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The use of critical conformal invariance in renormalization of twist-2 operators

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В столе ключи уснули
не вытянуть назад
и в силах тонкой пули
селения лежат.
Профессора снедает
графическая страсть
и он сопределяет
свою над богом власть.
Профессор, кроме сота
имея руконог –
я вынужден работать
твой теневой урок.
И в дрожи экспедиций
пружиною с меня
моя в шкале зарница
играющего дня.

— Стас Красовицкий

Abstract

This thesis explores the application of methods of Conformal Field Theory (CFT) in Quantum Field Theories where the classical conformal symmetry is broken. The main example and motivation for such studies is Quantum Chromodynamics (QCD), but other theories are discussed as well. In particular, we focus on the problem of the renormalization of twist-2 operators and the perturbative calculation of their anomalous dimensions.

The main approach implemented in this thesis is to consider the theory at its critical point, where the Poincaré symmetry is enhanced to the full conformal group, and subsequently relate the results to the physical theory. We exploit this approach in two ways. First, we argue that the structure of the analytic continuation of the CFT spectrum in spin constrains the analytic structure of anomalous dimensions. These constraints in turn allow us to predict the singular behavior and perform an all-order resummation of certain singular contributions. We verify this framework in a variety of theories, including six-dimensional φ^3-type models, the $O(N)$-symmetric φ^4 model, and the Gross–Neveu–Yukawa model. The second application concerns the collinear conformal symmetry, which is exact at the critical point, and constrains the evolution of exclusive QCD processes. These constraints, together with the use of conformal Ward identities, allow us to calculate the evolution kernel of the transversity quark operator at three-loop order.

Zusammenfassung

Diese Arbeit untersucht die Anwendung von Methoden der Konformen Feldtheorie (CFT) in Quantenfeldtheorien, in denen die klassische konforme Symmetrie gebrochen ist. Das wichtigste Beispiel und die Hauptmotivation für solche Untersuchungen ist die Quantenchromodynamik (QCD), aber auch andere Theorien werden diskutiert. Insbesondere konzentrieren wir uns auf das Problem der Renormierung von Twist-2-Operatoren und die perturbative Berechnung ihrer anomalen Dimensionen.

Der in dieser Arbeit verfolgte Hauptansatz besteht darin, die Theorie an ihrem kritischen Punkt zu betrachten, an dem die Poincaré-Symmetrie zur vollen Konformen Gruppe erweitert wird, und die Ergebnisse anschließend auf die physikalische Theorie zu übertragen. Wir nutzen diesen Ansatz auf zwei Arten. Erstens argumentieren wir, dass die Struktur der analytischen Fortsetzung des CFT-Spektrums im Spin die analytische Struktur der anomalen Dimensionen einschränkt. Diese Einschränkungen ermöglichen es uns, das singuläre Verhalten vorherzusagen und eine Resummation bestimmter singulärer Beiträge zu allen Ordnungen durchzuführen. Wir überprüfen diesen Rahmen in einer Reihe von Theorien, darunter sechsdimensionale Modelle vom φ^3-Typ, das $O(N)$-symmetrische φ^4-Modell sowie das Gross–Neveu–Yukawa-Modell. Die zweite Anwendung betrifft die kollineare konforme Symmetrie, die am kritischen Punkt exakt ist und die Evolution exklusiver QCD-Prozesse einschränkt. Diese Einschränkungen erlauben es uns, zusammen mit der Verwendung konformer Ward-Identitäten, den Evolutionskern des Transversitäts-Quark-Operators in Dreischleifenordnung zu berechnen.

List of Publications

This thesis is based on

[1] A. N. Manashov, S. Moch, **LS**, *Anomalous dimensions at small spins*, JHEP **09** (2025) 106.

[2] A. N. Manashov, S. Moch, **LS**, *Three-loop evolution kernel for transversity operator*, JHEP **09** (2024) 192, [2407.12696].

The material was also presented on the conference and published as proceeding

[3] A. N. Manashov, S. Moch, **LS**, *Resummation of small-spin singularities in anomalous dimensions of twist-two operators*, 17th International Symposium on Radiative Corrections: Applications of Quantum Field Theory to Phenomenology (RADCOR2025), [2602.14149].

Other publications during PhD

[4] S. E. Derkachov, A. P. Isaev, **LS**, *Conformal four-point ladder integrals in diverse dimensions and polylogarithms*, JHEP **04** (2026) 123, [2510.04235].

[5] A. N. Manashov, **LS**, *Correction exponents in the chiral Heisenberg model at* $1/N^2$*: singular contributions and operator mixing*, [2603.21989].

[6] M. Chakraborty, M. Klann, S. Moch, P. Mukherjee, T. Porsche, O. Schnetz, **LS**, *Graphical Functions by Examples*, [2604.25739].

Contents

Acknowledgements

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Introduction

The main goal of modern high-energy physics is to describe how the matter around us works by dividing it into the smallest building blocks, and the most successful and broadly confirmed framework for this is the Standard Model (SM). While all its sectors share the high precision with which they predict experimental results, each is unique in its complexities and unsolved problems. In the strong interactions, the central difficulty can be attributed to the absence of a clear link between the degrees of freedom observed in detectors and those appearing in the theoretical description. This problem dates back to M. Gell-Mann [7,8] (with the subsequent triumph of the experimental detection of Ω^- [9]) and remains unsolved to this day (see [10] for a review).

This difficulty, however, did not prevent the quantum field theory (QFT)-based description of strong interactions — Quantum Chromodynamics (QCD) — from developing into a well-established and successful theory (see [11] for a review of roughly 50 years of development), although it should be noted that the framework of perturbative QCD alone is not sufficient to describe the full range of observed phenomena. It is natural, then, that in the absence of simple solutions, scientists had to invent creative methods. The distinguished role of *conformal symmetry* in these studies is particularly remarkable. Indeed, the theoretical description of hadron scattering was driven by the discovery of Bjorken scaling [12, 13], which was tempting to describe through the lens of critical phenomena [14–16], although such a connection was not apparent at the time:

> *When I said that the boiling water may have something to do with the deep inelastic scattering, I received a very strange look. I shall add in the parenthesis that this was a beginning of the long series of "strange looks" which I keep receiving to this day.*
>
> — A. M. Polyakov, *From Quarks to Strings* [17]

It is also remarkable that the *operator product expansion*, introduced to this problem by K. G. Wilson [18], not only made this connection clear — reflecting on early advances in the use of conformal symmetry in QFT [19] — but also gave impuls to the future development of the subject [20, 21] (see also [22] for a review).

After intensive collaboration in the early days of these studies, the fields of CFT and QCD diverged. CFT development was largely influenced by the unprecedented success in two-dimensional theories [23] and played a significant role in connection with string theory and gravity (see e.g. [24] for a review). In QCD, the discovery of *asymptotic freedom* [25, 26] on the one hand clarified the connection between the observed scale invariance and the non-abelian QCD Lagrangian, but on the other hand established QCD as an explicitly non-conformal theory at the quantum level. As a result, the application of CFT methods in QCD fell out of favor for a long time. For example, the authors of [27] — the standard reference for the use of conformal symmetry in QCD — described the prevailing attitude in the community as follows:

> *It is usually thought, however, that the study of this behavior, QCD at small coupling, is not really interesting for a "hep-th" theorist, while using specific techniques based on conformal symmetry in the QCD phenomenology is complicated and is not worth the effort.*
>
> — V. M. Braun, G. P. Korchemsky, D. Müller, *The Uses of Conformal Symmetry in QCD* [27]

We can confidently say that this prejudice against applying conformal symmetry in the QCD context was overcome in recent years, and the subject has become quite popular. Such applications can be broadly divided according to the source of conformal invariance under consideration:

- **Celestial holographic description:**

  This approach (largely influenced by studies of gravity) originated from the idea that the scattering of particles in $d = 4$ Minkowski spacetime might be expressible in terms of a two-dimensional CFT defined on the celestial sphere, exploiting the isomorphism between the Lorentz group and $SL(2,\mathbb{C})$. This point of view gives an interesting interpretation of Weinberg soft theorems [28] as Ward identities [29] (see also [30]). The approach is naturally suited to the study of the infrared (IR) structure of scattering amplitudes, see e.g. [31–33], and suggests a promising direction for future development in connection with QCD [34].

- **Breaking of classical conformal invariance:**

  Although the classical conformal invariance of the QCD Lagrangian with massless quarks is broken by quantum corrections [35–37], it leaves traces even at the quantum level. This becomes especially clear in high-energy scattering when the relevant momenta approach the *light-cone*. Conformal symmetry manifests here through its subgroups. The DGLAP/ERBL [38–42] evolution turns out to possess the collinear $SL(2,\mathbb{R})$ symmetry at one-loop order [43–45]. In the BFKL framework [46, 47], the $SL(2,\mathbb{C})$ symmetry in the transverse plane plays an important role [48, 49]. [a] Expressing scattering amplitudes in a way that respects such symmetries turns out to be convenient even beyond one loop, since the breaking is highly constrained and allows one to predict higher-loop results from lower-loop data [52–54].

- **Exact conformal invariance at the critical point:**

  Moving from four-dimensional non-conformal QCD to a ($d = 4 - 2\epsilon$)-dimensional theory, one can consider a special value of the coupling, g^*, such that the β-function vanishes, $\beta(g^*) = 0$. This brings the discussion into the ϵ-expansion framework ($g^* = g^*(\epsilon)$) in the Wilson–Fisher (WF) fixed point CFT [55]. The non-trivial step is to translate the results back to the physical theory, which, for example, can be done effectively for renormalization problems due to specific properties of the $\overline{\text{MS}}$ scheme [27, 56, 57]. This approach has also been successfully applied in transverse-momentum factorization [58] and small-x physics [59, 60]. Note also that this approach can be used in the calculations of scattering amplitudes [61].

This thesis can be seen as a contribution to this topic, and it focuses primarily on the third approach mentioned above — tuning the coupling to the critical value and translating results back to the physical theory. The second point is also relevant to the present work, as it provides the language suitable for the description.

The object of study in this thesis is the renormalization of twist-2 operators [25] and the perturbative calculation of their anomalous dimensions. In QCD these anomalous dimensions are of particular phenomenological importance, since they govern the scale dependence of *parton distribution functions* (PDFs) (see [62] for a discussion of the *factorization* framework). The current state of the art is at

[a] It is remarkable that the $SL(2,\mathbb{C})$ symmetry in this case translates to the underlying integrable structure of the XXX spin chain [50, 51].

three loops for the singlet case [63] and was recently extended to four loops for the non-singlet operator [64]. Such quantities are also interesting from a purely CFT perspective, as they determine the leading contributions to the short-distance expansion of correlation functions. Despite all advances in computational techniques, high-order calculations remain extremely challenging (not least due to the complexity of applying the standard *integration-by-parts* (IBP)-based techniques [65]). Quite often the only feasible results are the anomalous dimensions of low-spin operators, see e.g. [66] and references therein. The situation becomes even more difficult when one considers the scale evolution of *exclusive processes*, which introduces non-local operators into the corresponding Feynman diagrams. In this context, any additional information about the structure of anomalous dimensions coming from CFT constraints is of particular interest, as it can serve as a practical tool for simplifying calculations.

The conceptual motivation for considering the theory at the critical point $\beta(g^*) = 0$ is also compelling. It naturally sidesteps the problem of confinement, since in the absence of an intrinsic scale the process of hadronization cannot occur [67]. A related motivation comes from the theory of *energy-energy correlators* (EEC) [68, 69] (see also [70] for a recent review). Indeed, since EECs are observables that operate with energy fluxes rather than particles in asymptotic states, they are less sensitive to the difficulties associated with those asymptotic states. Moreover, in the CFT setting they acquire a sensible interpretation in terms of well-defined operators [71, 72]. This description can be translated to physical QCD, see e.g. the application to jet structure studies [73]. Setting the theory at the critical point also plays a significant practical role in this context, see [74].

In this thesis we exploit critical conformal invariance in two ways. The first concerns the analytic structure of anomalous dimensions of twist-2 operators as functions of spin. Beyond purely theoretical interest, identifying specific structure can help in reconstructing the functional form of anomalous dimensions from a limited set of low moments. The "gold standard" here is the *generalized Gribov–Lipatov reciprocity* relation [75–77], which significantly constrains the functional space (see [78, 79] for a discussion of the space of *reciprocity-respecting* harmonic sums). Setting the theory at the critical point endows the anomalous dimensions with the meaning of *CFT observables* — as eigenvalues of the dilatation operator — which turns out to constrain their singular structure. Indeed, such singularities then represent singularities in the perturbative expansion of the CFT spectrum analytically continued in spin [80], and can be interpreted as manifestations of degeneracies in the free spectrum [81]. From the technical point of view, this picture provides a prescription for all-order resummation, which in the QCD case, for example, explains the existence of a generalization of the *double-logarithmic equation* (DLE) [82] (see also [83, 84] for the original DLE derivation). It also conceptually unifies this resummation with the BFKL-based resummation [85–87] in the gluon-gluon anomalous dimension, see [73]. We use this framework to derive resummation prescriptions in a variety of theories and test them against explicit perturbative results, finding perfect agreement. To make the tests more comprehensive, we specifically perform a number of explicit multi-loop calculations of anomalous dimensions of leading-twist operators in the theories under consideration.

The second application of critical conformal invariance concerns the description of exclusive processes. The evolution of the corresponding parton distributions can be effectively formulated in terms of *light-ray* operators [88] and their evolution kernels. The conformal symmetry at the critical point can here be used to efficiently characterize the pattern of symmetry breaking (see [52–54]) as a symmetry property of the corresponding evolution kernels [56, 89, 90]. Furthermore, the conformal Ward identities arising at the critical point serve as a practical tool for determining the generators of this symmetry [57] (see also [91]) and have delivered state-of-the-art three-loop results [92, 93]. We continue this line of work and calculate the three-loop evolution kernel of the *transversity* operator, whose nucleon matrix elements define the *chiral-odd generalised parton distributions* (GPDs), see [94, 95]. These are of potential interest for the future *Electron-Ion Collider* (EIC) [96–100]. The perspective of evolution kernels also turns out to be convenient for elucidating the structure of *forward* anomalous dimensions. The underlying symmetry gives an interesting perspective on Gribov–Lipatov reciprocity [101]. In this thesis we attempt to apply such ideas, originally developed in the non-singlet case, to the matrix of singlet anomalous dimensions, and to extend the known results [102, 103].

Thesis outline

The thesis is organised as follows. Chapters 1, 2 and 3 are intended to provide introductory material. In particular:

- In Chapter 1 we review the theoretical background for describing high-energy scattering involving hadrons. The presentation is centered around Wilson OPE, which naturally motivates the interest in twist-2 operators. We then move on to the subtleties arising beyond leading order and introduce UV renormalization, motivating the need for the renormalization of twist-2 operators and the calculation of their anomalous dimensions. At the end of the chapter we discuss the universality of Wilson OPE approach and its extension from deep-inelastic scattering (DIS) to exclusive processes, introducing the relevant concepts such as evolution kernels.

- In Chapter 2 we turn to conformal symmetry and its application in QFT. We begin with an abstract definition of the conformal algebra and discuss the classification of its representations. We then address the specific properties of QFTs possessing conformal symmetry. Additionally, we focus on the connection between the breaking of classical conformal symmetry due to renormalization and its restoration at fixed point. At the end of the chapter we apply the developed formalism to QCD, discussing the constraints that the $SL(2,\mathbb{R})$ collinear subgroup of the conformal group imposes on the renormalization of twist-2 operators.

- Chapter 3 is technical in nature. We review methods for the perturbative calculation of anomalous dimensions of twist-2 operators, centered around Feynman diagrams with operator insertions. We begin with the general approach and the associated Feynman rules, then review different methods for calculating diagrams with local and non-local leading-twist operator insertions, including the method of uniqueness and the use of computer algebra systems. At the end we illustrate these methods with two non-trivial three-loop diagrams.

The next two chapters present the original research results obtained by the author of this thesis:

- In Chapter 4 we discuss the application of critical conformal invariance to the analytic structure of anomalous dimensions as functions of spin. We begin with a review of the large-spin limit, touching on the relation to the cusp anomalous dimension and different perspectives on Gribov–Lipatov reciprocity. We then move to the small-spin case, mostly following the presentation of [1], interpreting the double-logarithmic equation in QCD and $\mathcal{N}=4$ SYM as a resolution of mixing with the *shadow operator* arising from the analytic continuation in spin. We illustrate this approach on the $O(N)$-symmetric φ^4 theory, the $SU(N)$-adjoint and complex φ^3 models, Gross–Neveu–Yukawa model, and Gross–Neveu model in the framework of $1/N$ expansion. At the end we outline the prospects and challenges of applying this approach in QCD.

- Chapter 5 is dedicated to non-forward evolution kernels and their symmetries, mostly following [2]. We begin by setting up the notation for the transversity non-local operator and proceed to the explicit calculation of the two-loop generators of the $SL(2,\mathbb{R})$ collinear conformal symmetry. We then construct the three-loop transversity evolution kernel dividing it into two parts by similarity transformation: the off-forward part is reconstructed based on the form of generators and two-loop evolution kernel, and the canonically invariant kernel is obtained by making use of forward results. At the end we note that the similarity transformation provides a promising perspective on the Gribov–Lipatov reciprocity relation, reviewing this connection in the non-singlet case and applying the analogous construction to the matrix of singlet anomalous dimensions.

After that we conclude in the summary section.

Chapter 1

Hadronic scattering and operator product expansion

One of the main outcomes of theoretical physics is the prediction of the results of experiments. In the context of high-energy scattering of elementary particles, the most developed and successful theory nowadays that allows one to make such predictions is perturbative Quantum Field Theory (QFT). Despite all the advances in the techniques, there is still a substantial gap when scattering particles take part in strong interactions — the problem of confinement (see [10] for a modern review). The theory of Quantum Chromodynamics (QCD), which is believed to describe such interactions, is formulated in terms of quarks and gluons, while the asymptotic states taking part in scattering are hadrons — their colorless bound states.

A remarkable aspect of the current state of development is that, in the absence of a rigorous bridge from the QCD Lagrangian, researchers have developed several ways to overcome this problem, ranging from low-energy chiral perturbation theory [104] (see also [105] for a review) to "brute-force" calculations on the lattice [106]. In this thesis, we focus on the Wilson OPE [18], an approach which is complementary to the use of factorization theorems [107] and allows one to conceptually separate the physics of the scattering event involving quarks from the process of their hadronisation.

This chapter is intended to give an overview of the use of operator product expansion, starting from the need to describe scattering events involving hadrons. Setting up such a framework inevitably raises further problems of a more technical nature, to the solution of which this thesis aims to contribute. The role of this material is therefore rather motivational, focusing mainly on two topics:

1. Motivating the definition and the study of twist-2 operators in QCD,

2. Describing the range of problems related to the use of twist-2 operators.

Before moving to the content of the chapter, let us note that a more complete discussion can be found in a variety of textbooks and lecture notes (see e.g. [108–112]) covering the topic of perturbative QCD, so we do not give detailed derivations and focus only on the conceptual links.

1.1 Deep-inelastic scattering

Deep-inelastic scattering (DIS) is one of the signature QCD processes, and serves as an excellent starting point for discussing the variety of problems arising in the study of hadron scattering. For the sake of specificity, let us focus on a particular version of DIS — electron-proton scattering

$$e(k) + H(P) \longrightarrow e(k') + X, \tag{1.1}$$

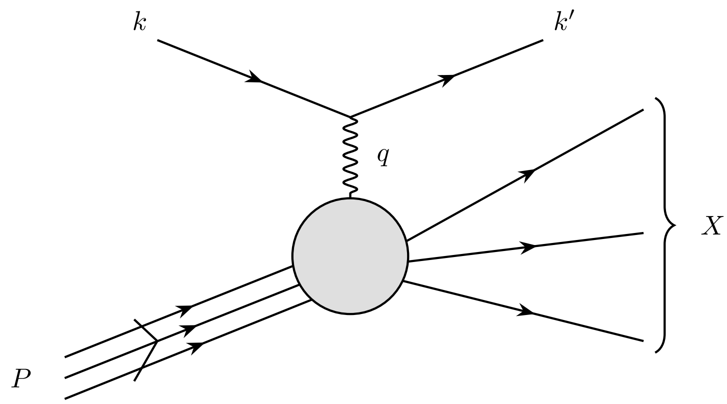


Figure 1.1: The DIS process (1.1) at leading order in QED interaction.

which should be read as follows: an electron with four-momentum k^μ and a proton (denoted by H) with four-momentum P^μ are the inputs of the scattering, while an electron with a different momentum k'^μ appears in the output together with *something* we denote as X. Let us focus on the *inclusive* process, where we are not interested in the outcome X and register only the electron in the final state.

Kinematically, such a process can be described as a $2 \to 2$ scattering and therefore depends on two independent Lorentz scalars. It is convenient to choose them as

$$Q^2 = -q^2, \qquad x_B = \frac{Q^2}{2P \cdot q}, \tag{1.2}$$

where $q^\mu = k^\mu - k'^\mu$. If we consider the scattering (1.1) at leading order in QED interactions, the momentum q^μ is then assigned to the virtual photon (see Fig. 1.1), and the variable Q^2 acquires the meaning of the resolution scale at which the photon "probes" the proton. Since $q^2 \leq 0$, [a] the range for this variable is $Q^2 \in [0, +\infty]$. It can be shown that the second kinematical variable, x_B, often called the "Bjorken x", measures the elasticity of the scattering. Indeed, analyzing the invariant mass of the outcome X we get

$$P_X^2 = (q+P)^2 = M^2 - Q^2 + 2q \cdot P = M^2 + \frac{1-x_B}{x_B} Q^2, \tag{1.3}$$

where $P^2 = M^2$ is the mass of the proton and P_X is the overall momentum associated with the outcome X. It is then clear that with respect to x_B we have several regimes: [b]

- If $x_B = 1$, the invariant mass of X coincides with the proton mass and the scattering is elastic.
- If $0 < x_B < 1$, the scattering is inelastic, and as $x_B \to 0$, the scattering becomes infinitely energetic.
- The regions $x_B < 0$ and $x_B > 1$ are kinematically forbidden, as they would imply that the invariant mass of the outcome is smaller than the proton mass, $P_X^2 < M^2$.

Thus, the physical range of the second kinematical variable is $x_B \in (0, 1]$.

A natural observable associated with the scattering (1.1) that can be measured experimentally is the double-differential cross-section with respect to the kinematical variables

$$\frac{d^2\sigma}{dx_B dQ^2} \propto \sum_X \int d\Pi_X \overline{|\mathcal{M}|}^2 (2\pi)^4 \delta^{(4)} \left(P + q - \sum_{x \in X} p_x \right), \tag{1.4}$$

[a] It is easy to show that if k^μ and k'^μ are the momenta of the incoming and outgoing on-shell particles, their difference $q^\mu = k^\mu - k'^\mu$ is always space-like, $q^2 \leq 0$.

[b] Here we omit the discussion of photoproduction $x_B = 0, Q^2 = 0$, although such processes can also be studied [113,114].

where $d\Pi_X$ represents the Lorentz-invariant phase-space measure, which under the assumption that the outcome X consists of distinct particles with momenta $\{p_x\}_{x\in X}$ takes the form

$$d\Pi_X = \prod_{x\in X} \frac{d^3 p_x}{(2\pi)^3 E_x}. \tag{1.5}$$

The bar over the squared amplitude represents the averaging over spin states, but this is not important for the further discussion and we simply omit this detail in order not to overload the presentation. The amplitude itself at leading order in the electromagnetic interaction can be written as

$$\mathcal{M} = 4\pi \frac{\alpha_e}{Q^2} \bar{u}(k')\gamma^\mu u(k) \big\langle X \big| J_\mu(0) \big| P \big\rangle + O(\alpha_e^2), \tag{1.6}$$

where $\bar{u}(k')$ and $u(k)$ are the spinors associated with the on-shell electron states and α_e is the *fine-structure constant*, which plays the role of the QED coupling.

Let us mention a certain assumption that was made in writing down such an expression for the amplitude. The introduction of states $\big|P\big\rangle$ and $\big|X\big\rangle$ requires the definition of a Hilbert space of QCD asymptotic states, $\mathcal{H}_{\mathrm{QCD}}$. In the framework of perturbative QCD such a definition is not possible (the confinement problem). However, it is often assumed that such a Hilbert space exists and one requires a few reasonable properties, namely:

- $\big|P\big\rangle$, $\big|X\big\rangle \in \mathcal{H}_{\mathrm{QCD}}$.
- $\big|P\big\rangle$, $\big|X\big\rangle$ are eigenstates of the translation operator $\widehat{P}^\mu = -\,\mathrm{i}\,\partial/\partial x_\mu$, i.e. $\widehat{P}^\mu \big|P\big\rangle = P^\mu \big|P\big\rangle$ and $\widehat{P}^\mu \big|X\big\rangle = P_X^\mu \big|X\big\rangle$.
- All possible outcomes of the DIS scattering (1.1), X, form a complete set in $\mathcal{H}_{\mathrm{QCD}}$, namely

$$\sum_X \int d\Pi_X \big|X\big\rangle\big\langle X\big| = \mathbb{1}. \tag{1.7}$$

With this set of assumptions one can drastically simplify (1.4) and write it in the form

$$\frac{d^2\sigma}{dx_B\, dQ^2} \propto \frac{\alpha_e^2}{Q^4} L^{\mu\nu}(k,k') W_{\mu\nu}(q,P) + O(\alpha_e^3), \tag{1.8}$$

where we have essentially separated the leptonic and hadronic parts of the interaction, introducing

$$L^{\mu\nu}(k,k') = \bar{u}(k')\gamma^\mu u(k) \bar{u}(k) \gamma^\nu u(k'), \tag{1.9}$$

which is called the *leptonic tensor*, and

$$W_{\mu\nu}(q,P) = \int d^4x\, e^{\mathrm{i}\, qx} \big\langle P \big| \big[J_\mu(x), J_\nu(0) \big] \big| P \big\rangle, \tag{1.10}$$

which is often called the *hadronic tensor*. The $J_\mu(x)$ is the electromagnetic current

$$J_\mu(x) = \sum_{i=1}^{n_f} Q_i \bar{q}_i(x) \gamma_\mu q_i(x), \tag{1.11}$$

where we assume that the quark and anti-quark fields, $q_i, \bar{q}_i$ ($i = 1, \dots, n_f$), transform under the global $SU(n_f)$ flavor group and Q_i is their electromagnetic charge. Note that due to the inclusive nature of DIS, the information about the final state $\big|X\big\rangle$ is absent from the final expression, which drastically simplifies the study of this process.

Assuming the general Lorentz structure of the matrix element of two currents, it is possible to introduce two dimensionless functions [115, 116]

$$W_{\mu\nu}(q,P) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x_B,Q^2) + \left(P_\mu - \frac{P\cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P\cdot q}{q^2} q_\nu\right)\frac{2x_B}{Q^2} F_2(x_B,Q^2). \quad (1.12)$$

Although the hadronic tensor itself is not an observable in the quantum-mechanical sense, the *structure functions* F_1 and F_2 can be effectively extracted from data (see e.g. [13, 117, 118]).

Thus, in order to deliver theoretical predictions for the structure functions, one needs theoretical access to the matrix element of two currents (1.10). At this point the framework of QCD as a perturbative QFT is no longer sufficient. The general assumptions about the properties of the state $|P\rangle$ allow us to simplify the final expressions, but in order to set up further perturbative calculations with respect to QCD interactions, one needs to know how the quark and gluon fields act on the state $|P\rangle$. However, it is possible to proceed by imposing specific kinematical limits. There are two which are usually discussed:

- The Bjorken limit: $Q^2 \to +\infty$, x_B – fixed.
- The Regge limit: $x_B \to 0$, Q^2 – fixed.

Both of these limits imply high-energy behavior, but in different senses. Physically, the Regge limit corresponds to an enormous total collision energy ($x_B \to 0$ implies $\mathbf{s} \to \infty$, where $\mathbf{s}$ is a Mandelstam variable) and leads to the rich "small-x" physics (see e.g. [119]). This goes beyond the scope of the thesis and in what follows we focus entirely on the Bjorken limit.

1.2 Wilson OPE

In the previous section we touched on the Bjorken limit ($Q^2 \to +\infty$ with x_B fixed) in the DIS process (1.1). The physical meaning of this limit can be understood by recalling that Q^2 corresponds to the virtuality of the photon. One can show that the spatial resolution of the photon scales as $\lambda \sim 1/\sqrt{Q^2}$, so as $Q^2 \to +\infty$ the interacting photon is very small compared to the proton and can genuinely probe its internal structure. This is, of course, only a rough picture, and in what follows we will see how it manifests in the formal description.

Before proceeding, let us mention one technical aspect that will be important in what follows. Due to the structure of the commutator of two currents in the definition of (1.10), the hadronic tensor itself does not possess good analytical properties as a function $W_{\mu\nu}(x_B, Q^2)$. It is convenient to introduce an object with better analytical properties, the so-called *forward Compton amplitude*

$$T_{\mu\nu}(x_B,Q^2) = \mathrm{i}\int d^4x\, e^{\mathrm{i}\,qx} \langle P|\, \mathrm{T}\left\{J_\mu(x), J_\nu(0)\right\} |P\rangle, \quad (1.13)$$

where $\mathrm{T}\{\ldots\}$ denotes the time-ordered product. For fixed $Q^2 > 0$, the forward Compton amplitude $T_{\mu\nu}(x_B,Q^2)$ is an analytic function of x_B defined on $\mathbb{C}\backslash[-1,1]$ (see [120, Section 14.1] for the discussion). The interval $x_B \in [-1,1]$ represents the physical cut. The connection to the hadronic tensor (1.10) is given by the simple formula representing the optical theorem

$$W_{\mu\nu}(x_B,Q^2) = 2\,\mathrm{Im}\, T_{\mu\nu}(x_B,Q^2). \quad (1.14)$$

It is therefore more convenient to work with the forward Compton amplitude $T_{\mu\nu}$ and then translate the results to the physical object $W_{\mu\nu}$.

Returning to the Bjorken limit, let us discuss what constraints it places on the Fourier integral (1.13). If the momentum in the exponent is large, $q^\mu \to \infty$, the integral is supported near $x^\mu \to 0$ in order to suppress the rapid oscillations (by the Riemann–Lebesgue lemma). The Bjorken limit is more subtle, and it can only be shown that in the limit $Q^2 \to +\infty$, the integral (1.13) is supported on $x^2 \to 0$ [121–124] (see also [125] for a mathematical review). In Minkowski space, $x^2 \to 0$ does not necessarily imply $x^\mu \to 0$, so the only restriction we can obtain is that the integral is supported on the *light-cone*. Let us nonetheless begin with an analysis of the behavior of the time-ordered product of two currents in the short-distance limit $x^\mu \to 0$, and then discuss the consequences of this simplification.

The crucial assumption related to this limit was made by K. G. Wilson in [18] (see also [126] for a historical overview), where he argued that the short-distance behavior of two currents is rather constrained. Summarizing the statement, we formulate the following version of the Wilson OPE:

Wilson OPE

Consider two local operators $\mathcal{O}_1$ and $\mathcal{O}_2$. In the short-distance limit $x^\mu \to 0$, their time-ordered product can be expressed as

$$\mathrm{T}\left\{\mathcal{O}_1(x)\mathcal{O}_2(0)\right\} = \sum_n C_n(x)\mathcal{O}_n(0), \tag{1.15}$$

where

- $\{\mathcal{O}_n\}$ is a set of local operators such that any local operator can be expanded as their linear combination.
- The operator $\mathcal{O}_n$ transforms under any symmetry in such a way that $C_n(x)\mathcal{O}_n$ has the same symmetry properties as the product $\mathcal{O}_1\mathcal{O}_2$.

We formulate the statement of the Wilson OPE here in a modest form and will return to a more precise statement in Section 1.3. Roughly speaking, since the matrix element of two currents is not well-defined due to the absence of a definition of the states $|P\rangle$, one can view (1.15) as an ansatz that allows one to separate the x-dependence in the Fourier integral (1.13). The true significance of Wilson's contribution in [18], however, lies not only in the ansatz itself, but in the assumption about the behavior of the functions (strictly speaking, distributions) $C_n(x)$, which are usually called *Wilson coefficients*.

Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ in (1.15) are local operators, with canonical scaling dimensions Δ_1 and Δ_2, respectively. The crucial assumption of [18] was then:

- In free QCD ($\alpha_s = 0$), [a] the Wilson coefficients scale exactly, i.e., for $\lambda \in \mathbb{R}$
$$C_n(\lambda x) = \lambda^{\Delta_n - \Delta_1 - \Delta_2} C_n(x), \tag{1.16}$$
where Δ_n is the canonical scaling dimension of the associated operator $\mathcal{O}_n$.
- In the interacting theory ($\alpha_s \neq 0$), the scaling behavior of the Wilson coefficients can only be corrected by logarithmic terms,
$$C_n(x) \underset{x\to 0}{=} \frac{1}{|x|^{\Delta_1+\Delta_2-\Delta_n}}\Big(1 + O(\alpha_s \ln x^2)\Big). \tag{1.17}$$

The remarkable nature of these assumptions is that they anticipated the discovery of asymptotic freedom in QCD [25, 26]. The approach of [18] brilliantly unified the empirical observation of structure functions scaling [12] with the idea that operator scaling dimensions are corrected in quantum theories that are classically scale invariant [19, 127]. Thus, Wilson not only assumed that the underlying theory of hadronic interactions is classically conformally invariant, but also that quantum corrections to scaling are logarithmic rather than power-law, and therefore subdominant in the UV-limit.

It can be shown that a convenient way to organise the set of local operators $\{\mathcal{O}_n\}$ is according to their *twist* [128], defined as the difference between the scaling dimension and the spin. Let us write down the elements of such a basis in more detail. An operator takes the form $\mathcal{O}_{\mu_1\ldots\mu_s}$, such that [b]

[a] Here and in what follows α_s abbreviates the strong coupling.
[b] Here S_s is the group of permutations of s elements.

- $\forall \sigma \in S_s$: $\mathcal{O}_{\mu_{\sigma(1)}\cdots\mu_{\sigma(s)}} = \mathcal{O}_{\mu_1\ldots\mu_s}$, so the operator is symmetric,
- $\forall i \neq j \in \{1,\ldots,s\}$: $g^{\mu_i\mu_j}\mathcal{O}_{\mu_1\ldots\mu_s} = 0$, meaning that the operator is traceless with respect to any pair of indices.

These requirements imply that the operator transforms in the $(s/2, s/2)$ representation of the Lorentz group (see e.g. [129] for the details), and we will refer to s as the *spin* of this operator. Suppose this operator has canonical scaling dimension Δ, its twist is then defined as $\tau = \Delta - s$. The notion of twist is important since it controls the contribution to $T_{\mu\nu}(x_B, Q^2)$. Indeed, the Wilson coefficient associated with the operator $\mathcal{O}_{\mu_1\ldots\mu_s}$ has the structure [c]

$$C^{\mu_1\cdots\mu_s}(x) = \widetilde{C}(x)\left(\frac{1}{x^2}\right)^{\frac{6-\Delta+s}{2}} x^{\mu_1\cdots\mu_s}, \tag{1.18}$$

where $x^{\mu_1\cdots\mu_s}$ denotes the symmetric traceless product, i.e. $x^{\mu_1\cdots\mu_s} = x^{\mu_1}\ldots x^{\mu_s} -$ traces. The corresponding contribution after the Fourier transform reads

$$\frac{x^{\mu_1\cdots\mu_s}}{x^{2(3-\tau/2)}} \xrightarrow{\text{Fourier}} \frac{q^{\mu_1\cdots\mu_s}}{(-Q^2)^{\tau/2-1}}. \tag{1.19}$$

In other words, the contribution of an operator on the right-hand side of the Wilson OPE is controlled not by its scaling dimension, but by its twist.

Applying this to the OPE of two currents in (1.13) and writing it in the form

$$\mathrm{T}\left\{J_\mu(x)J_\nu(0)\right\} = \sum_\tau \sum_{s=0}^{\infty} C^{\mu_1\ldots\mu_s}_{\mu\nu,\tau}(x)\mathcal{O}^{(\tau)}_{\mu_1\ldots\mu_s}(0), \tag{1.20}$$

gives the following expressions for the structure functions of the forward Compton amplitude [128] (with T_1 and T_2 defined in analogy with (1.12))

$$\begin{aligned} T_1(\omega, Q^2) &= \sum_{\tau=\tau_0}^{\infty}\sum_{s=0}^{\infty} \frac{c^{(\tau)}_{1,s}(Q^2)}{(Q^2)^{-1+\tau/2}} A^{(\tau)}_s \omega^s, \\ T_2(\omega, Q^2) &= \sum_{\tau=\tau_0}^{\infty}\sum_{s=0}^{\infty} \frac{c^{(\tau)}_{2,s}(Q^2)}{(Q^2)^{-1+\tau/2}} A^{(\tau)}_s \omega^{s-1}. \end{aligned} \tag{1.21}$$

Here we have introduced the variable $\omega = 1/x_B$ for convenience, and $A^{(\tau)}_s$ denotes the matrix element of the operator $\mathcal{O}^{(\tau)}_{\mu_1\ldots\mu_s}$,

$$\langle P|\mathcal{O}^{(\tau)}_{\mu_1\ldots\mu_s}(0)|P\rangle = (\mathrm{i})^s A^{(\tau)}_s P_{\mu_1\ldots\mu_s}. \tag{1.22}$$

The Lorentz structure is justified by the tensorial structure of the operator. The form of the Wilson coefficients (1.17) implies that the functions $c_{1,2}(Q^2)$ depend on Q^2 only through logarithms and are constants when $\alpha_s = 0$. Thus, the expansion (1.21) is organised in powers of $1/Q^2$, and contributions from operators with twist larger than some leading value, $\tau > \tau_0$, are suppressed by factors of $1/Q^2$. Truncating these series at a given twist value is therefore well-justified.

Let us comment on the role of the variable $\omega = 1/x_B$. In a real experiment one has a large but finite fixed value of Q^2. In this setup, the series (1.21) converge for $|\omega| < 1$. At first sight this appears problematic, as it would mean that the use of the Wilson OPE is justified only for $|x_B| > 1$, i.e. in the unphysical region. [d] This problem can be overcome if one is interested in restricted information about

[c] Here we consider the scaling properties of the Wilson coefficients appearing in the Wilson OPE of two electromagnetic currents in (1.13). In $d = 4$ QCD the canonical scaling dimension of the current is $\Delta_J = 3$.

[d] This problem is closely related to the difference between the short-distance and light-cone limits in the OPE. With the additional assumption that $\omega \to 0$, it is possible to show that $x^2 \to 0$ in (1.13) implies $x^\mu \to 0$, so the use of the Wilson OPE is fully justified.

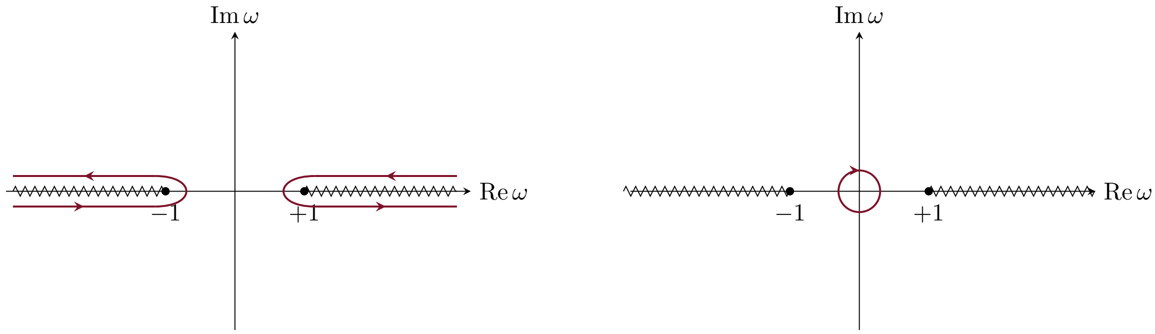


Figure 1.2: The deformation of the contour in (1.23).

the structure functions, such as their moments. Using the analytic properties of $T_{\mu\nu}$ as a function of x_B for fixed Q^2 (see Fig. 1.2), one can write, for example for T_1,

$$\frac{1}{2\pi\,\mathrm{i}} \int\limits_{|\omega|<\delta} \frac{d\omega}{\omega^{s+1}} T_1(\omega, Q^2) = \frac{1}{2\pi\,\mathrm{i}} \int\limits_{1}^{+\infty} \frac{d\omega}{\omega^{s+1}} \operatorname{disc} T_1(\omega, Q^2) + \frac{1}{2\pi\,\mathrm{i}} \int\limits_{-\infty}^{-1} \frac{d\omega}{\omega^{s+1}} \operatorname{disc} T_1(\omega, Q^2), \tag{1.23}$$

where $\delta \to 0$. This contour deformation allows one to relate the integral over the region where the Wilson OPE is justified to an integral over the physical cut. Using the fact that the left-hand side is a Cauchy integral together with the series representation (1.21), and simplifying the right-hand side, one arrives at

$$\frac{1}{\pi} \int\limits_0^1 dx_B\, x_B^{s-1} F_1(x_B, Q^2) = \sum_{\tau=\tau_0}^{\infty} \frac{c_{1,s}^{(\tau)}(Q^2)}{(Q^2)^{-1+\tau/2}} A_s^{(\tau)}. \tag{1.24}$$

We therefore conclude that the Wilson OPE provides well-justified predictions for the *Mellin moments* of the structure functions. The validity of this expansion for the structure functions themselves is a more subtle question — which is to be expected, since the reconstruction of a function from its moments is generally an ill-defined problem requiring additional assumptions about the structure of the function. The Mellin moments of the structure functions are interesting objects in their own right as they correspond to the so-called *sum rules* (see e.g. [15, 130, 131] and [132] for the review on the experimental studies in this direction).

Let us note that the derivation of the connection between the Wilson OPE and the Mellin moments of the structure functions is not as straightforward as presented here, and in principle requires certain assumptions, namely:

- To relate the discontinuity and the imaginary part, $\operatorname{disc} T_j(\omega, Q^2) = 2\,\mathrm{i}\,\operatorname{Im} T_j(\omega, Q^2)$, ($j \in \{1, 2\}$) the structure functions near the physical cut must satisfy the reflection identity, $T_j(x_B^*, Q^2) = T_j^*(x_B, Q^2)$.
- To discard the integral over the circle at infinity in (1.23), one requires certain behavior of $|T_j(x_B, Q^2)|$ as $x_B \to 0$, which relates to the Regge limit of DIS and the value of the so-called Regge intercept (see e.g. [133, Chapter 10]).
- To relate the two integrals in (1.23) over the physical cuts, one can use the *crossing* properties, namely $T_1(-x_B, Q^2) = T_1(x_B, Q^2)$ and $T_2(-x_B, Q^2) = -T_2(x_B, Q^2)$ (see e.g [134] and [135] for a recent review).

The last point also affects which values of spin can appear in the series (1.24). Each of these assumptions deserves closer treatment, and a detailed discussion is certainly beyond the scope of this thesis. What matters is the main message: the Wilson OPE yields sensible predictions for the phenomenologically relevant structure functions of the hadronic tensor (1.10), providing theoretical control over the set of contributing operators.

Let us now return to the discussion of the operator basis $\mathcal{O}^{(\tau)}_{\mu_1\ldots\mu_s}$ and the kinds of operators that can appear in the Wilson OPE (1.20). In fact, the general requirements on the twist τ can be sharpened by the symmetries of the product of two currents. For example, it is convenient to consider its properties under $SU(n_f)$ transformations. An arbitrary charge matrix $\widehat{Q} = \mathrm{diag}\{Q_1, \ldots, Q_{n_f}\}$ in the definition of the current $J_\mu = \sum_{i=1}^{n_f} Q_i \bar{q}_i \gamma_\mu q_i$ can always be decomposed into two contributions,

$$J_\mu = J^{(s)}_\mu + Q^A J^A_\mu, \qquad A \in \{1, \ldots, n_f^2 - 1\}, \tag{1.25}$$

one corresponding to the singlet representation of $SU(n_f)$, $\mathbf{1}$, and the other to the adjoint, $(\mathbf{n_f}^2 - \mathbf{1})$. The product of two currents can therefore be written as

$$J_\mu J_\nu = J^{(s)}_\mu J^{(s)}_\nu + Q^A \left(J^{(s)}_\mu J^A_\nu + J^{(s)}_\nu J^A_\mu \right) + Q^A Q^B J^A_\mu J^B_\nu . \tag{1.26}$$

The operators that can appear are therefore either flavor singlets, in the irreducible representations of $\mathbf{1} \otimes (\mathbf{n_f}^2 - \mathbf{1})$, or in the symmetric representations of the product $(\mathbf{n_f}^2 - \mathbf{1}) \otimes (\mathbf{n_f}^2 - \mathbf{1})$, since $Q^A Q^B$ is symmetric. For the standard case $n_f = 3$ with $\widehat{Q} = \mathrm{diag}\{\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\}$, one has $\mathrm{tr}\, \widehat{Q} = 0$, which leaves only the adjoint representation. We are thus left with the familiar decomposition

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27}, \tag{1.27}$$

and the symmetry requirement permits singlet operators $\mathbf{1}$, non-singlet operators $\mathbf{8}$, and operators transforming in the $\mathbf{27}$ representation. [e]

It is therefore convenient to further organize operators in QCD according to their representation under the $SU(n_f)$ flavor group. Analyzing the leading-twist operators leads to the following classification:

Twist-2 operators in QCD

The set of leading-twist operators, $\tau_0 = 2$, appearing in the Wilson OPE of two currents (1.20) can be organized as follows:

- Gluon singlet operator:

$$\mathcal{O}^g_{\mu_1\ldots\mu_s}(x) = F^a_{\nu(\mu_1}(x) D_{\mu_2} \ldots D_{\mu_{s-1}} F^{a,\nu}_{\mu_s)}(x) - \text{traces}. \tag{1.28}$$

- Quark singlet operator:

$$\mathcal{O}^q_{\mu_1\ldots\mu_s}(x) = \sum_{i=1}^{n_f} \bar{q}_i(x) \gamma_{(\mu_1} D_{\mu_2} \ldots D_{\mu_s)} q_i(x) - \text{traces}. \tag{1.29}$$

- Quark non-singlet operator:

$$\mathcal{O}^{q,A}_{\mu_1\ldots\mu_s}(x) = \bar{q}_i(x) (\lambda^A)_{ij} \gamma_{(\mu_1} D_{\mu_2} \ldots D_{\mu_s)} q_j(x) - \text{traces}, \tag{1.30}$$

  where λ^A is a generator of the $SU(n_f)$ group.

In generalizations of the process (1.1) (e.g. involving electroweak axial currents [136]), the following additional operators become relevant:

- Axial quark non-singlet operator:

$$\mathcal{O}^{q,A,(5)}_{\mu_1\ldots\mu_s}(x) = \bar{q}_i(x) (\lambda^A)_{ij} \gamma_5 \gamma_{(\mu_1} D_{\mu_2} \ldots D_{\mu_s)} q_j(x) - \text{traces}. \tag{1.31}$$

[e] Operators associated with this representation do not appear at the twist-2 level.

- Transversity quark non-singlet operator:

$$\mathcal{O}^{q,A,(T)}_{\alpha\mu_1\dots\mu_s}(x) = \bar{q}_i(x)(\lambda^A)_{ij}\sigma_{(\alpha\mu}\gamma_{\mu_1}D_{\mu_2}\dots D_{\mu_s)}q_j(x) - \text{traces}, \tag{1.32}$$

where $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$.

In all cases, round brackets denote symmetrization over indices, and traces are subtracted to ensure that the operators have spin s.

It can be shown explicitly that all these operators have twist $\tau = 2$, and for this reason they are usually referred to as *twist-2* (or leading-twist) operators. Restricting to operators of leading twist and retaining only the leading contribution in the expansion (1.21), one immediately obtains the following prediction for the behavior of the structure functions:

$$T_1(x_B, Q^2) = \sum_{s=0}^{+\infty} c^{(\tau_0=2)}_{1,s}(Q^2)A^{(\tau_0=2)}_s x_B^{-s} + O\left(\frac{1}{Q^2}\right),$$

$$T_2(x_B, Q^2) = \sum_{s=0}^{+\infty} c^{(\tau_0=2)}_{2,s}(Q^2)A^{(\tau_0=2)}_s x_B^{-s+1} + O\left(\frac{1}{Q^2}\right). \tag{1.33}$$

In the free-theory approximation ($\alpha_s = 0$), the functions $c_{j,s}(Q^2)$ are independent of Q^2, and this result expresses *Bjorken scaling* — the statement that structure functions are asymptotically independent of Q^2. When interactions are turned on ($\alpha_s \neq 0$), Bjorken scaling is violated, but only through small logarithmic corrections (see [137] for the discussion of experimental confirmation of such violations).

The physical intuition underlying Bjorken scaling was developed through the parton model [115,116]. The idea is the following: the photon actually interacts only with the individual quarks inside the proton. A kinematical analysis shows that x_B then corresponds to the momentum fraction carried by this quark (quark momentum $p^\mu = x_B P^\mu$). The statement of Bjorken scaling then becomes very transparent: if $Q^2 \to \infty$, the scattering is so localized that the quark "does not feel" the surrounding proton, and the cross-section depends on Q^2 only through the momentum fraction x_B. Let us introduce the *parton distribution function* (PDF), $q_i(x_B)$, defined as the probability density to find a quark of flavor i with momentum fraction x_B inside the proton. The structure functions can then be written as

$$F_1(x_B, Q^2) = \frac{1}{2}\sum_i Q_i^2\left(q_i(x_B) + \bar{q}_i(x_B)\right),$$

$$F_2(x_B, Q^2) = x_B\sum_i Q_i^2\left(q_i(x_B) + \bar{q}_i(x_B)\right), \tag{1.34}$$

where $\bar{q}(x_B)$ denotes the PDF for the antiquark. The expansion (1.33) (together with (1.24)) at leading order (LO) provides a clear connection between the matrix elements of twist-2 operators and the Mellin moments of PDFs,

$$A_{i,s} = \int_0^1 dx_B\, x_B^{s-1}\left(q_i(x_B) + \bar{q}_i(x_B)\right), \tag{1.35}$$

where [f]

$$\left\langle P\middle|\bar{q}_i\gamma_{(\mu_1}D_{\mu_2}\dots D_{\mu_s)}q_i - \text{traces}\middle|P\right\rangle = (\mathrm{i})^s A_{i,s}(P^{\mu_1}\dots P^{\mu_s} - \text{traces}). \tag{1.36}$$

[f] Note that here we do not imply the summation over flavor index, i.

The *factorization approach* [62] naturally extends the definition of PDFs to interacting QCD while keeping the connection (1.35) intact. It allows one to introduce a scale μ that separates the interaction between the quark and the photon from the dynamics of the quark inside the proton, leading to the expression

$$F_j(x_B, Q^2) = \sum_i \int_{x_B}^{1} \frac{dy}{y}\, \mathbf{C}_{j,i}\left(\frac{x_B}{y}, \frac{Q^2}{\mu^2}, \alpha_s\right) f_{q_i}(y, \mu) + O\left(\frac{1}{Q^2}\right), \tag{1.37}$$

where we use the notation $f_{q_i}(x_B, \mu)$ to emphasise its distinction from the PDF of the simple parton model and $\mathbf{C}(x_B, Q^2/\mu^2, \alpha_s)$ is called *coefficient function*. In Section 1.3 we will show how the connection (1.35) helps to study the properties of parton distribution functions.

To conclude this section, let us note that the Wilson OPE (1.15) does not, of course, resolve the problem of defining hadronic states $|P\rangle$ within perturbative QCD. This limitation manifests in the fact that the matrix elements $A_s^{(\tau)}$ in (1.21) cannot be computed in perturbation theory. Nevertheless, the OPE represents a significant theoretical advance, as it provides systematic control over hadronic scattering. The matrix elements can be extracted from experiment (see e.g. [138, 139]) or from lattice simulations (see [140, Section 10] and references therein). Even without knowing the precise values of these matrix elements, the general structure of the OPE already yields meaningful experimental predictions, such as Bjorken scaling and its violations. These are by no means the only applications and advances that one can extract from the Wilson OPE, and in the next chapter we will deepen this approach to address the scale dependence in hadronic scattering.

1.3 Renormalization and Zimmerman OPE

In the previous section we discussed the physical picture of the Wilson OPE (1.15) and its consequences for the particle phenomenology related to the study of deep inelastic scattering (1.1). Attempting to explore the structure of this expansion in more detail — e.g. calculating Wilson coefficients in perturbation theory — leads to further problems that were not discussed before. Let us illustrate these problems and how to address them in the following section.

Consider, for simplicity, a Euclidean four-dimensional theory with an action of the general form

$$S = \int d^4x\, \mathcal{L}(\Phi, g), \tag{1.38}$$

which describes the dynamics of a field $\Phi(x)$ with coupling g (the presentation generalises naturally to the case of multiple fields and couplings). In the previous section we stated that the main problem with the matrix elements (1.22) is the absence of a definition of $|P\rangle$, which prevents the use of perturbation theory. However, even if such a matrix element could be defined perturbatively, the definition (1.22) would still be insufficient. The reason is that the Green's function of n fields is divergent, [a]

$$G_n(x_1, \dots, x_n) = \langle \Phi(x_1) \dots \Phi(x_n) \rangle = \frac{1}{\mathcal{N}} \int D\Phi\, e^{-S} \Phi(x_1) \dots \Phi(x_n) \quad \longrightarrow \quad \infty. \tag{1.39}$$

These divergencies are associated with loop integrals in the corresponding Feynman diagrams and are usually referred to as UV divergencies.

There is, however, a well-defined procedure which allows one to remove such divergencies. The first step one can take is *regularization*, i.e. representing the singularity as a limit with respect to some parameter. There are many possible regularization schemes (cut-off, Pauli–Villars), but in this thesis we use *dimensional regularization* [141]. The idea is to rewrite the Feynman integrals corresponding to the Green's function (1.39) in arbitrary integer dimension $d \in \mathbb{Z}$, and then analytically continue the result to the complex plane $d \in \mathbb{C} \setminus \{\text{poles}\}$. Writing the dimension as $d = 4 - 2\epsilon$ in the analytically continued

[a] The normalizing factor $\mathcal{N}$ is chosen in a way to get $\langle 1 \rangle = 1$, but we do not provide the full expression for brevity.

expressions and expanding them around $\epsilon = 0$ we see that the original UV divergencies manifest as poles in ϵ.

The theory of renormalization (see e.g. [107, 142] for a detailed discussion) states that for a certain class of (multiplicatively renormalizable) theories, these poles in ϵ can be systematically removed by introducing *counterterms* [b]

$$\Phi(x) \mapsto \Phi_0(x) = Z_\Phi \Phi(x), \qquad\qquad g \mapsto g_0 = g\mu^{k\epsilon} Z_g, \tag{1.40}$$

where fields with the subscript 0 will be referred to as *bare*. We also introduce an additional scale μ to keep the coupling dimensionless in d-dimensional theory. The parameter k depends on the type of interaction.

A theory is multiplicatively renormalizable if the counterterms (Z_Φ, Z_g) can be chosen such that

$$\forall n \geq 0: \quad G_n^R(x_1, \ldots, x_n) = \frac{1}{\mathcal{N}} \int D\Phi \, e^{-S_R} \Phi(x_1) \ldots \Phi(x_n) \quad \overset{\epsilon \to 0}{\longrightarrow} \quad \text{finite}, \tag{1.41}$$

where $S_R(\Phi, g) = S(\Phi_0, g_0)$. The relation between bare and renormalized parameters then gives the connection between the renormalized (1.41) and bare (1.39) Green's functions,

$$G_n^R(x_1, \ldots, x_n) = Z_\Phi^{-n} G_n(x_1, \ldots, x_n). \tag{1.42}$$

The requirement (1.41) does not fix the counterterms uniquely. The remaining freedom is called the *renormalization scheme*. In what follows we always use the so-called $\overline{\text{MS}}$ scheme, unless explicitly stated otherwise. The renormalization constants in this scheme take the form ($j \in \{\Phi, g\}$)

$$Z_j(\epsilon, g) = 1 + \sum_{k=1}^{\infty} \frac{Z_{j,k}(g)}{\epsilon^k} = 1 + \sum_{n=1}^{\infty} g^n \sum_{k=1}^{n} \frac{z_{nk}}{\epsilon^k}, \tag{1.43}$$

where z_{nk} are ϵ-independent constants. The renormalization procedure thus allows one to pass from the UV-divergent Green's functions (1.39) to finite objects (1.41), at the cost of introducing two additional parameters: the dimensional regulator ϵ and the renormalization scale μ.

A remarkable feature of this procedure is that the choice of renormalization scheme is largely arbitrary and not fixed by physical considerations. Indeed, one can introduce two different systems of finite Green's functions related by

$$G_n^R(x_1, \ldots, x_n) = \widetilde{Z}_\Phi^n \widetilde{G}_n^R(x_1, \ldots, x_n), \qquad\qquad g = \widetilde{Z}_g \widetilde{g}, \tag{1.44}$$

where $\widetilde{Z}_\Phi(\epsilon)$ and $\widetilde{Z}_g(\epsilon)$ are finite as $\epsilon \to 0$. Such a transformation (with finite counterterms) is called a *finite renormalization*. The set of all finite renormalizations forms a group, usually called the *renormalization group* (RG).

The idea that the theory is invariant under renormalization group transformations is most conveniently expressed as a differential equation with respect to the renormalization scale μ. Since the bare Green's function (1.39) does not depend on μ, one can write

$$\mu \frac{d}{d\mu} G_n(x_1, \ldots, x_n) = 0 \quad \Rightarrow \quad \left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + n\gamma_\Phi \right) G_n^R(x_1, \ldots, x_n) = 0, \tag{1.45}$$

where we introduce two objects

$$\beta(g) = \mu \frac{dg}{d\mu} = -kg \left(\epsilon + \beta_0 g + \beta_1 g^2 + \ldots \right), \qquad \gamma_\Phi = \mu \frac{d \ln Z_\Phi}{d\mu} = -kg \frac{\partial}{\partial g} Z_{\Phi,1}(g). \tag{1.46}$$

[b] Note that different sources introduce renormalization constants Z in different ways, leading to differences in normalization. Here we use the notation of [142].

The first is called the *beta function*. It controls how the coupling changes with the scale. The second is the *anomalous dimension* of the field Φ, so named because γ_Φ contributes a quantum correction to the canonical scaling dimension of the field. The RG equation (1.45) in its infinitesimal form is called the *Callan–Symanzik equation* [16, 143].

From a technical perspective, the RG equation (1.45) can be used to study the asymptotics of Green's functions (or other objects) in specific limits. Let us elaborate on this. Consider the *propagator* of the field Φ in momentum space, $D(p)$, defined as

$$D(p) = \int d^4x\, e^{\mathrm{i}\,px} \big\langle \Phi(0)\Phi(x) \big\rangle. \tag{1.47}$$

After the renormalization procedure (1.40), the standard perturbative expansion of the propagator takes the form

$$D(p) \overset{\epsilon\to 0}{=} D_0(p) \left(1 + \sum_{k=1}^{\infty} g^k C_k \left(\ln \frac{p}{\mu} \right) \right), \tag{1.48}$$

where the coefficients $C_k(x)$ are polynomials of degree at most k in x, and $D_0(p)$ is the free-theory propagator. Suppose one is interested in a specific limit of the propagator (1.48) (typically $p \to \infty$ or $p \to 0$). The perturbative expansion (1.48) then breaks down, since the coefficients C_k grow in an uncontrolled manner regardless of the smallness of g. Expressing $D(p)$ as a solution of the Callan–Symanzik equation (1.45) allows one to overcome this problem by effectively resumming all the problematic logarithmic contributions. The information that can be extracted is the asymptotic behavior (e.g. the critical exponent η defined by $D(p \to 0) \sim 1/|p|^\eta$), rather than the exact value of the propagator at a specific $p = p_0$. Note also that, depending on the sign of β_0 (see Eq. (1.46)) [c], the Callan–Symanzik equation predicts the asymptotics in either the UV or the IR limit. Applying the resummed series outside its region of validity can lead to a misinterpretation of the critical scaling (see the problem of Landau pole in QED [144]).

After the procedure of renormalization, one might hope that with a perturbative definition of the states $\big|P\big\rangle$ it would be possible to compute the matrix elements (1.22). However, the problem of UV divergencies would still not be fully resolved. The reason lies in the introduction of local operators $\mathcal{O}(x)$. Indeed, the renormalization procedure (1.41) assumes that the coordinates x_j are in generic position. Taking the limit $x_i \to x_j$ for some $i \neq j \in \{1, \dots, n\}$, one obtains a Green's function of the form

$$G_n^R(x_1, \dots, x_n) \quad \longrightarrow \quad \big\langle \mathcal{O}(x)\Phi(x_1)\dots\Phi(x_n) \big\rangle, \tag{1.49}$$

where the local operator is $\mathcal{O}(x) = \Phi^2(x)$. The renormalization of such a Green's function is not covered by the multiplicative renormalization procedure (1.40), so in the limit $\epsilon \to 0$ it can still be divergent, since the coincidence limit $x_i \to x_j$ generally produces additional divergencies.

This problem motivates the introduction of a procedure of *operator renormalization* (see e.g. [145] for historical development and [142, Chapter 3, Section 23] for the modern presentation). Namely, given a local operator $\mathcal{O}_i(x)$, the renormalized operator is defined as

$$\big[\mathcal{O}_i(x)\big] = Z^{ij}(\epsilon, g)\mathcal{O}_j(x). \tag{1.50}$$

Note the crucial difference from the multiplicative renormalization of fields (1.40). The operator counterterms Z^{ij} are in general form a matrix, as they relate the divergencies generated by one operator $\mathcal{O}_i$ to those generated by a set of local operators $\{\mathcal{O}_j\}$. The form of the operator counterterms $Z^{ij}(\epsilon, g)$ in the $\overline{\text{MS}}$ scheme is the same as for the field counterterms (1.43). In a renormalizable theory it is possible to define the operator renormalization procedure (1.50) such that

$$\forall n : \quad \big\langle \big[\mathcal{O}_i(x)\big]\Phi(x_1)\dots\Phi(x_n) \big\rangle \quad \overset{\epsilon\to 0}{\longrightarrow} \quad \text{finite}, \tag{1.51}$$

[c] In the theory of critical behavior it is usually assumed that the qualitative behavior of the beta function (its zeros and the sign of each branch) can be read off from the first perturbative order and is unchanged by higher-order corrections.

however, in the special cases, such as non-integer dimensions, this procedure can be subtle, for example, the set of operator counterterms might be infinite.

The definition of operator renormalization (1.50) implies certain constraints on the set of operators $\{\mathcal{O}_j\}$ that can appear on the right-hand side, namely:

- The canonical scaling dimension Δ_j of the operator $\mathcal{O}_j(x)$ coincides with that of $\mathcal{O}_i(x)$, i.e. $\Delta_j = \Delta_i$. [d]
- The operator $\mathcal{O}_j(x)$ transforms under the same representation of any symmetry group as $\mathcal{O}_i(x)$.

The second condition acts as a restriction in a manner similar to our analysis of the operator basis in the Wilson OPE in Section 1.2. Symmetry-based arguments can be applied in specific examples. However, in what follows we wish to highlight certain cases where additional constraints on the structure of Z^{ij} are available, or where the symmetry requires more careful treatment:

1. **Equation-of-motion operators:**

   A special class of operators are the so-called *equation-of-motion* operators, defined as

   $$\mathrm{EOM}(x) = \frac{\delta S(\Phi, g)}{\delta \Phi(x)}. \tag{1.52}$$

   A Green's function with such an insertion can be expressed (using integration by parts) as

   $$\Big\langle \mathrm{EOM}(x) \prod_{j=1}^{n} \Phi(x_j) \Big\rangle = \sum_{k=0}^{n} \delta^{(d)}(x - x_k) \Big\langle \prod_{\substack{j=1 \\ j \neq k}}^{n} \Phi(x_j) \Big\rangle. \tag{1.53}$$

   The right-hand side consists of Green's functions of fields that are already finite when computed using the renormalized action $S_R(\Phi, g)$. We therefore conclude that

   $$\big[\mathrm{EOM}(x)\big] = \mathrm{EOM}(x), \tag{1.54}$$

   i.e. equation-of-motion operators do not require additional renormalization. Note, however, that EOM operators can appear as operator counterterms for physical operators.

2. **Conserved currents:**

   An analogous statement holds for conserved currents of the theory. Suppose the field transformation $\delta_\varepsilon \Phi$ corresponds to a symmetry of the action, $\delta_\varepsilon S = 0$. The Ward identity for the Noether current of such a transformation reads

   $$\partial_\mu \Big\langle j^\mu(x) \prod_{j=1}^{n} \Phi(x_j) \Big\rangle = -\sum_{k=0}^{n} \delta^{(d)}(x - x_k) \big\langle \Phi(x_1) \ldots \delta_\varepsilon \Phi(x_k) \ldots \Phi(x_n) \big\rangle. \tag{1.55}$$

   If the symmetry is such that $\delta_\varepsilon \Phi(x)$ is a linear combination of fields (the standard case for global symmetries), the right-hand side consists entirely of renormalized Green's functions. This means that the local operator $\partial_\mu j^\mu$ does not require additional renormalization. This property can be extended (see [107, Section 6.5] for details) to the current itself, and one concludes that

   $$\big[j_\mu(x)\big] = j_\mu(x). \tag{1.56}$$

3. **Gauge symmetry:**

[d] Note that in theories with dimensionful parameters this condition can be satisfied by absorbing them into the definition of the operator $\mathcal{O}_j$.

If gauge symmetry is present in (1.38), it requires special treatment. The natural assumption that gauge-invariant operators are closed under renormalization turns out to be incorrect. The reason is the introduction of gauge-fixing term in (1.38), which breaks gauge invariance. A more precise statement can be made by considering the larger set of BRST transformations rather than gauge transformations alone. The mixing matrix then takes the following block-triangular form (see [107, Section 12.6] for details)

$$\begin{pmatrix} [\mathcal{O}_A] \\ [\mathcal{O}_B] \end{pmatrix} = \begin{pmatrix} Z_{AA} & Z_{AB} \\ 0 & Z_{BB} \end{pmatrix} \begin{pmatrix} \mathcal{O}_A \\ \mathcal{O}_B \end{pmatrix}, \tag{1.57}$$

where $\{\mathcal{O}_A\}$ is the set of gauge-invariant operators and $\{\mathcal{O}_B\}$ are *BRST-exact*, meaning they can be written as $\mathcal{O}_B = \delta_{\mathrm{BRST}}\widetilde{\mathcal{O}}_B$.

With the operator renormalization procedure in place, one can work with the matrix elements (1.22) on a more rigorous footing. This allows us to pass from treating the Wilson OPE (1.15) as a physically motivated ansatz for the matrix element (1.13) to a well-defined statement about operator behavior. The first rigorous formulation of the OPE in perturbation theory can be attributed to W. Zimmermann [145]. The original treatment requires the definition of the so-called normal product prescription (not to be confused with normal ordering), but here we present the version in the notation used throughout this chapter (see [142, Chapter 3, Section 33]).

Zimmermann OPE

Consider two local operators $\mathcal{O}_1(x)$ and $\mathcal{O}_2(x)$ with canonical scaling dimension Δ_1 and Δ_2 defined in the renormalizable theory (1.38). Suppose we are interested in the asymptotics of their product in the limit $|x| \to 0$ up to terms of order $|x|^N$. The product can be expanded in a naive Taylor series:

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = H^{(N)}(x) + R^{(N)}(x), \tag{1.58}$$

where $H^{(N)}(x)$ is the part of the Taylor series up to $O(|x|^N)$ and $R^{(N)}(x)$ is the remainder. The statement of the Zimmermann OPE then reads

$$\forall n: \quad \big\langle [\mathcal{O}_1(x)]\,[\mathcal{O}_2(0)] \prod_{j=1}^{n} \Phi(x_j) \big\rangle = \sum_k C_k^{(N)}(x, g; \mu) \big\langle [\mathcal{O}_k^{(N)}(0)] \prod_{j=1}^{n} \Phi(x_j) \big\rangle + O(|x|^{N+1}), \tag{1.59}$$

where the set of local operators $\{\mathcal{O}_k^{(N)}\}$ is defined as the system of operators closed under renormalization (1.50), including all local operators from $H^{(N)}(x)$ and all operators with canonical scaling dimensions, $\Delta_k < \Delta_1 + \Delta_2$. [a] The Wilson coefficients $C_k^{(N)}(x, g; \mu)$ satisfy

$$C_k^{(N)}(x, g; \mu) \underset{|x|\to 0}{\sim} |x|^p \big(1 + O(g \ln |x|)\big), \qquad p \le N, \tag{1.60}$$

and are finite as $\epsilon \to 0$.

