \leq \eta \leq 1$, $\eta = 1$ in an open neighbourhood of 0, with support in an open ball with center at the origin contained in $y(V) \cap x(U')$. Note that this set is non-empty, as $y(i^{-1}(p)) = 0$. Define the smooth functions

$$\begin{aligned} \Theta^\mu_{\alpha\beta}\colon \mathbb{R}^{n+1} &\longrightarrow \mathbb{R}, & \Theta^\mu_{\alpha\beta}(w) &:= \bar{\Gamma}^\mu_{\alpha\beta} \circ y^{-1}[\eta(w)w] \\ \rho_{\mu\nu}\colon \mathbb{R}^{n+1} &\longrightarrow \mathbb{R}, & \rho_{\mu\nu}(w) &:= g_{\mu\nu} \circ x^{-1}[\eta(w)w] \end{aligned}$$

where $g_{\mu\nu}$ denotes the metric components in the x coordinates, and $\bar{\Gamma}^\mu_{\alpha\beta}$ denote the Christoffel symbols of $\bar{g}$ with respect to the y-coordinates. Note that ρ defines a smooth Lorentz metric on $\mathbb{R}^{n+1}$. All components $\rho_{\mu\nu}$ are uniformly bounded. By choosing the support of η sufficiently small, we can achieve $\rho_{00} < 0$ with uniform upper bound, and $(\rho_{ij}) > 0$ as a matrix and with uniform lower bound. It follows that, trivially extending ρ to a function $\mathbb{R}^{N+(n+1)N+n+1}$, ρ can be viewed as a C^∞ (N, n)-admissible metric for every $N \in \mathbb{N}$, cf. Definition A.2.

Take now another bump function $\chi \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \chi \leq 1$, $\chi = 1$ in an open neighbourhood of 0, with support in an open ball with center at the origin contained in $x[U'_\Sigma]$. We consider the following initial value problem for a smooth function $\bar{x}\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$:

$$\begin{aligned} \Box_\rho \bar{x}^\gamma|_w &= -\rho^{\alpha\beta}\Big|_{\bar{x}(w)} \frac{\partial \bar{x}^\mu}{\partial \xi^\alpha}\frac{\partial \bar{x}^\nu}{\partial \xi^\beta}\,\Theta^\gamma_{\mu\nu}\Big|_{\bar{x}(w)} \\ \bar{x}(0,\xi^1,...,\xi^n) &= \chi(\xi^1,...,\xi^n)(0,\xi^1,...,\xi^n) \\ \frac{\partial \bar{x}}{\partial \xi^0}(0,\xi^1,...,\xi^n) &= \chi(\xi^1,...,\xi^n)(1,0,...,0) \end{aligned} \tag{5.5}$$

Herein, we denoted by ξ the Cartesian coordinates on $\mathbb{R}^{n+1}$ (with indices starting at 0), and by $\Box_\rho$ the scalar wave operator

$$\Box_\rho = \rho^{\alpha\beta}(\partial_{\xi^\alpha}\partial_{\xi^\beta} - \Lambda^\mu_{\alpha\beta}\partial_{\xi^\mu})$$

with $\Lambda^\mu_{\alpha\beta}$ the Christoffel symbols of ρ.

We already observed that the metric ρ is (N,n)-admissible, where now $N = n+1$. Similarly, the non-linearity

$$f\colon \mathbb{R}^{N+(n+1)N+n+1} \longrightarrow \mathbb{R}^N, \qquad f^\gamma[\bar{x}](w) = -\rho^{\alpha\beta}(\bar{x})(\partial_\alpha \bar{x}^\mu)(\partial_\beta \bar{x}^\nu)\Theta^\gamma_{\mu\nu}(\bar{x})$$

is C^∞ (N,n)-admissible: Obviously $f[0] = 0$ is of locally x-compact support, and the global uniform bounds on $\rho^{\alpha\beta}$, $\Theta^\gamma_{\mu\nu}$ and their derivatives implies that for every multi-index $\alpha \in \mathbb{N}_0^{N+(n+1)N+n+1}$ there exists a continuous increasing function $h_\alpha\colon \mathbb{R} \to \mathbb{R}$ such that $|(\partial^\alpha f)(w,\xi)| \le h_\alpha(|\xi|)$.

Thus, the system (5.5) is of the form such that Theorem A.4 applies. Hence, we get the smooth local solution $\bar{x}$. By construction, $\partial \bar{x}^\mu/\partial \xi^\nu = \delta^\mu_\nu$ at the origin, hence $\bar{x}$ is locally a diffeomorphism. With $W' \subset U'$ an open neighbourhood of $i(p)$, we define the coordinate system

$$\tilde{x} \coloneqq \bar{x} \circ x|_{W'}\colon W' \longrightarrow \mathbb{R}^{n+1}$$

By choosing W' small enough, we achieve that the composition $\bar{x} \circ x|_{W'}$ is well-defined, and that $\eta|_{W'} = 1$ and $\chi|_{W'_\Sigma} = 1$, where $W'_\Sigma = W \cap i(\Sigma)$. Then

$$\rho_{\mu\nu} \circ x|_{W'} = g_{\mu\nu}|_{W'} \qquad \text{and} \qquad \frac{\partial \tilde{x}^\mu}{\partial x^\nu}\Big|_{W'_\Sigma} = \delta^\mu_\nu. \tag{5.6}$$

We can furthermore choose W' such that (W', g) is globally hyperbolic with Cauchy hypersurface W'_Σ. It is a technical observation that then

$$\frac{\partial \rho_{\mu\nu}}{\partial \xi^\alpha} \circ x\Big|_{W'} = \frac{\partial [g_{\mu\nu} \circ x^{-1}]}{\partial \xi^\alpha} \circ x\Big|_{W'} = \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\Big|_{W'}.$$

Hence, on W', $\Lambda^\mu_{\alpha\beta} \circ x$ gives the Christoffel symbols of g in the x coordinates. Acting on a function, we have

$$\Box_g = \nabla^\mu \nabla_\mu = g^{\mu\nu}\partial_{x^\mu}\partial_{x^\nu} - \Gamma^\gamma \partial_{x^\gamma}$$

where $\Gamma^\gamma = g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}$ denotes the contracted Christoffel symbols of g in the x-coordinates. Thus, acting on the function $\tilde{x}^\gamma\colon W' \to \mathbb{R}$, we obtain

$$\begin{aligned} (\Box_g \tilde{x}^\gamma) \circ x^{-1}\Big|_{x(W')} &= \Box_\rho \bar{x}^\gamma|_{x(W')} = -\left(\rho^{\alpha\beta}\frac{\partial \bar{x}^\mu}{\partial \xi^\alpha}\frac{\partial \bar{x}^\nu}{\partial \xi^\beta}\Theta^\gamma_{\mu\nu}\right) \circ \bar{x}|_{x(W')} \\ &= -\left(\tilde{g}^{\mu\nu} \circ x^{-1}\right)\Big|_{x(W')} \left(\bar{\Gamma}^\gamma_{\mu\nu} \circ y^{-1} \circ \bar{x}\right)\Big|_{x(W')}, \end{aligned}$$

where $\tilde{g}^{\mu\nu}$ are the inverse metric components in the $\tilde{x}$ coordinates, and in the last line we used that

$$\rho^{\alpha\beta}\frac{\partial\bar{x}^{\mu}}{\partial\xi^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial\xi^{\beta}} = (x^{-1*}g)^{\alpha\beta}\frac{\partial\bar{x}^{\mu}}{\partial\xi^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial\xi^{\beta}} = (\bar{x}^{-1}x^{-1*}g)^{\mu\nu}\circ\bar{x} = (\tilde{x}^{-1*}g)^{\mu\nu}\circ\bar{x}$$
$$= \tilde{g}^{\mu\nu}\circ\tilde{x}^{-1}\circ\bar{x}.$$

In particular,

$$\Box_g\tilde{x}^{\gamma} = -\tilde{g}^{\mu\nu}\,\bar{\Gamma}^{\gamma}_{\mu\nu}\circ y^{-1}\circ\tilde{x}.$$

The left hand side of this equation evaluates to $-\tilde{\Gamma}^{\gamma}$, the negative of the contracted Christoffel symbol of g in the $\tilde{x}$ coordinates. That is $\mathcal{D}[\tilde{x}^{-1*}g, y^{-1*}\bar{g}] = 0$.

Now, assume W' to be small enough for the composition $y^{-1}\circ\tilde{x}$ to be well-defined. We set $W := y^{-1}\circ\tilde{x}(W')$ and then consider the diffeomorphism $f := \tilde{x}^{-1}\circ y|_W\colon W\to W'$. Then, the pullback $\hat{g} := f^*g$ satisfies $\mathcal{D}[\hat{g},\bar{g}] = y^*\mathcal{D}[\tilde{x}^{-1*}g, y^{-1*}\bar{g}] = 0$, as desired.

Next, we see that on $W_\Sigma = W\cap\Sigma$

$$N' = \partial_{\tilde{x}^0} = (\tilde{x}^{-1})_*\partial_{\xi^0} = (\tilde{x}^{-1})_*y_*N = f_*N.$$

For the first equality, we used the definition of the coordinates x and (5.6). The third equality similarly follows from the definition of y.

Finally, we see that

$$\begin{aligned} i^{-1}\circ f|_{W_\Sigma} &= i^{-1}\circ x^{-1}\circ\bar{x}^{-1}\circ y|_{W_\Sigma} \\ &= (x\circ i)^{-1}\circ\bar{x}^{-1}\circ x\circ i|_{W_\Sigma} \\ &= \mathrm{id}_{W_\Sigma}. \end{aligned}$$

The last equality follows from the coordinates x being adapted to $i(\Sigma)$ and the initial conditions (5.5) for $\bar{x}$. It follows that $f_*|_{TW_\Sigma} = i_*$. □

5.3 Unions and Intersections of Globally Hyperbolic Subsets

We prove here two small auxiliary results employed in the patching together of local solutions (Theorem 6.28) and local gauge transformations (Theorem 6.31).

The following Lemma is implicitly established in [10, proof of Theorem 14.2].

Lemma 5.10. *Let (M,g) be a Lorentzian manifold, $\Sigma\subset M$ a spacelike hypersurface, and $t\colon M\to\mathbb{R}$ a smooth function such that* $\operatorname{grad} t$ *is timelike everywhere and $\Sigma = t^{-1}(0)$. For all i in some index set I, let $W_i\subset M$ be an open subset such that $(W_i, g|_{W_i})$ is globally hyperbolic with Cauchy hypersurface $W_i\cap\Sigma$. Then, $D = \bigcup_i W_i$ is globally hyperbolic with Cauchy hypersurface $D\cap\Sigma$.*

Proof. Let $\gamma\colon(0,1)\to D$ be an inextendible causal curve in D. Then, there has to be a W_i which γ intersects non-trivially. The restriction of γ onto W_i is a disjoint union of inextendible causal curves. By global hyperbolicity of W_i, each of these curves has to intersect $W_i\cap\Sigma$ exactly once, so γ intersects $D\cap\Sigma$ at least once.

At the same time, because $\operatorname{grad} t$ is timelike and γ is causal, $t\circ\gamma$ has to be a monotonously increasing function. Hence $t\circ\gamma(s) = 0$ can happen for at most one value of $s\in(0,1)$. It follows that γ intersects $D\cap\Sigma$ exactly once. Hence, $D\cap\Sigma$ is a Cauchy HS in D. □

A result similar to the following is established in [10, proof of Theorem 14.3].

Lemma 5.11. *Let* (M, g) *be globally hyperbolic with Cauchy hypersurface* Σ*. Let* U *and* V *be globally hyperbolic subsets with* $U \cap V \neq \emptyset$ *such that* $U_\Sigma = U \cap \Sigma$ *and* $V_\Sigma = V \cap \Sigma$ *are Cauchy hypersurfaces in* U *and* V*, respectively.*

Then $U \cap V$ *is globally hyperbolic with Cauchy hypersurface* $U_\Sigma \cap V_\Sigma$*. In particular, an isometry* $f\colon (U \cap V, g) \to (M', g')$ *is uniquely determined by the restriction of its differential to* $U_\Sigma \cap V_\Sigma$*.*

Proof. Let $p \in U \cap V$. Take a $U \cap V$-inextendible timelike curve $\gamma\colon I \to U \cap V$ through p. Extend it to an M-inextendible timelike curve $\gamma_M\colon I_M \to M$. The respective restrictions of γ_M to U and V are a disjoint union of inextendible curves. Each of these would have to intersect either U_Σ or V_Σ exactly once, hence each union contains exactly one inextendible curve. Denote them by $\gamma_U\colon I_U \to U$ and $\gamma_V\colon I_V \to V$ respectively. γ_U and γ_V go through p and intersect $U_\Sigma \cap V_\Sigma$ exactly once and in the same point. The same thus holds for $\gamma_\cap\colon I_U \cap I_V \to U \cap V$. $\gamma_\cap$ can only have one connected component, because the intersection of the two open intervals I_U and I_V is either another open interval or empty (which it can't be). Thus, $\gamma_\cap$ is a $U \cap V$-extension of γ, implying $\gamma_\cap = \gamma$. We conclude that γ intersects $U_\Sigma \cap V_\Sigma$ exactly once.

Finally, an isometry between two connected semi-Riemannian manifolds is uniquely determined by its differential at any point. From $U_\Sigma \cap V_\Sigma$ being Cauchy in $U \cap V$, it follows that every connected component of $U \cap V$ contains a point in $U_\Sigma \cap V_\Sigma$. The claim follows. □

6 The Initial Value Problem for the Generalised Einstein Equations

This chapter presents the results of [22]. These establish the well-posedness of the initial value problem (IVP) for the generalised Einstein equations (GEE) in the setting of closed divergence operators. In Section 6.1, we state the GEE and explain our focus on the case of closed divergence. We develop the initial value formulation of the GEE in Section 6.2. In Section 6.3, we describe the *Einstein frame GEE*, which is a well-defined Einstein-matter system equivalent to the GEE and thus of great utility to the analysis. We describe DeTurck's gauge condition, introduce the *generalised Lorenz gauge*, and state the hyperbolically reduced system of equations in Section 6.4. In Section 6.5, we explain the construction of analytic initial data to the Einstein frame GEE from the geometric initial data given in the IVP. We establish the existence of a globally hyperbolic development in Section 6.6, show geometric uniqueness in Section 6.7. Finally, we conclude the existence of an MGHD and thus establish the third main result of this work in Section 6.8.

In the following, we outline our approach to the IVP for the GEE, reproducing the outline of [22, chapter 5].

In solving the initial value problem, we closely follow [10], which proves the existence of a maximal globally hyperbolic development (MGHD) in the case of the Einstein equations coupled to a scalar field. It should be noted that [10] also provides a comprehensive introduction to the required mathematical theory, and for that reason is the main source for the present section.

The main ingredient to proving the existence of an MGHD in [10] is that locally, over a suitable coordinate patch, the Einstein equations become a system of PDEs for a vector valued function $u\colon \mathbb{R}^{n+1} \to \mathbb{R}^N$, $1 \leq n \in \mathbb{N}$ that takes, after some modification, the form

$$\begin{aligned} g[u]^{\mu\nu}\partial_\mu\partial_\nu u &= f[u], \\ u(T_0, \cdot) &= U_0, \\ \partial_t u(T_0, \cdot) &= U_1, \end{aligned} \tag{6.1}$$

where $T_0 \in \mathbb{R}$, $g \in C^\infty(\mathbb{R}^{N+(n+1)N+n+1}, \mathcal{C}_n)$ a so-called C^∞ (N,n)*-admissible metric* ($\mathcal{C}_n$ denotes the space of canonical Lorentz matrices), $f \in C^\infty(\mathbb{R}^{N+(n+1)N+n+1}, \mathbb{R}^N)$ a so-called C^∞ (N,n)*-admissible non-linearity*, and $U_0, U_1 \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$ initial data. We employed the notation

$$g[u](t,x) \coloneqq g(t, x, u(t,x), \partial_0 u(t,x), ..., \partial_n u(t,x))$$

and similarly for f. We provide the definitions of the two notions of C^∞ (N,n)-admissibility in appendix A.1. Of main importance to us is [10, Corollary 9.16.] (which we include for completeness in this work as Theorem A.4): The system of PDEs (6.1) admits a unique maximal solution.

Remark 6.1. It is known that, locally and in the right gauge, the vacuum Einstein equations [38, Theorem 7.1.], the Einstein-Maxwell equations [38, §10.1.], and the Einstein equations coupled to a scalar field [10, §14] are of the form (6.1). A sizeable part of this work will be to prove that, in fact, the GEE are also locally of this form.

Remark 6.2. As far as the author of this text is aware, [10] is the first work to provide a comprehensive and rigorous formulation of this approach to establish well-posedness.[18] For

[18] It seems to the author of this work that it was well-known that such a strategy could be made to work. A remark to that extent is made in [25, section 5.4.], and Ringström himself presents his foundational work as expository in the introduction to [10].

this reason, and because it is precisely the goal to establish basic well-posedness results for the system studied, we view it as desirable to follow Ringström's work. Its adaptation to other systems of matter is, at least in principle, straightforward. If one wants to discuss any given matter coupled to the Einstein equations, it should suffice to check that, in a local coordinate neighbourhood, the system is of a form such that a local existence and uniqueness result (such as Theorem A.4) applies. The entire rest of the proof is then expected to hold still.[19] However, this is not obvious, as [10] focuses on the particular case of the Einstein equations coupled to a scalar field and we consider a different matter model. It is furthermore complicated by the GEE only being of a good form after a locally defined conformal transformation (to the Einstein frame), which means that the background metric which was defined globally in [10] is in our setting only defined locally.

The GEE are the equations of generalised Ricci and scalar flatness, $\mathcal{R}c^{\pm}(\mathcal{G}, \mathrm{div}) = 0$ and $\mathcal{S}c(\mathcal{G}, \mathrm{div}) = 0$. In the context of the IVP, it is useful to express the pair $(\mathcal{G}, \mathrm{div})$ in terms of the data (g, H, e), where $g^{-1} = \mathcal{G}|_{\pi^* T^* M}$, $H \in \Omega^3_{\mathrm{cl}}(M)$ is the preferred representative of the Ševera class, and $e \in \Gamma(\mathbb{T}M)$ is the generalised dilaton, cf. Propositions 2.17 and 2.24. We will make the assumption that the dilaton is closed, i.e. $e = 2\xi \in \Gamma(T^*M)$, $\mathrm{d}\xi = 0$. Then the GEE can be written as, cf. Section 6.1,

$$\mathrm{d}^* H = -i_\xi H, \qquad \mathrm{Rc} = \frac{H^2}{4} - \nabla \xi, \qquad \frac{|H|^2}{6} = \mathrm{d}^*\xi + |\xi|^2 . \tag{6.2}$$

Evidently, the equations (6.1) are of second order. To view the GEE (6.2) as a second order system, we have to work with potentials B and ϕ that respectively encode the dynamics of H and ξ. Regarding the dilaton, we write locally $\xi = \mathrm{d}\phi$. Regarding the three-form, we take another approach and introduce a "background field" $\bar{H}$ and split $H = \bar{H} + \mathrm{d}B$. Then we have B as the dynamical object, meaning that its development will be defined by solving the equations of motion, while the background field $\bar{H}$ encodes global data, i.e. the cohomology class $[H]$. Note that all developments we consider will be defined on a tubular neighbourhood D of $\Sigma_0 := \{0\} \times \Sigma$ in $M = \mathbb{R} \times \Sigma$. In particular, Σ will be a deformation retract of the manifold D, implying that the initial value of H on Σ determines the cohomology class $[H]$. The approach allows for a simple geometric condition uniquely determining the initial value of the B-field, making it straightforward to obtain local uniqueness of solutions: Choosing the background field such that initially $\bar{H}\big|_\Sigma = H|_\Sigma$, one may demand $B|_\Sigma = 0$.

Implementing $H = \bar{H} + \mathrm{d}B$ and $\xi = \mathrm{d}\phi$ in the GEE (6.7), we obtain the system

$$\mathrm{d}^*\mathrm{d}B = -\mathrm{d}^*\bar{H} - i_\xi H, \qquad \mathrm{Rc} = \frac{H^2}{4} - \nabla^2\phi, \qquad \Box\phi = -\frac{|H|^2}{6} + |\xi|^2 , \tag{6.3}$$

where we used that $\Box\phi = \nabla^\mu\nabla_\mu\phi = -\mathrm{d}^*\mathrm{d}\phi$.

Note that, taken at its face, the system (6.3) is not of the form (6.1). In the Ricci equation, there are two issues, and a third issue is in the B-field equation.

The first issue arises from the second order term $\nabla^2\phi$. This causes the equation not to be locally of the desired form, since the vector-valued function u has to encompass all

[19] This adaptability should also be a feature of other approaches. For example, a claim to that extent is made in [25, section 5.1.], and DeTurck seemingly refers to this being the case in [23, remarks below Theorem 5.5].

dynamical objects $g_{\mu\nu}$, $B_{\mu\nu}$, and ϕ, i.e. roughly[20]

$$u = (g_{\mu\nu}, B_{\mu\nu}, \phi)$$

This can be overcome by a conformal transformation of the metric:

$$\tilde{g} := e^{-2\phi\,/(d-2)}g$$

Under this conformal transformation, the Ricci tensor absorbs the $\nabla^2\phi$-term. Considering then $(\tilde{g}_{\mu\nu}, B_{\mu\nu}, \phi)$ as our dynamical variables, the issue is solved. In the physics literature, this is known as "adopting the Einstein frame". The name is due to the fact that only in this "frame" energy-momentum is off-shell conserved, i.e. without imposing the Ricci-equation and only due to the dilaton and B-field equations one has

$$\mathrm{div}_{\tilde{g}}\ T[\tilde{g}, B, \phi] = 0$$

where T is the energy-momentum tensor. We discuss this in Section 6.3.

The other two issues are similar: In the Ricci and B-field equation, the respective differential operator acting on the metric or B-field is not locally of the form $g^{\mu\nu}\partial_\mu\partial_\nu +$ lower order derivatives. DeTurck's gauge (among others[21]) was developed precisely to solve this issue. Drawing from the DeTurck and Lorenz gauge, we introduce a gauge condition which achieves the same for the B-field equation. We describe the Lorenz and DeTurck gauge in Section 6.4. We show that, with both gauge conditions implemented, the Einstein frame GEE are equivalent to a system locally of the form (6.1). (In fact, they form a quasi-linear hyperbolic PDE with metric principal symbol as discussed in Section 5.1.)

Variant of the GEE	String Frame	Einstein Frame	Modified Einstein Frame
Relevance	study subject	auxiliary	metric principal symbol
Dynamical Objects	g, H, ξ	$\tilde{g}, H, \phi$	$\tilde{g}, B, \phi$
Defined	globally	locally w.r.t. a choice of ϕ	locally w.r.t. a choice of ϕ, $\bar{g}$, and $\bar{H}$
Relation to String Frame	identity	$\tilde{g} = \exp(-2\kappa\phi)g$	equivalent to Einstein frame if gauges are implemented

Table 1: This table showcases the three versions of the GEE relevant to this work and their main properties. By convention, g denotes the string frame metric, $\tilde{g}$ the Einstein frame metric, H a closed three-form, ξ a closed one-form (the dilaton), ϕ a real-valued function (a dilaton potential), $\bar{g}$ a "background" metric, and $\bar{H}$ a closed "background" three-form.

