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Chapter 1

Synopsis

1.1 Motivation

Structural credit risk models play a fundamental role in modern corporate finance theory. They support our understanding of how firms make financing decisions by providing a structured framework for identifying and quantifying financial risks. Beyond their theoretical significance, these models generate empirically testable predictions and form the basis for practical applications, such as the design of debt contracts and the assessment of creditworthiness in financial institutions.

The model developed by Leland (1994) represents a key methodological starting point in this field. As one of the first continuous-time models with endogenous default, it has profoundly shaped the academic discourse. Its framework allows for the determination of default thresholds and probabilities based on economic fundamentals and served as the foundation for numerous extensions (see e.g. Leland and Toft (1996), Goldstein et al. (2001) and Duffie and Lando (2001)).

This dissertation contributes to the literature by explicitly extending the Leland framework to incorporate realistic real-world frictions that have so far received little attention in theory. In three independent papers, I investigate the effect of behavioural factors on the firm's credit risk, the effect of uncertainty on optimal maturity choice, as well as the effect of information asymmetry and ambiguity on credit ratings. Each of these extensions analyses modern challenges whose impact on corporate behaviour and credit risk assessment has yet to be comprehensively analysed within a theoretical context.

The central research question of this dissertation is therefore: How do realistic and previously under-examined behavioural and informational frictions affect financing decisions, financial structures, and credit risk in firms? The aim is to extend the baseline Leland model with three problem settings that are of high practical relevance, and to analyse the impact of real-world frictions.

The contribution of this work lies in sharpening and differentiating classical credit risk models: By integrating behavioural and informational frictions, new empirically testable hypotheses are generated. The results provide novel perspectives for the academic literature and yield practical implications for contract design, risk management, and for regulatory issues such as credit rating processes.

1.2 Present Bias, Risk Management and Capital Structure

This paper advances the classical Leland (1994) credit risk framework by introducing present-biased (time-inconsistent) preferences into the dynamic risk and capital structure decisions of entrepreneurial firms. While traditional models assume rational agents, empirical research indicates that the assumption of time-consistent decision makers is questionable. These studies indicate that decision makers may be susceptible to present bias - overvaluing immediate gains at the expense of long-term outcomes. This paper addresses how this behavioural friction shapes optimal risk exposure, leverage, and default decisions, and explores the distinctions between naive and sophisticated entrepreneurs with respect to their awareness and handling of future self-control problems.

The model considers an entrepreneurial firm in which the entrepreneur can actively manage the volatility of cash flows with preferences described by a quasi-hyperbolic discount function. The entrepreneur may be naive, believing in future self-control, or sophisticated, anticipating future present bias. The analysis derives optimal strategies for risk-taking and default for both types, and investigates their interaction with unbiased, time-consistent debt-holders. Notably, the research reveals that a higher degree of present bias leads to higher optimal default thresholds but lower willingness to raise cash flow volatility in distress. These findings have direct implications for credit risk and contract outcomes since a higher degree of present bias can paradoxically increase the value of debt.

By incorporating both behavioural biases and dynamic volatility choice into the credit risk modelling framework, the paper advances our understanding of entrepreneurial financial decision-making under time inconsistency. The findings suggest that creditors may, under certain conditions, benefit from entrepreneurial present bias, while also highlighting new sources of credit risk arising from behavioural heterogeneity. These insights motivate future research on information asymmetry regarding entrepreneurial preferences and underscore the value of integrating behavioural finance into structural credit risk models.

1.3 Maturity Choice, Debt Rollover, and Asymmetric Information

This paper advances the structural credit risk literature by examining optimal debt maturity choices in the presence of information frictions and feedback effects in competitive debt markets. Specifically, we extend the Leland (2019) framework by introducing asymmetric information between firm insiders and outside creditors. After debt issuance, firm insiders learn the true asset value while creditors gradually update their beliefs based on the firm's observed decisions over time. Firms can strategically delay default to signal higher credit quality, which, in turn, leads the market to offer more favourable refinancing terms upon debt rollover.

The paper investigates how uncertainty about firm value influences the optimal choice of debt maturity. The analysis identifies a key trade-off: shorter maturities entail higher refinancing costs during distress and more frequent exposure to market reassessment, but allow firms to benefit more from positive market learning if they survive periods of apparent distress. Under asymmetric information, the analysis shows that firms tend to be overvalued immediately before default. This overvaluation lowers refinancing costs in times of distress - an effect particularly relevant for firms with shorter maturities that must roll over their debt more often - which in turn further delays default. Consequently, the optimal default threshold decreases with increased information asymmetry, especially for firms with shorter debt maturities.

Consistent with empirical evidence, the model predicts that greater uncertainty reduces the optimal debt maturity chosen by firms. The results thus provide a theoretical explanation for the tendency of companies facing greater informational frictions to favour shorter debt maturities. By integrating information dynamics and signalling into the Leland framework, this paper offers new insights into the interplay between capital structure, market learning, and endogenous credit risk, thereby helping to explain observed debt maturity patterns in practice.

1.4 Credit Rating Under Ambiguity

This paper advances the structural credit risk literature by analysing a dynamic interaction between a firm and a credit rating agency under ambiguity, with a particular focus on feedback effects and heterogeneous rating analyst beliefs. Building on the framework by Hilpert et al. (2024) with endogenous default and information dynamics, the model introduces Knightian uncertainty by allowing the rating agency to operate under multiple priors. These priors reflect the diverse and evolving opinions of analysts as the agency assesses a firm's creditworthiness based on imperfectly observed cash flows. The rating agency is assumed to be ambiguity averse in the sense of minimising the maximal expected reputational damage across all analyst priors.

A central feature of the model is a feedback loop: a firm's default strategy is shaped by the impact of ratings on financing costs, while the agency's assessments are dynamically updated as the firm's performance data accumulates. The rating agency aggregates analyst beliefs and adjusts the weights on each analyst's prior in response to new data, with beliefs showing higher variance exerting greater influence. The effect of ambiguity on ratings crucially depends on the direction of analyst opinions. For teams of jointly pessimistic analysts, ambiguity induces the firm to delay default relative to a benchmark without ambiguity, as the firm benefits from the agency's learning process and lower capital costs. Conversely, if analyst teams are jointly optimistic, firms may accelerate default under ambiguity compared to the benchmark without ambiguity.

These results yield several novel insights. First, ambiguity in the rating process does not simply induce conservatism but dynamically moderates rating outcomes and default behaviour through a feedback loop. Second, phases of high market-wide ambiguity - captured, for example, by elevated analyst forecast dispersion - can result in either higher or lower credit spreads, depending on the prevailing analyst sentiment. Overall, this paper enriches the credit risk literature by linking group decision-making under ambiguity to concrete predictions for firm behaviour and credit market outcomes, offering a new theoretical perspective on credit rating dynamics and credit spreads during periods of uncertainty.

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Chapter 2

Present Bias, Risk Management and Capital Structure

Abstract

Standard credit risk models assume entrepreneurs maximise long-term equity value, despite significant evidence that entrepreneurs overvalue short-term profits and undervalue future long-term returns. We extend a classical credit risk framework by entrepreneurial myopia and show that myopic entrepreneurs favour lower volatility, while defaulting at a higher cash flow level. Our analysis distinguishes between sophisticated and naive entrepreneurs and shows that naivety leads to lower cash flow volatility and delayed default. We analyse the interaction between unbiased debt-holders and myopic entrepreneurs. The desire for debt financing increases in the degree of the entrepreneur's bias. Surprisingly, the entrepreneur's myopia can actually increase the value of debt.

2.1 Introduction

Existing standard dynamic credit risk models generally assume time-consistent agents, who discount a firm's payout exponentially. However, several studies on time preferences suggest that the assumption of time-consistency is questionable (e.g. Thaler (1981); Ainslie (1992); Della Vigna and Malmedier (2006); Kaur et al. (2015) and Cheung et al. (2021)). These studies indicate that decision makers are present-biased, overvaluing short-term gains or losses at the expense of long-term profitability. The value of entrepreneurial firms, such as tech and biotech startups or early-stage companies in creative industries like fashion and entertainment - as well as consultancies - often primarily derives from the personal skills and expertise of the entrepreneur. While time-consistency is plausible for larger firms where decisions are made by a large board, entrepreneurial firms, operated by individuals or a single manager, are more likely to act time-inconsistently (Tian (2016)). Entrepreneurs, in particular, are especially vulnerable to the pitfalls of myopia due to the pressure they face in establishing and sustaining their firms. The demand for immediate results can lead entrepreneurs to make hasty decisions, often favouring immediate cash flow improvements over critical strategic long-term planning. Entrepreneurial susceptibility to biases is rooted in factors like high uncertainty, information overload, and time pressure (Zhang and Cueto (2017)), which exacerbate biases and result in less comprehensive decision-making (Simon et al. (2000)). While individual entrepreneurs may make sub-optimal decisions, the overall impact on the economy can be significant, potentially leading to higher rates of business failure and instability in entrepreneurial ecosystems. Consequently, understanding the implications of present bias in entrepreneurial decision-making is essential.

In entrepreneurial firms, the ability to actively manage cash flow volatility holds great significance, since flatter organisational structures facilitate faster decision-making, allowing entrepreneurial firms to rapidly adjust strategies, invest in new projects, or enter new markets. Their business models often entail cash flow patterns that can quickly adapt based on market feedback. With a strong culture of innovation and experimentation, entrepreneurs can actively shape product offerings, pricing, and market strategies, which directly influences their cash flow volatility. Additionally, entrepreneurs are more deeply involved in day-to-day operations and strategic decisions than managers in larger firms, enabling them to manage cash flows more actively. Serving niche markets provides them direct control over pricing and customer relations, which dynamically influences cash flow patterns.

My paper answers the question of how entrepreneurial myopia impacts a firm's optimal risk-taking strategy. It reveals a complex dynamic: a higher degree of entrepreneurial present bias can improve creditworthiness at similar leverage levels, but also drives entrepreneurs to seek higher leverage. This increased leverage, induced by the present bias, increases the probability of default, resulting in higher credit spreads.

Furthermore, my research contributes to the understanding of entrepreneurial behaviour by identifying that entrepreneurs with a present bias prefer to maintain lower levels of volatility. In financially distressed situations, we observe that entrepreneurs generally accept certain immediate costs to increase the volatility of their firm's cash flows. However, the extent to which they are willing to do so diminishes as the degree of their present bias increases. This reluctance to increase cash flow volatility in distressed times is pivotal because raising volatility can abbreviate periods of negative payout by accelerating a recovery or hastening a default. Despite the potential long-term benefit, a present-biased entrepreneur tends to postpone increasing the firm's cash flow volatility due to the occurrence of immediate costs and the deferred nature of potential cash flow recovery. This delay can extend the period of financial uncertainty, reflecting the substantial influence of present bias on entrepreneurial decision-making in times of financial distress.

This study pioneers an approach to analyse the impact of present-biased preferences on both the timing of optimal risk-taking and bankruptcy decisions. Baron (2004) suggests that a cognitive perspective, which includes the use of effective cognitive mechanisms for decision-making - such as counterfactual thinking and strategic foresight - can prove beneficial to researchers who wish to understand why some entrepreneurs perform better than others. Entrepreneurship encompasses individuals with a wide range of experience and varying degrees of awareness in their decision-making processes. Within the framework of our study, we identify two principal types of entrepreneurs, namely naive and sophisticated ones. This distinction, while not exhaustive, provides a useful framework for examining how different types of entrepreneurs handle strategic decisions and risk management. While the naive entrepreneur falsely believes that he can commit his future selves to act in the best interest of its current self, the sophisticated entrepreneur is aware that its future selves will act according to their own best interest. Interestingly, compared to the sophisticated entrepreneur, the naive entrepreneur reduces volatility and defaults at a lower cash flow value.

We consider an entrepreneurial firm that generates a stochastic cash flow. Following Grenadier and Wang (2007), we model the entrepreneur's present-biased preferences with a quasi-hyperbolic discount function. Its risk strategy is characterised by a continuous choice between a low and high cash flow volatility level. The present-biased entrepreneur distinguishes between the present and the future and is attributed decision power in the present. Its belief about its future selves' behaviour is dependent on the entrepreneur being naive or sophisticated. At each instant, the entrepreneur maximises its overall utility by implementing a present firm's risk and default strategy that is the best response to its belief about its future selves' implemented strategy.

In our analysis, we derive and examine the optimal risk and default strategies for naive and sophisticated present-biased entrepreneurs. We find that naive entrepreneurs default and increase the cash flow volatility at a lower level of cash flow compared to sophisticated entrepreneurs. Generally, the optimal default level increases and the willingness to increase risk decreases in the strength of the entrepreneur's present bias. Our findings regarding optimal default levels align with those of Tian (2016), who investigated optimal default decisions of present-biased entrepreneurs under constant volatility.

Additionally, we explore the interaction between present-biased entrepreneurs and unbiased time-consistent debt-holders. Despite the potential for present-biased preferences among entrepreneurs, entities such as banks, which often act as debt-holders, should ideally remain unbiased in their assessment of debt contracts. Our observations indicate that both naive and sophisticated entrepreneurs exhibit an increased inclination towards debt financing and are willing to accept higher credit spreads as the degree of present bias increases.

One of the seminal works by Merton (1974) introduced the concept that a firm's total asset value could be described by a diffusion-type stochastic process. This model laid the groundwork for numerous subsequent credit risk models. Leland (1998) has introduced a dynamic capital structure model that allows firms to choose their risk strategy by adjusting an optimal asset volatility level.

Grenadier and Wang (2007) have extended the real options framework and modelled optimal investment-timing decisions of a decision maker with present-biased preferences. They generally find that such decision maker's time-inconsistency leads to earlier investment. Optimal consumption under present-biased time preferences is considered by Harris and Laibson (2013), specifically focusing on the limiting case of instantaneous gratification.

Laverty (2004) defines myopia as a characteristic of a decision that overvalues short-term rewards and undervalues long-term consequences. This notion of myopia is closely related to managerial myopia, which is widely understood as the tendency of managers to prefer short-term profits to higher future returns that generate long-term shareholder value (Stein (1989); Mizik (2010)). The literature extensively discusses reasons for managerial myopia, including managerial opportunism, such as the aim to improve one's short-term prospects in the labor market (Nagarajan et al. (1995)), or information asymmetries between managers and shareholders (Antia et al. (2010)).

Tian (2016) investigates investment and capital structure decisions using quasi-hyperbolic discounting under constant volatility. In this study, we extend the framework created by Grenadier and Wang (2007) and further adapted by Tian (2016) by incorporating the entrepreneur's optimal dynamic volatility decision. In contrast to their studies, we do not focus on optimal investment timing. Instead, we emphasize the interplay between present-biased preferences, volatility management, and default decisions. Additionally, in contrast to Tian (2016), we analyse optimal debt contracts between present-biased entrepreneurs and unbiased debt-holders. Overall, only few recent credit risk models incorporate time-inconsistent preferences and little qualitative research has been done in this area.

The paper is organised as follows. Section 2.2 introduces the model, defines the stochastic discount function, and discusses the time-consistent benchmark case. Section 2.3 derives the optimal default and risk-strategy for the naive entrepreneur. Section 2.4 derives the optimal default and risk-strategy for the sophisticated entrepreneur. Section 2.5 studies the effect of entrepreneurial myopia on the value of debt and analyses the interaction between present-biased entrepreneurs and unbiased debt-holders. Section 2.6 concludes.

2.2 The Model

Let us consider an entrepreneur with present-biased preferences. The entrepreneur owns a firm and its skills and expertise enable the firm to generate an after-tax cash flow X with dynamics

$$\frac{dX_t}{X_t} = \mu dt + \sigma_{\nu_t} dB_t, \tag{2.1}$$

where μ is the deterministic cash flow drift, σ_{ν_t} its volatility and dB_t is the increment of a Wiener process. We assume that there are two levels of cash flow risk, high-risk and low-risk, specifically, $\nu_t \in \{H, L\}$. The entrepreneur can choose continuously between the low risk level σ_L and the high risk level σ_H with $\sigma_L \leq \sigma_H$. Following Hennessy and Tserlukevich (2008), Chen and Strebulaev (2019) and Gan and Yang (2024), we assume that risk-taking has a value-destroying effect and produces a flow cost at a rate proportional to the excess taking of risk. Specifically, whenever the firm chooses the higher risk level σ_H, it incurs flow costs of $\phi\epsilon X_t$, where $\phi \geq 0$ represents the costs proportional to the excess taking of risk $\epsilon = \sigma_H^2 - \sigma_L^2$. Further, we adopt the assumption by Grenadier and Wang (2007) that the entrepreneurial firm's asset is non-tradable and its payoff cannot be replicated by existing assets. The non-tradability feature is crucial to the model, as it allows us to exclude the possibility of time-inconsistent entrepreneurs selling the firm to time-consistent market participants. The lack of tradability may be a result by the firm's payout depending on the entrepreneur's specialised skills.

Further, the firm has risky debt and pays a constant coupon C until it calls for bankruptcy. Given an effective tax rate $\theta \in [0, 1)$, the entrepreneur's after-tax payout is given by

$$P(X_t) = \left(1 - 1_{\{\nu_t = H\}}\phi\epsilon\right) X_t - (1-\theta)\, C. \tag{2.2}$$

Note that Equation (2.2) presumes that the firm always benefits entirely from tax deductibility. However, negative payouts imply that the entrepreneur effectively pays into the firm, which introduces path dependency in the tax benefit, a complexity we will neglect for the sake of simplicity.

2.2.1 Entrepreneur's time preferences

We assume that at any time, the entrepreneur distinguishes between the present and the future and his preferences are biased towards the present. In this paper, we follow Grenadier and Wang (2007), Harris and Laibson (2013) and Tian (2016) to model the entrepreneur's quasi-hyperbolic discounting. We model the duration of the present as stochastic, since this allows for smoother solutions. Besides standard exponential discounting at a rate ρ with $\rho > \mu$, the entrepreneur additionally discounts the future by a discount factor β with $\beta \in [0, 1]$. Consider the following entrepreneur's present-biased (stochastic) discount function

$$D_n(t,s) = \begin{cases} e^{-\rho(s-t)} & \text{for} \quad s \in [t_n, t_{n+1}) \\ \beta e^{-\rho(s-t)} & \text{for} \quad s \in [t_{n+1}, \infty), \end{cases} \tag{2.3}$$

for $s \geq t$ and $t_n \leq t < t_{n+1}$. Here, t_n is the time of the n-th jump of a Poisson point process N with intensity λ. For all n, it holds that $t_{n+1} - t_n$ is exponentially distributed with intensity λ. Note, that the magnitude of the parameter β, along with the intensity parameter λ, determines the degree of the entrepreneur's time-inconsistency. We interpret the present-biased discount function in the following way: From the entrepreneur's perspective, the future is always (in expectation) λ^{-1} time units away, which results from the memorylessness of the exponential distribution. The future is discounted by the scalar factor β. We use stochastic duration of the present to keep the value functions simple.

As noted by Hennessy and Tserlukevich (2008), when risk-taking incurs costs, it would not take place in the absence of agency conflicts between equity and debt. However, the entrepreneurial firm is levered and in times of financial distress, it should aim to circumvent a period of a long-lasting negative payout. Consequently, in distressed times, where the cash flow is low and the overall payout might even be negative, it can be optimal to invest in a higher volatility to either achieve a rapid recovery or face default. In this context, we propose that the entrepreneur's risk strategy operates using a threshold mechanism: the firm selects the higher cash flow volatility σ_H whenever the cash flow is below the threshold and a lower volatility σ_L whenever it exceeds this level.

Decision-making power is attributed to the entrepreneur's current self and the entrepreneur chooses a risk and default strategy. More precisely, the entrepreneur's current self's strategy is a threshold strategy (X_B, X_S) with $X_B \leq X_S$, where X_B denotes the entrepreneur's bankruptcy threshold and X_S its risk-switch level below which the firm chooses the higher cash flow volatility. Following Grenadier and Wang (2007), the entrepreneur's current self's strategy must be a best response to its belief about its future selves strategy, in our case (X_B^F, X_S^F). Then, the present-biased entrepreneur's equity value is given by the expected discounted future payout

$$\mathbb{E}\left[\int_0^{\tau(X_B)\wedge T} e^{-\rho s} P(X_s)\mathrm{d}s + 1_{\{T<\tau(X_B)\}}\beta e^{-\rho T} F(X_T)\right], \tag{2.4}$$

where $T \sim \exp(\lambda)$ is independent of the firm's cash flow. The dynamics of the cash flow process in Equation (2.4) are given by Equation (2.1) with a low volatility σ_L, whenever the firm's cash flow is above the firm's risk-switch level X_S. On the other side, for cash flow levels below the firm's risk-switch threshold X_S, the cash flow has an increased volatility σ_H. Moreover, the continuation value F is given by the total expected value of future selves discounted cash flows with

$$F(X_t) = \mathbb{E}_{(t,X_t)}\left[\int_t^{\tau(X_B^F)} e^{-\rho(s-t)} P(X_s)\,\mathrm{d}s\right] \tag{2.5}$$

under the future selves default strategy $\tau(X_B^F)$. The future selves increase the firm's volatility whenever the cash flow falls below the future selves risk-switch threshold X_S^F. Therefore, the cash flow dynamics in Equation (2.5) are given by a low volatility σ_L, whenever the cash flow is above the future selves risk-switch level X_S^F, and a high volatility σ_H for cash flow levels below X_S^F.

In the following, we first consider the case of an entrepreneur with time-consistent preferences. This case is relevant both as a benchmark and for the subsequent calculation of the present-biased entrepreneur's optimal strategies.

2.2.2 The time-consistent benchmark case

The entrepreneur in the time-consistent case is not present-biased and his discount function is a special case of (2.3) with $\lambda \to 0$ (or alternatively $\beta = 1$). The resulting deterministic time-consistent discount function

$$D_n(t,s) = e^{-\rho(s-t)} \tag{2.6}$$

is widely used in the literature. We denote the firm's default threshold X_B and its risk-switching level X_S with $X_B \leq X_S$. Let the effective after-tax coupon be denoted $C_\theta = (1-\theta)C$. Under time-consistent preferences, the firm's equity value $V(X; X_B, X_S)$, which we abbreviate by $V(X)$, by standard arguments, must satisfy the following differential equation

$$\frac{1}{2}\sigma_\nu^2 X^2 V'' + \mu X V' - \rho V + (1 - 1_{\{\nu=H\}}\phi\epsilon)X - C_\theta = 0, \tag{2.7}$$

where $\nu = L$ denotes the lower volatility regime for $X \geq X_S$, while $\nu = H$ denotes the higher volatility regime for $X < X_S$. Now, any solution to the differential equation (2.7) is of the following form

$$V(X) = \begin{cases} VL(X) = d_{1L}X^{y_{1L}} + d_{2L}X^{y_{2L}} + \frac{X}{\rho-\mu} - \frac{C_\theta}{\rho} & \text{for} \quad X \geq X_S \\ VH(X) = d_{1H}X^{y_{1H}} + d_{2H}X^{y_{2H}} + \frac{(1-\phi\epsilon)X}{\rho-\mu} - \frac{C_\theta}{\rho} & \text{for} \quad X < X_S. \end{cases} \tag{2.8}$$

with

$$y_{1,j} = \frac{-(\mu - \frac{\sigma_j^2}{2}) + \sqrt{(\mu - \frac{\sigma_j^2}{2})^2 + 2\rho\sigma_j^2}}{\sigma_j^2} > 0 \tag{2.9}$$

and

$$y_{2,j} = \frac{-(\mu - \frac{\sigma_j^2}{2}) - \sqrt{(\mu - \frac{\sigma_j^2}{2})^2 + 2\rho\sigma_j^2}}{\sigma_j^2} < 0, \tag{2.10}$$

$j \in \{L, H\}$. The equity value has to satisfy the value matching condition at both the default and risk-switch level, respectively given by

$$VH(X_B) = 0 \quad \text{and} \quad VL(X_S) = VH(X_S). \tag{2.11}$$

Further, the following smoothness condition must hold at the risk-switch level $X = X_S$:

$$\frac{\partial VL(X_S)}{\partial X} = \frac{\partial VH(X_S)}{\partial X}. \tag{2.12}$$

We emphasize that the smoothness condition in (2.12) is not an optimality condition; it must be satisfied even for any arbitrary risk-switching threshold, as demonstrated by Dixit (1993) and further discussed by Hennessy and Tserlukevich (2008). Additionally, the fourth condition is, that the difference between levered and unlevered equity value must be bounded if the cash flow X goes to infinity:

$$\left| VL(X) - \frac{X}{\rho - \mu} \right| \quad \text{is bounded for} \quad X \to \infty. \tag{2.13}$$

This last condition implies $d_{1L} = 0$. The three conditions from Equations (2.11) and (2.12) provide the remaining coefficients and determine the firm's equity value $V(X; X_B, X_S)$ uniquely and in closed form for any strategy set (X_B, X_S), as shown in Appendix 2.A.2.

How is the entrepreneur's optimal strategy profile (X_B, X_S) characterised? As in Leland (1998), we assume that the default condition is such that default occurs when the entrepreneur is no longer willing to raise additional capital to cover the firm's losses. Given any risk-switch point X_S, this derives a firm's default threshold $X_B(X_S)$ as a solution to the smooth-pasting condition

$$\left.\frac{\partial V(X)}{\partial X}\right|_{X=X_B} = 0. \tag{2.14}$$

There is no closed-form for the optimal default threshold. However, we can calculate it numerically. As argued by Leland (1998), the function

$$Z_V(X_S) = \left.\frac{dV(X; X_B(X_S), X_S)}{dX_S}\right|_{X=X_S} \tag{2.15}$$

measures the change in equity value that would result from a small change of the switch point at $X = X_S$, recognizing that the firm's implemented default threshold X_B will change with X_S. We request the necessary condition for a firm's risk-switch point X_S to be that

$$Z_V(X_S) = 0. \tag{2.16}$$

When increasing the volatility is costly and $\sigma_H > \sigma_L$, our numerical studies suggest that in the time-consistent case, there always exists a unique strategy pair (X_B^{tc}, X_S^{tc}) consisting of an optimal default and risk-switch level. In line with the predictions by Jensen and Meckling (1976), we find that, due to to agency conflicts, when risk management can be made at no cost and $\sigma_H > \sigma_L$, the entrepreneur would always prefer the higher volatility to shift value from debt to equity.

In the following two sections, we analyse the entrepreneur's optimal strategy under present-biased preferences and distinguish between the naive and the sophisticated entrepreneur.

2.3 The Naive Entrepreneur

Let us consider the case of a naive entrepreneur. The naive entrepreneur's current strategy is based on the false belief that his future selves act in the interest of his current self. As Grenadier and Wang (2007) pointed out, the naive entrepreneur can be regarded as acting as if he can commit his future selves to behave according to his current preferences. From the entrepreneur's current self's perspective, all future payoffs are discounted by $\beta e^{-\rho t}$. By the structure of the present-biased discount function (2.3), naivety can alternatively be interpreted as follows: The naive entrepreneur mistakenly believes that all his future selves preferences are time-consistent and therefore assumes that his future selves will conduct exactly that strategy, which is given by the optimal solution to the time-consistent case. Hence, the naive entrepreneur incorrectly anticipates its total expected value of future selves discounted cash flows to be given by βV under a future selves' strategy profile

$(X_B^F, X_S^F) = (X_B^{tc}, X_S^{tc})$. Now, given this belief, the naive entrepreneur maximises his equity value by choosing his current self's default level X_B and risk-switch level X_S.

By standard arguments, the naive entrepreneur's equity value $N(X)$ must solve the differential equation

$$\frac{1}{2}\sigma_\nu^2 X^2 N'' + \mu X N' - \rho N + (1 - 1_{\{\nu=H\}}\phi\epsilon)X - C_\theta + \lambda[\beta V - N] = 0 \tag{2.17}$$

where $\nu = L$ denotes the lower volatility regime for $X \geq X_S$, while $\nu = H$ denotes the higher volatility regime for $X < X_S$. The final term in the differential equation assures that the current equity value is replaced by the value of future cash flows, when the future arrives at intensity λ. We formally demonstrate the validity of the differential equation in Appendix 2.A.1.

We are able to derive the firm's present-biased value function in closed form: We choose a representation

$$N(X) = (1-\beta)W(X) + \beta V(X), \tag{2.18}$$

where $W(X)$ is some function, that we yet have to determine. For demonstrating purposes, let us here assume $X_B^F \leq X_B \leq X_S \leq X_S^F$. Inserting (2.18) into (2.17), the general derivations in Appendix 2.A.2 show that $W(X)$ must solve

$$\frac{1}{2}\sigma_\nu^2 X^2 W'' + \mu X W' - (\rho + \lambda)W + (1 - 1_{\{\nu=H\}}\phi\epsilon)X - C_\theta + h = 0, \tag{2.19}$$

where as before, $\nu = L$ denotes the lower volatility regime for $X \geq X_S$, while $\nu = H$ denotes the higher volatility regime for $X < X_S$, and $h(X)$ is given by

$$h(X) = \begin{cases} \frac{\beta}{1-\beta}\left(\phi\epsilon X + \frac{1}{2}(\sigma_L^2 - \sigma_H^2)X^2 V''(X)\right), & \text{for} \quad X_S \leq X \leq X_S^F \\ 0, & \text{else.} \end{cases} \tag{2.20}$$

The general solution to the differential equation (2.19) is provided in Appendix 2.A.2. Further, for any strategy (X_B, X_S), the usual value-matching and smoothness conditions determine $W(X)$ and thereby the naive entrepreneur's equity value $N(X)$ uniquely. The naive entrepreneur's optimal strategy is obtained analogously to the time-consistent case. Given his (naive) beliefs about its future selves implemented strategy, the naive entrepreneur's current self's strategy is calculated by solving for the smooth-pasting conditions for both, default and risk-switching level:

$$\left.\frac{\partial N(X)}{\partial X}\right|_{X=X_B} = 0 \tag{2.21}$$

and

$$Z_N(X_S) = \left.\frac{dN(X; X_B(X_S), X_S)}{dX_S}\right|_{X=X_S} = 0. \tag{2.22}$$

Similar to the preceding section, $Z_N(X_S)$ measures the change in equity value that would result from a small change of the switch point at $X = X_S$, recognizing that the firm's implemented default threshold X_B will change with X_S. Solving (2.21) and (2.22) yields a unique solution, the naive entrepreneur's optimal strategy pair (X_B^n, X_S^n).

2.4 The Sophisticated Entrepreneur

The sophisticated entrepreneur correctly foresees that his future selves act according to their own preferences. Following the work of Grenadier and Wang (2007), we interpret the maximisation problem as an intra-personal game, where the entrepreneur's current self aims to find the best response risk and default strategy, given any strategies that its future selves will implement. The sophisticated entrepreneur is aware that by the structure of the time-inconsistent discount function from Equation (2.3), each future self faces the same time-invariant maximisation problem. Thus, as Grenadier and Wang (2007) pointed out, any intuitive equilibrium to the intra-personal game is moreover a fixed point in strategies. Therefore, we aim to find a set of strategies (X_B^s, X_S^s) with the following property: Given all future selves default at a cash flow level $X_B^F = X_B^s$ and increase the volatility at cash flow levels below $X_S^F = X_S^s$, it is the current self's best response to do so, too.

However, to determine a fixed point and equilibrium of the intra-personal game, we need to derive the entrepreneur's equity value given that present and future strategies might deviate from each other. Similar to the previous section, the sophisticated entrepreneur's equity value $S(X)$ must solve a differential equation

$$\frac{1}{2}\sigma_\nu^2 X^2 S'' + \mu X S' - \rho S + (1 - 1_{\{\nu=H\}}\phi\epsilon)X - C_\theta + \lambda[\beta F - S] = 0 \tag{2.23}$$

with $\nu = L$ denoting the lower volatility regime for $X \geq X_S$, while $\nu = H$ denotes the higher volatility regime for $X < X_S$. As before, the final term of the differential equation assures that the current firm value S is replaced by the value of future cash flows F, when the future arrives at intensity λ. The latter value is given by Equation (2.5) considering a future selves' strategy profile (X_B^F, X_S^F). The validity of the differential equation is shown in Appendix 2.A.1. Analogously to Section 2.3, a general solution to the differential equation (2.23) is calculated in Appendix 2.A.2 considering smoothness and value matching conditions.

Furthermore, if the entrepreneur's current self's strategy is a best response to the entrepreneur's future selves' strategy with $X_B = X_B^F$ and $X_S = X_S^F$, then $S(X)$ is an equilibrium equity value function of the sophisticated entrepreneur. How is the sophisticated entrepreneur's best response characterised?

As in Section 2.3, given any future selves' strategy profile (X_B^F, X_S^F), the current self's optimal strategy (X_B, X_S) is defined by solving for the smooth-pasting conditions for both, default and risk-switching level:

$$\left.\frac{\partial S(X; X_B, X_B^F, X_S, X_S^F)}{\partial X}\right|_{X=X_B} = 0 \tag{2.24}$$

and

$$Z_S(X_S) = \left.\frac{dS(X; X_B(X_S), X_B^F, X_S, X_S^F)}{dX_S}\right|_{X=X_S} = 0, \tag{2.25}$$

where $Z_S(X_S)$ measures the change in equity value that would result from a small change of the switch point at $X = X_S$, recognizing that the entrepreneur's current self's implemented default threshold X_B will change with X_S. Moreover, we aim to determine a fixed-point strategy (X_B^s, X_S^s) satisfying (2.24) and (2.25). Let us therefore define $b(X_S)$ as the (unique) solution to

$$\left.\frac{\partial S(X; X_B, X_B^F, X_S, X_S^F)}{\partial X}\right|_{X=X_B=X_B^F,\, X_S=X_S^F} = 0. \tag{2.26}$$

Assuming a risk-switch threshold X_S, $b(X_S)$ indicates the current self's default threshold that maximises value on the condition that future selves replicate the current self's strategy. Note that solving Equation (2.26) is necessary for any default threshold to be a fixed-point candidate. Thus, any fixed-point equilibrium candidate (X_B, X_S) necessarily needs to satisfy

$$b(X_S) = X_B. \tag{2.27}$$

More, given any fixed future selves risk-strategy X_S^F, we can define $l(X_S, X_S^F)$ as the unique solution to

$$\left.\frac{\partial S(X; X_B, b(X_S^F), X_S, X_S^F)}{\partial X}\right|_{X=X_B} = 0. \tag{2.28}$$

Then, $l(X_S, X_S^F)$ is the entrepreneur's current selves optimal default threshold under a current self's risk strategy X_S and a future selves strategy $(b(X_S^F), X_S^F)$. Now, a fixed-point equilibrium (X_B^s, X_S^s) necessarily needs to satisfy the Equation (2.27) and $Z^*(X_S^s) = 0$ with

$$Z^*(X_S) = \left.\frac{dS(X; l(X_S, X_S^F), b(X_S^F), X_S, X_S^F)}{dX_S}\right|_{X=X_S=X_S^F}. \tag{2.29}$$

Z^* measures the change in equity value that would result from a small change of the current self's risk-switch point while additionally containing the necessary conditions for the solution to be a fixed point. Equation (2.29) ensures that the risk-switch level is chosen optimally. Solving (2.27) and (2.29) yields a unique solution, the sophisticated entrepreneur's optimal strategy pair (X_B^s, X_S^s).

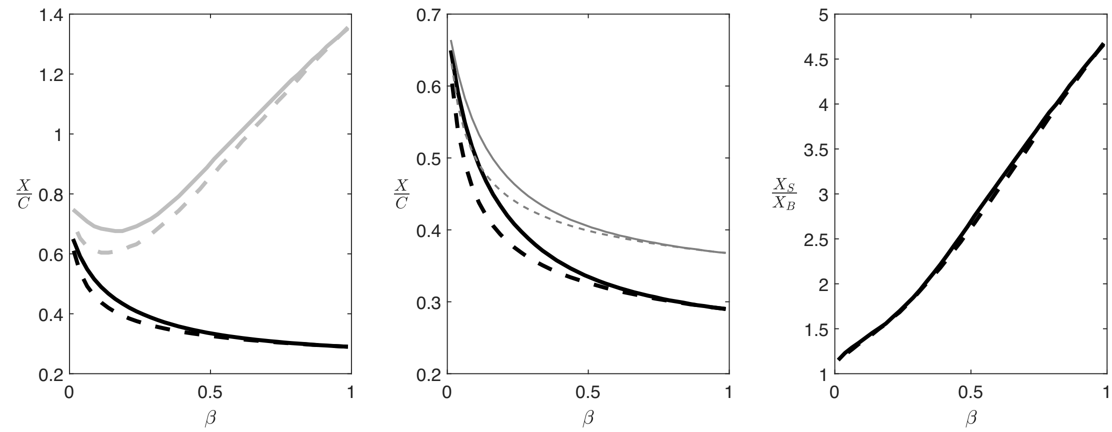
2.5 Strategies and Capital Structure

In the following, we present a numerical analysis of the previously derived results. Section 2.5.1 discusses the impact of the degree of the entrepreneur's present bias on its optimal strategies. Section 2.5.2 discusses how the degree of the entrepreneur's present bias affects the market value of debt. Surprisingly, we show that the value of debt can actually increase in the degree of the entrepreneur's bias. In Section 2.5.3, we examine present-biased entrepreneurs endogenous coupon choice, assuming they interact with unbiased debt-holders. Section 2.5.4 considers the comparative statics of our model implications.

2.5.1 Optimal strategies

The effect of present-biased preferences on the entrepreneur's optimal risk and default strategy depends on both the entrepreneur's degree of its bias and on the fact whether the entrepreneur is naive or sophisticated in his expectation about his future time-inconsistent behaviour.

For our subsequent analysis, in line with Leland (1998), we choose the parametrisation $\rho = 0.05$, $\mu = 0.01$, $\sigma_L = 0.2$, $\sigma_H = 0.3$, $C = 125$ and a tax rate $\theta = 0.2$. The cost rate



(a) *Risk and Default Strategies* ***(b)*** *Default Strategies* ***(c)*** *Risk to Default Ratio*

Figure 2.1: Present-biased entrepreneur's optimal strategies. *This figure analyses the naive (dashed) and sophisticated (solid) entrepreneur's optimal strategies for different degrees β of the entrepreneur's present bias. Panel 2.1a shows the optimal risk-switch (thick light grey) and default threshold (black). Panel 2.1b contrasts the situation in which the entrepreneur has access to risk management (black) with the situation in which the entrepreneur does not (thin dark grey) and where volatility is constantly given by σ_L. The thresholds are expressed relative to the coupon. Panel 2.1c plots the ratio of risk-switch threshold to the default threshold for the naive and sophisticated entrepreneur.*

ϕ associated with the excess risk, given by $\epsilon = \sigma_H^2 - \sigma_L^2$ is set to $\phi = 2$. This implies that the increase in cash flow volatility from σ_L to σ_H incurs flow costs equivalent to $\phi\epsilon = 0.1$ times the cash flow. Further, we set $\lambda = 1$. Now, the degree of the present-biased preferences is characterised by the scalar discount factor $\beta \in [0, 1]$. Figure 2.1 examines the optimal strategies for both the naive (*dashed*) and sophisticated (*solid*) entrepreneur across different degrees of present bias β. The thresholds are expressed in relation to the coupon, which allows us to generalise the findings to any level of leverage through a scaling argument. Panel 2.1a illustrates the optimal default and risk-switch thresholds. Throughout our analysis, we observe that both the sophisticated and naive entrepreneur's default threshold is increasing in the degree of its present bias. As a smaller β determines a stronger present bias, the default threshold X_B increases with the strength of the entrepreneur's bias. This result should not be surprising. Enduring in financially distressed periods comes with immediate costs, while the potential benefit of recovering firm value promises positive payouts in the future. As future payouts are weighted less with a stronger present bias, the default threshold increases in the strength of the bias. On the other side, we observe in Panel 2.1*a* that the risk-switch level is first decreasing and then increasing in the degree of present bias, for both types. However, Panel 2.1c shows that the risk-switch level relative to the default level is decreasing in the strength of the bias. Therefore, a stronger present bias leads to delayed (relative) risk-taking. This result seems intuitive, since increasing the volatility comes with immediate costs, while the benefit of greater risk is postponed. Note, that the benefit of increasing the firm's cash flow volatility is to circumvent long-lasting negative payouts by accelerating either the firm's recovery or bankruptcy. Panel 2.1b compares the firm's optimal default strategy when the firm can switch the cash flow volatility between σ_L and σ_H to a case where the firm does not have access to risk management and volatility is set to σ_L. We observe that access to risk management results in a lower default threshold for both the naive and sophisticated entrepreneur at any degree of present-bias. This effect can be attributed to the value created by access to risk management, which leads to a lower optimal bankruptcy threshold.

In addition, the optimal strategies for the time-consistent entrepreneur, who does not have a present bias, are given by the limiting case of β converging to 1. The time-consistent entrepreneur defaults at the lowest cash flow level while increasing risk at the highest cash flow level, compared to entrepreneurs with present-biased preferences.

Aside from analysing the effect of the degree of the present bias on the entrepreneur's optimal strategies, we can examine the difference in the strategies resulting from naivety or sophistication. Recall that the naive entrepreneur falsely believes that he can commit his future selves to behave in its current self's best interest and thus assumes its future selves implement the optimal strategy of an entrepreneur that is time-consistent and unbiased. The sophisticated entrepreneur, on the other side, is aware that its present bias will persist and that his future self will choose the same strategy as its current self. We observe in Panel 2.1a and 2.1b that the naive entrepreneur defaults at a lower cash flow value than the sophisticated entrepreneur for any given degree of present bias. Due to the false belief that his future selves are going to behave in its current self's best interest, the naive entrepreneur is willing to endure in financially distressed periods, where a sophisticated entrepreneur with the same present bias would already have defaulted. However, in contrast to its beliefs, the naive entrepreneur's future selves will default at the same cash flow level as its current self, since the present biased and naivety will persist. Therefore, due to its naivety and given its present biased preferences, the naive entrepreneur is incapable to maximise its true equity value near the default region. Moreover, we generally observe that the naive entrepreneur increases volatility at a lower level than the sophisticated entrepreneur. This difference is probably driven by the present-biased entrepreneur's preference to implement a riskier strategy in the future than in the present, as increasing the volatility comes with immediate costs and only potential future benefits. While the naive entrepreneur falsely believes that his future selves are going to increase risk at a higher cash flow level than its current self, the sophisticated entrepreneur is aware that its current risk strategy will persist in the future. Therefore, when choosing its risk strategy, the sophisticated entrepreneur balances its preference for higher risk in the future and lower risk in the present.

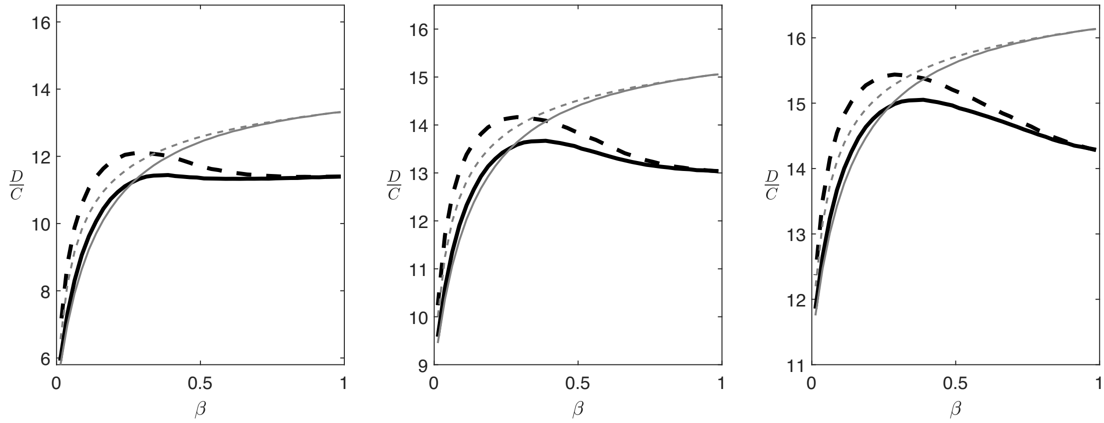
2.5.2 The value of debt

So far, our analysis has focused predominantly on the entrepreneurial perspective. However, the preceding deductions hold significant relevance for debt-holders, given that the entrepreneur's strategic decisions directly determine the value of debt. In contrast to Tian (2016), who does not consider the interaction between present-biased and unbiased agents, we assume that debt is issued by an agent without a present bias, for example a bank. From the debt-holders perspective, the value of debt is given by the future discounted coupon repayments and the firm's recovery value

$$D(X_0; C) = \mathbb{E}_{X_0}\left[\int_0^{\tau(X_B)} e^{-rt} C \,\mathrm{d}t + \alpha \frac{X_B}{r-\mu} e^{-r\tau(X_B)}\right] \tag{2.30}$$

under the firm's equilibrium strategy (X_B, X_S) with $r > \mu$ being the risk-free rate and α being the recovery rate. In Figure 2.2, the value of debt is displayed for different degrees of an entrepreneur's present bias. We utilise the baseline parametrisation from Section 2.5.1, set $r = \rho = 0.05$ and opt for a low recovery rate of $\alpha = 0.1$. We choose a low recovery rate motivated by our assumption that the firm's cash flow is non-tradable and the firm's value primarily stems from the personal skills and expertise of the entrepreneur. Figure 2.2 depicts the value of debt for the cases of a naive (*dashed*) and a sophisticated (*solid*) entrepreneur. The value of debt is presented in relation to the coupon, allowing the findings to be generalized to any level of leverage. Moreover, we compare the situation where the entrepreneur can switch between volatility levels $\sigma_L < \sigma_H$ to a situation where the entrepreneur does not have access to risk management and its volatility is constantly given by σ_L. Let us first analyse the case where the entrepreneur can switch between volatility levels.

Here, we obtain surprising results. Contrary to the common intuition that the value of debt decreases with the degree of the entrepreneur's present bias, we observe that debt-holders can actually benefit from an entrepreneur's present bias, when the entrepreneur has access



***(a)** Debt Value, $X/C = 0.8$* ***(b)** Debt Value, $X/C = 1$* ***(c)** Debt Value, $X/C = 1.2$*

***Figure 2.2: The effect of present bias on debt value.** This figure illustrates the value of debt from the perspective of a time-consistent debt-holder for varying degrees β of the entrepreneur's present bias at different solvency levels. The panels present scenarios where the cash flow is below, equal to, or above the nominal coupon. The plots contrast the situation in which the entrepreneur can switch between volatility levels (black) with the situation where the entrepreneur does not have access to risk management (thin grey), with volatility maintained at the constant level $\sigma_L = 0.2$. The value of debt is is analysed for the naive (dashed) and sophisticated (solid) entrepreneur and is expressed relative to the coupon. The recovery rate is set to $\alpha = 0.1$.*

to risk management. From the preceding analysis, we have learned that the entrepreneur's default level increases in the degree of present bias, which has a detrimental effect on the value of debt. Conversely, relative to default, the level below which the entrepreneur increases the firm's cash flow volatility decreases with the strength of the bias, resulting in a positive effect on the value of debt. As we observe in Figure 2.2, the value of debt can thereby increase in the degree of the entrepreneur's present bias.

In contrast, the value of debt monotonically decreases in the degree of the entrepreneur's present bias, when the entrepreneur does not have access to risk management and the volatility is constantly given by the lower level σ_L. Conventional wisdom states that when a firm has the option to increase its cash flow volatility, this will have a detrimental effect on the value of debt. For low to medium degrees of present bias, this result holds, while for higher degrees of present bias, we observe that firms with the option to increase their cash flow volatility actually exhibit a higher value of debt compared to firms who are required to stick to the lower volatility σ_L.

At strong levels of the entrepreneur's present bias, where β is close to zero, we observe a strong negative effect on the value of debt. Moreover, we discern from the figure that the value of debt of a naive entrepreneur's firm strictly dominates the value of debt of a sophisticated entrepreneur's firm for any given degree of present bias. This stems from the fact that the naive entrepreneur increases volatility and defaults at a lower cash flow value.

2.5.3 Endogenous coupon choice

So far, we have analysed the effect of entrepreneurial present-biased preferences on a levered firm's strategies relative to the coupon level. While the previous analysis is applicable to arbitrary leverage decisions, in practical terms, the choice of leverage itself significantly impacts the firm's creditworthiness. Therefore, in this section, we extend our analysis to investigate the effect of endogenous coupon choice on the entrepreneur's strategies, the value of debt, and credit spreads.

