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FAKULTÄT
FÜR MATHEMATIK, INFORMATIK
UND NATURWISSENSCHAFTEN

Schwarz Symmetrization and Prescribed Mean Curvature Problems

Dissertation presented for the degree of Doctor of Natural Sciences at the Faculty of Mathematics, Informatics and Natural Sciences

Submitted by

Jule Helena Schütt

University of Hamburg
Department of Mathematics

April 2026, Hamburg

Reviewers:

Prof. Dr. Thomas Schmidt	University of Hamburg
Prof. Dr. Vesa Julin	University of Jyväskylä
Prof. Dr. Filippo Cagnetti	University of Parma

Date of the defense: 15 June 2026

Acknowledgments

Above all, I would like to take this opportunity to express my gratitude to my advisor Thomas Schmidt. From day one, he has guided me through my studies, taught me so much, and supported me. Whatever he saw in me, I don't think I would have had the courage to pursue my childhood dream of becoming a scientist without him. Thank you so much!

I would also like to thank Eleonora Ficola for a small but no less wonderful work group where we were free to both laugh and groan. It always felt good to talk to someone in a similar situation.

I would like to thank my reviewers, Filippo Cagnetti and Vesa Julin, for agreeing to read my work without knowing me. I look forward to meeting you for the first time.

Thank you, Franz Gmeineder, for your tireless encouragement and genuine enthusiasm. In many situations, you've found just the right words and steered me in the right direction.

A bit further away professionally, but still in the same building: Many thanks to my lunch crew, Mai Trinh, Marianne Lawson, and Sven-Ake Wegner. Our interesting conversations were always a good opportunity to take a quick breather.

In particular, a big thank you goes to Marianne, who, with her tireless enthusiasm, provided many linguistic insights for this thesis. I truly value our frequent talks, whether about important politics or silly nonsense; they always cheer me up and make me feel less alone.

Many thanks also go to Sören Beckmann for brightening up every day simply by waving at me.

I would like to thank my parents, Iris and Paul Schütt, for their unwavering support of all my upcoming interests and talents. I would also like to thank my brother, Oskar Schütt, and his husband, Devin Feng, for their positive and open-minded attitude. I feel very fortunate that my family always supports me and my decisions.

A huge thank you goes to my partner, Tim Seifert, both for his emotional support and for his willingness to delve into technical details. It certainly helped that you're the person who understands me best.

Thanks to Dennis Breutigam for our Star Trek nights. Together with Lena and Floyd, I'd also like to thank the coolest pen-and-paper group. Thanks for all the support from Anna, Ali, Berit, Constantin, Finn, Johanna, Juliana, Leroy, Markus, Nadine, Paul K., Paul W., and Steven. Your friendships — some of which go all the way back to preschool — mean a great deal to me.

Last but not least, I'd like to thank my big band, the Fish Head Horns, for welcoming me into the group even though they were already at full capacity. This has created countless wonderful memories of concerts and rehearsal weekends, which have helped me relax and unwind while writing this thesis. I'd especially like to mention the saxophone section — Antje, Karin, Christian, Thomas, and Claudia — as well as Ole, Tanja, Markus, and Mino, and of course our fantastic bandleader Stefan Schröter.

Abstract

The Schwarz perimeter inequality states that rearranging the horizontal slices of a set into a body of revolution decreases its perimeter. In the first part of this thesis, a full characterization of the equality cases of the Schwarz perimeter inequality is given. Moreover, we prove that the Schwarz perimeter inequality applies for prescribed mean curvature functionals with measure data whenever the considered measure satisfies certain monotonicity and isoperimetric properties.
In the second part of this thesis, we consider Massari's well-known regularity theorem from 1975, which states that if a set $A \subseteq \mathbb{R}^n$ has a variational mean curvature in L^p for some $p > n$, then $\partial^* A$ is a $\mathrm{C}^{1,\alpha}$ submanifold with $\alpha = \frac{p-n}{4p}$. We show that every $\mathrm{C}^{1,\alpha}$ set admits a variational mean curvature in L^p for $p < \frac{1+\alpha}{1-\alpha}$, and prove the existence of counterexamples for the case $p > \frac{1+\alpha}{1-\alpha}$. We show that Massari's regularity theorem remains true for $\alpha = \frac{p-n}{p+1}$, which is the optimal relation. For this purpose, we transfer the parametric problem into a non-parametric one, which we treat via methods of non-parametric partial regularity theory.

Zusammenfassung

Die Schwarzungleichung besagt, dass der Perimeter einer Menge $A \subseteq \mathbb{R}^n$ größer ist als der Perimeter der entsprechenden Schwarzsymmetrisierung $A^\star$. Letztere entsteht durch Neuanordnung der horizontalen Schnitte von A zu einem Rotationskörper. Im ersten Teil dieser Arbeit wird eine vollständige Charakterisierung des Gleichheitsfalls der Schwarzungleichung hergeleitet. Zudem wird eine Verallgemeinerung der Schwarzungleichung für Funktionale mit vorgeschriebener mittlerer Krümmung bewiesen. Dabei sind die betrachteten Krümmungen Maße, welche bestimmte isoperimetrische Bedingungen und Monotonieeigenschaften erfüllen.
Im zweiten Teil betrachten wir den bekannten Regularitätssatz von Massari von 1975, der besagt, dass der reduzierte Rand $\partial^* A$ einer Menge $A \subseteq \mathbb{R}^n$ eine $\mathrm{C}^{1,\alpha}$-Untermannigfaltigkeit mit $\alpha = \frac{p-n}{4p}$ ist, wenn A eine variationelle mittlere Krümmung in L^p hat und $p > n$ gilt. Wir zeigen, dass auch jede $\mathrm{C}^{1,\alpha}$-Menge eine entsprechende Krümmung in L^p für alle $p < \frac{1+\alpha}{1-\alpha}$ besitzt und führen Gegenbeispiele für den Fall $p > \frac{1+\alpha}{1-\alpha}$ an. Zudem beweisen wir, dass Massaris Regularitätstheorem in einer verbesserten und optimalen Version mit $\alpha = \frac{p-n}{p+1}$ gilt. Dazu wird das parametrische Problem in ein nichtparametrisches überführt und Methoden der partiellen Regularitätstheorie angewandt.

Declaration of contributions

During the research phase of this thesis, four papers [79, 78, 82, 80] were written by the author in collaboration with Thomas Schmidt. The contributions to all of these papers were shared equally.
Chapter 3 processes the results from [79] in Section 3.2 and [78] in Section 3.3.
The theory of Chapter 5 is partially parallel to [82]. The main theorem of this part is stated for two-dimensional or convex sets and necessary adaptions for proving the non-convex higher dimensional case (which is treated in full detail in [82]) are pointed out.
Chapter 6 presents the results of the published paper [80].
The three papers [79, 78, 82] have not yet been submitted.
The given figures are all created by the author. Many of them are also included in the corresponding articles (partially in a slightly different form).

Contents

Chapter 1

Introduction

Curvatures are an essential tool to describe natural phenomena from tiny cell structure to massive black holes. As a notion in the field of differential geometry, it is defined on regular C^2 manifolds. The variational mean curvature is an L^1 function that aims to describe the curvature of the boundary of less regular sets. This generalization can further be extended to measures as curvatures. In this thesis, we introduce these concepts, answer open questions, and extend existing results.

We first consider the Schwarz symmetrization, a tool to find minimizers of geometric problems that involve the perimeter and which is closely connected to prescribed mean curvature problems. We characterize the equality cases of the Schwarz perimeter inequality and generalize this type of inequality to more general prescribed mean curvature functionals with measures as curvatures. In the second part of this thesis, we introduce variational mean curvatures for sets, show integrability results for variational mean curvatures of $\mathrm{C}^{1,\alpha}$ regular sets and prove the limit case of Massari's regularity theorem.

Schwarz symmetrization

The Schwarz symmetrization is a volume preserving symmetrization technique for sets which reduces the perimeter. These techniques are often used to find minimizers of perimeter-involving functionals. An early instance is the proof of the isoperimetric inequality by H. A. Schwarz in [83]. The Schwarz symmetrization of a measurable set $A \subseteq \mathbb{R}^n$ is defined as

$$A^\star := \left\{ x \in \mathbb{R}^n \,:\, |\overline{x}| < \left(\frac{1}{\omega_{n-1}} |A_{x_n}| \right)^{\frac{1}{n-1}} \right\},$$

where $\overline{x} := (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$.

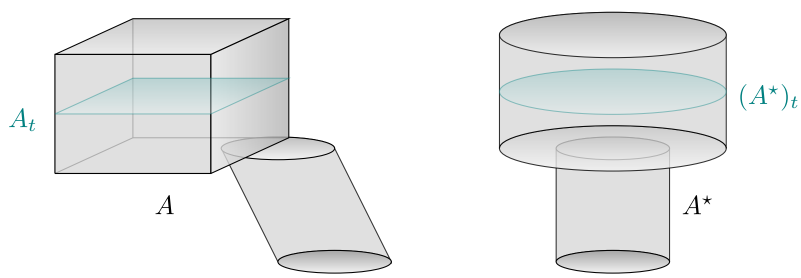


Figure 1.1: The Schwarz symmetrization

In particular, $A^\star$ is the body of revolution such that the horizontal t-slice $(A^\star)_t$ is the $(n-1)$-dimensional ball with volume equal to $|A_t|$ for all $t \in \mathbb{R}$. The Schwarz (perimeter) inequality then reads

$$\mathrm{P}(A^\star) \leqslant \mathrm{P}(A)\,, \tag{1.1}$$

where $\mathrm{P}(A) \in [0, \infty]$ denotes the perimeter of A.

Regarding the application to finding minimizing sets, it is important to know when a set A satisfies $\mathrm{P}(A) = \mathrm{P}(A^\star)$. The best possible outcome for A is to obtain rigidity, i.e., if only shifted copies of $A^\star$ satisfy equality in (1.1). This is not always satisfied. Indeed, there are many examples demonstrating the existence of compensation effects allowing for equality in the Schwarz inequality for sets which are not bodies of revolution, see Figure 1.2 for a basic instance in dimension 2.

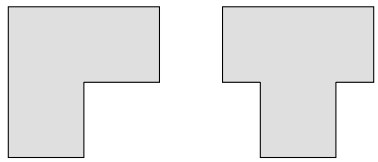


Figure 1.2: A set which does not satisfy rigidity with respect to the Schwarz symmetrization

A first rigidity result was shown for the Steiner symmetrization in [18, Theorem 1.3], which can be seen as the one-dimensional version of the Schwarz symmetrization. In particular, the Steiner symmetrization of A is the set $A^S := \left\{x \in \mathbb{R}^n : 2|x_1| < |A_{(x_2,\dots,x_n)}|\right\}$, and the Steiner (perimeter) inequality then reads $\mathrm{P}(A^S) \leqslant \mathrm{P}(A)$. Hence, instead of $(n-1)$-dimensional balls, one uses centered lines as rearranged slices. Results for higher dimensional symmetrizations, including the Schwarz symmetrization, were then obtained in [7], such as the fact that $\mathrm{P}(A) = \mathrm{P}(A^\star)$ implies that the slices A_t are $(n-1)$-dimensional balls for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$. Further rigidity results for the Steiner symmetrization have been established in [14] with the introduction of essential connectedness, and even an equality characterization of the Steiner inequality was achieved. The proof crucially exploits the advantages of rearranging lines. Refinements of the characterization result for the higher dimensional symmetrization have not yet been addressed. In fact, a difficulty of the higher dimensional case is the handling of the pathological horizontal parts, which simplifies for the one-dimensional Steiner symmetrization due to the fact that the projection of these parts onto the $(n-1)$-dimensional hyperspaces has vanishing Lebesgue measure. This is shown to be false for higher dimensional symmetrizations, see [7, Remark 3.2]. For more insight into symmetrizations, we refer to the survey [13] giving the state of the art from 2019.

As the first main result of this thesis, we give a full characterization of the equality cases in the Schwarz perimeter inequality, see Theorem 3.57. During the research, the paper [26] appeared, which covers rigidity results for the Schwarz symmetrization via a direct approach and not via a characterization, as it is given here. The characterization result specifically works for Schwarz symmetrization, exploiting that equality in (1.1) implies that slices are $(n-1)$-dimensional balls. In particular, the set A can be fully described by the radius and center function of these balls, i.e.,

$$\begin{aligned} \mathrm{rad}_A &: \mathbb{R} \to [0,\infty], t \mapsto \left(\frac{1}{\omega_n}|A_t|\right)^{\frac{1}{n-1}}, \\ \mathrm{cen}_A &: \mathbb{R} \to \mathbb{R}^{n-1}, t \mapsto \begin{cases} c & \text{if } \mathrm{rad}_A(t) \in (0,\infty) \text{ and } |A_t \triangle \mathrm{B}^{n-1}_{\mathrm{rad}(t)}(c)| = 0\,, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $\mathrm{B}^{n-1}_{\mathrm{rad}_A(t)}(c)$ denotes the $(n-1)$-dimensional ball of radius $\mathrm{rad}_A(t)$ and center $c \in \mathbb{R}^{n-1}$. In this case, we say that A is a stack of balls with radius function rad_A and center function cen_A. Once observing that, if A is a set of finite perimeter and finite volume, rad_A and cen_A are BV functions, the crucial simplification we use for further investigation is that single-variable BV functions are much easier to analyze than multivariable ones, due to the existence of one-sided limits at each point for good representatives. Therefore, we are able to prove that only certain compensation effects of the singular parts of rad_A and cen_A can lead to equality in (1.1), which is the following main result of Section 3.2:

Theorem (Theorem 3.57: Characterization of the equality cases in the Schwarz inequality)**.** *Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter and finite volume and let $I \subseteq \mathbb{R}$ open. Then the following statements are equivalent:*

i) $\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times I) = \mathrm{P}(A, \mathbb{R}^{n-1} \times I)$

ii) *The set A is a stack of balls in I and $|\mathrm{D}\mathrm{cen}_A| \leqslant |\mathrm{D}^s\mathrm{rad}_A|$ holds on $I \cap \{\mathrm{rad}_A^- > 0\}$.*

Here, $\mathrm{D}^s\mathrm{rad}_A$ denotes the singular part of the derivative measure $\mathrm{D}\mathrm{rad}_A$. To prove the full characterization result, we split the analysis into two parts. The first part considers the portion of A where the approximate lower limit rad_A^- of the radius function is not vanishing, which ensures that both the radius and the center function are locally of bounded variation. With this regularity at hand, we can then derive explicit formulas for the perimeter of $A^\star$, as well as for the perimeter of A in the special case when the center function has only one non-constant component. In fact, we can reduce the further analysis to this case by the Steiner symmetrization. In the second part, we consider sets on which rad_A^- vanishes. Then the behavior of the center function is negligible since the set A has no significant mass on slices with height t in $\{\mathrm{rad}_A^- = 0\}$, except for the case when $\mathrm{rad}_A^+(t) > 0$. This jump then implies that A has perimeter $\big|\mathrm{B}^{n-1}_{\mathrm{rad}_A^+(t)}\big|$ on $\mathbb{R}^{n-1} \times \{t\}$.
At this point, we want to mention that the barycenter function for general Steiner symmetrization (including the Schwarz symmetrization) has been introduced in [7] and that the regularity has been investigated under the general assumption that $\mathrm{rad}_A > 0$ and that A has no horizontal parts, resulting in the neglection of the jump and Cantor parts of cen_A.

As mentioned before, an extension of the Schwarz inequality is achieved for generalized prescribed mean curvature functionals of the form

$$A \mapsto \mathrm{P}(A) + \mu(A)\,,$$

where A is a set of finite perimeter and finite volume in $\mathbb{R}^n$ and μ is a signed Radon measure (not necessarily absolutely continuous with respect to $\mathcal{L}^n$). For this functional to be well-defined, we have to specify the representative of A, on which μ is evaluated. To this end, we define

$$\mathcal{P}(A) := \mathrm{P}(A) + \mu_+(A^1) - \mu_-(A^+)\,, \tag{1.2}$$

where $\mu_\pm$ are two mutually singular non-negative Radon measures on $\mathbb{R}^n$ and A^1 in the measure-theoretic interior, whereas A^+ is the measure-theoretic closure. With a result of [73], we ensure that $\mathcal{P}$ is well-defined and lower semicontinuous with respect to L^1 convergence if both measures μ_+ and μ_- satisfy some isoperimetric conditions with constant $c \in [0,1]$ (see Section 2.3.4). As it turns out, this is not enough to conclude a generalized Schwarz inequality for $\mathcal{P}$, which is the reason for the additional radial monotonicity assumption of the form

$$\mu(B^\star) \leqslant \mu(B) \qquad \text{for all bounded Borel sets } B \subseteq \mathbb{R}^n,\ \text{where } \mu := \mu_+ - \mu_-\,. \tag{1.3}$$

The the main result of Section 3.3 is the following theorem.

Theorem (Theorem 3.73: Generalized Schwarz inequality)**.** *Let μ_-, μ_+ be two mutually singular and non-negative Radon measures on $\mathbb{R}^n$ satisfying* (1.3) *and the small-volume isoperimetric condition with constant* 1*. Then*

$$\mathcal{P}(A^\star) \leqslant \mathcal{P}(A) \tag{1.4}$$

holds true for all sets $A \subseteq \mathbb{R}^n$ of finite perimeter and finite volume.

The result might seem immediate in view of the imposed monotonicity condition. However, the subtlety of (1.4) is in the transition from $(A^1)^\star$ to $(A^\star)^1$ and from $(A^+)^\star$ to $(A^\star)^+$, which creates overlaps which may be measured by μ, see Figure 1.3 for a simple but decisive case.

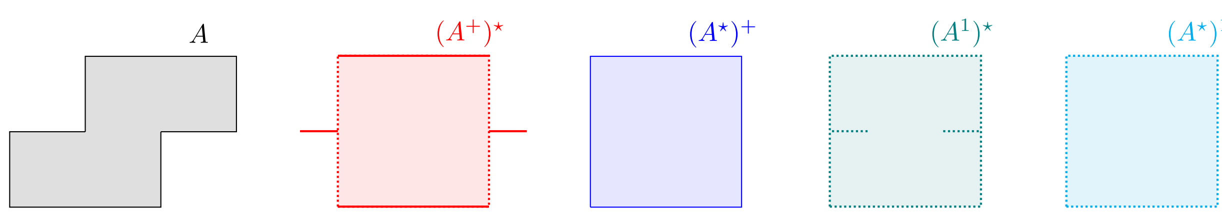


Figure 1.3: Difference of $(A^+)^\star$ and $(A^\star)^+$, and $(A^1)^\star$ and $(A^\star)^1$, respectively

Nonetheless, it is possible to show (1.4) by an analysis of the specified measures satisfying the radial monotonicity condition via disintegration. Heuristically, (1.3) forces μ to behave like $\mathcal{L}^{n-1}$ on every hypersurface

$H_t := \mathbb{R}^{n-1} \times \{t\}$, $t \in \mathbb{R}$. In particular, it implies $\mu_\pm = f^\pm(\mathcal{L}^{n-1} \otimes \mu_\pm^{\mathrm{V}})$ for certain normalized pushforwards $\mu_\pm^{\mathrm{V}}$ of $\mu_\pm$ with respect to the projection of $\mathbb{R}^n$ onto the last component and for densities $f^\pm$ which satisfy corresponding radial monotonicity conditions. Since $\mu_\pm^{\mathrm{V}}$ are non-negative Radon measures, $\mu_\pm = \mu_\pm^{\mathrm{dis}} + \mu_\pm^{\mathrm{reg}}$ can be split into the discrete parts $\mu_\pm^{\mathrm{dis}}$, where $\mu_\pm^{\mathrm{dis}}(H_t) > 0$ (which can only hold for at most countable many $t \in \mathbb{R}$), and the remaining part μ^{reg} with no atomic behavior in the vertical component. As Figure 1.3 indicates, it can be shown that $\mu_\pm^{\mathrm{reg}}$ vanish on the $(n-1)$-dimensional set $[(A^\star)^+ \triangle (A^+)^\star] \cup \big[(A^1)^\star \triangle (A^\star)^1\big]$, and $\mu_\pm^{\mathrm{dis}}$ can be investigated separately on each hypersurface H_t such that $\mu_\pm^{\mathrm{V}}(\{t\}) > 0$. In fact, via the imposed isoperimetric condition on $\mu_\pm$, we show that the measure of $[(A^+)^\star \triangle (A^\star)^+] \cup \big[(A^\star)^1 \triangle (A^1)^\star\big]$ on each hypersurface is compensated by the perimeter loss $\mathrm{P}(A) - \mathrm{P}(A^\star)$.

A more direct approach is also possible for reduced functionals of the form

$$\mathcal{P}_-(A) := \mathrm{P}(A) - \mu_-(A^+) \qquad \text{and} \qquad \mathcal{P}_+(A) := \mathrm{P}(A) + \mu_+(A^1)$$

via suitable approximation theorems that fail in the case of signed Radon measures μ.

Considering measures which only satisfy isoperimetric conditions with constant c strictly smaller than 1 prevents compensation effects of the perimeter and the measures (on hypersurfaces). Hence, equality in the generalized Schwarz inequality even transfers to the standard Schwarz inequality in these cases. With this at hand, we are able to state characterization and rigidity results also for the generalized case:

Corollary (Corollary 3.78: Equality case of the Schwarz inequality for functionals with measure data)**.** *Let μ_+ and μ_- be two mutually singular and non-negative Radon measures on $\mathbb{R}^n$ which both satisfy* (1.3) *and the small-volume isoperimetric condition on $\mathbb{R}^n$ with constant $c < 1$. Then*

$$\mathcal{P}(A^\star) = \mathcal{P}(A) \qquad \textit{implies} \qquad \mathrm{P}(A^\star) = \mathrm{P}(A)$$

for all sets $A \subseteq \mathbb{R}^n$ of finite perimeter and finite volume. In this case, A is a stack of balls with $|\mathrm{D}\,\mathrm{cen}_A| \leqslant |\mathrm{D}^s \mathrm{rad}_A|$ on $\{\mathrm{rad}_A^- > 0\}$.

As an application of the generalized Schwarz symmetrization results, we present three geometric obstacle problems for functionals of the form

$$\mathcal{P}_-[A] = \mathrm{P}(A) - \mu_-(A^+) \quad \text{in} \quad \mathcal{A}_{I,O} := \{A \in \mathcal{B}(\mathbb{R}^n) : |A| + \mathrm{P}(A) < \infty\,,\, I \subseteq A \subseteq O\} \tag{1.5}$$

in dimensions $n = 2$ and $n = 3$, where the subset relations are understood up to negligible sets. In fact, the well-known obstacle problem for the perimeter is extended by rewarding sets with large μ_--measure. We then specialize the planar problem to measures of the form $\mu_- = 2f\mathcal{H}^1 \llcorner \mathbb{R} \times \{t\}$ for some $t \in \mathbb{R}$ and radially decreasing functions $f = f(\,\cdot\,,t) \in \mathrm{C}(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})$ with values in $[0,1]$. We solve this problem for rectangles and discs as obstacles, where we reduce the latter to the case where f is a constant $\theta \in [0,1]$, due to the additional expense this problem requires. Depending on the height t and the weights f and θ, the resulting minimizer $A_{\min}$ can be shown to have the shape as in Figures 1.4 and 1.5, which can be found later in Figures 3.13 and 3.15 with some more details.

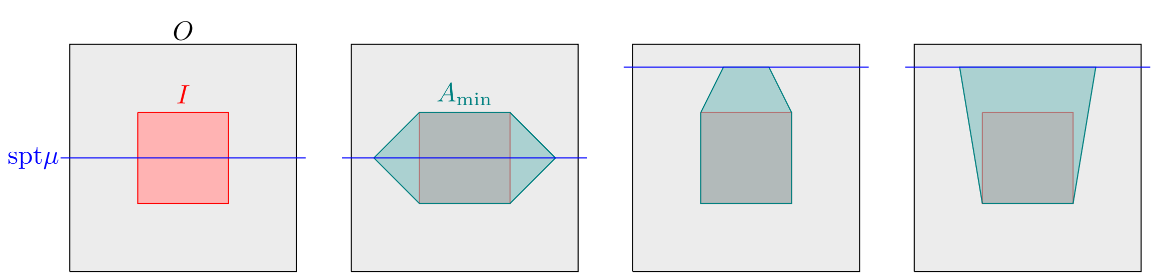


Figure 1.4: Obstacle problem with rectangles as obstacles in $\mathbb{R}^2$

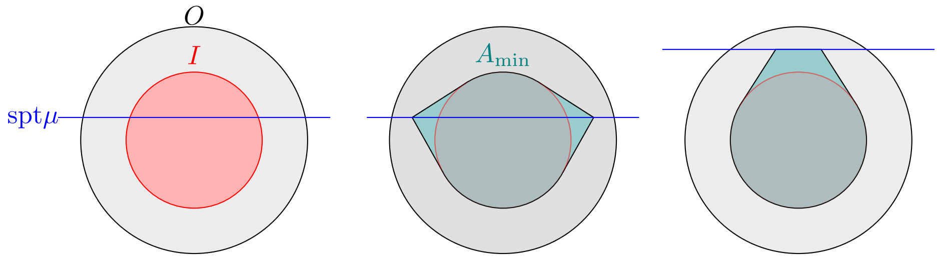


Figure 1.5: Obstacle problem with discs as obstacles in $\mathbb{R}^2$

In the three-dimensional case, we only consider balls as obstacles and measures of the form $\mu_- = 2\theta\mathcal{H}^2 \llcorner \mathbb{R}^2 \times \{t\}$ with $\theta \in [0,1]$, which gives minimizing sets of the form of Figures 1.6 and 3.16. Due to the considerable increase of complexity in higher dimensions, we further reduce the analysis to $|t| < r$ and neglect the uniqueness investigation in this case.

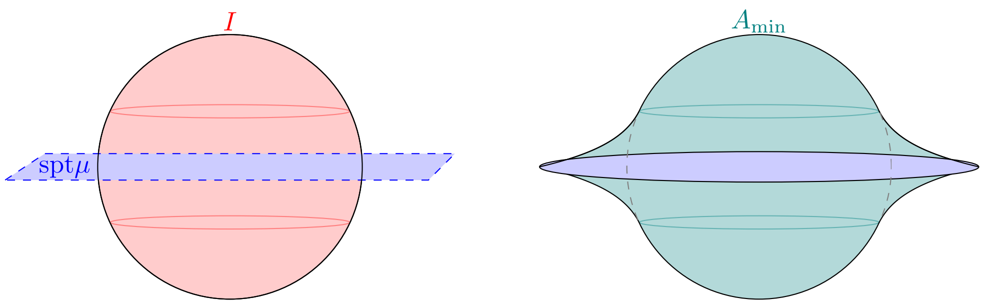


Figure 1.6: Obstacle problem with balls as obstacles in $\mathbb{R}^3$

For solving the described obstacle problems, we reduce to bodies of revolution as competitors by applying the generalized Schwarz inequality and the characterization of the equality cases. In turn, we have to consider only $(n-1)$-dimensional balls of some radius s to be measured via μ_-. Naturally, the minimizing set connects this ball and the obstacle via area-minimizing surfaces, i.e., lines in the two-dimensional case, and catenoid surfaces in the three-dimensional case, which can be shown by a transition of the parametric to a non-parametric obstacle problem. With this at hand, Euler's equation gives the exact dependence of s and f.

Variational mean curvature

If the boundary ∂E of some set $E \subseteq \mathbb{R}^n$ is an $(n-1)$-dimensional oriented C^2 submanifold in $\mathbb{R}^n$, then the mean curvature of ∂E is the continuous function $\pm\operatorname{div}\nu_E$. In [63], the author observes the following connection, see also Section 4.1. If E is the minimizer of the prescribed mean curvature functional

$$\mathcal{F}_H(F) := \mathrm{P}(F) - \int_F H \,\mathrm{d}x$$

among all measurable sets $F \subseteq \mathbb{R}^n$, where $H \in \mathrm{C}^0(\mathbb{R}^n) \cap \mathrm{L}^1(\mathbb{R}^n)$, then $\frac{1}{n-1}H|_{\partial E}$ is in fact the mean curvature of ∂E. As a generalization, the variational mean curvature of a measurable set $A \subseteq \mathbb{R}^n$ is defined as any function $H \in \mathrm{L}^1(\mathbb{R}^n)$ such that A is a minimizer of $\mathcal{F}_H$ among all measurable sets, see Definition 4.1. Consequently, variational mean curvatures are merely defined on the ambient space $\mathbb{R}^n$ without suitable restriction to ∂A. One can even find infinitely many variational mean curvatures for A, see Remark 4.2. This freedom of choice is limited, since the variational mean curvature cannot be arbitrarily good regarding its integrability. Indeed, if $x \in \partial^* A$ is a non-$\mathrm{C}^{1,1}$ point of the reduced boundary $\partial^* A$ of A, every variational mean curvature of A will blow up close to x. If this blow up is L^p-controlled, the following theorem provides conclusions regarding the regularity of $\partial^* A$.

Theorem (Theorem 4.4: Massari's regularity theorem, global version)**.** *For a set $A \subseteq \mathbb{R}^n$ of finite perimeter and with variational mean curvature $H \in \mathrm{L}^p(\Omega)$ for $p \in (n, \infty]$, the following properties hold true.*

i) *The set $\partial^* A$ is relatively open in ∂A.*

ii) *The set $\partial^* A$ is an $(n-1)$-dimensional $\mathrm{C}^{1,\alpha}$ manifold for $\alpha = \frac{p-n}{4p}$ if $p < \infty$ and for $\alpha = \frac{1}{4}$ if $p = \infty$.*

iii) *It holds $\mathcal{H}^s\big(\partial A \backslash \partial^* A\big) = 0$ for all $s \in (n-8, n]$.*

This is the famous regularity theorem of Massari, developed in the Italian sources [63, Teorema 5.1, Teorema 5.2], [64, Teorema 3.1, Teorema 3.2] and restated in [50, Theorem 3.6] in an English version. Later we will state the theorem for local variational mean curvatures – the exact notion can be found in Definition 4.1 – but for simplicity, we only present the global version here. Although the regularity analysis for the variational mean curvature was further developed, see for instance [85], Massari's conjecture from [50], stating that the optimal Hölder exponent for the theorem converges to 1 as $p \to \infty$, could not be confirmed until recently. In the author's Master thesis, it is shown that the exponent can be lifted to all $\alpha < \alpha_{\mathrm{opt}} := \frac{p-n}{p+1}$, see [81, Theorem 3.3]. Corresponding counterexamples from [81, Section 4] show the failure of the $\mathrm{C}^{1,\alpha}$ regularity conclusion for all $\alpha > \alpha_{\mathrm{opt}}$ and therefore confirm the optimality of the exponent and also Massari's conjecture.

Variational mean curvatures for $\mathbf{C^{1,\alpha}}$ sets

Given Massari's regularity theorem, it becomes natural to ask whether one can find 'good' curvatures for regular sets. This matter has only been partially addressed in the literature. For instance, in [9] and [8, Theorem 2.1], it is shown that every set of finite perimeter in $\mathbb{R}^n$ has a variational mean curvature in $\mathrm{L}^1(\mathbb{R}^n)$. This statement cannot be improved in general as there exist examples of sets of finite perimeter with no (local) variational mean curvature in L^p for all $p > 1$, see [50, Example 2.4, Example 2.5]. A fact that has not yet been stated in the literature but which might be known to experts is that every $\mathrm{C}^{1,1}$ set admits a variational mean curvature in $\mathrm{L}^\infty(\mathbb{R}^n)$. This is a straightforward special case of Theorem 5.8. The main result of Chapter 5 is the following theorem on local variational mean curvatures for the intermediate $\mathrm{C}^{1,\alpha}$ cases:

Theorem (Theorem 5.33: Integrability of variational mean curvatures of $\mathrm{C}^{1,\alpha}$ sets)**.** *Let $A \subseteq \mathbb{R}^n$ be a bounded open $\mathrm{C}^{1,\alpha}$ set with $\alpha \in (0,1)$. We assume either that A is convex or that $n = 2$ and A is connected. Then, for every ball $B \subseteq \mathbb{R}^n$ with $A \Subset B$, there exists a local variational mean curvature H of A in B with*

$$H \in \mathrm{L}^p(B) \qquad \text{for all } p \in \left[0, \frac{1+\alpha}{1-\alpha}\right) . \tag{1.6}$$

In a first approach, we state Theorem 5.8, which is a non-optimal subcase of the main result that only yields L^p regularity for all $p < \frac{1}{1-\alpha}$ (and the corresponding L^∞ result for $\mathrm{C}^{1,1}$ sets mentioned before). The advantage of Theorem 5.8 is that it holds for general $\mathrm{C}^{1,\alpha}$ sets A. The proof of this version exploits the strong connection between variational mean curvatures and (the divergence of) the unit normal of a set. Indeed, the motivation of the generalized curvatures carries over in the sense that the divergence of suitable extensions of the weak unit normal of ∂A is a variational mean curvature. Exploiting this and known trace results leads to the aforementioned subresult.

After deriving this first approach, the remainder of this chapter is dedicated to finding and estimating the best variational mean curvature. To this end, we use Barozzi's level set construction of variational mean curvatures for sets of finite perimeter from [8]. This curvature has optimal integrability properties among all variational mean curvatures, improving the result from the predecessor paper [9]. The construction uses the existence of nested subsets $\mathcal{C}_{A,\lambda}$ of a bounded set A with constant variational mean curvature $\lambda > 0$, i.e., the minimizers of

$$\mathcal{F}_\lambda(F) := \mathrm{P}(F) - \lambda |F|$$

among all measurable subsets F of A. In particular, it holds $\mathcal{C}_{A,\lambda} \subseteq \mathcal{C}_{A,\lambda+\varepsilon}$ for all $\varepsilon > 0$ and $\bigcup_{\lambda>0} \mathcal{C}_{A,\lambda} = A$. With this, the Barozzi curvature can be defined as

$$H_A(x) := \inf \{\lambda \,:\, x \in \mathcal{C}_{A,\lambda},\, \lambda \in \mathbb{Q}_{>0}\} \,.$$

In fact, for a.e. $\lambda \gg 1$, the boundaries of the sets $\mathcal{C}_{A,\lambda}$ of constant curvature λ represent the level set $\{H_A = \lambda\}$ of H_A as well as the constant mean curvature surfaces with curvature $\frac{1}{n-1}\lambda$ and outer obstacle A. It is essential to know the shape of $\mathcal{C}_{A,\lambda}$ in order to estimate H_A from both sides. In general, this turns

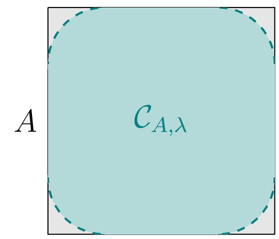


Figure 1.7: The set $\mathcal{C}_{A,\lambda}$ in a cube A

out to be quite technical. Similar to the analysis of Cheeger sets in the predecessor paper [58], the source [60, Theorem 2.3] verifies that $\mathcal{C}_{A,\lambda}$ is the union of balls of radius $\frac{1}{\lambda}$ in A if A is a two-dimensional Jordan domain without necks of radius $\frac{1}{\lambda}$ and with $|\partial A| = 0$. Hence, H_A blows up at non-$\mathrm{C}^{1,1}$ points, connecting the integrability of variational mean curvatures to the regularity of the corresponding set once more, compare Figure 1.7.

In higher dimensions there exist constant mean curvature surfaces besides the sphere and the sets $\mathcal{C}_{A,\lambda}$ are not necessarily the union of balls of corresponding radius $\frac{n-1}{\lambda}$ in A anymore. It is known that $\mathcal{C}_{A,\lambda}$ always contains all balls of radius $\frac{n}{\lambda}$ in A, see [86, Lemma 2.4]. In Proposition 4.31 we show an improved version of this result for the radius $\frac{n-1}{\lambda}$ instead of $\frac{n}{\lambda}$. Since (the boundaries of) balls of radius $r > 0$ in $\mathbb{R}^n$ have classical mean curvature $\frac{n-1}{r}$, the improved version fills a gap between the interpretation of the classical and the variational mean curvature.

Returning to the proof of the main statement of this section, the level-set construction of the Barozzi curvature leads to

$$\int_A |H_A|^p \,\mathrm{d}x = p \int_{\mathrm{c}_1}^{\infty} \lambda^{p-1} |A \backslash \mathcal{C}_{A,\lambda}| \,\mathrm{d}t + \mathrm{c}_2$$

for two positive constants $\mathrm{c}_1, \mathrm{c}_2$. As previously mentioned, the union $A_{\mathrm{B},\lambda}$ of balls in A with radius $\frac{n-1}{\lambda}$ is a subset of $\mathcal{C}_{A,\lambda}$, hence it is left to analyze the behavior of $|A \backslash A_{\mathrm{B},\lambda}|$ as $\lambda \to \infty$. To this end, a shift-enlargement result is presented, which claims that if a ball $\mathrm{B}_{\frac{n-1}{\lambda}}(y)$ is in a $\mathrm{C}^{1,\alpha}$ set A, then a new ball of radius $\frac{n-1}{\lambda-\varepsilon}$ is still contained in A and its Hausdorff distance to $\mathrm{B}_{\frac{n-1}{\lambda}}(y)$ is at most $\delta := \mathrm{c}\,\lambda^{-\frac{2}{1-\alpha}}\varepsilon$ for some $\mathrm{c} > 0$, all $\lambda \gg 1$ and all $\varepsilon \in \left(0, \frac{1}{3}\lambda\right]$. With this we show that the set $A_{\mathrm{B},\lambda+\varepsilon} \backslash A_{\mathrm{B},\lambda-\varepsilon}$ is contained in the δ-neighborhood of $\partial A_{\mathrm{B},\lambda}$. If $\partial A_{\mathrm{B},\lambda}$ behaves well, this leads to an upper bound of the derivative $\frac{\mathrm{d}}{\mathrm{d}\lambda}|A_{\mathrm{B},\lambda}|$ by $\mathrm{c}\,\lambda^{-\frac{2}{1-\alpha}}$, which in turn leads to the integrability of H_A in A as stated.

In order to verify the shift-enlargement result and the good behavior of $\partial A_{\mathrm{B},\lambda}$, we restrict the main statement of this chapter to either convex or two-dimensional sets. We can then show that $A_{\mathrm{B},\lambda}$ is a C^1 set for large λ, which eventually gives that the volume of the δ-neighborhood of $\partial A_{\mathrm{B},\lambda}$ is bounded suitably. This regularity result for $A_{\mathrm{B},\lambda}$ is not necessarily satisfied for higher dimensional non-convex $\mathrm{C}^{1,\alpha}$ sets A, which leads to additional technicalities for the shift-enlargement result. In particular, the shifting direction at non-C^1 points $x \in \partial A_{\mathrm{B},\lambda}$ cannot be chosen as the inner unit normal $-\nu_{A_{\mathrm{B},\lambda}}(x)$ anymore. Instead, it has to lie in the convex cone of the inner unit normals $-\nu_{\mathrm{B}}(x)$ of balls B in A with radius $\frac{n-1}{\lambda}$ such that $x \in \partial\mathrm{B}$. The volume of the δ-parallel set of ∂A can then be controlled by known results on the behavior of level sets of the distance function, see [56, Corollary 2]. A paper which covers the general case is in preparation, see [82].

Later in this chapter, we will derive counterexamples which show that the relation $p = \frac{1+\alpha}{1-\alpha}$ in the main theorem is optimal:

Theorem (Theorems 5.35 and 5.38: Counterexample to $\mathrm{C}^{1,\alpha}$ sets with good curvatures). *For all $\alpha \in (0,1)$ and all $\varepsilon > 0$ there exists a bounded, open and connected $\mathrm{C}^{1,\alpha}$ set $A \subseteq \mathbb{R}^n$ with no local variational mean curvature in $\mathrm{L}^p(B)$ for all $p \geqslant \frac{1+\alpha}{1-\alpha} + \varepsilon$ and for all open balls $B \subseteq \mathbb{R}^n$ with $A \Subset B$.*

Note that it is still open whether the limits case $p = \frac{1+\alpha}{1-\alpha}$ can be reached in (1.6).

In the proof of the previous theorem we introduce sets that have countably many $\mathrm{C}^{1,\alpha}$-points on the boundary. In the two-dimensional case, we combine countably many scalings of the supergraph of $x \mapsto |x|^{1+\alpha}$ to find a corresponding $\mathrm{C}^{1,\alpha}$ set A with pathological Barozzi curvature, see Figure 1.8. The latter can be estimated from below straightforwardly via the aforementioned result from [60] which gives $\mathcal{C}_{A,\lambda} = A_{\mathrm{B},\lambda}$ for large λ.

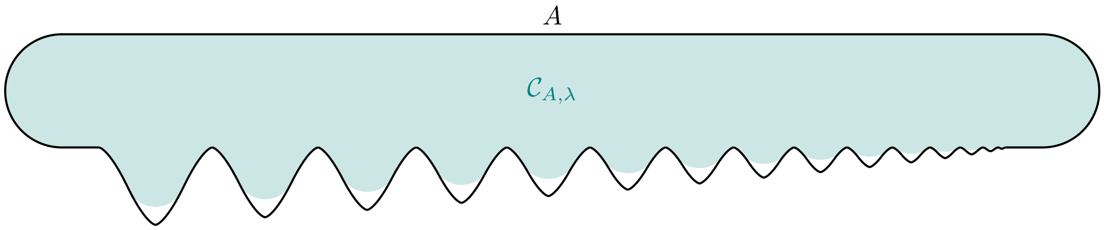


Figure 1.8: Two-dimensional counterexample to $\mathrm{C}^{1,\alpha}$ sets with L^p curvature

In the higher dimensional case, an $(n-1)$-dimensional hypercube is partitioned into infinitely many subcubes. On such a subcube with center a, we define functions which essentially behave like the $\mathrm{C}^{1,\alpha}$ functions $|x-a|^{1+\alpha}$. We then analyze the behavior of the Barozzi curvature on the supergraph A of this function. Only the subset relation $A_{\mathrm{B},\lambda} \subseteq \mathcal{C}_{A,\lambda}$ is available in this case and in general, the behavior of $\mathcal{C}_{A,\lambda}$ is hard to control. Hence, we put some effort into ensuring that the construction of A eventually gives $A_{\mathrm{B},\lambda} = \mathcal{C}_{A,\lambda}$ as in the two-dimensional case, see Proposition 5.36. To this end, we reduce the investigation to the supergraph of the function $x \mapsto |x|^{1+\alpha}$ (or a bounded variant of it) and consider a minimizer $\mathcal{C}_{A,\lambda}$ of $\mathcal{F}_\lambda$ among measurable subsets of A. According to the Schwarz and Steiner inequality, we may assume that $\mathcal{C}_{A,\lambda}$ satisfies good symmetry properties. Further (partially rather technical) arguments eventually lead to a smooth horizontal graph parametrization u of $\partial\mathcal{C}_{A,\lambda}$ in a small cylinder $\mathrm{B}^{n-1}_\varepsilon \times (0,\infty)$ with $u(x) > |x|^{1+\alpha}$. Since the latter ensures a positive distance of the graph of u and the outer obstacle boundary ∂A, we can then derive the Euler equation which results in $\operatorname{Graph}[u] = \partial A_{\mathrm{B},\lambda}$ in $\mathrm{B}^{n-1}_\varepsilon \times (0,\infty)$ and this finally leads to $\mathcal{C}_{A,\lambda} = A_{\mathrm{B},\lambda}$.

Partial regularity for non-parametric variational integrals

The predominant open question concerning Massari's regularity is if the theorem applies for $\alpha = \alpha_{\mathrm{opt}} = \frac{p-n}{p+1}$ as well. This is in fact the case, but the proof uses another strategy than [81], which reaches out for partial regularity results for local minimizers for variational integrals, see [80]. In particular, we leave the parametric framework in this chapter and translate the local minimization of the functional $\mathcal{F}_H$ among admissible sets for some $H \in \mathrm{L}^p(\mathbb{R}^n)$, $n \in \mathbb{N}_{\geqslant 2}$, $p \in (n,\infty)$ into a non-parametric minimization problem of the form

$$\mathbb{G}[w] := \int_V \sqrt{1+|\nabla w|^2} - \int_0^{w(x)} \tilde{H}(x,t)\,\mathrm{d}t\,\mathrm{d}x\,,$$

considering functions $w \in \mathrm{W}^{1,2}(V)$, where V is an open and bounded subset of $\mathbb{R}^{n-1}$ and $\tilde{H} \in \mathrm{L}^p(\mathbb{R}^n)$ is a (slight) modification of H. If the set $A \subseteq \mathbb{R}^n$ has variational mean curvature H, i.e., locally minimizes $\mathcal{F}_H$, we may apply Massari's regularity theorem on A and find local graph representations $u : V \to \mathbb{R}$ of $\partial^* A$. If chosen suitably, those functions then locally minimize $\mathbb{G}$ among functions $w \in \mathrm{W}^{1,2}_u(V)$. Due to the lack of continuity of $\tilde{H}$, it is not possible to determine the Euler equation of $\mathbb{G}$, so we are forced to use variational methods of partial regularity. During the transition from the parametric to the non-parametric setting, the perimeter converts into the area functional with integrand $f_{\mathbb{G}} : \mathbb{R}^{n-1} \to [0,\infty)$, $z \mapsto \sqrt{1+|z|^2}$, which has good properties, in principle. In fact, the function $f_{\mathbb{G}}$ satisfies local uniform ellipticity, which, combined with the a priori assumption that $u \in \mathrm{C}^1(\overline{V})$ from the application of Massari's regularity theorem, leads to partial regularity results for minimizers of the corresponding functional.
Throughout this chapter we generalize the partial regularity analysis to more general functionals. In order to avoid too many technical details in this introduction, we present the following special case of our main result.