Remark 6.3. We remark on the work [11] of Choquet-Bruhat establishing well-posedness of the Cauchy problem for 11-dimensional $N = 1$ supergravity, which is a system related but inequivalent to the one we study. Notably, with the so-called "three-index photon" A, it contains an analogue of the B-field (which could accordingly be referred to as the two-index photon). In [11], the coordinate-dependent generalisation $\partial^\lambda A_{\lambda\mu\nu} = 0$ of the Lorenz gauge is imposed on A in local harmonic coordinates.

[20] Of course, this would not yield a well-defined function on $\mathbb{R}^{n+1}$, as one needs to make use of a coordinate chart to be able to speak of the components. But, for now, to discuss the shape of the PDEs, we will ignore this technical detail.

[21] The *wave gauge* (or *harmonic coordinate condition*) is perhaps the most famous gauge condition rendering the Ricci tensor a hyperbolic operator. It is a condition on the set of coordinates in which the Ricci tensor is expressed. Notably, Choquet-Bruhat made use of the wave gauge in her famous work on the Cauchy problem for the vacuum Einstein equations [7].

The Cauchy Problem for General Einstein-Matter Systems. One can understand the GEE with closed divergence as an Einstein-matter system with matter given by H and ξ. Existence and uniqueness of solutions to Einstein-matter equations has been the subject of extensive investigation, and is established under varying assumptions of well-behavedness on the matter. In the following, we give a brief overview of relevant literature, outlining ways of obtaining well-posedness in different approaches, and motivating our own in the Sections 6.3-6.7.

Generally, the assumption of "well-behavedness" on the matter is needed to guarantee two properties. First, the energy-momentum tensor has to be divergence-free as a consequence of the matter equations alone (i.e. without requiring the energy-momentum tensor to equal the Einstein tensor). This assumption ensures that the gauge condition of choice for turning the Einstein equations into a hyperbolic system (e.g. DeTurck's gauge) is preserved under development of initial data by the Einstein equations, cf. Lemma 5.6. We see in Section 6.3 that the GEE only satisfy this requirement after a conformal transformation of the system (referred to in physics as a transformation to the Einstein frame). The Einstein frame requires a choice of potential ϕ for the dilaton, $\xi = \mathrm{d}\phi$, and can therefore only be implemented locally. Second, in a suitable gauge, the combined Einstein-matter system has to form a hyperbolic system with (in some notion) locally well-behaved IVP. Given these two properties, there are well-understood procedures to construct a development to initial data, prove that any two developments are an extension of a common development, and thus conclude the existence of an MGHD.

One of the most prominent treatments of the Cauchy problem for Einstein-matter systems may be found in Hawking and Ellis [39, section 7.7]. We will not pursue this approach, but would like to explain its main idea and difficulties. Fundamentally, Hawking and Ellis establish the existence and uniqueness of solutions via a fixed-point argument (based on the seminal work [7] by Choquet-Bruhat). Given initial data on a hypersurface in an ambient spacetime, the argument involves the map $\alpha\colon g_0 \mapsto g_1$ on the space of Lorentzian metrics, where g_1 is defined as follows. Solve the matter equations with g_0 as a background, then find a solution h to the linearised Einstein equations with given stress-energy tensor, and finally set $g_1 = g_0 + h$. Hawking and Ellis prove α to be a contraction, yielding the unique solution to the Einstein-matter equations.

The approach by Hawking and Ellis produces as main requirements on the matter equations that they have a well-defined Cauchy problem *given a background metric*, and that they have good stability behaviour under changes of the background metric. Both properties should be straightforward to show in our setting. What complicates things is the additional requirement that the stress-energy tensor is polynomial in the matter fields, their first covariant derivatives, and the metric. From Proposition 6.15, we know this requirement to be violated by the energy-momentum tensor associated to H and ϕ in the Einstein frame. However, this seems to be a purely technical point. A more substantial problem in adopting this approach here is that it works with the equations on the whole ambient spacetime, but we only know the GEE to be well-behaved in the locally defined Einstein frame. Therefore, we abandon the approach presented in [39] in favour of Ringström's approach as presented in [10].

We briefly note that one can also obtain well-posedness results in the ADM formalism. A comprehensive treatment of the Cauchy problem for general Einstein-matter systems in the ADM formalism can be found in [40].

6.1 The Generalised Einstein Equations

In this section, which reproduces [22, chapter 4], we discuss the generalised Einstein equations (GEE). From here on out, we do not describe them in terms of generalised geometry but in terms of objects naturally living on the manifold M.

Let (M,g) be a Lorentzian manifold, $H \in \Omega^3_{\mathrm{cl}}(M)$ a closed three-form, $X \in \Gamma(TM)$ a vector field and $\xi \in \Gamma(T^*M)$ a one-form. Denote by $H^2 \in \mathrm{Sym}^2(M)$ the symmetric two-tensor obtained from $H \otimes H$ by making two non-trivial contractions with the metric g, more precisely $H^2_{\mu\nu} = g^{\kappa\lambda} g^{\rho\pi} H_{\mu\kappa\rho} H_{\nu\lambda\pi}$.

Definition 6.4. The tuple (M,g,H,X,ξ) is called *generalised Ricci flat* if (cf. Corollary 3.11)

$$4\,\mathrm{Rc} = H^2 - 4[\nabla\xi]^{\mathrm{sym}}, \qquad \mathrm{d}^*H = 2[\nabla X^\flat]^{\mathrm{antisym}} - i_\xi H \tag{6.4}$$

Furthermore, (M,g,H,X,ξ) is called *generalised scalar flat* if (cf. Corollary 3.8)

$$\mathrm{Sc} = \frac{|H|^2}{12} + 2\mathrm{d}^*\xi + |X|^2_g + |\xi|^2_g \,.$$

Finally, (M,g,H,X,ξ) is called *generalised Einstein (in the string frame)* if it is both generalised Ricci and scalar flat.

Remark 6.5. Note that generalised Ricci flatness does not in general imply generalised scalar flatness. For a conceptual explanation of this, confer the definitions in Section 2.6.

Remark 6.6. Compared to the condition of generalised Ricci flatness found in [6, Proposition 3.30], our version does not include the compatibility equations $L_X g = 0$ and $\mathrm{d}\xi = i_X H$. They ensure compatibility of the divergence operator and the generalised metric in the sense that they are equivalent to the generalised vector field $X + \xi$ being an infinitesimal isometry, cf. Propostion 2.27. Note that the compatibility equations are implied by our assumption (6.5). An in-depth account of the different notions of curvature employed in the generalised geometry literature has been given in [33].

For our discussion of the initial value problem, we restrict our attention to the case of

$$X = 0 \qquad \text{and} \qquad \mathrm{d}\xi = 0 \tag{6.5}$$

Recall from Definition 2.28 that we refer to condition (6.5) also as the divergence operator being closed and, if $\xi = \mathrm{d}\phi$ for a function $\phi \in C^\infty(M)$, as the divergence operator being exact. The exact case is of particular importance for three reasons.

(1) The GEE for exact divergence can be derived from a variational principle. This is done by defining the generalised Einstein Hilbert action to be the integral of the generalised scalar curvature (with measure defined by the divergence operator), and then varying over the space of generalised metrics and exact divergence operators [6, § 3.7].

(2) In the case of exact divergence, $\xi = \mathrm{d}\phi$ for some $\phi \in C^\infty(M)$, the GEE are known to correspond to the equations of motion for the bosonic part of the NS-NS sector in type II ten dimensional theories of supergravity [2]. Under this correspondence, the function ϕ takes on the role of the *dilaton potential*, a physical field.

(3) The correspondence to supergravity allows us to borrow a trick well-known in the physics literature: going into the Einstein frame. This is a name for a conformal transformation of the metric with the dilaton ϕ. The trick is crucial to our analysis, as only in the Einstein frame we see the PDE to be a wave equation of a form that we recognise and can deal with. We elaborate on this in Section 6.3.

The reasons (1) and (3) given above indicate why working in the slightly more general case of a closed dilaton ξ is feasible. This is because in both cases, access to a potential $\xi = \mathrm{d}\phi$ is only required locally. For reason (1), this is because it suffices to define a generalised Einstein Hilbert action locally to obtain the GEE (which are local equations) from their variation. For reason (3), it is because we will only employ the Einstein frame locally.

We want to rewrite the GEE slightly. By (6.4) we have

$$\mathrm{Sc} = \frac{|H|^2}{4} + \mathrm{d}^*\xi. \tag{6.6}$$

Thus, $\mathcal{R}c = 0$ and $\mathcal{S}c = 0$ combined are equivalent to

$$\mathrm{d}^*H = -i_\xi H, \qquad \mathrm{Rc} = \frac{H^2}{4} - \nabla\xi, \qquad \frac{|H|^2}{6} = \mathrm{d}^*\xi + |\xi|^2 . \tag{6.7}$$

We refer to the first equation as *the H-field equation*, to the second equation as *the Ricci equation*, and to the third equation as *the dilaton equation*. Writing $H = \bar{H} + \mathrm{d}B$ for a closed three-form $\bar{H}$ and a two-form B, we also refer to the first equation as *the B-field equation*.

Remark 6.7. The GEE (6.7) are invariant under generalised diffeomorphisms on M in the sense of Definition 2.1. One can see this as follows. A generalised diffeomorphism $F = (f, B)$ acts on the generalised metric $\mathcal{G}_1 = (g_1, B_1)$ as well as on the twist $\bar{H}_1$ of the underlying exact Courant algebroid:

$$(g_2, B_2) \coloneqq F^*\mathcal{G}_1 = (f^*g_1, f^*B_1 - B), \qquad [\cdot,\cdot]_{\bar{H}_2} \coloneqq F^*[\cdot,\cdot]_{\bar{H}_1} = [\cdot,\cdot]_{f^*\bar{H}_1 + \mathrm{d}B}$$

Evidently, the metric g_1 transforms by pullback with the diffeomorphism, $g_2 = f^*g_1$. Similarly, the sum $H_1 = \bar{H}_1 + \mathrm{d}B_1$ transforms by pullback with the diffeomorphism, $H_2 = \bar{H}_2 + \mathrm{d}B_2 = f^*H_1$. Also, one can see (assuming closed divergence operator!) that the dilaton ξ transforms by pullback with the diffeomorphism. Thus, every object in (6.7) transforms by pullback. In other words, the equations are invariant.

Note that invariance under generalised diffeomorphisms encodes in particular the following two invariant aspects. One, we are free to choose any decomposition $H = \bar{H} + \mathrm{d}B$ of the twist H in terms of a background field $\bar{H}$ and a B-field. Two, given a fixed background field $\bar{H}$, we still have invariance under closed B-field transformations.

6.2 The Initial Value Formulation

This section reproduces [22, section 5.1].

Our goal is to show that the initial value problem to the GEE (6.7) is well-behaved. That is, we want to show that every choice of initial data for the equations uniquely admits a development. So, to begin with, we have to understand what constitutes initial data for the equations (6.7).

Initial data for the vacuum Einstein equations is well-understood (cf. e.g. [10, §14.2.]). It consists of a Riemannian manifold (Σ, g_Σ) and a symmetric two-tensor k on Σ (representing the second fundamental form), all such that the constraint equations are satisfied (cf. [10, cf. Proposition 13.3])

$$\begin{aligned} \mathrm{Sc}_\Sigma + (\operatorname{tr} k)^2 - |k|^2 &= 0 \\ \operatorname{div}_\Sigma k - \nabla^\Sigma \operatorname{tr} k &= 0 \end{aligned}$$

Herein, ∇^Σ denotes the LC connection of g_Σ, Sc_Σ its scalar curvature, and $\operatorname{div}_\Sigma k$ the contraction of the first two indices of $\nabla^\Sigma k$ with g_Σ. A *development* of this data is a semi-Riemannian embedding $\iota\colon (\Sigma, g_\Sigma) \hookrightarrow (M, g)$ into a Lorentzian manifold (M, g) satisfying the Einstein equations. The development is called *globally hyperbolic*, if (M, g) is.

For the GEE, we have with H and ξ two additional fields, for which the initial data has to determine the initial value. Notice that the equation of motion for H and ξ is first order. Hence, we expect initial data for (6.7) to additionally include the following objects defined on Σ: a closed three-form $H_0^{\|}$, a two-form h_0, a closed one-form $\xi_0^{\|}$ and a real-valued function x_0. A *development* of this data is then a semi-Riemannian embedding $\iota\colon (\Sigma, g_\Sigma) \hookrightarrow (M, g)$ into a Lorentzian manifold (M, g), together with a closed three-form $H \in \Omega^3_{\mathrm{cl}}(M)$ and a closed one-form $\xi \in \Omega^1_{\mathrm{cl}}(M)$, all such that the GEE are satisfied and on $\iota(\Sigma)$

$$\iota^* H = H_0^{\|}, \qquad \iota^*[H(N)] = h_0, \qquad \iota^* \xi = \xi_0^{\|}, \qquad \iota^*[\xi(N)] = x_0. \tag{6.8}$$

In the formalism of generalised geometry, we have established in Corollary 4.40 that for a tuple of objects $(\Sigma, g_\Sigma, k, H_0^{\|}, h_0, \xi_0^{\|}, x_0)$ as above to have a development, it is necessary that

$$\begin{aligned} \mathrm{Sc}_\Sigma + (\operatorname{tr} k)^2 - |k|^2 &= \frac{\left|H_0^{\|}\right|^2}{12} + \frac{|h_0|^2}{4} + 2(\mathrm{d}^\Sigma)^* \xi_0^{\|} + 2(\operatorname{tr} k) x_0 + \left|\xi_0^{\|}\right|^2 - x_0^2 \\ \operatorname{div}_\Sigma k - \mathrm{d}^\Sigma \operatorname{tr} k &= \frac{1}{4} C(h_0, H_0^{\|}) - \mathrm{d}^\Sigma x_0 + i_{\xi_0^{\|}} k \\ 0 &= \left((\mathrm{d}^\Sigma)^* + i_{\xi_0^{\|}} \right) h_0 \end{aligned} \tag{6.9}$$

Herein, we denoted by $C(h_0, H_0^{\|})$ the one-form obtained from h_0 and $H_0^{\|}$ by contracting indices as follows:

$$C_i(h_0, H_0^{\|}) = g_\Sigma^{kl} g_\Sigma^{mn} (h_0)_{km} (H_0^{\|})_{lni}$$

Equations (6.9) are the (string frame) constraint equations for the GEE.

Remark 6.8. Instead of obtaining these constraint equations from the generalised Gauß and Codazzi equations established in Section 4.4, one can also obtain them from a more down-to-earth viewpoint. In the GEE (6.7), the Ricci equation is known to produce constraint equations, and these are the first two equations in (6.9). The H-field equation is a form wave equation, and we see these to produce a constraint equation in Lemma A.8. This is the third equation in (6.9).

We formalise the notions of initial data, constraint equations, and the IVP for the GEE in the following two definitions. These are based on the analogous definition for the Einstein equations coupled to a scalar field in [10, Definition 14.1].

Definition 6.9. *Initial data (in terms of fields)* for the (string frame) GEE (6.7) in $n+1$ dimensions is a tuple $(\Sigma, g_0, k, H_0^{||}, h_0, \xi_0^{||}, x_0)$ consisting of an n-dimensional Riemannian manifold (Σ, g_Σ), and the following objects on Σ:

(1) a symmetric $(0,2)$-tensor k,

(2) a closed three-form $H_0^{||}$ and a two-form h_0,

(3) a closed one-form $\xi_0^{||}$ and a real-valued function x_0.

all such that the *constraint equations* (6.9) are satisfied.

Definition 6.10. Let $(\Sigma, g_0, k, H_0^{||}, h_0, \phi_0, \phi_1)$ be initial data. Then the *initial value problem* (IVP) is to find a *(string frame) development* of the initial data, that is a SuGra spacetime (M, g, H, ϕ) satisfying the (string frame) GEE and a semi-Riemannian embedding $i\colon (\Sigma, g_0) \hookrightarrow (M, g)$ such that k corresponds to the second fundamental form on $i(\Sigma)$ and H and ϕ induce on Σ the given initial data via the relations (6.8). A development (M, g, H, ϕ) is called globally hyperbolic, if $i(\Sigma)$ is a Cauchy hypersurface in (M, g).

Given a background three-form $\bar{H}$ and working locally, we can express the fields H and ξ via potentials B and ϕ:

$$H = \bar{H} + \mathrm{d}B, \qquad \xi = \mathrm{d}\phi. \tag{6.10}$$

Note that the initial data for ϕ is described by $\phi_0 = \phi|_\Sigma$ and $\phi_1 = N(\phi)$. It holds $\xi_0^{||} = \mathrm{d}^\Sigma \phi_0$ and $x_0 = \phi_1$.

To discuss initial data in terms of the potential B, we consider the following

Lemma 6.11. *Let $H = \mathrm{d}B$. Furthermore, express B and $\nabla_N B$ in terms of objects natural on Σ:*

$$B|_\Sigma = B_0^{||} - N^\flat \wedge b_0, \qquad \nabla_N B|_\Sigma = B_1^{||} - N^\flat \wedge b_1 \tag{6.11}$$

where $B_0^{||}, B_1^{||} \in \Omega^2(\Sigma)$ and $b_0, b_1 \in \Omega^1(\Sigma)$. Then, with $H_0^{||}$ and h_0 as in (6.8),

$$H_0^{||} = \mathrm{d}^\Sigma B_0^{||}, \qquad h_0 = B_1^{||} - \mathrm{d}^\Sigma b_0 - k \cdot B_0^{||}.$$

Herein, $\cdot$ is the natural action of endomorphisms on differential forms.

Proof. One finds

$$\mathrm{d}B = \mathrm{d}^\Sigma B_0^{||} - N^\flat \wedge \left[B_1^{||} - \mathrm{d}^\Sigma b_0 - k \cdot B_0^{||} \right].$$

as a special case of Lemma B.2. □

We summarise the interpretation of this statement in the following Remark.

Remark 6.12. In the decomposition (6.11), only the initial values of $B_0^{||}$ and the difference $B_1^{||} - \mathrm{d}^\Sigma b_0$ are constrained by the initial value of H. Note that, for initial data induced on a spacelike hypersurface one can always achieve $b_0 = 0$ via a closed B-field transformation on M, cf. Lemma 6.29. With this assumption, the initial data precisely constrains $B_0^{||}$ and $B_1^{||}$ - the former up to addition of a closed two-form, and then the latter uniquely (but depending on that freedom).

We conclude from the preceding discussion that *initial data (partially) in terms of potentials* differs from initial data in terms of fields in that one replaces $(H_0^{\|}, h_0^{\|})$ by $(B_0^{\|}, B_1^{\|})$ and/or $(\xi_0^{\|}, x_0)$ by (ϕ_0, ϕ_1), demanding then that with (6.10) the constraint equations (6.9) are satisfied.

6.3 The Einstein Frame

This section reproduces [22, section 5.2].

Recall that the $\nabla^2\phi$-term in (6.3) produces out-of-place second order derivatives, if we want to understand the system as being of the form (6.1). In this section, we discuss the GEE in the Einstein frame, and show that in this frame, there are no such out-of-place derivatives. Moreover, we see that in this frame, the energy-momentum tensor is divergence-free as a consequence of the B-field and dilaton equation alone.

We emphasise that the Einstein frame can (for non-exact dilaton) only be obtained locally, after writing $\xi = \mathrm{d}\phi$. In the context of the IVP, one should therefore imagine the SuGra spacetimes we discuss here to be a suitable small subset of the development one considers.

The Einstein frame is achieved by the conformal transformation described in the following Proposition.

Proposition 6.13. *Let (M, g, H, ϕ) be a SuGra spacetime of dimension $d = n + 1$. Denote $\xi = \mathrm{d}\phi$. Define with $\kappa = \frac{1}{d-2}$ the Einstein frame metric as $\tilde{g} := e^{-2\kappa\phi}g$. Then, combined generalised Ricci and scalar flatness (6.7) is equivalent to*

$$\begin{aligned}
\tilde{\mathrm{d}}^* H &= -\frac{4}{d-2}\tilde{\iota}_\xi H \\
\tilde{\mathrm{Rc}} &= \frac{1}{d-2}\left[\xi \otimes \xi - \frac{e^{-4\kappa\phi}}{6}\,|H|^2_{\tilde{g}}\,\tilde{g}\right] + e^{-4\kappa\phi}\frac{H^{2,\tilde{g}}}{4} \\
\tilde{\Box}\phi &= -\frac{e^{-4\kappa\phi}}{6}\,|H|^2_{\tilde{g}}
\end{aligned} \tag{6.12}$$

Herein, $\tilde{\iota}_\alpha H = H(\alpha^{\tilde{\sharp}})$, where the musical ismorphism $\tilde{\sharp}$ comes from the Einstein frame metric $\tilde{g}$. Furthermore, $\tilde{\Box} = \tilde{\nabla}^\mu\tilde{\nabla}_\mu$, and $H^{2,\tilde{g}} \in \mathrm{Sym}^2(M)$ denotes the symmetric two-tensor obtained by contracting with $\tilde{g}$, i.e. $H^{2,\tilde{g}}_{\mu\nu} = \tilde{g}^{\kappa\lambda}\tilde{g}^{\omega\pi}H_{\mu\kappa\omega}H_{\nu\lambda\pi} = e^{4\kappa\phi}H^2_{\mu\nu}$.