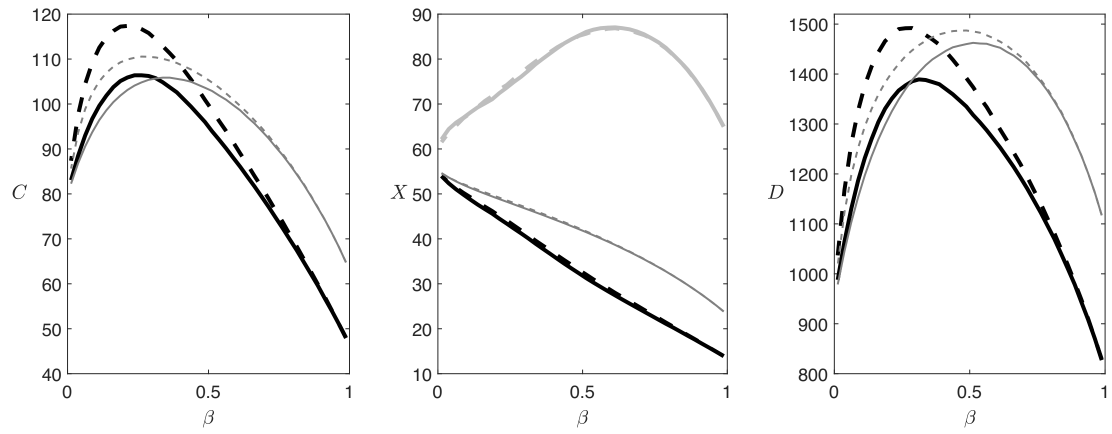
The present-biased entrepreneur interacts with unbiased debt-holders and initially chooses an optimal coupon C. To ensure the tractability of the interaction between debt-holders and the entrepreneur, we assume that debt-holders have perfect information about the entrepreneur and his preferences. We focus on the case that debt-holders wish to rule out the issuance of "junk" bonds, where the entrepreneurial firm's optimal strategy is to increase its cash flow volatility immediately after debt issuance. Therefore, we limit the set of eligible coupons by an upper boundary C^*. The upper boundary ensures that the entrepreneur's initial distance to the high volatility state is at least $X_0/X_S(C) > \psi > 1$ for any coupon $C \leq C^*$. In the extended analysis detailed in Appendix 2.B, we relax the leverage constraint, allowing for the issuance of such riskier bonds. Now, the naive entrepreneur selects its coupon C by maximising its naive perception of the total value of the firm

$$\max_{C \leq C^\star} \left(D(X_0; C) + N(X_0; C) \right), \tag{2.31}$$

whereas the sophisticated entrepreneur chooses the coupon that maximises the actual total value of the firm based on its preferences:

$$\max_{C \leq C^\star} \left(D(X_0; C) + S(X_0; C) \right). \tag{2.32}$$

From the entrepreneur's perspective, the total value of the firm is given by the sum of the debt value and the naive or sophisticated equity value that results after debt is in place. The debt value is not subject to additional discounting, assuming that this value is paid out to the firm immediately after issuance. Figure 2.3 illustrates the entrepreneur's endogenous

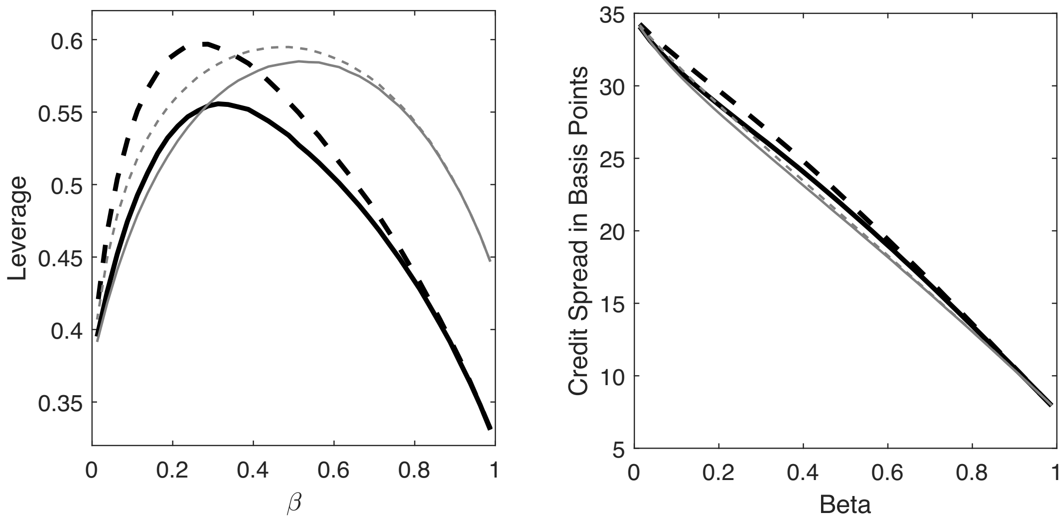


(a) Endog. Coupon Choice *(b) Entrepr. Strategies* *(c) Debt Value*

Figure 2.3: Endogenous coupon choice. *This figure illustrates the effect of endogenous coupon choice for the naive (dashed) and sophisticated (solid) entrepreneur for different degrees β of the entrepreneur's present bias. The panels contrast the situation in which the entrepreneur has access to risk management (black) with the situation in which the entrepreneur does not (thin dark grey) and where volatility is constantly given by σ_L. Panel 2.3a illustrates the entrepreneur's optimal coupon choice, while Panel 2.3c displays the resulting value of debt. Panel 2.3b represent the optimal default thresholds (black) and risk-switch thresholds (thick light grey) for the situation with risk management and compares them to the default thresholds (thin dark grey) for the situation with constant volatility. The initial cash flow is set to $X_0 = 100$.*

coupon choice and its effects on strategies and the value of debt. The analysis is based on an initial cash flow of $X_0 = 100$ and employs the baseline parameters from Section 2.5.1 with a recovery rate $\alpha = 0.1$, $r = 0.05$ and $\psi = 1.05$. The case of a naive (*dashed*) and sophisticated (*solid*) entrepreneur is depicted for varying degrees β of the entrepreneur's present bias. The panels compare the scenarios where the entrepreneur has access to risk management with those in which the entrepreneur does not, with the latter represented by the thinner grey lines with volatility constantly given by σ_L. Panel 2.3a displays the entrepreneur's endogenous coupon choice. We generally observe that the chosen coupon is hump-shaped. At strong degrees of the entrepreneur's present bias, where β is close to zero, the coupon is decreasing in the degree of the entrepreneur's bias. On the other side, at weak to medium degrees of the entrepreneur's present bias, the optimal coupon is increasing in the degree of the bias. The latter observation is driven by the notion that the desire for debt financing increases with the strength of the entrepreneur's present bias. While the debt is issued immediately, repayment occurs over time, and with a higher degree of present bias, the entrepreneur perceives coupon payments as less significant. On the other side, from the preceding analysis, it is evident that where β is close to zero, the value of debt experiences a substantial decline in the degree of the entrepreneur's present bias. Hence, entrepreneurs exhibiting a strong bias face unfavourable financing conditions, leading them to opt for lower coupon. Recall that a naive entrepreneur exhibits a higher value of debt than a sophisticated entrepreneur, both with and without access to risk management, for any given degree of time inconsistency. Therefore, when debt is accurately evaluated by the debt holders, the naive entrepreneur faces more favourable financing conditions and opts for higher coupon than the sophisticated entrepreneur. Moreover, we observe that, at low to medium degrees of present bias, entrepreneurs without access to risk management choose a higher coupon rate than those who can increase their cash flow volatility. However, at higher degrees of present bias, this reverses, which aligns with our previous analysis regarding the effect of present bias on debt value and reflects the debt valuation. Panel 2.3b provides the optimal strategies that result from the endogenous coupon choice. The black lines represent the entrepreneur's optimal default threshold in the scenario when the entrepreneur can increase cash flow volatility from $\sigma_L = 0.2$ to $\sigma_H = 0.3$. The thicker grey lines indicates the optimal threshold below which the firm opts to increase its volatility. The thinner grey lines depict the optimal default threshold when the entrepreneur's volatility is held constant at $\sigma_L = 0.2$. We have observed that, relative to the coupon, the naive entrepreneur tends to default later and increase volatility later than the sophisticated entrepreneur. However, since the naive entrepreneur chooses a higher coupon, the resulting effective strategies of both types of entrepreneurs with endogenous coupon choice almost align. We observe that access to risk management results in a lower default threshold and that the default threshold increases in the degree of the entrepreneur's present bias. We generally observed that the risk-switch threshold relative to the default level decreases with the strength of the present bias. However, in absolute terms, entrepreneurs with a medium to high degree of present bias choose the highest coupon, causing the risk-switch threshold to take on a hump-shaped profile. Panel 2.3c displays the markets valuation of debt after the firm's endogenous coupon choice.

Figure 2.4 analyses the optimal leverage and the resulting credit spreads for naive (*dashed*) and sophisticated (*solid*) entrepreneurs in dependence on the strength β of their present bias. The plots in Panel 2.4a depict the optimal leverage ratio. The leverage ratio is defined as the debt value divided by the total asset value, where the latter is calculated as $X/(r-\mu)$. In accordance with our preceding analysis about endogenous coupon choice, we observe that the optimal leverage is hump-shaped. At strong levels of the entrepreneurial bias, where β is close to zero, the optimal leverage ratio is decreasing in the degree of the entrepreneur's bias. At weaker levels of the entrepreneur's present bias, the optimal leverage ratio is increasing in the degree of the entrepreneur's present bias. Recall that, with and without



(a) Optimal Leverage Choice *(b) Credit Spreads*

***Figure 2.4:* Leverage and credit spreads.** *This figure illustrates the impact of endogenous coupon choice on leverage and credit spreads for naive (dashed) and sophisticated (solid) entrepreneurs across different degrees β of present bias. It contrasts scenarios with (black) and without (thin grey) access to risk management. Panel 2.4a shows the entrepreneur's optimal leverage ratio, while Panel 2.4b presents the corresponding credit spreads in basis points.*

access to risk management, the naive entrepreneur exhibits a higher value of debt than the sophisticated entrepreneur for any degree of time inconsistency. Therefore, when debt is correctly evaluated by debt holders, the naive entrepreneur benefits from more favourable financing conditions and opts for higher leverage. Panel 2.4b displays the respective credit spread in basis points for both the naive and the sophisticated entrepreneur, given by the coupon-to-debt ratio minus the risk-free rate r. Although naivety or sophistication, along with the ability to increase cash flow volatility, significantly influences the choice of leverage at any given degree of present bias β, the credit spreads remain relatively similar. This is because the leverage ratio directly reflects the market's valuation of debt. On the other side, credit spreads are more dependent on the entrepreneur's degree of present bias, as they indicate the maximal interest rate that entrepreneurs are willing to pay to take on an additional dollar of debt. We observe that the spread generally increases with the degree of present bias for both naive and sophisticated entrepreneurs, leading to the conclusion that the desire for debt financing increases with the strength of the entrepreneur's present bias.

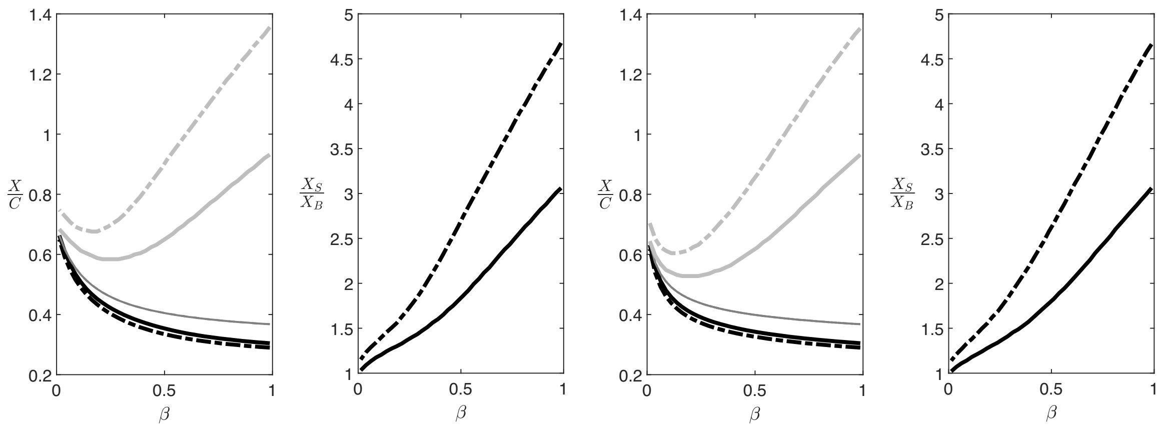
2.5.4 Comparative statics

In the following, we explore how the results are influenced by changes in the economic environment. First, we consider the cash flow process and investigate how an increase in the firm's costs to elevate the cash flow volatility influences the entrepreneur's optimal risk and default strategy. Afterwards, we analyse how a decrease in the firm's bankruptcy cost rate impacts the value of debt, the firm's optimal leverage and the resulting credit spread. We extend the sensitivity analysis in Appendix 2.B. In the subsequent analysis, we use the baseline parametrisation given by $\mu = 0.01$, $\sigma_L = 0.2$, $\sigma_H = 0.3$, $\rho = 0.05$, $r = 0.05$, $\lambda = 1$, $\theta = 0.2$, $\psi = 1.05$, a cost rate $\phi = 2$, such that the increase in volatility incurs costs of $\phi\epsilon = 0.1$ of the cash flow and a recovery rate of $\alpha = 0.1$.

2.5.4.1 Costs of volatility

The entrepreneur's levered equity value is a convex function of the firm's cash flow level. Consequently, if the firm could increase its cash flow volatility without incurring additional costs, choosing the higher volatility would increase the entrepreneur's equity value. Hence, it is reasonable to consider the firm's lower volatility level σ_L as the highest attainable volatility that the firm can reach without incurring additional expenses. Recall our assumption that the firm has the capacity to elevate its cash flow volatility to a higher level $\sigma_H > \sigma_L$ by paying flow costs of $\phi\epsilon$ times the current cash flow. The firm increases its cash flow volatility to σ_H whenever the cash flow falls below the implemented risk-switch threshold and operates at the lower volatility σ_L when the cash flow is above the risk-switch threshold.

In Figure 2.5, we study the sensitivity of the entrepreneur's optimal strategies regarding an increase in the cost rate ϕ from $\phi = 2$ to $\phi = 4$. This implies that the costs associated with an increase in volatility increase from $\phi\epsilon = 0.1$ to $\phi\epsilon = 0.2$ times the cash flow. The dashed-dotted lines in Figure 2.5 depict the base case and the solid lines present the case with increased costs of elevating the firm's cash flow volatility. The thinner dark grey lines present the default threshold for the scenario where the entrepreneur does not have access to risk management and the volatility is maintained at the constant level $\sigma_L = 0.2$. We observe that the entrepreneur increases its firm's cash flow volatility at a lower cash flow level and thereby later if the associated costs of elevating the volatility increase. Further, when costs of elevating the volatility increase, the firm defaults at a slightly higher cash flow



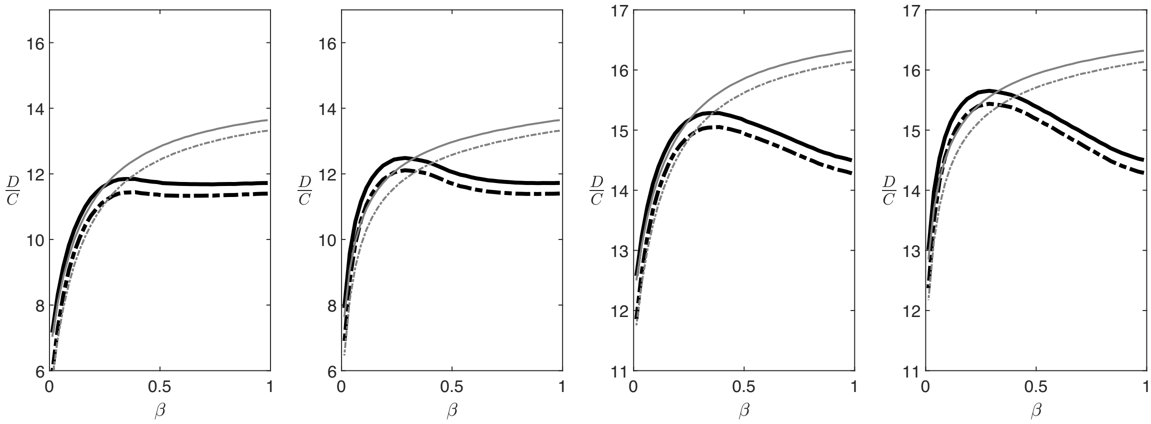
(a) Strategies, Soph. *(b) Relative Risk, Soph.* *(c) Strategies, Naive* *(d) Relative Risk, Naive*

Figure 2.5: Strategy sensitivity to costs of increasing volatility. *The plots illustrate how the entrepreneur's optimal strategies respond to an increase in the firm's cost rate ϕ to elevate the cash flow volatility from σ_L to the higher level σ_H for different degrees β of present bias. The dashed-dotted lines represent the base case, where increasing the volatility costs $\phi\epsilon = 0.1$ times the cash flow, while the solid lines show costs of $\phi\epsilon = 0.2$ times the cash flow. Panels 2.5a and 2.5c display the risk-switch (thick light grey) and default (black) thresholds for both the sophisticated and naive entrepreneur and compares them to the default thresholds (thin dark grey) of an entrepreneur without access to risk management and constant volatility σ_L. Panels 2.5b and 2.5d plot the ratio of the risk-switch to the default threshold for the sophisticated and naive entrepreneur, respectively.*

level. As a result, the risk-switch level relative to the default level decreases. The described results hold for both the naive and sophisticated entrepreneur with similar magnitude at all degrees of the entrepreneur's present bias.

2.5.4.2 Recovery rate

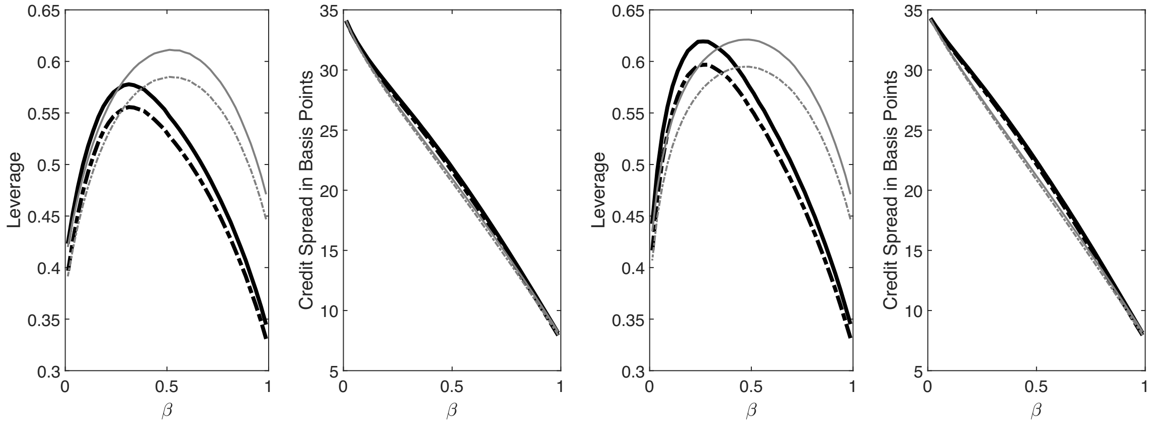
In the event of the firm's bankruptcy, the recovery rate represents the fraction of the firm's asset value that debt-holders can recover and which is not lost due to the costs associated with bankruptcy. After an establishment of the debt contract, the firm's future cash flows remain unaffected by the recovery rate. Consequently, the recovery rate does only implicitly exert an influence on the optimal risk and default strategy through the firm's leverage choice. However, it does play a pivotal role in shaping debt-holders' assessments of debt and, in turn, affects the firm's optimal capital structure decision. In line with Grenadier and Wang (2007), we have assumed that the firm's cash flow is non-tradable and the firm's value primarily stems from the personal skills and expertise of the entrepreneur. Hence, in our base parametrisation, we opted for a low recovery rate of $\alpha = 0.1$. In Figure 2.6, we analyse the sensitivity of the value of debt for to an increase in the firm's recovery rate



(a) Soph., $X/C = 0.8$ *(b) Naive, $X/C = 0.8$* *(c) Soph., $X/C = 1.2$* *(d) Naive, $X/C = 1.2$*

Figure 2.6: Debt value sensitivity to the recovery rate. *This figure illustrates the sensitivity of debt value to an increase in the firm's recovery rate α, contrasting the scenario where the entrepreneur can switch between volatility levels (black) with the situation where the entrepreneur does not have access to risk management (thin grey) with constant volatility σ_L. Panel 2.6b and 2.6a present a scenario where the cash flow is below the nominal coupon, whereas Panels 2.6c and 2.6d depict the scenario where the cash flow exceeds the nominal coupon. The dashed-dotted lines provide the base case with $\alpha = 0.1$, while the solid lines depict the case with $\alpha = 0.2$.*

from $\alpha = 0.1$ to $\alpha = 0.2$. We display the value of debt relative to the coupon level for both the naive and sophisticated entrepreneur at varying degrees of the entrepreneur's present bias. The scenario where the entrepreneur has access to risk management and can switch between the low and high volatility is contrasted with the scenario where the firm's cash flow volatility is held at the lower level. The dashed-dotted lines represent the baseline case with a recovery rate of $\alpha = 0.1$, while the solid lines depict the case with an increased recovery rate of $\alpha = 0.2$. Additionally, we analyse the value of debt for two scenarios, when the cash flow is below the nominal coupon with $X/C = 0.8$ and when the cash flow is above the nominal coupon with $X/C = 1.2$. An increased recovery rate directly increases debt-holder's default claim. Consequently, we observe that a higher recovery rate uniformly raises the value of debt relative to the coupon level, regardless of the degree of the entrepreneur's present bias. Figure 2.7 analyses the sensitivity of the entrepreneur's optimal capital structure choice to an increase of the firm's recovery rate from $\alpha = 0.1$ to $\alpha = 0.2$. Once again, the dashed-dotted lines in Figure 2.7 show the base case, and the solid lines present the scenario with an increased recovery rate. We compare the scenario where the entrepreneur can switch between volatility with the scenario where the entrepreneur does not have access to risk management and the volatility is held constant at σ_L. We observe that, across all degrees of the entrepreneur's present bias β, the optimal leverage



(a) Leverage, Soph. *(b) Credit Spread, Soph.* *(c) Leverage, Naive* *(d) Credit Spread, Naive*

Figure 2.7:* *Capital structure sensitivity to the recovery rate. *This figure shows the sensitivity of the entrepreneur's chosen leverage and resulting credit spread to an increase in the firm's recovery rate α. The dashed-dotted lines provide the base case with $\alpha = 0.1$ and the solid lines show the case with $\alpha = 0.2$. The black lines represent the scenario where the entrepreneur can switch between volatility levels, while the thin grey lines indicate the case where the entrepreneur does not have access to risk management and the volatility is held constant at σ_L. Panels 2.7a and 2.7c display the optimal leverage of the sophisticated and naive entrepreneur for different degrees β of the entrepreneur's present bias. Additionally, Panel 2.7b and Panel 2.7d respectively plot the resulting credit spreads for the sophisticated and naive entrepreneur.*

for both the sophisticated and the naive entrepreneur rises with an increased recovery rate. Interestingly, the resulting credit spread remains almost unaffected by the increased recovery rate. The entrepreneur benefits from an increased recovery rate as the value of debt rises, while the maximal interest rate that the entrepreneur is willing to pay to take on an additional dollar of debt remains almost unchanged.

2.6 Conclusion

This paper expands the continuous-time credit risk framework to incorporate time-inconsistent preferences, along with optimal volatility and default choice. In particular, we propose a model that incorporates entrepreneurial time-inconsistent preferences. In our model, a present-biased entrepreneur maximises its perception of equity value by choosing the firm's volatility and default strategy. We find that the entrepreneur has an incentive to incur costs to increase the firm's cash flow volatility in financially distressed periods. The impact of the present bias on the entrepreneur's optimal strategies is contingent upon factors such as whether the entrepreneur is naive or sophisticated in recognizing its future time-inconsistent behaviour. We generally observe that the entrepreneur's default threshold increases in the degree of its present bias. However, relative to the default threshold, a stronger present bias incentivises the entrepreneur to prefer a lower cash flow volatility. We analyse the interaction of a present-biased entrepreneur and unbiased, time-consistent debt-holders. Our analysis reveals a complex dynamic: Surprisingly, a higher degree of entrepreneurial present bias can actually increase debt value at similar leverage levels, suggesting that an unbiased debt-holder might benefit from an entrepreneur's time-inconsistent preferences. On the other hand, we find that a higher degree of present bias drives entrepreneurs to seek higher leverage. This increased leverage, induced by the present bias, raises the probability of default, resulting in higher credit spreads.

We have assumed debt-holders have perfect information about the entrepreneur during the process of debt issuance. In practice, debt-holders might not be able to distinguish the entrepreneur's types. Therefore, an extension of the model that includes information asymmetry about the entrepreneur's personal characteristics might be an interesting task for future research.

2.A Proof of Main Results

2.A.1 Differential equation

In the following we show the validity of the differential equations from Equations (2.17) and (2.23). Let the equity value of the present-biased entrepreneur be as defined in Equation (2.4), referred to as N in the section discussing the naive entrepreneur and S in the section discussing the sophisticated entrepreneur, be denoted by E. It is given by

$$E(X) = \mathbb{E}_X\Big[\int_0^{\tau(X_B)\wedge T} e^{-\rho t} P(X_t)\mathrm{d}t + 1_{\{\tau(X_B)>T\}} e^{-\rho T}\beta F(X_T)\Big], \tag{2.33}$$

where F is the continuation value function from Equation (2.5), $T \sim exp(\lambda)$ is independent of the firm's cash flow and

$$P(X) = X - C_\theta - 1_{\{X<X_S\}}\phi\epsilon X. \tag{2.34}$$

It holds that $E(X) = H(X) + G(X)$ with

$$H(X) = \mathbb{E}_X\Big[\int_0^{\tau(X_B)\wedge T} e^{-\rho t} P(X_t)\mathrm{d}t\Big] = \mathbb{E}_X\Big[\int_0^{\tau(X_B)} e^{-(\rho+\lambda)t} P(X_t)\mathrm{d}t\Big] \tag{2.35}$$

and

$$G(X) = \mathbb{E}_X\Big[1_{\{\tau(X_B)>T\}} e^{-\rho T}\beta F(X_T)\Big] = \mathbb{E}_X\Big[\int_0^{\tau(X_B)} \lambda e^{-(\rho+\lambda)t}\beta F(X_t)\mathrm{d}t\Big]. \tag{2.36}$$

Let $\nu = L$ denote the lower volatility regime for for $X \geq X_S$ and $\nu = H$ the higher volatility regime for $X < X_S$. By standard arguments in option pricing $H(X)$ satisfies

$$\frac{1}{2}\sigma_\nu^2 X^2 H'' + \mu X H' - (\rho+\lambda)H + P = 0 \tag{2.37}$$

and $G(X)$ satisfies

$$\frac{1}{2}\sigma_\nu^2 X^2 G'' + \mu X G' - (\rho+\lambda)G + (\lambda\beta)F = 0. \tag{2.38}$$

Since $E(X) = H(X) + G(X)$ this implies

$$\frac{1}{2}\sigma_\nu^2 X^2 E'' + \mu X E' - \rho E + P + \lambda(\beta F - E) = 0. \tag{2.39}$$

2.A.2 Closed-form solutions

In this part, we provide closed-form solutions for both, the equity value for the time-consistent entrepreneur and the equity value for a present-biased entrepreneur.

The time-consistent entrepreneur's equity value:

The time-consistent entrepreneur's equity value $V(X)$ is characterised in Section 2.2.2. The firm's equity value $V(X)$ must be a solution to the differential equation (2.7) and Equation (2.8) provides the general solution. The value matching condition for the default level (2.11) is:

$$d_{1H}X_B^{y_{1H}} + d_{2H}X_B^{y_{2H}} + \frac{(1-\phi\epsilon)X_B}{\rho-\mu} - \frac{C_\theta}{\rho} = 0. \tag{2.40}$$

The value matching and smoothness conditions at the risk-switching point $X = X_S$ from equation (2.12) are given by

$$d_{1H}X_S^{y_{1H}} + d_{2H}X_S^{y_{2H}} = d_{2L}X_S^{y_{2L}} + \frac{\phi\epsilon X_S}{\rho-\mu} \tag{2.41}$$

and

$$d_{1H}y_{1H}X_S^{y_{1H}-1} + d_{2H}y_{2H}X_S^{y_{2H}-1} = d_{2L}y_{2L}X_S^{y_{2L}-1} + \frac{\phi\epsilon}{\rho-\mu}. \tag{2.42}$$

Under the condition that the inverse matrix exists, solving for d determines the coefficients uniquely:

$$\begin{bmatrix} d_{1H} \\ d_{2H} \\ d_{2L} \end{bmatrix} = \begin{bmatrix} X_B^{y_{1H}} & X_B^{y_{2H}} & 0 \\ X_S^{y_{1H}} & X_S^{y_{2H}} & -X_S^{y_{2L}} \\ y_{1H}X_S^{y_{1H}-1} & y_{2H}X_S^{y_{2H}-1} & -y_{2L}X_S^{y_{2L}-1} \end{bmatrix}^{-1} \times \begin{bmatrix} \frac{-(1-\phi\epsilon)X_B}{\rho-\mu} + \frac{C_\theta}{\rho} \\ \frac{\phi\epsilon X_S}{\rho-\mu} \\ \frac{\phi\epsilon}{\rho-\mu} \end{bmatrix}. \tag{2.43}$$

The present-biased entrepreneur's equity value:

In the paper, the present-biased entrepreneur's equity value is denoted with N or S, depending on whether the naive or sophisticated entrepreneur is currently being considered. As before, we refer to the general present-biased equity value by E in this section, as the present-biased entrepreneur's type solely determines his expectation of his future behaviour. The goal of this section is to provide a closed form solution for the entrepreneur's equity value, given a strategy (X_B, X_S) of his current self and a persistent strategy (X_B^F, X_S^F) of his future selves. Let us here focus on the case $X_B^F \leq X_B \leq \min\{X_S, X_S^F\}$. In the following, we denote $\min\{X_S, X_S^F\} = X_S \wedge X_S^F$ and $\max\{X_S, X_S^F\} = X_S \vee X_S^F$. Note, that closed-form solutions for the alternative cases are obtained analogously. It is left to the interested reader to replicate those value functions in closed-form. In general, the present-biased entrepreneur's equity value $E(X)$ must satisfy the differential equation

$$\frac{1}{2}\sigma_\nu^2 X^2 E'' + \mu X E' - \rho E + (1 - 1_{\{\nu=H\}}\phi\epsilon)X - C_\theta + \lambda[\beta F - E] = 0. \tag{2.44}$$

where $\nu = L$ denotes the lower volatility regime for $X \geq X_S$, while $\nu = H$ denotes the higher volatility regime for $X < X_S$ and F is the continuation value from Equation (2.5) under the future selves strategy (X_B^F, X_S^F). Recall that the continuation value F equals the value function of the time-consistent entrepreneur, assuming the time-consistent entrepreneur implements the strategy (X_B^F, X_S^F). Therefore, for any strategy, a closed-form for F is derived by the calculations in Appendix 2.A.2. We use the representation

$$E(X) = (1-\beta)W(X) + \beta F(X), \tag{2.45}$$

This is equivalent to $W(X)$ satisfying

$$\frac{1}{2}\sigma_\nu^2 X^2 W'' + \mu X W' - (\rho + \lambda)W + (1 - 1_{\{\nu=H\}}\phi\epsilon)X - C_\theta + h = 0, \tag{2.46}$$

where h is given by

$$h(X) = \begin{cases} \frac{\beta}{1-\beta}\left(\phi\epsilon X + \frac{1}{2}(\sigma_L^2 - \sigma_H^2)X^2 F''(X)\right) & \text{for } X_S \leq X < X_S^F \\ \frac{-\beta}{1-\beta}\left(\phi\epsilon X + \frac{1}{2}(\sigma_L^2 - \sigma_H^2)X^2 F''(X)\right) & \text{for } X_S^F \leq X < X_S \\ 0 & \text{else.} \end{cases} \tag{2.47}$$

Now, it holds that for $X_S \leq X < X_S^F$ the term $X^2 F''(X)$ is given by

$$X^2 F''(X) = y_{1H}(y_{1H} - 1)d_{1H}X^{y_{1H}} + y_{2H}(y_{2H} - 1)d_{2H}X^{y_{2H}}, \tag{2.48}$$

while for $X_S^F \leq X < X_S$ it is given by

$$X^2 F''(X) = y_{2L}(y_{2L} - 1)d_{2L}X^{y_{2L}}. \tag{2.49}$$

Let $\underline{X}_S := X_S \wedge X_S^F$ and $\overline{X}_S := X_S \vee X_S^F$. Then, by Lemma 1, the general solution of the differential equation (2.46) for $W(X)$ is then given by

$$W(X) = \begin{cases} W_3(X) = f_{1,3}X^{x_{1L}} + f_{2,3}X^{x_{2L}} + \frac{X}{\rho+\lambda-\mu} - \frac{C_\theta}{\rho+\lambda} & \text{for } X \geq \overline{X}_S \\ W_2(X) = f_{1,2}X^{x_{1M}} + f_{2,2}X^{x_{2M}} + \frac{(1-1_{\{\nu=H\}}\phi\epsilon)X}{\rho+\lambda-\mu} - \frac{C_\theta}{\rho+\lambda} + k(X) & \text{for } X \in [\underline{X}_S, \overline{X}_S) \\ W_1(X) = f_{1,1}X^{x_{1H}} + f_{2,1}X^{x_{2H}} + \frac{(1-\phi\epsilon)X}{\rho+\lambda-\mu} - \frac{C_\theta}{\rho+\lambda} & \text{for } X < \underline{X}_S, \end{cases} \tag{2.50}$$

where for $j \in \{1, 2\}$ it holds that x_{jL} and x_{jH} is defined analogous to y_{jL} and y_{jH}, respectively, when replacing ρ by $\rho + \lambda$ in Equations (2.9) and (2.10). Further, in the previous equation let $M = L$ and

$$k(X) = \frac{\beta}{1-\beta}\left(\phi\epsilon\frac{X}{\rho+\lambda-\mu} + \frac{1}{2}(\sigma_L^2 - \sigma_H^2)\left(\alpha_{1H}\frac{X^{y_{1H}}}{\rho+\lambda-\mu_{y_{1H}}} + \alpha_{2H}\frac{X^{y_{2H}}}{\rho+\lambda-\mu_{y_{2H}}}\right)\right), \tag{2.51}$$

for the case $X_S \leq X < X_S^F$. In the case $X_S^F \leq X < X_S$ let $M = H$ and

$$k(X) = -\frac{\beta}{1-\beta}\left(\phi\epsilon\frac{X}{\rho+\lambda-\mu} + \frac{1}{2}(\sigma_L^2 - \sigma_H^2)\left(\alpha_{2N}\frac{X^{y_{2L}}}{\rho+\lambda-\mu_{y_{2L}}}\right)\right). \tag{2.52}$$

In Equations (2.51) and (2.52) it holds that for $j \in \{1, 2\}$ and $M \in \{L, H\} : \alpha_{jM} = y_{jM}(y_{jM} - 1)d_{jM}$ and $\mu_{y_{jM}}$ is the notation for the transformed drift from Lemma 1. Next, the value matching and smoothness conditions determine the coefficients uniquely. $W(X)$ must satisfy the value matching condition at the default level X_B:

$$W_1(X_B) = -\frac{\beta}{1-\beta}F(X_B). \tag{2.53}$$

Moreover, the following smoothness and value-matching conditions must be satisfied at $X = \underline{X}_S$:

$$W_1(\underline{X}_S) = W_2(\underline{X}_S) \quad \text{and} \quad \frac{\partial W_1(\underline{X}_S)}{\partial X} = \frac{\partial W_2(\underline{X}_S)}{\partial X}. \tag{2.54}$$

Finally, smoothness and value-matching conditions are further required at $X = \overline{X}_S$:

$$W_2(\overline{X}_S) = W_3(\overline{X}_S) \quad \text{and} \quad \frac{\partial W_2(\overline{X}_S)}{\partial X} = \frac{\partial W_3(\overline{X}_S)}{\partial X}. \tag{2.55}$$

In the time-consistent case, the final condition was that the absolute difference between unlevered and levered equity value is bounded for cash flow values $X \to \infty$. This condition still holds. Under present-biased preferences, the unlevered equity value is given by

$$\mathbb{E}_X\Big[\int_0^\infty D_0(0,t)X_t \mathrm{d}t\Big] = (1-\beta)\frac{X}{\rho+\lambda-\mu} + \beta\frac{X}{\rho-\mu}. \tag{2.56}$$

By the structure of $E(X)$ from (2.45) this is equivalent to:

$$|W_3(X) - \frac{X}{\rho + \lambda - \mu}| \quad \text{is bounded for} \quad X \to \infty. \tag{2.57}$$

This implies $f_{1,3} = 0$. Let us take a closer look at the conditions (2.53)-(2.54). The value matching condition for the default level (2.53) translates to

$$f_{1,1} X_B^{x_{1H}} + f_{2,1} X_B^{x_{2H}} + \frac{(1-\phi\epsilon)X_B}{\rho + \lambda - \mu} - \frac{C_\theta}{\rho + \lambda} \tag{2.58}$$

$$= -\frac{\beta}{1-\beta}\left(d_{1H} X_B^{y_{1H}} + d_{2H} X_B^{y_{2H}} + \frac{(1-\phi\epsilon)X_B}{\rho - \mu} - \frac{C_\theta}{\rho}\right). \tag{2.59}$$

From here on define $\epsilon^\star = 0$ and $M = L$ in the case $X_S \leq X < X_S^F$, while $\epsilon^\star = \epsilon$ and $M = H$ in the case $X_S^F \leq X < X_S$. The value matching and smoothness conditions at the risk-switching point $X = \underline{X}_S$ from equation (2.54) are given by

$$f_{1,1} \underline{X}_S^{x_{1H}} + f_{2,1} \underline{X}_S^{x_{2H}} = f_{1,2} \underline{X}_S^{x_{1M}} + f_{2,2} \underline{X}_S^{x_{2M}} + \frac{(\phi\epsilon - \phi\epsilon^\star)\underline{X}_S}{\rho + \lambda - \mu} + k(\underline{X}_S) \tag{2.60}$$

and

$$f_{1,1} x_{1H} \underline{X}_S^{x_{1H}-1} + f_{2,1} x_{2H} \underline{X}_S^{x_{2H}-1} \tag{2.61}$$

$$= f_{1,2} x_{1M} \underline{X}_S^{x_{1M}-1} + f_{2,2} x_{2M} \underline{X}_S^{x_{2M}-1} + \frac{\phi\epsilon - \phi\epsilon^\star}{\rho + \lambda - \mu} + k'(\underline{X}_S). \tag{2.62}$$

Finally, the value matching and smoothness conditions at the risk-switching point $X = \overline{X}_S$ from equation (2.55) are given by

$$f_{1,2} \overline{X}_S^{x_{1M}} + f_{2,2} \overline{X}_S^{x_{2M}} + k(\overline{X}_S) = \frac{\phi\epsilon^\star \overline{X}_S}{\rho + \lambda - \mu} + f_{2,3} \overline{X}_S^{x_{2L}} \tag{2.63}$$

and

$$f_{1,2} x_{1M} \overline{X}_S^{x_{1M}-1} + f_{2,2} x_{2M} \overline{X}_S^{x_{2M}-1} + k'(\overline{X}_S) = \frac{\phi\epsilon^\star}{\rho + \lambda - \mu} + f_{2,3} x_{2L} \overline{X}_S^{x_{2L}-1}. \tag{2.64}$$

Under the condition that the inverse matrix exists, solving for f gives

$$\begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{1,2} \\ f_{2,2} \\ f_{2,3} \end{bmatrix} = \begin{bmatrix} X_B^{x_{1H}} & X_B^{x_{2H}} & 0 & 0 & 0 \\ \underline{X}_S^{x_{1H}} & \underline{X}_S^{x_{2H}} & -\underline{X}_S^{x_{1M}} & -\underline{X}_S^{x_{2M}} & 0 \\ x_{1H}\underline{X}_S^{x_{1H}-1} & x_{2H}\underline{X}_S^{x_{2H}-1} & -x_{1M}\underline{X}_S^{x_{1M}-1} & -x_{2M}\underline{X}_S^{x_{2M}-1} & 0 \\ 0 & 0 & \overline{X}_S^{x_{1M}} & \overline{X}_S^{x_{2M}} & -\overline{X}_S^{x_{2L}} \\ 0 & 0 & x_{1M}\overline{X}_S^{x_{1M}-1} & x_{2M}\overline{X}_S^{x_{2M}-1} & -x_{2L}\overline{X}_S^{x_{2L}-1} \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} \frac{-(1-\phi\epsilon)X_B}{\rho+\lambda-\mu} + \frac{C_\theta}{\rho+\lambda} - \frac{\beta}{1-\beta}\left(d_{1H}X_B^{y_{1H}} + d_{2H}X_B^{y_{2H}} + \frac{(1-\phi\epsilon)X_B}{\rho-\mu} - \frac{C_\theta}{\rho}\right) \\ \frac{(\phi\epsilon-\phi\epsilon^\star)\underline{X}_S}{\rho+\lambda-\mu} + k(\underline{X}_S) \\ \frac{\phi\epsilon-\phi\epsilon^\star}{\rho+\lambda-\mu} + k'(\underline{X}_S) \\ \frac{\phi\epsilon^\star\overline{X}_S}{\rho+\lambda-\mu} - k(\overline{X}_S) \\ \frac{\phi\epsilon^\star}{\rho+\lambda-\mu} - k'(\overline{X}_S) \end{bmatrix}. \tag{2.65}$$

2.A.3 Auxiliary results

In this part, we provide and prove an important Lemma for the derivation of the closed-form solutions that were provided in the previous section of the Appendix.

Lemma 1. *Let $\alpha_0 \in \mathbb{R}, \alpha_1 \in \mathbb{R}/\{0\}$. The differential equation*

$$\frac{1}{2}\sigma^2 X^2 F''(X) + \mu X F'(X) - (\rho+\lambda)F(X) + \alpha_0 X^{\alpha_1} = 0 \tag{2.66}$$

has the general solution

$$F(X) = a_1 X^{x_1} + a_2 X^{x_2} + \alpha_0 \frac{X^{\alpha_1}}{\rho+\lambda-\mu_{\alpha_1}}, \tag{2.67}$$

whenever $\mu_{\alpha_1} := \alpha_1\left(\mu - \frac{1}{2}\sigma^2 + \alpha_1\frac{1}{2}\sigma^2\right) \neq \rho+\lambda$ and where

$$x_1 = \frac{-(\mu - \frac{\sigma^2}{2}) + \sqrt{(\mu - \frac{\sigma^2}{2})^2 + 2(\rho+\lambda)\sigma^2}}{\sigma^2} > 0 \tag{2.68}$$

and

$$x_2 = \frac{-(\mu - \frac{\sigma^2}{2}) - \sqrt{(\mu - \frac{\sigma^2}{2})^2 + 2(\rho+\lambda)\sigma^2}}{\sigma^2} < 0. \tag{2.69}$$

Proof. Inserting $F(X)$ into the differential equation (2.66) yields the claim, since

$$Q(X) = a_1 X^{x_1} + a_2 X^{x_2} \tag{2.70}$$

is the general solution to

$$\frac{1}{2}\sigma^2 X^2 Q''(X) + \mu X Q'(X) - (\rho + \lambda) Q(X) = 0 \tag{2.71}$$

□

2.B Extended Comparative Statics

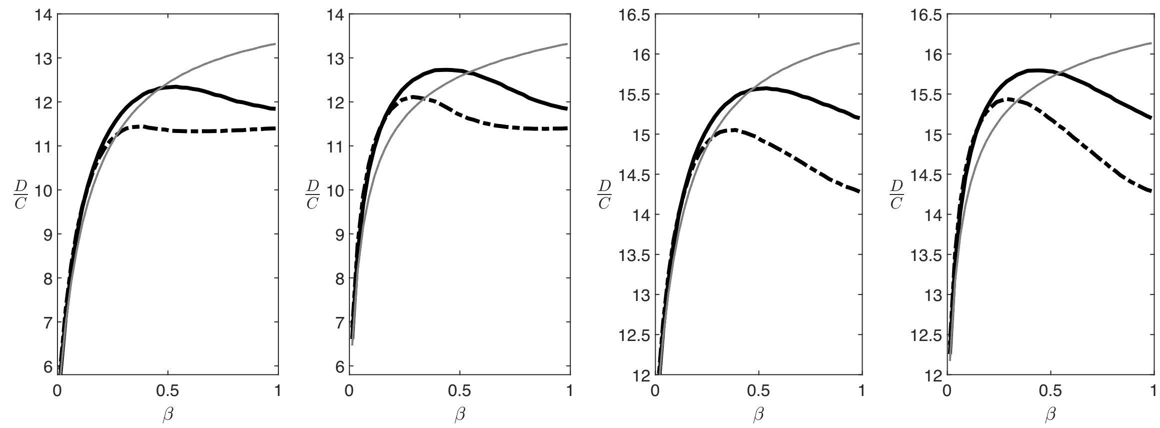
In the following, we investigate further into a comprehensive analysis of how alterations in the economic environment shape our research results. Within Appendix 2.B.1, we analyse how an increase of the firm's costs to elevate the cash flow volatility influences the value of debt and the firm's optimal capital structure decisions. Appendix 2.B.2 further investigates the effect of an increase in the firm's recovery rate and Appendix 2.B.3 studies how an increase in the entrepreneur's perception of the length of the present influences the entrepreneur's strategies as well as its financing decisions. Subsequently, Appendix 2.B.4 explores the sensitivity of the results to a decrease in the firm's cash flow drift. Afterwards, Appendix 2.B.5 analyses the sensitivity of our findings to a decrease of the firm's effective tax rate, while Appendix 2.B.6 examines the impact of an increased entrepreneur's exponential discount rate. Then, in Appendix 2.B.7, we analyse the sensitivity of the results to an increase of the firm's upper cash flow volatility level.

Throughout Appendices 2.B.1 - 2.B.7, we maintain a baseline parametrisation with values set at $\mu = 0.01$, $\sigma_L = 0.2$, $\sigma_H = 0.3$, $\rho = 0.05$, $r = 0.05$, $\lambda = 1$, $\theta = 0.2$, $\psi = 1.05$ and recovery rate of $\alpha = 0.1$. The cost rate ϕ associated with the excess risk, given by $\epsilon = \sigma_H^2 - \sigma_L^2$ is set to $\phi = 2$, which implies that the increase in cash flow volatility from σ_L to σ_H incurs flow costs of $\phi\epsilon = 0.1$ times the cash flow.

Finally, in Appendix 2.B.8, we relax the leverage constraint and examine the optimal capital structure for the present-biased entrepreneur in a scenario where we permit the issuance of "junk" bonds. These riskier bonds might involve an increase in the firm's cash flow volatility immediately after debt issuance.

2.B.1 The costs of volatility (continued)

During periods of financial distress, when the firm's cash flow drops below its coupon payments, the firm has an incentive to increase its cash flow volatility. An elevated cash flow volatility serves to either accelerate the firm's default or the firm's recovery, both of which being preferable to a long-lasting negative payout. We assumed that the firm can elevate its cash flow volatility from σ_L to a higher level $\sigma_H > \sigma_L$ as long as it pays flow costs of $\phi\epsilon$ times the current cash flow value. In Section 2.5.4, we analysed how an increase in the cost rate from $\phi = 2$ to $\phi = 4$ affects the firm's optimal default threshold and its risk-switch threshold, the level below which the entrepreneur elevates the firm's cash flow volatility. A cost rate of $\phi = 2$ implies that the costs associated with an increase in volatility are given by flow costs of $\phi\epsilon = 0.1$ times the cash flow, while a cost rate of $\phi = 4$ means that the costs associated with increase in volatility are given by flow costs of $\phi\epsilon = 0.2$. We observed that when the cost rate is higher both the naive and sophisticated entrepreneurs elevated the firm's cash flow volatility at a lower cash flow level and defaulted at a higher cash flow level for all degrees of the entrepreneur's present bias. We continue the analysis from Section 2.5.4 and analyse the effect of the increased costs to elevate the firm's cash flow volatility on the value of debt, the firm's endogenous coupon choice and thereby its optimal capital structure decisions. Figure B.1 illustrates the impact of heightened costs in raising the firm's cash flow volatility on the value of debt. We analyse the value of debt for two scenarios, when the cash flow is below the nominal coupon with $X/C = 0.8$ and when the cash flow is above the nominal coupon with $X/C = 1.2$. Figure B.1 depicts the value of debt for the naive and sophisticated entrepreneur, considering different degrees of the entrepreneur's present bias β. A lower β indicates a stronger present bias, while $\beta = 1$ represents an entrepreneur without a present bias. The dashed-dotted black lines represent the baseline case with a cost rate $\phi = 2$, while the solid black lines provide the case with a heightened cost rate of $\phi = 4$. The thinner grey line shows the value of debt in the situation where the entrepreneur does not have access to risk management and the cash flow volatility is constantly given by σ_L. It is evident that, at moderate levels of the entrepreneur's present bias, an increase in the firm's cost to raise the cash flow volatility corresponds to an increase in the debt value. We have previously observed that an increased cost rate leads to a higher default threshold and a lower threshold, below which the firm increases the cash flow volatility. Although a higher default threshold typically has a negative impact on the value of debt, we observe that the positive effect of reduced cash flow volatility outweighs it. Overall, the findings highlight that as the firm's



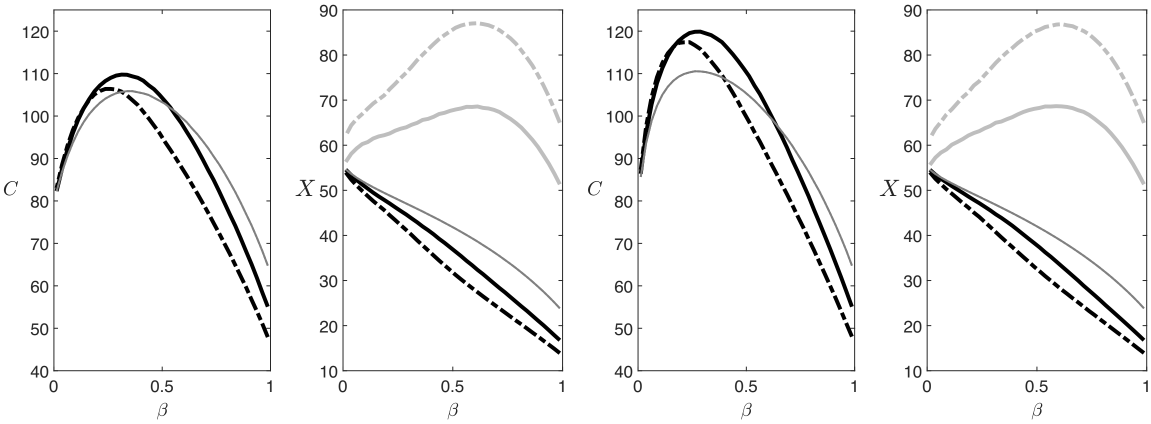
(a) *Soph., $X/C = 0.8$* ***(b)*** *Naive, $X/C = 0.8$* ***(c)*** *Soph., $X/C = 1.2$* ***(d)*** *Naive, $X/C = 1.2$*

Figure B.1: Debt value sensitivity to the costs of increasing volatility. *This figure illustrates the sensitivity of the debt value to an increase in the firm's cost rate ϕ to elevate the firm's cash flow volatility from σ_L to σ_H, contrasting the scenario where the entrepreneur can switch between volatility levels (black) with the situation where the entrepreneur does not have access to risk management (thin grey) with constant volatility σ_L. The dashed-dotted lines provide the base case with $\phi\epsilon = 0.1$ and the solid lines show the case with $\phi\epsilon = 0.2$. Panel B.1a and B.1b present a scenario where the cash flow is below the nominal coupon, whereas Panels B.1c and B.1d depict the scenario where the cash flow exceeds the nominal coupon.*

costs to elevate cash flow volatility increase, the value of debt tends to rise. Moreover, the findings support the result from the main analysis that an entrepreneurial present bias can paradoxically contribute to an increase in the market value of the firm's debt when the entrepreneur can choose between volatility levels.