Let us highlight the important differences from the Wilson OPE (1.15):

- The operator expansion is understood in a *weak* sense, i.e. as an insertion in all possible Green's functions.
- The operator product expansion is understood order by order in perturbation theory.
- The limit $x^\mu \to 0$ is taken in an arbitrary direction (ax^μ, with $a \to 0$), in contrast to the Wilson OPE, where the limit is taken from the space-like region.

[a] Note that in theories containing dimensionful parameters, this requirement can be relaxed to the minimal set of operators that includes $H^{(N)}(x)$ and is closed under renormalization (1.50).

W. Zimmermann first proved the identity (1.59) and established the smallness of the remainder [145]. This result was, however, not convenient from a practical point of view. The BPHZ renormalization procedure together with the MOM subtraction scheme used in [145] required the presence of a mass parameter to isolate IR divergencies. An unpleasant feature of this construction was the non-analytic dependence of $C_k(x, g, m; \mu)$ on m through logarithmic terms of the type $\ln(xm)$. In subsequent works it was shown (in [146–148] using the R^*-operation, and in [149,150] by extending Zimmermann's BPHZ analysis) that in MS-like schemes the Wilson coefficients depend on the mass analytically, so that one can take the limit $m \to 0$. This justifies the use of the OPE in theories with massless particles, such as QCD.

The introduction of the renormalization scale μ in (1.59) allows one to apply RG methods to study the asymptotics of objects presented in (1.59). In QCD, for example, it enables the resummation of logarithms in (1.17) in the UV limit and yields corrections to free scaling in the interacting theory. Applying RG invariance to the operators themselves leads to an analogue of the Callan–Symanzik equation,

$$\left(\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right)\delta^{ij} + \gamma^{ij}\right)\left[\mathcal{O}_j(x)\right] = 0, \tag{1.61}$$

where

$$\gamma^{ij} = -\mu\frac{dZ^{il}}{d\mu}\left(Z^{-1}\right)^{lj} = kg\frac{\partial}{\partial g}Z_1^{ij}(g) \tag{1.62}$$

is the matrix of *operator anomalous dimensions*. This equation is of considerable importance for the phenomenological application of the Wilson OPE approach, as it governs the scale evolution of operators, which can be translated into the scale evolution of the matrix elements (1.22). The Callan–Symanzik equation (1.61) applied to the twist-2 operators (see Section 1.2) is equivalent to the DGLAP equation [38–40]. Let us discuss this connection in more detail.

Consider the set of flavor singlet twist-2 QCD operators closed under renormalization (see Section 1.2 for details),

$$\left(\left(\mu\frac{\partial}{\partial\mu} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right)\mathbb{1} + \begin{pmatrix}\gamma_{qq}(s) & \gamma_{qg}(s)\\ \gamma_{gq}(s) & \gamma_{gg}(s)\end{pmatrix}\right)\begin{pmatrix}\left[\mathcal{O}_s^q(x)\right]\\ \left[\mathcal{O}_s^g(x)\right]\end{pmatrix} = 0. \tag{1.63}$$

Taking matrix elements of these operators (1.22) and representing them as Mellin transforms of PDFs via (1.35), we arrive at the following (DGLAP) equation [e]

$$\mu\frac{d}{d\mu}\begin{pmatrix}f_q(x_B,\mu)\\ f_g(x_B,\mu)\end{pmatrix} = \int\limits_{x_B}^{1}\frac{dy}{y}\begin{pmatrix}P_{qq}(y) & P_{qg}(y)\\ P_{gq}(y) & P_{gg}(y)\end{pmatrix}\begin{pmatrix}f_q(x_B/y,\mu)\\ f_g(x_B/y,\mu)\end{pmatrix}, \tag{1.64}$$

where we have introduced the flavor-singlet PDF

$$f_q(x_B,\mu) = \sum_{i=1}^{n_f}\left(f_{q_i}(x_B,\mu) + f_{\bar{q}_i}(x_B,\mu)\right), \tag{1.65}$$

and $f_g(x_B,\mu)$ denotes the gluon parton distribution function. Equation (1.64) is of particular importance as it allows one to resum contributions of the form $\ln(Q^2/\mu^2)$ in the coefficient functions in (1.37), which would otherwise spoil the perturbative expansion in the limit $Q^2 \to +\infty$. The integral kernels of equation (1.64) are called *splitting functions* and are related to the anomalous dimensions of twist-2 operators by a Mellin transform,

$$\gamma_{ij}(s) = -\int\limits_0^1 dx\, x^{s-1}P_{ij}(x), \tag{1.66}$$

[e] In principle, the scale appearing as a parameter of the PDFs in (1.37) and the scale in (1.40) have different origins and are sometimes referred to as the factorization scale μ_F and the renormalization scale μ_R, respectively. For the present discussion it is convenient to set them equal, $\mu_F = \mu_R$.

where $i, j \in \{q, g\}$.

In this section we have seen that the use of operator product expansion techniques (1.15) requires additional treatment of the UV singularities arising in operator matrix elements. We discussed how to define the operator renormalization procedure (1.50) and how the statement of the Wilson OPE should be refined (1.59). We also saw that RG invariance of the theory serves as a powerful tool for improving perturbative calculations, for example by expressing objects as solutions of the Callan–Symanzik equation (1.61). Thus, while Section 1.2 established the relevance of twist-2 operators for scattering involving hadrons, the present section has demonstrated the importance of their renormalization and the calculation of their anomalous dimensions (1.62).

1.4 Universality of OPE and off-forward scattering

So far the use of the Wilson OPE approach (1.15) has been limited to the case of deep inelastic scattering (1.1): in Section 1.2 we discussed its application to the prediction of Mellin moments of the structure functions (1.21), and in Section 1.3 we showed how accounting for the UV divergencies, arising in the matrix elements of twist-2 operators (1.22), leads to the evolution equation (1.64). A remarkable feature of the Wilson OPE, however, is its universality. Indeed, once established, the operator identity (1.20) can be used in a variety of contexts. Let us mention several cases where such matrix elements appear:

- **Vacuum matrix element:**

  The first application of the Wilson OPE beyond DIS was to the inclusive process of e^+e^- annihilation into hadrons. The corresponding matrix element takes the form

  $$\big\langle 0\big|\, \mathrm{T}\left\{J_\mu(x)J_\nu(0)\right\}\big|0\big\rangle. \tag{1.67}$$

  Applying the Wilson OPE in this context led to the derivation of the SVZ sum rules [151] (see also [152] for a review) and made it possible to predict hadron properties from high-energy scattering data. The same OPE can also be used to constrain the anomalous magnetic moment of the muon [153].

- **State-to-vacuum matrix element:**

  The Wilson OPE can be successfully applied to exclusive processes involving hadrons in the final state. For example, the cross-section for hard hadron production (see e.g. [154]) involves a matrix element of the form

  $$\big\langle 0\big|\, \mathrm{T}\left\{J_\mu(x)J_\nu(0)\right\}\big|P\big\rangle, \tag{1.68}$$

  where $\big|P\big\rangle$ corresponds to the state of the hadron (e.g. a pion). An analogous application of the Callan–Symanzik equation (see Section 1.3) leads to the derivation of the ERBL evolution equation [41, 42] for the analogue of PDFs known as *distribution amplitudes* (DAs).

- **Off-forward matrix element:**

  Another class of exclusive processes involves hadrons in both the initial and final states, such as *deeply-virtual Compton scattering* (DVCS), which leads to a matrix element of the form

  $$\big\langle P'\big|\, \mathrm{T}\left\{J_\mu(x)J_\nu(0)\right\}\big|P\big\rangle, \tag{1.69}$$

  where $P'^\mu - P^\mu \neq 0$. The factorization approach in this case gives rise to *generalized parton distributions* (GPDs), for details see [94] and references therein.

In principle, such universality means that one can take the result of the Wilson OPE for two currents, e.g. in DIS, evaluate it in a matrix element with different external momenta, and immediately obtain the result. In practice, however, this is far from straightforward. The reason is the following: at the operator level, the expansion (1.15) predicts the appearance of operators with a quite general structure, and the same is true for the operator mixing (1.50). When one wants to evaluate such an expansion in a given matrix element, it is convenient to discard mixing with operators that do not contribute. Consequently, switching from one matrix element to another may (and usually does) require additional consideration.

Let us illustrate this point by comparing the forward matrix element (1.13) with the off-forward matrix element (1.69). The difference lies in the treatment of *total-derivative* operators, by which we mean operators of the form $\mathcal{O}^\mu_D = \partial^\mu \widetilde{\mathcal{O}}$. These operators are closed under renormalization. Indeed, applying a derivative to the operator renormalization definition (1.50), one finds (see [62, Section 6.5] for discussion)

$$\left[\partial^\mu \mathcal{O}_i\right] = Z^{ij}(\epsilon,\mu)\partial^\mu \mathcal{O}_j. \tag{1.70}$$

Moreover, the renormalization matrix $Z^{ij}(\epsilon,\mu)$ coincides with that governing the mixing of the operators $\mathcal{O}_i$ themselves. This means that the renormalization of total-derivative operators requires no additional consideration beyond that of the operators $\mathcal{O}_i$. New information can arise, however, when considering the mixing of "physical" operators (those not reducible to total derivatives) with total-derivative operators. The corresponding mixing structure then takes the block-triangular form

$$\begin{pmatrix} [\mathcal{O}_P] \\ [\mathcal{O}_D] \end{pmatrix} = \begin{pmatrix} Z_{PP} & Z_{PD} \\ 0 & Z_{DD} \end{pmatrix} \begin{pmatrix} \mathcal{O}_P \\ \mathcal{O}_D \end{pmatrix}, \tag{1.71}$$

where the block Z_{DD} is fixed by (1.70), while the block Z_{PD} is left unconstrained.

In many cases one can simply neglect the total-derivative operators, owing to the triangular structure of (1.71). For example, when considering the Callan–Symanzik equation (1.61), if one is not interested in the scale evolution of total derivatives, this mixing can simply be dropped. This reflects the fact that the eigenvalues of the anomalous dimension matrix (1.62) are independent of the block Z_{PD}. The same simplification applies to the OPE in forward processes. Indeed, writing out the matrix element of a total derivative,

$$\left\langle P'\middle|\partial^\mu \widetilde{\mathcal{O}}\middle|P\right\rangle = \mathrm{i}\left\langle P'\middle|\left[\widehat{P}^\mu, \widetilde{\mathcal{O}}\right]\middle|P\right\rangle = \mathrm{i}\left(P'^\mu - P^\mu\right)\left\langle P'\middle|\widetilde{\mathcal{O}}\middle|P\right\rangle. \tag{1.72}$$

Thus, when $P'^\mu = P^\mu$, all total-derivative operators can be omitted, whereas when $P'^\mu \neq P^\mu$, they must be taken into account. In the latter case, instead of considering twist-2 operator $\mathcal{O}_{\mu_1\ldots\mu_s}$ one needs to consider the whole family

$$\{\mathcal{O}_{\mu_1\ldots\mu_s}, \partial_{(\mu_1}\mathcal{O}_{\mu_2\ldots\mu_s)}, \partial_{(\mu_1\mu_2}\mathcal{O}_{\mu_3\ldots\mu_s)}, \ldots\}. \tag{1.73}$$

In certain applications it is convenient to pass from the consideration of a set of local operators to a single non-local operator that plays the role of a generating function. Let us illustrate this on a simple example in a generic scalar theory (1.38). Consider the operator

$$\mathcal{O}(z_1,z_2) = \Phi(x+z_1 n)\Phi(x+z_2 n), \tag{1.74}$$

where n^μ is an auxiliary light-like vector ($n^2=0$) and $z_1, z_2 \in \mathbb{R}$ are coordinates along this light-like direction. [a] The operator (1.74) is called a *generating operator* for the twist-2 operators, since expanding in z_1 and z_2 one gets

$$\begin{aligned}\mathcal{O}(z_1,z_2) &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{z_1^m z_2^k}{m!k!}(n\cdot\partial)^m\Phi(x)(n\cdot\partial)^k\Phi(x) \\ &= \sum_{s=0}^{\infty}\sum_{j=0}^{s} \frac{z_1^j(z_2-z_1)^{s-j}}{j!(s-j)!}(n\cdot\partial)^j\Big(\Phi(x)\,(n\cdot\partial)^{s-j}\,\Phi(x)\Big). \end{aligned}\tag{1.75}$$

[a]The definition of a light-like direction can be subtle in Euclidean theory. However, when using it as an auxiliary vector for passing to a generating function, its existence is always justified. For example, one can define such a projection by analytic continuation from Minkowski space and verify that all physical results are independent of its precise Euclidean realization.

In connection with the light-like direction, it is convenient to define a second independent light-like vector $\bar{n}^\mu$ ($\bar{n}^2 = 0$) satisfying $(n \cdot \bar{n}) = 1$, and an associated coordinate system in which an arbitrary vector a^μ decomposes as

$$a^\mu = (n \cdot a)\bar{n}^\mu + (\bar{n} \cdot a)n^\mu + a_\perp^\mu = a^+\bar{n}^\mu + a^- n^\mu + a_\perp^\mu. \tag{1.76}$$

Remarkably, non-local operators of the type (1.74) can be applied directly. It can be shown (see [107, Eq. (78)]) that the analogue of the PDF in the factorization formula (1.37) in the case of a scalar field can be expressed as

$$f(x,\mu) = \frac{x(n \cdot P)}{2\pi} \int\limits_{-\infty}^{+\infty} dz\, e^{-\,\mathrm{i}\,xz(n\cdot P)} \big\langle P\big|\Phi(zn)\Phi(0)\big|P\big\rangle. \tag{1.77}$$

This formula clearly reflects the idea that the Wilson OPE of two electromagnetic currents (1.20) is dominated by the expansion in the light-like separation $x^2 \to 0$ rather than $x^\mu \to 0$, while at the technical level it resums the contributions from an infinite number of twist-2 operators.

The same formula can be written for *generalized parton distributions* (GPDs). Such distributions enter factorization theorems (analogous to (1.37)) for non-forward processes and admit the following light-ray representation [94]:

$$H(x,\xi,t,\mu) = \frac{x(n \cdot P)}{2\pi} \int\limits_{-\infty}^{+\infty} dz\, e^{-\,\mathrm{i}\,xz(n\cdot P)} \big\langle P'\big|\Phi\big(-\tfrac{zn}{2}\big)\,\Phi\big(\tfrac{zn}{2}\big)\,\big|P\big\rangle, \tag{1.78}$$

where the standard choice of additional kinematical variables is

$$t = (P' - P)^2, \qquad\qquad \xi = \frac{P^+ - P'^+}{P^+ + P'^+}. \tag{1.79}$$

Note that a crucial difference between the matrix elements in (1.78) and (1.77) is the absence of translational invariance, which corresponds, at the level of local operators, to the absence of contributions from total derivatives. In principle, one can rewrite the off-forward matrix element as

$$\big\langle P'\big|\Phi\big(-\tfrac{zn}{2}\big)\,\Phi\big(\tfrac{zn}{2}\big)\,\big|P\big\rangle = e^{\mathrm{i}\,\frac{z}{2}\left(n\cdot(P-P')\right)}\big\langle P'\big|\Phi(0)\,\Phi(zn)\,\big|P\big\rangle, \tag{1.80}$$

to separate the dependence on the momentum transfer and expand the operator in z without total-derivative contributions. The difficulty is that, as discussed in Section 1.3, one should use renormalized operators inside the matrix element, and this reintroduces total-derivative operators through the mixing (1.71). An advantage of the non-local formulation (1.78) is that renormalization can be organized without decomposing into an infinite tower of local operators. We will return to this point later in this section.

Let us discuss in more detail the expansion in local operators of (1.74). One might worry that by contracting with the light-like vector we lose information about the tensor structure of the underlying twist-2 operator. Let us address this issue. Suppose we have an arbitrary fully symmetric tensor $\psi_{\mu_1\ldots\mu_s}$, i.e. $\psi_{\mu_{\sigma(1)}\cdots\mu_{\sigma(s)}} = \psi_{\mu_1\ldots\mu_s}$ for all $\sigma \in S_s$. It is often convenient to contract this tensor with an auxiliary vector v^μ and consider the function $\psi(v)$,

$$\psi(v) = v^{\mu_1}\ldots v^{\mu_s}\psi_{\mu_1\ldots\mu_s}, \qquad\qquad \psi_{\mu_1\ldots\mu_s} = \frac{\partial}{\partial v^{\mu_1}}\cdots\frac{\partial}{\partial v^{\mu_s}}\psi(v)\Big|_{v=0}. \tag{1.81}$$

The second formula (which produces a symmetric tensor by construction) allows one to reconstruct the original tensor from the function $\psi(v)$. A remarkable property of this procedure is that if $\psi_{\mu_1\ldots\mu_s}$ is traceless, one may contract it with the light-like vector n^μ instead. Let us demonstrate this on the simple example of the symmetric traceless product $\psi_{\mu_1\ldots\mu_s} = x_{\mu_1}\ldots x_{\mu_s} -$ traces. Comparing contraction with the light-like, n^μ, versus an arbitrary vector, v^μ, we get

$$\psi(n) = (x\cdot n)^s, \tag{1.82}$$

$$\psi(v) = (x\cdot v)^s + a_1(x\cdot v)^{s-2}x^2v^2 + a_2(x\cdot v)^{s-4}x^4v^4 + \ldots. \tag{1.83}$$

At first sight it might appear that by contracting with the light-like vector we have lost information about the coefficients a_j. However, it can be shown that they are completely fixed by the requirement that $\psi_{\mu_1\ldots\mu_s}$ be traceless. Indeed, using (1.81) one finds

$$g^{\mu_i\mu_j}\psi_{\mu_1\ldots\mu_s} = 0 \quad \Longrightarrow \quad \partial^2\psi(v) = 0. \tag{1.84}$$

The requirement that $\psi(v)$ be harmonic can be viewed as a recurrence relation fixing the coefficients a_j in terms of a_{j+1}. Thus, knowing $\psi(n)$ together with the harmonicity of $\psi(v)$ completely determines $\psi(v)$.

The definition of GPDs (1.78) in QCD (as opposed to the scalar case) motivates us to introduce non-local generating operators for the QCD twist-2 operators discussed in Section 1.3. To avoid overloading the discussion, we do not give explicit formulas for the GPDs (which can be found e.g. in [94]), and instead focus on the non-local operators entering off-forward matrix elements in various cases. The system of flavor-singlet non-local operators takes the form

$$\mathcal{O}_q(z_1,z_2) = \frac{1}{2}\left(\sum_{i=1}^{n_f}\bar{q}_i(z_1n)\gamma_+\big[z_1n,z_2n\big]q_i(z_2n) - (z_1\leftrightarrow z_2)\right), \tag{1.85}$$

$$\mathcal{O}_g(z_1,z_2) = F^a_{+\mu}(z_1n)\big[z_1n,z_2n\big]^{ab}F^{b,\mu}_{+}(z_2n), \tag{1.86}$$

and the flavor non-singlet quark operator reads

$$\mathcal{O}^A_q(z_1,z_2) = \bar{q}_i(z_1n)(\lambda^A)_{ij}\gamma_+\big[z_1n,z_2n\big]q_j(z_2n). \tag{1.87}$$

The notable difference from the definition (1.74) is the presence of *Wilson lines*, defined as

$$\big[z_1n,z_2n\big] = \mathrm{Pexp}\left(igz_{12}\int_0^1 du\, A_+(z^u_{21}n)\right), \tag{1.88}$$

where g is the QCD coupling ($D_\mu = \partial_\mu - igA_\mu$) and we use the notation

$$z_{12} = z_1 - z_2, \qquad z^u_{12} = \bar{u}z_1 + uz_2, \qquad \bar{u} = 1-u. \tag{1.89}$$

The insertion of a Wilson line (1.88) in the definitions of non-local operators ensures their gauge invariance. The gluon field in the path ordered exponent (1.88) can be written as $A_\mu = A^a_\mu T^a$, where T^a are the generators of $SU(N_c)$. For quark operators these generators act in the fundamental representation, and for the gluon operator in the adjoint, $(T^a)_{bc} = \mathrm{i}\,f^{bac}$.

For gauge-invariant operators, the expansion in a basis of local operators is more involved (also due to the Wilson line insertions) and in general takes the form

$$\mathcal{O}(z_1,z_2) = \sum_s\sum_k \Psi_{s,k}(z_1,z_2)\mathcal{O}_{s,k}(0). \tag{1.90}$$

We do not present explicit formulas for $\Psi_{s,k}$ here, but merely mention the common choices of basis on the example of the quark operator (suppressing all indices not relevant for the discussion):

- **Derivative basis:**

  The straightforward Taylor expansion gives

$$\mathcal{O}_{s,k} = (n\cdot D)^{s-k}\bar{q}(0)\gamma_+(n\cdot D)^k q(0), \tag{1.91}$$

  which provides a simple way to write the coefficients in (1.90).

- **Total-derivative basis:**

  Analogously to the resummation in (1.75), one can separate total derivatives explicitly,

$$\mathcal{O}_{s,k} = (n\cdot\partial)^{s-k}\Big(\bar{q}(0)\gamma_+(n\cdot\overrightarrow{D})^k q(0)\Big). \tag{1.92}$$

  The advantage of this basis is the explicit triangular structure of the renormalization matrix, as guaranteed by (1.71).

- **Gegenbauer basis:**

  Another important choice with a triangular mixing structure is

$$\mathcal{O}_{s,s+k} = (-1)^k(n\cdot\partial)^{s+k}\left(\bar{q}(0)\gamma_+C_k^{3/2}\left(\frac{(n\cdot\overleftarrow{D})-(n\cdot\overrightarrow{D})}{(n\cdot\overleftarrow{D})+(n\cdot\overrightarrow{D})}\right)q(0)\right), \tag{1.93}$$

  where $C_k^{(\lambda)}(x)$ are the Gegenbauer polynomials, see e.g. [155, §6.6]. Further remarkable properties of this basis will be discussed in Section 2.4.

As discussed in Section 1.3, removing the UV divergencies present in operator matrix elements requires operator renormalization. This can be done for non-local operators as well. We define a renormalized non-local operator by

$$\big[\mathcal{O}(z_1,z_2)\big] = \mathbb{Z}\,\mathcal{O}(z_1,z_2), \tag{1.94}$$

where the renormalization factor $\mathbb{Z}$, in analogy with (1.43), takes the form

$$\mathbb{Z}(\epsilon,g) = \mathbb{1} + \sum_{k=1}^{\infty}\frac{\mathbb{Z}_k(g)}{\epsilon^k}. \tag{1.95}$$

The coefficients $\mathbb{Z}_k(g)$ are now integral operators acting on the variables z_1 and z_2. One can then write the non-local analogue of the Callan–Symanzik equation (1.61),

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\mathbb{H}\right)\big[\mathcal{O}(z_1,z_2)\big]=0, \tag{1.96}$$

where

$$\mathbb{H} = -\mu\frac{d\mathbb{Z}}{d\mu}\mathbb{Z}^{-1} \tag{1.97}$$

is the so-called *evolution kernel* [88]. The symmetry properties of the evolution equation (see Section 2.4 for a more detailed discussion) fix their general form:

$$\left[\mathbb{H}\,f\right](z_1,z_2) = \int_0^1 d\alpha\int_0^1 d\beta\,h(\alpha,\beta)f(z_{12}^{\alpha},z_{21}^{\beta}), \tag{1.98}$$

where $h(\alpha,\beta)$ is an integral kernel. Knowledge of the evolution kernel (1.98) is completely equivalent to knowledge of the anomalous dimension matrix of the system of operators (1.73) for all $s\in\mathbb{Z}_{\geq 0}$. The connection is established by applying the evolution kernel (1.98) to appropriate test functions $f(z_1,z_2)$. An important case is the *forward* twist-2 anomalous dimension $\gamma(s)$ (see (1.63) for the flavor-singlet case), which is an eigenvalue of the evolution kernel:

$$\mathbb{H}\,z_{12}^{s-1} = \gamma(s)\,z_{12}^{s-1}, \qquad \gamma(s) = \int_0^1 d\alpha\int_0^1 d\beta\,h(\alpha,\beta)(1-\alpha-\beta)^{s-1}, \tag{1.99}$$

with corresponding eigenfunctions $z_{12}^{s-1} = (z_1 - z_2)^{s-1}$.

In conclusion, let us summarize this section. We discussed how the Wilson OPE approach can be naturally extended from its application to DIS (1.1) to other QCD processes. We showed that in non-forward scattering one must account for the full set of twist-2 operators including their total derivatives. This consideration translates naturally to the language of non-local operators (1.74), which appear directly in the generalized parton distributions (1.78). This motivates the study of not only the renormalization of local twist-2 operators, but also of their non-local counterparts, whose renormalization can be conveniently expressed in terms of evolution kernels (1.98).

Chapter 2

Conformal symmetry and its applications

Without a doubt, symmetry plays a crucial role in physics. This role is typically constraining — a specific symmetry of the problem narrows the set of possible solutions and can sometimes serve as the only tool leading to an answer (see e.g. the solution for the hydrogen atom [156]). This applies, of course, to quantum field theories as well, and the most eloquent testament to this is the amount of time devoted to the study of representation theory of various groups (Poincaré, $SU(3)$, etc.) in a standard QFT course.

An interesting example of a symmetry that is sometimes not even obvious from the Lagrangian description is *conformal symmetry*. Its first connection to field theory apparently arose when it was noted [157, 158] that Maxwell's equations are invariant under *scale* transformations and *inversions*, which together with the Poincaré symmetry, already present, form a group. At the quantum level the situation was more subtle, as even a classically conformally invariant theory can have its symmetry broken by quantum corrections [16, 143]. Nevertheless, the importance of conformal symmetry became clear through its connection to renormalization and renormalization group analysis. The fixed points of the renormalization group flow turn out to be scale invariant [159], and a crucial further assumption was the conjecture that scale invariance implies conformal invariance [160] (the status of this assumption is, however, still not fully settled, see [161–164] for a review and [165] for an example of a non-unitary theory that enjoys scale but not conformal symmetry). In two-dimensional theories, the appearance of conformal invariance turns out to render the large class of theories exactly solvable [23]. In $d > 2$, conformal symmetry is not strong enough to solve the problem completely, but it allows one to fundamentally change the approach and enables the development of powerful tools such as the conformal bootstrap [21, 166–168] (see also [169] for a review).

In this chapter we give an introductory account of conformal field theories (CFTs), a more complete discussion of which can be found in a variety of textbooks [170] and lecture notes [171–173]. We will discuss the properties that distinguish CFTs from arbitrary QFTs and examine to what extent these structures can be found in QCD, specifically in the renormalization of twist-2 operators (see [27] for a detailed discussion of this part).

2.1 Conformal algebra and representation theory

Let us begin with the formal description of conformal symmetry. In what follows, we work in Euclidean d-dimensional space, $x^\mu \in \mathbb{R}^d$, and assume $d \geq 3$. Although the problems discussed in Chapter 1 were formulated in Minkowski space, CFT methods are most naturally applicable in Euclidean signature, e.g. assuming a Wick rotation [174]. The conformal group is the group of transformations $x \mapsto x'$ that preserve the metric $\delta_{\mu\nu}$ up to a (local) factor, [a]

$$\delta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x)\delta_{\mu\nu}. \tag{2.1}$$

It can be shown that the set of all such transformations consists of Poincaré transformations ($\Lambda(x) =$ const) together with dilatations, $x^\mu \mapsto \lambda x^\mu$, and inversions, $x^\mu \mapsto x^\mu/x^2$.

The set of transformations satisfying (2.1) is described by the conformal Lie algebra $\mathfrak{conf}(\mathbb{R}^d)$, which consists of $(d+1)(d+2)/2$ basis elements $\{\mathbf{P}_\mu, \mathbf{M}_{\mu\nu}, \mathbf{D}, \mathbf{K}_\mu\}$, where $\mathbf{P}_\mu$ and $\mathbf{M}_{\mu\nu}$ are the generators of translations and rotations respectively, $\mathbf{D}$ is the generator of dilatations, and $\mathbf{K}_\mu$ generates the special conformal transformations (SCTs). The latter can be viewed as the infinitesimal analogue of an inversion, which takes the form

$$x^\mu \mapsto x'^\mu = \frac{x^\mu + a^\mu x^2}{1 + 2(a\cdot x) + a^2 x^2}, \tag{2.2}$$

where a^μ is an auxiliary vector defining the direction of the SCT. The algebra $\mathfrak{conf}(\mathbb{R}^d)$ is defined by the commutation relations of the Poincaré algebra,

$$\begin{aligned} \big[\mathbf{P}_\mu, \mathbf{P}_\nu\big] &= 0, \qquad \big[\mathbf{P}_\lambda, \mathbf{M}_{\mu\nu}\big] = \mathrm{i}\left(\delta_{\lambda\mu}\mathbf{P}_\nu - \delta_{\lambda\nu}\mathbf{P}_\mu\right), \\ \big[\mathbf{M}_{\mu\nu}, \mathbf{M}_{\rho\sigma}\big] &= \mathrm{i}\left(\delta_{\nu\rho}\mathbf{M}_{\mu\sigma} + \delta_{\mu\sigma}\mathbf{M}_{\nu\rho} - \delta_{\mu\rho}\mathbf{M}_{\nu\sigma} - \delta_{\nu\sigma}\mathbf{M}_{\mu\rho}\right), \end{aligned} \tag{2.3}$$

together with the additional commutation relations

$$\begin{aligned} &\big[\mathbf{D}, \mathbf{P}_\mu\big] = \mathrm{i}\,\mathbf{P}_\mu, \qquad \big[\mathbf{D}, \mathbf{M}_{\mu\nu}\big] = 0, \qquad \big[\mathbf{D}, \mathbf{K}_\mu\big] = -\mathrm{i}\,\mathbf{K}_\mu, \\ &\big[\mathbf{K}_\mu, \mathbf{K}_\nu\big] = 0, \qquad \big[\mathbf{K}_\mu, \mathbf{P}_\nu\big] = -2\,\mathrm{i}\left(\delta_{\mu\nu}\mathbf{D} + \mathbf{M}_{\mu\nu}\right), \qquad \big[\mathbf{K}_\lambda, \mathbf{M}_{\mu\nu}\big] = \mathrm{i}\left(\delta_{\lambda\mu}\mathbf{K}_\nu - \delta_{\lambda\nu}\mathbf{K}_\mu\right). \end{aligned} \tag{2.4}$$

Remarkably, these commutation relations can be reorganized by introducing $\mathbf{J}_{MN} = -\mathbf{J}_{NM}$ for $M, N \in \{1, \ldots, d+2\}$ via

$$\mathbf{J}_{\mu\nu} = \mathbf{M}_{\mu\nu}, \qquad \mathbf{J}_{d+2,\,d+1} = \mathbf{D}, \qquad \mathbf{J}_{d+1,\,\mu} = \frac{1}{2}\big(\mathbf{P}_\mu + \mathbf{K}_\mu\big), \qquad \mathbf{J}_{d+2,\,\mu} = \frac{1}{2}\big(\mathbf{P}_\mu - \mathbf{K}_\mu\big), \tag{2.5}$$

with commutation relations

$$\big[\mathbf{J}_{MN}, \mathbf{J}_{RS}\big] = \mathrm{i}\left(\eta_{NR}\mathbf{J}_{MS} + \eta_{MS}\mathbf{J}_{NR} - \eta_{MR}\mathbf{J}_{NS} - \eta_{NS}\mathbf{J}_{MR}\right), \tag{2.6}$$

where η_{MN} is the Minkowski-signature metric $\eta = \mathrm{diag}\{1, 1, \ldots, 1, -1\}$. This establishes the Lie algebra isomorphism $\mathfrak{conf}(\mathbb{R}^d) \cong \mathfrak{so}(d+1, 1)$ [19, 127].

The representation theory of $\mathfrak{conf}(\mathbb{R}^d)$ is well-studied (see [175, 176] for the detailed discussion). Since the corresponding Lie group $\mathrm{Conf}(\mathbb{R}^d)$ is non-compact, the algebra admits unitary infinite-dimensional representations. The discrete series representations are labeled by a pair (Δ, ℓ), where $\Delta \in \mathbb{R}$ is the eigenvalue of the dilatation operator and $\ell \in \mathbb{Z}_{\geq 0}/2$ is a spin label associated with the rotational subalgebra $\mathfrak{so}(d) < \mathfrak{conf}(\mathbb{R}^d)$, we omit further details of this classification here. The representation space is built by introducing a lowest-weight vector $\big|\Delta, \ell\big\rangle$ satisfying

$$\mathbf{K}_\mu\big|\Delta, \ell\big\rangle = 0, \qquad -\mathrm{i}\,\mathbf{D}\big|\Delta, \ell\big\rangle = \Delta\big|\Delta, \ell\big\rangle, \tag{2.7}$$

so that Δ is the scaling dimension of the lowest-weight state. The state $\big|\Delta, \ell\big\rangle$ is usually called *primary*. The operator $\mathbf{P}_\mu$ then acts as a raising operator (see the commutation relations (2.5)),

$$-\mathrm{i}\,\mathbf{D}\,\mathbf{P}_\mu\big|\Delta, \ell\big\rangle = (\Delta + 1)\,\mathbf{P}_\mu\big|\Delta, \ell\big\rangle. \tag{2.8}$$

[a] The name "conformal" reflects the fact that such transformations preserve angles between arbitrary vectors x^μ and y^μ. Indeed, under (2.1), the angle $\cos\theta = x^\mu y^\mu/\sqrt{x^2 y^2} = x'^\mu y'^\mu/\sqrt{x'^2 y'^2}$.

The representation space from the discrete series is built from the *Verma module* $M(\Delta,\ell)$, defined as

$$M(\Delta,\ell) = \mathrm{span}\left\{\mathbf{P}_{\mu_1}\dots\mathbf{P}_{\mu_n}\big|\Delta,\ell\big\rangle \,\Big|\, n\in\mathbb{Z}_{\geq 0}\right\}. \tag{2.9}$$

In general, the Verma module $M(\Delta,\ell)$ may contain *null vectors*, i.e. states $\big|v\big\rangle \neq \big|\Delta,\ell\big\rangle$ such that $\mathbf{K}_\mu\big|v\big\rangle = 0$. One must therefore quotient out such subspaces in order to obtain irreducible representations. It can be shown that the representation (Δ,ℓ) from the discrete series is unitary if [177]

$$\begin{cases} \Delta \geq \dfrac{d}{2}-1, & \text{if } \ell = 0, \\ \Delta \geq \ell + d - 2, & \text{if } \ell \geq 1. \end{cases} \tag{2.10}$$

Additionally, the trivial representation is also unitary.

One of the simplest realizations of such a representation (the scalar case $(\Delta,0)$) can be constructed on the space of scalar functions, $f(x)$ $(x^\mu\in\mathbb{R}^d)$. The generators of the conformal group then take the form of differential operators,

$$\begin{aligned} \mathbf{P}_\mu &= -\,\mathrm{i}\,\partial_\mu, \\ \mathbf{M}_{\mu\nu} &= \mathrm{i}\,\big(x_\mu\partial_\nu - x_\nu\partial_\mu\big), \\ \mathbf{D} &= -\,\mathrm{i}\,\big(x^\mu\partial_\mu + \Delta\big), \\ \mathbf{K}_\mu &= \mathrm{i}\,\big(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 2\Delta x_\mu\big). \end{aligned} \tag{2.11}$$

The primary function $f_\Delta(x)$ must then satisfy the differential equations

$$\big(x^\mu\partial_\mu + \Delta\big)f_\Delta(x) = 0, \qquad \big(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 2\Delta x_\mu\big)f_\Delta(x) = 0. \tag{2.12}$$

One can show that in the scalar case, restricting to $SO(d)$-invariant solutions, this function can be identified as

$$f_\Delta(x) = \left(\frac{1}{x^2}\right)^{\Delta/2}. \tag{2.13}$$

The inner product on the space of functions should be chosen to reflect the Verma module structure and does not coincide with the standard L^2 norm.

Another important series of infinite-dimensional unitary representations is the *principal series*. These are again labeled by two numbers (Δ,ℓ), but now the scaling dimension is a continuous parameter of the form

$$\Delta = \frac{d}{2} + \mathrm{i}\,\nu, \quad \nu\in\mathbb{R}. \tag{2.14}$$

For the principal series representations there is no discrete structure analogous to the Verma module. In the case of representations built on scalar functions $f(x)$, the differential generators (2.11) retain their form. What changes is that the space of functions can now be taken to be $L^2(\mathbb{R}^d)$, on which the representation is unitary with respect to the standard inner product. The spectrum of the dilatation operator exhibits a remarkable symmetry [178],

$$\Delta = \frac{d}{2} + \mathrm{i}\,\nu \quad \xrightarrow{\nu\mapsto-\nu} \quad \widetilde{\Delta} = \frac{d}{2} - \mathrm{i}\,\nu = d - \Delta, \tag{2.15}$$

where $\widetilde{\Delta}$ is usually referred to as the *shadow scaling dimension*. It can be shown that the two principal series representations (Δ,ℓ) and $(\widetilde{\Delta},\ell)$ are equivalent, i.e. there exists an operator intertwining the generators in these two representations [179]. In the scalar case it can be constructed as

$$\big(S_\Delta\, f\big)(x) = \int\limits_{\mathbb{R}^d} \frac{d^dy}{(x-y)^{2(d-\Delta)}}\, f(y). \tag{2.16}$$

A similar intertwining operator can in principle be constructed for representations from the discrete series, but it intertwines the unitary representation (Δ, ℓ) with a non-unitary one, or one not belonging to the discrete series.

The conformal algebra $\mathfrak{conf}(\mathbb{R}^d)$ has one important subalgebra relevant to the discussion in Section 1.4 (see [27] for the details). One can define the generators $\{\mathbf{L}_+, \mathbf{L}_-, \mathbf{L}_0\}$ by

$$\mathbf{L}_+ = -\,\mathrm{i}\, n^\mu \mathbf{P}_\mu, \qquad \mathbf{L}_- = \frac{\mathrm{i}}{2}\bar{n}^\mu \mathbf{K}_\mu, \qquad \mathbf{L}_0 = \frac{\mathrm{i}}{2}\big(\mathbf{D} + n^\mu \bar{n}^\nu \mathbf{M}_{\mu\nu}\big), \tag{2.17}$$

where n^μ and $\bar{n}^\nu$ are two independent light-like vectors ($n^2 = \bar{n}^2 = 0$, $(n \cdot \bar{n}) = 1$), as in Eq. (1.76). Physically, this subalgebra generates transformations that preserve the chosen light-like direction n^μ. One can verify explicitly that these generators satisfy the commutation relations

$$\big[\mathbf{L}_0, \mathbf{L}_\pm\big] = \mp \mathbf{L}_\pm, \qquad \big[\mathbf{L}_-, \mathbf{L}_+\big] = 2\mathbf{L}_0, \tag{2.18}$$

which identify the collinear subalgebra as $\mathfrak{sl}(2,\mathbb{R}) < \mathfrak{conf}(\mathbb{R}^d)$.

Representations of the collinear subalgebra (2.17) [129] are labeled by the value of the $\mathfrak{sl}(2,\mathbb{R})$ quadratic Casimir, j, usually called the *conformal spin*,

$$\mathbf{C}_2 = \mathbf{L}_0^2 - \mathbf{L}_0 + \mathbf{L}_+\mathbf{L}_-, \qquad \mathbf{C}_2\big|j\big\rangle = j(j-1)\big|j\big\rangle, \tag{2.19}$$

where $\big|j\big\rangle$ spans the representation space. The representation (Δ, ℓ) of the full conformal algebra restricts to the subalgebra level via

$$j = \frac{\Delta + \ell}{2}. \tag{2.20}$$

For representations built on functions of one variable, it is convenient to restrict to the light-cone, setting $f(zn) \equiv f(z)$ with $z \in \mathbb{R}$. The collinear generators then take the form

$$\begin{aligned} \mathbf{L}_+ &= -\partial_z, \\ \mathbf{L}_- &= z^2\partial_z + 2jz, \\ \mathbf{L}_0 &= z\partial_z + j. \end{aligned} \tag{2.21}$$

Exponentiating these generators, one can see that the full set of transformations is described by the Möbius maps

$$z \mapsto \frac{az+b}{cz+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}). \tag{2.22}$$

At the level of functions, this transformation acts as

$$f(z) \mapsto f'(z) = \frac{1}{(cz+d)^{2j}} f\left(\frac{az+b}{cz+d}\right). \tag{2.23}$$

In this section we have introduced the conformal algebra of Euclidean space, $\mathfrak{conf}(\mathbb{R}^d)$, and discussed its representations, largely with the aim of setting up notation. Two interesting features of the conformal group have emerged along the way. The collinear subgroup $SL(2,\mathbb{R}) < \mathrm{Conf}(\mathbb{R}^d)$ appears directly relevant to the discussion in Chapter 1, as it parametrizes symmetries along the light-cone. The presence of the shadow transformation (2.15) in the spectrum has no immediate connection at this point, but its significance will become clear in Chapter 4.

2.2 Conformal field theory

In the previous section we discussed the conformal algebra of d-dimensional Euclidean space and its representations. In this section we go further and discuss quantum field theories that are invariant under conformal transformations. Let us begin with the abstract example introduced in Section 1.3. We consider a field theory described by an action

$$S = \int d^d x\, \mathcal{L}(\Phi, g). \tag{2.24}$$

Now let us assume the action is invariant under all transformations from the conformal group, $\mathrm{Conf}(\mathbb{R}^d)$. Conformal invariance of the action requires the fields to transform in a representation of the conformal group. Their infinitesimal transformations, $\Phi(x') = (1 + (\varepsilon \cdot \delta))\, \Phi(x)$, are given by [a]

$$\begin{aligned}
\delta_P^{\mu} \Phi(x) &= P^{\mu} \Phi(x) = \partial^{\mu} \Phi(x), \\
\delta_M^{\mu\nu} \Phi(x) &= M^{\mu\nu} \Phi(x) = \big(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu} - \Sigma^{\mu\nu}\big)\Phi(x), \\
\delta_D \Phi(x) &= D\Phi(x) = \big(x^{\mu}\partial_{\mu} + \Delta\big)\Phi(x), \\
\delta_K^{\mu} \Phi(x) &= K^{\mu} \Phi(x) = \big(2x^{\mu}\big(x\cdot\partial\big) - x^2\partial^{\mu} + 2\Delta x^{\mu} - 2x_{\nu}\Sigma^{\mu\nu}\big)\Phi(x),
\end{aligned} \tag{2.25}$$

where the representation (Δ, ℓ) is specified by the parameter Δ and the matrix $\Sigma^{\mu\nu}$. The latter encodes the spin of the field $\Phi(x)$ and takes the form

$$\Sigma^{\mu\nu}\varphi(x) = 0, \qquad \Sigma^{\mu\nu}\psi(x) = \frac{1}{4}\big[\gamma^{\mu}, \gamma^{\nu}\big]\psi(x), \qquad \Sigma^{\mu\nu} A^{\alpha} = \delta^{\nu\alpha} A^{\mu} - g^{\mu\alpha} A^{\nu}, \tag{2.26}$$

where φ, ψ, and A_{α} denote the scalar, fermionic, and spin-1 fields respectively. For a classical field $\Phi(x)$, the parameter Δ corresponds to its canonical scaling dimension, fixed by the requirement that the action (2.24) be dimensionless.

Let us now turn to the quantum theory. Quantizing the classical theory (2.24) means defining a Hilbert space of quantum states $\mathcal{H}$, with all local combinations of fields becoming operators acting on it, $\widehat{\mathcal{O}}(x) : \mathcal{H} \to \mathcal{H}$. The path integral formalism then provides a way of computing inner products in the Hilbert space, namely [b]

$$\big\langle 0\big|\, \mathrm{T}\left\{\widehat{\mathcal{O}}_1(x_1)\ldots\widehat{\mathcal{O}}_n(x_n)\right\}\big|0\big\rangle = \big\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\big\rangle \equiv \frac{1}{\mathcal{N}}\int D\Phi\, \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\, e^{-S}, \tag{2.27}$$

where $\big|0\big\rangle \in \mathcal{H}$ is the vacuum state. The time ordering in the correlation function (2.27) arises from the way states are prepared. For example, for the state obtained by acting with a local operator $\widehat{\mathcal{O}}_i(x_i)$ on the vacuum, one defines the wavefunctional

$$\big\langle \phi(\vec{x})\big|\widehat{\mathcal{O}}_i(x_i)\big|0\big\rangle = \int_{\substack{t\leq t_0,\\ \Phi(t_0,\vec{x})=\phi(\vec{x})}} D\Phi\, \mathcal{O}_i(x_i)\, e^{-S}, \tag{2.28}$$

where $x_i^{\mu} = (t_i, \vec{x}_i)$ and the reference time, t_0, is chosen such that $t_i < t_0$. Here $\phi(\vec{x})$ plays the role of the configuration space variable, in the same way as coordinates do in quantum mechanics. This essentially means that space is sliced into equal-time hypersurfaces, $\mathbb{R}^d = \mathbb{R}_{\geq 0} \times \mathbb{R}^{d-1}$, and every state lives on a specific slice $t = t_0$, absorbing all operator insertions in its past $t < t_0$.

A remarkable feature of this quantization procedure is that the choice of slicing is rather arbitrary. In CFTs, the natural choice turns out to be a slicing by concentric spheres, $\mathbb{R}^d = \mathbb{R}_{\geq 0} \times S^{d-1}$. The corresponding quantization procedure then gives

$$\big\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\big\rangle = \big\langle 0\big|\, \mathrm{R}\left\{\widehat{\mathcal{O}}_1(x_1)\ldots\widehat{\mathcal{O}}_n(x_n)\right\}\big|0\big\rangle, \tag{2.29}$$

[a] The difference between generators, e.g. $\mathbf{D}$, and differential operators, D, should be understood as the difference between elements of conformal algebra, $\mathfrak{conf}(\mathbb{R}^d)$, and their representation.

[b] In this section we explicitly distinguish operators $\widehat{\mathcal{O}}_i$ from the functions $\mathcal{O}_i(x)$ inserted inside the path integral. In what follows we drop this distinction, trusting that the nature of the object is clear from context.

where R$\{\ldots\}$ orders operators by their radial distance, $x_i^\mu = (r_i, \vec{n}_i)$. The analogue of the state definition (2.28) then reads

$$\langle \phi(\vec{n}) \big| \widehat{\mathcal{O}}_i(x_i) \big| 0 \rangle = \int_{\substack{r \le r_0, \\ \Phi(r_0,\vec{n}) = \phi(\vec{n})}} D\Phi \, \mathcal{O}_i(x_i) \, e^{-S}, \tag{2.30}$$

where $x^\mu = (r, \vec{n})$ and $|x_i| < r_0$ is implied. This procedure is called *radial quantization* and can in principle be performed in any QFT. The conformal invariance of the path integral in (2.30), however, has significant consequences for the structure of the Hilbert space, $\mathcal{H}_{\text{rad}}$, obtained by radial quantization. We do not discuss the derivation of these consequences in detail (see e.g. c [173, Chapter 6]) and simply present them here.

- **Dilatation operator:**

  In analogy with the way the Hamiltonian, as the generator of time translations, is Hermitian with respect to the standard inner product, one can show that the generator of translations in r, namely the dilatation operator $\mathbf{D}$, is Hermitian in a unitary CFT. [c] The Hilbert space therefore decomposes as

$$\mathcal{H}_{\text{rad}} = \bigoplus_{\Delta} \mathcal{H}_\Delta, \tag{2.31}$$

  where $\mathcal{H}_\Delta$ is the eigenspace of the dilatation operator with eigenvalue Δ.

- **State-operator correspondence:**

  The conformal symmetry of the path integral in (2.30) allows one to shrink the interior of the sphere $|x| < r_0$ to a point $r = 0$. In this way one can establish a one-to-one correspondence between local operators and states in the Hilbert space $\mathcal{H}_{\text{rad}}$,

$$\mathcal{O}(0) \longleftrightarrow \widehat{\mathcal{O}}(0) \big| 0 \rangle \equiv \big| \mathcal{O} \rangle. \tag{2.32}$$

- **Primaries and descendants:**

  The positive definiteness of the inner product in (2.29) in a unitary CFT implies a lower bound on the scaling dimensions, meaning that states must correspond discrete representations satisfying (2.10). This in turn implies that the set of all local operators in such a CFT is spanned by primary operators $\mathcal{O}_\Delta$, i.e.

$$-\mathrm{i} \left[\mathbf{D}, \mathcal{O}_\Delta(0) \right] = \Delta \mathcal{O}_\Delta(0), \qquad \mathrm{i} \left[\mathbf{K}^\mu, \mathcal{O}(0) \right] = 0, \tag{2.33}$$

  and their descendants,

$$\mathrm{i} \left[\mathbf{P}^{\mu_1}, \ldots \mathrm{i} \left[\mathbf{P}^{\mu_n}, \mathcal{O}(0) \right] \ldots \right] = \partial^{\mu_1} \ldots \partial^{\mu_n} \mathcal{O}(0). \tag{2.34}$$

- **Conformal OPE:**

  Consider the state $\big|\psi\rangle$ living on the sphere of radius r and prepared by operator insertions, $\big|\psi\rangle = \widehat{\mathcal{O}}_1(x) \widehat{\mathcal{O}}_2(0) \big| 0 \rangle$, where $|x| < r$. This state can be expanded in the basis at the origin,

$$\big|\psi\rangle = \sum_k \langle \mathcal{O}_k | \psi \rangle \big| \mathcal{O}_k \rangle, \qquad \big| \mathcal{O}_k \rangle = \widehat{\mathcal{O}}_k(0) \big| 0 \rangle, \tag{2.35}$$

  where the sum runs over primary operators and their descendants. Lifting this expansion to the level of operators justifies the existence of an OPE in conformal field theories. Note that in contrast

[c] Strictly speaking, in Euclidean theory unitarity should be replaced by the Osterwalder–Schrader reflection positivity [180], but we omit these details in the present discussion.

to the asymptotic expansions (1.15) and (1.59), the operator equality (2.35) is an exact statement, and the expansion

$$\big\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\prod_j \mathcal{O}_j(x_j)\big\rangle = \sum_k C_k(x)\big\langle \mathcal{O}_k(0)\prod_j \mathcal{O}_j(x_j)\big\rangle \tag{2.36}$$

converges inside the sphere $|x| < \min_j |x_j|$ [181].

The special structure of conformal theories thus allows one to pass from the consideration of states in the Hilbert space $\mathcal{H}$ to that of local operators forming a basis of this Hilbert space. These operators are conveniently organized into *conformal families*, labeled by two numbers (Δ, ℓ),

$$\big\{\partial^{\mu_1}\dots\partial^{\mu_n}\mathcal{O}_{\Delta,\ell}(x)\big\}_{n=0}^{\infty}, \tag{2.37}$$

where $\mathcal{O}_{\Delta,\ell}$ is a primary operator of spin ℓ satisfying the conditions (2.33) at $x = 0$. It is important to note, however, that the scaling dimension Δ does not coincide with the canonical scaling dimension in (2.25), but acquires an anomalous part, $\Delta(g) = \Delta_0 + \gamma(g)$. [d] In the theory (2.24) the anomalous dimension can generally be expressed as a series in the coupling g. Note that this definition of a CFT does not in principle require a Lagrangian formulation: one can view a CFT as a theory defined by a set of local operators with specified scaling dimensions and spins, which forms the basis of the bootstrap approach [21, 166, 167].

Let us now discuss what constitutes the complete information describing a CFT. Given the above, one might assume that a CFT is solved exactly once the correlation functions of all local operators are known. It is immediately clear, however, that such information is redundant, since correlators of descendant operators can be expressed as derivatives of correlators of primary operators,

$$\big\langle \partial^{\mu_1}\dots\partial^{\mu_n}\mathcal{O}_{\Delta,\ell}(x)\prod_j \mathcal{O}_j(x_j)\big\rangle = \partial^{\mu_1}\dots\partial^{\mu_n}\big\langle \mathcal{O}_{\Delta,\ell}(x)\prod_j \mathcal{O}_j(x_j)\big\rangle. \tag{2.38}$$

One therefore needs to know only the correlators of primary operators. The situation is in fact even simpler: the correlators of conformal primary operators can in principle be determined from a rather modest set of data [21, 160, 178, 182]. Let us discuss this in more detail.

It can be shown that the form of two-point functions is entirely fixed by conformal symmetry (see the discussion around (2.13)). For two primary operators $\mathcal{O}_{\Delta_i}$ and $\mathcal{O}_{\Delta_j}$ it fixes [e]

$$\big\langle \mathcal{O}_{\Delta_i}(x_i)\mathcal{O}_{\Delta_j}(x_j)\big\rangle = \frac{\delta_{ij}}{\big(x_{ij}^2\big)^{(\Delta_i+\Delta_j)/2}}, \tag{2.39}$$

where $x_{ij} = x_i - x_j$. The factor δ_{ij} implies that operators from different conformal families are orthogonal to each other. Moreover, the form of the three-point function is also fixed by conformal symmetry,

$$\big\langle \mathcal{O}_{\Delta_i}(x_i)\mathcal{O}_{\Delta_j}(x_j)\mathcal{O}_{\Delta_k}(x_k)\big\rangle = \frac{C_{ijk}}{(x_{ij}^2)^{(\Delta_i+\Delta_j-\Delta_k)/2}(x_{jk}^2)^{(\Delta_j+\Delta_k-\Delta_i)/2}(x_{ik}^2)^{(\Delta_i+\Delta_k-\Delta_j)/2}}, \tag{2.40}$$

where C_{ijk} is a non-trivial constant. The relations (2.39) and (2.40) are written for scalar operators. The form of correlators involving operators carrying spin was derived in [183] (see also [184] and [172, Chapter 2] for applications of the embedding formalism to this problem).

Conformal symmetry is not sufficient to fix the form of the four-point function entirely. The best one can write is (for simplicity restricting to operators with the identical scaling dimensions)

$$\big\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3)\mathcal{O}_\Delta(x_4)\big\rangle = \frac{g(u,v)}{x_{12}^{2\Delta}x_{34}^{2\Delta}}, \tag{2.41}$$

[d] From now on, we distinguish the canonical scaling dimension Δ_0, which can be read off from the Lagrangian and corresponds to the classical operator, from the full scaling dimension Δ, which corresponds to the quantized operator.

[e] It should be noted that conformal invariance does not fix the overall normalization constant, but primary operators can always be chosen so as to normalize it to δ_{ij}.

where u and v are the two *conformal cross-ratios*

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \tag{2.42}$$

which are invariant under conformal transformations. It is sometimes convenient to parametrize them with a single complex variable $z \in \mathbb{C}$ [23, 185],

$$u = z\bar{z}, \qquad v = (1-z)(1-\bar{z}), \tag{2.43}$$

which geometrically corresponds to mapping three of the four points conformally to $(0, 1, \infty)$ and working in the complex plane spanned by the remaining vector, 0 and 1.

The function $g(u, v)$ in (2.41) is, however, not the minimal data required to fix the four-point function. Indeed, one can apply the conformal OPE (2.36) to expand two of the operators as a sum of single operators,

$$\big\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3)\mathcal{O}_\Delta(x_4)\big\rangle = \sum_k C_k(x_{12})\big\langle \mathcal{O}_k\big(\tfrac{x_2}{2}\big)\,\mathcal{O}_\Delta(x_3)\mathcal{O}_\Delta(x_4)\big\rangle, \tag{2.44}$$

which means that four-point functions can be expressed as infinite sums of three-point functions. If all three-point constants C_{ijk} are known, the four-point function can be reconstructed via the conformal OPE. The Wilson coefficients C_k, due to the structure (2.35), can also be expressed in terms of three-point functions. For higher-point functions the situation is analogous, requiring multiple successive applications of the conformal OPE. This leads to the definition of *CFT data*,

$$\{\Delta_j, \ell_j, C_{ijk}\}, \tag{2.45}$$

i.e. the minimal set of information — scaling dimensions of primary operators together with their three-point function coefficients — that entirely defines the CFT. It should be noted, however, that not every such set (2.45) consistently defines a CFT, since additional constraints arise from properties such as unitarity and crossing symmetry. We, however, do not pursue this discussion further here.

To conclude this section, we summarize that the conformal symmetry of a quantum field theory leads to a significant simplification of the Hilbert space structure (2.31) and allows one to shift the discussion entirely to local operators (2.32), which can be organized into conformal families (2.37), i.e. primary operators together with their descendants. The emergence of conformal symmetry also clarifies the role of total-derivative operators (see (1.73)) and elevates the OPE from an asymptotic expansion (1.59) to an exact statement of the operator algebra (2.36). Let us also highlight an interesting consequence that will be important later: since in a unitary CFT the dilatation operator is Hermitian, its eigenvalues Δ are real and can be regarded as *CFT observables*, which constrains their structure. This in turn places constraints on the anomalous dimensions of operators, $\gamma(g)$.

2.3 Critical conformal invariance

In the previous section we discussed the rich structure of conformal theories. The general assumption throughout was that the theory is conformally invariant at the quantum level. Usually, a symmetry can be read directly from the Lagrangian and translates to a symmetry of the corresponding Hilbert space. Conformal symmetry is quite a different story, since quantum corrections generally break the conformal invariance of the theory. This breaking, however, turns out to be closely related to the renormalization procedure discussed in Section 1.3, as we will show in more detail in this section.

To make the discussion concrete, let us consider the simple example of the massless φ^4 scalar theory, described by the action

$$S(\varphi, g) = \int d^d x \left(\frac{1}{2}(\partial\varphi)^2 + \frac{g}{4!}\varphi^4\right). \tag{2.46}$$

We will use this theory to illustrate all the concepts that follow, but it should be noted that the conclusions can be translated to a larger class of theories. Let us first consider conformal symmetry at the classical level. It can be shown (see e.g. [164, Section 2.3]) that scale and conformal symmetry require, respectively,

$$\begin{aligned} T^\mu_\mu &= \partial_\mu V^\mu, \\ T^\mu_\mu &= 0, \end{aligned} \tag{2.47}$$

where V_μ is known as the *virial current*. Note that the first condition is weaker than the second, which reflects the fact that scale invariance does not in general imply conformal invariance. The canonical stress-energy tensor for the theory (2.46) takes the form

$$T^{\text{can}}_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \delta_{\mu\nu}\left(\frac{1}{2}(\partial\varphi)^2 - \frac{g}{4!}\varphi^4\right), \tag{2.48}$$

so the trace $T^{\text{can}}_{\mu\mu}$ is not automatically zero. The conditions (2.47) should, however, be imposed on-shell, and using the equation of motion one can show that

$$T^{\text{can}}_{\mu\mu}\Big|_{\text{on-shell}} = -(d/2-1)\partial_\mu\big(\varphi\partial^\mu\varphi\big) - (d-4)\frac{g}{4!}\varphi^4, \tag{2.49}$$

so the theory respects scale symmetry at $d=4$. This can in fact be read directly from the action (2.46), since in four dimensions the dimensionlessness of the action forces g to be dimensionless, and the theory therefore has no intrinsic scale.

The question of conformal symmetry at $d=4$ and the condition (2.47) is more subtle. At first sight only the first condition is satisfied, but the key observation is that the stress-energy tensor is not uniquely defined. It can be shown [186] (in analogy with the Belinfante tensor) that one can define the improved stress-energy tensor

$$T_{\mu\nu}\Big|_{d=4} = T^{\text{can}}_{\mu\nu} + \frac{1}{6}\big(\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\big)\varphi^2, \tag{2.50}$$

whose trace vanishes on-shell. One therefore concludes that the theory described by the action (2.46) is indeed classically conformally invariant.

Let us now turn to the quantum theory. After quantization, instead of the trace of the stress-energy tensor one must consider the correlator

$$\big\langle T^\mu_\mu(x)\prod_{j=1}^n\varphi(x_j)\big\rangle = \frac{1}{\mathcal{N}}\int D\varphi\, T^\mu_\mu(x)\prod_{j=1}^n\varphi(x_j)\, e^{-S(\varphi,g)}, \tag{2.51}$$

in which one cannot simply impose the on-shell condition for the trace of the stress-energy tensor. This trace measures the response of the action under a local Weyl transformation $\delta g_{\mu\nu}(x) = 2\sigma(x)g_{\mu\nu}(x)$, [a] namely

$$\delta_\sigma S = \int d^dx\,\sigma(x)T^\mu_\mu(x). \tag{2.52}$$

If the path integral in (2.51) is invariant under such a transformation, one can write

$$\big\langle T^\mu_\mu(x)\prod_{j=1}^n\varphi(x_j)\big\rangle = -\frac{\delta}{\delta\sigma(x)}\big\langle\prod_{j=1}^n\varphi(x_j)\big\rangle\Big|_{\sigma=0} = \Delta_\varphi\sum_{k=1}^n\delta^{(d)}(x-x_k)\big\langle\prod_{\substack{j=1\\ j\neq k}}^n\varphi(x_j)\big\rangle. \tag{2.53}$$

Since the right-hand side consists entirely of contact terms, we recognize that the symmetry at the quantum level follows from the classical one, and the theory is conformal. The invariance of the path

[a] Although we work in flat Euclidean space, it is convenient to imagine a coordinate-dependent metric to define this class of transformations.

integral (2.51) can be split into two requirements: the invariance of the action $S(\varphi, g)$ and the invariance of the measure $D\varphi$. The former is addressed by the classical analysis (2.47) and is satisfied in the theory (2.46). The latter is not guaranteed and is in general not true. Accounting for the change of the functional measure leads to terms known as the *Weyl anomaly* [187, 188] (see also [164, Chapter 3.2]), sometimes referred to as the *conformal anomaly* or *trace anomaly.* This approach leads to an elegant treatment in the style of the path integral derivation of the chiral anomaly [189], but we do not pursue this direction here.

Remaining within the framework of perturbative QFT, one can develop a different approach based entirely on the renormalization procedure discussed in Section 1.3, which can be shown to be consistent with the non-perturbative Weyl anomaly derivation [190]. The main idea can be stated very simply. In order to evaluate (2.51) perturbatively, one must renormalize the theory (2.46) via $S(\varphi, g) \mapsto S_R(\varphi, g)$, which introduces a dimensional parameter μ and thereby breaks scale invariance. In what follows, we discuss this approach in more detail and examine the resulting properties in the massless φ^4 theory (2.46). We also adopt here the more "pedestrian" approach, which may however be more convenient for specific applications.

The appearance of the trace of the stress-energy tensor in relation to scale invariance is given by the expression for the dilation current, J^D_μ, namely

$$J^D_\mu = x^\nu T_{\mu\nu} - V_\mu, \qquad \partial^\mu J^D_\mu = T_{\mu\mu} - \partial^\mu V_\mu, \tag{2.54}$$

where we have used the fact that the theory is translationally invariant and $\partial^\mu T_{\nu\mu} = 0$. Thus, instead of writing down the expression for the correlator with an insertion of T^μ_μ, one can directly check the Ward identity for the correlator of local fields under scale transformations, which we will refer to as the *scale Ward identity (SWI).* The pedestrian approach consists of the following: we consider the correlator of an arbitrary number of fields

$$G_n(x_1, \ldots, x_n) = \Big\langle \prod_{j=1}^n \varphi(x_j) \Big\rangle = \frac{1}{\mathcal{N}} \int D\varphi \prod_{j=1}^n \varphi(x_j)\, e^{-S(\varphi,g)}, \tag{2.55}$$

and view the scale transformation, $\varphi \mapsto \varphi + \varepsilon \left((x \cdot \partial) + \Delta^0_\varphi\right) \varphi$, as a change of variables in the path integral. The invariance of the measure [b] then leads to the statement

$$\sum_{k=1}^n \Big\langle \delta_D \varphi(x_k) \prod_{\substack{j=1 \\ j\neq k}}^n \varphi(x_j) \Big\rangle - \Big\langle \delta_D S \prod_{j=1}^n \varphi(x_j) \Big\rangle + O(\varepsilon) = 0. \tag{2.56}$$

Let us take a closer look at the scale variation of the action

$$\begin{aligned} \delta_D S(\varphi, g) &= \int d^d x \,\left(\partial_\mu \varphi \partial_\mu \delta_D \varphi + \frac{g}{6} \varphi^3 \delta_D \varphi \right) \\ &= \left(\Delta^0_\varphi - (d/2 - 1)\right) \int d^d x\, (\partial \varphi)^2 + \left(4\Delta^0_\varphi - d\right) \int d^d x \frac{g}{4!} \varphi^4, \end{aligned} \tag{2.57}$$

where we recall that Δ^0_φ is the canonical scaling dimension of the field φ, fixed by dimensional analysis as $\Delta^0_\varphi = d/2 - 1$, [c] so that at $d = 4$ both terms vanish. This result reproduces the fact that in $d = 4$ the action is scale invariant, and we obtain the simple scaling equation for the free Green's function

$$\left(\sum_{k=1}^n x^\mu_k \frac{\partial}{\partial x^\mu_k} + n \right) G_n(x_1, \ldots, x_n) \Big|_{d=4} = 0, \tag{2.58}$$

[b] Note that if we treat the change of variables in the path integral as only changing the field, the measure $D\varphi$ is invariant, in contrast with the Weyl transformation of the metric.

[c] Note that since we treat the scale transformation as a change of variables inside the path integral, one can, in principle, place an arbitrary number here.

which is nothing but the infinitesimal form of the condition $G_n(\lambda x_1,\dots,\lambda x_n)=\lambda^{-n}G_n(x_1,\dots,x_n)$.

Let us now show how renormalization changes this derivation. First, we introduce the renormalized action

$$S_R(\varphi,g)=\int d^dx\left(\frac{Z_1}{2}(\partial\varphi)^2+Z_3\mu^{2\epsilon}\frac{g}{4!}\varphi^4\right), \tag{2.59}$$

where μ is the renormalization scale and $Z_{1,3}=Z_{1,3}(\epsilon,g)$ are renormalization factors. Here we assume that we work in dimensional regularization, $d=4-2\epsilon$, and in the $\overline{\text{MS}}$-scheme (for details see Section 1.3). The Green's function under consideration is now assumed to be free of UV-divergencies and is defined as

$$G_n^R(x_1,\dots,x_n)=\big\langle\prod_{j=1}^n\varphi(x_j)\big\rangle=\frac{1}{\mathcal{N}}\int D\varphi\,\prod_{j=1}^n\varphi(x_j)\,e^{-S_R(\varphi,g)}, \tag{2.60}$$

but the further consideration of the scale Ward identity remains unchanged. What changes is the scale invariance of the action. Indeed, using result (2.57) we immediately arrive at

$$\delta_D S_R(\varphi,g)=-2\epsilon\int d^dx\left(Z_3\mu^{2\epsilon}\frac{g}{4!}\varphi^4\right)=-2\epsilon g\frac{\partial}{\partial g}\int d^dx\,\mathcal{L}_R(x), \tag{2.61}$$

and conclude that the scale invariance of the action is broken and the scaling behavior of the Green's function (2.58) is spoiled.