Proof. Equivalence between (6.7) and (6.12) follows from simple calculations, utilising the following well-known identities for conformal transformations $\tilde{g} = e^{2\varphi}g$:

$$\begin{aligned}
\tilde{\mathrm{Rc}} &= \mathrm{Rc} - (d-2)(\nabla^2\varphi - \mathrm{d}\varphi \otimes \mathrm{d}\varphi) - (\Box\varphi + (d-2)\,|\mathrm{d}\varphi|^2_g)g, \\
\tilde{\nabla}\alpha &= \nabla\alpha + \langle\alpha, \mathrm{d}\varphi\rangle_g\, g - (\mathrm{d}\varphi \otimes \alpha + \alpha \otimes \mathrm{d}\varphi), \\
e^{2\varphi}\tilde{\mathrm{d}}^*\omega &= \mathrm{d}^*\omega - (d-2p)i_{\mathrm{d}\varphi}\omega.
\end{aligned}$$

Herein, α denotes an arbitrary one-form, and ω a p-form. Inserting $\varphi = -\kappa\phi$, this becomes

$$\begin{aligned}
\tilde{\mathrm{Rc}} &= \mathrm{Rc} + (\nabla^2\phi + \frac{1}{d-2}\mathrm{d}\phi \otimes \mathrm{d}\phi) + \frac{1}{d-2}(\Box\phi - |\mathrm{d}\phi|^2_g)g, \\
\tilde{\nabla}\alpha &= \nabla\alpha - \frac{1}{d-2}\left[\langle\alpha, \mathrm{d}\phi\rangle_{\tilde{g}}\,\tilde{g} - (\mathrm{d}\phi \otimes \alpha + \alpha \otimes \mathrm{d}\phi)\right], \\
\tilde{\mathrm{d}}^*\omega &= e^{2\kappa\phi}\mathrm{d}^*\omega + \frac{d-2p}{d-2}\tilde{\iota}_{\mathrm{d}\phi}\omega.
\end{aligned}$$

We start with the first equation.

$$\begin{aligned}
&& \mathrm{d}^*H &= -i_\xi H \\
\iff && e^{-2\kappa\phi}\left(\tilde{\mathrm{d}}^*H - \frac{d-6}{d-2}\tilde{\imath}_\xi H\right) &= -i_\xi H \\
\iff && \tilde{\mathrm{d}}^*H &= \frac{-4}{d-2}\tilde{\imath}_{\mathrm{d}\phi}H
\end{aligned}$$

Noting that the dilaton equation can be stated as $\Box\phi - |\mathrm{d}\phi|_g^2 = -\frac{|H|^2}{6}$, we continue with the second equation.

$$\begin{aligned}
&& 4\,\mathrm{Rc} &= H^2 - 4\nabla\xi \\
\iff && \tilde{\mathrm{R}}\mathrm{c} &= \frac{1}{d-2}\left[\mathrm{d}\phi\otimes\mathrm{d}\phi + \left(\Box\phi - |\mathrm{d}\phi|_g^2\right)g\right] + \frac{H^2}{4} \\
\iff && \tilde{\mathrm{R}}\mathrm{c} &= \frac{1}{d-2}\left[\mathrm{d}\phi\otimes\mathrm{d}\phi - \frac{|H|_g^2}{6}\,g\right] + \frac{H^2}{4} \\
\iff && \tilde{\mathrm{R}}\mathrm{c} &= \frac{1}{d-2}\left[\xi\otimes\xi - e^{-4\kappa\phi}\frac{|H|_{\tilde g}^2}{6}\,\tilde g\right] + e^{-4\kappa\phi}\frac{H^{2,\tilde g}}{4}
\end{aligned}$$

Finally, we calculate

$$\begin{aligned}
\tilde\Box\phi &= \mathrm{tr}_{\tilde g}(\tilde\nabla^2\phi) = \mathrm{tr}_{\tilde g}(\tilde\nabla\mathrm{d}\phi) \\
&= \mathrm{tr}_{\tilde g}\left(\nabla^2\phi - \frac{1}{d-2}\left[\,|\mathrm{d}\phi|_{\tilde g}^2\,\tilde g - 2\,\mathrm{d}\phi\otimes\mathrm{d}\phi\right]\right) \\
&= e^{2\kappa\phi}\left(\Box\phi - |\mathrm{d}\phi|_g^2\right) \\
&= -e^{-4\kappa\phi}\frac{|H|_{\tilde g}^2}{6}.
\end{aligned}$$

We are done. □

Let us discuss the IVP formulation of the Einstein frame GEE. The notions of *Einstein frame initial data* (in fields and potentials) and *Einstein frame developments* are directly analogous to the corresponding notions in the string frame developed in Definitions 6.9 and 6.10, bearing in mind that one has to work with a potential for the dilaton for the Einstein frame to be defined. It remains to develop the Einstein frame constraint equations and the transformation that relates string frame and Einstein frame initial data. We start with the latter.

Lemma 6.14. *Let (M, g, H, ϕ), $H = \bar H + \mathrm{d}B$, be a SuGra spacetime with spacelike hypersurface Σ on which the initial data $(g_0, k, H_0, h_0, \phi_0, \phi_1)$ is induced. Denote by N the future-pointing unit normal on Σ. Then, after the conformal transformation $\tilde g = e^{-2\kappa\phi}g$, $\kappa = \frac{1}{d-2}$, (corresponding to a transformation from the string to the Einstein frame), the SuGra spacetime $(\tilde g, H, \phi)$ induces the initial data $(\tilde g_0, \tilde k, \tilde H_0, \tilde h_0, \tilde\phi_0, \tilde\phi_1)$ given by*

$$\begin{aligned}
\tilde g_0 &= e^{-2\kappa\phi}g_0, & \tilde k &= e^{-\kappa\phi}[k - \kappa N(\phi)g_0], \\
\tilde H_0 &= H_0, & \tilde h_0 &= e^{\kappa\phi}h_0 \\
\tilde\phi_0 &= \phi_0, & \tilde\phi_1 &= e^{\kappa\phi}\phi_1
\end{aligned}$$

Furthermore, the components B_0, B_1, b_0 and b_1 of a two-form $B \in \Omega^2(M)$ in the decomposition

$$B|_\Sigma = B_0 - N^\flat \wedge b_0, \qquad \nabla_N B|_\Sigma = B_1 - N^\flat \wedge b_1 \tag{6.13}$$

transform as

$$\tilde{B}_0 = B_0, \qquad \tilde{B}_1 = e^{\kappa\phi}[B_1 + 2\kappa N(\phi) B_0 - \kappa b_0 \wedge \mathrm{d}^\Sigma \phi],$$
$$\tilde{b}_0 = e^{\kappa\phi} b_0, \qquad \tilde{b}_1 = e^{2\kappa\phi}\left[b_1 - 2\kappa N(\phi) b_0 + \kappa i_{\mathrm{d}^\Sigma \phi} B_0\right].$$

Proof. The formulas for $\tilde{g}_0$, $\tilde{H}_0$, and $\tilde{\phi}_0$ are obvious. Denote by N and $\tilde{N}$ the future-pointing unit normal on Σ with respect to g and $\tilde{g}$, respectively. Then $\tilde{N} = e^{\kappa\phi} N$. The formulas for $\tilde{h}_0$ and $\tilde{\phi}_1$ follow trivially. Furthermore,

$$\begin{aligned}
\tilde{k} &= [\tilde{\nabla}\tilde{g}\tilde{N}]^\parallel = [\tilde{\nabla}(e^{-\kappa\phi} gN)]^\parallel = e^{-\kappa\phi}\left[\tilde{\nabla} gN - \kappa \mathrm{d}\phi \otimes gN\right]\Big|_{T\Sigma} \\
&= e^{-\kappa\phi}\left[\nabla gN - \kappa \langle gN, \mathrm{d}\phi\rangle_g\, g + \kappa(\mathrm{d}\phi \otimes gN + gN \otimes \mathrm{d}\phi) - \kappa \mathrm{d}\phi \otimes gN\right]\Big|_{T\Sigma} \\
&= e^{-\kappa\phi}\left[k - \kappa N(\phi) g_0\right].
\end{aligned}$$

Let us now discuss how the B-field components transform. Again, trivially $\tilde{B}_0 = B_0$ and $\tilde{b}_0 = e^{\kappa\phi} b_0$. Also,

$$\tilde{\nabla}_{\tilde{N}} B = e^{\kappa\phi}\tilde{\nabla}_N B = e^{\kappa\phi}\left[\nabla_N B + 2\kappa N(\phi) B - \kappa(i_N B) \wedge \mathrm{d}\phi - \kappa(gN) \wedge (i_{\mathrm{d}\phi} B)\right].$$

It follows that

$$\begin{aligned}
\tilde{B}_1 &= \tilde{\nabla}_{\tilde{N}} B\Big|_{T\Sigma} = e^{\kappa\phi}[B_1 + 2\kappa N(\phi) B_0 - \kappa b_0 \wedge \mathrm{d}^\Sigma \phi], \\
\tilde{b}_1 &= [\tilde{\nabla}_{\tilde{N}} B](\tilde{N}) = e^{+2\kappa\phi}\left[b_1 + 3\kappa N(\phi) b_0 + \kappa i_{\mathrm{d}\phi} B - \kappa b_0 (\operatorname{grad}_g \phi) gN\right] \\
&= e^{2\kappa\phi}\left[b_1 + 2\kappa N(\phi) b_0 + \kappa i_{\mathrm{d}^\Sigma \phi} B_0\right],
\end{aligned}$$

as claimed. □

Next, we compute the Einstein frame energy-momentum tensor and check that it is indeed divergence free as a consequence of the B-field and dilaton equation alone. For the rest of this section, we drop the tilde over the conformally transformed metric for ease of notation.

Proposition 6.15. *Let $(M, g, H, \phi), \xi = \mathrm{d}\phi$, be a SuGra spacetime satisfying the Einstein frame GEE (6.12). Then, the energy-momentum tensor $T = \mathrm{Rc} - \frac{\mathrm{Sc}}{2} g$ is given by*

$$T = \frac{1}{d-2}\left[\xi \otimes \xi - \frac{|\xi|^2}{2} g\right] + \frac{e^{-4\kappa\phi}}{4}\left[H^2 - \frac{|H|^2}{6}\, g\right]$$

Furthermore, assuming only the H-field and dilaton equation in (6.12), the energy-momentum tensor is divergence-free.

Proof. We calculate

$$\begin{aligned}
T &= \frac{1}{d-2}[\xi\otimes\xi - \frac{1}{6}e^{-4\kappa\phi}\,|H|_g^2\;g] + e^{-4\kappa\phi}\frac{H^2}{4}\\
&\quad - \frac{1}{2(d-2)}g\;\mathrm{tr}\,[\xi\otimes\xi - \frac{1}{6}e^{-4\kappa\phi}\,|H|_g^2\;g] - \frac{e^{-4\kappa\phi}}{8}\;g\;\mathrm{tr}(H^2)\\
&= \frac{1}{d-2}[\xi\otimes\xi - \frac{1}{6}e^{-4\kappa\phi}\,|H|_g^2\;g] + e^{-4\kappa\phi}\frac{H^2}{4}\\
&\quad - \frac{1}{2(d-2)}g\,[\,|\xi|^2 - \frac{d}{6}e^{-4\kappa\phi}\,|H|^2\,] - \frac{1}{8}\,g\,e^{-4\kappa\phi}\,|H|^2\\
&= \frac{1}{d-2}\left[\xi\otimes\xi - \frac{|\xi|^2}{2}g\right] + e^{-4\kappa\phi}\frac{H^2}{4} + \left(\frac{d-2}{12(d-2)} - \frac{1}{8}\right)e^{-4\kappa\phi}\,|H|^2\;g\\
&= \frac{1}{d-2}\left[\xi\otimes\xi - \frac{|\xi|^2}{2}g\right] + e^{-4\kappa\phi}\frac{H^2}{4} - \frac{1}{24}e^{-4\kappa\phi}\,|H|^2\;g.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\nabla^\mu T_{\mu\nu}\\
&= \frac{1}{2(d-2)}\nabla^\mu\left[2\xi_\mu\xi_\nu - \xi^\lambda\xi_\lambda g_{\mu\nu}\right]\\
&\quad + \frac{1}{4}\nabla^\mu(e^{-4\kappa\phi}H_\mu{}^{\lambda\kappa}H_{\nu\lambda\kappa}) - \frac{1}{24}\nabla_\nu(e^{-4\kappa\phi}\,|H|^2)\\
&= \frac{1}{2(d-2)}\Big[2(\nabla^\mu\xi_\mu)\xi_\nu \underbrace{+2\xi_\mu(\nabla^\mu\xi_\nu) - 2(\nabla_\nu\xi^\lambda)\xi_\lambda}_{=0}\Big]\\
&\quad + \frac{e^{-4\kappa\phi}}{4}\Big\{-4\kappa(\nabla^\mu\phi)H_\mu{}^{\lambda\kappa}H_{\nu\lambda\kappa} - \mathrm{d}^*H^{\lambda\kappa}H_{\nu\lambda\kappa} + H_\mu{}^{\lambda\kappa}\,\nabla^\mu H_{\nu\lambda\kappa}\\
&\quad \frac{2\kappa}{3}(\nabla_\nu\phi)\,|H|^2 - \frac{1}{6}\nabla_\nu\,|H|^2\Big\}\\
&\overset{(D)}{=} \frac{e^{-4\kappa\phi}}{6(d-2)}\left\{-\,|H|^2\;\xi_\nu - 6\xi^\mu H_\mu{}^{\lambda\kappa}H_{\nu\lambda\kappa} + \xi_\nu\,|H|^2\right\}\\
&\quad + \frac{e^{-4\kappa\phi}}{4}\left\{-\mathrm{d}^*H^{\lambda\kappa}H_{\nu\lambda\kappa} + H_\mu{}^{\lambda\kappa}\,\nabla^\mu H_{\nu\lambda\kappa} - \frac{1}{6}\nabla_\nu\,|H|^2\right\}\\
&\overset{(H)}{=} -\frac{e^{-4\kappa\phi}}{d-2}\xi^\mu H_\mu{}^{\lambda\kappa}H_{\nu\lambda\kappa}\\
&\quad + \frac{e^{-4\kappa\phi}}{4}\left\{\frac{4}{d-2}\xi^\mu H_\mu{}^{\lambda\kappa}H_{\nu\lambda\kappa} + H^{\mu\lambda\kappa}\nabla_\mu H_{\nu\lambda\kappa} - \frac{1}{3}H^{\mu\lambda\kappa}(\nabla_\nu H_{\mu\lambda\kappa})\right\}\\
&= \frac{e^{-4\kappa\phi}}{4}\left\{H^{\mu\lambda\kappa}\,\nabla_\mu H_{\nu\lambda\kappa} - \frac{1}{3}H^{\mu\lambda\kappa}\mathrm{d}H_{\nu\mu\lambda\kappa} - H^{\mu\lambda\kappa}\nabla_{[\mu}H_{\lambda\kappa]\nu}\right\}\\
&= 0.
\end{aligned}$$

The equality at the underbrace employs closedness of ξ. By (H) and (D) we denote that we made use of the H-field and dilaton equation respectively. The last equality employs closedness of H. □

Finally, for completeness, we determine the Einstein frame constraint equations.

Proposition 6.16. *The Einstein frame constraint equations are given by*

$$\begin{aligned} \mathrm{Sc}_\Sigma + (\mathrm{tr}\, k)^2 - |k|^2 &= \frac{1}{d-2}\left[\left|\mathrm{d}^\Sigma\phi_0\right|^2 + \phi_1^2\right] + \frac{e^{-4\kappa\phi_0}}{12}\left[|H_0|^2 + 3\,|h_0|^2\right], \\ \mathrm{div}_\Sigma\, k - \mathrm{tr}\, k &= \frac{1}{d-2}\phi_1 \mathrm{d}\phi_0 + \frac{e^{-4\kappa\phi_0}}{4} C(h_0, H_0), \\ 0 &= \left[(\mathrm{d}^\Sigma)^* + \frac{4}{d-2} i_{\mathrm{d}^\Sigma\phi_0}\right] h_0. \end{aligned}$$

Herein, $C(h_0, H_0)$ is the one-form obtained from contraction of h_0 and $H_0^{\|}$ as described below (6.9).

Proof. It is more efficient to obtain these equations from the Einstein frame GEE (6.12) directly than to apply the transformation rules for the initial data developed in Lemma 6.14 to the string frame constraint equations (6.9).

The first constraint equation in (6.9) is equivalent[22] to $G(N,N) = T(N,N)$, employing that $G(N,N) = \frac{1}{2}[\mathrm{Sc}_\Sigma + (\mathrm{tr}\, k)^2 - |k|^2]$. Noting that

$$\begin{aligned} T(N,N) &= \frac{1}{d-2}\left[\xi\otimes\xi - \frac{|\xi|^2}{2} g\right](N,N) + \frac{e^{-4\kappa\phi}}{4}[H^2 - \frac{|H|^2}{6}\, g](N,N) \\ &= \frac{1}{d-2}\left[(\phi_1)^2 + \frac{\left|\mathrm{d}^\Sigma\phi_0\right|^2 - \phi_1^2}{2}\right] + \frac{e^{-4\kappa\phi_0}}{4}\left[|h_0|^2 + \frac{|H_0|^2 - 3\,|h_0|^2}{6}\right] \\ &= \frac{1}{2(d-2)}\left[\left|\mathrm{d}^\Sigma\phi_0\right|^2 + \phi_1^2\right] + \frac{e^{-4\kappa\phi_0}}{24}\left[|H_0|^2 + 3\,|h_0|^2\right], \end{aligned}$$

the first Einstein frame constraint equation follows. Similarly, we note that the second constraint in (6.9) is equivalent to $G(N,X) = T(N,X)$ for $X \in \Gamma(T\Sigma)$, employing that $G(N,X) = (\mathrm{div}_\Sigma\, k)(X) - X(\mathrm{tr}\, k)$. We note that

$$\begin{aligned} T(N,X) &= \frac{1}{d-2}\left[\xi\otimes\xi - \frac{|\xi|^2}{2} g\right](N,X) + \frac{e^{-4\kappa\phi}}{4}[H^2 - \frac{|H|^2}{6}\, g](N,X) \\ &= \frac{1}{d-2}\phi_1 \mathrm{d}\phi_0(X) + \frac{e^{-4\kappa\phi_0}}{4} C(h_0, H_0)(X) \end{aligned}$$

and obtain the second constraint equation. Finally, the third string-frame constraint in (6.9) is equivalent to $i_N \mathrm{d}^* H = -\frac{4}{d-2} i_N i_\xi H$ (compare this to the first equation in Proposition 6.13). With Lemma B.2, we obtain the constraint equation

$$-(\mathrm{d}^\Sigma)^* h_0 = \frac{4}{d-2} i_{\mathrm{d}^\Sigma\phi_0} h_0.$$

□

[22]Note that the string frame equation $G[g] = T[g, B, \phi]$ and the Einstein frame equation $G[\tilde{g}] = \tilde{T}[\tilde{g}, B, \phi]$ are component-wise equivalent, as $\tilde{T}$ is defined by the equation $G[\tilde{g}] - G[g] = \tilde{T}[\tilde{g}, B, \phi] - T[g, B, \phi]$. Assuming the dilaton equation, we have computed an equivalent expression for $\tilde{T}$ in Proposition 6.15.

6.4 Gauge Conditions and Hyperbolic Reduction

This section reproduces [22, section 5.3].

Throughout this section, let $(M, g, H, \phi), \xi = \mathrm{d}\phi$, be a SuGra spacetime solving the Einstein frame GEE (6.12). Note that, for ease of notation, we drop the tilde in this section. We write $H = \bar{H} + \mathrm{d}B$ for some closed $\bar{H} \in \Omega^3_{\mathrm{cl}}(M)$ and a two-form $B \in \Omega^2(M)$. Again, we emphasise that in general, the Einstein frame can only be adopted locally after choosing a potential for the dilaton, $\xi = \mathrm{d}\phi$, and one should therefore imagine the SuGra spacetime we consider here to be a small subset of some development.

Crucial for the well-posedness of the Cauchy problem is the well-behavedness of the equations of motion (6.12) for g, B and ϕ. As explained before, we want to understand them as being of the form (6.1). In this section, we develop gauge conditions that achieve this.

The GEE are invariant under generalised diffeomorphisms (a combination of diffeomorphisms and B-field transformations, cf. Remark 6.7). Implementing a gauge condition means breaking this invariance. To that end, we introduce a background metric[23]. (Conceptually, one should think of fixing the background field $\bar{H}$ only after breaking diffeomorphism invariance with the DeTurck gauge. Then, only invariance under closed B-field transformations remains.) For now, it suffices to take any Lorentzian metric $\bar{g}$ on $M = \mathbb{R} \times \Sigma$. Define

$$A^{\mu}_{\alpha\beta} := (\nabla - \bar{\nabla})^{\mu}_{\alpha\beta} \in \Omega^1(\mathrm{End}\, TM)$$

where ∇ and $\bar{\nabla}$ denote the Levi Civita connection of g and $\bar{g}$ respectively. Notice that, in local coordinates

$$A^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\alpha\beta} - \bar{\Gamma}^{\mu}_{\alpha\beta}$$

Denoting

$$\Gamma_{\mu} := g_{\mu\nu} g^{\alpha\beta} \Gamma^{\nu}_{\alpha\beta}, \qquad F_{\mu} := g_{\mu\nu} g^{\alpha\beta} \bar{\Gamma}^{\nu}_{\alpha\beta}$$

one can check that

$$\mathrm{Rc}_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \partial_{\alpha} \partial_{\beta} g_{\mu\nu} + \partial_{(\mu} \Gamma_{\nu)} + \text{lower order}$$

It is now apparent how to modify the Ricci tensor. Following [10, Chapter 14.1], we define $\mathcal{D}_{\nu} \in \Gamma(T^*M)$ as

$$\mathcal{D}_{\nu} := -g_{\nu\mu} g^{\alpha\beta} A^{\mu}_{\alpha\beta} = F_{\nu} - \Gamma_{\nu} \tag{6.14}$$

and then

$$\hat{\mathrm{R}}\mathrm{c}_{\mu\nu} = \mathrm{Rc}_{\mu\nu} + \nabla_{(\mu} \mathcal{D}_{\nu)} = \mathrm{Rc}_{\mu\nu} + \frac{1}{2} L_{\mathcal{D}} g_{\mu\nu},$$

where $L_{\mathcal{D}}$ denotes the Lie derivative in direction of the vector field $g^{\mu\nu}\mathcal{D}_{\nu}$. Citing [10, eq. (14.3)], we see that

$$\hat{\mathrm{R}}\mathrm{c}_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \partial_{\alpha} \partial_{\beta} g_{\mu\nu} + \nabla_{(\mu} F_{\nu)} + g^{\alpha\beta} g^{\gamma\delta} [\Gamma_{\alpha\gamma\mu} \Gamma_{\beta\delta\nu} + \Gamma_{\alpha\gamma\mu} \Gamma_{\beta\nu\delta} + \Gamma_{\alpha\gamma\nu} \Gamma_{\beta\mu\delta}] \tag{6.15}$$

where $\Gamma_{\alpha\beta\gamma} := g_{\beta\mu} \Gamma^{\mu}_{\alpha\gamma}$, i.e. $\hat{\mathrm{R}}\mathrm{c}$ is indeed a hyperbolic differential operator of the form $g^{\mu\nu} \partial_{\mu} \partial_{\nu}$+lower order.

[23] It is more precise to say that we introduce a background connection $\bar{\nabla}$, as the metric itself will not appear in the gauge condition. However, we can assume this connection to be the Levi-Civita connection of a metric.