Theorem (Corollary 6.29: Partial regularity for variational integrands with L^p-Hölder zero-order integrands)**.** *Let $u \in \mathrm{W}^{1,\infty}(V,\mathbb{R}^N)$ be a local minimizer of the functional*

$$\mathbb{F}[w] := \int_V f(\mathrm{D}w) - g(\,\cdot\,,w)\,\mathrm{d}x\,, \qquad w \in \mathrm{W}^{1,2}(V,\mathbb{R}^N)\,,$$

where the first-order integrand $f \in \mathrm{C}^2(\mathbb{R}^{N\times(n-1)})$ is of at most quadratic growth and locally elliptic, and the zero-order integrand $g : V \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function that satisfies

$$|g(x,y) - g(x,\tilde{y})| \leqslant \Gamma(x)|y-\tilde{y}|^\beta \qquad \text{for all } x \in V \text{ and all } y,\tilde{y} \in \mathbb{R}^N$$

for a non-negative function $\Gamma \in \mathrm{L}^p_{\mathrm{loc}}(V)$ and parameters $\beta \in (0,1]$ and $p \in \left(\frac{n-1}{\beta}, \infty\right]$. Then u is in $\mathrm{C}^{1,\alpha}_{\mathrm{loc}}(V_{\mathrm{reg}}, \mathbb{R}^N)$ for some open set $V_{\mathrm{reg}} \subseteq V$ that satisfies $|V \backslash V_{\mathrm{reg}}| = 0$ and for

$$\alpha = \begin{cases} \frac{\beta p-(n-1)}{(2-\beta)p} & \text{if } p < \infty\,, \\ \frac{\beta}{2-\beta} & \text{if } p = \infty \text{ and } \beta < 1\,, \end{cases}$$

and any $\alpha \in (0,1)$ if $p = \infty$ and $\beta = 1$.

The regularity analysis is based on the integral characterization of Hölder continuity by Campanato, which states that a suitable behavior of the excess $\Phi := \fint_{\mathrm{B}_r(x)} |\mathrm{D}u - (\mathrm{D}u)_{x,r}|^2 \,\mathrm{d}y$ essentially translates to the Hölder continuity of $\mathrm{D}u$. We will estimate Φ via the intertwined A-harmonic approximation and Caccioppoli inequality strategy introduced by [33] and refined in [29, 27, 28, 30, 31, 32, 75, 76, 25, 24, 19, 49, 48]. During this, a special interest lies in finding the sharp dependencies between α, p and β in the case of the previous theorem, and between α, p, q and β in case of the generalized Settings 1, 2 and 3 of Chapter 6.
Results in this direction can be found, for instance, in [71, 45, 54, 72]. In [71, Theorem III], Phillips considers the special scalar case $f(z) = \frac{1}{2}|z|$ and $g(x,y) = y^\beta$ and derives full $\mathrm{C}^{1,\frac{\beta}{2-\beta}}$ regularity for minimizers of the corresponding functional, which is reproduced by our result in the case $\Gamma \in \mathrm{L}^\infty$. The example [71, p. 5] even shows that this result is sharp. In [45, Theorem 2.1], the authors generalize Phillips' result to a more general non-scalar setting with specific first-order integrand. This framework still addresses full regularity and assumes $\Gamma \in \mathrm{L}^\infty$, but in the subsequent remark, a generalization is predicted for the case $\Gamma \in \mathrm{L}^p$ and for Γ in the Morrey-Hölder space $\mathrm{L}^{p,\lambda}$. This is even more generalized by our result considering the exact configuration of f and g. The special case $\Gamma \in \mathrm{L}^\infty$, however, has been lifted to partial regularity results with optimal Hölder exponent $\frac{\beta}{2-\beta}$ in [54, Theorem 1] and [72, Theorem 1.1]. Partial regularity results for Γ in $\mathrm{L}^{p,\lambda}$ as in Theorem 6.27 have been addressed in [74, Corollary 3.3] in the context of almost-minimizers in BV, but the maximal Hölder exponent does not exceed $\frac{\beta}{2}$ and additional assumptions have been imposed on u.
In order to apply the non-parametric theory to Massari's regularity theorem, we a priori apply the same statement, which provides C^1 regularity for the minimizer u of the variational integral $\mathbb{G}$. This ensures that $u \in \mathrm{W}^{1,\infty}$, as assumed in the aforementioned theorem, as well as $V = V_{\mathrm{reg}}$ such that no additional singularities appear when transitioning the non-parametric framework back to the parametric framework. Hence, we can restate Massari's regularity theorem in the following optimal version, which includes the case of the limit exponent.

Theorem (Theorem 6.32: Optimal Massari's regularity theorem, global version)**.** *For a set $A \subseteq \mathbb{R}^n$ of finite perimeter and with variational mean curvature $H \in \mathrm{L}^p(\mathbb{R}^n)$ for $p \in (n,\infty]$, the following properties hold true.*

i) *The set $\partial^* A$ is relatively open in ∂A.*

ii) *The set $\partial^* A$ is an $(n-1)$-dimensional $\mathrm{C}^{1,\alpha}$ manifold for $\alpha = \frac{p-n}{p+1}$ if $p < \infty$ and for all $\alpha \in (0,1)$ if $p = \infty$.*

iii) *It holds $\mathcal{H}^s(\partial A \backslash \partial^* A) = 0$ for all $s \in (n-8, n]$.*

Due to the counterexamples of [81, Section 4], the theorem cannot be further improved, which leads to the conclusion that our non-parametric results are also optimal to some extent.
It is apparent that the relation between the integrability of variational mean curvature and the $\mathrm{C}^{1,\alpha}$ regularity in the optimal version of Massari's regularity theorem and the main result of Chapter 5 do not match. In particular, it holds that

$$\alpha_{\mathrm{M}} = \frac{p-n}{p+1} < \frac{p-1}{p+1} = \alpha_{\mathrm{I}} \qquad \text{and} \qquad p_{\mathrm{M}} = \frac{\alpha+n}{1-\alpha} > \frac{\alpha+1}{1-\alpha} = p_{\mathrm{I}} \qquad \text{for all } \alpha \in (0,1) \text{ and } p \in (n,\infty)\,.$$

The values $\alpha_{\mathrm{M}} = \alpha_{\mathrm{M}}(p)$ in $(0,1)$ and $p_{\mathrm{M}} = p_{\mathrm{M}}(\alpha)$ in (n,∞) are chosen such that the optimal relation of Massari's regularity theorem (with respect to p and α, respectively) holds true, whereas the values $\alpha_{\mathrm{I}} = \alpha_{\mathrm{I}}(p)$ in $(0,1)$ and $p_{\mathrm{I}} = p_{\mathrm{I}}(\alpha)$ in $(1,\infty)$ are such that the optimal relation in the integrability theorem is satisfied. This, however, is not a contradiction, since any $\mathrm{C}^{1,\alpha}$ set on which the integrability result for the variational mean curvature applies leads to a variational mean curvature in L^p for all $p < p_{\mathrm{I}}$. This, by the improved Massari's theorem, leads to a $\mathrm{C}^{1,\beta}$ regularity for all $\beta < \frac{\alpha+1-n(1-\alpha)}{2} < \alpha$, and the latter is a priori satisfied. The relation of p and α is formally the same if $n = 1$ (which is never the case since we consider at least two-dimensional sets) and drifts further apart for larger dimension.

Structure

This thesis is structured as follows. In Chapter 2, notation as well as general assumptions are given and overall preliminaries on BV functions and measure theory are presented. We point out that specific preliminaries are available in the corresponding sections. We will mostly renounce giving proofs of standard theorems throughout these parts. However, some selected proofs are carried out.

In Chapter 3, we introduce the Schwarz symmetrization and extend the existing theory by the announced characterization of the equality cases in the Schwarz perimeter inequality. Afterwards, the generalized Schwarz inequality for perimeter functionals with measure data together with some rigidity results and model cases are presented.

All results of the second part of this thesis are connected to variational mean curvatures. Thus, this subject is treated separately in Chapter 4, where we primarily collect known results, but also show new ones.

Building on this, we prove L^p-integrability of Barozzi's variational mean curvature for planar and convex $\mathrm{C}^{1,\alpha}$ sets for all $p \in \left[1, \frac{1+\alpha}{1-\alpha}\right)$ and present corresponding counterexamples for $p > \frac{1+\alpha}{1-\alpha}$ in Chapter 5.

Finally, in Chapter 6 we visit the non-parametric partial regularity theory, which we apply to the parametric case of variational mean curvatures in order to find the optimal Hölder exponent for Massari's regularity theorem.

Chapter 2

Preliminaries

In this chapter, we set general notation and assumptions and give some preliminary results on measure theory and BV functions.

2.1 Notation and overall assumption

2.1.1 Overall assumption

Throughout this thesis, we set $n, N \in \mathbb{N}$ and assume that Ω is an open subset of $\mathbb{R}^n$. We additionally assume $n \geqslant 2$ throughout the Chapters 3 –6, if not stated otherwise.

2.1.2 Notation

Vectors and sets

For some $x \in \mathbb{R}^n$, the $(n-1)$-dimensional vector $(x_1, \ldots, x_{n-1})$ is abbreviated by $\overline{x}$. For $i \in \{1, \ldots, n\}$, we write $\overline{x}^i$ for $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$. If additionally $r > 0$, the open ball of radius r and center x in $\mathbb{R}^n$ is denoted by $\mathrm{B}_r(x)$. In some cases, the dimension of the ball will be emphasized by using the notation $\mathrm{B}^n_r(x)$. We abbreviate $\mathrm{B}_r(x) \cap \mathbb{R}^{n-1} \times (0, \infty)$ by $\mathrm{B}_r(x)^+$ and $\mathrm{B}_r(x) \cap \mathbb{R}^{n-1} \times (-\infty, 0)$ by $\mathrm{B}_r(x)^-$. Instead of $\mathrm{B}^{(n)}_r(0)$, we write $\mathrm{B}^{(n)}_r$. The cylinder $\mathrm{B}^{n-1}_r(\overline{x}) \times (x_n - r, x_n + r)$ of radius and height r with center x is denoted by $C_r(x)$. For some $k \in \{1, \ldots, n-1\}$ and $x \in \mathbb{R}^k$, we denote the x-slice $\{y \in \mathbb{R}^{n-k} : (y, x) \in A\}$ of $A \subseteq \mathbb{R}^n$ by A_x. For some $\alpha \in [0, 1]$, the (Borel) set of all density-α points with respect to $A \subseteq \mathbb{R}^n$ is denoted by A^α. The measure-theoretic closure $\mathbb{R}^n \backslash A^0$ is denoted A^+ and we write $\partial^{\mathrm{e}} A$ for the essential boundary $A^+ \backslash A^1$. We write $H^+_{t,\nu}$ and $H^-_{t,\nu}$ for the half spaces $\{x \in \mathbb{R}^n : x \cdot \nu > t\}$ and $\{x \in \mathbb{R}^n : x \cdot \nu < t\}$. In the special case $\nu = e_n$, we put $H^\pm_t = H^\pm_{t,e_n}$, respectively. The hypersurfaces $\partial H^+_{t,\nu} = \partial H^-_{t,\nu}$ are denoted by $H_{t,\nu}$ and for $\nu = e_n$, we set $H_t := H_{t,\nu}$. For a set $A \subseteq \mathbb{R}^n$ and $\varepsilon > 0$, we denote the ε-neighborhood $\{x \in \mathbb{R}^n : \mathrm{dist}(x, A) < \varepsilon\}$ of A by $(A)_\varepsilon$ and the inner ε-parallel set $\{x \in A : \mathrm{dist}(x, \mathbb{R}^n \backslash A) > \varepsilon\}$ by $[A]_\varepsilon$.

Measures

The n-dimensional Lebesgue measure is denoted by $\mathcal{L}^n$. For some $\mathcal{L}^n$-measurable set $A \subseteq \mathbb{R}^n$, we may write $|A|$ instead of $\mathcal{L}^n(A)$. The volume $|\mathrm{B}^n_1|$ of the n-dimensional unit ball is abbreviated by ω_n. For $s \in [0, \infty)$, the s-dimensional Hausdorff measure is labeled $\mathcal{H}^s$ and for $s < 0$, we identify $\mathcal{H}^s = \mathcal{H}^0$. For some Borel set $B \subseteq \mathbb{R}^n$, we denote by $\mathcal{B}(B)$ the Borel-σ algebra of B. By $\mathcal{M}^n$ we denote the σ-algebra of all $\mathcal{L}^n$-measurable subsets of $\mathbb{R}^n$. The total variation measure of some (vector-valued signed) Radon measure μ is denoted by $|\mu|$. For a measure μ on $\mathbb{R}^n$ and some μ-measurable set $A \subseteq \mathbb{R}^n$, we write $\mu \llcorner A$ for the restriction of μ on A.

Functions and spaces

For every function space $\mathbb{H}$, the notation $\mathbb{H}(\Omega) = \mathbb{H}(\Omega, \mathbb{R})$ is used throughout this thesis. Moreover, we write $\mathbb{H}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and $\mathbb{H}_{\mathrm{loc}}(\Omega)$ for the localized function spaces, i.e., the set of all functions $u : \Omega \to \mathbb{R}^N$ with $u \in \mathbb{H}(K, \mathbb{R}^N)$ for all compact subsets $K \subseteq \Omega$. For $N = 1$ and a certain representative of u, we write u_- for the negative part of u and u_+ for its positive part. We say that a sequence of sets $(A_k)_k$ converges to the set A in $\mathrm{L}^1_{(\mathrm{loc})}$ if $\mathbb{1}_{A_k}$ converges to $\mathbb{1}_A$ in $\mathrm{L}^1_{(\mathrm{loc})}$. For $t \in \mathbb{R}$ and $\nu \in \mathbb{R}^n$, we denote the hyperplane $\{x \in \mathbb{R}^n : x \cdot \nu = t\}$ by $H_{t,\nu}$. For some $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and some measurable set $F \subseteq \Omega$ of finite volume, we write

$$\fint_F u \,\mathrm{d}x := \begin{cases} 0\,, & \text{if } |F| = 0 \\ \frac{1}{|F|} \int_F u \,\mathrm{d}x\,, & \text{else.} \end{cases}$$

We call the function which maps $x \in \Omega$ to $\lim_{r \searrow 0} \fint_{\mathrm{B}_r(x) \cap \Omega} u(y) \,\mathrm{d}y$ in case the limit exists, and to 0 otherwise the mean-value representative of u. If $F = \mathrm{B}_r(x)$, we abbreviate $\fint_F u \,\mathrm{d}x$ with $(u)_{x,r}$. The function π_i for $i \in \{1, \dots, n\}$ denotes the projection of $\mathbb{R}^n$ onto its i-th component. For some $u : \Omega \to \mathbb{R}$, we write $\mathrm{Sub}[u, \Omega]$ for the subgraph $\{(x,t) \in \Omega \times \mathbb{R} : x \in \Omega \text{ and } t < u(x)\}$ of u over Ω. In case the domain Ω of u is well-known, we will only write $\mathrm{Sub}[u]$ instead of $\mathrm{Sub}[u, \Omega]$. Similarly, we write $\mathrm{Sup}[u] = \mathrm{Sup}[u, \Omega]$ for the supergraph and $\mathrm{Graph}[u] = \mathrm{Graph}[u, \Omega]$ for the graph of u over Ω.

2.2 BV functions and sets of finite perimeter

As an extension of Sobolev functions, BV functions unite several advantages. Since they have Radon measures as derivatives, a version of the divergence theorem can still be applied. Moreover, the derivative even handles jumps or Cantor like parts. The compactness theorem and lower semicontinuity results allow for good existence results based on the direct method of calculus of variations. The strong connection to sets of finite perimeter makes it possible to easily change perspective between analytical and geometric measure theory perspectives.
For more background in this theory, we refer to the most common references [3, 47, 62, 38]. The source [62] mainly covers sets of finite perimeter, whereas [3] has deep insight into BV theory. The books [47, 38] cover both aspects compactly.

2.2.1 BV functions

In this section, we define BV functions and present useful results.

Definition 2.1 (Derivative measures, the space BV and variations)**.** For $\alpha \in \mathbb{N}_0^n$, the **weak α-th partial derivative measure** of a locally integrable function $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is the Radon measure $\mu \in \mathrm{RM}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ if

$$\int_\Omega u \cdot \partial^\alpha \varphi \,\mathrm{d}x = (-1)^{\sum_{i=1}^k \alpha_i} \int_\Omega \varphi \cdot \mathrm{d}\mu \qquad \text{holds for all test functions} \qquad \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)\,. \tag{2.1}$$

In this case, we denote $\mu = \mathrm{D}^\alpha u$.
For $k \in \{1, \dots, n\}$, we write $\mathrm{D}^{e_k} u = \mathrm{D}_k u$ and call this measure the **k-th (weak) partial derivative measure** of u.
Moreover, we denote $\mathrm{D}u := (\mathrm{D}_1 u, \dots, \mathrm{D}_n u)$ and call this the **(weak) derivative measure** of u.
The **variation** of u is defined as

$$\mathrm{Var}(u, \Omega) := \sup \left\{ \left| \int_\Omega u \cdot \operatorname{div} \varphi \,\mathrm{d}x \right| : \varphi \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega, \mathbb{R}^{N \times n}) \text{ with } \sup_\Omega |\varphi| \leqslant 1 \right\}.$$

The set $\mathrm{BV}(\Omega, \mathbb{R}^N)$ is defined as all $\mathrm{L}^1(\Omega, \mathbb{R}^N)$-functions u with $|\mathrm{D}u|(\Omega) < \infty$. In this case, u is called a function of **bounded variation** in Ω. The BV-norm of u is defined by

$$\|u\|_{\mathrm{BV}(\Omega, \mathbb{R}^N)} := \|u\|_{\mathrm{L}^1(\Omega, \mathbb{R}^N)} + |\mathrm{D}u|(\Omega)\,.$$

Remark 2.2. i) Weak derivative measures are well-defined, unique and locally determined.

ii) For $\alpha = e_k$, $k \in \{1, \dots, n\}$, a mollification argument shows that (2.1) is equivalent to

$$\int_\Omega u \, \partial^\alpha \varphi \,\mathrm{d}x = (-1)^{\sum_{i=1}^k \alpha_i} \int_\Omega \varphi \cdot \mathrm{d}\mu \qquad \text{for all test functions} \qquad \varphi \in \mathrm{W}^{1,\infty}_0(\Omega, \mathbb{R}^N)\,.$$

iii) The space $\mathrm{BV}(\Omega, \mathbb{R}^N)$ equipped with $\| \cdot \|_{\mathrm{BV}(\Omega,\mathbb{R}^N)}$ is a normed space.

iv) The inclusions $\mathrm{C}^1(\Omega, \mathbb{R}^N) \subseteq \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \subseteq \mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ hold true and the derivative of a function $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ coincides with its derivative measure in the following sense: If μ is the derivative measure of u, then there holds $\mu = \mathrm{D}u\mathcal{L}^n$ on Ω, where $\mathrm{D}u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times N})$ is the weak derivative of u in the Sobolev sense. Throughout this thesis, we identify $\mathrm{D}u$ with the Radon measure $\mathrm{D}u\mathcal{L}^n$ on Ω.

In what follows, we collect results concerning functions of bounded variations including computation rules as well as structural results.

Lemma 2.3 (Product rule for BV functions)**.** *For $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and $f \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega)$, the product fu is in $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ with derivative measure*

$$\mathrm{D}(fu) = f'u\,\mathcal{L}^n|_\Omega + f\,\mathrm{D}u\,.$$

Proof. According to Remark 2.2 and the definition of derivative measures, it holds

$$\int_\Omega u\,f'\,\varphi + u\,f\,\varphi'\,\mathrm{d}x = \int_\Omega u\,(f\varphi)'\,\mathrm{d}x = -\int_\Omega f\,\varphi\,\mathrm{d}\mathrm{D}u(x)\,,$$

for all $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$, which concludes the claim. □

Lemma 2.4 (Extended BV function)**.** *Let $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and*

$$v : V \to \mathbb{R},\ x \mapsto u(x_1, \ldots, x_n)$$

for an open set $V \subseteq \Omega \times \mathbb{R}^k$ and $k \in \mathbb{N}$. Then $v \in \mathrm{BV}_{\mathrm{loc}}(V, \mathbb{R}^N)$ with $\mathrm{D}_i v = \mathrm{D}_i u \otimes \mathcal{L}^k$ for all $i \in \{1, \ldots, n\}$ and $\mathrm{D}_i v = 0$ for $i \in \{n+1, \ldots, n+k\}$ on V.

Proof. Let $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}(V, \mathbb{R}^N)$, then

$$\int_V \partial_i\varphi \cdot v(x)\,\mathrm{d}x = \int_{\mathbb{R}^k}\int_\Omega \partial_i\varphi(y,z)\cdot u(z)\mathrm{d}z\,\mathrm{d}y = -\int_{\mathbb{R}^k}\int_\Omega \varphi(y,z)\cdot \mathrm{d}\mathrm{D}_i u(z)\,\mathrm{d}y = -\int_V \varphi(y,z)\cdot \mathrm{d}(\mathrm{D}_i u \otimes \mathcal{L}^k)$$

holds for all $i \in \{1, \ldots, n\}$. The other case is derived straightforwardly by the divergence theorem. □

The following proposition shows the connection of variation and BV functions.

Proposition 2.5 (BV functions and the variation)**.** *For $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, the following statements are equivalent*

i) *$u \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and $|\mathrm{D}u|(\Omega) < \infty$*

ii) *$\mathrm{Var}(u, \Omega) < \infty$*

and in this case, it holds

$$\mathrm{Var}(u, \Omega) = |\mathrm{D}u|(\Omega)\,.$$

The following lower semicontinuity result is the basis of existence results in BV due to the direct method of calculus of variations. We will use it mainly for the perimeter.

Proposition 2.6 (Lower semicontinuity)**.** *The variation is lower semicontinuous with respect to the $\mathrm{L}^1_{\mathrm{loc}}$ convergence. In particular, if u is in $\mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and $(u_k)_k$ is a sequence that converges to u in $\mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, then*

$$\mathrm{Var}(u, \Omega) \leqslant \liminf_{k\to\infty} \mathrm{Var}(u_k, \Omega)\,.$$

If additionally $u_k \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ for all $k \in \mathbb{N}$ with $\liminf_{k\to\infty} |\mathrm{D}u_k|(\Omega) < \infty$, then $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ and the previous lower semicontinuity transfers to the derivative measures.

In order to be able to transfer properties from smooth functions to BV functions, we need the following theorem.

Definition 2.7 (Strict convergence)**.** Let $u, u_k \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ for all $k \in \mathbb{N}$. We say that u_k **strictly converges to u in $\mathrm{BV}(\Omega, \mathbb{R}^N)$**, if

$$u_k \to u \qquad \text{in } \mathrm{L}^1(\Omega, \mathbb{R}^N) \qquad \text{and} \qquad |\mathrm{D}u_k|(\Omega) \to |\mathrm{D}u|(\Omega) \qquad \text{as } k \to \infty\,.$$

Theorem 2.8 (Smooth approximation)**.** *For any $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$, there exists a sequence $(u_k)_{k\in\mathbb{N}}$ of $\mathrm{C}^\infty(\Omega, \mathbb{R}^N)$ functions, such that $u_k \to u$ strictly in $\mathrm{BV}(\Omega, \mathbb{R}^N)$.*

Definition 2.9 (Mollification of a Radon measure)**.** Let $\eta \in \mathrm{C}^\infty(\mathbb{R}^n)$ be a non-negative function with support in the unit ball and $\int_{\mathbb{R}^n} \eta \,\mathrm{d}x = 1$. Denote $\eta_\varepsilon(x) := \varepsilon^{-n}\eta(\varepsilon^{-1}x)$ for all $x \in \mathbb{R}^n$ and $\varepsilon > 0$. The **η_ε-mollification** of the measure $\mu \in \mathrm{RM}(\Omega, \mathbb{R}^N)$ is the function μ_ε, defined by

$$\mu_\varepsilon(x) := \eta_\varepsilon * \mu(x) := \int_\Omega \eta_\varepsilon(x-y)\,\mathrm{d}\mu(y) \qquad \text{for all } x \in [\Omega]_\varepsilon := \{x \in \Omega : \operatorname{dist}(\mathbb{R}^n \setminus \Omega, x) > \varepsilon\}\,.$$

Throughout this thesis, we consider η to be fixed and call μ_ε the **mollification** of μ.

Lemma 2.10 (Properties of mollification of measures)**.** *Let $\mu \in \mathrm{RM}(\Omega, \mathbb{R}^N)$ and $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$.*

i) *For all $\varepsilon > 0$, the mollification μ_ε is in $\mathrm{C}^\infty([\Omega]_\varepsilon, \mathbb{R}^N) \cap \mathrm{L}^1((\Omega)_\varepsilon, \mathbb{R}^N)$ and satisfies $\int_B |\mu_\varepsilon|\,\mathrm{d}x \leqslant |\mu|((B)_\varepsilon)$ for all $B \in \mathcal{B}([\Omega]_\varepsilon)$.*

ii) *It holds that*

$$\mathrm{D}(u_\varepsilon) = (\mathrm{D}u)_\varepsilon \qquad \textit{a.e. on } [\Omega]_\varepsilon \qquad \textit{and} \qquad \int_B |\mathrm{D}(u_\varepsilon)|\,\mathrm{d}x \leqslant |\mathrm{D}u|((B)_\varepsilon)$$

for all $B \in \mathcal{B}([\Omega]_\varepsilon)$ and all $\varepsilon > 0$.

iii) *The functions u_ε converge to u strictly in $\mathrm{BV}(K, \mathbb{R}^N)$ for all open sets $K \Subset \Omega$ with $|\mathrm{D}u|(\partial K) = 0$ as $\varepsilon \searrow 0$.*

Remark 2.11. The strict convergence of the previous lemma can be lifted to area-strict convergence. In fact, for all $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$, it holds

$$\lim_{\varepsilon \searrow 0} |(\mathcal{L}^n, \mathrm{D}u_\varepsilon)|([\Omega]_\varepsilon) = |(\mathcal{L}^n, \mathrm{D}u)|(\Omega)$$

and

$$\lim_{\varepsilon \searrow 0} |(\mathcal{L}^n, \mathrm{D}u_\varepsilon)|(B) \leqslant |(\mathcal{L}^n, \mathrm{D}u)|(\overline{B})\,.$$

holds for all Borel sets $B \Subset \Omega$. Indeed, by the definition of mollification, we find $(\mathcal{L}^n, \mathrm{D}u_\varepsilon) = (\mathcal{L}^n_\varepsilon, \mathrm{D}u_\varepsilon) = (\mathcal{L}^n, \mathrm{D}u)_\varepsilon$, which, by Lemma 2.10, gives

$$\limsup_{\varepsilon \searrow 0} |(\mathcal{L}^n, \mathrm{D}u_\varepsilon)|(B) \leqslant \limsup_{\varepsilon \searrow 0} |(\mathcal{L}^n, \mathrm{D}u)|((B)_\varepsilon) = |(\mathcal{L}^n, \mathrm{D}u)|(\overline{B})\,.$$

By an exhaustion argument, this leads to $\limsup_{\varepsilon \searrow 0} |(\mathcal{L}^n, \mathrm{D}u_\varepsilon)|([\Omega]_\varepsilon) \leqslant |(\mathcal{L}^n, \mathrm{D}u)|(\Omega)$. On the other hand, for all $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}\big(\Omega, \mathrm{B}_1^{n+1}\big)$, we have

$$\lim_{\varepsilon \searrow 0} \int_{[\Omega]_\varepsilon} \varphi \cdot \mathrm{d}(\mathrm{D}u_\varepsilon, \mathcal{L}^n) = \lim_{\varepsilon \searrow 0} - \int_{[\Omega]_\varepsilon} \operatorname{div}(\overline{\varphi}) u_\varepsilon \,\mathrm{d}x + \int_{[\Omega]_\varepsilon} \varphi_{n+1}\,\mathrm{d}x = -\int_\Omega \operatorname{div}(\overline{\varphi}) u \,\mathrm{d}x + \int_\Omega \varphi_{n+1}\,\mathrm{d}x = \int_\Omega \varphi \cdot \mathrm{d}(\mathrm{D}u, \mathcal{L}^n)$$

by the $\mathrm{L}^1_{\mathrm{loc}}$-convergence of u_ε to u. Since $\mathrm{C}^1_{\mathrm{cpt}}\big(\Omega, \mathrm{B}_1^{n+1}\big)$ is dense in $\mathrm{C}^0_{\mathrm{cpt}}\big(\Omega, \mathrm{B}_1^{n+1}\big)$, the Riesz-Markov theorem implies

$$\begin{aligned}
\liminf_{\varepsilon \searrow 0} |(\mathcal{L}^n, \mathrm{D}u_\varepsilon)|([\Omega]_\varepsilon) &= \liminf_{\varepsilon \searrow 0} \sup \left\{ \int_{[\Omega]_\varepsilon} \varphi \cdot \mathrm{d}(\mathrm{D}u_\varepsilon, \mathcal{L}^n) : \varphi \in \mathrm{C}^1_{\mathrm{cpt}}\big([\Omega]_\varepsilon, \mathrm{B}_1^{n+1}\big) \right\} \\
&\geqslant \sup \left\{ \int_\Omega \varphi \cdot \mathrm{d}(\mathrm{D}u, \mathcal{L}^n) : \varphi \in \mathrm{C}^1_{\mathrm{cpt}}\big(\Omega, \mathrm{B}_1^{n+1}\big) \right\} \\
&= (\mathcal{L}^n, \mathrm{D}u)|(\Omega)\,.
\end{aligned}$$

We conclude with collecting standard results of BV theory.

Theorem 2.12 (Compactness theorem). *We consider $\Omega \subseteq \mathbb{R}^n$ as a bounded open set such that the Rellich-Kondrachov compactness theorem applies on Ω (for instance, this holds true if Ω is a Lipschitz domain). Then $\mathrm{BV}(\Omega, \mathbb{R}^N)$ is compactly embedded in $\mathrm{L}^1(\Omega, \mathbb{R}^N)$.*

Lemma 2.13 (Constancy theorem). *Let $u \in \mathrm{BV}(\mathbb{R}^n)$ with $\mathrm{D}_i u = 0$ for some $i \in \{1, \dots, n\}$. Then $u_i(t) := u(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$ is constant a.e. on $\mathbb{R}$ for $\mathcal{L}^{n-1}$-a.e. $\overline{x}^i \in \mathbb{R}^{n-1}$.*

Definition 2.14 (Approximate jump points). A function $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ has the **approximate jump point** $x \in \Omega$ if there exists a direction $\nu \in \partial \mathrm{B}_1$ and $y \neq z \in \mathbb{R}^N$ such that

$$\lim_{r \searrow 0} \fint_{\mathrm{B}_r(x) \cap H^+_{t,\nu}} |u - y| \, \mathrm{d}w = 0 = \lim_{r \searrow 0} \fint_{\mathrm{B}_r(x,\nu) \cap H^-_{t,\nu}} |u - z| \, \mathrm{d}w \,.$$

The triplet (y, z, ν) can be identified with $(z, y, -\nu)$ and is uniquely determined up to this relation. We denote this triplet by $(u^+(x), u^-(x), \nu_u(x))$.

Definition 2.15 (Decomposition of derivatives of BV functions). For any $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, we denote by $\mathrm{D}^a u$ the absolutely continuous part of $\mathrm{D}u$ with respect to $\mathcal{L}^n \llcorner \Omega$. This is called the **absolutely continuous part of *u***. In some cases, we identify the measure $\mathrm{D}^a u$ with its density with respect to the n-dimensional Lebesgue measure. Moreover, we write $\mathrm{D}^s u$ for the singular part of $\mathrm{D}u$ with respect to $\mathcal{L}^n \llcorner \Omega$, which is called **singular part of *u***. The measure $\mathrm{D}^j u := \mathrm{D}^s u \llcorner \mathrm{J}_u$ is called **jump part of *u***, where J_u denotes the set of approximate jump points of u. Furthermore, $\mathrm{D}^c u := \mathrm{D}^s u \llcorner \Omega \backslash \mathrm{J}_u$ is the **Cantor part of *u***. By $\tilde{\mathrm{D}} u$, we denote the **diffusive part $\mathrm{D}^a u + \mathrm{D}^c u$ of *u***.

Theorem 2.16 (Decomposition of derivatives of BV functions). *Let $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Then $\mathrm{D}^a u$, $\mathrm{D}^c u$ and $\mathrm{D}^j u$ are mutually singular measures and it holds*

$$\mathrm{D}u = \mathrm{D}^a u + \mathrm{D}^s u = \mathrm{D}^a u + \mathrm{D}^j u + \mathrm{D}^c u \,.$$

Moreover, $\mathrm{D}u \ll \mathcal{H}^{n-1}$ holds on Ω. For all σ-finite sets $\Gamma \subseteq \Omega$ with respect to $\mathcal{H}^{n-1}$, it holds $|\mathrm{D}u|(\Gamma) = |\mathrm{D}^j u|(\Gamma)$. There exists an $\mathcal{L}^n$-negligible Borel set $C \in \mathcal{B}(\Omega)$ with $\mathrm{D}^c u = \mathrm{D}u \llcorner C$.

Definition 2.17 (Approximate limits). Let $u : \Omega \to \mathbb{R}$ be a measurable scalar function. Then

$$u^+(x) := \sup\{t \in \mathbb{R} : x \in \{u > t\}^+\} \,, \quad x \in \Omega$$

is called **approximate upper limit of *u***. The **approximate lower limit of *u*** is therefore defined

$$u^-(x) := \sup\{t \in \mathbb{R} : x \in \{u > t\}^1\} \,, \quad x \in \Omega \,.$$

Unlike most references, we write u^+ and u^- instead of $u^\vee$ and $u^\wedge$ for the approximate upper and lower limit, respectively. Since the approximate upper and lower limits correspond $\mathcal{H}^{n-1}$-a.e. with the approximate jumps, this does not lead to ambiguity in notation.

Remark 2.18. For sets $A \subseteq \mathbb{R}^n$ of finite perimeter, it holds

$$(\mathbb{1}_A)^+ = \mathbb{1}_{A^+} \qquad \text{and} \qquad (\mathbb{1}_A)^- = \mathbb{1}_{A^1} \,.$$

Lemma 2.19. *For $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, the pointwise representatives u^+, u^-, as well as the mean-value and the Lebesgue representative of u, coincide $\mathcal{L}^n$-a.e. on Ω. If u is in $\mathrm{BV}(\Omega)$, then the functions even coincide $\mathcal{H}^{n-1}$-a.e. on $\Omega \backslash \mathrm{J}_u$.*

Proposition 2.20 (Chain rule for BV functions). *Let $u \in \mathrm{BV}(\Omega)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ a Lipschitz function. If either $|\Omega| < \infty$ or $\varphi(0) = 0$, then $\varphi \circ u \in \mathrm{BV}(\Omega)$ with*

$$\mathrm{D}(\varphi \circ u) = \varphi'(u^*) \tilde{\mathrm{D}} u + (\varphi(u^+) - \varphi(u^-)) \nu_u \mathcal{H}^{n-1} \llcorner \mathrm{J}_u \,,$$

where u^* is the mean-value representative of u. If $\varphi : \mathbb{R}^N \to \mathbb{R}^m$, $m \in \mathbb{N}$, is a C^1 Lipschitz function and $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, then $\varphi \circ u$ is in $\mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ with

$$\mathrm{D}(\varphi \circ u) = (\mathrm{D}\varphi \circ u^*) \cdot \tilde{\mathrm{D}} u + (\varphi(u^+) - \varphi(u^-)) \cdot \nu_u^T \mathcal{H}^{n-1} \llcorner \mathrm{J}_u \,.$$

The following lemma is a simplified version of [3, Remark 3.93].

Lemma 2.21 (Derivatives of BV functions on level sets). *If $u \in \mathrm{BV}(\Omega, \mathbb{R}^n)$, then $|\mathrm{D}u|$ vanishes on the set $\{u^* = 0\} \backslash \mathrm{J}_u$, where u^* denotes the mean-value representative of u.*

2.2.2 Single-variable BV functions

As mentioned in the introduction, the analysis of the Schwarz inequality simplifies in comparison to the lower dimensional Steiner symmetrizations, due to the fact that the symmetrized set $A^\star$ is a body of revolution. Hence, it can be represented solely by the radius function of A. Moreover, equality in the Schwarz perimeter inequality enforces A to be a stack of balls, i.e., $\mathcal{L}^1$-a.e. slice of A is an $(n-1)$-dimensional ball with center defined by the center function of A. It turns out that both of these functions are single-variable BV functions. As such, they have good representatives, which simplifies the analysis as they have one-sided limits at each point.

In this section, we discuss properties of single-variable BV functions.

Definition 2.22 (Pointwise variation)**.** Let $I = (a,b)$ with $a < b$ in $[-\infty,\infty]$, and let $u : I \to \mathbb{R}^N$. The **pointwise variation** of u in I is defined as

$$\mathrm{pVar}(u,I) := \sup\left\{\sum_{i=1}^{k-1} |u(t_{i+1}) - u(t_i)| \,:\, k \in \mathbb{N}_{\geqslant 2},\ a < t_1 < t_2 < \cdots < t_k < b\right\}.$$

For an open set $\Omega \subseteq \mathbb{R}$, the **pointwise variation** of u in I is defined as

$$\mathrm{pVar}(u,\Omega) := \sum_{k\in J} \mathrm{pVar}(u,\Omega_k)\,,$$

where $(\Omega_k)_{k\in J}$ is the family of connected components of Ω. For $\Omega \in \mathcal{B}(\mathbb{R})$, we define

$$\mathrm{pVar}(u,\Omega) = \inf\{\mathrm{pVar}(u,U) : U \text{ open with } \Omega \subseteq U\}\,.$$

Lemma 2.23 (Pointwise variation and variation)**.** *Let I be an open set in $\mathbb{R}$. For all $u \in \mathrm{L}^1_{\mathrm{loc}}(I,\mathbb{R}^N)$ it holds*

$$\mathrm{Var}(u,I) \leqslant \mathrm{pVar}(u,I)\,.$$

Theorem 2.24 (Good representatives)**.** *Let $I \subseteq \mathbb{R}$ be the open interval (a,b) for $a < b \in [-\infty,\infty]$ and let $u \in \mathrm{BV}(I,\mathbb{R}^N)$.*

i) *There exists a representative $\tilde{u}$ of u such that*

$$\mathrm{pVar}(\tilde{u},I) = \mathrm{Var}_\Omega(u,I) = |\mathrm{D}u|(I)\,.$$

ii) *Any such representative $\tilde{u}$ is a (pointwise) convex combination from u^ℓ and u^r, where*

$$u^\ell(t) := c + \mathrm{D}u((a,t)) \quad \textit{and} \quad u^r(t) := c + \mathrm{D}u((a,t])$$

for all $t \in I$ and for a certain constant $c \in \mathbb{R}^N$ (essentially $c = \tilde{u}(a+)$).

iii) *It holds that $\mathrm{J}_u = \{t \in I : |\mathrm{D}u|(\{t\}) > 0\}$, and each representative $\tilde{u}$ is continuous on $I \backslash \mathrm{J}_u$. Moreover,*

$$\tilde{u}(t-) = u^\ell(t) \in \{u^+(t), u^-(t)\} \quad \textit{and} \quad \tilde{u}(t+) = u^r(t) \in \{u^+(t), u^-(t)\} \qquad \textit{for all } t \in I\,.$$

In particular, u^ℓ is left continuous and u^r is right continuous and, for $N = 1$, it holds

$$u^+(t) = \max\{u^\ell(t), u^r(t)\} \quad \textit{and} \quad u^-(t) = \min\{u^\ell(t), u^r(t)\} \qquad \textit{for all } t \in I\,.$$

In this case, u^+ is upper semicontinuous and u^- is lower semicontinuous.

iv) *For arbitrary $N \in \mathbb{N}$, the jump set J_u is at most countable and for each $t \in \mathrm{J}_u$, we have*

$$|\mathrm{D}u|(\{t\}) = |\mathrm{D}^j u|(\{t\}) = |u^\ell(t) - u^r(t)| = |u^+(t) - u^-(t)|\,.$$

Definition 2.25 (Good representative)**.** For I and u as in Theorem 2.24, we call $\tilde{u}$ a **good representative of u**. More specifically, we call u^ℓ the **left-continuous representative of u** and u^r the **right-continuous representative of u**.

Remark 2.26. According to the previous theorem, good representatives of single-variable BV functions are regulated functions. As such, we can approximate these functions uniformly by pure jump functions on compact intervals.

Lemma 2.27 (Product rule)**.** *Let I be an open interval and let u, v be two functions in $\mathrm{BV}(I) \cap \mathrm{L}^\infty(I)$. Then the product $u\,v$ is in $\mathrm{BV}(I)$ with $\mathrm{D}(u\,v) = u^\ell \mathrm{D}v + v^r \mathrm{D}u$.*

A similar version of this result (for general bounded functions in $\mathrm{BV}(\Omega)$) can be found, for instance, in [3, Example 3.97]. For the convenience of the reader, we carry out the proof for this form of the statement.

Proof. In order to apply the chain rule for BV functions, we first record that $t \mapsto (u(t), v(t))$ is a function in $\mathrm{BV}(I, \mathbb{R}^2)$ with derivative measure $(\mathrm{D}u, \mathrm{D}v)$. Now, the function $\varphi : [-R, R]^2 \to \mathbb{R}$ defined by $\varphi(x) = x_1\, x_2$ is a C^1 Lipschitz function with Lipschitz constant $\sqrt{2}R$ for all $R > 0$. Setting $R := \max\left\{\|u\|_{\mathrm{L}^\infty(I)}, \|v\|_{\mathrm{L}^\infty(I)}\right\}$, we can apply Proposition 2.20 to obtain $\varphi \circ (u, v) = u\,v \in \mathrm{BV}_{\mathrm{loc}}(I)$ with

$$\mathrm{D}(u\,v) = (\mathrm{D}\varphi)((u^*, v^*)) \cdot \tilde{\mathrm{D}}(u, v) + \mathrm{D}^j(u\,v) = v^* \tilde{\mathrm{D}}u + u^* \tilde{\mathrm{D}}v + \sum_{j \in \mathrm{J}_{uv}} (u\,v)^r(j) - (u\,v)^\ell(j)\,, \tag{2.2}$$

where u^* and v^* denote the mean-value representative of u and v, respectively. We observe $\mathrm{J}_{uv} \subseteq \mathrm{J}_u \cup \mathrm{J}_v$ and, if we consider good representatives for u and v, we obtain

$$\begin{aligned}(u\,v)^r(j) - (u\,v)^\ell(j) &= u(j+)v(j+) - u(j-)v(j-)\\ &= u(j+)v(j+) - u(j-)v(j+) + u(j-)v(j+) - u(j-)v(j-)\\ &= v^r(j)\mathrm{D}u(\{j\}) + u^\ell(j)\mathrm{D}v(\{j\})\end{aligned} \tag{2.3}$$

for all $j \in \mathrm{J}_{uv}$, where we exploited that $u\,v$ is continuous on $I \backslash (\mathrm{J}_u \cup \mathrm{J}_v) \subseteq I \backslash \mathrm{J}_{uv}$ and therefore coincides with all good representatives on this set. Combining (2.2), (2.3) and the equalities $u^* = u^\ell$ and $v^* = v^r$ on $I \backslash (\mathrm{J}_u \cup \mathrm{J}_v)$, we conclude $\mathrm{D}(u\,v) = u^\ell \mathrm{D}v + v^r \mathrm{D}u$ on I with

$$|\mathrm{D}(u\,v)|(I) \leqslant \|u\|_{\mathrm{L}^\infty(I)} |\mathrm{D}v|(I) + \|v\|_{\mathrm{L}^\infty(I)} |\mathrm{D}u|(I) < \infty\,.$$

Together with the fact that $u\,v$ is in $\mathrm{L}^\infty(I)$, we obtain $u\,v \in \mathrm{BV}(I)$, which proves the claim. □

Theorem 2.28 (Decomposition of single-variable BV functions)**.** *Let $I = (a, b)$ for $a < b$ in $\mathbb{R}$. For all $u \in \mathrm{BV}(I, \mathbb{R}^N)$, there exist $u^a, u^c, u^j \in \mathrm{BV}(I, \mathbb{R}^N)$, uniquely defined up to addition of a constant, such that the following properties hold true*:

i) *$u = u^a + u^j + u^c$ holds a.e. on I,*

ii) *$u^a \in \mathrm{W}^{1,1}(I, \mathbb{R}^N)$ is an absolutely continuous function with density $\mathrm{D}^a u$,*

iii) *u^j is a pure jump function, that is, $\mathrm{D}^j u^j = \mathrm{D}u^j$ with jumps J_u and jump height $\mathrm{D}u(\{j\})$ for all $j \in \mathrm{J}_u$,*

iv) *u^c is a Cantor function, that is, a BV function with $\mathrm{D}^c u^c = \mathrm{D}u^c$ and $\mathrm{D}u^c = \mathrm{D}^c u$.*

In fact, it holds

$$\mathrm{D}u = \mathrm{D}^a u + \mathrm{D}^j u + \mathrm{D}^c u = \mathrm{D}u^a + \mathrm{D}u^j + \mathrm{D}u^c \qquad \text{on } I\,.$$

Remark 2.29. In the situation of Theorem 2.28, we additionally assume in the following that $\mathrm{D}^j u = 0$ on I. Then there exists a decomposition of u into the functions u^a, u^c and $u^j \equiv 0$ that satisfy the properties of Theorem 2.28.
Indeed, considering any decomposition of u as in the Theorem 2.28, we find $\mathrm{D}u^j \equiv 0$ and therefore $u^j \equiv c$ in I for some constant $c \in \mathbb{R}^N$ according to Lemma 2.13. Since the decomposition is invariant under the addition of corresponding constants to u^j, u^c and u^a, we can choose $u^j_{\mathrm{new}} \equiv 0$, $u^c_{\mathrm{new}} := u^c$ and $u^a_{\mathrm{new}} := c + u^a$, which satisfy the properties of Theorem 2.28.
Similarly, if $\mathrm{D}^c u = 0$ or $\mathrm{D}^a u = 0$ holds on I, we can choose a decomposition of u with $u^c \equiv 0$ and $u^a \equiv 0$, respectively.