A remarkable property of Eq. (2.61) is that the variation of the action is given by a quite specific operator, $g\partial_g\mathcal{L}_R(x)$ (see [142, chapter 3, section 29]). The first observation is that this operator does not require additional renormalization. Indeed, for any number of fields

$$\int d^dx\,\big\langle g\partial_g\mathcal{L}_R(x)\prod_{j=1}^n\varphi(x_j)\big\rangle=-g\partial_g\big\langle\prod_{j=1}^n\varphi(x_j)\big\rangle, \tag{2.62}$$

which means that it is expressed through the ϵ-finite Green's function (taking the derivative in g does not spoil this property). On the other hand, this operator can be conveniently rewritten (see e.g. [57, Appendix B] for details of the derivation), relating it to the bare quantities, namely [d]

$$\partial_g\mathcal{L}_R(x)=\frac{\delta S}{\delta\varphi_0(x)}\frac{\partial\varphi_0(x)}{\partial g}+\frac{\partial\mathcal{L}(x)}{\partial g_0}\frac{\partial g_0}{\partial g}. \tag{2.63}$$

A simple RG analysis shows that

$$\frac{\partial\varphi_0}{\partial g}=\frac{\gamma_\varphi}{\beta(g)}\varphi_0, \qquad\qquad \frac{\partial g_0}{\partial g}=-\frac{2\epsilon}{\beta(g)}g_0, \tag{2.64}$$

so that

$$\beta(g)\partial_g\mathcal{L}_R(x)=\gamma_\varphi\varphi_0(x)\frac{\delta S}{\delta\varphi_0(x)}-2\epsilon g_0\partial_{g_0}\mathcal{L}(x). \tag{2.65}$$

The left-hand side of this equation is finite by virtue of (2.62), so on the right-hand side all singular contributions from counterterms must cancel, and we can replace the operators with their renormalized versions. Substituting this relation into (2.56) we get:

Scale Ward identity

The scale Ward identity (SWI) written for the renormalized Green's function, $G_n^R(x_1,\dots,x_n)$

[d] We recall that the connection between the renormalized and bare Lagrangian is in the variables and not the functional form, $\mathcal{L}_R(\varphi,g)=\mathcal{L}(\varphi_0,g_0)=\mathcal{L}(Z_\varphi\varphi,\mu^{2\epsilon}Z_g g)$.

takes the form

$$\left(\sum_{k=1}^{n} x_k^\mu \frac{\partial}{\partial x_k^\mu} + n\Delta_\varphi^0\right) G_n^R(x_1,\dots,x_n) = \beta(g)\int d^dx \big\langle \partial_g \mathcal{L}_R(x) \prod_{j=1}^{n}\varphi(x_j)\big\rangle - \gamma_\varphi \int d^dx \big\langle \varphi(x)\frac{\delta S_R}{\delta\varphi(x)}\prod_{j=1}^{n}\varphi(x_j)\big\rangle. \quad (2.66)$$

The classical scaling property (2.58) is broken on the quantum level.

This relation is of significant importance for two reasons:

- **Connection to RG-equation:**

  It is instructive to see that the SWI (2.66) is nothing more than the Callan–Symanzik equation (1.45) [191]. Indeed, rewriting the right-hand side in the form

$$\beta(g)\int d^dx \big\langle \partial_g \mathcal{L}_R(x)\prod_{j=1}^{n}\varphi(x_j)\big\rangle = -\beta(g)\frac{\partial}{\partial g} G_n^R(x_1,\dots,x_n), \quad (2.67)$$

$$\int d^dx \big\langle \varphi(x)\frac{\delta S_R}{\delta\varphi(x)}\prod_{j=1}^{n}\varphi(x_j)\big\rangle = \int d^dx \sum_{k=1}^{n}\delta^{(d)}(x-x_k)\big\langle\varphi(x)\prod_{\substack{j=1\\ j\neq k}}^{n}\varphi(x_j)\big\rangle = nG_n^R(x_1,\dots,x_n), \quad (2.68)$$

  we arrive at

$$\left(\sum_{k=1}^{n} x_k^\mu \frac{\partial}{\partial x_k^\mu} + \beta(g)\frac{\partial}{\partial g} + n(\Delta_\varphi^0+\gamma_\varphi)\right) G_n^R(x_1,\dots,x_n) = 0. \quad (2.69)$$

  The final step is to obtain the form of equation (1.45). We consider the scaling properties of the renormalized Green's function, namely, under rescaling it behaves as

$$G_n^R(\lambda x_1,\dots\lambda x_n;\mu,g) = \lambda^{-n\Delta_\varphi^0} G_n^R(x_1,\dots,x_n;\lambda\mu,g), \quad (2.70)$$

  which in infinitesimal form can be written as

$$\sum_{k=1}^{n} x_k^\mu \frac{\partial}{\partial x_k^\mu} G_n^R(x_1,\dots,x_n) = \left(\mu\frac{\partial}{\partial\mu} - n\Delta_\varphi^0\right) G_n^R(x_1,\dots,x_n). \quad (2.71)$$

  Thus, the Callan–Symanzik equation (1.45), which was derived in Section 1.3 as an equation expressing the independence of the bare Green's function from the renormalization scale, μ, has a deeper nature. It can be seen as a broken scale Ward identity in the renormalized theory.

- **Critical point:**

  The broken SWI (2.66) has the structure

$$\mathrm{SWI} = \beta(g)\big\langle[\varphi^4]\dots\big\rangle + \big\langle \mathrm{EOM}\dots\big\rangle. \quad (2.72)$$

  The appearance of EOM operators does not spoil scale symmetry at the quantum level relative to the classical one. Such contributions only shift the canonical scaling dimensions, see (2.67). The real problem is the emergent contribution from the local operator $\varphi^4(x)$. However, one can clearly see that if we consider the theory at the special value of the coupling, g^*, such that $\beta(g^*) = 0$, this term vanishes and scale symmetry is restored.

The corresponding value g^*, describing fixed points of the RG equation for the coupling, is usually referred to as *critical*. At this point, the Euclidean theory describes the fluctuation theory of a second-order phase transition and is indeed scale invariant. The idea of scaling in phase transitions (and the subsequent introduction of *critical exponents*) was introduced at an early stage of the field [192–194] (see also [195] for the application to correlation functions) and influenced QFT renormalization [196]. The above calculation makes this idea explicit.

Performing such a calculation, we have derived the main idea of this section: a renormalized QFT in which classical scale invariance is broken by quantum corrections (and by the renormalization procedure) enjoys scale invariance at the critical value of the coupling, g^*, such that $\beta(g^*) = 0$. This idea manifests as the breaking of the scale Ward identity by additional terms (2.66), which are however proportional to $\beta(g)$. In principle, the same analysis can be performed for the *conformal Ward identity (CWI)* [142, Chapter 4, Section 41], and the operator violating the Ward identity, $N^\mu(x)$, turns out to be related to the one violating the scale Ward identity, $N(x)$ ($\varphi^4(x)$ in (2.66)), as $N^\mu(x) = -2x^\mu N(x)$. It is natural to assume that at $\beta(g^*) = 0$ the CWI also holds, but it turns out to be sensitive to contributions of operators of the type $\partial^\mu \mathcal{O}$ (which can be associated with the virial current), which do not contribute to the SWI, and the conformal Ward identity is therefore not guaranteed. In some cases it is however possible to derive the absence of such contributions from dimensional analysis (e.g. a theory with one scalar field, or fermion and scalar without derivatives in the interaction). In full generality this question is not settled (see [164]), and only partial positive results are available, e.g. in two-dimensional unitary theories [162]. [e] In general, it seems reasonable to assume conformal invariance of critical points in unitary theories.

At the end of this section, let us focus on the critical condition $\beta(g^*) = 0$ in more detail and illustrate the corresponding ideas on the same example of (2.46). The β-function in this theory has been calculated to high accuracy in perturbation theory (see [198] for the 7-loop result) and has the form

$$\beta(g) = -2g\left(\epsilon - \frac{3}{2}g + \frac{17}{6}g^2 - \left(\frac{145}{16} + 6\zeta_3\right)g^3 + O(g^4)\right). \tag{2.73}$$

One critical point is trivial, $g^* = 0$, representing the fact that free theory is scale invariant. Another solution can be obtained perturbatively

$$g^*(\epsilon) = \frac{2}{3}\epsilon + \frac{68}{81}\epsilon^2 + \left(\frac{9433}{4374} + \frac{16}{9}\zeta_3\right)\epsilon^3 + O(\epsilon^4). \tag{2.74}$$

This is the IR critical point of *Wilson–Fisher* type [55]. Thus, the model (2.46) taken at the coupling (2.74) becomes scale invariant, and the other parameters evaluated at the critical coupling (e.g. $\gamma_\varphi^* = \gamma_\varphi\big|_{g=g^*}$) describe the critical scaling of different objects. The practical significance of the WF fixed point (2.74) is that the model (2.46) lies in the same universality class [195] as the Ising model (see e.g. [199] for a review), and all critical exponents of (2.46) therefore coincide with those of the Ising model. The true power of this connection is that the expansions for the critical exponents as series in ϵ [200] allow one to continue to $\epsilon \to 1/2$ [f] and obtain results for the physical model at $d = 3$. Remarkably, these results agree with experiment [201] and numerical simulations [202].

Given the previous discussion, the model (2.46) at the WF critical point (2.74) enjoys not only scale but also conformal symmetry. This allows one to view it as a conformal theory at the quantum level, with ϵ as the parameter of the perturbative expansion. This in turn allows one to use CFT methods

[e] The proof that scale invariance implies conformal invariance in two-dimensional unitary theories relies heavily on Zamolodchikov's c-theorem [161]. In four-dimensional unitary theories the analogous theorem has been proved (the a-theorem [197]), which provides strong evidence, but a fully rigorous proof is not yet available.

[f] The series expansion (2.74) is asymptotic, with radius of convergence equal to zero, so this step must be performed with a suitable resummation procedure, such as Borel resummation or Padé approximation.

to calculate critical exponents. For example, the numerical conformal bootstrap [168] yields predictions for the critical indices of the $d = 3$ Ising model with remarkable accuracy [203].

Let us also note another interesting property of the conformal theory at the critical coupling: in Section 2.2 we mentioned that the dilation operator $\mathbf{D}$ is Hermitian in CFT, so its eigenvalues correspond to CFT observables. This might seem contradictory with the anomalous dimension γ_φ, which in (2.46) depends on the subtraction scheme and therefore cannot be called an observable even in the formal sense. It can be shown (see [142, Chapter 1, Section 40]) that anomalous dimensions at the critical point are scheme independent. Indeed, consider a finite renormalization of the field in (2.46), $\varphi \mapsto Z'_\varphi(g,\epsilon)\varphi$, where Z'_φ is finite as $\epsilon \to 0$ (see (1.44)). The corresponding change in the field anomalous dimension is

$$\gamma_\varphi \mapsto \gamma_\varphi + \gamma'_\varphi, \qquad \gamma'_\varphi = \beta(g)\partial_g \ln Z'_\varphi(g,\epsilon), \tag{2.75}$$

which vanishes since $\beta(g^*) = 0$. The finiteness of the renormalization factor $Z'_\varphi(g,\epsilon)$ ensures that its derivative does not diverge as $g \to g^*$. This argument is of course not specific to φ^4, and critical anomalous dimensions are scheme independent in general.

In this section we discussed the mechanism by which classical conformal invariance is broken in QFT, through the violation of the scale (2.66) and conformal Ward identities. Scale invariance is however restored once the theory is placed at the critical value of the coupling, g^*, such that $\beta(g^*) = 0$. Under quite mild assumptions one can also expect that the corresponding critical theory is conformal. Here we discussed the case of the WF fixed point in φ^4 theory (2.46), but the same considerations apply in other theories, even those with a richer structure of RG flow (see e.g. the conformal window in QCD [67] or $\mathcal{N} = 1$ SYM [204]). From a technical point of view, this also leads to the following idea: one can tune the coupling in the QFT under consideration to its critical value (e.g. using the ϵ-expansion as in (2.74)) and thereby enhance the symmetry to conformal, which allows one to employ the full machinery of CFT (see Section 2.2). The question of to what extent one can translate results back to the physical coupling is, of course, problem dependent and in every situation requires additional consideration.

2.4 QCD and collinear conformal symmetry

In this section we will return to the problem discussed in Chapter 1 and focus our attention on QCD. In the case of massless quarks the QCD action (in Euclidean space) takes the form

$$S_{\rm QCD} = \int d^d x \left(\bar{q} \not{D} q + \frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} - \bar{c}^a \partial_\mu \left(D^\mu c\right)^a + \frac{1}{2\xi}(\partial_\mu A^{a,\mu})^2 \right), \tag{2.76}$$

where $q(x), \bar{q}(x)$ are quark fields, $D_\mu = \partial_\mu - \mathrm{i}\, g A^a_\mu T^a$ is the covariant derivative with T^a the generators of the $SU(N_c)$ group, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, $c(x)$ and $\bar{c}(x)$ are the ghost fields introduced by the Faddeev–Popov quantization, and $\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ is the gauge-fixing term. One can see that (despite the special treatment of gauge symmetry, see [27, Section 2.4.1 and Appendix B] for discussion) the action (2.76) is classically conformally invariant at $d = 4$. The UV renormalization procedure

$$q \mapsto Z_q q, \qquad A \mapsto Z_A A, \qquad c \mapsto Z_c c, \qquad g \mapsto \mu^\epsilon Z_g g, \qquad \xi \mapsto Z_\xi \xi, \tag{2.77}$$

where $Z_\xi = Z_A^2$ [107, Chapter 12], breaks scale (and therefore conformal) invariance, and we find ourselves precisely in the setting discussed in Section 2.3. Indeed, one can show [35–37] (see also [205] for the non-trivial application of this result) that the trace anomaly in QCD takes the form [a]

$$T^\mu_\mu = \frac{\beta(g)}{2g} F^a_{\mu\nu} F^{a,\mu\nu}, \qquad \beta(g) = \mu \frac{dg}{d\mu} = -g \left(\epsilon + \beta_0 \frac{\alpha_s}{4\pi} + \beta_1 \left(\frac{\alpha_s}{4\pi}\right)^2 + \dots \right). \tag{2.78}$$

[a] This statement should be understood via the insertion of the corresponding renormalized operators into an arbitrary Green's function.

The QCD beta function is non-vanishing [25, 26, 206, 207] (see also [208–210] for the state-of-the-art five-loop results) and

$$\beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f, \qquad\qquad \beta_1 = \frac{2}{3}\left(17C_A^2 - 5C_A n_f - 3C_F n_f\right), \tag{2.79}$$

where C_A, C_F are the Casimirs of the $SU(N_c)$ group in the adjoint and fundamental representations respectively. It is also convenient to introduce the normalized coupling $a = \alpha_s/4\pi$.

If one considers QCD in $d = 4 - 2\epsilon$ dimensions, the β-function acquires a non-trivial fixed point a^* such that $\beta(a^*) = 0$,

$$a^*(\epsilon) = -\frac{\epsilon}{\beta_0} - \left(\frac{\epsilon}{\beta_0}\right)^2 \frac{\beta_1}{\beta_0} + O(\epsilon^3). \tag{2.80}$$

If one wishes to use critical conformal invariance as a practical tool for calculations with the aim of translating results back to the physical coupling, $a^*(\epsilon) \mapsto a$, the condition of a large number of flavors (see [67] for details) is in principle not necessary. Indeed, in perturbative calculations the dependence on n_f is polynomial, so one can treat it as a free parameter and substitute the desired value only at the last step. We therefore conclude that unless one considers QCD (2.76) at the critical point (2.80), classical conformal symmetry is broken.

Although conformal symmetry is broken, one can observe that this breaking has a quite particular structure. The corresponding SWI (see [27, Eq. (B.12)] for the reference and also for the CWI) takes the form

$$\sum_{k=1}^{n}\left(x_k^\mu \frac{\partial}{\partial x_k^\mu} + \Delta^0_{\Phi_k}\right)\langle X_n\rangle = -\sum_{k=1}^{n}\gamma_{\Phi_k}\langle X_n\rangle - \frac{\beta(g)}{g}\big\langle\big[N^g\big]X_n\big\rangle - \sigma\big\langle\big[N^\xi\big]X_n\big\rangle, \tag{2.81}$$

where $X_n(x_1,\ldots,x_n)$ is a product of n local gauge-invariant operators $\Phi_j(x_j)$ with scaling dimensions $\Delta_{\Phi_j} = \Delta^0_{\Phi_j} + \gamma_{\Phi_j}$. The operators N^g and N^ξ describe the variation of the renormalized action (2.76) under scale transformations, and σ is a number expressible in terms of the anomalous dimension of the gauge-fixing parameter. The operator $\big[N^\xi\big]$ consists of EOM and BRST-exact operators, so it does not genuinely spoil the Ward identity (see e.g. [107, Chapter 12] and [211] for a discussion of correlators between BRST-exact and gauge-invariant operators). The remarkable property of the other violating term, N^g, is that it is accompanied by a factor which, at $\epsilon = 0$, is proportional to the coupling. This means that if one treats (2.81) perturbatively, the violating term at ℓ-th loop order contributes only at $(\ell+1)$-th loop order and higher. In particular, this implies that in one-loop calculations the conformal symmetry of the theory can be expected to hold exactly.

This statement is of course quite strong and in every case should be treated with care (especially when one considers Green's functions with composite operators), but the consequences can be read from direct calculations. A problem in which exact one-loop conformal symmetry plays an important role is the renormalization of non-local operators discussed in Section 1.4, known as the ERBL evolution equation [41, 42]. [b] Consider for simplicity the quark non-singlet operator (see (1.87))

$$\mathcal{O}_q(z_1,z_2) = \bar{q}(z_1 n)\gamma_+\big[z_1 n, z_2 n\big]q(z_2 n), \tag{2.82}$$

where we drop the flavor indices but assume the quark fields to be of different flavors to exclude mixing with the gluon operators. Suppose that QCD is exactly conformally invariant, so that the quark field $q(x)$ transforms as a primary operator (2.25). This fact, restricted to the light-cone (see (2.17)), should impose the transformation law

$$\mathcal{O}(z_1,z_2) \quad\mapsto\quad \mathcal{O}'(z_1,z_2) = \frac{1}{(cz_1+d)^{2j_q}}\frac{1}{(c'z_2+d')^{2j_q}}\mathcal{O}\left(\frac{az_1+b}{cz_1+d}, \frac{a'z_2+b'}{c'z_2+d'}\right), \tag{2.83}$$

[b] These results translates also on the evolution equations for GPDs, but we do not focus on these details here.

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2,\mathbb{R}), \tag{2.84}$$

and $j_q = (3/2 + 1/2)/2 = 1$ is the conformal spin of the quark field.

To be rigorous, we should note one subtlety with quark operators. The quark field $q(x)$ is not an eigenstate of the projection of the generator of rotations, Σ_{+-}, entering (2.17), so one should first decompose it as

$$q(x) = q_+(x) + q_-(x) = \frac{1}{2}\gamma_-\gamma_+ q(x) + \frac{1}{2}\gamma_+\gamma_- q(x). \tag{2.85}$$

The two projections then have different *collinear spins*, $s_q = \{1/2, -1/2\}$, defined by

$$\Sigma_{+-} q_+(x) = \frac{1}{2} q_+(x), \qquad \Sigma_{+-} q_-(x) = -\frac{1}{2} q_-(x). \tag{2.86}$$

This discussion does not, however, affect the transformation property (2.83), as the operator (2.82) consists only of one type of contribution due to $\gamma_+ q_-(x) = \bar{q}_-(x)\gamma_+ = 0$.

The natural further assumption is that the symmetry (2.83) imposes constraints on the mixing of local operators appearing in the expansion of the operator (1.90). Let us discuss this question in more detail. Consider the bilocal operator

$$\mathcal{O}(z_1, z_2) = \Phi_{j_1}(z_1)\Phi_{j_2}(z_2), \tag{2.87}$$

where $\Phi_{j_1}(z_1)$ and $\Phi_{j_2}(z_2)$ are primary conformal operators labeled by conformal spin j_k, i.e.

$$\left[\mathbf{L}_-^{(k)}, \Phi_{j_k}(0)\right] = 0, \qquad \left[\mathbf{L}_0^{(k)}, \Phi_{j_k}(0)\right] = j_k \Phi_{j_k}(0), \tag{2.88}$$

where the generators of the collinear conformal group are given by (2.21) and the superscript refers to the variable they act on. Each of these operators spans a set of descendants corresponding to the representation $\mathcal{D}^+_{j_k}$ [212] from the discrete series of representations of $SL(2,\mathbb{R})$,

$$\mathcal{O}_n^{(j_k)}(0) = \left[\mathbf{L}_+^{(k)}, \left[\mathbf{L}_+^{(k)}, \ldots, \left[\mathbf{L}_+^{(k)}, \Phi(0)\right]\right]\right] = (-\partial_+)^n \Phi_{j_k}(0). \tag{2.89}$$

The operator $\Phi_{j_k}(z_k)$ can then be expanded in the series of local operators as

$$\Phi_{j_k}(x_k) = \sum_{n=0}^{\infty} \frac{(-z_k)^n}{n!} \mathcal{O}_n^{(j_k)}(0). \tag{2.90}$$

The product of two bilocal operators (2.87) decomposes according to the tensor product of two representations, namely

$$\mathcal{D}^+_{j_1} \otimes \mathcal{D}^+_{j_2} = \bigoplus_{n=0}^{\infty} \mathcal{D}^+_{j_1+j_2+n}. \tag{2.91}$$

Thus, in the expansion of the product of two local operators we expect to find an infinite series of primary operators, $\{\mathbb{O}_n^{(j_1,j_2)}\}_{n=0}^{\infty}$, being lowest-weight states in the representation $\mathcal{D}^+_{j_1+j_2+n}$, which corresponds to

$$\begin{aligned}
\left[\mathbf{L}^2, \mathbb{O}_n^{(j_1,j_2)}(0)\right] &= (j_1+j_2+n)(j_1+j_2+n-1)\mathbb{O}_n^{(j_1,j_2)}(0), \\
\left[\mathbf{L}_0, \mathbb{O}_n^{(j_1,j_2)}(0)\right] &= (j_1+j_2+n)\mathbb{O}_n^{(j_1,j_2)}(0), \\
\left[\mathbf{L}_-, \mathbb{O}_n^{(j_1,j_2)}(0)\right] &= 0,
\end{aligned} \tag{2.92}$$

where the generators are given by the sum of the corresponding generators in (2.88), $\mathbf{L}_\alpha = \mathbf{L}_\alpha^{(1)} + \mathbf{L}_\alpha^{(2)}$, $\alpha = \{+,-,0\}$. Such operators can be explicitly constructed (see [27, Section 2.2] for details) and the result, translated to arbitrary x, reads [43]

$$\mathbb{O}_n^{(j_1,j_2)}(x) = \partial_+^n \left(\Phi_{j_1}(x) P_n^{(2j_1-1,2j_2-1)} \left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+} \right) \Phi_{j_2}(x) \right), \tag{2.93}$$

where $P_n^{(2j_1-1,2j_2-1)}$ is a Jacobi polynomial [155, §6] appearing here as the Clebsch–Gordan coefficient for representations from the discrete series of $SL(2,\mathbb{R})$. The conformal operators (2.93) together with their descendants

$$\mathbb{O}_{n,n+k}^{(j_1,j_2)} = \left[\mathbf{L}_+, \left[\mathbf{L}_+, \ldots, \left[\mathbf{L}_+, \mathbb{O}_n^{(j_1,j_2)}\right]\right]\right] = (-\partial_+)^k \mathbb{O}_n^{(j_1,j_2)}, \tag{2.94}$$

span the entire expansion of the bilocal operator (2.87) in terms of local operators.

Let us now return to the quark non-local operator (2.82). As discussed in (1.90), this operator generates the local twist-2 operators

$$\mathcal{O}_q(z_1,z_2) = \sum_{s,k} \Psi_{s,k}(z_1,z_2)\mathcal{O}_{s,k}(0). \tag{2.95}$$

The above discussion then shows that one can choose a basis of local operators corresponding to the conformal operators (2.93) and their descendants. In the case of the quark operator with conformal spin $j_q = 1$ for each quark field, the resulting basis takes the form

$$\mathbb{O}_{n,n+k}^q(x) = (-1)^k (\partial_+)^{n+k} \left(\bar{q}(x) C_n^{(3/2)} \left(\frac{\overrightarrow{D}_+ - \overleftarrow{D}_+}{\overrightarrow{D}_+ + \overleftarrow{D}_+} \right) q(x) \right), \tag{2.96}$$

which coincides with the Gegenbauer basis (1.93) discussed in Section 1.4. It therefore admits the interpretation of operators transforming covariantly under collinear conformal transformations. Let us describe the consequences of working in this basis for renormalization. If the collinear conformal symmetry is exact, the anomalous dimension matrix in the basis (2.96) is diagonal. Indeed, consider the operators $\mathbb{O}_{n,n+k}^q$ and $\mathbb{O}_{n+k,n+k}^q$. They have the same canonical scaling dimension and the same number of total derivatives and can therefore mix with each other. However, the $SL(2,\mathbb{R})$ symmetry prohibits such mixing, as they belong to different discrete series, $\mathcal{D}_{2+n}^+$ and $\mathcal{D}_{2+n+k}^+$, respectively. The diagonal structure of mixing in this basis is indeed realized in QCD at one-loop level [43, 45, 213] (the diagonalization by Gegenbauer polynomials was already noticed in [44]) and breaks down at two loops and beyond [214, 215] (see [216, 217] for the direct two-loop ERBL kernel calculations).

Starting from two loops, conformal symmetry can no longer be used directly in calculations. Nevertheless, working in the Gegenbauer basis (2.96) still provides considerable advantages. As discussed in relation to the breaking of the SWI (2.81), the off-diagonal part of the two-loop anomalous dimension matrix must be proportional to the β-function and can therefore be read from the one-loop result. There has been substantial work in this direction [52–54, 218–220] (see also [221] for a discussion in the six-dimensional φ^3 model), and it becomes clear that working in the basis of local operators (2.96) is not the most transparent way to exploit this structure. In what follows, we adopt a different approach [56, 89, 90, 92, 222], working directly at the level of non-local operators and considering evolution kernels (1.98) instead of the anomalous dimension matrix.

Let us begin by reviewing the one-loop conformal symmetry in this language. The evolution kernel corresponding to (2.82) can be expanded in a perturbative series

$$\mathbb{H}(a) = a\mathbb{H}^{(1)} + a^2\mathbb{H}^{(2)} + O(a^3), \tag{2.97}$$

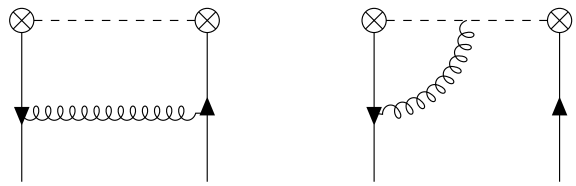


Figure 2.1: Feynman diagrams contributing to the one-loop evolution kernel $\mathbb{H}^{(1)}$.

and the corresponding one-loop result (see Fig. 2.1 for the contributing diagrams) reads [88]

$$\begin{aligned}\left[\mathbb{H}^{(1)} f\right](z_1,z_2) = 4C_F\Bigg(&\int_0^1 d\alpha \frac{\bar\alpha}{\alpha}\Big(2f(z_1,z_2) - f(z_{12}^\alpha, z_2) - f(z_1, z_{21}^\alpha)\Big)\\ &-\int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\, f(z_{12}^\alpha, z_{21}^\beta) + \frac{1}{2} f(z_1,z_2)\Bigg),\end{aligned} \tag{2.98}$$

where we use the notation (1.89). The infinitesimal version of the transformation (2.83) can be written by considering the set of differential operators forming the representation (2.21)

$$\begin{aligned} S_-^{(0)} &= -\partial_{z_1} - \partial_{z_2},\\ S_0^{(0)} &= z_1\partial_{z_1} + z_2\partial_{z_2} + 2,\\ S_+^{(0)} &= z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + 2(z_1+z_2).\end{aligned} \tag{2.99}$$

The meaning of the superscript (0) will become clear shortly. One can verify explicitly that the one-loop evolution kernel (2.98) commutes with the collinear conformal generators (2.99). This can also be seen from the fact that (apart from the $\delta(\alpha)\delta(\beta)$ contributions, which correspond to the identity operator and obviously commute with the generators) the integral kernel of (2.98), $h^{(1)}(\alpha,\beta)$, depends only on the *conformal ratio* τ, namely

$$h^{(1)}(\alpha,\beta) = -4C_F\Big(\delta_+(\tau) + \theta(1-\tau) - \frac{1}{2}\delta(\alpha)\delta(\beta)\Big), \qquad \tau = \frac{\alpha\beta}{\bar\alpha\bar\beta}, \tag{2.100}$$

where the regularized δ-function, $\delta_+(\tau)$, is defined by

$$\begin{aligned}\int_0^1 d\alpha\int_0^1 d\beta\,\delta_+(\tau)\,f(z_{12}^\alpha, z_{21}^\beta) &= \int_0^1 d\alpha\int_0^1 d\beta\,\delta(\tau)\,\big(f(z_{12}^\alpha, z_{21}^\beta) - f(z_1,z_2)\big)\\ &= -\int_0^1 d\alpha\frac{\bar\alpha}{\alpha}\big(2f(z_1,z_2) - f(z_{12}^\alpha,z_2) - f(z_1,z_{21}^\alpha)\big).\end{aligned} \tag{2.101}$$

This is consistent with the anomalous dimension matrix being diagonal in the conformal basis. Indeed, using the property that the diagonal entries of the anomalous dimension matrix are eigenvalues of the evolution kernel (see (1.99)), one can reconstruct the integral kernel (2.100) from its moments (see Chapter 5 and specifically Section 5.4.3 for an explicit example of such a reconstruction). If the integral kernel is a genuine function of two variables, such a reconstruction is not possible, which corresponds to the fact that the evolution kernel carries more information (i.e. the off-diagonal parts).

Thus, in the language of non-local operators, collinear conformal symmetry manifests as the commutativity of the evolution kernel (2.98) with all generators of this symmetry (2.99). As already discussed, this property is broken starting from two loops (in particular, it is no longer possible to express $h^{(2)}(\alpha,\beta)$

solely as a function of the conformal ratio), but traces of this symmetry are still present. Here the idea of considering the theory at the critical point yields a significant advantage. Indeed, let us fix the coupling to its critical value, $a \mapsto a^*$, (2.80). According to the discussion in Section 2.3, exact conformal symmetry is restored and all arguments about the properties of the evolution kernel apply again. One can show (see [89, Section 3] for the detailed discussion) that the generators (2.99) then receive quantum corrections $\Delta S_\alpha(a^*)$, and the symmetry under collinear conformal transformations is expressed as

$$\big[S_\alpha(a^*), \mathbb{H}(a^*)\big] = 0, \qquad S_\alpha(a^*) = S_\alpha^{(0)} + \Delta S_\alpha(a^*), \qquad \alpha \in \{+,-,0\}. \tag{2.102}$$

A remarkable property of the $\overline{\mathrm{MS}}$ scheme (see the more detailed discussion in Section 4.1) is that the operators $\mathbb{H}^{(k)}$ in the perturbative expansion (2.97) are d-independent, so one can extend equation (2.102) to arbitrary coupling.

Deformed generators of collinear $SL(2,\mathbb{R})$ symmetry

Generators of $SL(2,\mathbb{R})$ symmetry of the evolution equation (2.106) at arbitrary coupling a have the form

$$\begin{aligned}
S_-(a) &= S_-^{(0)}, \\
S_0(a) &= S_0^{(0)} + \bar\beta(a) + \frac{1}{2}\mathbb{H}(a), \\
S_+(a) &= S_+^{(0)} + (z_1+z_2)\left(\bar\beta(a) + \frac{1}{2}\mathbb{H}(a)\right) + z_{12}\Delta(a),
\end{aligned} \tag{2.103}$$

where $\Delta(a)$ is an integral operator usually called the *conformal anomaly*. It can be calculated perturbatively (see [57, 91] for the detailed calculations).

In the one-loop approximation the conformal anomaly reads

$$\big[\Delta^{(1)} f\big](z_1,z_2) = -2C_F\int_0^1 d\alpha\, \left(\frac{\bar\alpha}{\alpha} + \ln\alpha\right)\big(f(z_{12}^\alpha, z_2) - f(z_1, z_{21}^\alpha)\big). \tag{2.104}$$

Let us comment on the result obtained. The name "conformal anomaly" for the correction term in the generators (2.103) may be somewhat misleading. The symmetry of the evolution kernel $\mathbb{H}(a)$ under transformations generated by (2.103) is exact. Moreover, one can show explicitly [92, 101] that there exists an operator $\mathrm{V}(a)$ intertwining the representations of the deformed generators (2.103) and the canonical generators (2.21),

$$S_\alpha^{(0)} = \mathrm{V}(a) S_\alpha(a) \mathrm{V}^{-1}(a). \tag{2.105}$$

The symmetry is not broken by the conformal anomaly, rather, the appearance of this term is precisely what ensures the presence of the symmetry. The important point, however, is to interpret this symmetry correctly. It would be wrong to say that the collinear conformal symmetry of the theory is exact beyond one-loop order, what is correct to say is that the evolution equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a} + \mathbb{H}(a)\right)\big[\mathcal{O}_q(z_1,z_2)\big] = 0 \tag{2.106}$$

with the evolution kernel $\mathbb{H}(a)$ written in the $\overline{\mathrm{MS}}$ scheme possesses a hidden symmetry that is exact at any loop order.

Thus, in this section we have shown how the classical conformal symmetry of the QCD action (2.76) can be exploited as a significant advantage in the problem of renormalizing non-local operators of the type (2.82). The specific pattern of conformal symmetry breaking (2.81) allows one to effectively decompose the evolution kernel (2.97) at ℓ-th loop order into two parts: $\mathbb{H}^{(\ell)} = \mathbb{H}^{(\ell)}_{\text{inv}} + \mathbb{H}^{(\ell)}_{\text{non-inv}}$, where the "conformal" part $\mathbb{H}^{(\ell)}_{\text{inv}}$ is symmetric under the exact transformations of the collinear conformal group (2.21), while the "broken" part $\mathbb{H}^{(\ell)}_{\text{non-inv}}$ is entirely determined by the evolution kernel at the previous loop order, $\mathbb{H}^{(\ell-1)}$. The idea of considering the theory at the critical point, $a = a^*(\epsilon)$, (2.80) allows one to organize this structure in the form of an exact symmetry of the evolution equation (2.106) under the deformed generators (2.103). Let us also mention that this is far from exhausting all applications of conformal symmetry in QCD. One can, for example, consider the analogue of (2.87) for three-parton operators [223] (see also [224, 225] for a discussion of the integrable structures that arise). For further details and additional examples see [27].

Chapter 3

Perturbative calculations and Feynman diagrams

So far we have been focusing on the conceptual aspects of the problems related to the study of twist-2 operators. While Chapter 2 provided a promising viewpoint on the connection of twist-2 operators with conformal symmetry, Chapter 1 resulted in a quite practical demand for their renormalization. This need manifests as a concrete question: how to calculate anomalous dimensions for twist-2 operators in the perturbative expansion. The suitable framework for such calculations is, of course, well-known — Feynman diagrams.

The calculation of Feynman diagrams is a very broad and well-established field of study (see [226] for one of the most comprehensive books on the topic). In precision high-energy physics, to which this thesis is naturally related, a certain canon has been established. One first attacks Feynman integrals with *integration-by-parts* (IBP) identities [65] to reduce them to a set of master integrals. This step is highly algorithmized [227] and a large market of automated solutions exists (see e.g. `FIRE` [228], `Reduze` [229], `LiteRed` [230], `Kira` [231]). The remaining set of master integrals is then evaluated primarily through the *differential equation* method [232–235]. This approach has proven extremely successful and, with sufficient computational resources, produces state-of-the-art results. The problem of calculating twist-2 anomalous dimensions is, however, quite specific. On the one hand, it requires the calculation of massless diagrams [236] (accounting for masses in Feynman integrals often leads to elliptic structures, see e.g. [237, 238]) and only their divergent contributions, which naturally simplifies the problem. On the other hand, twist-2 operators introduce tensor structures at the level of Feynman graphs, which drastically complicates the calculations. [a] These two properties make the "pen and paper" approach competitive, especially in situations involving not so many but quite complex Feynman integrals, where one can exploit methods and identities that are still difficult to automatize (see e.g. [142, 236, 242] for a pedagogical overview).

In this chapter we give an overview of the techniques that can be used for the calculation of twist-2 anomalous dimensions. For the forward anomalous dimensions, the main tool is the *star-triangle relation* [243–245]. We will also briefly discuss implementations of computer algebra tools such as `Forcer` [246] and their limitations. The case of diagrams with non-local operator insertions will also be addressed, as it likewise does not fit well into the standard IBP-based approach. The purpose of this chapter is to introduce the formalism and provide some instructive examples.

[a] For completeness it should be noted that it is possible to overcome this problem by introducing a generating function for different spins and treating it as an additional linear propagator, see [239–241] for details, though this of course introduces a different kind of complexity.

3.1 Setup for perturbative calculations and Feynman rules

Let us begin by setting up the problem of calculating the anomalous dimensions of twist-2 operators. In order to illustrate all the techniques in a more concrete way, let us consider the $(d = 6-2\epsilon)$-dimensional Euclidean φ^3 model

$$S = \int d^d x \left(\frac{1}{2}(\partial\varphi)^2 + \frac{g}{3!}\varphi^3\right). \tag{3.1}$$

This is sufficient to introduce all the relevant procedures and to organize an introductory discussion, though it will not cover some peculiarities of gauge theories (see for example the discussion on the so-called alien operators [247–254]). The leading-twist operators in this model take the form

$$\mathcal{O}_s(x) = \varphi(x)\partial_+^s\varphi(x) - \text{traces}. \tag{3.2}$$

In the theory (3.1), the operators (3.2) have twist $\tau = 4$, in contrast to four-dimensional theories, but as leading-twist operators they share the same properties. We have already projected the operators onto the light-cone vector for convenience (see (1.76) and the discussion around (1.81)). Introducing the renormalization procedure gives the action

$$S_R = \int d^d x \left(\frac{Z_1}{2}(\partial\varphi)^2 + Z_3\mu^\epsilon\frac{g}{3!}\varphi^3\right), \tag{3.3}$$

where $\varphi_0 = Z_\varphi\varphi$ and $g_0 = Z_g\mu^\epsilon g$, so that $Z_1 = Z_\varphi^2$ and $Z_3 = Z_\varphi^3 Z_g$. It is also convenient to introduce the normalized coupling, $u = g^2/(4\pi)^3$. According to the discussion in Section 1.3, the operators (3.2) require additional renormalization

$$\big[\mathcal{O}_s(x)\big] = Z(s)\mathcal{O}_s(x), \tag{3.4}$$

where the factor $Z(s)$ in the $\overline{\text{MS}}$ scheme takes the form

$$Z(s) = 1 + \sum_{k=1}^{\infty}\frac{Z_k(s)}{\epsilon^k} = 1 + \sum_{n=1}^{\infty}u^n\sum_{k=1}^{n}\frac{Z_k^{(n)}(s)}{\epsilon^k}. \tag{3.5}$$

In practice we are interested in the anomalous dimension of the operator (3.2), which reads [b]

$$\gamma(s) = -\mu\frac{d\ln Z(s)}{d\mu} = 2u\partial_u Z_1(s). \tag{3.6}$$

This formula provides a clear recipe: one calculates the renormalization factor, $Z(s)$ (3.4), extracts the simple pole ($\sim 1/\epsilon$), and uses (3.6) to obtain the anomalous dimension $\gamma(s)$.

Let us comment on the presence of mixing (see (1.50) and the discussion therein). The only operators that (3.2) can mix with are total-derivative operators. However, due to the block-diagonal structure of mixing (1.71) we can omit these details and consider multiplicative renormalization (3.4). This also places a constraint on the values of s. Indeed,

$$\varphi\partial_+^s\varphi = (-1)^s(\partial_+^s\varphi)\varphi + \text{ total derivatives.} \tag{3.7}$$

Thus, if s is *odd*, the operator (3.2) is a total derivative and can therefore be omitted. In what follows we always assume that s is *even*.

The renormalization constant $Z(s)$ is defined by the requirement that (see (1.51))

$$\forall N: \quad Z(s)\big\langle\mathcal{O}_s(x)\prod_{j=1}^{N}\varphi(x_j)\big\rangle \quad \overset{\epsilon\to 0}{\longrightarrow} \quad \text{finite}, \tag{3.8}$$

[b] Note that $\beta(u))/u = 2\beta(g)/g$.

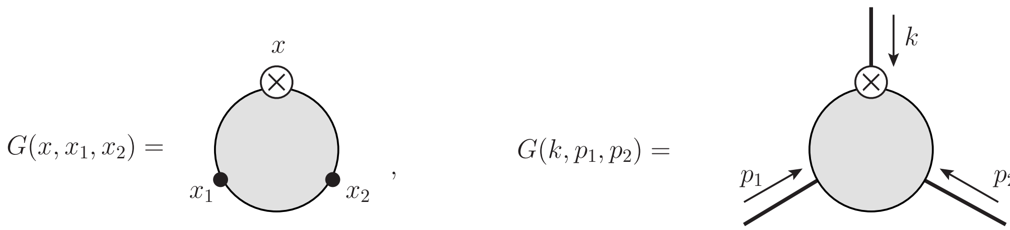


Figure 3.1: Representation of the Green's function (3.11) in coordinate and momentum space. The symbol $\otimes$ denotes an operator insertion.

where the matrix element is taken with the renormalized action (3.3). This formula gives the practical recipe for calculating $Z(s)$. Indeed, expanding (3.8) one obtains

$$\sum_{n=0}^{\infty} u^n \sum_{k=0}^{n} Z^{(k)}(s) \big\langle \mathcal{O}_s(x) \prod_{j=1}^{N} \varphi(x_j) \big\rangle_{n-k} = O(\epsilon^0), \tag{3.9}$$

where $\langle \dots \rangle_\ell$ denotes the Green's function at ℓ-th order in the perturbative expansion and $Z^{(0)}(s) = 1$. The expansion (3.9) can then be viewed as a system of equations for $Z^{(\ell)}(s)$, to be solved order by order.

In practical calculations it is of course impossible to consider infinite number of Green's functions, but this is not necessary. It can be shown (see [142, Chapter 1]) that to satisfy (3.8) it is sufficient to consider only *one-particle-irreducible* (1PI) graphs and then add the contribution absorbed by the field renormalization, namely for the operator (3.2)

$$\gamma(s) = 2\gamma_\varphi - \mu \frac{d}{d\mu} \ln Z(s)\Big|_{1\mathrm{PI}}, \tag{3.10}$$

where $Z(s)\big|_{1\mathrm{PI}}$ denotes the factor derived by considering only 1PI graphs. Specifically for the operator (3.2) it is then sufficient to consider [c]

$$G(x, x_1, x_2) = \big\langle \mathcal{O}_s(x) \varphi(x_1) \varphi(x_2) \big\rangle_{1\mathrm{PI}}. \tag{3.11}$$

Indeed, taking into account Green's functions with a higher number of fields $\varphi(x_j)$ leads to the same set of 1PI graphs. In practice, Feynman diagrams are usually considered in momentum space, so it is convenient to define

$$G(k, p_1, p_2) = \int d^d x \, e^{\mathrm{i}\, kx} \int d^d x_1 \, e^{\mathrm{i}\, p_1 x_1} \int d^d \, x_2 e^{\mathrm{i}\, p_2 x_2} \big\langle \mathcal{O}_s(x) \varphi(x_1) \varphi(x_2) \big\rangle. \tag{3.12}$$

The choice of signs in the exponents corresponds to all momenta being incoming (see Fig. 3.1).

A convenient feature of renormalization factors $Z(s)$ in the $\overline{\mathrm{MS}}$ scheme is that they are independent of external momenta or coordinates (see the locality of counterterms [255]), so one can in principle assign arbitrary values to the coordinates and momenta. Why not then set the external momenta in (3.12) to zero in order to simplify calculations? The reason is the possible appearance of IR divergencies, since $k \to 0$ corresponds to the on-shell condition for a massless particle. These additional divergencies are regularized in $d = 6 - 2\epsilon$, but they interfere with the ϵ-dependence of the renormalization factors. One must therefore proceed with care and impose specific constraints only after analysing potential additional divergencies. In practice it is usually sufficient to consider

$$\begin{aligned} G(x) &= \int d^d z \big\langle \mathcal{O}_s(z) \varphi(x) \varphi(0) \big\rangle, \\ G(p) &= \int d^d x \, e^{\mathrm{i}\, px} G(x), \end{aligned} \tag{3.13}$$

[c]Here we highlight the 1PI contribution to the Green's function, but in what follows we omit this subscript, assuming all Green's functions to be 1PI unless stated otherwise.

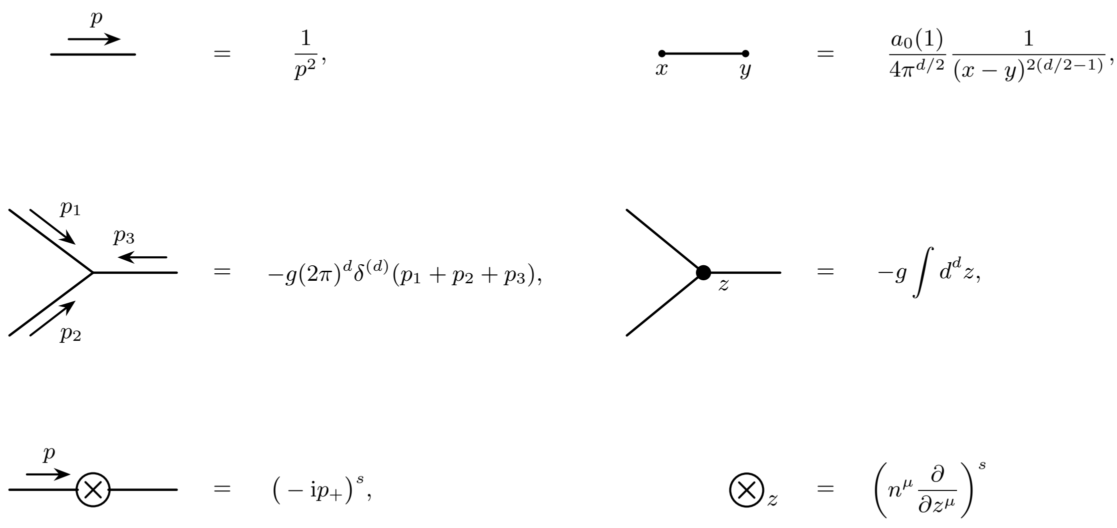


Figure 3.2: Feynman rules for the theory (3.1), together with operator insertions, in both momentum and coordinate space. In coordinate space, external vertices are represented by small dots, while internal vertices are depicted by larger dots.

which restricts our consideration to diagrams of *self-energy* (two-point) type. Another way of dealing with the same issue is to introduce masses in the propagators as an IR regulator (see e.g. [256–258] for applications of this method). This choice allows one to freely set external momenta to zero, but in turn requires working with massive Feynman diagrams.

A convenient feature of massless theories is that the propagator has the same functional form in both coordinate and momentum representations. Indeed, calculating the Fourier transform gives

$$\int d^d x \, \frac{1}{x^{2\alpha}} e^{\mathrm{i}\,px} = \pi^{d/2} 2^{d-2\alpha} a_0(\alpha) \frac{1}{p^{2(d/2-\alpha)}}, \tag{3.14}$$

where we introduce the convenient notation

$$a_n(\alpha) = \frac{\Gamma(d/2-\alpha+n)}{\Gamma(\alpha)}. \tag{3.15}$$

Thus, if we generalize the Feynman diagram technique by allowing propagators to carry an arbitrary power α, the integrals in coordinate and momentum space take the same functional form, so depending on the calculation it may be convenient to switch between the two representations. The parameter $\alpha \in \mathbb{C}$ can in principle be any complex number, and in all problematic cases the Feynman integral, as a function of α, is defined via analytic continuation. In what follows we refer to α as an *index* of the line. This connection between momentum and coordinate integrals works particularly well in combination with the *dual transform*, which is available for planar graphs (see [242, Section 1.6] and [236, Section 4.5]).

Using the definition (3.13) and the standard construction of perturbative expansions, it is straightforward to derive the Feynman rules, see Fig. 3.2. Let us point out one important feature of calculations involving operator insertions. Consider (3.13) in the *leading order* (LO) approximation (where

$\lambda = d/2 - 1$):

$$G_0(x) = \int d^d z \left(\langle \varphi(z)\partial_+^s \varphi(z)\varphi(x)\varphi(0)\rangle_0 + \langle \varphi(z)\partial_+^s \varphi(z)\varphi(x)\varphi(0)\rangle_0 \right)$$
$$= \frac{a_0(1)}{4\pi^{d/2}} \int d^d z \left(\frac{1}{(x-z)^{2\lambda}} \partial_+^s \frac{1}{z^{2\lambda}} + \frac{1}{z^{2\lambda}} \partial_+^s \frac{1}{(x-z)^{2\lambda}} \right)$$
$$= (-1)^s 2^s \frac{\Gamma(\lambda+s)}{\Gamma(\lambda)} \frac{a_0(1)}{4\pi^{d/2}} \int d^d z \left(\frac{z_+^s}{z^{2s}} + \frac{(z-x)_+^s}{(z-x)^{2s}} \right) \frac{1}{(z-x)^{2\lambda} z^{2\lambda}}. \tag{3.16}$$

This calculation shows that one must be careful with operator insertions treated as derivatives, with respect to which leg the derivative acts (this also translates to the momentum representation). Note, however, that the two contributions in (3.16) are related by a total derivative. In the renormalization scheme (3.4) we omit mixing with total derivatives, so it is convenient to take only one contribution from (3.16). This does not lead to any problem as long as it is done consistently.

Recipe for calculation of leading-twist anomalous dimensions

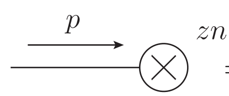
Given all the considerations, we are ready to present the procedure for calculating the anomalous dimension $\gamma(s)$ at ℓ-th perturbative order:

1. Consider the *amputated* Green's function (3.13) at ℓ-th order
$$\widetilde{G}^{(\ell)}(p) = p^4 G^{(\ell)}(p)\Big|_{1\mathrm{PI}}. \tag{3.17}$$

2. For every Feynman diagram $\mathcal{D}(s,p)$ appearing in the expansion of $\widetilde{G}^{(\ell)}$, calculate its contribution to $Z^{(\ell)}(s)$:
$$\Delta Z^{(\ell)}(s) = -\,\mathrm{KR}'\left\{\mathcal{D}(s,p)\right\}/\widetilde{G}^{(0)}(p), \tag{3.18}$$
where the operation $\mathrm{KR}'\{\dots\}$ stands for extracting all parts singular in ϵ after subtracting all subdivergencies (see [142, Chapter 3] for details on the R-operation).

3. Collect the result for the anomalous dimension:
$$\gamma^{(\ell)}(s) = 2\gamma_\varphi^{(\ell)} + 2\ell\, Z_1^{(\ell)}(s). \tag{3.19}$$

The same procedure translates very straightforwardly to the level of non-local operators. Indeed, consider the operator ($z_{1,2} \in \mathbb{R}$)
$$\mathcal{O}(z_1, z_2) = \varphi(z_1 n)\varphi(z_2 n), \tag{3.20}$$
where we recall that n^μ is an auxiliary light-like vector ($n^2 = 0$). The operator renormalization is then defined as
$$\big[\mathcal{O}(z_1,z_2)\big] = \mathbb{Z}\,\mathcal{O}(z_1,z_2) = \int_0^1 d\alpha \int_0^1 d\beta\, z(\alpha,\beta)\mathcal{O}(z_{12}^\alpha, z_{21}^\beta), \tag{3.21}$$
using the notation of Section 1.4. The defining property of the renormalization operator $\mathbb{Z}$ reads analogously to (3.8):
$$\forall N: \quad \mathbb{Z}\left\langle \mathcal{O}(z_1,z_2) \prod_{j=1}^N \varphi(x_j) \right\rangle \quad \overset{\epsilon\to 0}{\longrightarrow} \quad \text{finite}. \tag{3.22}$$
All the analogous considerations then apply, leading to the formula for the evolution kernel
$$\mathbb{H} = 2\gamma_\varphi \mathbb{1} - \mu \frac{d\mathbb{Z}}{d\mu}\mathbb{Z}^{-1}\Big|_{1\mathrm{PI}}. \tag{3.23}$$



$$= e^{-\mathrm{i}p_+ z}$$

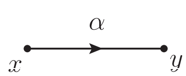
Figure 3.3: Additional element of the diagrammatic technique arising from the insertion of the non-local operator (3.20).

The crucial difference from the forward case is that the non-locality of the operator (3.20) introduces non-locality into the Feynman integrals. As a result, there is an additional Feynman rule in momentum space, see Fig. 3.3. In coordinate space, this insertion simply corresponds to an additional external vertex.

3.2 Integral identities and one-loop sample calculations

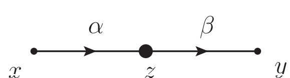
In the previous section we set up the conventions for the calculations of leading-twist operators in the theory (3.1), and additionally discussed that the overall treatment of non-local operators (3.20) can be carried out in essentially the same way. In this section we proceed to the actual integral identities that allow one to evaluate the corresponding Feynman diagrams.

The difference between the usual Feynman diagram technique for a scalar field and calculations with an operator insertion of the type (3.2) is the presence of the following additional lines [a]



$$= \frac{(y-x)_+^s}{(y-x)^{2\alpha}} = (\mathrm{i})^s \pi^{d/2} 2^{d-2\alpha+s} a_s(\alpha) \int \frac{d^dp}{(2\pi)^d} \frac{p_+^s}{p^{2(d/2-\alpha+s)}}. \tag{3.24}$$

Note that the direction of the arrow matters here, as $(y-x)_+^s = (-1)^s(x-y)_+^s$. The basic integral identity involving such vector lines is the so-called *chain rule*



$$= \int d^dz \frac{(z-x)_+^{s_1}(y-z)_+^{s_2}}{(z-x)^{2\alpha}(y-z)^{2\beta}} = \pi^{d/2} \frac{a_{s_1}(\alpha)a_{s_2}(\beta)}{a_{s_1+s_2}(\alpha+\beta-d/2)} \frac{(y-x)_+^{s_1+s_2}}{(y-x)^{2(\alpha+\beta-d/2)}}. \tag{3.25}$$

Applying the dual transform to such an integral yields the expression for the corresponding one-loop momentum integral. The relation (3.25) can be proved in a rather straightforward way. Indeed, one can recognize it as a convolution of two propagators, so in Fourier space it is represented by a simple product. The inverse Fourier transform can then be performed using the same formula (3.24), yielding the result (3.25). Note also that it is important that the numerators on the left-hand side of (3.25) are contracted with the same light-like vector n^μ. Attempting to extend this result to two independent traceless tensor structures is a more involved problem, as it requires a non-trivial intertwining of indices on the right-hand side (see [259, Appendix D] for the construction in the four-dimensional case).

The integral identity (3.25) is sufficient to calculate the anomalous dimension (3.6) at one loop. At

[a] For the graphical representation we follow [236]. The vector lines depicted as arrows should not be confused with the fermionic propagators.

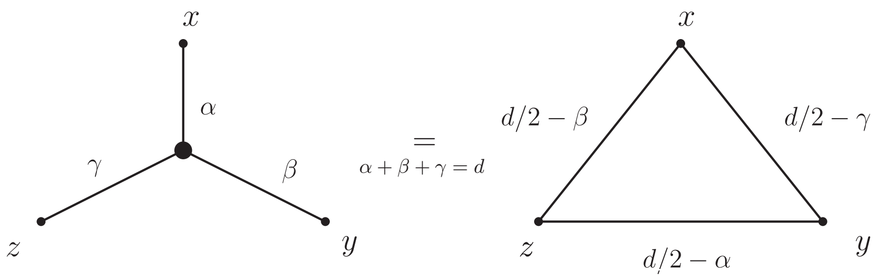
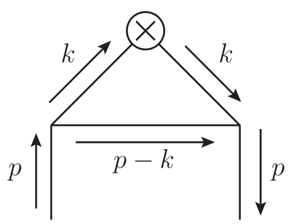


Figure 3.4: Star-triangle relation (3.28).

this order only one 1PI diagram contributes. Using the Feynman rules (see Fig. 3.2) [b]



$$= (-g)^2 \int \frac{d^d k}{(2\pi)^d} \frac{k_+^s}{(p-k)^2 k^4} = \frac{g^2}{(4\pi)^{d/2}} \frac{a_s(2)a_0(1)}{a_s(3-d/2)} \frac{p_+^s}{p^{2(3-d/2)}}, \tag{3.26}$$

where we treat the diagram without external legs. Expanding (3.26) as a series in ϵ, one concludes that

$$Z_1(s)\Big|_{1\text{PI}} = -\frac{u}{(s+1)(s+2)}\frac{1}{\epsilon} + O(u^2), \qquad \gamma(s) = 2\gamma_\varphi^{(1)} - \frac{2u}{(s+1)(s+2)} + O(u^2). \tag{3.27}$$

Going beyond the one-loop approximation of course cannot be accomplished with the chain rule (3.25) alone. In higher-loop calculations we follow what is usually referred to in the literature as the *method of uniqueness* [242, 245] (see also [236, Section 4.3]). It should be noted that the word "method" here does not denote a well-algorithmized procedure, as e.g. IBP reduction does. We believe the method of uniqueness should rather be seen as a set of tools that one can apply to a specific diagram. We do not present the full range of these tools in this thesis, as that would overcomplicate the presentation, and instead focus on the main one: the *star-triangle relation* (see Fig. 3.4): [c]

Star-triangle relation

For the three-valent vertex the following identity holds

$$\int d^d w \frac{1}{(w-x)^{2\alpha}(w-y)^{2\beta}(w-z)^{2\gamma}} = \pi^{d/2} \frac{a_0(\alpha)a_0(\beta)a_0(\gamma)}{(x-y)^{2(d/2-\gamma)}(y-z)^{2(d/2-\alpha)}(z-x)^{2(d/2-\beta)}}, \tag{3.28}$$

$$\text{if } \alpha+\beta+\gamma = d.$$

This relation allows one to perform the integration over a vertex in the diagram whose line indices satisfy the condition $\alpha+\beta+\gamma=d$. Such a vertex is called *unique*, which gives the method its name.

Let us comment on the application of this relation (3.28) in the theory (3.1). The index of the propagator in the six-dimensional model in coordinate space is $\alpha = d/2-1 = 2$, which means that the

[b]In the calculations we omit factors of $(\mathrm{i})^s$ coming from the Feynman rules, Fig. 3.2, as they would inevitably cancel by the LO diagram, $\widetilde{G}^0(p)$, see (3.18).

[c]This relation first appeared in [243], but its first non-trivial application to multi-loop calculations was given in [244].

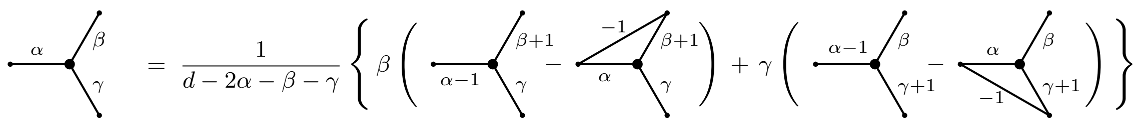


Figure 3.5: Integration-by-parts identity (IBP).

usual trivalent vertex (see Fig. 3.2) is unique: $2+2+2=6=d$. The problem is that once we pass to dimensional regularization, the sum of indices becomes $3(d/2-1)=6-3\epsilon\neq 6-2\epsilon=d$. The method of uniqueness then consists of applying various transformations of indices and symmetries of the diagram until the uniqueness condition is restored and (3.28) can be applied. In particular, in the case of *one step from uniqueness*, $\alpha+\beta+\gamma=d-1$, one can apply an IBP identity (see Fig. 3.5). This does not, of course, provide a concrete recipe for calculating a particular diagram. Instead, we illustrate several applications in Section 3.4. For a more algorithmized exposition the reader may refer to [142, Chapter 4, Section 36].

These considerations show why the method of uniqueness can be powerful in theories where the basic vertex (at $\epsilon=0$) is unique, particularly when only a few terms in the ϵ-expansion are needed (see e.g. [260, 261] for applications in QCD), since one can often slightly modify the indices — without affecting the required coefficients — while restoring uniqueness. The renormalization problem fits this scenario perfectly, see for example the five-loop β-function calculation in the φ^4 model [245, 262] or the four-loop renormalization of twist-2 operators in the same model [263]. As a somewhat separate field of application we would like to mention calculations in the so-called $1/N$-expansion (see [264] and [142, Chapter 2, Section 45] for a pedagogical introduction). This technique allows one to express the critical indices as series in the parameter $1/N$, [d] while d is treated as a genuine free parameter rather than as a small deviation from an integer value. This makes standard IBP-based techniques harder to apply here, as they require master integrals to be known analytically at general d with non-integer propagator indices. The star-triangle relation (3.28), by contrast, is insensitive to these details and leads to practical results, see [244, 265–271].

Let us mention another important property of the star-triangle relation. The integral on the left-hand side of (3.28) is an example of a so-called *conformal integral*. This name stems from the fact that the integral (3.28), viewed as a function of the external variables x, y, and z, is conformally covariant, i.e. it transforms in a representation of the conformal algebra on functions of three variables (see Section 2.1 for details). Indeed, performing an inversion of the integration variable gives [e]

$$F\left(1/x,1/y,1/z\right)=\int d^dw\,\frac{\cancel{w^{2(\alpha+\beta+\gamma)}}}{\cancel{w^{2d}}}\frac{x^{2\alpha}y^{2\beta}z^{2\gamma}}{(w-x)^{2\alpha}(w-y)^{2\beta}(w-z)^{2\gamma}}=x^{2\alpha}y^{2\beta}z^{2\gamma}F(x,y,z), \tag{3.29}$$

where we used the transformation rules under inversion

$$d^dw\mapsto\frac{d^dw}{w^{2d}}, \qquad\qquad (x-y)^{2\alpha}\mapsto\frac{(x-y)^{2\alpha}}{x^{2\alpha}y^{2\alpha}}. \tag{3.30}$$

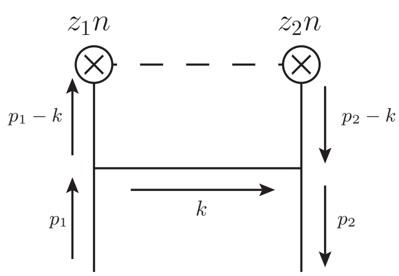
Thus, the function $F(x,y,z)$ transforms covariantly under conformal transformations (see the infinitesimal form in (2.12)) and the functional form of result is fully determined. The connection between the uniqueness condition and conformal invariance was noted already in the early stages of the field [272], and conformal integrals themselves continue to attract attention, see e.g. [4, 6, 273–277]. In fact the structure of the star-triangle relation runs even deeper, as it can be interpreted as a conformally invariant solution of the Yang–Baxter equation and underlies an integrability structure (see [278–280] for

[d]The parameter N can have different meanings: for example, the number of degrees of freedom in $O(N)$-symmetric models, the number of fermion flavors in QCD, or in models of Gross–Neveu type.

[e]The behaviour under other transformations from the conformal group is obvious.

details and e.g. [281–284] for applications). The fact that the right-hand side of (3.28) takes the form of a three-point conformal correlator (see (2.40)) is also not a coincidence. Indeed, the fact that the basic vertex of the theory is unique naturally translates into the conformal covariance of the three-point Green's function of the fields. If one is interested in the critical values of the anomalous dimensions (see (2.75)), one can use the fixed form of the three-point correlator at the critical point and recover the corresponding powers by comparing with the perturbative expansion. This approach forms the basis of the powerful technique of conformal bootstrap, [f] with particular success in $1/N$ expansions, see [285–289].

Unfortunately, the method of uniqueness does not translate naturally to the level of non-local integrals (3.22). The main obstacle is that the non-locality carries over to the level of Feynman integrals. To illustrate this, let us present the one-loop calculation for the operator (3.20). At one-loop only one diagram contributes: [g]



$$= (-g)^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{(p_1-k)^2(p_2-k)^2k^2} e^{-\,\mathrm{i}(p_1-k|n)z_1} e^{\mathrm{i}(p_2-k|n)z_2}, \tag{3.31}$$

where we used the Feynman rule in Fig. 3.3 for the non-local insertion. The presence of non-local exponentials also lies beyond the standard IBP-based machinery, although it should be noted that some progress in this direction has been achieved in recent years, see [60, 290]. Remarkably, the most basic application of Feynman parameter techniques [291] provides a natural way to proceed. The reason is that one wants to obtain the divergent part of the diagram (3.31) as an integral operator acting on the leading-order matrix element. The Feynman parameter integral therefore does not require full simplification, and one wishes to retain the final double integral in a form matching that of the integral operator. Let us illustrate this idea more explicitly:

$$\int \frac{d^dk}{(2\pi)^d} \frac{e^{\mathrm{i}\,z_{12}k_+}}{(p_1-k)^2(p_2-k)^2k^2} = 2\int \frac{d^dk}{(2\pi)^d} \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta \frac{e^{\mathrm{i}\,z_{12}k_+}}{\big(k^2-2k(\alpha p_1+\beta p_2)+m^2\big)^3}, \tag{3.32}$$

where we have introduced a mass parameter, m, as an IR regulator and set $p_1^2 = p_2^2 = 0$. Performing the change of variables $l = k - \alpha p_1 - \beta p_2$, the integral (3.32) transforms to

$$\int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\, e^{\mathrm{i}\,z_{12}(\alpha(p_1|n)+\beta(p_2|n))} \int \frac{d^dl}{(2\pi)^d} \frac{e^{\mathrm{i}\,z_{12}l_+}}{(l^2+m^2)^3}. \tag{3.33}$$

A remarkable feature of the resulting integral is that expanding $e^{\mathrm{i}\,z_{12}l_+}$ in a Taylor series, one can see that only the zeroth-order term contributes. Indeed,

$$\int \frac{d^dl}{(2\pi)^d} \frac{l_+^k}{(l^2+m^2)^n} = 0, \qquad\qquad \forall k \neq 0, \tag{3.34}$$

[f] This name should not be confused with the conformal bootstrap in the sense of [166, 168]. Both techniques use constraints which conformal invariance puts on correlators in CFT, but in different way on the technical level.

[g] Here we connect the two non-local insertion points, $z_1 n$ and $z_2 n$, by a dashed line. In the theory (4.112) this is purely cosmetic, but in QCD such a line represents a Wilson line, see Chapter 5 for details.

by symmetry. The remainder of the calculation in (3.31) is then straightforward, yielding

$$\left[\mathbb{Z}_1\big|_{1\mathrm{PI}} f\right](z_1, z_2) = -\frac{u}{\epsilon}\int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\, f(z_{12}^{\alpha}, z_{21}^{\beta}) + O(u^2), \tag{3.35}$$

and

$$h^{(1)}(\alpha,\beta) = 2\gamma_\varphi^{(1)}\delta(\alpha)\delta(\beta) - 2\theta(1-\tau), \tag{3.36}$$

where $\tau = \alpha\beta/\bar\alpha\bar\beta$ and the integral kernel $h(\alpha,\beta)$ is defined by (1.98).

3.3 IBP-based approach and harmonic-sum bootstrap

In the previous section we discussed the application of the method of uniqueness to the calculation of Feynman diagrams for the anomalous dimensions of twist-2 operators. In simple terms, the difference from standard IBP reduction can be described as follows: at every step one tries to apply a certain set of tools based on the star-triangle relation (3.28), instead of generating the full variety of intermediate integrals. This approach is, of course, difficult to use as a sole method in calculations where the number of diagrams is too large to handle by hand.