Remark 6.17. The gauge condition that we impose in this formalism is $\mathcal{D} = 0$, in which case $\hat{\mathrm{Rc}} = \mathrm{Rc}$. To highlight the dependence of $\mathcal{D}$ on the dynamical metric g and the background metric $\bar{g}$, we denote $\mathcal{D} = \mathcal{D}[g, \bar{g}]$. Then, of course we have the diffeomorphism invariance $f^*(\mathcal{D}[g, \bar{g}]) = \mathcal{D}[f^*g, f^*\bar{g}]$. However, viewing the background metric as fixed, we also denote $\mathcal{D}[g] = \mathcal{D}[g, \bar{g}]$. Then, the condition $\mathcal{D} = 0$ breaks diffeomorphism invariance, because in general

$$f^*(\mathcal{D}[g]) = \mathcal{D}[f^*g, f^*\bar{g}] \neq \mathcal{D}[f^*g, \bar{g}] = \mathcal{D}[f^*g].$$

Inspired by how we modified the Ricci-tensor, we take a look at the operator $\mathrm{d}^*\mathrm{d}$. Acting on B, and summarising all terms which do not contain second derivatives of $g_{\mu\nu}$ or $B_{\mu\nu}$ under the term "lower order", we get

$$\begin{aligned}
\mathrm{d}^*\mathrm{d}B_{\mu\nu} &= -3\nabla^\lambda \nabla_{[\lambda} B_{\mu\nu]} \\
&= -3\nabla^\lambda \partial_{[\lambda} B_{\mu\nu]} \\
&= -3\partial^\lambda \partial_{[\lambda} B_{\mu\nu]} + \text{lower order} \\
&= -\partial^\lambda \partial_\lambda B_{\mu\nu} - 2\partial^\lambda \partial_{[\mu} B_{\nu]\lambda} + \text{lower order} \\
&= -\partial^\lambda \partial_\lambda B_{\mu\nu} - 2\partial_{[\mu}(\partial^\lambda B_{\nu]\lambda}) + \text{lower order.}
\end{aligned}$$

We denoted $\partial^\mu = g^{\mu\nu}\partial_\nu$.

The second term contains precisely those second derivatives of the B-field that are problematic, as they cause the operator to not be of the form (6.1). Notice that these terms equal

$$\mathrm{d}\mathrm{d}^*B = 2\nabla_{[\mu}\nabla^\lambda B_{\nu]\lambda}$$

up to second order derivatives of the metric. These, however, we may not add to our equation, as this would only result in the equation differing from the desired form (6.1) in a different way. Luckily, we have another metric at our disposal: $\bar{g}$. Its second derivatives may enter our equation. Thus, we identify as the quantity by which we want to modify the B-field equation

$$\begin{aligned}
\mathcal{C}_\nu &:= \mathrm{d}^*_{g,\bar{g}} B_\nu := -g^{\kappa\lambda}\bar{\nabla}_\kappa B_{\lambda\nu} \in \Gamma(T^*M) \\
&= \mathrm{d}^*B_\nu + \mathcal{D}^\mu B_{\mu\nu} + g^{\kappa\lambda} A^\mu_{\kappa\nu} B_{\lambda\mu}.
\end{aligned} \tag{6.16}$$

To highlight the dependence of $\mathcal{C}$ on g, B, and $\bar{g}$, we sometimes write $\mathcal{C} = \mathcal{C}[g, B]$ or even $\mathcal{C} = \mathcal{C}[g, B, \bar{g}]$.

Finally, by replacing in (6.12)

$$\begin{aligned}
\mathrm{Rc} &\longrightarrow \hat{\mathrm{Rc}}, \\
\mathrm{d}^*\mathrm{d}B &\longrightarrow \mathrm{d}^*\mathrm{d}B + \mathrm{d}\mathcal{C} =: -\hat{\Box}_{\mathrm{Hd}} B,
\end{aligned}$$

we obtain the modified system

$$\begin{aligned}
\hat{\Box}_{\mathrm{Hd}} B &= \mathrm{d}^*\bar{H} + \frac{4}{d-2} i_\xi H \\
\hat{\mathrm{Rc}} &= \frac{1}{d-2}\left[\xi \otimes \xi - \frac{e^{-4\kappa\phi}}{6} |H|^2_g \, g\right] + e^{-4\kappa\phi}\frac{H^{2,g}}{4} \\
\Box\phi &= -\frac{e^{-4\kappa\phi}}{6} |H|^2_g
\end{aligned} \tag{6.17}$$

where $H = \bar{H} + \mathrm{d}B$, $\xi = \mathrm{d}\phi$.

Proposition 6.18. *The modified Einstein frame GEE (6.17) form a quasi-linear hyperbolic system with principal symbol given by $g^{\mu\nu}$. Furthermore, writing in local coordinates the system as $g^{\mu\nu}\partial_\mu\partial_\nu - f[u] = 0$ where $u = (g_{\mu\nu}, B_{\mu\nu}, \phi)$, we see the non-linearity $f[u](t,x)$ to vanish if the following expressions are all set to vanish at (t,x):*

- $\partial_\alpha g_{\mu\nu}$, $B_{\mu\nu}$, $\partial_\alpha B_{\mu\nu}$, ϕ, *and* $\partial_\alpha\phi$,
- $\bar{H}_{\mu\nu\lambda}$, $\partial_\alpha \bar{H}_{\mu\nu\lambda}$, $\partial_\alpha \bar{g}_{\mu\nu}$, *and* $\partial_\alpha\partial_\beta\bar{g}_{\mu\nu}$.

In particular, the modified GEE (6.17) are of the form (5.1) required for the local existence and uniqueness result Proposition 5.1 to apply.

Proof. In the following, we refer to the demands on the principal symbol and the non-linearity as the equation "being of the right form". For the Ricci-equation, the right form can be seen from the expression (6.15) for the modified Ricci tensor $\hat{\mathrm{Rc}}$ in local coordinates. For the dilaton equation, the right form follows from the local expression of the d'Alembert operator $\Box\phi = g^{\mu\nu}\partial_\mu\partial_\nu\phi - g^{\lambda\kappa}\Gamma^\alpha_{\lambda\kappa}\partial_\alpha\phi$. Finally, for the B-field equation, we compute

$$\begin{aligned}\hat{\Box}_{\mathrm{Hd}} B_{\mu\nu} = \partial^\lambda\partial_\lambda B_{\mu\nu} - \Gamma^\alpha\partial_\alpha B_{\mu\nu} + 2g^{\lambda\kappa}\partial_{[\mu}(\bar{\Gamma}^\alpha_{\nu]\lambda}B_{\alpha\kappa} + B_{\nu]\alpha}\bar{\Gamma}^\alpha_{\lambda\kappa}) - 2g^{\lambda\kappa}B_{\beta\kappa}\Gamma^\alpha_{\lambda[\mu}\bar{\Gamma}^\beta_{\nu]\alpha} \\ - 4g^{\lambda\kappa}B_{\beta[\mu}\Gamma^\alpha_{\nu]\lambda}\bar{\Gamma}^\beta_{\alpha\kappa} + 2g^{\lambda\kappa}\Gamma^\beta_{\kappa[\mu}\partial_{|\lambda|}B_{\nu]\beta} - 2g^{\lambda\kappa}B_{\alpha\beta}\Gamma^\beta_{\kappa[\mu}\bar{\Gamma}^\alpha_{\nu]\lambda},\end{aligned}$$

where we denoted $\partial^\lambda := g^{\lambda\kappa}\partial_\kappa$. The right form follows. □

In the end, the modified system has to agree with the unmodified system, i.e. $\mathrm{d}\mathcal{C} = 0$, since our actual interest lies in finding solutions to the latter. (For the same reason, we will also have $\mathcal{D} = 0$.) Thus, the gauge that we implement can be understood as

$$0 = \mathrm{d}\mathcal{C} = \mathrm{d}\mathrm{d}^*_{g,\bar{g}}B \tag{6.18}$$

We call this condition the *generalised Lorenz gauge*. Note that it breaks invariance of the equations under closed B-field transformations, as $\mathrm{d}\mathcal{C}[g,B] \neq \mathrm{d}\mathcal{C}[g,B+\beta]$ for β a closed two-form. We prove in Lemma 6.29 that one can always implement $\mathrm{d}\mathcal{C} = 0$ by virtue of a closed B-field transformation.

Remark 6.19. We can compare the generalised Lorenz gauge (6.18) to the Lorenz gauge $\mathrm{d}^*A = 0$ as it is imposed on the electromagnetic field potential[24] A in the study of the Cauchy Problem for the Einstein-Maxwell system in [38]. For a one-form, one can calculate that $\mathrm{d}^*A = -\partial^\mu A_\mu + \Gamma^\mu A_\mu$. That is, the difference $\mathrm{d}^*_{g,\bar{g}}A - \mathrm{d}^*A$ is given by $\mathcal{D}^\mu A_\mu$. As one also has to implement the gauge $\mathcal{D} = 0$ in the study of the Cauchy problem, one can equivalently state the Lorenz gauge as $\mathrm{d}^*_{g,\bar{g}}A = 0$. Noting that for a 0-form being closed implies being constant, we see that, for a one form, the generalised Lorenz gauge $\mathrm{d}\mathrm{d}^*_{g,\bar{g}}A = 0$ is a mild generalisation of the Lorenz gauge.

Remark 6.20. Instead of finding a gauge condition casting the GEE into a hyperbolic system with symbol given by the metric, one could find a PDE-theoretic result general enough to be applicable to the Einstein frame GEE (6.12). For example, employing that H is closed, one can derive the equation $\Box_{\mathrm{Hd}}H = \frac{4}{d-2}\mathrm{d}^*(i_\xi H)$ for H (an analogous observation is made for the generalised Ricci flow in [6]). One obtains a second order system that is structurally analogous to the Einstein-Maxwell system as it is presented in [41, chapter 18.8]. Therefore, transferring the methods presented in [41] to the Einstein frame GEE,

[24] For the purposes of this remark, we forget that the letter A is already occupied by the object $A = \nabla - \bar{\nabla}$.

one should be able to establish local well-posedness of the IVP.[25] So, while it may not be necessary to employ the generalised Lorenz gauge to obtain the well-posedness results established in this work, the plethora of applications for the Lorenz gauge provides a strong case for the utility of its generalised cousin; we view the elegant form of the modified system (6.17) as indicative.

We now establish that one can obtain solutions to the original system (6.12) by studying the modified one (6.17). Recall that under the assumptions of Lemma 5.6, $\mathcal{D}$ vanishes on the entire Cauchy development of any subset Ω of the initial hypersurface Σ on which $\mathcal{D} = 0$ and $\nabla\mathcal{D} = 0$. However, this result does not immediately apply to a solution of the modified GEE (6.17). While we know from Proposition 6.15 that the energy-momentum tensor T obtained in the Einstein frame is divergence-free if one assumes the H-field and dilaton equation, a priori this does not hold after one passes to the modified system (6.17), because then one also modifies the H-field equation. The solution to this is to first prove the vanishing of the modifying term $\mathrm{d}\mathcal{C}$ of the H-field equation under suitable conditions.

Proposition 6.21. *Let (M, g, H, ϕ), $H = \bar{H} + \mathrm{d}B$, $\xi = \mathrm{d}\phi$, be a globally hyperbolic SuGra spacetime such that (6.17) is satisfied. Let $\bar{g}$ be an arbitrary Lorentz metric on M, and define $\mathcal{D}$ and $\mathcal{C}$ as in (6.14) and (6.16), respectively.*

Then, assuming $\mathcal{D} = 0$, and $\nabla\mathcal{D} = \mathrm{d}\mathcal{C} = 0$ on some subset $\Omega \subset \Sigma$, we have $\mathcal{D} = 0$ and $\mathrm{d}\mathcal{C} = 0$ on the entire Cauchy development $D(\Omega)$.

Proof. We begin by deriving a source-free wave equation for $\mathcal{C}$. We take the co-differential of the modified B-field equation:

$$\begin{aligned} 0 &= \mathrm{d}^* \left[\mathrm{d}^* H + \mathrm{d}\mathcal{C} + \frac{4}{d-2} i_\xi H \right] \\ &= \mathrm{d}^*\mathrm{d}\mathcal{C} - \frac{4}{d-2} i_\xi \mathrm{d}^* H \\ &= \mathrm{d}^*\mathrm{d}\mathcal{C} + \frac{4}{d-2} i_\xi \left(\mathrm{d}\mathcal{C} + \frac{4}{d-2} i_\xi H \right) \\ &= \mathrm{d}^*\mathrm{d}\mathcal{C} + \frac{4}{d-2} i_\xi \mathrm{d}\mathcal{C} \end{aligned} \tag{6.19}$$

Herein, we have used the antisymmetry of H to conclude $i_\xi i_\xi H = 0$ and

$$\mathrm{d}^*(i_\xi H)_\nu = -\nabla^\mu[(\nabla^\lambda\phi) H_{\lambda\mu\nu}] = (\nabla^\lambda\phi)\nabla^\mu H_{\mu\lambda\nu} = -(i_\xi \mathrm{d}^* H)_\nu.$$

We note that (6.19) has indeed no source terms. Thus, by Lemma A.9, we must have $\mathrm{d}\mathcal{C} = 0$ on $D(\Omega) = M$.

Finally, the tensor $T = \hat{\mathrm{Rc}} - \frac{1}{2} \mathrm{tr}_g \hat{\mathrm{Rc}}$ agrees with the energy momentum tensor obtained in Proposition 6.15 in the Einstein frame. It follows from the (now effectively unmodified) H-field and dilaton equations that $\operatorname{div} T = 0$. Hence, by Lemma 5.6, $\mathcal{D} = 0$ on $D(\Omega)$. ☐

6.5 Setting Up Initial Values

In this section, which reproduces [22, section 5.4], we set up initial values for the modified system (6.17). This means that, given initial data $\mathcal{I} = (\Sigma, g_0, k, H_0, h_0, \phi_0, \phi_1)$ to the Einstein frame GEE (6.12), we construct a manifold M on which the development may take

[25] In yet another approach, Choquet-Bruhat [38, section 10.4.1.] shows that the Einstein-Maxwell system as we present it is Leray-hyperbolic and hence well-behaved, which should also apply to the GEE.

place, equipped with a background metric $\bar{g}_{\mathcal{I}}$, a background field $\bar{H}_{\mathcal{I}} \in \Omega^3_{\mathrm{cl}}(M)$, and on the initial hypersurface Σ the metric g, the B-field, the dilaton ϕ, and their respective first derivatives, all such that with $H = \bar{H}_{\mathcal{I}} + \mathrm{d}B$ the initial data induced on Σ by (M, g, H, ϕ) reproduces the given initial data. We again drop the tildes over the objects in the Einstein frame. (We remark on this as we reinstate them in subsequent sections.)

We choose to highlight the dependence of the background fields on the initial data in our notation. This is motivated by the Einstein frame initial data only existing locally and with respect to a choice of potential for the dilaton, in contrast to the situation present in the Einstein equations coupled to a scalar field as discussed in [10] where well-defined global initial data and background fields exist.

Note we don't have initial data for the B-field directly, but only for its exterior derivative $\mathrm{d}B$. However, in the discussion of uniqueness of solutions, it is necessary to have better control over B, and so we artificially introduce two-forms $B_0, B_1 \in \Omega^2(\Sigma)$ that are compatible with the requirements on $\mathrm{d}B$ and demand $B^{\parallel} = B_0$ and $(\nabla_N B)^{\parallel} - \mathrm{d}^{\Sigma}(B(N)) = B_1$, where N is the unit normal on Σ. With these additional requirements, we can determine most degrees of freedom in the initial values of g, B, and ϕ.

However, in the specification of the normal parts of g and B and their normal derivatives, there still appear degrees of freedom that are not determined by the initial data. For the metric, this is observed e.g. in [10, Chapter 14.2]. The reason is that the induced initial data is invariant under the action of diffeomorphisms on M which restrict to the identity on Σ. The freedom in specifying the normal parts of g and its normal derivative can be used to specify the unit normal vector and force $\mathcal{D} = 0$ initially. For the B-field, the initial data induced is invariant even under the action of closed B-field transformations which restrict to the identity on the generalised tangent bundle over Σ. That is, we can add to B any closed two-form $\beta \in \Omega^2(M)$ such that $\beta^{\parallel} = 0$. We use this freedom to achieve $B(N) = 0$ and $\mathcal{C}^{\parallel} = 0$ on Σ.

Crucially, the way we set up the initial data is geometrical and does not depend on a choice of coordinates. Following [10], this allows the glueing together of developments defined only on a small neighbourhood of points in Σ, as it forces these developments to agree on the initial hypersurface (where both are defined). Taking this one step further, we have to set up the initial data independent of the choice of Einstein frame, so that we can glue together developments defined in different Einstein frames. Note that two dilaton potentials ϕ and ϕ' to a given dilaton ξ defined over the same connected set differ by a constant.

Let us define the background manifold and the background fields. Let $M = \mathbb{R} \times \Sigma$, and identify $\Sigma \cong \{0\} \times \Sigma$. Denote by $t \colon \mathbb{R} \to \Sigma$ the coordinate coming from projection onto the first factor. We define the background fields as

$$\bar{g}_{\mathcal{I}} \coloneqq -e^{-2\kappa\phi_0}\mathrm{d}t^2 + g_0, \qquad \bar{H}_{\mathcal{I}} \coloneqq H_0 + \mathrm{d}t \wedge h_0 + t\,\mathrm{d}^{\Sigma} h_0 \tag{6.20}$$

where one can check $\bar{H}_{\mathcal{I}}$ to indeed be closed.

In [10, Chapter 14.2], the initial values of g and ϕ were fixed by imposing the requirements stated in the following Lemma - except for the factor $e^{\kappa\phi_0}$ in the first condition. This factor is needed to achieve invariance of the conditions under a change of Einstein frame.

Lemma 6.22. *The initial values for g, ϕ, and their first time-derivatives are uniquely determined (and not overdetermined) by the following requirements.*

(1) $e^{\kappa\phi_0}\partial_t$ agrees with the unit normal N on Σ.

(2) g induces the initial data (g_0, k) on Σ, i.e. $g^{\parallel} = g_0$ and $2k = L_N g$.

(3) ϕ induces the initial data (ϕ_0, ϕ_1) on Σ, i.e. $\phi|_\Sigma = \phi_0$ and $N(\phi) = \phi_1$.

(4) Initially, g satisfies DeTurck's gauge, i.e. $\mathcal{D}[g, \bar{g}_{\mathcal{I}}]|_\Sigma = 0$.

Proof. To discuss this more explicitly, pick a coordinate chart (U_Σ, x^m) in Σ and extend it to coordinates $(x^0 = t, x^m)$ on $\mathbb{R} \times U_\Sigma$. We specify the metric g and its first derivatives on U_Σ in these coordinates. The spatial part of g is fixed by (2):

$$g_{mn}|_{t=0} = (g_0)_{mn} \tag{6.21}$$

Requirement (1) is equivalent to

$$g_{00}|_{t=0} = -e^{-2\kappa\phi_0}, \qquad g_{0n}|_{t=0} = 0. \tag{6.22}$$

From the formula for the second fundamental form $K_{mn} = \frac{1}{2}\,|g_{00}|^{-1/2}\, g_{mn,0}$, it is immediate that (2) is equivalent to

$$g_{mn,0}|_{t=0} = 2e^{-\kappa\phi_0} k_{mn} \tag{6.23}$$

It remains to specify $g_{00,0}$ and $g_{0k,0}$, whose values we will see to be determined by (4). We make the following auxiliary computations for $t = 0$:

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{2} g^{00} g_{00,0} = -\frac{1}{2} e^{2\kappa\phi_0} g_{00,0},\\
\Gamma^k_{00} &= -\frac{1}{2} g^{kl}(g_{00,l} - 2g_{0l,0}) = g^{kl}(-\kappa e^{-2\kappa\phi_0}\partial_l\phi + g_{0l,0}),\\
\Gamma^0_{mn} &= \frac{1}{g_{00}} g(\nabla_m\partial_n, \partial_0) = -e^{\kappa\phi_0} g(\nabla_m\partial_n, N) = e^{\kappa\phi_0} k_{mn},\\
\Gamma^k_{mn} &= -\frac{1}{2} g^{kl}(g_{mn,l} - g_{lm,n} - g_{nl,m}) = (\Gamma^\Sigma)^k_{mn}.
\end{aligned}$$

Herein, $(\Gamma^\Sigma)^k_{mn}$ denote the Christoffel symbols for g_0. From this, it follows that

$$\begin{aligned}
\Gamma_0 &= g_{00} g^{\alpha\beta}\Gamma^0_{\alpha\beta} = \Gamma^0_{00} - e^{-\kappa\phi_0}\operatorname{tr} k = -\frac{1}{2} e^{2\kappa\phi_0} g_{00,0} - e^{-\kappa\phi_0}\operatorname{tr} k,\\
\Gamma_l &= g_{lk} g^{\alpha\beta}\Gamma^k_{\alpha\beta} = \Gamma^\Sigma_l + g_{lk} g^{00}\Gamma^k_{00} = \Gamma^\Sigma_l + \kappa\partial_l\phi_0 - e^{2\kappa\phi_0} g_{0l,0}.
\end{aligned}$$

Herein, Γ^Σ_l denotes the contracted Christoffel symbol for g_0. We see that initially $0 = \mathcal{D}_\mu = F_\mu - \Gamma_\mu$ can be easily achieved by requiring

$$\begin{aligned}
g_{00,0}|_{t=0} &= -2e^{-2\kappa\phi_0}[F_0 + e^{-\kappa\phi_0}\operatorname{tr} k]\Big|_{t=0},\\
g_{0k,0}|_{t=0} &= e^{-2\kappa\phi_0}\left[\Gamma^\Sigma_k - F_k + \kappa\partial_l\phi_0\right]\Big|_{t=0}.
\end{aligned}$$

We finish the proof by noting that, trivially, (3) uniquely determines the initial value of ϕ and its first time-derivative. □

We denote by ∇ the Levi-Civita connection of g, and by $N = e^{\kappa\phi_0}\partial_t$ the unit normal on Σ. We denote by $B_0, B_1 \in \Omega^2(\Sigma)$ initial data for the B-field. Note that $\bar{H}_{\mathcal{I}}$ induces the initial data (H_0, h_0), so that the initial requirement for B is $\mathrm{d}B = 0$. Thus, we have the canonical choice $B_0 = B_1 = 0$ available.

Lemma 6.23. *The initial values for B and its first derivative are uniquely determined (and not overdetermined) by the following requirements.*

(1) B induces the initial data (B_0, B_1), i.e. $B^{\|} = B_0$ and $(\nabla_N B)^{\|} = B_1$.

(2) Initially, B has no normal component, i.e. $B(N) = 0$.

(3) Initially, B satisfies $(\mathcal{C}[g, B, \bar{g}_{\mathcal{I}}])^{\|} = 0$.

Finally, in the case $B_0 = B_1 = 0$, these requirements are solved by demanding B and its time derivative to vanish initially.