Lemma 2.30 (Positive decomposition)**.** *Let $I = (a, b)$ for $a < b$ in $\mathbb{R}$ and let u be a function in $\mathrm{BV}(I) \cap \mathrm{C}^0(I)$ that satisfies $u > \varepsilon$ on I for some $\varepsilon > 0$. For all $t \in I$ there exists a neighborhood $I_t^\delta := (t-\delta, t+\delta) \subseteq I$ with $\delta > 0$ and a decomposition of u in I_t^δ into $u^j \equiv 0$ and positive and continuous functions u^a and u^c that satisfy the properties from Theorem 2.28 on I_t^δ.*

Proof. In the following, we consider a good representative of u. By Theorem 2.28, we find for all $t \in I$ some $\delta > 0$ such that I_t^δ is contained in I and such that $|\mathrm{D}^a u|(I_t^\delta) < \frac{\varepsilon}{2}$ and $|\mathrm{D}^c u|(I_t^\delta) < \frac{\varepsilon}{2}$. Furthermore, there exists a decomposition $u = u^a + u^c$ on I_t^δ for continuous functions u^a and u^c with

$$u^a(s) = c + \mathrm{D}^a u((t-\delta, s)) \quad \text{and} \quad u^c(s) = \hat{c} + \mathrm{D}^c u((t-\delta, s)) \quad \text{for all } s \in I_t^\delta \quad \text{and} \quad c + \hat{c} = u((t-\delta)+)\,,$$

according to Theorems 2.24 and 2.28 and Remark 2.29. Since $u((t-\delta)+) \geqslant \varepsilon$ by the assumptions on u, we can set $c = \hat{c} = \frac{u((t-\delta)+)}{2}$ and still consider all properties of Theorem 2.28 on I_t^δ, including

$$u^a(s) \geqslant \frac{\varepsilon}{2} - |\mathrm{D}^a u|((t-\delta, s)) > 0 \qquad \text{and} \qquad u^c(s) \geqslant \frac{\varepsilon}{2} - |\mathrm{D}^c u|((t-\delta, s)) > 0 \qquad \text{for all } s \in I_t^\delta\,,$$

where we exploited the initial choice of δ. □

It is clear that we need fixed representatives in order to take limits in the classical sense. Since we will show that any representative of the center function is BV, we need a suitable notion for limit in this case.

Definition 2.31 (Almost everywhere (one-sided) limit)**.** Let $I \subseteq \mathbb{R}$ be open and let $u : I \to \mathbb{R}^N$ be measurable. The **a.e. taken limit of u** at $t \in I$ is $x \in \mathbb{R}^N$ if there exists a negligible subset N of I such that

$$\lim_{I \setminus N \ni s \to t} u(s) = x\,.$$

In this case, we say that the a.e. taken limit exists and write $u_{\text{a.e.}}(t) = x$.
The **a.e. taken right** or **left limit of u** is x if the same property as above holds true with the additional assumption that s is strictly greater or smaller than t, respectively. We will write $u_{\text{a.e.}}(t\pm) = x$ for the a.e. taken right and left limit.

Remark 2.32. Almost everywhere (one-sided) limits are well-defined for measurable functions, which are identified under null set changes.
Moreover, the existence of an a.e. taken limit of a function u at t is equivalent to the following: For all $\varepsilon > 0$, there exists some $\delta > 0$ and a negligible set $N \subseteq I$ such that $|u(s) - u(s')| < \varepsilon$ for all $s, s' \in (t-\delta, t+\delta) \setminus N$.

In the following, we collect some properties of a.e. taken limits, which are needed in Section 3.2.1.

Lemma 2.33 (Properties of a.e. taken limits)**.** *Let $u : I \to \mathbb{R}^N$ be a measurable function on the open set $I \subseteq \mathbb{R}$, and let $t \in I$.*

i) *Both $u_{a.e.}(t-)$ and $u_{a.e.}(t+)$ exist and equal $x \in \mathbb{R}^N$ if and only if $u_{a.e.}(t) = x$.*

ii) *If $u_{a.e.}(t)$ exists in $\mathbb{R}^N$ for some $t \in I$, then there exists $\delta > 0$ with $u \in \mathrm{L}^\infty((t-\delta, t+\delta), \mathbb{R}^N)$.*

iii) *For $u \in \mathrm{L}^1_{\mathrm{loc}}(I)$, the following holds true for all t in I and $x \in \mathbb{R}^N$:*

$$u_{a.e.}(t) = x \quad \textit{if and only if} \quad \lim_{I \ni s \to t} u^*(s) = u^*(t) = x\,,$$

where u^* denotes the mean-value representative of u. In fact, the backwards implication applies for all representatives of u. The same equivalence holds true for the a.e. taken one-sided limits.

iv) *If both $u_{a.e.}(t-)$ and $u_{a.e.}(t+)$ exist for some $t \in I$, then it holds*

$$\lim_{r \searrow 0} \fint_{t-r}^{t+r} u(s)\,\mathrm{d}s = \frac{1}{2}(u_{a.e.}(t-) + u_{a.e.}(t+))\,.$$

Proof. Showing i) is a straightforward application of the fact that the one-sided (standard) limits of a function f exist at t with the same value if and only if the (standard) limit of f at t exists.
Property ii) follows from Remark 2.32.
We now prove the first implication of iii) and assume $u_{\text{a.e.}}(t) = x$. In fact, for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $|u(s) - x| < \varepsilon$ for a.e. $s \in (t-\delta, t+\delta)$. Hence, applying the imposed property on the mean-value representative u^* of u, we obtain

$$|u^*(t) - x| = \left| \lim_{r \searrow 0} \fint_{t-r}^{t+r} u \,\mathrm{d}s - x \right| \leqslant \lim_{r \searrow 0} \fint_{t-r}^{t+r} |u - x| \,\mathrm{d}s < \varepsilon \,.$$

Similarly, it holds

$$\lim_{s \to t} |u^*(s) - x| \leqslant \lim_{s \to t} \lim_{r \searrow 0} \fint_{s-r}^{s+r} |u - x| \,\mathrm{d}s < \varepsilon \,.$$

The other implication follows from the definition.
We turn to the proof of iv) and assume that $u_{\text{a.e.}}(t\pm)$ exist. As in Remark 2.32, there exists for every $\varepsilon \in (0, \operatorname{dist}(\partial I, t))$ some $\delta > 0$ such that

$$|u(s) - u_{\text{a.e.}}(t-)| < \varepsilon \qquad \text{for a.e. } s \in (t-\delta, t) \,.$$

Hence, for $r \in (0, \delta)$, we obtain

$$\left| \frac{1}{r} \int_{t-r}^{t} u(s) \,\mathrm{d}s - u_{\text{a.e.}}(t-) \right| \leqslant \frac{1}{r} \int_{t-r}^{t} |u(s) - u_{\text{a.e.}}(t-)| \,\mathrm{d}s \leqslant \varepsilon \,,$$

such that $\lim_{r \searrow 0} \frac{1}{r} \int_{t-r}^{t} u(s) \,\mathrm{d}s = u_{\text{a.e.}}(t-)$ follows. The analogous estimate holds true for $u_{\text{a.e.}}(t+)$. Therefore, we conclude

$$\lim_{r \searrow 0} \fint_{t-r}^{t+r} u(s) \,\mathrm{d}s = \lim_{r \searrow 0} \frac{1}{2r} \left(\int_{t-r}^{t} u(s) \,\mathrm{d}s + \int_{t}^{t+r} u(s) \,\mathrm{d}s \right) = \frac{1}{2}(u_{\text{a.e.}}(t-) + u_{\text{a.e.}}(t+)) \,. \qquad \square$$

For one-dimensional restrictions of BV functions, we can also obtain suitable properties, compare [3, Theorem 3.107, Theorem 3.108, Remark 3.109].

Definition 2.34 (One-dimensional restrictions)**.** Let $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$, $\nu \in \partial \mathrm{B}_1$ and $x \in \mathbb{R}^n$. By $\Omega_{\nu,x}$ we denote the set $\{t \in \mathbb{R} : x + t\nu \in \Omega\}$, and by Ω_ν we denote the orthogonal projection of Ω onto $\nu^\perp$. We define the **(ν, x)-restriction of u** as

$$u_{\nu,x} : \Omega_{\nu,x} \to \mathbb{R}^N,\, t \mapsto u(x + t\nu) \,.$$

Proposition 2.35 (Properties of one-dimensional restrictions)**.** *Let $u \in \mathrm{BV}(\Omega)$ and $\nu \in \partial \mathrm{B}_1$. Then $u_{\nu,x}$ is in $\mathrm{BV}(\Omega_{\nu,x})$ for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$ and the pointwise defined function $(\tilde{u})_{\nu,y}$ is a good representative of $u_{\nu,x}$, where $\tilde{u} := \frac{u^+ + u^-}{2}$. Moreover, it holds*

$$((\mathrm{D}u) \cdot \nu)(\Omega) = \int_{\Omega_\nu} \mathrm{D}u_{\nu,x}(\Omega_{\nu,x}) \,\mathrm{d}\mathcal{H}^{n-1}(x)$$

and the same formula remains true when replacing $\mathrm{D}u$ *with the absolutely continuous parts, jump or Cantor parts of u and $u_{\nu,x}$. Furthermore, it holds that $(u_{\nu,x})^\pm(t) \in \{u^-(x,t), u^+(x,t)\}$ for all $t \in \Omega_{\nu,x}$ and for a.e. $x \in \Omega_\nu$.*

2.2.3 Sets of finite perimeter

The perimeter of a set is a measure for its boundary, even if it contains pathological parts.

Definition 2.36 (Perimeter)**.** Let $A \in \mathcal{M}^n$. The **perimeter** of A in Ω is defined as

$$\mathrm{P}(A, \Omega) := \operatorname{Var}(\mathbb{1}_A, \Omega) \,.$$

If $\mathrm{P}(A,\Omega) < \infty$, the set A is called a **set of finite perimeter in Ω**. If $\Omega = \mathbb{R}^n$, we just say that A is a **set of finite perimeter** and write $\mathrm{P}(A) = \mathrm{P}(A,\mathbb{R}^n)$. The set A is called a set of **locally finite perimeter (in Ω)** if $\mathrm{P}(A,K) < \infty$ for every open and bounded set K in $\mathbb{R}^n$ or Ω, respectively. If B is a general (not necessarily open) Borel set in $\mathbb{R}^n$, and A has finite perimeter in an open neighborhood of B, we define

$$\mathrm{P}(A,B) := |\mathrm{D}\mathbb{1}_A|(B)\,.$$

Remark 2.37. i) The perimeter is well-defined with respect to the choice of representative of A. In particular, if $A, F \in \mathcal{M}^n$ and $|(A \triangle F) \cap \Omega| = 0$, then $\mathrm{P}(A,\Omega) = \mathrm{P}(F,\Omega)$.

ii) The previous analysis of the variation applies to the perimeter according to the definition. In particular, the perimeter is lower semicontinuous with respect to $\mathrm{L}^1_{\mathrm{loc}}$ convergence.

iii) The operator $\mathrm{P}(A,\,\cdot\,)$ is a non-negative Borel measure on $(\Omega,\mathcal{B}(\Omega))$ for all sets A of locally finite perimeter in Ω.

iv) For Lipschitz sets $A \subseteq \mathbb{R}^n$, we have $\mathrm{P}(A,\Omega) = \mathcal{H}^{n-1}(\partial A \cap \Omega)$.

v) If $\mathcal{H}^{n-1}(\partial A) < \infty$ for a measurable set $A \subseteq \mathbb{R}^n$, then $\mathrm{P}(A) < \infty$.

vi) The perimeter is invariant under translation and rotation (in both arguments).

Lemma 2.38 (Properties of the perimeter)**.** *Let $A, F \in \mathcal{M}^n$ be sets of finite perimeter in Ω. Then the following hold true*:

i) $\mathrm{P}(A,\Omega) = \mathrm{P}(\mathbb{R}^n \backslash A,\Omega)$,

ii) *$A \cap F$ and $A \cup F$ are sets of finite perimeter in Ω with*

$$\mathrm{P}(A \cup F,\Omega) + \mathrm{P}(A \cap F,\Omega) \leqslant \mathrm{P}(A,\Omega) + \mathrm{P}(F,\Omega)\,.$$

Lemma 2.39 (Sets of finite perimeter in $\mathbb{R}$)**.** *A set $I \subseteq \mathbb{R}$ is of finite perimeter if and only if there exist $k \in \mathbb{N}_0$ and $t_1 < t_2 < \cdots < t_{2k}$ in $[-\infty,\infty]$ such that $I = \bigcup_{i=1}^k (t_{2i-1},t_{2i})$ except for a null set. In this case, it holds $\partial^* I = \{t_1,\ldots,t_{2k}\} \backslash \{-\infty,\infty\}$.*

Approximation theorems allow for the transfer of results for more regular sets to sets of finite perimeter, as we will see, for instance, in the proof of Proposition 3.71.

Theorem 2.40 (Smooth approximation)**.** *For every bounded set $A \subseteq \mathbb{R}^n$ of finite perimeter, there exists a sequence $(A_k)_{k\in\mathbb{N}}$ of bounded, open and smooth sets in $\mathbb{R}^n$ such that A_k converges to A in L^1 as $k \to \infty$, $\mathrm{P}(A_k,B)$ converges to $\mathrm{P}(A,B)$ for all $B \in \mathcal{B}(\mathbb{R}^n)$ with $\mathrm{P}(A,\partial B) = 0$, and ∂A_k converges to ∂A with respect to the Hausdorff distance as for some representative of A as $k \to \infty$.*

Theorem 2.41 (Coarea formula)**.** *Let $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ with $\mathrm{Var}(u,\Omega) < \infty$. For $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, the super level sets $\{u > t\}$ are sets of finite perimeter and it holds that*

$$|\mathrm{D}u|(B) = \int_{\mathbb{R}} \mathrm{P}(\{u > t\},B)\,\mathrm{d}t \qquad \textit{and} \qquad \mathrm{D}u(B) = \int_{\mathbb{R}} \mathrm{D}\mathbb{1}_{\{u>t\}}(B)\,\mathrm{d}t$$

for all $B \in \mathcal{B}(\Omega)$.

As mentioned in the introduction, the isoperimetric inequality can be proven via symmetrization techniques, which we will introduce and analyze in Section 3.1.

Theorem 2.42 (Isoperimetric inequality)**.** *Let A be a set of finite perimeter. Then either $|\mathbb{R}^n \backslash A| < \infty$ or $|A| < \infty$. In the latter case, it holds that*

$$(n^n \omega_n)^{\frac{1}{n-1}} |A| \leqslant \mathrm{P}(A)^{\frac{n}{n-1}}\,.$$

Equality in the previous inequality is equivalent to $|A \triangle \mathrm{B}| = 0$ for some ball $\mathrm{B} \subseteq \mathbb{R}^n$.

Definition 2.43 (Reduced boundary)**.** For a set $A \in \mathcal{M}^n$, the (weak outer) unit normal is defined by

$$\nu_A(x) := -\lim_{r \searrow 0} \frac{\mathrm{D}\mathbb{1}_A(\mathrm{B}_r(x))}{\mathrm{P}(A, \mathrm{B}_r(x))}$$

for all $x \in \mathbb{R}^n$ such that the right-hand side is defined. Moreover, the **reduced boundary** of A is defined as

$$\partial^* A := \{x \in \Omega_A : |\nu_A(x)| = 1\},$$

where Ω_A denotes the largest open set such that A has locally finite perimeter in Ω_A.

Remark 2.44. i) The weak unit normal is a Borel function on Ω_A, and $\partial^* A$ is a Borel set.

ii) If A is Lipschitz, then $\mathcal{H}^{n-1}(\partial^* A \triangle \partial A)$ vanishes.

iii) The reduced boundary is a subset of the topological boundary. In fact, $\partial^* A$ is invariant under null set changes of A. However, we can always find the best possible representative A^1 for the topological boundary of a set A of locally finite perimeter in the following sense: for all representatives $\tilde{A}$ of A, it holds $\overline{\partial^* A} = \partial A^1 \subseteq \partial \tilde{A}$.

The following structure theorems of De Giorgi and Federer are important for the analysis of sets of finite perimeter.

Theorem 2.45 (De Giorgi's structure theorem)**.** *For every set $A \in \mathcal{M}^n$ of locally finite perimeter in Ω, the reduced boundary of A is countably $(n-1)$-rectifiable and $\lim_{r \searrow 0} \frac{\mathcal{H}^{n-1}(\partial^* A \cap \mathrm{B}_r(x))}{|\mathrm{B}_r^{n-1}|} = 1$ for all $x \in \partial^* A$. Moreover, it holds*

$$\mathrm{P}(A, \cdot\,) = \mathcal{H}^{n-1} \llcorner \partial^* A \qquad \text{and} \qquad \mathrm{D}\mathbb{1}_A = -\nu_A \mathcal{H}^{n-1} \llcorner \partial^* A \qquad \text{on } \mathcal{B}(\Omega)\,.$$

Theorem 2.46 (Divergence theorem)**.** *For a set $A \in \mathcal{M}^n$ of locally finite perimeter in Ω, it holds*

$$\int_A \operatorname{div} \varphi \, \mathrm{d}x = \int_{\partial^* A} \varphi \cdot \nu_A \, \mathrm{d}\mathcal{H}^{n-1}$$

for all $\varphi \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^n) \cap \mathrm{C}^0_{\mathrm{cpt}}(\Omega, \mathbb{R}^n)$.

Theorem 2.47 (Federer's structure theorem)**.** *For every set $A \subseteq \mathbb{R}^n$ of locally finite perimeter in Ω, it holds*

$$\Omega \cap \partial^* A = \Omega \cap A^{\frac{1}{2}} = \Omega \cap \partial^{\mathrm{e}} A$$

up to $\mathcal{H}^{n-1}$-null sets.

Remark 2.48. For a set $A \subseteq \mathbb{R}^n$ of finite perimeter, we have

$$\mathrm{P}(A, H_t) = 0 \qquad \text{and} \qquad \mathcal{H}^{n-1}(\partial^{\mathrm{e}} A \cap H_t) = 0$$

for all but countably many $t \in \mathbb{R}$. Indeed, otherwise, the perimeter would sum up to infinity by Theorem 2.47.

Lemma 2.49 (Perimeter of unions and intersections)**.** *For sets A, F in $\mathcal{M}^n$ of finite perimeter, it holds*

$$\partial^*(A \cup F) = \big(\partial^* A \cap F^0\big) \cup \big(\partial^* F \cap A^0\big) \cup \{\nu_A = \nu_F\} \quad \text{and} \quad \partial^*(A \cap F) = \big(\partial^* A \cap F^1\big) \cup \big(\partial^* F \cap A^1\big) \cup \{\nu_A = \nu_F\}$$

up to $\mathcal{H}^{n-1}$-null sets, where $\{\nu_A = \nu_F\} := \{x \in \partial^* A \cap \partial^* F : \nu_A(x) = \nu_F(x)\}$. In fact, we have $\nu_{A \cap F} = \nu_A = \nu_F$ on $\{\nu_A = \nu_F\}$,

$$\mathrm{P}(A \cup F, B) = \mathrm{P}(A, F^0 \cap B) + \mathrm{P}(F, A^0 \cap B) + \mathcal{H}^{n-1}(\{\nu_A = \nu_F\} \cap B)\,,$$

and

$$\mathrm{P}(A \cap F, B) = \mathrm{P}(A, F^1 \cap B) + \mathrm{P}(F, A^1 \cap B) + \mathcal{H}^{n-1}(\{\nu_A = \nu_F\} \cap B)$$

for all $B \in \mathcal{B}(\mathbb{R}^n)$. In the special case $F \subseteq A$, it holds $\nu_F = \nu_A$ on $\partial^* F \cap \partial^* A$.

The perimeter decreases under intersection with convex sets and in some cases, this might be even strict. We will use this fact to find unique minimizers of perimeter functionals. The proof relies on [62, Exercise 15.13, Exercise 15.14, Proposition 19.22].

Lemma 2.50 (Perimeter of convex sets). *Let C be a convex set in $\mathbb{R}^n$ and let $A \subseteq \mathbb{R}^n$ a set of finite perimeter and finite volume. Then it holds*

$$\mathrm{P}(C \cap A) \leqslant \mathrm{P}(A)\,. \tag{2.4}$$

If, moreover, $\mathrm{P}(A, \mathbb{R}^n \backslash C) > 0$, then the inequality in (2.4) is strict.

Proof. We first prove (2.4) as a strict inequality in the special case, where C is the half-space $H := \{x \in \mathbb{R}^n : \nu \cdot x > t\}$ with generating vector $\nu \in \partial \mathrm{B}_1$ and constant $t \in \mathbb{R}$. According to Remark 2.44 ii), ∂H coincides with $\partial^* H$, and $\nu = -\nu_H = \nu_{\mathbb{R}^n \backslash H}$ on ∂H.
Now we show $\mathrm{P}(A \cap H, \partial H) < \mathrm{P}(A \cap H, H)$ for all sets $A \subseteq \mathbb{R}^n$ of finite perimeter and finite volume with $\mathrm{P}(A, H) > 0$, and therefore $|A \cap H| > 0$.
To this end, for $r > 0$, let φ be an auxiliary function in $\mathrm{C}^\infty_{\mathrm{cpt}}([0, \infty), [0, 1])$ with $\varphi \equiv 1$ on $[0, r]$ and $\varphi \equiv 0$ on $[r + 1, \infty)$, such that $|\varphi'| \leqslant 2$ on $[r, r + 1]$ and $\varphi' = 0$ on $[0, r] \cup [r + 1, \infty]$. Then, the function

$$\Psi : \mathbb{R}^n \to \overline{\mathrm{B}}_1,\, x \mapsto \varphi(|x|)\nu$$

is C^∞ with support in B_{r+1}. By construction, $\operatorname{div} \Psi(x)$ vanishes at $x = 0$, and equals $\varphi'(|x|)\frac{x \cdot \nu}{|x|}$ elsewhere. Applying Theorem 2.46 provides

$$\begin{aligned}
\int_{A \cap H} \varphi'(|x|)\frac{x \cdot \nu}{|x|}\,\mathrm{d}x &= \int_{A \cap H} \operatorname{div} \Psi \,\mathrm{d}x \\
&= \int_{\partial^*(A \cap H)} \Psi \cdot \nu_{A \cap H}\,\mathrm{d}\mathcal{H}^{n-1}(x) \\
&= \int_{\partial^*(A \cap H) \cap H} \Psi \cdot \nu_{A \cap H}\,\mathrm{d}\mathcal{H}^{n-1}(x) + \int_{\partial^*(A \cap H) \cap \partial H} \Psi \cdot \nu_{A \cap H}\,\mathrm{d}\mathcal{H}^{n-1}(x)\,.
\end{aligned}$$

According to Lemma 2.49 and Theorem 2.47, we have $\partial^*(A \cap H) \cap \partial H = \{\nu_{A \cap H} = \nu_H\}$ with $\nu_{A \cap H} = \nu_H = \nu_A$ on this set. Thus, we obtain

$$\int_{\partial^*(A \cap H) \cap \partial H} \varphi(|x|)\,\mathrm{d}\mathcal{H}^{n-1}(x) + \int_{A \cap H} \varphi'(|x|)\frac{x \cdot \nu}{|x|}\,\mathrm{d}x = \int_{\partial^*(A \cap H) \cap H} \Psi \cdot \nu_{A \cap H}\,\mathrm{d}\mathcal{H}^{n-1}(x)\,.$$

Applying the assumptions on φ, we derive

$$\begin{aligned}
\mathcal{H}^{n-1}(\partial^*(A \cap H) \cap \partial H \cap \mathrm{B}_r) \leqslant \int_{\partial^*(A \cap H) \cap H \cap \mathrm{B}_r} \nu \cdot \nu_{A \cap H}\,\mathrm{d}\mathcal{H}^{n-1}(x) &+ \mathcal{H}^{n-1}(\partial^*(A \cap H) \cap (\mathrm{B}_{r+1} \backslash \mathrm{B}_r)) \\
&+ 2|A \cap H \cap (\mathrm{B}_{r+1} \backslash \mathrm{B}_r)|\,.
\end{aligned}$$

Exploiting the fact that A has finite perimeter and finite volume, we take the limit of both sides as $r \to \infty$ and obtain

$$\mathcal{H}^{n-1}(\partial^*(A \cap H) \cap \partial H) \leqslant \int_{\partial^*(A \cap H) \cap H} \nu \cdot \nu_{A \cap H}\,\mathrm{d}\mathcal{H}^{n-1}(x)\,. \tag{2.5}$$

The claim follows from the observation $\nu \neq \nu_{A \cap H}$ on some set $\Gamma \subseteq \partial^*(A \cap H) \cap H$ with positive $\mathcal{H}^{n-1}$-measure. Indeed, if this is not the case, we can w.l.o.g. assume $\nu = e_n$ and obtain $\mathrm{D}\mathbb{1}_{A \cap H} = e_n \mathcal{H}^{n-1} \llcorner A \cap H$. Hence, Lemma 2.13 implies that $\mathbb{1}_{A \cap H}(x)$ is only depending on the value of x_n for a.e. $x \in \mathbb{R}^{n-1}$. By Fubini's theorem and $|A \cap H| \in (0, \infty)$, there exists some set $I \in \mathcal{M}^1$ with $|I| > 0$ and $\mathbb{1}_{A \cap H}(x) = 1$ for all $x \in \mathbb{R}^{n-1} \times I$, which therefore implies the contradiction $|A \cap H| = \infty$ by Fubini again. Thus, $\nu \cdot \nu_{A \cap H} < 1$ on a set $\Gamma \subseteq \partial^*(A \cap H) \cap H$ of positive $\mathcal{H}^{n-1}$-measure, and (2.5) turns into

$$\mathrm{P}(A \cap H, \partial H) = \mathcal{H}^{n-1}(\partial^*(A \cap H) \cap \partial H) < \mathcal{H}^{n-1}(\partial^*(A \cap H) \cap H) = \mathrm{P}(A \cap H, H)\,.$$

Hence, we showed the claimed estimate. Now, Lemma 2.49 implies in this case

$$\mathrm{P}(H, A^1) + \mathcal{H}^{n-1}(\{\nu_A = \nu_H\}) = \mathrm{P}(A \cap H, \partial H) < \mathrm{P}(A \cap H, H) = \mathrm{P}(A, H)\,. \tag{2.6}$$

In the following, we will apply the shown inequality for the half space $\mathbb{R}^n \backslash \overline{H}$. In fact, this and Lemma 2.49 once again implies

$$\begin{aligned}
\mathrm{P}(A \cap H) &= \mathrm{P}(A, H) + \mathrm{P}(H, A^1) + \mathcal{H}^{n-1}(\{\nu_{A\cap H} = \nu_H\}) \\
&= \mathrm{P}(A, H) + \mathrm{P}(\mathbb{R}^n \backslash H, A^1) + \mathcal{H}^{n-1}(\{\nu_A = \nu_H\}) \\
&< \mathrm{P}(A, H) + \mathrm{P}(A, \mathbb{R}^n \backslash H) \\
&= \mathrm{P}(A)
\end{aligned}$$

for sets A chosen as in the theorem. Note that (2.4) for $C = H$ follows from (2.5) for sets A of finite perimeter and finite volume with $\mathrm{P}(A, \mathbb{R}^n \backslash H) = 0$ analogously.
We now turn to the case of general convex sets C. Since $|\partial C| = 0$, we may assume w.l.o.g. that C is closed. According to [12, Proposition 7.5.6 (a)], C is a countable intersection of closed half-spaces $\{H_1, H_2, \dots\}$. In particular, $H(k) := \bigcap_{i=1}^k H_i$ converges to C in $\mathrm{L}^1_{\mathrm{loc}}$ such that the lower semicontinuity of the perimeter along with the first part of the proof yield

$$\mathrm{P}(A \cap C) \leqslant \liminf_{k\to\infty} \mathrm{P}(A \cap H(k)) \leqslant \mathrm{P}(A)\,.$$

If we additionally assume $\mathrm{P}(A, \mathbb{R}^n \backslash C) > 0$, we find $i \in \mathbb{N}$ such that $\mathrm{P}(A, \mathbb{R}^n \backslash H_i) > 0$ since, otherwise, $\mathrm{P}(A, \mathbb{R}^n \backslash C) = \lim_{k\to\infty} \mathrm{P}(A, \bigcup_{i=1}^k \mathbb{R}^n \backslash H_i) = 0$. Hence, we even deduce from the first part of the proof that

$$\mathrm{P}(A \cap H(k)) \leqslant \mathrm{P}(A \cap H_i) < \mathrm{P}(A)$$

for all $k \geqslant i$ in $\mathbb{N}$, which concludes the claim to the full extent. □

The perimeter coincides with the area functional, making it possible to transfer results from parametric to non-parametric settings and vice versa.

Definition 2.51 (Area functional)**.** The **area functional on Ω** is defined as the operator

$$\mathcal{A}^{\mathrm{area}}_\Omega : \mathrm{BV}(\Omega) \to [0, \infty],\ u \mapsto |(\mathcal{L}^n, \mathrm{D}u)|(\Omega)\,.$$

In Section 3.1.2, we will introduce the area functional for surfaces of revolution, and show similar properties as in the following proposition, which can be found, for instance, in [67, Teorema 1.10] and [47, Theorem 14.6].

Proposition 2.52 (Perimeter-area-functional relation)**.** *Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$. The subgraph $\mathrm{Sub}\,u$ of u is a set of finite perimeter in $K \times \mathbb{R}$ for all compact sets $K \subseteq \Omega$ if and only if $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. In this case, it holds*

$$\mathrm{P}(\mathrm{Sub}\,u, B \times \mathbb{R}) = \mathcal{A}^{\mathrm{area}}_B u = \sup\left\{ \int_B u \,\mathrm{div}\,(\overline{\varphi}) + \varphi_{n+1}\,\mathrm{d}x : \varphi \in \mathrm{C}^1_{\mathrm{cpt}}(B, \mathbb{R}^{n+1}),\ \sup_B |\varphi| \leqslant 1 \right\}$$

for all open sets $B \subseteq \Omega$.

Throughout the argumentation in Chapter 3, we will need the following variant of the previous theorem. The proof is based on the proof of [47, Theorem 14.6].

Proposition 2.53 (Perimeter-area-functional relation on the positive half space)**.** *Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, [0, \infty))$. It holds*

$$\mathrm{P}(\mathrm{Sub}\,u, \Omega \times (0, \infty)) = \sup\left\{ \int_\Omega u \,\mathrm{div}\,(\overline{\varphi}) + \mathbb{1}_{\{u>0\}}\varphi_{n+1}\,\mathrm{d}x : \varphi \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega, \mathbb{R}^{n+1}),\ \sup_\Omega |\varphi| \leqslant 1 \right\}.$$

The subgraph $\mathrm{Sub}\,u$ of u is a set of finite perimeter in $\Omega \times (0, \infty)$ if and only if u is a $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ function with $|\mathrm{D}u|(\Omega) + |\{u > 0\}| < \infty$. In this case, it holds

$$\mathrm{P}(\mathrm{Sub}\,u, \Omega \times (0, \infty)) = \big|(\mathbb{1}_{\{u>0\}}\mathcal{L}^n, \mathrm{D}u)\big|(\Omega)\,.$$

Remark 2.54. If $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega, [0,\infty))$ with $|\mathrm{D}u|(\Omega) + |\{u>0\}| < \infty$, then it holds

$$\mathrm{P}(\mathrm{Sub}\,u, B \times (0,\infty)) = \big|(\mathcal{L}^n, \mathrm{D}u)\big|(B \cap \{u^+ > 0\}) = \big|(\mathcal{L}^n, \mathrm{D}u)\big|(B \cap \{u^- > 0\}) \qquad \text{for all } B \in \mathcal{B}(\mathbb{R}^n \backslash \mathrm{J}_u)\,.$$

Indeed, if we define the set $L := \{u^+ = 0\} \backslash \mathrm{J}_u$, Proposition 2.53 (transferred to general Borel sets in Ω) and Lemma 2.21 imply

$$\begin{aligned}
\mathrm{P}(\mathrm{Sub}\,u, B \times (0,\infty)) &= |(\mathbb{1}_{\{u^+>0\}}\mathcal{L}^n, \mathrm{D}u)|(B) \\
&\leqslant |(\mathcal{L}^n, \mathrm{D}u)|(B \backslash L) + |(\mathbb{1}_{\{u^+>0\}}\mathcal{L}^n, \mathrm{D}u)|(L) \\
&= |(\mathcal{L}^n, \mathrm{D}u)|(B \cap \{u^+>0\}) + |\mathrm{D}u|(L) \\
&= |(\mathcal{L}^n, \mathrm{D}u)|(B \cap \{u^+>0\}) \\
&= \mathrm{P}(\mathrm{Sub}\,u, (B \cap \{u^+>0\}) \times (0,\infty))\,.
\end{aligned}$$

The claim follows from $\mathrm{P}(\mathrm{Sub}\,u, (B \cap \{u^+ > 0\}) \times (0,\infty)) \leqslant \mathrm{P}(\mathrm{Sub}\,u, B \times (0,\infty))$ and by observing that $\mathcal{H}^{n-1}\big((\{u^+>0\} \triangle \{u^->0\}) \backslash \mathrm{J}_u\big) = 0$ holds by Lemma 2.19, while taking into account Theorem 2.16.

Proof of Proposition 2.53. We first treat the special case of u being a bounded function and consider $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega, \mathbb{R}^{n+1})$ with $\sup_\Omega |\varphi| \leqslant 1$. For $\varepsilon > 0$ we choose $\eta_\varepsilon \in \mathrm{C}^1_{\mathrm{cpt}}((0,\infty),[0,1])$ with $\eta_\varepsilon = 1$ on $(\varepsilon, \sup_\Omega u)$ and $|\eta_\varepsilon'| \leqslant \frac{2}{\varepsilon}$ on $(0,\infty)$. We then define the vector field $\Phi(x,t) := \varphi(x)\eta_\varepsilon(t)$ for $(x,t) \in \Omega \times (0,\infty)$, which has compact support in $\Omega \times (0,\infty)$ and satisfies $|\Phi| \leqslant 1$. We conclude

$$\mathrm{P}(\mathrm{Sub}\,u, \Omega \times (0,\infty)) \geqslant \int_{\Omega \times (0,\infty) \cap \mathrm{Sub}\,u} \mathrm{div}\,\Phi \,\mathrm{d}z = \int_\Omega \mathrm{div}\,\overline{\varphi}(x) \int_0^{u(x)} \eta_\varepsilon(t)\,\mathrm{d}t\,\mathrm{d}x + \int_\Omega \varphi_{n+1}(x) \int_0^{u(x)} \eta_\varepsilon'(t)\,\mathrm{d}t\,\mathrm{d}x\,.$$

We record that $\int_0^{u(x)} \eta_\varepsilon(t)\,\mathrm{d}t$ converges to $u(x)$ in $\mathrm{L}^\infty(\Omega)$ and that $\int_0^{u(x)} \eta_\varepsilon'(t)\,\mathrm{d}t = 1$ if $u(x) > \varepsilon$. If $u(x) \in (0,\varepsilon]$, the construction of η_ε gives $\int_0^{u(x)} |\eta_\varepsilon'(t)|\,\mathrm{d}t \leqslant 2$. In particular, it holds $\int_0^{u(x)} \eta_\varepsilon'(t)\,\mathrm{d}t \to \mathbb{1}_{\{u>0\}}(x)$ as $\varepsilon \searrow 0$ for a.e. $x \in \Omega$. The dominated convergence theorem yields

$$\mathrm{P}(\mathrm{Sub}\,u, \Omega \times (0,\infty)) \geqslant \int_\Omega \mathrm{div}\,\overline{\varphi}\,u + \varphi_{n+1}\mathbb{1}_{\{u>0\}}\,\mathrm{d}x\,.$$

Now, if $\mathrm{P}(\mathrm{Sub}\,u, \Omega \times (0,\infty))$ is finite, we conclude

$$\begin{aligned}
\infty &> \mathrm{P}(\mathrm{Sub}\,u, \Omega \times (0,\infty)) \\
&\geqslant \sup\left\{ \int_\Omega u\,\mathrm{div}\,(\overline{\varphi}) + \mathbb{1}_{\{u>0\}}\varphi_{n+1}\,\mathrm{d}x : \varphi \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega, \mathbb{R}^{n+1}),\ \sup_\Omega |\varphi| \leqslant 1 \right\} \\
&\geqslant \max\{|\{u>0\}|, \mathrm{Var}(u,\Omega)\}\ ,
\end{aligned}$$

which implies the claimed properties of u by Proposition 2.5. Furthermore, the equality

$$\sup\left\{ \int_\Omega u\,\mathrm{div}\,(\overline{\varphi}) + \mathbb{1}_{\{u>0\}}\varphi_{n+1}\,\mathrm{d}x : \varphi \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega, \mathbb{R}^{n+1}),\ \sup_\Omega |\varphi| \leqslant 1 \right\} = \big|(\mathbb{1}_{\{u>0\}}\mathcal{L}^n, \mathrm{D}u)\big|(\Omega) \tag{2.7}$$

follows from the definition of vector-valued Radon measures, the Riesz-Markov representation theorem and the density of $\mathrm{C}^1_{\mathrm{cpt}}(\Omega)$ in $\mathrm{C}^0_{\mathrm{cpt}}(\Omega)$. We hence conclude $\mathrm{P}(\mathrm{Sub}\,u, \Omega \times (0,\infty)) \geqslant \big|(\mathbb{1}_{\{u>0\}}\mathcal{L}^n, \mathrm{D}u)\big|(\Omega)$.
In order to show the other inequality, we assume w.l.o.g. that Ω is bounded and that

$$\sup\left\{ \int_\Omega u\,\mathrm{div}\,(\overline{\varphi}) + \mathbb{1}_{\{u>0\}}\varphi_{n+1}\,\mathrm{d}x : \varphi \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega, \mathbb{R}^{n+1}),\ \sup_\Omega |\varphi| \leqslant 1 \right\} < \infty\,.$$

As before, we find $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ with $|\mathrm{D}u|(\Omega) < \infty$ and $|\{u>0\}| < \infty$, as well as the equality (2.7). Then we consider the mollifications u_ε in $\mathrm{C}^\infty([\Omega]_\varepsilon)$ for all $\varepsilon > 0$ which, according to Lemma 2.10, converge to u strictly in $\mathrm{BV}(\Omega)$ as $\varepsilon \searrow 0$. As $\partial^*\mathrm{Sub}\,u_\varepsilon = \mathrm{Graph}\,u_\varepsilon$ is an $(n-1)$-dimensional C^∞ submanifold in $\mathbb{R}^{n+1}$, one finds

$$\begin{aligned}
\mathrm{P}(\mathrm{Sub}\,u_\varepsilon, K \times (\delta,\infty)) &= \mathcal{H}^{n-1}(\mathrm{Graph}\,u_\varepsilon \cap (K \times (\delta,\infty))) \\
&= |(\mathbb{1}_{\{u_\varepsilon>\delta\}}\mathcal{L}^n, \mathrm{D}u_\varepsilon)|(K) \\
&= \sup\left\{ \int_K u_\varepsilon\,\mathrm{div}\,(\overline{\varphi}) + \mathbb{1}_{\{u_\varepsilon>\varepsilon\}}\varphi_{n+1}\,\mathrm{d}x : \varphi \in \mathrm{C}^1_{\mathrm{cpt}}(K, \mathbb{R}^{n+1}),\ \sup_K |\varphi| \leqslant 1 \right\}
\end{aligned}$$

for all $\delta > 0$, $K \Subset \Omega$, and for all $\varepsilon > 0$ such that $K \subseteq [\Omega]_\varepsilon$. Since $\mathrm{Sub}\,u_\varepsilon \to \mathrm{Sub}\,u$ in $\mathrm{L}^1_{\mathrm{loc}}$ can be concluded from the corresponding $\mathrm{L}^1_{\mathrm{loc}}$ convergence of u_ε to u, the lower semicontinuity of the perimeter implies

$$\begin{aligned} \mathrm{P}(\mathrm{Sub}\,u, \Omega \times (0,\infty)) &= \lim_{\delta \searrow 0} \mathrm{P}(\mathrm{Sub}\,u, \Omega \times (\delta,\infty)) \\ &\leqslant \liminf_{\delta \searrow 0} \liminf_{\varepsilon \searrow 0} \mathrm{P}(\mathrm{Sub}\,u_\varepsilon, [\Omega]_\varepsilon \times (\delta,\infty)) \\ &= \liminf_{\delta \searrow 0} \liminf_{\varepsilon \searrow 0} \big|(\mathbb{1}_{\{u_\varepsilon > \delta\}}\mathcal{L}^n, \mathrm{D}u_\varepsilon)\big|([\Omega]_\varepsilon)\,. \end{aligned} \tag{2.8}$$

As $\{u_\varepsilon > \delta\}$ is open for all $\varepsilon, \delta > 0$, we may use Lemma 2.10 in order to estimate

$$\begin{aligned} \big|(\mathbb{1}_{\{u_\varepsilon > \delta\}}\mathcal{L}^n, \mathrm{D}u_\varepsilon)\big|([\Omega]_\varepsilon) &= \big|(\mathcal{L}^n, \mathrm{D}u_\varepsilon)\big|([\Omega]_\varepsilon \cap \{u_\varepsilon > \delta\}) + |\mathrm{D}u_\varepsilon|([\Omega]_\varepsilon \backslash \{u_\varepsilon > \delta\}) \\ &= \big|(\mathcal{L}^n, \mathrm{D}u)_\varepsilon\big|([\Omega]_\varepsilon \cap \{u_\varepsilon > \delta\}) + |\mathrm{D}u_\varepsilon|([\Omega]_\varepsilon \backslash \{u_\varepsilon > \delta\}) \\ &\leqslant \big|(\mathcal{L}^n, \mathrm{D}u)\big|(\Omega \cap \{u_\varepsilon > \delta\}) + \big|(\mathcal{L}^n, \mathrm{D}u)\big|\big(\Omega \backslash [\Omega]_\varepsilon\big) \\ &\quad + |\mathrm{D}u|(\Omega \backslash \{u_\varepsilon > \delta\}) + |\mathrm{D}u|\big(\Omega \backslash [\Omega]_\varepsilon\big) \\ &= \big|(\mathbb{1}_{\{u_\varepsilon > \delta\}}\mathcal{L}^n, \mathrm{D}u)\big|(\Omega) + \big|(\mathcal{L}^n, \mathrm{D}u)\big|(\Omega \backslash [\Omega]_\varepsilon) + |\mathrm{D}u|(\Omega \backslash [\Omega]_\varepsilon)\,. \end{aligned} \tag{2.9}$$

Since Ω is a bounded set, it holds $|(\mathcal{L}^n, \mathrm{D}u)|(\Omega) + |\mathrm{D}u|(\Omega) < \infty$, hence

$$\lim_{\varepsilon \searrow 0} \big|(\mathcal{L}^n, \mathrm{D}u)\big|(\Omega \backslash [\Omega]_\varepsilon) + |\mathrm{D}u|(\Omega \backslash [\Omega]_\varepsilon) = 0 \tag{2.10}$$

follows. For each null sequence $(\varepsilon_k)_{k \in \mathbb{N}}$, there exists $k_0 \in \mathbb{N}$ such that for all $k \in \mathbb{N}_{\geqslant k_0}$, it holds

$$\big|\{u_{\varepsilon_k} > \delta\} \backslash \{u > 0\}\big| < \delta\,,$$

since $u_{\varepsilon_k} \to u$ in $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ as $k \to \infty$ and $|\Omega| < \infty$. Hence,

$$\begin{aligned} \big|(\mathbb{1}_{\{u_{\varepsilon_k} > \delta\}}\mathcal{L}^n, \mathrm{D}u)\big|(\Omega) &\leqslant \big|(\mathbb{1}_{\{u > 0\} \cap \{u_{\varepsilon_k} > \delta\}}\mathcal{L}^n, \mathrm{D}u)\big|(\Omega) + \big|(\mathbb{1}_{\{u_{\varepsilon_k} > \delta\} \backslash \{u > 0\}}\mathcal{L}^n, 0)\big|(\Omega) \\ &\leqslant \big|(\mathbb{1}_{\{u > 0\}}\mathcal{L}^n, \mathrm{D}u)\big|(\Omega) + \delta \end{aligned} \tag{2.11}$$

follows by the triangle inequality for all $k \in \mathbb{N}_{\geqslant k_0}$. Together, (2.8), (2.9), (2.10), and (2.11) yield

$$\mathrm{P}(\mathrm{Sub}\,u, \Omega \times (0,\infty)) \leqslant |(\mathbb{1}_{\{u > 0\}}\mathcal{L}^n, \mathrm{D}u)|(\Omega) + \delta\,.$$

Taking into account the arbitrariness of δ, we established $\mathrm{P}(\mathrm{Sub}\,u, \Omega \times (0,\infty)) \leqslant |(\mathbb{1}_{\{u > 0\}}\mathcal{L}^n, \mathrm{D}u)|(\Omega)$.
If u is unbounded, we can define the cut-off function $u_k := \min\{u, k\}$ for all $k \in \mathbb{N}$ and repeat the argument for all u_k. We directly record $u_k \to u$ and $\mathrm{Sub}\,u_k \to \mathrm{Sub}\,u$ in $\mathrm{L}^1_{\mathrm{loc}}$ as $k \to \infty$ and $\{u > 0\} = \{u_k > 0\}$. Furthermore, once finding $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ with $|\mathrm{D}u|(\Omega) < \infty$, it follows $|\mathrm{D}(u - u_k)|(\Omega) \to 0$ as $k \to \infty$. With this, we conclude the claim in full generality. □

2.2.4 Traces

Traces are an important tool to handle the behavior of BV functions on lower dimensional sets.