In this section we discuss a method for calculating the forward anomalous dimensions (3.6) based on reduction to scalar integrals, though it should be noted that other options also exist. Let us first write down the kind of integrals we are interested in. The structure of (3.26) naturally generalizes to ℓ-th loop order as

$$I_s^{(\ell)}(p) = \int \frac{d^dk_1 \dots d^dk_{\ell-1} d^dq}{(2\pi)^{d\ell}} \frac{q_+^s}{\prod_{j=1}^{3\ell} D_j(p, \{k_i\}_{i=1}^{\ell-1}, q)}, \tag{3.37}$$

where the index $j = 1, \dots, 3\ell$ [a] labels all propagators $D_j(p, \{k_i\}_{i=1}^{\ell-1}, q)$. The momentum q represents the momentum incoming in the operator vertex. One can apply the IBP-reduction procedure to the integral (3.37) and consider the integral family

$$I_{s,\{b_j\}}^{(\ell)}(p) = \int \frac{d^dk_1 \dots d^dk_{\ell-1} d^dq}{(2\pi)^{d\ell}} \frac{q_+^s}{\prod_{j=1}^{3\ell} D_j^{b_j}(p, \{k_i\}_{i=1}^{\ell-1}, q)}, \qquad I_s^{(\ell)}(p) = I_{s,\{1,\dots,1\}}^{(\ell)}(p), \tag{3.38}$$

where b_j is the integer power of the propagator. It can be shown that in order to calculate the integral (3.38) for given s and specific values of $\{b_1^0, \dots, b_{3\ell}^0\}$, it is sufficient to know the integral families $\{I_{s-1,\{b_j\}}^{(\ell)}, I_{s-2,\{b_j\}}^{(\ell)}, \dots, I_{0,\{b_j\}}^{(\ell)}\}$ with the finite set of values $\{b_j\}$ for each s, depending on $\{b_1^0, \dots, b_{3\ell}^0\}$. This imposes a recursive structure, and in the end what is needed is the scalar family $I_{0,\{b_j\}}^{(\ell)}$. Thus one concludes that knowledge of all master integrals for the family (3.38) at $s = 0$, combined with a recursive procedure, can be translated into knowledge of the family (3.38) for arbitrary s.

Let us illustrate this idea with the simple example of the one-loop integral

$$I_{2,\{a,b\}}^{(1)}(p) = \int \frac{d^dk}{(2\pi)^d} \frac{k_+^2}{k^{2a}(p-k)^{2b}}. \tag{3.39}$$

Let us consider the integral without contraction with the light-like vector and write down the most general ansatz consistent with the symmetries (see the Passarino–Veltman reduction [292])

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{k^{2a}(p-k)^{2b}} = A(p)p^\mu p^\nu + B(p)\delta^{\mu\nu}p^2. \tag{3.40}$$

[a] Here we consider the case of the theory (3.1). In other theories the connection between the number of loops and the number of propagators can be different.

Contracting the numerator with the light-like vector n^μ, with the external momentum p^μ, or with the Lorentz invariant $\delta^{\mu\nu}$, and using $2\big(k\cdot p\big) = p^2 + k^2 - (p-k)^2$, we obtain

$$\begin{aligned} I^{(1)}_{2,\{a,b\}}(p) &= A(p)p_+^2, \\ p^2 I^{(1)}_{1,\{a,b\}}(p) + I^{(1)}_{1,\{a-1,b\}}(p) - I^{(1)}_{1,\{a,b-1\}}(p) &= 2\big(A(p)+B(p)\big)p_+p^2, \\ I^{(1)}_{0,\{a-1,b\}}(p) &= \big(A(p)+dB(p)\big)p^2. \end{aligned} \tag{3.41}$$

Thus, the answer for $I^{(1)}_{2,\{a,b\}}$ is given by $\{I^{(1)}_{1,\{a,b\}}, I^{(1)}_{1,\{a-1,b\}}, I^{(1)}_{1,\{a,b-1\}}, I^{(1)}_{0,\{a-1,b\}}\}$. The next step of recursion allows one to express integrals with $s=1$ through the scalar integrals, $s=0$. This simple calculation highlights two sources of complication for this approach:

- The parameter s cannot be treated analytically, as the calculation requires the explicit solution of the recursion procedure.
- The complexity grows rapidly with s: on the one hand, the number of terms in the Passarino–Veltman reduction is $\lfloor s/2\rfloor + 1$, on the other hand, the higher the value of s we consider the deeper the recursion.

Once we know that the set of master integrals corresponding to the fully scalar case $s=0$ is sufficient, we can use procedures in which the corresponding IBP reduction is implemented. A particularly convenient tool is `Forcer` [246] (the four-loop `FORM` [293–295] version of the program `Mincer` [296,297]). Using `Forcer`, one can in principle handle arbitrary massless integrals of two-point topology up to four loops — exactly matching the setup of the calculation of (3.6). Here we do not present the specific workflow of any particular implementation, as our interest is mostly conceptual. Especially given that the strategy described around (3.41) is presented primarily for clarity of exposition, while real implementations can employ considerably more sophisticated ways of organizing the recursion.

Regardless of the way the reduction to scalar integrals is implemented, there is one complication that cannot be avoided: the inability to treat s as an analytic parameter. The typical output of such a computational procedure then takes the following form: [b]

$$\begin{aligned} \gamma(1) &= \frac{15}{16} - 3\zeta_3 - \frac{47}{4}N + 3N^2, \\ \gamma(2) &= -\frac{1421}{3888} + \frac{1}{3}\zeta_3 - \frac{469}{108}N - \frac{1}{27}N^2, \\ \gamma(3) &= \frac{79733}{62208} - \frac{7}{4}\zeta_3 - \frac{1067}{576}N - \frac{73}{216}N^2, \\ \gamma(4) &= -\frac{121273}{2400000} + \frac{3}{100}\zeta_3 - \frac{77849}{120000}N - \frac{321}{1000}N^2, \\ \gamma(5) &= \frac{3352333}{12150000} - \frac{19}{25}\zeta_3 - \frac{60637}{135000}N - \frac{901}{3375}N^2, \\ \gamma(6) &= \ldots, \end{aligned} \tag{3.42}$$

from which one can clearly see the pattern: the anomalous dimensions at fixed values of s are expressed in terms of rational numbers and ζ-values. The natural goal is then to restore the full s-dependence from a restricted (in complex calculations typically $s_{\max} \lesssim 20$) set of moments (3.42). At first sight this problem looks hopeless, but knowledge of the underlying structure allows one to perform such a reconstruction quite efficiently.

[b] The actual theory behind these values as well as the meaning of the parameter N are not relevant here. The results are written merely to illustrate the general pattern.

It is well known that anomalous dimensions as functions of s are expressible in terms of so-called *harmonic sums* [298] (see also [299, Chapter 2] for a comprehensive discussion of their properties), which can be defined as follows:

$$S_{\pm a}(s) = \sum_{k=1}^{s} \frac{(\pm 1)^k}{k^a},$$

$$S_{\pm a_1, a_2, \ldots, a_n}(s) = \sum_{k=1}^{s} \frac{(\pm 1)^k}{k^{a_1}} S_{a_2,\ldots,a_n}(k). \tag{3.43}$$

From a "pedestrian" point of view, this fact is justified by the result of perturbative calculations typically being represented by ratios of Γ-functions, $\Gamma(s+a)/\Gamma(s+b)$. These ratios produce logarithmic derivatives of Γ-functions, $\psi^{(k)}(s)$, when expanded near integer values of s. The latter can then be expressed in terms of the functions (3.43), which are closed in the sense of forming a *quasi-shuffle algebra*, see [300] for details. In principle, the same idea can be put on a more rigorous footing by considering the class of possible functions in x-space (where x is the Mellin conjugate of s). It can be shown [301] that as long as one stays within the class of harmonic polylogarithms (HPLs) [302] (see also [303] for comprehensive lectures on multiple polylogarithms), Mellin space is fully described by harmonic sums (3.43). In relation to (3.43) it is convenient to define the *transcendental weight* of a harmonic sum as

$$\text{for } S_{a_1,\ldots,a_n}(s): \qquad w = \sum_{i=1}^{n} |a_i|. \tag{3.44}$$

We therefore expect the anomalous dimensions to have the following general structure. At perturbative order ℓ, the anomalous dimension $\gamma^{(\ell)}(s)$ consists of:

- Harmonic sums (3.43) with transcendental weights $w \leq 2\ell - 1$.
- Each harmonic sum of transcendental weight w may be accompanied by a factor $1/(s+a)^k$, where a is an integer and $k + w \leq 2\ell - 1$.

In practical calculations the set of possible values of a is rather modest and can be quite accurately determined by analyzing the prime decomposition of the denominators in (3.42). This gives an upper bound on the number of terms that can appear in the expression for $\gamma(s)$ and justifies a "bootstrap" approach: one writes down a general ansatz bounded by the maximal transcendental weight $2\ell - 1$ and fixes the coefficients using the set of moments (3.42).

The number of independent harmonic sums at a given transcendental weight w is $2 \cdot 3^{w-1}$, which already at two loops gives 18, so the size of an ansatz consisting of pure harmonic sum terms (not even including rational coefficients) is already comparable with the number of moments one can feasibly compute, and grows exponentially with the loop order. It is therefore not possible to fully constrain all the coefficients in the ansatz from moments alone. However, the following procedure is possible: consider an ansatz $\{F_1(s), \ldots, F_k(s)\}$ with $k > s_{\max}$, where $s_{\max}$ is the largest moment available. The data (3.42) then yields the system of equations

$$\begin{cases} a_1 F_1(1) + \ldots + a_k F_k(1) = \gamma(1), \\ \quad \ldots \\ a_1 F_1(s_{\max}) + \ldots + a_k F_k(s_{\max}) = \gamma(s_{\max}), \end{cases} \tag{3.45}$$

for the coefficients $\{a_1, \ldots, a_k\}$. Since both $F_k(s)$ and $\gamma(s)$ take rational values, [c] one can multiply through by the overall GCD to obtain a system of *Diophantine equations*. Various techniques are

[c] Note that in this approach one organizes a separate system of equations for each coefficient structure, e.g. for N, N^2, and ζ_3 in (3.42).

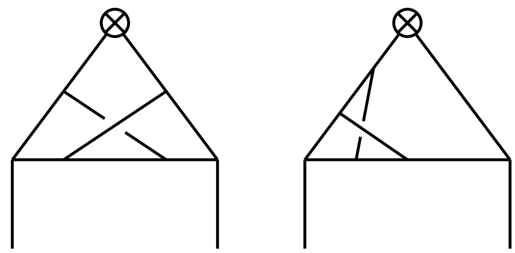


Figure 3.6: Two non-trivial three-loop diagrams contributing to the anomalous dimensions $\gamma(s)$ in the theory (3.1).

available for solving such systems, one of the most efficient is the so-called *LLL* algorithm [304], which has a wide variety of implementations, such as the `LatticeReduce` method in `Wolfram Mathematica` or the `fplll` package. [d] Remarkably, this approach turns out to be quite efficient and allows one to recover the full analytic s-dependence of anomalous dimensions from a very restricted set of input data, see [305–311] for applications in QCD and [312–318] in $\mathcal{N}=4$ SYM.

Let us also comment on the justification for the ansatz construction (3.45). In this construction we explicitly assume that the functional space of anomalous dimensions consists only of harmonic sums, which is an assumption rather than the result of a rigorous argument. Indeed, as mentioned above, this relies on the belief that amplitudes in the Mellin-conjugate x-space consist only of HPLs, which seems reasonable. Going to massive corrections already requires considering *generalized polylogarithms* (GPLs) with a larger alphabet in x-space, corresponding to so-called *cyclotomic harmonic sums* (see [319] for a discussion). Going further, one can in principle expect elliptic integrals to appear, which would require a yet more general class of "elliptic" harmonic sums. The fact that we are working with massless integrals does not protect them from developing elliptic (or in principle even more complicated) structures, as happens for example in the renormalization of φ^4 theory [320]. However, no such structures have yet been found in the anomalous dimensions of twist-2 operators in the literature, so the natural approach is to maintain this assumption until a loop order is reached at which they first appear.

3.4 Examples of calculations

In the previous sections we discussed two different approaches to the calculation of Feynman diagrams arising from twist-2 operator insertions. The strengths and weaknesses of the IBP-based approach of Section 3.3 are quite clear: on the one hand, one can perform standard, well-developed, and algorithmized calculations using computer algebra systems, but on the other hand one must pay the price of a reconstruction procedure to obtain an analytic result in s. The disadvantages of the method of uniqueness, Section 3.2, are also clear — the approach is not algorithmized, every diagram requires separate treatment, and no existing computer implementation of this procedure is available. The advantage of this method, namely the analytic dependence on s at all intermediate steps, is conceptually transparent, but may not be so obvious in practice, being obscured by the unlisted set of additional tools that were not specified in Section 3.2. In order to correct this and provide a concrete demonstration of the method, in this section we perform the calculation of two rather complex diagrams.

Both diagrams are shown in Fig. 3.6 and contribute at three-loop order to the anomalous dimensions $\gamma(s)$ in the theory (3.1). Let us focus on the right diagram first, CV. By simple dimensional analysis

[d] See the github link github.com/fplll.

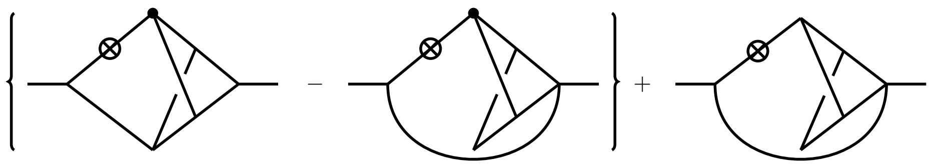


Figure 3.7: Graphical representation of the decomposition (3.47).

we expect the answer to have the structure

$$\mathrm{CV}_s(p) = \left(\frac{A_2(s)}{\epsilon^2} + \frac{A_1(s)}{\epsilon} + O(\epsilon) \right) \frac{p_+^s}{p^{2(3\epsilon)}}. \tag{3.46}$$

The double pole is expected because the diagram has a single divergent subdiagram (the two-loop non-planar vertex insertion) and also diverges as a whole. It is more convenient to work with superficially divergent diagrams, so let us perform the following: we subtract and add another diagram carrying the same subdivergence (see Fig. 3.7), namely

$$\mathrm{CV}_s(p) = \underbrace{\mathrm{CV}_s(p) - \delta\mathrm{CV}_s(p)}_{\Delta\mathrm{CV}_s(p)} + \delta\mathrm{CV}_s(p). \tag{3.47}$$

The advantage of this decomposition is clear: the difference $\Delta\mathrm{CV}_s(p)$ has only a superficial divergence, while the diagram $\delta\mathrm{CV}_s(p)$ is simpler. This way of isolating the subdivergent part as a practical tool was used in [244, Eq. (17)] (see also [142, Chapter 4, Section 37]). A similar idea appears in the theory of graphical functions (see rerouting [6, Section 12.3]).

Let us now address each part in turn. The part $\Delta\mathrm{CV}_s(p)$ has only a superficial divergence. We now demonstrate one of the advantages of working in a massless theory, which allows one to organize an interplay between the coordinate and momentum representations. Again by dimensional analysis, the contribution $\Delta\mathrm{CV}_s(x)$ takes the form

$$\Delta\mathrm{CV}_s(x) = B(\epsilon, s) \frac{x_+^s}{x^{2(2d-9+s)}}, \tag{3.48}$$

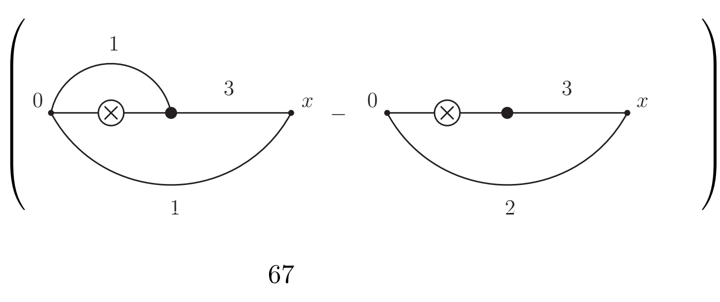
so performing the Fourier transform gives

$$\Delta\mathrm{CV}_s(p) = B(\epsilon, s) \int d^d x \frac{x_+^s}{x^{2(2d-9+s)}} = (\mathrm{i})^s \frac{\pi^3}{2^s} B(\epsilon, s) \left(\frac{1}{3\Gamma(3+s)\epsilon} + O(\epsilon^0) \right) \frac{p_+^s}{p^{2(9-3d/2)}}. \tag{3.49}$$

Thus we conclude that if the resulting diagram has only a superficial divergence, the coefficient $B(\epsilon, s)$ must be regular as $\epsilon \to 0$. This fully justifies the convenience of the decomposition (3.47): one can now calculate $\Delta\mathrm{CV}_s(x)$ at $\epsilon = 0$ to obtain $B(0, s)$ and then simply use Eq. (3.49) to get the answer. The only subtlety is that the subdivergence cancels only for the full difference $\Delta\mathrm{CV}_s(x)$, so one must keep track of the order of integrations to ensure cancellation at every step. In particular, we denote the final integration by a black dot in Fig. 3.7.

Setting $\epsilon = 0$ is significant, as all vertices in the diagram become unique and we can perform the integrations using (3.28). After three integrations we are left with

$$\Delta\mathrm{CV}_s(x) = \pi^6 \quad (3.50)$$



At this point each of the terms is individually divergent, so to perform the final integration we introduce an additional regulator ξ, shifting the index of the line as $3 \mapsto 3-\xi$. The final integration can then be performed using (3.25):

$$
\begin{aligned}
\Delta\mathrm{CV}_s(x) &= \pi^{-3} 2^s \frac{\Gamma(s+2)}{\Gamma(2)} \frac{a_0(2) a_s(s+2)}{a_s(s+1)} a_0(3-\xi) \left(\frac{a_s(s+2)}{a_s(s+2-\xi)} - \frac{a_s(s+1)}{a_s(s+1-\xi)} \right) \frac{x_+^s}{x^{2(s+3-\xi)}} \\
&= \pi^{-3} 2^s \Gamma(s+2) \frac{s}{2(s+1)^2} \frac{x_+^s}{x^{2(s+3)}} + O(\xi^0).
\end{aligned} \tag{3.51}
$$

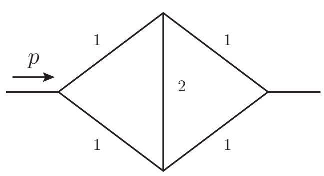
Reading off $B(0,s)$ and substituting back into (3.49) we obtain (here we introduce the variable $j \equiv s+1$ to match the expressions in Appendix A)

$$
\mathrm{KR}'\left(\Delta\mathrm{CV}_j(p)\right) = \frac{j-1}{j^2(j+1)} \frac{1}{\epsilon} p_+^{j-1}. \tag{3.52}
$$

The remaining part $\delta\mathrm{CV}_s(p)$ is simpler and can be calculated directly in momentum space. The subdiagram factorizes, and performing the remaining integration using the chain rule (3.25) one obtains

$$
\delta\mathrm{CV}_s(p) = \frac{a_0(1) a_s(2+2\epsilon)}{a_s(3\epsilon)} C(\epsilon) \frac{p_+^s}{p^{2(3\epsilon)}}, \tag{3.53}
$$

where the factorized contribution $C(\epsilon)$ reads



$$
\Bigg|_{p^2=1} = \frac{1}{4\epsilon} + \left(\frac{47}{24} - \zeta_3 \right) + O(\epsilon). \tag{3.54}
$$

This diagram, known in the literature as the *two-loop master diagram*, has a remarkable history of calculations (see [236, Chapter 3]), owing to its frequent occurrence in multi-loop calculations. An interesting question is to obtain an analytic expression for this diagram with arbitrary line indices. Progress in this direction was achieved by the so-called *Gegenbauer polynomial* technique, which provided results for two [321] and three [322] arbitrary parameters. For more complicated cases only series representations are available (see [262, 323–325]).

Let us now proceed to the second diagram in Fig. 3.6, $\mathrm{CC}_s(p)$. This diagram has only a superficial divergence, so the previous consideration, (3.49), directly applies and one can immediately switch to coordinate space, $\mathrm{CC}_s(x)$, and set $\epsilon = 0$. Considering the diagram in exactly six dimensions restores the uniqueness of all vertices. However, performing all possible integrations via the star-triangle relation (3.28) is not sufficient to obtain the final result. Here we wish to exploit a different advantage of working with unique vertices, namely the behavior of the diagram under conformal inversion (3.29). The diagram $\mathrm{CC}_s(x)$ is almost conformally covariant as a whole, except for the operator insertion, but this obstacle can be overcome since

$$
\mathcal{O}_s(x) = \mathbb{O}_s(x) + \text{total derivatives}, \tag{3.55}
$$

where $\mathbb{O}_s(x)$ is a conformal operator (see (2.93) for its explicit form and the discussion therein). One can therefore relate the diagram to a conformal three-point function whose form is known (see (2.40) and the discussion of the generalization to operators with spin [183]) [a]

$$
\left\langle \mathbb{O}_s(x) \varphi(x_1) \varphi(x_2) \right\rangle = \frac{C_{\mathbb{O}\varphi\varphi}}{\left[(x-x_1)^2 (x-x_2)^2\right]^2} \left(\frac{(x-x_1)_+}{(x-x_1)^2} - \frac{(x-x_2)_+}{(x-x_2)^2} \right)^s, \tag{3.56}
$$

where the constant $C_{\mathbb{O}\varphi\varphi}$ is part of the CFT data (see (2.45)).

[a] Here we adopt the notation of (3.12) and (3.11).

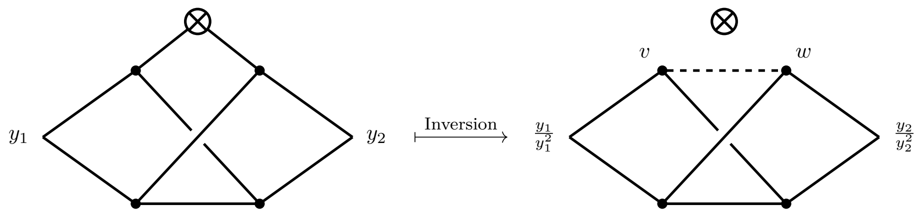


Figure 3.8: Graphical representation of the conformal inversion (3.59). The dashed line represents the factor $(v_+ - w_+)^s$.

This fixes the form of the diagram with the integrated variable x:

$$\mathrm{CC}_s(x_1, x_2) = \int d^d x\, \mathrm{CC}_s(x, x_1, x_2) = \int d^d x\, \mathrm{CC}_s(0, x_1 - x, x_2 - x). \tag{3.57}$$

The correlator with two shifted coordinates, $\mathrm{CC}_s(0, y_1, y_2)$, then reads

$$\mathrm{CC}_s(0, y_1, y_2) = R(s) \frac{C^{(0)}_{\mathbb{O}\varphi\varphi}}{\left(y_1^2 y_2^2 (y_1 - y_2)^2\right)^2} \left(\frac{y_{1,+}}{y_1^2} - \frac{y_{2,+}}{y_2^2}\right)^s, \tag{3.58}$$

where $C^{(0)}_{\mathbb{O}\varphi\varphi}$ is the LO three-point constant, which can be seen as the factor absorbing the transition from the original operator insertion (3.2) to the conformal operator insertion. We are therefore interested in the factor $R(s)$, which gives the contribution to the renormalization factor. The remarkable property of the representation (3.58) is that it allows one to perform an inversion of the external coordinates simultaneously with an inversion of all integration variables (see Fig. 3.8):

$$\begin{aligned} \mathrm{CC}_s(0, y_1, y_2) &= \left(\int \ldots\right) \int d^6 v\, d^6 w\, \frac{1}{(v^2 w^2)^4} \left(\frac{v_+}{v^2} - \frac{w_+}{w^2}\right)^s, \\ \mathrm{CC}_s\left(0, \tfrac{1}{y_1}, \tfrac{1}{y_2}\right) &= \left(\int \ldots\right) \int d^6 v\, d^6 w\, \left(v_+ - w_+\right)^s. \end{aligned} \tag{3.59}$$

Here we write explicitly the integration around the operator insertion and observe that under a conformal inversion (see (3.30) for the change of variables), the operator insertion effectively "decouples" from the diagram. We still have to deal with the factor $(v_+ - w_+)^s$, but it can be handled by expanding in a binomial series. The remaining integrals are fully solvable by means of the star-triangle relation (3.28) and the vector form of the chain rule (3.25). Extracting the result one obtains

$$\mathrm{R}(s) = \frac{1}{s+1} \sum_{k=0}^{s} \frac{2}{k - 2\epsilon} \left(A^{sk} + B^{sk} - C^{sk} - \frac{1}{k - 1 - 2\epsilon} \left(2D^{sk} - 2E^{sk} + 3F^{sk} - 3G^{sk}\right)\right) \tag{3.60}$$

where the terms $A^{sk}, \dots, G^{sk}$ have the form

$$
\begin{aligned}
A^{sk}(\epsilon) &= \sum_{p=0}^{k} (-1)^p \binom{k}{p} \frac{a_{s-p}(1)a_0(2)}{a_{s-p}(\epsilon)} \frac{a_p(3)a_0(2)}{a_p(2+\epsilon)}, \\
B^{sk}(\epsilon) &= \sum_{p=0}^{k} (-1)^p \binom{k}{p} \frac{a_{s-p}(1)a_0(2)}{a_{s-p}(\epsilon)} \frac{a_p(2)a_0(3)}{a_p(2+\epsilon)}, \\
C^{sk}(\epsilon) &= \sum_{p=0}^{s-k} \binom{s-k}{p} \frac{a_{s-p}(1)a_0(2)}{a_{s-p}(\epsilon)} \frac{a_p(3)a_{s-p}(2+\epsilon)}{a_s(2+2\epsilon)}, \\
D^{sk}(\epsilon) &= \sum_{p=0}^{k} (-1)^p \binom{k}{p} \frac{a_{s-p}(1)a_0(1)}{a_{s-p}(-1+\epsilon)} \frac{a_p(3)a_0(3)}{a_p(3+\epsilon)}, \\
E^{sk}(\epsilon) &= \sum_{p=0}^{s-k} \binom{s-k}{p} \frac{a_{s-p}(1)a_0(1)}{a_{s-p}(-1+\epsilon)} \frac{a_p(3)a_{s-p}(2+\epsilon)}{a_s(2+2\epsilon)}, \\
F^{sk}(\epsilon) &= \sum_{p=0}^{k} (-1)^p \binom{k}{p} \frac{a_{s-p}(1)a_0(1)}{a_{s-p}(-1+\epsilon)} \frac{a_p(2)a_0(4)}{a_p(3+\epsilon)}, \\
G^{sk}(\epsilon) &= (-1)^k \frac{a_{s-k}(1)a_k(1)}{a_s(-1+\epsilon)} \frac{a_s(1+\epsilon)a_0(4)}{a_s(2+2\epsilon)},
\end{aligned} \tag{3.61}
$$

Note that in order to calculate the remaining diagram we have again introduced dimensional regularization, but the singularities that arise should be treated as intermediate and the final answer is finite.

It is of course difficult to perform the summations in (3.61) analytically. Instead one can use the "bootstrap" approach similar to that described in Section 3.3. The difference is that producing the values $R(s)$ for fixed s is straightforward, since (3.61) is a finite sum known analytically. Building an ansatz of harmonic sums of weight $w \le 3$ (see (3.44)) and generating a sufficient number of terms, one can uniquely fix all coefficients. Returning to the full diagram, the result is (here as before $j = s+1$)

$$
\begin{aligned}
\mathrm{KR}'\left(\mathrm{CC}_j(p)\right) = & \frac{1}{\epsilon}\Bigg[\frac{1}{3j(j+1)}\Bigg(\frac{1-(-1)^j}{2}\left(\frac{4}{(j-1)(j+2)}\left(1+S_{-2}(j)\right) - \frac{4}{j(j+1)}\right) \\
& + \frac{1+(-1)^j}{2}\left(\frac{4}{j^2(j+1)^2}S_{-2}(j) + \frac{4}{j(j+1)}S_{-2}(j)\right) \\
& + \frac{6\zeta_3}{j(j+1)} - \frac{2}{j(j+1)}\left(S_1(j)+S_3(j)+2\left(S_{1,-2}(j) - \frac{1}{2}S_{-3}(j)\right)\right)\Bigg)\Bigg]p_+^{j-1}.
\end{aligned} \tag{3.62}
$$

Chapter 4

Singularities of anomalous dimensions of twist-2 operators

We believe that the previous sections have established sufficient context for the need in perturbative calculations of anomalous dimensions of twist-2 operators. In QCD they govern the scale dependence of PDFs (see (1.64) and the discussion in Chapter 1). In CFTs they play an important role as part of the CFT data (2.45), and their precise knowledge is relevant for the use of the conformal OPE (see (2.36)). However, the methods for their calculation (see Chapter 3) can be quite involved and sometimes require additional information about the structure of anomalous dimensions as functions of spin (see Section 3.3). This motivates a particular interest in their analytic properties.

Over the past decades a considerable number of studies have focused on the large-spin asymptotics ($s \to \infty$, or $x \to 1$ in the Mellin-conjugate space), see [76, 77, 102, 326, 327]. Another interesting limit concerns the singularities arising at small values of spin in analytically continued expressions. Constraining the asymptotic behavior of anomalous dimensions near these points not only provides input (or a cross-check) for perturbative calculations, but also has direct physical relevance. A distinguished role is played by the *right-most* singularity. [a] For example, the gluon–gluon anomalous dimension, $\gamma_{gg}(s)$ (see (1.63)), acquires its first pole at $s = 1$ (see [25, 26, 63, 328–330] for explicit results), and the behavior near this pole turns out to be related to BFKL physics [46, 47]. In particular, resummation of these singularities allows one to predict the position of the pomeron intercept [48, 331]. Remarkably, Regge theory also has given rise to interest in singularities in spin in conformal field theories [332], as they control a similar kinematic limit of four-point functions. This is complementary to the studies of analytic continuation in spin in CFTs [80], which makes it possible to properly account for the basis of mixing operators at singular points and organise a resummation procedure (see e.g. [73, 81, 333] for details).

In this chapter we consider the right-most singularity in certain theories where it shares similar properties to the $s \to 0$ singularity of the quark non-singlet anomalous dimension, $\gamma_{\rm ns}(s)$ (see (1.30) for the corresponding operator). On the practical side, this leads to the following results:

- An all-order resummation procedure for the singular contributions at the right-most singularity.

- Predictions for the higher-loop singular contributions based on lower-loop data.

The content of this chapter largely follows Ref. [1] (see also [3]). The following new results, obtained with the direct participation of the author of this thesis, are presented:

[a] Singularities of anomalous dimensions are typically organized as an infinite series starting at some $s = s_0$ and continuing at integer points for all $s < s_0$.

1. Four-loop anomalous dimensions [b] of all types of twist-2 operators in the $O(N)$-symmetric φ^4 model, together with the resummation of their right-most singularities.

2. Three-loop anomalous dimensions of all types of leading-twist operators in the complex φ^3 model and resummation of their right-most singularities.

3. Two-loop anomalous dimensions of all types of leading-twist operators in the Gross–Neveu–Yukawa model and resummation of their right-most singularities at the critical point.

4. Resummation of the right-most singularities of leading-twist operators at order $1/N^2$ in the Gross–Neveu model in the $1/N$-expansion, leading to a correction of the $s \to 0$ limit of the $1/N$-order anomalous dimensions.

4.1 The large spin limit

In Section 3.3 we briefly touched on the properties of anomalous dimensions as functions of spin: namely, we discussed the well-motivated observation that at ℓ-th loop order the anomalous dimension, $\gamma^{(\ell)}(s)$, is given by harmonic sums (see (3.43)) with transcendental weight $w \leq 2\ell - 1$ and rational functions with bounded denominator degree. However, this is not the whole story, and the underlying physics further constrains the functional space. In this section we review such arguments, related to the behavior in the limit $s \to \infty$ and their consequences. As the main outcome of this discussion we keep in mind the restrictions for the QCD case (see Section 1.2 and specifically Eqs. (1.28), (1.29), (1.30), (1.31), (1.32)), but the discussion is quite general and applicable in other theories. These results are in direct interplay with perturbative calculations. Let us mention the available results from a historical perspective: [a]

1. The one-loop twist-2 anomalous dimensions in QCD were worked out in the foundational papers on the topic [25, 26] (see also [39] for the formulation in terms of splitting functions).

2. The two-loop non-singlet case was calculated independently by different groups [335–337], and these calculations were extended to the singlet case [328, 329]. Remarkably, there was a discrepancy [330] in the gluon–gluon anomalous dimension $\gamma^{(2)}_{gg}(s)$, related to incorrect accounting of the mixing of gauge-invariant operators, see (1.57).

3. The three-loop result was obtained in the non-singlet case in [338] and in the singlet case in [63].

4. The complete four-loop result is a long-standing open problem: a series of partial results (for fixed moments) are available [66, 339–343], and partial analytic results have been obtained by the procedure described in Section 3.3: for the non-singlet case in the planar limit [308], proportional to ζ-values [310], and in the fermionic contributions [307,309,311]. Remarkable progress was recently achieved when the procedure applied in [241, 344] was extended to yield the fully analytical four-loop non-singlet result [64]. The singlet case is still not fully covered.

Let us also mention the key references for analogous calculations in the case of the axial operator (1.31) [39,306,345–351] and the transversity operator (1.32) [305,352–358]. It is remarkable that the complexity of state-of-the-art perturbative calculations is so high that going from three-loop to four-loop results took more than 20 years. This certainly justifies the study of the analytic properties of these functions if this can reduce the complexity of calculations.

Before proceeding to the discussion of the large-s limit, let us point out the following subtlety. The definition of harmonic sums presented in Eq. (3.43) is valid only for positive integers, $s \in \mathbb{Z}_{>0}$. However,

[b] It should be noted that this result was obtained earlier in the critical model in [334] using the Lorentzian inversion formula.

[a] Note that anomalous dimensions of twist-2 operators and splitting functions are in one-to-one correspondence, see (1.66).

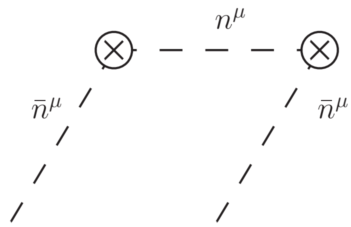


Figure 4.1: Wilson lines arising from the non-local operator (1.87) in the $s \to \infty$ limit.

the discussion of the continuous limit requires an analytic continuation, which is possible from $s \in \mathbb{Z}_{>0}$ to $s \in \mathbb{C} \setminus \{-1, -2, \ldots\}$. The simplest example of such a continuation is given by

$$S_1(s) = \psi(s+1) - \psi(1), \qquad\qquad \psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}. \tag{4.1}$$

For the generalization of this procedure, as well as for the corresponding `Mathematica` implementations, see [359,360]. Rational functions do not require any special continuation procedure and can be straightforwardly extended to the complex plane, except at the poles of their denominators. In what follows we do not specify this detail and always assume that analytically continued functions are used when continuous values of s are implied.

4.1.1 Cusp anomalous dimension

To begin the discussion of large-spin asymptotics, let us adopt the language of evolution kernels for non-local operators (see Section 1.4) and consider the one-loop evolution kernel (2.98) for the quark non-local operator (1.87). Using the connection with the forward anomalous dimensions, see (1.99), we obtain

$$\gamma^{(1)}_{\rm ns}(s) = 4C_F\left(2\int_0^1 d\alpha\,\frac{\bar\alpha}{\alpha}\left(1-\bar\alpha^{s-1}\right) - \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\,(1-\alpha-\beta)^{s-1} + \frac{1}{2}\right). \tag{4.2}$$

The asymptotics of both integrals in the limit $s \to \infty$ are easy to identify, namely

$$\begin{aligned} \int_0^1 d\alpha\,\frac{\bar\alpha}{\alpha}\left(1-\bar\alpha^{s-1}\right) \quad &\xrightarrow{s\to+\infty} \quad \ln s + \gamma_E - 1 + O(1/s), \\ \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\,(1-\alpha-\beta)^{s-1} \quad &\xrightarrow{s\to+\infty} \quad O(1/s^2), \end{aligned} \tag{4.3}$$

where γ_E is the Euler–Mascheroni constant. Thus, we find that the leading term in the large-spin asymptotics is logarithmic. It turns out that this fact extends beyond the one-loop result, and the separation of the evolution kernel into the two pieces in (4.3) reveals the physical picture behind it.

The single-integral kernel (the first integral in (4.3)) comes from the right diagram (b) in Fig. 2.1, where the gluon line connects the quark line and the Wilson line between points $z_1 n$ and $z_2 n$. The dominant contribution to this integral comes from the region $\alpha \to 0$. From the physical point of view, this corresponds to a soft gluon, and one can therefore consider the problem in the eikonal approximation, where the quarks are replaced by Wilson lines in the direction $\bar{n}^\mu$ [361–363]. The renormalization of the non-local operator thus reduces to the renormalization of Wilson lines with cusps [326], see Fig. 4.1. It should be noted that the cusp is formed by two light-like vectors, so the angle between them is formally ill-defined, $\phi \to \infty$, but it can be shown (see [364, Section 4.2]) that the value of s plays exactly the role of a regulator, resulting in the asymptotics [b]

$$\gamma_{\rm ns}^{(1)}(s) = 2\left(\Gamma_{\rm cusp}^F\right)^{(1)} \ln s + O\left(1/s^0\right), \tag{4.4}$$

where $\Gamma_{\rm cusp}^F$ in QCD is currently known to four-loop accuracy [365, 366] ($a = \alpha_s/4\pi$):

$$\begin{aligned}
\Gamma_{\rm cusp}^F(a) = a\,4C_F + a^2 C_F &\left[C_A\left(\frac{268}{9} - 8\zeta_2\right) - \frac{40}{9}n_f\right] \\
&+ a^3 C_F\left[C_A^2\left(\frac{176}{5}\zeta_2^2 + \frac{88}{3}\zeta_3 - \frac{1072}{9}\zeta_2 + \frac{490}{3}\right) + C_A n_f\left(-\frac{64}{3}\zeta_3 + \frac{160}{9}\zeta_2 - \frac{1331}{27}\right)\right. \\
&\left. + \frac{n_f}{N_c}\left(-16\zeta_3 + \frac{55}{3}\right) - \frac{16}{27}n_f^2\right] + O(a^4)\,.
\end{aligned} \tag{4.5}$$

It can be shown [327, 364, 367] that this consideration generalizes to all loop orders, as well as to the gluon–gluon channel (where one only needs to consider the Wilson line in a different color representation):

Logarithmic scaling of anomalous dimensions

Each entry of the singlet anomalous dimension matrix (1.63) has the following leading asymptotics in the large-s limit

$$\begin{aligned}
\gamma_{qq}(s) &= 2\Gamma_{\rm cusp}^F(a)\ln s + O(1/s^0), \\
\gamma_{qg}(s) &= O(1/s^0), \\
\gamma_{gq}(s) &= O(1/s^0), \\
\gamma_{gg}(s) &= 2\Gamma_{\rm cusp}^A(a)\ln s + O(1/s^0),
\end{aligned} \tag{4.6}$$

where $\Gamma_{\rm cusp}^A$ stands for the cusp anomalous dimension in the adjoint representation.

Let us also note that from the point of view of the functional space, this logarithmic asymptotic fixes the presence of harmonic sums with logarithmic growth, for example,

$$\begin{aligned}
S_1(s) &= \ln\bar{s} + O(1/s), \\
S_{1,1}(s) &= \frac{1}{2}\ln^2\bar{s} + \frac{\zeta_2}{2} + O(1/s), \\
S_{1,2}(s) &= \zeta_2\ln\bar{s} - \zeta_3 + O(1/s), \\
&\ldots
\end{aligned} \tag{4.7}$$

where $\bar{s} = se^{\gamma_E}$. Using (4.6) one can therefore not only restrict the functional space to asymptotically non-vanishing harmonic sums, but also predict their coefficients.

4.1.2 Reciprocity in the non-singlet case

The appearance of the cusp anomalous dimension as the leading term in the asymptotics (4.6) is a quite powerful result, which not only allows one to predict the coefficients in the ansatz (3.45) but also reveals hidden connections with other objects exhibiting the same universal behavior (see e.g. [368] and references therein for its appearance in the IR structure of QCD amplitudes). In fact, it turns out that

[b]The subscript F refers to the Wilson line in the fundamental representation of the color group, see (1.88) for details.

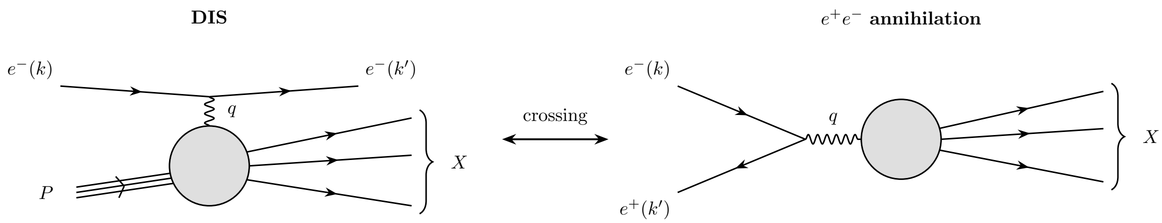


Figure 4.2: Crossing between the DIS process and e^+e^- annihilation.

the large-s asymptotics of anomalous dimensions is constrained even beyond the leading term. In this section we review the statement of *Gribov–Lipatov reciprocity*, starting from the historical results and presenting the modern formulation.

To formulate the origin of these constraints, let us switch to the language of splitting functions. Already at an early stage of the field it was noted [369, 370] that DIS (see Section 1.1) is related by crossing to the e^+e^- annihilation process, see Fig. 4.2. In the latter, instead of PDFs (see (1.37) for the definition) one considers *fragmentation functions*, $D(z, Q^2)$, and their scale evolution is controlled by *time-like* splitting functions, $P_T(z)$, in contrast to the *space-like* splitting functions (1.66). The crossing kinematic map suggests that the two are connected by

$$P_T(z) = -\frac{1}{z} P_S(1/z). \tag{4.8}$$

This relation, however, poses a problem due to the domain of these functions, $0 < z \leq 1$. Physically, the variable z denotes the fraction of parton momentum carried by the hadron after fragmentation. Therefore, to obtain $P_T(z)$ for $0 < z \leq 1$ one must extend the domain of the space-like splitting functions $P_S(x)$ to the region $1 \leq x < \infty$. This procedure requires an analytic continuation, and the first natural assumption, made by Gribov and Lipatov in [371], was that within the physical domain the splitting functions coincide, $P_T(x) = P_S(x)$, which implies the reciprocity property

$$P(x) = -\frac{1}{x} P\left(1/x\right), \qquad\qquad \gamma_{\rm ns}(s) = \gamma_{\rm ns}(-s-1). \tag{4.9}$$

Once the two-loop result became available, it was noted [337] that the original Gribov–Lipatov reciprocity (4.9) breaks at two-loop order for kinematic reasons (see [75] for details). However, the connection between the space-like and time-like channels persists beyond one loop and places a constraint on the anomalous dimension $\gamma_{\rm ns}(s)$. This constraint is conveniently formulated in terms of the large-s asymptotic expansion (see the MMV relations [372]), and in the modern language it can be stated as follows (see [76, 77] and [373] for a review):

Generalized Gribov-Lipatov reciprocity

- Consider the function $f_{\rm ns}(j)$, related to the anomalous dimension by the change of variables
$$\gamma_{\rm ns}(s) = f_{\rm ns}\underbrace{\left(s + \bar{\beta}(a) + \frac{1}{2}\gamma_{\rm ns}(s)\right)}_{j}, \tag{4.10}$$
where $\bar{\beta}(a) = \beta_0 a + \beta_1 a^2 + \ldots$, see (2.79).
- The asymptotic expansion of $f_{\rm ns}(j)$ in the large-j limit is invariant under the transfor-

mation $j \mapsto -j-1$, [a] i.e. it is organized as

$$f_{\rm ns}(j) = 2\Gamma^F_{\rm cusp}(a)\ln\bar{j} + \# + \frac{\#}{j(j+1)} + \frac{\#}{j^2(j+1)^2} + O\left(\frac{1}{j^3(j+1)^3}\right), \tag{4.11}$$

where $\bar{j} = je^{\gamma_E}$.

The function $f_{\rm ns}(j)$ whose asymptotic expansion is given by (4.11) is called *reciprocity-respecting* (RR). For brevity we will also say that the anomalous dimension $\gamma_{\rm ns}(s)$ possesses the reciprocity property. Note that the requirement for a function to be RR is rather constraining: for a generic function, the asymptotic expansion analogous to (4.11) also contains terms of the form $\sqrt{j(j+1)}$, $\sqrt{j(j+1)}^3$, etc. The connection between the RR function and the usual anomalous dimension (4.10) is non-linear and in practice is expanded perturbatively:

$$\begin{aligned}
f^{(1)}_{\rm ns}(j) =& \gamma^{(1)}_{\rm ns}(j),\\
f^{(2)}_{\rm ns}(j) =& \gamma^{(2)}_{\rm ns}(j) - \partial_j\gamma^{(1)}_{\rm ns}(j)\left(\beta_0 + \frac{1}{2}\gamma^{(1)}_{\rm ns}(j)\right),\\
f^{(3)}_{\rm ns}(j) =& \gamma^{(3)}_{\rm ns}(j) - \partial_j\gamma^{(2)}_{\rm ns}(j)\left(\beta_0 + \frac{1}{2}\gamma^{(1)}_{\rm ns}(j)\right) - \partial_j\gamma^{(1)}_{\rm ns}(j)\left(\beta_1 + \frac{1}{2}\gamma^{(2)}_{\rm ns}(j)\right)\\
&+ \frac{1}{2}\left(\left(\partial_j\gamma^{(1)}_{\rm ns}(j)\right)^2 + \partial_j^2\gamma^{(1)}_{\rm ns}(j)\left(\beta_0 + \frac{1}{2}\gamma^{(1)}_{\rm ns}(j)\right)\right)\left(\beta_0 + \frac{1}{2}\gamma^{(1)}_{\rm ns}(j)\right).
\end{aligned} \tag{4.12}$$

The form of the asymptotic expansion (4.11) places quite serious constraints on the space of harmonic sums present in the RR anomalous dimension $f_{\rm ns}(j)$: their large-j asymptotic expansion must be consistent with (4.11). The basis of such RR harmonic sums was worked out in [78, 79, 374] (see also [375, 376] for an alternative construction in terms of *binomial sums*). The RR harmonic sums $\Omega_{\vec{m}}(j)$ are defined recursively, $\Omega_{a,\vec{m}}(j) = \omega_a(\Omega_{\vec{m}})(j)$, where the map ω_a acting on the harmonic sums in $\Omega_{\vec{m}}$ gives

$$\omega_a(S_{a,b,\vec{m}})(j) = S_{a,b,\vec{m}}(j) - \frac{1}{2}S_{a\wedge b,\vec{m}}(j), \qquad a\wedge b = \mathrm{sign}(ab)\big(|a|+|b|\big). \tag{4.13}$$

This rule is to be applied together with the restriction on the initial indices in $\Omega_{\vec{m}}$: the vector $\vec{m}$ consists of positive odd and negative even integers. This gives, for example, the following low-weight RR harmonic sums:

$$\begin{aligned}
\Omega_a &= S_a,\\
\Omega_{1,-2} &= S_{1,-2} - \frac{1}{2}S_{-3},\\
\Omega_{1,3,-2} &= S_{1,3,-2} - \frac{1}{2}\big(S_{1,-5} + S_{4,-2}\big) + \frac{1}{4}S_{-6},\\
\Omega_{1,-2,3,1} &= S_{1,-2,3,1} - \frac{1}{2}\big(S_{-3,3,1} + S_{1,-2,4} + S_{1,-5,1}\big) + \frac{1}{4}\big(S_{-6,1} + S_{-3,4} + S_{1,-6}\big) - \frac{1}{8}S_{-7}.
\end{aligned} \tag{4.14}$$

One can thus give an equivalent characterization of a reciprocity-respecting function, $f_{\rm ns}(j)$ is reciprocity-respecting if:

- All harmonic sums in $f_{\rm ns}(j)$ can be expressed in terms of RR harmonic sums (4.13).

[a] The symmetry $j \mapsto -j-1$ applies here to the power-series part of the expansion.

- All rational functions are exactly symmetric under the transformation $j \to -j-1$.

The number of RR harmonic sums at a given weight w is greatly reduced compared to the number of ordinary harmonic sums, $2 \cdot 3^{w-1} \mapsto 2^{w-1}$. This gives a significant advantage for the reconstruction procedure described in Section 3.3: at ℓ-th loop order one can first reconstruct only the RR part of the anomalous dimension and then use (4.10) to restore the full anomalous dimension from the lower-loop results.

4.1.3 Reciprocity from the CFT perspective

From the formulation of the reciprocity relation in (4.10) and (4.11) one can clearly see a connection between this statement and conformal invariance. Indeed, if one considers QCD at the critical point (see (2.80) and the discussion therein), the value of the variable

$$j^* = s - \epsilon + \frac{1}{2}\gamma^*_{\text{ns}}(s) \tag{4.15}$$

coincides with the conformal spin [c] (see (2.20) and Section 2.4 for the discussion) of the quark non-singlet twist-2 operator (1.30), and moreover the expansion in (4.11) is organized in powers of the $\mathfrak{sl}(2,\mathbb{R})$ Casimir (2.19). This connection was already exploited in [76] and was further developed in the approach of [102] (see also [377] for earlier work in this direction). In this section we briefly review the arguments leading to the reciprocity relation in this language.

Consider for simplicity (see [102, Appendix A] for the generalization) a general CFT with a scalar field denoted $\Phi(x)$. The leading-twist operators enter the conformal OPE (2.36) of these fields, namely

$$\Phi(x)\Phi(0) = \sum_s C_{\tau_0,s}|x|^{\tau_0-2\Delta_\Phi} x^{\mu_1}\dots x^{\mu_s}\mathcal{O}_{\mu_1\dots\mu_s}(0) + (\text{higher twists}), \tag{4.16}$$

where τ_0 is the leading twist and Δ_Φ is the scaling dimension of the field Φ. The next step is to use the crossing symmetry of correlators, following the approach of [378, 379]. For this we consider the four-point correlator (see (2.41) for discussion)

$$G(u,v) = x_{12}^{2\Delta_\Phi} x_{34}^{2\Delta_\Phi}\big\langle\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\big\rangle, \tag{4.17}$$

where u and v are conformal cross-ratios, see (2.42). Using the conformal OPE (4.16), one can expand the four-point function (4.17) in two different ways, corresponding kinematically to the s- and t-channels. Restricting to the leading-twist contribution one obtains

$$\begin{aligned} G(u,v)\Big|_{\text{twist }\tau_0} &\sim \sum_s a_s u^{\tau_0/2+\gamma(s)/2}(1-v)^s\big(C_s \ln v + \dots\big) \\ &\sim u^{\tau_0/2}v^{-\tau_0/2}, \end{aligned} \tag{4.18}$$

where a_s and C_s are constants related to the structure of four-dimensional conformal blocks [185]. The main idea is that on the one hand the limit $v \to 0$ is given by a power-like singularity, while on the other hand it is controlled by an infinite sum of logarithmic contributions from the leading-twist operators (4.16). This can happen only if the structure of the expansion — and specifically its tail, i.e. the $s \to \infty$ limit — is constrained in a specific way. This is the key idea, and the remainder of [102] is dedicated to a careful derivation of the reciprocity property (4.11) by analyzing the expansion (4.18).

In the rest of this section we do not pursue the complete derivation but instead discuss an important point in the application of this consideration to QCD. The expansion (4.18) is of course valid only at

[c] It should be noted that the value of the conformal spin in definition (2.20) differs by 1. In the literature on collinear conformal symmetry in QCD, the definition (4.15) is more usual. It also corresponds to organizing the expansion in powers of $j(j+1)$ in (4.11) rather than $j(j-1)$ as in definition (2.19).

the critical point (2.80). The remarkable property of the $\overline{\text{MS}}$ scheme is that the perturbative expansions for $\gamma_{\text{ns}}(s)$ at the critical point and at arbitrary coupling,

$$\begin{aligned}\gamma_{\text{ns}}(s) &= a\gamma_{\text{ns}}^{(1)}(s) + a^2\gamma_{\text{ns}}^{(2)}(s) + \dots,\\ \gamma^*_{\text{ns}}(s) &= a^*\gamma_{\text{ns}}^{(1)}(s) + (a^*)^2\gamma_{\text{ns}}^{(2)}(s) + \dots,\end{aligned} \tag{4.19}$$

differ only through the value of the coupling, while the coefficients in the perturbative expansion $\gamma_{\text{ns}}^{(\ell)}(s)$ are ϵ-independent. This establishes a one-to-one correspondence at the level of non-perturbative functions, [d]

$$\gamma^*_{\text{ns}}(s;\epsilon) = \gamma_{\text{ns}}(s; a^*(\epsilon)), \qquad \gamma_{\text{ns}}(s;a) = \gamma^*_{\text{ns}}(s; -\bar{\beta}(a)), \tag{4.20}$$

which allows one to trade the conformal spin (4.15) back for the physical coupling and thereby obtain the relation (4.10). This approach was used in [76] to connect reciprocity in conformal theories with that in QCD. We also referred to it in Section 2.4 when discussing the constraints that collinear conformal symmetry imposes on the evolution kernels, see (2.102).

4.1.4 Reciprocity in the singlet case

In the previous sections we discussed the Gribov–Lipatov reciprocity in the non-singlet case, which is conceptually simpler. Indeed, the operator (1.30) is multiplicatively renormalizable (if we do not consider mixing with total derivatives) and the corresponding anomalous dimension $\gamma_{\text{ns}}(s)$ directly controls the scaling. In the flavor-singlet case the situation is more complicated due to mixing, recalling the RG equation (1.63) we have

$$\left(\left(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a}\right)\mathbb{1} + \begin{pmatrix}\gamma_{qq}(s) & \gamma_{qg}(s)\\ \gamma_{gq}(s) & \gamma_{gg}(s)\end{pmatrix}\right)\begin{pmatrix}[\mathcal{O}^q_s(x)]\\ [\mathcal{O}^g_s(x)]\end{pmatrix} = 0. \tag{4.21}$$

The individual entries of the singlet anomalous dimension matrix $\widehat{\gamma}(s)$ do not possess the reciprocity property, as was noted already at two-loop order [380].

The consideration in Section 4.1.3 clearly suggests how to identify which objects should be reciprocal. Since reciprocity can be traced to the behavior of the four-point correlator at the critical point, what possesses reciprocity are the eigenvalues of the dilatation operator (the CFT data (2.45)). In particular, for the anomalous dimension matrix $\widehat{\gamma}(s)$ one expects the eigenvalues

$$\gamma_\pm(s) = \frac{1}{2}\left(\operatorname{tr}\widehat{\gamma}(s) \pm \sqrt{(\operatorname{tr}\widehat{\gamma}(s))^2 - 4\det\widehat{\gamma}(s)}\right) \tag{4.22}$$

to be reciprocal. For each of them one can thus define

$$\gamma_\pm(s) = f_\pm\left(s + \bar{\beta}(a) + \frac{1}{2}\gamma_\pm(s)\right), \tag{4.23}$$

where the RR function $f_\pm(j)$ satisfies the expansion (4.11). The reciprocity property of the eigenvalues was observed in the literature (see [76, Section 2.2]) and is confirmed by more recent considerations [103].

The eigenvalues (4.22) are inconvenient to work with in the perturbative expansion due to the presence of square roots. However, since reciprocity is a property of the scaling dimensions regardless of the basis, we expect all matrix invariants of the anomalous dimension matrix $\widehat{\gamma}(s)$ to possess the

[d] By non-perturbative here we mean a hypothetical function providing (4.19) as an expansion in the coupling. This consideration is, of course, not sensitive to genuinely non-perturbative corrections, e.g. of the form e^{-1/a^2}.

reciprocity property. For example, one can choose the sum and product of the eigenvalues (4.22) and derive the perturbative expansion for the sum:

$$f_+^{(1)} + f_-^{(1)} = T^{(1)}, \tag{4.24}$$

$$f_+^{(2)} + f_-^{(2)} = T^{(2)} - \partial_j T^{(1)}\left(\beta_0 + \frac{1}{2}T^{(1)}\right) + \frac{1}{2}\partial_j D^{(2)}, \tag{4.25}$$

$$\begin{aligned} f_+^{(3)} + f_-^{(3)} =& T^{(3)} - \partial_j T^{(2)}\left(\beta_0 + \frac{1}{2}T^{(1)}\right) + \frac{1}{2}\partial_j D^{(3)} - \partial_j T^{(1)}\left(\beta_1 + \frac{1}{2}T^{(2)} + \frac{1}{4}\partial_j D^{(2)}\right) \\ &+ \frac{1}{2}\left(\partial_j T^{(1)}\right)^2\left(\beta_0 + \frac{1}{2}T^{(1)}\right) + \frac{1}{2}\partial_j^2 T^{(1)}\left(\left(\beta_0 + \frac{1}{2}T^{(1)}\right)^2 - \frac{1}{4}D^{(2)}\right) \\ &- \frac{1}{4}\partial_j^2 D^{(2)}\left(2\beta_0 + \frac{1}{2}T^{(1)}\right), \end{aligned} \tag{4.26}$$

and for the product:

$$(f_+ \cdot f_-)^{(1)} = D^{(2)}, \tag{4.27}$$

$$(f_+ \cdot f_-)^{(2)} = D^{(3)} - \beta_0 \partial_j D^{(2)} - \frac{1}{2}\partial_j T^{(1)} D^{(1)}, \tag{4.28}$$

$$\begin{aligned} (f_+ \cdot f_-)^{(3)} =& D^{(4)} - \frac{1}{2}\partial_j T^{(1)}\left(D^{(3)} - \beta_0 \partial_j D^{(2)}\right) - \beta_0 \partial_j D^{(3)} - \beta_1 \partial_j D^{(2)} \\ &+ D^{(2)}\left(\left(\frac{1}{2}\partial_j T^{(1)}\right)^2 - \frac{1}{2}\partial_j T^{(2)}\right) + \frac{1}{2}\partial_j^2 D^{(2)}\left(\beta_0^2 - \frac{1}{4}D^{(2)}\right) \\ &+ \frac{1}{4}\partial_j^2 T^{(1)} D^{(2)}\left(2\beta_0 + \frac{1}{2}T^{(1)}\right), \end{aligned} \tag{4.29}$$

where we use $T = \operatorname{tr}\widehat{\gamma}(j)$ and $D = \det\widehat{\gamma}(j)$ to simplify the notation. We also use notation for the perturbative expansion, $T = T^{(1)}a + T^{(2)}a^2 + \ldots, D = D^{(2)}a^2 + D^{(3)}a^3 \ldots$ [e]. The choice of matrix invariants is rather arbitrary and, depending on the application, can alternatively be made as e.g. $\operatorname{tr}\widehat{\gamma}(s)$, $\det\widehat{\gamma}(s)$ or $\operatorname{tr}\widehat{\gamma}(s)$, $\operatorname{tr}\widehat{\gamma}^2(s)$. We will return to the question of reciprocity of the singlet matrix in Chapter 5 and will show that some additional information can be derived.

4.2 Double-logarithmic equation

Section 4.1 was dedicated to the limit $s \to \infty$ of the anomalous dimensions of twist-2 operators, and we saw that a quite large set of constraints can be derived from different physical considerations. In particular, in Section 4.1.3 we discussed how the CFT arising at the QCD critical point (2.80) can provide non-trivial input. In this section we consider the opposite limit of small spins. For concreteness we again focus on the non-singlet case (1.30), $\gamma_{\rm ns}(s)$, and the limit $s \to 0$. It can be noted that such a limit is non-physical: indeed, from the definition (1.30) we see that it effectively corresponds to "-1" derivatives. However, from the point of view of perturbative QCD a resummation is possible, and we start this section by reviewing corresponding procedures.

First, let us recall the one-loop result for the anomalous dimension:

$$\gamma_{\rm ns}^{(1)}(s) = 2C_F\left(4S_1(s) - \frac{2}{s(s+1)} - 3\right). \tag{4.30}$$

[e]Note that the expansion of the determinant starts at order a^2

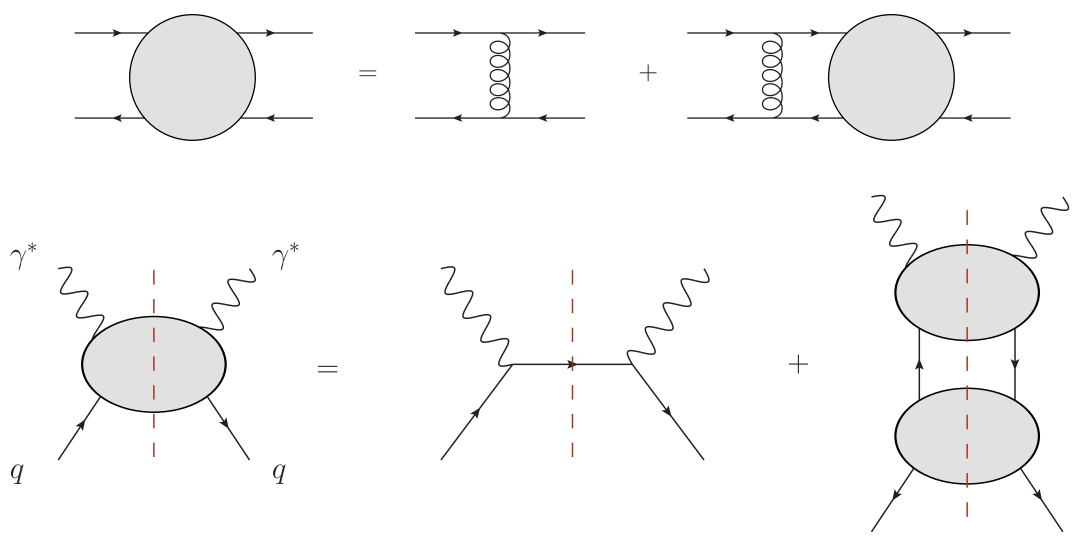


Figure 4.3: Graphical representation of the Bethe–Salpeter equation for the resummation of double-logarithmic contributions, $\alpha_s \ln^2 x$. The upper picture represents the resummation of ladder diagrams corresponding to the double-logarithmic contributions in $2 \to 2$ scattering, and the lower picture represents the use of the corresponding kernel for the forward Compton scattering, where the dashed line denotes the physical cut. For details of this derivation see [83].

The harmonic sum S_1 is regular in the $s \to 0$ limit,

$$S_1(s) = \zeta_2 s - \zeta_3 s^2 + O(s^3), \tag{4.31}$$

so the asymptotics of the full one-loop anomalous dimension reads

$$\gamma_{\rm ns}^{(1)}(s) = -\frac{4C_F}{s} - 2C_F + O(s), \tag{4.32}$$

and indeed the one-loop anomalous dimension acquires a right-most pole at $s \to 0$. Going to higher loops one can see how this pattern generalizes to

$$\gamma_{\rm ns}^{(\ell)}(s) = \frac{\#}{s^{2\ell-1}} + \frac{\#}{s^{2\ell-2}} + O\left(\frac{1}{s^{2\ell-3}}\right). \tag{4.33}$$

The presence of such singular terms can be motivated by considering the corresponding splitting function. The leading terms correspond to the Mellin transform

$$\int_0^1 dx\, x^{s-1} \big(\ln x\big)^{2\ell-2} = \frac{(2\ell-2)!}{s^{2\ell-1}}. \tag{4.34}$$

Thus, at ℓ-th loop order the singular contributions arise from logarithmically divergent terms in the $x \to 0$ limit, starting from $\alpha_s^\ell (\ln x)^{2\ell-2}$ and continuing through to the constant. The physical origin of such terms in the splitting functions is the so-called *double-logarithmic limit* of the four-point amplitude.

This limit was first considered in QED [381–383], where it was shown that the four-point amplitude in the kinematic limit $\mathbf{s} \to \infty$ ($\mathbf{s}$ here is a Mandelstam variable) acquires double-logarithmic behavior $\ln^2 \mathbf{s}$. This result was then generalized to QCD [384–386], [a] and the use of the *Bethe–Salpeter equation* (see Fig. 4.3) made it possible to resum contributions of the type $(\alpha_s \ln^2 \mathbf{s})^\ell$. Applying this resummation

[a]Note, however, that the authors of [384–386] used a somewhat more advanced approach to the resummation, based on isolating the softest momentum of the quark line, rather than a straightforward Bethe–Salpeter equation.

procedure to DIS and subsequently extracting the scale dependence [83, 84] (see also [387, 388] for the discussion on the non-singlet polarized case) allows one to resum the corresponding double-logarithmic contributions $(\alpha_s \ln^2 x)^\ell$ in the splitting functions (1.66). Translated to Mellin space, the corresponding Bethe–Salpeter equation takes the following form, which we will call the *double-logarithmic equation* (DLE):

Double-logarithmic equation

$$2\gamma_{\rm ns}(s)\left(s+\frac{1}{2}\gamma_{\rm ns}(s)\right) = -8C_F a. \tag{4.35}$$

This equation effectively resums all leading singular contributions in the $s \to 0$ limit. Indeed, solving (4.35) as a quadratic equation on the one hand yields a square-root branch point instead of poles, and on the other hand reproduces terms of the form $a^\ell/s^{2\ell-1}$ to all orders:

$$\gamma_{\rm ns}(s) = -s+\sqrt{s^2-8C_F a} = -4\frac{C_F a}{s} - 8\frac{C_F^2 a^2}{s^3} - 32\frac{C_F^3 a^3}{s^5} + O(a^4). \tag{4.36}$$

One can verify by taking the explicit limit $s \to 0$ in perturbative results that the expansion (4.36) reproduces the leading singularity at each perturbative order. Let us also comment on the relation of this resummation procedure to the BFKL approach. Passing to the singlet channel one can consider the analogous Bethe–Salpeter-based derivation, now based on resumming gluon ladders. Two things change when the propagating quark is replaced by a gluon:

- Each rung of the ladder produces a single logarithm, $\ln \mathbf{s}$, instead of a double logarithm.
- The overall amplitude is $\mathbf{s}$-enhanced, which translates into an overall factor of $1/x$.

Together these translate into contributions of the form $\alpha_s^\ell (\ln x)^{\ell-1}/x$ in the splitting functions, which under the Mellin transform behave as

$$\int_0^1 dx\, x^{s-1}\frac{\big(\ln x\big)^{\ell-1}}{x} = (-1)^{\ell-1}\frac{(\ell-1)!}{(s-1)^\ell}, \tag{4.37}$$

shifting the position of the right-most singularity and giving the dominant contribution (in the limit $x \to 0$) in the singlet case. These singularities can also be resummed using methods other than (4.35), see [85–87] for applications in QCD and [314, 389, 390] for applications in $\mathcal{N}=4$ SYM.

The generalization of the DLE (4.35) to higher-loop orders (corresponding to next-to-leading logarithm orders in splitting functions) via the Bethe–Salpeter approach is a highly non-trivial problem. Remarkably, the equation (4.35) itself turns out to have a much larger domain of validity. In [391] (see also [392] for additional details) it was noted that upon simply replacing the QCD non-singlet anomalous dimension by the $\mathcal{N}=4$ SYM *universal anomalous dimension*, [b] $\gamma_{\mathcal{N}=4}(s)$, the same combination of anomalous dimensions exhibits regular behavior:

$$2\gamma_{\mathcal{N}=4}(s)\left(s+\frac{1}{2}\gamma_{\mathcal{N}=4}(s)\right) = \sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\left(\mathbf{C}_{\mathcal{N}=4}\right)^k_m a^k s^m, \tag{4.38}$$

where $\left(\mathbf{C}_{\mathcal{N}=4}\right)^k_m$ are some constants (see [391, Appendix A] for the specific values). Interestingly, this combination can be expanded around different negative integer points and still yields a regular result. We, however, focus on the case corresponding to $s \to 0$ in the quark non-singlet operator, so we do not continue this discussion. Equation (4.38) holds up to seven loops [314] (higher-loop corrections are

[b] Due to the presence of supersymmetry, the spectrum of $\mathcal{N}=4$ SYM is largely degenerate, causing the anomalous dimensions of different operators to coincide, see [390] for details.

not available) in the planar limit, and begins to break at four-loop order [314, 316] in the non-planar contributions. It should however be noted that the generalized DLE (4.38) still fully resums the leading and subleading poles, $s^{1-2\ell}$ and $s^{2-2\ell}$, at ℓ-th loop order.