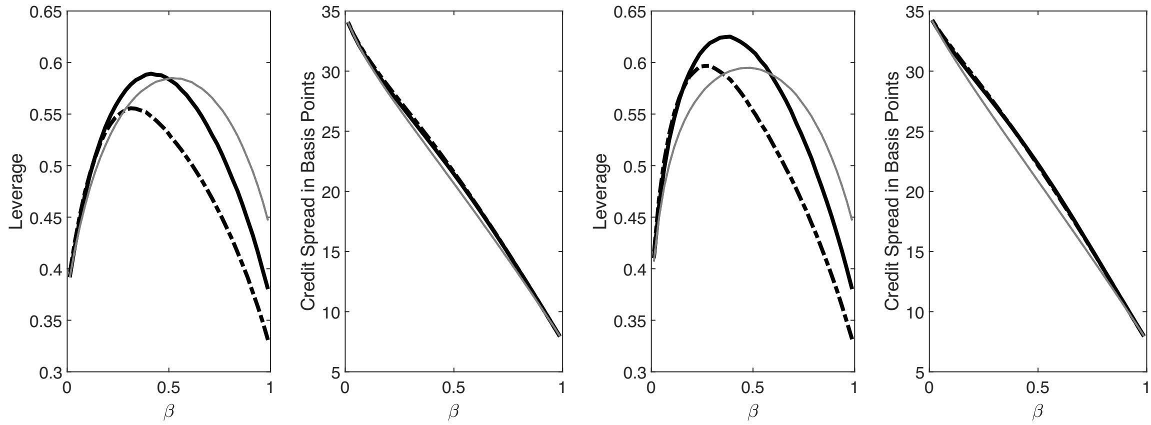
The plots in Figure B.2 illustrate how the firm's endogenous optimal coupon choice and resulting optimal strategies change with rising costs associated to an increase in the firm's cash flow volatility. The baseline scenario, represented by dashed-dotted lines with a costs of $\phi\epsilon = 0.1$ times the cash flow, is contrasted with the case of heightened costs, indicated by solid lines with costs of $\phi\epsilon = 0.2$ times the cash flow in each of the panels. We observe that with a moderate degree of the entrepreneur's present bias and increased costs, both sophisticated and naive entrepreneurs choose a higher coupon. This results in higher default thresholds, while the willingness to increase the cash flow volatility decreases if the associated costs increase.

The plots in Figure B.3 illustrate how the optimal capital structure of the firm responds to rising costs associated with increasing the firm's cash flow volatility. Once again, the baseline scenario, represented by dashed-dotted lines with a costs of $\phi\epsilon = 0.1$ times the cash flow, is contrasted with the case of heightened costs, indicated by solid lines with costs



(a) Coupon, Soph. *(b) Strategies, Soph.* *(c) Coupon, Naive* *(d) Strategies, Naive*

Figure B.2: Coupon choice sensitivity to the costs of increasing volatility. *This figure shows the sensitivity of the entrepreneur's chosen coupon and the resulting strategies to an increase in the firm's cost rate ϕ to elevate the firm's cash flow volatility for different degrees β of the entrepreneur's present bias, when $X_0 = 100$. The dashed-dotted lines provide the base case with $\phi\epsilon = 0.1$ and the solid lines show the case with $\phi\epsilon = 0.2$. In Panel B.2a and B.2c, the coupon choice when the entrepreneur can switch between volatility levels (black) is compared to the coupon choice (thin grey) when volatility is held constant at σ_L. In Panel B.2b and B.2d, the resulting default (black) and risk-switch thresholds (thick light grey), when volatility can be changed, are compared to the default threshold (thin dark grey) when volatility is maintained constantly at σ_L.*



(a) Leverage, Soph. *(b) Credit Spread, Soph.* *(c) Leverage, Naive* *(d) Credit Spread, Naive*

Figure B.3: Capital structure sensitivity to the costs of increasing volatility*. This figure shows the sensitivity of the entrepreneur's chosen leverage and resulting credit spread to an increase in the firm's cost rate ϕ to elevate the firm's cash flow volatility. The dashed-dotted lines provide the base case with $\phi\epsilon = 0.1$ and the solid lines show the case with $\phi\epsilon = 0.2$. The black lines represent the scenario where the entrepreneur can switch between volatility levels, while the thin grey lines indicate the case where the entrepreneur does not have access to risk management and volatility is held constant at σ_L. Panel B.3a and Panel B.3c display the optimal leverage of the sophisticated and naive entrepreneur for different degrees β of the entrepreneur's present bias, while Panel B.3b and Panel B.3d plot the resulting credit spreads.*

of $\phi\epsilon = 0.2$ times cash flow in each of the panels. Specifically, Panel B.3a and Panel B.3c respectively depict the sophisticated and naive entrepreneur's optimal leverage ratio, while Panel B.3b and Panel B.3d show the corresponding credit spreads. Notably, with a moderate degree of the entrepreneur's present bias and increased costs, both sophisticated and naive entrepreneurs opt for higher leverage, which results from the higher coupon choice. Interestingly, the credit spreads remain relatively stable, indicating that despite the firms issue more debt, their interest rates do not visibly change. This coincides with our previous finding that higher costs for increasing cash flow volatility positively affect the debt value.

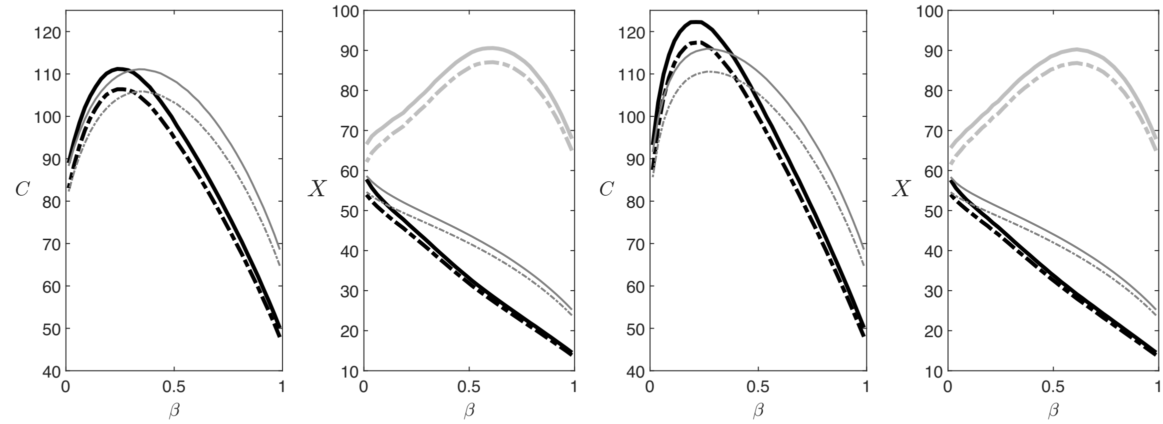
2.B.2 The recovery rate (continued)

The recovery rate represents the fraction of the firm's asset value that debt-holders can recover in case of bankruptcy. In our main analysis, we opted for a low recovery rated motivated by the assumption that the firm's value primarily stems from the personal skills and expertise of the entrepreneur. In Section 2.5.4, we have already analysed the effect of an increase in the firm's recovery rate on the value of debt as well as the leverage ratio and credit spreads. Once debt is issued and the coupon rate is settled, a change in the recovery rate does not change the entrepreneur's equity value nor its optimal strategy. However, since the recovery rate affects the debt value, it influences the entrepreneur's coupon choice, which then determines the optimal default and risk strategy. Therefore, in Figure B.4, we further investigate the effect of an increase in the firm's recovery rate on endogenous coupon choice and the resulting optimal strategies. For both cases with and without access to risk management, we observe that a higher recovery rate increases the optimal coupon. This, in turn, increases the optimal default and risk-switch threshold.

2.B.3 The duration of the present

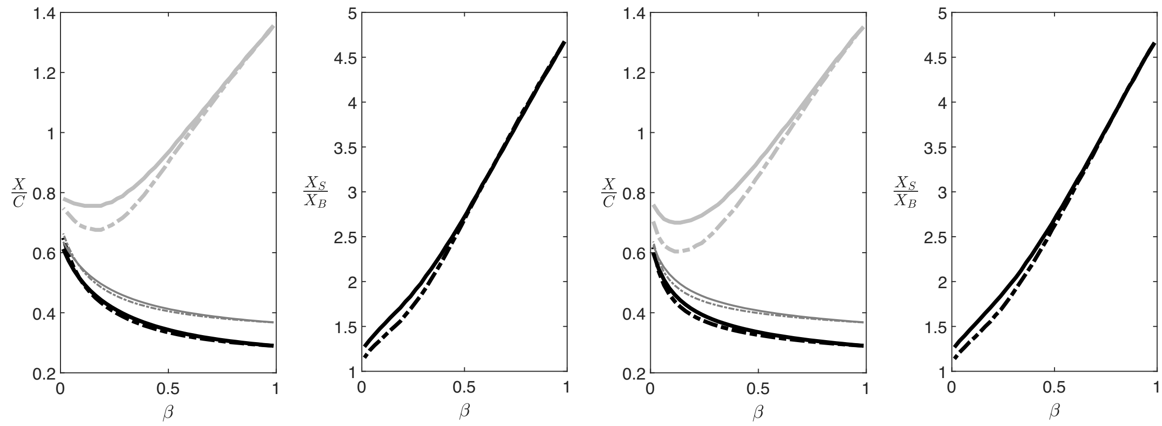
At each time, the entrepreneur distinguishes between the future and the present and its preferences are biased towards the present. From the entrepreneur's perspective, the expected duration of the present is given by λ^{-1}. Besides standard exponential discounting at the rate ρ, the entrepreneur additionally discounts the future by a discount factor $\beta \in [0, 1]$.

In Figure B.5, the robustness of the entrepreneur's decision-making strategies to changes



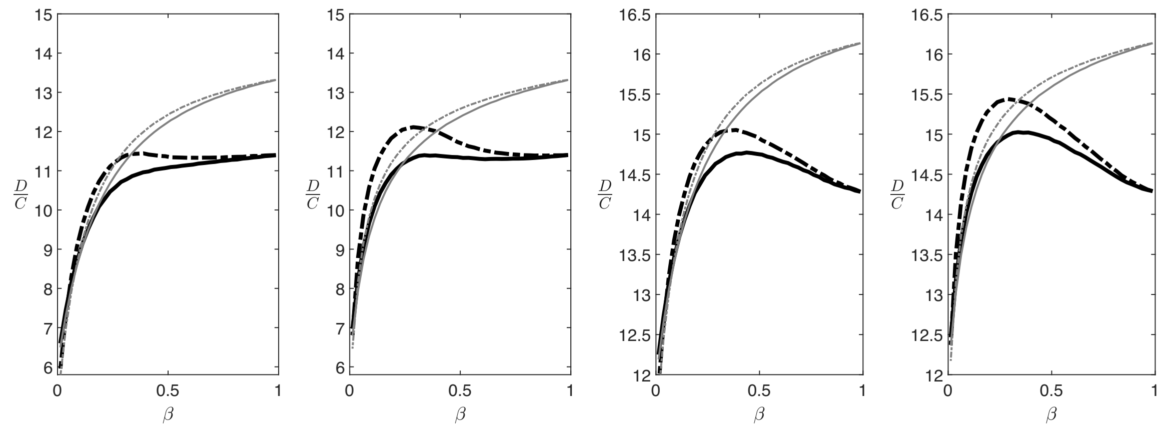
***(a)** Coupon, Soph.* ***(b)** Strategies, Soph.* ***(c)** Coupon, Naive* ***(d)** Strategies, Naive*

Figure B.4:* *Coupon choice sensitivity to the recovery rate. *This figure shows the sensitivity of the entrepreneur's chosen coupon and the resulting strategies to an increase in the firm's recovery rate α for different degrees β of the entrepreneur's present bias , when $X_0 = 100$. The dashed-dotted lines provide the base case with $\alpha = 0.1$ and the solid lines show the case with $\alpha = 0.2$. In Panel B.4a and B.4c, the coupon choice when the entrepreneur can switch between volatility levels (black) is compared to the coupon choice (thin grey) when volatility is held constant at σ_L. In Panel B.4b and B.4d, the resulting default (black) and risk-switch thresholds (thick light grey), when volatility can be changed, are compared to the default threshold (thin dark grey) when volatility is maintained constantly at σ_L.*



***(a)** Strategies, Soph.* ***(b)** Relative Risk, Soph.* ***(c)** Strategies, Naive* ***(d)** Relative Risk, Naive*

Figure B.5:* *Strategy sensitivity to the duration of the present. *The plots illustrate how the entrepreneur's optimal strategies respond to an increase in the entrepreneur's perceived duration λ^{-1} of the present for different degrees β of present bias. The dashed-dotted lines represent the base case $\lambda^{-1} = 1$, while the solid lines show the case with a duration of the present given by $\lambda^{-1} = 2$. Panels B.5a and B.5c display the risk-switch (thick light grey) and default (black) thresholds for both the sophisticated and naive entrepreneur and compares them to the default thresholds (thin dark grey) of an entrepreneur without access to risk management and constant volatility σ_L. Panels B.5b and B.5d plot the ratio of the risk-switch to the default threshold for the sophisticated and naive entrepreneur, respectively.*

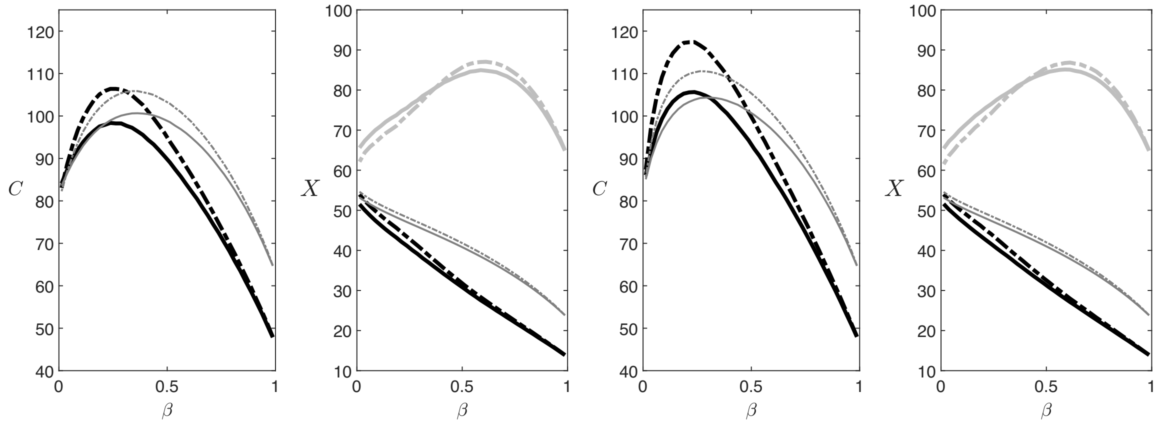


(a) Soph., $X/C = 0.8$ *(b) Naive, $X/C = 0.8$* *(c) Soph., $X/C = 1.2$* *(d) Naive, $X/C = 1.2$*

Figure B.6: Debt value sensitivity to the duration of the present. *This figure illustrates the sensitivity of the debt value to an increase in the entrepreneur's perceived duration λ^{-1} of the present, contrasting the scenario where the entrepreneur can switch between volatility levels (black) with the situation where the entrepreneur does not have access to risk management (thin grey) with constant volatility σ_L. The dashed-dotted lines provide the base case with $\lambda^{-1} = 1$ and the solid lines show the case with $\lambda^{-1} = 2$. Panel B.6a and B.6b present a scenario where the cash flow is below the nominal coupon, whereas Panels B.6c and B.6d depict the scenario where the cash flow exceeds the nominal coupon.*

in the expected duration of the present from $\lambda^{-1} = 1$ to $\lambda^{-1} = 2$ is explored. The dashed-dotted lines in Figure B.5 depict the base case and the solid lines present the case with a longer duration of the present. We observe that especially entrepreneurs with a strong degree of present bias, where β is close to zero, increase their firm's cash flow volatility at a higher cash flow level and thereby earlier when their perception of the length of the present increases. Additionally, changes in the default level are minimal. Consequently, among entrepreneurs with significant present bias, the risk threshold relative to the default threshold increases when their perceived duration of the present is longer. These observations are consistent across both naive and sophisticated entrepreneurs to a similar extent. Note that increasing the firm's cash flow volatility comes with immediate costs while its potential benefits of accelerated recovery or default manifest over time. Nevertheless, from a present-biased entrepreneur's perspective, an elongation in the perceived duration of the present results in these benefits appearing more immediate. Thereby, a present-biased entrepreneur is more willing to increase the firm's cash flow risk when its perception of the present is longer.

Figure B.6 demonstrates the impact of increasing the entrepreneur's perceived duration of the present on the value of the firm's debt for varying degrees β of the entrepreneur's present bias. Recall that smaller degrees of β suggest a stronger present bias, whereas $\beta = 1$

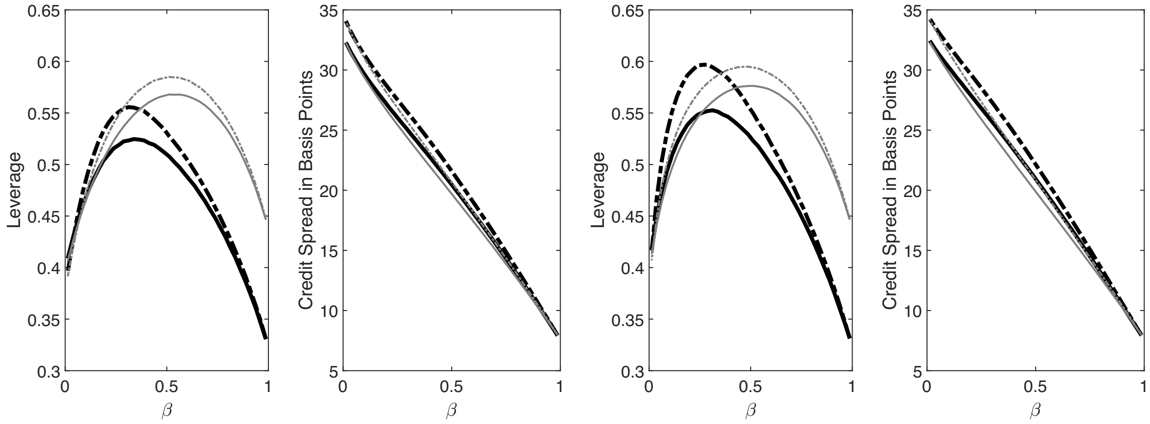


(a) Coupon, Soph. *(b) Strategies, Soph.* *(c) Coupon, Naive* *(d) Strategies, Naive*

Figure B.7: Coupon choice sensitivity to the duration of the present. *This figure shows the sensitivity of the entrepreneur's chosen coupon and the resulting strategies to an increase in the entrepreneur's perceived duration λ^{-1} of the present for different degrees β of the entrepreneur's present bias, when $X_0 = 100$. The dashed-dotted lines provide the base case with $\lambda^{-1} = 1$ and the solid lines show the case with $\lambda^{-1} = 2$. In Panel B.7a and B.7c, the coupon choice when the entrepreneur can switch between volatility levels (black) is compared to the coupon choice (thin grey) when volatility is held constant at σ_L. In Panel B.7b and B.7d, the resulting default (black) and risk-switch thresholds (thick light grey), when volatility can be changed, are compared to the default threshold (thin dark grey) when volatility is maintained constantly at σ_L.*

characterises an entrepreneur without a present bias. We analyse the value of debt in two situations, when the cash flow is below the nominal coupon with $X/C = 0.8$ and when the cash flow is above the nominal coupon with $X/C = 1.2$. The dashed-dotted lines represent the baseline case when the duration of the present is $\lambda^{-1} = 1$, while the solid lines provide the case with a longer duration of the present with $\lambda^{-1} = 2$. The black lines represent the case where the entrepreneur has access to risk management, while the thinner grey lines show the situation where the volatility is constantly given by the lower volatility σ_L. We observe that, at stronger degrees of the entrepreneur's present bias, a longer duration of the present corresponds to a decrease in debt value, especially when the entrepreneur can switch volatility levels. We have previously observed that a longer duration of the present leads to earlier risk-taking for entrepreneurs who exhibit a stronger degree of present bias. This decreases the debt value for both the naive and the sophisticated entrepreneur.

The plots in Figure B.7 illustrate how the firm's endogenous optimal coupon choice and resulting optimal strategies change with an increase in the entrepreneur's perceived duration of the present. As before, the baseline scenario, represented by dashed-dotted lines with a duration $\lambda^{-1} = 1$ is contrasted with the case $\lambda^{-1} = 2$, indicated by solid lines in each of the panels. We observe that at all degrees of the entrepreneur's present bias both

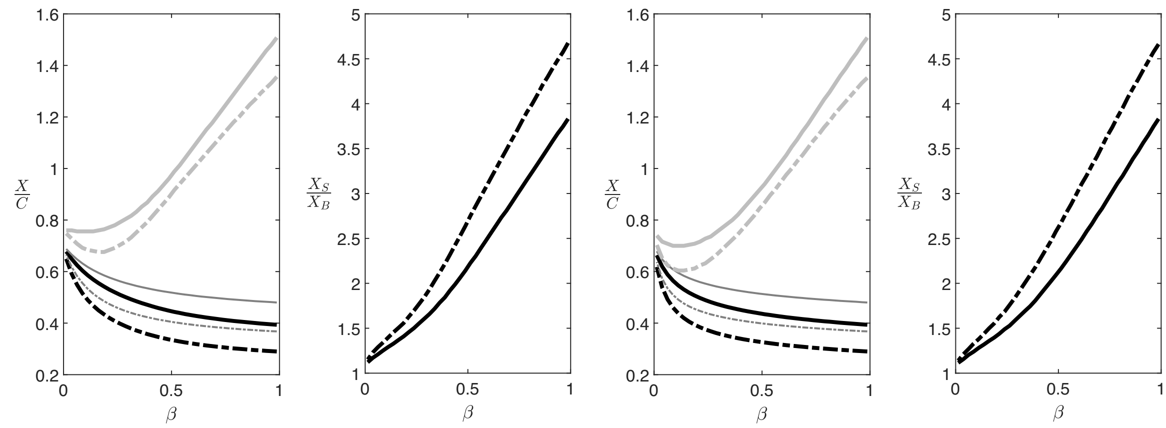


(a) Leverage, Soph. *(b) Credit Spread, Soph.* *(c) Leverage, Naive* *(d) Credit Spread, Naive*

Figure B.8: Capital structure sensitivity to the duration of the present. *This figure shows the sensitivity of the entrepreneur's chosen leverage and resulting credit spread to an increase in the entrepreneur's perceived duration λ^{-1} of the present. The dashed-dotted lines provide the base case with $\lambda^{-1} = 1$ and the solid lines show the case with $\lambda^{-1} = 2$. The black lines represent the scenario where the entrepreneur can switch between volatility levels, while the thin grey lines indicate the case where the entrepreneur does not have access to risk management and the volatility is held constant at σ_L. Panel B.8a and Panel B.8c display the optimal leverage of the sophisticated and naive entrepreneur for different degrees β of the entrepreneur's present bias, while Panel B.8b and Panel B.8d plot the resulting credit spreads.*

sophisticated and naive entrepreneurs choose a lower coupon if they perceive the present to be longer. This observation can be attributed to the fact that a longer perception of the present decreases the value of debt.

Finally, Figure B.8 examines how the leverage ratio and the credit spreads respond to an increase in the expected duration of the present. Once again, the dashed-dotted lines in the figure represent the baseline scenario, while the solid lines depict the case with a prolonged duration of the present. The analysis reveals that, regardless of the degree of present bias, both sophisticated and naive entrepreneurs show a decrease in optimal leverage when the duration of the present is increased. Simultaneously, the associated credit spreads also diminish as the duration of the present extends. These results are predominantly influenced by the entrepreneur's increased willingness to elevate the firm's cash flow volatility, which in turn reduces the value of the firm's debt and consequently the optimal leverage.



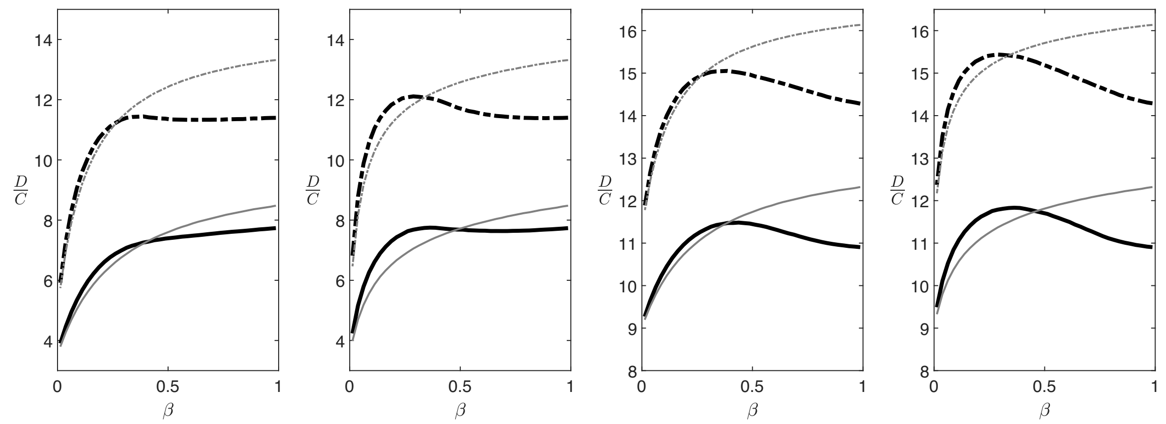
(a) Strategies, Soph. *(b) Relative Risk, Soph.* *(c) Strategies, Naive* *(d) Relative Risk, Naive*

Figure B.9: Strategy sensitivity to the cash flow drift. *The plots illustrate how the entrepreneur's optimal strategies respond to a decrease in the firm's cash flow drift μ for different degrees β of present bias. The dashed-dotted lines represent the base case $\mu = 0.01$, while the solid lines show the case with a cash flow drift given by $\mu = -0.01$. Panels B.9a and B.9c display the risk-switch (thick light grey) and default (black) thresholds for both the sophisticated and naive entrepreneur and compares them to the default thresholds (thin dark grey) of an entrepreneur without access to risk management and constant volatility σ_L. Panels B.9b and B.9d plot the ratio of the risk-switch to the default threshold for the sophisticated and naive entrepreneur, respectively.*

2.B.4 The cash flow drift

Figure B.9 examines how changes in the firm's cash flow drift, from $\mu = 0.01$ to $\mu = -0.01$, affect the decision-making strategies of entrepreneurs. The dashed-dotted lines in the figure represent the baseline scenario where $\mu = 0.01$, and the solid lines illustrate the scenario with a reduced cash flow drift of $\mu = -0.01$. Across all levels of present bias, it is observed that both naive and sophisticated entrepreneurs default and increase their cash flow volatility at higher levels of cash flow when the cash flow drift is lower. Additionally, the risk threshold relative to the default threshold is reduced among entrepreneurs, regardless of them being sophisticated or naive and their degree of present bias. These findings align with expectations, as enduring periods of financial distress becomes less appealing with a decreased cash flow drift. Moreover, the incentive to incur costs to increase the firm's cash flow volatility appears to be more attractive when the cash flow drift is higher.

Figure B.10 displays the effect of a decreased cash flow drift on the value of debt. The dashed-dotted lines in these figures represent the baseline scenario with a cash flow drift of $\mu = 0.01$, and the solid lines illustrate the scenario with a reduced cash flow drift of $\mu = -0.01$. It is observed that across all levels of present bias, a decrease in cash flow



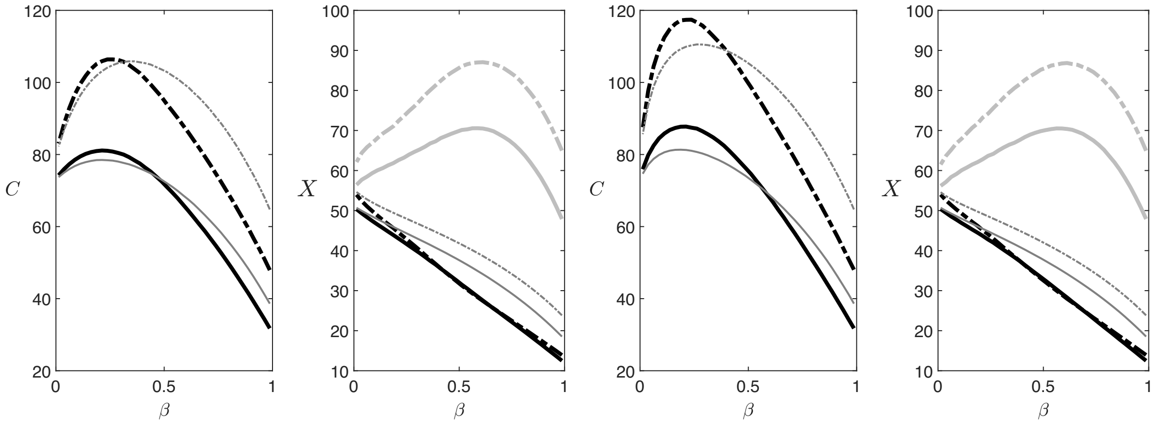
(a) Soph., $X/C = 0.8$ *(b) Naive, $X/C = 0.8$* *(c) Soph., $X/C = 1.2$* *(d) Naive, $X/C = 1.2$*

Figure B.10: Debt value sensitivity to the cash flow drift. *This figure illustrates the sensitivity of the debt value to a decrease in the in the firm's cash flow drift μ, contrasting the scenario where the entrepreneur can switch between volatility levels (black) with the situation where the entrepreneur does not have access to risk management (thin grey) with constant volatility σ_L. The dashed-dotted lines provide the base case with $\mu = 0.01$ and the solid lines show the case with $\mu = -0.01$. Panel B.10a and B.10b present a scenario where the cash flow is below the nominal coupon, whereas Panels B.10c and B.10d depict the scenario where the cash flow exceeds the nominal coupon.*

drift results in a lower debt value. Besides the detrimental effects of the change in the cash flow dynamics, this outcome can be attributed to the fact that a reduced cash flow drift increases the default threshold and the threshold, at which the firm increases its cash flow volatility. All of those factors contribute to a decrease in debt value.

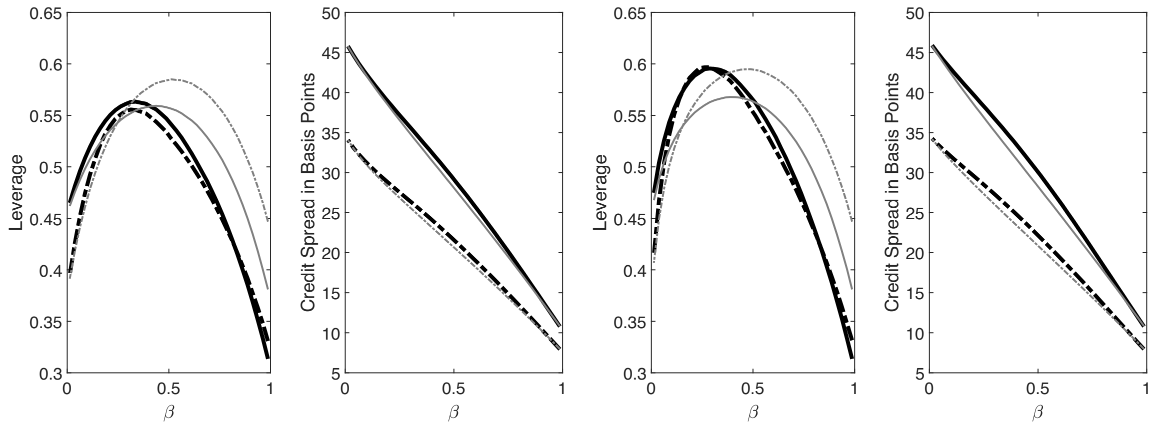
The plots in Figure B.11 illustrate how the firm's endogenous optimal coupon choice and resulting optimal strategies change with a decrease in the firm's cash flow drift. As before, the baseline scenario, represented by dashed-dotted lines with a duration $\mu = 0.01$ is contrasted with the case $\mu = -0.01$, indicated by solid lines in each of the panels. We observe that at all degrees of the entrepreneur's present bias both sophisticated and naive entrepreneurs choose a lower coupon when the cash flow drift decreases. This observation can be attributed to the fact that a lower cash flow drift leads to earlier default and this, in turn, decreases the value of debt.

Figure B.12 explores how changes in the firm's cash flow drift, specifically a decrease from $\mu = 0.01$ to $\mu = -0.01$, affect the optimal capital structure of entrepreneurs. The dashed-dotted lines in the figure indicate the baseline scenario with $\mu = 0.01$, while the solid lines illustrate the scenario with the decreased cash flow drift. This decrease directly impacts both the value of debt and the overall value of the equity, leading to reductions



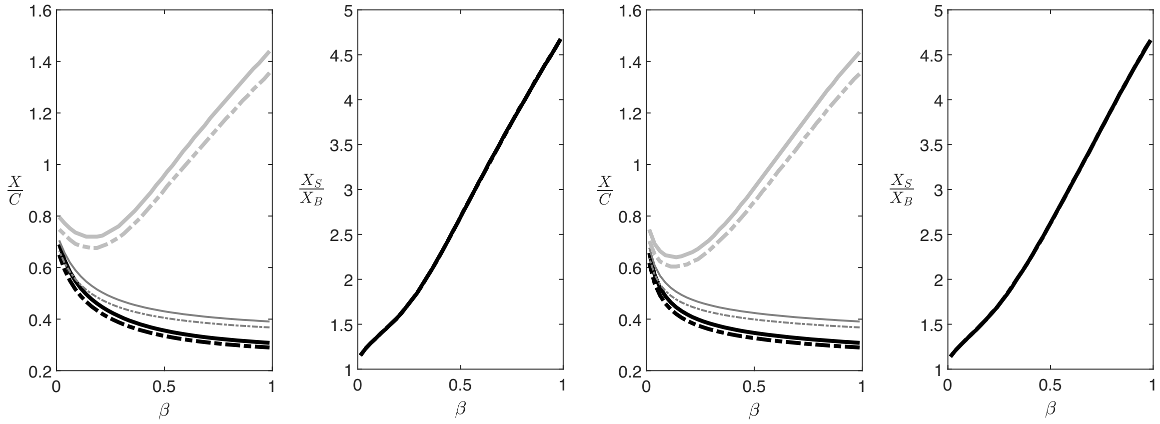
(a) Coupon, Soph. *(b) Strategies, Soph.* *(c) Coupon, Naive* *(d) Strategies, Naive*

Figure B.11: Coupon choice sensitivity to the cash flow drift. *This figure shows the sensitivity of the entrepreneur's chosen coupon and the resulting strategies to a decrease in the firm's cash flow drift μ for different degrees β of the entrepreneur's present bias, when $X_0 = 100$. The dashed-dotted lines provide the base case with $\mu = 0.01$ and the solid lines show the case with $\mu = -0.01$. In Panel B.11a and B.11c, the coupon choice when the entrepreneur can switch between volatility levels (black) is compared to the coupon choice (thin grey) when volatility is held constant at σ_L. In Panel B.11b and B.11d, the resulting default (black) and risk-switch thresholds (thick light grey), when volatility can be changed, are compared to the default threshold (thin dark grey) when volatility is maintained constantly at σ_L.*



(a) Leverage, Soph. *(b) Credit Spread, Soph.* *(c) Leverage, Naive* *(d) Credit Spread, Naive*

Figure B.12: Capital structure sensitivity to the cash flow drift. *This figure shows the sensitivity of the entrepreneur's chosen leverage and resulting credit spread to a decrease in the firm's cash flow drift μ. The dashed-dotted lines provide the base case with $\mu = 0.01$ and the solid lines show the case with $\mu = -0.01$. The black lines represent the scenario where the entrepreneur can switch between volatility levels, while the thin grey lines indicate the case where the entrepreneur does not have access to risk management and the volatility is held constant at σ_L. Panel B.12a and Panel B.12c display the optimal leverage of the sophisticated and naive entrepreneur for different degrees β of the entrepreneur's present bias, while Panel B.12b and Panel B.12d plot the resulting credit spreads.*



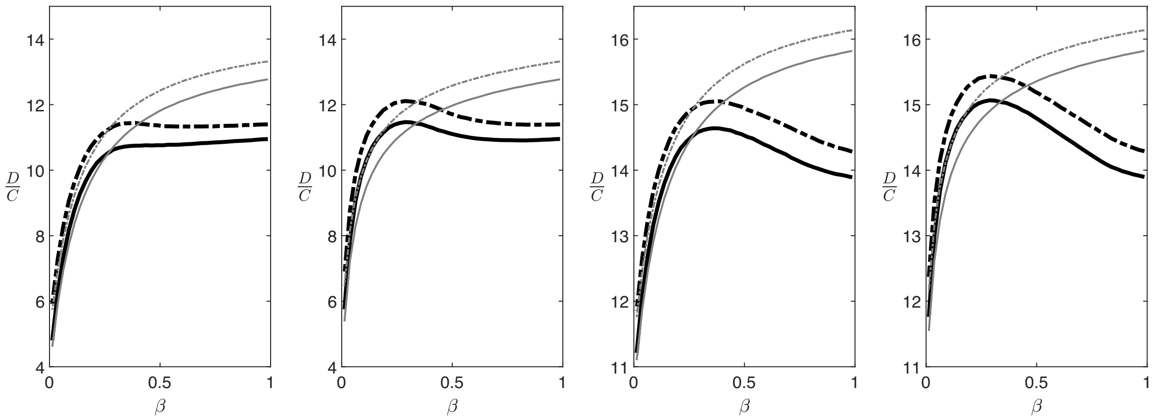
***(a)** Strategies, Soph.* ***(b)** Relative Risk, Soph.* ***(c)** Strategies, Naive* ***(d)** Relative Risk, Naive*

***Figure B.13: Strategy sensitivity to the tax rate.** The plots illustrate how the entrepreneur's optimal strategies respond to a decrease in the tax rate θ for different degrees β of present bias. The dashed-dotted lines represent the base case $\theta = 0.2$, while the solid lines show the case with a tax rate given by $\theta = 0.15$. Panels B.13a and B.13c display the risk-switch (thick light grey) and default (black) thresholds for both the sophisticated and naive entrepreneur and compares them to the default thresholds (thin dark grey) of an entrepreneur without access to risk management and constant volatility σ_L. Panels B.13b and B.13d plot the ratio of the risk-switch to the default threshold for the sophisticated and naive entrepreneur, respectively.*

in each. Despite these direct decreases in debt and equity value, the analysis indicates that the overall impact on optimal leverage, especially when the firm can switch between volatility levels, is relatively minor. This suggests that the reductions in debt and firm value somewhat offset each other. However, the associated credit spreads experience an increase for both sophisticated and naive entrepreneurs across all degrees of present bias. This is primarily due to the increase of the default threshold resulting from the decreased cash flow drift.

2.B.5 The tax rate

Figure B.13 explores the impact of reducing the tax rate from $\theta = 0.2$ to $\theta = 0.15$ on the decision-making strategies of entrepreneurs. The dashed-dotted lines depict the baseline scenario with $\theta = 0.2$, while solid lines represent the modified scenario with a tax rate of $\theta = 0.15$. Across all levels of present bias, it is observed that both naive and sophisticated entrepreneurs default and increase their cash flow volatility at a slightly higher level of cash flow when the tax rate is decreased. This effect arises solely from an increase in the effective coupon due to the tax reduction. As a result, the optimal thresholds for default



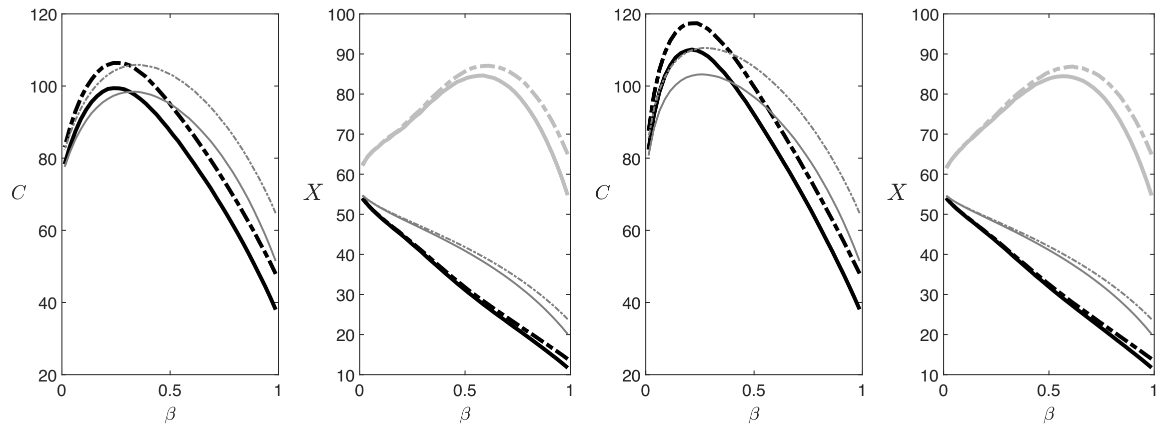
(a) Soph., $X/C = 0.8$ *(b) Naive, $X/C = 0.8$* *(c) Soph., $X/C = 1.2$* *(d) Naive, $X/C = 1.2$*

***Figure B.14:* Debt value sensitivity to the tax rate.** *This figure illustrates the sensitivity of the debt value to a decrease in the tax rate θ, contrasting the scenario where the entrepreneur can switch between volatility levels (black) with the situation where the entrepreneur does not have access to risk management (thin grey) with constant volatility σ_L. The dashed-dotted lines provide the base case with $\theta = 0.2$ and the solid lines show the case with $\theta = 0.15$. Panel B.14a and B.14b present a scenario where the cash flow is below the nominal coupon, whereas Panels B.14c and B.14d depict the scenario where the cash flow exceeds the nominal coupon.*

and risk-taking rise proportionally to the increase in the effective coupon. Across all levels of present bias for both types of entrepreneur, the risk threshold relative to the default threshold remains unchanged by the decrease in the tax rate as both scale up by the same factor.

Figure B.14 illustrates the effect of a reduced tax rate on debt value across varying degrees of present bias β. In these panels, dashed-dotted lines represent the baseline scenario with a tax rate of $\theta = 0.2$, and solid lines represent a lowered tax rate of $\theta = 0.15$. Across all levels of present bias, the decrease in the tax rate consistently leads to a lower debt value. This observation can be linked to the increase in both the default and risk-taking thresholds, as both increases contribute to a reduction in the debt value.

The plots in Figure B.15 illustrate how the firm's endogenous optimal coupon choice and resulting optimal strategies change with a decrease in the tax rate. As before, the baseline scenario, represented by dashed-dotted lines with a duration $\theta = 0.2$ is contrasted with the case $\theta = 0.15$, indicated by solid lines in each of the panels. We observe that at all degrees of the entrepreneur's present bias both sophisticated and naive entrepreneurs choose a lower coupon when the tax rate decreases. This observation can be fully attributed to the fact that a lower tax rate decreases the tax benefit of debt and therefore lowers the



(a) Coupon, Soph. *(b) Strategies, Soph.* *(c) Coupon, Naive* *(d) Strategies, Naive*

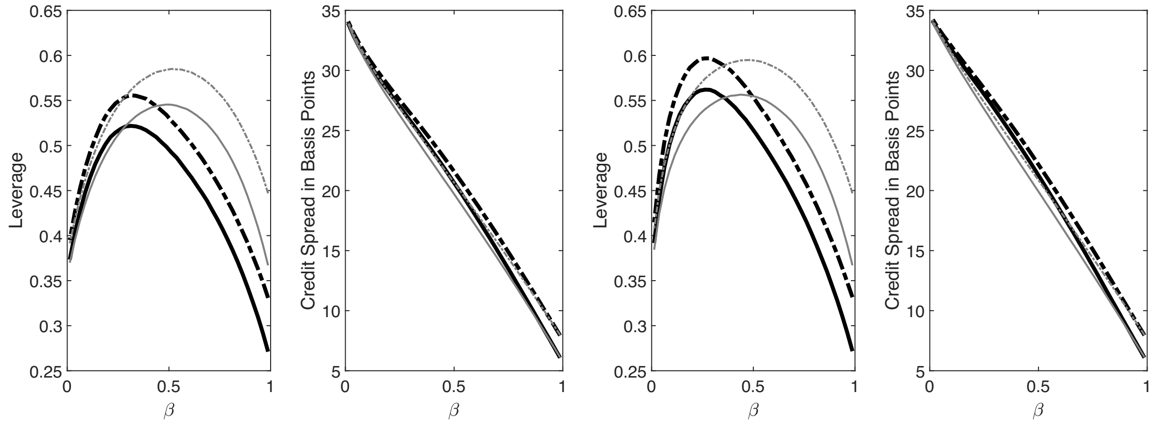
Figure B.15: Coupon choice sensitivity to the tax rate. *This figure shows the sensitivity of the entrepreneur's chosen coupon and the resulting strategies to a decrease in the tax rate θ for different degrees β of the entrepreneur's present bias, when $X_0 = 100$. The dashed-dotted lines provide the base case with $\theta = 0.2$ and the solid lines show the case with $\theta = 0.15$. In Panel B.15a and B.15c, the coupon choice when the entrepreneur can switch between volatility levels (black) is compared to the coupon choice (thin grey) when volatility is held constant at σ_L. In Panel B.15b and B.15d, the resulting default (black) and risk-switch thresholds (thick light grey), when volatility can be changed, are compared to the default threshold (thin dark grey) when volatility is maintained constantly at σ_L.*

incentive for debt issuance.

Figure B.16 examines the impact of a tax rate reduction from $\theta = 0.2$ to $\theta = 0.15$ on the optimal capital structure of entrepreneurs. Dashed-dotted lines show the baseline scenario with $\theta = 0.2$, and solid lines depict the scenario with the reduced tax rate. The panels depict that both the optimal leverage and the related credit spreads decrease across all levels of present bias for both naive and sophisticated entrepreneurs. This decrease in optimal leverage results from the diminished tax benefit of debt under a lower tax rate. Consequently, lower leverage reduces credit risk, which in turn leads to lower credit spreads.

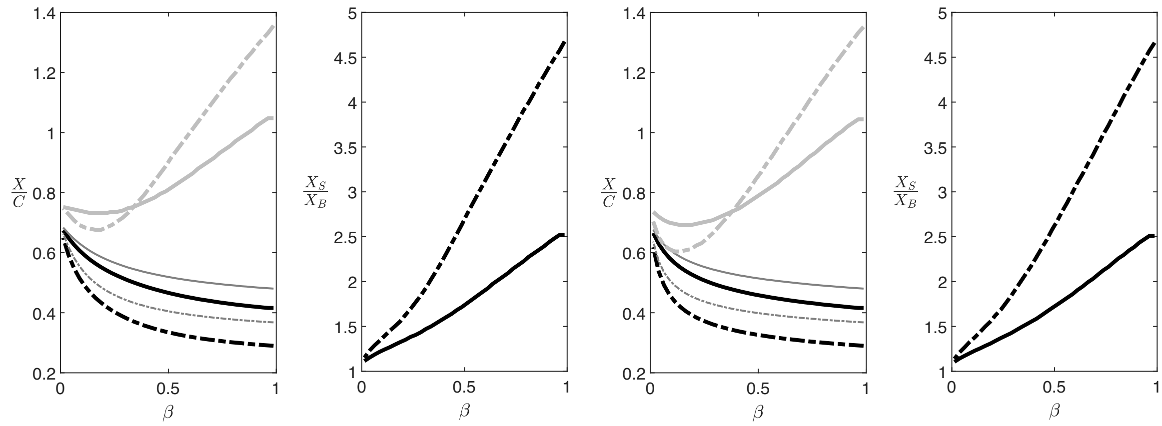
2.B.6 The exponential discount rate

Figure B.17 investigates the effects of increasing the exponential discount rate from $\rho = 0.05$ to $\rho = 0.1$ on entrepreneurial decision-making strategies. The dash-dotted lines illustrate the baseline scenario at $\rho = 0.05$, and the solid lines indicate the adjusted scenario at a higher rate of $\rho = 0.1$. Across various levels of present bias, an increase in the discount rate prompts both naive and sophisticated entrepreneurs to opt for default at increased levels of cash flow. This outcome aligns with expectations, as a heightened discount rate



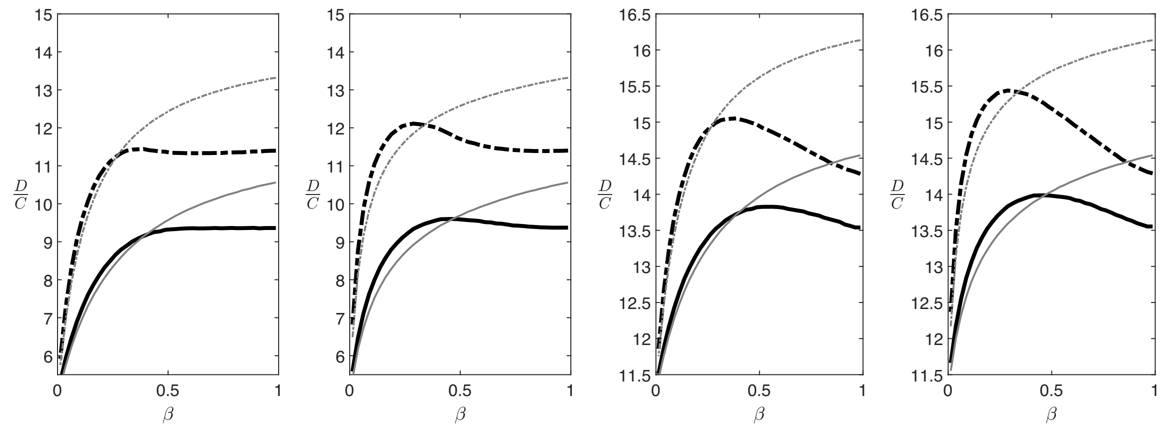
***(a)** Leverage, Soph.* ***(b)** Credit Spread, Soph.* ***(c)** Leverage, Naive* ***(d)** Credit Spread, Naive*

Figure B.16: Capital structure sensitivity to the tax rate. *This figure shows the sensitivity of the entrepreneur's chosen leverage and resulting credit spread to a decrease in the tax rate θ. The dashed-dotted lines provide the base case with $\theta = 0.2$ and the solid lines show the case with $\theta = 0.15$. The black lines represent the scenario where the entrepreneur can switch between volatility levels, while the thin grey lines indicate the case where the entrepreneur does not have access to risk management and the volatility is held constant at σ_L. Panel B.16a and Panel B.16c display the optimal leverage of the sophisticated and naive entrepreneur for different degrees β of the entrepreneur's present bias, while Panel B.16b and Panel B.16d plot the resulting credit spreads.*



***(a)** Strategies, Soph.* ***(b)** Relative Risk, Soph.* ***(c)** Strategies, Naive* ***(d)** Relative Risk, Naive*

Figure B.17: Strategy sensitivity to the entrepreneur's discount rate. *The plots illustrate how the entrepreneur's optimal strategies respond to an increase of the entrepreneur's discount rate ρ for different degrees β of present bias. The dashed-dotted lines represent the base case $\rho = 0.05$, while the solid lines show the case with a discount rate given by $\rho = 0.1$. Panels B.17a and B.17c display the risk-switch (thick light grey) and default (black) thresholds for both the sophisticated and naive entrepreneur and compares them to the default thresholds (thin dark grey) of an entrepreneur without access to risk management and constant volatility σ_L. Panels B.17b and B.17d plot the ratio of the risk-switch to the default threshold for the sophisticated and naive entrepreneur, respectively.*



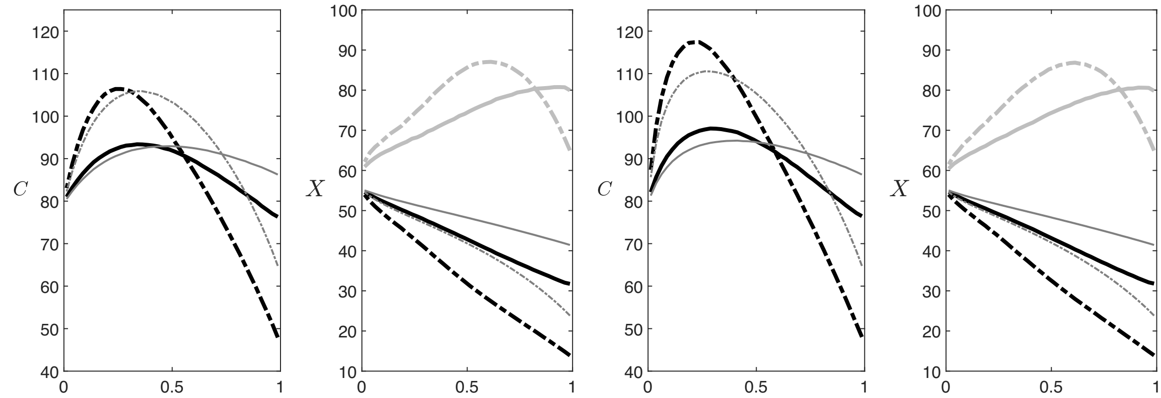
(a) Soph., $X/C = 0.8$ *(b) Naive, $X/C = 0.8$* *(c) Soph., $X/C = 1.2$* *(d) Naive, $X/C = 1.2$*

Figure B.18: Debt value sensitivity to the entrepreneur's discount rate. *This figure illustrates the sensitivity of the debt value to an increase in the entrepreneur's discount rate ρ, contrasting the scenario where the entrepreneur can switch between volatility levels (black) with the situation where the entrepreneur does not have access to risk management (thin grey) with constant volatility σ_L. The dashed-dotted lines provide the base case with $\rho = 0.05$ and the solid lines show the case with $\rho = 0.1$. Panel B.18a and B.18b present a scenario where the cash flow is below the nominal coupon, whereas Panels B.18c and B.18d depict the scenario where the cash flow exceeds the nominal coupon.*

diminishes the perceived value of the firm at any particular level of cash flow. Additionally, the observations generally reveal a decrease in the relative distance from the risk threshold to the default threshold as the discount rate rises. Therefore, an increased discount rate reduces the firm's willingness to increase the volatility. This phenomenon occurs because a heightened firm volatility introduces immediate costs, whereas the prospective benefits accrue more gradually.