Proof. Write, implementing (1),

$$B|_{t=0} = B_0 - N^\flat \wedge b_0, \qquad \nabla_N B|_{t=0} = B_1 - N^\flat \wedge b_1.$$

Requirement (2) yields $b_0 = 0$. It remains to determine b_1. Consider that by definition of $\mathcal{C}$ (6.16) and Lemma B.2

$$\begin{aligned}\mathcal{C}^{\|} &= (\mathrm{d}^*_{g,\bar{g}_{\mathcal{I}}} B)^{\|} = (\mathrm{d}^* B)^{\|} + E[B] \\ &= (\mathrm{d}^\Sigma)^* B_0^{\|} + b_1 + E[B]\end{aligned}$$

where $E[B]$ is some covector valued expression homogeneous and linear in B. To achieve that $\mathcal{C}^{\|}$ vanishes initially, we thus set

$$b_1 = -(\mathrm{d}^\Sigma)^* B_0^{\|} - E[B].$$

Finally, considering the special case $B_0 = B_1 = 0$, we arrive at $B|_{t=0} = \nabla_N B|_{t=0} = 0$. □

The requirements of Lemmas 6.22 and 6.23 leave no degrees of freedom left to specify. However, due to the constraint equations, we get the following

Lemma 6.24. *Let $\mathcal{I} = (\Sigma, g_0, k, H_0, h_0, \phi_0, \phi_1)$ be initial data for the Einstein frame system (6.12). Assume that we can find a development (M, g, H, ϕ) of the initial data under the modified system (6.17) satisfying the assumption of Lemmas 6.22 and 6.23. Then we have initially*

$$\nabla \mathcal{D}|_\Sigma = 0, \qquad \mathrm{d}\mathcal{C}|_\Sigma = 0.$$

Proof. Since we assume the initial data to solve the constraint equations, we have on Σ

$$G_{0m} = T_{0m}, \qquad G_{00} = T_{00}, \qquad \mathrm{d}^*\mathrm{d}B_{0n} = -\mathrm{d}^*(\bar{H}_{\mathcal{I}})_{0n} - \frac{4}{d-2}\xi^\lambda H_{\lambda 0n}$$

Denote $\hat{G}_{\mu\nu} = \hat{\mathrm{R}}\mathrm{c}_{\mu\nu} - \frac{1}{2}(\mathrm{tr}_g \hat{\mathrm{R}}\mathrm{c}) g_{\mu\nu}$. Since we assume (M, g, H, ϕ) to be a solution of the modified system (6.17), we can conclude that on Σ

$$0 = \hat{G}_{0m} - T_{0m} = \hat{G}_{0m} - G_{0m} = \nabla_{(0}\mathcal{D}_{m)} - \frac{1}{2} g_{0m} \nabla^\lambda \mathcal{D}_\lambda = \frac{1}{2}\partial_0 \mathcal{D}_m$$

where the last equality follows from $\partial_t|_\Sigma$ being normal to Σ and $\mathcal{D}|_\Sigma = 0$ initially. Furthermore on Σ

$$0 = \hat{G}_{00} - T_{00} = \hat{G}_{00} - G_{00} = \nabla_{(0}\mathcal{D}_{0)} - \frac{1}{2} g_{00} \nabla^\lambda \mathcal{D}_\lambda = \frac{1}{2}\partial_0 \mathcal{D}_0$$

where, in the last equality, we employed again that $\mathcal{D} = 0$ initially. Therefore, $\nabla \mathcal{D}|_\Sigma = 0$.

By a similar argument, we see that on Σ

$$0 = \hat{\Box}_{\mathrm{Hd}} B_{0n} + \mathrm{d}^*\mathrm{d}B_{0n} = -(i_{\partial_t}\mathrm{d}\mathcal{C})_n.$$

From the required $\mathcal{C}^{\|} = 0$ (cf. Lemma 6.23), we get that

$$(\mathrm{d}\mathcal{C})^{\|} = 0$$

Put together, $\mathrm{d}\mathcal{C}|_\Sigma = 0$ as claimed. □

Let us conclude the discussion of the initial data by checking that our constructions are in fact invariant under a change of Einstein frame.

Lemma 6.25. *Let* $\mathcal{I} = (g_0, k, H_0, h_0, \phi_0, \phi_1)$ *be initial data in an Einstein frame, and denote by* $\bar{g}_{\mathcal{I}}$ *and* $\bar{H}_{\mathcal{I}}$ *the associated background fields. Let* (g, H, ϕ), $H = \bar{H}_{\mathcal{I}} + \mathrm{d}B$, *be a SuGra spacetime. Consider with* $c \in \mathbb{R}$ *a change of Einstein frame*

$$(g', B', \phi') := (e^{-2\kappa c} g, B, \phi + c)$$
$$\mathcal{I}' = (g_0', k', H_0', h_0', \phi_0', \phi_1') := (e^{-2\kappa c} g_0, e^{-\kappa c} k, H_0, e^{\kappa c} h_0, \phi_0 + c, e^{\kappa c} \phi_1).$$

Then, the following hold.

(i) $\mathcal{D}[g', \bar{g}_{\mathcal{I}'}] = \mathcal{D}[g, \bar{g}_{\mathcal{I}}]$. *In particular, DeTurck's gauge condition* $\mathcal{D} = 0$ *is invariant under a change of Einstein frame.*

(ii) $\mathcal{C}[g', B', \bar{g}_{\mathcal{I}'}] = e^{2\kappa c} \mathcal{C}[g, B, \bar{g}_{\mathcal{I}}]$. *In particular, the generalised Lorenz gauge condition* $\mathrm{d}\mathcal{C} = 0$ *is invariant under a change of Einstein frame.*

(iii) If (g, B, ϕ) *satisfies the conditions from Lemma 6.22 with respect to* $\mathcal{I}$, *then* (g', B', ϕ') *satisfies those conditions with respect to* $\mathcal{I}'$.

(iv) If (g, B, ϕ) *satisfies the conditions from Lemma 6.23 with respect to* $\mathcal{I}$, *then* (g', B', ϕ') *satisfies those conditions with respect to* $\mathcal{I}'$.

Proof. Note first that

$$\bar{g}_{\mathcal{I}'} = e^{-2\kappa c} \bar{g}_{\mathcal{I}}, \qquad \bar{H}_{\mathcal{I}'} = \bar{H}_{\mathcal{I}}$$

(i) & (ii): Recall that a conformal change with constant conformal factor leaves the associated LC connection invariant. The claims respectively follow from the definitions (6.14) and (6.16) of $\mathcal{D}$ and $\mathcal{C}$.

(iii): We check conditions (1-4) for (g', B', ϕ'). Condition (4) follows from assertion (i). Conditions (2) and (3) follow from Lemma 6.14. Finally, (1) demands that $N' = e^{\kappa \phi_0'} \partial_t$, where N' is the unit normal on Σ with respect to g'. This holds because orthogonality is preserved by conformal transformations and by (1) for (g, B, ϕ)

$$g'(e^{\kappa \phi_0'} \partial_t, e^{\kappa \phi_0'} \partial_t) = g(e^{\kappa \phi_0} \partial_t, e^{\kappa \phi_0} \partial_t) = g(N, N) = -1,$$

where N denotes the unit normal on Σ with respect to g.

(iv): We check conditions (1-3) for (g', B', ϕ'). Condition (1) follows from Lemma 6.14. Condition (3) follows from assertion (ii). Finally, (2) follows from $B' = B$ and $N' = e^{\kappa c} N$. □

6.6 Existence of a Globally Hyperbolic Development

In this section, which reproduces [22, section 5.5], we prove the existence of a globally hyperbolic development to initial data for the GEE in the string frame (6.7). In the setting of the Einstein equations coupled to a scalar field, the corresponding result can for example be found in [10, Theorem 14.2.], and we closely follow the proof strategy established there. To minimise repetition and highlight our contributions, we present the results established in the proof of [10, Theorem 14.2.] in Section 5.

From Proposition 6.18, we know the modified Einstein frame GEE to be locally of the form required for Proposition 5.1 to apply. Proposition 5.1 states that, in a neighbourhood of any point $p \in \Sigma$, there is a solution to the modified Einstein frame GEE. We apply it to obtain the following local existence result for the string frame GEE. Note that the result remembers the gauge-fixed Einstein frame development associated to the obtained string frame development. This is useful for patching together of solutions, as we only have good control over solutions of the modified Einstein frame GEE.

Lemma 6.26. *Let $\mathcal{I} = (g_0, k, H_0, h_0, \xi_0, x_0)$ be string frame initial data on Σ. Let $U_\Sigma \subset \Sigma$ be a coordinate neighbourhood of $p \in \Sigma$ with coordinates $(x^1, ..., x^n)$, and define coordinates $(x^0 = t, x^1, ..., x^n)$ on $U = \mathbb{R} \times U_\Sigma$.*

Then, for every open neighbourhood $O \subset U$ of p, there exists an open neighbourhood $W \subset O$ of p on which the string frame GEE (6.7) admit a globally hyperbolic development (g, H, ξ) of the initial data on $W \cap \Sigma$. Furthermore, the development can be assumed to be such that the following properties are satisfied.

(1) There exists a dilaton potential ϕ, $\xi = \mathrm{d}\phi$, and thus an associated Einstein frame development $(\tilde{g}, H, \phi)$ and Einstein frame initial data $\tilde{\mathcal{I}}$. There also exists a B-field potential, $H = \bar{H}_{\tilde{\mathcal{I}}} + \mathrm{d}B$. These potentials satisfy the gauge conditions $\mathcal{D}[\tilde{g}, \bar{g}] = 0$ and $\mathrm{d}\mathcal{C}[\tilde{g}, B, \bar{g}] = 0$ as well as the initial conditions from Lemmas 6.22 and 6.23 with an initial vanishing condition for B.

(2) The component-matrix $(\tilde{g}_{\mu\nu})_{\mu\nu}$ (or equivalently $(g_{\mu\nu})_{\mu\nu}$) takes values in the space of canonical Lorentz matrices $\mathcal{C}_n$. Then, in particular, $\operatorname{grad} t$ is timelike on W.

Proof. We take an open contractible coordinate neighbourhood $V_\Sigma \subset \Sigma$ of p, so that we can write $\xi_0 = \mathrm{d}^\Sigma \phi_0$ for some $\phi_0 \in C^\infty(V_\Sigma)$. We consider as initial data $(g_0, k, H_0, h_0, \phi_0, \phi_1)$ with $\phi_1 = x_0$. We transform this data to Einstein frame initial data $\tilde{\mathcal{I}} = (\tilde{g}_0, \tilde{k}, \tilde{H}_0, \tilde{h}_0, \tilde{\phi}_0, \tilde{\phi}_1)$ according to the formulas from Lemma 6.14. We define on $\mathbb{R} \times V_\Sigma$ the background fields $\bar{g}_{\tilde{\mathcal{I}}}$ and $\bar{H}_{\tilde{\mathcal{I}}}$ according to the formulas (6.20). Due to Proposition 6.18, we can apply Proposition 5.1 to obtain a globally hyperbolic development $(W, \tilde{g}, B, \phi)$ of the Einstein frame initial data under the modified Einstein frame system (6.17) with respect to the background fields $\bar{H}_{\tilde{\mathcal{I}}}$ and $\bar{g}_{\tilde{\mathcal{I}}}$. Note that Proposition 5.1 allows us to impose on the development that, one, the component-matrix $(\tilde{g}_{\mu\nu})_{\mu\nu}$ takes values in the space of canonical Lorentz matrices $\mathcal{C}_n$ and, two, it satisfies on $W \cap \Sigma$ the initial conditions from Lemmas 6.22 and 6.23 with initial vanishing condition for the B-field. In particular, $\mathcal{D}[\tilde{g}, \bar{g}]|_{W\cap\Sigma} = 0$.

By construction, the development $(\tilde{g}, B, \phi)$ is such that Lemma 6.24 applies. It follows that $\nabla\mathcal{D}[\tilde{g}, \bar{g}]|_{W\cap\Sigma} = 0$ and $\mathrm{d}\mathcal{C}[\tilde{g}, B, \bar{g}]|_{W\cap\Sigma} = 0$. From Proposition 6.21, we see that the gauge conditions $\mathcal{D} = 0$ and $\mathrm{d}\mathcal{C} = 0$ are implemented on W. Therefore, with $H = \bar{H}_{\tilde{\mathcal{I}}} + \mathrm{d}B$, $(W, \tilde{g}, H, \phi)$ solves the unmodified Einstein frame system (6.12). Noting that conformal transformations preserve global hyperbolicity, we see that a transformation of $(\tilde{g}, H, \phi)$ to the string frame yields the desired development (g, H, ξ), where $\xi = \mathrm{d}\phi$. □

The next Lemma, a local uniqueness result in our setting, is built on a local uniqueness result established (implicitly) in the proof of [10, Theorem 14.2.], presented in the preliminary Section 5.1 as Proposition 5.5.

Lemma 6.27. *Let $\mathcal{I}$ be string frame initial data on Σ. For $i = 1, 2$, let $W_i \subset O_i \subset M = \mathbb{R} \times \Sigma$ be open subsets such that $W_1 \cap W_2 \cap \Sigma \neq \emptyset$ is simply connected, and denote $\Sigma_i := W_i \cap \Sigma$. Assume that we have coordinates on O_i that are adapted to Σ. Assume also that on W_i we have string frame developments (g_i, H_i, ξ_i) of the initial data satisfying conditions (1) and (2) from Lemma 6.26. Denoting by ϕ_i and B_i the associated potentials, and by $\tilde{g}_i$ the associated Einstein frame metrics, assume that we have smooth continuations of $(\tilde{g}_i, B_i, \phi_i)$ to O_i, and assume that O_i is convex with respect to $\tilde{g}_i$ and that Σ_i is spacelike Cauchy in $(W_i, \tilde{g}_i)$. Assume finally that $\overline{W}_i$ is compact and contained in O_i, $i = 1, 2$.*

Then $(g_1, H_1, \xi_1) = (g_2, H_2, \xi_2)$ on the whole intersection $W_1 \cap W_2$.

Proof. By assumption, $\xi_1|_{\Sigma_1\cap\Sigma_2} = \xi_2|_{\Sigma_1\cap\Sigma_2}$. We define $c = [\phi_1 - \phi_2]|_{\Sigma_1\cap\Sigma_2}$. Then $\mathrm{d}^\Sigma c = [\xi_1 - \xi_2]|_{\Sigma_1\cap\Sigma_2} = 0$, hence we can view c as a real number. We set $\hat{\phi}_1 = \phi_1$ and $\hat{\phi}_2 = \phi_2 + c$, and denote by $\hat{g}_i$ the Einstein frame metric associated to the solution $(g_i, B_i, \hat{\phi}_i)$.

Due to the invariance of the initial conditions from Lemmas 6.22 and 6.23 under a change of Einstein frame, cf. Lemma 6.25, we see that $(\hat{g}_1, B_1, \hat{\phi}_1)$ and $(\hat{g}_2, B_2, \hat{\phi}_2)$ take the same initial values on $\Sigma_1 \cap \Sigma_2$. Due to the invariance of the gauge conditions $\mathcal{D} = 0$ and $\mathrm{d}\mathcal{C} = 0$ under a change of Einstein frame, cf. Lemma 6.25, $(\hat{g}_1, B_1, \hat{\phi}_1)$ and $(\hat{g}_2, B_2, \hat{\phi}_2)$ are both solutions of the modified GEE (6.17). Because conformal transformations with constant conformal factor leave geodesics invariant, $(O_2, \hat{g}_2)$ is convex. Thus, we are in the setting to apply Proposition 5.5. The result follows. □

With these results, we establish the existence of a globally hyperbolic development in our setting, following again the ideas of [10, Theorem 14.2.].

Theorem 6.28. *Let $(\Sigma, g_0, k, H_0, h_0, \xi_0, x_0)$ be initial data for the string frame GEE (6.7). Then there exists a globally hyperbolic string frame development of the data.*

Proof. From Lemma 6.26, we know that around every point $p \in \Sigma$ we can construct a local string frame development with an associated Einstein frame development satisfying initial and gauge conditions determining it uniquely, and which takes metric values in $\mathcal{C}_n$. To patch the local solutions together, we associate to every point $p \in \Sigma$ multiple neighbourhoods so as to arrive in the setting of Lemma 6.27.

First, let (U_p^Σ, x_p^m) be a coordinate neighbourhood of p in Σ, so that we obtain coordinates x_p^μ on $\mathbb{R} \times U_p^\Sigma \subset M$, where $x_p^0 = t$. Then, we choose $W_p' \subset \mathbb{R} \times U_p^\Sigma$ with string frame development (g_p, H_p, ξ_p) and associated Einstein frame development $(\tilde{g}_p, H_p, \phi_p)$ as provided by Lemma 6.26. In particular, we ask the development to satisfy the additional demands (1) and (2) from Lemma 6.26. We choose $O_p \subset W_p'$ convex with respect to $\tilde{g}_p$. Finally, we choose an open subset $W_p \subset O_p$ with the properties that, one, its compact closure is contained in O_p, two, $(W_p, \tilde{g}_p)$ has $\Sigma_p := W_p \cap \Sigma$ as Cauchy hypersurface, three, $(\Sigma_q)_{q\in\Sigma}$ is a good open cover of Σ in the sense that the intersection of any two of its members is simply connected.

From the local data above, one can patch together a development on $D := \cup_{p\in\Sigma} W_p$. By assumption, $\Sigma_p \cap \Sigma_q$ is simply connected for all $p, q \in \Sigma$, and conditions (1) and (2) from Lemma 6.26 hold. Hence, it follows from Lemma 6.27 that the constructed solutions agree on $W_p \cap W_q$. We obtain a well-defined string frame solution (g, H, ξ) on D. By Lemma 5.10, because for all $p \in \Sigma$ the local development (W_p, g_p) is globally hyperbolic and g_p takes values in $\mathcal{C}_n$, (D, g) is globally hyperbolic with Cauchy hypersurface Σ. Thus, with the embedding $i\colon \Sigma \hookrightarrow M$ given by $p \mapsto (0, p)$, we have that (g, H, ξ) is a globally hyperbolic string frame development of the initial data $(g_0, k, H_0, h_0, \xi_0, x_0)$ on Σ. □

6.7 Local Geometric Uniqueness

In this section, which reproduces [22, section 5.6], we prove local geometric uniqueness of string frame developments, following the proof strategy of [10, Theorem 14.3]. That is, we show that any two developments are extensions of a common development. The proof relies on comparing any given development (M', g', H', ξ') with the development (D, g, H, ξ) constructed in Theorem 6.28.

In the construction of the development on D, one defines in a local Einstein frame $(\tilde{g}, B, \phi)$ background fields $\bar{g}$ and $\bar{H}$ (cf. (6.20)), and decomposes $H = \bar{H} + \mathrm{d}B$. Crucially, $(\tilde{g}, B, \phi)$ are constructed such that they satisfy the gauge conditions $\mathcal{D} = 0$ and $\mathrm{d}\mathcal{C} = 0$ and hence solve the modified system (6.17) whose solutions we know to be locally unique by Proposition 5.5. Thus, the idea is to locally relate (g', H', ϕ') to $(\tilde{g}, H, \phi)$ by a diffeomorphism $f\colon D \supset W \to W' \subset M'$ and then decompose $f^*H' = \bar{H} + \mathrm{d}\hat{B}'$ such that

$$(\hat{g}', \hat{B}', \hat{\phi}') = (f^*g', \hat{B}', f^*\phi)$$

implements the gauge conditions $\mathcal{D}[\hat{g}', \bar{g}] = 0$ and $\mathrm{d}\mathcal{C}[\hat{g}', \hat{B}', \bar{g}] = 0$. Then, from the aforementioned uniqueness property, we can conclude that our two solutions coincide. In a final step, one has to combine the local diffeomorphisms to a global one.

Before we carry out the full construction, we focus on the existence of a diffeomorphism f and a decomposition $f^*H' = \bar{H} + \mathrm{d}\hat{B}'$ as above.

Recall that by Proposition 5.9, one can locally find a diffeomorphism f implementing the DeTurck gauge condition $\mathcal{D}[f^*g, \bar{g}]$, as well as the initial conditions on the metric discussed in Lemma 6.22. We now prove a similar result for the B-field. That is, we show that by a closed B-field transformation, one can implement on any given B-field the generalised Lorenz gauge $\mathrm{d}\mathcal{C} = 0$ as well as the initial conditions from Lemma 6.23.

Lemma 6.29. *Let (M, g) be a globally hyperbolic smooth spacetime, $B \in \Omega^2(M)$ a two-form, and Σ a smooth spacelike hypersurface. Denote the future-pointing unit normal on Σ by N. Let $\bar{g}$ be a second Lorentzian metric on M, the background metric. Let $B_0 \in \Omega^2(\Sigma)$ be a two-form on Σ such that $\mathrm{d}^\Sigma(B^\parallel - B_0) = 0$, and denote $B_1 = (\mathrm{d}B)(N) - k \cdot B_0 \in \Omega^2(\Sigma)$.*

Then, there exists a closed two-form $\beta \in \Omega^2_{\mathrm{cl}}(M)$ such that the gauge transformed quantity $\hat{B} = B + \beta$ satisfies the generalised Lorenz gauge $\mathrm{d}\mathcal{C}[g, \hat{B}] = 0$ and the initial conditions from Lemma 6.23.

Proof. Let us define for some $\beta \in \Omega^2_{\mathrm{cl}}(M)$

$$\hat{B} := B + \beta, \qquad \hat{\mathcal{C}} := -\mathrm{d}^*_{g,\bar{g}}\hat{B}$$

Then $\mathrm{d}\hat{\mathcal{C}} = 0$ is equivalent to

$$\begin{aligned} 0 &= \mathrm{d}\mathrm{d}^*_{g,\bar{g}}\hat{B} = \mathrm{d}\mathrm{d}^*_{g,\bar{g}}B + \mathrm{d}\mathrm{d}^*_{g,\bar{g}}\beta \\ &= \mathrm{d}\mathrm{d}^*\beta + \mathrm{d}[F(\beta)] + \mathrm{d}\mathrm{d}^*_{g,\bar{g}}B \end{aligned} \tag{6.24}$$

where $F \in \Gamma(\mathrm{Hom}(\Lambda^2T^*M, T^*M))$, explicitly $F(\beta) = (\mathrm{d}^*_{g,\bar{g}} - \mathrm{d}^*)\beta$. We claim that we obtain a solution to (6.24), if we define β to be, as provided by Theorem A.6, a solution to the following equation:

$$0 = \Box_{\mathrm{Hd}}\beta - \mathrm{d}[F(\beta)] - \mathrm{d}\mathrm{d}^*_{g,\bar{g}}B$$

Assume that we manage to set up initial conditions for β such that $\mathrm{d}\beta = 0$ on Σ. Taking the exterior derivative of this equation, we then find that

$$0 = -\mathrm{d}\mathrm{d}^*\mathrm{d}\beta = \Box_{\mathrm{Hd}}\mathrm{d}\beta$$

By Theorem A.6, $\mathrm{d}\beta \equiv 0$ globally, so that indeed β solves 6.24.

Finally, we note that by Lemma 6.23, the initial values of $\hat{B}$ and its time-derivative are uniquely determined by the requirements in the statement. The same then holds for β. It remains to check that this is consistent with the requirement $\mathrm{d}\beta = 0$. Clearly, $(\mathrm{d}\beta)^{\parallel} = \mathrm{d}^{\Sigma}(\hat{B} - B)^{\parallel} = 0$. Also, by Lemma B.2 and definition of B_1,

$$(\mathrm{d}\hat{B})(N) = B_1 + k \cdot B_0 = (\mathrm{d}B)(N),$$

so that also $(\mathrm{d}\beta)(N) = 0$. □

Combining Proposition 5.9 and Lemma 6.29, we can prove that any given Einstein frame development of initial data on a hypersurface Σ can be locally related to a solution on $M = \mathbb{R} \times \Sigma$ with background fields as defined in (6.20) such that the gauge conditions $\mathcal{D} = 0$ and $\mathrm{d}\mathcal{C} = 0$ as well as the conditions on the initial data from Lemmas 6.22 and 6.23 are implemented.