This simple form of the generalized DLE for $\mathcal{N}=4$ SYM motivated a search for a similar generalization in QCD, and in [82] it was observed that the following generalization holds:

Generalized double-logarithmic equation in QCD

$$\delta m^2_{\rm ns}(s) = 2\gamma_{\rm ns}(s)\left(s+\bar{\beta}(a)+\frac{1}{2}\gamma_{\rm ns}(s)\right) = \sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\mathbf{C}^k_m a^k s^m. \tag{4.39}$$

We have introduced the notation $\delta m^2_{\rm ns}(s)$ for this combination of anomalous dimensions, the origin of this name will become clear later. Using the available perturbative data, the generalized DLE in QCD has been verified to hold up to four-loop order [308] in the planar limit and begins to break at three loops [82, 393]. In particular, one obtains

$$\begin{aligned}
\delta m^{2,+}_{\rm ns}(s) = {}& -8aC_F + O(s) + a^2\Bigg\{\left(52-32\zeta_2\right)C_F^2 - \frac{1340}{9}C_AC_F + \frac{200}{9}C_Fn_f + O(s)\Bigg\}\\
&+a^3\Bigg\{\Bigg(576C_F^3 - 768C_F^2C_A + 240C_FC_A^2\Bigg)\frac{\zeta_2}{s^2} + \Bigg(-960C_F^3 + 1216C_F^2C_A - 368C_FC_A^2\\
&-64C_F^2n_f + 32C_AC_Fn_f\Bigg)\frac{\zeta_2}{s} + \left(4+1360\zeta_2+512\zeta_3-224\zeta_2^2\right)C_F^3 + \Bigg(\frac{3542}{9} - \frac{19168}{9}\zeta_2\\
&-\frac{2944}{3}\zeta_3 + \frac{2176}{5}\zeta_2^2\Bigg)C_F^2C_A + \Bigg(-\frac{30910}{27} + 448\zeta_2 + 464\zeta_3 - \frac{648}{5}\zeta_2^2\Bigg)C_FC_A^2 - \frac{152}{27}C_Fn_f^2\\
&+\Bigg(\frac{616}{9} + \frac{1408}{9}\zeta_2 - \frac{128}{3}\zeta_3\Bigg)C_F^2n_f + \Bigg(\frac{2432}{9} - 32\zeta_2 + 32\zeta_3\Bigg)C_FC_An_f + O(s)\Bigg\} + O(a^4).
\end{aligned} \tag{4.40}$$

Here we note an important feature of the non-singlet anomalous dimension. One can perform two types of analytic continuation of $\gamma_{\rm ns}(s)$ from $s\in\mathbb{Z}_{>0}$: from even values of spin, $\gamma^+_{\rm ns}(s)$, or from odd ones, $\gamma^-_{\rm ns}(s)$. The reason these cannot be unified into a single continuation is the presence of $(-1)^s$ in the functional form of $\gamma_{\rm ns}(s)$. To obtain (4.40) we continued from the even integers, which is the natural choice for the consideration of $s=0$. One can also construct the same combination (4.39) for the continuation from the odd integers:

$$\delta m^{2,-}_{\rm ns}(s) = -8aC_F + a^2\Bigg\{\left(64C_F^2 - 32C_AC_F\right)\frac{1}{s^2} + \Bigg(-64C_F^2 + 32C_FC_A\Bigg)\frac{1}{s} + O(s^0)\Bigg\} + O(a^3), \tag{4.41}$$

which becomes divergent already at two loops. The regularity of $\delta m^{2,-}_{\rm ns}(s)$ at one-loop order is expected, since $\left(\gamma^+_{\rm ns}(s)\right)^{(1)} = \left(\gamma^-_{\rm ns}(s)\right)^{(1)}$. The original double-logarithmic analysis [83, 84, 386, 394] in principle allows one to derive an analogue of the one-loop double-logarithmic equation, whose form is considerably more complicated (see [393, Eq. (2.20)] for the specific result). We do not pursue this line of thought further, especially given that its generalization to higher-loop orders is not available. Let us point out one remarkable property shared by both results (4.40) and (4.41). Although both acquire singular terms starting from three- and two-loop order respectively, these singular contributions vanish in the planar limit. Specifically, upon setting

$$C_A \to N_c, \qquad\qquad C_F \to N_c/2, \tag{4.42}$$

one observes a complete cancellation of *all* singular terms. The cancellation of the subleading terms for $\gamma^-_{\rm ns}(s)$ in the planar limit is anticipated as in this limit the anomalous dimension itself coincides with the even part, $\gamma^+_{\rm ns}(s)$.

Let us comment on the technical applications of equation (4.39). Solving it as a quadratic equation gives

$$\gamma_{\rm ns}(s) = -s - \bar\beta(a) \pm \sqrt{\left(s+\bar\beta(a)\right)^2 + \delta m^2_{\rm ns}(s)}. \tag{4.43}$$

The limit $\lim_{s\to 0}\gamma_{\rm ns}(s)$ can now be defined via the resummed expression (4.43). Indeed, since $\lim_{s\to 0}\delta m^2_{\rm ns}(s)$ is well-defined (up to the order where (4.39) breaks), one can first take this limit and then expand in a perturbative series. As we saw in (4.36), this equation also predicts the singular contributions at higher loop orders from lower-loop input. This is conveniently seen by expanding (4.39) in a perturbative series:

$$\begin{aligned}
\left(\delta m^2_{\rm ns}(s)\right)^{(1)} &= 2s\gamma^{(1)}_{\rm ns}(s),\\
\left(\delta m^2_{\rm ns}(s)\right)^{(2)} &= 2s\gamma^{(2)}_{\rm ns}(s) + 2\gamma^{(1)}_{\rm ns}(s)\left(\beta_0 + \frac{1}{2}\gamma^{(1)}_{\rm ns}(s)\right),\\
\left(\delta m^2_{\rm ns}(s)\right)^{(3)} &= 2s\gamma^{(3)}_{\rm ns}(s) + 2\gamma^{(2)}_{\rm ns}(s)\left(\beta_0 + \frac{1}{2}\gamma^{(1)}_{\rm ns}(s)\right) + 2\gamma^{(1)}_{\rm ns}(s)\left(\beta_1 + \frac{1}{2}\gamma^{(2)}_{\rm ns}(s)\right),\\
&\ldots .
\end{aligned} \tag{4.44}$$

The regularity of $\delta m^2_{\rm ns}(s)$ at a given perturbative order can be viewed as a set of equations on the coefficients in the expansion around $s = 0$. For example, denoting the coefficient of the leading singularity at ℓ-th loop order by X_ℓ, the system (4.44) yields the recursion relation

$$X_\ell = -\frac{1}{2}\sum_{k=1}^{\ell-1} X_{\ell-k}X_k, \tag{4.45}$$

which can be solved explicitly: [c]

$$X_\ell = (-1)^{\ell-1}\frac{2}{\ell}\binom{2\ell-2}{\ell-1}\left(\frac{X_1}{2}\right)^\ell, \qquad \gamma^{(\ell)}_{\rm ns}(s) = -\frac{2}{\ell}\binom{2\ell-2}{\ell-1}\frac{(2C_F)^\ell}{s^{2\ell-1}} + O\left(1/s^{2\ell-2}\right). \tag{4.46}$$

This result is of course derived under the assumption that (4.39) resums the leading singularity at every perturbative order, which, given the three-loop violation, is not guaranteed. However, it can be shown that the resummation of the leading singularity (as well as the subleading one) is a simple consequence of the reciprocity relation (see Section 4.1.2) for $\gamma_{\rm ns}(s)$.

4.3 Mixing with the shadow operator

The situation regarding the double-logarithmic equation in the QCD community was rather interesting following the discovery of the generalization (4.39). Of course, the cancellation of a large number of subleading poles could in principle be accidental, but the number of accidentally matched parameters in that case would be far too high. A satisfactory explanation for the validity of such a generalization was lacking. On the other hand, the presence of the β-function was quite suggestive and hinted at

[c] This solution, of course, coincides with the expansion of the square root in (4.36).

the importance of conformal symmetry. Indeed, rewritten at the critical point using the conformal spin (2.20), the combination of anomalous dimensions reduces to

$$\delta m^{2,*}_{\rm ns}(s) = 2\gamma^*_{\rm ns}(s)\left(s-\epsilon+\frac{1}{2}\gamma^*_{\rm ns}(s)\right) = 2j^* f^*_{\rm ns}(j^*). \tag{4.47}$$

It therefore seems natural to expect that considering the problem at the critical point with a subsequent translation to the physical coupling (see (4.20) for details) could provide additional insight.

Once one has passed to the critical point and established the CFT framework, it is possible to take the next step. The nature of the problem itself suggests the right way of thinking: the presence of a singularity as $s\to 0$ in the anomalous dimension signals some incompleteness of the description. It turns out that this incompleteness is related to operator mixing. This may sound confusing, since we argued that the non-singlet operator does not mix (aside from total derivatives), and this is precisely where the CFT setup begins to play a role. In brief, the additional mixing arises when one performs the analytic continuation $s\in\mathbb{Z}_{>0}$ to $s\in\mathbb{C}\setminus\{0,-1,-2,\ldots\}$. The point is that one cannot simply analytically continue the CFT spectrum (see [80] for the construction) and expect to remain with a description purely in terms of local twist-2 operators (see [71] for the discussion of continuation on the operator level). In this section we follow [81] on the question of how to apply these ideas in practice. We begin with the somewhat pedagogical example of the $O(N)$-symmetric φ^4 model (the direct generalization of the Wilson–Fisher CFT considered in [81]) and then proceed to other theories. At the end we also discuss how this framework operates in QCD (see [73] for a more detailed discussion).

4.3.1 $O(N)$-symmetrical φ^4 model

Let us begin with the simple example of the $O(N)$-symmetric φ^4 model, which in $(d=4-2\epsilon)$-dimensional Euclidean space is described by the renormalized action

$$S_R = \int d^dx\left(\frac{Z_1}{2}\left(\partial\varphi\right)^2 + Z_3\mu^{2\epsilon}\frac{g}{4!}\left(\varphi^2\right)^2\right). \tag{4.48}$$

The field φ_a $(a=1,\ldots,N)$ transforms under the fundamental representation of the $O(N)$ group. Here $Z_{1,3}$ are the renormalization constants in the $\overline{\rm MS}$ scheme and μ is the renormalization scale (see Section 1.3 for the detailed discussion). The model (4.48) plays an important role in the theory of critical phenomena [102, 168, 286, 395] and also serves as an ideal testing ground for the application of new methods and techniques, see e.g. [396, 397]. We also considered a specific example of this model $(N=1)$ in the discussion of critical conformal invariance in Section 2.3 and its relation to the phase transition in the Ising model. The RG functions in this model are known with high precision, see [142, 198, 398–400] for recent developments:

$$\begin{aligned}\beta(u) &= -2\epsilon u + \frac{N+8}{3}u^2 - \frac{3N+14}{3}u^3\\ &\quad + \frac{u^4}{216}\Big(33N^2+922N+2960+96(5N+22)\zeta_3\Big) + O(u^5) \equiv -2u(\epsilon+\bar\beta(u)),\\ \gamma_\varphi(u) &= (N+2)\left(\frac{u^2}{36} - \frac{u^3(N+8)}{432} + \frac{5u^4}{5184}(-N^2+18N+100)\right) + O(u^5),\end{aligned} \tag{4.49}$$

where we use the convenient change of variables $u=g/(4\pi)^2$ for the coupling.

The set of leading-twist operators ($\tau_0=2$, as in QCD) in the model (4.48) is defined as

$$\mathcal{O}^{ab}_s(x) \equiv \mathcal{O}^{ab}_{\mu_1\ldots\mu_s}(x) = \varphi^a(x)\partial_{\mu_1}\ldots\partial_{\mu_s}\varphi^b(x) - \text{traces}, \tag{4.50}$$

and their renormalization reads

$$\left[\mathcal{O}^{ab}_s(x)\right] = Z^{ab}_{a'b'}(s)\mathcal{O}^{a'b'}_s(x). \tag{4.51}$$

As in QCD with the flavor group, it is convenient to project onto the irreducible representations of the $O(N)$ group in order to obtain multiplicatively renormalizable operators:

Twist-2 operators in the $O(N)$-symmetric model

The multiplicatively renormalizable leading-twist operators in the model (4.48) are organized with respect to irreducible representations of the $O(N)$ group:

- Scalar operator ($s \in 2\mathbb{Z}_{\geq 0}$):
$$\mathcal{O}_s(x) = \delta^{ab}\mathcal{O}_s^{ab}(x), \tag{4.52}$$
- Symmetric-traceless operator ($s \in 2\mathbb{Z}_{\geq 0}$):
$$\mathcal{O}_s^+(x) = \frac{1}{2}\big(\mathcal{O}_s^{ab}(x) + \mathcal{O}_s^{ba}(x)\big) - \frac{1}{N}\delta^{ab}\mathcal{O}_s(x), \tag{4.53}$$
- Anti-symmetric operator ($s \in 2\mathbb{Z}_{\geq 0} + 1$):
$$\mathcal{O}_s^-(x) = \frac{1}{2}\big(\mathcal{O}_s^{ab}(x) - \mathcal{O}_s^{ba}(x)\big). \tag{4.54}$$

Note that the definitions of the operators (4.52) and (4.53) are appropriate for *even* values of s, for odd s these operators reduce to total derivatives. The same argument justifies the definition of the operator (4.54) for *odd* values of s. The anomalous dimensions of the operator (4.52) for $N = 1$ were obtained at four-loop order in [263] and subsequently generalized to arbitrary N in [270]. They read

$$\begin{aligned}
\gamma(s) = 2\gamma_\varphi - u^2\frac{N+2}{9s(s+1)}\Bigg\{&3 + \frac{u}{3}(N+8)\left(2S_1 - 5 + \frac{3}{2}\frac{2s+1}{s(s+1)}\right) + \frac{u^2}{9}\Bigg((N+8)^2 S_2\\
&+ \big(N^2+11N+42\big)S_1^2 + \big(5N+22\big)\left(1 - \frac{4}{s(s+1)}\right)S_{-2} + \frac{3(N+2)}{2}\Bigg[S_1 - \frac{11}{2}\\
&+ 3\frac{2s+1}{s^2(s+1)^2} + \frac{s-3}{s(s+1)}\Bigg] + \big(N^2+6N+20\big)\Bigg[\left(\frac{1+2s}{s(s+1)} - 4\right)S_1 + \frac{3}{4s^2(s+1)^2}\\
&- \frac{20s+1}{4s(s+1)} + 2\Bigg] + \big(5N+22\big)\Bigg[S_1\left(2\frac{2s+1}{s(s+1)} - 11\right) + \frac{3}{2s^2(s+1)^2}\\
&- \frac{32s+7}{2s(s+1)} + 21\Bigg]\Bigg)\Bigg\} + O(u^5)\,.
\end{aligned} \tag{4.55}$$

Here we also generalize the four-loop result of [263] to the symmetric and antisymmetric cases by restoring the symmetry factors for the individual graphs (see Table 1). The results for the corresponding operators read

$$\begin{aligned}
\gamma^+(s) = 2\gamma_\varphi - \frac{u^2}{9s(s+1)}\Bigg\{&N + 6 + \frac{u}{6}\Bigg(8\,(N+4)\left[2S_1 + \frac{2s+1}{s(s+1)} - 4\right] - \big(N^2+6N\\
&+16\big)\left[2 - \frac{2s+1}{s(s+1)}\right]\Bigg) + \frac{u^2}{9}\Bigg(4\big(N+4\big)\big(N+8\big)S_2 + \big(N^2+28N+84\big)S_1^2\\
&+ \left(N^2+16N+44 - 8\frac{5N+22}{s(s+1)}\right)S_{-2} + \big(N^3+8N^2+24N+40\big)\Bigg[\frac{1}{4s^2(s+1)^2}\\
&- \frac{4s-1}{4s(s+1)}\Bigg] + \big(N^2+16N+44\big)\Bigg[\left(\frac{5}{3}\frac{2s+1}{s(s+1)} - \frac{32}{3}\right)S_1 + \frac{5}{6s^2(s+1)^2}
\end{aligned}$$

$$
\begin{aligned}
& - \frac{64s+17}{6s(s+1)} + \frac{52}{3}\Bigg] + \frac{3N^2+20N+44}{3}\Bigg[\bigg(\frac{2s+1}{s(s+1)} - 1\bigg)S_1 + \frac{2}{s^2(s+1)^2} \\
& - 2\frac{8s+1}{s(s+1)} + 11\Bigg] + \frac{N+6}{2}\bigg\{(N+2)\left[S_1 + \frac{s}{s(s+1)} - \frac{11}{2}\right] + \big(N+6\big)\bigg[\frac{2s+1}{s^2(s+1)^2} \\
& - \frac{1}{s(s+1)}\bigg] + \frac{8(N+5)}{3}\Bigg[\bigg(\frac{2s+1}{s(s+1)} - 4\bigg)S_1 + \frac{1}{2s^2(s+1)^2} - \frac{8s+1}{2s(s+1)} \\
& + 2\Bigg]\bigg\}\Bigg)\Bigg\} + O(u^5)\,,
\end{aligned}
\tag{4.56}
$$

$$
\begin{aligned}
\gamma^-(s) = 2\gamma_\varphi - u^2\frac{N+2}{9s(s+1)}\Bigg\{& 1 + u\Bigg(\frac{2}{3}\left[2S_1 + \frac{2s+1}{s(s+1)} - 4\right] - \frac{N+4}{6}\left[2 - \frac{2s+1}{s(s+1)}\right]\Bigg) \\
& + \frac{u^2}{9}\Bigg(2\big(N+8\big)S_2 - \big(N-6\big)S_1^2 - \big(N-2\big)S_{-2} + \big(N^2+6N+12\big)\bigg(\frac{1}{4s^2(s+1)^2} \\
& - \frac{4s-1}{4s(s+1)}\bigg) + \frac{4\big(6+N\big)}{3}\Bigg[\bigg(\frac{2s+1}{s(s+1)} - 7\bigg)S_1 + \frac{1}{2s^2(s+1)^2} - \frac{7s+2}{s(s+1)} + \frac{25}{2}\Bigg] \\
& + \frac{10+3N}{3}\Bigg[\bigg(\frac{2s+1}{s(s+1)} - 1\bigg)S_1 + \frac{2}{s^2(s+1)^2} - 2\frac{8s+1}{s(s+1)} + 11\Bigg] \\
& + \frac{14-N}{3}\Bigg[\bigg(\frac{2s+1}{s(s+1)} - 4\bigg)S_1 + \frac{1}{2s^2(s+1)^2} - \frac{8s+1}{2s(s+1)} + 2\Bigg] + \frac{N+2}{2}\Bigg[S_1 \\
& + \frac{2s+1}{s^2(s+1)^2} + \frac{s-1}{s(s+1)} - \frac{11}{2}\Bigg]\Bigg)\Bigg\} + O(u^5)\,.
\end{aligned}
\tag{4.57}
$$

The results of our direct calculation (4.56) and (4.57) coincide with the independent treatment using the Lorentzian inversion formula [334] (see also (4.67)). Another way to check the validity of these results is to relate them to conserved currents (see (1.56) for the discussion). Indeed, the action (4.48) is invariant under the transformation $\varphi^a \mapsto \varphi^a + \omega^{ab}\varphi^b$ ($\omega^{ab} = -\omega^{ba}$) is the generator of the $O(N)$ group. The corresponding Noether current reads

$$
J_\mu^{ab} = \varphi^a \partial_\mu \varphi^b - \varphi^b \partial_\mu \varphi^a, \tag{4.58}
$$

which coincides with $\mathcal{O}^-_{s=1}$. One can explicitly check that $\gamma^-(1) = 0$, consistent with the conservation of the $O(N)$ current. Another conserved current is the stress-energy tensor

$$
T^{\mu\nu} = -\varphi^a \partial_\mu \partial_\nu \varphi^a - \delta^{\mu\nu}\mathcal{L}, \tag{4.59}
$$

which under projection onto the light-like vector reproduces the scalar operator $\mathcal{O}_{s=2}$. An explicit check confirms that $\gamma(2) = 0$, as expected. One can also verify the reciprocity relation (see Section 4.1.2 for details). Considering the RR anomalous dimension $f^\alpha(j)$ for each operator [a] and applying the formulas (4.11), we obtain

$$
f(j) = 2\gamma_\varphi - u^2\frac{N+2}{9j(j+1)}\Bigg\{3 + \frac{2u}{3}\big(N+8\big)\bigg(S_1 - \frac{5}{2}\bigg) + \frac{u^2}{9}\Bigg(\big(22+5N\big)\bigg(1 - \frac{4}{j(j+1)}\bigg)S_{-2}
$$

[a] The index α here and in what follows labels the type of operator: scalar, symmetric, or antisymmetric.

	Scalar	Symmetric	Antisymmetric
A_1	$\frac{5N^2}{81}+\frac{32N}{81}+\frac{44}{81}$	$\frac{N^2}{81}+\frac{16N}{81}+\frac{44}{81}$	$\frac{N^2}{81}+\frac{8N}{81}+\frac{4}{27}$
A_2	$\frac{5N^2}{324}+\frac{8N}{81}+\frac{11}{81}$	$\frac{N^2}{108}+\frac{5N}{81}+\frac{11}{81}$	$\frac{N^2}{108}+\frac{4N}{81}+\frac{5}{81}$
A_3	$\frac{5N^2}{162}+\frac{16N}{81}+\frac{22}{81}$	$\frac{5N}{81}+\frac{22}{81}$	0
B_1	$\frac{5N^2}{81}+\frac{32N}{81}+\frac{44}{81}$	$\frac{N^2}{81}+\frac{16N}{81}+\frac{44}{81}$	$\frac{N^2}{81}+\frac{8N}{81}+\frac{4}{27}$
B_2	$\frac{5N^2}{162}+\frac{16N}{81}+\frac{22}{81}$	$\frac{N^2}{54}+\frac{10N}{81}+\frac{22}{81}$	$\frac{N^2}{54}+\frac{8N}{81}+\frac{10}{81}$
B_3	$\frac{5N^2}{324}+\frac{8N}{81}+\frac{11}{81}$	$\frac{N^2}{324}+\frac{4N}{81}+\frac{11}{81}$	$-\frac{N^2}{324}+\frac{1}{81}$
C_1	$\frac{N^3}{648}+\frac{N^2}{81}+\frac{4N}{81}+\frac{5}{81}$	$\frac{N^3}{648}+\frac{N^2}{81}+\frac{N}{27}+\frac{5}{81}$	$\frac{N^3}{648}+\frac{N^2}{81}+\frac{N}{27}+\frac{1}{27}$
C_2	$\frac{N^3}{162}+\frac{4N^2}{81}+\frac{16N}{81}+\frac{20}{81}$	$\frac{N^2}{81}+\frac{8N}{81}+\frac{20}{81}$	$\frac{2N}{81}+\frac{4}{81}$
C_3	$\frac{N^3}{324}+\frac{2N^2}{81}+\frac{8N}{81}+\frac{10}{81}$	$\frac{N}{27}+\frac{10}{81}$	$\frac{N}{81}+\frac{2}{81}$
D_1	$\frac{N^2}{54}+\frac{2N}{27}+\frac{2}{27}$	$\frac{N^2}{162}+\frac{4N}{81}+\frac{2}{27}$	$\frac{N^2}{162}+\frac{2N}{81}+\frac{2}{81}$
D_2	$\frac{N^2}{54}+\frac{2N}{27}+\frac{2}{27}$	$\frac{N^2}{162}+\frac{4N}{81}+\frac{2}{27}$	$\frac{N^2}{162}+\frac{2N}{81}+\frac{2}{81}$
D_3	$\frac{N^2}{36}+\frac{N}{9}+\frac{1}{9}$	$\frac{N^2}{324}+\frac{N}{27}+\frac{1}{9}$	$\frac{N^2}{324}+\frac{N}{81}+\frac{1}{81}$

Table 1: Symmetry factors for the diagrams contributing to the anomalous dimensions of the operators (4.52), (4.53), and (4.54). We use the notation of [263, Appendix A] to describe the corresponding integrals.

$$
\begin{aligned}
&+\big(N^2+11N+42\big)S_1^2-\frac{16N^2+310N+1276}{4}S_1+\big(N+8\big)^2\zeta_2-\frac{21(N+2)}{4j(j+1)}\\
&+\frac{8N^2+435N+1942}{4}\Bigg)\Bigg\}+O(u^5),
\end{aligned}
\tag{4.60}
$$

$$
\begin{aligned}
f^+(j)=2\gamma_\varphi-u^2\frac{1}{9j(j+1)}\Bigg\{&N+6+\frac{u}{3}\Bigg(8(N+4)S_1-N^2-22N-80\Bigg)\\
&+\frac{u^2}{9}\Bigg(\Bigg(N^2+16N+44-8\frac{22+5N}{j(j+1)}\Bigg)S_{-2}+\big(N^2+28N+84\big)S_1^2\\
&-\frac{33N^2+464N+1276}{2}S_1+4\big(N+4\big)\big(N+8\big)\zeta_2-\frac{3N^2+32N+84}{4j(j+1)}\\
&+\frac{113N^2+1432N+3884}{4}\Bigg)\Bigg\}+O(u^5),
\end{aligned}
\tag{4.61}
$$

$$f^{-}(j) = 2\gamma_\varphi - u^2 \frac{N+2}{9j(j+1)} \Bigg\{ 1 + \frac{u}{3}\Bigg(4S_1 - 12 - N \Bigg) - \frac{u^2}{9}\Bigg((N-2)S_{-2} + (N-6)S_1^2 + \frac{17N+154}{2} S_1 - 2(N+8)\zeta_2 + \frac{3\,(N+2)}{4j(j+1)} - \frac{97N+562}{4} \Bigg) \Bigg\} + O(u^5). \tag{4.62}$$

Indeed, all the j-dependence is organized in powers of $j(j+1)$, which is manifestly invariant under the transformation $j \mapsto -j-1$, and the harmonic sums that appear, S_1, S_1^2, S_{-2}, are reciprocity-respecting, see (4.13) for details. Note also that there are no terms proportional to S_1 (without supressing factors $s(s+1)$) in either the anomalous dimensions $\gamma^\alpha(s)$ or the RR parts $f^\alpha(s)$. This is perfectly consistent with the leading large-s asymptotics discussed in Section 4.1.1, since there is no cusp anomalous dimension in non-gauge theories such as (4.48).

Let us now return to the original problem and consider the $s \to 0$ limit. For concreteness let us focus on the scalar operator (4.52) first. Expanding the anomalous dimension (4.55) around $s = 0$ we obtain

$$\gamma(s) = (N+2)\Bigg\{ \Bigg(-\frac{1}{3s} + \frac{7}{18} + O(s) \Bigg) u^2 + (N+8)\Bigg(-\frac{1}{18s^2} + \frac{5}{27s} - \frac{61}{216} + O(s) \Bigg) u^3 + \\ + \Bigg(-\frac{N^2+22N+76}{108s^3} + \frac{5N^2+128N+488}{162s^2} - \Bigg(2\frac{5N+22}{27}\zeta_3 + \frac{16N^2+605N+2538}{324} \Bigg)\frac{1}{s} \\ - \frac{2N^2-103N-466}{162}\zeta_3 + \frac{1295N^2+50410N+215316}{12960} + O(s) \Bigg) u^4 \Bigg\} + O(u^5), \tag{4.63}$$

so the anomalous dimension is indeed divergent in this limit. The leading singular term at ℓ-th loop order has a lower power, $\ell - 1$, compared to the QCD case (see Section 4.2), where it is $2\ell - 1$. This reflects the generally lower transcendental weight structure of the perturbative expansion in this theory. [b] The presence of a singularity in this limit is even more striking than in QCD, since the corresponding limit at the operator level is perfectly well-defined:

$$\lim_{s\to 0} \mathcal{O}_s(x) = \varphi^2(x), \tag{4.64}$$

with a known anomalous dimension (see [142, Chapter 4, Section 2]):

$$\gamma_{\varphi^2} = u\frac{N+2}{3}\Bigg\{ 1 - \frac{5u}{6} + u^2\frac{(37+5N)}{12} \Bigg\} + O(u^4). \tag{4.65}$$

In [81] it was argued that in corresponding CFT (at the critical point) this inconsistency arises due to the specific behavior of the spectrum under the analytical continuation in spin, in case of operator (4.52), from $s \in 2\mathbb{Z}_{>0}$ to a meromorphic function on $\mathbb{C}\setminus\{0,-1,\ldots\}$. In what follows we review these arguments and see how they affect a limit $s \to 0$ in (4.63).

4.3.2 Consequences of analytical continuation in spin

Let us now turn our attention to the critical point $u \to u^*$, such that $\beta(u^*) = 0$. This equation can be solved perturbatively using (4.49), namely

$$u^*(\epsilon) = \frac{6\epsilon}{N+8} + 36(3N+14)\frac{\epsilon^2}{(N+8)^3} - \Bigg(3\left(33N^3 - 110N^2 - 1760N - 4544\right) +$$

[b] It would be interesting to identify the origin of the corresponding logarithmic terms in the Mellin-conjugate x-space. However, the analogue of the Bethe–Salpeter treatment described in Fig. 4.3 for amplitudes in φ^4 theory is not known to the author. Some hints on the use of the Bethe–Salpeter equation in this context can be found in [401, Section 3.3].

$$288(N+8)(5N+22)\zeta_3\Bigg)\frac{\epsilon^3}{(N+8)^5}+\Bigg(-5N^5-2670N^4-5584N^3+52784N^2$$
$$+309312N+529792+96(N+8)\left(63N^3-82N^2-3796N-9064\right)\zeta_3$$
$$-288(N+8)^3(5N+22)\zeta_4+1920(N+8)^2\left(2N^2+55N+186\right)\zeta_5\Bigg)\frac{\epsilon^4}{(N+8)^7}+O(\epsilon^5). \quad (4.66)$$

As we discussed in Section 2.3, the theory (4.48) evaluated at the coupling (4.66) corresponds to a proper CFT (often called the *Wilson–Fisher* CFT in the literature), and the anomalous dimensions of the twist-2 operators (4.56), (4.55) and (4.57) together with the canonical scaling dimensions, $\Delta_0(s)=2(d/2-1)+s$, belong to the CFT spectrum (2.45). So far, when considering the continuous limit $s\to 0$ in (4.63), we performed the analytic continuation from $s\in 2\mathbb{Z}_{>0}$ to $s\in\mathbb{C}\setminus\{0,-1,-2,\ldots\}$ explicitly at the level of functions. The advantage of working in the CFT context is that one can approach this question in more detail and continue $\gamma(s)$ not as a function, but as part of the CFT spectrum. The technical means by which this can be done is expressed by the *Lorentzian inversion formula* [80] (see also [402] for a more pedagogical overview and its use in the large-spin limit)

$$C(\Delta,\ell)=\left(1\pm(-1)^\ell\right)\frac{\kappa_{\Delta+\ell}}{4}\int_0^1 dz\int_0^1 d\bar z\,\mu(z,\bar z)\,G^{(d)}_{d-1+\ell,1-d+\Delta}(z,\bar z)\,\mathrm{dDisc}\left[G(z,\bar z)\right], \quad (4.67)$$

where the forms of the prefactor $\kappa_{\Delta+\ell}$ and the measure $\mu(z,\bar z)$ are not relevant for the present discussion. Here $G(z,\bar z)$ is the four-point function (2.41) (with proper normalization) of identical fields parametrized by (2.43), and dDisc computes its *double discontinuity*, namely [c]

$$\mathrm{dDisc}\left[G(z,\bar z)\right]=G(z,\bar z)-\frac{1}{2}G^{\circlearrowright}(z,\bar z)-\frac{1}{2}G^{\circlearrowleft}(z,\bar z), \quad (4.68)$$

where $G^{\circlearrowright,\circlearrowleft}(z,\bar z)$ denote the monodromy around $\bar z=1$ in the clockwise and anticlockwise directions, respectively. The final ingredient is the *conformal block* $G^{(d)}_{d-1+\ell,1-d+\Delta}(z,\bar z)$, see e.g. [173, Section 9] for a detailed discussion. Formula (4.67) is of particular importance in CFT, see [403, 404] for examples of its use (see also the discussion on reciprocity (4.10) in [334, Section 2.5.3]). We specifically mention it here because it is explicitly analytic in ℓ. The spectrum is extracted from it as poles of $C(\Delta,\ell)$, which fixes $\Delta=\Delta(\ell)$, and if the quantity under consideration is analytic in spin ℓ, this justifies considering the spectrum $\Delta(\ell)$ for $\ell\in\mathbb{C}$.

The inversion formula (4.67) has one remarkable property, which stems from the structure of $\kappa_{\Delta+\ell}$ and, more importantly, the conformal block $G^{(d)}_{d-1+\ell,1-d+\Delta}(z,\bar z)$, namely its behavior under the shadow transform (see Section 2.1 for the discussion of the underlying representation theory). If $\Delta(\ell)$ describes a pole in $C(\Delta,\ell)$, the shadow scaling dimension $\widetilde{\Delta}(\ell)=d-\Delta(\ell)$ also describes a pole. The latter do not contribute to the physical operators. The straightforward way to see this is to reproduce the expansion of the correlator

$$G(z,\bar z)=\sum_{\ell=0}^{\infty}\int\limits_{d/2-\mathrm{i}\,\infty}^{d/2+\mathrm{i}\,\infty}\frac{d\Delta}{2\pi\,\mathrm{i}}\,C(\Delta,\ell)\Psi_{\Delta,\ell}(z,\bar z), \quad (4.69)$$

where $\Psi_{\Delta,\ell}(z,\bar z)$ is a *conformal partial wave*. Closing the contour in one of the half-planes then picks up only the physical poles, avoiding the redundancy related to the symmetry of the poles. But what if we consider the theory at a non-physical point of arbitrary complex ℓ — could the shadow poles somehow give a contribution?

Let us now adopt a "pedestrian" approach and simply assume the existence of two operators: $\mathbf{O}_s$ with scaling dimension $\Delta(s)=2(d/2-1)+s+\gamma(s)$, and $\widetilde{\mathbf{O}}_s$ with scaling dimension $\widetilde{\Delta}(s)=d-\Delta(s)$.

[c] This justifies the word "Lorentzian" in the name, as the calculation of the monodromy around $\bar z=1$ requires analytically continuing the correlator from $\bar z=z^*$ in Euclidean space to $(z,\bar z)\in\mathbb{C}^2$.

Here we also assume $s \in \mathbb{C}$. Let us now go beyond the example of the theory (4.48) and treat them as belonging to some arbitrary CFT. This of course goes beyond the classification of conformal operators described in Section 2.2, but let us proceed with such a consideration and then briefly mention a more specific footing. Consider now the standard UV renormalization of the operators $\mathbf{O}_s$ and $\widetilde{\mathbf{O}}_s$. For generic s, these operators do not mix, and accounting simultaneously for their divergencies leads to the diagonal mixing matrix

$$\begin{pmatrix} [\mathbf{O}_s] \\ [\widetilde{\mathbf{O}}_s] \end{pmatrix} = \begin{pmatrix} Z(s) & 0 \\ 0 & \widetilde{Z}(s) \end{pmatrix} \begin{pmatrix} \mathbf{O}_s \\ \widetilde{\mathbf{O}}_s \end{pmatrix}. \tag{4.70}$$

However, if we look at the specific point $s = 0$, we see that the canonical scaling dimensions (in the free theory, $\epsilon = 0$) coincide:

$$\Delta_0(0) = 2 = 4 - 2 = \widetilde{\Delta}_0(0). \tag{4.71}$$

This means that if we treat s as a small parameter, the renormalization matrix (4.70) acquires a non-diagonal structure [d]

$$\begin{pmatrix} [\mathbf{O}_{s\to 0}] \\ [\widetilde{\mathbf{O}}_{s\to 0}] \end{pmatrix} = \begin{pmatrix} Z_{11}(s) & Z_{12}(s) \\ Z_{21}(s) & Z_{22}(s) \end{pmatrix} \begin{pmatrix} \mathbf{O}_{s\to 0} \\ \widetilde{\mathbf{O}}_{s\to 0} \end{pmatrix}. \tag{4.72}$$

At the same time, we have a well-defined description of the theory exactly at $s = 0$, where the set of local operators is known. To match the analytically continued description with the usual local CFT data, it is natural to assume that the analytically continued operator $\mathbf{O}_s$ at the point $s = 0$ coincides in some sense with the operator $\mathcal{O}_s$, while the shadow operator $\widetilde{\mathbf{O}}_s$ drops out of the consideration. This mixing structure implies the following situation for the anomalous dimensions, which can be described by the schematic pattern

$$\begin{pmatrix} \gamma(s) & 0 \\ 0 & \widetilde{\gamma}(s) \end{pmatrix} \qquad \longrightarrow \qquad \begin{pmatrix} \gamma_{11}(s) & \gamma_{12}(s) \\ \gamma_{21}(s) & \gamma_{22}(s) \end{pmatrix}\Bigg|_{s\to 0} \qquad \longrightarrow \qquad \gamma(0), \tag{4.73}$$

where the first matrix describes the independent renormalization of $\mathbf{O}_s$ and $\widetilde{\mathbf{O}}_s$ for generic complex s (with the *shadow anomalous dimension* defined as $\widetilde{\gamma}(s) \equiv \widetilde{\Delta}(s) - \Delta_0(s) = 4 - d - 2s - \gamma(s)$), the second entry describes their mixing in a neighbourhood of the point $s = 0$ with $s \neq 0$, and the last one describes the situation exactly at the integer point $s = 0$, where there is a single local operator $\mathcal{O}_{s=0}$.

From the schematic picture (4.73), it is clear why taking the naive limit of the anomalous dimension,

$$\lim_{s\to 0} \gamma(s) \neq \gamma(0), \tag{4.74}$$

does not correspond to the anomalous dimension of the local operator $\mathcal{O}_{s=0}$, and in the perturbative expansion would contribute as a pole. The anomalous dimension $\gamma(0)$ should correspond to the *eigenvalues* of the mixing matrix rather than to the individual entries. The technical difficulty is of course the middle step of (4.73), since computing the full matrix requires a proper definition of the operator $\widetilde{\mathbf{O}}_s$ and, moreover, a prescription for its renormalization, given that it cannot be constructed as a local operator. [e] Taking as given that this difficulty is related to the incomplete specification of the operator basis in the vicinity of $s = 0$, we can, however, bypass this step entirely. Indeed, the invariants of the anomalous dimension matrix are insensitive to the choice of basis, which means that their analytical continuation should not create a problem. This idea is conveniently expressed in terms of the characteristic equation for the matrix $\widehat{\gamma}(s)$:

[d] In principle, one should introduce additional scaling factors for the operators, depending on s, to match their scaling dimensions, and then expand in s if it is treated as an additional perturbative parameter, but we omit these details for brevity.

[e] This can be justified by Mack's classification of positive-energy operators in CFT [181] (see also [71, Eq. (1.6)] and the subsequent discussion).

Characteristic equation for the mixing matrix

$$x^2 - \operatorname{tr}\widehat{\gamma}(s)\, x + \det\widehat{\gamma}(s) = 0. \tag{4.75}$$

Away from $s = 0$, this equation written for the first matrix in (4.73) has $\gamma(s)$ and $\widetilde{\gamma}(s)$ as solutions, while after analytic continuation to $s \to 0$ it should yield $\gamma(0)$ as a solution.

Let us now return to the specific problem of the anomalous dimension (4.52) and the divergencies of (4.63), and apply the result (4.75). We first consider the anomalous dimension at the critical point (4.66), $\gamma^*(s) = \gamma(s; u^*)$. The coefficients in the characteristic equation (4.75) then read

$$\begin{cases} \operatorname{tr}\widehat{\gamma}^*(s) = 2\epsilon - 2s \\ \det\widehat{\gamma}^*(s) = -2\gamma^*(s)\left(s - \epsilon + \dfrac{1}{2}\gamma^*(s)\right) \end{cases}. \tag{4.76}$$

The limit $s \to 0$ for the first coefficient causes no problem, as it is explicitly regular. The second coefficient at the same time coincides with (4.39). Indeed, returning to arbitrary coupling (see (4.20) and the surrounding discussion for details), one defines

$$\delta m^2(s) = 2\gamma(s)\left(s + \bar{\beta}(u) + \frac{1}{2}\gamma(s)\right). \tag{4.77}$$

Using the available perturbative data (see (4.55)), we can explicitly verify

$$\delta m^2(0) = 2u^2(N+2)\left\{-\frac{1}{3} + u\frac{13(N+8)}{108} - u^2\left(\frac{2(22+5N)}{27}\zeta_3 + \frac{3N^2+950N+4588}{1296}\right)\right\} + O(u^5). \tag{4.78}$$

In analogy with the QCD case (see Section 4.2), this allows us to resum the $s \to 0$ singularities by interpreting the anomalous dimension (4.55) at $s = 0$ as the solution of the quadratic equation

$$\gamma(0) = -\bar{\beta}(u) + \sqrt{\left(\bar{\beta}(u)\right)^2 + \delta m^2(0)}. \tag{4.79}$$

The square root should then be understood as a function on a two-sheeted Riemann surface (see Fig. 4.4). One can explicitly verify that one branch of (4.79) (corresponding to the solution with the "+" sign) coincides with the anomalous dimension of the local operator φ^2, see (4.65). This fact provides a justification for the resummation procedure and confirms that the logic described by (4.73) is consistent.

The same construction can be carried out for the anomalous dimensions of the two other types of operators, (4.53) and (4.54). Defining the analogous quantity (4.77), we derive

$$\begin{aligned}\delta m_+^2(0) = 2u^2\Bigg\{&-\frac{N+6}{9} + u\frac{N^2+50N+208}{108} \\ &- u^2\left(\frac{4(22+5N)}{27}\zeta_3 - \frac{5N^3-164N^2-3112N-9176}{1296}\right)\Bigg\} + O(u^5),\end{aligned} \tag{4.80}$$

$$\delta m_-^2(0) = 2u^2(N+2)\left\{-\frac{1}{9} + u\frac{24+N}{108} + u^2\left(\frac{5N^2-158N-1148}{1296}\right)\right\} + O(u^5). \tag{4.81}$$

Using (4.80) and writing the analogue of (4.79), we reproduce

$$\gamma^+(0) = \frac{u}{3}\left\{2 - u\frac{10+N}{6} - u^2\frac{5N^2-84N-444}{72}\right\} + O(u^4), \tag{4.82}$$

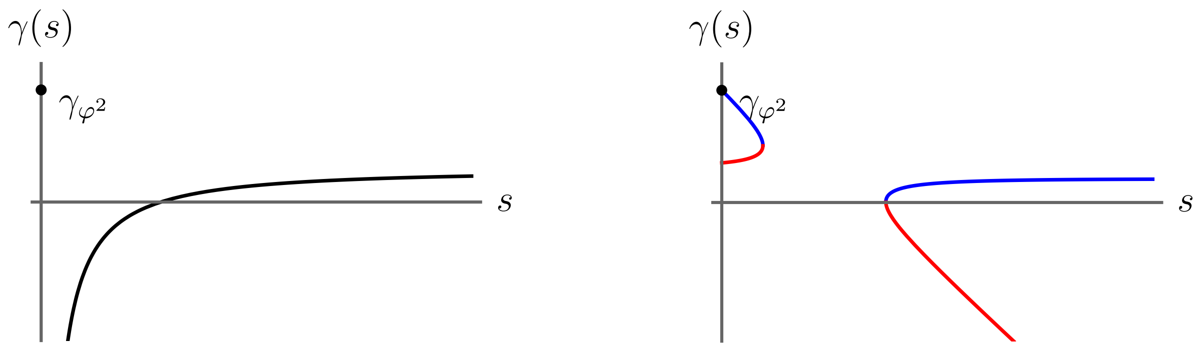


Figure 4.4: The "naive" perturbation theory result for the anomalous dimension is shown on the left: $\gamma(s)$ diverges as $s \to 0$ and γ_{φ^2} does not lie on the trajectory. The result of the resummation (4.79) is shown on the right: the blue and red lines represent the two branches of the square root, and γ_{φ^2} belongs to one of them.

which coincides with the anomalous dimension of the operator $\varphi^a\varphi^b - 1/N\delta^{ab}\varphi^2$, see [142] for the result. It is remarkable that (4.81) is also regular at $s = 0$, although the expansion (4.57) is valid for $s \in 2\mathbb{Z}_{\geq 0}+1$, and the point $s = 0$ does not naturally fit such an analytic continuation. In the QCD case (see (4.41)), this led to singularities that are not cancelled in the combination of anomalous dimensions (4.77). As discussed in Section 4.2, the cancellation of singularities can be used to reconstruct the singular behavior at higher loop orders. For example, the result (4.78) leads to the prediction

$$\begin{aligned}\gamma^{(5)}(s) = (N+2)\Bigg\{&-\frac{(N+8)(N^2+34N+100)}{648s^4} + \frac{(N+8)(5N^2+236N+776)}{972s^3} \\ &- \Bigg(\frac{(14+N)(22+5N)}{81}\zeta_3 + \frac{(8+N)(N^2+28N+88)}{486}\zeta_2 \\ &+ \frac{16N^3+2067N^2+24270N+63152}{3888}\Bigg)\frac{1}{s^2} + \Bigg(\frac{40N^2+1100N+3720}{243}\zeta_5 \\ &+ \frac{(32+N)(22+5N)}{810}\zeta_2^2 - \frac{5N^3-365N^2-5278N-16880}{972}\zeta_3 \\ &+ \frac{(N+8)(N+26)(8N+35)}{972}\zeta_2 + \frac{371N^2+6286N+18192}{648}\Bigg)\frac{1}{s}\Bigg\} + O(s^0),\end{aligned} \tag{4.83}$$

for the five-loop anomalous dimension. This prediction for $N = 1$ is consistent with [81, Eq. (2.28)].

Theory (4.48) can also be analyzed in the framework of the $1/N$ expansion (see the discussion in Section 3.2 for references). The corresponding theory is a proper CFT, in which all anomalous dimensions are calculated as perturbative series in $1/N$ with the dimension d kept as an analytic parameter. The perturbative results (4.55), (4.56) and (4.57) evaluated at the critical point (4.66) with the subsequent limit $N \to \infty$ should then match the $1/N$ expansion considered around four dimensions, $d = 4-2\epsilon$. This is indeed the case, and we perform cross-checks against the $1/N$-expansion results for the symmetric, antisymmetric [264] and scalar [270] operators. Another remarkable property of the $1/N$ expansion is that at $d = 3$ it is conjectured [405] (see also [406, 407]) to be dual to higher-spin theory on AdS_4 (see [408, 409] and references therein for a discussion of higher-spin theory). The duality manifests as the fact that the spectra of both models coincide, and the connection is given by [410–413]

$$m^2(s) = \Delta(s)\big(\Delta(s) - d\big) - s, \tag{4.84}$$

where $m^2(s)$ is the mass of the spin-s field expressed in the units of cosmological constant. Corrections in $1/N$ correspond to radiative corrections in the AdS language. Writing the higher-spin mass in general AdS_{d+1} as $m^2(s) = (d-2+s)(s-d) + \delta m^2(s)$ and using the relation (4.84) then leads to

$$\delta m^2(s) = 2\left(s + \frac{d}{2} - 2 + \frac{1}{2}\gamma(s)\right)\gamma(s), \tag{4.85}$$

which, expanded at $d = 4 - 2\epsilon$, gives precisely (4.77). This is the reason why in Section 4.2 (and specifically in (4.39)) we adopted this name for the combination of anomalous dimensions appearing in the double-logarithmic equation, and subsequently used it in (4.77). From the physical point of view, the cancellation of singularities as $s \to 0$ in (4.78) can then be interpreted as the finiteness of radiative corrections to the mass of a scalar field in AdS_4, which is itself a natural requirement. The explicit validity of the double-logarithmic equation (4.39) in the planar limit, $N_c \to \infty$, thereby acquires an interesting possible explanation from the side of AdS/CFT correspondence in QCD (see e.g. [414–416] for developments in this direction). It would be highly interesting to explore the role of the dual description in this context.

As we saw in this section, the use of (4.75) for the resummation of small-spin singularities gives a meaningful ($\lim_{s\to 0}\gamma(s) = \gamma_{\varphi^2}$) way of resumming singular contributions, but the reasoning (see (4.73)) may be questionable. In particular:

- Can the operators $\mathbf{O}_s$, $\widetilde{\mathbf{O}}_s$ be properly defined?
- What is the procedure for their renormalization, and how does one properly calculate the mixing matrix in (4.73)?
- Why do we consider precisely the operator $\widetilde{\mathbf{O}}_s$, and are there other “continued” operators that mix with $\mathbf{O}_s$?

The answers to these questions can be found in [71], where analytically continued operators were explicitly constructed as non-local, *light-ray*, [f] operators smeared along the light-cone, using a special set of integral transforms. We do not review this construction here and refer the reader to the original paper (see also [333] for the treatment of the model (4.48) in the $1/N$ expansion), but it is useful to introduce some notation. Light-ray operators diagonalize [g] the dilatation operator and yield $\Delta(s)$, with $s \in \mathbb{C}$, as eigenvalues. The resulting function can be associated with Regge trajectories (see [332] for the discussion of conformal Regge theory), in analogy with Regge trajectories in the more traditional 4-point scattering.

It is convenient to use a representation via *Chew–Frautschi* plot (see Fig. 4.5), i.e. the real slice of $\big(s, \Delta(s)\big)$ in analogy with $\big(J, m^2(J)\big)$ [419, 420], and examine the specific role of the point $s = 0$. In [81] a comprehensive study was performed on which light-ray operators should appear on this Chew–Frautschi plot. This analysis shows that the free trajectories (defined through $\Delta_0(s)$) of leading-twist operators, (4.52), (4.53) and (4.54), intersect the shadow trajectories (see [81, Section 2.4]) exactly at the point $s = 0$. This situation strongly resembles the setup for the *Wigner–von Neumann non-crossing rule* [421]: in quantum mechanics, if the free spectrum of a Hermitian Hamiltonian is degenerate for a specific value of a parameter of the theory, it is highly unlikely that this degeneracy would not be resolved in the interacting theory (unless some specific symmetry protects it). The characteristic equation (4.75) can then be seen as the defining equation for the trajectories $\Delta(s)$ around $s = 0$:

$$x^2 - \big(\Delta(s) + \widetilde{\Delta}(s)\big)x + \Delta(s)\cdot\widetilde{\Delta}(s) = 0. \tag{4.86}$$

Indeed, away from $s = 0$, Vieta's theorem gives the leading-twist, $\Delta(s)$, and shadow, $\widetilde{\Delta}(s) = d - \Delta(s)$, trajectories as the roots, while near $s = 0$ the trajectory acquires a branch point and is given by

[f] The construction of light-ray operators in CFT can be compared with light-ray operators in gauge theories, see [27,88] (and also [417,418] for the discussion in $\mathcal{N} = 4$ SYM) and the discussion in Section 2.4.

[g] The connection is more subtle since the light transform changes the quantum numbers of the operator, $(\Delta, \ell) \mapsto (1-\ell, 1-\Delta)$, but we omit these details here.

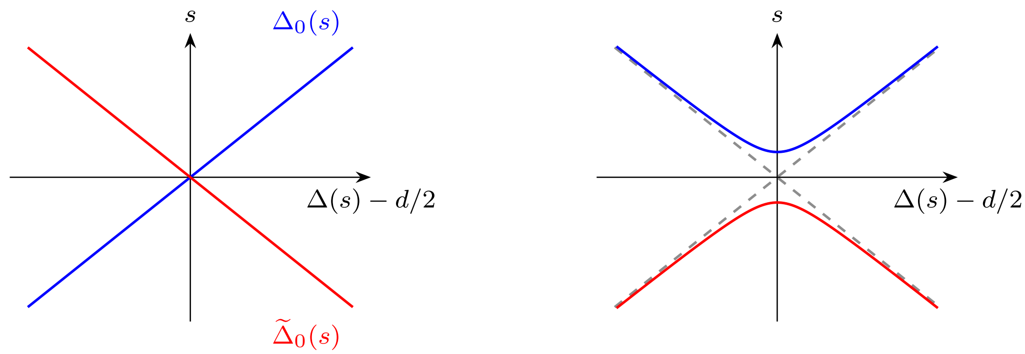


Figure 4.5: Chew–Frautschi plot around $s = 0$ in the scalar sector of (4.48) at the critical point. On the left the trajectories in the free theory ($\epsilon = 0$) are depicted. On the right are the trajectories in the interacting theory, represented as solutions of (4.86). The red and blue curves represent the two branches of the square root. The horizontal axis is shifted by $d/2$ for convenience, showing $\Delta(s) - d/2$.

a single-valued function on a two-sheeted Riemann surface. A similar consideration was performed in [422] (see also analogous patterns in [5,266,423,424]) in the case of local operators in a unitary CFT. The straightforward application of this principle to the spectrum of non-local operators is conceptually more subtle, since the applicability of the non-crossing rule relies on Hermiticity in a Hilbert space whose unitarity structure for light-ray operators is itself nontrivial.

Let us also briefly mention the generalization of this picture. The anomalous dimensions (4.55), (4.56) and (4.57) have an infinite series of poles at all negative integer points (the harmonic sums diverge at $s \in \{-1, -2, \dots\}$). In the language of [81], these poles can be associated with intersections of the leading-twist trajectories with trajectories of higher-twist operators (and more precisely their shadows). Indeed, consider the family of operators with canonical scaling dimension $\Delta_{0,\tau}(s) = \tau + s$, where τ is their canonical twist. The simple equation

$$\widetilde{\Delta}_{0,\tau}(s) \equiv d - \Delta_{0,\tau}(s) = \Delta_0(s) \tag{4.87}$$

gives $s = 1 - \tau/2$ as a solution. Moreover, one should also consider the infinite set of horizontal (so-called BFKL) trajectories, see [81, Section 2.6]. It is believed that no such trajectory exists at $s = 0$ for the leading-twist operators in the model (4.48), and therefore (4.77) should be regular to all orders of perturbation theory. The same is not true, for example, in QCD, where the horizontal trajectory at $s = 1$ plays an important role, see [73] for details. Another interesting question concerns the generalization to the non-perturbative picture, which requires a non-perturbative treatment of light-ray operators. Works in which non-perturbative methods (such as integrability) yield the exact form of the spectrum $\Delta(s)$ are therefore of great interest, see e.g. [425] for the discussion in planar $\mathcal{N} = 4$ SYM and [426, 427] in fishnet CFT [428–431].

4.3.3 $SU(N)$-adjoint φ^3 model

In this section we begin to explore the application of the resummation of small-spin singularities, based on (4.75), in other theories. We start with different realizations of a ($d = 6 - 2\epsilon$)-dimensional cubic model. The first example is the model described by the renormalized action

$$S_R = \int d^d x \left(\frac{Z_1}{2} (\partial\varphi^a)^2 + Z_3 \mu^\epsilon \frac{g}{6} d^{abc} \varphi^a \varphi^b \varphi^c \right), \tag{4.88}$$

where the field φ_a ($a \in \{1, \dots, N^2-1\}$) transforms under the adjoint representation of the $SU(N)$ group. Here we assume $N \geq 3$, since otherwise the interaction is not defined. The symmetric tensor in the interaction is $d^{abc} = 2\,\mathrm{tr}\left(t^a\left\{t^b, t^c\right\}\right)$, where t^a are the generators of the $SU(N)$ group. The renormalization functions of this model are known to four-loop accuracy [432] (see also [433] for a review)

$$\begin{aligned}\beta(u) &= -2\epsilon u - \frac{N^2-20}{2N}u^2 - \frac{5N^4-496N^2+5360}{72N^2}u^3 + O(u^4),\\ \gamma_\varphi &= \frac{N^2-4}{12N}u + \frac{(N^2-4)(N^2-100)}{432N^2}u^2 + O(u^3),\end{aligned} \tag{4.89}$$

where $u = g^2/(4\pi)^3$, and we present results up to two loops, which is sufficient for our purposes.

The leading-twist operators in this model take the form

$$\mathcal{O}_s^{ab}(x) = \varphi^a \partial_{\mu_1} \dots \partial_{\mu_{s-1}} \varphi^b(x). \tag{4.90}$$

Note that we shift the number of derivatives in order to make the final expressions more consistent with the previous discussion. Let us comment on this choice: the scaling dimension, $\Delta(s)$, of the operator (4.90) has the form $\Delta(s) = 2(d/2-1)+s-1+\gamma(s)$, which in the free theory gives $\Delta_0(s) = 3+s$. Mixing with the shadow trajectory (see Fig. 4.5 and the discussion in Section 4.3.2) then occurs at $s=0$, whereas without this shift it would occur at $s=-1$. These operators were studied in [89], where the corresponding anomalous dimensions were calculated to two loops. This accuracy is, of course, not high enough to provide a powerful check of the pole cancellations in (4.77), but is of interest because the structure of the multiplicatively renormalizable operators is quite rich. We introduce projectors onto the irreducible components of (4.90)

$$\mathcal{O}_{\alpha,s}^{ab}(x) = \left(P_\alpha\right)^{ab}_{a'b'} \mathcal{O}_s^{a'b'}(x), \tag{4.91}$$

where $\alpha = 1, \dots, 7$ labels the irreducible representations. The corresponding projectors take the form (see [89, Appendix B])

$$(P_1)^{ab}_{a'b'} = \frac{1}{N^2-1}\delta^{ab}_{a'b'}, \qquad (P_2)^{ab}_{a'b'} = \frac{1}{N}f^{abc}f^{a'b'c}, \qquad (P_3)^{ab}_{a'b'} = \frac{N}{N^2-4}d^{abc}d^{a'b'c}, \tag{4.92}$$

where $f^{abc} = -2\,\mathrm{i}\,\mathrm{tr}\left([t^a, t^b]t^c\right)$ is the antisymmetric tensor. The remaining projectors have a more complicated form:

$$P_4 = S_+\Pi_S, \qquad P_5 = S_-\Pi_S, \qquad P_6 = A_+\Pi_A, \qquad P_7 = A_-\Pi_A, \tag{4.93}$$

where

$$\begin{aligned}\Pi_S &= \frac{1}{2}\left(1+\mathbb{P}\right) - P_1 - P_3, & S_\pm &= \frac{1}{2N}\left(N \pm 2 \pm N\mathbb{R}\right),\\ \Pi_A &= \frac{1}{2}\left(1-\mathbb{P}\right) - P_2, & A_\pm &= \frac{1}{2}\left(\mathbb{1} \pm \mathrm{i}\,\mathbb{K}\right).\end{aligned} \tag{4.94}$$

Here we introduced

$$\mathbb{P}^{ab}_{a'b'} = \delta^a_{b'}\delta^b_{a'}, \qquad \mathbb{R}^{ab}_{a'b'} = d^{aa'c}d^{bb'c}, \qquad \mathbb{K}^{ab}_{a'b'} = d^{aca'}f^{bcb'}. \tag{4.95}$$

After projecting onto the irreducible representations, the anomalous dimensions of the multiplicatively renormalizable operators read

$$\gamma_\alpha(s) = 2\gamma_\varphi - \frac{2\lambda_\alpha}{s(s+1)}u - \Bigg\{2\lambda_\alpha\frac{16-N^2}{3N}\left(S_1 - \frac{92-5N^2}{2(16-N^2)}\right)\frac{1}{s(s+1)} \tag{4.96}$$

$$+\left(2\nu_\alpha - 4\lambda_\alpha^2 - \lambda_\alpha\left(\frac{N^2-4}{6N} - 2\frac{16-N^2}{3N}(1+2s)\right)\right)\frac{1}{s^2(s+1)^2} \tag{4.97}$$

$$\left.+ 2\lambda_\alpha^2\frac{s^2+3s+1}{s^3(s+1)^3}\right\}u^2 + O(u^3). \tag{4.98}$$

Here we introduced the representation-dependent constants λ_α and ν_α:

$$\lambda_1 = \frac{N^2-4}{N}, \quad \lambda_2 = \frac{N^2-4}{2N}, \quad \lambda_3 = \frac{N^2-12}{2N}, \quad \lambda_4 = 1-\frac{2}{N}, \quad \lambda_5 = -1-\frac{2}{N}, \quad \lambda_6 = \lambda_7 = -\frac{2}{N}. \tag{4.99}$$

$$\nu_1 = \frac{(N^2-4)(N^2-12)}{2N^2}, \quad \nu_3 = -4\frac{N^2-10}{N^2}, \quad \nu_{4,5} = \pm\frac{(N\mp 2)(N^2\pm 4N-8)}{2N^2}, \quad \nu_2 = \nu_6 = \nu_7 = 0. \tag{4.100}$$

As a check, one can verify that in (4.90), $\gamma_{\alpha=1}(3) = \gamma_{\alpha=2}(2) = 0$. The first equality corresponds to conservation of the stress-energy tensor and the second to conservation of the isotopic current (see (1.56) and (4.58) for the discussion).

The right-most singularity in the anomalous dimensions (4.96) is located at $s = 0$. With the example of the model (4.48) in mind, we construct the mass correction [h]

$$\delta m_\alpha^2(s) = 2\gamma_\alpha(s)\left(s + \bar\beta(u) + \frac{1}{2}\gamma_\alpha(s)\right). \tag{4.101}$$

Using the perturbative results (4.96) in the small-s limit, we obtain

$$\delta m_\alpha^2(s) = \Big(-4\lambda_\alpha + O(s)\Big)u + \left(-\frac{4\nu_\alpha}{s} + 4\lambda_\alpha^2 + \frac{174-7N^2}{3N}\lambda_\alpha + 8\nu_\alpha \right.$$
$$\left. + \frac{N^4+64}{9N^2} - \frac{20}{9} + O(s)\right) + O(u^3). \tag{4.102}$$

Thus, although the leading and subleading ($\sim 1/s^3$, $1/s^2$) poles cancel at two-loop order, a simple pole $\sim 1/s$ remains. An interesting feature is, however, the appearance of ν_α as the coefficient. [i] Reviewing the irreducible representations (4.91), we conclude that representations $2, 6$ and 7 are *antisymmetric*, while the others are *symmetric*, and according to (4.100), $\nu_2 = \nu_6 = \nu_7 = 0$. Leading-twist operators (4.90) are originally defined for *even* values of s for representations $1, 3, 4, 5$ and for *odd* values of s for representations $2, 6, 7$. Thus, the theory (4.88) reproduces the same effect as observed in QCD (see (4.41)), where the double-logarithmic equation behaved differently depending on the parity of the analytic continuation. [j] In the language of Regge trajectories (see Section 4.3.2 for the discussion), this, in principle, can be explained as the presence of a horizontal trajectory at $s = 0$, which should, however, be associated only with the parity-odd continued operators..

4.3.4 Complex φ^3 model

The next six-dimensional cubic model, belonging to the same general class (see [433, 435–439]), is the following

$$S_R = \int d^dx\left(Z_1\partial\varphi\partial\bar\varphi + Z_3\mu^\epsilon\frac{g}{6}\left(\varphi^3+\bar\varphi^3\right)\right), \tag{4.103}$$

[h] Here we omit the step of considering the model (4.88) at the critical point and the subsequent restoration of the physical coupling using the properties of the $\overline{\text{MS}}$ scheme. However, it should be noted that model (4.103) possesses an IR-stable fixed point, see (4.104).

[i] It is remarkable that the terms proportional to ν_α come from the one non-planar diagram, see [89, Appendix C].

[j] Note that in the case of the six-dimensional φ^3 model, the consideration in terms of the double-logarithmic equation is also possible, see [434] for details.

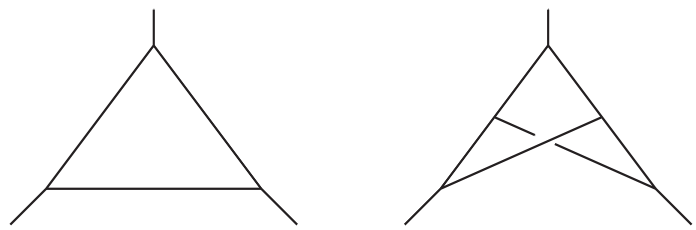


Figure 4.6: The one- and two-loop corrections to the three-point function in the model of φ^3 type. In the complex model (4.103) the left diagram is absent and the diagram on the right gives the first non-trivial correction.

where φ is a complex scalar field. This model has the global symmetry group $S_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ ($\varphi \mapsto e^{\pm \mathrm{i}\, 2\pi/3}$, $\varphi \leftrightarrow \bar{\varphi}$), which makes it multiplicatively renormalizable. Indeed, the possible six-dimensional interaction terms $\{\bar{\varphi}^2\varphi,\, \bar{\varphi}\varphi^2\}$ are prohibited by the $\mathbb{Z}_3 < S_3$ symmetry. The renormalization group coefficients in this model can be calculated quite easily ($u = g^2/(4\pi)^3$):

$$\beta(u) = -2\epsilon u + \frac{1}{2}u^2 - \frac{83}{72}u^3 + O(u^4), \tag{4.104}$$

$$\gamma_\varphi(u) = \frac{u}{12} - \frac{11}{3}\left(\frac{u}{12}\right)^2 + \frac{u^3}{24}\left(\frac{5357}{2592} - \zeta_3\right) + O(u^4). \tag{4.105}$$

A remarkable property of the model (4.103) in comparison to the standard real φ^3 model (or the modification considered in Section 4.3.3) is its generally simpler perturbative structure. The one-loop vertex correction in the model (4.103) is absent (see Fig. 4.6). Moreover, the one-loop vertex correction cannot appear as a subgraph, which significantly reduces the complexity of calculations, especially given that in the calculational framework discussed in Chapter 3 (see Section 3.2 for the specific discussion) vertex corrections are usually the hardest to compute.

The set of leading-twist operators in the model (4.103) consists of several types:

Leading-twist operators in the complex φ^3 model

Multiplicatively renormalizable leading-twist operators in the model (4.103) can be organized according to their transformation properties under $\mathbb{Z}_3$:

- Non-singlet operator ($s \in 2\mathbb{Z}_{\geq 0} + 1$):

$$\mathcal{O}_s^{\mathrm{ns}}(x) = \varphi(x)\partial_{\mu_1}\ldots\partial_{\mu_{s-1}}\varphi(x) - \text{traces}. \tag{4.106}$$

- Two singlet operators ($s \in 2\mathbb{Z}_{\geq 0} + 1$ for $\mathcal{O}_s^-$ and $s \in 2\mathbb{Z}_{\geq 0}$ for $\mathcal{O}_s^+$):

$$\mathcal{O}_s^{\pm}(x) = \bar{\varphi}\partial_{\mu_1}\ldots\partial_{\mu_{s-1}}\varphi \mp (\varphi \leftrightarrow \bar{\varphi}) - \text{traces}. \tag{4.107}$$

Note that there are also operators related to these by complex conjugation, but their treatment is completely analogous.

The underlying symmetry again restricts the values of s compatible with the definitions (4.106) and (4.107). In particular, the non-singlet operators are defined for odd s, and the singlet operators $\mathcal{O}_s^+$ and $\mathcal{O}_s^-$ for even and odd s, respectively. As in the case described in Section 4.3.3, this affects the mass correction (4.101). In the non-singlet case this is already visible at two loops, using the perturbative result

$$\gamma^{\rm ns}(s) = 2\gamma_\varphi - \frac{2u^2}{s^2(s+1)^2} + \mathcal{O}(u^3), \tag{4.108}$$

so the corresponding mass correction $\delta m^2_{\rm ns}(s)$ diverges as $s \to 0$. The situation with the singlet operators is more involved. We calculate their anomalous dimensions at three-loop order:

$$\begin{aligned}
\gamma^{\pm}(s) = 2\gamma_\varphi &\pm \frac{2u}{s(s+1)} \pm \frac{u^2}{3s(s+1)}\left(S_1 - \frac{1}{2s(s+1)} - 4 + \frac{2s+1}{s(s+1)}\right)\\
&+ \frac{2u^2}{s^2(s+1)^2}\left(1 - \frac{2s+1}{s(s+1)}\right)\\
&+ \frac{u^3}{3s^2(s+1)^2}\Bigg\{12S_3 + 24S_{1,-2} - 12S_{-3} - S_2 - 12\left(2 + \frac{1}{s(s+1)}\right)S_{-2}\\
&+ 2\left(7 - \frac{1+2s}{s(s+1)}\right)S_1 - 36\zeta_3 - \frac{24\left(S_{-2}+1\right)}{(s-1)(s+2)} + \frac{6s-3}{2s^2(s+1)^2} + \frac{20s-1}{s(s+1)} - 7\Bigg\}\\
&\pm \frac{u^3}{36s(s+1)}\Bigg\{2S_2 + S_1^2 - 144\left(\frac{1}{s^2(s+1)^2} + \frac{1}{s(s+1)}\right)S_{-2}\\
&- 72\zeta_3 + \left(\frac{1+4s}{s(s+1)} - \frac{143}{3}\right)S_1 + \frac{144\big(S_{-2}+1\big)}{(s-1)(s+2)} + \frac{144}{s^4(s+1)^4} + 144\frac{2-3s}{s^3(s+1)^3}\\
&+ \frac{289-8s}{2s^2(s+1)^2} - \frac{538+286s}{3s(s+1)} + \frac{4927}{24}\Bigg\} + O(u^4).
\end{aligned} \tag{4.109}$$

The details of the calculation can be found in Chapter 3, and the results for the individual diagrams are listed in Appendix A. Note that factors of $(s-1)$ appear in the denominators of (4.109). These poles are, however, spurious: the corresponding combination cancels as $s \to 1$, since $S_{-2}(s) = -1 + (2 - 3/2\,\zeta_3)(s-1) + O\big((s-1)^2\big)$. One can verify that (4.109) possesses the reciprocity property (see (4.10) and the discussion in Section 4.1). The corresponding mass corrections behave differently depending on parity:

$$\delta m^2_-(s) = \left\{8u^3\frac{2-3\zeta_3}{s} + O(s^0)\right\} + O(u^4), \tag{4.110}$$

$$\delta m^2_+(0) = 4u + \frac{41}{18}u^2 + \left(-\frac{16}{3}\zeta_3 - \frac{44}{5}\zeta_2^2 + 8\zeta_2 + \frac{19721}{1296}\right)u^3 + O(u^4), \tag{4.111}$$

which again underlines the exclusive possible role of the parity-odd horizontal trajectory at $s = 0$.