Proposition 2.55 (Trace operator existence). *If $\Omega \subseteq \mathbb{R}^n$ is an open set with bounded Lipschitz boundary, there exists a bounded linear and strictly continuous operator*

$$\mathrm{Tr}_\Omega : \mathrm{BV}(\Omega, \mathbb{R}^N) \to \mathrm{L}^1(\partial\Omega, \mathbb{R}^N; \mathcal{H}^{n-1})$$

such that for all $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$, it holds

$$\frac{1}{|\mathrm{B}_r|} \int_{\Omega \cap \mathrm{B}_r(x)} |u - \mathrm{Tr}_\Omega u(x)|\,\mathrm{d}y \to 0 \quad \text{as } r \searrow 0 \tag{2.12}$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$.

Remark 2.56. If Ω is an open Lipschitz set with possibly unbounded boundary and $u \in \mathrm{BV}(B, \mathbb{R}^N)$ for all bounded open sets $B \subseteq \Omega$, we can still find a function $\mathrm{Tr}_\Omega u$ such that (2.12) holds for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$. This is because for every $x \in \partial\Omega$, we can find some bounded Lipschitz set Ω_x with $\Omega_x \cap N = \Omega \cap N$ for some neighborhood N of x. In this case, $\mathrm{Tr}_\Omega u$ is in the space $\mathrm{L}^1_{\mathrm{loc}}(\partial\Omega, \mathbb{R}^N; \mathcal{H}^{n-1})$.

Definition 2.57 (Trace operator)**.** For Ω as in Proposition 2.55 or Remark 2.56, we call Tr_Ω the **(inner) trace operator** on Ω. For a function u which is in $\mathrm{BV}(B, \mathbb{R}^N)$ for all bounded and open sets $B \subseteq \Omega$, we call $\mathrm{Tr}_\Omega u$ the **(inner) trace** of u in Ω. For sets $A \subseteq \mathbb{R}^n$ of finite perimeter and finite volume in Ω, we abbreviate $\mathrm{Tr}_\Omega \mathbb{1}_A$ by $\mathrm{Tr}_\Omega A$.

Remark 2.58. i) Traces of BV functions coincide with traces of $\mathrm{W}^{1,1}$ functions and therefore, they also coincide with restrictions of continuous functions on the corresponding boundary.

ii) Let Ω be an open set with bounded Lipschitz boundary and let F, A be subsets of $\mathbb{R}^n$ of finite perimeter and finite volume in Ω that satisfy $F \subseteq A$. Then $\mathrm{Tr}_\Omega F \leqslant \mathrm{Tr}_\Omega A$ on $\partial\Omega$. Indeed, since $\mathcal{H}^{n-1}(\Omega^{\frac{1}{2}} \triangle \partial\Omega)$, it holds

$$\mathrm{Tr}_\Omega A(x) = 2 \lim_{r \searrow 0} \frac{1}{|\mathrm{B}_r|} \int_{\mathrm{B}_r(x) \cap \Omega} \mathbb{1}_A(y)\, \mathrm{d}y \geqslant 2 \lim_{r \searrow 0} \frac{1}{|\mathrm{B}_r|} \int_{\mathrm{B}_r(x) \cap \Omega} \mathbb{1}_F(y)\, \mathrm{d}y = \mathrm{Tr}_\Omega F(x)\,.$$

Proposition 2.59 (Trace coincides with jumps)**.** *For Ω as in Proposition 2.55, it holds*

$$|\mathrm{D}u|(\partial\Omega) = \int_{\partial\Omega} |\mathrm{Tr}_\Omega u - \mathrm{Tr}_{\mathbb{R}^n \setminus \Omega} u|\, \mathrm{d}\mathcal{H}^{n-1}(x) \qquad \text{and} \qquad \mathrm{P}(A, \partial\Omega) = \int_{\partial\Omega} |\mathrm{Tr}_\Omega \mathbb{1}_A - \mathrm{Tr}_{\mathbb{R}^n \setminus \overline{\Omega}} \mathbb{1}_A|\, \mathrm{d}\mathcal{H}^{n-1}(x)$$

for all $u \in \mathrm{BV}(\mathbb{R}^n, \mathbb{R}^N)$ and sets of finite perimeter and finite volume $A \subseteq \mathbb{R}^n$.

The surjectivity of the trace operator is a well-known fact due to [42, Teorema 1.II]. The following theorem can be found in this form in [47, Proposition 2.15].

Theorem 2.60 (Trace operator surjectivity)**.** *Let $\varphi \in \mathrm{L}^1(H_t; \mathcal{H}^{n-1})$ such that there exists a representative of φ with compact support. Then there exists for every $\delta > 0$ a Sobolev function $u \in \mathrm{W}^{1,1}(H_t^+)$ satisfying the properties*

i) $\mathrm{Tr}_{H_t^+} u = \varphi\,,$

ii) $\int_{H_t^+} |u|\, \mathrm{d}x \leqslant \delta \int_{H_t} |\varphi|\, \mathrm{d}\mathcal{H}^{n-1}\,,$

iii) $\int_{H_t^+} |\mathrm{D}u| \mathrm{d}x \leqslant (1+\delta) \int_{H_t} |\varphi|\, \mathrm{d}\mathcal{H}^{n-1}\,.$

We will need a special version of surjectivity for perimeters with small volume as follows.

Proposition 2.61 (Surjectivity of the trace operator)**.** *Let $t \in \mathbb{R}$ and T^+, T^- be two $\mathcal{H}^{n-1}$-finite and bounded sets in $\mathcal{B}(H_t)$. For every $\varepsilon > 0$ there exists a set $A \in \mathcal{M}^n$ of finite perimeter and volume in $(0, \varepsilon)$ such that*

i) $\mathrm{Tr}_{H_t^+} A = \mathbb{1}_{T^+}\,,\ \mathrm{Tr}_{H_t^-} A = \mathbb{1}_{T^-}\,,$

ii) $\mathrm{P}(A, H_t) = \mathcal{H}^{n-1}(T^+ \triangle T^-)\,,$

iii) $\mathrm{P}(A, H_t^+) < \mathcal{H}^{n-1}(T^+) + \varepsilon\,,\ \mathrm{P}(A, H_t^-) < \mathcal{H}^{n-1}(T^-) + \varepsilon\,,$

where H_t^+ and H_t^- are the open half spaces $\{x \in \mathbb{R}^n : x_n > 0\}$ and $\{x \in \mathbb{R}^n : x_n < 0\}$, respectively.

Proof. We first choose $\delta \in (0,1)$ small enough such that the estimates $\frac{1+\delta^2}{1-\delta}\mathcal{H}^{n-1}(T^+) < \mathcal{H}^{n-1}(T^+) + \varepsilon$ and $\delta\mathcal{H}^{n-1}(T^+) < \varepsilon$ are satisfied. By applying Theorem 2.60 to $\varphi = \mathbb{1}_{T^+}$, we find $u \in \mathrm{W}^{1,1}(H_t^+)$ with trace φ and

$$\int_{H_t^+} |u|\, \mathrm{d}x \leqslant \delta^2 \mathcal{H}^{n-1}(T^+) \qquad \text{and} \qquad \int_{H_t^+} |\mathrm{D}u| \mathrm{d}x \leqslant (1+\delta^2)\mathcal{H}^{n-1}(T^+)\,. \tag{2.13}$$

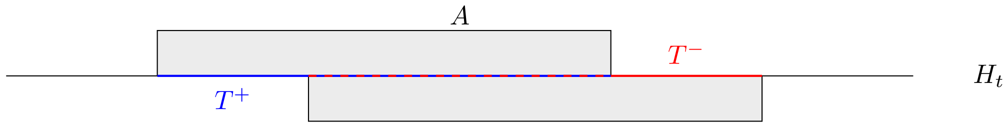


Figure 2.1: Surjectivity of the trace operator

In order to define the set A, we choose a suitable superlevel set of u. Applying the coarea formula from Theorem 2.41 and taking into account (2.13) yields

$$\int_\delta^1 \mathrm{P}(\{u>s\},H_t^+)\,\mathrm{d}s \leqslant \int_{\mathbb{R}} \mathrm{P}(\{u>s\},H_t^+)\,\mathrm{d}s = \int_{H_t^+} |\mathrm{D}u|\,\mathrm{d}x \leqslant (1+\delta^2)\,\mathcal{H}^{n-1}(T^+)\,.$$

Hence, there exists $s \in (\delta,1)$ such that $\mathrm{P}(\{u>s\},H_t^+) \leqslant \frac{1+\delta^2}{1-\delta}\,\mathcal{H}^{n-1}(T^+) < \mathcal{H}^{n-1}(T^+)+\varepsilon$. We now define the set

$$\tilde{A} = A^\varepsilon_{H_t^+}(T^+) := \{u>s\}\cap H_t^+$$

satisfying the corresponding perimeter estimate in iii). The first inequality of (2.13) gives

$$|\tilde{A}| = |\{u>s\}\cap H_t^+| \leqslant \frac{1}{s}\int_{H_t^+} |u|\,\mathrm{d}x \leqslant \frac{\delta^2}{s}\mathcal{H}^{n-1}(T^+) \leqslant \delta\mathcal{H}^{n-1}(T^+) < \varepsilon\,,$$

where the choice of s was exploited. We now show that A^+ has trace T^+ in H_t^+. Indeed, for $\mathcal{H}^{n-1}$-a.e. $x \in H_t$, it holds

$$\begin{aligned}\frac{1}{r^n}\int_U |\mathbb{1}_{\tilde{A}} - \mathbb{1}_{T^+}(x)|\,\mathrm{d}y &= \frac{1}{r^n}\int_{U\cap\{u>s\}} \mathbb{1}_{H_t^+\setminus T^+}(x)\,\mathrm{d}y + \frac{1}{r^n}\int_{U\setminus\{u>s\}} \mathbb{1}_{T^+}(x)\,\mathrm{d}y\\ &\leqslant \frac{1}{r^n}\int_U \frac{|u-\mathbb{1}_{T^+}(x)|}{s} + \frac{|u-\mathbb{1}_{T^+}(x)|}{1-s}\,\mathrm{d}y \to 0\end{aligned}$$

as $r \searrow 0$, where we abbreviated $U := \mathrm{B}_r(x)\cap H_t^+$ and exploited the fact that the trace of u on H_t^+ is T^+. We can now choose $A^\varepsilon_{H_t^-}(T^-)$ analogously and finally define the set of finite perimeter A by $A^{\frac{\varepsilon}{2}}_{H_t^+}(T^+)\cup A^{\frac{\varepsilon}{2}}_{H_t^-}(T^-)$. By construction, we conclude $|A|<\varepsilon$, as well as i) and iii). Finally, property ii) follows from i) and Proposition 2.59. $\square$

2.3 Measure theory

In this section, we give some specific results in measure theory. For further details in the general topic, we refer the reader to [3, 62]. We will first give a result from [14, Lemma 3.7], giving a characterization of a measure estimate.

Lemma 2.62. *Let $\mu,\nu \in \mathrm{RM}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$. For all $B \in \mathcal{B}(\mathbb{R}^n)$, it holds*

$$2|\mu|(B) \leqslant |\mu+\nu|(B) + |\mu-\nu|(B) \quad \textit{for all } B \in \mathcal{B}(\mathbb{R}^n)\,.$$

Equality in the previous inequality for all $B \in \mathcal{B}(\mathbb{R}^n)$ is equivalent to the existence of a Borel function $f : \mathbb{R}^n \to [-1,1]$ such that

$$\nu(B) = \int_B f\,\mathrm{d}\mu$$

is satisfied for all bounded sets $B \in \mathcal{B}(\mathbb{R}^n)$.

Remark 2.63. According to the regularity of Radon measures, this lemma remains true in the following localized version: Let B be open in $\mathbb{R}^n$ and $\mu,\nu \in \mathrm{RM}_{\mathrm{loc}}(B,\mathbb{R}^N)$. Then there exists some function $f : \mathbb{R}^n \to [-1,1]$ such that $\nu = f\mu$ holds on $\mathcal{B}(B)$ if and only if

$$2|\mu|(B') = |\mu+\nu|(B') + |\mu-\nu|(B') \quad \text{holds for all open sets } B' \subseteq B\,.$$

2.3.1 Disintegration

In Section 3.3, we will extend the Schwarz perimeter inequality to functionals of the form (1.2) with measures $\mu = \mu_+ - \mu_-$ that satisfy the isoperimetric conditions from Section 2.3.4 as well as the radial monotonicity assumptions of the form (1.3). The crucial step for this result is the characterization of such measures as generalized product measures, which is introduced in the following. Indeed, the radial monotonicity of μ implies that it is absolutely continuous with respect to the Hausdorff measure. This property enforces μ to act like the $(n-1)$-dimensional Hausdorff measure only on horizontal parts in $\mathbb{R}^n$, i.e., it vanishes on $(n-1)$-dimensional sets without horizontal parts. Via the disintegration theory, we can write μ as a generalized product of weighted Hausdorff measures and its pushforward $\pi_{n\#}\mu$ on $\mathbb{R}$. This fact decisively allows controlling μ on the sets $(A^\star)^1\triangle(A^1)^\star$ and $(A^\star)^+\triangle(A^+)^\star$, respectively, via the imposed isoperimetric conditions on μ. For more information about disintegration theory, we refer to [3, Chapter 2.5].

Definition 2.64 (μ-measurable functions)**.** Let μ be a non-negative Radon measure on Ω and $U \subseteq \mathbb{R}^N$ open. The map $f : \Omega \to \mathrm{RM}(U, \mathbb{R}^N)$ is ***μ*-measurable** if for every open set $V \subseteq U$, the function $\Omega \to \mathbb{R}^N$, $x \mapsto (f(x))(V)$ is μ-measurable.

Lemma 2.65 (μ-measurability of integrals)**.** *Let U, f and μ be as in Definition 2.64. If f is μ-measurable, then*

$$\Omega \to \mathbb{R}^N,\, x \mapsto \int_V g(x,y)\,\mathrm{d}(f(x))(y)$$

is μ-measurable for every bounded Borel function $g : \Omega \times V \to \mathbb{R}$.

Lemma 2.66 (Disintegration)**.** *Let μ be a non-negative Radon measure on $\mathbb{R}^n$ with $\mu(\mathbb{R}^{n-1}\times K) < \infty$ for all compact sets $K \subseteq \mathbb{R}$. For $\pi_{n\#}\mu$-a.e. $t \in \mathbb{R}$, there exists a Radon measure $\mu_{\mathrm{H},t}$ on $\mathbb{R}^{n-1}$ with $\mu_{\mathrm{H},t}(\mathbb{R}^{n-1}) = 1$ such that*

$$\mathbb{R} \to \mathrm{RM}(\mathbb{R}^{n-1}, [0,\infty]),\, t \mapsto \mu_{\mathrm{H},t}$$

is $\pi_{n\#}\mu$-measurable and such that

$$\mu(B) = \int_{\mathbb{R}} \mu_{\mathrm{H},t}(B_t)\,\mathrm{d}(\pi_{n\#}\mu)(t) \qquad \textit{holds true for all } B \in \mathcal{B}(\mathbb{R}^n)\,. \tag{2.14}$$

Furthermore, for every $f \in \mathrm{L}^1(\mathbb{R}^n;\mu)$ the following properties hold true:

i) *$f(\,\cdot\,,t)$ is in $\mathrm{L}^1(\mathbb{R}^{n-1};\mu_{\mathrm{H},t})$ for $\pi_{n\#}\mu$-a.e. $t \in \mathbb{R}$,*

ii) *$t \mapsto \int_{\mathbb{R}^{n-1}} f(x,t)\,\mathrm{d}\mu_{\mathrm{H},t}(x)$ is in $\mathrm{L}^1(\mathbb{R},\pi_{n\#}\mu)$,*

iii)
$$\int_{\mathbb{R}^n} f\,\mathrm{d}\mu(x) = \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} f(x,t)\,\mathrm{d}\mu_{\mathrm{H},t}(x)\,\mathrm{d}\pi_{n\#}\mu(t)\,. \tag{2.15}$$

Definition 2.67 (Disintegration)**.** Let μ be a non-negative Radon measure on $\mathbb{R}^n$ such that $\mu(\mathbb{R}^{n-1}\times K) < \infty$ for all compact sets $K \subseteq \mathbb{R}$. We say that μ can be **disintegrated** if (2.14) holds true. In this case, we write $\mu = \mu_{\mathrm{H},t} \otimes \pi_{n\#}\mu$.

Definition 2.68 (Vertical parts)**.** Let μ be a non-negative Radon measure on $\mathbb{R}^n$. The measure

$$\mu^{\mathrm{V}} := \sum_{k=1}^{\infty} \frac{1}{2^k}\frac{\mu_k}{\mu_k(\mathbb{R})}$$

is called the **vertical part of μ**, where $\mu_k = \pi_{n\#}(\mu\llcorner(-k,k)^n)$ is the pushforward of the restriction of μ onto $(-k,k)^n$, and $\frac{\mu_k}{\mu_k(\mathbb{R})}$ is identified with 0 if $\mu_k(\mathbb{R}) = 0$.

Remark 2.69. i) For μ as in the definition, the measures μ_k are finite measures for all $k \in \mathbb{N}$. Since $\mu^{\mathrm{V}}(\mathbb{R}) = \sum_{k=1}^{\infty}\frac{1}{2^k}\frac{\mu_k(\mathbb{R})}{\mu_k(\mathbb{R})} \leqslant \sum_{k=1}^{\infty}\frac{1}{2^k} = 1$, the measure μ^{V} is even a finite Radon measure.

ii) We could also define μ^{V} as $\sum_{k=1}^{\infty} t_k \frac{\mu_k}{\mu_k(\mathbb{R})}$ if $\mu_k(\mathbb{R}) > 0$ for any positive sequence $(t_k)_{k\in\mathbb{N}} \in \ell^1$. However, we are mainly interested in finding a suitable one-dimensional reference measure of the pushforward.

iii) In Section 3.3, we want to apply Lemma 2.66 for general Radon measures μ on $\mathbb{R}^n$ and for μ^{V} instead of $\pi_{n\#}\mu$.

To this end, we define the restrictions $\nu^R := \mu \llcorner \mathrm{B}_R^{n-1} \times \mathbb{R}$ for all $R > 0$. The probability measures $\nu_{\mathrm{H},t}^R$ from Lemma 2.66 exist for all $t \in \mathbb{R}\backslash N_R$, where N_R is a $\pi_{n\#}\nu^R$-null set. Setting $\nu_{\mathrm{H},t}^R = 0$ for all $t \in N_R$, we find

$$\mu \llcorner \mathrm{B}_R^{n-1} \times \mathbb{R} = \nu^R = \nu_{\mathrm{H},t}^R \otimes \pi_{n\#}\nu^R$$

for all $R > 0$.

In the following, we show $\nu_{\mathrm{H},t}^R \otimes \pi_{n\#}\nu^R \ll \tilde{\mu}_{\mathrm{H},t} \otimes \mu^{\mathrm{V}}$ for all $R > 0$, where $\tilde{\mu}_{\mathrm{H},t}$ is defined as the measure $\sum_{k=1}^{\infty} \nu_{\mathrm{H},t}^k \llcorner \left(\mathrm{B}_k^{n-1}\backslash \mathrm{B}_{k-1}^{n-1}\right)$ for all $t \in \mathbb{R}$. To this end, we argue by contradiction and assume that there exists a Borel set $B \in \mathcal{B}(\mathbb{R}^n)$ with

$$\tilde{\mu}_{\mathrm{H},t} \otimes \mu^{\mathrm{V}}(B) = 0 \qquad \text{and} \qquad \nu_{\mathrm{H},t}^R \otimes \pi_{n\#}\nu^R(B) > 0\,.$$

We find $k \in \mathbb{N}$ with $\nu_{\mathrm{H},t}^R \otimes \pi_{n\#}\nu^R(\tilde{B}) > 0$, where $\tilde{B} = \tilde{B}(k) := ((\mathrm{B}_k^{n-1}\backslash \mathrm{B}_{k-1}^{n-1}) \times \mathbb{R}) \cap B$. The definition of $\tilde{\mu}_{\mathrm{H},t}$ yields $\nu_{\mathrm{H},t}^k(\tilde{B}_t) = 0$ for μ^{V}-a.e. $t \in \mathbb{R}$. From $\pi_{n\#}\nu^s \ll \mu^{\mathrm{V}}$ for all $s > 0$ it follows that $\nu_{\mathrm{H},t}^k(\tilde{B}_t)$ vanishes for $\pi_{n\#}\nu^k$-a.e. $t \in \mathbb{R}$. We conclude

$$0 < \nu_{\mathrm{H},t}^R \otimes \pi_{n\#}\nu^R(\tilde{B}) = \nu^R(\tilde{B}) \leqslant \mu(\tilde{B}) = \nu^k(\tilde{B}) = \nu_{\mathrm{H},t}^k \otimes \pi_{n\#}\nu^k(\tilde{B}) = 0\,,$$

which is a contradiction. Finally, $\nu_{\mathrm{H},t}^R \otimes \pi_{n\#}\nu^R \ll \tilde{\mu}_{\mathrm{H},t} \otimes \mu^{\mathrm{V}}$ follows from the fact that the considered measures are Radon measures.

We showed $\mu \ll \tilde{\mu}_{\mathrm{H},t}\otimes\mu^{\mathrm{V}}$ such that we find a Radon-Nikodym density $f : \mathbb{R}^n \to \mathbb{R}$ with $\mu = f\, \tilde{\mu}_{\mathrm{H},t}\otimes\mu^{\mathrm{V}}$. Note that in this setting we cannot hope for the corresponding lower dimensional Radon measures $\mu_t^{\mathrm{H}} = f(\cdot,t)\, \tilde{\mu}_{\mathrm{H},t}$ to be probability measures.

iv) If we additionally assume local finiteness of $\pi_{n\#}\mu$ as in Lemma 2.66, we obtain $\mu^{\mathrm{V}} \ll \pi_{n\#}\mu \ll \mu^{\mathrm{V}}$, which gives a Radon-Nikodym density $f : \mathbb{R} \to \mathbb{R}$ with $\pi_{n\#}\mu = f\,\mu^{\mathrm{V}}$, leading to $\mu = \mu_{\mathrm{H},t} \otimes (f\,\mu^{\mathrm{V}}) = \mu_t^{\mathrm{H}} \otimes \mu^{\mathrm{V}}$ with $\mu_t^{\mathrm{H}} = f(t)\,\mu_{\mathrm{H},t}$ for μ^{V}-a.e. $t \in \mathbb{R}$.

Definition 2.70 (Disintegration into vertical and horizontal parts)**.** Let μ be a non-negative Radon measure on $\mathbb{R}^n$ and μ_t^{H} a non-negative Radon measure on $\mathbb{R}^{n-1}$ for all $t \in \mathbb{R}$. We say that μ can be **disintegrated into the horizontal part μ_t^{H} and the vertical part μ^{V}** if it holds

$$\mu(B) = \int_{\mathbb{R}} \mu_t^{\mathrm{H}}(B_t)\,\mathrm{d}\mu^{\mathrm{V}}(t) \qquad \text{holds true for all } B \in \mathcal{B}(\mathbb{R}^n)\,,$$

and in this case, we write $\mu = \mu_t^{\mathrm{H}} \otimes \mu^{\mathrm{V}}$.

2.3.2 Box and Hausdorff dimension

In Section 3.2.1, we need to estimate the Hausdorff dimension of a Cartesian product of certain sets, in order to show the BV regularity of the center function of Definition 3.9. Since the Hausdorff measure does not behave as simple as the Lebesgue measure when it comes to products, we introduce the box-counting dimension from [88] as an auxiliary tool to bound the Hausdorff dimension of such sets. The results of this section can be derived from [88, Corollary 2, Theorem 3, Proposition 4] and they are stated in the terminology of this thesis in [39, Chapter 3, Product formula 7.3].

Definition 2.71 (Box-counting dimension)**.** Let $A \in \mathcal{M}^n$ and

$$N_A(\delta) := \min\left\{ k \in \mathbb{N} :\ A \subseteq \bigcup_{i=1}^{k} \overline{\mathrm{B}}_\delta(x_i) \text{ for some } x_i \in \mathbb{R}^n,\ i = 1,\ldots,k \right\}$$

for $\delta > 0$. The **box-counting dimension of A** is the number

$$\dim_{\mathcal{B}}(A) := \limsup_{\delta \searrow 0} \left[-\frac{\log(N_A(\delta))}{\log(\delta)} \right] .$$

Definition 2.71 can be reformulated equivalently using closed cubes instead of balls, which inspires the name of this notion.

Lemma 2.72 (Box-counting dimension for smooth manifolds)**.** *Let $M \subseteq \mathbb{R}^n$ be a bounded smooth k-dimensional submanifold of $\mathbb{R}^n$ for some $k \in \{1, \dots, n\}$. Then $\dim_{\mathcal{B}}(M) = k$.*

Proposition 2.73 (Hausdorff dimension of products)**.** *For $A \in \mathcal{B}(\mathbb{R}^k)$ and $B \in \mathcal{B}(\mathbb{R}^{n-k})$ with $k \in \{1, \dots, n-1\}$, it holds*

$$\dim_{\mathcal{H}}(A \times B) \leqslant \dim_{\mathcal{B}}(A) + \dim_{\mathcal{H}}(B) .$$

2.3.3 Minkowski content

In the following, we introduce the Minkowski content, which will be used in Section 5.2.4 in order to estimate the variational mean curvature of $\mathrm{C}^{1,\alpha}$ sets. The statements of this section can be found in [3, Definition 2.100, Theorem 2.104, Theorem 2.106].

Definition 2.74 (Minkowski content)**.** Let $K \subseteq \mathbb{R}^n$ be a closed set and $k \in \{1, \dots, n-1\}$. If it exists, we call the limit

$$\mathrm{Mink}^k(K) := \lim_{\rho \searrow 0} \frac{|(K)_\rho|}{\omega_{n-k}\rho^{n-k}}$$

the ***k*-dimensional Minkowski content** of K.

Remark 2.75. In general, the box-counting dimension and the Minkowski content are connected via the equality $\dim_{\mathcal{B}} A = \limsup_{\delta \searrow 0} \left[-\frac{\log\left(\frac{|(A)_\delta|}{\delta^n}\right)}{\log(\delta)} \right]$ for all $A \in \mathcal{M}^n$, according to [39, Proposition 3.2].
Indeed, if A has finite and positive k-dimensional Minkowski content, then it holds

$$\dim_{\mathcal{B}} A = \limsup_{\delta \searrow 0} -\frac{\log\left(\frac{|(A)_\delta|}{\delta^{n-k}}\right)}{\log(\delta)} + \frac{\log(\delta^k)}{\log(\delta)} = k .$$

In good cases, the Minkowski content coincides with the k-dimensional Hausdorff measure, which makes it a useful tool to handle surface areas.

Theorem 2.76. *Let $K \subseteq \mathbb{R}^n$ be a countably rectifiable compact set and let $k \in \{1, \dots, n-1\}$. If there exists $\lambda > 0$ such that*

$$\mathcal{H}^k(K \cap \mathrm{B}_\rho(x)) \geqslant \lambda \rho^k \qquad \text{for all } x \in K,\ \rho \in (0,1) ,$$

then $\mathrm{Mink}^k(K) = \mathcal{H}^k(K)$.

Remark 2.77. With the theorem at hand, one can easily show that $\mathrm{Mink}^k(K) = \mathcal{H}^k(K)$ for a k-dimensional compact set K in $\mathbb{R}^n$, which can be locally represented as the graph of a Lipschitz function defined on an open subset of $\mathbb{R}^k$ and with values in $\mathbb{R}^{n-k}$. Indeed, according to the compactness of K, we find some uniform bound $C \geqslant 1$ and some $\rho_0 > 0$ such that for each $x \in K$, there exists a Lipschitz function f with Lipschitz constant in $[0, C]$ and which is the (rotated and shifted) graph representation of K near x on $\big[-\frac{1}{C\sqrt{n}}\rho_0, \frac{1}{C\sqrt{n}}\rho_0\big]^k$ with values in $\big[-\frac{1}{\sqrt{n}}\rho_0, \frac{1}{\sqrt{n}}\rho_0\big]^{n-k}$. For those x, the fact that Hausdorff measures do not increase under projection gives

$$\mathcal{H}^k(\mathrm{B}_\rho(x) \cap K) \geqslant \mathcal{H}^k\left(\mathrm{Graph} f \cap \left[-\frac{1}{\sqrt{n}}\rho_0, \frac{1}{\sqrt{n}}\rho_0\right]^n\right) \geqslant \mathcal{H}^k\left(\left[-\frac{1}{C\sqrt{n}}\rho_0, \frac{1}{C\sqrt{n}}\rho_0\right]^k\right) = \left(\frac{1}{C\sqrt{n}}\rho\right)^k$$

for all $\rho \in (0, \rho_0)$. Hence, for $\lambda = \left(\frac{1}{C\sqrt{n}}\right)^k \min\{\rho_0^k, 1\}$, the condition from Theorem 2.76 is satisfied.

2.3.4 Isoperimetric conditions

This section is based on [73], which considers isoperimetric conditions on non-negative Radon measures. These conditions, imposed on μ_- and μ_+, provide lower semicontinuity results with respect to the convergence in L^1 for (localized) functionals of the form

$$\mathcal{P}(A) := \mathrm{P}(A) + \mu_+(A^1) - \mu_-(A^+)\,, \qquad A \in \{F \in \mathcal{M}^n : \mathrm{P}(F) + |F| < \infty\}\,,$$

as well as existence results of corresponding geometric minimization problems. We use these results throughout the proof of the generalized Schwarz inequality for functionals of the form $\mathcal{P}$ in Section 3.3.
The functional $\mathcal{P}$ is a generalization of the prescribed mean curvature functional of Massari-type from [63, 64], see Section 4. In the variational sense, minimizers of $\mathcal{P}$ have generalized mean curvature measure $\mu := \mu_+ - \mu_-$. We will see that measures which satisfy the (small-volume) isoperimetric condition are at least $(n-1)$-dimensional, i.e., they are absolutely continuous with respect to the $(n-1)$-dimensional Hausdorff measure, such that $\mathcal{P}(A)$ becomes invariant under null set changes of A. The standard examples for measures $\mu_\pm$ which satisfy isoperimetric conditions are weighted $(n-1)$-dimensional Hausdorff measures restricted to (hyper)surfaces.
To understand the connection of the isoperimetric condition and the lower semicontinuity of the functional $\mathcal{P}$, we look at the following example: If we consider the reduced functional $\mathcal{P}_-(\,\cdot\,) := \mathrm{P}(\,\cdot\,) - \mu_-((\,\cdot\,)^+)$ with $\emptyset \neq \operatorname{spt}(\mu_-) \subseteq (-1,1)^{n-1} \times \{0\}$, then the sequence of sets $A_k = (-1,1)^{n-1} \times \left(-\frac{1}{k}, \frac{1}{k}\right)$, $k \in \mathbb{N}$, converge to $A := \emptyset$ in L^1 and it holds $\limsup_{k\to\infty} \mu_-((A_k)^+) = \mu_-(\mathbb{R}^n) > 0 = \mu_-(\emptyset)$. Strict inequality appears also when considering the lower semicontinuity of the perimeter, i.e., it holds $\mathrm{P}(A) = 0 < \liminf_{k\to\infty} \mathrm{P}(A_k)$. Without any connection between μ_- and the perimeter, the functional $\mathcal{P}_-$ is not lower semicontinuous. The same effect applies for functionals of the form $\mathcal{P}_+(\,\cdot\,) := \mathrm{P}(\,\cdot\,) + \mu_+((\,\cdot\,)^1)$, taking for instance $A_k = (-1,1)^{n-1} \times \left(\left(-1, -\frac{1}{k}\right) \cup \left(\frac{1}{k}, 1\right)\right)$ and $A = (-1,1)^n$.
It turns out that imposing the small-volume isoperimetric condition with constant 1 on both μ_- and μ_+ provides a sharp compensation effect between μ and the perimeter in the limit, hence giving the desired lower semicontinuity result. For a similar approach of this theory from the non-parametric perspective, we refer to [57].

Definition 2.78 (Isoperimetric conditions)**.** Let μ be a non-negative Radon measure on Ω and $C > 0$. Then we say that

i) μ satisfies the **(strong) isoperimetric condition in Ω with constant C** if

$$\mu(A^+) \leqslant C\mathrm{P}(A) \qquad \text{for all } A \in \mathcal{M}^n \text{ such that } \overline{A} \subseteq \Omega \text{ and } |A| < \infty\,,$$

ii) μ satisfies the **small-volume isoperimetric condition in Ω with constant C** if

$$\text{for all } \varepsilon > 0 \text{ we find } \delta > 0 \text{ with} \quad \mu(A^+) \leqslant C\mathrm{P}(A) + \varepsilon \quad \text{for all } A \in \mathcal{M}^n \text{ such that } |A| < \delta \text{ and } \overline{A} \subseteq \Omega\,.$$

In the following, we collect some auxiliary results for measures satisfying isoperimetric conditions, before stating the main semicontinuity result.

Proposition 2.79 (Properties of measures with isoperimetric condition)**.** *Let μ, ν be non-negative Radon measures on Ω which satisfy the small-volume isoperimetric condition with constant $C > 0$. Then the following properties are satisfied.*

i) *The value $\mu(A^+)$ is finite for all sets of finite volume and finite perimeter $\overline{A} \subseteq \Omega$.*

ii) *The measure μ is absolutely continuous with respect to $\mathcal{H}^{n-1}$ on $\mathcal{B}(\Omega)$*

iii) *If it holds $\mu \perp \nu$ with $\operatorname{dist}(\operatorname{spt}\mu, \operatorname{spt}\nu) > 0$, then $\mu + \nu$ satisfies the small-volume isoperimetric condition with constant C.*

Property i) of the previous proposition can be found in [73, Lemma 3.3], ii) in [73, Lemma 3.2] and iii) in [73, Proposition 7.4].
The model case for measures satisfying isoperimetric inequalities are Hausdorff measures restricted to hyperplanes. Indeed, due to the decrease of the perimeter under intersection with half-spaces, one even obtain strong isoperimetric conditions. The following two lemmas can be derived by applying [73, Proposition 8.1] on suitable measures.

Lemma 2.80 (Model case for isoperimetric condition). *The measure $2\theta\mathcal{H}^{n-1}\llcorner B\times\{t\}$ satisfies the strong isoperimetric condition in $\mathbb{R}^n$ with constant $\theta>0$ for all $t\in\mathbb{R}$ and all $B\in\mathcal{B}(\mathbb{R}^{n-1})$.*

Lemma 2.81. *Let $\theta>0$, $t\in\mathbb{R}$, $B\in\mathcal{B}(\mathbb{R}^{n-1})$ and $f\in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{n-1}\times\{t\},\mathcal{H}^{n-1}\llcorner B\times\{t\})$. If the small-volume isoperimetric condition with constant θ holds for $f\mathcal{H}^{n-1}\llcorner B\times\{t\}$, then $f\leqslant 2\theta$ is verified $\mathcal{H}^{n-1}$-a.e. on $B\times\{t\}$.*

The next statement summarizes the aforementioned lower semicontinuity results from [73, Theorem 4.1(c), Theorem 9.1(c)].

Theorem 2.82 (Global and local lower semicontinuity under isoperimetric conditions). *In what follows, we consider the functional*

$$\mathcal{P}(A;\Omega) := \mathrm{P}(A,\Omega)+\mu_+(A^1)-\mu_-(A^+)\,,$$

where $A\subseteq\mathbb{R}^n$ is a set of finite perimeter in $\Omega\in\mathcal{B}(\mathbb{R}^n)$ with $\min\{\mu_+(A^1),\mu_-(A^+)\}<\infty$, and both μ_+ and μ_- are non-negative Radon measures on Ω. Then the following properties are satisfied.

i) Global version: *If $\Omega=\mathbb{R}^n$ and both μ_+ and μ_- satisfy the small-volume isoperimetric condition with constant 1, then*

$$\mathcal{P}(A;\mathbb{R}^n)\leqslant\liminf_{k\to\infty}\mathcal{P}(A_k;\mathbb{R}^n)\qquad \textit{for all sets } A_k, A \textit{ as above with } A_k\to A \textit{ in } \mathrm{L}^1 \textit{ as } k\to\infty\,.$$

ii) Local version: *If $\Omega\neq\mathbb{R}^n$ and if the following properties are satisfied:*

   a) *Ω is a set of locally finite perimeter,*
   b) *$\mu_-(\mathbb{R}^n\setminus\Omega^1)=0=\mu_+(\mathbb{R}^n\setminus\Omega^1)$,*
   c) *μ_+ and the Radon measure $\mu_-(\,\cdot\,)+\mathrm{P}(\Omega,\,\cdot\,)$ both satisfy the small-volume isoperimetric condition in $\mathbb{R}^n$ with constant 1,*

   then it holds

$$\min\{\mu_-(A^+),\mu_+(A^1)\}<\infty\qquad \textit{and}\qquad \mathcal{P}(A;\Omega^1)\leqslant\liminf_{k\to\infty}\mathcal{P}(A_k;\Omega^1)$$

   for all sets $A_k, A\in\mathcal{M}^n$ with $\mathrm{P}(A_k\cap\Omega)+\mathrm{P}(A\cap\Omega)<\infty$ for all $k\in\mathbb{N}$, $A_k\to A$ in $\mathrm{L}^1(\Omega)$ as $k\to\infty$ and with $\min\{\mu_-(A_k^+),\mu_+(A_k^1)\}<\infty$ for all $k\in\mathbb{N}$.

In Section 3.3.5, we consider obstacle problems as model cases for the generalized Schwarz perimeter inequality. The following existence result from [73, Theorem 5.1] will be essential.

Theorem 2.83 (Existence for obstacle problems). *Let μ_-, μ_+ be two non-negative Radon measures which both satisfy the small-volume isoperimetric condition in $\mathbb{R}^n$ with constant 1. We consider the inner and outer obstacles $I\subseteq O$ in $\mathcal{M}^n$ such that either $\mu_-(O^+)$ is finite or μ_- satisfies the strong isoperimetric condition with constant $C<1$. Then there exists a minimizer of the functional*

$$\mathcal{P}(A) := \mathrm{P}(A)+\mu_+(A^1)-\mu_-(A^+)$$

among sets $A\in\mathcal{B}(\mathbb{R}^n)$ of finite perimeter and finite volume with $I\subseteq A\subseteq O$ if there exists at least one set satisfying these assumptions.*

*Here, the subset relations should be understood as a relation which holds true up to $\mathcal{L}^n$-null sets.

Chapter 3

Schwarz symmetrization results

In this chapter, we present a characterization of the equality case of the Schwarz perimeter inequality. Moreover, a generalization of the inequality to prescribed mean curvature functionals with measure data is given.
We use some results of the current part later in Chapter 5.3 in order to analyze sharp counterexamples to the integrability result for variational mean curvatures of $\mathrm{C}^{1,\alpha}$ sets in higher dimensions.
At this point, we remind the reader that, if not otherwise stated, we assume $n \geqslant 2$ from now on.

3.1 Symmetrization techniques and auxiliary results

In this section, we introduce different symmetrization techniques, collect the specific preliminaries for Sections 3.2 and 3.3 and present some auxiliary results.

3.1.1 Symmetrization

In many contexts, geometric optimization problems can be solved via symmetrization techniques. We are particularly interested in the Steiner and Schwarz symmetrization of sets, which are both volume-preserving. For more information about other symmetrization techniques and the current state of the art, we refer to the survey [13] from 2019. For information about the Steiner and Schwarz symmetrization, we refer to [18, 7, 14] and [62, Section 19.2].

Definition 3.1 (Steiner and Schwarz symmetrization)**.** Let $A \in \mathcal{M}^n$ and $i \in \{1, \ldots, n\}$. The **Steiner symmetrization of A with respect to the i-th component** is the set

$$A^{S_i} := \left\{x \in \mathbb{R}^n : 2|x_i| < \left|\{t \in \mathbb{R} : (x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n) \in A\}\right|\right\}.$$

We put $A^S := A^{S_1}$ and call this the **Steiner symmetrization of A**.
Moreover, the set

$$A^\star := \left\{x \in \mathbb{R}^n : |\overline{x}| < \left(\frac{1}{\omega_{n-1}}|A_{x_n}|\right)^{\frac{1}{n-1}}\right\}$$

is called the **Schwarz symmetrization of A**.

Remark 3.2. i) If a set is Borel or Lebesgue measurable, then the Steiner and the Schwarz symmetrizations are likewise Borel or Lebesgue measurable.

We sketch the proof of this only for the Schwarz symmetrization (the argument for the Steiner symmetrization is analogous) and concentrate on showing the claim for Borel sets since the proof simplifies for measurable sets.

Indeed, the argument for Borel sets $B \in \mathcal{B}(\mathbb{R}^n)$ requires more work since the fact that $t \mapsto |B_t|$ is a Borel function on $\mathbb{R}$ (and not only measurable) is not covered by Fubini's theorem. To show this

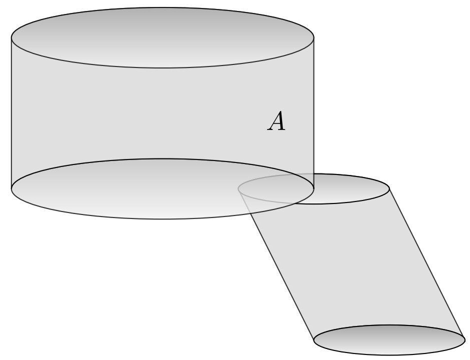


Figure 3.1: Illustration of the Schwarz symmetrization

property, we consider w.l.o.g. bounded sets in $[-R,R]^n$ for some $R>0$ and define the set systems

$$\mathcal{Q} := \left\{ Q \subseteq [-R,R]^n : Q \text{ is a rectangle with } \overline{Q} = \prod_{k=1}^{n} [a_k,b_k] \text{ for some } a_k,b_k \in [-R,R],\ k \in \{1,\ldots,n\} \right\},$$
$$\mathcal{X} := \{ B \subseteq [-R,R]^n : t \mapsto |B_t| \text{ is a Borel function on } \mathbb{R} \} .$$

The following properties hold true:

a) $\mathcal{Q}$ is a generator of $\mathcal{B}([-R,R]^n)$,
b) for $Q_1, Q_2 \in \mathcal{Q}$, the intersection $Q_1 \cap Q_2$ is in $\mathcal{Q}$,
c) $\mathcal{Q} \subseteq \mathcal{X}$,
d) $\mathcal{X}$ is a Dynkin system on $[-R,R]^n$ (i.e., $[-R,R]^n \in \mathcal{X}$ and $\mathcal{X}$ is closed under complements in $[-R,R]^n$ and under countable disjoint unions).