Figure B.18 explores the impact of an increased entrepreneur's discount rate ρ on the value of debt across different levels of present bias β. The dash-dotted lines in these figures represent the baseline scenario with a tax rate of $\rho = 0.05$, and the solid lines show the scenario with an increased discount rate of $\rho = 0.1$. Across all levels of present bias, raising the discount rate consistently leads to a reduction in debt values. Prior findings indicated that a higher discount rate increases the default threshold but reduces the willingness to enhance the firm's risk. Although an elevated default threshold generally decreases debt value, a reduced inclination to increase the firm's risk would normally increase it. Yet, it is evident that the effect of the increased default threshold is more significant, resulting in an overall decrease in debt value.

The plots in Figure B.19 illustrate how the firm's endogenous optimal coupon choice and

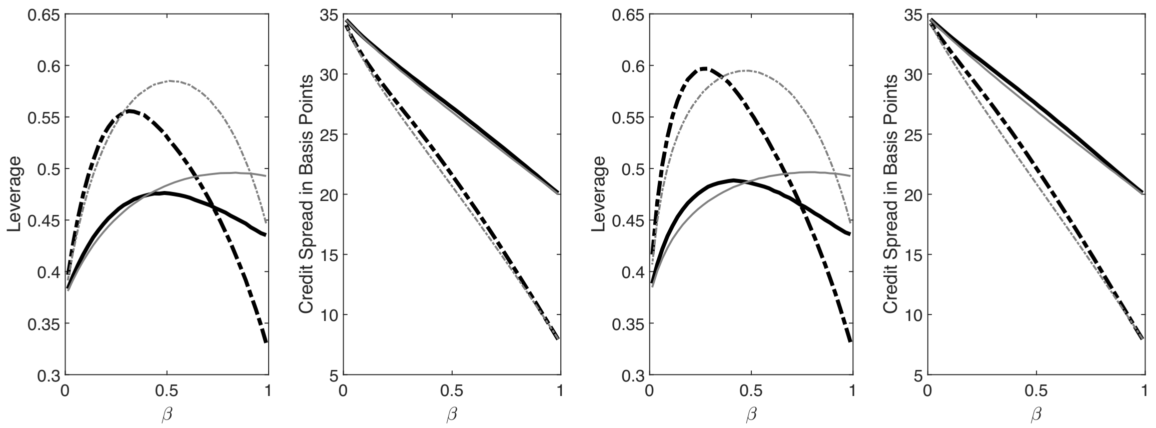


(a) Coupon, Soph. *(b) Strategies, Soph.* *(c) Coupon, Naive* *(d) Strategies, Naive*

Figure B.19: Coupon choice sensitivity to the entrepreneur's discount rate. *This figure shows the sensitivity of the entrepreneur's chosen coupon and the resulting strategies to an increase in the entrepreneur's discount rate ρ for different degrees β of the entrepreneur's present bias, when $X_0 = 100$. The dashed-dotted lines provide the base case with $\rho = 0.05$ and the solid lines show the case with $\rho = 0.1$. In Panel B.19a and B.19c, the coupon choice when the entrepreneur can switch between volatility levels (black) is compared to the coupon choice (thin grey) when volatility is held constant at σ_L. In Panel B.19b and B.19d, the resulting default (black) and risk-switch thresholds (thick light grey), when volatility can be changed, are compared to the default threshold (thin dark grey) when volatility is maintained constantly at σ_L.*

resulting optimal strategies change with an increase in the entrepreneur's discount rate. As before, the baseline scenario, represented by dashed-dotted lines with a duration $\rho = 0.05$ is contrasted with the case $\rho = 0.1$, indicated by solid lines in each of the panels. We observe that at lower degrees of the entrepreneur's present bias both sophisticated and naive entrepreneurs choose a higher coupon when their discount rate increases. On the other side, for entrepreneurs with a higher degree of present bias, the increase in the discount rate leads to a lower coupon choice. We have previously observed that an increase in the entrepreneur's discount rate decreases the value of debt, which should lead to unfavourable financing conditions and could make the entrepreneur initially choose a lower coupon. Despite that, a higher discount rate can lead to a higher desire in debt financing, since future coupon payments are weighted less by the entrepreneur. This effect is more significant for entrepreneur's with a smaller degree of bias as entrepreneurs with a stronger present bias already initially weighted the future coupon payments less.

Figure B.20 investigates the effect of raising the entrepreneur's discount rate from $\rho = 0.05$ to $\rho = 0.1$ on entrepreneurs' optimal capital structure. Our findings indicate that credit spreads rise for both sophisticated and naive entrepreneurs across all levels of present bias. In essence, the maximum interest rate the entrepreneur is ready to pay for an additional



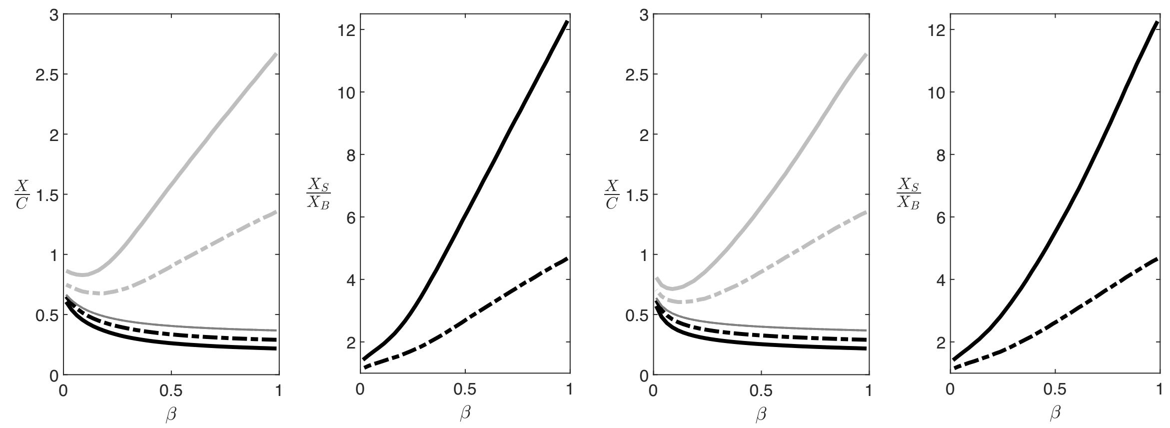
***(a)** Leverage, Soph.* ***(b)** Credit Spread, Soph.* ***(c)** Leverage, Naive* ***(d)** Credit Spread, Naive*

Figure B.20: Capital structure sensitivity to the entrepreneur's discount rate. *This figure shows the sensitivity of the entrepreneur's chosen leverage and resulting credit spread to an increase in the entrepreneur's discount rate ρ. The dashed-dotted lines provide the base case with $\rho = 0.05$ and the solid lines show the case with $\rho = 0.1$. The black lines represent the scenario where the entrepreneur can switch between volatility levels, while the thin grey lines indicate the case where the entrepreneur does not have access to risk management and the volatility is held constant at σ_L. Panel B.20a and Panel B.20c display the optimal leverage of the sophisticated and naive entrepreneur for different degrees β of the entrepreneur's present bias, while Panel B.20b and Panel B.20d plot the resulting credit spreads.*

dollar of debt grows with the discount rate. This outcome is expected since the equity value directly declines with a higher discount rate, whereas debt remains valued by debt holders at a rate of $r = 0.05$ and its value is only indirectly affected by the change in the entrepreneur's discount rate by alterations in the firm's strategies.

2.B.7 The upper volatility level

Until now, we have operated under the assumption that debt-holders are averse to the issuance of "junk" bonds, characterised by an entrepreneurial strategy of increasing the firm's cash flow volatility immediately after debt issuance. It has been presumed that debt-holders restrict the offering of coupons such that the entrepreneur's initial relative distance to the high volatility state is at least $\psi > 1$. Within previous parameter settings, the firm's optimal decision consistently favoured the issuance of non-junk bonds. Going forward, we will explore the ramifications of increasing the upper volatility limit on the firm's strategic decisions, the valuation of debt, and the choices regarding optimal capital structure. We observe that entrepreneurs with a moderate present bias will find it optimal to issue "junk" bonds, electing to increase cash flow volatility directly after debt issuance.



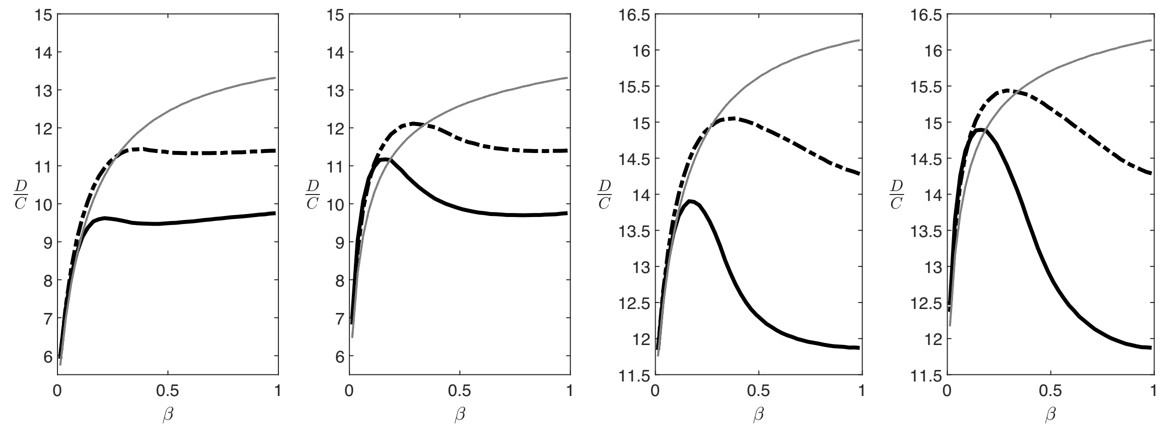
(a) *Strategies, Soph.* ***(b)*** *Relative Risk, Soph.* ***(c)*** *Strategies, Naive* ***(d)*** *Relative Risk, Naive*

Figure B.21: Strategy sensitivity to the upper volatility level σ_H. *The plots illustrate how the entrepreneur's optimal strategies respond to an increase in the upper volatility level σ_H for different degrees β of present bias. The dashed-dotted lines represent the base case $\sigma_H = 0.3$, while the solid lines show the case with a discount rate given by $\sigma_H = 0.4$, when the costs for increasing the volatility are fixed at $\phi\epsilon = 0.1$. Panels B.21a and B.21c display the risk-switch (thick light grey) and default (black) thresholds for both the sophisticated and naive entrepreneur and compares them to the default thresholds (thin dark grey) of an entrepreneur without access to risk management and constant volatility σ_L. Panels B.21b and B.21d plot the ratio of the risk-switch to the default threshold for the sophisticated and naive entrepreneur, respectively.*

The analysis of capital structure choices, particularly with regard to the existence of a leverage constraint, is further detailed here in Appendix 2.B.7. The outcomes of this analysis are then compared to scenarios where the leverage constraint is removed, which is discussed in Appendix 2.B.8. This comparative approach allows us to assess the impact of such constraints on the strategic financial decisions of the firm.

The equity value of an entrepreneur's firm is a convex function of the cash flow level. This convexity leads entrepreneurs to favour higher levels of volatility when it can be achieved without extra costs. Increased volatility in cash flows typically reallocates value from debt to equity. We have assumed that the entrepreneurs aim to maximise value by selecting the highest volatility they can attain without incurring additional costs, and we have denoted this volatility by σ_L. However, recall that entrepreneurs can choose to further elevate their cash flow volatility to $\sigma_H > \sigma_L$ by incurring costs at a rate $\phi\epsilon$ times the firm's cash flow.

Figure B.21 analyses the effects of elevating the upper volatility level from $\sigma_H = 0.3$ to $\sigma_H = 0.4$ on the strategic decisions of entrepreneurs. For both the benchmark case with the higher volatility level $\sigma_H = 0.3$ and the case with $\sigma_H = 0.4$, we fix the flow costs for increasing the volatility at $\phi\epsilon = 0.1$ times the cash flow. Since $\epsilon = \sigma_H^2 - \sigma_L^2$, we therefore

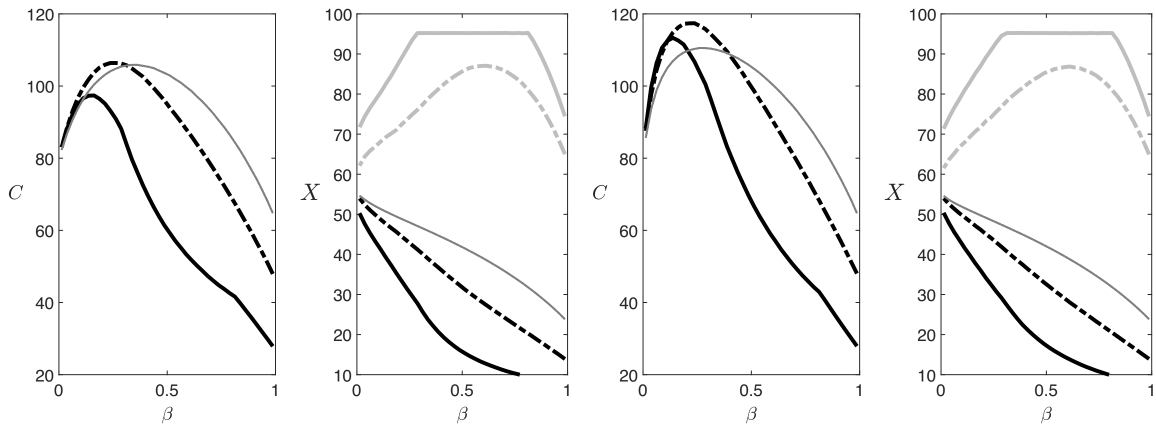


(a) Soph., $X/C = 0.8$　*(b) Naive, $X/C = 0.8$*　*(c) Soph., $X/C = 1.2$*　*(d) Naive, $X/C = 1.2$*

Figure B.22: Debt value sensitivity to the the upper volatility level σ_H. *This figure illustrates the sensitivity of the debt value to an increase in the upper volatility level σ_H, when the costs for increasing the volatility are fixed at $\phi\epsilon = 0.1$. It contrasts the scenario where the entrepreneur can switch between volatility levels (black) with the situation where the entrepreneur does not have access to risk management (thin grey) with constant volatility σ_L. The dashed-dotted lines provide the base case with $\sigma_H = 0.3$ and the solid lines show the case with $\sigma_H = 0.4$. Panel B.22a and B.22b present a scenario where the cash flow is below the nominal coupon, whereas Panels B.22c and B.22d depict the scenario where the cash flow exceeds the nominal coupon.*

set $\phi = 2$ in the benchmark case and $\phi = 5/6$ for the case with $\sigma_H = 0.4$. The dash-dotted lines show the baseline scenario with $\sigma_H = 0.3$, and the solid lines provide the adjusted scenario where entrepreneurs can achieve a higher upper volatility level of $\sigma_H = 0.4$ for the same price. Across various levels of present bias, an increase in the upper volatility level prompts both naive and sophisticated entrepreneurs to opt for default at lower cash flow levels. This adjustment is in line with theoretical expectations, given that entrepreneurs generally prefer higher volatility levels. Being able to reach a higher volatility level at the same cost makes enduring financial distress more appealing, hence lowering the threshold for default. Moreover, we observe that a higher upper volatility level increases the firm's threshold below which the firm increases its volatility. Thereby, firms that can achieve the higher volatility level of $\sigma_H = 0.4$ aim for the higher volatility level at a higher cash flow level. This especially means that firms who can achieve a higher upper volatility level are willing to pay more for the increase in volatility.

Figure B.22 examines the effects of increasing the upper volatility level, σ_H, on the value of debt, when the costs for increasing the volatility are fixed at $\phi\epsilon = 0.1$. In the figure, the dashed-dotted lines represent the baseline scenario with an upper volatility level of $\sigma_H = 0.3$, and the solid lines depict the scenario with an increased upper volatility of

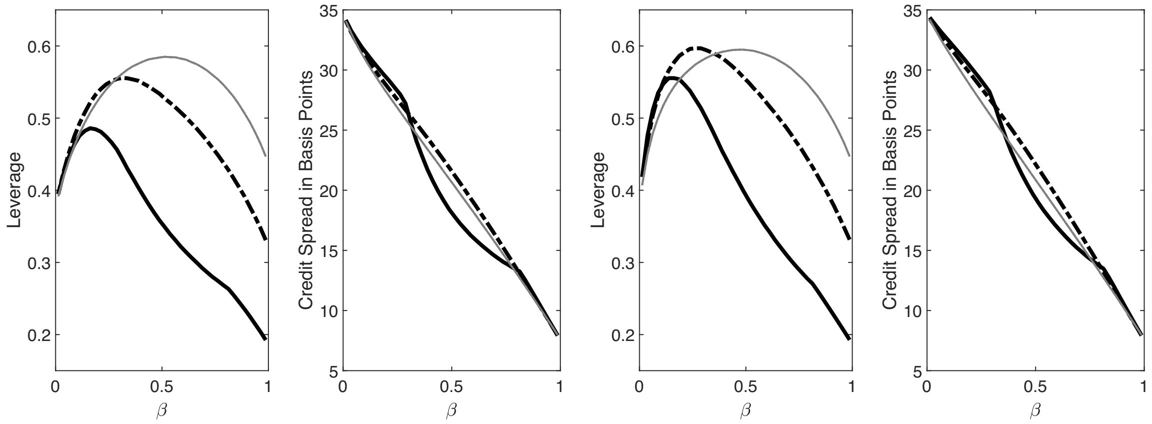


***(a)** Coupon, Soph.* ***(b)** Strategies, Soph.* ***(c)** Coupon, Naive* ***(d)** Strategies, Naive*

Figure B.23: Coupon choice sensitivity to the upper volatility level σ_H. *This figure shows the sensitivity of the entrepreneur's chosen coupon and the resulting strategies to an increase in the upper volatility level σ_H for different degrees β of the entrepreneur's present bias, when the costs for increasing the volatility are fixed at $\phi\epsilon = 0.1$ and $X_0 = 100$. The dashed-dotted lines provide the base case with $\sigma_H = 0.3$ and the solid lines show the case with $\sigma_H = 0.4$. In Panel B.23a and B.23c, the coupon choice when the entrepreneur can switch between volatility levels (black) is compared to the coupon choice (thin grey) when volatility is held constant at σ_L. In Panel B.23b and B.23d, the resulting default (black) and risk-switch thresholds (thick light grey), when volatility can be changed, are compared to the default threshold (thin dark grey) when volatility is maintained constantly at σ_L.*

$\sigma_H = 0.4$. Across a spectrum from low to medium-high present bias, increasing the upper volatility level consistently results in a decrease in debt values. Previous insights from Figure B.21 suggested that an increased upper volatility level reduces the default threshold, which theoretically should have a positive effect on the value of debt. Conversely, a higher upper volatility also raises the threshold below which the firm decides to increase cash flow volatility and the actual volatility itself, both factors reducing debt value. The analysis indicates that the reductive impact of increased volatility on debt value outweighs the potential positive effects of a lower default threshold, resulting in an overall reduction in the value of debt.

The plots in Figure B.23 illustrate how the firm's endogenous optimal coupon choice and the resulting optimal strategies change with an increase in the upper volatility level, while the costs of increasing the volatility are unchanged. As before, the baseline scenario, represented by dashed-dotted lines with an upper volatility level $\sigma_H = 0.3$ is contrasted with the case $\sigma_H = 0.4$, indicated by solid lines in each of the panels. We observe that at medium degrees of the entrepreneur's present bias both for sophisticated and naive entrepreneurs the leverage constraint prevents the entrepreneurs to issue higher amounts of debt. The leverage restriction induces a minimum initial relative distance to the high

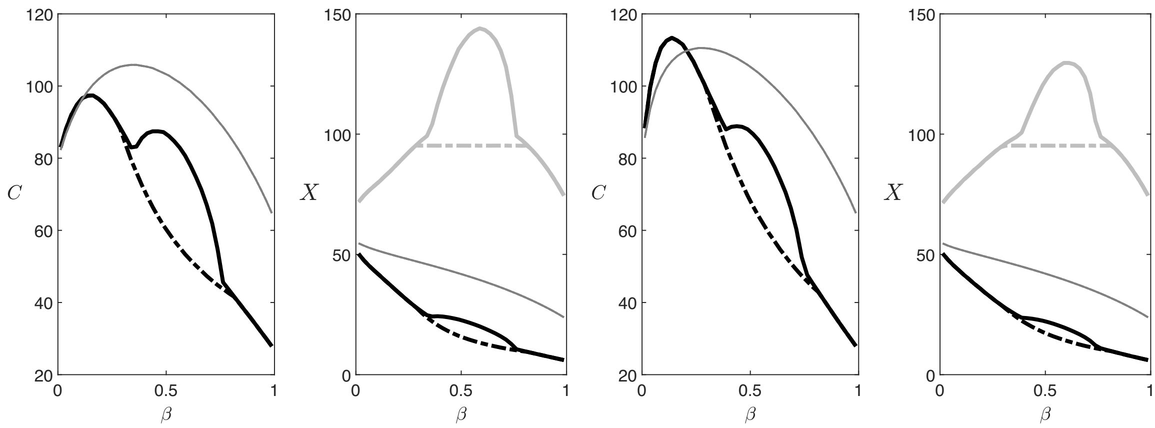


***(a)** Leverage, Soph.* ***(b)** Credit Spread, Soph.* ***(c)** Leverage, Naive* ***(d)** Credit Spread, Naive*

Figure B.24: Capital structure sensitivity to the upper volatility level σ_H. *This figure shows the sensitivity of the optimal leverage and resulting credit spread to an increase in the entrepreneur's upper volatility level σ_H, when the costs for increasing volatility are fixed at $\phi\epsilon = 0.1$. The dashed-dotted lines provide the base case with $\sigma_H = 0.3$ and the solid lines show the case with $\sigma_H = 0.4$. The black lines represent the scenario where the entrepreneur can switch between volatility levels, while the thin grey lines indicate the case where the entrepreneur does not have access to risk management and the volatility is held constant at σ_L. Panel B.24a and Panel B.24c display the optimal leverage of the sophisticated and naive entrepreneur for different degrees β of the entrepreneur's present bias, while Panel B.24b and Panel B.24d plot the resulting credit spreads.*

volatility state of $\psi = 1.05 > 1$, prohibiting the issuance of "junk" bonds which would which would exhibit an immediate increase in cash flow volatility after debt issuance.

Figure B.24 investigates the effect of raising the upper volatility level from $\sigma_H = 0.3$ to $\sigma_H = 0.4$ on entrepreneurs' optimal capital structure, when the costs for increasing the volatility are fixed at $\phi\epsilon = 0.1$ times the cash flow level. The baseline scenario (dashed-dotted lines) with $\sigma_H = 0.3$ is compared to the scenario with an increased upper volatility level $\sigma_H = 0.4$ (solid lines). We generally observe that initial optimal leverage decreases when the upper volatility level is increased for both the sophisticated and naive entrepreneur within a spectrum from low to medium-high present bias. This result stems from the fact that within this spectrum, the value of debt heavily decreases by the increased upper volatility level. Additionally, both naive and sophisticated entrepreneurs exhibit a convex dip in their optimal leverage and respective credit spread curves at medium levels of present bias. This deviation is influenced by the leverage constraint. In the absence of this constraint, the optimal strategy for these entrepreneurs would involve issuing high-volatility "junk" bonds. Appendix 2.B.8 provides further discussion of the results of Figure B.21 and compares the entrepreneur's optimal leverage to a scenario where the leverage constraint is removed.

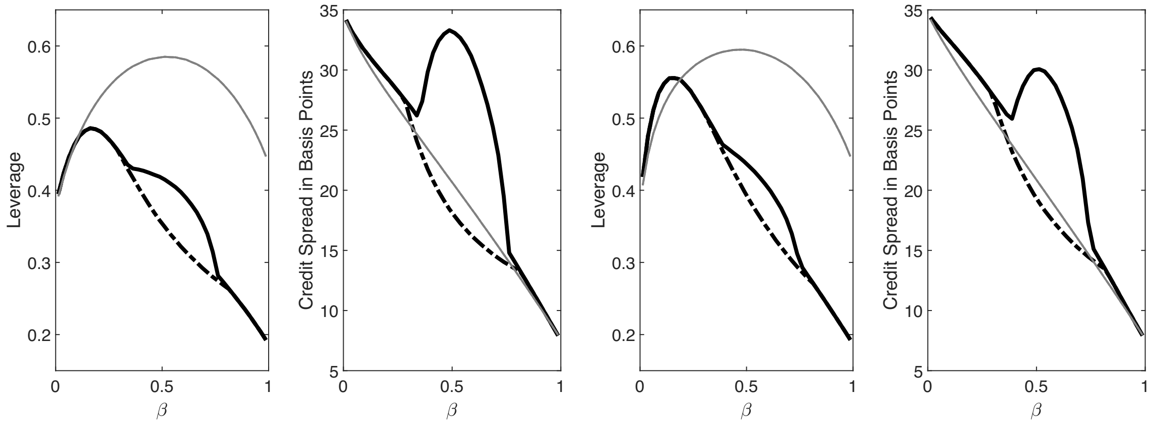


***(a)** Coupon, Soph.* ***(b)** Strategies, Soph.* ***(c)** Coupon, Naive* ***(d)** Strategies, Naive*

***Figure B.25: Impact of leverage constraint removal on coupon choice.** The figure shows the effect of a removal of the leverage constraint on coupon choice and the resulting strategies for different degrees β of the entrepreneur's present bias. It assumes the baseline parametrisation with an upper volatility level of $\sigma_H = 0.4$ The dashed-dotted lines provide the case with the leverage constraint in place and the solid lines show the case without the constraint. In Panel B.25a and B.25c, the coupon choice when the entrepreneur can switch between volatility levels (black) is compared to the coupon choice (thin grey) when volatility is held constant at σ_L. In Panel B.25b and B.25d, the resulting default (black) and risk-switch thresholds (thick light grey), when volatility can be changed, are compared to the default threshold (thin dark grey) when volatility is maintained constantly at σ_L.*

2.B.8 The leverage constraint

Figure B.25 compares the effect of the presence and absence of the leverage constraint on the firm's optimal coupon choice as well as the resulting optimal strategies after debt issuance. Further, Figure B.26 analyses the effect of the presence and absence of the leverage constraint on the firm's leverage ratio and the resulting credit spreads. In this particular analysis, we use the parameters from the base case in the previous sections, except the upper volatility level is given by $\sigma_H = 0.4$ and the costs for increasing the volatility are given by $\phi\epsilon = 0.1$ times the cash flow. The leverage restriction mandates a minimum initial distance to the high volatility state of $\psi > 1$, prohibiting the issuance of "junk" bonds which would entail an immediate increase in cash flow volatility post-debt issuance. The dashed-dotted lines represent scenarios with the leverage constraint in place, while the solid lines depict scenarios without the constraint. As noted previously in the analysis of Figure B.24, entrepreneurs who exhibit a medium level of present bias are notably influenced by the leverage constraint. Without this constraint, they tend to issue greater amounts of debt, increase their cash flow volatility immediately with debt issuance, and consequently face higher credit spreads. Conversely, the capital structure choices for



(a) Leverage, Soph. *(b) Credit Spread, Soph.* *(c) Leverage, Naive* *(d) Credit Spread, Naive*

Figure B.26: Impact of leverage constraint removal on capital structure. *This figure shows the difference in the chosen leverage and resulting credit spread when the leverage constraint is removed. It assumes the baseline parametrisation with an upper volatility level of $\sigma_H = 0.4$. The dashed-dotted lines provide the base case with the leverage constraint, while solid lines show the case without the constraint. The black lines provide the case where the entrepreneur can switch between volatility levels, while the thin grey lines indicate the case where the entrepreneur has no access to risk management and volatility is held constant at σ_L. Panel B.26a and Panel B.26c display the optimal leverage of the sophisticated and naive entrepreneur for different degrees β of the entrepreneur's present bias, while Panel B.26b and Panel B.26d plot the resulting credit spreads.*

entrepreneurs with either low or high present bias remain unaffected by this constraint.

This observation prompts further investigation into why entrepreneurs with a medium present bias are inclined to issue "junk" bonds, unlike entrepreneurs with lower or higher present biases. From the insights of Figure B.21, it appears that at any set coupon level, the optimal threshold for increasing cash flow volatility is decreasing in the strength of the entrepreneur's present bias. On the other hand, the preference for debt financing increases with the strength of the present bias and the firm's optimally chosen coupon level increases in the degree of the entrepreneur's present bias. These dynamics counterbalance each other to a certain degree. Entrepreneurs without or with a small present bias have a strong preference for risk but opt for the lowest coupon rates and thus, position their optimal risk threshold below their initial cash flow. Conversely, those with a strong present bias choose higher coupon rates while having a small preference for increasing the volatility, also setting their risk threshold beneath the initial cash flow level. In scenarios lacking a leverage constraint, entrepreneurs with a moderately strong present bias possess adequate motivation for debt financing, maintaining their willingness to pursue higher volatility at a sufficiently high cash flow level that justifies increasing volatility after issuing debt.

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Chapter 3

Maturity Choice, Debt Rollover, and Asymmetric Information

with Christian Hilpert, Stefan Hirth and Alexander Szimayer

Abstract

This paper investigates debt maturity choices under default risk and feedback effects on competitive debt markets. Debt market investors face asymmetric information about the true asset value of the firm. The firm strategically delays default to signal credit quality backed by a high true asset value. Subsequently, surviving apparent distress carries value: debt market investors, observing non-default, offer improved financing conditions once the firm refinances its debt. Consistent with empirical evidence, optimal maturity decreases in the degree of uncertainty about the firm's true asset value.

3.1 Introduction

This paper examines the value of a firm's debt with default risk and finite maturity in a continuous-time framework. We account for information asymmetry by assuming that investors on the financial market can observe the firm's true asset value only imperfectly. The market learns continuously about the firm's true asset value, and debt replacement occurs at the market's fair price. Firms optimally delay default and can thereby signal a higher true asset value, which reduces their future costs of debt.

Considering information asymmetry, we examine firm's optimal maturity choice. Typically, information asymmetry gives rise to signaling issues. A firm that knows it is of higher quality may want to issue short-term debt to signal its type (Flannery, 1986). We are interested in the effect of information asymmetry on maturity choice beyond what can be explained by signaling motivations. Therefore, we assume that upon issuance, both parties, the firm's insiders and its potential creditors, have the same expectations about the firm's quality. Right after issuance, the firm's insiders learn the firm's type. Creditors do so only over time, and as a consequence of strategic actions. The latter, in our context, consist of the firm's decision to roll over its debt rather than to default.

Our main finding is that optimal maturity decreases with the degree of initial uncertainty about the firm's asset value, which evolves into information asymmetry once the firm learns its own quality. This is due to the following trade-off: On the one hand, short maturity is costly, as it means higher interest for rollover at low asset values, which implies earlier default and higher expected bankruptcy costs. On the other hand, firms of all quality are overvalued immediately prior to default under information asymmetry. This results in lower effective interest payments during periods of severe distress, which delays default and thereby reduces expected bankruptcy costs. The second effect becomes more important for higher asymmetric information, and thus, optimal maturity decreases in information asymmetry.

According to Colla et al. (2020), there are four factors that determine firms' choice of debt maturity, namely, agency costs (Myers, 1977), taxes (Brick and Ravid, 1985; Kane et al., 1985), the maturity of firm's assets (Morris, 1976), and asymmetric information. We focus on the latter. In that respect, Flannery (1986) shows a separating equilibrium in which borrowers with favourable private information issue short-term debt and roll it over, whereas bad-quality borrowers use long-term debt. Diamond (1991) considers liquidation risk, which implies that borrowers choose debt maturity dependent on credit quality. Here, both high and low credit quality firms borrow short term, while medium credit quality firms borrow long term. We also focus on the impact of asymmetric information on firms' maturity choice, however, with the twist that asymmetric information first arises after debt issuance.

In terms of modelling, we build on seminal work by Leland (1994) and co-authors. Leland (2019)[1] and Leland and Toft (1996); Leland (1998); Goldstein et al. (2001) introduce models that include finite maturity and endogenous default, which is optimally triggered by equity holders at low asset values. However, these models neither allow for asymmetric information nor account for bond liquidity in secondary markets, and therefore suggest the optimal maturity to be infinite. By contrast, He and Milbradt (2014) discuss that shorter-maturity bonds are more liquid on secondary markets - a benefit that can be valued by debt holders - which, when incorporated into the modelling framework, can lead to a finite optimal debt maturity.

Substantial empirical evidence indicates that corporate bond yield spreads contain a liquidity premium exceeding what can be attributed to credit risk alone, see (among others): Chen et al. (2018), Helwege et al. (2014), Huang and Huang (2012) and Chen et al. (2007). Furthermore, studies such as He and Milbradt (2014), Friewald et al. (2012), Dick-Nielson et al. (2012), and Bao et al. (2011) find that the liquidity premium increases with bond maturity. To capture this effect in a tractable manner, we introduce a maturity-dependent costs on the side of debt-holders.

Duffie and Lando (2001) elaborate a continuous-time model to valuate risky debt under asymmetric information regarding the firm's true asset value.[2] Secondary markets of corporate debt dynamically receive exogenous news about the unknown true asset value. Unlike ours, the model does not include feedback effects, since the firm's credit rating does not affect the firm's default decision. Duffie and Lando (2001) restrict their analysis of asymmetric information to firm's with debt of infinite maturity, which conflicts with empirical research, stating that information asymmetry is related to debt of shorter maturity. In contrast to Duffie and Lando (2001), our paper focuses on the reduction of information asymmetry resulting from strategic firm actions. The firm has incentive

[1] Leland (2019) is the published version of the long-term working paper typically cited as "Leland (1994b)".

[2] Other subsequent papers on debt valuation under asymmetric information include Chen (2003); Shibata and Tian (2010, 2012).

to delay bankruptcy and therefore signal a higher true asset value. Delaying bankruptcy improves a firm's financing prospects. In our model, the firm's credit rating affects their optimal default decision. Therefore, our model can be considered as a further development of Duffie and Lando (2001): We consider debt of arbitrary finite maturity and include feedback effects of information asymmetry.

Hilpert et al. (2024) have provided the first continuous-time credit risk model that is incorporating feedback of dynamically adjusted information asymmetry, and we model information asymmetry in their fashion. They use performance sensitive debt, where debt payments depend on the firm's credit rating. Nevertheless, their setting does not involve the valuation of specific debt maturities. We model learning from strategic actions, as in Grenadier and Malenko (2011), Grenadier et al. (2016) and Hilpert et al. (2024).

Empirical research has analysed conditions of asymmetric information. Among others, Barclay and Smith (1995), Stohs and Mauer (1996), Ozkan (2000), Berger et al. (2005), García-Teruel et al. (2010) and Custódio et al. (2013) have found that greater information asymmetries are related to shorter debt maturities. Those findings are well in line with our model prediction.

3.2 A Model of Debt Rollover under Asymmetric Information

In this section, we present a structural model of credit risk of a firm's debt under dynamic information and feedback effects. The firm continuously rolls over parts of its debt, creating a feedback channel between the firm and its creditors who constantly assess the firm's credit quality. The firm's management acts in the interest of its shareholders and optimises the equity value as the total asset value net of debt. The firm rolls over its debt on a competitive debt market. Debt market investors are price takers who require zero profits in expectation for any newly issued debt and gradually update their beliefs about the firm's default threshold as information unfolds.

3.2.1 Information structure, debt rollover and learning from distress

Consider a firm that acquires productive assets, whose true value V follows a continuous diffusion process with risk-neutral dynamics

$$\mathrm{d}V_t = \mu\, V_t\, \mathrm{d}t + \sigma\, V_t \mathrm{d}B_t. \tag{3.1}$$

Here μ and $\sigma > 0$ describe the assets' drift and volatility, respectively. The constant risk-free rate is $r > \mu$, and B is a standard Brownian motion under the risk-neutral measure.

The firm finances these assets by issuing debt with a total principal P, selecting a maturity M. We consider a stationary debt policy whereby, at any point in time, the firm maintains a constant total debt principal P, paying a constant coupon rate C. As in Leland (2019), the firm continuously rolls over a fraction $m = 1/M$ of its debt, meaning it continuously retires debt at the rate mP and simultaneously issues new debt of equal coupon, principal and seniority.

Before initial debt issuance, both the firm and debt-holders share the same uncertainty regarding the firm's true asset value. Both observe the firm's true asset value V only imperfectly due to a persistent multiplicative distortion $\tilde{\theta}$, independent of B. Thus, they observe a distorted asset value W, which relates to the true asset value as

$$V_t = \tilde{\theta}\, W_t. \tag{3.2}$$

Immediately after debt issuance, the firm learns its true type θ while the market remains uninformed. Note that the the firm's true quality increases in its type θ. The observed asset value W exhibits the same stochastic dynamics - namely, the same drift and volatility - as the true asset value V, due to the multiplicative structure given in (3.2). Therefore, observing W does not reveal information about the distortion $\tilde{\theta}$. Moreover, we assume the density ϕ of $\tilde{\theta}$ to be nonnegative and continuous on some support $[\underline{\theta}, \overline{\theta}]$ with $0 < \underline{\theta} < \overline{\theta} < \infty$. Suppose the firm's type-dependent default strategy is given by default thresholds $W_B = (W_B(\theta))_{\theta\in\Theta}$ and thus the optimal default strategy is a first hitting time[3]

$$\tau(\theta) = \inf\{t \geq 0 : W_t \leq W_B(\theta)\}. \tag{3.3}$$

[3] We show that this assumption is harmless. Under reasonable assumptions on debt valuation, the firm's default strategy indeed turns out to be a cut-off strategy; see Appendix 3.A.

Accordingly, the running minimum of the observed asset value, $\underline{W}$, with

$$\underline{W}_t = \inf_{0 \leq s \leq t} W_s \tag{3.4}$$

captures the firm's default information by determining which type θ would have already defaulted at time $t \geq 0$. In the following discussion the dependence on the specific choice of W_B is suppressed. Note that the optimal strategy is later indicated by $W_B^\star$. As is standard in a Markovian setup, $\mathbb{E}_W$ denotes the expectation conditional on the initial value $W_0 = W$. When the quantity inside the expectation depends on W and its running minimum $\underline{W}$, we write $\mathbb{E}_{(W,\underline{W})}$ accordingly. Now, from the perspective of the firm, the equity value depends on its type θ and is given by holding the asset minus debt payments until default

$$E(\theta, W, \underline{W}) = \theta W - \mathbb{E}_{(W,\underline{W})}\left[\int_0^{\tau(\theta)} e^{-rt}\left(C{+}mP{-}d(W_t, \underline{W}_t)\right) \mathrm{d}t + e^{-r\tau(\theta)}\theta W_B(\theta)\right], \tag{3.5}$$

where, prior to default, debt-holders receive the coupon stream C and the proceeds from rolling over the debt. Rolling over debt at rate m entails repaying debt with principal P while issuing new debt of the same principal and maturity structure, receiving its market value $d(W, \underline{W})$.

In a setting inspired by the framework of Duffie and Singleton (1999), we account for certain liquidity-related costs for debt-holders that are independent of the inherent firm's credit risk. Empirical studies such as Friewald et al. (2012), Dick-Nielson et al. (2012) and Bao et al. (2011) document that the liquidity premium embedded in bond yields increases with bond maturity. To capture this empirically observed effect in a tractable way, as proposed by He and Milbradt (2014), we introduce maturity-dependent costs on the side of debt-holders. These costs not only reflect preference for liquidity but also may capture other benefits associated with shorter maturities, such as increased flexibility for debt-holders.

Specifically, we model these costs as a continuous cash flow amounting to $k(m)P$, where P is the debt principal. This flow cost captures the maturity-dependent costs for debt-holders that go beyond credit risk and $k(m)$ is assumed to be decreasing in the rollover rate m and thus increasing in the average maturity $M = m^{-1}$, which reflects lower costs associated with shorter maturities.

Now, from the perspective of a firm with type θ, the total value of debt is given by

$$D(\theta, W, \underline{W}) = \mathbb{E}_{(W,\underline{W})}\left[\int_0^{\tau(\theta)} e^{-rt}\left(C{+}mP{-}k(m)P{-}d(W_t, \underline{W}_t)\right) \mathrm{d}t + e^{-r\tau(\theta)}(1{-}\alpha)\theta W_B(\theta)\right], \tag{3.6}$$

where, if bankruptcy occurs, a fraction $0 \leq \alpha \leq 1$ of value will be lost to bankruptcy costs, leaving debt-holders with value $(1-\alpha)\,\theta\,W_B(\theta)$ and stockholders with nothing. Taken together, for $W_B(\theta) < \underline{W} \leq W$, the firm value is given by the sum of equity and debt

$$v(\theta, W) = E(\theta, W, \underline{W}) + D(\theta, W, \underline{W}) = \theta\,W - (BC(\theta, W) + MC(\theta, W))\,, \tag{3.7}$$

which can further be represented as the asset value minus the sum of bankruptcy costs and maturity costs with

$$BC(\theta, W) = \alpha\,\mathbb{E}_W\left[e^{-r\,\tau(\theta)}\,\theta\,W_B(\theta)\right] \text{ and } MC(\theta, W) = \mathbb{E}_W\left[\int_0^{\tau(\theta)} e^{-r\,t}\,k(m)P\,\mathrm{d}t\right]. \tag{3.8}$$

The market value of newly issued debt depends on the current observed asset value W and its running minimum $\underline{W}$. Note that cash flows to debt-holders generated from newly issued debt will include future coupon payments plus principal repayments minus maturity-dependent costs of $m\,e^{-mt}(C + mP - k(m)P)$ until bankruptcy occurs. Using the law of iterated expectation conditioning on θ, the rolled-over debt can be computed by the discounted expected payoff to debt-holders $d(\theta, W)$ for each possible firm type θ, weighted by the corresponding probability $\phi(\theta, \underline{W})$, which is continuously updated through the information generating process $\underline{W}$:

$$d(W, \underline{W}) = \int_\Theta d(\theta, W)\,\phi(\theta, \underline{W})\,\mathrm{d}\theta \tag{3.9}$$

with

$$d(\theta, W) = m\mathbb{E}_W\left[\int_0^{\tau(\theta)} e^{-(r+m)t}\,(C + mP - k(m)P)\,\mathrm{d}t + e^{-(r+m)\tau(\theta)}(1-\alpha)\,\theta\,W_B(\theta)\right] \tag{3.10}$$

$$\phi(\theta, \underline{W}) = \frac{\mathbb{1}_{\{W_B(\theta) < \underline{W}\}}\,\phi(\theta)}{\int_\Theta \mathbb{1}_{\{W_B(\theta') < \underline{W}\}}\,\phi(\theta')\,\mathrm{d}\theta'}\,, \quad \text{for } \theta \in \Theta. \tag{3.11}$$

To illustrate how learning and belief updating by debt holders affects debt valuation, we present an example in Figure 3.1. The figure illustrates the evolution of observed assets and the corresponding debt valuation for a sample trajectory of the observed asset value. The solid black line represents the firms' asset value W as observed by the debt market and the dashed black line captures its running minimum $\underline{W}$. Initially, debt holders expect the firm's default threshold to fall in $[W_B(\underline{\theta}), W_B(\overline{\theta})] = [44.5, 67.5]$, displayed by the horizontal grey lines. The dotted horizontal thin line represents the upper bound for the default threshold $W_B(\overline{\theta}) = 67.5$ and the solid horizontal thin line depicts the lower bound with

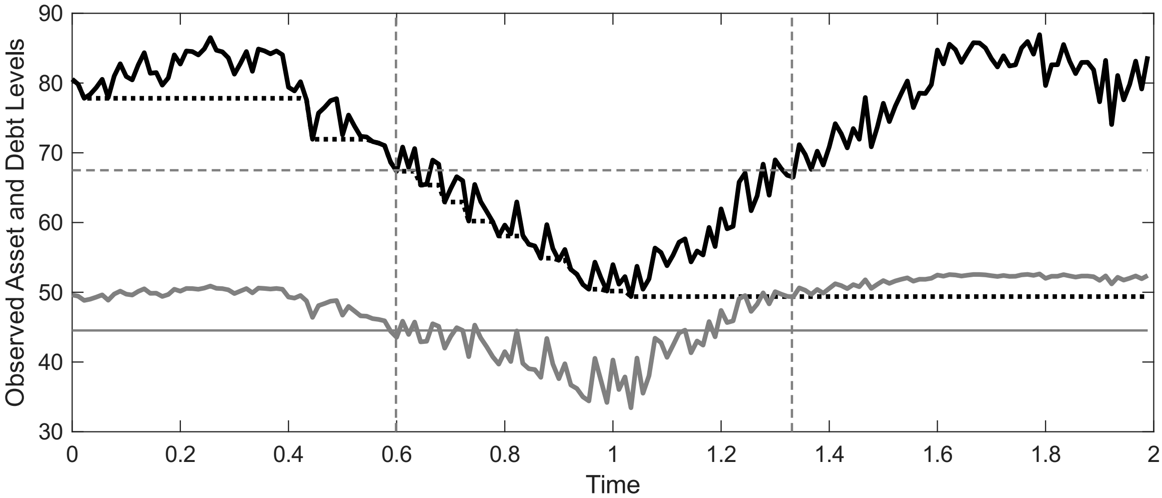


Figure 3.1: Observed asset and debt value trajectory *This figure depicts the path of the observed asset value W_t (solid black line) and its running minimum $\underline{W}_t$ (dotted black line) over the time period $t \in [0, 2]$. Debt holders observe W_t and initially believe the true asset value $\tilde{\theta}W_0$ is uniformly distributed on the interval $[\underline{\theta}W_0, \overline{\theta}W_0]$ with $\underline{\theta} = 0.75$ and $\overline{\theta} = 1.25$. They expect the firm's default threshold - given in terms of the observed asset value W_t - to be within $[W_B(\underline{\theta}), W_B(\overline{\theta})] = [44.5, 67.5]$. As time progresses and no default occurs, debt-holders update their belief about the firm's quality and revise their estimate of the total outstanding debt $d(W_t, \underline{W}_t)/m$ (solid grey line). Thin vertical dotted grey lines at $t_1 = 0.60$ and $t_2 = 1.33$ illustrate points in time where the debt value corresponds to the same observed asset value, allowing for comparison of the firm's perceived state before and after market learning. Relevant parameters used for this plot are: $P = 50$, $C = 2$, $m = 1/20$, $\mu = 0.015$, $\sigma = 0.2$, $r = 0.03$, $k(m) = 0.0002/m$ and $\alpha = 0.5$.*

$W_B(\underline{\theta}) = 44.5$.

The grey line traces the expected value of the firm's outstanding debt, reflecting how this value evolves with the observed firm value. The figure notably illustrates the learning process of debt holders: At time $t_1 = 0.6$, the observed asset value first deteriorates to a level where certain low quality firm types would optimally default, which both decreases the overall debt value and increases its volatility, as the debt value becomes sensitive to new information under distress. Between $t_1 = 0.6$ and $t_2 = 1.33$, the observed asset value declines further while the firm does not default.