4.3.5 $O(N)$-symmetric complex φ^3 model

In this section we continue the discussion of six-dimensional cubic models by considering the $O(N)$-symmetric generalization of the model in Section 4.3.4. The renormalized action is given by

$$S_R = \int d^dx \left(Z_1\partial\varphi\partial\bar\varphi + Z_2\partial\sigma\partial\bar\sigma + Z_3\mu^\epsilon\frac{g}{2}\Big(\varphi^2\sigma + \bar\varphi^2\bar\sigma\Big)\right), \tag{4.112}$$

where the complex field φ^a ($a = 1, \ldots, N$) transforms under the fundamental representation of the $O(N)$ group and we have introduced an additional complex field σ. In addition to the $O(N)$ symmetry, the model (4.112) is invariant under the global $U(1)$ rotations $\varphi \mapsto e^{\mathrm{i}\,\alpha}\varphi$, $\sigma \mapsto e^{-2\,\mathrm{i}\,\alpha}\sigma$, where $\alpha \in \mathbb{R}$.

Remarkably, for $N = 2$ this symmetry is enhanced to $U(1) \times U(1) \times S_3$, and the bare action can be written as

$$S = \int d^d x \left(\sum_{k=1}^{3} (\partial \varphi_k)^2 + g(\varphi_1 \varphi_2 \varphi_3 + \text{c.c.}) \right). \tag{4.113}$$

We mention this because it provides a valuable cross-check for the perturbative calculations: for $N = 2$ the RG functions must respect this symmetry and be insensitive to the difference between the fields φ and σ. The same applies to the anomalous dimensions of the leading-twist operators. The renormalization group coefficients in the model (4.112) are given by ($u = g^2/(4\pi)^3$):

$$\gamma_\sigma(u) = \frac{1}{12} N u - \frac{11}{216} N u^2 + \frac{1}{12} N u^3 \left(\frac{103N}{648} + \frac{2677}{1296} - \zeta_3 \right) + \mathcal{O}(u^4),$$

$$\gamma_\varphi(u) = \frac{1}{6} u - \frac{11(N+2)}{432} u^2 + \frac{u^3}{6} \left(-\frac{13N^2}{5184} + \frac{641N}{2592} + \frac{2461}{1296} - \zeta_3 \right) + \mathcal{O}(u^4),$$

$$\beta(u) = -2\epsilon u + \frac{1}{6}(N+4) u^2 - \frac{11N + 119}{54} u^3 + \mathcal{O}(u^4). \tag{4.114}$$

Note that indeed for $N = 2$, $\gamma_\sigma = \gamma_\varphi$, as required by the symmetry (4.113).

The set of leading-twist operators is quite large, and in this section we consider only operators that are not reducing to total derivatives for the odd values of spin (i.e. continued from even values of s), in order to exclude the discussion of parity-odd operators around the point of mixing with the shadow trajectory.

Leading-twist operators in the $O(N)$-symmetric complex φ^3 model

Leading-twist operators in the model (4.112) can be organized with respect to the number of holomorphic and antiholomorphic fields [a] ($U(1)$ charge, with $Q_\varphi = 1$, $Q_\sigma = -2$) and by the irreducible representation of the $O(N)$ group:

- $Q = 3$ operator ($s \in \mathbb{Z}_{>0}$):

$$\overline{\mathcal{O}}^a_s(x) = \bar\sigma(x) \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \varphi^a(x) - \text{traces}. \tag{4.115}$$

- $Q = 2$ operator ($s \in 2\mathbb{Z}_{>0}$):

$$\mathcal{O}^{[ab]}_s(x) = \frac{1}{2} \big(\varphi^a(x) \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \varphi^b(x) - (a \leftrightarrow b) \big) - \text{traces}. \tag{4.116}$$

- $Q = 0$ operators:
  - $O(N)$-singlet operators (mix under renormalization) ($s \in 2\mathbb{Z}_{>0}$):

$$\overline{\mathcal{O}}_s(x) = \sum_{a=1}^{N} \bar\varphi^a(x) \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \varphi^a(x) - \text{traces},$$

$$\Sigma_s(x) = \bar\sigma(x) \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \sigma(x) - \text{traces}. \tag{4.117}$$

  - $O(N)$-non-singlet operators (do not mix under renormalization) ($s \in \mathbb{Z}_{>0}$):

$$\overline{\mathcal{O}}^{(ab)}_s(x) = \frac{1}{2} \big(\bar\varphi^a(x) \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \varphi^b(x) + (a \leftrightarrow b) \big) - \frac{1}{N} \delta^{ab} \overline{\mathcal{O}}_s - \text{traces}, \tag{4.118}$$

$$\overline{\mathcal{O}}^{[ab]}_s(x) = \frac{1}{2} \big(\bar\varphi^a(x) \partial_{\mu_1} \ldots \partial_{\mu_{s-1}} \varphi^b(x) - (a \leftrightarrow b) \big) - \text{traces}. \tag{4.119}$$

- $Q = -1$ operator ($s \in \mathbb{Z}_{>0}$):

$$\mathcal{O}_s^a = \frac{1}{2}\big(\sigma(x)\partial_{\mu_1}\dots\partial_{\mu_{s-1}}\varphi^a(x)\big) - \text{traces}. \tag{4.120}$$

We calculate the anomalous dimensions of these operators up to three loops, using the integrals listed in Appendix A. The anomalous dimensions of the operators (4.120) and (4.116) take a particularly simple form, since the quantum corrections vanish at the first two orders of perturbation theory:

$$\begin{aligned}
\gamma^a(s) &= \gamma_\sigma + \gamma_\varphi - \frac{u^3}{6}(25N-2)\frac{S_1(s)}{s^2(s+1)^2} + O(u^4)\,,\\
\gamma^{[ab]}(s) &= 2\gamma_\varphi + \frac{u^3}{3}(N-26)\frac{S_1(s)}{s^2(s+1)^2} + O(u^4)\,.
\end{aligned} \tag{4.121}$$

Expanding $S_1(s) \sim \zeta_2 s + O(s^2)$, we see that the anomalous dimensions (4.121) diverge only as $\sim 1/s$, so the corresponding mass corrections (see (4.77)) are finite. The anomalous dimensions of the non-singlet operators read:

$$\begin{aligned}
\bar{\gamma}^a(s) &= \gamma_\varphi + \gamma_\sigma + (-1)^s\frac{2u}{s(s+1)} + \frac{2u^2}{s^2(s+1)^2}\left(1 - \frac{1+2s}{s(s+1)}\right)\\
&+ (-1)^s\frac{u^2}{6s(s+1)}\left((N+2)S_1 + \frac{2+s(6+N)}{s(s+1)} - \frac{8(N+4)}{3}\right)\\
&+ \frac{u^3}{s^2(s+1)^2}\Bigg\{(N+1)\Bigg(4S_3 - 4S_{-3} + 8S_{1,-2} - 12\zeta_3 - 4\left(\frac{1}{s(s+1)} + 2\right)S_{-2}\\
&- \frac{8\,(S_{-2}+1)}{(s-1)(s+2)}\Bigg) - \frac{1}{3}\left(\frac{N+(4+2N)s+2}{s(s+1)} - (13N+14)\right)S_1 - \frac{N+2}{6}S_2\\
&+ \frac{(N+2)s-2}{2s^2(s+1)^2} + \frac{N+2s(19N+82)-14}{18s(s+1)} - \frac{13N+58}{18}\Bigg\}\\
&+ (-1)^s\frac{u^3}{s(s+1)}\Bigg\{\frac{(N+2)(N+6)}{144}S_2 + \frac{(N+2)^2}{144}S_1^2 - 4(N+1)\Bigg(\frac{1}{s^2(s+1)^2}\\
&+ \frac{1}{s(s+1)}\Bigg)S_{-2} - 4\zeta_3 + \left((N+2)\frac{(N+6)s+2}{72s(s+1)} - \frac{8N^2+81N+518}{216}\right)S_1\\
&+ \frac{4(N+1)\,(S_{-2}+1)}{(s-1)(s+2)} + \frac{4}{s^4(s+1)^4} + \frac{4(2-3s)}{s^3(s+1)^3} - \frac{(N^2+8N+12)s-292}{72s^2(s+1)^2}\\
&- \frac{(8N^2+135N+1122)s - 3N^2+861N+1298}{216s(s+1)} + \frac{12N^2+631N+7099}{648}\Bigg\} + O(u^4).
\end{aligned} \tag{4.122}$$

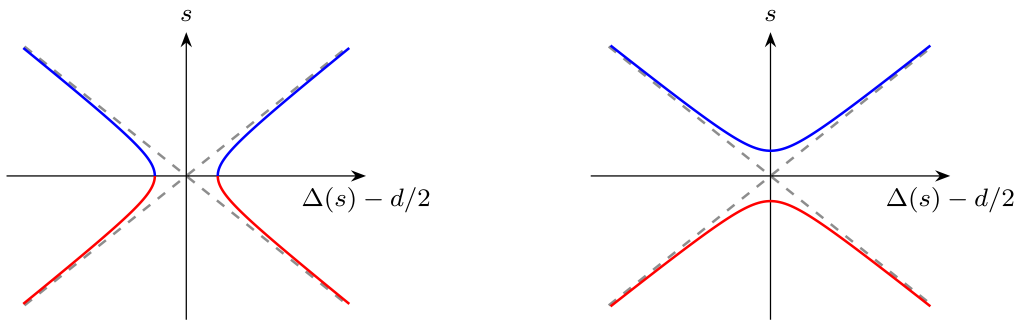
$$\bar{\gamma}^{(ab)}(s) = 2\gamma_\varphi + \frac{2u(-1)^s}{s(s+1)} + \frac{2u^2}{s^2(s+1)^2}\left(1 - \frac{1+2s}{s(s+1)}\right)$$

[a]Note that, as in Section 4.3.4, the treatment of the fully holomorphic sector is completely analogous to the fully antiholomorphic one.

$$
\begin{aligned}
&+\frac{2u^2(-1)^s}{3s(s+1)}\left(S_1+\frac{2(N+2)s+N}{4s(s+1)}-\frac{2N+8}{3}\right)\\
&+\frac{u^3}{s^2(s+1)^2}\Bigg\{12S_3+24S_{1,-2}-12S_{-3}-\frac{2}{3}S_2-12\left(\frac{1}{s(s+1)}+2\right)S_{-2}-36\zeta_3\\
&+\frac{4}{3}\left(10-\frac{1+2s}{s(s+1)}\right)S_1-\frac{24\left(S_{-2}+1\right)}{(s-1)(s+2)}+\frac{4s-N}{2s^2(s+1)^2}+\frac{(22N+76)s+2-4N}{9s(s+1)}\\
&-\frac{2(4N+13)}{9}\Bigg\}\\
&+\frac{u^3(-1)^s}{s(s+1)}\Bigg\{\frac{N+2}{18}S_2+\frac{1}{9}S_1^2-12\left(\frac{1}{s^2(s+1)^2}+\frac{1}{s(s+1)}\right)S_{-2}-4\zeta_3\\
&+\left(\frac{N+(2N+4)s}{18s(s+1)}-\frac{302+27N}{108}\right)S_1+\frac{12(S_{-2}+1)}{(s-1)(s+2)}+\frac{4}{s^4(s+1)^4}\\
&+\frac{N^2-8(N+2)s+288}{72s^2(s+1)^2}+\frac{N^2(1-16s)-18N(1+9s)-4(259s+744)}{216s(s+1)}\\
&+\frac{8-12s}{s^3(s+1)^3}+\frac{12N^2+631N+7099}{648}\Bigg\}+O(u^4),
\end{aligned}
\tag{4.123}
$$

$$
\begin{aligned}
\bar{\gamma}^{[ab]}(s)=2\gamma_\varphi&-\frac{2u(-1)^s}{s(s+1)}+\frac{2u^2}{s^2(s+1)^2}\left(1-\frac{1+2s}{s(s+1)}\right)\\
&-\frac{2u^2(-1)^s}{3s(s+1)}\left(S_1+\frac{2(N+2)s+N}{4s(s+1)}-\frac{2N+8}{3}\right)\\
&+\frac{u^3}{s^2(s+1)^2}\Bigg\{4S_3+8S_{1,-2}-4S_{-3}-\frac{2}{3}S_2-4\left(\frac{1}{s(s+1)}+2\right)S_{-2}\\
&-12\zeta_3+\frac{4}{3}\left(4-\frac{1+2s}{s(s+1)}\right)S_1-\frac{8\left(S_{-2}+1\right)}{(s-1)(s+2)}+\frac{4s-N}{2s^2(s+1)^2}\\
&+\frac{(22N+76)s+2-4N}{9s(s+1)}-\frac{2(4N+13)}{9}\Bigg\}\\
&-\frac{u^3(-1)^s}{s(s+1)}\Bigg\{\frac{N+2}{18}S_2+\frac{1}{9}S_1^2+4\left(\frac{1}{s^2(s+1)^2}+\frac{1}{s(s+1)}\right)S_{-2}\\
&-4\zeta_3+\left(\frac{N+(2N+4)s}{18s(s+1)}-\frac{302+27N}{108}\right)S_1-\frac{4\left(S_{-2}+1\right)}{(s-1)(s+2)}+\frac{4}{s^4(s+1)^4}\\
&+\frac{N^2-8(N+2)s+288}{72s^2(s+1)^2}+\frac{N^2(1-16s)-18N(1+9s)-4(259s-120)}{216s(s+1)}\\
&+\frac{8-12s}{s^3(s+1)^3}+\frac{12N^2+631N+7099}{648}\Bigg\}+O(u^4).
\end{aligned}
\tag{4.124}
$$

Several checks are available for these results. For example, the operator (4.119) at $s=2$ coincides



$\delta m^2(0) > 0$ $\delta m^2(0) < 0$

Figure 4.7: Regge trajectories around $s = 0$ for different values of $\delta m^2(0)$. The blue and red curves represent the two branches of the square root.

with the $O(N)$ current (see (1.56) and (4.58)), and one can explicitly verify that $\overline{\gamma}^{[ab]}(s = 2) = 0$. Furthermore, the symmetry (4.113) implies $\overline{\gamma}^{a}(s) = \overline{\gamma}^{(ab)}(s)$ for $N = 2$, which can also be verified explicitly. All mass corrections for the non-singlet operators are finite up to three loops, namely

$$
\begin{aligned}
\delta m_a^2(0) = 4u - &\frac{N^2+88N-244}{144}u^2 + \left(-\frac{2(N+14)}{3}\zeta_3 - \frac{44(N+1)}{5}\zeta_2^2\right. \\
&\left. + 8(N+1)\zeta_2 + \frac{10N^2+1255N+29422}{1296}\right)u^3 + O(u^4), \\
\delta m_{(ab)}^2(0) = 4u + &\frac{30-11N}{18}u^2 + \left(-\frac{32}{3}\zeta_3 - \frac{132}{5}\zeta_2^2 + 24\zeta_2\right. \\
&\left. - \frac{N^2-618N-30740}{1296}\right)u^3 + O(u^4), \\
\delta m_{[ab]}^2(0) = -4u + &\frac{9N+110}{18}u^2 + \left(\frac{16}{3}\zeta_3 - \frac{44}{5}\zeta_2^2 + 8\zeta_2\right. \\
&\left. + \frac{23N^2-2414N-33444}{1296}\right)u^3 + O(u^4).
\end{aligned}
\tag{4.125}
$$

It is instructive to examine the sign of the leading correction in (4.125). Indeed, the first correction to $\delta m_\alpha^2(0)$ can in principle take either sign. This is the case here: two operators have $\delta m^2(0) > 0$, while the antisymmetric operator (4.119) has $\delta m^2(0) < 0$. This leads to different types of level repulsion (see Section 4.3.2 for the discussion), see Fig. 4.7. In principle, the type of branching can also depend on the parameters of the theory, see (4.102).

Discussing the mixing of the singlet operators (4.117), we introduce the matrix of anomalous dimensions

$$
\widehat{\gamma}(s) = \begin{pmatrix} \gamma_{\varphi\varphi}(s) & \gamma_{\varphi\sigma}(s) \\ \gamma_{\sigma\varphi}(s) & \gamma_{\sigma\sigma}(s) \end{pmatrix},
\tag{4.126}
$$

where the entries are also calculated to three-loop accuracy (see Appendix A for the details):

$$
\begin{aligned}
\gamma_{\varphi\varphi}(s) = {}& 2\gamma_\varphi + (-1)^s \frac{2u}{s(s+1)} + \frac{2(N+1)u^2}{s^2(s+1)^2}\left(1 - \frac{1+2s}{s(s+1)}\right) \\
&+ (-1)^s \frac{2u^2}{3s(s+1)}\left(S_1 - \frac{2N+8}{3} + \frac{2(N+2)s+N}{4s(s+1)}\right) \\
&+ \frac{u^3}{s^2(s+1)^2}\Bigg\{12S_3 + 24S_{1,-2} - 12S_{-3} - \frac{(N+2)(N+4)}{12}S_2 \\
&- 12\left(\frac{1}{s(s+1)} + 2\right)S_{-2} - 36\zeta_3 - \frac{N(N-2)}{12}S_1^2 + \Bigg(\frac{7N^2-2N+120}{9} \\
&- \frac{(5N^2+6N+16)s + 2(N^2+2N+4)}{6s(s+1)}\Bigg)S_1 - \frac{24\left(S_{-2}+1\right)}{(s-1)(s+2)} \\
&+ \frac{3(N^2+2N+4)s - N(N+7)}{6s^2(s+1)^2} + \frac{4(N+2)(13N+19)s - 5N^2 - 10N + 4}{18s(s+1)} - \\
&\frac{2(25N^2+25N+39)}{27}\Bigg\} \\
&+ (-1)^s \frac{u^3}{s(s+1)}\Bigg\{\frac{N+2}{18}S_2 + \frac{1}{9}S_1^2 - 12\left(\frac{1}{s^2(s+1)^2} + \frac{1}{s(s+1)}\right)S_{-2} \\
&- 4\zeta_3 + \left(\frac{N+(2N+4)s}{18s(s+1)} - \frac{302+27N}{108}\right)S_1 + \frac{12(S_{-2}+1)}{(s-1)(s+2)} + \frac{4(2N+1)}{s^4(s+1)^4} \\
&- 4\frac{(2N+1)(3s-2)}{s^3(s+1)^3} + \frac{N^2+576N+288-8(N+2)s}{72s^2(s+1)^2} \\
&+ \frac{N^2(1-16s) - 18N(1+9s) - 4(259s+744)}{216s(s+1)} + \frac{12N^2+631N+7099}{648}\Bigg\} + O(u^4),
\end{aligned}
\tag{4.127}
$$

$$
\begin{aligned}
\gamma_{\varphi\sigma}(s) = {}& (-1)^s \frac{2uN}{s(s+1)} + \frac{2u^2N}{s^2(s+1)^2}\left(1 - \frac{1+2s}{s(s+1)}\right) \\
&+ (-1)^s \frac{2u^2N}{3s(s+1)}\left(S_1 + \frac{1+4s}{2s(s+1)} - 4\right) \\
&+ \frac{u^3N}{s^2(s+1)^2}\Bigg\{4S_3 + 8S_{1,-2} - 4S_{-3} - \frac{2}{3}S_2 - 4\left(\frac{1}{s(s+1)} + 2\right)S_{-2} - 12\zeta_3 \\
&+ \frac{4}{3}\left(4 - \frac{1+2s}{s(s+1)}\right)S_1 - \frac{8\left(S_{-2}+1\right)}{(s-1)(s+2)} + \frac{12s-N-4}{6s^2(s+1)^2} + \frac{N(22s-7)+436s-10}{36s(s+1)} \\
&+ \frac{N-128}{27}\Bigg\} \\
&+ (-1)^s \frac{u^3N}{s(s+1)}\Bigg\{\frac{N+14}{72}S_2 + \frac{N+6}{72}S_1^2 - 4\left(\frac{1}{s^2(s+1)^2} + \frac{1}{s(s+1)}\right)S_{-2}
\end{aligned}
$$

$$
\begin{aligned}
&- 4\zeta_3 + \left(\frac{(N+14)s+4}{36s(s+1)} - \frac{19N+318}{108}\right) S_1 + \frac{4\left(S_{-2}+1\right)}{(s-1)(s+2)} + \frac{4(N+1)}{s^4(s+1)^4} \\
&- \frac{4(N+1)(3s-2)}{s^3(s+1)^3} + \frac{N(289-2s)-28s+290}{72s^2(s+1)^2} - \frac{(19N+318)s+N+318}{54s(s+1)} \\
&\left. + \frac{583N+10046}{864}\right\} + O(u^4), \qquad (4.128)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\sigma\varphi}(s) = {} & (-1)^s \frac{2u}{s(s+1)} + \frac{2u^2}{s^2(s+1)^2}\left(1 - \frac{1+2s}{s(s+1)}\right) \\
&+ (-1)^s \frac{u^2}{3s(s+1)}\left(NS_1 + \frac{(N+2)s+1}{s(s+1)} - \frac{8(N+1)}{3}\right) \\
&+ \frac{u^3}{s^2(s+1)^2}\left\{4S_3 + 8S_{1,-2} - 4S_{-3} + \frac{(N-2)}{12}S_1^2 - \frac{3N+2}{12}S_2 - 12\zeta_3\right. \\
&- 4\left(\frac{1}{s(s+1)} + 2\right)S_{-2} - \left(\frac{N(3s+2)+10s+4}{6s(s+1)} + \frac{N-50}{9}\right)S_1 \\
&\left. - \frac{8\left(S_{-2}+1\right)}{(s-1)(s+2)} + \frac{(3s-1)N+6s-4}{6s^2(s+1)^2} + \frac{N(114s+5)+252s-34}{36s(s+1)} - \frac{14(N+7)}{27}\right\} \\
&+ (-1)^s \frac{u^3}{s(s+1)}\left\{\frac{N(3N+10)}{144}S_2 + \frac{N(3N+2)}{144}S_1^2 - 4\left(\frac{1}{s^2(s+1)^2}\right.\right. \\
&\left. + \frac{1}{s(s+1)}\right)S_{-2} - 4\zeta_3 + \left(\frac{N((3N+10)s+4)}{72s(s+1)} - \frac{6N^2+23N+108}{54}\right)S_1 \\
&+ \frac{4\left(S_{-2}+1\right)}{(s-1)(s+2)} + \frac{4(N+1)}{s^4(s+1)^4} - \frac{4(N+1)(3s-2)}{s^3(s+1)^3} - \frac{3N^2s - N(287-10s) - 294}{72s^2(s+1)^2} \\
&+ \frac{N^2(9-24s) - 2N(81s+1) - 4(251s+328)}{216s(s+1)} \\
&\left. + \frac{144N^2+3107N+26846}{2592}\right\} + O(u^4), \qquad (4.129)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\sigma\sigma}(s) = {} & 2\gamma_\sigma + \frac{2u^2N}{s^2(s+1)^2}\left(1 - \frac{1+2s}{s(s+1)}\right) + \frac{4u^3N}{s^2(s+1)^2}\left\{S_3 + 2S_{1,-2} - S_{-3} - \frac{3N+2}{48}S_2\right. \\
&- \left(\frac{1}{s(s+1)} + 2\right)S_{-2} - 3\zeta_3 + \frac{N-2}{48}S_1^2 - \left(\frac{(3N+10)s+2N+4}{24s(s+1)} + \frac{N-50}{36}\right)S_1 \\
&\left. - \frac{2\left(S_{-2}+1\right)}{(s-1)(s+2)} + \frac{3s(N+2)+N-8}{24s^2(s+1)^2} + \frac{24(N+8)s+7N-26}{72s(s+1)} + \frac{11N-148}{108}\right\} \\
&+ (-1)^s \frac{4u^3N}{s^3(s+1)^3}\left\{S_{-2} + \frac{4\left(S_{-2}+1\right)}{(s-1)(s+2)} + \frac{1}{s^2(s+1)^2} + \frac{2-3s}{s(s+1)} + 3\right\}. \qquad (4.130)
\end{aligned}
$$

The available checks on these results are related to the conserved $U(1)$ current and the stress-energy tensor. Namely, one can contract the vector of operators $\left(\overline{\mathcal{O}}_s, \Sigma_s\right)^{\mathrm{T}}$ with the two vectors $v_1^{\mathrm{T}} = (1, -2)$ and $v_2^{\mathrm{T}} = (1, 1)$. Conservation of these currents manifests as (see (1.56))

$$\begin{aligned} v_1^{\mathrm{T}} \cdot \widehat{\gamma}(s=2) &= \begin{pmatrix} 1, & -2 \end{pmatrix} \cdot \widehat{\gamma}(s=2) = 0, \\ v_2^{\mathrm{T}} \cdot \widehat{\gamma}(s=3) &= \begin{pmatrix} 1, & 1 \end{pmatrix} \cdot \widehat{\gamma}(s=3) = 0. \end{aligned} \tag{4.131}$$

We now turn to the question of how to generalize the mass correction (4.77) to the case of mixing operators. The most natural idea is to introduce the matrix

$$\delta\widehat{m}^2(s) = 2\left(\left(s + \bar{\beta}(u)\right)\mathbb{1} + \frac{1}{2}\widehat{\gamma}(s)\right)\widehat{\gamma}(s), \tag{4.132}$$

where $\mathbb{1}$ is the identity matrix. This matrix is regular at $s = 0$ up to two loops:

$$\delta\widehat{m}^2(0) = 4u\begin{pmatrix} 1 & N \\ 1 & 0 \end{pmatrix} + \frac{u^2}{18}\begin{pmatrix} 61N+30 & 12N \\ -20N+52 & 70N \end{pmatrix} + O(u^3)\,. \tag{4.133}$$

This regularity, however, breaks at three loops. Such a breaking is nevertheless anticipated if we approach the problem from the perspective of Regge trajectories (see Section 4.3.2). Indeed, what defines a trajectory on a Chew–Frautschi plot is not the individual entries of the matrix (4.126), but its eigenvalues (see the analogous discussion for reciprocity in Section 4.1.4). The eigenvalues $\delta m_1^2(s)$ and $\delta m_2^2(s)$ are not convenient to work with directly (due to the presence of square roots), and we use the trace and determinant (as in (4.24) and (4.27)) instead:

$$\begin{aligned} \delta m_1^2 + \delta m_2^2 = \operatorname{tr}\left[\delta\widehat{m}^2\right](0) = 4u + \frac{131N+30}{18}u^2 + \Bigg(&-\frac{4\left(N^2+2N+8\right)}{3}\zeta_3 - \frac{44}{5}(N+3)\zeta_2^2 \\ &+ 8(N+3)\zeta_2 - \frac{1131N^2 - 25950N - 30740}{1296}\Bigg)u^3 + O(u^4), \end{aligned} \tag{4.134}$$

$$\begin{aligned} \delta m_1^2 \cdot \delta m_2^2 = \det\left[\delta\widehat{m}^2\right](0) = N\Bigg\{&-16u^2 + \frac{4(10N+3)}{9}u^3 + \Bigg(\frac{224}{3}\zeta_3 + \frac{176}{5}\zeta_2^2 - 32\zeta_2 \\ &- \frac{28N^2 + 4421N + 12504}{81}\Bigg)u^4 + O(u^5)\Bigg\}. \end{aligned} \tag{4.135}$$

In principle, for a fully consistent picture one should first construct the eigenvalues of the anomalous dimension matrix (4.126) and then substitute them into the combination (4.77). We note, however, that the matrix (4.132) diagonalizes in the same basis as the anomalous dimension matrix (4.126), so the order does not matter. The branching points for the trajectories defined by $\delta m_{1,2}^2(0)$ are the same as in Fig. 4.7, namely

$$\delta m_{1,2}^2(0) = 2u\left(1 \pm \sqrt{1+4N}\right) + O(u^2), \tag{4.136}$$

so $\delta m_1^2(0) > 0$ and $\delta m_2^2(0) < 0$ for $N > 0$.

4.3.6 Gross-Neveu-Yukawa model

After discussing scalar theories in this section, we turn to models involving fermions. A suitable example in this regard is the Gross–Neveu–Yukawa model [440] defined in Euclidean ($d = 4 - 2\epsilon$)-dimensional space by the renormalized action

$$S_R = \int d^dx\left\{Z_1\bar{q}\partial\!\!\!/ q + \frac{Z_2}{2}(\partial\sigma)^2 + Z_3 g\mu^\epsilon\bar{q}\sigma q + Z_4\mu^{2\epsilon}\frac{\bar{\lambda}}{4!}\sigma^4\right\}, \tag{4.137}$$

where $q_i(x)$ $(i = 1, \ldots, N)$ is the fermion field with additional global $SU(N)$ symmetry and $\sigma(x)$ is a scalar field. In what follows we will sometimes refer to $q(x)$ as a quark field, using the analogy with the case of QCD. The model (4.137) is important in the field of critical phenomena as a representative of the so-called *Gross–Neveu universality class* (see [441–444] for the discussion of critical behavior), as it (sometimes with certain generalizations) is believed to describe the phase transition in graphene [445–447]. This model also serves as a valuable setting for the numerical conformal bootstrap [448–450] and diagrammatic calculations [451, 452]. The RG functions of the model (4.137) are known to five-loop accuracy [444]:

$$\begin{aligned}\beta_u(u,\lambda) &= -2u\epsilon + 2(3+2N)u^2 + O(u^3,\lambda^3),\\ \beta_\lambda(u,\lambda) &= -2\lambda\epsilon + 3\lambda^2 + 8Nu\lambda - 48Nu^2 + O(u^3,\lambda^3),\end{aligned} \tag{4.138}$$

$$\begin{aligned}\gamma_q(u,\lambda) &= \frac{u}{2} - \frac{u^2}{8}(12N+1) + O(u^3,\lambda^3),\\ \gamma_\sigma(u,\lambda) &= 2Nu + \frac{1}{12}\lambda^2 - 5Nu^2 + O(u^3,\lambda^3),\end{aligned} \tag{4.139}$$

where we introduced a convenient normalization for the couplings

$$u = \frac{g^2}{(4\pi)^2}, \qquad \lambda = \frac{\bar{\lambda}}{(4\pi)^2}. \tag{4.140}$$

Since the model (4.137) has two coupling constants, in contrast to the examples considered before, the straightforward translation of our CFT-based considerations (see Section 4.3.2) to the physical coupling (see (4.20) for the discussion) is not possible, and we must explicitly consider the theory at the critical point. The critical behavior of this model is also richer than in the previous cases, and (4.138) yields four possible critical points (u^*, λ^*) [142, 440, 443]: $(0,0)$, $(0, \lambda^*_{\mathrm{WF}})$, (u^*_-, λ^*_-), (u^*_+, λ^*_+). The first two are the trivial point and the one corresponding to the critical point of the φ^4 interaction, λ^*_{WF} (see Section 4.3.1 and (2.74) for the discussion). The latter two are non-trivial, and we focus on the IR-stable one, (u^*_+, λ^*_+): [k]

$$\begin{aligned}u_* &= u_1\epsilon + u_2\epsilon^2 + \ldots = \frac{\epsilon}{3+2N}\\ &\quad - \frac{8N^2 - 1032N - 441 - (4N+66)\sqrt{4N^2+132N+9}}{108(3+2N)^3}\epsilon^2 + O(\epsilon^3),\\ \lambda_* &= \lambda_1\epsilon + \lambda_2\epsilon^2 + \ldots = \frac{3 - 2N + \sqrt{4N^2+132N+9}}{9+6N}\epsilon + O(\epsilon^2).\end{aligned} \tag{4.141}$$

The set of leading-twist operators in this model is analogous to that in QCD, see (1.30), (1.29), (1.28):

Leading-twist operators in the Gross–Neveu–Yukawa model

Leading-twist operators in the model (4.137) can be classified according to the irreducible representations of the $SU(N)$ flavor group:

- Flavor non-singlet operator ($s \in \mathbb{Z}_{>0}$):

$$\mathcal{O}^{q,a}_s(x) = \bar{q}_i(x)\gamma_{(\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_s)}t^a_{ij}q_j(x) - \text{traces}. \tag{4.142}$$

[k] Note, however, that the same analysis can be carried out for the other critical point (u^*_-, λ^*_-), yielding the same regularity of the mass corrections (4.149) and (4.151), but we do not pursue this discussion and present results only for the IR-stable critical point.

- Flavor singlet operators (mix under renormalization) ($s \in 2\mathbb{Z}_{>0}$):

$$
\begin{aligned}
&\mathcal{O}_s^q(x) = -s\sum_{i=1}^{N} \bar{q}_i(x)\gamma_{(\mu_1}\partial_{\mu_2}\dots\partial_{\mu_s)}q_i(x) - \text{traces}, \\
&\mathcal{O}_s^\sigma(x) = \sigma(x)\partial_{\mu_1}\dots\partial_{\mu_s}\sigma(x) - \text{traces}.
\end{aligned} \tag{4.143}
$$

The round brackets on the indices of the quark operators denote symmetrization.

Let us comment on the additional normalization of the quark-singlet operator. This factor only affects the off-diagonal entries of the singlet anomalous dimension matrix (see (4.146)), and therefore does not affect the analysis of the singular structure (since only the product of off-diagonal elements enters the relevant formulas). It is, however, convenient for translating the symmetries arising at the level of generating operators (see Section 2.4 for a discussion), namely

$$
\begin{aligned}
\sigma(z_1 n)\sigma(z_2 n) &= \sum_{s=0}^{\infty} \frac{z_{12}^s}{s!}\sigma(-\partial_+)^s\sigma + \text{total derivatives}, \\
\bar{q}(z_1 n)\gamma_+ q(z_2 n) &= \sum_{s=1}^{\infty} \frac{z_{12}^{s-1}}{(s-1)!}\bar{q}\gamma_+(-\partial_+)^{s-1}q + \text{total derivatives},
\end{aligned} \tag{4.144}
$$

so that operators with the same scaling dimensions are related by the coefficient $s!/(s-1)! = s$.

We calculate the anomalous dimensions of the leading-twist operators up to two-loop order. We do not present the details of the calculation here, as we employed the general approach described in Chapter 3 and specifically in Section 3.2. The diagrams contributing at two loops are listed in Figs. 4.8, 4.9, 4.10, 4.11. The results read

$$
\begin{aligned}
\gamma^q(s) = 2\gamma_q - \frac{2u}{s(s+1)} - \frac{2u^2}{s(s+1)}\Bigg(3S_1 + \frac{1+2s}{s^2(s+1)^2} + \frac{1+5s+2N(1+2s)}{s(1+s)} \\
- 8 - 6N + (1-(-1)^s)\left(1 - \frac{1}{s(s+1)}\right)\Bigg) + O(u^3,\lambda^3),
\end{aligned} \tag{4.145}
$$

for the anomalous dimension of the operator (4.142), and

$$
\widehat{\gamma}(s) = \begin{pmatrix} 2\gamma_q & 0 \\ 0 & 2\gamma_\sigma \end{pmatrix} - 2u\begin{pmatrix} \frac{1}{s(s+1)} & 4N \\ \frac{1}{s(s+1)} & 0 \end{pmatrix} + \begin{pmatrix} \gamma_{qq}^{(2)}(s) & \gamma_{q\sigma}^{(2)}(s) \\ \gamma_{\sigma q}^{(2)}(s) & \gamma_{\sigma\sigma}^{(2)}(s) \end{pmatrix} + O(u^3,\lambda^3), \tag{4.146}
$$

where

$$
\begin{aligned}
\gamma_{qq}^{(2)}(s) &= -\frac{2u^2}{s(s+1)}\left(3S_1 + \frac{1+2s}{s^2(s+1)^2} + \frac{1+5s+2N(3+2s)}{s(1+s)} - 8 - 10N\right), \\
\gamma_{q\sigma}^{(2)}(s) &= -8Nu^2\left(3S_1 + \frac{1+2s}{s^2(s+1)^2} + \frac{3}{2s(s+1)} - 7\right), \\
\gamma_{\sigma q}^{(2)}(s) &= -\frac{4u^2}{s(s+1)}\left((1+2N)S_1 + \frac{1+2s}{2s^2(s+1)^2} + \frac{7+2(5+4N)s}{4s(s+1)} - 4N - \frac{5}{2}\right), \\
\gamma_{\sigma\sigma}^{(2)}(s) &= -\frac{8Nu^2}{s(s+1)}\left(\frac{1+2s}{s(s+1)} - 2\right) - \frac{\lambda^2}{s(s+1)},
\end{aligned} \tag{4.147}
$$

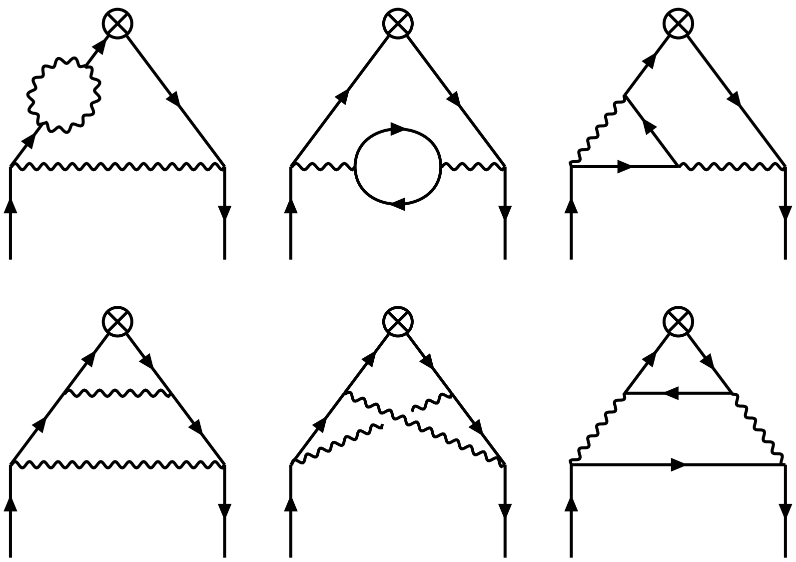


Figure 4.8: Two-loop diagrams contributing to the $\gamma_{qq}(s)$ anomalous dimension. Note that only the last (right-bottom) diagram give contribution to the pure-singlet anomalous dimension, $\gamma_{\rm ps}(s) = \gamma_{qq}(s) - \gamma_{\rm ns}(s)$.

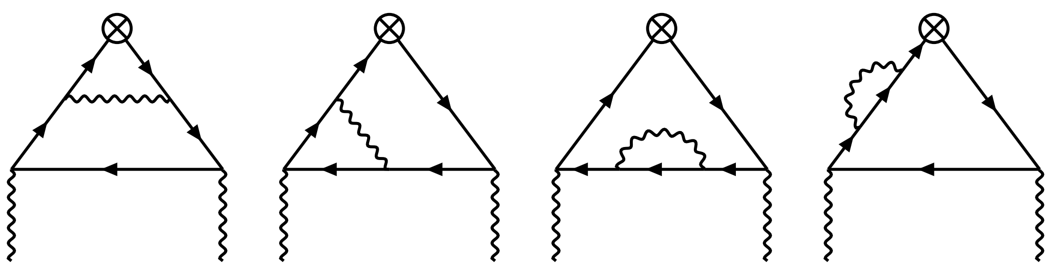


Figure 4.9: Two-loop diagrams contributing to the $\gamma_{q\sigma}(s)$ anomalous dimension.

for the singlet operators (4.143). One available cross-check (see (1.56) for the discussion) is the fact that $\gamma^q(1) = 0$, corresponding to the conserved $SU(N)$ current. Another cross-check for the singlet matrix is provided by the fact that projecting the matrix $\widehat{\gamma}$ with the vector $v^T = (1, 2)$ gives

$$v^T \cdot \widehat{\gamma}(2) = \begin{pmatrix} 1, & 2 \end{pmatrix} \cdot \widehat{\gamma}(2) = 0, \tag{4.148}$$

which corresponds to the conserved stress-energy tensor.

Proceeding to the discussion of the mass correction (4.77), we define

$$\delta m^2_{\pm}(s) = 2\left(s - \epsilon + \frac{1}{2}\gamma^q_*(s)\right)\gamma^q_*(s), \tag{4.149}$$

where the $+$ and $-$ functions are defined for even and odd values of s, respectively, and $\gamma^q_*(s) \equiv \gamma^q(s, u_*, \lambda_*)$ (see (4.141) for the critical couplings). The function $\delta m^2_+(s)$ is regular at $s = 0$:

$$\delta m^2_+(0) = -4u_1\epsilon + \big((24N + 37)u_1^2 - 6u_1 - 4u_2\big)\,\epsilon^2 + O(\epsilon^3), \tag{4.150}$$

while δm^2_- is singular, $\delta m^2_-(s) \simeq 8u_1^2/s$. For the singlet anomalous dimension (4.146) we define the matrix

$$\delta\widehat{m}^2(s) = 2\left((s - \epsilon)\mathbb{1} + \frac{1}{2}\widehat{\gamma}_*(s)\right)\widehat{\gamma}_*(s), \tag{4.151}$$

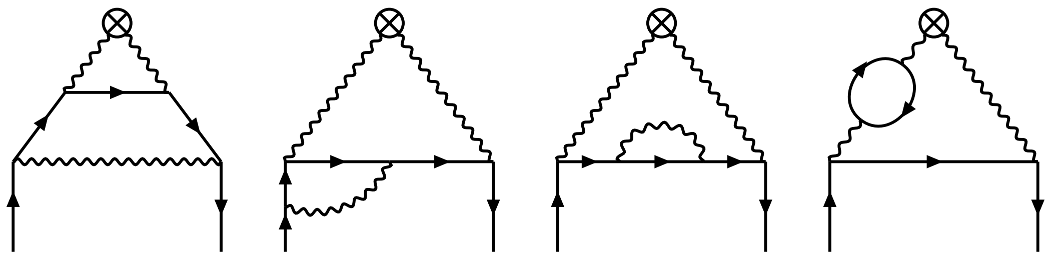


Figure 4.10: Two-loop diagrams contributing to the $\gamma_{\sigma q}(s)$ anomalous dimension.

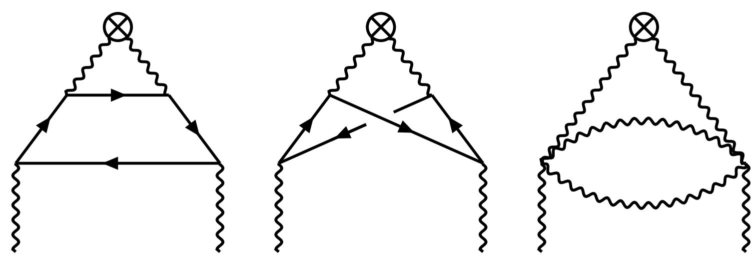


Figure 4.11: Two-loop diagrams contributing to the $\gamma_{\sigma\sigma}(s)$ anomalous dimension.

following the logic presented around (4.132). The eigenvalues of the matrix (4.151) are finite as $s \to 0$:

$$\delta\widehat{m}_1^2(0) = -4u_1\epsilon + \Big((16N-6)u_1 - (32N^2+24N-37)u_1^2 - 4u_2\Big)\epsilon^2 + O(\epsilon^3),$$

$$\delta\widehat{m}_2^2(0) = \Big(48N(N+2)u_1^2 - 24Nu_1 - 2\lambda_1^2\Big)\epsilon^2 + O(\epsilon^3). \tag{4.152}$$

Let us point out a special feature of the model (4.137) that was absent in the case of the singlet anomalous dimension (4.132) in Section 4.3.5. In Section 4.3.1 we saw that continuing the resummed anomalous dimension to the point $s=0$ reproduces the anomalous dimension of a local operator with the appropriate scaling dimension. In the cubic models, the point $s=0$ corresponds to effectively "-1" derivatives, and there is no local operator whose scaling dimension matches this limit. In the Gross–Neveu–Yukawa model, however, we again have a local operator, $\sigma^2(x)$, whose scaling dimension $d-2$, together with its other quantum numbers, matches the $s \to 0$ limit in the definition (4.143). It is therefore natural to expect that the analytically continued resummed anomalous dimension

$$\gamma_{1,2}(s) = \epsilon - s + \sqrt{(\epsilon-s)^2 + \delta\widehat{m}_{1,2}^2(s)} \tag{4.153}$$

should coincide with the anomalous dimension of the operator $\sigma^2(x)$ at $s=0$. Indeed, evaluating its anomalous dimension at the critical point (4.141), we obtain

$$\lim_{s\to 0}\gamma_2(s) = \gamma^*_{\sigma^2}, \qquad \gamma^*_{\sigma^2} = (\lambda_1 + 4Nu_1)\,\epsilon + O(\epsilon^2). \tag{4.154}$$

Considering the branch points of the resummed anomalous dimensions (4.153) (and the corresponding Regge trajectories on a Chew–Frautschi plot), we obtain

$$s_1^{\pm} = \pm 2\sqrt{\epsilon/(2N+3)} + O(\epsilon), \qquad s_2^{\pm} = \left(\frac{9-30N}{9+6N} \pm a_N\right)\epsilon + O(\epsilon^2), \tag{4.155}$$

where $a_N \in \mathbb{R}$ for $N \geq 1$. All of these branch points lie on the real axis.

4.3.7 Gross-Neveu model in $1/N$ expansion

The CFT arising at the critical point (4.141) of the model (4.137) can be analyzed from a different angle by choosing another theory from the Gross–Neveu universality class, namely the Gross–Neveu model itself:

$$S = \int d^d x \left(\bar{q}\partial\!\!\!/ q + \sigma \bar{q} q - \frac{N}{2g}\sigma^2 \right), \tag{4.156}$$

where the fermion field $q_i(x)$ again transforms in the fundamental representation of $SU(N)$ [441]. In the asymptotic regime the latter is critically equivalent to the theory

$$S = \int d^d x \left(\bar{q}\partial\!\!\!/ \sigma - \frac{1}{2}\sigma L_\Delta q + \bar{q}\sigma q + \frac{1}{2}\sigma L \sigma \right), \tag{4.157}$$

where the kernel $L(x)$ is chosen to cancel the one-loop contribution to the σ-field propagator and $L_\Delta(x)$ denotes its regularization (see [142, 264, 453]). This approach (see [142, Chapter 2, Section 45] for a pedagogical introduction and the discussion in Section 3.2) allows one to calculate the critical exponents (e.g. (4.139)) as perturbative series in $1/N$ with analytic dependence on d. The requirement of renormalizability of (4.157) still imposes some constraints, and in this section we assume $2 < d < 4$. Such calculations are available, for example, for the critical index η (determining the scaling dimension of the field q, [1] $\Delta_q = \mu - 1 + \eta/2$) at order $1/N^3$ [287, 454], for the critical dimension of the σ-field ($\Delta_s = 1+\gamma_\sigma$) and of the operator σ^2 ($\Delta_{\sigma^2} = 2+\gamma_{\sigma^2}$) at $1/N^2$ [288,455,456], see also [457,458]. We also note the work on determining *correction exponents* (which are represented by the scaling dimensions of specific local operators) in the $1/N^2$ approximation [271].

The analogous expansion can be performed for the anomalous dimensions of the leading-twist operators (4.143), which at order $1/N$ are given by [423, 459]:

$$\gamma_\sigma(s) = -\frac{\eta_1}{n}\frac{2(2\mu-1)}{(\mu-1)}\left(1 - \frac{\mu}{2\mu-1}\frac{\Gamma(\mu)}{\Gamma(s+\mu)}\frac{\Gamma(s+2-\mu)}{\Gamma(3-\mu)}\right) + O(1/n^2)\,, \tag{4.158}$$

$$\gamma_q(s) = \frac{\eta_1}{n}\left(1 - \frac{\mu(\mu-1)}{(s+\mu-1)(s+\mu-2)}\left(1 + \frac{\Gamma(2\mu-1)\Gamma(s+1)}{(\mu-1)\Gamma(2\mu-3+s)}\right)\right) + O(1/n^2)\,, \tag{4.159}$$

where $n = N \times \mathrm{tr}\,\mathbb{1} \equiv 4N$ and the index η_1 is

$$\eta_1 = -\frac{2\Gamma(2\mu-1)}{\Gamma(\mu+1)\Gamma(\mu)\Gamma(\mu-1)\Gamma(1-\mu)}\,. \tag{4.160}$$

Note that, in contrast to the model (4.137), the singlet operators $\mathcal{O}^q_s(x)$ and $\mathcal{O}^\sigma_s(x)$ defined in (4.143) do not mix under renormalization in the Gross-Neveu model (4.157), since their canonical scaling dimensions, $\Delta^0_q = 2\mu - 2 + s$ and $\Delta^0_\sigma = 2 + s$, differ for generic d. By virtue of the critical equivalence discussed above, we expect the anomalous dimensions (4.158), when expanded as a perturbative series in ϵ at $d = 4-2\epsilon$, to coincide with the eigenvalues of the anomalous dimension matrix (4.146) evaluated at the critical point (4.141) and expanded in $1/N$. This is indeed confirmed by explicit calculation. Note also that the quark anomalous dimension $\gamma_q(s)$ in (4.158) is known to $1/N^2$ accuracy [269].

The difference between the scaling dimension of the σ-field in the $1/N$ expansion framework (4.157) and in the Gross–Neveu–Yukawa model (4.137) leads to an interesting interplay in the structure of singularities in s between the two expansion regimes. Once $\epsilon = 2 - \mu$ is no longer treated as a small parameter, the intersection point of the leading-twist trajectory defined by (4.153) and its shadow shifts, giving rise to four intersection points instead of one, see Fig. 4.12. We therefore identify four intersection points:

- The quark trajectory $\Delta^0_q(s)$ intersects its shadow $\widetilde{\Delta}^0_q(s) = d - \Delta^0_q(s)$ at $s = 2 - \mu$.

[1] In this section we use the standard notation of the $1/N$ expansion literature, $\mu = d/2$.

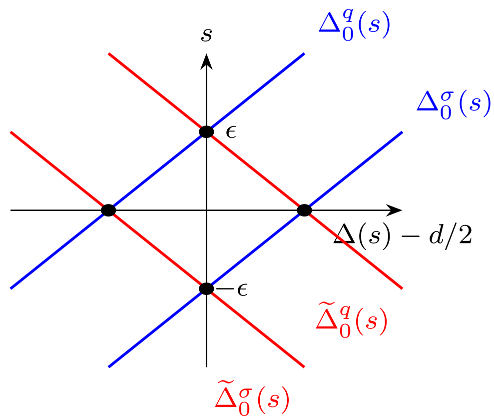


Figure 4.12: The Chew–Frautschi plot depicting the free trajectories ($N \to \infty$) for the singlet leading-twist operators in (4.157). We stress that ϵ is an arbitrary parameter here.

- The trajectory $\Delta_\sigma^0(s)$ intersects its shadow $\widetilde{\Delta}_\sigma^0(s)$ at $s = \mu - 2$.
- The trajectory $\Delta_q^0(s)$ intersects $\widetilde{\Delta}_\sigma^0(s)$, and $\Delta_\sigma^0(s)$ intersects $\widetilde{\Delta}_q^0(s)$, at $s = 0$.

The consequences of these intersections are clearly visible in (4.158). Indeed, $\gamma_\sigma(s)$ and $\gamma_q(s)$ have poles at $s = \mu - 2$ and $s = 2 - \mu$, respectively. The situation at $s = 0$ is more subtle. The anomalous dimensions (4.158) are regular at this point, but as was noted in [423],

$$\lim_{s\to 0} \gamma_\sigma(s) \neq \gamma_{\sigma^2}, \tag{4.161}$$

without any explanation being offered. Let us now address this problem using the resummation prescription derived in the previous sections. Applying (4.75) at the points $s_{q,\sigma} = \pm(2-\mu)$, we define two mass corrections:

$$\delta m_q^2(s) = 2\left(s + \mu - 2 + \frac{1}{2}\gamma_q(s)\right)\gamma_q(s)\,, \tag{4.162}$$

$$\delta m_\sigma^2(s) = 2\left(s + 2 - \mu + \frac{1}{2}\gamma_\sigma(s)\right)\gamma_\sigma(s). \tag{4.163}$$

Using the perturbative results (4.158), we find that these combinations of anomalous dimensions are indeed regular, and

$$\begin{aligned}
\delta m_q^2(s_q) &= -\frac{\eta_1}{n}\mu(\mu-1)\left(1 + \frac{\Gamma(2\mu-1)\Gamma(3-\mu)}{\Gamma(\mu)}\right) + O\left(1/n^2\right),\\
\delta m_\sigma^2(s_\sigma) &= \frac{\eta_1}{n}\frac{8\Gamma(\mu+1)}{\Gamma(2\mu-1)\Gamma(3-\mu)} + O\left(1/n^2\right).
\end{aligned} \tag{4.164}$$

Using the $1/N^2$ result for $\gamma_q(s)$ [269], one can verify the regularity for the quark trajectory at $1/N^2$. Here we have tacitly assumed that the anomalous dimensions $\gamma_\sigma(s)$ and $\gamma_q(s)$ are determined by analytic continuation from *even* integer s. However, the mass correction $\delta m_q^2(s)$ is also regular at $s = 2 - \mu$ for the anomalous dimension $\gamma_q(s)$ continued from *odd* spins, as well as for the non-singlet anomalous dimensions of both signatures. [m] Thus, in the vicinity of the points $s_{q,\sigma}$, the anomalous dimensions

[m] Operators of positive and negative signature are understood as continuations from even and odd values of s, respectively.

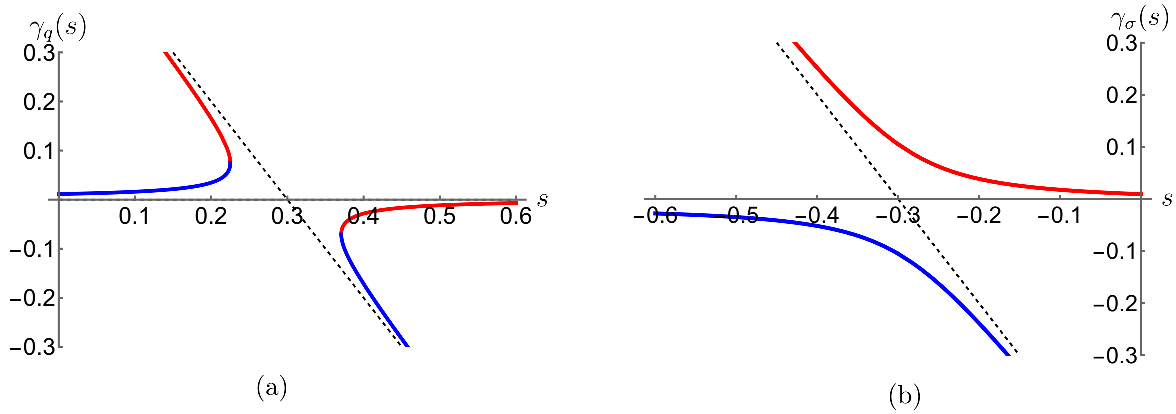


Figure 4.13: The anomalous dimensions defined by (4.165) around the points $s = s_\alpha$. Panel (a) shows $\gamma_q(s)$ and panel (b) shows $\gamma_\sigma(s)$. Red and blue trajectories correspond to the two branches of the square root, while the dashed black line represents the trajectories in the free theory $N \to \infty$. The plot is made for $\epsilon = 0.3$, $N = 40$.

acquire square-root branch points:

$$\gamma_\alpha(s) = s_\alpha - s + \sqrt{(s - s_\alpha)^2 + \delta m_\alpha^2(s)}, \tag{4.165}$$

where $\alpha = q, \sigma$. Using Eq. (4.164), one finds that the mass corrections have opposite signs in the region $2 < d < 4$, namely $\delta m_q^2(s_q) < 0$ and $\delta m_\sigma^2(s_\sigma) > 0$. Consequently, the branch points of the square root lie on the real axis for $\gamma_q(s)$, and become imaginary for $\gamma_\sigma(s)$. The trajectories of Eq. (4.165) around the points $s = s_\alpha$ are shown in Figure 4.13. Note the different types of behavior depending on whether the branch points lie on the real axis (a) or have a non-zero imaginary part (b).

At the point $s = 0$ the quark trajectory intersects the shadow σ-trajectory, and vice versa. The analyticity of the coefficients in the characteristic equation (4.75) is equivalent to the analyticity of the following functions at $s = 0$:

$$\omega_1(s) = \frac{1}{2}\left(\gamma_q(s) - \gamma_\sigma(s)\right), \tag{4.166}$$

$$\omega_2(s) = s(\gamma_q(s) + \gamma_\sigma(s)) + \gamma_q(s)\gamma_\sigma(s). \tag{4.167}$$

Remarkably, the first equation states that the singularities of the quark and σ anomalous dimensions (4.158) must coincide to all orders of perturbation theory. Solving Eqs. (4.166), one derives for the anomalous dimensions in the vicinity of $s = 0$:

$$\gamma_{q,\sigma}(s) = \pm\omega_1(s) - s + \sqrt{s^2 + \omega_2(s) + \omega_1^2(s)}. \tag{4.168}$$

Taking into account that

$$\omega_1(0) = -\frac{\eta_1}{n}\frac{(2\mu-3)(\mu-1)}{\mu-2} + O\left(1/n^2\right), \tag{4.169}$$

and

$$\omega_2(0) = -\left(\frac{\eta_1}{n}\right)^2 \frac{4(2\mu-1)}{\mu-2} + O\left(1/n^3\right), \tag{4.170}$$

one finds that the anomalous dimension of the operator σ^2 is approached by the trajectory $\gamma_\sigma(s)$ on the first Riemann sheet:

$$\lim_{s\to 0}\gamma_\sigma(s) = \gamma_{\sigma^2} = \frac{\eta_1}{n}2(2\mu-1) + O\left(1/n^2\right)\,, \tag{4.171}$$

while

$$\lim_{s\to 0}\gamma_q(s) = \frac{\eta_1}{n}\frac{2}{2-\mu} + O\left(1/n^2\right)\,. \tag{4.172}$$

Let us comment on the calculation of $\omega_2(0)$ in (4.170), which also sheds light on the nature of the inconsistency (4.161). To compute the leading order of $\omega_2(0)$, one may discard the first term in the definition (4.166) unless the sum of anomalous dimensions $\gamma_q(s)+\gamma_\sigma(s)$ has a pole at $s=0$. This pole is absent at order $1/N$, but is present at order $1/N^2$, which must be taken into account to match the accuracy of (4.170). Analyzing the results of [423], we obtain

$$\gamma_q(s) = \left(-\frac{\eta_1^2}{n^2}\frac{2\mu(2\mu-1)(2\mu-3)}{(\mu-2)^2}\times\frac{1}{s} + O(s^0)\right) + O(1/n^3)\,. \tag{4.173}$$

The anomalous dimension $\gamma_\sigma(s)$ is not known at $1/N^2$ order, but we can use the regularity of $\omega_1(s)$: the singular contribution to $\gamma_\sigma(s)$ simply coincides with (4.173). This allows us to compute (4.170) correctly and recover the limit (4.171). This explains the apparent inconsistency of (4.161): the σ anomalous dimension (4.158) is indeed regular at order $1/N$, but acquires singular behavior at order $1/N^2$, which through the resummation procedure (4.168) corrects the leading-order result. The branch points of the trajectory (4.168) are located at

$$s^\pm = \frac{\eta_1}{n}\frac{(\mu-1)(2\mu+1)\pm 2\sqrt{\mu(2\mu-1)(2\mu-3)}}{\mu-2} + O(1/n^2), \tag{4.174}$$

to $1/N$ accuracy. In the region $2<d<3$ they lie on the real axis and acquire a non-zero imaginary part for $3<d<4$. At $d=3$ the two branch points coincide, so the $1/N^2$ order must be taken into account.

We plot the function $\gamma_q(s)$ around the point $s=0$ in Figure 4.14. Note the change in behavior for $d<3$ (a) and $d>3$ (b). The branch points (4.174) are the same for the quark and σ anomalous dimensions, so the trajectory $\gamma_\sigma(s)$ has the same appearance as shown in Figure 4.14, up to small corrections of order $1/N$. Before closing, we remark on the limit $d\to 3$, in which the Gross–Neveu model describes a real physical system in three dimensions. The pole contribution (4.173) vanishes at $d=3$. Therefore, the anomalous dimensions $\gamma_q(s)$, $\gamma_\sigma(s)$, and the mass correction $\delta m_q^2(s)$ in Eq. (4.162) are analytic functions at $s=0$. Thus, if higher orders in the $1/N$ expansion are not taken into account, the anomalous dimensions $\gamma_q(s)$ and $\gamma_\sigma(s)$ cross each other at $s=0$ without branching.

4.4 QCD and concluding remarks

In Section 4.3 we saw how taking into account the mixing of two trajectories — one defined by the leading-twist scaling dimension, $\Delta(s)=\Delta^0(s)+\gamma(s)$, and another by its shadow, $\widetilde{\Delta}(s)=d-\Delta(s)$ — leads to predictions for the singular structure of the anomalous dimensions. A remarkable feature is that the definition of the shadow trajectory requires no additional information beyond the knowledge of $\gamma(s)$, yet imposing general analyticity requirements on the function (see (4.75)) allows one to resum all singular contributions. This resummation gives a very natural agreement with the structure of the spectrum: in Sections 4.3.1, 4.3.6 and 4.3.7 we saw that taking the limit of the resummed anomalous dimension reproduces the anomalous dimension of the local operator located at that point. The natural next step is to apply this formalism to QCD and examine its correspondence with what was discussed in Section 4.2 in the context of the double-logarithmic equation.

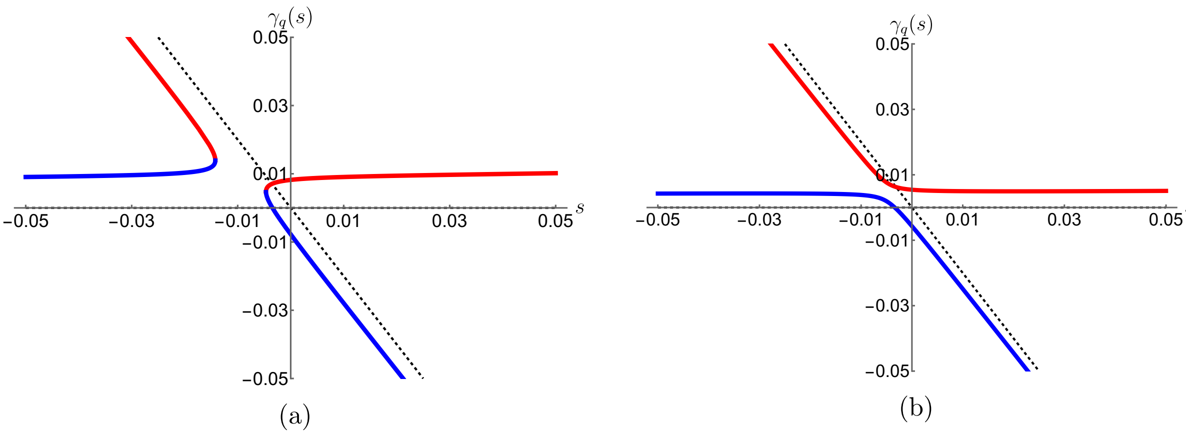


Figure 4.14: The anomalous dimension of the quark operator, $\gamma_q(s)$, defined by (4.168) around the point $s = 0$. Panel (a) is made for $\epsilon = 0.45$, $N = 40$, and panel (b) for $\epsilon = 0.55$, $N = 40$. Red and blue trajectories correspond to the two branches of the square root, while the dashed black line represents the trajectories in the free theory $N \to \infty$.

We can first separate the singlet (1.28), (1.29) and non-singlet (1.30) cases. In the singlet case, the corresponding Regge trajectories are defined by the eigenvalues of the singlet anomalous dimension matrix (4.22), $\Delta_\pm(s) = d - 2 + \gamma_\pm(s)$. In the non-singlet case, the Regge trajectory is defined by the scaling dimension $\Delta_{\rm ns}(s) = d - 2 + \gamma_{\rm ns}(s)$. The one-loop analysis of the corresponding Chew–Frautschi plot (see Fig. 4.15) was carried out in [73], using the explicit renormalization of *detector operators* [81] — a special class of non-local operators that, at the critical point, coincide with the light-ray operators, see Section 4.3.2 for the discussion. The remarkable difference from the non-gauge theories discussed in Section 4.3 is the presence of horizontal trajectories at $s = 1$, connected to the poles in $\gamma_{gg}(s)$ and therefore present in the eigenvalues $\gamma_\pm(s)$. Resolving this intersection leads to predictions analogous to the BFKL resummation, see Section 4.2 and specifically (4.37). It is believed (see [81, Section 2.6] for the discussion) that there is an infinite number of trajectories starting from $s = 1$ and continuing in the $s \to -\infty$ direction. It is therefore natural to assume that at the point $s = 0$ one can have an intersection of three trajectories: the leading-twist, its shadow, and a subleading BFKL trajectory (or multiple BFKL trajectories). This situation should be analogous to what we encountered in the cubic models (Sections 4.3.3, 4.3.4, 4.3.5) and the Gross–Neveu–Yukawa model (Section 4.3.6), where trajectories continued from points of different parity than $s = 0$ still possessed singularities even after the resummation (4.75). The difference here is that even the "right-parity" non-singlet trajectory acquires unresummed singular contributions at three loops, see (4.40). It would be interesting to investigate whether extending the detector operator approach [73, 81] to higher loops can make this resolution of the intersection more explicit. It would also be interesting to explore the relation with the long-standing problem of the three-loop BFKL kernel associated with the appearance of cuts [460–462] and whether the difficulties therein can be overcome using the approach of [73].

With all of this in mind, it is clear that the framework described in Section 4.3 is quite powerful and yields specific results that are valuable both for the technical calculation of anomalous dimensions of leading-twist operators and for the understanding of Regge behavior. At the same time, the situation in QCD raises many open questions and suggests directions for future development. Some of these were mentioned above and we would like to add several more:

- **What is the most efficient way to calculate Regge trajectories?**

  In this thesis we focused exclusively on leading-twist trajectories and their shadows. In several examples (including QCD) we saw that analytic continuation from the results for local operators is not sufficient, and one must also define horizontal trajectories. There are several ways to extract

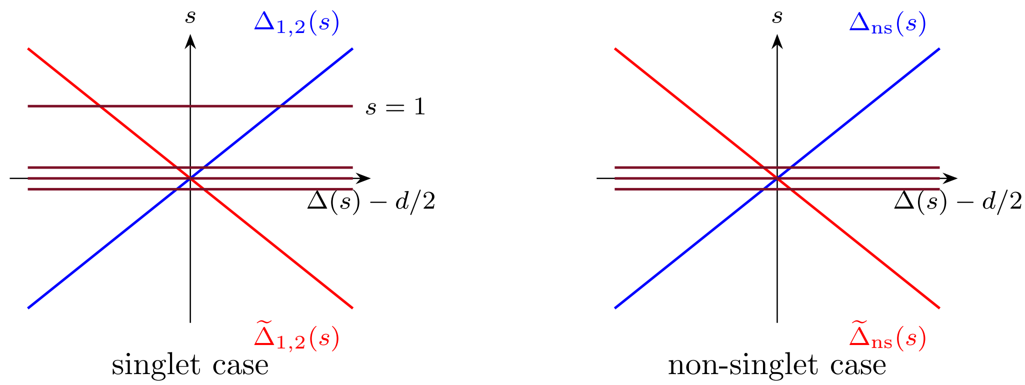


Figure 4.15: Regge trajectories on the Chew–Frautschi plot in QCD. On the left we depict the trajectories defined by the singlet operators and on the right we present the non-singlet case. The blue line represents the leading-twist trajectory and the red one represents its shadow. The horizontal lines correspond to BFKL-like trajectories.

these (see [81, Section 5]), for example using the Lorentzian inversion formula (see Eq. (4.67)) or direct renormalization of non-local light-ray operators (or detector operators in non-conformal QFT). It would be interesting to identify the most convenient formalism for the purposes of high-order perturbative calculations and to possibly automate it.