Proposition 6.30. *Let $\tilde{\mathcal{I}} = (\Sigma, \tilde{g}_0, \tilde{k}, H_0, \tilde{h}_0, \phi_0, \tilde{\phi}_1)$ be initial data for the Einstein frame GEE (6.7). Let $(M', \tilde{g}', H', \phi')$ be an arbitrary Einstein frame development. Denote the associated embedding by $i \colon \Sigma \hookrightarrow M'$. Denote furthermore $M = \mathbb{R} \times \Sigma$ and by $\bar{g} = \bar{g}_{\tilde{\mathcal{I}}}$ and $\bar{H} = \bar{H}_{\tilde{\mathcal{I}}}$ the background fields defined in (6.20).*

Then, for every $p \in \Sigma$ there exist neighbourhoods $p \in W \subset M$ and $i(p) \in W' \subset M'$ and a diffeomorphism $f \colon W \to W'$ such that with $\hat{g}' = f^*\tilde{g}'$, $\hat{H}' = f^*H'$, and $\hat{\phi}' = f^*\phi'$, one can decompose $\hat{H}' = \bar{H} + \mathrm{d}\hat{B}'$ such that the gauge conditions $\mathcal{D}[\hat{g}', \bar{g}] = 0$ and $\mathrm{d}\mathcal{C}[\hat{g}', \hat{B}', \bar{g}] = 0$ hold and the initial conditions from Lemmas 6.22 and 6.23 are satisfied with initial vanishing condition for the B-field.

Proof. Take $p \in \Sigma$ and a diffeomorphism $f \colon W \to W'$ such that $\mathcal{D}[f^*\tilde{g}', \bar{g}] = 0$, where $p \in W \subset M$. By Proposition 5.9, such a diffeomorphism exists and can be chosen such that $f_*\partial_t = N'$ and $f_*|_{TW_\Sigma} = i_*$, where we set $W_\Sigma = W \cap \Sigma$ and denoted by N' the future-pointing unit-normal on $i(\Sigma)$. By restricting, we can assume that $(W', \tilde{g}')$ is globally hyperbolic with Cauchy hypersurface $W' \cap \Sigma$.

Set $\hat{g}' = f^*\tilde{g}'$, $\hat{H}' = f^*H'$, and $\hat{\phi}' = f^*\phi'$. Note that the pullback of the data $(\tilde{g}', H', \phi')$ by f induces on $f^{-1} \circ i(W_\Sigma) = W_\Sigma$ the pullback of the initial data $(\tilde{g}_0, \tilde{k}, H_0, h_0, \phi_0, \phi_1)$ by $i^* \circ (f^{-1})^* = \mathrm{id}_{TW_\Sigma}$, so the initial data itself. Since also $f_*N = N'$, we see that $(\hat{g}', \hat{\phi}')$ satisfies the conditions from Lemma 6.22.

We want to define a two-form $\hat{B}'$ that satisfies the gauge condition $\mathrm{d}\mathcal{C}[\hat{g}', \hat{B}'] = 0$ and the initial conditions from Lemma 6.23. First, consider that, since $W \subset M$, $\bar{H}$ is defined on W. We can assume W to be contractible, so that all closed forms are exact. Hence, we can find a two-form $B' \in \Omega^2(W)$ such that $\hat{H}' = \bar{H} + \mathrm{d}B'$. Global hyperbolicity of $(W', \tilde{g}')$ is equivalent to global hyperbolicity of $(W, \hat{g}')$. Hence, by Lemma 6.29, we can find a closed two-form β' such that the gauge-transformed quantity $\hat{B}' := B' + \beta'$ satisfies the gauge-condition $\mathrm{d}\mathcal{C}[\hat{g}', \hat{B}'] = 0$ and the initial conditions from Lemma 6.23 with $B_0 = B_1 = 0$. Note that this choice is allowed because, by $f_*N = N'$ and agreement of the initial data induced by H and $\hat{H}'$, $\hat{H}'\big|_\Sigma = H|_\Sigma$ and thus $\mathrm{d}B'|_\Sigma = 0$. This finishes the proof. □

Finally, we can adapt [10, Theorem 14.3] to our setting, following closely the proof strategy presented there. We show that any two developments of the initial data are extensions of a common development by showing that any development is an extension of a development as constructed in Theorem 6.31.

Theorem 6.31. *Let $(\Sigma, g_0, k, H_0, h_0, \xi_0, x_0)$ be initial data for the string frame equations (6.7). Denote by (D, g, H, ξ) a globally hyperbolic development constructed as in Theorem 6.28, $D \subset M = \mathbb{R} \times \Sigma$. Assume that we have another development (M', g', H', ξ') with embedding $i\colon \Sigma \hookrightarrow M'$.*

Then, there is a tubular neighbourhood $\tilde{D} \subset D$ of Σ and a smooth time orientation preserving diffeomorphism $\psi\colon \tilde{D} \to \psi(\tilde{D}) \subset M'$ with $\psi|_\Sigma = i$ relating the two solutions, i.e. $\psi^* g' = g$, $\psi^* H' = H$, and $\psi^* \xi' = \xi$.

Proof. Recall that in the construction performed in Theorem 6.28, we associate to every point $p \in \Sigma$ a coordinate neighbourhood $U_p = \mathbb{R} \times U_p^\Sigma$ with coordinates $(x_p^0 = t, x_p^m)$ and then an open subset $W_p' \subset U_p$ on which there is associated to the string frame development a globally hyperbolic Einstein frame development $(\tilde{g}_p, H_p, \phi_p)$ satisfying conditions (1) and (2) from Lemma 6.26. In particular, we have background fields $\bar{g}_p$ and $\bar{H}_p$ on W_p'. We also picked a $\tilde{g}_p$-convex subset $p \in O_p \subset W_p'$ and a globally hyperbolic neighbourhood $(W_p, \tilde{g}_p)$ with compact closure contained in O_p.

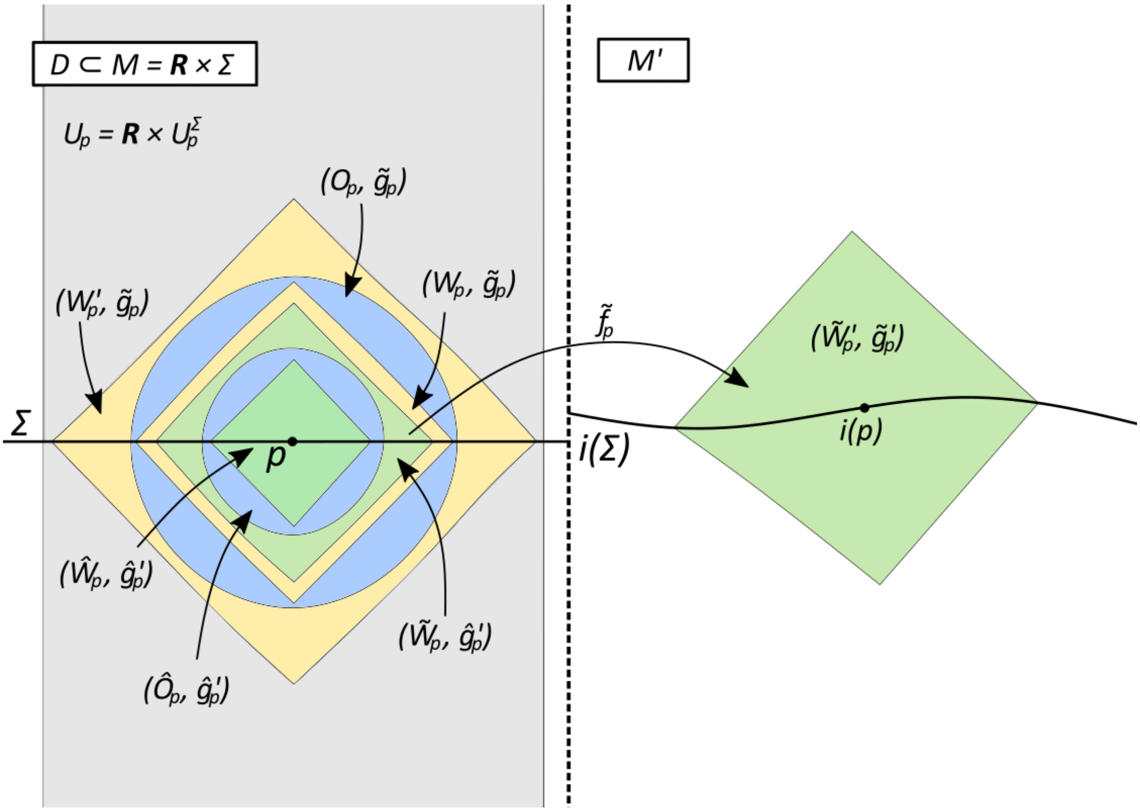


Figure 1: An illustration of the data associated to a point p of the initial hypersurface Σ in the proof of Theorem 6.31. On the left hand side is the gauge-fixed development D of the initial data constructed in Theorem 6.28. On the right hand side is an arbitrary development M'. The two developments are compared (in a local Einstein frame) via a diffeomorphism $\tilde{f}_p\colon \widetilde{W}_p \to \widetilde{W}_p'$.

Let us associate additional data to every point $p \in \Sigma$. We choose a diffeomorphism $\widetilde{f}_p \colon \widetilde{W}_p \to \widetilde{W}'_p$ defined as in Proposition 6.30 with the additional requirement that $\widetilde{W}_p \subset W_p$. We obtain the gauge-fixed solution $(\widehat{g}'_p, \widehat{B}'_p, \widehat{\phi}'_p)$ as in that Proposition. We pick a $\widehat{g}'_p$-convex neighbourhood $\widehat{O}_p \subset \widetilde{W}_p$ and a subset $\widehat{W}_p \subset \widehat{O}_p$ such that, one, $(\widehat{W}_p, \widehat{g}'_p)$ is globally hyperbolic with Cauchy hypersurface $\widehat{W}_p \cap \Sigma$, two, $\widehat{W}_p$ has compact closure contained in $\widehat{O}_p$, three, $\widehat{g}'_p$ takes on $\widehat{W}_p$ values in $\mathcal{C}_n$ and, four, $\widehat{W}_p \cap \Sigma$ is simply-connected. We define $f_p := \widetilde{f}_p\big|_{\widehat{W}_p}$.

By their construction, both $(W_p, \tilde{g}_p, B_p, \phi_p)$ and $(\widehat{W}_p, \widehat{g}'_p, \widehat{B}'_p, \widehat{\phi}'_p)$ satisfy (1) and (2) from Lemma 6.26. Also, the intersection $W_p \cap \widehat{W}_p \cap \Sigma = \widehat{W}_p \cap \Sigma$ is simply connected. Thus, we are in the setting to apply Lemma 6.27. We conclude that $(g, H, \xi)|_{\widehat{W}_p} = (f_p^* g', f_p^* H', f_p^* \xi')$, so f_p is an isometry relating the string frame solutions locally.

To finish the proof, we have to patch the local isometries together to a global isometry. For every $p \in \Sigma$, we find a neighbourhood $\tilde{D}_p \subset W_p$ such that $(\tilde{D}_p, g)$ is globally hyperbolic with Cauchy HS $\tilde{D}_p \cap \Sigma$. Then, by Lemma 5.11, the isometries f_p and f_q are determined on $\tilde{D}_p \cap \tilde{D}_q$ by the value of their differential on $\Sigma \cap \tilde{D}_p \cap \tilde{D}_q$. These agree, and hence $f_p|_{\tilde{D}_p \cap \tilde{D}_q} = f_q|_{\tilde{D}_p \cap \tilde{D}_q}$. We conclude that we can define a global isometry ψ on $\tilde{D} = \bigcup_p \tilde{D}_p$, as desired. □

6.8 The Maximal Globally Hyperbolic Development

This section reproduces [22, section 5.6].

In the famous work [8], Choquet-Bruhat and Geroch established for the Einstein equations the existence of a geometrically unique globally hyperbolic development which extends any other development of a given set of initial data.[26] We formalise the notion of the maximal globally hyerbolic development for the string frame GEE as follows (we adapt [10, Definition 16.5.]).

Definition 6.32. Let $\mathcal{I} = (\Sigma, g_0, k, H_0, h_0, \xi_0, x_0)$ be initial data for the string frame GEE. A *maximal globally hyperbolic development (MGHD)* of $\mathcal{I}$ is a development (M, g, H, ξ) with embedding $i \colon \Sigma \hookrightarrow M$ such that for every other globally hyperbolic development (M', g', H', ξ') with embedding $i' \colon \Sigma \hookrightarrow M'$ there exists a map $\psi \colon M' \to \psi(M') \subset M$ which is a time-orientation preserving diffeomorphism onto its image relating the two developments, i.e. $\psi^* g = g'$, $\psi^* H = H'$, $\psi^* \xi = \xi'$, and $\psi \circ i' = i$.

If (M, g, H, ξ) and (M', g', H', ξ') are two developments of given initial data for which a map ψ as in the definition exists, we also call (M, g, H, ξ) an *extension* of (M', g', H', ξ'). Thus, an MGHD (M, g, H, ξ) is an extension of every other globally hyperbolic development.

It follows from the properties of the MGHD that it is unique up to diffeomorphism, i.e. for two MGHDs (M, g, H, ξ) and (M', g', H', ξ') one can see any map ψ as provided by Definition 6.32 to be a diffeomorphism between M and M'. [10]

The explicit proof of existence for the MGHD given in [8] focuses on the Einstein vacuum system. Remarks preceding the proof explain[27] why the results extend to general Einstein

[26] We note that many details absent from the proof in [8] are provided by Ringström in [42, § 23] in the context of the Einstein-Vlasov-nonlinear scalar field system. A text similar to [42], but in the context of the Einstein-nonlinear scalar field system, is provided in Ringström's errata [43] to his book [10] (which contains an erroneous proof of the existence of an MGHD). A proof that does not (in contrast to the other mentioned proofs) rely on Zorn's Lemma was given by Sbierski [44].

[27] The same remarks apply to the more detailed proof due to Ringström mentioned in a previous footnote.

matter systems for which the notions of solutions, initial data, developments, and extensions are as in (1-4) and satisfy the properties (i-iii).

(1) A *solution* is a tuple (M, g, Φ) consisting of an $n+1$-dimensional time-oriented Lorentzian manifold (M, g), a section $\Phi \in \Gamma(E)$ of a diffeomorphism-invariant vector subbundle E of the tensor bundle $\mathcal{T}M$, all such that a diffeomorphism-invariant set of conditions (the Einstein-matter equations) is satisfied.

(2) *Initial data* is a tuple (Σ, g_0, k, Φ_0) consisting of an n-dimensional Riemannian manifold (Σ, g_0), a symmetric two-tensor k, and a section $\Phi_0 \in \Gamma(E_0)$ of a diffeomorphism-invariant vector subbundle of the tensor bundle $\mathcal{T}\Sigma$ satisfying a given diffeomorphism-invariant set of constraint conditions.[28]

(i) A solution (M, g, Φ) naturally induces on every spacelike hypersurface Σ a set of initial data $\mathcal{I} = (\Sigma, g_0, k, \Phi_0)$, where g_0 is the inherited Riemannian metric and k the second fundamental form on Σ. Naturality means that, if $\psi\colon M' \to M$ is a diffeomorphism and $i\colon \Sigma \hookrightarrow M$ the embedding associated to $\mathcal{I}$, then $(M', \psi^* g, \psi^* \Phi)$ induces on $\psi(i(\Sigma))$ the initial data $(\psi(i(\Sigma)), (\psi \circ i)_* g_0, (\psi \circ i)_* k, (\psi \circ i)_* \Phi_0)$.

(3) A *development* of initial data (Σ, g_0, k, Φ_0) is a solution (M, g, Φ) and a semi-Riemannian embedding $i\colon \Sigma \hookrightarrow M$ such that the pullback by i of the initial data induced on $i(\Sigma)$ yields the initial data (Σ, g_0, k, Φ_0).

(4) Given initial data $\mathcal{I}$, a development (M, g, Φ) of $\mathcal{I}$ with embedding i is an *extension* of another development (M', g', Φ') of $\mathcal{I}$ with embedding i' if there exists a map $\psi\colon M' \to \psi(M') \subset M$ which is a time-orientation preserving diffeomorphism onto its image such that $\psi^* g = g'$, $\psi^* \Phi = \Phi'$, and $\psi \circ i' = i$.

(ii) Every set of initial data admits a globally hyperbolic development.

(iii) Any two developments of given initial data are extensions of a common development.

We see that our notions of string frame solutions, string frame initial data (see Definition 6.9), string frame developments (see Definition 6.10), and extensions (see above) fit the template (1-4). In our setting, property (i) is trivial, and properties (ii) and (iii) respectively correspond to Theorems 6.28 and 6.31. Thus,

Theorem 6.33. *Let $\mathcal{I}$ be initial data for the string frame GEE. Then there exists a string frame MGHD of $\mathcal{I}$. It is unique up to diffeomorphism.*

[28] The constraint conditions do not have to be of a specific form; their purpose is to encode conditions necessary and sufficient to guarantee property (ii).

– Part III –
Appendix

A Results on Wave Equations

This chapter reproduces [22, appendix A]. However, Section A.3 is new.

We summarise here the results on symmetric hyperbolic PDEs, also called wave equations, which we employ in our study of the initial value problem for the generalised Einstein equations.

A.1 Non-Linear Wave Equations

In this section, we give appropriate definitions for the discussion of and the main result on non-linear wave equations. Specifically, we discuss systems of PDEs of the form (6.1). Recall that this is a system of PDEs for a vector valued function $u\colon \mathbb{R}^{n+1} \to \mathbb{R}^N$, where $n, N \in \mathbb{N}_{\geq 1}$.

Working on $\mathbb{R}^{n+1}$, we employ the following notion of a canonical Lorentz matrix, cf. [10, Definition 8.4.].

Definition A.1. Let $A \in \mathrm{Sym}^2(n+1)$ be a symmetric matrix. We call A a *canonical Lorentz matrix* if $A_{00} < 0$ and (A_{ij}) positive definite. We denote the space of canonical Lorentz matrices by $\mathcal{C}_n$. Finally, given $a = (a_0, a_1, a_2) \in \mathbb{R}^3_{>0}$, we denote by $\mathcal{C}_{n,a}$ the space of canonical Lorentz matrices A such that $A_{00} < -a_0$, $(A_{ij}) > a_1$, and $\|A\|_1 > a_2$, where $\|\cdot\|_1$ denotes the 1-norm on $\mathbb{R}^{(n+1)^2}$.

One can show that a canonical Lorentz matrix A has n positive eigenvalues and one negative eigenvalue [10, Lemma 8.3.].

Recall now more specifically that (6.1) is the following system of PDEs:

$$\begin{aligned} g[u]^{\mu\nu}\partial_\mu\partial_\nu u &= f[u], \\ u(T_0, \cdot) &= U_0, \\ \partial_t u(T_0, \cdot) &= U_1. \end{aligned} \tag{A.1}$$

Herein, T_0 some real value, $U_0, U_1 \in C_0^\infty(\mathbb{R}^n, N)$ compactly supported initial data, and

$$g \in C^\infty(\mathbb{R}^{N+(n+1)N+n+1}, \mathcal{C}_n), \qquad f \in C^\infty(\mathbb{R}^{N+(n+1)N+n+1}, \mathbb{R}^N)$$

respectively a C^∞ (N,n)-admissible metric and a C^∞ (N,n)-admissible non-linearity. Furthermore, we employed the notation

$$\begin{aligned} g[u](t,x) &:= g(t,x,u(t,x),\partial_0 u(t,x),\dots,\partial_n u(t,x)), \\ f[u](t,x) &:= g(t,x,u(t,x),\partial_0 u(t,x),\dots,\partial_n u(t,x)). \end{aligned}$$

In particular, $g[u] \in C^\infty(\mathbb{R}^{n+1}, \mathcal{C}_n)$ and $f[u] \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$.

We explain now the two notions of a C^∞ (N,n)-admissibility, cf. [10, Definitions 9.1 and 9.4.].

Definition A.2. Let $g \in C^\infty(\mathbb{R}^{N+(n+1)N+n+1}, \mathcal{C}_n)$. We call g a C^∞ (N,n)-*admissible metric* if for all compact intervals $I \subset \mathbb{R}$

(i) there exists $a \in \mathbb{R}^3_{>0}$ such that $g(t,\cdot) \in \mathcal{C}_{n,a}$ for all $t \in I$, and

(ii) for all multi-indices $\alpha \in \mathbb{N}_0^{N+(n+1)N+n+1}$, there exists a continuous, monotonously increasing function $h\colon \mathbb{R} \to \mathbb{R}$ such that

$$\|\partial^\alpha g(t,x,\xi)\| \leq h(\|\xi\|), \qquad t \in I, x \in \mathbb{R}^n, \xi \in \mathbb{R}^{N+(n+1)N},$$

where $\|\cdot\|$ is the maximum norm on $\mathbb{R}^k$ for appropriate k.

Definition A.3. Let $f \in C^\infty(\mathbb{R}^{N+(n+1)N+n+1}, \mathbb{R}^N)$. We call f a C^∞ (N,n)-*admissible non-linearity*, if

(i) $f[0]$ is of locally x-compact support, i.e. for all $t \in \mathbb{R}$ $f[0](t)$ is of compact support, and

(ii) for all compact intervals $I \subset \mathbb{R}$ and all multi-indices $\alpha \in \mathbb{N}_0^{N+(n+1)N+n+1}$, there exists a continuous, monotonously increasing function $h\colon \mathbb{R} \to \mathbb{R}$ such that

$$\|\partial^\alpha f(t,x,\xi)\| \le h(\|\xi\|), \qquad t \in I, x \in \mathbb{R}^n, \xi \in \mathbb{R}^{N+(n+1)N},$$

where $\|\cdot\|$ is the maximum norm on $\mathbb{R}^k$ for appropriate k.

We can now cite [10, Corollary 9.16.], which for us is the main theorem on non-linear wave equations. Essentially verbatim, it states the following.

Theorem A.4. *Consider the system (6.1), where $T_0 \in \mathbb{R}$, $g \in C^\infty(\mathbb{R}^{N+(n+1)N+n+1}, \mathcal{C}_n)$ a C^∞ (N,n)-admissible metric, $f \in C^\infty(\mathbb{R}^{N+(n+1)N+n+1}, \mathbb{R}^N)$ a C^∞ (N,n)-admissible non-linearity, and $U_0, U_1 \in C_0^\infty(\mathbb{R}^n, N)$ initial data.*

Then there are $T_1 < T_0 < T_2$ and a unique solution $U \in C^\infty[(T_1,T_2) \times \mathbb{R}^n, \mathbb{R}^N]$ to (6.1). The solution is of locally x-compact support. Moreover, T_2 can be chosen such that $T_2 = \infty$ or

$$\lim_{\tau \to T_2-} \sup_{T_0 \le t \le \tau} \sum_{|\alpha|+j \le 2} \sup_{x \in \mathbb{R}^n} \left|\partial^\alpha \partial_t^j u(t,x)\right| = \infty.$$

The statement concerning T_1 is similar.