If a set system is closed under intersections, its generated Dynkin system coincides with its generated σ-algebra. Hence, by a) and b), the Dynkin system generated by $\mathcal{Q}$ is equal to $\mathcal{B}([-R,R]^n)$. The properties c) and d) then imply $\mathcal{B}([-R,R]^n) \subseteq \mathcal{X}$, hence showing that $t \mapsto |B_t|$ is a Borel function for every Borel set. The remaining argument is straightforward.

ii) According to Fubini's theorem, the symmetrization techniques are volume preserving. Moreover, for $A,\ F \in \mathcal{M}^n$, it holds $|A^\star \triangle F^\star| \leqslant |A \triangle F|$.

iii) Let $t \in \mathbb{R}$, $i \in \{1,\ldots,n-1\}$ and $A \in \mathcal{M}^n$ with $A_t = \mathrm{B}_r^{n-1}(x)$ for some $x \in \mathbb{R}^{n-1}$ and $r>0$. Then,

$$(A^{S_i})_t = \mathrm{B}_r^{n-1}(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_{n-1}), \quad (A^\star)_t = \mathrm{B}_r^{n-1} \quad \text{and} \quad (A^\star)_t = (((A^{S_1})^{\cdots})^{S_{n-1}})_t .$$

The following notion characterizes a necessary condition for equality in the Schwarz inequality.

Definition 3.3 (Stack of balls)**.** Let $I \subseteq \mathbb{R}$ be a Borel set and $A \subseteq \mathbb{R}^n$ a measurable set. The set A is called **a stack of balls in *I*** if there exist two measurable functions $r : I \to [0,\infty)$ and $c : I \to \mathbb{R}^{n-1}$ such that it holds

$$\left| A_t \triangle \mathrm{B}_{r(t)}^{n-1}(c(t)) \right| = 0 \qquad \text{for a.e. } t \in I .$$

In this case, the function r is called the **radius function** and c is called the **center function** of A in I. In the special case $I = \mathbb{R}$, we call A a **stack of balls**.

Remark 3.4. If A is a stack of balls in some Borel set $I \subseteq \mathbb{R}^n$, then the radius and center function r and c are uniquely defined on $I \cap \{r>0\}$ up to null sets.

The following theorem is proven in [7, Theorem 1.1] for the Steiner symmetrization of any dimension.

Theorem 3.5 (Steiner and Schwarz inequality)**.** *Let $A \in \mathcal{M}^n$ be a set of finite perimeter and finite volume and let $i \in \{1,\ldots,n\}$.*

i) *For all Borel sets* $B_1 \in \mathcal{B}(\mathbb{R}^{i-1})$, $B_2 \in \mathcal{B}(\mathbb{R}^{n-i})$, *we have the Steiner inequality*

$$\mathrm{P}(A^{S_i}, B_1 \times \mathbb{R} \times B_2) \leqslant \mathrm{P}(A, B_1 \times \mathbb{R} \times B_2)\,.$$

If equality in the previous inequality holds true, then, for $\mathcal{L}^{n-1}$*-a.e.* $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in B_1 \times B_2$, *the one-dimensional slice* $\{t \in \mathbb{R} : (x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n) \in A\}$ *is a line segment up to an* $\mathcal{L}^1$*-null set.*

ii) *For all* $B \in \mathcal{B}(\mathbb{R})$, *we have the Schwarz inequality*

$$\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times B) \leqslant \mathrm{P}(A, \mathbb{R}^{n-1} \times B)\,. \tag{3.1}$$

If equality in the previous inequality holds true, then A *is a stack of balls in* B.

The Schwarz inequality is usually stated for open sets B only. The open version of the Schwarz inequality can be directly transferred to the Borel version, thanks to the Radon measure property of the perimeter. However, we want to apply the result in the equality case on general Borel sets B, which is why we revisit the proof based on [62, Theorem 19.11]. This version is already stated in [14, Theorem A] and [70, Theorem 5.9] for the case of Steiner symmetrization of dimension 1, but with reference to statements which only consider equality on all of $\mathbb{R}^n$ or to the indirect proof via spherical symmetrization. We focus on the parts of the proof which differ from the case $B = \mathbb{R}$.
In order to prove this version, we need to apply the following slice theorems, which provide the basis for analyzing the behavior of symmetrizations with respect to the perimeter. The statements can be found in [7, Theorem 2.4] and [62, Theorem 18.11, Exercise 18.15] as improved versions of [89], see also [18, Theorem G].

Definition 3.6 (Volume function)**.** For a measurable set $A \subseteq \mathbb{R}^n$, the function

$$\mathrm{vol}_A : \mathbb{R} \to [0, \infty],\ t \mapsto |A_t|$$

is called the **volume function of $\boldsymbol{A}$**.

Theorem 3.7 (Slice theorem)**.** *Let* $A \subseteq \mathbb{R}^n$ *be a set of finite perimeter and* $k \in \{1, \dots, n-1\}$. *Then there exists an* $\mathcal{L}^{n-k}$*-null set* $N \subseteq \mathbb{R}^{n-k}$ *such that the following properties hold true.*

i) *For all* $y \in \mathbb{R}^{n-k} \backslash N$, *the slice* A_y *is a set of finite perimeter.*

ii) *For all* $y \in \mathbb{R}^{n-k} \backslash N$, *it holds* $\partial^*(A_y) = (\partial^* A)_y$ *up to an* $\mathcal{H}^{k-1}$*-null set.*

iii) *For all* $y \in \mathbb{R}^{n-k} \backslash N$, *it holds*
$$\tilde{\nu}_A^k(x, y) = \nu_{A_y}(x)\, |\tilde{\nu}_A^k(x, y)| \neq 0$$
for $\mathcal{H}^{k-1}$*-a.e.* $x \in \partial^*(A_y) \cap (\partial^* A)_y$, *where* $\tilde{\nu}_A^k(x, y) := (\nu_A(x, y)_i)_{i \in \{1, \dots, k\}}$.

Theorem 3.8 (Slice theorem for $(n-1)$-dimensional slices)**.** *Let* $A \subseteq \mathbb{R}^n$ *be a set of finite perimeter and define* $\Sigma := \big\{x \in \partial^* A : \nu_A(x) \in \{-e_n, e_n\}\big\}$. *Then the following properties hold true.*

i) *If* A *is additionally a set of finite volume and* $\mathcal{H}^{n-1}(\Sigma) = 0$, *then* $\mathrm{vol}_A \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R})$ *and, for* $\mathcal{L}^1$*-a.e.* $t \in \mathbb{R}$, *it holds that*
$$\mathrm{vol}_A'(t) = -\int_{(\partial^* A)_t} \frac{(\nu_A(y,t))_n}{\big|\overline{\nu_A}(y,t)\big|}\, \mathrm{d}\mathcal{H}^{n-2}(y)\,,$$
where $\overline{\nu_A}(y, t) = ((\nu_A(y, t))_k)_{k=1,\dots,n-1}$.

ii) *The equality*
$$\int_{\partial^* A} g\, \mathrm{d}\mathcal{H}^{n-1}(x) = \int_\Sigma g\, \mathrm{d}\mathcal{H}^{n-1}(x) + \int_{\mathbb{R}} \int_{(\partial^* A)_t} \frac{g(y,t)}{\big|\overline{\nu_A}(y,t)\big|}\, \mathrm{d}y\, \mathrm{d}t$$
holds true for all Borel functions $g : \mathbb{R}^n \to [0, \infty]$.

Proof of Theorem 3.5 ii). The proof is divided into Step 1, which involves the introduction of some notation and collecting some properties, and the main proof, which is Steps 2 to 7.

Step 1: We first define

$$\Sigma_A := \big\{x \in \partial^* A \,:\, \nu_A(x) \in \{-e_n, e_n\}\big\} \quad \text{and} \quad p_A(t) := \mathrm{P}(A_t) \quad \text{for } t \in \mathbb{R}\,.$$

According to Theorem 3.7, the set A_t is of finite perimeter with $p_A(t) = \mathcal{H}^{n-2}((\partial^* A)_t)$ for $\mathcal{L}^1$-a.e. t in $\mathbb{R}$. Moreover, $\mathrm{vol}_A \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R})$ whenever $\mathcal{H}^{n-1}(\Sigma_A) = 0$ by Theorem 3.8.

The first three steps of the main proof can be carried out analogously to the corresponding steps of the proof of [62, Theorem 19.11], hence we only collect the results at this point.

Step 2: The estimate

$$\int_B \sqrt{p_A(t)^2 + \mathrm{vol}_A'(t)^2}\,\mathrm{d}t \leqslant \mathrm{P}(A, \mathbb{R}^{n-1} \times B)$$

holds for all open sets $B \subseteq \mathbb{R}$ such that $\mathcal{H}^{n-1}(\Sigma_A \cap (\mathbb{R}^{n-1} \times B)) = 0$, with equality if and only if $(\nu_A(\cdot, t))_n$ is $\mathcal{H}^{n-2}$-a.e. constant on $(\partial^* A)_t$ for a.e. $t \in B$. In the case of equality, we have

$$\frac{(\nu_A)_n}{|\overline{\nu_A}|} = \frac{(\nu_A)_n}{\sqrt{1 - (\nu_A)_n^2}} \equiv c \qquad \mathcal{H}^{n-2}\text{-a.e. on } (\partial^* A)_t$$

for some $c = c(t) \in \mathbb{R}$.

Step 3: If $\mathrm{vol}_A \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, then $A^\star$ is a set of finite perimeter in $\mathbb{R}^n$.

Step 4: If A is an open, bounded and polyhedral set (see Definition 3.23) such that $\mathcal{H}^{n-1}(\Sigma_A) = 0$, then $A^\star$ is a set of finite perimeter with

$$\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times B) = \int_B \sqrt{p_{A^\star}(t)^2 + \mathrm{vol}_A'(t)^2}\,\mathrm{d}t, \quad \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times B) \leqslant \mathrm{P}(A, \mathbb{R}^{n-1} \times B)$$

and

$$2\,\mathrm{P}(A, \mathbb{R}^{n-1} \times B)\,[\mathrm{P}(A, \mathbb{R}^{n-1} \times B) - \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times B)] \geqslant \left(\int_B \sqrt{p_A(t)^2 - p_{A^\star}(t)^2}\,\mathrm{d}t\right)^2$$

for all open sets $B \subseteq \mathbb{R}$.

Step 5: Conclusion of the theorem for open sets $B \subseteq \mathbb{R}$.
In this step, the main adaptions to the original source occur. For arbitrary bounded sets $A \subseteq \mathbb{R}^n$ of finite perimeter, we apply the approximation result of the later Theorem 3.25 to obtain a sequence $(A_k)_{k \in \mathbb{N}}$ of bounded polyhedral sets with

$$A_k \to A \quad \text{in } \mathrm{L}^1(\mathbb{R}^n), \quad \text{and} \quad \mathrm{P}(A_k, F) \to \mathrm{P}(A, F) \quad \text{as } k \to \infty$$

whenever $F \in \mathcal{B}(\mathbb{R}^n)$ satisfies $\mathrm{P}(A, \partial F) = 0$. Since we can slightly rotate the sets, we can even assume that $\mathcal{H}^{n-1}(\Sigma_{A_k}) = 0$ for all $k \in \mathbb{N}$, compare [62, Exercise 15.10, Exercise 17.3]. By Remark 3.2, $A_k^\star \to A^\star$ in $\mathrm{L}^1(\mathbb{R}^n)$ holds true as well. With

$$\int_{\mathbb{R}} |\mathrm{vol}_A(t) - \mathrm{vol}_{A_k}(t)|\,\mathrm{d}t = \int_{\mathbb{R}} \big|\,|A_t| - |(A_k)_t|\,\big|\,\mathrm{d}t \leqslant \int_{\mathbb{R}} |A_t \triangle (A_k)_t|\,\mathrm{d}t = |A \triangle A_k| \to 0 \quad \text{as } k \to \infty\,,$$

we even find a subsequence (without relabeling) such that, additionally to the other properties, $\mathrm{vol}_{A_k} \to \mathrm{vol}_A$ a.e. on $\mathbb{R}$. For those points, the approximation transfers to

$$p_{A_k^\star} = (n-1)\omega_{n-1}\left(\frac{1}{\omega_{n-1}}\mathrm{vol}_{A_k}\right)^{\frac{n-2}{n-1}} \to (n-1)\omega_{n-1}\left(\frac{1}{\omega_{n-1}}\mathrm{vol}_A\right)^{\frac{n-2}{n-1}} = p_{A^\star} \qquad \text{a.e. on } \mathbb{R}\,.$$

Now, for all open sets $B \subseteq \mathbb{R}^n$ with $\mathrm{P}(A, \mathbb{R}^{n-1} \times \partial B) = 0$, by the lower semicontinuity of the perimeter and Step 4, we obtain

$$\mathrm{P}(A, \mathbb{R}^{n-1} \times B) = \lim_{k\to\infty} \mathrm{P}(A_k, \mathbb{R}^{n-1} \times B) \geqslant \liminf_{k\to\infty} \mathrm{P}(A_k^\star, \mathbb{R}^{n-1} \times B) \geqslant \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times B)\,.$$

For a general open set B, we can approximate the set by an increasing sequence $(B_\ell)_{\ell\in\mathbb{N}}$ of open sets with $B_\ell \Subset B$, $B_\ell \Subset B_{\ell+1}$ for all $\ell \in \mathbb{N}$ and $\bigcup_{\ell\in\mathbb{N}} B_\ell = B$. We take a suitable choice of $(B_\ell)_{\ell\in\mathbb{N}}$, so that we can assume $\mathrm{P}(A, \mathbb{R}^{n-1} \times \partial B_\ell) = 0$ for all $\ell \in \mathbb{N}$ and conclude

$$\begin{aligned}
\mathrm{P}(A, \mathbb{R}^{n-1} \times B) &= \lim_{\ell\to\infty} \mathrm{P}(A, \mathbb{R}^{n-1} \times B_\ell)\\
&= \lim_{\ell\to\infty}\lim_{k\to\infty} \mathrm{P}(A_k, \mathbb{R}^{n-1} \times B_\ell)\\
&\geqslant \liminf_{\ell\to\infty}\liminf_{k\to\infty} \mathrm{P}(A_k^\star, \mathbb{R}^{n-1} \times B_\ell)\\
&\geqslant \lim_{\ell\to\infty} \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times B_\ell)\\
&= \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times B)\,.
\end{aligned} \tag{3.2}$$

This confirms (3.1) for open sets $B \subseteq \mathbb{R}$. Since $\mathrm{P}(A, \cdot\,)$ is a Radon measure, the Schwarz inequality for all Borel sets $B \subseteq \mathbb{R}$ follows from the regularity of Radon measures.
In the remainder of the proof, we treat the equality case of (3.1). If equality in (3.2) is attained, we consider a suitable subsequence of $(B_\ell)_{\ell\in\mathbb{N}}$ that satisfies

$$\liminf_{\ell\to\infty}\liminf_{k\to\infty} \mathrm{P}(A_k^\star, \mathbb{R}^{n-1} \times B_\ell) = \lim_{\ell\to\infty}\liminf_{k\to\infty} \mathrm{P}(A_k^\star, \mathbb{R}^{n-1} \times B_\ell)$$

such that, for each $h \in \mathbb{N}$, we find $\ell_h \in \mathbb{N}$ with

$$\frac{1}{h} > \limsup_{k\to\infty} \mathrm{P}(A_k, \mathbb{R}^{n-1} \times B_\ell) - \mathrm{P}(A_k^\star, \mathbb{R}^{n-1} \times B_\ell)$$

for all $\ell \geqslant \ell_h$. Taking into account the results of Step 4 and the convergence of A_k with respect to the perimeter on $\mathbb{R}^{n-1} \times B_\ell$, we conclude

$$\frac{2\,\mathrm{P}(A)}{h} \geqslant \limsup_{k\to\infty} \left(\int_{B_\ell} \sqrt{p_{A_k}(t)^2 - p_{A_k^\star}(t)^2}\,\mathrm{d}t\right)^2 \quad \text{for large } \ell\,. \tag{3.3}$$

According to Fatou's lemma, it follows

$$\frac{2\,\mathrm{P}(A)}{h} \geqslant \limsup_{k\to\infty} \left(\int_{B_\ell} \sqrt{p_{A_k}(t)^2 - p_{A_k^\star}(t)^2}\,\mathrm{d}t\right)^2 \geqslant \left(\int_{B} \liminf_{k\to\infty} \sqrt{p_{A_k}(t)^2 - p_{A_k^\star}(t)^2}\,\mathrm{d}t\right)^2 - \frac{1}{h} \qquad \text{for large } \ell\,.$$

The inequality follows from the integrability of $\liminf_{k\to\infty} \sqrt{p_{A_k}^2 - p_{A_k^\star}^2}$ and the convergence $|B_\ell \triangle B| \to 0$ as $\ell \to \infty$. Indeed, the former can be verified by Step 4 and

$$\begin{aligned}
\left(\int_{\mathbb{R}} \liminf_{k\to\infty} \sqrt{p_{A_k}(t)^2 - p_{A_k^\star}(t)^2}\,\mathrm{d}t\right)^2 &\leqslant \liminf_{k\to\infty} \left(\int_{\mathbb{R}} \sqrt{p_{A_k}(t)^2 - p_{A_k^\star}(t)^2}\,\mathrm{d}t\right)^2\\
&\leqslant \liminf_{k\to\infty} 2\,\mathrm{P}(A_k)\,(\mathrm{P}(A_k) - \mathrm{P}(A_k^\star))\\
&\leqslant 2\,\mathrm{P}(A)\,(\mathrm{P}(A) - \liminf_{k\to\infty} \mathrm{P}(A_k^\star))\\
&\leqslant 2\,\mathrm{P}(A)\,(\mathrm{P}(A) - \mathrm{P}(A^\star)) < \infty\,.
\end{aligned}$$

Since h in (3.3) is arbitrary, we thus come out with

$$\limsup_{\ell\to\infty}\limsup_{k\to\infty} \left(\int_{B_\ell} \sqrt{p_{A_k}(t)^2 - p_{A_k^\star}(t)^2}\,\mathrm{d}t\right)^2 = 0\,.$$

Hence, for $\mathcal{L}^1$-a.e. $t \in B$, we have

$$0 = \liminf_{k\to\infty} |p_{A_k}(t) - p_{A_k^\star}(t)| \geqslant \liminf_{k\to\infty} p_{A_k}(t) - \limsup_{k\to\infty} p_{A_k^\star}(t) = \liminf_{k\to\infty} p_{A_k}(t) - p_{A^\star}(t)\,,$$

where we applied the convergence of $p_{A_k^\star}$ to $p_{A^\star}$ almost everywhere on $\mathbb{R}$ from the beginning of this step. With the observations from Step 1 and from this step so far, it follows

$$p_A(t) = \mathrm{P}(A_t) \leqslant \liminf_{k\to\infty} \mathrm{P}((A_k)_t) = \liminf_{k\to\infty} p_{A_k}(t) \leqslant p_{A^\star}(t)$$

for $\mathcal{L}^1$-a.e. $t \in B$. Since $A_t^\star = \mathrm{B}_r^{n-1}$ for some $r \geqslant 0$, the isoperimetric inequality implies that A_t is $\mathcal{H}^{n-1}$-equivalent to an $(n{-}1)$-dimensional ball.

Step 6 : Conclusion for bounded Borel sets B.
For a bounded Borel set $B \subseteq \mathcal{B}(\mathbb{R})$, there exist open sets B_ℓ such that

$$|B_\ell \triangle B| \to 0, \quad \mathrm{P}(A, \mathbb{R}^{n-1} \times B_\ell) \to \mathrm{P}(A, \mathbb{R}^{n-1} \times B) \quad \text{and} \quad \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times B_\ell) \to \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times B)$$

as $\ell \to \infty$. As in Step 5, we find for each $\ell \in \mathbb{N}$ an increasing sequence of open sets $(B_k^\ell)_{k\in\mathbb{N}}$ with $\bigcup_{k\in\mathbb{N}} B_k^\ell = B_\ell$ and $\mathrm{P}(A, \partial B_k^\ell) = 0$ for all $k, \ell \in \mathbb{N}$. By taking a diagonal sequence, we additionally assume $\mathrm{P}(A, \partial B_\ell) = 0$ and again find

$$\begin{aligned}
\mathrm{P}(A, \mathbb{R}^{n-1} \times B) &= \lim_{\ell\to\infty} \mathrm{P}(A, \mathbb{R}^{n-1} \times B_\ell) \\
&= \lim_{\ell\to\infty} \lim_{k\to\infty} \mathrm{P}(A_k, \mathbb{R}^{n-1} \times B_\ell) \\
&\geqslant \liminf_{\ell\to\infty} \liminf_{k\to\infty} \mathrm{P}(A_k^\star, \mathbb{R}^{n-1} \times B_\ell) \\
&\geqslant \lim_{\ell\to\infty} \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times B_\ell) \\
&= \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times B)
\end{aligned}$$

for a sequence of sets A_k of finite perimeter that approximate A, as in the previous step. The conclusion follows as in Step 5.

Step 7 : We show the theorem in full generality.
Let $B \subseteq \mathbb{R}^n$ be an arbitrary Borel set. In case of equality, we can evaluate the perimeter on the open intervals $(-k, k)$ with length $2k$ for all $k \in \mathbb{N}$, and on the exterior of these sets. By the Schwarz inequality on both of these partitions, we get

$$\mathrm{P}(A, \mathbb{R}^{n-1} \times (B \cap \mathrm{B}_k^1)) = \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times (B \cap \mathrm{B}_k^1))\,.$$

Hence, A_t has the required shape for $\mathcal{L}^1$-a.e. $t \in B \cap \mathrm{B}_k^1$. By increasing the radius k, the claim follows. □

The Schwarz symmetrization of $A \in \mathcal{M}^n$ is by definition a body of revolution, which can be described by a profile curve, which we call the radius function. In the case of equality, Theorem 3.5 implies also the existence of a center function to describe A, representing the dislocation of the $(n{-}1)$-dimensional slice balls.

Definition 3.9 (Radius and center function for general sets)**.** Let $A \subseteq \mathbb{R}^n$ be a measurable set. The function

$$\mathrm{rad}_A : \mathbb{R} \to [0, \infty],\, t \mapsto \left(\frac{1}{\omega_{n-1}} |A_t|\right)^{\frac{1}{n-1}}$$

is called the **radius function of A**. The **center function of A** is the function $\mathrm{cen}_A : \mathbb{R} \to \mathbb{R}^{n-1}$ defined by

$$\mathrm{cen}_A(t) := \begin{cases} x & \text{if } \mathrm{rad}_A(t) < \infty \text{ and } \left|A_t \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(t)}(x)\right| = 0 \text{ for some } x \in \mathbb{R}^{n-1}\,, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.10. i) We draw attention to the fact that, if A is a stack of balls, then the radius and center functions of A from Definitions 3.3 and 3.9 coincide a.e. on $\mathbb{R}$. Since the latter is defined for more general sets, we may always consider this definition in the remainder of this thesis.

ii) If $A \in \mathcal{M}^n$ has infinite volume and finite perimeter, then, according to Theorem 2.42, $\mathbb{R}^n \backslash A$ is a set of finite volume, which implies $\mathrm{rad}_{\mathbb{R}^n\backslash A} < \infty$ a.e. on $\mathbb{R}$ and therefore $\mathrm{rad}_A = \infty$ a.e. on $\mathbb{R}$. Hence, in the further analysis we will often assume that $A \in \mathcal{M}^n$ is a set of finite volume, which implies $\mathrm{rad}_A \in \mathrm{L}^{n-1}(\mathbb{R})$, since

$$|A| = \int_{-\infty}^{\infty} |A_t| \,\mathrm{d}t = \int_{-\infty}^{\infty} \omega_{n-1} \mathrm{rad}_A^{n-1}(t) \,\mathrm{d}t = \omega_{n-1} \|\mathrm{rad}_A\|_{\mathrm{L}^{n-1}(\mathbb{R})}^{n-1} .$$

In the equality case of the Schwarz inequality, we can describe A solely via its radius and center function. As a crucial result, we show that both of these functions are of bounded variation on $\{r^- > 0\}$, hence the existence of good representatives is ensured. This simplifies the analysis in comparison with Steiner symmetrizations of dimension in $\{2, n-2\}$. In fact, a characterization of the equality cases of the one-dimensional Steiner symmetrization is given in [14, Theorem 1.9] and a paper on the higher dimensional case was announced but has not appeared.

Definition 3.11 (Rigidity)**.** Let $A \in \mathcal{M}^n$ and $B \in \mathcal{B}(\mathbb{R})$. We say that **rigidity is satisfied for $\boldsymbol{A}$ in $\boldsymbol{B}$** if for all sets $F \in \mathcal{M}^n$ with $\mathrm{rad}_F = \mathrm{rad}_A$ on B and with

$$\mathrm{P}(F^\star, \mathbb{R}^{n-1} \times B) = \mathrm{P}(F, \mathbb{R}^{n-1} \times B) ,$$

there exists $x \in \mathbb{R}^{n-1} \times \{0\}$ such that $|(F \triangle (x + A)) \cap (\mathbb{R}^{n-1} \times B)| = 0$.

Rigidity results are important for ensuring uniqueness of minimizers of perimeter involving optimization problems. There are, however, examples showing that rigidity fails in simple cases due to compensation effects of the singular part of the radius and the center function.

Example 3.12. Let

$$A := \mathrm{B}_1^{n-1} \times [-1, 0] \cup \mathrm{B}_2^{n-1} \times [0, 1].$$

Then rigidity is not satisfied for A in $\mathbb{R}$, as can be seen by defining the set $F := \mathrm{B}_1^{n-1}(e_1) \times [-1, 0] \cup \mathrm{B}_2^{n-1} \times [0, 1]$.

In [18, Theorem 1.3], a rigidity result for the one-dimensional Steiner symmetrization was given – a breakthrough after the slice characterization of the equality cases in [22, Teorema II] (see [21, p. 191-192] for an English translation), which was restated in Theorem 3.5.

Rigidity results for higher dimensional cases are analyzed in [7, Theorem 1.2] under the additional assumption that there exists no flat boundary part of A and that the corresponding slices of A have positive volume. The analysis is based on the introduction of the (bary)center function. During the preparation of this thesis, [26] was published, covering a full analysis of rigidity for the Schwarz symmetrization, see Theorem 1.3 in the reference. However, a full characterization of the equality cases of the Schwarz inequality is not given.

3.1.2 The area functional for surfaces of revolution

In the following, we introduce the area functional for surfaces of revolution and prove corresponding results based on [67].

Definition 3.13 (Area functional for surfaces of revolution)**.** Let $I \subseteq \mathbb{R}$ be an open set. The **area functional for $\boldsymbol{(n-1)}$-dimensional surfaces of revolutions in $\mathbb{R}^{\boldsymbol{n-1}} \times \boldsymbol{I}$** is defined as

$$\mathcal{A}_{n,I}^{\mathrm{rev}}[r] := \sup\left\{ \int_I \varphi_0 \, r^{n-2} - \frac{\varphi'}{n-1} r^{n-1} \,\mathrm{d}t : \, \varphi_0, \varphi \in \mathrm{C}^1_{\mathrm{cpt}}(I), \, \|(\varphi_0, \varphi)\|_{\mathrm{L}^\infty(I, \mathbb{R}^2)} \leqslant 1 \right\} \quad \text{for } r \in \mathrm{L}^{n-1}_{\mathrm{loc}}(I, [0, \infty)) ,$$

where, in the case $n = 2$, we identify $r^0 := \mathbb{1}_{\{r>0\}}$.

Proposition 3.14 (Properties of the area functional for surfaces of revolution)**.** *Let $I \subseteq \mathbb{R}$ be an open set and $r \in \mathrm{L}^{n-1}_{\mathrm{loc}}(I, [0, \infty))$. The following properties are satisfied.*

i) *It holds that $\mathcal{A}_{n,I}^{\mathrm{rev}}[r] < \infty$ if and only if $r^{n-1} \in \mathrm{BV}_{\mathrm{loc}}(I)$ with $|\mathrm{D}r^{n-1}|(I) < \infty$ and $r^{n-2} \in \mathrm{L}^1(I)$. In this case, the following equality holds true*

$$\mathcal{A}_{n,I}^{\mathrm{rev}}[r] = \left| \left(r^{n-2} \mathcal{L}^1, \frac{1}{n-1} \mathrm{D}(r^{n-1}) \right) \right| (I) .$$

ii) *For $r^{n-1} \in \mathrm{W}^{1,1}_{\mathrm{loc}}(I)$, it holds*

$$\mathcal{A}^{\mathrm{rev}}_{n,I}[r] = \int_I r^{n-2}\sqrt{1+(r')^2}\,\mathrm{d}t\,.$$

iii) *If $\mathcal{A}^{\mathrm{rev}}_{n,I}[r] < \infty$ then there exists a representative of r such that the one-sided limits exist at each $t \in I$, and $r \in \mathrm{BV}_{\mathrm{loc}}(\{r^- > 0\}) \cap \mathrm{L}^\infty(I)$.*

iv) *If $\{r > 0\}$ is a set of finite length and $r^{n-1} \in \mathrm{BV}(I)$, then $\mathcal{A}^{\mathrm{rev}}_{n,I}[r] < \infty$. The backward implication holds true in the case $n = 2$.*

Example 3.15. For $n > 2$, there exists $r \in \mathrm{L}^1_{\mathrm{loc}}(I)$ with $\mathcal{A}^{\mathrm{rev}}_{n,I}[r] < \infty$ and $r \notin \mathrm{BV}(I)$. Indeed, take for instance $I = (-1,1)$ and $r(t) = \left|t\sin\left(\frac{1}{t}\right)\right| \in \mathrm{L}^\infty(I)$. Then $r^{n-1} \in \mathrm{W}^{1,1}(I)$ and the corresponding body of revolution in $\mathbb{R}^n$ has area functional

$$\mathcal{A}^{\mathrm{rev}}_{n,I}[r] = \int_I r^{n-2}\sqrt{1+(r')^2}\,\mathrm{d}t \;= \int_I \left|t\sin\left(\frac{1}{t}\right)\right|^{n-2}\sqrt{1+\left(\sin\left(\frac{1}{t}\right)-\frac{1}{t}\cos\left(\frac{1}{t}\right)\right)^2}\,\mathrm{d}t < \infty$$

for all $n > 2$, but $r \notin \mathrm{BV}_{\mathrm{loc}}(-1,1)$.

Proof of Proposition 3.14. We first prove statement i) and consider some non-negative $r \in \mathrm{L}^1_{\mathrm{loc}}(I)$ with $r^{n-1} \in \mathrm{BV}_{\mathrm{loc}}(I)$, $|\mathrm{D}r^{n-1}|(I) < \infty$, and $r^{n-2} \in \mathrm{L}^1(I)$. Then, for φ_0, φ in $\mathrm{C}^1_{\mathrm{cpt}}(I)$ with $\|(\varphi_0,\varphi)\|_{\mathrm{L}^\infty(I,\mathbb{R}^2)} \leqslant 1$, we estimate

$$\left|\int_I \varphi_0 r^{n-2} - \frac{\varphi'}{n-1}r^{n-1}\,\mathrm{d}t\right| = \left|\int_I \varphi_0 r^{n-2}\,\mathrm{d}t + \int_I \frac{\varphi}{n-1}\,\mathrm{d}\mathrm{D}(r^{n-1})(t)\right| \leqslant \|r^{n-2}\|_{\mathrm{L}^1(I)} + \frac{1}{n-1}|\mathrm{D}(r^{n-1})|(I) < \infty\,.$$

Hence, $\mathcal{A}^{\mathrm{rev}}_{n,I}[r]$ is finite by definition. If, on the other hand, it holds $\mathcal{A}^{\mathrm{rev}}_{n,I}[r] < \infty$, we find

$$\|r^{n-2}\|_{\mathrm{L}^1(I)} = \sup\left\{\int_I \varphi r^{n-2}\,\mathrm{d}t : \varphi \in \mathrm{C}^1_{\mathrm{cpt}}(I),\ \|\varphi\|_{\mathrm{L}^\infty(I)} \leqslant 1\right\} \leqslant \mathcal{A}^{\mathrm{rev}}_{n,I}(r) < \infty$$

and

$$\frac{1}{n-1}\mathrm{Var}(r^{n-1},I) = \sup\left\{\int_I \frac{\varphi'}{n-1}r^{n-1}\,\mathrm{d}t : \varphi \in \mathrm{C}^1_{\mathrm{cpt}}(I),\ \|\varphi\|_{\mathrm{L}^\infty(I)} \leqslant 1\right\} \leqslant \mathcal{A}^{\mathrm{rev}}_{n,I}(r) < \infty\,.$$

The equality follows from the Riesz-Markov representation theorem, which gives

$$\left|\left(r^{n-2}\mathcal{L}^1, \frac{1}{n-1}\mathrm{D}(r^{n-1})\right)\right|(I) = \sup\left\{\int_I \varphi_0 r^{n-2}\,\mathrm{d}t + \int_I \frac{\varphi}{n-1}\,\mathrm{d}\mathrm{D}(r^{n-1}) : \varphi_0,\varphi \in \mathrm{C}^0_{\mathrm{cpt}}(I),\ \sup_I|(\varphi_0,\varphi)| \leqslant 1\right\},$$

the density of $\mathrm{C}^1_{\mathrm{cpt}}(I)$ in $\mathrm{C}^0_{\mathrm{cpt}}(I)$ and the definition of derivative measure.
Statement ii) follows from i) by the chain rule for Sobolev functions, and the definition of vector-valued measures.
Property iii) can be concluded from the first observation, Theorem 2.24 and Proposition 2.20 locally applied for $\varphi(t) = |t|^{\frac{1}{n-1}}$. The essential boundedness of r follows from the definition of good representatives and $|\mathrm{D}r|(I) < \infty$.
The first part of iv) follows from $\mathrm{L}^{n-1}(\{r > 0\}) \subseteq \mathrm{L}^{n-2}(\{r > 0\})$ and i). In the case $n = 2$, $\mathbb{1}_{\{r>0\}} = r^0 \in \mathrm{L}^1(I)$ is equivalent to $|\{r > 0\}| < \infty$. Hence, r is a $\mathrm{BV}_{\mathrm{loc}}(I)$ function with $|\mathrm{D}r|(I) < \infty$. The latter implies $r \in \mathrm{L}^1(I)$ by Theorem 2.24. □

Definition 3.16 (Body of revolution)**.** Let $r : I \to [0,\infty)$ be a function on some open set $I \subseteq \mathbb{R}$. Then we call $\mathrm{R}^n_r := \{(x,t) \in \mathbb{R}^{n-1} \times I : |x| < r(t)\}$ the **body of revolution with profile curve r on I**.

Remark 3.17. The body of revolution R^n_r is measurable, whenever r is measurable.

The next statement can be found in a more general form, but with some slightly different viewpoint in [7, Proposition 3.4].

Proposition 3.18 (Perimeter-area relation for bodies of revolution)**.** *For an open set $I \subseteq \mathbb{R}$ and some $r \in \mathrm{L}^{n-1}_{\mathrm{loc}}(I,[0,\infty))$, it holds*
$$\mathrm{P}(\mathrm{R}^n_r, \mathbb{R}^{n-1}\times I) = (n-1)\omega_{n-1}\mathcal{A}^{\mathrm{rev}}_{n,I}[r]\,.$$

Example 3.19. In the following, we consider $n=3$ and define the set R^3_r with profile curve $r(t) := \sum_{k=2}^{\infty}\frac{1}{k}\mathbb{1}_{[\frac{1}{k-1},\frac{1}{k})}$. Clearly, $r, r^2 \in \mathrm{L}^\infty(\mathbb{R})$, and r^2 is even $\mathrm{BV}(\mathbb{R})$ as a jump function with converging jump heights. However, since the harmonic series is diverging to ∞, we find $r \notin \mathrm{BV}_{\mathrm{loc}}(-\varepsilon,\varepsilon)$ for all $\varepsilon > 0$. By Propositions 3.14 and 3.18, the set R^3_r is a set of finite perimeter and finite volume.

The following proof is based on [67, Teorema 1.10], compare also [47, Theorem 14.6].

Proof of Proposition 3.18. First, we notice that for $n=2$, the body of revolution is a rotation of the reflected subgraph of r, intersected with H_0^+ on I. Thus, in this case, we apply Proposition 2.53 and obtain that if R^n_r is a set of finite perimeter, then r is in $\mathrm{BV}_{\mathrm{loc}}(I)$ with $|\mathrm{D}r|(I)<\infty$ and $|\{r>0\}|<\infty$, which, in particular, implies $r\in\mathrm{BV}(I)$ according to Theorem 2.24. On the other hand, if $\mathcal{A}^{\mathrm{rev}}_{n,I}[r]<\infty$, then Proposition 3.14 iv) implies $r\in\mathrm{BV}(I)$ and $|\{r>0\}|<\infty$. We can therefore assume that r is a good representative with existing one-sided limits at each point. Proposition 2.53 gives

$$\begin{aligned}\mathrm{P}(\mathrm{R}^n_r,\mathbb{R}\times I) &= \mathrm{P}(\mathrm{Sup}[-r], I\times(-\infty,0)) + \mathrm{P}(\mathrm{Sub}\,r, I\times(0,\infty)) + \mathrm{P}(\mathrm{Sub}\,r\cap\mathrm{Sup}[-r], I\times\{0\})\\ &= 2\mathcal{A}^{\mathrm{rev}}_{n,I}[r] + \mathrm{P}(\mathrm{Sub}\,r\cap\mathrm{Sup}[-r], I\times\{0\})\,.\end{aligned}$$

The claim follows from the fact that the set $\partial^*(\mathrm{Sub}\,r\cap\mathrm{Sup}[-r])\cap(I\times\{0\})$ is an $\mathcal{H}^1$-null set. Indeed, by Lemma 2.49 and since $I\times(-\infty,0)\subseteq\mathrm{Sub}\,r$, it holds $\nu_{\mathrm{Sub}\,r}=e_2$ on $\partial^*\mathrm{Sub}\,r\cap(I\times\{0\})$ and $\nu_{\mathrm{Sup}\,r}=-e_2$ on $\partial^*\mathrm{Sup}[-r]\cap(I\times\{0\})$. Hence, taking into account the symmetry of $\mathrm{Sub}\,r$ and $\mathrm{Sub}[-r]$, the same statement gives $\partial^*(\mathrm{Sub}\,r\cap\mathrm{Sup}[-r])\cap I\times\{0\}\subseteq\{e_2=-e_2\}=\emptyset$ up to $\mathcal{H}^1$-null sets.
In the following, we consider the case $n\geqslant 3$. Let $\varphi_0,\varphi\in\mathrm{C}^1_{\mathrm{cpt}}(I)$ and $\eta\in\mathrm{C}^1_{\mathrm{cpt}}((0,\infty))$ such that both $\sup_I|(\varphi_0,\varphi)|$ and $\sup_{(0,\infty)}|\eta|$ are bounded from above by 1. Then the vector field V_η, defined by

$$V_\eta:\,\mathbb{R}^{n-1}\times I\to\mathbb{R}^n,\,(x,t)\mapsto\left(\eta(|x|)\varphi_0(t)\frac{x}{|x|},-\eta(|x|)\varphi(t)\right)\,,$$

is in $\mathrm{C}^1_{\mathrm{cpt}}(\mathbb{R}^{n-1}\times I,\mathbb{R}^n)$ with

$$\sup_{\mathbb{R}^{n-1}\times I} V_\eta(x,t)\leqslant\sup_{(0,\infty)}|\eta|\,\sup_I|(\varphi_0,\varphi)|\leqslant 1$$

and
$$\operatorname{div}V_\eta(x,t)=\eta'(|x|)\varphi_0(t)+\eta(|x|)\left[\varphi_0(t)\frac{n-2}{|x|}-\varphi'(t)\right]\qquad\text{for all }(x,t)\in\mathbb{R}^{n-1}\times I.$$

We now consider some function $\eta_k\in\mathrm{C}^1_{\mathrm{cpt}}((0,\infty))$ with $\sup_{(0,\infty)}|\eta_k|\leqslant 1$, $\eta_k=1$ on the interval $\left[\frac{1}{k},k\right]$, $\eta_k=0$ on $[k+2,\infty)$, $|\eta_k'|\leqslant 2k$ on $\left(0,\frac{1}{k}\right]$ and $|\eta_k'|\leqslant 1$ on $[k,\infty)$. According to Fubini's theorem, it follows

$$\begin{aligned}\left|\int_{\mathrm{R}^n_r\cap(\mathbb{R}^{n-1}\times I)}\eta_k'(|x|)\,\varphi_0(t)\,\mathrm{d}(x,t)\right| &= \left|\int_{\mathrm{B}^{n-1}_{\frac{1}{k}}\times I}\eta_k'(|x|)\,\varphi_0(t)\,\mathrm{d}(x,t)+\int_{\mathrm{R}^n_r\cap(\mathbb{R}^{n-1}\setminus\mathrm{B}^{n-1}_k\times I)}\eta_k'(|x|)\,\varphi_0(t)\,\mathrm{d}(x,t)\right|\\ &\leqslant 2\frac{\omega_{n-1}}{k^{n-2}}|\mathrm{spt}\varphi_0|+\left|\mathrm{R}^n_r\cap((\mathbb{R}^{n-1}\setminus\mathrm{B}^{n-1}_k)\times\mathrm{spt}\varphi_0)\right|\\ &\leqslant 2\frac{\omega_{n-1}}{k^{n-2}}|\mathrm{spt}\varphi_0|+\int_{\mathrm{spt}\varphi_0}|\mathrm{B}^{n-1}_{r(t)}\setminus\mathrm{B}^{n-1}_k|\,\mathrm{d}t\to 0\qquad\text{as }k\to\infty\,,\end{aligned}$$

where the latter term vanishes in the limit since $\mathrm{spt}\varphi_0\Subset I$ and $r\in\mathrm{L}^{n-1}_{\mathrm{loc}}(I)$ by assumption. Similarly, we estimate

$$\left|\int_{\mathrm{R}^n_r\cap\setminus(\mathrm{B}^{n-1}_k\times I)}\eta_k(|x|)\left[\varphi_0(t)\frac{n-2}{|x|}-\varphi'(t)\right]\mathrm{d}(x,t)\right|\leqslant\frac{n-2}{k}\int_{\mathrm{spt}\varphi_0}|\mathrm{B}^{n-1}_{r(t)}|\,\mathrm{d}t+\int_{\mathrm{spt}\varphi}|\varphi'(t)|\,|\mathrm{B}^{n-1}_{r(t)}\setminus\mathrm{B}^{n-1}_k|\,\mathrm{d}t\to 0$$

and

$$\left|\int_{\mathrm{R}^n_r\cap\left(\mathrm{B}^{n-1}_{\frac{1}{k}}\times I\right)}\eta_k(|x|)\left[\varphi_0(t)\frac{n-2}{|x|}-\varphi'(t)\right]\mathrm{d}(x,t)\right|\leqslant\left|\mathrm{B}^{n-1}_{\frac{1}{k}}\right|\left[(n-2)k|\mathrm{spt}\varphi_0|+\int_{\mathrm{spt}\varphi}|\varphi'(t)|\,\mathrm{d}t\right]\to 0$$

as $k\to\infty$. By definition of the perimeter, the inequality $\mathrm{P}(\mathrm{R}^n_r,\mathbb{R}^{n-1}\times I)\geqslant\int_{\mathrm{R}^n_r\cap(\mathbb{R}^{n-1}\times I)}\operatorname{div}V_{\eta_k}\,\mathrm{d}(x,t)$ is satisfied for all $k\in\mathbb{N}$. In total, we obtain

$$\begin{aligned}\mathrm{P}(\mathrm{R}^n_r,\mathbb{R}^{n-1}\times I)&\geqslant\liminf_{k\to\infty}\int_{\mathrm{R}^n_r\cap(\mathbb{R}^{n-1}\times I)}\eta_k'(|x|)\varphi_0(t)+\eta_k(|x|)\left[\varphi_0(t)\frac{n-2}{|x|}-\varphi'(t)\right]\mathrm{d}(x,t)\\&=\int_{\mathrm{R}^n_r\cap(\mathbb{R}^{n-1}\times I)}\varphi_0(t)\frac{n-2}{|x|}-\varphi'(t)\,\mathrm{d}(x,t)\,.\end{aligned}$$

Applying Fubini's theorem provides

$$\mathrm{P}(\mathrm{R}^n_r,\mathbb{R}^{n-1}\times I)\geqslant\int_I\int_{\mathrm{B}^{n-1}_{r(t)}}\varphi_0(t)\frac{n-2}{|x|}-\varphi'(t)\,\mathrm{d}x\,\mathrm{d}t=(n-1)\omega_{n-1}\int_I r^{n-2}(t)\varphi_0(t)-\frac{r^{n-1}(t)}{n-1}\varphi'(t)\,\mathrm{d}t\,.$$

Since φ_0, φ were chosen arbitrarily, it follows $\mathrm{P}(\mathrm{R}^n_r,\mathbb{R}^{n-1}\times I)\geqslant(n-1)\omega_{n-1}\mathcal{A}^{\mathrm{rev}}_{n,I}[r]$.
To show the remaining estimate, we assume w.l.o.g. $\mathcal{A}^{\mathrm{rev}}_{n,I}[r]<\infty$, which implies $r^{n-1}\in\mathrm{BV}_{\mathrm{loc}}(I)\cap\mathrm{L}^\infty(I)$ by Proposition 3.14. Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a null sequence and consider the mollifications $(r^{n-1})_{\varepsilon_k}\in\mathrm{C}^\infty([I]_{\varepsilon_k})\cap\mathrm{L}^\infty([I]_{\varepsilon_k})$. We define the bounded functions $s_k:=\left((r^{n-1})_{\varepsilon_k}\right)^{\frac{1}{n-1}}$ on $[I]_{\varepsilon_k}$ for all $k\in\mathbb{N}$ and, using Lemma 2.10, estimate

$$\begin{aligned}\mathcal{A}^{\mathrm{rev}}_{n,[I]_\varepsilon}[s_k]&=\int_{[I]_\varepsilon}\left|\left(s_k^{n-2},\frac{(s_k^{n-1})'}{n-1}\right)\right|\mathrm{d}t\\&\leqslant\int_{[I]_\varepsilon}\left|s_k^{n-2}-(r^{n-2})_{\varepsilon_k}\right|\mathrm{d}t+\int_{[I]_\varepsilon}\left|\left((r^{n-2})_{\varepsilon_k},\frac{(r^{n-1})'_{\varepsilon_k}}{n-1}\right)\right|\mathrm{d}t\\&\leqslant\int_{[I]_\varepsilon}\left|s_k^{n-2}-(r^{n-2})_{\varepsilon_k}\right|\mathrm{d}t+\int_{[I]_\varepsilon}\left|\left(r^{n-2},\frac{(r^{n-1})'}{n-1}\right)_{\varepsilon_k}\right|\mathrm{d}t\\&\leqslant\int_{[I]_\varepsilon}\left|s_k^{n-2}-(r^{n-2})_{\varepsilon_k}\right|\mathrm{d}t+\mathcal{A}^{\mathrm{rev}}_{n,I}[r]\end{aligned}$$

for fixed $\varepsilon>0$ and $k\in\mathbb{N}$ large enough such that $\varepsilon_k<\varepsilon$. By the convergence of s_k to r in $\mathrm{L}^1(I)$ and the convergence of both s_k^{n-2} and $(r^{n-2})_{\varepsilon_k}$ to r^{n-2} in $\mathrm{L}^1(I)$ (where we exploit the essential boundedness of r), we obtain

$$\limsup_{k\to\infty}\mathcal{A}^{\mathrm{rev}}_{n,[I]_\varepsilon}[s_k]\leqslant\mathcal{A}^{\mathrm{rev}}_{n,I}[r]\,.$$

Next, we notice

$$|(\mathrm{R}^n_{s_k}\triangle\mathrm{R}^n_r)\cap(\mathbb{R}^{n-1}\times[I]_\varepsilon)|=\int_{[I]_\varepsilon}|\mathrm{B}^{n-1}_r\triangle\mathrm{B}^{n-1}_{s_k}|\,\mathrm{d}t=\omega_{n-1}\int_{[I]_\varepsilon}|r^{n-1}-s_k^{n-1}|\,\mathrm{d}t\to 0$$

as $k\to\infty$. We conclude

$$\mathrm{P}(\mathrm{R}^n_r,\mathbb{R}^{n-1}\times[I]_\varepsilon)\leqslant\liminf_{k\to\infty}\mathrm{P}(\mathrm{R}^n_{s_k},\mathbb{R}^{n-1}\times[I]_\varepsilon)\leqslant\limsup_{k\to\infty}(n-1)\omega_{n-1}\mathcal{A}^{\mathrm{rev}}_{n,[I]_\varepsilon}[s_k]\leqslant(n-1)\omega_{n-1}\mathcal{A}^{\mathrm{rev}}_{n,I}[r]\,,$$

where we applied that

$$\mathrm{P}(\mathrm{R}^n_s,\mathbb{R}^{n-1}\times I)\leqslant(n-1)\omega_{n-1}\mathcal{A}^{\mathrm{rev}}_{n,I}[s]$$

holds for functions $s:I\to\mathbb{R}$ such that $s^{n-1}\in\mathrm{C}^\infty(I)$, which can be shown by applying the area formula. □

3.1.3 Approximation results

In this section, we collect auxiliary non-standard approximation results for BV functions and sets of finite perimeter.
The first result attempts to suitably control the jump part of the radius and center function of a stack of balls. To this end, we prove some approximation result for the corresponding BV functions as well as for the corresponding sets of finite perimeter.