Subsequently, at $t_2 = 1.33$, the observed asset value recovers to its level from time $t_1 = 0.6$. However, since firm's of low or medium quality would have defaulted during the period of distress, surviving firms are now perceived as being of higher quality. This updating of beliefs leads to an increase in the debt value for the same observed fundamentals.

Recall the equity value from Equation (3.5). Equity holders retain the total asset value until default, at which point it is transferred to debt holders. Thus, equity benefits from holding the asset and receives financial gains quantified by the payout rate $\delta = r - \mu$ up to the time of default. Therefore,[4]

$$E(\theta, W, \underline{W}) = \mathbb{E}_{(W,\underline{W})}\left[\int_0^{\tau(\theta)} e^{-r\,t}\left(\theta\,\delta\,W_t + d(W_t, \underline{W}_t) - (C + m\,P)\right)\,\mathrm{d}t\right]. \tag{3.12}$$

We impose the following requirement on the value of continuously reissued debt. If a firm experiences an exogenous increase in its asset value, this surplus should lead to higher rollover proceeds in all possible future scenarios, relative to the case without such an increase. Sufficient conditions are:

$$d(W, \underline{W}) \;\geq\; d(W, \underline{W}'), \quad \text{for } 0 \leq \underline{W} \leq \underline{W}' \leq W, \tag{3.13}$$

$$d(W, \underline{W}) \;\leq\; d(\lambda W, \lambda \underline{W}), \; \text{for } 0 \leq \underline{W} \leq W \text{ and } \lambda \geq 1. \tag{3.14}$$

The first condition in (3.13) states that the value of the rollover debt decreases with the running minimum of the total asset value. Survival of apparent distress and subsequent recovery to a previous asset value signals higher firm quality and reduces information asymmetry, which is then reflected in a higher valuation of the rollover debt. The second condition in (3.14) requires that a uniform scaling up of the firm's asset path cannot reduce

[4] Alternatively, the equality can be established using stochastic calculus. Observe that $e^{-r\,\tau(\theta)}\,W_B(\theta) = e^{-r\,\tau(\theta)} W_{\tau(\theta)}$, and application of the product rule yields the desired result.

the value of rollover debt. Thus, if the asset value reaches a new historical minimum, the firm should not be able to refinance on better terms simply because its entire asset trajectory has shifted downward.

Suppose the conditions in (3.13–3.14) hold. Then the firm's optimal default strategy is indeed a threshold strategy $\tau^\star(\theta) = \inf\{t \geq 0 : W_t \leq W_B^\star(\theta)\}$, where $W_B(\theta)$ for $\theta \in \Theta$ is strictly decreasing in θ, see Proposition 1 in Appendix 3.A. The optimal default strategy satisfies the smooth pasting condition

$$\frac{\partial E}{\partial W}(\theta, W_B^\star(\theta), W_B^\star(\theta)) = 0\,, \text{ for } \theta \in \Theta\,. \tag{3.15}$$

We show in Appendix 3.B that the equity value can be computed in closed-form:

$$\begin{aligned} E(\theta, W, W_B^\star(\theta)) = {} & \theta\, W - \theta\, W_B^\star(\theta) \left(\frac{W}{W_B^\star(\theta)}\right)^{-x} - \left(\frac{C{+}mP(1{+}k(m)/r)}{r+m}\right)\left(1 - \left(\frac{W}{W_B^\star(\theta)}\right)^{-x}\right) \\ & + \frac{\hat{g}^\star(\theta)\, W_B^\star(\theta)^{-y}}{m} \left(\left(\frac{W}{W_B^\star(\theta)}\right)^{-x} - \left(\frac{W}{W_B^\star(\theta)}\right)^{-y}\right)\,. \end{aligned} \tag{3.16}$$

with

$$x = \frac{1}{\sigma^2}\left(\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\,r\,\sigma^2}\right) \tag{3.17}$$

$$y = \frac{1}{\sigma^2}\left(\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2(r+m)\sigma^2}\right) \tag{3.18}$$

$$\hat{g}^\star(\theta) = \frac{\displaystyle\int_\theta^{\overline{\theta}} \phi(\theta') \left(m(1-\alpha)\theta'\, W_B^\star(\theta')^{y+1} - \frac{m\,(C{+}(m{-}k(m))P)}{r+m} W_B^\star(\theta')^y\right) \mathrm{d}\theta'}{\Phi(\theta)}\,, \tag{3.19}$$

where $\Phi(\theta) = \int_\theta^{\overline{\theta}} \phi(\theta')\,\mathrm{d}\theta'$, for $\theta \in \Theta$. Moreover, the value of rollover debt is given by

$$d(W, W_B^\star(\theta)) = \frac{m(C + (m - k(m))P)}{r+m} + W^{-y}\hat{g}^\star(\theta) \text{ for } W \geq W_B^\star(\theta) \text{ and } \theta \in \Theta. \tag{3.20}$$

The smooth-pasting condition in (3.15) becomes

$$0 = \frac{\partial E}{\partial W}(\theta, W_B^\star(\theta), W_B^\star(\theta))$$

$$= \quad (1+x)\,\theta - \left(x\,\frac{C + mP(1+k(m)/r)}{(r+m)} - \frac{\hat{g}^\star(\theta)\, W_B^\star(\theta)^{-y}}{m}\,(y-x) \right) W_B^\star(\theta)^{-1}\,.$$

Multiplying by $W_B^\star(\theta)$ and solving for the latter gives

$$W_B^\star(\theta) = \frac{1}{\theta} \left(\frac{x}{1+x}\,\frac{C + mP(1+k(m)/r)}{(r+m)} - \frac{y-x}{1+x}\,\frac{\hat{g}^\star(\theta)\, W_B^\star(\theta)^{-y}}{m} \right)\,, \text{ for } \theta \in \Theta\,. \quad (3.21)$$

Equation (3.21) is a fixed point equation characterising $W_B^\star(\theta)$, and has to hold for all $\theta \in \Theta$ at the same time. In particular, $\hat{g}^\star(\theta)$ depends on $W_B^\star(\theta')$ for $\theta' \in [\theta, \overline{\theta}]$, see (3.19). Solving this equation system with continuous parameter θ is, generally speaking, not straightforward.

In fact, $\hat{g}^\star(\theta)$ embeds the information that still can be learned from observing the firm and is thus reflected in the valuation of rollover debt. If lastly the best type $\overline{\theta}$ has survived, nothing more can be learned and the information asymmetry has vanished. The optimal strategy can therefore be established recursively starting at $\overline{\theta}$. Suppose that the uncertainty around $\tilde{\theta}$ has been eliminated. This is identical to the case without information asymmetry with $\Theta = \{\overline{\theta}\}$. The optimal default threshold can then be calculated using the limiting case of (3.19) at $\overline{\theta}$:

$$\hat{g}^\star(\overline{\theta}) = m\,(1-\alpha)\,\overline{\theta}\,W_B^\star(\overline{\theta})^{y+1} - \frac{m\,(C + (m - k(m))P)}{r+m}\,W_B^\star(\overline{\theta})^{y}\,. \quad (3.22)$$

Plugging this in (3.21) for $\theta = \overline{\theta}$ and solving for $W_B^\star(\overline{\theta})$ results in

$$W_B^\star(\overline{\theta}) = \frac{1}{\overline{\theta}}\,\frac{y(C+mP) + k(m)P\,(xm/r - (y-x))}{((1+x) + (y-x)\,(1-\alpha))\,(r+m)}\,, \quad (3.23)$$

where $V_B^\star = \overline{\theta} W_B^\star(\overline{\theta})$ is equity's optimal default threshold in the scenario without information asymmetry.

Now, if we differentiate (3.19) and (3.21), we obtain the following ordinary differential equation on Θ[5], which allows us to effectively calculate the optimal solution:

$$\begin{pmatrix} W_B'(\theta) \\ \hat{g}'(\theta) \end{pmatrix} = \begin{pmatrix} -\dfrac{(1+x)m\,W_B(\theta) + (y-x)\,\hat{g}'(\theta)\,W_B(\theta)^{-y}}{(1+x)m\,\theta - y\,(y-x)\,\hat{g}(\theta)\,W_B(\theta)^{-y-1}} \\ \dfrac{\phi(\theta)}{\Phi(\theta)}\left(\hat{g}(\theta) + \dfrac{m(C+(m-k(m))P)}{r+m} W_B(\theta)^y - m(1-\alpha)\theta W_B(\theta)^{y+1}\right) \end{pmatrix} \tag{3.24}$$

with boundary conditions given in (3.23) and (3.22). If the solution of the ODE further satisfies

$$0 \leq \hat{g}'(\theta) \leq \, y\,W_B'(\theta)\frac{\hat{g}(\theta)}{W_B(\theta)} \quad \text{and} \quad W_B'(\theta) < 0 \quad \text{for all} \quad \theta \in (\underline{\theta}, \overline{\theta}), \tag{3.25}$$

then $W_B^\star = W_B$ is the firm's optimal default strategy, $\hat{g}^\star = \hat{g}$ describes the value of the rollover debt via Equation (3.20) and the conditions (3.13–3.14) hold, which we discuss in more detail in Appendix 3.C.

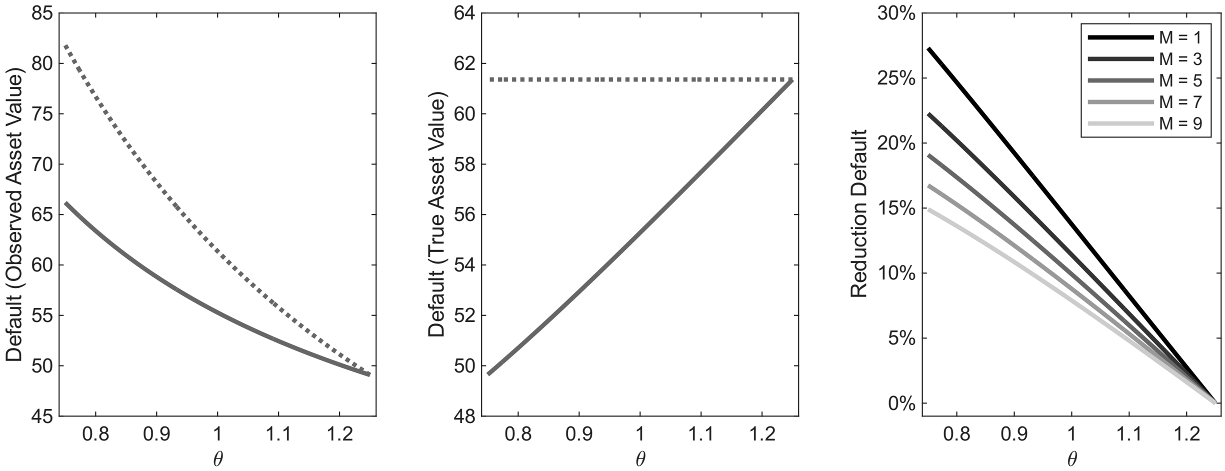
3.3 Feedback Effects and Maturity Choice under Asymmetric Information

This section examines the firm's optimal strategies, the feedback effect of information asymmetry and the optimal maturity choice under different degrees of uncertainty. Section 3.3.1 investigates further into how the firm's optimal strategy is affected by asymmetric information for a given capital structure. Afterwards, Section 3.3.2 then further analyses what maturity a firm should optimally choose under varying degrees of uncertainty.

Throughout this analysis, we maintain a base parametrisation of $\mu = 0.015$, $\sigma = 0.1$, $r = 0.03$, bankruptcy costs $\alpha = 0.5$, an initially observed asset value $W(0) = 100$ and debt principal $P = 50$.

We set maturity-related cost rate to $k(m) = 0.0002/m$. This implies that holding debt with a maturity of $M = 1/m$ entails annual costs of $k(m) = 0.0002 \cdot M$ times the principal for debt-holders. When the coupon rate is initially determined by debt-holders, they pass these maturity-related costs on to the firm through the coupon rate. Therefore, each additional

[5] At the boundaries $\underline{\theta}$ and $\overline{\theta}$, derivatives are understood as the respective limits from the interior. In the case of $\hat{g}(\overline{\theta})$, L'Hôpital's rule must be applied, see Appendix 3.C.



(a) Default (Observed Asset Value) *(b) Default (True Asset Value)* *(c) Reduction Default (in %)*

Figure 3.2: Impact of information asymmetry on the firm's default strategy. *This figure analyses the default thresholds of a firm when the market's initial belief about the scalar distortion factor $\tilde{\theta}$ is uniformly distributed over $\Theta = [0.75, 1.25]$. The debt principal is set to $P = 50$ and the firm pays a coupon $C = 2$. Panels 3.2a and 3.2b focus on the case of a maturity $M = 5$, showing the thresholds as functions of the observed and true asset values, respectively. Solid lines depict thresholds under information asymmetry, while dotted lines provide the corresponding benchmarks under perfect information. Panel 3.2c illustrates how the default thresholds are reduced due to the feedback effects of information asymmetry compared to perfect information across different maturities.*

year of debt maturity increases the firm's effective coupon rate by 2 basis points.

3.3.1 Feedback effects of asymmetric information

In this section, we investigate how, for a given capital structure, the degree of information asymmetry between the firm and debt-holders affects the firm's optimal default strategy.

We identify a notable feedback mechanism: under asymmetric information, immediately prior to default, the firm is overvalued by the market. This overvaluation results in more favourable refinancing conditions compared to a scenario with perfect information. Consequently, the firm has incentive to delay default in order to capitalize on these advantageous refinancing terms, which further improves its refinancing terms. As a result, asymmetric information effectively postpones default. Furthermore, firms with shorter debt maturities refinance larger amounts of debt more frequently, making this feedback mechanism more relevant.

In Figure 3.2, we illustrate how information asymmetry influences the firm's optimal default threshold, considering a debt principal of $P = 50$ and a coupon rate of $C = 2$. The initial market belief assumes that the scalar distortion factor $\tilde{\theta}$ is uniformly distributed on $\Theta = [0.75, 1.25]$. For a debt maturity $M = 5$, Panels 3.2a and 3.2b depict the firm's optimal default thresholds as functions of the observed and true asset values, V and W respectively from Equation (3.2), for each type θ. The solid lines depict the optimal default thresholds under information asymmetry, while the dash-dotted lines provide the corresponding benchmarks under perfect information. We observe that, owing to the information asymmetry, the firm tends to delay default compared to the perfect information scenario. Notably, the default threshold for the highest quality firm type $\overline{\theta} = 1.25$ coincides with the perfect information case. Since this best type defaults at the lowest observed asset value, it implies that all other types have been previously ruled out by the market. Therefore, just before the best type defaults, the information asymmetry effectively vanishes, and the market conditions match those under perfect information.

Conversely, the largest gap between the default thresholds under asymmetric information and perfect information is observed for the lowest quality firm type $\underline{\theta} = 0.75$. In this case, the debt market has not yet learned about the firm's true quality, leading to a persistent overvaluation of the firm quality until default. Consequently, as in Hilpert et al. (2024), the firm benefits from better refinancing conditions than under perfect information, which delays default. This illustrates the feedback mechanism by which asymmetric information leads to delayed default, especially for lower quality firms, by creating an overestimation of their asset values prior to default.

Panel 3.2c shows the reduction in the default threshold resulting from asymmetric information and the inherent feedback mechanism, expressed as a percentage for each type and different maturities. We observe that the reduction is larger for shorter maturities. This is because the effect is driven by better refinancing conditions, which are more relevant for shorter-term debt that is refinanced more frequently, leading to a greater impact on delaying default.

3.3.2 Optimal maturity choice under uncertainty

So far, we have analysed the effect of information asymmetry on the firm's optimal default threshold for a given capital structure. We have observed that the overvaluation immediately before default as well as the induced feedback mechanism results in lower default thresholds, especially for shorter debt maturities. Assuming the firm can freely choose its debt maturity, this raises the question of how these effects influence the firm's optimal maturity choice. To analyse the firm's optimal maturity choice in the simplest manner, we make a few key assumptions. Initially, both the firm and the debt market share symmetric uncertainty regarding the firm's true quality. Specifically, both agree that the firm's true asset value is given by $V_0 = \tilde{\theta} W_0$, where W_0 is common knowledge, and both believe that $\tilde{\theta}$ is uniformly distributed on $[\underline{\theta}, \overline{\theta}]$. Immediately after issuance, the firm learns its true type $\theta \in [\underline{\theta}, \overline{\theta}]$, while the market remains uninformed. This transition from symmetric uncertainty to asymmetric information occurs after the debt is issued. This setup yields the advantage that the initial maturity choice does not reveal any information about the firm's type, simplifying the analysis by avoiding the complexities associated with signaling effects during debt issuance.

The firm is required to issue debt with principal P. Importantly, the firm can freely choose its debt maturity. The coupon is set, such that the market value of initially issued debt equals the the principal P:

$$D(W_0; m, P, C, \phi) = P. \tag{3.26}$$

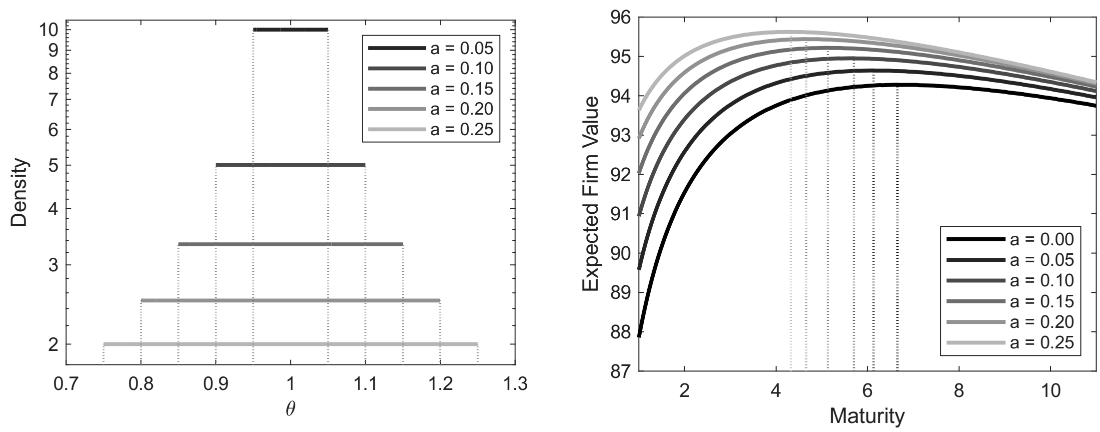
Here, the market value of initially issued debt is given by the expected future cash flows from equity repaying debt issued at time $t = 0$, given by $D(W_0; m) = d(W_0, W_0; m)/m$, as derived from Equations (3.9) - (3.11). We impose the restriction that debt-holders do not offer a debt contract, where it is optimal for firm's of certain quality to default immediately after issuance.

The firm's optimal maturity choice $M = 1/m$ is determined by maximising the (expected) firm value $v(W_0; m)$, given by the sum of the firm's expected equity value and the market value of initially issued debt:

$$v(W_0; m) = \mathbb{E}[E(\tilde{\theta}, W_0, W_0; m)] + D(W_0; m) \tag{3.27}$$

We show in Appendix 3.E that the market value of initially issued debt equals the expected value of total cash flow to debt-holders including payments resulting from rolling over the debt, as given in Equation (3.6). That is, it holds $D(W_0; m) = \mathbb{E}[D(\tilde{\theta}, W_0, W_0; m)]$. This yields another representation of firm value as the expected asset value less the sum of expected bankruptcy and maturity-related costs:

$$v(W_0; m) = \mathbb{E}\left[\tilde{\theta} W_0\right] - \left(\mathbb{E}\left[BC(\tilde{\theta}, W_0; m)\right] + \mathbb{E}\left[MC(\tilde{\theta}, W_0; m)\right]\right), \tag{3.28}$$

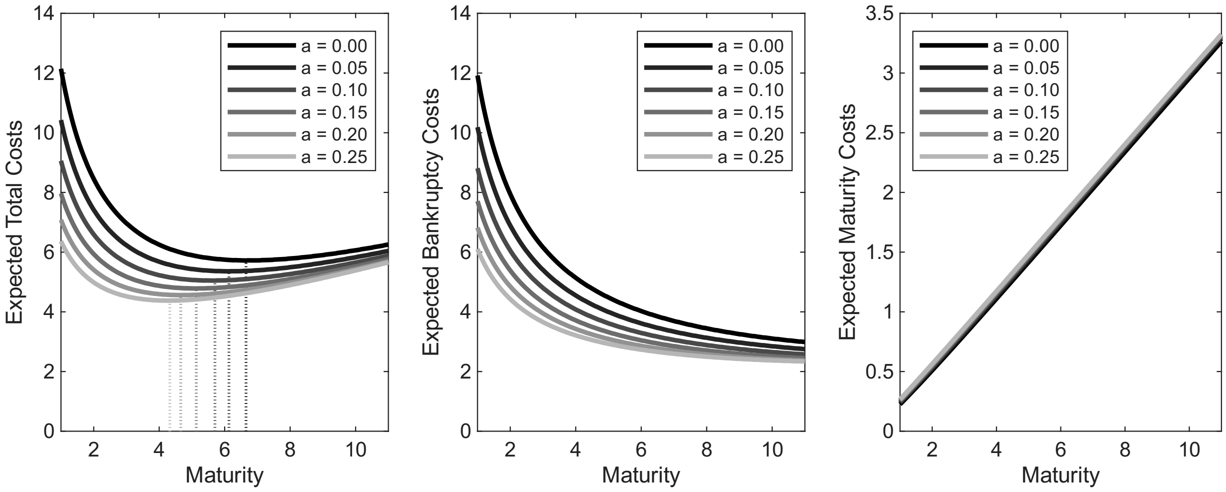


(a) Densities of Uncertainty *(b) Firm Value*

Figure 3.3: Optimal maturity under uncertainty. *Panel 3.3a shows densities for the initial uncertainty about the scalar distortion factor $\tilde{\theta}$ that is uniformly distributed on $[1-a, 1+a]$ for $a \in \{0.05, 0.10, 0.15, 0.20, 0.25\}$, where $a = 0$ depicts the benchmark case of perfect information. Panel 3.3b plots the respective total firm value given for different maturities M given $W_0 = 100$ and a debt principal of $P = 50$.*

where the bankruptcy and maturity costs are defined in (3.8). At time $t = 0$, the firm chooses the maturity $M = 1/m$ that maximises the expected firm value. Figure 3.3 presents the firm's expected firm value across different debt maturities and varying degrees of uncertainty regarding the firm's true asset value. The analysis considers cases with $P = 50$ and an initial symmetric belief that the firm's asset value is $\tilde{\theta}W_0$, where $W_0 = 100$ and $\tilde{\theta}$ is uniformly distributed on $[1-a, 1+a]$ for $a \in \{0.05, 0.10, 0.15, 0.20, 0.25\}$. The benchmark scenario of perfect information with $a = 0$ is shown for comparison. In line with the observations from Section 3.3.1, we observe that uncertainty leads to a lower optimal maturity compared to the perfect information case. The induced feedback effect that reduces the firm's optimal default thresholds is stronger at lower debt maturities. Moreover, we observe that the optimal maturity decreases as the degree of uncertainty increases, reflecting a stronger feedback effect associated with greater information asymmetry. Additionally, in this case, higher uncertainty raises the firm value, since higher information asymmetry further delays default and reduces expected bankruptcy costs.

Recall that the firm value is given by the expected asset value minus the sum of expected bankruptcy and maturity costs. Notably, the expected asset value does not depend on maturity and is $\mathbb{E}[\tilde{\theta}W_0] = 100$ for all degrees of uncertainty considered in Figure 3.3. Consequently, the firm's optimal maturity choice reduces to selecting the maturity that minimises the sum of expected bankruptcy and maturity costs. Figure 3.4 examines how uncertainty affects these costs in greater detail. Panel 3.4a displays the sum of expected



(a) Expected total costs *(b) Expected bankruptcy costs* *(c) Expected maturity costs*

***Figure 3.4:* Bankruptcy and maturity costs.** *This figure breaks down the firm's optimal maturity choice and analyses the leverage costs in more detail. Panel 3.4a shows the sum of expected bankruptcy and maturity costs, while Panel 3.4b shows the expected bankruptcy costs and Panel 3.4c shows the respective maturity costs. The initial uncertainty about the scalar distortion factor* $\tilde{\theta}$ *that is that it is uniformly distributed on* $[1-a, 1+a]$ *for* $a \in \{0.05, 0.10, 0.15, 0.20, 0.25\}$*, where* $a = 0$ *shows the benchmark case under perfect information.*

bankruptcy and maturity costs, while Panel 3.4b and Panel 3.4c present the expected bankruptcy costs and expected maturity costs separately. We observe that expected maturity costs increase with longer maturities, but are relatively insensitive to the degree of initial uncertainty about the asset value.

Conversely, we observe that within the examined maturity range, expected bankruptcy costs rise as maturity shortens, but significantly decline with greater uncertainty, especially at shorter maturities. This reduction is driven by the greater information asymmetry following debt issuance and the resulting feedback effect. Overall, we find that the optimal maturity that minimises the firm's total costs decreases as the level of uncertainty about the firm's true asset value increases.

3.4 Conclusion

We provide a structural analysis of debt maturity choices in the presence of default risk and asymmetric information, incorporating feedback effects in competitive debt markets. By generalizing the classical credit risk framework to allow for imperfect and evolving market beliefs about a firm's true asset value, we uncover novel strategic incentives for firms to delay default, thereby signalling credit quality and benefiting from improved refinancing conditions over time.

Our equilibrium characterisation demonstrates that asymmetric information induces a feedback loop: creditors, observing non-default in periods of apparent distress, update beliefs in favour of higher firm quality, which in turn reduces the firm's financing costs upon refinancing and further delays default. This effect is particularly pronounced for firms with shorter debt maturities, as more frequent refinancing allows them to capitalize repeatedly on favourable market reassessments. As a result, asymmetric information lowers optimal default thresholds, with the effect being more substantial for lower-quality firms and shorter maturities.

Examining the firm's optimal maturity choice, our analysis shows that increased initial uncertainty about firm value systematically reduces the maturity length that maximises total firm value. This is driven by the interaction of reduced bankruptcy costs and relatively stable maturity costs with respect to uncertainty. Thus, under greater uncertainty, firms optimally favour shorter-term debt to capitalize on the beneficial effects of market learning during distress.

Overall, our findings highlight the importance of information dynamics in shaping the optimal capital structure, and suggest that observed patterns of debt maturity may reflect sophisticated strategic responses to underlying informational frictions in credit markets. Importantly, our results are consistent with empirical evidence showing that greater information asymmetries are associated with shorter debt maturities Future work could extend the model to incorporate endogenous information acquisition, richer signalling environments, or the effects of market power among creditors.

3.A characterising the Firm's Optimal Strategy

Proposition 1. *In the setting of Section 3.2, consider the optimal stopping problem*

$$E(\theta, W, \underline{W}) = \sup_{\tau(\theta)\geq 0} \mathbb{E}_{(W,\underline{W})}\left[\int_0^{\tau(\theta)} e^{-rt}\left(\theta(r-\mu)\, W_t + d(W_t, \underline{W}_t) - (C+mP)\right) \mathrm{d}t\right]. \quad (3.29)$$

Suppose the conditions (3.13-3.14) imposed on $d(W, \underline{W})$ hold, then the optimal stopping time $\tau^\star(\theta) = \inf\{t \geq 0 : (W_t, \underline{W}_t) \in \mathcal{E}(\theta)\}$ is characterised by its early exercise region

$$\mathcal{E}(\theta) = \{(W, \underline{W}) : 0 < \underline{W} \leq W_B(\theta), \underline{W} \leq W \leq B_\theta(\underline{W})\},$$

where B_θ is increasing on $(0, W_B(\theta)]$. In particular, when starting at (W, W), for some $W > 0$, the optimal stopping time is a threshold strategy, that is,

$$\tau^\star(\theta) = \inf\{t \geq 0 : W_t \leq W_B(\theta)\}\,.$$

Furthermore, W_B is strictly decreasing in θ.

Proof. The state process for the optimal stopping problem in (3.29) is the Markov process $(W_t, \underline{W}_t)$. Accordingly, the early exercise region is characterised by a subset $\mathcal{E}(\theta)$ of the state space $\{(W, \underline{W}) : 0 \leq \underline{W} \leq W\} \subseteq \mathbb{R}^2$. Our next objective is to determine the shape of $\mathcal{E}(\theta)$. To this end, let $(W_t^{(W,\underline{W})}, \underline{W}_t^{(W,\underline{W})})_{t\geq 0}$ denote the process consisting of the observed asset value and its running minimum, with starting value $(W_0^{(W,\underline{W})}, \underline{W}_0^{(W,\underline{W})}) = (W, \underline{W})$, where we explicitly allow $0 < \underline{W} \leq W$. We proceed in three steps.

First, consider $(W, \underline{W})$ with $0 < \underline{W} \leq W$, and increase the observed asset value to $W' \geq W$. Using (3.13–3.14), we show that the value function is increasing, that is, $E(\theta, W, \underline{W}) \leq E(\theta, W', \underline{W})$. This implies that for $(W', \underline{W}) \in \mathcal{E}(\theta)$, it holds that $(W, \underline{W}) \in \mathcal{E}(\theta)$, for all $W \in [\underline{W}, W']$, since $\mathcal{E}(\theta) = \{(W, \underline{W}) : 0 < \underline{W} \leq W, E(\theta, W, \underline{W}) = 0\}$ and in particular $(\underline{W}, \underline{W}) \in \mathcal{E}(\theta)$.

To show that $E(\theta, W, \underline{W}) \leq E(\theta, W', \underline{W})$, for $\underline{W} \leq W \leq W'$, consider the original process $(W_t^{(W,\underline{W})}, \underline{W}_t^{(W,\underline{W})})_{t\geq 0}$ and the one with the improved asset value $(W_t^{(W',\underline{W})}, \underline{W}_t^{(W',\underline{W})})_{t\geq 0}$, where we set

$$W_t^{(W',\underline{W})} = \frac{W'}{W}\, W_t^{(W,\underline{W})}, \text{ for } t \geq 0\,.$$

Then
$$\underline{W}_t^{(W',\underline{W})} = \frac{W'}{W}\,\underline{W}_t^{(W,\underline{W})} \wedge \underline{W}, \text{ for } t \geq 0\,.$$
With $\lambda = \frac{W'}{W} \geq 1$, write
$$\begin{aligned}
d(W_t^{(W',\underline{W})}, \underline{W}_t^{(W',\underline{W})}) &= d(\lambda\, W_t^{(W,\underline{W})}, \lambda\, \underline{W}_t^{(W,\underline{W})} \wedge \underline{W}) \\
&= \mathbb{1}_{\lambda\, \underline{W}_t^{(W,\underline{W})} \geq \underline{W}}\, d(\lambda\, W_t^{(W,\underline{W})}, \underline{W}) + \mathbb{1}_{\lambda\, \underline{W}_t^{(W,\underline{W})} < \underline{W}}\, d(\lambda\, W_t^{(W,\underline{W})}, \lambda\, \underline{W}_t^{(W,\underline{W})}) \\
&\geq \mathbb{1}_{\lambda\, \underline{W}_t^{(W,\underline{W})} \geq \underline{W}}\, d(W_t^{(W,\underline{W})}, \lambda^{-1}\underline{W}) + \mathbb{1}_{\lambda\, \underline{W}_t^{(W,\underline{W})} < \underline{W}}\, d(W_t^{(W,\underline{W})}, \underline{W}_t^{(W,\underline{W})}) \\
&\geq \mathbb{1}_{\lambda\, \underline{W}_t^{(W,\underline{W})} \geq \underline{W}}\, d(W_t^{(W,\underline{W})}, \underline{W}_t^{(W,\underline{W})}) + \mathbb{1}_{\lambda\, \underline{W}_t^{(W,\underline{W})} < \underline{W}}\, d(W_t^{(W,\underline{W})}, \underline{W}_t^{(W,\underline{W})}) \\
&= d(W_t^{(W,\underline{W})}, \underline{W}_t^{(W,\underline{W})})\,,
\end{aligned}$$

where, in the third line, we use (3.14) for both terms; in the fourth line, we apply (3.13). The payoff stream in (3.29) with starting value $(W', \underline{W})$ dominates the payoff stream with starting value $(W, \underline{W})$ for all $t \geq 0$ and in all possible scenarios. This dominance is reflected in the corresponding value functions, that is, $E(\theta, W', \underline{W})) \geq E(\theta, W, \underline{W}))$, as claimed.

Second, focus on the diagonal in the state space, and consider $(W, \underline{W})$ with $\underline{W} = W$. For $\lambda \geq 1$, consider the upward-shifted starting value $\lambda\,(W, W)$. Then, we can write $\lambda\,(W_t^{(W,W)}, \underline{W}^{(W,W)}) = (W_t^{\lambda\,(W,W)}, \underline{W}_t^{\lambda\,(W,W)})$. By (3.14) we have $d(\lambda\, W, \lambda W) \geq d(W, W)$, and using the same arguments as in the first part, we obtain $E(\theta, \lambda\, W, \lambda\, W) \geq E(\theta, W, W)$. Thus, the equity value function is increasing along the diagonal, for each $\theta \in \Theta$. Accordingly, for the critical value $W_B(\theta) = \sup\{W > 0 : E(\theta, W, W) = 0\}$ we have $\{(W, W) : 0 < W \leq W_B(\theta)\} \subseteq \mathcal{E}(\theta)$ and $\{(W, W) : W > W_B(\theta)\} \cap \mathcal{E}(\theta) = \emptyset$.

Moreover, for each $\underline{W} < W_B(\theta)$ there exists a critical value $\underline{W} \leq B_\theta(\underline{W}) \leq W_B(\theta)$ separating the stopping region from the continuation region; that is, $\{(W, \underline{W}) : \underline{W} \leq W \leq B_\theta(\underline{W})\} \subseteq \mathcal{E}(\theta)$ and $\{(W, \underline{W}) : W > B_\theta(\underline{W})\} \cap \mathcal{E}(\theta) = \emptyset$.

Combining this result with the first part, it follows that
$$\mathcal{E}(\theta) = \{(W, \underline{W}) : 0 < \underline{W} \leq W_B(\theta), \underline{W} \leq W \leq B_\theta(\underline{W})\}.$$

Using mainly (3.13) and the same arguments as before, it follows that B_θ is increasing.

Third, since the integrand specifying the payoff in (3.29) is strictly increasing in θ, it follows that $W_B(\theta)$ strictly decreasing in θ. $\square$

3.B Value Function Representation

In the following, we show that the equity value can be represented as shown in Equation (3.16). Set $\underline{W} = W_B^\star(\theta)$, then $\underline{W}_t = W_B^\star(\theta)$, for $0 \le t \le \tau^\star(\theta)$, and for $W \ge W_B^\star(\theta)$

$$\begin{aligned}
E(\theta, W, W_B^\star(\theta)) =& \,\theta\, W - \mathbb{E}_W\left[e^{-r\,\tau^\star(\theta)}\theta\, W_B^\star(\theta)\right] \\
&+ \mathbb{E}_W\left[\int_0^{\tau^\star(\theta)} e^{-r\,t}\left(d(W_t, W_B^\star(\theta)) - (C{+}mP)\right)\,\mathrm{d}t\right] \\
=& \,\theta\, W - \theta\, W_B^\star(\theta)\left(\frac{W}{W_B^\star(\theta)}\right)^{-x} - \frac{C{+}mP}{r} + \frac{C{+}mP}{r}\left(\frac{W}{W_B^\star(\theta)}\right)^{-x} \\
&+ \mathbb{E}_W\left[\int_0^{\tau^\star(\theta)} e^{-r\,t} d(W_t, W_B^\star(\theta))\,\mathrm{d}t\right].
\end{aligned}$$

Next, for $W \ge W_B^\star(\theta)$, compute

$$\begin{aligned}
d(\theta', W) &= m\mathbb{E}_W\left[\int_0^{\tau^\star(\theta')} e^{-(r+m)\,t}\left(C{+}mP{-}k(m)P\right)\,\mathrm{d}t + e^{-(r+m)\,\tau^\star(\theta')}(1{-}\alpha)\,\theta'\, W_B^\star(\theta')\right] \\
&= \frac{m(C{+}(m{-}k(m))P)}{r+m} + \left(m(1-\alpha)\theta' W_B^\star(\theta') - \frac{m(C{+}(m{-}k(m))P)}{r+m}\right)\left(\frac{W}{W_B^\star(\theta')}\right)^{-y} \\
\phi(\theta', W_B^\star(\theta)) &= \frac{\mathbb{1}_{\{W_B^\star(\theta') < W_B^\star(\theta)\}}\phi(\theta')}{\int_\Theta \mathbb{1}_{\{W_B^\star(\theta'') < W_B^\star(\theta)\}}\phi(\theta'')\,\mathrm{d}\theta''} = \frac{\mathbb{1}_{\{\theta' > \theta\}}\phi(\theta')}{\int_\theta^{\overline{\theta}}\phi(\theta'')\,\mathrm{d}\theta''},
\end{aligned}$$

where it assumed that $W_B^\star$ is strictly decreasing in θ to write $\int_\Theta \mathbb{1}_{\{W_B^\star(\theta'') < W_B^\star(\theta)\}}\,\phi(\theta'')\,\mathrm{d}\theta'' = \int_\theta^{\overline{\theta}}\phi(\theta'')\,\mathrm{d}\theta''$. With $\Phi(\theta) = \int_\theta^{\overline{\theta}}\phi(\theta')\,\mathrm{d}\theta'$, for $\theta \in \Theta$, it holds that

$$\begin{aligned}
d(W, W_B^\star(\theta)) &= \int_\theta^{\overline{\theta}} d(\theta', W)\,\phi(\theta', W_B^\star(\theta))\,\mathrm{d}\theta' \\
&= \frac{m(C{+}(m{-}k(m))P)}{r+m} + \frac{\int_\theta^{\overline{\theta}}\phi(\theta')\left(m(1{-}\alpha)\theta' W_B^\star(\theta') - \frac{m(C+(m-k(m))P)}{r+m}\right)\left(\frac{W}{W_B^\star(\theta')}\right)^{-y}\mathrm{d}\theta'}{\Phi(\theta)} \\
&= \frac{m(C{+}(m{-}k(m))P)}{r+m} + \frac{W^{-y}\int_\theta^{\overline{\theta}}\phi(\theta')\left(m(1{-}\alpha)\theta' W_B^\star(\theta')^{y+1} - \frac{m(C+(m-k(m))P)}{r+m}W_B^\star(\theta')^{y}\right)\mathrm{d}\theta'}{\Phi(\theta)}
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbb{E}_W\left[\int_0^{\tau^\star(\theta)} e^{-r\,t} d(W_t, W_B^\star(\theta))\,\mathrm{d}t\right] &= \frac{m\,(C{+}(m{-}k(m))P)}{r\,(r+m)}\left(1 - \mathbb{E}_W\left[e^{-r\,\tau^\star(\theta)}\right]\right) \\
&\quad + \hat{g}^\star(\theta)\,\mathbb{E}_W\left[\int_0^{\tau^\star(\theta)} e^{-r\,t}\,W_t^{-y}\,\mathrm{d}t\right],
\end{aligned}$$

with

$$\hat{g}^\star(\theta) = \frac{\int_\theta^{\overline{\theta}} \phi(\theta') \left(m\,(1-\alpha)\,\theta'\,W_B^\star(\theta')^{y+1} - \frac{m\,(C+(m-k(m))P)}{r+m}\,W_B^\star(\theta')^{y}\right)\,\mathrm{d}\theta'}{\Phi(\theta)}$$

Define

$$F(W) = \mathbb{E}_W\left[\int_0^{\tau^\star(\theta)} e^{-r\,t}\,W_t^{-y}\,\mathrm{d}t\right],$$

and use the ansatz

$$F(W) = a_1\,W^{-y} + a_2\,W^{a_3}\,,\ \text{for } W \geq W_B^\star(\theta)\,,$$

and $F(W) = 0$, for $0 \leq W \leq W_B^\star(\theta)$, for $a_1, a_2, a_3 \in \mathbb{R}$. Noting that $F \leq \frac{W_B^\star(\theta)^{-y}}{r}$, gives $a_3 \leq 0$. On $(W_B^\star(\theta), \infty)$, F satisfies

$$0 \;=\; \mu\,W\,\frac{\partial F}{\partial W}(W) + \frac{1}{2}\,\sigma^2\,W^2\,\frac{\partial^2 F}{\partial W^2}(W) + W^{-y} - r\,F(W)\,.$$

Computing the partials, comparing the coefficients and using $F(W_B^\star(\theta)) = 0$, gives

$$\begin{aligned}
a_1 &= \frac{1}{r - \left(-y\left(\mu - \frac{1}{2}\,\sigma^2\right) + \frac{1}{2}\,y^2\,\sigma^2\right)}\,,\\
a_2 &= -\frac{W_B^\star(\theta)^{-y-a_3}}{r - \left(-y\left(\mu - \frac{1}{2}\,\sigma^2\right) + \frac{1}{2}\,y^2\,\sigma^2\right)}\,,\ \text{and}\\
a_3 &= -\frac{1}{\sigma^2}\left(\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\,r\,\sigma^2}\right) = -x\,,
\end{aligned}$$

where the positive solution corresponding to a_3 is ruled out, because $a_3 \leq 0$. Putting together the pieces, results in

$$\mathbb{E}_W\left[\int_0^{\tau^\star(\theta)} e^{-r\,t}\,W_t^{-y}\,\mathrm{d}t\right] = \frac{W^{-y} - W_B^\star(\theta)^{-y+x}\,W^{-x}}{r - \left(-y\left(\mu - \frac{1}{2}\,\sigma^2\right) + \frac{1}{2}\,y^2\,\sigma^2\right)}$$

From $-y\left(\mu - \frac{1}{2}\,\sigma^2\right) + \frac{1}{2}\,y^2\,\sigma^2 = r + m$, we finally obtain

$$\mathbb{E}_W\left[\int_0^{\tau^\star(\theta)} e^{-r\,t}\,W_t^{-y}\,\mathrm{d}t\right] = \frac{W_B^\star(\theta)^{-y}}{m}\left(\left(\frac{W}{W_B^\star(\theta)}\right)^{-x} - \left(\frac{W}{W_B^\star(\theta)}\right)^{-y}\right).$$

The equity value with $\underline{W} = W_B^\star(\theta)$ is

$$E(\theta, W, W_B^\star(\theta)) = \theta\, W - \theta\, W_B^\star(\theta) \left(\frac{W}{W_B^\star(\theta)}\right)^{-x} - \left(\frac{C+mP(1+k(m)/r)}{r+m}\right)\left(1 - \left(\frac{W}{W_B^\star(\theta)}\right)^{-x}\right)$$
$$+ \frac{\hat{g}^\star(\theta)\, W_B^\star(\theta)^{-y}}{m} \left(\left(\frac{W}{W_B^\star(\theta)}\right)^{-x} - \left(\frac{W}{W_B^\star(\theta)}\right)^{-y}\right). \tag{3.30}$$

3.C Calculating the Optimal Solution

The optimal solution $(W_B^\star, \hat{g}^\star)$ characterising the firm's optimal default strategy as well as the valuation of rollover debt must satisfy the ODE from Equation (3.24). Further, we provide necessary initial condition.

Initial values for the derivative are obtained by taking the limit in Equation (3.24). We can represent

$$\hat{g}(\theta) = \frac{1}{\Phi(\theta)} \int_\theta^{\overline{\theta}} \phi(\theta') f(\theta') \mathrm{d}\theta' \quad \text{and} \quad \hat{g}'(\theta) = \frac{\phi(\theta)}{\Phi(\theta)} \left(\hat{g}(\theta) - f(\theta)\right) \tag{3.31}$$

with

$$f(\theta) = m\,(1-\alpha)\theta W_B(\theta)^{y+1} - \frac{m\,(C + (m - k(m))P)}{r+m} W_B(\theta)^y.$$

Applying L'Hôpital's rule for $\hat{g}'$ in (3.31) gives

$$\hat{g}'(\overline{\theta}) = \lim_{\theta \nearrow \overline{\theta}} \hat{g}'(\theta) = \phi(\overline{\theta}) \lim_{\theta \nearrow \overline{\theta}} \frac{\hat{g}(\theta) - f(\theta)}{\Phi(\theta)} = \phi(\overline{\theta}) \lim_{\theta \nearrow \overline{\theta}} \frac{\hat{g}'(\theta) - f'(\theta)}{-\phi(\theta)} = f'(\overline{\theta}) - \hat{g}'(\overline{\theta})$$

and thus $\hat{g}'(\overline{\theta}) = \frac{1}{2} f'(\overline{\theta})$. Taking the limit of the derivatives, we can represent

$$\hat{g}'(\overline{\theta}) = \alpha_1 W_B'(\overline{\theta}) + \alpha_0 \quad \text{and} \quad W_B'(\overline{\theta}) = \frac{\beta_0 + \beta_2 \hat{g}'(\overline{\theta})}{\beta_1}$$

with $\alpha_0 = \frac{1}{2} m(1-\alpha) W_B(\overline{\theta})^{y+1}$ and

$$\alpha_1 = \frac{1}{2}\left(m(1-\alpha)(y+1)\overline{\theta} W_B(\overline{\theta})^y - \frac{m\,(C + (m-k(m))P)}{r+m} y W_B(\overline{\theta})^{y-1}\right),$$

while $\beta_0, \beta_1, \beta_2$ are directly given by the limiting case in Equation (3.24). Thus, we have an initial value for the derivative:

$$W_B'(\overline{\theta}) = \frac{\gamma_0}{1-\gamma_1}$$

for $\gamma_0 = \beta_0/\beta_1 + (\beta_2\alpha_0)/\beta_1$ and $\gamma_1 = (\beta_2\alpha_1)/\beta_1$. Now, let $(W_B, \hat{g})$ be the solution of the differential equation (3.24) and satisfy (3.25). Then

$$W_B^\star(\theta) = W_B(\theta) \quad \text{and} \quad d(W, \underline{W}) = \frac{m(C+(m-k(m))P)}{r+m} + W^{-y}\hat{g}(W_B^{-1}(\underline{W})) \tag{3.32}$$

is indeed providing the optimal default strategy and the the corresponding valuation of the rollover debt, since Equations (3.13–3.14) hold: Equation (3.25) states that $W_B(\theta)$ is strictly decreasing in θ and

$$0 \leq \hat{g}'(\theta) \leq yW_B'(\theta)\frac{\hat{g}(\theta)}{W_B(\theta)}. \tag{3.33}$$

We show that the value of the rollover debt $d(W, \underline{W})$ satisfies Equations (3.13) and (3.14): From $\hat{g}'(\theta) \geq 0$ and $W_B'(\theta) < 0$, which gives $W_B^{-1\prime}(\underline{W}) < 0$, we conduct $\hat{g}(W_B^{-1}(\underline{W}))' \leq 0$, which implies that $d(W, \underline{W})$ satisfies Equation (3.13). Moreover, the second inequality in Equation (3.33) is equivalent to $d(W_B(\theta), W_B(\theta))$ being non-increasing in θ. Since $W_B(\theta)$ is decreasing, $d(\underline{W}, \underline{W})$ is non-decreasing in W. By the structure of $d(W, \underline{W})$ this implies that Equation (3.14) is satisfied.

3.D Calculating Equity Value

Following argumentation from Heinricher and Stockbridge (1991) and Barron (1993), the firm's equity value $E(\theta, W, \underline{W})$ is a solution to a non-standard system of differential equations. It is characterised by

$$\frac{1}{2}\sigma^2 W^2 E_{WW} + \mu W E_W + \theta(r-\mu)W + d(W, \underline{W}) - (C+mP) - rE = 0 \tag{3.34}$$

in the region $W_B(\theta) \leq \underline{W} \leq W$ satisfying terminal condition

$$E(\theta, W_B(\theta), W_B(\theta)) = 0, \tag{3.35}$$

as well as the boundary condition

$$\frac{\partial E}{\partial \underline{W}}(W, \underline{W})\bigg|_{W=\underline{W}} = 0 \quad (\underline{W} \geq W_B(\theta)). \tag{3.36}$$

Let us define $f(\underline{W}) = E(\theta, \underline{W}, \underline{W})$. For $W_B(\theta) \leq \underline{W} \leq W$, we can represent the equity value by

$$E(\theta, W, \underline{W}) = \mathbb{E}_{(W,\underline{W})}\left[\int_0^{\tau(\underline{W})} e^{-rt}\left(\theta(r-\mu)W + d(W,\underline{W}) - (C+mP)\right)\mathrm{d}t\right] + f(\underline{W})\left(\frac{W}{\underline{W}}\right)^{-x}$$

For $W_B^\star(\theta) \leq \underline{W} \leq W_B^\star(\underline{\theta})$, similar to argumentation in Appendix 3.B, we can provide the equity value in closed form

$$\begin{aligned} E(\theta, W, \underline{W}) \quad = \quad & \theta\, W - \theta\, \underline{W}\left(\frac{W}{\underline{W}}\right)^{-x} - \frac{C + m\,P(1+k(m)/r)}{(r+m)}\left(1 - \left(\frac{W}{\underline{W}}\right)^{-x}\right) \\ & + \frac{\hat{g}^\star(W_B^{\star\,-1}(\underline{W}))\,\underline{W}^{-y}}{m}\left(\left(\frac{W}{\underline{W}}\right)^{-x} - \left(\frac{W}{\underline{W}}\right)^{-y}\right) + f(\underline{W})\left(\frac{W}{\underline{W}}\right)^{-x}. \end{aligned}$$

To calculate arbitrary equity value, we have to find the function $f(\underline{W})$. For this, we implement and solve for the normal reflection condition: $E_{\underline{W}}(\underline{W}, \underline{W}) = 0$ for $\underline{W}$ with $W_B^\star(\underline{\theta}) \geq \underline{W} \geq W_B^\star(\theta)$. We obtain

$$f'(\underline{W}) = \theta(1+x) - \frac{(x-y)}{m}\hat{g}^\star\Big(W_B^{\star-1}(\underline{W})\Big)\underline{W}^{-y-1} - \frac{x}{\underline{W}}\left(f(\underline{W}) + \frac{C+mP(1+k(m)/r)}{r+m}\right) \tag{3.37}$$

and f is the solution to an explicit differential equation with initial condition $f(W_B(\theta)) = 0$.