A.2 Inhomogeneous Linear Tensor Wave Equations

Given a fixed smooth Lorentzian metric g on the manifold M, we often deal with tensor wave equations. To discuss them, it is convenient to make the following

Definition A.5. Given a tensor $A \in \Gamma(T^r_s M)$, we define with with $r \ge l$, $s \ge k$ the following map:

$$A\colon \Gamma(T^k_l M) \longrightarrow \Gamma(T^{r-l}_{s-k} M), \qquad A(B)^{\alpha_1\dots\alpha_{r-l}}_{\beta_1\dots\beta_{s-k}} := A^{\gamma_1\dots\gamma_l\alpha_1\dots\alpha_{r-l}}_{\delta_1\dots\delta_k\beta_1\dots\beta_{s-k}} B^{\delta_1\dots\delta_k}_{\gamma_1\dots\gamma_l}.$$

If $l < r$ and $s < k$, we also set $A(B) := B(A)$.

Furthermore, given a metric g and two tensors $A \in \Gamma(T_s M)$, $B \in \Gamma(T_l M)$, w.l.o.g. $s \ge l$, we denote

$$\langle A, B\rangle_{\beta_1\dots\beta_{s-l}} \equiv \langle B, A\rangle_{\beta_1\dots\beta_{s-l}} := B^{\gamma_1\dots\gamma_l} A_{\gamma_1\dots\gamma_l\beta_1\dots\beta_{s-l}} \in \Gamma(T_{s-l}M).$$

The main result on tensor wave equations that we employ is the following. It is a combination of [10, Theorem 12.17. and Corollary 12.12].

Theorem A.6. *Let (M,g) be a globally hyperbolic spacetime with spacelike Cauchy hypersurface $i\colon S \hookrightarrow M$. Assume $B \in T^{r+s+1}_{r+s}(M)$, $C \in T^{r+s}_{r+s}(M)$, $E \in T^r_s(M)$. Given smooth tensor fields $A_0, A_1 \in T^r_s(M)$, there exists a unique smooth tensor field $A \in T^r_s(M)$ solving the initial value problem*

$$\begin{aligned} \Box_g A + B(\nabla A) + C(A) &= E, \\ i^* A &= i^* A_0, \\ i^*(\nabla_N A) &= i^* A_1, \end{aligned} \tag{A.2}$$

where $\Box_g = \nabla^\lambda \nabla_\lambda$ and N is the future directed normal to S.

Furthermore, if $E \equiv 0$, and also $i^* A_0 = i^* A_1 = 0$ on $\Omega \subset S$, then $A = 0$ on the Cauchy development $D(\Omega)$.

Assuming $H = \mathrm{d}B$, the B-field has to satisfy the equation (cf. (6.12) with $\bar{H} = 0$)

$$\mathrm{d}^*\mathrm{d}B = -\frac{4}{d-2} i_{\xi^\sharp} \mathrm{d}B$$

for some closed one-form $\xi \in \Omega^1_{\mathrm{cl}}(M)$. This almost looks like a tensor wave equation as above, but differs from it in two ways:

(i) The Beltrami wave operator $\Box_g$ is replaced by the co-exact wave operator $\mathrm{d}^*\mathrm{d}B$.

(ii) The equation only depends on $\mathrm{d}B$.

Due to the second point in particular, we expect the PDE to exhibit fundamentally different uniqueness behaviour than the tensor wave equation (A.2): For any solution B to such an equation and any closed one-form $b \in \Omega^1_{\mathrm{cl}}(M)$, also $B + b$ is a solution. In physics, this is referred to as the B-field having a "gauge freedom".

We will call equations that satisfy (i) and (ii) *co-exact wave equations*. These are equations for a p-form $A \in \Omega^p(M)$ that can be stated as

$$\mathrm{d}^*\mathrm{d}A + B(\mathrm{d}A) = E, \tag{A.3}$$

wherein $B \in \Gamma(\mathrm{Hom}(\Lambda^{p+1}T^*M, \Lambda^p T^*M))$, and $E \in \Omega^p(M)$ is co-closed, i.e. $\mathrm{d}^*E = 0$.

We investigate this type of PDE in more detail in Section A.3. In particular, motivated by the Lorenz gauge, we want to find a set of equations which always admit a co-closed solution. We find that it is sufficient to impose the following condition:

(iii) Applying the co-differential to the equation gives an equation directly proportional to the original equation.

Note that this condition depends on the choice of Lorentz metric used to define the co-differential. In Lemma A.10, we find that any co-exact wave equation that satisfies condition (iii) can be stated as

$$\mathrm{d}^*\mathrm{d}A + \langle B, \mathrm{d}A \rangle + \langle \beta, \mathrm{d}A \rangle = E, \tag{A.4}$$

where $\beta \in \Omega^1_{\mathrm{cl}}(M)$ is closed, $B \in \Omega^{2p+1}(M)$ and $E \in \Omega^p(M)$ such that

$$\mathrm{d}^*E = -\langle \beta, E \rangle, \qquad \mathrm{d}^*B = -\langle \beta, B \rangle.$$

Note that the string frame and Einstein frame B-field equation with arbitrary background field $\bar{H}$ is of this form, cf. (6.7) and (6.12). (Note that in the string frame $\beta = \xi$ while in the Einstein frame $\beta = \frac{4}{d-2}\xi$.) This may shed some light on the B-field equation and its coupling to the metric and dilaton.

Our main result for co-exact wave equations is Theorem A.11, which we shall repeat here. We use it to show that the generalised Lorenz gauge propagates well, cf. Proposition 6.21.

Theorem A.7. *Let (M, g) be a globally hyperbolic sapcetime with spacelike Cauchy hypersurface $i\colon S \hookrightarrow M$. Let $\beta \in \Omega^1_{\mathrm{cl}}(M)$ closed, $B \in \Omega^{2p+1}(M)$ and $E \in \Omega^p(M)$ such that*

$$\mathrm{d}^* E = -\langle \beta, E \rangle, \qquad \mathrm{d}^* B = -\langle \beta, B \rangle .$$

Let $A_0 \in \Omega^p(M)$ be smooth p-form; the initial data. Consider the following system of PDEs:

$$\begin{aligned} \mathrm{d}^*\mathrm{d}A + \langle B, \mathrm{d}A\rangle + \langle \beta, \mathrm{d}A\rangle &= E, \\ i^*\mathrm{d}A &= i^*\mathrm{d}A_0. \end{aligned} \tag{A.5}$$

Herein, N is the future directed normal to S.

If and only if A_0 satisfies on S the constraint equation

$$i_N \left[\mathrm{d}^*\mathrm{d}A_0 + \langle B, \mathrm{d}A_0\rangle + \langle \beta, \mathrm{d}A_0\rangle - E\right] = 0$$

does there exist a smooth p-form $A \in \Omega^p(M)$ solving the initial value problem to equation (A.5). In this case, solutions are unique up to addition of a closed p-form, and there exist co-closed solutions. The latter are uniquely determined by their initial value i^*A on S.

A.3 Inhomogeneous Linear Co-Exact Wave Equations

We investigate here in more detail *co-exact wave equations*, that is equations for a p-form $A \in \Omega^p(M)$ that can be stated as

$$\mathrm{d}^*\mathrm{d}A + B(\mathrm{d}A) = E, \tag{A.6}$$

wherein $B \in \Gamma(\mathrm{Hom}(\Lambda^{p+1}T^*M, \Lambda^p T^*M))$, and $E \in \Omega^p(M)$ is co-closed, i.e. $\mathrm{d}^*E = 0$. In particular, we will investigate a set of equations which satisfy a condition that guarantees their admitting a co-closed solution. This set includes the string and Einstein frame B-field equation with arbitrary background field $\bar{H}$. In this way, we hope to shed some light on the B-field equation and its coupling to the metric and dilaton.

The first result that we obtain is that the co-exact wave equation (A.6) exhibits a behaviour that the tensor wave equation (A.2) does not: It gives rise to a constraint equation.

Lemma A.8. *Let (M, g) be a globally hyperbolic spacetime with Cauchy hypersurface $i\colon S \hookrightarrow M$. Denote by N the future directed unit normal on S. Let $A \in \Omega^p(M)$. Then $i_N(\mathrm{d}^*\mathrm{d}A)$ depends only on the initial value $A_0 = i^*\mathrm{d}A$. Thus (A.3) gives rise to the constraint equation*

$$i_N \left[\mathrm{d}^*\mathrm{d}A_0 + B(\mathrm{d}A_0) - E\right] = 0. \tag{A.7}$$

Proof. Total antisymmetry of $\mathrm{d}A$ implies that the normal derivative in

$$i_N\mathrm{d}^*\mathrm{d}A = -\nabla^\mu \mathrm{d}A_{\mu 0 \dots}$$

gets projected out. □

The following statement about the homogeneous case with vanishing initial data proves that solutions (provided they exist) are unique up to addition of a closed form.

Lemma A.9. *Let (M, g) be a globally hyperbolic spacetime with spacelike Cauchy hypersurface $i\colon S \hookrightarrow M$, and take $B \in \Gamma(\mathrm{Hom}(\Lambda^{p+1}T^*M, \Lambda^p T^*M))$. Let finally the initial data $A_0 \in \Omega^p(M)$ be such that $\mathrm{d}A_0 = 0$ on $\Omega \subset S$. Consider the following system of PDEs:*

$$\begin{aligned} \mathrm{d}^*\mathrm{d}A + B(\mathrm{d}A) &= 0, \\ i^*\mathrm{d}A &= 0, \end{aligned} \tag{A.8}$$

where N is the future directed normal to S.

Then, given a solution A to (A.8), it holds that $\mathrm{d}A = 0$ on the Cauchy development $D(\Omega)$. Furthermore, if $\mathrm{d}^*A = 0$ on $D(\Omega)$ and $A = 0$ on S, then $A = 0$ on $D(\Omega)$.

Proof. Taking the exterior derivative of (A.8), we get that $\mathrm{d}A$ satisfies the source free wave equation

$$-\Box_{\mathrm{Hd}}\mathrm{d}A + \mathrm{d}[B(\mathrm{d}A)] = 0.$$

Implicitly via the Weitzenböck formula, this is a source free tensor wave equation. If initially $\mathrm{d}A = 0$ and $\nabla_N \mathrm{d}A = 0$, we can conclude with Theorem A.6 that $\mathrm{d}A = 0$ on $D(\Omega)$.

We already have that initially $\mathrm{d}A = 0$. Furthermore on S

$$\begin{aligned} 0 = \mathrm{d}\mathrm{d}A &\overset{\text{L B.2}}{=} \mathrm{d}^S\mathrm{d}A^\parallel - N^\flat \wedge \left[(\nabla_N \mathrm{d}A)^\parallel - \mathrm{d}^S(i_N\mathrm{d}A)^\parallel + k \cdot \mathrm{d}A^\parallel\right] \\ &= -N^\flat \wedge \nabla_N \mathrm{d}A^\parallel \end{aligned}$$

and equation (A.8) simplifies on S to give

$$0 = \mathrm{d}^*\mathrm{d}A = i_N \nabla_0 \mathrm{d}A.$$

Thus, $\nabla_N \mathrm{d}A = 0$ initially, and we conclude $\mathrm{d}A = 0$ globally.

Finally, assume $\mathrm{d}^*A = 0$ on $D(\Omega)$, as well as $A = 0$ on S. Note that Lemma B.2 implies that $\mathrm{d}A = 0$ forces initially $(\nabla_N A)^\parallel$, and $\mathrm{d}^*A = 0$ forces initially $(\nabla_N A)^\perp = 0$. That is, $\nabla A = 0$ on S.

Due to co-closedness of A, we can add $\mathrm{d}\mathrm{d}^*A$ to the left hand side of (A.8) to obtain that on $D(\Omega)$

$$-\Box_{\mathrm{Hd}}A + B(\mathrm{d}A) = 0.$$

Just as before, implicitly via the Weitzenböck formula, this is a source free tensor wave equation, and we obtain $A = 0$ on $D(\Omega)$ from Theorem A.6. □

To find a solution to the co-exact wave equation (A.6), we want to assume the existence of a co-closed solution, which allows us to relate the co-exact wave equation to a tensor wave equation. We will see that the following condition is a sufficient guarantee for the existence of a co-closed solution.

Lemma A.10. *Precisely those co-exact wave equations that satisfy*

(iii) Applying the co-differential to the equation gives an equation directly proportional to the original equation.

can be stated as

$$\mathrm{d}^*\mathrm{d}A + \langle B, \mathrm{d}A\rangle + \langle \beta, \mathrm{d}A\rangle = E, \tag{A.9}$$

where $\beta \in \Omega^1_{\mathrm{cl}}(M)$ is closed, $B \in \Omega^{2p+1}(M)$ and $E \in \Omega^p(M)$ such that

$$\mathrm{d}^*E = -\langle \beta, E\rangle, \qquad \mathrm{d}^*B = -\langle \beta, B\rangle.$$

In this case,

$$\begin{aligned}\mathrm{d}^*\left[\langle B, \mathrm{d}A\rangle + \langle \beta, \mathrm{d}A\rangle - E\right] &= \langle \beta, \mathrm{d}^*\mathrm{d}A + \langle B, \mathrm{d}A\rangle + \langle \beta, \mathrm{d}A\rangle - E\rangle \\ &= \langle \beta, \mathrm{d}^*\mathrm{d}A + \langle B, \mathrm{d}A\rangle - E\rangle\,.\end{aligned}$$

Proof. We make the general ansatz

$$\mathrm{d}^*\mathrm{d}A + \tilde{B}(\nabla A) + C(A) = E \tag{A.10}$$

with $\tilde{B} \in \Gamma(T_p^{p+1}M)$, $C \in \Gamma(T_p^p M)$ and $E \in \Gamma(T_p M)$.

Since the C-term does not depend on $\mathrm{d}A$, we have to require $C = 0$. To make the $\tilde{B}$-term only depend on $\mathrm{d}A$, we have to require complete antisymmetry of $\tilde{B}$ in the contravariant entries. Furthermore, for the equation to be well-defined as an equation over the space of p-forms, we have to ask $\tilde{B}$ and E to be completely antisymmetric in the covariant entries.

Now, taking the co-differential of the resulting equation yields

$$\begin{aligned}\mathrm{d}^*E &= \mathrm{d}^*(\tilde{B}(\mathrm{d}A)) \\ &= -(\operatorname{div}\tilde{B})(\mathrm{d}A) - \tilde{B}^\sharp(\nabla\mathrm{d}A),\end{aligned} \tag{A.11}$$

where $(\tilde{B}\#)^{\alpha_1\gamma_1\ldots\gamma_{p+1}}_{\alpha_2\ldots\alpha_p} := g^{\alpha_1\beta}\tilde{B}^{\gamma_1\ldots\gamma_{p+1}}_{\beta\alpha_2\ldots\alpha_p}$. Thus, A satisfies the second order PDE (A.11). This cannot be compatible with (iii) if $\tilde{B}^\sharp(\nabla\mathrm{d}A)$ is not proportional to $\mathrm{d}^*\mathrm{d}A$. We thus have to require that

$$\tilde{B}^{\gamma_1\ldots\gamma_{p+1}}_{\alpha_1\ldots\alpha_p} = \delta^{[\gamma_1}_{\alpha_1}(\tilde{B}')^{\gamma_2\ldots\gamma_{p+1}]}_{\alpha_2\ldots\alpha_p} + \hat{B}^{\gamma_1\ldots\gamma_{p+1}}_{\alpha_1\ldots\alpha_p}$$

for some $\tilde{B}' \in \Omega^{p-1}(T^pM)$ and $\hat{B} \in \Omega^p(T^{p+1}M)$, both totally antisymmetric in their contravariant entries, and $\hat{B}$ additionally such that $\hat{B}^\sharp$ is totally antisymmetric in its contravariant entries. Note that then

$$B^\sharp(\nabla\mathrm{d}A)_{\alpha_2\ldots\alpha_p} = \left(g^{\beta[\gamma_1}(\tilde{B}')^{\gamma_2\ldots\gamma_{p+1}]}_{\alpha_2\ldots\alpha_p} + (\hat{B}^\sharp)^{\beta\gamma_1\ldots\gamma_{p+1}}_{\alpha_2\ldots\alpha_p}\right)\nabla_\beta\mathrm{d}A_{\gamma_1\ldots\gamma_{p+1}} = -\tilde{B}'(\mathrm{d}^*\mathrm{d}A)_{\alpha_2\ldots\alpha_p}$$

Contracting the first indices of $\tilde{B}$ yields

$$\begin{aligned}(p+1)(\tilde{B}-\hat{B})^{\beta\gamma_2\ldots\gamma_{p+1}}_{\beta\alpha_2\ldots\alpha_p} &= \delta^\beta_\beta(\tilde{B}')^{\gamma_2\ldots\gamma_{p+1}}_{\alpha_2\ldots\alpha_p} + p(-1)^p\delta^{[\gamma_2}_\beta(\tilde{B}')^{\gamma_3\ldots\gamma_{p+1}]\beta}_{\alpha_2\ldots\alpha_p} \\ &= (d-p)(\tilde{B}')^{\gamma_2\ldots\gamma_{p+1}}_{\alpha_2\ldots\alpha_p}\end{aligned}$$

Due to the antisymmetry requirements for $\tilde{B}$, we can compare this to

$$\begin{aligned}(p+1)(\tilde{B}-\hat{B})^{\beta\gamma_2\ldots\gamma_{p+1}}_{\alpha_2\ldots\alpha_p\beta} &= \delta^\beta_{\alpha_2}(\tilde{B}')^{\gamma_2\ldots\gamma_{p+1}}_{\alpha_3\ldots\alpha_p\beta} + p(-1)^p\delta^{[\gamma_2}_{\alpha_2}(\tilde{B}')^{\gamma_3\ldots\gamma_{p+1}]\beta}_{\alpha_2\ldots\alpha_p\beta} \\ &= (-1)^p(\tilde{B}')^{\gamma_2\ldots\gamma_{p+1}}_{\alpha_2\ldots\alpha_p} + p(-1)^p\delta^{[\gamma_2}_{\alpha_2}(\tilde{B}')^{\gamma_3\ldots\gamma_{p+1}]\beta}_{\alpha_2\ldots\alpha_p\beta}\end{aligned}$$

From $0 = (\tilde{B}-\hat{B})^{\beta\gamma_2\ldots\gamma_{p+1}}_{\beta\alpha_2\ldots\alpha_p} + (-1)^p(\tilde{B}-\hat{B})^{\beta\gamma_2\ldots\gamma_{p+1}}_{\alpha_2\ldots\alpha_p\beta}$, we obtain

$$0 = (d-p+1)(\tilde{B}')^{\gamma_2\ldots\gamma_{p+1}}_{\alpha_2\ldots\alpha_p} + p\delta^{[\gamma_2}_{\alpha_2}(\tilde{B}')^{\gamma_3\ldots\gamma_{p+1}]\beta}_{\alpha_2\ldots\alpha_p\beta}$$

We can repeatedly apply similar arguments to obtain

$$\tilde{B}^{\gamma_1\ldots\gamma_{p+1}}_{\alpha_1\ldots\alpha_p} = \hat{B}^{\gamma_1\ldots\gamma_{p+1}}_{\alpha_1\ldots\alpha_p} + \delta^{[\gamma_1}_{\alpha_1}\ldots\delta^{\gamma_p}_{\alpha_p}\hat{\beta}^{\gamma_{p+1}]}$$

for some $\hat{\beta} \in \Gamma(TM)$. Inserting this back into (A.11) yields

$$\begin{aligned}\mathrm{d}^*E &= -(\operatorname{div}\tilde{B})(\mathrm{d}A) + \tilde{B}'(\mathrm{d}^*\mathrm{d}A) \\ &= -(\operatorname{div}\tilde{B})(\mathrm{d}A) - \tilde{B}'(\tilde{B}(\mathrm{d}A) - E) + \tilde{B}'(\mathrm{d}^*\mathrm{d}A + \tilde{B}(\mathrm{d}A) - E)\end{aligned} \tag{A.12}$$

This is a violation of (iii) if the dependence of the first two terms on $\mathrm{d}A$ is non-trivial. Therefore, we have to demand

$$-\operatorname{div}\tilde{B}^{\gamma_1\cdots\gamma_{p+1}}_{\alpha_2\ldots\alpha_p} = (\tilde{B}')^{\delta_1\ldots\delta_p}_{\alpha_2\ldots\alpha_p}\tilde{B}^{\gamma_1\cdots\gamma_{p+1}}_{\delta_1\ldots\delta_p} \tag{A.13}$$

We compute

$$\begin{aligned}\operatorname{div}\tilde{B}^{\gamma_1\cdots\gamma_{p+1}}_{\alpha_2\ldots\alpha_p} &= \operatorname{div}\hat{B}^{\gamma_1\cdots\gamma_{p+1}}_{\alpha_2\ldots\alpha_p} + \nabla^\mu\delta^{[\gamma_1}_\mu\delta^{\gamma_2}_{\alpha_2}\ldots\delta^{\gamma_p}_{\alpha_p}\hat{\beta}^{\gamma_{p+1}]}\\ &= \operatorname{div}\hat{B}^{\gamma_1\cdots\gamma_{p+1}}_{\alpha_2\ldots\alpha_p} - (-1)^p(\nabla^{[\gamma_1}\hat{\beta})^{\gamma_2}\delta^{\gamma_3}_{\alpha_2}\ldots\delta^{\gamma_{p+1}]}_{\alpha_p}\end{aligned} \tag{A.14}$$

and also

$$\begin{aligned}\tilde{B}^{\delta_1\ldots\delta_p}_{\alpha_2\ldots\alpha_p}B^{\gamma_1\cdots\gamma_{p+1}}_{\delta_1\ldots\delta_p} &= \delta^{[\delta_1}_{\alpha_1}\ldots\delta^{\delta_{p-1}}_{\alpha_p}\hat{\beta}^{\delta_p]}\left(\hat{B}^{\gamma_1\cdots\gamma_{p+1}}_{\delta_1\ldots\delta_p} + \delta^{[\gamma_1}_{\delta_1}\ldots\delta^{\gamma_p}_{\delta_p}\hat{\beta}^{\gamma_{p+1}]}\right)\\ &= -(-1)^p\hat{B}(\hat{\beta})^{\gamma_1\cdots\gamma_{p+1}}_{\alpha_2\ldots\alpha_p}\end{aligned} \tag{A.15}$$

We imagine pulling all indices downstairs on the individual terms in (A.14) and (A.15). Notice that $\operatorname{div}\hat{B}_{\alpha_2\ldots\alpha_p\gamma_1\ldots\gamma_{p+1}}$ and $\hat{B}(\hat{\beta})_{\alpha_2\ldots\alpha_p\gamma_1\ldots\gamma_{p+1}}$ are totally antisymmetric, while the complete antisymmetrisation $\mathrm{d}\hat{\beta}_{[\gamma_1\gamma_2}g_{\gamma_3\alpha_2}\ldots g_{\gamma_{p+1}\alpha_p]}$ vanishes. Thus, to satisfy (A.13), we have to have

$$\mathrm{d}\hat{\beta}^\flat = 0, \qquad \operatorname{div}\hat{B} = (-1)^{p-1}\hat{B}(\hat{\beta}).$$

Going now back to (A.12), we obtain finally that (iii) is satisfied if and only if

$$\mathrm{d}^*E = \tilde{B}'(E) = (-1)^{p-1}E(\hat{\beta}).$$

Defining

$$\begin{aligned}\beta &:= (-1)^p\hat{\beta}^\flat \in \Omega^1_{\mathrm{cl}}(M),\\ B_{\beta_1\ldots\beta_{p+1}\alpha_1\ldots\alpha_p} &:= g_{\beta_1\gamma_1}\ldots g_{\beta_{p+1}\gamma_{p+1}}\hat{B}^{\gamma_1\cdots\gamma_{p+1}}_{\alpha_1\ldots\alpha_p} \in \Omega^{2p+1}(M),\end{aligned}$$

we obtain the result. □

Theorem A.11. *Let (M,g) be a globally hyperbolic sapcetime with spacelike Cauchy hypersurface $i\colon S \hookrightarrow M$. Let $\beta \in \Omega^1_{\mathrm{cl}}(M)$ closed, $B \in \Omega^{2p+1}(M)$ and $E \in \Omega^p(M)$ such that*

$$\mathrm{d}^*E = -\langle\beta, E\rangle, \qquad \mathrm{d}^*B = -\langle\beta, B\rangle.$$

Let $A_0 \in \Omega^p(M)$ be smooth p-form; the initial data. Consider the following system of PDEs:

$$\begin{aligned}\mathrm{d}^*\mathrm{d}A + \langle B, \mathrm{d}A\rangle + \langle\beta, \mathrm{d}A\rangle &= E,\\ i^*\mathrm{d}A &= i^*\mathrm{d}A_0.\end{aligned} \tag{A.16}$$

Herein, N is the future directed normal to S.