Proposition 3.20 (Approximation by functions with finitely many jumps). *We consider an open set $I \subseteq \mathbb{R}$ and a set $A \subseteq \mathbb{R}^n$ of finite perimeter in $\mathbb{R}^{n-1} \times I$ with $|A \cap (\mathbb{R}^{n-1} \times I)| < \infty$. Moreover, we assume that A is a stack of balls in I with radius function $r \in \mathrm{BV}(I)$ and center function $c \in \mathrm{BV}(I, \mathbb{R}^{n-1})$ such that $r^- > \delta$ for some $\delta > 0$ on I. Then, for any fixed decomposition of r and c as in Theorem 2.28, there exist two sequences $(r_k)_{k\in\mathbb{N}} \subseteq \mathrm{BV}(I)$ and $(c_k)_{k\in\mathbb{N}} \subseteq \mathrm{BV}(I, \mathbb{R}^{n-1})$, such that $\mathrm{J}_{r_k} \cup \mathrm{J}_{c_k}$ is a finite set and such that there exist decompositions of r_k and c_k that satisfy*

$$
\begin{aligned}
&\mathrm{J}_{r_k} \subseteq \mathrm{J}_r\,, && \mathrm{D}^j r \llcorner \mathrm{J}_{r_k} = \mathrm{D}^j r_k\,, && r_k^a = r^a\,, && r_k^c = r^c\,,\\
&\mathrm{J}_{c_k} \subseteq \mathrm{J}_c\,, && \mathrm{D}^j c \llcorner \mathrm{J}_{c_k} = \mathrm{D}^j c_k\,, && c_k^a = c^a\,, && c_k^c = c^c\,,
\end{aligned}
$$

for all $k \in \mathbb{N}$, where J_r, J_{r_k}, J_c and J_{c_k} denote the jump set of the corresponding function on I. Furthermore, the sequences satisfy

$$
|r - r_k| + |c - c_k| < \frac{1}{k} \qquad \mathcal{L}^1\text{-a.e. on } I \qquad \text{and} \qquad |\mathrm{D}^j r|(\mathrm{J}_r \backslash \mathrm{J}_{r_k}) + |\mathrm{D}^j c|(\mathrm{J}_c \backslash \mathrm{J}_{c_k}) \leqslant \frac{1}{k}
$$

for all $k \in \mathbb{N}$. For sets A_k, defined as the stack of balls in I with radius function r_k and center function c_k, it holds

$$
A_k \to A \quad \text{in } \mathrm{L}^1(\mathbb{R}^{n-1} \times I) \qquad \text{and} \qquad \mathrm{P}(A_k^\star, \mathbb{R}^{n-1} \times I) \to \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times I) \quad \text{as } k \to \infty\,.
$$

Proof. According to Theorem 2.28, we can decompose r and c into the absolutely continuous part r^a, c^a, the Cantor part r^c, c^c, and the jump part r^j, c^j, respectively. Each decomposition is uniquely defined up to a constant and the jump part has at most countably many jumps on I. Indeed, otherwise, the jump heights would sum up to infinity, in contradiction with r and c being BV. With the notation from the aforementioned theorem, we have

$$
\sum_{i=1}^{\infty} |r^r(j_{r,i}) - r^\ell(j_{r,i})| < \infty \qquad \text{and} \qquad \sum_{m=1}^{\infty} |c^r(j_{c,m}) - c^\ell(j_{c,m})| < \infty\,,
$$

where $\mathrm{J}_r \cap I = \{j_{r,1}, j_{r,2}, \dots\}$ and $\mathrm{J}_c \cap I = \{j_{c,1}, j_{c,2}, \dots\}$. Thus, for every $k \in \mathbb{N}$ there exists k_0^*, k_0 in $\mathbb{N}$ with

$$
\sum_{i=k_0^*}^{\infty} |r^r(j_{r,i}) - r^\ell(j_{r,i})| < \frac{1}{2k} \qquad \text{and} \qquad \sum_{m=k_0}^{\infty} |c^r(j_{c,m}) - c^\ell(j_{c,m})| < \frac{1}{2k}\,, \tag{3.4}
$$

and by removing the jumps $j_{r,i}$ and $j_{c,m}$ for $i \geqslant k_0^*$ and $m \geqslant k_0$, respectively, we find pure jump functions

$$
\tilde{r}_k^j := r^j - \sum_{i=k_0^*}^{\infty} [r^r(j_{r,i}) - r^\ell(j_{r,i})]\mathbb{1}_{[j_{r,i},\infty]} \qquad \text{and} \qquad \tilde{c}_k^j := c^j - \sum_{m=k_0}^{\infty} [c^r(j_{c,m}) - c^\ell(j_{c,m})]\mathbb{1}_{[j_{c,i},\infty]}
$$

on I with finitely many jumps $\mathrm{J}_{\tilde{r}_k^j} = \{j_{r,1}, \dots, j_{r,k_0^*}\}$ and $\mathrm{J}_{\tilde{c}_k^j} = \{j_{c,1}, \dots, j_{c,k_0}\}$. Moreover, the construction yields

$$
r^r(j_{r,i}) - r^\ell(j_{r,i}) = (\tilde{r}_k^j)^r(j_{r,i}) - (\tilde{r}_k^j)^\ell(j_{r,i}) \qquad \text{and} \qquad c^r(j_{r,m}) - c^\ell(j_{r,m}) = (\tilde{c}_k^j)^r(j_{r,m}) - (\tilde{c}_k^j)^\ell(j_{r,m})
$$

for all $i \in \{1, \dots, k_0^*\}$ and for all $m \in \{1, \dots, k_0\}$, and

$$
\sup_I |r^j - \tilde{r}_k^j| + \sup_I |c^j - \tilde{c}_k^j| \leqslant \sum_{i=k_0^*}^{\infty} |r^r(j_{r,i}) - r^\ell(j_{r,i})| + \sum_{m=k_0}^{\infty} |c^r(j_{c,m}) - c^\ell(j_{c,m})| < \frac{1}{k}\,.
$$

Finally, we define
$$r_k := r^a + r^c + \tilde{r}_k^j \qquad \text{and} \qquad c_k := c^a + c^c + \tilde{c}_k^j$$
and obtain $r_k^j = \tilde{r}_k^j$, $c_k^j = \tilde{c}_k^j$ and the stated properties of the first part of the claim.
In order to ensure that A_k is well-defined, we consider $k > 0$ large enough such that $\frac{1}{k} < \delta$ and therefore $r_k > 0$ on I. Then we obtain
$$|(A \triangle A_k)_t| \leqslant |A_t| + |(A_k)_t| \leqslant 2\omega_{n-1}\left(r(t) + \frac{1}{k}\right)^{n-1} \leqslant 2^n |A_t|$$
for all $t \in I$. By the dominated convergence theorem which we can apply due to the volume restriction of A, Fubini's theorem and the uniform convergence of r_k and c_k, we conclude
$$\lim_{k\to\infty} |A \triangle A_k| = \lim_{k\to\infty} \int_I |(A\triangle A_k)_t| \,\mathrm{d}t = \int_I \lim_{k\to\infty} |(A\triangle A_k)_t| \,\mathrm{d}t = 0\,.$$
In order to show the last property of the claimed proposition, we observe that Propositions 3.18 and 3.14 and the assumptions on A imply $r \in \mathrm{L}^\infty(I)$, which, combined with $0 \leqslant |\mathrm{D}^j r|(I) - |\mathrm{D}^j r_k|(I) \leqslant \frac{1}{k}$, gives
$$\begin{aligned}\big||\mathrm{D}^j(r_k^{n-1})|(I) - |\mathrm{D}^j(r^{n-1})|(I)\big| &= (n-1)\big|r_k^{n-2}|\mathrm{D}^j r_k|(I) - r^{n-2}|\mathrm{D}^j r|(I)\big| \\ &\leqslant (n-1)\left(r^{n-2}\big||\mathrm{D}^j r_k|(I) - |\mathrm{D}^j r)|(I)\big| + \big|r_k^{n-2} - r^{n-2}\big||\mathrm{D}^j r_k|(I)\right) \\ &\leqslant (n-1)\left(\|r\|_{\mathrm{L}^\infty(I)}^{n-2} + (n-2)\big(1+\|r\|_{\mathrm{L}^\infty(I)}\big)^{n-3}|\mathrm{D}^j r|(I)\right)\frac{1}{k}\end{aligned} \tag{3.5}$$
by Proposition 2.20 and the Lipschitz continuity of $t \mapsto t^{n-2}$ on $[0, \|r\|_{\mathrm{L}^\infty(I)} + 1]$ with Lipschitz constant $(n-2)(1 + \|r\|_{\mathrm{L}^\infty(I)})^{n-3}$ in the case $n > 2$. Similarly, we find
$$\begin{aligned}\big||\mathrm{D}^c(r_k^{n-1})|(I) - |\mathrm{D}^c(r^{n-1})|(I)\big| &= \big||(n-1)r_k^{n-2}\mathrm{D}^c r|(I) - |(n-1)r^{n-2}\mathrm{D}^c r|(I)\big| \\ &\leqslant (n-1)(n-2)\big(1+\|r\|_{\mathrm{L}^\infty(I)}\big)^{n-3}|\mathrm{D}^c r|(I)\frac{1}{k}\end{aligned} \tag{3.6}$$
and, taking into account that $\big|((n-1)\hat{r}^{n-2}\mathcal{L}^1, \mathrm{D}^a(\hat{r}^{n-1}))\big|(I) = \int_I \sqrt{|(n-1)\hat{r}^{n-2}(t)|^2 + |\mathrm{D}^a(\hat{r}^{n-1})(t)|^2}\,\mathrm{d}t$ holds for $\hat{r} = r$ and $\hat{r} = r_k$,
$$\begin{aligned}&\Big|\big|((n-1)r^{n-2}\mathcal{L}^1, \mathrm{D}^a(r^{n-1}))\big|(I) - \big|((n-1)r_k^{n-2}\mathcal{L}^1, \mathrm{D}^a(r_k^{n-1}))\big|(I)\Big| \\ &\qquad\leqslant (n-1)\int_I \Big||r^{n-2}(t)|\sqrt{1+|\mathrm{D}^a r(t)|^2} - |r_k^{n-2}(t)|\sqrt{1+|\mathrm{D}^a r(t)|^2}\Big|\,\mathrm{d}t \\ &\qquad\leqslant (n-1)(n-2)\big(1+\|r\|_{\mathrm{L}^\infty(I)}\big)^{n-3}|(\mathcal{L}^1, \mathrm{D}^a r)|(I)\frac{1}{k}\end{aligned} \tag{3.7}$$
holds for a.e. $t \in I$. Moreover, we observe $A^\star \cap (\mathbb{R}^{n-1} \times I) = \mathrm{R}_r^n$ and $A_k^\star \cap (\mathbb{R}^{n-1} \times I) = \mathrm{R}_{r_k}^n$. Propositions 3.14 and 3.18 imply
$$\begin{aligned}\frac{\mathrm{P}(A^\star, \mathbb{R}^{n-1}\times I)}{\omega_{n-1}} &= \big|((n-1)r^{n-2}\mathcal{L}^1, \mathrm{D}(r^{n-1}))\big|(I) \\ &= \big|((n-1)r_k^{n-2}\mathcal{L}^1, \mathrm{D}(r_k^{n-1}))\big|(I) \\ &\quad + |\mathrm{D}^j(r^{n-1})|(I) - |\mathrm{D}^j(r_k^{n-1})|(I) + |\mathrm{D}^c(r^{n-1})|(I) - |\mathrm{D}^c(r_k^{n-1})|(I) \\ &\quad + \big|((n-1)r^{n-2}\mathcal{L}^1, \mathrm{D}^a(r^{n-1}))\big|(I) - \big|((n-1)r_k^{n-2}\mathcal{L}^1, \mathrm{D}^a(r_k^{n-1}))\big|(I)\end{aligned}$$
for all $k \in \mathbb{N}$. Together with (3.5), (3.6) and (3.7), we conclude
$$\frac{\mathrm{P}(A^\star, \mathbb{R}^{n-1}\times I)}{\omega_{n-1}} = \lim_{k\to\infty} \big|((n-1)r_k^{n-2}\mathcal{L}^1, \mathrm{D}(r_k^{n-1}))\big|(I) = \lim_{k\to\infty} \frac{\mathrm{P}(A_k^\star, \mathbb{R}^{n-1}\times I)}{\omega_{n-1}}\,,$$
where Proposition 3.18 gives the last equality. □

Remark 3.21. i) In the context of Proposition 3.20, we may additionally assume $|\mathrm{D}^j c| \leqslant |\mathrm{D}^j r|$ on J. In this case, we ensure $|\mathrm{D}^j c_k| \leqslant |\mathrm{D}^j r_k|$ on $\mathrm{J}_{r_k} \cup \mathrm{J}_{c_k}$ for the approximations by potentially adding more jumps to r_k. Indeed, during the choice of k_0^* and k_0 in (3.4), we can w.l.o.g. consider $k_0^* \in \mathbb{N}$ large enough such that each $j_{c,i}$ is contained in $\{j_{r,i} : i \in \mathbb{N}_{\leqslant k_0^*}\}$ for all $i \in \{1, \dots, k_0\}$. Then $|\mathrm{D}^j c_k| \leqslant |\mathrm{D}^j r_k|$ follows from the fact that $\mathrm{D}^j r$ and $\mathrm{D}^j c$ coincide with $\mathrm{D}^j r_k$ and $\mathrm{D}^j c_k$ on J_{r_k} and J_{c_k}, respectively.

ii) As in Proposition 3.20, we can show that, if $I \subseteq \mathbb{R}$ is open, any function $u \in \mathrm{BV}(I, \mathbb{R}^N)$ can be approximated by functions $u_k \in \mathrm{BV}(I, \mathbb{R}^N)$ with only finitely many jumps such that $\mathrm{J}_{u_k} \subseteq \mathrm{J}_u$, $|\mathrm{D}^j u|(\mathrm{J}_u \backslash \mathrm{J}_{u_k}) \leqslant \frac{1}{k}$, $|u - u_k| \leqslant \frac{1}{k}$ on I, and such that $u^a = u_k^a$ and $u^c = u_k^c$ hold for a certain choice of decomposition of u and u_k.

In the following, we collect approximation results from [62, 77, 53, 87]. The pathological parts of sets are those of 'horizontal boundary' when dealing with symmetrizations. A fruitful approach to deal with such regions is to obtain the corresponding results for polyhedral sets, then transfer this to arbitrary sets of finite perimeter via the following theorem.

Definition 3.22 (Piecewise affine function)**.** A continuous function $u : \Omega \to \mathbb{R}^N$ is called **piecewise affine** if there exist finitely many open Lipschitz subsets $\Omega_1, \dots, \Omega_k$ of Ω such that $|\Omega \backslash \bigcup_{i=1}^k \Omega_i| = 0$ and $u|_{\Omega_i}$ is an affine function for all $i \in \{1, \dots, k\}$.

Definition 3.23 (Polyhedral set)**.** A set $A \subseteq \mathbb{R}^n$ is called a **polyhedral set** if for each $x \in \partial A$, there exists a radius $r > 0$ and a piecewise affine function $u : \mathrm{B}_r^{n-1}(\overline{x}) \to \mathbb{R}$ such that

$$C_r(x) \cap \partial A = \mathrm{Graph}[u] \qquad \text{and} \qquad C_r(x) \cap A = \mathrm{Sub}[u]$$

up to rotation.

Remark 3.24. Clearly, a polyhedral set A is also Lipschitz, and the union $\partial A \backslash \partial^* A$ of the edges of A is an $\mathcal{H}^{n-1}$-null set. Moreover, the unit normal of A is locally constant $\mathcal{H}^{n-1}$-a.e. on $\partial^* A$.

Theorem 3.25 (Approximation by polyhedral sets)**.** *Let A be a bounded set of finite perimeter and finite volume in $\mathbb{R}^n$. There exists a representative of A and a sequence of bounded open polyhedral sets $A_k \subseteq \mathbb{R}^n$, $k \in \mathbb{N}$ of finite perimeter such that*

$$A_k \to A \textit{ in } \mathrm{L}^1(\mathbb{R}^n), \quad \mathrm{P}(A_k) \to \mathrm{P}(A), \quad \mathrm{dist}_{\mathcal{H}}(\partial A, \partial A_k) \to 0 \textit{ as } k \to \infty\,,$$

where $\mathrm{dist}_{\mathcal{H}}$ *denotes the Hausdorff distance. Furthermore, it holds that* $\mathrm{P}(A_k, B) \to \mathrm{P}(A, B)$ *for all Borel sets* $B \in \mathcal{B}(\mathbb{R}^n)$ *with* $\mathrm{P}(A, \partial B) = 0$.

In [62, Remark 13.13], a proof of the previous result for the convergence in L^1 and with respect to the perimeter is sketched. In order to derive $\lim_{k\to\infty} \sup\{\mathrm{dist}(x, \partial A) : x \in \partial A_k\} = 0$ as well, one can exploit the fact that smooth functions and their gradient can be approximated locally uniformly by piecewise affine functions, see, for instance, [34, Chapter X Proposition 2.1].
For the proof of a generalized version of the Schwarz inequality, we need to overcome horizontal boundary parts of even smooth sets. The following theorem from [77, Theorem 1.1] provides an approximation which contains a positive distance to the original set in order to be able to rotate horizontal boundary parts which may arise during a second approximation by polyhedral sets.

Theorem 3.26 (Approximation by compactly contained smooth sets)**.** *Let $A \subseteq \mathbb{R}^n$ be a bounded open set with $\mathcal{H}^{n-1}(\partial A) = \mathrm{P}(A)$. Then there exists a sequence of smooth open sets $A_k \Subset A$ with $\mathrm{P}(A_k) \to \mathrm{P}(A)$, $A_k \to A$ in L^1, and $\partial A_k \to \partial A$ with respect to the Hausdorff distance as $k \to \infty$.*

Remark 3.27. We can reformulate Theorem 3.26 with a strict exterior instead of a strict inner approximation, i.e., such that the sets A_k in the theorem are bounded and satisfy $A \Subset A_k$.
To this end, we apply Theorem 3.26 on $\mathrm{B}_{2R} \backslash A$ for $R > 0$ large enough such that $A \Subset \mathrm{B}_R$ and find a strict interior approximation of $\mathrm{B}_{2R} \backslash A$ by smooth and bounded sets $(\hat{A}_k)_{k \in \mathbb{N}}$ in L^1 and with respect to the perimeter. Furthermore, $\partial \hat{A}_k$ converges to $\partial(\mathrm{B}_{2R} \backslash A)$ with respect to the Hausdorff distance. We observe that $A \Subset \mathrm{B}_R \backslash \hat{A}_k$ for all $k \in \mathbb{N}$, and that $\mathrm{B}_R \backslash \hat{A}_k$ converges to A in L^1 as $k \to \infty$. Now, by the choice of R, we

find $k_0 \in \mathbb{N}$ such that $\operatorname{dist}(\partial \hat{A}_k, \partial \mathrm{B}_R) > \frac{1}{k_0}$ holds for all $k \in \mathbb{N}_{\geqslant k_0}$, hence $\mathrm{B}_R \backslash \hat{A}_k$ is a smooth and bounded set for all $k \geqslant k_0$. By the lower semicontinuity of the perimeter, we have

$$\begin{aligned}
\mathrm{P}(\mathrm{B}_{2R}\backslash A, \mathrm{B}_R) + \mathrm{P}(\mathrm{B}_{2R}\backslash A, \mathbb{R}^n\backslash \overline{\mathrm{B}}_R) &\leqslant \liminf_{k\to\infty} \mathrm{P}(\hat{A}_k, \mathrm{B}_R) + \liminf_{k\to\infty} \mathrm{P}(\hat{A}_k, \mathbb{R}^n\backslash \overline{\mathrm{B}}_R) \\
&\leqslant \liminf_{k\to\infty} \mathrm{P}(\hat{A}_k) \\
&= \mathrm{P}(\mathrm{B}_{2R}\backslash A) \\
&= \mathrm{P}(\mathrm{B}_{2R}\backslash A, \mathrm{B}_R) + \mathrm{P}(\mathrm{B}_{2R}\backslash A, \mathbb{R}^n\backslash \overline{\mathrm{B}}_R)\,,
\end{aligned}$$

which leads to the convergence of $\mathrm{P}(\mathrm{B}_R\backslash \hat{A}_k) = \mathrm{P}(\hat{A}_k, \mathrm{B}_R)$ to $\mathrm{P}(A)$ and of $\mathrm{P}(\hat{A}_k, \mathbb{R}^n\backslash \overline{\mathrm{B}}_R)$ to $\mathrm{P}(\mathrm{B}_{2R})$ as $k \to \infty$. We finally define $A_k := \mathrm{B}_R\backslash \hat{A}_{k+k_0}$ to obtain the claimed properties for A_k.

As a continuation of the previous result, the following theorem from [53, Theorem 1.8] allows for the conclusion of the generalized Schwarz inequality for sets without pathological boundary portion.

Theorem 3.28 (Approximation of non-pathological sets)**.** *Let $A \subseteq \mathbb{R}^n$ be a bounded set of finite perimeter. It holds that $\mathcal{H}^{n-1}(\partial A \cap A^1) = 0$ if and only if there exist smooth open sets $A_k \Subset A$, $k \in \mathbb{N}$, with $A_k \to A$ in $\mathrm{L}^1(\mathbb{R}^n)$ and $\mathrm{P}(A_k) \to \mathrm{P}(A)$ as $k \to \infty$.*

Remark 3.29. For A as in the previous theorem, we show that $\mathcal{H}^{n-1}(\partial A \cap A^0) = 0$ holds if and only if there exist bounded smooth open sets A_k, $k \in \mathbb{N}$, with $A \Subset A_k$ for all $k \in \mathbb{N}$, $A_k \to A$ in $\mathrm{L}^1(\mathbb{R}^n)$ and $\mathrm{P}(A_k) \to \mathrm{P}(A)$ as $k \to \infty$.
Indeed, if $\mathcal{H}^{n-1}(\partial A \cap A^0) = 0$ holds true, we can use a similar approach as in Remark 3.27 and consider the strict inner approximation $(\hat{A}_k)_{k\in\mathbb{N}}$ of $\mathrm{B}_{2R}\backslash A$ for $R > 0$ large enough such that $A \Subset \mathrm{B}_R$ from Theorem 3.28. This is possible since $\mathcal{H}^{n-1}(\partial(\mathrm{B}_{2R}\backslash A) \cap (\mathrm{B}_{2R}\backslash A)^1) = 0$ follows from the assumption on A. We find some $r \in (R, 2R)$ such that $\mathrm{P}(\hat{A}_k, \partial \mathrm{B}_r) = 0$ for all $k \in \mathbb{N}$ and consider the sets $\mathrm{B}_r\backslash \hat{A}_k$ which contain A compactly and approximate it with respect to the L^1-convergence. As in Remark 3.27, we can now conclude that $\mathrm{B}_r\backslash \hat{A}_k$ converges to A with respect to the perimeter. In order to ensure that we can choose suitable smooth approximations, we apply Theorem 2.40 to $\mathrm{B}_r\backslash \hat{A}_k$ for all $k \in \mathbb{N}$. Since the theorem ensures the convergence of the boundaries with respect to the Hausdorff distance, we find a diagonal sequence that approximates A from the exterior as claimed.
The argument for the other implication is similar.

Finally, the result of [87, Teorema 2] allows an approximation of general sets of finite perimeter by sets without pathological boundary parts.

Theorem 3.30 (Approximation by non-pathological sets)**.** *Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter and finite and positive volume. Then there exist sets $A_k \subseteq A$, $k \in \mathbb{N}$ such that $\partial A_k \cap A_k^0 = \emptyset$ and*

$$A_k \to A \quad \text{in } \mathrm{L}^1(\mathbb{R}^n), \quad \mathrm{P}(A\backslash A_k) \to 0 \quad \text{and} \quad \partial A_k \to \partial A \text{ with respect to the Hausdorff distance as } k \to \infty\,.$$

Remark 3.31. Let A be a bounded set of finite perimeter. Then there exist bounded sets $A_k \subseteq \mathbb{R}^n$, $k \in \mathbb{N}$, with $A \subseteq A_k$ and $\partial A_k \cap A_k^1 = \emptyset$ for all $k \in \mathbb{N}$ such that $A_k \to A$, $\mathrm{P}(A\backslash A_k) \to 0$ and $\partial A_k \to \partial A$ with respect to the Hausdorff measure as $k \to \infty$.
Indeed, taking into account $F^0 = (\mathbb{R}^n\backslash F)^1$ for all measurable sets $F \subseteq \mathbb{R}^n$, we can repeat the argumentation of Remark 3.27 to obtain the claim.

3.1.4 Auxiliary results

In this section, we show further auxiliary results for this chapter.
The following intuitive result on the Hausdorff dimension of extended graphs may be known to experts, but no statement was found by the author in the literature, hence we here give details of the proof.

Definition 3.32 (Extended graph)**.** For $u \in \mathrm{BV}((a,b), \mathbb{R}^N)$ for $a < b$ in $[-\infty, \infty]$, the **extended graph of *u*** is defined as the set

$$\Gamma_u := \left\{(t, u^\ell(t)) : t \in (a,b)\right\} \cup \bigcup_{j\in \mathrm{J}_u} \left[\{j\} \times \left\{u^\ell(j) + s(u^r(j) - u^\ell(j)) : s \in [0,1]\right\}\right]\,.$$

Lemma 3.33 (Hausdorff dimension of extended graphs). *Let $u \in \mathrm{BV}((a,b),\mathbb{R}^N)$ for some numbers $a < b$ in $[-\infty,\infty]$. The Hausdorff dimension of the extended graph of u is equal to 1, and it holds*

$$\mathcal{H}^1(\Gamma_u) = |(\mathcal{L}^1, \mathrm{D}u)|((a,b)) .$$

Proof. The proof is divided into several steps, where the level of regularity for u is weakened iteratively. If not otherwise stated, we will always consider the representative $(\,\cdot\,)^\ell$ from Theorem 2.24.

Step 1: Show the claim for all $u \in \mathrm{C}^1((a,b),\mathbb{R}^N)$:
In this case, the function $v : (a,b) \to (a,b) \times \mathbb{R}^N$, $t \mapsto (t, u(t))$ is a C^1 curve with image Γ_u, which leads to

$$\mathcal{H}^1(\Gamma_u \cap ((a,b) \times \mathbb{R}^N)) = |\mathrm{D}v|((a,b)) = |(\mathcal{L}^1, \mathrm{D}u)|((a,b)) .$$

Step 2: Show the claim for all $u \in \mathrm{BV}((a,b),\mathbb{R}^N)$ with $|\mathrm{D}^j u|((a,b)) = 0$:
We consider $c < d$ in (a,b) and note that u is a continuous function. Hence, we find suitable approximations $u_k \in \mathrm{C}^\infty((c,d),\mathbb{R}^N) \cap \mathrm{C}^0([c,d],\mathbb{R}^N)$ of u, such that u_k converges to u uniformly and with respect to the strict and area-strict convergence in $\mathrm{BV}((c,d),\mathbb{R}^N)$, and such that

$$\liminf_{k\to\infty} |(\mathcal{L}^1, \mathrm{D}u_k)|([c,d]) \leqslant |(\mathcal{L}^1, \mathrm{D}u)|([c,d])$$

is satisfied according to Remark 2.11. The uniform convergence transfers to the extended graphs. In particular, $\Gamma_{u_k} \cap ([c,d] \times \mathbb{R}^N)$ converges to $\Gamma_u \cap ([c,d] \times \mathbb{R}^N)$ with respect to the Hausdorff distance as $k \to \infty$. Since $\Gamma_{u_k} \cap ([c,d] \times \mathbb{R}^N)$ and $\Gamma_u \cap ([c,d] \times \mathbb{R}^N)$ are connected and compact sets, we can apply Gołąb's theorem*, which yields $\mathcal{H}^1(\Gamma_u \cap ([c,d] \times \mathbb{R}^N)) \leqslant \liminf_{k\to\infty} \mathcal{H}^1(\Gamma_{u_k} \cap ([c,d] \times \mathbb{R}^N))$. Additionally, for v defined as in Step 1, it holds by the continuity of u that $|v(t) - v(s)| \leqslant \mathcal{H}^1(\Gamma_u \cap ((s,t) \times \mathbb{R}^N))$ for all $s < t$ in (a,b) and with this we conclude $\mathrm{pVar}(v,(c,d)) \leqslant \mathcal{H}^1(\Gamma_u \cap ([c,d] \times \mathbb{R}^N))$. Finally, with the result of the first step, we estimate

$$\begin{aligned}
|(\mathcal{L}^1, \mathrm{D}u)|((c,d)) &= |\mathrm{D}v|((c,d)) \\
&= \mathrm{pVar}(v,(c,d)) \\
&\leqslant \mathcal{H}^1(\Gamma_u \cap ([c,d] \times \mathbb{R}^N)) \\
&\leqslant \liminf_{k\to\infty} \mathcal{H}^1(\Gamma_{u_k} \cap ([c,d] \times \mathbb{R}^N)) \\
&= \liminf_{k\to\infty} |(\mathcal{L}^1, \mathrm{D}u_k)|([c,d]) \\
&\leqslant |(\mathcal{L}^1, \mathrm{D}u)|([c,d]) .
\end{aligned} \tag{3.8}$$

Again, we use the continuity of u to deduce $|(\mathcal{L}^1, \mathrm{D}u)|(\{c,d\}) = 0$, which leads to equality in the above inequality. In order to transfer this result to the whole interval (a,b), we use a standard exhaustion argument, which yields

$$\mathcal{H}^1\big(\Gamma_u \cap (a,b)\big) = \lim_{\varepsilon\searrow 0} \mathcal{H}^1\big(\Gamma_u \cap [a+\varepsilon, b-\varepsilon]\big) = \lim_{\varepsilon\searrow 0} \big|(\mathcal{L}^1, \mathrm{D}u)\big|([a+\varepsilon, b-\varepsilon)] = \big|(\mathcal{L}^1, \mathrm{D}u)\big|((a,b)) ,$$

confirming the claim in the general case.

Step 3: Show the claim for $u \in \mathrm{BV}((a,b),\mathbb{R}^N)$ with only finitely many jumps:
In this case, we can consider finitely many connected components $I_1, \dots, I_k$ of $(a,b)\backslash \mathrm{J}_u$ and obtain the claim on each I_i, instead of (a,b) as in Step 2. For the jumps, we consider $j \in \mathrm{J}_u$ and obtain

$$\mathcal{H}^1(\Gamma_u \cap (\{j\} \times \mathbb{R}^N)) = |u(j+) - u(j-)| = |(\mathcal{L}^1, \mathrm{D}u)|(\{j\}) .$$

By adding up, the claim follows.
Step 4: Show the claim for all $u \in \mathrm{BV}((a,b),\mathbb{R}^N)$:
With Remark 3.21, we can approximate u by functions u_k with finitely many jumps such that it holds $u^a = u_k^a$, $u^c = u_k^c$, $\mathrm{J}_{u_k} \subseteq \mathrm{J}_u$, $|u - u_k| \leqslant \frac{1}{k}$ and $|\mathrm{D}^j u|(\mathrm{J}_u \backslash \mathrm{J}_{u_k}) \leqslant \frac{1}{k}$ for all $k \in \mathbb{N}$. In fact, u_k approximates u

*Gołąb's theorem can be found, for instance, in [40, Theorem 3.18].

uniformly and area-strict in $\mathrm{BV}((a,b),\mathbb{R}^n)$, such that the same strategy as in Step 2 applies. Indeed, since $\mathrm{J}_u = \mathrm{J}_v$ is a countable set and the jumps coincide with the distance of the one-sided limits of v, we can verify

$$\mathrm{pVar}(v,(c,d)) = \sup\left\{\sum_{i=1}^{k-1} |v(t_{i+1}) - v(t_i)| : k \in \mathbb{N}_{\geqslant 2},\, t_1 < t_2 < \cdots < t_k \text{ in } (c,d)\backslash \mathrm{J}_v\right\},$$

which leads to $|\mathrm{D}v|((c,d)) \leqslant \mathcal{H}^1(\Gamma_u \cap ((c,d)\times\mathbb{R}^N))$ and to (3.8) for all $c<d$ in (a,b) as before. Furthermore, for all $c<d$ in $(a,b)\backslash \mathrm{J}_u$ it holds $|(\mathcal{L}^1,\mathrm{D}u)|((c,d)) = |(\mathcal{L}^1,\mathrm{D}u)|([c,d])$. With this, we find $\mathcal{H}^1\big(\Gamma_u \cap [c,d]\big) = \big|(\mathcal{L}^1,\mathrm{D}u)\big|([c,d])$ for all $c<d$ in $(a,b)\backslash \mathrm{J}_u$. The claim follows by a exhaustion argument with respect to the interval (a,b), which works in this case as well since J_u is countable. □

Lemma 3.34 (Basic regularity of the radius function)**.** *Let I be an open set in $\mathbb{R}$ and let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter and finite volume in $\mathbb{R}^{n-1}\times I$. Then rad_A is a measurable function on I and rad_A^+ and rad_A^- both are representatives of rad_A such that the one-sided limits exist at each point in I and it holds*

$$\mathrm{rad}_A^+(t) := \max\{\mathrm{rad}_A(t+), \mathrm{rad}_A(t-)\} \qquad \textit{and} \qquad \mathrm{rad}_A^-(t) := \min\{\mathrm{rad}_A(t+), \mathrm{rad}(t-)\} \qquad \textit{for all } t \in I\,.$$

In particular, rad_A^- is lower semicontinuous on I and $\{\mathrm{rad}_A^- > 0\}\cap I$ is an open set. Moreover, the radius function of A is in $\mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\}\cap I)\cap \mathrm{L}^\infty(I)$.

Proof. With the definition of the Schwarz symmetrization and Fubini's theorem, it holds $(\mathbb{R}^{n-1}\times I)\cap A^\star = (\mathbb{R}^{n-1}\times I)\cap \mathrm{R}^n_{\mathrm{rad}_A}$ up to null sets for each representative of rad_A. The Schwarz inequality implies that $\mathrm{R}^n_{\mathrm{rad}_A}$ is a set of finite perimeter in $\mathbb{R}^{n-1}\times I$. By Proposition 3.14, the function rad_A^{n-1} is in $\mathrm{BV}_{\mathrm{loc}}(I)\cap\mathrm{L}^\infty(I)$ with $|\mathrm{D}\mathrm{rad}_A^{n-1}|(I) < \infty$. Hence, we find representatives $(\mathrm{rad}_A^{n-1})^\pm$ of rad_A^{n-1}, such that the one-sided limits exist at each point in I and by Theorem 2.24, they coincide with the jumps as claimed. Then, by definition of the approximate limits, we obtain $\mathrm{rad}_A^\pm = ((\mathrm{rad}_A^{n-1})^\pm)^{\frac{1}{n-1}}$, which gives representatives of rad_A with existing one-sided limits at each point in I. In fact, $(\mathrm{rad}_A^{n-1})^-$ is a lower semicontinuous function, which transfers also to rad_A^-, thus follows the openness of $\{\mathrm{rad}_A^- > 0\}$. With $\mathrm{rad}_A \in \mathrm{L}^\infty(I)$ and Proposition 2.20 we conclude $\mathrm{rad}_A \in \mathrm{BV}_{\mathrm{loc}}(\{(\mathrm{rad}_A^{n-1})^- > 0\}) = \mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\})$. □

Lemma 3.35. *Let $A\subseteq\mathbb{R}^n$ be a set of locally finite perimeter. Moreover, let $r\colon I\to[0,\infty)$ and $c\colon I\to\mathbb{R}^{n-1}$ be two regulated functions defined on the open interval I, i.e., for every $t\in I$, the one-sided limits $r(t\pm)$ and $c(t\pm)$ exist. If A is a stack of balls in I with radius function r and center function c, then it holds*

$$\mathrm{Tr}_{H_t^-}\mathbb{1}_A = \mathbb{1}_{\mathrm{B}^{n-1}_{r(t-)}(c(t-))\times\{t\}} \qquad \textit{and} \qquad \mathrm{Tr}_{H_t^+}\mathbb{1}_A = \mathbb{1}_{\mathrm{B}^{n-1}_{r(t+)}(c(t+))\times\{t\}}$$

for all $t\in I$. In particular, it holds

$$\mathrm{P}(A,H_t) = \big|\mathrm{B}^{n-1}_{r(t+)}(c(t+))\triangle\mathrm{B}^{n-1}_{r(t-)}(c(t-))\big|\,.$$

Remark 3.36. For an open interval $I\subseteq\mathbb{R}$ and a set $A\subseteq\mathbb{R}^n$ of finite perimeter in $\mathbb{R}^{n-1}\times I$, Lemma 3.34 implies that rad_A is a regulated function on I. Moreover, $\mathrm{cen}_{A^\star}\equiv 0$ on I, which, according to the previous lemma, leads to

$$\mathrm{P}(A^\star,H_t) = \big|\mathrm{B}^{n-1}_{r(t+)}\triangle\mathrm{B}^{n-1}_{r(t-)}\big|$$

for all $t\in I$.

Proof. Throughout the proof, we abbreviate $B := \mathrm{B}^{n-1}_{r(t-)}(c(t-))$ if $r(t-)>0$ and $B := \{c(t-)\}$ if $r(t-)=0$. Let $x\in(\mathbb{R}^{n-1}\backslash\overline{B})\times\{t\}$ and choose some $\varepsilon>0$ such that $\mathrm{dist}(x,B)>3\varepsilon$. By the one-sided continuity of r and c, we find $\delta>0$ such that

$$|r(t-)-r(s)|\leqslant\varepsilon \qquad \text{and} \qquad |c(t-)-c(s)|\leqslant\varepsilon \qquad \text{for all } s\in(t-\delta,t)\,. \tag{3.9}$$

For $R<\min\{\delta,\varepsilon\}$, we show $\mathrm{B}_R(x)\cap A\cap H_t^- = \emptyset$. Indeed, for each $y=(\overline{y},y_n)\in\mathrm{B}_R(x)\cap A\cap H_t^-$, it holds $\mathrm{dist}(\overline{y},B)>0$ by the choice of R. For $p\in\partial B$ on the line between $c(t-)$ and $\overline{y}$ (i.e., p is the nearest point of

∂B to $\overline{y}$), we estimate

$$\begin{aligned}
\operatorname{dist}(\overline{y}, B) &= |p - \overline{y}| \\
&= |c(t-) - \overline{y}| - |p - c(t-)| \\
&= |c(t-) - \overline{y}| - r(t-) \\
&\leqslant |c(t-) - c(y_n)| + |c(y_n) - \overline{y}| - r(t-) \\
&\leqslant \varepsilon + r(y_n) - r(t-) \leqslant 2\varepsilon\,,
\end{aligned}$$

where we use that the shape of A is stack of balls. This is a contradiction to the choice of ε, hence $\mathrm{B}_R(x) \cap A \cap H_t^- = \emptyset$.
We conclude that, for $\rho < R$, it holds

$$\fint_{\mathrm{B}_\rho(x)\cap H_t^-} \mathbb{1}_A \,\mathrm{d}y = 0\,.$$

With this, the case $r(t-) = 0$ is proven.

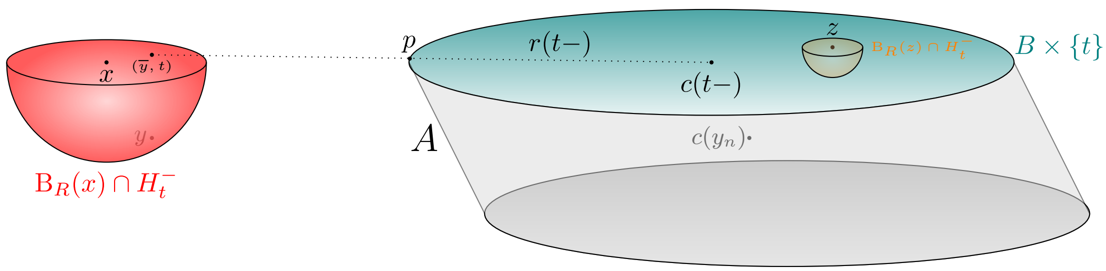


Figure 3.2: Illustration of the proof of Lemma 3.35

If $r(t-) > 0$, we find for each $z \in B \times \{t\}$ some $\varepsilon > 0$ such that $\operatorname{dist}(\overline{z}, \partial B) > 3\varepsilon$. Again, we choose $\delta > 0$ such that (3.9) is satisfied. For $R < \min\{\delta, \varepsilon\}$, we observe

$$|v - (c(v_n), v_n)| \leqslant |\overline{v} - \overline{z}| + |\overline{z} - c(t-)| + |c(t-) - c(v_n)| \leqslant \varepsilon + r(t-) - 3\varepsilon + \varepsilon \leqslant r(v_n)$$

for all $v \in \mathrm{B}_R(z) \cap H_t^-$. Thus, for all $\rho \leqslant R$, it holds

$$\fint_{\mathrm{B}_\rho(z)\cap H_t^-} \mathbb{1}_A \,\mathrm{d}y = 1\,.$$

Since $\partial B \times \{t\}$ is an $\mathcal{H}^{n-1}$-null set, we proved $\operatorname{Tr}_{H_t^-} \mathbb{1}_A = \mathbb{1}_B$. The proof of $\operatorname{Tr}_{H_t^+} \mathbb{1}_A = \mathbb{1}_{\mathrm{B}^{n-1}_{r(t+)}(c(t+))\times\{t\}}$ works analogously, and the perimeter equality follows from Proposition 2.59. $\square$

The following result can be found for the one-dimensional Steiner symmetrization in [14, Corollary 3.4]. For the Schwarz symmetrization, we refer to [26, Lemma 3.1].