3.E Initial Debt Valuation

In the following, we demonstrate that the market value of initially issued debt at time $t = 0$ equals the expected total cash flows to debt-holders including payments resulting from rolling over the debt. Formally, we establish that $D(W_0; m) = \mathbb{E}[D(\tilde{\theta}, W_0, W_0; m)]$, where $D(W_0; m) = d(W_0, W_0; m)/m$ equals the expected cash flows to debt-holders generated by the initial issuance at time $t = 0$, as defined by Equations (3.9) - (3.11). Meanwhile, $\mathbb{E}[D(\tilde{\theta}, W_0, W_0; m)]$ captures the expected sum of all future cash flows to debt-holders over time, including proceeds from debt rollover, where the term $D(\theta, W_0, W_0; m)$ is given in Equation (3.6).

In the following derivations, as in the main analysis, we omit the dependence on m in the value functions for simplicity. To establish the result, we first demonstrate that for value of continuously issued new debt $d(W, \underline{W})$ from (3.9), it holds

$$\int_\Theta \mathbb{E}_W \left[\int_0^{\tau(\theta)} e^{-r\,t} d(\theta, W_t)\mathrm{d}t\right] \phi(\theta)\mathrm{d}\theta = \int_\Theta \mathbb{E}_{(W,W)} \left[\int_0^{\tau(\theta)} e^{-r\,t} d(W_t, \underline{W}_t)\mathrm{d}t\right] \phi(\theta)\mathrm{d}\theta. \quad (3.38)$$

This holds true since

$$\begin{aligned}
&\int_\Theta \mathbb{E}_W\left[\int_0^{\tau(\theta)} e^{-r\,t} d(\theta, W_t)\mathrm{d}t\right]\phi(\theta)\,\mathrm{d}\theta = \mathbb{E}_{(W,W)}\left[\int_0^{\infty} e^{-r\,t}\int_\Theta d(\theta, W_t)\mathbb{1}_{\{W_B(\theta)\leq \underline{W}_t\}}\phi(\theta)\,\mathrm{d}\theta\,\mathrm{d}t\right]\\
=&\,\mathbb{E}_{(W,W)}\left[\int_0^{\infty} e^{-r\,t}\int_\Theta d(\theta, W_t)\frac{\int_\Theta \mathbb{1}_{\{W_B(\theta')<\underline{W}_t\}}\,\phi(\theta')\,\mathrm{d}\theta'}{\int_\Theta \mathbb{1}_{\{W_B(\theta')<\underline{W}_t\}}\,\phi(\theta')\,\mathrm{d}\theta'}\mathbb{1}_{\{W_B(\theta)\leq \underline{W}_t\}}\phi(\theta)\,\mathrm{d}\theta\,\mathrm{d}t\right]\\
=&\,\mathbb{E}_{(W,W)}\left[\int_0^{\infty} e^{-r\,t}\left(\int_\Theta d(\theta, W_t)\phi(\theta, \underline{W}_t)\,\mathrm{d}\theta\right)\left(\int_\Theta \mathbb{1}_{\{W_B(\theta')\leq \underline{W}_t\}}\phi(\theta')\,\mathrm{d}\theta'\right)\mathrm{d}t\right]\\
=&\,\mathbb{E}_{(W,W)}\left[\int_0^{\infty} e^{-r\,t} d(W_t, \underline{W}_t)\left(\int_\Theta \mathbb{1}_{\{W_B(\theta')\leq \underline{W}_t\}}\phi(\theta')\,\mathrm{d}\theta'\right)\mathrm{d}t\right]\\
=&\,\mathbb{E}_{(W,W)}\left[\int_\Theta\int_0^{\tau(\theta')} e^{-r\,t} d(W_t, \underline{W}_t)\,\mathrm{d}t\phi(\theta')\mathrm{d}\theta'\right] = \int_\Theta \mathbb{E}_{(W,W)}\left[\int_0^{\tau(\theta)} e^{-r\,t} d(W_t, \underline{W}_t)\mathrm{d}t\right]\phi(\theta)\mathrm{d}\theta.
\end{aligned}$$

Further, with

$$d(\theta, W) = m\left(\frac{C + mP - k(m)P}{r+m} - \left(\frac{W}{W_B(\theta)}\right)^{-y}\left((1-\alpha)\theta W_B(\theta) - \frac{C + mP - k(m)P}{r+m}\right)\right)$$

it holds

$$\begin{aligned}
&\mathbb{E}_W\left[\int_0^{\tau(\theta)} e^{-rt}\left(C{+}mP{-}k(m)P{-}d(\theta, W_t)\right)\mathrm{d}t + e^{-r\tau(\theta)}(1{-}\alpha)\theta W_B(\theta)\right]\\
=&\,\mathbb{E}_W\left[\int_0^{\tau(\theta)} e^{-rt}\left(C + mP - k(m)P - \frac{m}{r+m}\left(C + mP - k(m)P\right)\right)\mathrm{d}t\right]\\
+&\,\mathbb{E}_W\left[e^{-r\tau(\theta)}(1{-}\alpha)\theta W_B(\theta) {-} \int_0^{\tau(\theta)} e^{-rt} m\left(\frac{W(t)}{W_B(\theta)}\right)^{-y}\left((1{-}\alpha)\theta W_B(\theta) {-} \frac{C{+}mP{-}k(m)P}{r+m}\right)\mathrm{d}t\right]\\
=&\,\frac{C + mP - k(m)P}{r+m}\left(1 - \left(\frac{W}{W_B(\theta)}\right)^{-x}\right) + (1-\alpha)\theta W_B(\theta)\left(\frac{W}{W_B(\theta)}\right)^{-x}
\end{aligned}$$

$$- mW_B(\theta)^y \left((1-\alpha)\theta W_B(\theta) - \frac{C{+}mP{-}k(m)P}{r+m}\right) \mathbb{E}_W \left[\int_0^{\tau(\theta)} e^{-rt} W(t)^{-y} \mathrm{d}t\right]$$

and we have shown in Appendix 3.B that

$$\mathbb{E}_W \left[\int_0^{\tau(\theta)} e^{-r\,t}\, W_t^{-y}\, \mathrm{d}t\right] = \frac{W_B(\theta)^{-y}}{m} \left(\left(\frac{W}{W_B(\theta)}\right)^{-x} - \left(\frac{W}{W_B(\theta)}\right)^{-y}\right).$$

Therefore, we conduct

$$\begin{aligned}
&\mathbb{E}_W \left[\int_0^{\tau(\theta)} e^{-rt}\, (C{+}mP{-}k(m)P{-}d(\theta, W_t))\, \mathrm{d}t + e^{-r\tau(\theta)}(1{-}\alpha)\theta W_B(\theta)\right] \\
&= \frac{C+mP-k(m)P}{r+m} \left(1 - \left(\frac{W}{W_B(\theta)}\right)^{-y}\right) + (1-\alpha)\theta W_B(\theta) \left(\frac{W}{W_B(\theta)}\right)^{-y} \\
&= \mathbb{E}_W \left[\int_0^{\tau(\theta)} e^{-(r+m)\,t}\, (C{+}mP{-}k(m)P)\, \mathrm{d}t + e^{-(r+m)\tau(\theta)}(1{-}\alpha)\theta W_B(\theta)\right] = d(\theta, W_t)/m.
\end{aligned}$$

Now, taking the expectation with respect to $\tilde{\theta}$ together with equation (3.38) yield that

$$\mathbb{E}[D(\tilde{\theta}, W_0, W_0)] = d(W_0, W_0)/m$$

and thus

$$\mathbb{E}[D(\tilde{\theta}, W_0, W_0; m)] = d(W_0, W_0; m)/m = D(W_0; m). \tag{3.39}$$

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Chapter 4

Credit Rating Under Ambiguity

with Christian Hilpert, Stefan Hirth and Alexander Szimayer

Abstract

We analyse a dynamic credit rating game under ambiguity with feedback effects. A firm signals its quality by surviving phases of apparent distress. A rating agency has multiple priors, representing analysts holding heterogeneous beliefs. Analysts adjust their beliefs as information unfolds. Contrasting classical min-max results under ambiguity, we show that the rating agency dynamically adjusts a weighted average of beliefs, overweighing beliefs with higher variance. Ambiguity affects ratings if analysts' beliefs have a common direction. Firms facing jointly pessimistic (optimistic) analysts strategically delay (accelerate) default, which significantly impacts credit spreads.

4.1 Introduction

This paper considers how ambiguity influences the assessment of credit quality by a rating agency that aims for accurate ratings and holds multiple priors about a firm's cash flow, while the firm maximises its value by selecting its default strategy. The rating agency faces a range of plausible measurement errors of the firm's asset value. The latter is not perfectly observable, for example, due to disagreement on how to assess a firm's intangible assets. Matching the observation that rating agencies assess rated entities with a changing group of analysts (Fracassi et al., 2016; Kempf, 2020; Kempf and Tsoutsoura, 2021), we interpret our setting such that the rating agency needs to aggregate the opinions of several analysts into one rating, as it is the case for all major rating agencies.[1] We thus model an ambiguous decision problem of a group with heterogeneous beliefs as in Garlappi et al. (2017).

The rating agency is unable to specify exact probabilities for each state of the world; instead, it faces ambiguity in the form of a range of possible scenarios for each plausible measurement error, that is, it holds multiple priors (Gilboa and Schmeidler, 1989). In our application, each prior corresponds to one analyst opinion. To address the question of how to rate credit risk under ambiguity with feedback effects, we consider ambiguity similar to recent work on security design (Malenko and Tsoy, 2023) and corporate capital structure (Izhakian et al., 2022). Our second building block consists of the credit rating with feedback effects. Specifically, the firm's financing costs are connected to the rating agency through a feedback loop: The rating agency's assessment influences the firm's capital costs, which affects the firm's default considerations, which in turn feed back to the rating. Thus, the firm and the rating agency are interconnected via performance sensitive debt (Manso et al., 2010; Manso, 2013), and the rating agency learns the firm's credit quality over time, as in Hilpert et al. (2024).

The rating agency aims for accurate and time-consistent ratings to maximise its reputation over time. If the estimated default threshold departs from the actual one, we assume that its reputation suffers equally whether an under- or overestimation occurs. Either failure

[1] For example, Moody's (2025) describe a two-stage process. First, a lead analyst aggregates the assessment of the analyst team and gives a recommendation to a rating committee. The latter then takes a majority vote on the rating. In our model, either the lead analyst or the rating committee can be seen as instances that represent the rating agency as a whole, deciding on a credit rating based on the assessments of individual analysts. See also Fracassi et al. (2016); Kempf (2020). We implicitly follow the assumptions of the Condorcet Jury Theorem, as stated in Gerling et al. (2005). That is, the analysts reveal their true opinions, rather than considering any strategic voting.

is indeed costly, as underestimating the distance-to-default implies losses for the investors, whereas a too conservative perspective may prevent firms' access to financing. The rating agency aims to maximise its reputation, based on its expectation of the stream of worst-case reputational damages that its rating implies. Such a strategy allows the rating agency to include multiple analyst opinions in a time-consistent strategy. We choose this approach instead of selecting a reputation-optimising strategy over the worst-case expectation of its reputation, as our approach avoids reputation optimisation that induces time-inconsistent rating strategies that occur, for example, in Garlappi et al. (2017).[2] As Kisgen et al. (2020) show, the rating agency's objective for accuracy is handed through to the analysts, as accurate analysts are more likely to be promoted and less likely to depart.

The firm's manager-owner maximises the firm value by choosing the default strategy. The rating agency observes the firm's cash flow only with measurement error as a basis for its rating. The firm accounts for the rating, and the subsequent capital costs, in its strategy. In particular, the firm anticipates the learning of the rating agency over time: If the observed cash flow implies that the firm is in distress, then the rating agency infers that the current situation is costly for the firm. Once the cash flow deteriorates enough, the firm cannot sustain the costs and defaults. Therein lies a learning opportunity for the rating agency. If it observes apparent distress but no default, the rating agency updates its beliefs, for each prior,[3] to exclude possible measurement error levels for which the firm would have defaulted already.

We show that under this learning mechanism, the firm follows a cash flow based threshold strategy. The rating agency's best response then turns out as a dynamic updating of the estimated distance-to-default, based on whichever measurement errors it can already exclude from observing non-default. Hence, for the estimated default threshold, the running minimum of the observed cash flow generates new information because it identifies the measurement errors that the rating agency can rule out. However, although the rating depends on the distance between the observed cash flow to the estimated default threshold, the firm determines the actual default threshold based on the cash flow alone. The rating agency can never obtain perfect information before default, as it only gets to know the true firm's quality at the time the firm defaults. The credit rating game generally features

2 This formulation allows us to avoid time inconsistency which from a game-theoretic perspective is challenging to manage. In discrete time, Pahlke (2022) proposes a general approach to rule out dynamic inconsistency given max-min expected utility for games featuring ambiguity.

3 For tractability, we assume that the priors have identical support, i.e., possible and impossible events are identical across priors.

a unique equilibrium.

We show that ambiguity has a profound impact on credit ratings. First, we argue that the rating agency does not simply take over the prediction of the most pessimistic analyst to account for the worst case. Rather, avoiding the worst case in our context can result in averaging between analysts and taking a less extreme stance on the firm's outlook. This perhaps surprising result roots in the agency's aim to estimate the firm's distance to default as precise as possible. Reputation concerns symmetrically affect the rating agency: Both systematic deviations to over- or underestimate the default threshold are equally unattractive. We observe that ambiguity aversion leads the rating agency to give more weight to less informative beliefs over pronounced opinions that are clearly positive or negative.

Second, contrasting classical min-max results rooting in ambiguity aversion in models of multiple priors, the rating agency aggregates multiple prior beliefs in a weighted average. As information unfolds, the rating agency adjusts the weights it places on each prior with the shifts in direction within each prior. For example, if the rating agency considers a pessimistic distribution featuring high probability mass on overvaluation and an optimistic distribution with a lot of probability mass on undervaluation, learning from survival in apparent distress implies a substantial shift in the pessimistic distribution whereas the optimistic belief remains largely stable. The reason lies in the rating agency's learning: Under the pessimistic prior, the rating agency learns more, that is, for the pessimistic distribution, the part of the belief that is initially revised carries more probability mass compared to the optimistic distribution. It turns out that when pessimistic and optimistic beliefs are equally present, the learning raises the informativeness of the pessimistic belief compared to the optimistic one, which in turn causes the rating agency to sharply underweigh the pessimistic belief.

Third, we show that the impact of the feedback effect under ambiguity critically hinges on whether rating analysts show a common direction of disagreement. In case of pessimistic and optimistic beliefs being equally present, ambiguity has little impact on either the firm's or the rating agency's aggregated strategy. In contrast, if the analysts' beliefs are jointly pessimistic, that is, if they place high probability mass on the actual cash flow being below the currently observed one, then the rating agency tends to select the most moderate belief it can justify as its sole prior.

This extreme response to ambiguity with jointly pessimistic beliefs affects both the rating agency's and the firm's strategy via the feedback loop: The rating agency now has a belief that is more moderate than a weighted average. Consequently, the firm delays default compared to the case with asymmetric information and without ambiguity, which we take as the average distribution across priors (Halevy, 2007). Economically, the firm waits for the rating agency to learn its quality because the overall belief, although balanced, remains pessimistic. Thus, once the rating agency starts to learn, its belief adjusts substantially once it updates its priors. Hence, the firm's capital costs reduce in a meaningful way once it recovers, making the delayed default appealing. In equilibrium, the rating strategy responds accordingly. As the flip side of the coin, the feedback loop with jointly optimistic beliefs accelerates firm defaults because the rating agency takes a more conservative belief as the foundation of its choice, making it more expensive for the firm to wait for the rating agency to learn its quality compared to the case without ambiguity.

Our paper contributes, firstly, to the literature of credit rating and credit risk in dynamic settings. Building on structural credit risk models with endogenous default, such as Black and Cox (1976), Leland (1994), and Goldstein et al. (2001), Manso et al. (2010) consider performance-sensitive debt to capture feedback loops. In turn, Manso (2013) considers credit rating with feedback effects. In a seminal paper, Duffie and Lando (2001) consider asymmetric information between the firm and its creditors and allow for learning of credit risk over time. While Duffie and Lando (2001) features exogenous learning, Hilpert et al. (2024) analyse the learning of credit risk from strategic default.

Secondly, our paper contributes to the impact of group decisions in corporate finance. Corporate boards of directors jointly decide on key decisions, with board members diverging in their information as insiders and outsiders, as well as their preferences regarding the choice in question. Baranchuck and Dybvig (2009) introduce a consensus equilibrium to argue that diverse boards featuring preferences across multiple dimensions benefit from a majority of outside directors and cannot be moved substantially by individual extreme opinions. Chemmanur and Fedaseyeu (2018) show that dissent costs can prevent board members from enforcing the efficient majority opinion to fire a low-quality CEO and coordination frictions prevent effective information aggregation within diverse boards. Contrasting previously developed static models of corporate group decisions, Donaldson et al. (2020) build a dynamic voting model of a corporate board which hesitates to implement pareto-dominant policies because such policies prevent board members to later replace them with their preferred, more controversial policies. Garlappi et al. (2022) consider the exercise of a sequence of standard real option problems to invest in and abandon a project by a group of diverse decision makers and show that the exercise behaviour deviates from both a representative median member and the majority opinion. Similar to our approach, Garlappi et al. (2017) focus on investment dynamics and study how multiple prior specifications differ in their investment predictions, emphasizing the importance of ambiguity with a group of heterogeneous decision-makers.

Besides presenting a natural approach to capture heterogenous opinions in a group of decision makers, multiple priors present a fruitful approach to address financial questions. Malenko and Tsoy (2023) consider the optimal security design under ambiguity and show that ambiguity increases the attractiveness of equity over debt to finance new projects. In a related paper, Izhakian et al. (2022) develop a static 2-period model of the trade-off theory of capital structure under ambiguity and show that ambiguity-averse managers tend to increase leverage. Baillon et al. (2018) consider learning for initial public offerings with ambiguous information. Furthermore, ambiguity can also explain why investors refrain from investing in seemingly profitable investment opportunities in the first place, as in Easley and O'Hara (2009).

Finally, our notion of ambiguity within a credit analyst team can be related to measures of market-wide ambiguity, which capture the empirically well-documented fact that financial information lacks exactly known probabilistic models. Ambiguity turns out to be a driving factor of credit risk, in particular, it can explain the credit spread increases during the 2007-2008 U.S. financial crisis (Boyarchenko, 2012) and significantly influences the pricing of credit default swaps (Augustin and Izhakian, 2020).

4.2 A Model of Credit Rating under Ambiguity

In this section, we gradually develop a model of a credit rating agency's learning about a firm's credit risk under ambiguous and dynamic information with feedback effects. The firm's payoff consists of its expected discounted stream of cash flows net of capital costs up to default. The rating agency considers its reputation, given by the discounted stream of its reputation costs. Reputation costs are measured as the deviation of the estimated default threshold to the actual one. The strategy space consists of the firm's default strategy and the rating agency's estimated default strategy with belief updating for the rating agency.

We set off by introducing a base-case model under asymmetric information without feedback effects by holding coupons constant. This special case corresponds to Hilpert et al. (2024; Section 2.2). Then, we add the ambiguous information structure using multiple priors. Finally, we add performance-sensitive debt to the model to connect the firm and rating agency by a feedback loop and discuss economically reasonable strategies to allow for a tractable analysis of an ambiguity-extended version of Perfect Bayesian Nash Equilibria (Hanany et al., 2020; Malenko and Tsoy, 2023).

4.2.1 Credit rating with learning under asymmetric information

We consider a levered firm with one outstanding consol bond, managed by the sole owner of the firm's equity in a structural model of credit risk (Leland, 1994; Goldstein et al., 2001). It generates a cash flow $X = (X_t)_{t \geq 0}$ and pays a constant coupon C on its outstanding debt. The cash flow satisfies

$$\frac{dX_t}{X_t} = \mu dt + \sigma dB(t), \tag{4.1}$$

where $\mu < r$ and σ represents the cash-flow's growth rate and volatility, respectively, with r being the risk-free rate and $B = (B_t)_{t \geq 0}$ a Wiener process. The cash flow's drift μ, its volatility σ and r are constants and common knowledge. The firm's cash flow X is the firm's manager-owner's private information. The rating agency observes the firm's cash flow only imperfectly due to a persistent multiplicative measurement error $\tilde{\theta}$, that is, the observed cash flow follows

$$D_t = \tilde{\theta} X_t \tag{4.2}$$

for $t \geq 0$ and it has the same dynamics as the true cash flow X.[4,5] The firm knows the realised θ. We refer to a downward (multiplicative) measurement error, when $\theta < 1$, and the observed cash flow is below the firm's true cash flow and to an upward (multiplicative) measurement error, when $\theta > 1$, and the observed cash flow exceeds the firm' true cash flow.

The rating agency determines the rating as the estimated distance to default given its

[4] This multiplicative formulation stems from Grenadier et al. (2016) and is also used in Hilpert et al. (2024). This specific formulation of the persistent measurement error excludes statistical learning as in David (2008) and Pastor and Veronesi (2009).

[5] In structural models, the firm's asset value V is typically a scalar multiple of the firm's cash flow, $\delta V = X$ with $\delta = r - \mu$. This connection allows for an alternative interpretation of our framework: Uncertainty regarding the firm's overall asset value can be viewed as uncertainty about its intangible assets.

current information. Formally, the rating equals

$$R_t = \frac{D_t}{\hat{D}_t^\star}, \tag{4.3}$$

the ratio of the current observed cash flow to the predicted default cash flow level $\hat{D}_t^\star$ at each time $t \geq 0$. A rating of $R = 1$ implies immediate default, whereas the rating diverges to infinity as the observed cash flows approach infinity, that is, the default risk vanishes.

The firm chooses the default strategy τ to maximise its equity value. The firm owner is risk-neutral and maximises the present value of the cash flows net the interest payments on its outstanding debt. In this particular model case, excluding feedback effects, the firm endogenously selects the default strategy to maximises its value in isolation. Any action of the rating agency cannot influence the firm value and subsequently its choice. Because the coupon is constant, the true default threshold is a commonly known constant

$$X_C^\star = \frac{\eta}{\eta+1}\frac{r-\mu}{r}C \quad \text{with} \quad \eta = \frac{1}{\sigma^2}\left[\left(\mu - \frac{\sigma^2}{2}\right) + \sqrt{\left(\mu - \frac{\sigma^2}{2}\right) + 2r\sigma^2}\right]. \tag{4.4}$$

The rating agency learns the observed default threshold $D^\star(\theta) = \theta X_C^\star$ by observing the firm's decision not to default for low observed cash flows, containing the valuable information that the firm's measurement error does not exceed a particular level. Consequently, the minimum observed cash flow $E = (E_t)_{t\geq 0}$, with $E_t = \inf_{0\leq s\leq t} D_s, t \geq 0$, carries value on top of the observed current cash flow D. The extended state space thus consists of the current observed cash flow D_t and its running minimum E_t, as the rating agency takes E as the information generating process and the default threshold is a function of this variable, that is, $\hat{D}^\star = g(E)$.

The rating agency forms a belief $\pi = (\pi_t)_{t\geq 0}$ about the law of $\tilde{\theta}$. The belief is common knowledge and fully described by the density $\phi^{\pi_0} := \phi$, which we call the prior. The prior is bounded away from zero and above and has support $\Theta = [\underline{\theta}, \overline{\theta}]$ with $0 < \underline{\theta} < \overline{\theta} < \infty$. The rating agency dynamically updates its belief over $t \geq 0$ according to Bayes' rule whenever possible until default. The firm's best solution is to play a threshold based default strategy $\tau(\theta) = \inf\{t \geq 0 : D_t \leq D^\star(\theta)\}$ for each type $\theta \in \Theta$. To ensure compatibility with latter more general formulations, we denote the firm's default threshold by a function $f(\theta) = D^\star(\theta)$. The updating presumes that the density ϕ^{π_t} characterises the belief about the distribution of the persistent measurement error.[6] In combination with the firm's default strategy τ, satisfying $0 \leq t < \tau$, the consistent belief equals

$$\phi^{\pi_t}(\theta) = \frac{1_{\{f(\theta)<E_t\}}}{\int_\Theta 1_{\{f(\theta')<E_t\}}\phi(\theta')d\theta'}\phi(\theta), \text{ for } \theta \in \Theta. \tag{4.5}$$

The rating agency selects the predicted default level $\hat{D}^\star = (\hat{D}^\star_t)_{t\geq 0}$ as its strategy to minimise its reputation costs conditional on its current information consisting of the belief π and the firm's default strategy τ. Specifically, the rating agency minimises the squared distance between the estimated default threshold given its belief and the true one, $f(\theta) = D^\star(\theta)$, yielding the cost rate

$$k^\pi_t = \int_\Theta \left(\hat{D}^\star_t - f(\theta)\right)^2 \phi^{\pi_t}(\theta)\mathrm{d}\theta, \text{ for } t \geq 0 \tag{4.6}$$

at any given time. The rating agency aggregates and discounts it to an overall reputation

$$U^\pi_{RA}(\tau, \hat{D}^\star) = \mathbb{E}\Big[\int_0^\tau e^{-\rho t}k^\pi_t \ \mathrm{d}t\Big]. \tag{4.7}$$

Here, $\rho > 0$ is the rating agency's internal discount rate. This optimisation results in an estimated default threshold

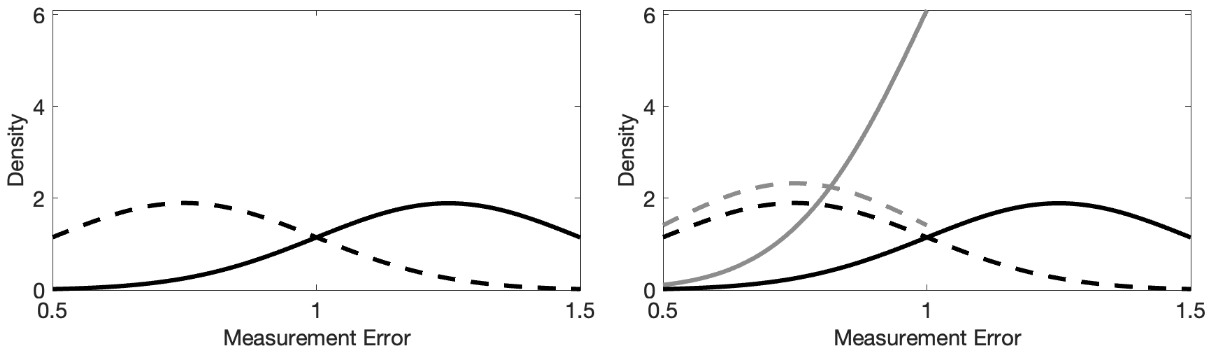
$$\hat{D}^\star_{C,t} = \mathbb{E}^{\pi_t}\left[D^\star(\tilde{\theta})\right] = X^\star_C\mathbb{E}^{\pi_t}\left[\tilde{\theta}\right]. \tag{4.8}$$

4.2.2 Credit rating with learning under ambiguity

This section expands the base-case model by introducing ambiguity in the guise of multiple priors. Extending the information structure of Grenadier et al. (2016) and Hilpert et al. (2024), we consider an ambiguity averse rating agency which has n priors, representing n internal credit analysts of the rating agency, with differing beliefs $\pi_i = (\pi_{i,t})_{t\geq 0}$ about the law of $\tilde{\theta}$ for $i = 1, ..., n$.[7] The beliefs are common knowledge and initial beliefs are fully described by the densities $\phi^{\pi_{i,0}} := \phi_i$, which we call the priors. All priors are bounded away from zero and above and have the same support $\Theta = [\underline{\theta}, \overline{\theta}]$ with $0 < \underline{\theta} < \overline{\theta} < \infty$. The

6 $\tilde{\theta}$ is independent of the Wiener process W. The filtrations $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ capture the information generated by X and D, respectively. Formally, the firm's information set at t is given by $\sigma(\tilde{\theta}) \vee \mathcal{F}_t$, for $t \geq 0$. Since $\tilde{\theta}$ is known to the firm at $t = 0$, we can condition on $\tilde{\theta} = \theta$ and work with $\mathbb{F}$.

7 The special case $n = 1$ simplifies to the model in Hilpert et al. (2024).



(a) Initial beliefs *(b) Updated beliefs*

Figure 4.1: Belief updating: *This figure shows two analysts' beliefs, a pessimistic one (solid) and an optimistic one (dashed), about the persistent multiplicative cash flow measurement error $\tilde{\theta}$. Panel 4.1a shows both analysts' beliefs at time zero in black. Panel 4.1b shows the same beliefs and the update analysts' beliefs (grey) after measurement errors above one have been ruled out. The parametrisation follows the neutral case in Table 4.1.*

rating agency, for each prior i, dynamically updates its beliefs over $t \geq 0$ and presumes that the densities $\phi^{\pi_{i,t}}$ characterise the different beliefs about the distribution of the persistent measurement error.

As the rating agency aims to assess the firm's default probability, the running minimum of the observed cash flows E_t continues to carry information value under ambiguity. In combination with the firm's default strategy τ, given by its default threshold f, each consistent belief i equals

$$\phi^{\pi_{i,t}}(\theta) = \frac{1_{\{f(\theta)<E_t\}}}{\int_\Theta 1_{\{f(\theta')<E_t\}}\phi_i(\theta')d\theta'}\phi_i(\theta), \text{ for } \theta \in \Theta, \tag{4.9}$$

with $0 \leq t < \tau$ and $i \in \{1, ..., n\}$.

Figure 4.1 illustrates the belief updating from Equation (4.9). Panel 4.1a shows the initial situation of two analysts holding initial beliefs about the persistent multiplicative cash flow measurement error $\tilde{\theta}$ as introduced in Equation (4.2).[8] In this case, the pessimistic analyst places more probability mass on an upward measurement errors (solid line), which implies that the analyst expects the firm's observed cash flow to be above the true level. On the other side, the optimistic analyst thinks downward measurement errors are more likely (dashed line), such that the optimistic analyst expects the observed cash flow to be

[8] Formally, the priors are described by the densities ϕ_1 and ϕ_2 as truncated Gaussian distributions on $[0.5, 1.5]$ with mean parameters $\xi_1 = 1.25$ (pessimist) and $\xi_2 = 0.75$ (optimist). The standard deviation parameter for both analysts is $\kappa_1 = \kappa_2 = 0.25$.

below the true level. Panel 4.1b displays the case in which the minimum of the observed cash flows has decreased sufficiently to allow the rating agency to eliminate measurement errors exceeding $\theta > 1$. Thus, both analysts include this information and update their beliefs according to Equation (4.9). Panel 4.1b shows their initial and updated beliefs in black and grey, respectively.

The rating agency selects the predicted default level $\hat{D}^\star = (\hat{D}_t^\star)_{t\geq 0}$ as its strategy to minimise the costs of the rating agency under ambiguity aversion, expands the asymmetric information reputation costs to

$$U_{RA}^{\pi}(\tau, \hat{D}^\star) = \mathbb{E}\Big[\int_0^\tau e^{-\rho t}\left(\max_{i=1,\dots,n} k_t^{\pi_i}\right) dt\Big]. \tag{4.10}$$

Here $k_t^{\pi_i}$ is the cost rate for the predicted default level $\hat{D}^\star$ of belief i, given by[9]

$$k_t^{\pi_i} = \int_\Theta \left(\hat{D}_t^\star - f(\theta)\right)^2 \phi^{\pi_{i,t}}(\theta) d\theta. \tag{4.11}$$

As under asymmetric information, the rating agency's aim is to minimise its reputation damage. The latter occurs when its estimated default threshold deviates from the correct one. Due to ambiguity aversion, it faces the maximum cost rate at any given time. Equation (4.10) deviates from the standard max-min expected utility framework (Gilboa and Schmeidler, 1989), as it implies a time-consistent rating agency that maximises its reputation by considering the worst-case reputation costs for every point in time. Thus, our formulation avoids an optimisation in which the rating agency considers worst-case expectation over the entire horizon, which may cause inconsistencies due to a switching worst case as in Pahlke (2022).

As we assume a constant coupon so far, this setting still precludes feedback effects. The firm's true default threshold remains $X_C^\star$ and the default threshold in the observed strategy space is $f(\theta) = \theta X_C^\star$. The absence of a feedback loop isolates the firm's payoff from the rating agency's strategy. Consequently, any ambiguity in the rating agency's side remains an issue internal for the rating agency. For each prior i, the expected default threshold of the default threshold equals $E^{\pi_{i,t}}\left[f(\tilde{\theta})\right] = X_C^\star E^{\pi_{i,t}}\left[\tilde{\theta}\right]$. The agency aggregates its belief by predicting a default threshold g_t that minimises the maximal mean squared error given

[9] Instead of $f(\theta)$, the correct specification would be $\mathbb{E}[D_{\tau(\theta)} \mid \mathscr{F}_t]$, where $D_{\tau(\theta)}$ is potentially not measurable with respect to the rating agency's filtration $\mathscr{F}$. However, later in Proposition 2 we show under some mild assumptions that the default time $\tau(\theta)$ is the first hitting time of a default threshold $f(\theta)$, and hence $\mathbb{E}[D_{\tau(\theta)} \mid \mathscr{F}_t] = f(\theta)$.

the analysts beliefs

$$g_t = \arg\min_{D>0} \left\{ \max_{i=1,\ldots,n} \mathbb{E}^{\pi_{i,t}} \left[(D - f(\tilde{\theta}))^2 \right] \right\}. \tag{4.12}$$

Since the default threshold is linear in θ, the rating can equivalently be described by the analysts trying to minimise the maximal mean squared error of the measurement error

$$g_t = \arg\min_{\theta^\star \in \Theta} \left\{ \max_{i=1,\ldots,n} \mathbb{E}^{\pi_{i,t}} \left[(\theta^\star - \tilde{\theta})^2 \right] \right\} X_C^\star. \tag{4.13}$$

4.2.3 Credit rating with learning under ambiguity and feedback effects

This section introduces our full model featuring the learning of the firm's credit risk by the rating agency under ambiguity, which connects the strategies of rating agency and the firm in a game via a feedback loop. Specifically, we model the firm's liability as a performance-sensitive debt claim (see Manso et al., 2010; Manso, 2013). Therefore, the firm's cost of capital, that is, the debt coupon, reacts to the rating. Formally, the coupon follows a non-decreasing function $C : [1, \infty] \mapsto \mathbb{R}^+$ depending on the firm's rating:

$$C_t = C(R_t) \tag{4.14}$$

We employ the following coupon structure to ensure that C is sufficiently smooth to allow for an equilibrium.[10]

Assumption 1. *Assume that the interest payment rate C satisfies for some $0 < L_C < 1$ that*

$$C(z) \leq C(z') \leq (z/z')^{L_C} C(z)\,, \text{ for } 1 \leq z' \leq z\,. \tag{4.15}$$

Under the performance-sensitive debt induced feedback loop, the firm strategically accounts for the effect of its default strategy τ on the rating agency. Since the firm is aware of the measurement error at the start, we denote $\tau = (\tau(\theta))_{\theta \in \Theta}$, where $\tau(\theta)$ is a stopping time

[10] Assumption 1 imposes Lipschitz continuity on the log-log scale; that is, for some L_C, with $0 < L_C < 1$, it holds that $|\log C(z) - \log C(z')| \leq L_C\, |\log(z) - \log z'|$, for all $z, z' \geq 1$.

implemented by type $\theta \in \Theta$. With $X = D/\tilde{\theta}$, the firm's equity value is given by

$$U_F^{(\theta)}(\tau, \hat{D}^\star) = \mathbb{E}\Big[\int_0^{\tau(\theta)} e^{-rt}\Big(D_t/\theta - C(D_t/\hat{D}_t^\star)\Big)dt\Big]. \tag{4.16}$$

Here, the coupon function C, depending on the observed distance to default, highlights that the firm's strategy τ influences the estimated default threshold $\hat{D}^\star$, and subsequently, its capital costs, feeding back into its equity value.

The firm plays a default strategy of the type[11]

$$\tau(\theta) = \inf\{s \geq 0 : D(s) \leq f(\theta)\}. \tag{4.17}$$

The rating agency's beliefs π_i are updated via Bayes-Rule whenever possible. The rating agency's strategy equals the predicted default level, must be sequentially optimal, and is a function of the observed cash flow's running minimum $E_t = \inf_{s\leq t} D_s$, that is $\hat{D}_t^\star = g(E_t)$ for some measurable $g : \mathbb{R}^+ \to \mathbb{R}^+$. By Equation (4.3), the firm's rating is then given by $R_t = D_t/g(E_t)$.

To obtain a tractable model, we focus on Markov strategies. Because beliefs in our model are formed under ambiguity, unlike Grenadier et al. (2016), we cannot directly employ Perfect Bayesian Nash equilibria in Markov strategies. Instead, we use the sequential-optimality expansion by Hanany et al. (2020) in the implementation for multiple priors by Malenko and Tsoy (2023).

Moreover, we restrict our analysis to pure strategies, and additionally focus on rating strategies that are economically reasonable. We call a rating strategy g economically reasonable, if it satisfies the following conditions:

$$g(e) \leq e, \quad g(e) \text{ is non-decreasing}, \quad \text{and} \quad g(e)/e \text{ is non-increasing}. \tag{4.18}$$

The first inequality states that any predicted default level must be below the observed running-minimum of the cash flow. The second condition ensures that firm's survival of a distressed period and recovery to an initial higher cash flow level improves the rating. Furthermore, the third condition ensures that arrival at a new all-time low cash flow does not improve the rating. We denote the set of rating strategies satisfying Equation (4.18) by $\mathscr{A}_g$. Similarly, a firm's default strategy f is called economically reasonable if it is strictly

[11] It is possible to show that for economically reasonable rating strategies, which satisfy the conditions in Equation (4.18), the firm strategy indeed takes the form of a cut-off strategy.

increasing and continuous. We denote the set of such strategies by $\mathscr{A}_f$.

4.3 Best Responses

In this section, we consider the best responses of the firm and the rating agency based on their information. If the rating agency follows an economically reasonable strategy, the firm's best response is to default at a type-dependent default barrier. The rating agency's best response is to predict the cost-minimising expected default threshold given its current information. Depending on the economic situation, the rating agency bases its default threshold prediction on the most moderate prior assessment or a weighted average of beliefs. The rating agency weighs the estimated default thresholds under each prior to obtain its overall estimate, where the weight for a particular prior increases if it features higher residual variance, that is, the prior is less informative.[12]

4.3.1 Firm's best response

Although the main focus of our analysis lies on the rating agency's strategy, the subsequent equilibrium analysis requires the firm's best response. The firm's best response equals a cut-off strategy regardless of the exact information structure; in particular, the firm neglects how the rating agency internally determines its strategy with or without ambiguity but formulates a general response. Thus, we characterise the firm's best response default strategy τ for any economically reasonable rating strategy g as under asymmetric information precluding ambiguity with the following proposition from Hilpert et al. (2024).[13]

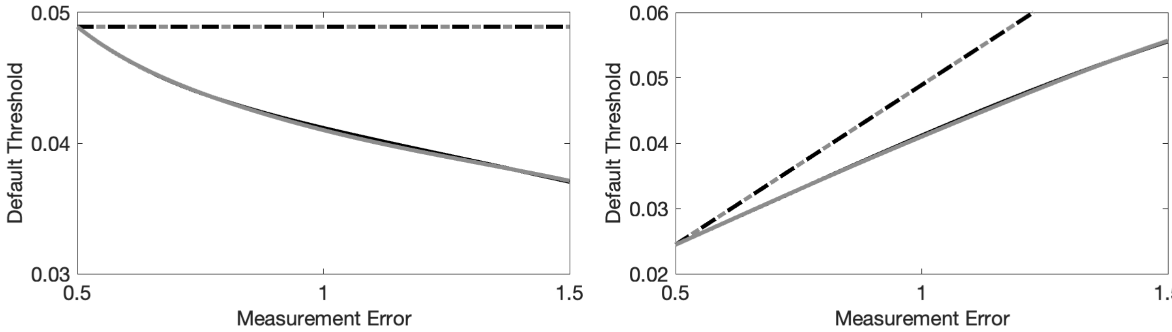
Proposition 2. *For rating strategies $g \in \mathscr{A}_g$, $\theta \in \Theta$, the optimal stopping time of Equation* (4.16) *is given by*

$$\tau(\theta; g) = \inf\{s \geq 0 : D(s) \leq f(\theta; g)\}, \tag{4.19}$$

where $f(\theta; g)$ is some positive real constant. Moreover, $f(\cdot; g)$ is strictly increasing and

[12] Recall that both the presented best responses of firm and rating agency in this section are general responses to any strategy the other player can use independent of whether this strategy is actually played in equilibrium. We relegate the discussion of equilibrium effects to Section 4.5.

[13] Formally, this shows that for rating strategies $g \in \mathscr{A}_g$, the firm's best response default strategy is indeed of the form (4.17) and described by a default threshold function $f : \Theta \to \mathbb{R}^+$.



(a) True cash flow *(b) Observed cash flow*

Figure 4.2: Firm's best response. *This figure illustrates the firm's best response in terms of the true cash flow (Panel 4.2a) and the observed cash flow (Panel 4.2b) for the range of multiplicative measurement errors. The black lines indicate the default threshold under ambiguity; grey lines represent the case with asymmetric information. Solid lines show the default threshold including feedback effects, whereas dashed lines represent constant coupons, i.e., no feedback effects. The parametrisation follows the neutral case in Table 4.1.*

continuous.

The firm's optimal default strategy is the first hitting time of the imperfectly observed cash flow D with respect to the default barrier $f(\theta; g)$. Figure 4.2 displays the firm's best response in terms of the true cash flow (Panel 4.2a) and observed one (Panel 4.2b) as a function of the multiplicative measurement errors θ. The solid black line indicates the firm's default threshold under both ambiguity and feedback effects, whereas the grey solid line displays the default threshold with asymmetric information and feedback effects.[14] Likewise, the dashed lines show the default thresholds with constant coupons, that is, without feedback effects, with and without ambiguity in black and grey, respectively.

In case of feedback effects, both with and without ambiguity, for downward multiplicative measurement errors, the firm's observed cash flow is below the true cash flow, which worsens the rating and, subsequently, implies high capital costs (Panel 4.2a). Consequently, a firm with a downward measurement error defaults for a higher true cash flow compared to a firm with an upward measurement error given identical true cash flows. Panel 4.2b presents the flip-side of the coin: For a fixed observed cash flow, a downward measurement error $\theta < 1$ implies that the firm's true cash flow is higher than the observed cash flow indicates. Thus, the firm finds it optimal to delay default further: It can afford to wait for the rating agency to learn and profit from the following reduction in capital costs. In this particular

[14] The case with asymmetric information (and without ambiguity) is characterised by a single rating agencies belief, which consists of the average density of the analysts' priors.

neutral setting, ambiguity does not affect choices as the analysts are set not to shift the rating agency's assessment. We relegate the discussion of ambiguity effects on the firm's default threshold to the equilibrium analysis in Section 4.5.

In general, without feedback effects, the firm's true default threshold is independent of ambiguity; the dashed black and grey lines are identical and constant. In this case, ambiguity is a rating agency-internal issue which the firm ignores. As the firm's coupons are constant, the firm also neglects the measurement error. Any assessment on the rating agency's side does not affect the firm's equity value and, subsequently, its default timing. However, in terms of observed cash flows, downward measurement errors imply a lower default threshold.

4.3.2 Rating agency's best response

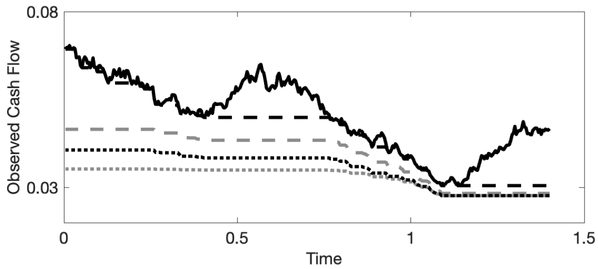
Now, we consider dynamic learning and the best response of the ambiguity-averse rating agency.

Proposition 3. *Let a firm's default strategy τ be given by some default threshold function $f \in \mathscr{A}_f$, with $\tau(\theta) = \inf\{s \geq 0 : D_s \leq f(\theta)\}$. The rating agency's ambiguity averse best-response to f is given by $\hat{D}^\star = g(E; f)$ with*

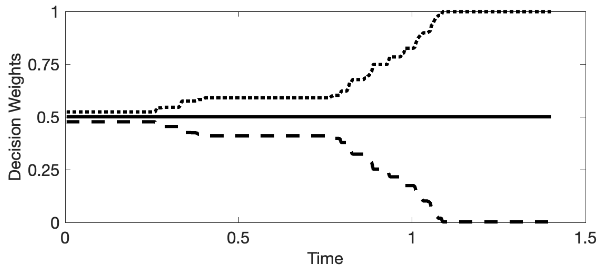
$$g(e; f) = \begin{cases} \arg\min\limits_{g>0} \left\{ \max\limits_{i=1,\dots,n} \mathbb{E}^{\pi_{i,0}} \left[(g - f(\tilde{\theta}))^2 \mid f(\tilde{\theta}) < e \right] \right\}, & \text{for } e > \inf_{\theta\in\Theta} f(\theta)\,, \\ e\,, & \text{else,} \end{cases}$$

and $g(e) \leq e$, for $e \geq 0$.

To better grasp the intuition of the rating agency's learning, consider Figure 4.3. It shows a trajectory of the observed cash flow (Panel 4.3a) and the rating agency's weights (Panel 4.3b) for the sample density Figure 4.1 displays. Panel 4.3a shows the path of an observed cash flow (solid black line) and its observed running-minimum (dashed black line). Based on the observed running minimum, the rating agency postulates an expected default level for an optimistic (grey dotted line) and a pessimistic prior (grey dashed line). Its strategy equals its expected default threshold (black dotted line). Panel 4.3b displays the weights the rating agency places on the pessimistic (dashed line) and optimistic (dotted line) priors. The exact parametrisation follows Table 4.1 and will be discussed in detail in



(a) Trajectory



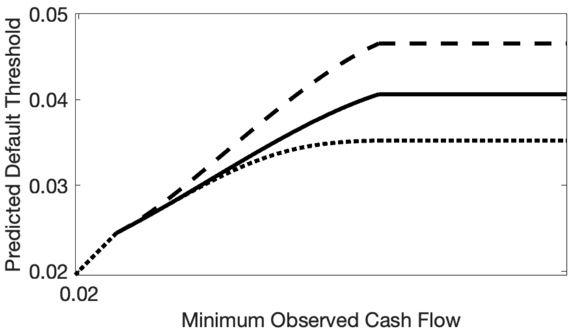
(b) Weights

Figure 4.3: Learning for sample path. *This figure illustrates the rating agency's learning of the firm's default threshold (Panel 4.3a) and the evolution of its weights (Panel 4.3b) for a sample trajectory of the observed cash flow. Panel 4.3a shows the path of an observed cash flow (solid black line) and its observed running-minimum (dashed black line). Based on the observed running minimum, the expected default threshold for the pessimistic prior and an optimistic prior follow the grey dashed and dotted lines, respectively. The agency's estimated default threshold is captured by the black dotted line. Panel 4.3b displays the weights the rating agency places on the pessimistic (dashed line) and optimistic (dotted line) priors. The black line shows an equal weighting for comparison. The parametrisation follows the neutral case in Table 4.1.*

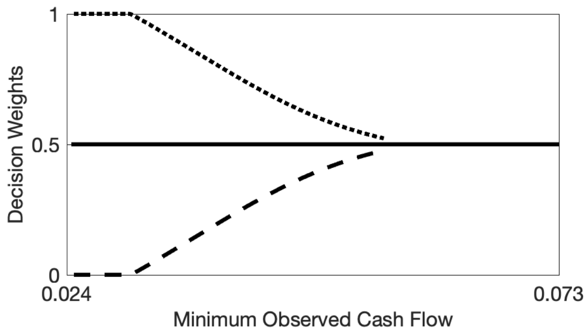
Section 4.5.

For this trajectory of the observed cash flow, each prior individually implies a default threshold for the firm optimising the stream of the agency's expected reputation damage conditional on their prior, which it updates as information unfolds. The optimistic prior postulates a lower expected default threshold because the probability mass on downward measurement errors for which the true cash flow exceeds the currently observed one are higher. Conversely, the pessimistic prior implies a firm default for a higher observed cash flow. Initially, the rating agency overweighs the optimistic prior slightly (Panel 4.3b).

Approximately at time 0.3, the observed cash flow deteriorates sufficiently for both priors to discard the highest upward measurement error as implausible (recall that both priors have the same support). Consequently, both adjust their expectation of the firm's default threshold downwards. The learning affects the pessimistic prior's information stronger because the eliminated measurement errors carry high weight in its belief, causing a higher relative shift in its belief. As the perceived distress of the firm deepens, the rating agency updates its expectation of the default threshold based on both priors. However, its weights for the analysts shift with this updating: After learning, the rating agency increasingly overweighs the optimistic belief in its overall assessment. From time 1.1 onwards, the rating agency ignores the pessimistic analyst and exclusively bases its rating on the optimistic prior.



(a) Rating strategy, plain



(b) Weights

***Figure 4.4:* Rating agency's best response.** *This figure illustrates the best response of the rating agency. Panel 4.4a presents the rating agency's best response as a solid line. The pessimistic prior expects the default threshold indicated by the dashed line, whereas the optimistic prior expects default at the dotted line. Panel 4.4b displays the weights the rating agency places on the pessimistic (dashed line) and optimistic (dotted line) prior. The solid line indicates equal weights for comparison. The parametrisation follows the neutral case in Table 4.1.*

Figure 4.4 presents the rating strategy in more detail. Panel 4.4a shows the rating strategy (solid line) as well as the expected default levels under the pessimistic (dashed line) and optimistic (dotted line) priors, respectively. Panel 4.4b presents the weights the rating agency places on the pessimistic (dashed line) and optimistic (dotted line) priors. The solid line indicates an equal weighting. In this example, for simplicity, we plot the rating agency's best response for a firm strategy determined under feedback effects; however, note that the rating agency's best response remains structurally unchanged for alternative firm default thresholds. We relegate the discussion of equilibrium effects to Section 4.5.

As our initial example for the sample trajectory indicates, rating under ambiguity does not lead to what might be expected as a classical "worst-case" behaviour. That is, in the equilibrium above, the first intuition might be that a rating agency that minimises the costs of the worst case should adapt the worst case prior, that is, the prior giving the most pessimistic estimate of the firm's prospect. After all, this seems to minimise the costs of deviating from the true default threshold, if the most pessimistic prior indeed has the most appropriate assessment of the measurement error distribution.