- **How to account for higher-twist operators?**

  It would be interesting to try to generalize the picture to the subleading singularities in the leading-twist anomalous dimensions ($s = -1, -2, \ldots$). As we saw in Section 4.3.2, this requires consideration of higher-twist operators (see (4.87) for details). The situation with higher-twist operators is more subtle, since they generally cannot be represented as a single family parametrized by one variable and span several Regge trajectories. There might, however, be a possibility to account for them effectively without a complete treatment. In this context, the generalization of the DLE to negative integers [392] and works on anomalous dimensions of higher-twist families [374] in $\mathcal{N} = 4$ SYM are of potential significance.

- **Which trajectories mix with each other?**

  In the preceding discussion we saw explicitly that the resummation procedure is very sensitive to the parity of the operators under consideration (see especially Section 4.3.3, where this effect is most obvious). In general, we found that anomalous dimensions continued from even values of spin are resummed by considering the shadow trajectory at $s = 0$, while those continued from odd values of spin require additional consideration. If one wishes to connect this effect to the resolution of mixing among Regge trajectories, it seems that the appearance of mixing trajectories can be anticipated by studying the set of possible non-local operators at the relevant point, with the parity of these non-local operators playing a role. It would be interesting to develop a framework for systematically accounting for all such operators.

- **What minimal information about a trajectory is needed to resolve the mixing?**

  We saw that in principle only restricted information from the full Regge trajectory is needed to predict the singular structure of the leading-twist anomalous dimension. If one is interested only in the latter, there may be a way to use not the full information about the other trajectories, but only certain restricted data. For example, in the case of mixing operators, the Lorentzian inversion formula (see (4.67) for details) operates with "averaged" anomalous dimensions (see [334, Section 3.4] for details). Since Regge trajectories can in principle be predicted from it, one may ask whether knowledge of this "averaged" trajectory is sufficient for predictions in perturbative calculations.

Chapter 5

Conformal anomaly and evolution kernels

In the previous chapter we discussed how the application of the conformal symmetry arising at the critical point of certain theories allows one to predict the analytic structure of the *forward* anomalous dimensions. In short, this connection can be summarized by the statement that at the critical point anomalous dimensions become CFT observables, whose structure is significantly constrained by the requirements on the analytically continued spectrum (see [80, 81, 332]). The reason why this matters for non-conformal physical theories is the specific structure of the $\overline{\mathrm{MS}}$ scheme, which in many cases allows one to translate results to arbitrary coupling (see (4.20) and e.g. [76, Eq. (1.15)]).

In this chapter we want to exploit the same property of the $\overline{\mathrm{MS}}$ scheme but apply it to the *non-forward* anomalous dimensions. In Section 2.4 we discussed that the non-forward part of the evolution kernel (corresponding to the off-forward anomalous dimensions in the case of local operators) at ℓ-th loop order can be determined from the $(\ell-1)$-th order result together with a special operator called the conformal anomaly (see (2.103)). This approach (complementary to D. Müller's works on the explicit breaking of classical conformal invariance [52, 219]) has been actively used to obtain state-of-the-art results for the evolution kernel of the non-singlet vector [57, 92] and axial [93] operators. Note that conformal invariance is used here not only as a conceptual idea justifying the existence of the quantum-deformed generators (2.103), but also as a practical tool: conformal Ward identities arising at the critical point provide a convenient way for calculating the conformal anomaly. In this section we continue the line of works [57, 92, 93, 463] and apply these techniques to the *transversity* operator — the last independent Dirac structure among the twist-2 quark operators.

Another way in which the conformal anomaly can be used is to characterize the conformal properties of the *forward* anomalous dimensions. In particular, the differential operators (2.103) satisfying the $\mathfrak{sl}(2,\mathbb{R})$ collinear conformal algebra commutation relations give an interpretation of the function $f(j)$, see (4.10), as an eigenvalue of the operator commuting with the $\mathfrak{sl}(2,\mathbb{R})$ Casimir. This approach was explored in [101], and in this chapter we attempt to generalize it to the singlet case.

The content of this chapter largely follows [2] and contains original results obtained with the participation of the author of this thesis:

1. Two-loop quantum corrections to the deformed generators of the collinear $SL(2,\mathbb{R})$ symmetry (2.103) for the transversity non-local operator.

2. The three-loop evolution kernel (1.98) for the non-local transversity operator.

3. A proposed matrix transform for the singlet anomalous dimension matrix (1.63) that makes all four entries of the matrix possess the reciprocity property, in contrast to just the two eigenvalues (see Section 4.1.4 for details).

5.1 Transversity operator

We begin by considering the twist-2 transversity non-local operator. The purpose of this section is mainly to set up the notation and make the presentation self-consistent, so we will partially repeat the discussion of Section 2.4. The reader may additionally consult the original paper [2] and other papers establishing the approach [57, 92, 93, 463]. We start by recalling the bare QCD action in ($d = 4 - 2\epsilon$)-dimensional Euclidean space:

$$S_{\mathrm{QCD}} = \int d^d x \left(\bar{q} \not{D} q + \frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} - \bar{c}^a \partial_\mu \left(D^\mu c\right)^a + \frac{1}{2\xi} (\partial_\mu A^{a,\mu})^2 \right), \tag{5.1}$$

where $q(x), \bar{q}(x)$ are quark fields, $D_\mu = \partial_\mu - \mathrm{i}\, g A^a_\mu T^a$ is the covariant derivative with T^a the generators of the $SU(N_c)$ group, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, $c(x)$ and $\bar{c}(x)$ are the ghost fields introduced by the Faddeev–Popov quantization, and $\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ is the gauge-fixing term. The operator of interest is defined as follows:

Transversity non-local operator

The transversity flavor non-singlet non-local operator is defined as

$$\mathcal{O}(z_1, z_2) = \bar{q}(z_1 n)\, [z_1 n, z_2 n]\, \sigma_{\perp +} q(z_2 n), \tag{5.2}$$

where the quark fields are assumed to be of different flavor (and we omit their flavor indices), the Wilson line between the quark fields is defined in (1.88), and $\sigma_{\perp +}$ denotes the corresponding projection of

$$\sigma_{\mu\nu} \equiv \frac{1}{2} \left[\gamma_\mu, \gamma_\nu\right] \tag{5.3}$$

in the notations of coordinate system (1.76).

The nucleon matrix elements of these operators (see Section 1.4 and Eq. (1.78)) define the chiral-odd GPDs, see e.g. [94, 95]. In deeply-virtual Compton scattering (DVCS) processes, transversity operators contribute only to power-suppressed helicity-flip amplitudes, making quark-helicity flip subprocesses strongly suppressed and chiral-odd GPDs difficult to access experimentally. Nevertheless, their experimental determination appears to be feasible in photo- or electroproduction or deeply-virtual meson production processes at the energies of the Electron-Ion Collider (EIC), see e.g. [96–100]. Here we are interested in the evolution (RG) equation for the operator (5.2) (see (1.96) for the discussion):

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(a) \frac{\partial}{\partial a} + \mathbb{H}(a) \right) \left[\mathcal{O}(z_1, z_2)\right] = 0, \tag{5.4}$$

where $\left[\mathcal{O}\right]$ denotes the renormalized version of the operator (5.2) (see Section 1.4 for the detailed discussion). The evolution kernel, $\mathbb{H}(a) = a\mathbb{H}^{(1)} + a^2 \mathbb{H}^{(2)} + a^3 \mathbb{H}^{(3)} + \ldots$, which in the non-forward case can be represented as an integral operator

$$\mathbb{H}^{(\ell)} f(z_1, z_2) = \int_0^1 d\alpha \int_0^1 d\beta\, h^{(\ell)}(\alpha, \beta) f(z_{12}^\alpha, z_{21}^\beta), \tag{5.5}$$

was known to two-loop accuracy prior to the work [2]:

1. The one-loop result was obtained in [53].

2. The two-loop result was first obtained in [54, 464] using the conformal anomaly technique and subsequently confirmed by direct calculation [465].

Let us also mention the work [466], where the evolution kernel for the transversity operator (5.2) was calculated to all orders in the leading asymptotics in the large-n_f limit.

The specific transversity projection of the gamma matrix, $\gamma_\perp$, appearing in the definition of the operator (5.2) via (5.3), anticommutes with $\gamma_\pm$. This determines the collinear spin (and conformal spin $j_q = 1$) of the quark fields involved (see (2.86)), and we conclude that the canonical $SL(2,\mathbb{R})$ generators (see (2.99)) are

$$\begin{aligned} S_-^{(0)} &= -\partial_{z_1} - \partial_{z_2}, \\ S_0^{(0)} &= z_1\partial_{z_1} + z_2\partial_{z_2} + 2, \\ S_+^{(0)} &= z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + 2(z_1 + z_2). \end{aligned} \tag{5.6}$$

Following the discussion in Section 2.4 (which can of course be verified by inspecting the results of explicit calculations), we conclude that the one-loop evolution kernel is invariant under these transformations:

$$\left[S_{\pm,0}^{(0)}, \mathbb{H}^{(1)}\right] = 0. \tag{5.7}$$

We do not repeat the full discussion of Section 2.4 on what happens beyond one loop [a] and simply state the results. It is convenient to decompose the full evolution kernel into a canonically invariant and a non-invariant part: [b]

$$\mathbb{H}^{(\ell)} = \mathbb{H}^{(\ell)}_{\text{inv}} + \mathbb{H}^{(\ell)}_{\text{non-inv}}, \qquad [S_\alpha^{(0)}, \mathbb{H}^{(\ell)}_{\text{inv}}] = 0. \tag{5.8}$$

The non-invariant part in turn satisfies the following set of equations:

$$\left[S_+^{(0)}, \mathbb{H}^{(1)}_{\text{non-inv}}\right] = 0\,, \tag{5.9a}$$

$$\left[S_+^{(0)}, \mathbb{H}^{(2)}_{\text{non-inv}}\right] = \left[\mathbb{H}^{(1)}, \Delta S_+^{(1)}\right]\,, \tag{5.9b}$$

$$\left[S_+^{(0)}, \mathbb{H}^{(3)}_{\text{non-inv}}\right] = \left[\mathbb{H}^{(1)}, \Delta S_+^{(2)}\right] + \left[\mathbb{H}^{(2)}, \Delta S_+^{(1)}\right]\,, \tag{5.9c}$$

where $\Delta S_+^{(\ell)}$ represents the quantum corrections to the canonical generators (5.6) (see (2.103)):

$$\begin{aligned} S_-(a) &= S_-^{(0)}, \\ S_0(a) &= S_0^{(0)} + \bar\beta(a) + \frac{1}{2}\mathbb{H}(a), \\ S_+(a) &= S_+^{(0)} + \underbrace{(z_1 + z_2)\left(\bar\beta(a) + \frac{1}{2}\mathbb{H}(a)\right) + z_{12}\Delta(a)}_{\Delta S_+(a)}, \end{aligned} \tag{5.10}$$

and we recall that $\Delta(a)$ is an integral operator called the conformal anomaly (see e.g. (2.104) for the one-loop example in the case of the vector operator). Thus, if one knows the evolution kernel $\mathbb{H}^{(\ell-1)}$ and the conformal anomaly $\Delta^{(\ell-1)}$ at $(\ell-1)$-th perturbative order, one can use (5.9) to fully constrain the non-invariant part of the evolution kernel, $\mathbb{H}^{(\ell)}_{\text{non-inv}}$, at ℓ-th loop order. This also summarizes the approach we follow in this chapter. After reconstructing the non-invariant part, the invariant part $\mathbb{H}^{(\ell)}_{\text{inv}}$ can be expressed using the known forward anomalous dimension $\gamma^{(\ell)}(s)$ and the relation (1.99). The three-loop forward transversity anomalous dimension was obtained in [305] using the approach described in Section 3.3, and later confirmed by direct calculation [358].

[a] Note the conceptually important intermediate step of putting the theory (5.1) at the critical point and subsequently translating to arbitrary coupling via (4.20).

[b] Here and in what follows $\alpha \in \{+, -, 0\}$.

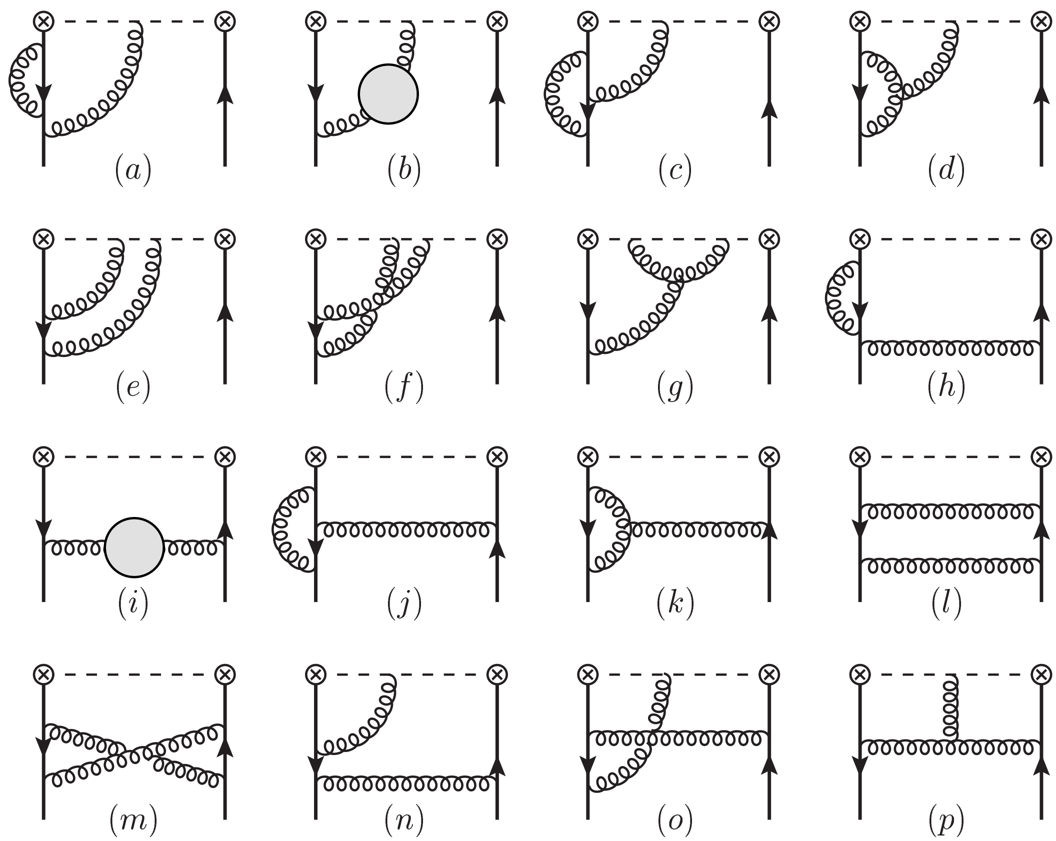


Figure 5.1: Feynman diagrams of different topology contributing to the two-loop evolution kernel and the two-loop conformal anomaly. The grey blob stands for the gluon self-energy insertion

5.2 Two-loop transversity evolution kernel

In the previous section we outlined the plan to obtain the three-loop anomalous dimension for the transversity operator (5.2), which requires two steps in the sense of calculations: two-loop evolution kernel, $\mathbb{H}^{(2)}$, and two-loop conformal anomaly, $\Delta^{(2)}$. Here we start with the calculations of the evolution kernel. The technique of calculations of diagrams involving non-local operators of the type (5.2) is discussed in Section 3.4. However, we would like to highlight the important feature of the transversity Dirac structure (5.3). The algebra of gamma matrices in d dimensions leads to the identity

$$\gamma_\mu \sigma_{\perp +} \gamma^\mu = -2\epsilon \sigma_{\perp +}, \tag{5.11}$$

which significantly simplifies the calculations. For example, at one-loop level (see Fig. 2.1) the diagram with gluon exchange (a) does not contribute to the evolution kernel at all and the one-loop evolution kernel reads

$$\mathbb{H}^{(1)} f(z_1, z_2) = 4C_F \left(\int_0^1 \frac{d\alpha}{\alpha} \Big(2f(z_1, z_2) - \bar{\alpha}\big(f(z_{12}^\alpha, z_2) + f(z_1, z_{21}^\alpha) \big) \Big) - \frac{3}{2} f(z_1, z_2) \right). \tag{5.12}$$

It is also convenient that some diagrams (e.g. diagram (b) in Fig. 2.1) are not sensitive to the Dirac structure involved and therefore can be read from the vector case [57].

Diagrams contributing to the two-loop evolution kernel, $\mathbb{H}^{(2)}$, are shown in Fig. 5.1. Answers for the individual diagrams are given in the Appendix B.1. Note that the answers for the diagrams without gluon exchange between the quark lines, namely the diagrams (a) – (g) in Fig. 5.1, are exactly the same as in the vector case and we have taken the corresponding results from [57]. The result for the evolution kernel can be written in the form

$$\mathbb{H}(a) = \Gamma_{\rm cusp}(a)\widehat{\mathcal{H}}(a) + A(a) + \mathcal{H}(a), \tag{5.13}$$

where $\Gamma_{\rm cusp}(a)$ stands for the cusp anomalous dimension in the fundamental representation (4.5). The first term is completely determined by the large spin asymptotic of the anomalous dimensions (see Section 4.1.1 for discussion). The kernel $\widehat{\mathcal{H}}$ has the form

$$\widehat{\mathcal{H}}f(z_1,z_2) = \int_0^1 \frac{d\alpha}{\alpha}\left(2f(z_1,z_2) - \bar\alpha\big(f(z_{12}^\alpha,z_2) + f(z_1,z_{21}^\alpha)\big)\right). \tag{5.14}$$

It is a canonically invariant operator, $[S_\alpha^{(0)},\widehat{\mathcal{H}}] = 0$, whose eigenvalues represent the logarithmically growing harmonic sum

$$\widehat{\mathcal{H}}\, z_{12}^{s-1} = 2S_1(s) z_{12}^{s-1}. \tag{5.15}$$

Next, $A(a)$ is a constant and $\mathcal{H}(a)$ is an integral operator of the following form

$$\begin{aligned}\mathcal{H}f(z_1,z_2) = &\int_0^1 d\alpha\,\varphi(\alpha)\Big(f(z_{12}^\alpha,z_2) + f(z_1,z_{21}^\alpha)\Big) + \\ &\int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\,\big(\chi(\alpha,\beta) + \overline{\chi}(\alpha,\beta)\mathbb{P}_{12}\big)\Big(f(z_{12}^\alpha,z_{21}^\beta) + f(z_{12}^\beta,z_{21}^\alpha)\Big),\end{aligned} \tag{5.16}$$

where the permutation operator $\mathbb{P}_{12}$ interchanges the variables z_1, z_2, i.e.

$$\mathbb{P}_{12}f(z_1,z_2) = f(z_2,z_1), \qquad \Big(\mathbb{P}_{12}f(z_{12}^\alpha,z_{21}^\beta) = f(z_{21}^\alpha,z_{12}^\beta)\Big). \tag{5.17}$$

The representation (5.13) is unique if one supposes that the eigenvalues of the kernel, $\mathcal{H}(a)z_{12}^{s-1} \mapsto \mathcal{H}(s)$, vanish at $s\to\infty$. Using the results for the diagrams in App. B.1 we obtain for the constant $A(a) = aA^{(1)} + a^2A^{(2)} + \dots$

$$\begin{aligned}A^{(1)} &= -6C_F\,,\\ A^{(2)} &= -\frac{8}{3}C_F^2\left(\frac{43}{8} + 13\zeta_2\right) + 8C_Fn_f\left(\frac{1}{12} + \frac{2}{3}\zeta_2\right) + \frac{8C_F}{N_c}\left(-\frac{17}{24} - \frac{11}{3}\zeta_2 + 3\zeta_3\right),\end{aligned} \tag{5.18}$$

while for the integral kernels φ, χ, and $\overline{\chi}$ we get

$$\begin{aligned}\varphi^{(2)}(\alpha) &= -4C_F\beta_0\frac{\bar\alpha}{\alpha}\ln\bar\alpha + 8C_F^2\frac{\bar\alpha}{\alpha}\ln\bar\alpha\left(\frac{3}{2} - \ln\bar\alpha + \frac{1+\bar\alpha}{\bar\alpha}\ln\alpha\right),\\ \chi^{(2)}(\alpha,\beta) &= 8C_F^2\left(\frac{1}{\bar\alpha}\ln\alpha - \frac{1}{\alpha}\ln\bar\alpha\right) + \frac{4C_F}{N_c}\left(\frac{\bar\tau}{\tau}\ln\bar\tau + \frac{1}{2}\right),\\ \overline{\chi}^{(2)}(\alpha,\beta) &= \frac{4C_F}{N_c}\left(-\bar\tau\ln\bar\tau + \frac{1}{2}\right),\end{aligned} \tag{5.19}$$

where $\tau = \alpha\beta/\bar\alpha\bar\beta$. These expressions are consistent with the result for the two-loop kernel in momentum fraction representation in Refs. [54, 464, 465]. As an additional check one can calculate the eigenvalue of the integral operator $\mathbb{H}^{(2)}$ and obtain

$$\begin{aligned}\gamma^{(2)}(s) = &-8C_F\beta_0\left(S_2 - \frac{5}{3}S_1 + \frac{1}{8}\right) + 8C_F^2\left(-2S_2\left(2S_1 - \frac{3}{2}\right) + \frac{8}{3}S_1 - \frac{7}{8}\right)\\ &+ \frac{8C_F}{N_c}\left(2S_3 - 2S_{-3} + 4S_{1,-2} + \frac{4}{3}S_1 - \frac{1}{4} + \frac{1-(-1)^s}{2s(s+1)}\right),\end{aligned} \tag{5.20}$$

which is in perfect agreement with the results of [305,353–355]. We have also checked that the kernel $\mathbb{H}^{(2)}$ satisfies the consistency relation (5.9b). This implies that although the two-loop kernel was obtained by direct calculation, it is uniquely determined by the conformal anomaly $\Delta^{(1)}$, and the two-loop anomalous dimensions, Eq. (5.20).

5.3 Two-loop transversity conformal anomaly

In this section we proceed to the calculation of the conformal anomaly, i.e. the quantum correction to the generator $S_+(a)$ (see (5.10) for details). As announced at the beginning, this can be conveniently done by making use of the conformal Ward identity. The corresponding procedure is described in considerable detail in [57, Section 3]. The different Dirac structure, $\gamma_+ \mapsto \sigma_{\perp+}$, does not affect the analysis, so all results are directly applicable in our case. We do not reproduce all the calculations, but outline the general plan in order to highlight the important conceptual details.

Calculation of the conformal anomaly

To calculate the conformal anomaly $z_{12}\Delta(a)$, one can follow the steps below:

- Consider the Green's function of two light-ray operators:

$$\mathcal{G}(x; z_1, z_2, w_1, w_2) = \big\langle \big[\mathcal{O}(z_1, z_2)\big] \big[\overline{\mathcal{O}}(x, w_1, w_2)\big] \big\rangle, \tag{5.21}$$

  where the second operator is given by

$$\overline{\mathcal{O}}(x, w_1, w_2) = \bar{q}(x + w_1\bar{n})\big[x + w_1\bar{n}, x + w_2\bar{n}\big]\sigma_{\perp+} q(x + w_2\bar{n}), \tag{5.22}$$

  and the coordinate x is chosen such that $\bar{n} \cdot x = 0$.

- Consider the conformal Ward identity for $\mathcal{G}(x; z_1, z_2, w_1, w_2)$ obtained by the change of variables (see the discussion in Section 2.3 and the derivation of (2.56)), $\Phi \mapsto \Phi + \bar{n}^\mu \delta^\mu_K \Phi$, where δ^μ_K is the infinitesimal special conformal transformation, see Section 2.1 for details. This yields a formula involving the canonical generator $S_+^{(0)}$ and the operator breaking conformal invariance: [a]

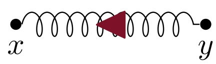
$$\mathcal{N}(x) = 2\epsilon \left(\frac{1}{4} F^2 + \frac{1}{2\xi} (\partial A)^2 \right). \tag{5.23}$$

- Consider the conformal Ward identity at the critical point (2.80) for the Green's function $\mathcal{G}(x; z_1, z_2, w_1, w_2)$. By virtue of the critical conformal invariance (see Section 2.3 for details), the variation of the action vanishes, and the resulting expression involves the critical value of the full deformed generator $S_+(a^*)$.

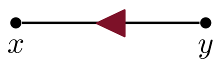
- Comparing the structure of these Ward identities and translating the appropriate expressions back to the physical coupling using (4.20), applied both to the evolution kernel $\mathbb{H}(a)$ and the conformal anomaly $\Delta(a)$, obtain the connection between the deformed and undeformed generators and the operator breaking conformal invariance, $\mathcal{N}(x)$.

From the technical point of view, this procedure can be conveniently packaged into the following concise prescription. Consider the QCD action (5.1) perturbed by the insertion of the operator breaking

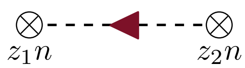
[a] Note that in Section 2.3 the operator breaking conformal invariance was obtained as an interaction term of the Lagrangian. In [57] the shifted parameters of canonical scaling dimensions in the special conformal transformation are used instead, which changes the form of CWI and makes the calculations technically more convenient.



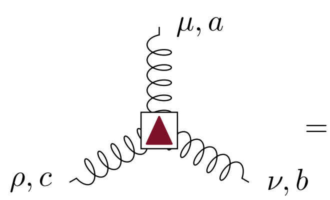
$$= \quad 2\mathrm{i}\delta_{\mu\nu} \int \frac{d^d k}{(2\pi)^d}\, e^{-\mathrm{i}k(x-y)}\, \frac{(\bar n \cdot k)}{k^4},$$



$$= \quad -\int \frac{d^d k}{(2\pi)^d}\, e^{-\mathrm{i}k(x-y)}\, \frac{\not k\, \not{\bar n}\, \not k}{k^4},$$



$$= \quad (\bar n \cdot n) z_{12} \big[z_1 n, z_2 n\big],$$



$$= \quad g f^{abc}\big(\delta^{\mu\rho}\bar n^\nu - \delta^{\mu\nu}\bar n^\rho\big).$$

Figure 5.2: Additional Feynman rules arising from the inclusion of the operator (5.23) into the QCD action (5.1).

conformal invariance (5.23):

$$S_{QCD} \mapsto S_\omega = S_{QCD} + \delta^\omega S = S_{QCD} - 2\omega \int d^d y (\bar n \cdot y) \left(\frac{1}{4}F^2 + \frac{1}{2\xi}(\partial A)^2\right), \tag{5.24}$$

where we are interested in the modification at leading order in the parameter ω. The renormalization of the operator (5.2) in the theory defined by (5.24) follows the same procedure described in Section 1.3, with the renormalization factor modified to

$$Z \mapsto Z_\omega = Z + \omega(\bar n \cdot n)\widetilde Z, \qquad \widetilde Z = \frac{1}{\epsilon}\widetilde Z_1(a) + \frac{1}{\epsilon^2}\widetilde Z_2(a) + \dots . \tag{5.25}$$

The residues $\widetilde Z_k$ are integral operators, and the conformal anomaly is determined by $\widetilde Z_1$:

$$\widetilde Z_1(a) = z_{12}\Delta_+(a) + \frac{1}{2}\left[\mathbb{H}(a) - 2\gamma_q(a)\right](z_1 + z_2), \tag{5.26}$$

where the operator $\Delta_+(a)$ is related to the operator $\Delta(a)$ appearing in the deformed generators (5.10) at the first two orders as (see [57, Eq. (3.47)] for the discussion)

$$\begin{aligned} z_{12}\Delta^{(1)} &= z_{12}\Delta_+^{(1)}, \\ z_{12}\Delta^{(2)} &= z_{12}\Delta_+^{(2)} + \frac{1}{4}\Big[\mathbb{H}^{(2)}, z_1 + z_2\Big]. \end{aligned} \tag{5.27}$$

Thus, the calculation of the conformal anomaly reduces to the standard renormalization of the operator (5.2) (see Section 3.1) in the presence of the additional Feynman rules generated by the modification (5.24), see Fig. 5.2. Note also that all contributions from mixing with the BRST and EOM operators (see Section 1.3 and specifically (1.57) for the discussion) are conveniently absorbed into the form of the Ward identities and require no additional treatment.

Calculating the one-loop conformal anomaly requires consideration of the two diagrams in Fig. 2.1, and one obtains

$$\Delta_+^{(1)} f(z_1,z_2) = -2C_F \int_0^1 d\alpha \left(\frac{\bar\alpha}{\alpha} + \ln\alpha\right) \Big(f(z_{12}^\alpha, z_2) - f(z_1, z_{21}^\alpha)\Big)\,. \tag{5.28}$$

Note that the diagram with gluon exchange (a) does not contribute, and the one-loop conformal anomaly (5.28) is exactly the same as in the vector case [54, 57]. The two-loop diagrams coincide with those presented in Fig. 5.1 supplemented by the additional Feynman rules in Fig. 5.2. The complete results for the contribution of each Feynman diagram can be found in Appendix B.2. Technical details and some examples can be found in [57,91]. We note that the diagrams without gluon exchange between the quark lines, diagrams (a) – (g) in Fig. 5.1, give the same contribution to Δ_+ as in the vector case. The kernel $\Delta_+^{(2)}$ can be written in the following form:

$$\begin{aligned}
[\Delta_+^{(2)} f](z_1,z_2) &= \int_0^1 du \int_0^1 dt\, \varkappa(t) \Big[f(z_{12}^{ut}, z_2) - f(z_1, z_{21}^{ut})\Big] \\
&\quad + \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta \Big[\omega(\alpha,\beta) + \overline{\omega}(\alpha,\beta)\mathbb{P}_{12}\Big] \Big[f(z_{12}^\alpha, z_{21}^\beta) - f(z_{12}^\beta, z_{21}^\alpha)\Big].
\end{aligned} \tag{5.29}$$

The function $\varkappa(t)$ is exactly the same as in the vector case, see [57,91]:

$$\varkappa(t) = C_F^2\, \varkappa_P(t) + \frac{C_F}{N_c}\, \varkappa_{FA}(t) + C_F\beta_0 \varkappa_{bF}(t), \tag{5.30}$$

where

$$\begin{aligned}
\varkappa_{bF}(t) &= -2\frac{\bar t}{t}\Big(\ln\bar t + \frac{5}{3}\Big), \\
\varkappa_{FA}(t) &= \frac{2\bar t}{t}\Big\{(2+t)\Big[\mathrm{Li}_2(\bar t) - \mathrm{Li}_2(t)\Big] - (2-t)\Big(\frac{t}{\bar t}\ln t + \ln\bar t\Big) - \frac{\pi^2}{6}t - \frac{4}{3} - \frac{t}{2}\left(1 - \frac{t}{\bar t}\right)\Big\}, \\
\varkappa_P(t) &= 4\bar t\Big[\mathrm{Li}_2(\bar t) - \mathrm{Li}_2(1)\Big] + 4\left(\frac{t^2}{\bar t} - \frac{2\bar t}{t}\right)\Big[\mathrm{Li}_2(t) - \mathrm{Li}_2(1)\Big] - 2t\ln t\ln\bar t - \frac{\bar t}{t}(2-t)\ln^2\bar t \\
&\quad + \frac{t^2}{\bar t}\ln^2 t - 2\left(1 + \frac{1}{t}\right)\ln\bar t - 2\left(1 + \frac{1}{\bar t}\right)\ln t - \frac{16}{3}\frac{\bar t}{t} - 1 - 5t\,.
\end{aligned} \tag{5.31}$$

For the functions ω and $\overline{\omega}$ we obtain

$$\overline{\omega}(\alpha,\beta) = \frac{C_F}{N_c}\overline{\omega}_{NP}(\alpha,\beta), \tag{5.32}$$

with

$$\overline{\omega}_{NP}(\alpha,\beta) = -2\left\{\frac{\alpha}{\bar\alpha}\left[\mathrm{Li}_2\left(\frac{\alpha}{\bar\beta}\right) - \mathrm{Li}_2(\alpha)\right] - \alpha\bar\tau\ln\bar\tau - \frac{1}{\bar\alpha}\ln\bar\alpha\ln\bar\beta - \frac{\beta}{\bar\beta}\ln\bar\alpha - \frac{1}{2}\beta\right\} \tag{5.33}$$

and

$$\omega(\alpha,\beta) = C_F^2\,\omega_P(\alpha,\beta) + \frac{C_F}{N_c}\omega_{NP}(\alpha,\beta), \tag{5.34}$$

where

$$\omega_P(\alpha,\beta) = \frac{4}{\alpha}\Big[\mathrm{Li}_2(\bar\alpha) - \zeta_2 + \frac{1}{4}\bar\alpha\ln^2\bar\alpha + \frac{1}{2}(\beta-2)\ln\bar\alpha\Big]$$

$$+\frac{4}{\bar\alpha}\Big[\mathrm{Li}_2(\alpha)-\zeta_2+\frac{1}{4}\alpha\ln^2\alpha+\frac{1}{2}(\bar\beta-2)\ln\alpha\Big],$$

$$\omega_{NP}(\alpha,\beta)=2\bigg\{\frac{\bar\alpha}{\alpha}\left[\mathrm{Li}_2\left(\frac{\beta}{\bar\alpha}\right)-\mathrm{Li}_2(\beta)-\mathrm{Li}_2(\alpha)+\mathrm{Li}_2(\bar\alpha)-\zeta_2\right]-\ln\alpha-\frac{1}{\alpha}\ln\bar\alpha$$
$$+\alpha\left(\frac{\bar\tau}{\tau}\ln\bar\tau+\frac{1}{2}\right)\bigg\}. \tag{5.35}$$

5.4 Three-loop evolution kernel

In the previous sections we obtained both ingredients needed to reconstruct the three-loop evolution kernel, according to the plan introduced in Section 5.1: in Section 5.2 we presented the results of the direct calculation of the two-loop evolution kernel, and in Section 5.3 we reproduced the two-loop conformal anomaly and thereby obtained the two-loop approximation for the deformed generators (5.10). In this section we proceed to the reconstruction of the evolution kernel from this data, organizing it in two steps: restoration of the non-forward part $\mathbb{H}_{\text{non-inv}}$, and then the forward part $\mathbb{H}_{\text{inv}}$, see (5.8).

5.4.1 Similarity transformation

In Section 5.1 we discussed that Eq. (5.9c) completely determines the three-loop off-forward evolution kernel, $\mathbb{H}^{(3)}_{\text{non-inv}}$. However, directly using this equation is technically inconvenient, and we instead employ the approach described in full detail in [101]. In short, we want to construct an intertwining operator $\mathrm{V}(a)$ that connects the deformed generators (5.10) with the canonical ones (5.6) (see also (2.105) for discussion), $S_\alpha(a)\mapsto S^{(0)}_\alpha$. This implies the intertwining of the evolution kernel:

$$S^{(0)}_\alpha=\mathrm{V}S_\alpha(a)\mathrm{V}^{-1}, \qquad\qquad \mathbf{H}_{\text{inv}}(a)=\mathrm{V}\,\mathbb{H}(a)\,\mathrm{V}^{-1}, \tag{5.36}$$

with the canonically invariant kernel $\mathbf{H}_{\text{inv}}(a)$ satisfying

$$\big[S^{(0)}_\alpha,\mathbf{H}_{\text{inv}}(a)\big]=0. \tag{5.37}$$

This property fixes the form of this integral operator:

$$\mathbf{H}_{\text{inv}}(a)=\Gamma_{\text{cusp}}(a)\widehat{\mathcal{H}}+\mathcal{A}(a)+\mathcal{H}(a), \tag{5.38}$$

where the kernel $\widehat{\mathcal{H}}$ is defined in (5.14), $\mathcal{A}(a)$ is a constant, and

$$\mathcal{H}(a)f(z_1,z_2)=\int_0^1 d\alpha\int_0^{\bar\alpha}d\beta\,\big(h(\tau)+\overline{h}(\tau)\mathbb{P}_{12}\big)\,f(z_{12}^\alpha,z_{21}^\beta). \tag{5.39}$$

We will return to determining the invariant kernel (5.38) in Section 5.4.3. For now let us assume that it is known and discuss how to recover the full evolution kernel $\mathbb{H}$.

The construction of the intertwining operator V can be naturally divided into two steps. Writing $\mathrm{V}=\mathrm{V}_2\mathrm{V}_1$, the first transform V_1 brings the symmetry generators to the “covariant” form $\mathbf{S}_\alpha(a)=\mathrm{V}_1S_\alpha(a)\mathrm{V}_1^{-1}$:

$$\mathbf{S}_-(a)=S^{(0)}_-,$$
$$\mathbf{S}_0(a)=S^{(0)}_0+\bar\beta(a)+\frac{1}{2}\mathbf{H}(a),$$
$$\mathbf{S}_+(a)=S^{(0)}_++(z_1+z_2)\left(\bar\beta(a)+\frac{1}{2}\mathbf{H}(a)\right), \tag{5.40}$$

where $\mathbf{H}(a) = \mathrm{V}_1\mathbb{H}(a)\mathrm{V}_1^{-1}$. Note that the new generators have the form (5.10) with the conformal anomaly set to zero, $\Delta(a) \mapsto 0$. An attractive feature of this representation is that when the generators act on an eigenfunction of the kernel $\mathbf{H}$, one can replace the kernel by the corresponding eigenvalue, $\mathbf{H} \mapsto \gamma(s)$. Seeking the operator V_1 in the form

$$\mathrm{V}_1(a) = \exp\big\{\mathrm{X}(a)\big\}, \qquad \text{where} \qquad \mathrm{X}(a) = a\mathrm{X}^{(1)} + a^2\mathrm{X}^{(2)} + O(a^3), \tag{5.41}$$

one obtains the following equations for $\mathrm{X}^{(k)}$:

$$[S_-^{(0)}, \mathrm{X}^{(k)}] = [S_0^{(0)}, \mathrm{X}^{(k)}] = 0 \tag{5.42}$$

and

$$\Big[S_+^{(0)}, \mathrm{X}^{(1)}\Big] = z_{12}\Delta^{(1)}, \tag{5.43a}$$

$$\Big[S_+^{(0)}, \mathrm{X}^{(2)}\Big] = z_{12}\Delta^{(2)} + \Big[\mathrm{X}^{(1)}, z_1 + z_2\Big]\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right) + \frac{1}{2}\Big[\mathrm{X}^{(1)}, z_{12}\Delta^{(1)}\Big]. \tag{5.43b}$$

These equations determine the operators $\mathrm{X}^{(k)}$ up to a canonically invariant operator, reflecting the arbitrariness in the definition of V_1, which may be multiplied on the left by an arbitrary operator depending only on the kernel $\mathbf{H}$:

$$\mathrm{V}_1 \mapsto \mathrm{V}_1' = \mathrm{U}(\mathbf{H})\mathrm{V}_1. \tag{5.44}$$

Equations (5.43) then lead to differential equations for the integral kernels, which are straightforward to solve. We will return to determining the explicit expressions for $\mathrm{X}^{(1)}$ and $\mathrm{X}^{(2)}$ in Section 5.4.2.

Let us now discuss the operator V_2 intertwining the generators (5.40) with the canonical ones (5.6):

$$\mathrm{V}_2\, \mathbf{S}_\alpha(a) = S_\alpha^{(0)}\, \mathrm{V}_2, \qquad\qquad \mathrm{V}_2\, \mathbf{H}(a) = \mathbf{H}_{\mathrm{inv}}(a)\, \mathrm{V}_2\,. \tag{5.45}$$

The construction of this operator requires no additional calculations and was worked out in the perturbative form in [101]:

$$\mathrm{V}_2 = \sum_{k=0}^{\infty}\frac{1}{k!}\mathrm{L}^k\left(\bar\beta(a) + \frac{1}{2}\mathbf{H}(a)\right)^k, \qquad \mathrm{V}_2^{-1} = \sum_{k=0}^{\infty}\frac{1}{k!}(-\mathrm{L})^k\left(\bar\beta(a) + \frac{1}{2}\mathbf{H}_{\mathrm{inv}}(a)\right)^k, \tag{5.46}$$

where $\mathrm{L} = \ln|z_{12}|$. Substituting this expression into (5.45), we obtain the connection between $\mathbf{H}_{\mathrm{inv}}$ and $\mathbf{H}$:

$$\mathbf{H}(a) = \mathbf{H}_{\mathrm{inv}}(a) + \sum_{n=1}^{\infty}\frac{1}{n!}\mathrm{T}_n(a)\left(\bar\beta(a) + \frac{1}{2}\mathbf{H}(a)\right)^n, \tag{5.47}$$

where the operators $\mathrm{T}_n(a)$ are defined by the recursion

$$\mathrm{T}_n(a) = [\mathrm{T}_{n-1}(a), \mathrm{L}]\,, \qquad\qquad \mathrm{T}_0(a) = \mathbf{H}_{\mathrm{inv}}(a)\,. \tag{5.48}$$

Taking into account Eqs. (5.38) and (5.39), one obtains for $\mathrm{T}_n(a)$ with $n > 0$:

$$\begin{aligned}\mathrm{T}_n(a)f(z_1,z_2) = {}& -\Gamma_{\mathrm{cusp}}(a)\int_0^1 d\alpha\frac{\bar\alpha}{\alpha}\ln^n\bar\alpha\,\big(f(z_{12}^\alpha, z_2) + f(z_1, z_{21}^\alpha)\big)\\ &+ \int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\,\ln^n(1-\alpha-\beta)\Big(h(\tau) + \bar h(\tau)\mathbb{P}_{12}\Big)f(z_{12}^\alpha, z_{21}^\beta)\,.\end{aligned} \tag{5.49}$$

Since the n-th term in the sum in (5.47) is of order $O(a^{n+1})$, one can obtain an approximation for $\mathbf{H}(a)$ to arbitrary precision. For example,

$$\mathbf{H}(a) = \mathbf{H}_{\rm inv}(a) + \mathrm{T}_1(a)\left(1+\frac{1}{2}\mathrm{T}_1(a)\right)\left(\bar\beta(a)+\frac{1}{2}\mathbf{H}_{\rm inv}(a)\right)$$
$$+\frac{1}{2}\mathrm{T}_2(a)\left(\bar\beta(a)+\frac{1}{2}\mathbf{H}_{\rm inv}(a)\right)^2 + O(a^4)\,. \tag{5.50}$$

Expanding all operators in power series, $\mathbf{H}_{\rm inv}(a)=\sum_k a^k\mathbf{H}^{(k)}_{\rm inv}$, $\mathrm{T}_n(a)=\sum_k a^k\mathrm{T}^{(k)}_n$, one derives

$$\mathbf{H}^{(1)} = \mathbf{H}^{(1)}_{\rm inv}, \tag{5.51a}$$

$$\mathbf{H}^{(2)} = \mathbf{H}^{(2)}_{\rm inv} + \mathrm{T}^{(1)}_1\left(\beta_0+\frac{1}{2}\mathbf{H}^{(1)}_{\rm inv}\right), \tag{5.51b}$$

$$\mathbf{H}^{(3)} = \mathbf{H}^{(3)}_{\rm inv} + \mathrm{T}^{(1)}_1\left(\beta_1+\frac{1}{2}\mathbf{H}^{(2)}_{\rm inv}\right)+\frac{1}{2}\mathrm{T}^{(1)}_2\left(\beta_0+\frac{1}{2}\mathbf{H}^{(1)}_{\rm inv}\right)^2+\left(\mathrm{T}^{(2)}_1+\frac{1}{2}\left(\mathrm{T}^{(1)}_1\right)^2\right)\left(\beta_0+\frac{1}{2}\mathbf{H}^{(1)}_{\rm inv}\right). \tag{5.51c}$$

To conclude this section, let us present the explicit connection between the kernel $\mathbf{H}(a)$ and the full kernel $\mathbb{H}(a)$, which follows from the intertwining relation involving $\mathrm{X}(a)$:

$$\mathbb{H}^{(1)} = \mathbf{H}^{(1)}\,, \tag{5.52a}$$

$$\mathbb{H}^{(2)} = \mathbf{H}^{(2)} + [\mathbf{H}^{(1)},\mathrm{X}^{(1)}]\,, \tag{5.52b}$$

$$\mathbb{H}^{(3)} = \mathbf{H}^{(3)} + [\mathbf{H}^{(2)},\mathrm{X}^{(1)}] + [\mathbf{H}^{(1)},\mathrm{X}^{(2)}] + \frac{1}{2}[[\mathbf{H}^{(1)},\mathrm{X}^{(1)}],\mathrm{X}^{(1)}]\,. \tag{5.52c}$$

Thus, the kernel $\mathbb{H}$ is determined through the successive chain of equations (5.52) and (5.51), where the operators $\mathrm{T}_n(a)$ are given in (5.49). The results for the kernels $\mathrm{X}^{(1)}$ and $\mathrm{X}^{(2)}$ are presented in Section 5.4.2, and the invariant kernel $\mathbf{H}_{\rm inv}(a)$ in Section 5.4.3.

5.4.2 X kernel and off-forward part

As mentioned above, Eqs. (5.43) can be viewed as differential equations for the integral kernels of the operators $\mathrm{X}^{(k)}$. Let us discuss this in more detail. First, the invariance of the operator X under the two other canonical generators, see (5.42), implies that the operators $\mathrm{X}^{(k)}$ can be represented as integral operators analogous to (5.16). More generally, it corresponds to the arbitrary operator F of the form

$$\big[\mathrm{F}\, f\big](z_1,z_2) = F_{\rm const}f(z_1,z_2) + \int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\, h(\alpha,\beta) f(z_{12}^\alpha, z_{21}^\beta)$$
$$+\int_0^1 d\alpha\,\frac{\bar\alpha}{\alpha}h^\delta(\alpha)\left(2f(z_1,z_2)-f(z_{12}^\alpha,z_2)-f(z_1,z_{21}^\alpha)\right). \tag{5.53}$$

The equation (5.43) represented through the commutator with the canonical generator $S_+^{(0)}$ then involves

$$\left[S_+^{(0)},\mathrm{F}\right] f = z_{12}\int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\,\left(\alpha\bar\alpha\partial_\alpha-\beta\bar\beta\partial_\beta\right)h(\alpha,\beta)f(z_{12}^\alpha,z_{21}^\beta)$$
$$- z_{12}\int_0^1 d\alpha\,\bar\alpha^2\partial_\alpha h^\delta(\alpha)\left(f(z_{12}^\alpha,z_2)-f(z_1,z_{21}^\alpha)\right). \tag{5.54}$$

Solving the differential equation for the integral kernel provides a more straightforward way to determine the kernel $\mathrm{X}^{(k)}$. For example,

$$\mathrm{X}^{(1)}f(z_1,z_2) = 2C_F\int_0^1 d\alpha\,\frac{\ln\alpha}{\alpha}\Big(2f(z_1,z_2)-f(z_{12}^\alpha,z_2)-f(z_1,z_{21}^\alpha)\Big), \tag{5.55}$$

which is exactly the same as in the vector case (see [92, Eq. (B.3)]).

The kernel $\mathrm{X}^{(2)}$ can be written as a sum of three terms corresponding to the three contributions on the right-hand side of Eq. (5.43b):

$$\mathrm{X}^{(2)} = \mathrm{X}_{\mathrm{I}}^{(2)} + \mathrm{X}^{(2,1)}\left(\beta_0 + \frac{1}{2}\mathbf{H}_{\mathrm{inv}}^{(1)}\right) - \frac{1}{2}\mathrm{X}^{(2,2)}. \tag{5.56}$$

The operators $\mathrm{X}^{(2,1)}$ and $\mathrm{X}^{(2,2)}$ are exactly the same as in the vector case [92, Eqs. (B.11) and (B.12)] (see also [91] for details):

$$\mathrm{X}^{(2,1)} f(z_1, z_2) = -2C_F \int_0^1 d\alpha \left[\frac{\bar\alpha}{\alpha}\ln\bar\alpha + \ln\alpha\right]\Big[2f(z_1, z_2) - f(z_{12}^{\alpha}, z_2) - f(z_1, z_{21}^{\alpha})\Big], \tag{5.57}$$

and

$$\begin{aligned}
&\mathrm{X}^{(2,2)} f(z_1, z_2) = \\
&= 4C_F^2 \Bigg\{\int_0^1 d\alpha \int_0^1 du \left[\frac{\ln\bar\alpha}{\alpha}\left(\frac{1}{2}\ln\bar\alpha + 2\right) + \frac{\bar u}{u}\frac{\vartheta(\alpha)}{\bar\alpha}\right]\Big[2f(z_1, z_2) - f(z_{12}^{\alpha u}, z_2) - f(z_1, z_{21}^{\alpha u})\Big] \\
&+ \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta \Bigg[\frac{\vartheta_+(\alpha) + \vartheta_+(\beta)}{\tau}\Big(f(z_{12}^{\alpha}, z_{21}^{\beta}) - f(z_1, z_{21}^{\beta}) - f(z_{12}^{\alpha}, z_2) + f(z_1, z_2)\Big) \\
&+ \Big(\vartheta_0(\alpha) + \vartheta_0(\beta)\Big) f(z_{12}^{\alpha}, z_{21}^{\beta})\Bigg]\Bigg\},
\end{aligned} \tag{5.58}$$

where

$$\begin{aligned}
\vartheta_+(\alpha) &= -\frac{1}{\bar\alpha}\Big[\ln\alpha\ln\bar\alpha + 2\alpha\ln\alpha + 2\bar\alpha\ln\bar\alpha\Big], \\
\vartheta_0(\alpha) &= 2\Big[\mathrm{Li}_3(\bar\alpha) - \mathrm{Li}_3(\alpha) - \ln\bar\alpha\,\mathrm{Li}_2(\bar\alpha) + \ln\alpha\,\mathrm{Li}_2(\alpha)\Big] + \frac{1}{\alpha}\ln\alpha\ln\bar\alpha + \frac{2}{\alpha}\ln\bar\alpha, \\
\vartheta(\alpha) &= \frac{\alpha}{\bar\alpha}\Big[\mathrm{Li}_2(\bar\alpha) - \ln^2\alpha\Big] - \frac{\bar\alpha}{2\alpha}\ln^2\bar\alpha + \left[\alpha - \frac{2}{\alpha}\right]\ln\alpha\ln\bar\alpha - \left[3 + \frac{1}{\bar\alpha}\right]\ln\alpha - (\alpha - \bar\alpha)\frac{\bar\alpha}{\alpha} - 2.
\end{aligned} \tag{5.59}$$

The operator $\mathrm{X}_{\mathrm{I}}^{(2)}$ satisfies the following equation:

$$\begin{aligned}
\Big[S_+^{(0)}, \mathrm{X}_{\mathrm{I}}^{(2)}\Big] &= z_{12}\Delta_+^{(2)} + \frac{1}{4}\Big[\mathbb{H}^{(2)}, z_1 + z_2\Big] \\
&= z_{12}\Delta_+^{(2)} + \frac{1}{4}\left[\frac{1}{2}\mathrm{T}_1^{(1)}\mathbf{H}_{\mathrm{inv}}^{(1)} + \Big[\mathbf{H}_{\mathrm{inv}}^{(1)}, \mathrm{X}^{(1)}\Big], z_1 + z_2\right] + \frac{1}{4}\Big[\mathbf{H}_{\mathrm{inv}}^{(2)} + \beta_0\mathrm{T}_1^{(1)}, z_1 + z_2\Big].
\end{aligned} \tag{5.60}$$

The solution can be written as

$$\mathrm{X}_{\mathrm{I}}^{(2)} = \mathrm{X}_{\mathrm{IAB}}^{(2)} + \frac{1}{4}\left(\mathrm{T}_1^{(2)} + \frac{1}{2}\beta_0\mathrm{T}_2^{(1)}\right). \tag{5.61}$$

The last term corresponds to the third term in Eq. (5.60).[a] We also note that the combination $\mathbf{H}_{\mathrm{ninv}} = \frac{1}{2}\mathrm{T}^{(1)}\mathbf{H}_{\mathrm{inv}}^{(1)} + \Big[\mathbf{H}_{\mathrm{inv}}^{(1)}, \mathrm{X}^{(1)}\Big]$ corresponds to the non-invariant C_F^2 part of the two-loop kernel and takes the form

$$\mathbf{H}_{\mathrm{ninv}} f(z_1, z_2) = 8C_F^2\left(\int_0^1 d\alpha\frac{\bar\alpha}{\alpha}\ln\bar\alpha\left(\frac{3}{2} - \ln\bar\alpha + \frac{1+\bar\alpha}{\bar\alpha}\ln\alpha\right)\Big(f(z_{12}^{\alpha}, z_2) + f(z_1, z_{21}^{\alpha})\Big)\right.$$

[a] We note that our definition of the operators $\mathrm{T}_n^{(k)}$ differs from that in Ref. [92].

$$+\int_0^1 d\alpha \int_0^{\bar\alpha} d\beta \left(\frac{1}{\bar\alpha}\ln\alpha - \frac{1}{\alpha}\ln\bar\alpha + (\alpha\leftrightarrow\beta)\right) f(z_{12}^{\alpha}, z_{21}^{\beta})\Bigg). \tag{5.62}$$

Since the two-loop anomaly $\Delta_+^{(2)}$ is also known, one can readily find $\mathrm{X}_{\mathrm{IAB}}^{(2)}$, which is convenient to split into two terms:

$$\mathrm{X}_{\mathrm{IAB}}^{(2)} = \mathrm{X}_{\mathrm{IA}}^{(2)} + \mathrm{X}_{\mathrm{IB}}^{(2)}. \tag{5.63}$$

The first term, $\mathrm{X}_{\mathrm{IA}}^{(2)}$, contains all contributions in which at least one argument of the function remains intact. Moreover, this term is exactly the same as in the vector case:

$$\begin{aligned}\mathrm{X}_{\mathrm{IA}}^{(2)} f(z_1,z_2) &= \int_0^1 du \frac{\bar u}{u}\int_0^1 \frac{d\alpha}{\bar\alpha}\Big(\varkappa(\alpha)-\varkappa(1)\Big)\Big(2f(z_1,z_2) - f(z_{12}^{\alpha u}, z_2) - f(z_1, z_{21}^{\alpha u})\Big)\\ &\quad + \int_0^1 d\alpha\, \xi_{\mathrm{IA}}(\alpha)\Big(2f(z_1,z_2) - f(z_{12}^{\alpha},z_2) - f(z_1,z_{21}^{\alpha})\Big),\end{aligned} \tag{5.64}$$

where $\varkappa(\alpha)$ can be found in (5.31) and

$$\begin{aligned}\xi_{\mathrm{IA}}(\alpha) &= 2C_F^2 \frac{\bar\alpha}{\alpha}\Bigg(-\mathrm{Li}_3(\bar\alpha) + \ln\bar\alpha\,\mathrm{Li}_2(\bar\alpha) + \frac{1}{3}\ln^3\bar\alpha + \mathrm{Li}_2(\alpha) + \frac{1}{\bar\alpha}\ln\alpha\ln\bar\alpha - \frac{1}{4}\ln^2\bar\alpha\\ &\quad - \frac{3\alpha}{\bar\alpha}\ln\alpha - 3\ln\bar\alpha\Bigg) + \frac{C_F}{N_c}\left(\ln\alpha + \frac{\bar\alpha}{\alpha}\ln\bar\alpha\right).\end{aligned} \tag{5.65}$$

The result for the second term $\mathrm{X}_{\mathrm{IB}}^{(2)}$ reads

$$\mathrm{X}_{\mathrm{IB}}^{(2)} f(z_1,z_2) = \int_0^1 d\alpha\int_0^{\bar\alpha} d\beta \left(C_F^2\,\xi_P(\alpha,\beta) + \frac{C_F}{N_c}\left(\xi_{NP}(\alpha,\beta) + \bar\xi_{NP}(\alpha,\beta)\mathbb{P}_{12}\right)\right) f(z_{12}^{\alpha}, z_{21}^{\beta}), \tag{5.66}$$

where

$$\begin{aligned}\xi_P(\alpha,\beta) &= 4\left(-\mathrm{Li}_3(\alpha) + \ln\alpha\,\mathrm{Li}_2(\alpha) + \frac{1}{\bar\alpha}\left(\mathrm{Li}_2(\alpha) - \zeta_2 + \frac{1}{4}\ln^2\alpha - \ln\alpha\right) + (\alpha\leftrightarrow\beta)\right)\\ &\quad - (\alpha,\beta\leftrightarrow\bar\alpha,\bar\beta),\\ \xi_{NP}(\alpha,\beta) &= -\frac{2}{\alpha}\left(\mathrm{Li}_2\left(\frac{\beta}{\bar\alpha}\right) - \mathrm{Li}_2(\beta) - \mathrm{Li}_2(\alpha) + \mathrm{Li}_2(\bar\alpha) - \zeta_2\right) - \ln\bar\alpha + (\alpha\leftrightarrow\beta),\\ \bar\xi_{NP}(\alpha,\beta) &= \frac{2}{\bar\alpha}\left(\mathrm{Li}_2\left(\frac{\alpha}{\bar\beta}\right) - \mathrm{Li}_2(\alpha) - \ln\bar\alpha\ln\bar\beta\right) - \ln\bar\alpha + (\alpha\leftrightarrow\beta).\end{aligned} \tag{5.67}$$

Note that the integral kernel $\xi_P(\alpha,\beta)$ corresponds to $z_{12}\Delta_+^{(2)}$ and the second term in Eq. (5.60), while $\xi_{NP}(\alpha,\beta)$ and $\bar\xi_{NP}(\alpha,\beta)$ correspond only to $z_{12}\Delta_+^{(2)}$.

5.4.3 Forward part

The final part of the calculation is the restoration of the invariant evolution kernel $\mathbf{H}_{\mathrm{inv}}$, see (5.38). As announced at the beginning, this can be done using the forward anomalous dimensions $\gamma(s)$, which in the case of the transversity operator are known to three-loop accuracy [305] (see also [358] for direct calculations). The results of these calculations, however, do not correspond directly to the eigenvalues of the operator (5.38) and require some additional consideration. Let us elaborate on this. Using the connection between the forward anomalous dimensions and the evolution kernels (see (1.99) for the discussion), we introduce two functions:

$$\mathbf{H}(a)\psi_s^{\pm}(z_1,z_2) = \gamma_{\pm}(s)\psi_s^{\pm}(z_1,z_2), \qquad \text{and} \qquad \mathbf{H}_{\mathrm{inv}}(a)\psi_s^{\pm}(z_1,z_2) = f_{\pm}(s)\psi_s^{\pm}(z_1,z_2), \tag{5.68}$$

where

$$\psi_s^+(z_1,z_2) = |z_{12}|^{s-1}, \qquad \psi_s^-(z_1,z_2) = \mathrm{sign}(z_{12})|z_{12}|^{s-1}. \tag{5.69}$$

It is easy to see that both functions define invariant eigenspaces for both operators, which is justified by the fact that both operators commute with the permutation operator $\mathbb{P}_{12}$. In comparison with Section 1.4, here we explicitly introduce two sets of functions to distinguish between anomalous dimensions defined for even, $\gamma_+(s)$, and odd, $\gamma_-(s)$, values of s.

It can be shown (see Section 5.5 for a more detailed discussion) that the function $f_\pm(s)$ is precisely the reciprocity-respecting part of the anomalous dimensions. It is related to the full forward anomalous dimension by

$$\gamma_\pm(s) = f_\pm\left(s + \bar\beta(a) + \frac{1}{2}\gamma_\pm(s)\right), \tag{5.70}$$

and all the properties of the reciprocity-respecting function (4.11) apply in this case. [b] This also simplifies working with the forward anomalous dimension, as it allows one to express everything in terms of RR harmonic sums, see (4.13). The structure of the function $f_\pm(s)$ (following the definition (5.38)) takes the form

$$f_\pm(s) = 2\Gamma_{\mathrm{cusp}}(a)S_1(s) + \mathcal{A}(a) + m_\pm(s). \tag{5.71}$$

The constant term $\mathcal{A}(a)$ translates directly to the constant term in the operator $\mathbf{H}_{\mathrm{inv}}(a)$, and the real challenge is to identify the integration kernels $h(\tau)$ and $\overline{h}(\tau)$, see (5.39). The remarkable property is that since these integration kernels depend on only one variable, $\tau = \alpha\beta/\bar\alpha\bar\beta$, this identification can be performed uniquely. Indeed, using (5.68) one obtains

$$m(s) = \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\, h(\tau)(1-\alpha-\beta)^{s-1} = \int_0^1 \frac{d\tau}{(1-\tau)^2}\, h(\tau)\, Q_s\left(\frac{1+\tau}{1-\tau}\right), \tag{5.72}$$

where Q_s is the Legendre function of the second kind. This integral transform can be inverted:

$$h(\tau) = \frac{1}{2\pi\,\mathrm{i}} \int_C ds\,(2s+1)\, m(s)\, P_s\left(\frac{1+\tau}{1-\tau}\right), \tag{5.73}$$

where P_s is the Legendre function of the first kind and the integration contour C runs parallel to the imaginary axis with all singularities of $m(s)$ lying to its left. Here we presented the formula connecting $m(s)$ and $h(\tau)$, to reconstruct the second integration kernel, one defines $\overline{m}(s)$ in analogy with (5.72), and sets

$$m_\pm(s) = m(s) \mp \overline{m}(s). \tag{5.74}$$

In practice, it is not very convenient to use the integral transform (5.73) directly, as it requires calculation of rather involved integrals, so we adopt a slightly different approach. A convenient property of the moments $m_\pm(s)$ is that when expressed in an appropriate set of functions, the integral transform (5.73) reduces to simple building blocks. The one-loop result is trivial:

$$m_\pm^{(1)}(s) = 0, \qquad \mathcal{A}^{(1)} = -6C_F, \tag{5.75}$$

so let us illustrate these ideas at two loops:

$$\mathcal{A}^{(2)} = -C_F^2\left(\frac{43}{3} + \frac{104}{3}\zeta_2\right) + C_F n_f\left(\frac{2}{3} + \frac{16}{3}\zeta_2\right) + \frac{C_F}{N_c}\left(-\frac{17}{3} - \frac{88}{3}\zeta_2 + 24\zeta_3\right),$$

$$m_\pm^{(2)}(s) = \frac{2C_F}{N_c}\Bigg(32S_1\left(S_{-2} + \frac{\zeta_2}{2}\right) + 16\left(S_3 - \zeta_3\right) - 32\left(S_{-2,1} - \frac{1}{2}S_{-3} + \frac{1}{3}\zeta_3\right)$$

[b] In principle the entire discussion was carried out for the non-singlet anomalous dimensions in the vector case, but the change of Dirac structure $\gamma_+ \mapsto \sigma_{\perp+}$ does not affect the analysis.

$$+\frac{2\left(1-(-1)^s\right)}{s(s+1)}\Bigg). \tag{5.76}$$

As discussed in Section 4.1, these functions can be expressed solely in terms of RR harmonic sums, see (4.13). Here we take one step further and define

$$\overline{\Omega}_1(s) = S_1(s), \qquad \overline{\Omega}_{-2}(s) = (-1)^s\left(S_{-2}(s)+\frac{\zeta_2}{2}\right),$$
$$\overline{\Omega}_3(s) = S_3(s)-\zeta_3, \qquad \overline{\Omega}_{-2,1}(s) = (-1)^s\left(S_{-2,1}(s)-\frac{1}{2}S_{-3}(s)+\frac{1}{3}\zeta_3\right). \tag{5.77}$$

The difference between these functions and the RR harmonic sums (4.13) lies in the large-s asymptotics:

$$\lim_{s\to\infty}\Omega_{\vec m}(s) = \# + O(1/s), \qquad \lim_{s\to\infty}\overline{\Omega}_{\vec m}(s) = O(1/s). \tag{5.78}$$

The construction of these functions can be found in [374, Appendix A], and the list of such harmonic sums needed for the three-loop transversity reconstruction is presented in [2, Appendix C]. [c] This allows us to rewrite the two-loop result (5.76) as

$$m_\pm^{(2)}(s) = \frac{2C_F}{N_c}\left(16\overline{\Omega}_3 \pm 32\left(\overline{\Omega}_1\overline{\Omega}_{-2}+\overline{\Omega}_{-2,1}\right)+\frac{2(1\mp 1)}{s(s+1)}\right). \tag{5.79}$$

Each RR harmonic sum with corrected asymptotics corresponds to a simple integration kernel, e.g. $\overline{\Omega}_{-2}\mapsto\bar\tau/2$, $\overline{\Omega}_3\mapsto\bar\tau/(2\tau)\ln\bar\tau$ [101]. The complete list of such correspondences is given in [2, Appendix C]. The only complication that can arise is the appearance of products of two harmonic sums, $\overline{\Omega}_{\vec a}\times\overline{\Omega}_{\vec b}$. In view of (5.73), the integration kernel of such a product is given by the convolution of the corresponding kernels, which can be computed efficiently using the `HyperInt` package [467]. For the two-loop integration kernels $h^{(2)}(\tau)$ and $\overline{h}^{(2)}(\tau)$, the result reads

$$h^{(2)}(\tau) = \frac{8C_F}{N_c}\left(\frac{\bar\tau}{\tau}\ln\bar\tau+\frac{1}{2}\right), \qquad \overline{h}^{(2)}(\tau) = \frac{8C_F}{N_c}\left(-\bar\tau\ln\bar\tau+\frac{1}{2}\right), \tag{5.80}$$

which is in full agreement with the result of the explicit calculation, Eq. (5.19).