Lemma 3.37. *Let I be an open set in $\mathbb{R}$ and let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter and finite volume in $\mathbb{R}^{n-1} \times I$. Then, it holds*

$$\mathrm{B}^{n-1}_{\mathrm{rad}_A^-(t)} \subseteq ((A^\star)^1)_t \subseteq \overline{\mathrm{B}}^{n-1}_{\mathrm{rad}_A^-(t)} \qquad \textit{and} \qquad \mathrm{B}^{n-1}_{\mathrm{rad}_A^+(t)} \subseteq ((A^\star)^+)_t \subseteq \overline{\mathrm{B}}^{n-1}_{\mathrm{rad}_A^+(t)} \qquad \textit{for all } t \in I\,,$$

where we identify $\overline{\mathrm{B}}^{n-1}_0 := \{0\}$.

Remark 3.38. Let A and I be as in Lemma 3.37. Then, by the symmetry of $A^\star$, it holds that

$$((A^\star)^1)_t \in \left\{\mathrm{B}^{n-1}_{\mathrm{rad}_A^-(t)}, \overline{\mathrm{B}}^{n-1}_{\mathrm{rad}_A^-(t)}\right\} \qquad \text{and} \qquad ((A^\star)^+)_t \in \left\{\mathrm{B}^{n-1}_{\mathrm{rad}_A^+(t)}, \overline{\mathrm{B}}^{n-1}_{\mathrm{rad}_A^+(t)}\right\} \qquad \text{for all } t \in I\,.$$

In general, we can neither verify that $((A^\star)^1)_t = \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(t)}$ nor $((A^\star)^1)_t = \overline{\mathrm{B}}^{n-1}_{\mathrm{rad}_A^-(t)}$ holds for all $t \in \mathbb{R}$. Similarly, $((A^\star)^+)_t = \mathrm{B}^{n-1}_{\mathrm{rad}_A^+(t)}$ and $((A^\star)^+)_t = \overline{\mathrm{B}}^{n-1}_{\mathrm{rad}_A^+(t)}$ cannot be expected to hold for all $t \in I$, compare the following example.

Example 3.39. Let $I = \left(-\frac{1}{4}, \frac{1}{4}\right)$ and $A = A^\star := (-1,1) \times I$, then $(A^1)_t = (-1,1)$ and $(A^+)_t = [-1,1]$ for all $t \in I$. For the same I, we define

$$\tilde{A} = \tilde{A}^\star := \bigcup_{t\in I} (-\mathrm{rad}_{\tilde{A}}(t), \mathrm{rad}_{\tilde{A}}(t)) \times \{t\} \qquad \text{with} \qquad \mathrm{rad}_{\tilde{A}}(t) := 1 + \frac{1}{\log|t|}\,.$$

It holds $\mathrm{rad}_{\tilde{A}} \in \mathrm{W}^{1,1}(I)$, and since $\mathrm{rad}_{\tilde{A}}$ is decreasing on $\left(0, \frac{1}{4}\right)$ and $\mathrm{rad}_{\tilde{A}}\left(\mathrm{e}^{-\frac{1}{\rho}}\right) = 1 - \rho$ for small $\rho > 0$, we conclude

$$\frac{|Q_\rho \cap \tilde{A}|}{|Q_\rho|} = \frac{2\int_0^{\mathrm{e}^{-\frac{1}{\rho}}} 1 + \frac{1}{\log t}\,\mathrm{d}t}{4\rho^2} \leqslant \frac{\mathrm{e}^{-\frac{1}{\rho}}}{\rho^2} \to 0 \qquad \text{as } \rho \searrow 0$$

for $Q_\rho := (-\rho,\rho)^2 + (1,0)$.

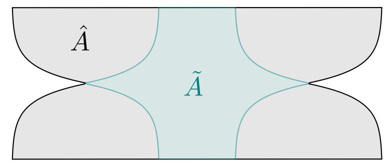


Figure 3.3: The sets $\tilde{A}$ and $\hat{A}$

Hence, the slice $(\tilde{A}^+)_0$ is the open interval $(-1,1)$. Similarly, if we define $\hat{A} = \hat{A}^\star$ via $\mathrm{rad}_{\hat{A}}(t) := 1 - \frac{1}{\log|t|}$ as before, we can verify $(\hat{A}^1)_0 = [-1,1]$.

Proof of Lemma 3.37. We first notice that rad_A is a regulated function on I according to Lemma 3.34. Let $t \in I$ and $x \in \mathrm{B}^{n-1}_{\mathrm{rad}^-_A(t)}$. W.l.o.g., we assume $\mathrm{rad}^-_A(t) = \mathrm{rad}_A(t-)$ and define the value $\varepsilon := \frac{1}{2}\mathrm{dist}\left(x, \partial\mathrm{B}^{n-1}_{\mathrm{rad}^-_A(t)}\right)$, such that, by Lemma 3.34, there exists some $\delta > 0$ with $|x| < \mathrm{rad}^-_A(s) - \varepsilon$ for all $s \in (t-\delta, t+\delta) \subseteq I$. By this choice, it follows $\mathrm{B}^{n-1}_\varepsilon(x) \backslash \mathrm{B}^{n-1}_{\mathrm{rad}^-_A(s)} = \emptyset$ for all $s \in (t-\delta, t+\delta)$. Fubini's theorem yields

$$|\mathrm{B}_{\min\{\varepsilon,\delta\}}(x,t) \backslash A^\star| = \left|\mathrm{B}_{\min\{\varepsilon,\delta\}}(x,t) \backslash \mathrm{R}^n_{\mathrm{rad}^-_A}\right| = 0\,,$$

which leads to $(x,t) \in (A^\star)^1$.
In order to prove the other relation, we consider $x \in \mathbb{R}^{n-1} \backslash \overline{\mathrm{B}}^{n-1}_{\mathrm{rad}^-_A(t)}$. Then, for ε as before, there exists $\delta > 0$ such that $|x| > \mathrm{rad}^-_A(s) + \varepsilon$ for all $s \in (t-\delta, t)$. For $\rho < \min\{\varepsilon, \delta\}$ it follows

$$|\mathrm{B}_\rho(x,t) \cap A^\star| \leqslant |\mathrm{B}_\rho(x,t) \cap H^+_t| = \frac{1}{2}\omega_n \rho^n\,,$$

which implies $x \notin A^1$.
For $x \in \mathrm{B}^{n-1}_{\mathrm{rad}^+_A(t)}$, we find for $\varepsilon := \frac{1}{2}\mathrm{dist}\left(x, \partial\mathrm{B}^{n-1}_{\mathrm{rad}^+_A(t)}\right)$ some $\delta > 0$ satisfying $|x| < \mathrm{rad}^+_A(s) - \varepsilon$, for all $s \in (t, t+\delta)$ such that $|\mathrm{B}_\rho(x,t) \cap A^\star| \geqslant \frac{1}{2}|\mathrm{B}_\rho(x,t)|$ follows for all $\rho \in (0, \min\{\varepsilon,\delta\})$. This implies $x \in (A^\star)^+$.
Finally, for $x \in \mathbb{R}^{n-1} \backslash \overline{\mathrm{B}}^{n-1}_{\mathrm{rad}^+_A(t)}$ and ε as before, there exists $\delta > 0$ with $|x| > \mathrm{rad}^+_A(s) + \varepsilon$ for all $s \in (t-\delta, t+\delta)$. Hence, $\mathrm{B}_{\min\{\varepsilon,\delta\}}(x,t) \subseteq \mathbb{R}^n \backslash A^\star$ holds up to a null set. □

Lemma 3.40. *Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter. Moreover, let $s < t$ in $\mathbb{R}$ and consider two open sets B_s, B_t in $\mathbb{R}^{n-1}$. The equalities*

$$(A^+)_t = (A^1)_t = B_t \qquad \textit{and} \qquad (A^+)_s = (A^1)_s = B_s\,,$$

up to $\mathcal{L}^{n-1}$-null sets, imply

$$|B_s \triangle B_t| \leqslant \mathrm{P}(A, \mathbb{R}^{n-1} \times (s,t))\,.$$

Proof. Let π be the projection to $\mathbb{R}^{n-1}$, defined by $\pi(x) = \overline{x}$ for $x \in \mathbb{R}^n$. Since the Hausdorff measure decreases under projections, it holds

$$\mathcal{H}^{n-1}(\pi(\partial^* A \times (\mathbb{R}^{n-1} \times (s,t)))) \leqslant \mathcal{H}^{n-1}(\partial^* A \times (\mathbb{R}^{n-1} \times (s,t))) = \mathrm{P}(A, \mathbb{R}^{n-1} \times (s,t))\,.$$

Therefore, it is left to show

$$|(B_s \triangle B_t) \backslash \pi(\partial^* A \times (\mathbb{R}^{n-1} \times (s,t)))| = 0\,.$$

We argue for the latter by contradiction and assume that we find some set $F \in \mathcal{M}^{n-1}$ with $F \subseteq B_s \triangle B_t$, $0 < |F|$, and $\partial^* A \cap (F \times (s,t)) = \emptyset$. Then, the De Giorgi structure theorem implies

$$|\mathrm{D}\mathbb{1}_A|(F \times (s,t)) = \mathrm{P}(A, F \times (s,t)) = 0\,.$$

In the following, we assume w.l.o.g. $|F\cap(B_t\backslash B_s)|>0$. By the regularity of Radon measures, we find an open set $\tilde F\supseteq F$ which satisfies $\mathrm{P}(A,\tilde F\times(s,t))<|F\cap(B_t\backslash B_s)|$. By Proposition 2.35, the function $t\mapsto \mathbb{1}_A(x,t)$ is in $\mathrm{BV}((s,t))$ for a.e. $x\in\mathbb{R}^{n-1}$ with

$$\mathrm{D}_n\mathbb{1}_A(\tilde F\times(s,t))=\mathrm{D}\mathbb{1}_A\cdot e_n(\tilde F\times(s,t))=\int_{\tilde F}\mathbb{1}_A(x,t-)-\mathbb{1}_A(x,s+)\,\mathrm{d}x\,. \tag{3.10}$$

By Remark 2.18 and Theorem 2.24, it holds

$$\mathbb{1}_{A^1}(x,\cdot)=(\mathbb{1}_A(x,\cdot))^-\leqslant\mathbb{1}_A(x,\cdot)\leqslant(\mathbb{1}_A(x,\cdot))^+=\mathbb{1}_{A^+}(x,\cdot)$$

for a.e. $x\in\mathbb{R}^{n-1}$ and every good representative of $\mathbb{1}_A$. Together with the assumptions on A and (3.10), we conclude $\mathrm{D}_n\mathbb{1}_A(\tilde F\times(s,t))=|\tilde F\cap B_t|-|\tilde F\cap B_s|=|\tilde F\cap(B_t\backslash B_s)|\geqslant|F\cap(B_t\backslash B_s)|$, which contradicts the assumptions on $\tilde F$. ☐

Lemma 3.41. *For every set $A\subseteq\mathbb{R}^n$ of finite perimeter and finite volume, there exists a sequence $(R_k)_{k\in\mathbb{N}}$ of positive numbers with $R_k\to\infty$ and*

$$\left|\left((A\cap\mathrm{B}_{R_k})^\star)^+\right)_t\right|\to\left|((A^\star)^+)_t\right|$$

as $k\to\infty$ for every $t\in\mathbb{R}$.

Proof. Since A is a set of finite perimeter and finite volume we can consider a sequence $(R_k)_{k\in\mathbb{N}}$ of positive numbers with $R_k\to\infty$ as $k\to\infty$ such that $\mathcal{H}^{n-1}(A^+\cap\partial\mathrm{B}_{R_k})\to0$ as $k\to\infty$. Indeed, the formula for radial integration applied on $\mathbb{1}_{A^+}$ gives $\infty>|A|=\int_0^\infty\mathcal{H}^{n-1}(A^+\cap\partial\mathrm{B}_R)\,\mathrm{d}R$ and therefore $\lim_{r\to\infty}\int_r^\infty\mathcal{H}^{n-1}(A^+\cap\partial\mathrm{B}_R)\,\mathrm{d}R=0$. The latter implies the existence of the sequence $(R_k)_{k\in\mathbb{N}}$ as stated. In the following, we show that $|((A\backslash\mathrm{B}_{R_k})^+)_t|\to0$ uniformly in $t\in\mathbb{R}$ as $k\to\infty$.
To this end, we observe that

$$\mathcal{H}^{n-1}(\partial^*(A\backslash\mathrm{B}_{R_k}))=\mathrm{P}(A\backslash\mathrm{B}_{R_k})\leqslant\mathrm{P}(A,\mathbb{R}^n\backslash\mathrm{B}_{R_k})+\mathcal{H}^{n-1}(A^+\cap\partial\mathrm{B}_{R_k})\to0\qquad\text{as }k\to\infty$$

holds by Lemma 2.49, Theorem 2.47, the choice of R_k and the assumptions on A. Hence, for every $\varepsilon>0$, we find $k_0\in\mathbb{N}$ large enough so that

$$\mathcal{H}^{n-1}(\partial^*(A\backslash\mathrm{B}_{R_k}))<\frac{\varepsilon}{2} \tag{3.11}$$

is verified for all $k\in\mathbb{N}_{\geqslant k_0}$. We now argue that this implies

$$\left|((A\backslash\mathrm{B}_{R_k})^+)_t\right|\leqslant2\varepsilon\qquad\text{for all }t\in\mathbb{R}\text{ and all }k\in\mathbb{N}_{\geqslant k_0}\,. \tag{3.12}$$

Indeed, if we assume that we have

$$2\varepsilon<\left|((A\backslash\mathrm{B}_{R_k})^+)_t\right|=\left|(\partial^*(A\backslash\mathrm{B}_{R_k}))_t\right|+\left|((A\backslash\mathrm{B}_{R_k})^1)_t\right|,$$

then the choice of k implies $\left|((A\backslash\mathrm{B}_{R_k})^1)_t\right|>\varepsilon$. Since $\infty>|A\backslash\mathrm{B}_{R_k}|=\int_{-\infty}^\infty|(A\backslash\mathrm{B}_{R_k})_s|\,\mathrm{d}s$, we find some $s\in\mathbb{R}\backslash\{t\}$ such that

$$\left|((A\backslash\mathrm{B}_{R_k})^+)_s\right|\leqslant\frac{\varepsilon}{2}\,.$$

W.l.o.g. we assume $s<t$ and conclude as in the proof of Lemma 3.40 that the choice of k_0 and s implies

$$\mathrm{P}(A\backslash\mathrm{B}_{R_k},\mathbb{R}^{n-1}\times(s,t))\geqslant\left|((A\backslash\mathrm{B}_{R_k})^1)_t\right|-\left|((A\backslash\mathrm{B}_{R_k})^+)_s\right|\geqslant\varepsilon-\frac{\varepsilon}{2}=\frac{\varepsilon}{2}\,,$$

in contradiction to (3.11). Hence, we have (3.12). By Lemma 3.37, Proposition 3.14 and Theorem 2.24, we can estimate

$$\begin{aligned}\left|\left((A^\star)^+\right)_t\right|-\left|\left(((A\cap\mathrm{B}_{R_k})^\star)^+\right)_t\right|&=\mathrm{vol}_A^+(t)-\mathrm{vol}_{A\cap\mathrm{B}_{R_k}}^+(t)\\&\leqslant\left(\mathrm{vol}_{A^+}(t)-\mathrm{vol}_{(A\cap\mathrm{B}_{R_k})^+}(t)\right)^+\\&=\left(|(A^+)_t|-|((A\cap\mathrm{B}_{R_k})^+)_t|\right)^+\\&\leqslant\left(|((A\backslash\mathrm{B}_{R_k})^+)_t|\right)^+\,.\end{aligned}$$

Together with (3.12), we conclude $|((A^\star)^+)_t| - \left|(((A\cap \mathrm{B}_{R_k})^\star)^+)_t\right| \leqslant 2\varepsilon$ independently of t and for all $k \in \mathbb{N}_{\geqslant k_0}$. By the arbitrariness of ε, we find

$$\lim_{k\to\infty} \left|\left(((A\cap \mathrm{B}_{R_k})^\star)^+\right)_t\right| = \left|\left((A^\star)^+\right)_t\right|$$

for all $t \in \mathbb{R}$. □

Lemma 3.42. *For a set $A \subseteq \mathbb{R}^n$ of finite perimeter, the estimate*

$$\int_K \mathrm{P}(A_t, F)\,\mathrm{d}t \leqslant \mathrm{P}(A, \overline{F}\times K)$$

holds true for all compact sets $K \subseteq \mathbb{R}$ and open sets $F \subseteq \mathbb{R}^{n-1}$.

The proof is analogous to the one in [62, Proposition 14.5] but as the frameworks are not the exactly same, we provide the proof for the altered statement.

Proof. We first note that, according to Theorem 3.7, the slices A_t are sets of finite perimeter for a.e. $t \in \mathbb{R}$. Moreover, since for the mollification $(\mathbb{1}_A)_\varepsilon$ of $\mathbb{1}_A$, it holds that

$$\int_{-R}^{R}\int_{\mathrm{B}_R} |\mathbb{1}_A(x,t) - (\mathbb{1}_A)_\varepsilon(x,t)|\,\mathrm{d}x\,\mathrm{d}t \to 0 \qquad \text{as } \varepsilon \searrow 0$$

for all $R > 0$, we conclude (for a subsequence, which we will denote as the original sequence) that $(\mathbb{1}_A)_\varepsilon(\,\cdot\,,t) \to \mathbb{1}_A(\,\cdot\,,t)$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{n-1})$ for $\mathcal{L}^1$-a.e. $t\in\mathbb{R}$. Taking now some vector field $\Phi \in \mathrm{C}^1_{\mathrm{cpt}}(F, \mathbb{R}^{n-1})$ with $\sup_{\mathbb{R}^{n-1}} |\Phi| \leqslant 1$, the convergence transfers to $(\mathbb{1}_A)_\varepsilon\,(\,\cdot\,,t)\mathrm{div}\,\Phi(\cdot\,)$, leading to

$$\left|\int_{A_t} \mathrm{div}\,\Phi\,\mathrm{d}x\right| = \lim_{\varepsilon\searrow 0}\int_F |\nabla_x(\mathbb{1}_A)_\varepsilon(x,t)\cdot\Phi(x)|\,\mathrm{d}x \leqslant \liminf_{\varepsilon\searrow 0}\int_F |\nabla(\mathbb{1}_A)_\varepsilon(x,t)|\,\mathrm{d}x$$

for a.e. $t \in \mathbb{R}$. An application of Lemma 2.10 and the lemma of Fatou yields

$$\begin{aligned}
\int_K \mathrm{P}(A_t,F)\,\mathrm{d}t &= \int_K \sup\left\{\left|\int_{F\cap A_t}\mathrm{div}\,\Phi\,\mathrm{d}x\right| \,:\, \Phi\in \mathrm{C}^1_{\mathrm{cpt}}(F,\mathbb{R}^{n-1}) \text{ with } \sup_F|\Phi|\leqslant 1\right\}\mathrm{d}t\\
&\leqslant \int_K \liminf_{\varepsilon\searrow 0}\int_F |\nabla(\mathbb{1}_A)_\varepsilon(x,t)|\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant \liminf_{\varepsilon\searrow 0} |\mathrm{D}(\mathbb{1}_A)_\varepsilon|(F\times K)\\
&\leqslant \liminf_{\varepsilon\searrow 0} \mathrm{P}(A,(F\times K)_\varepsilon)\\
&= \mathrm{P}(A,\overline{F}\times K)\,,
\end{aligned}$$

which proves the claim. □

3.2 Characterization of the equality cases in the Schwarz inequality

In this section, we derive the full characterization of the equality cases in the Schwarz inequality. According to Theorem 3.5, we know that if a set $A \subseteq \mathbb{R}^n$ of finite perimeter satisfies $\mathrm{P}(A^\star) = \mathrm{P}(A)$, then A is a stack of balls. In turn, the analysis of the radius and center function of A is crucial for showing the main result.

3.2.1 Regularity of the radius and center function

In Section 3.1.4, we saw that the radius function rad_A of a set of finite perimeter A is locally of bounded variation in $\{\mathrm{rad}_A^- > 0\}$. The purpose of this section is to show that if A is a stack of balls, then the center function cen_A is in $\mathrm{BV}_{\mathrm{loc}}$ as well.

Notation. For all $F \subseteq \mathbb{R}^k$, $k \in \mathbb{N}$ and $I \subseteq \mathbb{R}$, we abbreviate

$$F_I := F \cap (\mathbb{R}^{k-1} \times I) .$$

If $k \geqslant 2$, then we write

$$F' := \{(x, s, t) \in \mathbb{R}^{k-2} \times \mathbb{R} \times \mathbb{R} : (x, t, s) \in F\}$$

for the set that exchanges the positions of the last two components.

Proposition 3.43 (The perimeter formula for Schwarz symmetrizations)**.** *Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter and finite volume. Then, the function*

$$\rho : \mathbb{R}^{n-2} \times \mathbb{R} \to \mathbb{R},\ (x, t) \mapsto \sqrt{(\mathrm{rad}_A^-(t)^2 - |x|^2)_+}$$

is in $\mathrm{BV}_{\mathrm{loc}}(D)$ for the open set $D := \{(x, t) \in \mathbb{R}^{n-2} \times \mathbb{R} : \mathrm{rad}_A^-(t) > |x|\}$. Moreover, for $I \in \mathcal{B}(\{\mathrm{rad}_A^- > 0\})$ and $F \in \mathcal{B}(\mathbb{R}^{n-2})$, it holds

$$\mathrm{P}(A^\star, F \times \mathbb{R} \times I) = 2|(\mathcal{L}^{n-1}, \tilde{\mathrm{D}}\rho)|(D \cap (F \times I)) + \sum_{j \in \mathrm{J}_{\mathrm{rad}_A} \cap I} \left|\left(\mathrm{B}^{n-1}_{\mathrm{rad}_A^+(j)} \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(j)}\right) \cap (F \times \mathbb{R})\right| . \tag{3.13}$$

The corresponding statement for the Steiner symmetrization can be found in [14, Corollary 3.4].

Remark 3.44. We draw attention to the fact that, with the assumptions and the notation from the previous proposition, $\mathrm{D}^j\rho(D)$ does not fully measure $\partial^* A^\star \cap (\mathbb{R}^{n-1} \times \mathrm{J}_{\mathrm{rad}_A})$ because of the definition of D.

Proof of Proposition 3.43. First, we record that the essential boundedness and the lower semicontinuity of rad_A from Lemma 3.34 transfers to ρ on all of $\mathbb{R}^{n-1}$.
The subgraph of ρ over $\mathbb{R}^{n-1}$ coincides with $(A^\star)' \cup \mathbb{R}^n_-$ up to null sets, where we set $\mathbb{R}^k_- := \mathbb{R}^{k-1} \times (-\infty, 0)$ for all $k \in \mathbb{N}_{\geqslant 2}$. Indeed, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, it holds $(A^\star \cup \mathbb{R}^{n}_-{}')_t = \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(t)} \cup \mathbb{R}^{n-1}_-$, which correlates to the corresponding (rotated) slice of $\mathrm{Sub}\,\rho$. According to Proposition 2.53, we find $\rho \in \mathrm{BV}_{\mathrm{loc}}(D)$ with $|\mathrm{D}\rho|(D) < \infty$ and $|\{\rho > 0\}| < \infty$. Moreover, ρ is continuous on $\mathbb{R}^{n-2} \times (I \backslash \mathrm{J}_{\mathrm{rad}_A})$ since $\mathrm{J}_\rho \subseteq (\overline{D})_{\mathrm{J}_{\mathrm{rad}_A}}$. Now, let

$$S_+ := \{(x, s, t) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R} : s < \rho(x, t)\} = \mathrm{Sub}[\rho]'$$

and

$$S_- := \{(x, s, t) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R} : s > -\rho(x, t)\} = \mathrm{Sup}[-\rho]'$$

denote the rotated sub and supergraph of ρ and $-\rho$, respectively. Then, we can identify

$$A^\star = S_- \cap S_+$$

up to null sets. This gives

$$\begin{aligned} \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times I) &= \mathrm{P}(A^\star, \mathbb{R}^{n-2} \times (0, \infty) \times I) + \mathrm{P}(A^\star, \mathbb{R}^{n-2} \times (-\infty, 0) \times I) + \mathrm{P}(A^\star, \mathbb{R}^{n-2} \times \{0\} \times I) \\ &= \mathrm{P}(S_+, \mathbb{R}^{n-2} \times (0, \infty) \times I) + \mathrm{P}(S_-, \mathbb{R}^{n-2} \times (-\infty, 0) \times I) + \mathrm{P}(A^\star, \mathbb{R}^{n-2} \times \{0\} \times I) \end{aligned}$$

for all Borel sets $I \subseteq \{\mathrm{rad}_A^- > 0\}$.
In the following, we show $\mathrm{P}(A^\star, \mathbb{R}^{n-2} \times \{0\} \times I) = 0$ by arguing for $n = 2$ and $n > 2$ separately.
For $n = 2$ and all $\varepsilon > 0$, we define the set $I_\varepsilon := I \cap \{\mathrm{rad}_A^- > \varepsilon\}$, such that $\{0\} \times I_\varepsilon \subseteq (A^\star)^1$ follows by the definition of $A^\star$. Hence, we have $\mathrm{P}(A^\star, \{0\} \times I_\varepsilon) = \mathcal{H}^1(\partial^* A^\star \cap (\{0\} \times I_\varepsilon)) = 0$, and by sending ε to 0, the same holds true for I instead of I_ε in the two-dimensional case.
In the case $n > 2$, we argue by contradiction and assume $\mathrm{P}(A^\star, \mathbb{R}^{n-2} \times \{0\} \times I) > 0$. The invariance of the perimeter under rotation gives $\mathrm{P}(TA^\star, T[\mathbb{R}^{n-2} \times \{0\} \times I]) > 0$ for all rotations T that leave the n-th axis fixed, i.e., with $T(0, t) = (0, t)$ for all $t \in \mathbb{R}$. For such rotations, the definition of the Schwarz symmetrization gives $TA^\star = A^\star$, and therefore, $\mathrm{P}(A^\star, T[\mathbb{R}^{n-2} \times \{0\} \times I]) > 0$ holds for all those T. In fact, we find infinitely many rotations $T \neq \tilde{T}$ that satisfy $T[\mathbb{R}^{n-2} \times \{0\} \times I] \cap \tilde{T}[\mathbb{R}^{n-2} \times \{0\} \times I] = \{0\} \times I$, which is an $\mathcal{H}^{n-1}$-negligible set. We therefore conclude

$$\mathrm{P}(A^\star) \geqslant \sum_{k=1}^{\infty} \mathrm{P}(A^\star, T_i[\mathbb{R}^{n-2} \times \{0\} \times I]) = \sum_{k=1}^{\infty} \mathrm{P}(A^\star, \mathbb{R}^{n-2} \times \{0\} \times I) = \infty$$

for any rotations T_i chosen as described with $T_k \neq T_i$ for $k \neq i$ in $\mathbb{N}$, in contradiction to the choice of A. According to Remark 2.54 and the definition of D, it follows

$$\begin{aligned} \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times (I \backslash \mathrm{J}_{\mathrm{rad}_A})) &= \mathrm{P}(S_+, \mathbb{R}^{n-2} \times (0,\infty) \times (I \backslash \mathrm{J}_{\mathrm{rad}_A})) + \mathrm{P}(S_-, \mathbb{R}^{n-2} \times (-\infty, 0) \times (I \backslash \mathrm{J}_{\mathrm{rad}_A})) \\ &= \mathrm{P}(\mathrm{Sub}\,\rho, \mathbb{R}^{n-2} \times (I \backslash \mathrm{J}_{\mathrm{rad}_A}) \times (0,\infty)) + \mathrm{P}(\mathrm{Sup}[-\rho], \mathbb{R}^{n-2} \times (I \backslash \mathrm{J}_{\mathrm{rad}_A}) \times (-\infty, 0)) \\ &= 2|(\mathcal{L}^{n-1}, \mathrm{D}\rho)|(D_{I \backslash \mathrm{J}_{\mathrm{rad}_A}}), \end{aligned} \tag{3.14}$$

where we crucially exploited the continuity of ρ on $\mathbb{R}^{n-2} \times I \backslash \mathrm{J}_{\mathrm{rad}_A}$, compare Remark 3.44.
Now, for $j \in \mathrm{J}_{\mathrm{rad}_A} \cap I$, it holds

$$\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times \{j\}) = \mathcal{H}^{n-1}((\partial^* A^\star)_t) = \left|\mathrm{B}^{n-1}_{\mathrm{rad}^+_A(t)} \backslash \mathrm{B}^{n-1}_{\mathrm{rad}^-_A(t)}\right| \tag{3.15}$$

according to Lemma 3.37. Together, (3.14) and (3.15) yield (3.13) in the special case $F = \mathbb{R}^{n-2}$. The argument for arbitrary sets $F \in \mathcal{B}(\mathbb{R}^{n-2})$ works analogously. $\square$

Lemma 3.45. *We consider A, I, ρ and D as in Proposition 3.43. Then the equations*

i) $|\mathrm{D}^c\rho|(D_I) = \frac{\omega_{n-1}}{2}|\mathrm{D}^c(\mathrm{rad}^{n-1}_A)|(I)$,

ii) $\sum_{j \in \mathrm{J}_{\mathrm{rad}_A} \cap I} \left|\mathrm{B}^{n-1}_{\mathrm{rad}^+_A(j)} \triangle \mathrm{B}^{n-1}_{\mathrm{rad}^-_A(j)}\right| = \omega_{n-1}|\mathrm{D}^j(\mathrm{rad}^{n-1}_A)|(I)$,

iii) $\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times I) = 2|(\mathcal{L}^{n-1}, \mathrm{D}^a\rho)|(D_I) + \omega_{n-1}|\mathrm{D}^s(\mathrm{rad}^{n-1}_A)|(I)$,

iv) $2|(\mathcal{L}^{n-1}, \mathrm{D}^a\rho)|(D_I) = (n-1)\omega_{n-1}\mathrm{rad}^{n-2}_A|(\mathcal{L}^1, \mathrm{D}^a\mathrm{rad}_A)|(I)$

hold true.

Remark 3.46. Lemma 3.45 implies $\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times I) = (n-1)\omega_{n-1}\mathrm{rad}^{n-2}_A|(\mathcal{L}^1, \mathrm{D}^a\mathrm{rad}_A)|(I) + \omega_{n-1}|\mathrm{D}^s(\mathrm{rad}^{n-1}_A)|(I)$ for sets A of finite perimeter and finite volume and for Borel sets $I \subseteq \{\mathrm{rad}^-_A > 0\}$, which is also a consequence of [7, Proposition 3.4], compare also [26, Corollary 2.4].

Proof of Lemma 3.45. For showing i), we first apply Proposition 2.20 to obtain

$$\mathrm{D}^c\rho = \frac{(0,r)}{\rho}(\mathcal{L}^{n-2} \otimes \mathrm{D}^c\mathrm{rad}_A),$$

where $r(x,t) = \mathrm{rad}^-_A(t)$ for all $(x,t) \in \mathbb{R}^{n-2} \times I$. Fubini's theorem yields

$$\begin{aligned} |\mathrm{D}^c\rho|(D_I) &= \frac{r}{\rho}(\mathcal{L}^{n-2} \otimes |\mathrm{D}^c\mathrm{rad}_A|)(D_I) \\ &= \int_{D_I} \frac{\mathrm{rad}^-_A(t)}{\sqrt{\mathrm{rad}^-_A(t)^2 - |x|^2}} \,\mathrm{d}(\mathcal{L}^{n-2} \otimes |\mathrm{D}^c\mathrm{rad}_A|)(x,t) \\ &= \int_I \int_{\mathrm{B}^{n-2}_{\mathrm{rad}^-_A(t)}} \frac{\mathrm{rad}^-_A(t)}{\sqrt{\mathrm{rad}^-_A(t)^2 - |x|^2}} \,\mathrm{d}x \,\mathrm{d}|\mathrm{D}^c\mathrm{rad}_A|(t). \end{aligned}$$

We draw attention to the fact that in the case $n = 2$, the computation above simplifies since $\rho = \mathrm{rad}^-_A$, hence $\mathrm{D}^c\rho = \mathrm{D}^c\mathrm{rad}_A$. In the general case, a suitable substitution gives

$$|\mathrm{D}^c\rho|(D_I) = \int_I (\mathrm{rad}^-_A)^{n-2} \,\mathrm{d}|\mathrm{D}^c\mathrm{rad}_A|(t) \int_{\mathrm{B}^{n-2}_1} \frac{1}{\sqrt{1 - |x|^2}} \,\mathrm{d}x. \tag{3.16}$$

We now show via an inductive argument that the latter integral equals $\pi\omega_{n-3} = \frac{(n-1)}{2}\omega_{n-1}$. To this end, we record that

$$\int_{\mathrm{B}^k_1} \frac{1}{\sqrt{1 - |x|^2}} \,\mathrm{d}x = \frac{k+1}{2}\omega_{k+1} \tag{3.17}$$

follows for $k = 1$ and $k = 2$ by standard integration methods. We assume that (3.17) holds true for all $k \in \{1, 2, \dots, n-3\}$, where we consider $n \geqslant 5$ from now on. With Fubini's theorem, we determine

$$\begin{aligned}
\int_{\mathrm{B}_1^{n-2}} \frac{1}{\sqrt{1-|x|^2}} \,\mathrm{d}x &= \int_{\mathrm{B}_1^2} \int_{\mathrm{B}^{n-4}_{\sqrt{1-|v|^2}}} \frac{1}{\sqrt{1-|v|^2-|y|^2}} \,\mathrm{d}y \,\mathrm{d}v \\
&= \int_{\mathrm{B}_1^2} \sqrt{1-|v|^2}^{\,n-5} \,\mathrm{d}v \int_{\mathrm{B}_1^{n-4}} \frac{1}{\sqrt{1-|z|^2}} \,\mathrm{d}z \\
&= \frac{n-3}{2}\omega_{n-3} \int_{\mathrm{B}_1^2} \sqrt{1-|v|^2}^{\,n-5} \,\mathrm{d}v \\
&= \frac{n-3}{2}\omega_{n-3}\, 2\pi \int_0^1 t\sqrt{1-t^2}^{\,n-5} \,\mathrm{d}t \\
&= \pi\omega_{n-3}\frac{n-3}{2} \int_0^1 s^{\frac{n-5}{2}} \,\mathrm{d}s \\
&= \pi\omega_{n-3}\,.
\end{aligned}$$

Hence, by another application of the chain rule for BV functions, (3.16) can be rewritten as

$$|\mathrm{D}^c\rho|(D_I) = \frac{\omega_{n-1}}{2} \int_I (n-1)(\mathrm{rad}_A^-)^{n-2} \,\mathrm{d}|\mathrm{D}^c \mathrm{rad}_A|(t) = \frac{\omega_{n-1}}{2} |\mathrm{D}^c(\mathrm{rad}_A^{n-1})|(I)\,.$$

The analysis of the jump parts gives

$$\begin{aligned}
\sum_{j \in \mathrm{J}_{\mathrm{rad}_A} \cap I} \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A^+(j)} \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(j)}\right| &= \omega_{n-1} \sum_{j \in \mathrm{J}_{\mathrm{rad}_A} \cap I} |(\mathrm{rad}_A^+)^{n-1}(j) - (\mathrm{rad}_A^-)^{n-1}(j)| \\
&= \omega_{n-1} \sum_{j \in \mathrm{J}_{\mathrm{rad}_A} \cap I} |\mathrm{D}^j(\mathrm{rad}_A^{n-1})|(\{j\}) \\
&= \omega_{n-1} |\mathrm{D}^j(\mathrm{rad}_A^{n-1})|(I)\,.
\end{aligned}$$

Therefore, the estimate for the perimeter of the symmetrization is verified by Proposition 3.43. To show iv), we apply Proposition 3.18 to obtain

$$\omega_{n-1}\left|((n-1)\mathrm{rad}_A^{n-2}\mathcal{L}^1, \mathrm{D}(\mathrm{rad}_A^{n-1})\right|(I) = \mathrm{P}(A^\star, \mathbb{R}^{n-1}\times I) = 2|(\mathcal{L}^{n-1}, \mathrm{D}^a\rho)|(D_I) + \omega_{n-1}|\mathrm{D}^s(\mathrm{rad}_A^{n-1})|(I)\,.$$

In particular, applying Theorem 2.16 yields

$$\omega_{n-1}\left|((n-1)\mathrm{rad}_A^{n-2}\mathcal{L}^1, \mathrm{D}^a(\mathrm{rad}_A^{n-1})\right|(I) = 2\left|(\mathcal{L}^{n-1}, \mathrm{D}^a\rho)\right|(D_I)$$

and since $\mathrm{rad}_A \in \mathrm{BV}_{\mathrm{loc}}(I)$, we can further apply Proposition 2.20 and obtain

$$(n-1)\omega_{n-1}\mathrm{rad}_A^{n-2}|(\mathcal{L}^1, \mathrm{D}^a\mathrm{rad}_A)|(I) = 2|(\mathcal{L}^{n-1}, \mathrm{D}^a\rho)|(D_I)\,. \qquad \square$$

Our next goal is to conclude the announced regularity result for the center function and to find a perimeter formula for general stacks of balls similar to the one in Proposition 3.43, that involves the center function. The following remark derives this formula for the special case $n = 3$ and with rad_A, cen_A in C^1.

Remark 3.47. Let

$$A := \{(y,t) \in \mathbb{R}^2 \times \mathbb{R} \,:\, |y - \mathrm{cen}(t)| < \mathrm{rad}(t)\}$$

be a set of finite perimeter, where both $\mathrm{cen}(t) = (0, c(t)) : \mathbb{R} \to \mathbb{R}^2$ and $\mathrm{rad} : \mathbb{R} \to [0,\infty)$ are C^1 functions on $\mathbb{R}$. For any open set $I \subseteq \{\mathrm{rad} > 0\}$, we define the bijective C^1 function

$$F : [0, 2\pi) \times I \to (\partial A)_I,\ (\vartheta, t) \mapsto (\mathrm{rad}(t)\sin\vartheta, \mathrm{rad}(t)\cos\vartheta + c(t), t)\,.$$

Hence, the transformation formula and Fubini's theorem give

$$\begin{aligned}
\mathrm{P}(A, \mathbb{R}^2 \times I) &= \mathcal{H}^2(\partial A \cap (\mathbb{R}^2 \times I)) \\
&= \mathcal{H}^2(F([0,2\pi) \times I)) \\
&= \int_I \mathrm{rad}(t) \int_0^{2\pi} \sqrt{1 + |c'(t)\cos\vartheta + \mathrm{rad}'(t)|^2} \,\mathrm{d}\vartheta \,\mathrm{d}t\,.
\end{aligned} \tag{3.18}$$

We define the functions $\hat{c}(y,t) := c(t)$ for all $(y,t) \in \mathbb{R} \times I$ and $\rho(y,t) := \sqrt{(\mathrm{rad}(t)^2 - y^2)_+}$ for $(y,t) \in \mathbb{R} \times I$. For the set $D := \{\rho > 0\}$, it holds $D_t = (-\mathrm{rad}(t), \mathrm{rad}(t))$ for all $t \in I$, which yields

$$\begin{aligned}
|(\mathcal{L}^2, \mathrm{D}(\rho \pm \hat{c}))|(D_I) &= \int_{D_I} \sqrt{1 + |\mathrm{D}(\rho(y,t) \pm \hat{c}(y,t))|^2}\,\mathrm{d}(y,t) \\
&= \int_I \int_{-\mathrm{rad}(t)}^{\mathrm{rad}(t)} \sqrt{\frac{\mathrm{rad}(t)^2}{\mathrm{rad}(t)^2 - y^2} + \frac{(\mathrm{rad}'(t))^2 \mathrm{rad}(t)^2}{\mathrm{rad}(t)^2 - y^2} + (c'(t))^2 \pm 2c'(t)\frac{\mathrm{rad}'(t)\mathrm{rad}(t)}{\sqrt{\mathrm{rad}(t)^2 - y^2}}}\,\mathrm{d}y\,\mathrm{d}t \\
&= \int_I \mathrm{rad}(t) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\vartheta \sqrt{\frac{1}{(\cos\vartheta)^2} + \frac{(\mathrm{rad}'(t))^2}{(\cos\vartheta)^2} + (c'(t))^2 \pm 2c'(t)\frac{\mathrm{rad}'(t)}{\cos\vartheta}}\,\mathrm{d}\vartheta\,\mathrm{d}t \\
&= \int_I \mathrm{rad}(t) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + |c'(t)\cos\vartheta \pm \mathrm{rad}'(t)|^2}\,\mathrm{d}\vartheta\,\mathrm{d}t\,.
\end{aligned}$$

Hence,

$$|(\mathcal{L}^2, \mathrm{D}(\rho + \hat{c}))|(D_I) + |(\mathcal{L}^2, \mathrm{D}(\rho - \hat{c}))|(D_I) = \mathrm{P}(A, \mathbb{R}^2 \times I)$$

follows by (3.18) and symmetry properties.

Theorem 3.48 (Regularity of radius and center functions)**.** *Let $A \subseteq \mathbb{R}^n$ be a stack of balls which has finite perimeter and finite volume. Then the following properties hold true.*

i) *The radius function rad_A of A is in $\mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\}) \cap \mathrm{L}^\infty(\mathbb{R})$.*

ii) *The center function cen_A of A is in $\mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\}, \mathbb{R}^{n-1}) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\}, \mathbb{R}^{n-1})$.*

iii) *If $\mathrm{cen}_A = ce_{n-1}$ on some open set that contains $I \in \mathcal{B}(\{\mathrm{rad}_A^- > 0\})$ and for a function $c : \{\mathrm{rad}_A^- > 0\} \to \mathbb{R}$, then it holds that*

$$\begin{aligned}
\mathrm{P}(A, F \times \mathbb{R} \times I) &= |(\mathcal{L}^{n-1}, \tilde{\mathrm{D}}(\hat{c} + \rho))|(D \cap (F \times I)) + |(\mathcal{L}^{n-1}, \tilde{\mathrm{D}}(\hat{c} - \rho))|(D \cap (F \times I)) \\
&\quad + \sum_{j \in \mathrm{J} \cap I} \left|\left(\mathrm{B}^{n-1}_{\mathrm{rad}_A(j+)}(\mathrm{cen}_A(j+)) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(j-)}(\mathrm{cen}_A(j-))\right) \cap (F \times \mathbb{R})\right|,
\end{aligned} \tag{3.19}$$

for all $F \in \mathcal{B}(\mathbb{R}^{n-2})$, where we identify $\hat{c}(x,t) = c(t)$ for all $(x,t) \in D$, define $\mathrm{J} := \mathrm{J}_{\mathrm{rad}_A} \cup \mathrm{J}_{\mathrm{cen}_A}$, consider good representatives of rad_A and cen_A, and define the function ρ as in Proposition 3.43.

The main result of this theorem is the regularity of the center function. To ensure this, we must show that cen_A is indeed a regulated function on $\{r^- > 0\}$. Then, the continuous part and the jump part of the center function can be treated separately. Lemma 3.35 gives the summability of the jumps of cen_A, whereas, for continuous center functions, we argue similar to the proof of Proposition 3.43. Then, as in Proposition 3.20, the approximation of cen_A by functions with countably many jumps gives the claim in full generality.
The formula (3.19) will be crucial in the proof of Proposition 3.53, where we consider the cases of equality of the Schwarz inequality on the non-vanishing radius parts. Thanks to the one-dimensional Steiner inequality, we will be able to reduce the general case to center functions that vary in only one variable.
Since the center function can become pathological when the radius approaches 0, we cannot extend the regularity of cen_A from the previous proposition to larger sets. The following example from [7, Example 4.2] illustrates that the given regularity is optimal.

Example 3.49. We consider the set

$$A := \left\{(x,t) \in \mathbb{R}^2 \times (-1,1) : \left|x - \left(\frac{1}{t}, 0\right)\right| < t^2\right\}.$$

According to Remark 3.47 and Lemma 3.35, we obtain

$$\begin{aligned}
\mathrm{P}(A) &= \mathrm{P}(A, \mathbb{R}^2 \times (-1,1)) + \mathrm{P}(A, \mathbb{R}^2 \times \{-1,1\}) \\
&= \int_{-1}^{1} t^2 \int_0^{2\pi} \sqrt{1 + \left|-\frac{\cos\vartheta}{t^2} + 2t\right|^2}\,\mathrm{d}\vartheta\,\mathrm{d}t + 2\omega_{n-1} \\
&\leqslant 2\pi \int_{-1}^{1} t^2 + 1 + 2t^3\,\mathrm{d}t + 2\omega_{n-1} < \infty\,.
\end{aligned}$$

Furthermore, the center function $\mathrm{cen}_A = \left(0, \frac{1}{t}\right)$ is not even locally integrable on $(-1,1)$, but it is in the space $\mathrm{BV}_{\mathrm{loc}}((-1,1)\backslash\{0\}, \mathbb{R}^2) = \mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\}, \mathbb{R}^2)$.