As is obvious in the example above, this is not the case (Panel 4.4b). For the given case, the rating agency does not follow a strategy that assigns a single worst-case prior. Rather, the rating agency chooses a strategy that considers the expectations under each prior. Generalizing our previous example, the rating agency does not place equal weight on each prior but balances them, and, as a result of learning on both beliefs, considerably shifts the weights over time. The reason is that the rating agency's objective function is symmetric in nature, and the rating agency's objective is to produce a rating that is as accurate as possible. We will discuss this in more detail below.

We now explore the structure of the learning presented so far. Focusing on a specific prior $\pi_{i,0}$ in its own right, the best response g_i to any firm strategy f is given by the conditional mean.[15] Specifically, analyst i selects his best estimate of the firm's default strategy f, conditional that the firm has not defaulted at the information available. Denote by ν_i the corresponding conditional variance, then

$$\begin{aligned} g_i(e;f) &= E^{\pi_{i,0}}\left[f(\tilde{\theta}) \middle| f(\tilde{\theta}) < e \right], && (4.20)\\ \nu_i(e;f) &= E^{\pi_{i,0}}\left[\left(g_i(e;f) - f(\tilde{\theta})\right)^2 \middle| f(\tilde{\theta}) < e \right], && (4.21) \end{aligned}$$

for $e > \inf_\theta f(\theta)$, and else $g_i(e;f) = e$ and $\nu_i(e;f) = 0$, $i = 1, ..., n$. Then the central equation of Proposition 3 reads

$$g(e;f) = \arg\min_{g>0} \left\{ \max_{i=1,\dots,n} \left[(g - g_i(e;f))^2 + \nu_i(e;f) \right] \right\}, \tag{4.22}$$

and accordingly, the rating agency strives to minimise the worst case mean squared error over all priors. It can be shown that the minimising argument is either the intersection point of two mean squared error curves of two priors, say, i and j, or it is the expected value of a single prior, say, $i = j$, and is given by

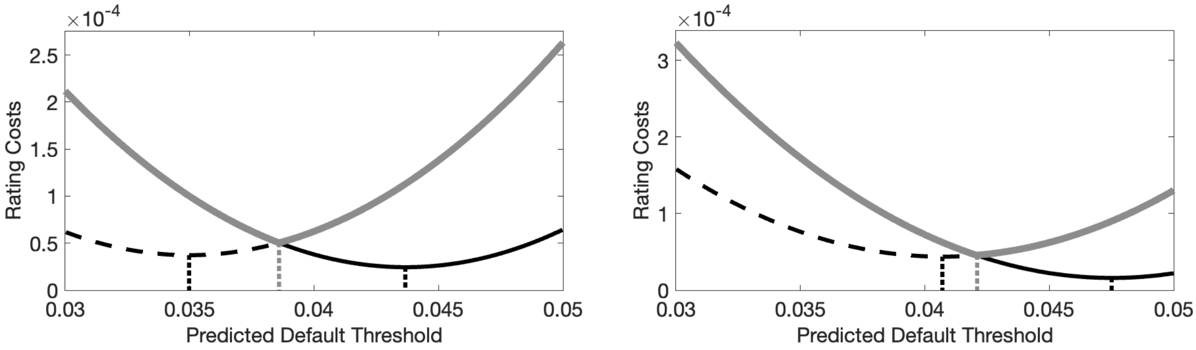
$$g(e;f) = \frac{g_i(e;f) + g_j(e;f)}{2} + \mathbb{1}_{i \neq j} \frac{1}{2} \frac{\nu_i(e;f) - \nu_j(e;f)}{g_i(e;f) - g_j(e;f)}, \tag{4.23}$$

where $(i, j) = (i(e;f), j(e;f))$ depends on the running minimum cash flow e. For better understanding, one may think of i being the most pessimistic and j the most optimistic prior, that is, $g_i(e;f) \geq g_k(e;f) \geq g_j(e;f)$, for all $e \geq 0$ and $k = 1, ..., n$.[16] Equation (4.23) can be restated in terms of weighted average of the best responses $g_i(\cdot;f)$ and $g_j(\cdot;f)$ as follows

$$\begin{aligned} g(e;f) &= w(e;f,i,j)\, g_i(e;f) + (1 - w(e;f,i,j))\, g_j(e;f), && (4.24)\\ w(e;f,i,j) &= \frac{1}{2}\left(1 + \mathbb{1}_{i \neq j} \frac{\nu_i(e;f) - \nu_j(e;f)}{[g_i(e;f) - g_j(e;f)]^2}\right). && (4.25) \end{aligned}$$

[15] See Proposition 1 in Hilpert et al. (2024).

[16] This is not gnerally accurate but holds true in case the conditional variances are all identical, that is, $\nu_i(e;f) = \nu_j(e;f)$, for $1 \leq i, j, \leq n$.



(a) Balanced analysts *(b) Pessimistic analysts*

Figure 4.5: Cost rates and prior weights: *This figure shows how the rating agency balances the priors of two analysts based on the reputation costs their prior implies. Panel 4.5a presents an optimistic analyst (solid) and a pessimistic one (dashed). Panel 4.5b shows a pessimistic analyst (solid) and a super-pessimistic one (dashed). The grey line indicates the maximum of both lines, i.e., the maximal cost rate. The black dotted lines mark the minima of the individual analyst's reputation cost functions. The grey dotted line marks the minimum of reputation cost across analysts.*

For $i \neq j$, the weight w for prior i increases (decreases) relative to prior j, if it has a higher (lower) residual variance, that is, it is less (more) informative.

For the case of two analysts, Equation (4.25) offers an intuitive interpretation. Suppose that one analyst features higher conditional variance than the other; that is, $\nu_1 > \nu_2$. The rating agency's best response in this case is to follow the analyst with higher conditional variance as long as their default predictions are not too divergent. Specifically, using $\hat{D}_i^\star = g_i(e; f)$, $i = 1, 2$, we obtain that

$$w(e; f, 1, 2) = 1 \text{ if } |\hat{D}_1^\star - \hat{D}_2^\star| < \sqrt{\nu_1 - \nu_2}. \tag{4.26}$$

In contrast, if the analysts' estimated default thresholds diverge sufficiently compared to their conditional variances, this condition is violated and the rating agency's best response weighs both analysts' opinions.

Figure 4.5 illustrates the above weighting of the analysts' priors by the rating agency for a fixed observed cash flow and its running minimum. The solid and dashed black lines present the reputation costs to the rating agency based on a relatively pessimistic and optimistic analyst's beliefs, respectively, conditional on the current information in from of the observed cash flow and its current running minimum. The grey line indicates the maximum of both lines, that is, the maximal cost rate across analysts; see Equation (4.22). The black dotted lines mark the minima of the individual analyst's reputation cost functions. The grey dotted line marks the minimum of reputation cost across analysts. The optimistic one predicts a lower default threshold than the pessimistic one.

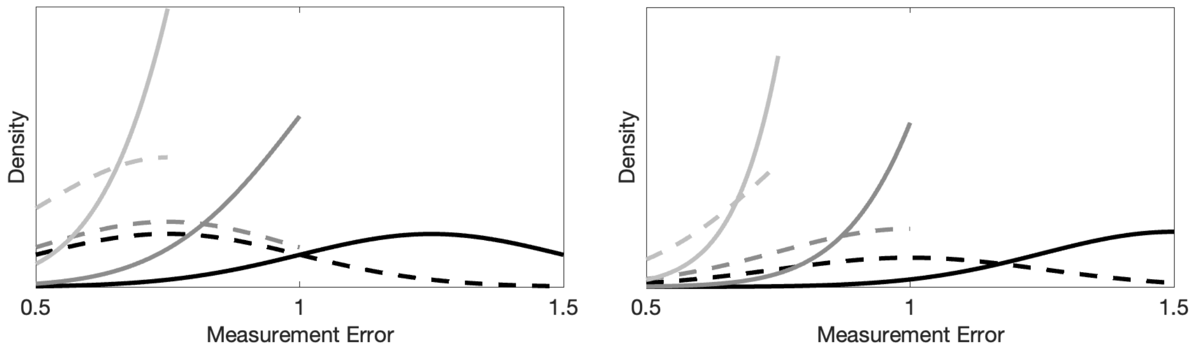
Panel 4.5a presents a balanced case with a pessimistic analyst (solid line) and an optimistic one (dashed line). Figure 4.6 displays the matching densities of the priors for both cases. Consider Panel 4.5a. The rating agency aims to minimise the worst case reputation costs based on both beliefs: to do this, it selects the minimum of the maximal costs it can justify. For example, if the rating agency considers a very low predicted default threshold, the solid line exceeds the dashed one, whereas for a high predicted default threshold the relations change. The overall minimum is obtained by an intermediate predicted default threshold. Note that for this minimum, the rating agency follows neither the optimistic nor the pessimistic analyst but its choices falls between their estimates. In contrast, in Panel 4.5b the rating agency obtains minimal reputation costs by effectively taking a predicted default threshold that alomst minimises the less pessimistic analysts reputation costs (black), that exceeds the costs from balancing both.[17]

While Figure 4.5 explains why the rating agency balances the two agents' priors for a particular information, that is, a state of the state space, it lacks a dynamic perspective. Figure 4.6 illustrates how the rating agency's learning shifts its prior beliefs over time. It shows the initial belief, that is, the prior (black) and after two phases of learning (dark and bright grey). Panel 4.6a shows the belief for a pessimistic (solid lines) and an optimistic (dashed lines) prior, rescaled to be comparable. Panel 4.6b shows a case of a common pessimistic direction.

Consider the case of two priors ($n = 2$) having symmetric initial beliefs. One prior, the pessimist, assigns high likelihood to cases in which the true cash flow has an upward measurement error, that is, the firm is worse off than it looks. Conversely, the likelihood for the downward measurement errors is low, that is, those in which the firm is better off than it looks. In contrast, the other prior, called optimist, assigns high likelihood to cases in which the true cash flow has a downward measurement error, and low likelihood to the upward measurement errors. The rating agency places weights on each prior, where the weights are given by Equation (4.25).

Figure 4.6 indicates that the weight the rating agency places on each of the analysts shifts over time for both the neutral common direction (Panel 4.6a) and the pessimistic one

[17] The pessimist features a truncated normal distribution on $[0.5, 1.5]$ with mean $\xi_1 = 1.25$ and standard deviation of $\kappa_1 = 0.5$. The super-pessimist features mean $\xi_2 = 1.5$ and standard deviation of $\kappa_2 = 0.25$.



(a) Balanced analysts *(b) Pessimist analysts*

Figure 4.6: Belief updating*. This figure illustrates the updating of the beliefs for a pessimistic (solid lines) and optimistic prior (dashed lines). Both panels show the initial belief, i.e., the prior (black) and after two phases of learning (dark and light grey). Panel 4.6a shows the case of a neutral common direction, whereas Panel 4.6b shows the case of a pessimistic common direction.*

(Panel 4.6b). Because the rating agency aims to optimise its reputation costs under ambiguity, it minimises the maximal reputation damage, that is, the reputation damage with the highest reputation cost rate. From this perspective, as the perceived distress worsens and more and more measurement errors are excluded, the optimist's belief is becomes increasingly uninformative: It distributes the remaining probability mass substantially more evenly across the remaining measurement errors. In contrast, the pessimistic belief shifts to a considerably more extreme position over time and places most of its probability mass on default in the immediate future. Consequently, the optimistic prior is more challenging from the rating agency's perspective because the beliefs allow for meaningful and likely deviations of the actual default threshold from the estimate. The pessimistic prior, on the opposite, allows for high confidence for the default to happen in a narrow corridor, allowing for a precise estimate with low reputation cost. Hence, the optimistic prior presents actually the worst case for a rating agency that aims for accurate ratings.

4.4 Equilibrium

The best responses of the firm and rating agency allow us to discuss perfect Bayesian equilibria in Markov strategies augmented for beliefs under ambiguity via sequential optimality as derived by Hanany et al. (2020) in the implementation for multiple priors by Malenko and Tsoy (2023). Both the rating agency and the firm use Markov strategies on a suitably augmented state space, characterised by real-valued functions in Propositions 2 and 3. We obtain an equilibrium by the Schauder fixed-point theorem for a feedback loop with sufficiently smooth interest payments, building on Hilpert et al. (2024).

In the following, we first provide the existence of an equilibrium candidate in Proposition 4. Then, Proposition 5 characterises the solution and verifies the existence and uniqueness, provided that specific technical conditions hold. To establish our equilibrium, we apply the Schauder fixed-point theorem to the best responses. To do this, we must ensure that the best response $g(\cdot; f)$ is an economically reasonable strategy by a suitable transform $\mathscr{R}$. In particular, the transform $\mathscr{R}$ is defined on the set of measurable functions $g : \mathbb{R}_+ \to \mathbb{R}_+$ bounded by id, that is, $g(e) \leq e$, for all $e \geq 0$, by

$$\mathscr{R}(g)(e) = \begin{cases} e \inf \left\{ \sup \left\{ g(s)/z : 0 < s \leq z \right\} : 0 < z \leq e \right\}, & \text{for } e > 0\,, \\ 0, & \text{for } e = 0\,. \end{cases} \tag{4.27}$$

Then $\mathscr{R}(g)$ is economically reasonable as desired, see Lemma 2. We apply the Schauder fixed-point theorem to the mapping $T : (f, g) \mapsto (f(\cdot; g), \mathscr{R}(g(\cdot; f)))$, where $f(\cdot; g)$ is the firm's best response given in Proposition 2 and $g(\cdot; f)$ is the rating agency's best response given in Proposition 3. The proof of Proposition 4 is detailed in Appendix 4.A.

Proposition 4. *Suppose Assumption 1 holds. Then $T : (f, g) \mapsto (f(\cdot; g), \mathscr{R}(g(\cdot; f)))$ has at least one fixed point in the space of economically reasonable strategies. Let $(f^\star, g^\star)$ be such a fixed point, if $\mathscr{R} \circ g(\cdot; f^\star) = g(\cdot; f^\star)$, then $(f^\star, g^\star)$ is an equilibrium.*

Proposition 4 is an existence result. It ensures that at least one candidate for an equilibrium exists and this candidate is indeed an equilibrium if the condition $\mathscr{R} \circ g(\cdot; f^\star) = g(\cdot; f^\star)$ holds. However, no particular guidance is given on how to actually compute such an equilibrium candidate. Proposition 5 characterises an equilibrium candidate as the solution to a $2 \times (n + 1)$-dimensional ordinary differential equation (ODE) under some technical assumptions. Specifically, we require the firm's equity value function to be sufficiently differentiable.[18] The strategies f and g depend on g_i and ν_i, the conditional mean and variance of each prior, see (4.23), explaining the dimension of the setup. For the subsequent Proposition 5, the following transforms are useful $\hat{g}_i(\theta; f) = g_i(f(\theta); f)$ and $\hat{\nu}_i(\theta; f) = \nu_i(f(\theta); f)$, for $\theta \in \Theta$, $i = 1, ..., n$. Both transforms are collected in a column vector each,

[18] Following Hilpert et al. (2024), we call the collection of the firm's equity value functions $(v(\cdot, \cdot; \theta, g))_{\theta \in \Theta}$ sufficiently differentiable, if (i) $v(\cdot, \cdot; \cdot, g)$ is continuously differentiable in θ, (ii) $v(\cdot, \cdot; \cdot, g)$ allows for interchanging the order of differentiation with respect to d and θ, and (iii) the collection of boundary functions $(b(\cdot, \theta; g))_{\theta \in \Theta}$ is continuously differentiable with respect to e and θ.

that is, $\hat{G}(\cdot; f) = (\hat{g}_1(\cdot; f), ..., \hat{g}_n(\cdot; f))^\top$ and $\hat{V}(\cdot; f) = (\hat{\nu}_1(\cdot; f), ..., \hat{\nu}_n(\cdot; f))^\top$, and denote by $\odot$ the pointwise product of vectors and matrices and by $\mathbf{1}_n = (1, ..., 1)^\top$ the one vector of dimension n. With η as given in Equation (4.4), $\Phi_i(\theta) = \int_{\underline{\theta}}^{\theta} \phi_i(t)\,\mathrm{d}t$, $i = 1, ..., n$, then set $\Pi = (\frac{\phi_1}{\Phi_1}, ..., \frac{\phi_n}{\Phi_n})^\top$, and denote by $b(\cdot; \theta, g)$ the boundary that separates the default region from the nondefault region depending on measurement error θ and rating agency strategy g. Finally, define the function

$$\begin{aligned}\mathscr{L}(f, \hat{g}, \hat{g}_i, \hat{g}_j, \hat{\nu}_i, \hat{\nu}_j, i, j, \theta) \quad &= \quad \mathbb{1}_{i=j}\,\frac{\phi_i(\theta)}{\Phi_i(\theta)}(f - \hat{g}_i) + \\ \frac{\mathbb{1}_{i\neq j}}{2(\hat{g}_i - \hat{g}_j)} &\left(\frac{\phi_i(\theta)}{\Phi_i(\theta)}\left[(f - \hat{g})^2 - (\hat{g} - \hat{g}_i)^2 - \hat{\nu}_i\right] - \frac{\phi_j(\theta)}{\Phi_j(\theta)}\left[(f - \hat{g})^2 - (\hat{g} - \hat{g}_j)^2 - \hat{\nu}_j\right]\right).\end{aligned} \tag{4.28}$$

Beside characterising an equilibrium candidate, Proposition 5 provides a condition for it being an actual equilibrium. Its proof is provided in Appendix 4.A.

Proposition 5. *Given the setting of Proposition 4, denote by $(f^\star, g^\star)$ a fixed point of T. Suppose $f^\star, g^\star$, $(\phi_i)_i$ are continuously differentiable, and the collection of the firm's equity values, denoted by $(v(\cdot, \cdot; \theta, g^\star))_{\theta\in\Theta}$, is sufficiently differentiable. Denote by $(f, \hat{g}, \hat{G}, \hat{V})$ the solution of the ODE*

$$\begin{pmatrix} f'(\theta) \\ \hat{g}'(\theta) \\ \hat{G}'(\theta) \\ \hat{V}'(\theta) \end{pmatrix} = \begin{pmatrix} \frac{(1+\eta)\sigma^2}{2(r-\mu)}\frac{(f(\theta)/\theta)^2}{C(f(\theta)/\hat{g}(\theta)) - f(\theta)/\theta} + \frac{\partial \hat{b}(\theta,\theta;g)}{\partial \hat{\theta}} \\ \mathscr{L}(f(\theta), \hat{g}(\theta), \hat{g}_{i(\theta)}(\theta), \hat{g}_{j(\theta)}(\theta), \hat{\nu}_{i(\theta)}(\theta), \hat{\nu}_{j(\theta)}(\theta), i(\theta), j(\theta), \theta) \\ \Pi(\theta) \odot (f(\theta)\,\mathbf{1}_n - \hat{G}(\theta)) \\ \Pi(\theta) \odot \left([f(\theta)\,\mathbf{1}_n - \hat{G}(\theta)] \odot [f(\theta)\,\mathbf{1}_n - \hat{G}(\theta)] - \hat{V}(\theta)\right) \end{pmatrix}, \tag{4.29}$$

on $(\underline{\theta}, \overline{\theta})$ with initial condition $f(\underline{\theta}) = \underline{\theta}\, f_1^\star$, $\hat{g}(\underline{\theta}) = \underline{\theta}\, g_1^\star$, $\hat{G}(\underline{\theta}) = \mathbf{1}_n\,\underline{\theta}\, g_1^\star$, and $\hat{V}(\underline{\theta}) = 0$, where $(f_1^\star, g_1^\star)$ denotes the unique equilibrium of the case $\Theta = \{1\}$ with $f_1^\star = g_1^\star$, $\hat{b}(\hat{\theta}, \theta; g) = b(f(\hat{\theta}), \theta; g)$, for $\hat{\theta} \leq \theta$, and $(i(\theta), j(\theta))$ are the minimising indices of (4.22) as given in (4.23). If

$$0 \leq \hat{g}'(\theta) \leq f'(\theta)\frac{\hat{g}(\theta)}{f(\theta)}\,, \quad \textit{for all } \theta \in (\underline{\theta}, \overline{\theta})\,, \tag{4.30}$$

then the fixed point is characterised by $(f^\star, g^\star) = (f, \hat{g} \circ f^{-1})$ and is moreover an equilibrium.

The ODE in Equation (4.29) is explicit and admits a unique solution $(f, \hat{g})$ that can be used for computations. If Condition (4.30) holds, then $(f^\star, g^\star) = (f, \hat{g} \circ f^{-1})$ is the unique equilibrium satisfying the differentiability conditions in Proposition 5.

Parameter	Value	Parameter	Value
Risk-free rate	$r = 2.11\%$	Expected growth rate	$\mu = 0$
Volatility	$\sigma = 30\%$	Measurement error support	$\Theta \in [0.5, 1.5]$
	Neutral team	Pessimistic team	Optimistic team
Mean	$\xi_1 = 0.75$, $\xi_2 = 1.25$	$\xi_1 = 1$, $\xi_2 = 1.5$	$\xi_1 = 0.5$, $\xi_2 = 1$
Std. dev.	$\kappa_1 = 0.25$, $\kappa_2 = 0.25$	$\kappa_1 = 0.25$, $\kappa_2 = 0.25$	$\kappa_1 = 0.25$, $\kappa_2 = 0.25$

Table 4.1: Parametrisation of the base case. *This table summarizes the parameters for the base case of our numerical analysis. It considers first an overall neutral analyst team, second a pessimistic analyst team, and third an optimistic analyst team, each consisting of two analysts. The parameters ξ and κ represent the mean and standard deviation parameters of the truncated Gaussian distribution. The interest payment function C follows the specification in Assumption 1. The interest payment function is calibrated to the default probabilities and time horizon in the rating classes for Moody's default probabilities. The interest rate is calibrated to the Federal Reserve Economic Data, using the Average Rate 3-Month T-Bill (Constant Maturity, 1996-2016).*

4.5 Economic Implications of Credit Rating with Ambiguity

In this section, we shed light on the implications of ambiguity for the equilibrium strategies of the firm and rating agency with an emphasis on learning. In Appendix 4.C, we state general analytical implications of the equilibrium. In our numerical equilibrium analysis, we initially consider a team of two analysts and discuss the influence of the analysts' overall direction on the strategies of the firm and rating agency. Here, we distinguish whether they are overall optimistic or pessimistic on average, or potentially balance each other. We analyse how feedback effects affect the influence of ambiguity on both strategies. Next, we quantify the equilibrium impact of ambiguity on credit spreads. Contrasting the previous discussion, this analysis critically depends on the existence of feedback effects. Finally, we consider the surprise effect upon default, and how it is affected by ambiguity and feedback effects.

The following numerical analysis captures the specific effect of ambiguity on the unique market equilibrium. Here, we use the parametrisation as well as the calibration of both the risk-free rate and the interest payment rate as in Hilpert et al. (2024). Specifically, the firm's debt face value is scaled to unity; the cash flow features the expected growth rate of $\mu = 0$ and a volatility of $\sigma = 0.30$; the risk-free rate equals $r = 0.0211$; the interest payment rate C follows the specification in Assumption 1. The interest payment function is calibrated to the default probabilities and time horizon in the rating classes for Moody's default probabilities. The interest rate is calibrated to the Federal Reserve Economic Data, using the Average Rate 3-Month T-Bill (Constant Maturity, 1996-2016). We summarize this parametrisation in Table 4.1. We model the analyst's priors as truncated Gaussian distributions, where ξ and κ represent the mean and standard deviation parameters of this distribution.

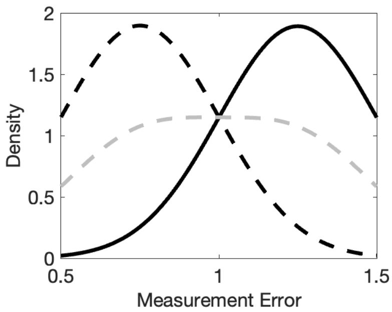
4.5.1 The impact of a common direction of disagreement

In this section, we discuss how ambiguity affects the rating agency's learning of the firm's default threshold and rating over time. Specifically, we study the impact of the common direction of analyst disagreement as the driving factor for if and how ambiguity affects the equilibrium of the rating game. We capture the analysts' common direction by the respective means of their priors. Yet, the analysts' beliefs about the firm's true cash flow level remain heterogeneous. Additionally, we observe that, while firm's default threshold depends on ambiguity only under feedback effects, the rating strategy accounts for ambiguity both with and without the feedback loop.

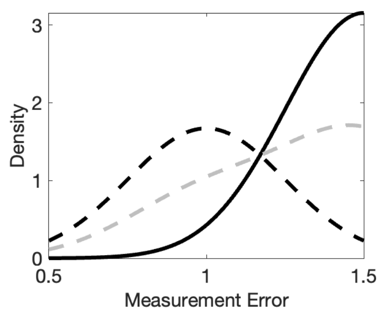
Consider the previously discussed case of an optimistic and a pessimistic analyst (see Section 4.3.2). Their two priors differ because they perceive that the firm likely features cash flows either above or below the currently observed ones, respectively. In contrast, consider the case of a jointly pessimistic and a jointly optimistic team of analysts, respectively. The pessimistic team on average perceives the firm's observed cash flow overstated. One of the analysts has a strongly pessimistic prior, while the other one has a neutral prior. The optimistic team is composed of one strongly optimistic and one neutral prior. On average, the optimistic team perceives the firm's observed cash flow understated.[19]

Figure 4.7 illustrates the three cases. Panel 4.7a presents the case of an optimist and a pessimist, whereas Panel 4.7b and 4.7c shows the case of a pessimistic and optimistic team of analysts, respectively (see Section 4.3). The solid line in Panel 4.7b indicates a relatively more pessimistic belief, whereas the dashed black line displays a more optimistic belief. The reverse is true for Panel 4.7c. The dashed grey line marks the average density as the asymmetric information case without ambiguity. Both analysts in Panel 4.7a present a

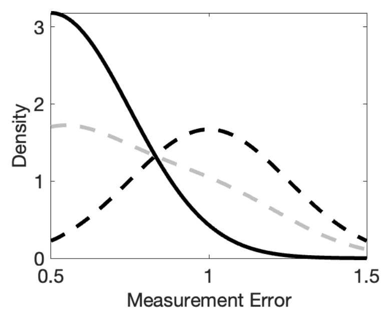
[19] Specifically, for the pessimistic team, we increase the mean of each analyst prior by 0.25 to $\xi_1 = 1$ and $\xi_2 = 1.5$ compared to the neutral base case in Table 4.1, respectively. For the optimistic analysts, we select $\xi_1 = 0.5$ and $\xi_2 = 1$ The standard deviation remains unchanged at $\kappa_1 = \kappa_2 = 0.25$ for both teams.



(a) Balanced priors



(b) Jointly pessimistic team



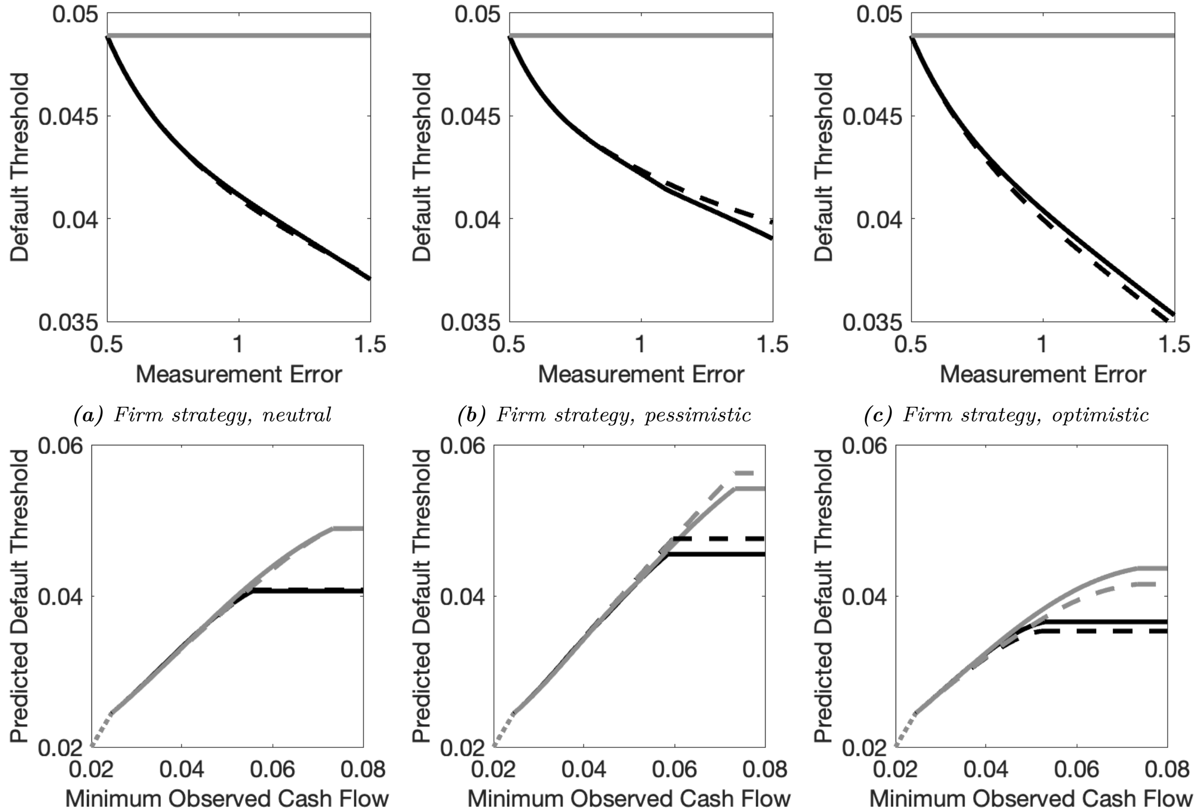
(c) Jointly optimistic team

***Figure 4.7: Jointly neutral, pessimistic, and optimistic analyst teams**. This figure displays the prior beliefs for an optimistic and a pessimistic prior (Panel 4.7a), a jointly pessimistic team (Panel 4.7b), and a jointly optimistic team (Panel 4.7c). The solid line in Panel 4.7b indicates a relatively more pessimistic belief, whereas the dashed black line displays a more optimistic belief. The reverse is true for Panel 4.7c. The grey line marks the average density as the asymmetric information case without ambiguity. The parametrisation follows Table 4.1.*

balanced average view without common direction. In contrast, the positive slope of the grey line in Panel 4.7b implies a high probability mass for upward measurement error cases, that is, the average belief is that the observed cash flow is above of the true one. Panel 4.7c, the optimist case, reverses the pessimistic case.

Figure 4.8 shows the equilibrium strategies for the rating game. Panels 4.8a, 4.8b and 4.8c present the firm's equilibrium strategy in terms of the true cash flow X for the neutral, pessimistic and optimistic common direction, respectively. Panels 4.8d, 4.8e and 4.8f show the rating agency's equilibrium rating strategy. The case under ambiguity and feedback effects is indicated by the solid black line. The case without ambiguity but with feedback effects is captured by the dashed black line and consists of the equilibrium from a rating agency with one single prior, given by the average density of analysts' beliefs. The grey solid and dotted lines indicate the cases with and without ambiguity but without feedback effects, that is, constant coupons, respectively. The dotted line in the bottom panels indicate immediate default of all possible firm types. The flat part of the rating strategy in the bottom panels on the right part of Panels 4.8d and 4.8e represent the area before the rating agency starts learning.

In Panels 4.8a and 4.8d, the optimistic and pessimistic priors are balanced, that is, the priors overall do not have a clear common direction. Then, ambiguity has minimal influence on the resulting firm and rating strategies. Hence, the resulting rating strategy balances these two views and their implied costs (see Figure 4.5) by not deviating from the no-ambiguity strategy which coincides with Hilpert et al. (2024). In this case, ambiguity has



(a) Firm strategy, neutral *(b) Firm strategy, pessimistic* *(c) Firm strategy, optimistic*

(d) Rating strategy, neutral *(e) Rating strategy, pessimistic* *(f) Rating strategy, optimistic*

Figure 4.8: ***Ambiguity impact on equilibrium.*** *This figure illustrates the impact of ambiguity on the equilibrium. Panels 4.8a, 4.8b and 4.8c present the firm's equilibrium strategy in terms of the true cash flow for the neutral, pessimistic, and optimistic common direction, respectively. Panels 4.8d, 4.8e and 4.8f show the rating agency's equilibrium rating strategy. The case under ambiguity is indicated by the solid line. The case without ambiguity is captured by the dashed line. The dotted line in the bottom panels show the case all possible firm types already defaulted. Black lines indicate strategies with feedback effects. grey lines show the case without feedback, i.e., constant coupons. The parametrisation follows Table 4.1.*

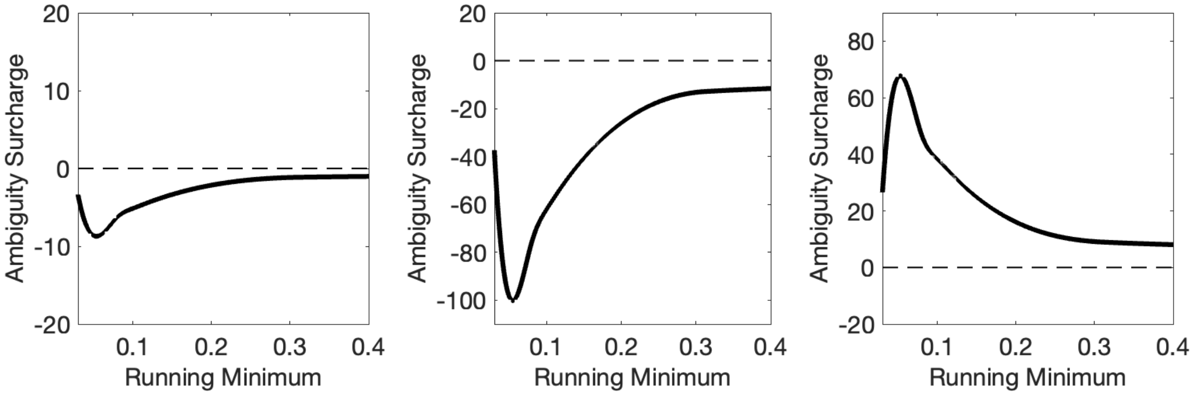
no influence on either the firm or the rating agency as the information is too dispersed for the rating agency to move its aggregated belief. Hence, the rating agency, and, in turn, the firm account for ambiguity in their strategies but do not deviate from the assessment under asymmetric information alone in a meaningful way. The presence of feedback effects considerably delays learning compared to the case of constant coupons. Specifically, the firm strategically delays default to benefit from the rating agency's learning. The rating agency anticipates this behaviour and expects a delayed default under feedback effects.

In contrast, Panels 4.8b and 4.8e show that under a common pessimistic direction, ambiguity changes both the firm and the rating agency's equilibrium strategy compared to the case of pure asymmetric information. In equilibrium, ambiguity in combination with overall jointly pessimistic beliefs systematically delays firm default compared to the case without ambiguity. The rating agency, as in Section 4.3.2, takes a conservative view. With two pessimistic analysts, following the more nuanced prior is the prudent strategy. The rating agency ignores the more pessimistic first prior. As the perspective, and subsequently the strategy, of the rating agency now deviates from a best response to an average distribution, the firm delays its default compared to the case without ambiguity to benefit from the rating that builds on the less pessimistic perspective on the market. Conversely, we observe in Panels 4.8c and 4.8f that ambiguity in combination with overall jointly optimistic beliefs systematically accelerates firm default compared to the case without ambiguity.

4.5.2 Credit spreads

The calculation of the equilibrium with feedback effects allows us to investigate how learning and uncertainty affects the firm's credit spread, which is defined as the coupon minus the risk-free rate. When the cash flow deteriorates, the firm has to pay a higher coupon. This, by definition, raises the firm's credit spread. On the other hand, the recovery from distressed periods signals firm quality by reducing uncertainty, which decreases the firm's credit spread. We observe that ambiguity profoundly affects credit spreads. Note that this analysis mandates a feedback effect; constant coupons preclude any learning effect on credit spreads.

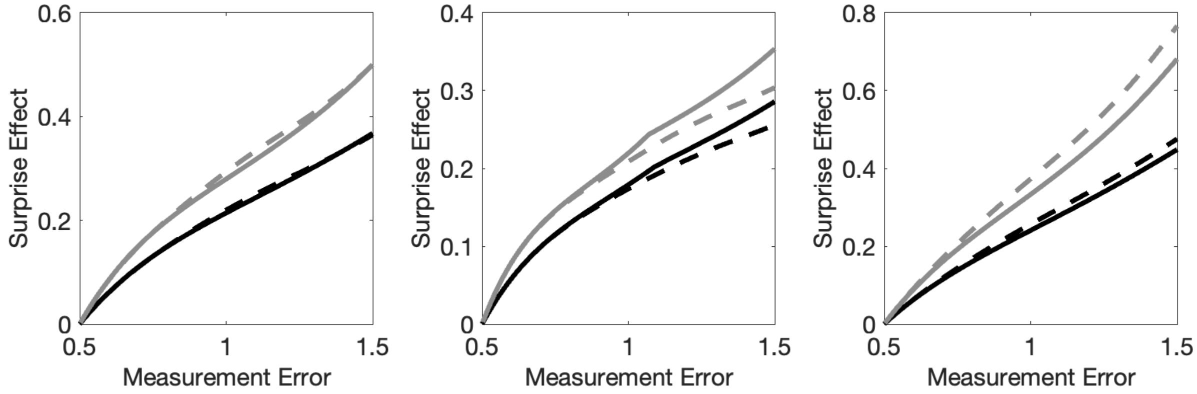
Figure 4.9 displays the impact of ambiguity on credit spreads in basis points for non-defaulted firms at levels where the firm's observed cash flow matches its running minimum. Panel 4.9a displays balanced priors, while Panels 4.9b and 4.9c show the cases with jointly pessimistic and optimistic priors, respectively. Panel 4.9a confirms the intuition that for the neutral common direction, ambiguity does not significantly affect credit spreads; that is, the two lines coincide. However, and in line with the empirical observation by Augustin and Izhakian (2020), ambiguity can both increase or decrease credit spreads dependent on the common direction of analysts in our model. First, we observe that the common direction direction of priors influence the absolute level of the ambiguity surcharge on credit spreads, which is calculated as the difference of the credit spreads under ambiguity and asymmetric information without ambiguity. In Panel 4.9b, we notice that the ambiguity surcharge is negative and increases in absolute value up to 100 basis points as the observed cash flow decreases and perceived bankruptcy approaches. Once bankruptcy looms closely, the ambiguity impact vanishes. This hump-shaped effect originates in two underlying



(a) Neutral team *(b) Pessimistic team* *(c) Optimist team*

Figure 4.9: Uncertainty surcharge in credit spreads (basis points). *This figures display the uncertainty surcharge in credit spreads for non-defaulted firms at levels where the firm's observed cash flow matches its running minimum. Panel 4.9a displays the cases with balanced priors, while Panel 4.9b presents the case with jointly pessimistic priors. Panel 4.9c shows the optimistic analyst team. The ambiguity surcharge on credit spreads is calculated as the difference of the credit spreads under ambiguity and asymmetric information at various levels of the observed cash flow's running minimum. The solid black line represents the ambiguity surcharge on credit spreads over the asymmetric information case in basis points. The dashed line indicates zero. The parametrisation aligns with Table 4.1.*

economic arguments. First, as the firm is financially sound, a moderate decrease of the running minimum of the cash flow has little effect. In our model, learning appears in periods of distress; hence, a small cash flow decrease of a healthy firm does not contain meaningful information. Once the firm experiences a significant distress, learning appears. In the case of Panel 4.9b, the rating analyst team is pessimistic. The firm anticipates that, should it survive a period of apparent distress, the rating agency quickly adjusts the predicted default threshold, making a delayed default attractive. As the rating agency expects a lower default threshold under ambiguity, see Panel 4.8e, the credit spread is lower compared to the asymmetric information case. Finally, as the firm is severely distressed, the credit spread of the firm becomes identical under both ambiguity and asymmetric information because the firm is essentially bankrupt and the rating agency has already learned essentially all information, rendering further learning moot. For the optimist rating agency presented in Panel 4.9c, the economic argument reverses. The firm accelerates default compared to the asymmetric information case, causing an ambiguity surcharge of up to 70 basis points.



(a) Neutral common direction *(b) Pessimistic team* *(c) Optimistic team*

Figure 4.10: Surprise effects upon default. *This figure displays the surprise effect upon default in Panel 4.10a, Panel 4.10b, and Panel 4.10c for a neutral, pessimistic, and optimistic team, respectively. The case under ambiguity is indicated by the solid line. The case without ambiguity is captured by the dashed line. Black lines indicate surprise effects in the equilibrium under feedback effects with performance-sensitive debt. grey lines display the surprise effect in the equilibrium precluding feedback effects with a constant coupon. The parametrisation aligns with Table 4.1*

4.5.3 Surprise effects

The rating agency's estimated default point $\hat{D}^\star(D^\star(\theta))$ deviates from the actual threshold $D^\star(\theta)$, giving a surprise effect introduced in Hilpert et al. (2024) as the relative difference of

$$\frac{D^\star(\theta) - \hat{D}^\star(D^\star(\theta))}{\hat{D}^\star(D^\star(\theta))}.$$

Figure 4.10 presents the surprise effect upon default. Panel 4.10a, Panel 4.10b, and Panel 4.10c show the surprise effect for a neutral, pessimistic, and optimistic team, respectively. The solid lines indicate the case with ambiguity. The dashed lines capture the benchmark of asymmetric information alone. As previously, black lines display the surprise effect arising in an equilibrium featuring feedback effects. In contrast, the grey lines show results based on an equilibrium without feedback effects, that is, with constant coupons.

All three panels indicate a sizable surprise effect, particularly for firms with a high measurement error. In line with its influence on credit spreads, ambiguity hardly affects the surprise effect in case of a neutral common direction (Panel 4.10a). In contrast, both pessimistic and optimistic analyst direction modify the surprise effect upon default. For the pessimist team, the rating agency takes a less pessimistic view relative to the case under asymmetric information alone, causing an amplified surprise effect upon default. Following the mirrored argument, Panel 4.10c indicates that ambiguity in an overall optimistic team moderates the surprise effect. The feedback effect moderates the surprise effect upon default as learning influences the firm's default choices; yet, the qualitative impact of ambiguity remains robust to the existence of feedback effects.

4.6 Conclusion

In this paper, we consider the impact of ambiguity, or Knightian uncertainty, on credit rating in presence of a feedback loop. The feedback loop connects a rating agency, aiming for an accurate and unbiased estimate of a firm's distance-to-default, and a firm that maximises its equity value over its default strategy, via performance-sensitive debt.

We show that the rating agency, learning the firm's true quality from observing survival of periods of apparent distress, deviates from a classical min-max response to ambiguity. Instead, to satisfy its aim of accurate ratings, it balances different priors. When rating analysts show a common direction of disagreement, the impact of ambiguity on the equilibrium of the credit rating game has important implications for the rated firm's default policy. For a pessimistic analyst team, the firm strategically delays default compared to the results under classical asymmetric information without ambiguity to profit from the feedback loop in form of lower capital costs, as the rating agency rapidly shifts its priors. In contrast, for an optimistic analyst team, the firm strategically accelerates default to cut higher capital costs, as the rating agency under ambiguity sluggishly updates its prior. We show that the effect of ambiguity on the timing of default also influences the firm's credit spread. For a pessimistic analyst team, ambiguity decreases credit spreads while for an optimistic analyst team ambiguity increases credit spreads compared to a scenario without ambiguity. We predict that in phases of high ambiguity — measured by the level of market-wide ambiguity or equity analyst forecast dispersion — the rating agency's policy is to dampen its analysts with pronounced optimistic or pessimistic opinions.

We hope that our work can further inspire empirical research on ambiguity in groups with heterogeneous beliefs.

4.A Proof of Main Results

Proof of Proposition 3. For a given f, the rating agency maximises its expected utility given by Equation (4.10). To do so, the rating agency minimises the maximal cost-rate, where the cost-rate is given in Equation (4.11) at each time t. Thereby, at $E_t = e$, the rating agency predicts the default level that minimises the maximal costs, as claimed in the proposition. Next, $g(e) \leq e$ follows from the representation of g in Equation (4.22) resulting in $g(e; f) \leq \max_{i=1,\dots,n} g_i(e; f)$, and Proposition 1 of Hilpert et al. (2024) stating that $g_i(e; f) \leq e$. □

Proof of Proposition 4. A fixed point is obtained by the Schauder fixed point theorem: Let $\mathcal{K}$ be a nonempty, convex, compact subset of a Banach space $\mathcal{V}$, if $\tilde{T} : \mathcal{K} \to \mathcal{K}$ is continuous, then $\tilde{T}$ has a fixed point. This result extends Proposition 3 of Hilpert et al. (2024) concerning the rating agency's best response in the presence of ambiguity. The specification of $\mathcal{V}$ and $\mathcal{K} = \mathscr{K}_f \times \mathscr{K}_g$ as well as of $\tilde{T}$ is detailed in their Internet Appendix, but note that here $\mathscr{R}$ is defined differently, see Equation (4.27). It remains to show that $f \mapsto \mathscr{R} \circ g(\cdot; f)$ is continuous in the sup-norm, for all $f \in \mathscr{K}_f$, that is, for all functions $f \in C(\Theta, \mathbb{R}_0^+)$ that satisfy

$$l_f(\theta - \theta') \leq f(\theta) - f(\theta') \leq L_f(\theta - \theta'), \underline{\theta}\, \underline{f} \leq f(\theta) \leq \overline{\theta}\, \overline{f}, \text{ for } \theta, \theta' \in \Theta \text{ with } \theta' \leq \theta\,,$$

where the bounds $\underline{f}$ and $\overline{f}$ result from the firm's endogenous default thresholds for the best and worst case, respectively, as well as $l_f = (1 - L_C)\,\underline{f}$ and $L_f = \overline{f}$, see Lemma 2 of Hilpert et al. (2024).[20]

To establish the continuity of $f \mapsto \mathscr{R} \circ g(\cdot; f)$, in the first step, continuity of $f \mapsto g(\cdot; f)$ is established. Suppose $\|f - f'\|_\infty \leq \varepsilon$, for $f, f' \in \mathscr{K}_f$. Recall Equation (4.22), to see that

$$g(e; f) - g(e, f') = \underset{g>0}{\arg\min}\, Q(g; e, f) - \underset{g'>0}{\arg\min}\, Q(g'; e, f')\,,$$

where

$$Q(g; e, f) = \max_{i=1,\dots,n} \{Q_i(g; e, f)\}\,, \qquad \text{with} \qquad Q_i(g; e, f) = (g - g_i(e; f))^2 + \nu_i(e; f)\,,$$

[20] These cases correspond to constant interest payments with minimal coupon $\underline{C} = \inf_{R\geq 1} C(R) \geq r > 0$ and maximal coupon $\overline{C} = \sup_{R\geq 1} C(R) < \infty$, respectively, where ambiguity and hence the feedback loop are switched off as the interest rates are not sensitive to rating changes.

$$Q(g'; e, f') = \max_{i=1,\dots,n} \{Q_i(g'; e, f')\}\,, \qquad \text{with} \qquad Q_i(g'; e, f') = (g' - g_i(e; f'))^2 + \nu_i(e; f')\,.$$

For prior $i \in \{1, \dots, n\}$, $Q_i(\cdot; e, f)$ and $Q_i(\cdot; e, f')$ are the mean squared error functions for the rating agency, given respective firm strategies f and f', and given that the observed cashflow has reached the minimal level e. Further, $Q(\cdot; e, f)$ and $Q(\cdot; e, f')$ are their worst case values over all priors that the rating agency minimises by varying the predicted default threshold g and g', respectively. The coefficients g_i and ν_i determining Q_i are uniformly continuous in f, that is,

$$\|g_i(\cdot; f) - g_i(\cdot; f')\|_\infty \le K_i\,(\varepsilon + \varepsilon^{1/2})\,, \text{ and } \|\nu_i(\cdot; f) - \nu_i(\cdot; f')\|_\infty \le \tilde{K}_i\,(\varepsilon + \varepsilon^{1/2})\,,$$

for some $K_1, ..., K_n, \tilde{K}_1, ..., \tilde{K}_n > 0$. The first part follows from Equation (III.13) in the Online Appendix of Hilpert et al. (2024). For the second part, Equation (4.20) and Equation (4.21) give

$$\nu_i(e; f) = \mathbb{E}^{\pi_i,0}\left[\left(g_i(e; f) - f(\tilde{\theta})\right)^2 \middle| f(\tilde{\theta}) < e\right] = \mathbb{E}^{\pi_i,0}\left[f(\tilde{\theta})^2 \middle| f(\tilde{\theta}) < e\right] - g_i(e; f)^2\,.$$

According to Lemma 3 in Appendix 4.D and $|g_i(e; f)^2 - g_i(e; f')^2| \le 2\,\overline{f}\,\overline{\theta}\,|g_i(e; f) - g_i(e; f')|$, the second part follows. For $i \in \{1, \dots, n\}$,

$$\begin{aligned}|Q_i(g; e, f) - Q_i(g; e, f')| &= \left|(g - g_i(e; f))^2 + \nu_i(e; f) - (g - g_i(e; f'))^2 - \nu_i(e; f')\right| \\ &\le (2\,g + g_i(e; f) + g_i(e; f'))\,|g_i(e; f) - g_i(e; f')| + |\nu_i(e; f) - \nu_i(e; f')|\,.\end{aligned}$$

The relevant values for g are those between the minimal and maximal value of $g_i(e; f)$ and given by $[\underline{g}, \overline{g}]$, with $\underline{g} = \underline{\theta}\,\underline{f}$ and $\overline{g} = \overline{\theta}\,\overline{f}$, for all $f \in \mathscr{K}_f$. Then

$$\begin{aligned}\sup_{\underline{g} \le g \le \overline{g}} |Q(g; e, f) - Q(g; e, f')| &\le \max_{i=1,\dots,n} \sup_{\underline{g} \le g \le \overline{g}} |Q_i(g; e, f) - Q_i(g; e, f')| \\ &\le 4\,\overline{g}\,|g_i(e; f) - g_i(e; f')| + |\nu_i(e; f) - \nu_i(e; f')| \\ &\le K\,(\varepsilon + \varepsilon^{1/2})\,,\end{aligned}$$

with $K = 4\,\overline{g}\,\max_{i=1,\dots,n} K_i + \max_{i=1,\dots,n} \tilde{K}_i$, giving a uniform bound in e. The rating agency's best responses $g(e; f)$ and $g(e; f')$ minimise the maximal mean squared error $Q(\cdot; e, f)$ and $Q(\cdot; e, f')$, respectively. Without loss of generality, let $Q(g(e; f); e, f) \le Q(g(e; f'); e, f')$. Then $Q(g(e; f'); e, f') \le Q(g(e; f); e, f) + 4\,\overline{g}\,K\,(\varepsilon + \varepsilon^{1/2})$. Since Q is the pointwise maximum of parabolas in g of the form $g^2 + \dots$, the minimising value $g(e; f)$ gives the following lower bound $\underline{Q}(g;e,f) = Q(g(e,f);e,f) + (g - g(e,f))^2 \le Q(g;e,f)$. This gives also a lower bound for $Q(\cdot;e,f')$ of the form $\underline{Q}(g;e,f') = \underline{Q}(g;e,f) - 4\,\overline{g}\,K\,(\varepsilon + \varepsilon^{1/2}) \le Q(g;e,f')$. Combing the lower and upper bound for $Q(\cdot;e,f')$ at the best response $g(e;f')$ gives

$$\begin{aligned} Q(g(e,f);e,f) + (g(e;f') - g(e,f))^2 - 4\,\overline{g}\,K\,(\varepsilon + \varepsilon^{1/2}) &\le Q(g(e;f');e,f') \\ &\le Q(g(e;f);e,f) + 4\,\overline{g}\,K\,(\varepsilon + \varepsilon^{1/2})\,. \end{aligned}$$

This implies $(g(e;f') - g(e,f))^2 \le 8\,\overline{g}\,K\,(\varepsilon + \varepsilon^{1/2})$, or, as the bound is uniform in e,

$$\|g(\cdot;f') - g(\cdot,f)\|_\infty \le \Big(8\,\overline{g}\,K\,(\varepsilon + \varepsilon^{1/2})\Big)^{1/2}\,.$$

The continuity of $f \mapsto \mathscr{R}(g(\cdot;f))$ follows from Lemma 5 in Appendix 4.D. $\square$

Lemma 2. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function that is bounded by id. For the transformed strategy it holds that $\mathscr{R}(g) \in \mathscr{A}_g$. Moreover, $\mathscr{R}(g) = g$, for $g \in \mathscr{A}_g$.*

Proof. Since

$$\mathscr{R}(g)(e) = e \inf_{0<z\le e} \frac{\sup\big\{g(s) \mid 0 < s \le z\big\}}{z}$$

it follows from the boundedness by the identity that

$$\mathscr{R}(g)(e) \le e \inf\{e/z \mid 0 < z \le e\} = e.$$

We define $g^*(e) = \sup\{g(s) \mid 0 < s \le e\}$ and observe that $\mathscr{R}(g) \le g^* \le id$. To show that $\mathscr{R}$ is non-decreasing take $e' \ge e > 0$. Then

$$\begin{aligned} \mathscr{R}(g)(e') &= e' \inf\{g^*(z)/z \mid 0 < z \le e'\} \\ &= e'\Big(\inf\{g^*(z)/z \mid 0 < z \le e\} \wedge \inf\{g^*(z)/z \mid e < z \le e'\}\Big) \\ &\ge \Big(\frac{e'}{e}\mathscr{R}(g)(e)\Big) \wedge \Big(e' \inf\{g^*(e)/z \mid e < z \le e'\}\Big) \\ &= \Big(\frac{e'}{e}\mathscr{R}(g)(e)\Big) \wedge g^*(e) \\ &= \mathscr{R}(g)(e) + \Big(\frac{e'-e}{e}\mathscr{R}(g)(e) \wedge (g^*(e) - \mathscr{R}(g)(e))\Big) \ge \mathscr{R}(g)(e). \end{aligned}$$

The property that $\mathscr{R}(g)(e)/e$ is non-increasing holds by definition. $\square$

4.B ODE characterisation of Equilibrium

Proposition 5 provides an ODE characterisation of an equilibrium, provided technical conditions hold. In the following, this result is derived. First, the best response of the rating agency is described by a solution to an ODE system. Building on this, the equilibrium characterisation is established.