Applying the same procedure to the three-loop anomalous dimensions, we obtain

$$\begin{aligned}
\mathcal{A}^{(3)} &= C_F\, n_f^2\left(\frac{34}{9}-\frac{160}{27}\zeta_2+\frac{32}{9}\zeta_3\right)+C_F^2\, n_f\left(-34+\frac{4984}{27}\zeta_2-\frac{512}{15}\zeta_2^2+\frac{16}{9}\zeta_3\right)\\
&+\frac{C_F n_f}{N_c}\left(-40+\frac{2672}{27}\zeta_2-\frac{8}{5}\zeta_2^2-\frac{400}{9}\zeta_3\right)\\
&+C_F^3\left(\frac{1694}{9}-\frac{22180}{27}\zeta_2+\frac{2464}{15}\zeta_2^2+\frac{1064}{9}\zeta_3-320\zeta_5\right)\\
&+\frac{C_F^2}{N_c}\left(\frac{5269}{18}-\frac{28588}{27}\zeta_2+\frac{2216}{15}\zeta_2^2+\frac{7352}{9}\zeta_3-32\zeta_2\zeta_3-560\zeta_5\right)\\
&+\frac{C_F}{N_c^2}\left(\frac{1657}{18}-\frac{8992}{27}\zeta_2+4\zeta_2^2+\frac{3104}{9}\zeta_3-80\zeta_5\right),
\end{aligned} \tag{5.81}$$

for the constant term, and for the integration kernels $h^{(3)}(\tau)$ and $\overline{h}^{(3)}(\tau)$ we obtain

$$h^{(3)}(\tau) = -C_F n_f^2\frac{16}{9}+C_F^2 n_f\left(\frac{352}{9}-\frac{8}{3}\mathrm{H}_0+\frac{16}{3}\frac{\bar\tau}{\tau}\left(\mathrm{H}_2-\mathrm{H}_{10}\right)\right)$$

[c] Note that a different notation is used there, with the harmonic sums of corrected asymptotics (5.77) denoted $\Omega_{\vec m}(s)$.

$$
\begin{aligned}
&+\frac{C_F n_f}{N_c}\left(8-\frac{8}{3}\mathrm{H}_1-\frac{4}{3}\mathrm{H}_0+\frac{\bar\tau}{\tau}\left(8\mathrm{H}_2-\frac{8}{3}\mathrm{H}_{10}+\frac{16}{3}\mathrm{H}_{11}+\frac{160}{9}\mathrm{H}_1\right)\right)\\
&+C_F^3\left(-\frac{1936}{9}+\frac{88}{3}\mathrm{H}_0+32\frac{\bar\tau}{\tau}\left(\mathrm{H}_3+\mathrm{H}_{12}-\mathrm{H}_{110}-\mathrm{H}_{20}-\frac{1}{3}\mathrm{H}_2+\frac{1}{3}\mathrm{H}_{10}+\frac{1}{2}\mathrm{H}_1\right)\right)\\
&+\frac{C_F^2}{N_c}\Bigg(-\frac{152}{3}-96\zeta_3-\left(\frac{8}{3}-48\zeta_2\right)\mathrm{H}_0+\frac{76}{3}\mathrm{H}_1-32\mathrm{H}_{10}+4\mathrm{H}_2-48\mathrm{H}_{20}-16\mathrm{H}_{11}\\
&-24\mathrm{H}_{21}+\frac{\tau}{\bar\tau}\Big(-24\zeta_2-48\zeta_3+64\mathrm{H}_0\Big)+\frac{\tau+1}{\bar\tau}\Big(-\big(32-16\zeta_2\big)\mathrm{H}_0\\
&+12\mathrm{H}_2-16\mathrm{H}_{20}-8\mathrm{H}_{21}\Big)+\frac{\bar\tau}{\tau}\Big(-\left(\frac{2000}{9}+16\zeta_2\right)\mathrm{H}_1+\frac{32}{3}\mathrm{H}_{10}-\frac{208}{3}\mathrm{H}_2\\
&-64\mathrm{H}_{20}-\frac{32}{3}\mathrm{H}_{11}-32\mathrm{H}_{110}+64\mathrm{H}_3+80\mathrm{H}_{12}+64\mathrm{H}_{21}+96\mathrm{H}_{111}\Big)\Bigg)\\
&+\frac{C_F}{N_c^2}\Bigg(\frac{544}{9}+16\zeta_2-96\zeta_3-\left(\frac{68}{3}-36\zeta_2\right)\mathrm{H}_0+\frac{68}{3}\mathrm{H}_1-24\mathrm{H}_{10}+4\mathrm{H}_2-36\mathrm{H}_{20}\\
&+\frac{\tau}{\bar\tau}\Big(-8\zeta_2-48\zeta_3+48\mathrm{H}_0\Big)+\frac{\tau+1}{\bar\tau}\Big(\big(-24+12\zeta_2\big)\mathrm{H}_0+4\mathrm{H}_2-12\mathrm{H}_{20}\Big)\\
&+\frac{\bar\tau}{\tau}\Big(-\left(\frac{1072}{9}+16\zeta_2\right)\mathrm{H}_1+\frac{44}{3}\mathrm{H}_{10}-44\mathrm{H}_2-32\mathrm{H}_{20}-\frac{16}{3}\mathrm{H}_{11}-16\mathrm{H}_{110}\\
&+32\mathrm{H}_3+32\mathrm{H}_{12}+48\mathrm{H}_{21}+32\mathrm{H}_{111}\Big)\Bigg),
\end{aligned}
\tag{5.82a}
$$

$$
\begin{aligned}
\bar h^{(3)}(\tau)=&-\frac{C_F n_f}{N_c}\left(\frac{104}{9}+\frac{8}{3}\mathrm{H}_0+\frac{8}{9}\Big(23-20\tau\Big)\mathrm{H}_1+\frac{16}{3}\bar\tau\Big(\mathrm{H}_{11}+\mathrm{H}_{10}\Big)\right)\\
&+\frac{C_F^2}{N_c}\Bigg(\frac{1480}{9}-40\zeta_2-48\zeta_3+\left(\frac{28}{3}+24\zeta_2\right)\mathrm{H}_0+\frac{76}{3}\mathrm{H}_1+16\mathrm{H}_{10}-4\mathrm{H}_2-24\mathrm{H}_{20}\\
&-16\mathrm{H}_{11}+24\mathrm{H}_{21}+\frac{\tau}{\bar\tau}\Big(-24\zeta_2+48\zeta_3-32\mathrm{H}_0\Big)+\frac{\tau+1}{\bar\tau}\Big(\Big(16-8\zeta_2\Big)\mathrm{H}_0+12\mathrm{H}_2\\
&+8\mathrm{H}_{20}-8\mathrm{H}_{21}\Big)+\bar\tau\Big(-24+48\zeta_2+48\zeta_3-16\zeta_2\mathrm{H}_0+\left(\frac{2144}{9}+16\zeta_2\right)\mathrm{H}_1+\frac{104}{3}\mathrm{H}_{10}\\
&-24\mathrm{H}_2+16\mathrm{H}_{20}+\frac{32}{3}\mathrm{H}_{11}-16\mathrm{H}_{110}-32\mathrm{H}_{12}-32\mathrm{H}_{21}-96\mathrm{H}_{111}\Big)\Bigg)\\
&+\frac{C_F}{N_c^2}\Bigg(\frac{1028}{9}-24\zeta_2-48\zeta_3+\left(\frac{44}{3}+36\zeta_2\right)\mathrm{H}_0+\frac{68}{3}\mathrm{H}_1+24\mathrm{H}_{10}-4\mathrm{H}_2-36\mathrm{H}_{20}\\
&+\frac{\tau}{\bar\tau}\Big(-8\zeta_2+48\zeta_3-48\mathrm{H}_0\Big)+\frac{\tau+1}{\bar\tau}\Big(\Big(24-12\zeta_2\Big)\mathrm{H}_0+4\mathrm{H}_2+12\mathrm{H}_{20}\Big)\\
&+\bar\tau\Big(-24+24\zeta_2+48\zeta_3-32\zeta_2\mathrm{H}_0+\left(\frac{1072}{9}+16\zeta_2\right)\mathrm{H}_1+\frac{88}{3}\mathrm{H}_{10}
\end{aligned}
$$

$$-24\mathrm{H}_2+32\mathrm{H}_{20}+\frac{16}{3}\mathrm{H}_{11}-32\mathrm{H}_{110}+16\mathrm{H}_{12}+16\mathrm{H}_{21}-32\mathrm{H}_{111}\Big)\Big), \tag{5.82b}$$

where $\mathrm{H}_{\vec{a}}(\tau)\equiv\mathrm{H}_{a_1\ldots a_k}$ are harmonic polylogarithms (HPLs) [302].

5.5 Reciprocity from the non-local perspective

The previous sections were dedicated to the solution of a specific practical problem — the calculation of the three-loop evolution kernel for the transversity operator (5.2). This problem was solved with extensive use of conformal symmetry, and even though the final result is quite involved (one has to assemble the evolution kernel starting from (5.51c) with (5.52c), using the results of Sections 5.4.2 and 5.4.3), it is explicit and can be used in practical calculations (for example in the form of the anomalous dimension matrix in the Gegenbauer basis (1.93), see [2, Section 5]).

What is interesting in this calculation beyond its practical phenomenological application is the method employed, and specifically the discussion of Section 5.4.1. Using the similarity transformation, we obtained a perspective on the reciprocity relation (5.70) that is conceptually different from the one discussed in Section 4.1. This perspective suggests, perhaps counter-intuitively, that one should consider non-local rather than local operators and view anomalous dimensions as eigenvalues of evolution kernels. This apparent complexification turns out to be quite convenient, since the conformal symmetry — which is related to the reciprocity property (see Section 4.1.3) [a] — is formulated in a particularly natural way in this framework. This is not surprising if one views non-local operators as generating operators for local ones (see (1.90) and the discussion in Section 1.4). In this section we first review the arguments leading to the generalized Gribov–Lipatov reciprocity property, following [101] in the case of a non-singlet evolution kernel. We then attempt to generalize this derivation to the case of a singlet evolution kernel. The main question we address is whether this perspective offers any advantage over what was discussed in Section 4.1.4, where the reciprocity property was argued to hold for the eigenvalues of the singlet anomalous dimension matrix.

5.5.1 Non-singlet case

We begin by reviewing Gribov–Lipatov reciprocity in the language of non-local operators. Suppose we are interested in a non-singlet evolution kernel $\mathbb{H}_{\mathrm{ns}}$, which can be taken as the quark non-singlet vector operator (1.30) as in the original discussion of [101], or as the transversity evolution kernel (5.2) — the main requirement is the absence of operator mixing. Consider the corresponding anomalous dimension as the eigenvalue of this operator:

$$\mathbb{H}_{\mathrm{ns}}(a)\psi_s(z_1,z_2)=\gamma_{\mathrm{ns}}(s)\psi_s(z_1,z_2),\qquad\qquad \psi_s(z_1,z_2)=z_{12}^{s-1}. \tag{5.83}$$

Here we focus only on parity-even functions (corresponding to $\psi_+(s)$ in (5.69)), so we assume $z_{12}>0$ throughout, the general case is treated in [101].

In Section 5.4.1 we discussed the chain of similarity transformations, which in the non-singlet case can be summarized as

$$\mathbb{H}_{\mathrm{ns}}\quad\xrightarrow{\mathrm{V}_1}\quad\mathbf{H}\quad\xrightarrow{\mathrm{V}_2}\quad\mathbf{H}_{\mathrm{inv}}, \tag{5.84}$$

where the intertwining operator is indicated above each arrow. Each evolution kernel is invariant under a set of generators forming an $\mathfrak{sl}(2,\mathbb{R})$ algebra:

$$\big[\mathbb{H}_{\mathrm{ns}}(a),S_\alpha(a)\big]=0,\qquad\big[\mathbf{H}(a),\mathbf{S}_\alpha(a)\big]=0,\qquad\big[\mathbf{H}_{\mathrm{inv}}(a),S_\alpha^{(0)}\big]=0, \tag{5.85}$$

[a] Note also the alternative perspective on generalized Gribov–Lipatov reciprocity related to the connection between light-ray operators and local operators in the CFT sense [81, Section 3.1.1] and [73, 468] in the QCD case.

where $S_\alpha(a)$ are the fully deformed generators (5.10), $\mathbf{S}_\alpha(a)$ are the same generators with the conformal anomaly set to zero, $\Delta(a) \mapsto 0$ (see (5.40)), and $S_\alpha^{(0)}$ are the canonical generators (5.6).

Let us now discuss the eigenvalues of each operator in (5.84). The key observation is that since all three operators are related by similarity transformations V_1 and V_2, *their spectra are identical.* What can change is the labeling of the eigenfunctions $\psi_s(z_1, z_2)$. We argue, however, that V_1 does not change the eigenfunctions, so that

$$\mathbf{H}\psi_s(z_1, z_2) = \gamma_{\rm ns}(s)\psi_s(z_1, z_2). \tag{5.86}$$

Indeed, since V_1 is constructed to non-trivially modify only the generator $S_+(a)$, it can be easily confirmed that

$$\big[\mathrm{V}_1, S_-^{(0)}\big] = 0, \qquad\qquad \big[\mathrm{V}_1, S_0^{(0)}\big] = 0, \tag{5.87}$$

which shows that $\psi_s(z_1, z_2)$ is also an eigenfunction of V_1, and therefore intertwining with it does not change the functional form of the eigenvalues. The explicit calculation of V_1 (via the exponentiation $\mathrm{V}_1 = e^{\mathrm{X}(a)}$) in Section 5.4.2 and in [91–93] confirms this.

The situation with the operator V_2 is more involved. Recalling its explicit expression (see (5.46)),

$$\mathrm{V}_2 = \sum_{k=0}^{\infty} \frac{1}{k!}\mathrm{L}^k \left(\bar\beta(a) + \frac{1}{2}\mathbf{H}(a)\right)^k, \qquad \mathrm{V}_2^{-1} = \sum_{k=0}^{\infty} \frac{1}{k!}(-\mathrm{L})^k \left(\bar\beta(a) + \frac{1}{2}\mathbf{H}_{\rm inv}(a)\right)^k, \tag{5.88}$$

where $L = \ln z_{12}$ (recall we assume $z_{12} > 0$), let us apply V_2 to the eigenfunction $\psi_s(z_1, z_2)$. The first observation is that we can immediately act with $\mathbf{H}(a)$ on the eigenfunction using (5.86):

$$\mathrm{V}_2\psi_s(z_1, z_2) = \sum_{k=0}^{\infty} \frac{1}{k!}\mathrm{L}^k \left(\bar\beta(a) + \frac{1}{2}\gamma_{\rm ns}(s)\right)^k \psi_s(z_1, z_2). \tag{5.89}$$

The second observation is that logarithms can be represented as derivatives of the eigenfunctions:

$$L^k\psi_s(z_1, z_2) = \partial_s^k\psi_s(z_1, z_2). \tag{5.90}$$

One can therefore identify the series as a Taylor expansion:

$$\mathrm{V}_2\psi_s(z_1, z_2) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\bar\beta(a) + \frac{1}{2}\gamma_{\rm ns}(s)\right)^k \partial_s^k\psi_s(z_1, z_2) = \psi_{s+\bar\beta(a)+\frac{1}{2}\gamma_{\rm ns}(s)}(z_1, z_2). \tag{5.91}$$

Let us now define $f_{\rm ns}(s)$ as the eigenvalue of the evolution kernel $\mathbf{H}_{\rm inv}$:

$$\mathbf{H}_{\rm inv}\psi_s(z_1, z_2) = f_{\rm ns}(s)\psi_s(z_1, z_2), \tag{5.92}$$

and apply the intertwining relation $\mathrm{V}_2\mathbf{H} = \mathbf{H}_{\rm inv}\mathrm{V}_2$ to the eigenfunction $\psi_s(z_1, z_2)$:

$$\begin{aligned}
\mathrm{V}_2\mathbf{H}\psi_s(z_1, z_2) &= \mathbf{H}_{\rm inv}\mathrm{V}_2\psi_s(z_1, z_2),\\
\gamma_{\rm ns}(s)\,\mathrm{V}_2\psi_s(z_1, z_2) &= \mathbf{H}_{\rm inv}\psi_{s+\bar\beta(a)+\frac{1}{2}\gamma_{\rm ns}(s)}(z_1, z_2),\\
\gamma_{\rm ns}(s)\psi_{s+\bar\beta(a)+\frac{1}{2}\gamma_{\rm ns}(s)}(z_1, z_2) &= f_{\rm ns}\left(s + \bar\beta(a) + \frac{1}{2}\gamma_{\rm ns}(s)\right)\psi_{s+\bar\beta(a)+\frac{1}{2}\gamma_{\rm ns}(s)}(z_1, z_2).
\end{aligned} \tag{5.93}$$

This establishes a relation of the form (4.10) and shows that $f_{\rm ns}(s)$ is indeed the RR part of the anomalous dimension $\gamma_{\rm ns}(s)$.

Let us discuss the interpretation of this result. We have shown that the function $f_{\rm ns}(s)$, now fully identified with the RR part of the anomalous dimension (see Section 4.1.2 for details), is the eigenvalue of the operator $\mathbf{H}_{\text{inv}}$. The remarkable feature of this operator is that it commutes with the canonical generators (5.6), which by Schur's lemma implies

$$\mathbf{H}_{\text{inv}} \sim C_2 = s(s+1). \tag{5.94}$$

It is tempting to conclude that this proves the generalized Gribov–Lipatov reciprocity. Indeed, Eq. (5.94) tells us that the eigenvalue of the invariant evolution kernel $\mathbf{H}_{\text{inv}}$ is a function of the Casimir, $f_{\text{ns}}(s) \equiv f_{\text{ns}}(s(s+1))$. The combination $s(s+1)$ is invariant under the parity transform $s \mapsto -s-1$, so the naive conclusion is that $f_{\text{ns}}(s)$ is invariant under this transformation. The subtlety is that Schur's lemma fixes the action of $\mathbf{H}_{\text{inv}}$ only within one irreducible representation, whereas the transform $s \mapsto -s-1$, despite leaving the Casimir unchanged, generally corresponds to a non-equivalent representation. At the technical level, this is reflected in the fact that $f_{\text{ns}}(s(s+1))$ can involve terms $\sim \sqrt{1+4s(s+1)}$, which introduce a branch cut. The parity transform, $s \mapsto -s-1$ translates the argument to another sheet. Examining even the one-loop result (which is RR itself, see Section 2.4 for the discussion) confirms that this is indeed the case and, e.g. terms like $S_1(s)$ are not directly parity invariant, despite being reciprocity-respecting in the sense (4.10).

5.5.2 Singlet case

As the main result of the previous section, we concluded that the RR part of the anomalous dimension can be identified with the eigenvalue of the operator $\mathbf{H}_{\text{inv}}$, which commutes with the canonical $SL(2,\mathbb{R})$ generators. Although this fact does not automatically prove Gribov–Lipatov reciprocity via Schur's lemma, the reciprocity property of $f_{\text{ns}}(s)$ still holds, as justified by explicit results and other considerations, see e.g. Section 4.1.3. The chain of similarity transforms (5.84) and the analysis of the corresponding eigenvalues therefore constitutes a promising framework to apply in the singlet case.

Let us begin by reviewing the evolution kernels in the singlet case, mostly following the notation of [463]. The singlet evolution kernel is defined as

$$\widehat{\mathbb{H}}(a) = \begin{pmatrix} \mathbb{H}_{qq}(a) & \mathbb{H}_{qg}(a) \\ \mathbb{H}_{gq}(a) & \mathbb{H}_{gg}(a) \end{pmatrix}, \tag{5.95}$$

where each entry $\mathbb{H}_{ij}$ $(i,j \in \{q,g\})$ is an integral operator of the form (1.98). The same considerations as in Section 2.4 (see [463] for the original discussion) apply, and the evolution kernel (5.95) is invariant under a set of generators (now represented as matrices):

$$\big[\widehat{S}_\alpha(a), \widehat{\mathbb{H}}(a)\big] = 0, \tag{5.96}$$

where

$$\begin{aligned}
(S_-)_{ij}(a) &= (S_-^{(0)})_{ij}, \\
(S_0)_{ij}(a) &= (S_0^{(0)})_{ij} + \bar\beta(a)\delta_{ij} + \frac{1}{2}\mathbb{H}_{ij}(a), \\
(S_+)_{ij}(a) &= (S_+^{(0)})_{ij} + (z_1+z_2)\left(\bar\beta(a)\delta_{ij} + \frac{1}{2}\mathbb{H}_{ij}(a)\right) + z_{12}\Delta_{ij}(a).
\end{aligned} \tag{5.97}$$

The integral operator $\Delta_{ij}(a)$ is the singlet analogue of the conformal anomaly (2.104) and is currently known at one-loop order [54, 463]:

$$\widehat{\Delta}^{(1)} \begin{pmatrix} f(z_1,z_2) \\ g(z_1,z_2) \end{pmatrix} = -2\int_0^1 d\alpha \begin{pmatrix} C_F\big(\frac{\bar\alpha}{\alpha} + \ln\alpha\big) & 0 \\ z_{12}^{-1} C_F & C_A\big(\frac{\bar\alpha}{\alpha} + 2\ln\alpha\big) \end{pmatrix} \begin{pmatrix} f(z_{12}^\alpha, z_2) - f(z_1, z_{21}^\alpha) \\ g(z_{12}^\alpha, z_2) - g(z_1, z_{21}^\alpha) \end{pmatrix}. \tag{5.98}$$

The canonical generators $S_{ij}^{(0)}$ are given by diagonal matrices of differential operators:

$$\widehat{S}_-^{(0)} = \begin{pmatrix} -\partial_{z_1} - \partial_{z_2} & 0 \\ 0 & -\partial_{z_1} - \partial_{z_2} \end{pmatrix}, \qquad \widehat{S}_0^{(0)} = \begin{pmatrix} z_1\partial_{z_1} + z_2\partial_{z_2} + 2 & 0 \\ 0 & z_1\partial_{z_1} + z_2\partial_{z_2} + 3 \end{pmatrix},$$

$$\widehat{S}_+^{(0)} = \begin{pmatrix} z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + 2(z_1+z_2) & 0 \\ 0 & z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + 3(z_1+z_2) \end{pmatrix}. \tag{5.99}$$

The different numerical coefficients (2 and 3) on the diagonal correspond to the different conformal spins of the quark and gluon fields, $j_q = 1$ and $j_g = 3/2$.

Since the evolution kernel now acts in two spaces — as an integral operator on functions of two variables z_1 and z_2, and in the two-dimensional flavor space — it is important to distinguish operator eigenvalues from matrix eigenvalues. The plan of this section is to repeat the derivation of Section 5.5.1, diagonalizing the operator part while leaving the flavor space untouched. To this end, we introduce the two vectors of eigenfunctions:

$$e_s^q = \begin{pmatrix} z_{12}^{s-1} \\ 0 \end{pmatrix}, \qquad\qquad e_s^g = \begin{pmatrix} 0 \\ z_{12}^{s-2} \end{pmatrix}. \tag{5.100}$$

Note that these vectors are not eigenvectors of the matrix (5.95) in flavor space. The analogue of (1.99) then reads

$$\widehat{\mathbb{H}}\, e_s^q = \gamma_{qq}(s) e_s^q + \big(s-1\big)\gamma_{gq}(s) e_s^g, \qquad\qquad \widehat{\mathbb{H}}\, e_s^g = \frac{1}{s-1}\gamma_{qg}(s) e_s^q + \gamma_{gg}(s) e_s^g, \tag{5.101}$$

so that one recovers the entries of the singlet anomalous dimension matrix $\widehat{\gamma}(s)$, see (1.63). As can be seen, the correspondence is not exact and a special normalization is introduced for the off-diagonal entries. It is therefore convenient to define the matrix $\widehat{\mathrm{h}}(s)$ formally as

$$\widehat{\mathrm{h}}(s) \equiv \begin{pmatrix} \mathrm{h}_{qq}(s) & \mathrm{h}_{qg}(s) \\ \mathrm{h}_{gq}(s) & \mathrm{h}_{gg}(s) \end{pmatrix} = \begin{pmatrix} z_{12}^{-s+1} & 0 \\ 0 & z_{12}^{-s+2} \end{pmatrix} \begin{pmatrix} \mathbb{H}_{qq}(a) & \mathbb{H}_{qg}(a) \\ \mathbb{H}_{gq}(a) & \mathbb{H}_{gg}(a) \end{pmatrix} \begin{pmatrix} z_{12}^{s-1} & 0 \\ 0 & z_{12}^{s-2} \end{pmatrix}. \tag{5.102}$$

Care must be taken when using this definition in practical calculations, since the left-hand side is defined as a matrix in flavor space, while the right-hand side also implicitly represents a (constant) function in operator space.

Let us now define the chain of two similarity transformations analogous to (5.84):

$$\widehat{\mathbb{H}} \quad \xrightarrow{\widehat{\mathrm{V}}_1} \quad \widehat{\mathbf{H}} \quad \xrightarrow{\widehat{\mathrm{V}}_2} \quad \widehat{\mathbf{H}}_{\mathrm{inv}}. \tag{5.103}$$

Both intertwining operators $\widehat{\mathrm{V}}_1$ and $\widehat{\mathrm{V}}_2$ are also matrices in flavor space. The operator $\widehat{\mathrm{V}}_1$ transforms the generators (5.97) into generators without conformal anomaly ($\Delta_{ij}(a) = 0$):

$$\begin{aligned}
(\mathbf{S}_-)_{ij}(a) &= (S_-^{(0)})_{ij}, \\
(\mathbf{S}_0)_{ij}(a) &= (S_0^{(0)})_{ij} + \bar{\beta}(a)\delta_{ij} + \frac{1}{2}\mathbf{H}_{ij}(a), \\
(\mathbf{S}_+)_{ij}(a) &= (S_+^{(0)})_{ij} + (z_1+z_2)\left(\bar{\beta}(a)\delta_{ij} + \frac{1}{2}\mathbf{H}_{ij}(a)\right),
\end{aligned} \tag{5.104}$$

where $\widehat{\mathbf{H}}(a) = \widehat{\mathrm{V}}_1 \widehat{\mathbb{H}}(a)\, \widehat{\mathrm{V}}_1^{-1}$. Recall that in the non-singlet case, the first similarity transformation does not change the eigenvalues of the operator. However, the introduction of the additional flavor space spoils this property. Indeed, writing the operator $\widehat{\mathrm{V}}_1$ in exponential form,

$$\widehat{\mathrm{V}}_1(a) = e^{\widehat{\mathrm{X}}(a)}, \tag{5.105}$$

where at one-loop order the operator $\widehat{\mathrm{X}}^{(1)}$ is determined from (5.98) (see [463, Eq. (3.14)]):

$$\begin{aligned}
\mathrm{X}_{qq}^{(1)} f(z_1,z_2) =& 2C_F \int_0^1 d\alpha\, \frac{\ln\alpha}{\alpha}\Big(2f(z_1,z_2) - f(z_{12}^\alpha, z_2) - f(z_1, z_{21}^\alpha)\Big), \\
\mathrm{X}_{qg}^{(1)} f(z_1,z_2) =& 0,
\end{aligned}$$

$$
\begin{aligned}
\mathrm{X}^{(1)}_{gq} f(z_1,z_2) =& -2C_F \frac{1}{z_{12}} \int_0^1 d\alpha \left(f(z_{12}^\alpha, z_2) + f(z_1, z_{21}^\alpha) \right), \\
\mathrm{X}^{(1)}_{gg} f(z_1,z_2) =& 2C_A \int_0^1 d\alpha \, \frac{\ln\alpha}{\alpha} \Big(2f(z_1,z_2) - f(z_{12}^\alpha, z_2) - f(z_1, z_{21}^\alpha) \Big).
\end{aligned} \tag{5.106}
$$

The property (5.101) allows us to define the matrix of operator eigenvalues: [b]

$$
\widehat{\mathrm{x}}(s) = \begin{pmatrix} \mathrm{x}_{qq}(s) & \mathrm{x}_{qg}(s) \\ \mathrm{x}_{gq}(s) & \mathrm{x}_{gg}(s) \end{pmatrix} = \begin{pmatrix} -4C_F S_{1,1}(s-1) & 0 \\ -4C_F/s & -4C_A S_{1,1}(s-2) \end{pmatrix}, \tag{5.107}
$$

and express the matrix of operator eigenvalues corresponding to $\widehat{\mathbf{H}}(a)$ as

$$
\widehat{\mathbf{h}}(s) = \begin{pmatrix} \mathbf{h}_{qq}(s) & \mathbf{h}_{qg}(s) \\ \mathbf{h}_{gq}(s) & \mathbf{h}_{gg}(s) \end{pmatrix} = e^{\widehat{\mathrm{x}}(s)} \widehat{\mathrm{h}}(s) e^{-\widehat{\mathrm{x}}(s)}. \tag{5.108}
$$

At one-loop order this transformation does not affect the matrix of operator eigenvalues,

$$
\widehat{\mathbf{h}}^{(1)}(s) = \widehat{\mathrm{h}}^{(1)}(s), \tag{5.109}
$$

but it starts to modify it from two loops onwards:

$$
\begin{aligned}
\widehat{\mathbf{h}}^{(2)} &= \widehat{\mathrm{h}}^{(2)} + \left[\widehat{\mathrm{x}}^{(1)}, \widehat{\mathrm{h}}^{(1)} \right], \\
\widehat{\mathbf{h}}^{(3)} &= \widehat{\mathrm{h}}^{(3)} + \left[\widehat{\mathrm{x}}^{(1)}, \widehat{\mathrm{h}}^{(2)} \right] + \left[\widehat{\mathrm{x}}^{(2)}, \widehat{\mathrm{h}}^{(1)} \right] + \frac{1}{2} \left[\left[\widehat{\mathrm{h}}^{(1)}, \widehat{\mathrm{x}}^{(1)} \right], \widehat{\mathrm{x}}^{(1)} \right].
\end{aligned} \tag{5.110}
$$

We thus identify the first difference between the singlet and non-singlet cases: the matrix that can possess the reciprocity property is $\widehat{\mathbf{h}}(s)$ rather than the original $\widehat{\gamma}(s)$ (or the normalized version (5.102)).

The second difference arises when we consider the second intertwining operator $\widehat{\mathrm{V}}_2$, which in the singlet case is generalized to

$$
\widehat{\mathrm{V}}_2 = \sum_{k=0}^{\infty} \frac{1}{k!} L^k \left(\bar{\beta}(a) \mathbb{1} + \frac{1}{2} \widehat{\mathbf{H}}(a) \right)^k, \tag{5.111}
$$

where $\mathbb{1}$ is the identity matrix. Applying this operator to the vectors of operator eigenfunctions (5.100), we cannot simply replace $\widehat{\mathbf{H}}(a)$ by a number, since it acts as

$$
\widehat{\mathbf{H}}\, e_s^q = \mathbf{h}_{qq}(s) e_s^q + \mathbf{h}_{gq}(s) e_s^g, \qquad \widehat{\mathbf{H}}\, e_s^g = \mathbf{h}_{qg}(s) e_s^q + \mathbf{h}_{gg}(s) e_s^g. \tag{5.112}
$$

It is more convenient to use the relation between $\widehat{\mathbf{H}}(a)$ and $\widehat{\mathbf{H}}_{\mathrm{inv}}(a)$ in the form of the series (see [101, Eq. (13) and Appendix A] for the discussion):

$$
\widehat{\mathbf{H}}(a) = \widehat{\mathbf{H}}_{\mathrm{inv}}(a) + \sum_{k=1}^{\infty} \frac{1}{k!} \widehat{\mathrm{T}}_k(a) \left(\bar{\beta}(a) \mathbb{1} + \frac{1}{2} \widehat{\mathbf{H}}(a) \right)^k, \tag{5.113}
$$

where $\widehat{\mathrm{T}}_k(a) = \left[\widehat{\mathrm{T}}_{k-1}(a), L \right]$ and $\widehat{\mathrm{T}}_0(a) = \widehat{\mathbf{H}}_{\mathrm{inv}}(a)$. It is straightforward to show that $\widehat{\mathrm{T}}_k$ acts on the operator eigenfunctions (5.100) as

$$
\widehat{\mathrm{T}}_k\, e_s^q = \partial_s^k f_{qq}(s) e_s^q + \partial_s^k f_{gq}(s) e_s^g, \qquad \widehat{\mathrm{T}}_k\, e_s^g = \partial_s^k f_{qg}(s) e_s^q + \partial_s^k f_{gg}(s) e_s^g, \tag{5.114}
$$

where $\widehat{f}(s)$ denotes the matrix of operator eigenvalues of the invariant kernel $\widehat{\mathbf{H}}_{\mathrm{inv}}(a)$. This allows us to translate (5.113) into the matrix of operator eigenvalues:

[b] Recall that this operator commutes with the canonical generators $S_-^{(0)}$ and $S_0^{(0)}$, see (5.42), and therefore diagonalizes in the same basis of eigenfunctions.

Generalization of the singlet Gribov–Lipatov reciprocity relation

The reciprocity property stated for the eigenvalues of the singlet anomalous dimension matrix $\widehat{\gamma}(s)$ in Section 4.1.4 can be enhanced by means of the series

$$\widehat{\mathbf{h}}(s) = \widehat{f}(s) + \sum_{k=1}^{\infty} \frac{1}{k!} \partial_s^k \widehat{f}(s) \left(\bar{\beta}(a) \mathbb{1} + \frac{1}{2} \widehat{\mathbf{h}}(s) \right)^k , \tag{5.115}$$

where $\widehat{\mathbf{h}}(s)$ is the matrix of operator eigenvalues of $\widehat{\mathbf{H}}(a)$. We *expect* all entries of the matrix $\widehat{f}(s)$ to be reciprocity-respecting, see (4.11).

We note again that the derivation of the series (5.115) cannot be regarded as a formal proof of a stronger version of the reciprocity relation, since the fact that $\widehat{f}(s)$ is the matrix of operator eigenvalues of an evolution kernel commuting with the canonical generators (5.99) does not by itself imply this. However, it appears to be a well-motivated assumption, confirmed by current perturbative calculations. Indeed, using (5.115) and explicitly solving the connection (5.110), one derives

$$\begin{aligned}
\mathrm{h}_{qq}^{(2)} =& f_{qq}^{(2)} + \mathrm{h}_{qg}^{(1)} \mathrm{x}_{gq}^{(1)} - \mathrm{h}_{gq}^{(1)} \mathrm{x}_{qg}^{(1)} + \partial_s \mathrm{h}_{qq}^{(1)} \left(\beta_0 + \frac{1}{2} \mathrm{h}_{qq}^{(1)} \right) + \frac{1}{2} \mathrm{h}_{gq}^{(1)} \partial_s \mathrm{h}_{qg}^{(1)} , \\
\mathrm{h}_{qg}^{(2)} =& f_{qg}^{(2)} + \mathrm{h}_{qg}^{(1)} \left(\frac{1}{2} \partial_s \mathrm{h}_{qq}^{(1)} + \mathrm{x}_{gg}^{(1)} - \mathrm{x}_{qq}^{(1)} \right) + \partial_s \mathrm{h}_{qg}^{(1)} \left(\beta_0 + \frac{1}{2} \mathrm{h}_{gg}^{(1)} \right) + \mathrm{x}_{qg}^{(1)} \left(\mathrm{h}_{qq}^{(1)} - \mathrm{h}_{gg}^{(1)} \right) , \\
\mathrm{h}_{gq}^{(2)} =& f_{gq}^{(2)} + \mathrm{h}_{gq}^{(1)} \left(\frac{1}{2} \partial_s \mathrm{h}_{gg}^{(1)} + \mathrm{x}_{qq}^{(1)} - \mathrm{x}_{gg}^{(1)} \right) + \partial_s \mathrm{h}_{gq}^{(1)} \left(\beta_0 + \frac{1}{2} \mathrm{h}_{qq}^{(1)} \right) + \mathrm{x}_{gq}^{(1)} \left(\mathrm{h}_{gg}^{(1)} - \mathrm{h}_{qq}^{(1)} \right) , \\
\mathrm{h}_{gg}^{(2)} =& f_{gg}^{(2)} + \mathrm{h}_{gq}^{(1)} \mathrm{x}_{qg}^{(1)} - \mathrm{h}_{qg}^{(1)} \mathrm{x}_{gq}^{(1)} + \partial_s \mathrm{h}_{gg}^{(1)} \left(\beta_0 + \frac{1}{2} \mathrm{h}_{gg}^{(1)} \right) + \frac{1}{2} \mathrm{h}_{qg}^{(1)} \partial_s \mathrm{h}_{gq}^{(1)} .
\end{aligned} \tag{5.116}$$

All four entries of the matrix $\widehat{f}(s)$ are constructed entirely from RR harmonic sums (see (4.13)), namely S_1 and S_{-2}, and rational functions of s that are invariant under the transform $s \mapsto -s-1$. We do not reproduce these results here, for the explicit formulas the reader may consult [463, Section 5]. It would be very interesting to test this property at three loops, however, this requires knowledge of the singlet conformal anomaly $\Delta_{ij}(a)$ at two-loop order, which is currently not available.

Summary

It would not be a mistake — albeit with some simplification — to say that theoretical progress on complex problems always proceeds in two directions: while some effort goes into advancing and mastering current methods, other effort is devoted to situating the problem in a broader context, in the hope that it will thereby dissolve naturally. A. Grothendieck, in his memoirs [469], used the metaphor of a *rising sea* to describe this second approach: instead of cracking a nut with a hammer, one can submerge it in water and let the shell soften until the nut can be opened by hand.

This thesis can be seen as the author's attempt to raise the sea level around high-order perturbative calculations of twist-2 anomalous dimensions, both in the forward (Chapter 4) and non-forward (Chapter 5) cases. Continuing the metaphor, the water in this attempt is drawn from developments in the field of CFT, which has long cultivated its own language and methods, making it a separate and highly successful branch of mathematical physics. The bridge connecting these communicating vessels is the idea of considering the theory at the critical coupling (g^* such that $\beta(g^*) = 0$), enhancing the symmetry group to the conformal group. The connection to the physical coupling is then established by the specific properties of the $\overline{\text{MS}}$ scheme, in which the coefficients in the perturbative expansion of anomalous dimensions are d-independent. This approach has already been implemented in the literature in various contexts (see e.g. [27, 56–61, 74]), and we believe its potential applications are far from exhausted.

The first application of these ideas discussed in the thesis concerns the constraints that critical conformal invariance places on the analytic structure of forward anomalous dimensions as functions of spin. At the critical point, the analytic continuation in spin ($s \in \mathbb{Z}_{>0}$ to $s \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$) can be performed not only at the level of functions, but also at the level of the CFT spectrum [80]. Studying the properties of this analytically continued spectrum [81] provides non-trivial information about the singularities of analytically continued anomalous dimensions. From a practical point of view, this information is of particular value as it enables the reconstruction of the functional form of anomalous dimensions from data on a limited number of low-spin operators (see Section 3.3 for a more detailed discussion). It also allows one to resum these singular contributions, yielding the values of the corresponding Regge intercepts (see [46, 47] for the QCD case and [332] for the general CFT context). It should be noted, however, that the type of singularity considered in this thesis corresponds, in the QCD case (via the double-logarithmic equation [83, 84]), to the resummation of contributions subleading with respect to the Pomeron intercept (corresponding to $s \to 1$ in $\gamma_{gg}(s)$) [48, 331].

In the second application we did not establish a new conceptual connection, but rather applied a well-established method [57, 89, 91–93, 463] to the study of the off-forward evolution equation. This approach (complementary to the use of classical conformal symmetry breaking [52–54]) allowed us to calculate the three-loop evolution kernel of the transversity non-local operator, whose nucleon matrix elements define the chiral-odd GPDs. We also discussed how the modern understanding of the $SL(2, \mathbb{R})$ symmetry of evolution kernels (see [27, 101]) provides a convenient perspective on the generalized Gribov–Lipatov reciprocity [76, 77].

From a broader perspective, this work can be seen as a natural development within high-energy physics, where approaches and methods originally developed in toy-model settings gradually and steadily find application in physical theories (see e.g. the use of conformal bootstrap ideas [166, 168] in the study of energy-energy correlators in $\mathcal{N} = 4$ SYM [470]). This again fits naturally into the metaphor of

communicating vessels, as the "water level" in different branches of theoretical physics rises unevenly. The abovementioned field of energy-energy correlators [70] serves as a good example where those interested in formal abstract theories and those focused on phenomenological applications have been working in concert in recent years [73, 471]. As regards the topics covered in this thesis, there remain many open problems and conceptual gaps (see e.g. the discussion in Section 4.4), and it is tempting to continue research in this direction.

Appendix A

Individual diagrams in the cubic model

In this appendix we list the results for the three-loop six-dimensional diagrams relevant for the renormalization of the leading-twist operators. We give the results for the simple φ^3 model (see Chapter 3 for the discussion and Figure 3.2 for the Feynman rules) without additional global symmetry, so the results given in Sections 4.3.4 and 4.3.5 can be obtained by restoring the corresponding symmetry factors. We give the result for the KR$'$ operation acting on the diagram, i.e. the divergent part after subtracting all subdivergences.

In the Figure A.1 we list all diagrams with two self-energy insertions:

$$\begin{aligned}
A_1(j) =& \frac{1}{\epsilon^3}\left[\frac{1}{108j(j+1)}\right] \\
&+ \frac{1}{\epsilon^2}\left[\frac{1}{108}\left(\frac{S_1(j)}{j(j+1)} + \frac{j}{j^2(j+1)^2} - \frac{8}{3j(j+1)}\right)\right] \\
&+ \frac{1}{\epsilon}\left[\frac{1}{216}\left(\frac{S_2(j)}{j(j+1)} + \frac{S_1^2(j)}{j(j+1)} - S_1(j)\left(\frac{16}{3j(j+1)} - \frac{2j}{j^2(j+1)^3}\right)\right.\right. \\
&\left.\left. - \frac{2j}{j^3(j+1)^3} + \frac{6-16j}{3j^2(j+1)^2} + \frac{8}{3j(j+1)}\right)\right],
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
A_2(j) =& \frac{1}{\epsilon^3}\left[\frac{1}{108j(j+1)}\right] \\
&+ \frac{1}{\epsilon^2}\left[\frac{1}{108}\left(\frac{1+2j}{j^2(j+1)^2} - \frac{8}{3j(j+1)}\right)\right] \\
&+ \frac{1}{\epsilon}\left[\frac{1}{324}\left(\frac{3}{j^3(j+1)^3} + \frac{1-16j}{j^2(j+1)^2} + \frac{4}{j(j+1)}\right)\right],
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
B_1(j) =& \frac{1}{\epsilon^3}\left[\frac{1}{216j(j+1)}\right] \\
&+ \frac{1}{\epsilon^2}\left[\frac{1}{216}\left(\frac{S_1(j)}{j(j+1)} + \frac{j}{j^2(j+1)^2} - \frac{19}{3j(j+1)}\right)\right] \\
&+ \frac{1}{\epsilon}\left[\frac{1}{432}\left(\frac{S_2(j)}{j(j+1)} + \frac{S_1^2(j)}{j(j+1)} - S_1(j)\left(\frac{38}{3j(j+1)} + \frac{2j}{j^2(j+1)^2}\right)\right.\right.
\end{aligned}$$

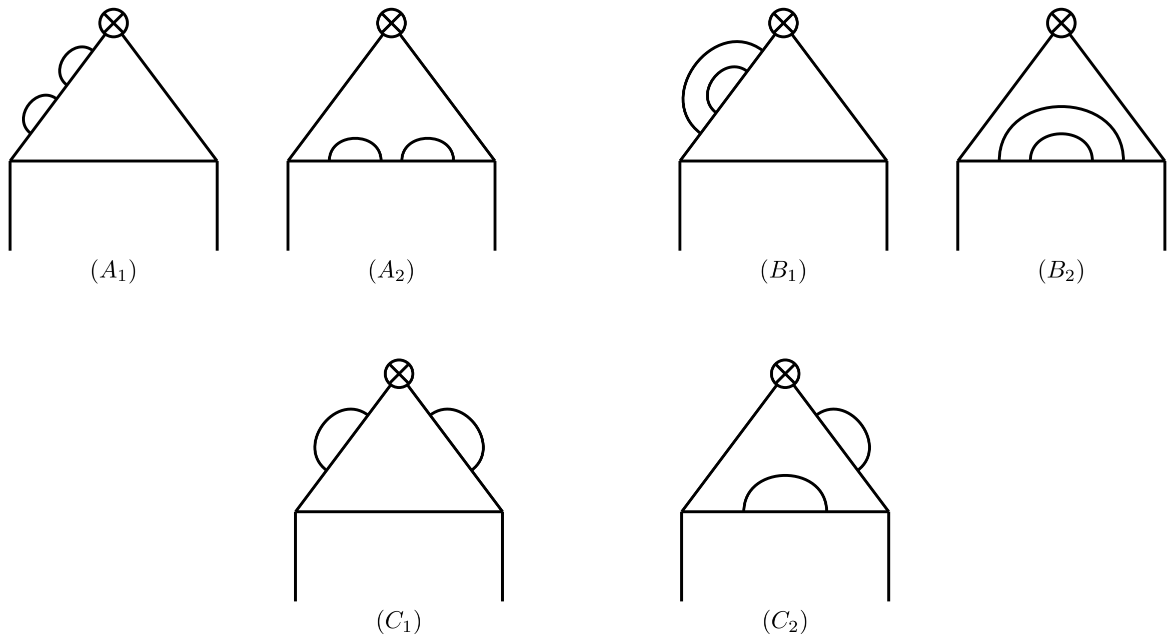


Figure A.1: One-loop diagrams with two self-energy insertions.

$$
- \frac{2j}{j^3(j+1)^3} + \frac{6-38j}{3j^2(j+1)^2} + \frac{583}{18j(j+1)}\Bigg)\Bigg], \tag{A.3}
$$

$$
\begin{aligned}
B_2(j) =& \frac{1}{\epsilon^3}\left[\frac{1}{216j(j+1)}\right] \\
&+ \frac{1}{\epsilon^2}\left[\frac{1}{216}\left(\frac{1+2j}{j^2(j+1)^2} - \frac{19}{3j(j+1)}\right)\right] \\
&+ \frac{1}{\epsilon}\left[\frac{1}{108}\left(\frac{1}{2j^3(j+1)^3} - \frac{5+19j}{3j^2(j+1)^2} + \frac{583}{72j(j+1)}\right)\right],
\end{aligned} \tag{A.4}
$$

$$
\begin{aligned}
C_1(j) =& \frac{1}{\epsilon^3}\left[\frac{1}{108j(j+1)}\right] \\
&+ \frac{1}{\epsilon^2}\left[\frac{1}{108}\left(\frac{S_1(j)}{j(j+1)} + \frac{j}{j^2(j+1)^2} - \frac{8}{3j(j+1)}\right)\right] \\
&+ \frac{1}{\epsilon}\Bigg[\frac{1}{216}\Bigg(\frac{S_2(j)}{j(j+1)} + \frac{S_1^2(j)}{j(j+1)} - \frac{2}{3}S_1(j)\left(-\frac{3j}{j^2(j+1)^2} + \frac{8}{j(j+1)}\right) \\
&+ \frac{1}{3}\left(-\frac{6j}{j^3(j+1)^3} + \frac{6-16j}{j^2(j+1)^2} + \frac{8}{j(j+1)}\right)\Bigg)\Bigg],
\end{aligned} \tag{A.5}
$$

$$
\begin{aligned}
C_2(j) =& \frac{1}{\epsilon^3}\left[\frac{1}{108j(j+1)}\right] \\
&+ \frac{1}{\epsilon^2}\left[\frac{1}{216}\left(\frac{S_1(j)}{j(j+1)} + \frac{1+3j}{j^2(j+1)^2} - \frac{16}{3j(j+1)}\right)\right]
\end{aligned}
$$

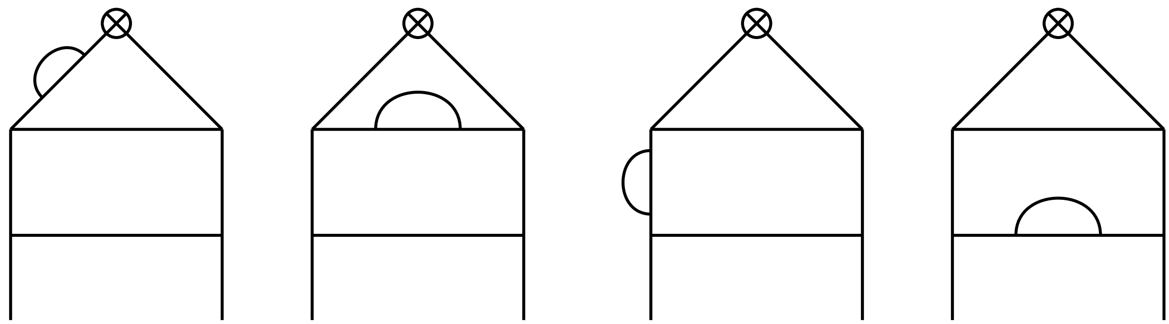


Figure A.2: Two-loop ladder diagrams with one self-energy insertion.

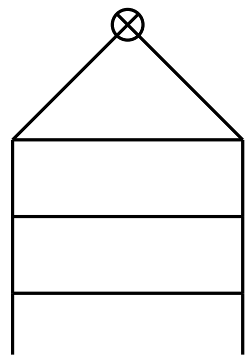
$$+\frac{1}{\epsilon}\left[\frac{1}{144}\left(\frac{S_2(j)}{j(j+1)}-\frac{S_1^2(j)}{3j(j+1)}-\frac{2}{9}S_1(j)\left(-\frac{6+9j}{j^2(j+1)^2}+\frac{8}{j(j+1)}\right)\right.\right.$$
$$\left.\left.+\frac{2}{9}\left(-\frac{3+9j}{j^3(j+1)^3}-\frac{2+24j}{j^2(j+1)^2}+\frac{8}{j(j+1)}\right)\right)\right]. \tag{A.6}$$

In the Figure A.2 we list all two-loop ladder diagrams with the one self-energy insertion:

$$\begin{aligned}
D_1(j) =& \frac{1}{\epsilon^3}\left[-\frac{1}{36j^2(j+1)^2}\right] \\
&+\frac{1}{\epsilon^2}\left[-\frac{1}{18}\left(\frac{S_1(j)}{j^2(j+1)^2}-\frac{1}{2j^3(j+1)^3}-\frac{13}{6j^2(j+1)^2}\right)\right] \\
&+\frac{1}{\epsilon}\left[\frac{1}{24}\left(\frac{S_2(j)}{j^2(j+1)^2}-\frac{S_1^2(j)}{3j^2(j+1)^2}+\frac{2}{9}S_1(j)\left(\frac{6+9j}{j^3(j+1)^3}+\frac{2}{j^2(j+1)^2}\right)\right.\right. \\
&\left.\left.-\frac{2}{27}\left(\frac{9+27j}{j^4(j+1)^4}+\frac{21+72j}{j^3(j+1)^3}+\frac{22}{j^2(j+1)^2}\right)\right)\right],
\end{aligned} \tag{A.7}$$

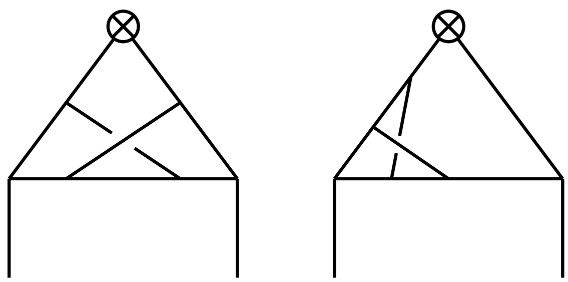
$$\begin{aligned}
D_2(j) =& \frac{1}{\epsilon^3}\left[-\frac{1}{36j^2(j+1)^2}\right] \\
&+\frac{1}{\epsilon^2}\left[\frac{1}{36}\left(-\frac{1+2j}{j^3(j+1)^3}+\frac{13}{3j^2(j+1)^2}\right)\right] \\
&+\frac{1}{\epsilon}\left[\frac{1}{324}\left(\frac{18}{j^4(j+1)^4}+\frac{21-66j}{j^3(j+1)^3}-\frac{4}{j^2(j+1)^2}\right)\right],
\end{aligned} \tag{A.8}$$

$$\begin{aligned}
D_3(j) =& \frac{1}{\epsilon^3}\left[-\frac{1}{18j^2(j+1)^2}\right] \\
&+\frac{1}{\epsilon^2}\left[-\frac{1}{36}\left(\frac{S_1(j)}{j^2(j+1)^2}-\frac{1+j}{j^3(j+1)^3}-\frac{5}{3j^2(j+1)^2}\right)\right] \\
&+\frac{1}{\epsilon}\left[\frac{1}{72}\left(\frac{S_2(j)}{j^2(j+1)^2}+\frac{S_1^2(j)}{j^2(j+1)^2}-\frac{2}{3}S_1(j)\left(-\frac{6+15j}{j^3(j+1)^3}+\frac{14}{j^2(j+1)^2}\right)\right.\right.
\end{aligned}$$



(L_3)

Figure A.3: Three-loop ladder diagram.



(CC) (CV)

Figure A.4: Non-planar three-loop diagrams.

$$+\frac{1}{9}\left(\frac{18-54j}{j^4(j+1)^4}+\frac{30-312j}{j^3(j+1)^3}+\frac{200}{j^2(j+1)^2}\right)\Bigg)\Bigg], \tag{A.9}$$

$$\begin{aligned}D_4(j) =&\frac{1}{\epsilon^3}\left[-\frac{1}{18j^2(j+1)^2}\right]\\&+\frac{1}{\epsilon^2}\left[\frac{5}{108j^2(j+1)^2}\right]\\&+\frac{1}{\epsilon}\left[\frac{1}{324}\left(\frac{36}{j^4(j+1)^4}+\frac{27-198j}{j^3(j+1)^3}+\frac{100}{j^2(j+1)^2}\right)\right].\end{aligned}\tag{A.10}$$

In the Figure A.3 we present the three-loop ladder diagram:

$$\begin{aligned}L_3 =&\frac{1}{\epsilon^3}\left[\frac{1}{6j^3(j+1)^3}\right]\\&+\frac{1}{\epsilon^2}\left[-\frac{1}{2}\left(\frac{1+2j}{j^4(j+1)^4}-\frac{1}{j^3(j+1)^3}\right)\right]\\&+\frac{1}{\epsilon}\left[\frac{2}{3}\left(\frac{1}{j^5(j+1)^5}+\frac{2-3j}{j^4(j+1)^4}+\frac{1}{j^3(j+1)^3}\right)\right],\end{aligned}\tag{A.11}$$

In the Figure A.4 we present two non-trivial diagrams with non-planar topology arising at the three-loop order:

$$
\begin{aligned}
CV(j) =& -\frac{1}{\epsilon^2}\frac{1}{6j(j+1)} \\
&+\frac{1}{\epsilon}\left[-\frac{1}{6}\left(\frac{S_1(j)}{j(j+1)}+\frac{2\zeta_3}{j(j+1)}+\frac{1+2j}{j^2(j+1)^2}-\frac{59}{12j(j+1)}\right)\right],
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
CC(j) =& \frac{1}{\epsilon}\left[\frac{1}{3j(j+1)}\left(\frac{1-(-1)^j}{2}\left(\frac{4}{(j-1)(j+2)}\left(1+S_{-2}(j)\right)-\frac{4}{j(j+1)}\right)\right.\right. \\
&+\frac{1+(-1)^j}{2}\left(\frac{4}{j^2(j+1)^2}S_{-2}(j)+\frac{4}{j(j+1)}S_{-2}(j)\right) \\
&\left.\left.+\frac{6\zeta_3}{j(j+1)}-\frac{2}{j(j+1)}\left(S_1(j)+S_3(j)+2\left(S_{1,-2}(j)-\frac{1}{2}S_{-3}(j)\right)\right)\right)\right].
\end{aligned}
\tag{A.13}
$$

The details on the calculation of diagrams listed in Fig. A.4 can be found in Section 3.4. Note that here we list not all the diagrams contributing to the leading-twist operators in the simple cubic model, but the subset sufficient for the calcualations in the models described in Section 4.3.4 and Section 4.3.5.

Appendix B

Two-loop diagrams for the transversity non-local operator

B.1 Individual diagrams for the two-loop evolution kernel

The contributions to the evolution kernel from the diagrams in Fig. 5.1 (a)–(p) (including symmetric diagrams with the interchange of the quark and the antiquark) can be written in the following form:

$$
\begin{aligned}
[\mathbb{H}\, f](z_1 z_2) = {} & -4\int_0^1 d\alpha \int_0^{\bar\alpha} d\beta \Big[\chi(\alpha,\beta) + \chi^{\mathbb{P}}(\alpha,\beta)\mathbb{P}_{12}\Big]\Big[f(z_{12}^{\alpha}, z_{21}^{\beta}) + f(z_{12}^{\beta}, z_{21}^{\alpha})\Big] \\
& - 4\int_0^1 du\, h(u)\Big[2f(z_1,z_2) - f(z_{12}^{u}, z_2) - f(z_1, z_{21}^{u})\Big]\,,
\end{aligned}
\tag{B.1}
$$

where $\mathbb{P}_{12}$ is the permutation operator. For any function $f(z_1, z_2)$

$$
\mathbb{P}_{12} f(z_1,z_2) = f(z_2,z_1), \qquad\qquad \Big(\mathbb{P}_{12}\mathcal{O}(z_{12}^{\alpha}, z_{21}^{\beta}) = \mathcal{O}(z_{21}^{\alpha}, z_{12}^{\beta})\Big).
\tag{B.2}
$$

One obtains (only the non-vanishing contributions are listed):

$$
\begin{aligned}
h_{(a)}(u) &= C_F^2 \frac{\bar u}{u}\left[\ln u + 1\right], \\
h_{(b)}(u) &= C_F \frac{\bar u}{u}\Big[(2C_A - \beta_0)\ln\bar u + \frac{8}{3}C_A - \frac{5}{3}\beta_0\Big], \\
h_{(c)}(u) &= \Big[C_F^2 - \frac{1}{2}C_F C_A\Big]\frac{\bar u}{u}\Big[\ln^2\bar u - 3\frac{u}{\bar u}\ln u + 3\ln\bar u - \ln u - 1\Big], \\
h_{(d)}(u) &= \frac{1}{2}C_F C_A \frac{\bar u}{u}\left[\frac{1}{2}\Big(1 - \frac{u}{\bar u}\Big)\ln^2 u + \ln\bar u - 3\right], \\
h_{(e+f)}(u) &= 2\,C_F^2 \frac{\bar u}{u}\left[2\big(\mathrm{Li}_2(1) - \mathrm{Li}_2(\bar u)\big) - \ln^2\bar u + 2\frac{u}{\bar u}\ln u\right] \\
&\quad + C_F C_A \frac{\bar u}{u}\left[2\big(\mathrm{Li}_2(\bar u) - \mathrm{Li}_2(u)\big) + \frac{1}{2}\ln^2\bar u - \frac{1}{2}\ln^2 u - \frac{1+u}{\bar u}\ln u - 2\right], \\
h_{(g)}(u) &= -C_F C_A \frac{\bar u}{u}\left[\mathrm{Li}_2(\bar u) - \mathrm{Li}_2(1) + 1 + \frac{1}{4}\ln^2\bar u + \ln\bar u - \frac{1+u}{2\bar u}\ln u\left(\frac{1}{2}\ln u + 1\right)\right], \\
h_{(j)}(u) &= \Big[C_F^2 - \frac{1}{2}C_F C_A\Big]\,\ln u\,,
\end{aligned}
$$

$$h_{(o)}(u) = 2\Big[C_F^2 - \frac{1}{2}C_FC_A\Big]\frac{\bar u}{u}\Big[-2\,\mathrm{Li}_2(u) + \frac{u}{\bar u}\ln u\ln\bar u - \frac{1}{2}\ln^2\bar u - \frac{u}{\bar u}\ln u\Big],$$

$$h_{(p)}(u) = C_FC_A\frac{\bar u}{u}\Big[\mathrm{Li}_2(u) + \frac{1}{\bar u}\ln u\ln\bar u - \frac{1}{4}\ln^2\bar u - \frac{u}{4\bar u}\ln^2 u - \frac{u}{\bar u}\ln u\Big], \tag{B.3}$$

and

$$\chi_{(i)}(\alpha,\beta) = \frac{1}{6}C_F(C_A-\beta_0)\delta(\alpha)\delta(\beta),$$

$$\chi_{(j)}(\alpha,\beta) = \Big[C_F^2 - \frac{1}{2}C_FC_A\Big]\delta(\alpha)\delta(\beta),$$

$$\chi_{(m)}(\alpha,\beta) = \Big[C_F^2 - \frac{1}{2}C_FC_A\Big],$$

$$\chi_{(o)}(\alpha,\beta) = -2\Big[C_F^2 - \frac{1}{2}C_FC_A\Big]\left[\frac{1}{\bar\alpha}\ln\alpha - \frac{1}{\alpha}\ln\bar\alpha - \frac{\bar\tau}{\tau}\ln\bar\tau + \big[2+\zeta_2-3\zeta_3\big]\delta(\alpha)\delta(\beta)\right],$$

$$\chi_{(p)}(\alpha,\beta) = C_FC_A\left[\frac{1}{\alpha}\ln\bar\alpha - \frac{1}{\bar\alpha}\ln\alpha + \big[\zeta_2-2\big]\delta(\alpha)\delta(\beta)\right]. \tag{B.4}$$

The nonvanishing contributions to $\overline{\chi}(\alpha,\beta)$ originate from two diagrams only:

$$\overline{\chi}_{(m)}(\alpha,\beta) = \Big[C_F^2 - \frac{1}{2}C_FC_A\Big],$$

$$\overline{\chi}_{(o)}(\alpha,\beta) = -2\Big[C_F^2 - \frac{1}{2}C_FC_A\Big]\,\bar\tau\ln\bar\tau\,. \tag{B.5}$$

We note here that the results for the h functions are exactly the same as in the vector case [57].

B.2 Individual diagrams for the two-loop conformal anomaly

Terms due to the conformal variation of the action can be written in the form

$$\Delta S_+ = \frac{1}{2}\mathbb{H}(z_1+z_2) + z_{12}\Delta_+\,, \tag{B.6}$$

where $\mathbb{H}$ is the corresponding contribution to the evolution kernel. The contributions to Δ_+ from the diagrams in Fig. 5.1 (including symmetric diagrams with the interchange of the quark and the antiquark) can be brought to the following form: below we list the non-vanishing contributions only

$$\varkappa_{(a)}(t) = C_F^2\left[\frac{1}{t} + \frac{1+\bar t}{t}\ln t\right],$$

$$\varkappa_{(b)}(t) = -2C_F\frac{\bar t}{t}\Big[(\beta_0-2C_A)\ln\bar t - \frac{8}{3}C_A + \frac{5}{3}\beta_0\Big],$$

$$\varkappa_{(c)}(t) = \Big[C_F^2 - \frac{1}{2}C_FC_A\Big]\Big[t\ln^2 t + \frac{2\bar t}{t}\ln^2\bar t + \frac{6\bar t}{t}\ln\bar t - \frac{\bar t}{t}(3t+2)\ln t - 9t + 8 - \frac{1}{t}\Big],$$

$$\varkappa_{(d)}(t) = C_FC_A\Big\{\frac{\bar t}{t}\left[\frac{1-2t}{2\bar t}\ln^2 t + \ln\bar t - 3\right]$$
$$+\frac{1}{2}\Big[\frac{1}{2}\ln^2 t - \bar t\ln^2\bar t + \frac{t^2-\bar t}{t}\ln t - 2\bar t\ln\bar t - 1 - \bar t\Big]\Big\},$$

$$\varkappa_{(e+f)}(t) = -4C_F^2\Big\{t\Big[\mathrm{Li}_2(t) - \mathrm{Li}_2(1)\Big] + 2\frac{\bar t}{t}\Big[\mathrm{Li}_2(\bar t) - \mathrm{Li}_2(1)\Big] + \frac{\bar t}{t}\ln^2\bar t + \frac{1}{2}t\ln^2 t + 2\bar t\ln\bar t$$

$$
\begin{aligned}
&- \frac{3}{2}(1-2t)\ln t + 2\bigg\} + C_F C_A \frac{\bar{t}}{t}\bigg\{4\Big[\mathrm{Li}_2(\bar{t}) - \mathrm{Li}_2(t)\Big] + \frac{1}{2}(2+t)\ln^2\bar{t} \\
&- \left(1 - \frac{t^2}{2\bar{t}}\right)\ln^2 t - 2(1-2t)\ln\bar{t} - \left(5t + \frac{1}{\bar{t}}\right)\ln t - 3 + 2t\bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\varkappa_{(g)}(t) &= C_F C_A \frac{\bar{t}}{t}\bigg\{t\Big[\mathrm{Li}_2(\bar{t}) - \mathrm{Li}_2(1)\Big] + \frac{1}{4}t\ln^2\bar{t} + \frac{1}{4}(2+t)\ln^2 t - (3-t)\ln\bar{t} \\
&\quad + \frac{1}{2}\left(1 - \frac{t^2}{\bar{t}}\right)\ln t - \bar{t} - \frac{3}{2}\bigg\}, \\
\varkappa_{(j)}(t) &= \Big[C_F^2 - \frac{1}{2}C_F C_A\Big]\Big[-t\ln t - 1\Big], \\
\varkappa_{(o)}(t) &= \Big[C_F^2 - \frac{1}{2}C_F C_A\Big]\bigg\{\frac{4}{\bar{t}}\Big[\mathrm{Li}_2(t) - \mathrm{Li}_2(1)\Big] - 4t\Big[\mathrm{Li}_2(t) - \mathrm{Li}_2(1)\Big] + 4\bar{t}\,\mathrm{Li}_2(1) \\
&\quad - 2t\ln t\ln\bar{t} + \frac{t}{\bar{t}}\ln^2 t + \bar{t}\ln^2\bar{t} - 4t\ln\bar{t} + \frac{2t}{\bar{t}}(2-3t)\ln t + 2\bigg\}, \\
\varkappa_{(p)}(t) &= C_F C_A\bigg\{\frac{2t}{\bar{t}}\Big[\mathrm{Li}_2(t) - \mathrm{Li}_2(1)\Big] + \bar{t}\Big[\mathrm{Li}_2(\bar{t}) - \mathrm{Li}_2(1)\Big] - t\ln t\ln\bar{t} \\
&\quad + \frac{1}{4}\bar{t}\ln^2\bar{t} + \frac{1}{4}\frac{t(3-t)}{\bar{t}}\ln^2 t - \frac{t^2}{\bar{t}}\ln t + \frac{1}{2}\ln t - \frac{1+t}{t}\ln\bar{t} + 1\bigg\}. \qquad \text{(B.7)}
\end{aligned}
$$

Note here that all $\varkappa$ functions are exactly the same as in the vector case. The function $\varpi(\alpha,\beta)$ receives contributions from two diagrams only:

$$
\begin{aligned}
\varpi_{(m)}(\alpha,\beta) &= -2\left[C_F^2 - \frac{1}{2}C_F C_A\right]\beta, \\
\varpi_{(o)}(\alpha,\beta) &= 4\left[C_F^2 - \frac{1}{2}C_F C_A\right]\left\{\frac{\alpha}{\bar{\alpha}}\left[\mathrm{Li}_2\left(\frac{\alpha}{\bar{\beta}}\right) - \mathrm{Li}_2(\alpha)\right] - \alpha\bar{\tau}\ln\bar{\tau} - \frac{1}{\bar{\alpha}}\ln\bar{\alpha}\ln\bar{\beta} - \frac{\beta}{\bar{\beta}}\ln\bar{\alpha}\right\}. \qquad \text{(B.8)}
\end{aligned}
$$

The non-vanishing contributions to $\omega(\alpha,\beta)$ are

$$
\begin{aligned}
\omega_{(m)}(\alpha,\beta) &= 2\Big[C_F^2 - \frac{1}{2}C_F C_A\Big]\beta, \\
\omega_{(o)}(\alpha,\beta) &= -4\Big[C_F^2 - \frac{1}{2}C_F C_A\Big]\bigg\{\frac{\bar{\alpha}}{\alpha}\Big[\mathrm{Li}_2(\beta/\bar{\alpha}) - \mathrm{Li}_2(\alpha) - \mathrm{Li}_2(\beta)\Big] - \frac{\alpha}{\bar{\alpha}}\Big[\mathrm{Li}_2(\alpha) - \zeta_2\Big] \\
&\quad - \frac{1}{4}\frac{\bar{\alpha}}{\alpha}\ln^2\bar{\alpha} - \frac{1}{4}\frac{\alpha}{\bar{\alpha}}\ln^2\alpha + \ln\alpha\ln\bar{\alpha} + \alpha\frac{\bar{\tau}}{\tau}\ln\bar{\tau} + \frac{\alpha}{\bar{\alpha}}\ln\alpha - \frac{1}{2}\frac{\bar{\beta}}{\bar{\alpha}}\ln\alpha - \frac{1}{2}\frac{\beta}{\alpha}\ln\bar{\alpha}\bigg\}, \\
\omega_{(p)}(\alpha,\beta) &= C_F C_A\bigg\{\frac{2}{\bar{\alpha}}\Big[\mathrm{Li}_2(\alpha) - \mathrm{Li}_2(1)\Big] + \frac{2}{\alpha}\Big[\mathrm{Li}_2(\bar{\alpha}) - \mathrm{Li}_2(1)\Big] + \frac{1}{2}\frac{\bar{\alpha}}{\alpha}\ln^2\bar{\alpha} + \frac{1}{2}\frac{\alpha}{\bar{\alpha}}\ln^2\alpha \\
&\quad - \frac{2}{\bar{\alpha}}\ln\alpha - \frac{2}{\alpha}\ln\bar{\alpha} + \frac{\beta}{\alpha}\ln\bar{\alpha} + \frac{\bar{\beta}}{\bar{\alpha}}\ln\alpha\bigg\}. \qquad \text{(B.9)}
\end{aligned}
$$

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