If and only if A_0 satisfies on S the constraint equation

$$i_N\left[\mathrm{d}^*\mathrm{d}A_0 + \langle B, \mathrm{d}A_0\rangle + \langle\beta, \mathrm{d}A_0\rangle - E\right] = 0$$

does there exist a smooth p-form $A \in \Omega^p(M)$ solving the initial value problem to equation (A.16). In this case, solutions are unique up to addition of a closed p-form, and there exist co-closed solutions. The latter are uniquely determined by their initial value i^*A on S.

Proof. We start by proving the existence part of this Theorem. Define A to be the solution to the following wave equation subject to yet to be specified initial conditions, as provided by Theorem A.6, to

$$-\Box_g A + R(A) + \langle B, \mathrm{d}A\rangle + \langle \beta, \mathrm{d}A\rangle = E. \tag{A.17}$$

Herein, we defined $R \in \Omega^p(T^pM)$ such that its contraction with A gives all curvature terms in the Weitzenböck formula, i.e. with $\Box_{\mathrm{Hd}} = -\mathrm{d}^*\mathrm{d} - \mathrm{d}\mathrm{d}^*$

$$(\Box_g - \Box_{\mathrm{Hd}})A \equiv R(A).$$

Note that taking the co-differential of equation (A.17) yields by Lemma A.10

$$\begin{aligned}
0 &= \mathrm{d}^*\mathrm{d}\mathrm{d}^*A + \mathrm{d}^*\left[\langle B, \mathrm{d}A\rangle + \langle \beta, \mathrm{d}A\rangle - E\right] \\
&= -\Box_{\mathrm{Hd}}\mathrm{d}^*A + \langle \beta, \mathrm{d}^*\mathrm{d}A + \langle B, \mathrm{d}A\rangle + \langle \beta, \mathrm{d}A\rangle - E\rangle \\
&\overset{(A.17)}{=} -\Box_{\mathrm{Hd}}\mathrm{d}^*A - \langle \beta, \mathrm{d}\mathrm{d}^*A\rangle \\
&= -\Box_g \mathrm{d}^*A + R(\mathrm{d}^*A) - \langle \beta, \mathrm{d}\mathrm{d}^*A\rangle\,.
\end{aligned}$$

We see that d^*A satisfies a tensor wave equation without source terms. If it were to satisfy initial conditions $\mathrm{d}^*A = 0$ and $\nabla\mathrm{d}^*A = 0$, we could conclude $\mathrm{d}^*A = 0$ everywhere.

To enforce $\mathrm{d}^*A = 0$ initially, we prescribe as initial conditions to (A.17) that $A = A_0$, $\nabla_N A = \nabla_N A_0 - N^\flat \wedge \mathrm{d}^*A_0$. This yields indeed that on S

$$\begin{aligned}
\mathrm{d}^*A_{\alpha_2\ldots\alpha_p} &= -\nabla^\mu A_{\mu\alpha_2\ldots\alpha_p} = \nabla_0 A_{0\alpha_2\ldots\alpha_p} - \nabla^m A_{m\alpha_2\ldots\alpha_p} \\
&= -(\nabla_N A)_{0\alpha_2\ldots\alpha_p} - \nabla^m A_{m\alpha_2\ldots\alpha_p} \\
&= -(\nabla_N A_0 - N^\flat \wedge \mathrm{d}^*A_0)_{0\alpha_2\ldots\alpha_p} - \nabla^m (A_0)_{m\alpha_2\ldots\alpha_p} \\
&= -(A_0)_{0\alpha_2\ldots\alpha_p} - \mathrm{d}^*(A_0)_{\alpha_2\ldots\alpha_p} - \nabla^m (A_0)_{m\alpha_2\ldots\alpha_p} \\
&= 0
\end{aligned}$$

To see that also $\nabla\mathrm{d}^*A = 0$ initially, note that A satisfies on S the normal component of the evolution equation in (A.16) - as this is nothing other than the constraint equation for $\mathrm{d}A$. Taking the normal component of the difference between (A.16) and (A.17), we obtain that on S

$$0 = i_N \mathrm{d}\mathrm{d}^*A \overset{\mathrm{L}\,B.2}{=} (\nabla_N \mathrm{d}^*A)^\parallel - \mathrm{d}^S(i_N \mathrm{d}^*A)^\parallel + k \cdot (\mathrm{d}^*A)^\parallel = (\nabla_N \mathrm{d}^*A)^\parallel,$$

where we used in the last equality that $\mathrm{d}^*A = 0$ on S. Employing this anew, we get

$$0 = \mathrm{d}^*\mathrm{d}^*A_{\alpha_3\ldots\alpha_p} = -\nabla^\mu \mathrm{d}^*A_{\mu\alpha_3\ldots\alpha_p} = i_N \nabla_N \mathrm{d}^*A$$

and thus, indeed $\nabla\mathrm{d}^*A = 0$ on S.

Finally, we obtain that A solves

$$\mathrm{d}^*\mathrm{d}A = -\Box_g A + R(A)$$

and therefore also our PDE (A.16). By Lemma B.2, the exterior differential does not depend on the value of $i_N \nabla_N A$ on S. Aside from this normal component of the normal derivative, A agrees initially with A_0. Therefore, $\mathrm{d}A = \mathrm{d}A_0$.

The uniqueness statements follow from Lemma A.9 because, given any two solutions A, $\hat{A}$, their difference satisfies an equation with vanishing source and initial data. And, if both solutions are co-closed, then so is their difference. □

B Decomposition of the (Co-)Differential of Forms over Hypersurfaces

This chapter reproduces [22, appendix B].

We collect here a few trivial results on the decomposition of forms, their differential, and their co-differential into a tangent and normal part over hypersurfaces. We work in abstract index notation to have a convenient and unambiguous way of denoting contractions. We remind ourselves that Greek indices refer to arbitrary components, while Latin indices refer to components of a parallel projection of the tensor onto a spacelike hypersurface, usually denoted by Σ.

Lemma B.1. *Let $A_{\mu_1\dots\mu_k}$ be a k-form on a Lorentzian manifold (M, g) with spacelike hypersurface (Σ, g_Σ). Denote by $N = \partial_0$ a (local) unit normal on Σ, and by ∇ and D the LC connections on M and Σ respectively. Then, on Σ*

$$\nabla_m A_{0m_2\dots m_k} = D_m A^\perp_{m_2\dots m_k} - k^l_m A_{lm_2\dots m_k},$$

where $A^\perp = A(N) \in \Omega^{k-1}(\Sigma)$.

Proof.

$$[(\nabla A)(N)]^\| = [\nabla(A(N)) - A(\nabla N)]^\| = D(A^\perp) - A(k).$$

□

Lemma B.2. *Let A be a p-form on a Lorentzian manifold (M, g) with spacelike hypersurface (Σ, g_Σ). Let N be a (local) unit normal on Σ. Denote by $i\colon \Sigma \hookrightarrow M$ the inclusion map, and by d and d^Σ the exterior derivative on M and Σ respectively. Furthermore, denote*

$$A = A_0 = A_0^\| - N^\flat \wedge A_0^\perp, \qquad \nabla_N A = A_1 = A_1^\| - N^\flat \wedge A_1^\perp$$

Then, on Σ

$$\begin{aligned} \mathrm{d}A &= \mathrm{d}^\Sigma A_0^\| - N^\flat \wedge \left[A_1^\| - \mathrm{d}^\Sigma A_0^\perp + k \cdot A_0^\|\right], \\ \mathrm{d}^* A &= (\mathrm{d}^\Sigma)^* A_0^\| + A_1^\perp + A_0^\perp \operatorname{tr} k + k \cdot A_0^\perp - N^\flat \wedge [-(\mathrm{d}^\Sigma)^* A_0^\perp]. \end{aligned}$$

Proof. Denote by ∇ and D the LC connections on M and Σ, respectively. One,

$$\begin{aligned} \mathrm{d}A_{m_1\dots m_{p+1}} &= (p+1)\nabla_{[m_1} A_{m_2\dots m_{p+1}]} = (p+1)\partial_{[m_1} A_{m_2\dots m_{p+1}]} = D_{[m_1} A_{m_2\dots m_{p+1}]} \\ &= \mathrm{d}^\Sigma A_{m_1\dots m_{p+1}}. \end{aligned}$$

Two,

$$\begin{aligned} \mathrm{d}A_{0m_1\dots m_p} &= (p+1)\nabla_{[0} A_{m_1\dots m_p]} \\ &= \nabla_0 A_{m_1\dots m_p} - p\nabla_{[m_1} A_{|0|m_2\dots m_p]} \\ &= \nabla_N A_{m_1\dots m_p} - \mathrm{d}^\Sigma (i_N A)^\|_{m_1\dots m_p} - (-1)^p p k^l_{[m_1} A_{m_2\dots m_p]l} \\ &= (A_1^\|)_{m_1\dots m_p} - \mathrm{d}^\Sigma (A_0)^\perp_{m_1\dots m_p} - k \cdot (A_0)_{m_1\dots m_p}. \end{aligned}$$

Three,

$$\begin{aligned}
\mathrm{d}^* A_{m_2 \ldots m_p} &= -\nabla^\lambda A_{\lambda m_2 \ldots m_p} \\
&= \nabla_0 A_{0 m_2 \ldots m_p} - \nabla^l A_{l m_2 \ldots m_p} \\
&= (A_1^\perp)_{m_2 \ldots m_p} + (\mathrm{d}^\Sigma)^* (A_0^\parallel)_{m_2 \ldots m_p} + k_l^l A_{0 m_2 \ldots m_p} \\
&\quad + k_{m_2}^l A_{l 0 m_3 \ldots m_p} + \ldots + k_{m_p}^l A_{l m_2 \ldots m_p 0} \\
&= (A_1^\perp)_{m_2 \ldots m_p} + (\mathrm{d}^\Sigma)^* (A_0^\parallel)_{m_2 \ldots m_p} - \operatorname{tr} k \, (A_0^\perp)_{m_2 \ldots m_p} \\
&\quad + (-1)^{p-1} k^l_{[m_2} A^\perp_{m_3 \ldots m_p] l} \\
&= (A_1^\perp)_{m_2 \ldots m_p} + (\mathrm{d}^\Sigma)^* (A_0^\parallel)_{m_2 \ldots m_p} + \operatorname{tr} k \, (A_0^\perp)_{m_2 \ldots m_p} \\
&\quad + (k \cdot A_0^\perp)_{m_2 \ldots m_p}.
\end{aligned}$$

Four,

$$\begin{aligned}
\mathrm{d}^* A_{0 m_3 \ldots m_p} &= -\nabla^\lambda A_{\lambda 0 m_3 \ldots m_p} \\
&= \nabla^l A_{0 l m_3 \ldots m_p} \\
&= D^l A_{0 l m_3 \ldots m_p} + k^{ln} A_{l n m_3 \ldots m_p} \\
&= -(\mathrm{d}^\Sigma)^* (A_0^\perp)_{m_3 \ldots m_p}.
\end{aligned}$$

□

C The Divergence of Generalised Tensors

In this section, which reproduces [17, appendix B], we investigate the divergence of generalised tensors. The main result is that the divergence of a mixed-type tensor with respect to a generalised Levi-Civita connection D is invariant in the sense that it depends only on the divergence operator of D and the generalised metric. This is relevant for the proper interpretation of the generalised Codazzi equations in the context of the generalised momentum constraint, see Corollary 4.38. Indeed the constraints should only involve the fields of the considered gravitational theory, which involve the generalised metric and the divergence operator but no further components of the generalised connection.

Definition C.1. Let E be an exact Courant algebroid, let $D \in \mathcal{D}(E)$ be a generalised connection on E. Then, we define the following divergence operator on the space of generalised tensors:

$$\operatorname{div}_D \colon \Gamma(E_s^{r+1}) \longrightarrow \Gamma(E_s^r); \qquad \operatorname{div}_D(T) \coloneqq \operatorname{tr}(DT) \tag{C.1}$$

or, in index notation,

$$\operatorname{div}_D(T)^{B_1\ldots B_r}_{C_1\ldots C_s} = D_A T^{AB_1\ldots B_r}_{C_1\ldots C_s}$$

Let now $\mathcal{G}$ be a generalised metric on E. Then, we can define the canonical divergence operators

$$\begin{aligned} &\operatorname{div}^{\mathcal{G},\pm} \colon \Gamma(E_\mp \otimes (E_\pm)^r_s) \longrightarrow \Gamma((E_\pm)^r_s), \\ &\operatorname{div}^{\mathcal{G},\pm}(T) = \sigma_\pm \circ \operatorname{tr}(\nabla^\pm[\pi T \sigma_\pm]) \circ \pi, \end{aligned} \tag{C.2}$$

where $\nabla^\pm$ are the Bismut connections from (2.13). In index notation, this reads

$$\operatorname{div}^{\mathcal{G},\pm}(T)^{B_1\ldots B_r}_{C_1\ldots C_s} = (\sigma_\pm)^{B_1}_{b_1} \ldots (\sigma_\pm)^{B_r}_{b_r} \nabla^\pm_a [\pi T \sigma_\pm]^{ab_1\ldots b_r}_{c_1\ldots c_s} (\pi\pi_\pm)^{c_1}_{C_1} \ldots (\pi\pi_\pm)^{c_s}_{C_s}$$

Note that this formula defines tensors over E as opposed to $E_\pm$, hence the appearance of the projection $\pi_\pm$ on the right hand side.

The next Lemma investigates the meaning and compatibility of these definitions. Crucially, it asserts that the divergence operator div_D is not uniquely determined by the requirement that D be generalised LC, i.e. $D \in \mathcal{D}^0(\mathcal{G}, \operatorname{div})$. However, restricted to the right subspace of generalised tensors, this property is achieved.

Lemma C.2. *Let E be an exact Courant algebroid equipped with a generalised metric $\mathcal{G} \cong (g, F)$ and a compatible divergence operator $\operatorname{div} = \operatorname{div}^{\mathcal{G}} - \langle e, \cdot\rangle$. Then, the divergence operator div_D from (C.1) depends non-trivially on the choice of divergence compatible generalised connection $D \in \mathcal{D}(\operatorname{div})$.*

However, if D is assumed to be metric compatible and torsion-free, then the action of div_D on the "mixed-type" sub-spaces $\Gamma(E_\mp \otimes (E^*_\pm)^s)$ is independent of the choice of generalised connection. It then holds

$$\operatorname{div}(T) = \operatorname{div}^{\mathcal{G},\pm}(T) - \langle e, T\rangle\,, \qquad T \in \Gamma(E_\mp \otimes (E^*_\pm)^s)$$

where $\operatorname{div}^{\mathcal{G},\pm}$ is as in (C.2).

Remark C.3. Note that an analogous result applies to generalised tensors of arbitrary contravariant rank. This is due to compatibility of generalised connections with the inner product $\langle\cdot,\cdot\rangle$, which provides the isomorphism $E \cong E^*$ used to identify these spaces.

Proof. We calculate, taking an orthonormal frame $\{e_A\}$ of E and setting $\epsilon_A \coloneqq \langle e_A, e_A \rangle$,

$$\begin{aligned}
&(\operatorname{div}_D\ T)(e_{A_1}, ..., e_{A_s}) \\
&= \sum_B \epsilon_B \left\langle e_B, (D_{e_B} T)(e_{A_1}, ..., e_{A_s}) \right\rangle \\
&= \sum_B \epsilon_B \left\{ \langle e_B, D_{e_B}[T(e_{A_1}, ..., e_{A_s})] \rangle \right. \\
&\qquad \left. - \langle e_B, T(D_{e_B} e_{A_1}, ..., e_{A_s}) + ... + T(e_{A_1}, ..., D_{e_B} e_{A_s}) \rangle \right\} \\
&= \operatorname{div}(T(e_{A_1}, ..., e_{A_s})) \\
&\qquad - \sum_B \epsilon_B \left\langle e_B, T(D_{e_B} e_{A_1}, ..., e_{A_s}) + ... + T(e_{A_1}, ..., D_{e_B} e_{A_s}) \right\rangle \\
&= \operatorname{div}^{\mathcal{G}}(T(e_{A_1}, ..., e_{A_s})) - \langle e, T(e_{A_1}, ..., e_{A_s}) \rangle \\
&\qquad - \sum_B \epsilon_B \left\langle e_B, T(D_{e_B} e_{A_1}, ..., e_{A_s}) + ... + T(e_{A_1}, ..., D_{e_B} e_{A_s}) \right\rangle
\end{aligned}$$

Note that, without further assumptions, this expression is in general not independent of the choice of divergence compatible generalised connection D. However, assuming $D \in \mathcal{D}^0(\mathcal{G}, \operatorname{div})$ and $T \in \Gamma(E_+ \otimes (E_-^*)^s)$ (the case of flipped signs is analogous), the expression is uniquely defined, since the mixed-type operators $D^{\pm}_{\mp}$ are, cf. Corollary 2.36. Using $D_{\sigma_+ X} \sigma_- Y = \sigma_-(\nabla^-_X Y)$, cf. (2.13), we see that

$$\begin{aligned}
&(\operatorname{div}_D\ T)(e^-_{a_1}, ..., e^-_{a_s}) \\
&= \operatorname{div}^{\mathcal{G}}(T(e^-_{a_1}, ..., e^-_{a_s})) - \langle e, T(e^-_{a_1}, ..., e^-_{a_s}) \rangle \\
&\qquad - \sum_b \left\langle e^+_b, T(D_{e^+_b} e^-_{a_1}, ..., e^-_{a_s}) + ... + T(e^-_{a_1}, ..., D_{e^+_b} e^-_{a_s}) \right\rangle \\
&= \sum_b \Big\{ g\left(e_b, \nabla_{e_b}[\pi T \sigma_-(e_{a_1}, ..., e_{a_s})]\right) \\
&\qquad - g\left(e_b, \pi T \sigma_-(\nabla^-_{e_b} e_{a_1}, ..., e_{a_s}) + ... + \pi T \sigma_-(e_{a_1}, ..., \nabla^-_{e_b} e_{a_s})\right) \Big\} \\
&\qquad - \langle e, T(e_{A_1}, ..., e_{A_s}) \rangle \\
&= \sum_b \Big\{ g\left(e_b, \nabla_{e_b} \pi T \sigma_-)(e_{a_1}, ..., e_{a_s})\right) \\
&\qquad + g\left(e_b, (\pi T \sigma_-)((\nabla - \nabla^-)_{e_b} e_{a_1}, ..., e_{a_s}) + ... + T(e_{a_1}, ..., (\nabla - \nabla^-)_{e_b} e_{a_s})\right) \Big\} \\
&\qquad - \langle e, T(e^-_{a_1}, ..., e^-_{a_s}) \rangle \\
&= \sum_b \Big\{ g\left(e_b, \nabla_{e_b} \pi T \sigma_-)(e_{a_1}, ..., e_{a_s})\right) \\
&\qquad + g\left(e_b, (\pi T \sigma_-)(H(e_b, e_{a_1})^\sharp, ..., e_{a_s}) + ... + T(e_{a_1}, ..., H(e_b, e_{a_s})^\sharp)\right) \Big\} \\
&\qquad - \langle e, T(e^-_{a_1}, ..., e^-_{a_s}) \rangle \\
&= \sum_b g\left(\nabla^-_{e_b} \pi T \sigma_-)(e_{a_1}, ..., e_{a_s}), e_b\right) - \langle e, T(e^-_{a_1}, ..., e^-_{a_s}) \rangle \\
&= (\operatorname{div}^{\mathcal{G},-}\ T)(e^-_{a_1}, ..., e^-_{a_s}) - \langle e, T \rangle (e^-_{a_1}, ..., e^-_{a_s})
\end{aligned}$$

This proves the claim. □

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Publication List

This thesis is based in part on the following preprints (co-)authored by the author of this thesis:

1. *Exterior Generalised Geometry.*
   Vicente Cortés and Oskar Schiller.
   www.doi.org/10.48550/arXiv.2507.12362

2. *The Canonical Generalised Levi-Civita Connection and its Curvature.*
   Vicente Cortés, Matas Mackevicius, Thomas Mohaupt, and Oskar Schiller.
   www.doi.org/10.48550/arXiv.2507.17604

3. *The Initial Value Problem for the Generalised Einstein Equations.*
   Oskar Schiller.
   www.doi.org/10.48550/arXiv.2601.05041

Declaration of Personal Contribution

Chapters 1, 2, and 5 are introductory material and do not contain original results.

My contribution to the preprint [16], which is reproduced in Sections 3.1-3.3, is comparable to that of my co-authors.

My contribution to the preprint [17], which is reproduced in Section 3.4 and Chapter 4, is comparable to that of my co-author.

Chapters 5 and 6 are based on the independently authored preprint [22].

Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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Hamburg, 2026

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