Proposition 6. *Given a continuous and strictly increasing default strategy f, the ambiguity averse rating agency's best response rating satisfies*

$$\hat{g}'(\theta) = \mathscr{L}(f(\theta), \hat{g}(\theta), \hat{g}_{i(\theta)}(\theta), \hat{g}_{j(\theta)}(\theta), \hat{\nu}_{i(\theta)}(\theta), \hat{\nu}_{j(\theta)}(\theta), i(\theta), j(\theta), \theta)\,, \tag{4.31}$$

almost everywhere on $(\underline{\theta}, \overline{\theta})$, where $(i(\theta), j(\theta))$ are the minimising indices of (4.22) *as given in* (4.23) *and $\mathscr{L}$ given by Equation* (4.28) *with initial conditions*

$$\hat{g}'(\underline{\theta}) = \frac{1}{2} f'(\underline{\theta})\,, \quad \textit{and} \quad \hat{g}(\underline{\theta}) = f(\underline{\theta}). \tag{4.32}$$

Additionally, it holds that

$$\hat{g}_i'(\theta) = \frac{\phi_i(\theta)}{\Phi_i(\theta)}\Big(f(\theta) - \hat{g}_i(\theta)\Big)\,, \quad \textit{and} \quad \hat{\nu}_i'(\theta) = \frac{\phi_i(\theta)}{\Phi_i(\theta)}\Big((f(\theta) - \hat{g}(\theta))^2 - \hat{\nu}_i(\theta)\Big)\,, \tag{4.33}$$

for $i = 1, ..., n$ on $(\underline{\theta}, \overline{\theta})$ with initial conditions

$$\hat{g}_i(\underline{\theta}) = f(\underline{\theta}), \quad \hat{g}_i'(\underline{\theta}) = \frac{1}{2} f'(\underline{\theta}), \quad \hat{\nu}_i(\underline{\theta}) = 0\,, \ \textit{and}\ \hat{\nu}_i'(\underline{\theta}) = f(\underline{\theta}) f'(\underline{\theta}) - \frac{1}{4} f'(\underline{\theta})^2. \tag{4.34}$$

Proof. Recall that $\hat{g}(\theta)$ is the argument that minimises the upper envelope of parabolas of the form $(\hat{g} - \hat{g}_i(\theta))^2 + \hat{\nu}_i(\theta)$, with $i = 1, ..., n$. Observe that the minimising argument $\hat{g}(\theta)$ is either minimising a specific parabola, say, given by i, or, it sits at an intersection of two parabolas, say, given by i and j. Denote these indices by $i(\theta)$ and $j(\theta)$, including $i(\theta) = j(\theta)$. In the first case we have $\hat{g}(\theta) = \hat{g}_{i(\theta)}(\theta) = \hat{g}_{j(\theta)}(\theta)$. For simplicity, we further assume that the minimising indices are well-defined and continuous up to a set of finite cardinality. The intersection point $\hat{g}(\theta)$ of parabola $i(\theta)$ and parabola $j(\theta)$, with $i(\theta) \neq j(\theta)$ is straightforward, and, covering also the case $i(\theta) = j(\theta)$

$$\hat{g}(\theta) = \frac{1}{2}\left(\Big(\hat{g}_{i(\theta)} + \hat{g}_{j(\theta)}\Big) + \mathbb{1}_{i(\theta) \neq j(\theta)} \frac{\hat{\nu}_{i(\theta)} - \hat{\nu}_{j(\theta)}}{\hat{g}_{i(\theta)} - \hat{g}_{j(\theta)}}\right)\,,$$

see also Equation (4.23). Here, we suppress the dependence of all quantities on f for convenience as the latter is fixed.

To establish (4.31), with $\mathscr{L}$ given in Equation (4.28), the partial derivative of $\hat{g}(\theta)$ has to be calculated. For doing so, observe that ingredients of $\hat{g}(\theta)$ are given by

$$\hat{g}_i(\theta) = \frac{1}{\Phi_i(\theta)} \int_{\underline{\theta}}^{\theta} f(t)\phi_i(t)dt\,, \quad \text{and} \quad \hat{m}_{2,i}(\theta) = \frac{1}{\Phi_i(\theta)} \int_{\underline{\theta}}^{\theta} f(t)^2\phi_i(t)dt\,,$$

where $\hat{\nu}_i = \hat{m}_{2,i} - \hat{g}_i^2$, for $i = 1, ..., n$. Applying the quotient rule yields first order ODEs

$$\hat{g}_i'(\theta) = \frac{\phi_i(\theta)}{\Phi_i(\theta)}\Big(f(\theta) - \hat{g}_i(\theta)\Big)\,, \quad \text{and} \quad \hat{m}_{2,i}'(\theta) = \frac{\phi_i(\theta)}{\Phi_i(\theta)}\Big(f(\theta)^2 - \hat{m}_{2,i}(\theta)\Big)\,,$$

and therefore

$$\hat{\nu}_i'(\theta) = \frac{\phi_i(\theta)}{\Phi_i(\theta)}\Big((f(\theta) - \hat{g}_i(\theta))^2 - \hat{\nu}_i(\theta)\Big)$$

on $(\underline{\theta}, \overline{\theta})$, for $i = 1, ..., n$. With these results Equation (4.31) follows by application of the quotient rule. The boundary conditions at $\underline{\theta}$ for $\hat{g}_i$, $\hat{m}_i$, $\hat{\nu}_i$ and $\hat{g}$, follow directly from the above representation and application of the L'Hôpital's rule. Similarly, the boundary conditions for derivatives at $\underline{\theta}$, that is, $\hat{g}_i'$, $\hat{m}_i'$, $\hat{\nu}_i'$, and $\hat{g}'$, can be established. By these arguments, Equation (4.32), Equation (4.33) and Equation (4.34) follow. □

Proposition 7. *Let $f \in \mathscr{K}_f$. For the ambiguity averse best response $g = g(\cdot; f)$ and its transformation $\mathscr{R}(g) = \mathscr{R}(g(\cdot; f))$ denote $\hat{g} = g \circ f$ and $\tilde{g} = \mathscr{R}(g) \circ f$. Then, $\tilde{g}$ is Lebesgue almost everywhere differentiable and satisfies on $(\underline{\theta}, \overline{\theta})$:*

$$\tilde{g}' = f'\frac{\tilde{g}}{f}\mathbf{1}_{\{\tilde{g} \neq \hat{g}\}} + \hat{g}'\mathbf{1}_{\{\tilde{g} = \hat{g}\}}. \tag{4.35}$$

The initial conditions are given by

$$\tilde{g}(\underline{\theta}) = \hat{g}(\underline{\theta}) = f(\underline{\theta}) \quad \text{and} \quad \tilde{g}'(\underline{\theta}) = \hat{g}'(\underline{\theta}) = f'(\underline{\theta})/2. \tag{4.36}$$

Proof. Define $\hat{g}^*(\theta) = \sup_{\underline{\theta} \le \tilde{\theta} \le \theta} \hat{g}(\tilde{\theta})$, then

$$\begin{aligned}\tilde{g}(\theta) = \mathscr{R}(g)(f(\theta)) &= e \inf_{0 < z \le e} \frac{\sup_{0 < s \le z} g(s)}{z}\bigg|_{e = f(\theta)} \\ &= f(\theta) \inf_{f(\underline{\theta}) < z \le f(\theta)} \frac{\sup_{0 < s \le z} g(s)}{z} = f(\theta) \inf_{\underline{\theta} < \theta' \le \theta} \frac{\hat{g}^*(\theta')}{f(\theta')}.\end{aligned}$$

Thus, $\tilde{g}(\underline{\theta}) = \hat{g}(\underline{\theta})$. And $\hat{g}(\underline{\theta}) = f(\underline{\theta})$ by Equation (4.32) of Proposition 6. By Lemma 2, $\tilde{g}$ is continuous, non-decreasing and bounded by f. For $\theta \geq \theta'$, it holds that

$$\begin{aligned} 0 \leq \tilde{g}(\theta) - \tilde{g}(\theta') &= f(\theta) \inf_{\underline{\theta} < z \leq \theta} \frac{\hat{g}^*(z)}{f(z)} - f(\theta') \inf_{\underline{\theta} < z \leq \theta'} \frac{\hat{g}^*(z)}{f(z)} \\ &\leq (f(\theta) - f(\theta')) \inf_{\underline{\theta} < z \leq \theta} \frac{\hat{g}^*(z)}{f(z)} \leq (f(\theta) - f(\theta')) \leq L_f |\theta - \theta'| \end{aligned}$$

Then, $\tilde{g}$ is Lipschitz continuous and has a derivative Lebegues almost everywhere on Θ. By the same argument, f is differentiable almost everywhere. Define $E_f \subset \Theta$ the set where $\tilde{g} = \hat{g}$, that is $E_f = \{\theta \in \Theta : \hat{g}(\theta) = \tilde{g}(\theta)\}$. Since $\hat{g}$ and $\tilde{g}$ are both continuous and Θ bounded, E_f is compact. On the interior of E_f it holds almost everywhere that $\tilde{g}' = \hat{g}'$. Since the boundary of E_f hast Lebegues measure 0, $\tilde{g}' = \hat{g}'$, almost everywhere on E_f. Next, consider $(\underline{\theta}, \overline{\theta}) \setminus E_f$, which is open. Take $\theta \in (\underline{\theta}, \overline{\theta}) \setminus E_f$ and by continuity of $\tilde{g}, \hat{g}$ there exists some $\epsilon > 0$, such that either $\tilde{g} < \hat{g}$ or $\tilde{g} > \hat{g}$ everywhere on $B_\epsilon(\theta) \subset (\underline{\theta}, \overline{\theta}) \setminus E_f$. In either case, there exists some $\theta^\star < \theta$ with

$$\tilde{g}(\theta) = f(\theta) \inf_{\underline{\theta} < \theta' \leq \theta} \frac{\hat{g}^*(\theta')}{f(\theta')} = f(\theta) \frac{\hat{g}^*(\theta^\star)}{f(\theta^\star)}, \tag{4.37}$$

and with continuity of $f, \hat{g}^*$ some $\epsilon^\star < \epsilon$, such that

$$\tilde{g}(\theta') = f(\theta') \frac{\hat{g}^*(\theta^\star)}{f(\theta^\star)} \quad \text{for} \quad \theta' \in B_{\epsilon^\star}(\theta).$$

Therefore, if f is differentiable in θ, which holds almost surely on Θ, and with (4.37),

$$\tilde{g}'(\theta) = f'(\theta) \frac{\tilde{g}(\theta)}{f(\theta)}, \tag{4.38}$$

implying Equation (4.35).

For the initial conditions of derivatives, observe that $\hat{g}'(\underline{\theta}) = \frac{1}{2} f'(\underline{\theta})$, by Equation (4.32) of Proposition 6. Now, $\tilde{g}'(\underline{\theta})$ is analysed. First, observe that $\hat{g}$ is strictly increasing in the neighborhood of $\underline{\theta}$ as $f' \geq l_f > 0$ Lebesgues almost sure, and thus, there exists an $\varepsilon > 0$, such that $\hat{g}$ is strictly increasing on $[\underline{\theta}, \underline{\theta} + \varepsilon]$. Then

$$\tilde{g}(\theta) = f(\theta) \inf_{\underline{\theta} \leq \theta' \leq \theta} \frac{\hat{g}(\theta')}{f(\theta')}, \text{ for } \theta \in [\underline{\theta}, \underline{\theta} + \varepsilon].$$

On $[\underline{\theta}, \underline{\theta} + \varepsilon]$, both, $\hat{g}$ and f are increasing and have identical boundary value $\hat{g}(\underline{\theta}) = f(\underline{\theta})$.

However, the slope of $\hat{g}$ is strictly smaller than that of f, in an $\tilde{\varepsilon}$-neighborhood around $\underline{\theta}$, with $\tilde{\varepsilon} \geq 0$. Therefore, $\frac{\hat{g}}{f}$ is decreasing and

$$\tilde{g}(\theta) = f(\theta)\,\frac{\hat{g}(\theta)}{f(\theta)} = \hat{g}(\theta)\,,\ \text{for } \theta \in [\underline{\theta}, \underline{\theta} + \varepsilon \wedge \tilde{\varepsilon}]\,.$$

And finally, $\tilde{g}'(\underline{\theta}) = \hat{g}'(\underline{\theta}) = \frac{1}{2}\,f'(\underline{\theta})$. □

Proposition 8. *Let $(f^\star, g^\star)$ be a fixed point of T. Suppose $f^\star, g^\star$, $(\phi_i)_i$ and the collection of the equity value function are sufficiently differentiable. Then, the fixed point is fully characterised, via $f^\star = f$ and $g^\star = \tilde{g} \circ f^{-1}$, by the solution $(f, \tilde{g}, \hat{g}, \hat{G}, \hat{V})$ of the differential equation:*

$$\begin{pmatrix} f'(\theta) \\ \tilde{g}'(\theta) \\ \hat{g}'(\theta) \\ \hat{G}'(\theta) \\ \hat{V}'(\theta) \end{pmatrix} = \begin{pmatrix} \frac{(1+\eta)\sigma^2}{2(r-\mu)}\frac{(f(\theta)/\theta)^2}{C(f(\theta)/\tilde{g}(\theta)) - f(\theta)/\theta} + \frac{\partial \hat{b}(\theta,\theta;g^\star)}{\partial \hat{\theta}} \\ f'(\theta)(\tilde{g}(\theta)/f(\theta))\mathbb{1}_{\tilde{g}(\theta)\neq\hat{g}(\theta)} + \hat{g}'(\theta)\mathbb{1}_{\tilde{g}(\theta)=\hat{g}(\theta)} \\ \mathscr{L}(f(\theta), \hat{g}(\theta), \hat{g}_{i(\theta)}(\theta), \hat{g}_{j(\theta)}(\theta), \hat{\nu}_{i(\theta)}(\theta), \hat{\nu}_{j(\theta)}(\theta), i(\theta), j(\theta), \theta) \\ \Pi(\theta) \odot (f(\theta)\,\mathbf{1}_n - \hat{G}(\theta)) \\ \Pi(\theta) \odot \Big([f(\theta)\,\mathbf{1}_n - \hat{G}(\theta)] \odot [f(\theta)\,\mathbf{1}_n - \hat{G}(\theta)] - \hat{V}(\theta)\Big) \end{pmatrix} \tag{4.39}$$

on $(\underline{\theta}, \overline{\theta})$ with $\hat{g}(\theta) = g \circ f$, where $(i(\theta), j(\theta))$ are the minimising indices of (4.22) *as given in* (4.23)*. The partial derivative of the boundary describing function $\hat{b}(\hat{\theta}, \theta; g^\star) = b(f(\hat{\theta}), \theta; g^\star)$ satisfies*

$$\frac{\partial \hat{b}(\theta, \theta; g^\star)}{\partial \hat{\theta}} = h(\theta, f(\theta), \tilde{g}(\theta), \tilde{g}'(\theta)) \tag{4.40}$$

for a specific real-valued functions $h : \Theta \times \mathbb{R}^3_+ \to \mathbb{R}_{\geq 0}$. The initial conditions are

$$\begin{pmatrix} f(\underline{\theta}) \\ \tilde{g}(\underline{\theta}) \\ \hat{g}(\underline{\theta}) \end{pmatrix} = \underline{\theta} \begin{pmatrix} f_1^\star \\ f_1^\star \\ f_1^\star \end{pmatrix}, \tag{4.41}$$

with $\hat{G}(\underline{\theta}) = \mathbb{1}_n \underline{\theta} g_1^\star$ and $\hat{V}(\underline{\theta}) = 0$, where $(f_1^\star, g_1^\star)$ denotes the equilibrium of the perfect information case, that is $\Theta = \{1\}$; hence $D = X$ with $f_1^\star = g_1^\star$, which exists and is unique under the given assumptions.

Proof. Propositions 6 and 7 characterise the best response of the rating agency. Hilpert et al. (2024) show in Proposition 6 of their Internet Appendix that, for any $k \in \mathscr{A}_g$, the firms best response strategy $f = f(\cdot; k)$ satisfies

$$f'(\theta) = \frac{(1+\eta)\sigma^2}{2(r-\mu)} \frac{(f(\theta)/\theta)^2}{C(f(\theta)/\hat{k}(\theta)) - f(\theta)/\theta} + \frac{\partial \hat{b}(\theta, \theta; k)}{\partial \hat{\theta}} \tag{4.42}$$

for $\theta \in (\underline{\theta}, \overline{\theta})$, where $\hat{k} = k \circ f$ and the partial derivative of the boundary describing function $\hat{b}(\hat{\theta}, \theta; k) = b(f(\hat{\theta}), \theta; k)$ for $\hat{\theta} \leq \theta$ in (θ, θ) is a function of $f(\theta), \hat{k}'(\theta)$ and $\hat{k}(\theta)$, that is

$$\frac{\partial \hat{b}(\theta, \theta; k)}{\partial \hat{\theta}} = h(\theta, f(\theta), \hat{k}(\theta), \hat{k}'(\theta)) \tag{4.43}$$

for some function $h : \Theta \times \mathbb{R}^3_+ \to \mathbb{R}_{\geq 0}$. By those Propositions the best responses satisfy $f(\underline{\theta}) = \hat{g}(\underline{\theta}) = \tilde{g}(\underline{\theta})$ and $\hat{g}'(\underline{\theta}) = \tilde{g}'(\underline{\theta})$. Now, for $\theta \searrow \underline{\theta}$ the information asymmetry vanishes and the limit must be the unique equilibrium of the perfect information case of Manso (2013). □

Proof of Proposition 5. Let $(f^\star, g^\star)$ be a fixed point of T from Proposition 8. The fixed-point $(f^\star, g^\star)$ is an equilibrium, if $g(\cdot; f^\star) = \mathscr{R} \circ g(\cdot; f^\star)$. By Proposition 7, the latter is equivalent to $\tilde{g} = \hat{g}$, which is follows from $0 \leq \hat{g}' \leq f' \frac{\hat{g}}{f}$ on $(\underline{\theta}, \overline{\theta})$ by Lemma 6. □

4.C General Implications

Our equilibrium allows general implications for the firm's equity value and its default strategy. Hilpert et al. (2024) have shown that those hold in a setting with information asymmetry and feedback, but without ambiguity. It is important to point out that ambiguity within the rating agency's analyst team changes the way that the firm's observed cash flow path is translated into a rating. However, from the rated firm's point of view, best responses to a given rating agency strategy (see Section 4.3.1) are the same, irrespective of the internal process that generated the rating. Therefore, it holds that the following three equilibrium results continue to hold in a setting with ambiguity. Let us denote by $(f^\star, g^\star)$ a unique equilibrium and a corresponding firm's equity value by $v^\star(d, e; \theta) = v^\star(d, e; \theta, g^\star)$ for $\theta \in \Theta$ in compliance with Proposition 5.

First, a firm benefits from surviving distressed periods, that is, recovering from a new all time low cash flow level e' to the initial observed cash flow level d increases the firm's equity value. Let $D^\star(\theta) = f^\star(\theta)$ be the firm's observed cash flow based default threshold, d the currently observed cash flow and e, e' two minimum observed cash flows with $D^\star(\theta) \leq e' \leq e \leq d$. Then, it holds that $v^\star(d, e'; \theta) \geq v^\star(d, e; \theta)$.

Second, if two firms display identical observed cash flows but differing measurement errors, the firm with a downward measurement error has higher true cash flow and therefore will choose to survive longer. Let $X^\star(\theta) = D^\star(\theta)/\theta$ be the firm's default threshold in terms of its true cash flow. Then, it holds for $\theta, \theta' \in \Theta$ with $\theta' \leq \theta$ that $X^\star(\theta') \geq X^\star(\theta)$ while $D^\star(\theta') \leq D^\star(\theta)$.

Third, consider a fixed observed cash flow d and a fixed observed running minimum e. For two firms with different measurement errors, the equity value of the firm who has a larger downward measurement error and thus exhibits a higher true cash flow must be higher. That is, for $\theta' \leq \theta$, the firm's equity value satisfies $v^\star(d, e; \theta') \geq v^\star(d, e; \theta)$.

Finally, for a firm with fixed true cash flow level x and a fixed true running minimum y, a larger upward measurement error level - implying that the firm is more overrated - leads to a higher equity value. Formally, for an observed cash flow $d = \theta x$ and observed running minimum by $e = \theta y$ for any measurement error θ, we have $v^\star(\theta' x, \theta' y; \theta') \leq v^\star(\theta x, \theta y; \theta)$ for $\theta' \leq \theta$.

4.D Auxiliary Results

Lemma 3. *The mapping $f \mapsto m_{2,i}(\cdot; f)$ is defined by*

$$m_{2,i}(e; f) = \begin{cases} \mathbb{E}^{\pi_{i,0}}\left[f(\tilde{\theta})^2 \mid f(\tilde{\theta}) < e\right] & \text{for} \quad e \geq f(\underline{\theta}), \\ e^2 & \text{for} \quad e < f(\underline{\theta}), \end{cases} \tag{4.44}$$

for $f \in \mathscr{K}_f$. Then $f \mapsto m_{2,i}(\cdot; f)$ is continuous with respect to the sup-norm.

Proof. Suppose $f, f' \in \mathscr{K}_f$ with $||f - f'||_\infty \leq \epsilon$, for some $\epsilon > 0$. Consider $e \in [0, f(\underline{\theta}) \wedge f'(\underline{\theta}))$. Then $m_{2,i}(e; f) = m_{2,i}(e; f') = e^2$ and

$$|m_{2,i}(e; f) - m_{2,i}(e; f')| = 0, \qquad \text{for} \quad e \in [0, f(\underline{\theta}) \wedge f'(\underline{\theta})).$$

Now, let $e \in [f(\underline{\theta}) \wedge f'(\underline{\theta}), f(\underline{\theta}) \vee f'(\underline{\theta}))$. Without loss of generality, assume $f(\underline{\theta}) < f'(\underline{\theta})$ and thus $f(\underline{\theta}) \leq e < f'(\underline{\theta})$. It holds that $f(\underline{\theta})^2 \leq m_{2,i}(e; f) \leq e^2$. Note that $m_{2,i}(e; f') =$

$e^2 < f'(\underline{\theta})^2$. As $f \in \mathscr{K}_f$ implies a uniform upper bound $\overline{f}$, $f^2 - f'^2 = (f - f')(f + f')$ results in $\|f^2 - f'^2\|_\infty \leq 2\,\overline{f}\,\overline{\theta}\,\epsilon$. Therefore, it follows that

$$|m_{2,i}(e; f) - m_{2,i}(e; f')| \leq 2\,\overline{f}\,\overline{\theta}\,\epsilon, \quad \text{for} \quad e \in [f(\underline{\theta}) \wedge f'(\underline{\theta}), f(\underline{\theta}) \vee f'(\underline{\theta})).$$

Consider $e \in [f(\underline{\theta}) \vee f'(\underline{\theta}), f(\underline{\theta}) \vee f'(\underline{\theta}) + \epsilon^{\frac{1}{2}})$. Then

$$f(\underline{\theta})^2 \leq m_{2,i}(e; f) \leq e^2\,, \text{ and } f'(\underline{\theta})^2 \leq m_{2,i}(e; f') \leq e^2\,,$$

as well as

$$e^2 \leq \Big((f(\underline{\theta})^2 + 2\,f(\underline{\theta})\epsilon^{\frac{1}{2}}) \vee (f'(\underline{\theta})^2 + 2\,f'(\underline{\theta})\epsilon^{\frac{1}{2}}) \Big) + \epsilon.$$

Recall again, $f(\theta), f'(\theta) \leq \overline{f}\,\overline{\theta}$, and thus

$$|m_{2,i}(e; f) - m_{2,i}(e; f')| \leq 2\,\overline{f}\,\overline{\theta}\,\epsilon^{\frac{1}{2}} + (1 + 2\,\overline{f}\,\overline{\theta})\epsilon\,, \text{ for } e \in [f(\underline{\theta}) \vee f'(\underline{\theta}), f(\underline{\theta}) \vee f'(\underline{\theta}) + \epsilon^{\frac{1}{2}}).$$

Consider $e \geq (f(\underline{\theta}) \vee f'(\underline{\theta})) + \epsilon^{\frac{1}{2}}$. Since $f, f' \in \mathscr{K}_f$ both functions are continuous and strictly increasing with minimum slope l_f and thus their respective inverse functions exist and are well-defined. Without loss of generality, assume $f^{-1}(e) \leq f'^{-1}(e)$. Note that $m_{2,i}(e; f')$ has the following representation:

$$m_{2,i}(e; f') = \frac{\int_{\underline{\theta}}^{f'^{-1}(e)} f'(\theta)^2 \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta) d\theta}.$$

This equation can be rearranged to

$$\begin{aligned}
m_{2,i}(e; f') =& \frac{\int_{\underline{\theta}}^{f^{-1}(e)} f'(\theta)^2 \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta) d\theta} + \frac{\int_{f^{-1}(e)}^{f'^{-1}(e)} f'(\theta)^2 \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta) d\theta} \\
=& \frac{\int_{\underline{\theta}}^{f^{-1}(e)} f(\theta)^2 \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta) d\theta} \\
& + \frac{\int_{\underline{\theta}}^{f^{-1}(e)} (f'(\theta)^2 - f(\theta)^2) \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta) d\theta} + \frac{\int_{f^{-1}(e)}^{f'^{-1}(e)} f'(\theta)^2 \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta) d\theta} \\
=& \frac{\int_{\underline{\theta}}^{f^{-1}(e)} f(\theta)^2 \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f^{-1}(e)} \phi_i(\theta) d\theta} \frac{\int_{\underline{\theta}}^{f^{-1}(e)} \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta) d\theta} \\
& + \frac{\int_{\underline{\theta}}^{f^{-1}(e)} (f'(\theta)^2 - f(\theta)^2) \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta) d\theta} + \frac{\int_{f^{-1}(e)}^{f'^{-1}(e)} f'(\theta)^2 \phi_i(\theta) d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta) d\theta}
\end{aligned}$$

$$\begin{aligned}
&= m_{2,i}(e;f) - m_{2,i}(e;f)\frac{\int_{f^{-1}(e)}^{f'^{-1}(e)} \phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta)d\theta} \\
&\quad + \frac{\int_{\underline{\theta}}^{f^{-1}(e)}(f'(\theta)^2 - f(\theta)^2)\phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta)d\theta} + \frac{\int_{f^{-1}(e)}^{f'^{-1}(e)} f'(\theta)^2\phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta)d\theta}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|m_{2,i}(e;f') - m_{2,i}(e;f)| \leq & m_{2,i}(e;f)\frac{\int_{f^{-1}(e)}^{f'^{-1}(e)} \phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta)d\theta} + \frac{\int_{f^{-1}(e)}^{f'^{-1}(e)} f'(\theta)^2\phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta)d\theta} \\
&+ \frac{\int_{\underline{\theta}}^{f^{-1}(e)} |f'(\theta)^2 - f(\theta)^2|\phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta)d\theta}. \qquad (4.45)
\end{aligned}$$

For all three parts upper bounds are subsequently derived. Observe that $0 \leq f'^{-1}(e) - f^{-1}(e) \leq l_f^{-1}\epsilon$, where the first inequality holds by assumption and the second follows from the fact that f, f' are strictly increasing with a minimum slope $l_f > 0$ and $\|f - f'\|_\infty < \varepsilon$ by assumption. Also $f, f' \in \mathscr{K}_f$ and thus are bounded by $\overline{f}\,\overline{\theta}$ and have maximal slope L_f. Now, ϕ_i is bounded away from zero and bounded from above by assumption $0 < \underline{\phi}_i \leq \phi_i(\theta) \leq \overline{\phi}_i$, for $\theta \in \Theta$, set $\underline{\phi} = \min_{i=1,\ldots,n} \underline{\phi}_i$ and $\overline{\phi} = \max_{i=1,\ldots,n} \overline{\phi}_i$. Then

$$\begin{aligned}
\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta)d\theta \geq \int_{\underline{\theta}}^{f'^{-1}(e)} \underline{\phi}d\theta \geq \int_{\underline{\theta}}^{f'^{-1}(f'(\underline{\theta})+\epsilon^{\frac{1}{2}})} \underline{\phi}d\theta \\
= \underline{\phi}\Big(f'^{-1}(f'(\underline{\theta}) + \epsilon^{\frac{1}{2}}) - \underline{\theta}\Big) \geq \underline{\phi}L_f^{-1}\epsilon^{\frac{1}{2}}.
\end{aligned}$$

Using this, we can bound the first expression on the right-hand side of Equation (4.45):

$$m_{2,i}(e;f)\frac{\int_{f^{-1}(e)}^{f'^{-1}(e)} \phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta)d\theta} \leq \overline{f}^2\overline{\theta}^2\frac{\overline{\phi}l_f^{-1}\epsilon}{\underline{\phi}L_f^{-1}\epsilon^{\frac{1}{2}}} = \frac{\overline{f}^2\overline{\theta}^2 L_f\overline{\phi}}{l_f\underline{\phi}}\epsilon^{\frac{1}{2}}.$$

For the second expression the same arguments apply and

$$\frac{\int_{f^{-1}(e)}^{f'^{-1}(e)} f'(\theta)^2\phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta)d\theta} \leq \overline{f}^2\overline{\theta}^2\frac{\int_{f^{-1}(e)}^{f'^{-1}(e)} \phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta)d\theta} = \frac{\overline{f}^2\overline{\theta}^2 L_f\overline{\phi}}{l_f\underline{\phi}}\epsilon^{\frac{1}{2}}.$$

The third expression can be estimated as follows:

$$\frac{\int_{\underline{\theta}}^{f^{-1}(e)} |f'(\theta)^2 - f(\theta)^2|\phi_i(\theta)d\theta}{\int_{\underline{\theta}}^{f'^{-1}(e)} \phi_i(\theta)d\theta} \leq \frac{2\overline{f}\,\overline{\theta}\|f - f'\|_\infty}{\underline{\phi}L_f^{-1}\epsilon^{\frac{1}{2}}} \leq \frac{2\overline{f}\,\overline{\theta}\,L_f}{\underline{\phi}}\epsilon^{\frac{1}{2}}.$$

Putting together the pieces gives

$$|m_{2,i}(e;f) - m_{2,i}(e;f')| \leq \left(\frac{2\overline{f}\,\overline{\theta}(\underline{\phi} + L_f)}{\underline{\phi}} + \frac{\overline{f}^2\overline{\theta}^2 L_f \overline{\phi}}{l_f \underline{\phi}} \right) \varepsilon^{\frac{1}{2}} + (1 + 2\overline{f}\,\overline{\theta})\varepsilon \,. \tag{4.46}$$

□

Lemma 4. *There exist* $0 < \underline{e} < \overline{e} < \infty$ *such that for all* $f \in \mathscr{K}_f$*, the ambiguity averse best response rating strategy* $g = g(\cdot; f)$ *satisfies*

$$g(e) = e\,, \ \textit{for } e \in (0, \underline{e}]\,, \quad \textit{and} \quad \mathscr{R}(g)(e) = \sup_{0<s\leq \overline{e}} g(s)\,, \ \textit{for } e \geq \overline{e}\,.$$

Proof. Recall the definition of $\mathscr{K}_f$ in the Proof of Proposition 4. Observe that $g_i(e;f) = e$, for $0 < e \leq f(\underline{\theta})$ and $i = 1, ..., n$. Equation (4.24) immediately gives $g(e) = g(e;f) = e$, for $0 < e \leq f(\underline{\theta})$. Set $\underline{e} = \underline{\theta}\,\underline{f} \leq f(\underline{\theta})$, for all $f \in \mathscr{K}_f$, then the first claim of the lemma is established.

For $e \geq f(\overline{\theta})$, $g_i(e;f) = g_i(f(\overline{\theta});f)$ and $\nu_i(e;f) = \nu_i(f(\overline{\theta});f)$. Then Equation (4.22) gives $g(e) = g(e;f) = g(f(\overline{\theta});f)$, for $e \geq f(\overline{\theta})$. Recall, $g^*(e) = \sup_{0<s\leq e} g(s)$. Then, $g^*(e) = g^*(f(\overline{\theta}))$, for $e \geq f(\overline{\theta})$. Turning to the intermediate values, then $f(\underline{\theta}) \leq g_i(e;f) \leq f(\overline{\theta})$ and hence $f(\underline{\theta}) \leq g(e) \leq f(\overline{\theta})$ by Equation (4.24), for $e \in [f(\underline{\theta}), f(\overline{\theta})]$. This implies $f(\underline{\theta}) \leq g^*(e) \leq f(\overline{\theta})$ by Equation (4.24), for $e \in [f(\underline{\theta}), f(\overline{\theta})]$. Motivated by these insights for the three distinct regions, write for $e \geq f(\overline{\theta})$

$$\begin{aligned}
\mathscr{R}(g)(e) &= e \left(\left(\inf_{0<z\leq f(\underline{\theta})} \frac{g^*(z)}{z} \right) \wedge \left(\inf_{f(\underline{\theta})<z\leq f(\overline{\theta})} \frac{g^*(z)}{z} \right) \wedge \left(\inf_{f(\overline{\theta})<z\leq e} \frac{g^*(z)}{z} \right) \right) \\
&= e \left(\left(\inf_{0<z\leq f(\underline{\theta})} \frac{z}{z} \right) \wedge \left(\inf_{f(\underline{\theta})<z\leq f(\overline{\theta})} \frac{g^*(z)}{z} \right) \wedge \left(\inf_{f(\overline{\theta})<z\leq e} \frac{g^*(f(\overline{\theta}))}{z} \right) \right) \\
&= e \wedge \left(e \inf_{f(\underline{\theta})<z\leq f(\overline{\theta})} \frac{g^*(z)}{z} \right) \wedge g^*(f(\overline{\theta}))
\end{aligned}$$

Now, $g^*(f(\overline{\theta})) \leq f(\overline{\theta}) \leq e$ and

$$\left(e \inf_{f(\underline{\theta})<z\leq f(\overline{\theta})} \frac{f(\underline{\theta})}{z} \right) \wedge g^*(f(\overline{\theta})) \leq \mathscr{R}(g)(e) \leq \left(e \inf_{f(\underline{\theta})<z\leq f(\overline{\theta})} \frac{f(\overline{\theta})}{z} \right) \wedge g^*(f(\overline{\theta}))\,,$$

resulting in

$$\left(e\,\frac{f(\underline{\theta})}{f(\overline{\theta})}\right)\wedge g^*(f(\overline{\theta}))\leq \mathscr{R}(g)(e)\leq g^*(f(\overline{\theta}))\,,\ \text{for } e\geq f(\overline{\theta})\,.$$

Let $\overline{e}_f = f(\overline{\theta})\,\frac{f(\overline{\theta})}{f(\underline{\theta})}$, then $\mathscr{R}(g)(e) = g^*(f(\overline{\theta})) = g^*(e) = \sup_{0<s\leq \overline{e}_f} g(s)$, for $e \geq \overline{e}_f$. Finally, set $\overline{e} = \overline{\theta}\,\overline{f}\,\frac{\overline{\theta}\,\overline{f}}{\underline{\theta}\,\underline{f}}$, then $\overline{e} \geq \overline{e}_f$, for all $f \in \mathscr{K}_f$, and the second claim of the lemma is established. □

Lemma 5. *There exist* $0 < \underline{e} < \overline{e} < \infty$*, such that for all ambiguity averse best responses* g, g'*, with* $g = g(\cdot; f)$ *and* $g' = g(\cdot; f')$*, where* $f, f' \in \mathscr{K}_f$*, it holds that*

$$\left\|\mathscr{R}(g)-\mathscr{R}(g')\right\|_\infty \leq \frac{\overline{e}}{\underline{e}}\,\left\|g-g'\right\|_\infty\,.$$

Proof. We start with the following straight-forward observation. For bounded functions $h, h' : A \to \mathbb{R}$, with $A \subset \mathbb{R}$, it holds that $|\sup_{x\in A} h(x) - \sup_{x\in A} h'(x)| \leq \sup_{x\in A} |h(x) - h'(x)|$ and $|\inf_{x\in A} h(x) - \inf_{x\in A} h'(x)| \leq \sup_{x\in A} |h(x) - h'(x)|$. Then Lemma 4 applies and with $0 < \underline{e} < \overline{e} < \infty$ given there, it holds

$$\begin{aligned}
|\mathscr{R}(g)(e)-\mathscr{R}(g')(e)| &= |g(e)-g'(e)| = 0\,,\ \text{for } 0<e\leq \underline{e}\,,\ \text{and}\\
|\mathscr{R}(g)(e)-\mathscr{R}(g')(e)| &= \left|\sup_{0<s\leq\overline{e}} g(s) - \sup_{0<s\leq\overline{e}} g'(s)\right| \leq \|g-g'\|_\infty\,,\ \text{for } e\geq\overline{e}\,.
\end{aligned}$$

For $e \in [\underline{e}, \overline{e}]$, first observe that $z \mapsto \frac{\sup_{0<s\leq z} g(s)}{z}$ is bounded by 1, as by Proposition 3, $g(e) \leq e$ for all $e > 0$, and, for $0 < e \leq \underline{e}$ it holds that $\frac{\sup_{0<s\leq z} g(s)}{z} = 1$. Then for $\underline{e} \leq e \leq \overline{e}$

$$\mathscr{R}(g)(e) = e\inf_{0<z\leq e}\frac{\sup\limits_{0<s\leq z} g(s)}{z} = e\left(\inf_{0<z\leq \underline{e}}\frac{\sup\limits_{0<s\leq z} g(s)}{z}\wedge\inf_{\underline{e}\leq z\leq e}\frac{\sup\limits_{0<s\leq z} g(s)}{z}\right) = e\inf_{\underline{e}\leq z\leq e}\frac{\sup\limits_{0<s\leq z} g(s)}{z},$$

and

$$\begin{aligned}
|\mathscr{R}(g)(e)-\mathscr{R}(g')(e)| &= e\left|\inf_{\underline{e}\leq z\leq e}\frac{\sup_{0<s\leq z} g(s)}{z} - \inf_{\underline{e}\leq z\leq e}\frac{\sup_{0<s\leq z} g'(s)}{z}\right|\\
&\leq e\sup_{\underline{e}\leq z\leq e}\frac{1}{z}\left|\sup_{0<s\leq z} g(s) - \sup_{0<s\leq z} g'(s)\right| \leq \frac{\overline{e}}{\underline{e}}\,\|g-g'\|_\infty\,.
\end{aligned}$$

A bound for each of the three cases is established, finishing the proof. □

We here state the inequality, which implies that the solution to the ODE to be economically reasonable:

Lemma 6. *Let f be a strictly increasing, differentiable and non-negative firm strategy and g the rating agency's ambiguity averse best-response, which is almost everywhere differentiable. Define $\hat{g}$ by $\hat{g}(\theta) = (g \circ f)(\theta)$ on Θ, and suppose that $0 \leq \hat{g}'(\theta) \leq f'(\theta)\frac{\hat{g}(\theta)}{f(\theta)}$, for $\theta \in (\underline{\theta}, \overline{\theta})$, then $\mathscr{R}(g) = g$ on $f(\Theta)$.*

Proof. Note, $\hat{g}'(\theta) \geq 0$ and f being strictly increasing implies that g is non-decreasing, and

$$\mathscr{R}(g)(e) = e \inf_{0<z\leq e} \frac{g(z)}{z},$$

and by the assumed upper bound for $\hat{g}'$ it holds

$$\left(\frac{\hat{g}}{f}\right)' = \frac{\hat{g}' f - \hat{g} f'}{f^2} \leq \frac{f'\frac{\hat{g}}{f} f - \hat{g} f'}{f^2} \leq 0\,.$$

Now,

$$\mathscr{R}(g)(f(\theta)) = f(\theta) \inf_{f(\underline{\theta})\leq z\leq f(\theta)} \frac{g(z)}{z} = f(\theta) \inf_{\underline{\theta}\leq\theta'\leq\theta} \frac{\hat{g}(\theta')}{f(\theta')} = f(\theta)\,\frac{\hat{g}(\theta)}{f(\theta)} = \hat{g}(\theta)\,,$$

gives the claimed result $\mathscr{R}(g) = g$ as f is strictly increasing. $\square$

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Appendix A

Summaries

A.1 Summary

This dissertation advances the literature on structural credit risk models by explicitly integrating realistic and previously under-examined frictions - namely, behavioural biases, information asymmetry, and ambiguity. The three independent studies presented here examine how these features affect firms' financing decisions, capital structures, and credit risk.

The first paper explores the impact of time-inconsistent preferences on risk-taking and leverage decisions in entrepreneur-led firms. The analysis distinguishes between naive and sophisticated entrepreneurs, revealing that myopic decision-makers not only display differing preferences with respect to volatility and default, but may actually increase the value of debt from the perspective of lenders in certain cases. These findings have direct implications for credit contract design and the assessment of borrower risk.

The second paper focuses on uncertainty regarding the true value of firm assets and its effect on the optimal choice of debt maturity. The model demonstrates that greater uncertainty systematically leads to shorter optimal maturities, as repeated market interactions and the opportunity for market learning become especially valuable under high uncertainty. This mechanism helps explain why firms facing higher informational frictions are empirically observed to favour shorter debt maturities.

The third paper analyses the interplay between credit-relevant ambiguity and heterogeneous, potentially conflicting, beliefs within rating agencies. The study finds that the the direction of prevailing uncertainty plays a decisive role for credit ratings and default behaviour. Depending on whether analysts are jointly optimistic or pessimistic, ambiguity can result in either accelerated or delayed firm's default compared to a benchmark without ambiguity within the rating agency.

Collectively, these papers sharpen our understanding of how behavioural factors, market learning, and information constraints interact to shape firms' financing and risk profiles. The findings emphasize the importance of accounting for behavioural aspects and information dynamics in both the theoretical and empirical study of credit markets, as well as in regulatory frameworks. For practice, the results highlight new practical guidance for the structuring of credit contracts, risk management strategies, and rating methodologies.

Looking ahead, the modelling approaches developed here open up further avenues for

research. In particular, extending the analysis of information frictions to settings with endogenous information acquisition, and empirically validating the model predictions, would strengthen the link between theory and financial market practice.

A.2 Zusammenfassung

Die vorliegende Dissertation leistet einen Beitrag zur Literatur über strukturelle Kreditrisikomodelle, indem sie realistische und bislang wenig untersuchte Friktionen - wie verhaltensökonomische Verzerrungen, Informationsasymmetrien und Ambiguität - explizit integriert. Die drei unabhängigen Studien, die hier vorgestellt werden, untersuchen, wie diese Merkmale die Finanzierungsentscheidungen, Kapitalstrukturen und das Kreditrisiko von Unternehmen beeinflussen.

Die erste Arbeit untersucht die Auswirkungen zeitinkonsistenter Präferenzen auf Risiko- und Verschuldungsentscheidungen in gründergeführten Unternehmen. Die Analyse unterscheidet zwischen naiven und sich ihrer Zeitinkonsistenz bewussten Unternehmern und zeigt, dass kurzsichtige Entscheidungsträger nicht nur unterschiedliche Präferenzen hinsichtlich Volatilität und Ausfall zeigen, sondern in bestimmten Fällen sogar den Wert der Schuldtitel aus Sicht der Gläubiger erhöhen können. Diese Erkenntnisse haben direkte Implikationen für die Gestaltung von Kreditverträgen und die Beurteilung des Kreditnehmerrisikos.

Die zweite Arbeit konzentriert sich auf die Unsicherheit bezüglich des wahren Wertes der Vermögenswerte eines Unternehmens und deren Einfluss auf die optimale Wahl der Laufzeit von Schuldtiteln. Das Modell zeigt, dass größere Unsicherheit systematisch zu kürzeren optimalen Laufzeiten führt, da wiederholte Marktinteraktionen und die Möglichkeit des Marktlernens unter hoher Unsicherheit besonders wertvoll werden. Dieser Mechanismus hilft zu erklären, warum Unternehmen mit höheren Informationsasymmetrien empirisch beobachtet tendenziell kürzere Laufzeiten bevorzugen.

Die dritte Arbeit analysiert das Zusammenspiel von kreditrelevanter Ambiguität und heterogenen, potenziell widersprüchlichen Überzeugungen innerhalb von Ratingagenturen. Die Studie zeigt, dass die Richtung der vorherrschenden Unsicherheit eine entscheidende Rolle für Kreditratings und Ausfallverhalten spielt. Je nachdem, ob Analysten gemeinsam optimistisch oder pessimistisch sind, kann Ambiguität im Vergleich zum Referenzfall ohne

Ambiguität innerhalb der Ratingagentur entweder zu beschleunigten oder zu verzögertem Ausfall des Unternehmens führen.

Insgesamt schärfen diese Arbeiten das Verständnis dafür, wie Verhaltensfaktoren, marktbasierte Lernprozesse und Informationsbeschränkungen zusammenwirken und die Risiko- und Finanzierungsprofile von Unternehmen prägen. Die Erkenntnisse unterstreichen die Bedeutung, verhaltensökonomische Aspekte und Informationsdynamiken sowohl in der theoretischen als auch in der empirischen Analyse von Kreditmärkten sowie in regulatorischen Rahmenbedingungen zu berücksichtigen. Für die Praxis bieten die Ergebnisse neue Ansätze für die Gestaltung von Kreditverträgen, Risikomanagementstrategien und Ratingmethoden.

Mit Blick auf die Zukunft eröffnen die hier entwickelten Modellierungsansätze weitere Forschungsfelder. Insbesondere könnte die Analyse von Informationsfriktionen auf Kontexte mit endogener Informationsakquisition ausgeweitet werden, und eine empirische Validierung der Modellergebnisse würde die Verbindung zwischen Theorie und Finanzmarktpraxis weiter stärken.

List of publications

In August 2025, I published the article “Present Bias, Risk Management and Capital Structure“ in the *Journal of Economic Behavior & Organization* (Volume 236, Article 107094). The version included here has been slightly corrected from the original publication to fix minor typographical errors.

Erklärung

Hiermit erkläre ich, Jan Pape, dass ich keine kommerzielle Promotionsberatung in Anspruch genommen habe. Die Arbeit wurde nicht schon einmal in einem früheren Promotionsverfahren angenommen oder als ungenügend beurteilt.

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Ich, Jan Pape, versichere an Eides statt, dass ich die Dissertation mit dem Titel

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Selbstdeklaration

Die folgende Tabelle stellt meinen Anteil bei der Konzeption, Durchführung und Berichtsabfassung der in meiner kumulativen Dissertation enthaltenen Artikel dar, welche den Kapiteln 2, 3 und 4 entsprechen. Die Einschätzung des geleisteten Anteils erfolgt mittels Punkteinschätzung von 1-100%. Bei dem Artikel in Kapitel 2 liegt die Eigenleistung bei 100%.

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