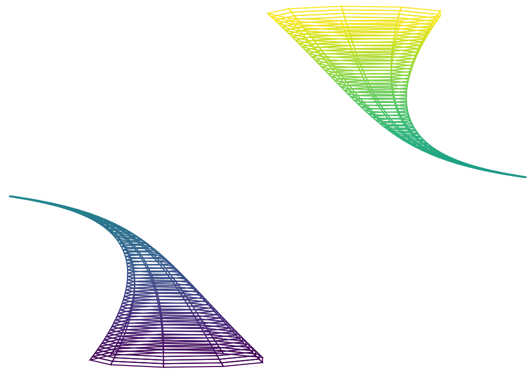


Figure 3.4: The surface of the set A in Example 3.49

The following example shows that the regularity analysis of the radius function also cannot be improved.

Example 3.50. Consider the bounded body of revolution

$$A := \mathrm{R}^3_r\,,$$

where the radius function $r = \mathrm{rad}_A$ of A is defined as $\sum_{k=1}^{\infty} \frac{1}{k} \mathbb{1}_{\left(\frac{1}{2k}, \frac{1}{2k-1}\right]}$. The function r is in $\mathrm{L}^\infty(\mathbb{R})$ and

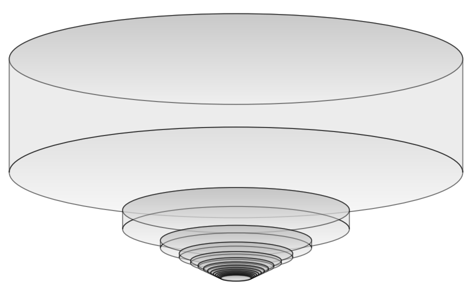


Figure 3.5: The set $A \cap \mathbb{R}^2 \times (\varepsilon, 1)$ in Example 3.50

vanishes at the origin. However, since each jump at $\frac{1}{2k}$ has height $\frac{1}{k}$, it is not a BV function in any neighborhood of 0. Nonetheless, we record $r \in \mathrm{BV}_{\mathrm{loc}}(\{r^- > 0\})$, $|A| \leqslant \pi$ and

$$\mathrm{P}(A) = 2\pi \sum_{k=1}^{\infty} \frac{1}{k^2} + \frac{1}{k}\left(\frac{1}{2k-1} - \frac{1}{2k}\right) < \infty\,.$$

Note that any stack of balls with radius function r and a center function which is a.e. constant on each interval $\left(\frac{1}{2k}, \frac{1}{2k-1}\right)$ has the same perimeter as A.

In the case of Steiner symmetrizations in arbitrary dimension, the regularity of the barycenter function is only shown to be GBV on the sets $\{\mathrm{rad}_A^- > \delta\}$ for a.e. $\delta > 0$ according to [14, Theorem 1.7]. Certainly, for $n = 2$, it holds $A^S = A^\star$, and therefore, the regularity can be lifted in this case. In general, the behavior of barycenter functions of Steiner symmetrizations at infinity might not have a direct effect on the perimeter, compare [14, Remark 3.5], whereas the center functions for Schwarz symmetrizations do. This fact will be exploited in Step 1 of the following theorem, showing that cen_A is in fact a regulated function.

Proof of Theorem 3.48. The first statement concerning the regularity of the radius function is covered by Lemma 3.34. Turning to the center function, we notice $\mathrm{cen}_A(t) = \fint_{A_t} x \,\mathrm{d}x \mathbb{1}_{\{\mathrm{rad}_A^- > 0\}}$ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, showing the measurability of cen_A. The remaining proof is divided into four steps.

Step 1: Show that cen_A is a regulated function on $\{\mathrm{rad}_A^- > 0\}$:
Let $t \in \{\mathrm{rad}_A^- > 0\}$. By the regularity of rad_A, the set $\{\mathrm{rad}_A^- > 0\}$ is open and the one-sided limits of rad_A^- exist at t as positive values. In fact, for all sequences $(t_k)_{k\in\mathbb{N}}$ with $t_k \nearrow t$ as $k \to \infty$ and all $\varepsilon \in (0, \mathrm{rad}_A^-(t-))$, there exists $k_0 \in \mathbb{N}$ such that $|\mathrm{rad}_A^-(t-) - \mathrm{rad}_A^-(t_k)| < \varepsilon$ for all $k \geqslant k_0$.
We now assume that the a.e. taken left limit of cen_A does not exist at t and record that for every representative $\widetilde{\mathrm{cen}}_A$ of cen_A and every null set $N \subseteq \mathbb{R}$, there exists a sequence $t_k \nearrow t$ in $\mathbb{R}\backslash N$ such that $\widetilde{\mathrm{cen}}_A(t_k)$ does not converge in $\mathbb{R}^{n-1}$ as $k \to \infty$. Since A is a set of finite perimeter, $\mathcal{H}^{n-1}(\partial^* A \cap H_t)$ vanishes for all but countably many $t \in \mathbb{R}$. By Lemma 3.35, for a fixed representative of cen_A, we can consider a null set $N \subseteq \mathbb{R}$ such that

$$A^1 \cap H_s = A^+ \cap H_s = \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(s)}(\mathrm{cen}_A(s)) \times \{s\}$$

holds except for $\mathcal{H}^{n-1}$-negligible sets for all $s \in \mathbb{R}\backslash N$. In particular, there exists a sequence $(s_k)_{k\in\mathbb{N}}$ in $\mathbb{R}$ with $s_k \nearrow t$ as $k \to \infty$ and some $\varepsilon > 0$ such that for all $k_0 \in \mathbb{N}$, there exists $k, i \in \mathbb{N}_{>k_0}$ which satisfy the following properties:

i) $s_k, s_i \in \mathbb{R}\backslash N$,

ii) $|\mathrm{rad}_A^-(t-) - \mathrm{rad}_A^-(s_i)| < \varepsilon$,

iii) $|\mathrm{cen}_A(s_k) - \mathrm{cen}_A(s_i)| \geqslant 3\varepsilon$.

In the following, we assume w.l.o.g. $s_i > s_k$, such that Lemma 3.40 and the definition of N imply

$$\mathrm{P}(A, \mathbb{R}^{n-1} \times (s_k, s_i)) \geqslant \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A^-(s_k)}(\mathrm{cen}_A(s_k)) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(s_i)}(\mathrm{cen}_A(s_i))\right| \geqslant \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A^-(t-)-\varepsilon}(3\varepsilon e_1) \backslash \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(t-)+\varepsilon}\right| > 0$$

for all $k, i \in \mathbb{N}_{>k_0}$. In particular, the perimeter of A sums up to ∞, in contradiction to the assumptions. Hence, the a.e. taken left limit of cen_A at t exists in $\mathbb{R}^{n-1}$. By an analogous argument, this transfers to the a.e. taken right limit as well. Lemma 2.33 particularly implies $\mathrm{cen}_A \in \mathrm{L}^\infty_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\}, \mathbb{R}^{n-1})$ and that the mean-value representative of cen_A is a regulated function on $\{\mathrm{rad}_A^- > 0\}$.
From now on, we only consider the mean-value representative of cen_A.

Step 2: Show that the function $h := \sum_{t\in I}(\mathrm{cen}_A(t+) - \mathrm{cen}_A(t-))\mathbb{1}_{[t,\infty)}$ is in $\mathrm{BV}(I, \mathbb{R}^{n-1})$ for all $I \Subset \{\mathrm{rad}_A^- > 0\}$:
Since we showed that cen_A is a regulated function, there exist at most countably many points $t \in \{\mathrm{rad}_A^- > 0\}$ with $\mathrm{cen}_A(t+) \neq \mathrm{cen}_A(t-)$. By Step 1, h is a pure jump function, hence

$$\mathrm{pVar}(h, I) \leqslant \sum_{t\in I} |\mathrm{cen}_A(t+) - \mathrm{cen}_A(t-)|\,.$$

Taking into account Lemma 2.23 and Proposition 2.5, it is left to show $\sum_{t\in I} |\mathrm{cen}_A(t+) - \mathrm{cen}_A(t-)| < \infty$ for all $I \Subset \{\mathrm{rad}_A^- > 0\}$. To this end, we exploit the lower semicontinuity of rad_A^- to find $\varepsilon > 0$ with $I \subseteq \{\mathrm{rad}_A^- > \varepsilon\}$ and define

$$\hat{\mathrm{J}} := \{t \in I : \mathrm{cen}_A(t+) \neq \mathrm{cen}_A(t-)\} \qquad \text{and} \qquad s_j := |\mathrm{cen}_A(j+) - \mathrm{cen}_A(j-)| \quad \text{for all } j \in \hat{\mathrm{J}}\,. \tag{3.20}$$

Then, Lemma 3.35 implies

$$\infty > \mathrm{P}(A) \geqslant \sum_{j\in\hat{\mathrm{J}}} \mathrm{P}(A, H_t) = \sum_{j\in\hat{\mathrm{J}}} \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A^-(j+)}(\mathrm{cen}_A(j+)) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(j-)}(\mathrm{cen}_A(j-))\right|. \tag{3.21}$$

We further define the set of jumps that are smaller than ε by $\mathrm{J}_\varepsilon := \{j \in \hat{\mathrm{J}} : s_j < \varepsilon\}$. In order to estimate the total variation of the jumps of cen_A in J_ε, we consider the case where $\mathrm{rad}_A^-(j+) \geqslant \mathrm{rad}_A^-(j-)$ and derive

$$\left|\mathrm{B}^{n-1}_{\mathrm{rad}_A^-(j+)}(\mathrm{cen}_A(j+)) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(j-)}(\mathrm{cen}_A(j-))\right| \geqslant \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A^+(j)}(\mathrm{cen}_A(j+)) \backslash \mathrm{B}^{n-1}_{\mathrm{rad}_A^+(j)}(\mathrm{cen}_A(j-))\right|$$

and therefore

$$\left|\mathrm{B}^{n-1}_{\mathrm{rad}_A^-(j+)}(\mathrm{cen}_A(j+)) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(j-)}(\mathrm{cen}_A(j-))\right| \geqslant \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A^+(j)}(s_j e_1) \backslash \mathrm{B}^{n-1}_{\mathrm{rad}_A^+(j)}\right| \tag{3.22}$$

for all $j \in \hat{\mathrm{J}}$, where we used suitable translations and rotations in the final step. By symmetry, (3.22) remains true in the case $\mathrm{rad}_A^-(j+) < \mathrm{rad}_A^-(j-)$. We continue with computing the volume of $\mathrm{B}_r^{n-1}(se_1)\backslash\mathrm{B}_r^{n-1}$ for $r, s > 0$ in general. Clearly, if $s \geqslant 2r$, the considered balls are disjoint, hence $\left|\mathrm{B}_r^{n-1}(se_1)\backslash\mathrm{B}_r^{n-1}\right| = \left|\mathrm{B}_r^{n-1}\right|$ follows. If, on the other hand, $s < 2r$, it holds

$$x \in \mathrm{B}_r^{n-1}(se_1) \cap \mathrm{B}_r^{n-1} \qquad \text{if and only if} \qquad se_1 - x \in \mathrm{B}_r^{n-1}(se_1) \cap \mathrm{B}_r^{n-1}\,.$$

In particular, we have $x_1 \in (s-r, r)$, and by symmetry, we obtain

$$\begin{aligned}\left|\mathrm{B}_r^{n-1}(se_1) \cap \mathrm{B}_r^{n-1}\right| &= 2\left|\mathrm{B}_r^{n-1} \cap \left[\left(\frac{s}{2}, r\right) \times \mathbb{R}^{n-2}\right]\right| \\ &= \left|\mathrm{B}_r^{n-1} \cap \left[\left(\frac{s}{2}, r\right) \times \mathbb{R}^{n-2}\right]\right| + \left|\mathrm{B}_r^{n-1} \cap \left[\left(-r, -\frac{s}{2}\right) \times \mathbb{R}^{n-2}\right]\right|,\end{aligned} \tag{3.23}$$

compare Figure 3.6. Hence, we can estimate

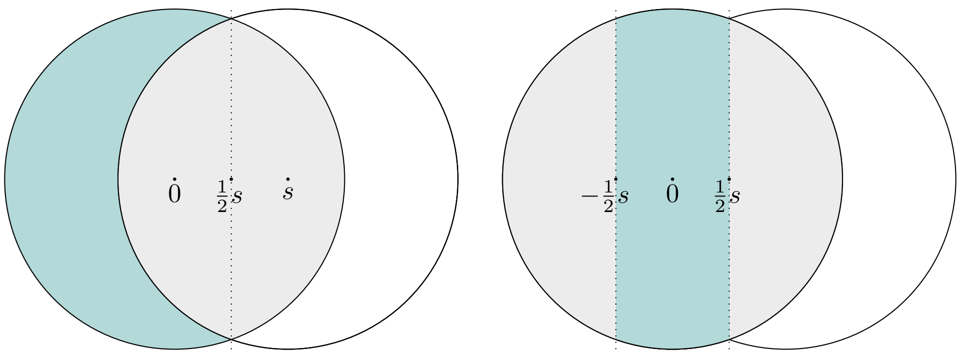


Figure 3.6: Illustration of the argument of equation (3.23)

$$\begin{aligned}\left|\mathrm{B}^{n-1}_{\mathrm{rad}_A^-(j+)}(\mathrm{cen}_A(j+)) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(j-)}(\mathrm{cen}_A(j-))\right| &\geqslant \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A^+(j)} \cap \left[\left(-\frac{s_j}{2}, \frac{s_j}{2}\right) \times \mathbb{R}^{n-2}\right]\right| \\ &= \int_{-\frac{s_j}{2}}^{\frac{s_j}{2}} \left|\mathrm{B}^{n-2}_{\sqrt{(\mathrm{rad}_A^+(j))^2 - t^2}}\right| \mathrm{d}t \\ &= \int_{-\frac{s_j}{2}}^{\frac{s_j}{2}} \omega_{n-2} \sqrt{(\mathrm{rad}_A^+(j))^2 - t^2}^{\,n-2} \mathrm{d}t \\ &\geqslant s_j \omega_{n-2} \sqrt{(\mathrm{rad}_A^+(j))^2 - \frac{s_j^2}{4}}^{\,n-2} \\ &\geqslant s_j \omega_{n-2} \sqrt{\varepsilon^2 - \frac{s_j^2}{4}}^{\,n-2} \\ &\geqslant s_j \omega_{n-2} \varepsilon^{n-2} \left(\frac{3}{4}\right)^{\frac{n-2}{2}}\end{aligned}$$

for all $j \in \mathrm{J}_\varepsilon$. Now, (3.21) yields

$$\infty > \omega_{n-2}\varepsilon^{n-2}\left(\frac{3}{4}\right)^{\frac{n-2}{2}} \sum_{j \in \mathrm{J}_\varepsilon} s_j + \sum_{j \in \hat{\mathrm{J}}\backslash\mathrm{J}_\varepsilon} \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A^-(j+)}(\mathrm{cen}_A(j+)) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(j-)}(\mathrm{cen}_A(j-))\right|.$$

In particular, $\sum_{j \in \mathrm{J}_\varepsilon} s_j$ is finite. Furthermore, $\hat{\mathrm{J}}\backslash\mathrm{J}_\varepsilon$ is a finite set, since otherwise, the right hand side of (3.21) sum would equal infinity and create a contradiction. Thus, $\sum_{j \in \hat{\mathrm{J}}} s_j$ is indeed finite.

Step 3: Show that (3.19) holds under the special assumption $\mathrm{cen}_A = c\, e_{n-1}$:

We first consider the special case $F = \mathbb{R}^{n-2}$ and assume w.l.o.g. that $I \Subset \{\mathrm{rad}_A^- > 0\}$ is an open set. By the previous analysis, we obtained that c is a regulated function in $\mathrm{L}^\infty_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\})$ and that the jumps of c in I sum up finitely. This step is divided into further substeps considering different numbers of jumps.

Step 3.i): Show the claim if c is continuous on I:
Similar to the proof of Proposition 3.43, we denote

$$S_+ := \mathrm{Sub}[\hat{c} + \rho] \cap (D \times \mathbb{R}) , \qquad S_- := \mathrm{Sup}[\hat{c} - \rho] \cap (D \times \mathbb{R}) \tag{3.24}$$

and identify

$$\begin{gathered} A' \cap (D \times \mathbb{R}) = S_+ \cap S_- , \qquad (A' \cup \mathrm{Sub}[\hat{c}]) \cap (D \times \mathbb{R}) = S_+ , \qquad (A' \cup \mathrm{Sup}[\hat{c}]) \cap (D \times \mathbb{R}) = S_- \\ A' \cap \mathrm{Sup}[\hat{c}] \cap (D \times \mathbb{R}) = S_+ \cap \mathrm{Sup}[\hat{c}] , \qquad A' \cap \mathrm{Sub}[\hat{c}] \cap (D \times \mathbb{R}) = S_- \cap \mathrm{Sub}[\hat{c}] . \end{gathered} \tag{3.25}$$

As we have to take into account the center function, we replace the sets $\mathbb{R}^n_\pm$ from the proof of Proposition 3.43 with the sub and supergraph of $\hat{c}$, which is the main difficulty of this step.
In order to conclude the perimeter formula as in Proposition 3.43, we record

$$\partial \mathrm{Sub}[\hat{c}] = \partial \mathrm{Sup}[\hat{c}] = \mathrm{Graph}[\hat{c}] , \qquad \mathrm{Sub}[\hat{c}] \subseteq S_+^1 \qquad \text{and} \qquad \mathrm{Sup}[\hat{c}] \subseteq S_-^1 \tag{3.26}$$

by the continuity of c. Moreover, we can show

$$S_+^1 \supseteq (D \times \mathbb{R}) \cap (A')^1 \supseteq (D \times \mathbb{R}) \cap \mathrm{Graph}[\hat{c}] . \tag{3.27}$$

Indeed, $(D \times \mathbb{R}) \cap S_+^1 \supseteq (D \times \mathbb{R}) \cap (A')^1$ follows from (3.25) and the other subset relation comes from the fact that for $(x, t, c(t)) \in (D \times \mathbb{R}) \cap \mathrm{Graph}[\hat{c}]$, we find $|x| < \mathrm{rad}_A^-(t)$ and $(A^1)_t \supseteq \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(t)}(\mathrm{cen}_A(t))$ for all $t \in \{\mathrm{rad}_A^- > 0\}$, which can be proven analogously to the proof of Lemma 3.37 due to the continuity of the center function. Hence, by (3.25), (3.26), (3.27) and by taking into account the openness of $\mathrm{Sup}[\hat{c}]$ and D, we obtain

$$(\partial^* S_+) \cap (D \times \mathbb{R}) = (\partial^* S_+) \cap \mathrm{Sup}[\hat{c}] \cap (D \times \mathbb{R}) = (\partial^* A') \cap \mathrm{Sup}[\hat{c}] \cap (D \times \mathbb{R})$$

and similarly for S_- instead of S_+.
Moreover, $\mathrm{P}(A', (D \times \mathbb{R}) \cap \mathrm{Graph}[\hat{c}]) = 0$ follows from (3.27). We record by Proposition 2.52 that $\hat{c} \pm \rho \in \mathrm{BV}_{\mathrm{loc}}(\mathrm{D}_I)$ with

$$\begin{aligned} \infty > \mathrm{P}(A', D_{\tilde{I}} \times \mathbb{R}) &= \mathrm{P}(A', \mathrm{Sub}[\hat{c}] \cap (D_{\tilde{I}} \times \mathbb{R})) + \mathrm{P}(A', \mathrm{Sup}[\hat{c}] \cap (D_{\tilde{I}} \times \mathbb{R})) + \mathrm{P}(A', \mathrm{Graph}[\hat{c}] \cap (D_{\tilde{I}} \times \mathbb{R})) \\ &= \mathrm{P}(S_-, D_{\tilde{I}} \times \mathbb{R}) + \mathrm{P}(S_+, D_{\tilde{I}} \times \mathbb{R}) \\ &= |(\mathcal{L}^{n-1}, \mathrm{D}(\hat{c} + \rho))|(D_{\tilde{I}}) + |(\mathcal{L}^{n-1}, \mathrm{D}(\hat{c} - \rho))|(D_{\tilde{I}}) \end{aligned} \tag{3.28}$$

for each set $\tilde{I} \in \mathcal{B}(I)$. Since $\rho \in \mathrm{BV}_{\mathrm{loc}}(D)$, we find $\hat{c} \in \mathrm{BV}_{\mathrm{loc}}(\mathrm{D}_I)$, which implies the claimed regularity for c. Furthermore, for $t \in I \backslash \mathrm{J}_{\mathrm{rad}_A}$, the continuity of cen_A implies $\partial A \cap H_t = \partial \mathrm{B}^{n-1}_{\mathrm{rad}_A(t)}(\mathrm{cen}_A(t)) \times \{t\}$, which leads to

$$\partial^* A' \cap ((\mathbb{R}^{n-1} \backslash D) \times \mathbb{R}) \cap (\mathbb{R}^{n-1} \times I \backslash \mathrm{J}_{\mathrm{rad}_A})' \subseteq \{(x, t, s) \in \mathbb{R}^{n-1} \times I \times \mathbb{R} : \, s = c(t), \, |x| = \mathrm{rad}_A^-(t)\} =: G .$$

The set G is in fact the image of the Lipschitz map

$$g : \partial \mathrm{B}_1^{n-2} \times \Gamma_f \to \mathbb{R}^n, \, (x, y) \mapsto (x y_3, \overline{y}) ,$$

where Γ_f is the extended graph of the function $f \in \mathrm{BV}(I, \mathbb{R}^2) \cap \mathrm{L}^\infty(I, \mathbb{R}^2)$, defined by $f(t) := (c(t), \mathrm{rad}_A^-(t))$ and where $\overline{y}$ denotes (y_1, y_2). Now, Proposition 2.73 and Lemmas 2.72 and 3.33 imply

$$\begin{aligned} \dim_{\mathcal{H}}(G) = \dim_{\mathcal{H}}(g(\partial \mathrm{B}_1^{n-2} \times \Gamma_f)) &\leqslant \dim_{\mathcal{H}}(\partial \mathrm{B}_1^{n-2} \times \Gamma_f) \\ &\leqslant \dim_{\mathcal{B}}(\partial \mathrm{B}_1^{n-2}) + \dim_{\mathcal{H}}(\Gamma_f) \\ &= n - 2 . \end{aligned}$$

This gives

$$\mathrm{P}(A, \mathbb{R}^{n-1} \times I \backslash \mathrm{J}_{\mathrm{rad}_A}) = \mathrm{P}(A', D_{I\backslash \mathrm{J}_{\mathrm{rad}_A}} \times \mathbb{R})\,.$$

We continue with applying (3.28) for $\tilde{I} = I\backslash \mathrm{J}_{\mathrm{rad}_A}$ and Lemma 3.35, which yields

$$\begin{aligned} \mathrm{P}(A, \mathbb{R}^{n-1} \times I) = |(\mathcal{L}^{n-1}, \tilde{\mathrm{D}}(\hat{c}+\rho))|(D_{I\backslash \mathrm{J}_{\mathrm{rad}_A}}) + |(\mathcal{L}^{n-1}, \tilde{\mathrm{D}}(\hat{c}-\rho))|(D_{I\backslash \mathrm{J}_{\mathrm{rad}_A}}) \\ + \sum_{j\in \mathrm{J}\cap I} \left|\left(\mathrm{B}^{n-1}_{\mathrm{rad}_A(j+)}(\mathrm{cen}_A(j)) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(j-)}(\mathrm{cen}_A(j))\right)\right|\,. \end{aligned}$$

Formula (3.19) then follows from Theorem 2.16.

Step 3.ii): Show the claim if c has only finitely many jumps in I:
For such a function, we can apply the previous step on the open set $I\backslash \mathrm{J}_c$ and apply Lemma 3.35 onto the remaining part.

Step 3.iii): Show the claim if c has countably many jumps in I:
Since we do not have suitable regularity of c at hand, we cannot apply Proposition 3.20 directly, but we will argue similarly. According to Step 2, the pure jump function $h := \sum_{t\in I}(\mathrm{cen}_A(t+) - \mathrm{cen}_A(t-))\mathbb{1}_{[t,\infty)}$ is in $\mathrm{BV}(I, \mathbb{R}^{n-1})$. Moreover, by Step 1, the function $\mathrm{cen}_A - h$ is continuous on I. Thus, transferring this to the $(n-1)$-th component, we can decompose c into a continuous function c_{cont} and a pure jump function $c_j = h$ with countably many summable jumps on I. In order to handle c_j, we rewrite the set $\hat{\mathrm{J}}$ from (3.20) by $\{j_1, j_2, \dots\}$. Then, by Step 1, it holds $c_j = \sum_{k=1}^{\infty}(c(j_k+) - c(j_k-))\mathbb{1}_{[j_k,\infty)}$ and $\sum_{k=1}^{\infty}|c(j_k+) - c(j_k-)| < \infty$. As in the proof of Proposition 3.20, we can find for every $k \in \mathbb{N}$, some $k_0 \in \mathbb{N}$, such that

$$\sum_{i=k_0}^{\infty} |c(j_i+) - c(j_i-)| < \frac{1}{k}\,.$$

In particular, the function $c_k := c_{\mathrm{cont}} + \sum_{i=1}^{k_0}(c(j_i+) - c(j_i-))\mathbb{1}_{[j_i,\infty)}$ has only finitely many jumps on I, and satisfies $|c - c_k| \leqslant \frac{1}{k}$ on I. Now, we define the sets

$$A_k := A_{\mathbb{R}\backslash I} \cup \bigcup_{t\in I} \mathrm{B}^{n-1}_{\mathrm{rad}_A(t)}(e_{n-1}c_k(t)) \times \{t\}$$

and find $A_k \to A$ in $\mathrm{L}^1(\mathbb{R}^n)$ as $k \to \infty$ by dominated convergence as in the proof of Proposition 3.20. In particular, the lower semicontinuity of the perimeter gives

$$\mathrm{P}(A, \mathbb{R}^{n-1} \times I) \leqslant \liminf_{k\to\infty} \mathrm{P}(A_k, \mathbb{R}^{n-1} \times I)\,. \tag{3.29}$$

In order to ensure that the perimeter of A_k converges, we define another sequence of sets that converge to A in $\mathbb{R}^{n-1} \times I$ but which only differ from A by finitely many shifted sections. To this end, we choose for every $\ell > k$ in $\mathbb{N}$ some $\ell_0 \in \mathbb{N}$, such that we have

$$\sum_{i=\ell_0+k_0}^{\infty} |c(j_i+) - c(j_i-)| < \frac{1}{\ell}\,.$$

For $c_k^\ell := c_k + \sum_{i=k_0+\ell_0}(c(j_i+) - c(j_i-))\mathbb{1}_{[j_i,\infty)}$, it holds $|c_k - c_k^\ell| < \frac{1}{\ell}$ and therefore, we conclude

$$A_k^\ell \to A_k \quad \text{in } \mathrm{L}^1(\mathbb{R}^n) \text{ as } \ell \to \infty\,, \quad \text{where} \quad A_k^\ell := A_{\mathbb{R}\backslash I} \cup \bigcup_{t\in I} \mathrm{B}^{n-1}_{\mathrm{rad}_A(t)}(e_{n-1}c_k^\ell(t)) \times \{t\}\,.$$

If, for simplicity of notation, we further assume $j_{k_0+1} < j_{k_0+2} < \cdots < j_{k_0+\ell_0-1}$ in the following, we obtain that the set $(A_k^\ell)_{I\cap(j_{k_0+i}, j_{k_0+i+1})}$ is a shift of $A_{I\cap(j_{k_0+i}, j_{k_0+i+1})}$ into the $(n-1)$-th direction for all $i \in \{1, \dots, \ell_0-2\}$. Analogously, the sets $(A_k^\ell)_{I\cap(-\infty, j_{k_0+1})}$ and $(A_k^\ell)_{I\cap(j_{k_0+\ell_0-1},\infty)}$ are shifts of $A_{I\cap(-\infty, j_{k_0+1})}$ and $A_{I\cap(j_{k_0+\ell_0-1},\infty)}$, respectively.

By Lemma 3.35, it holds

$$\begin{aligned}\mathrm{P}(A_k^\ell, \mathbb{R}^{n-1}\times\{j_{k_0+i}\}) &= \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A^-(j_{k_0+i}+)} \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(j_{k_0+i}-)}\right| \\ &\leqslant \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A^-(j_{k_0+i}+)}(c(j_{k_0+i}+)e_{n-1}) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(j_{k_0+i}-)}(c(j_{k_0+i}-)e_{n-1})\right| \\ &= \mathrm{P}(A, \mathbb{R}^{n-1}\times\{j_{k_0+i}\})\end{aligned}$$

for all $i \in \{1, \dots, \ell_0 - 1\}$. Hence, we find

$$\begin{aligned}\mathrm{P}(A_k^\ell, \mathbb{R}^{n-1}\times I) &= \mathrm{P}(A, \mathbb{R}^{n-1}\times(I\backslash\{j_{k_0+1}, \dots, j_{k_0+\ell_0-1}\})) + \sum_{i=1}^{\ell_0-1} \mathrm{P}(A_k^\ell, \mathbb{R}^{n-1}\times\{j_{k_0+i}\}) \\ &\leqslant \mathrm{P}(A, \mathbb{R}^{n-1}\times I)\end{aligned} \tag{3.30}$$

by the translation invariance of the perimeter. Now, the lower semicontinuity of the perimeter, (3.29), and (3.30) give

$$\mathrm{P}(A, \mathbb{R}^{n-1}\times I) \leqslant \limsup_{k\to\infty} \mathrm{P}(A_k, \mathbb{R}^{n-1}\times I) \leqslant \limsup_{k\to\infty} \liminf_{\ell\to\infty} \mathrm{P}(A_k^\ell, \mathbb{R}^{n-1}\times I) \leqslant \mathrm{P}(A, \mathbb{R}^{n-1}\times I)\,.$$

By Step 3.ii), we have $c_k \in \mathrm{BV}_{\mathrm{loc}}(I)$ for all $k \in \mathbb{N}$, such that Step 2 implies $c \in \mathrm{BV}_{\mathrm{loc}}(I)$. Furthermore, the summability of $(\mathrm{P}(A, \mathbb{R}^{n-1}\times\{j_k\}))_{k\in\mathbb{N}}$, the construction of c_k, and Theorem 2.16 give

$$\begin{aligned}\mathrm{P}(A, \mathbb{R}^{n-1}\times I) &= \lim_{k\to\infty} \mathrm{P}(A_k, \mathbb{R}^{n-1}\times I) \\ &= |(\mathcal{L}^{n-1}, \tilde{\mathrm{D}}(c+\rho))|(D_I) + |(\mathcal{L}^{n-1}, \tilde{\mathrm{D}}(c-\rho))|(D_I) \\ &+ \lim_{k\to\infty} \sum_{j\in \mathrm{J}_k\cap I} \left|\left(\mathrm{B}^{n-1}_{\mathrm{rad}_A(j+)}(\mathrm{cen}_A(j+)) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(j-)}(\mathrm{cen}_A(j-))\right)\right| \\ &= |(\mathcal{L}^{n-1}, \tilde{\mathrm{D}}(c+\rho))|(D_I) + |(\mathcal{L}^{n-1}, \tilde{\mathrm{D}}(c-\rho))|(D_I) \\ &+ \sum_{j\in \mathrm{J}\cap I} \left|\left(\mathrm{B}^{n-1}_{\mathrm{rad}_A(j+)}(\mathrm{cen}_A(j+)) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(j-)}(\mathrm{cen}_A(j-))\right)\right|,\end{aligned}$$

where we abbreviate $\mathrm{J}_k := \mathrm{J}_{\mathrm{rad}_A} \cup \{j_1, \dots j_{k_0}\}$. The claim follows for $F = \mathbb{R}^{n-2}$ and for arbitrary Borel sets $I \in \mathcal{B}(\{\mathrm{rad}_A^- > 0\})$ due to the regularity of Radon measures.
Finally, for open sets $F \subseteq \mathbb{R}^{n-2}$, the argumentation for Step 3 is analogous and the formula can be transferred to general Borel sets in $\mathbb{R}^{n-2}$ via the regularity of Radon measures.

Step 4: Show $\mathrm{cen}_A \in \mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\}, \mathbb{R}^{n-1})$:
We argue via contradiction. If $\mathrm{cen}_A \notin \mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\}, \mathbb{R}^{n-1})$, then we find some $i \in \{1, \dots, n-1\}$ such that $(\mathrm{cen}_A)_i \notin \mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\})$. W.l.o.g., we may assume $i = n-1$. According to Remark 3.2 and Theorem 3.5 i), the set $\tilde{A} := (((A)^{S_1})^{\cdots})^{S_{n-2}}$ satisfies the assumptions of the theorem with $\mathrm{rad}_A = \mathrm{rad}_{\tilde{A}}$ and $\mathrm{cen}_{\tilde{A}} = (\mathrm{cen}_A)_{n-1}e_{n-1}$. By the previous argument, we find $(\mathrm{cen}_A)_{n-1} \in \mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\})$, in contradiction to the assumption. □

Remark 3.51. Let $A \subseteq \mathbb{R}^n$ be a set of finite volume and a stack of balls in $\mathbb{R}$ such that $A^\star$ is a set of finite perimeter. In particular, Lemma 3.34 applies for rad_A. Assume further that A has finite perimeter in $\mathbb{R}^{n-1}\times\{\mathrm{rad}_A^- > 0\}$. If $\mathrm{rad}_A^+(t+) = \mathrm{rad}_A^+(t) > 0$ for some $t \in \mathbb{R}$, then the a.e. taken right limit of cen_A exists at t.
Indeed, in this framework, there exist $\varepsilon > 0$ and $\delta > 0$ such that $(t, t+\delta)$ is a subset of $\{\mathrm{rad}_A^- > \varepsilon\}$ and with this, we can repeat Step 1 from the previous proof, which implies the claim. In particular, cen_A has a bounded and regulated representative on the interval $[t, t+\delta]$.
Analogously, cen_A has a bounded and regulated representative on the interval $[t-\delta, t]$ for some $\delta > 0$ if $\mathrm{rad}_A^+(t-) = \mathrm{rad}_A^+(t) > 0$.

The theorem can be localized as follows.

Corollary 3.52 (Localized radius and center regularity)**.** *Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter and finite volume. Moreover, we assume that A is a stack of balls in some open set $O \subseteq \mathbb{R}$. Then the following properties hold true.*

i) *The radius function* rad_A *of* A *is in* $\mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\}) \cap \mathrm{L}^\infty(\{\mathrm{rad}_A^- > 0\})$.

ii) *The center function* cen_A *of* A *is in* $\mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\} \cap O, \mathbb{R}^{n-1}) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\} \cap O, \mathbb{R}^{n-1})$.

iii) *If* $\mathrm{cen}_A = ce_{n-1}$ *on some open set that contains* $I \in \mathcal{B}(\{\mathrm{rad}_A^- > 0\} \cap O)$ *and for a function* $c : \{\mathrm{rad}_A^- > 0\} \cap O \to \mathbb{R}$, *then*

$$\begin{aligned} \mathrm{P}(A, F \times \mathbb{R} \times I) = &|(\mathcal{L}^{n-1}, \tilde{\mathrm{D}}(\hat{c} + \rho))|(D \cap (F \times I)) + |(\mathcal{L}^{n-1}, \tilde{\mathrm{D}}(\hat{c} - \rho))|(D \cap (F \times I)) \\ &+ \sum_{j \in \mathrm{J} \cap I} \left| \left(\mathrm{B}^{n-1}_{\mathrm{rad}_A(j+)}(\mathrm{cen}_A(j+)) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(j-)}(\mathrm{cen}_A(j-))\right) \cap (F \times \mathbb{R}) \right|, \end{aligned} \tag{3.31}$$

for $F \in \mathcal{B}(\mathbb{R}^{n-2})$, *where we identify* $\hat{c}(x,t) = c(t)$ *for all* $(x,t) \in D$, *define* $\mathrm{J} := \mathrm{J}_{\mathrm{rad}_A} \cup \mathrm{J}_{\mathrm{cen}_A} \cap O$, *consider good representatives of* rad_A *and* cen_A, *and define the function* ρ *as in Proposition 3.43.*

Proof. For every connected and compactly contained open set $K \Subset O$, Lemma 2.49 and the fact that $\mathrm{rad}_A \in \mathrm{L}^\infty(\mathbb{R})$ give that $\tilde{A} := A \cap (\mathbb{R}^{n-1} \cap K)$ is a set of finite perimeter and finite volume. Moreover, $\mathrm{cen}_{\tilde{A}} = \mathrm{cen}_A|_K$ holds by definition, therefore we verify $\mathrm{cen}_A \in \mathrm{BV}_{\mathrm{loc}}(K \cap \{\mathrm{rad}_A^- > 0\}, \mathbb{R}^{n-1}) \cap \mathrm{L}^\infty(\{\mathrm{rad}_A^- > 0\} \cap K)$ by Theorem 3.48.
For all $I \in \mathcal{B}(\{\mathrm{rad}_A^- > 0\} \cap K)$ and $F \in \mathcal{B}(\mathbb{R}^{n-2})$, we have

$$\mathrm{P}(A, F \times \mathbb{R} \times I) = \mathrm{P}(\tilde{A}, F \times \mathbb{R} \times I)$$

according to Lemma 2.49, which, together with Theorem 3.48 and considering the identification of the radius and center functions of A and $\tilde{A}$ on K, concludes the claim. Taking into account the regularity of Radon measures, we conclude the statement in full generality by an exhaustion argument. □

3.2.2 The non-vanishing radius part

In this section, we give a characterization of the equality cases in the Schwarz inequality on the non-vanishing radius part $\{\mathrm{rad}_A^- > 0\}$.

Proposition 3.53 (Equality characterization on the non-vanishing radius part)**.** *Let* A *be a set of finite perimeter and finite volume and* $I \subseteq \{\mathrm{rad}_A^- > 0\}$ *an open set.*

i) *If* A *satisfies equality in the Schwarz inequality on* I, *i.e., if*

$$\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times I) = \mathrm{P}(A, \mathbb{R}^{n-1} \times I)\,, \tag{3.32}$$

then A *is a stack of balls in* I *with radius function* $\mathrm{rad}_A \in \mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\}) \cap \mathrm{L}^\infty(\{\mathrm{rad}_A^- > 0\})$ *and center function* $\mathrm{cen}_A \in \mathrm{BV}_{\mathrm{loc}}(I, \mathbb{R}^{n-1}) \cap \mathrm{L}^\infty_{\mathrm{loc}}(I, \mathbb{R}^{n-1})$. *Moreover, it holds*

$$|\mathrm{D}^s \mathrm{cen}_A| \leqslant |\mathrm{D}^s \mathrm{rad}_A| \qquad \text{and} \qquad \mathrm{D}^a \mathrm{cen}_A = 0 \qquad \text{on } I\,. \tag{3.33}$$

ii) *If* A *is a stack of balls in* I, *then* $\mathrm{rad}_A \in \mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\}) \cap \mathrm{L}^\infty(\{\mathrm{rad}_A^- > 0\})$ *and* $\mathrm{cen}_A \in \mathrm{BV}_{\mathrm{loc}}(I, \mathbb{R}^{n-1}) \cap \mathrm{L}^\infty_{\mathrm{loc}}(I, \mathbb{R}^{n-1})$. *Moreover, in this case,* (3.33) *implies* (3.32).

Remark 3.54. Due to the mutual singularity of the jump and the Cantor part of BV functions according to Theorem 2.16, $|\mathrm{D}^s \mathrm{cen}_A| \leqslant |\mathrm{D}^s \mathrm{rad}_A|$ is equivalent to $|\mathrm{D}^j \mathrm{cen}_A| \leqslant |\mathrm{D}^j \mathrm{rad}_A|$ and $|\mathrm{D}^c \mathrm{cen}_A| \leqslant |\mathrm{D}^c \mathrm{rad}_A|$.

The proposition is comparable to [14, Theorem 1.9] with similarities in the proof. In particular, for showing property i), an application of Theorem 3.48 leads to (3.33) via Lemma 2.62 and via the strict convexity of the area functional for the Cantor part and for the absolutely continuous part of the radius and center function, respectively. The same strategy is also used in the aforementioned reference.
For showing property ii), we control the Cantor part of both radius and center function via pure step functions, which has parallels to the proof of [26, Proposition 3.4]. In the reference, this method is used to construct a stack of balls E which violates rigidity in the case $\mathrm{D}^c \mathrm{rad}_E \neq 0$. To this end, it is w.l.o.g. assumed that rad_E is a pure Cantor function, i.e., that $\mathrm{Drad}_E = \mathrm{D}^c \mathrm{rad}_E$ holds true. Here, we can only reduce the argument to continuous radius and center functions since the behavior of Drad_A (and not only of Dvol_A as in [26]) is crucial for ensuring (3.33) (up to some small ε) for the approximations.

Proof of Proposition 3.53. We first prove the regularity statement in both cases. To this end, we record that if (3.32) is satisfied, Theorem 3.5 implies A is a stack of balls on I. Thus, we can apply Corollary 3.52 in both cases to conclude that the radius and center functions are in fact regular, as stated. If not otherwise indicated, we consider the mean-value representatives of rad_A and cen_A in the following.

Proof of statement i):
We first consider the jump parts of the radius and center function. For $t \in I$, Theorem 3.5 ii) applies to $B = \{t\}$ and $B = I \backslash \{t\}$ and therefore, (3.32) gives

$$\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times \{t\}) = \mathrm{P}(A, \mathbb{R}^{n-1} \times \{t\})\,.$$

Applying Lemma 3.35 provides

$$\begin{aligned}
\left|\mathrm{B}^{n-1}_{\mathrm{rad}^+_A(t)} \backslash \mathrm{B}^{n-1}_{\mathrm{rad}^-_A(t)}\right| &= \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A(t-)} \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(t+)}\right| \\
&= \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A(t-)}(\mathrm{cen}_A(t-)) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(t+)}(\mathrm{cen}_A(t+))\right| \\
&= \left|\mathrm{B}^{n-1}_{\mathrm{rad}^-_A(t)}(s_t e_1) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}^-_A(t)}\right|
\end{aligned}$$

for $s_t := |\mathrm{cen}_A(t+) - \mathrm{cen}_A(t-)|$, where the latter equality follows by symmetry. Furthermore, we can verify

$$0 = \left|\mathrm{B}^{n-1}_{\mathrm{rad}^-_A(t)}(s_t e_1) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}^-_A(t)}\right| - \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A(t-)} \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(t+)}\right| = 2\left|\mathrm{B}^{n-1}_{\mathrm{rad}^-_A(t)}(s_t e_1) \backslash \mathrm{B}^{n-1}_{\mathrm{rad}^+_A(t)}\right| \tag{3.34}$$

by elementary geometric considerations. Now, if $s_t > \mathrm{rad}^+_A(t) - \mathrm{rad}^-_A(t)$, there exists $\varepsilon > 0$ such that $(s_t - \delta + \mathrm{rad}^-_A(t))e_1 \notin \overline{\mathrm{B}}^{n-1}_{\mathrm{rad}^+_A(t)}$ holds for all $\delta \in [0, \varepsilon)$. In particular, $\mathrm{B}^{n-1}_{\mathrm{rad}^-_A(t)}(s_t e_1) \backslash \overline{\mathrm{B}}^{n-1}_{\mathrm{rad}^+_A(t)}$ is a non-empty open set, which gives

$$0 < \left|\mathrm{B}^{n-1}_{\mathrm{rad}^-_A(t)}(s_t e_1) \backslash \overline{\mathrm{B}}^{n-1}_{\mathrm{rad}^+_A(t)}\right| = \left|\mathrm{B}^{n-1}_{\mathrm{rad}^-_A(t)}(s_t e_1) \backslash \mathrm{B}^{n-1}_{\mathrm{rad}^+_A(t)}\right|,$$

in contradiction to (3.34). Hence,

$$|\mathrm{D}^j \mathrm{cen}_A|(\{t\}) = |\mathrm{cen}_A(t+) - \mathrm{cen}_A(t-)| \leqslant |\mathrm{rad}_A(t+) - \mathrm{rad}_A(t-)| = |\mathrm{D}^j \mathrm{rad}_A|(\{t\})$$

can be verified by Theorem 2.24. Since $\mathrm{J}_{\mathrm{cen}_A}$ is at most countable, we find $|\mathrm{D}^j \mathrm{cen}_A| \leqslant |\mathrm{D}^j \mathrm{rad}_A|$ on I.
We now turn to the analysis of the absolutely continuous part and the Cantor part. For this, we assume without loss of generality $I \Subset \{\mathrm{rad}^-_A > 0\}$, such that, by the lower semicontinuity of rad_A, there exists $\varepsilon > 0$ with $I \subseteq \{\mathrm{rad}^-_A > \varepsilon\}$. Similar to the proof of Theorem 3.48, we first consider the special case $\mathrm{cen}_A = c e_{n-1}$ on I. Applying Proposition 3.43 and Theorem 3.48 gives

$$\begin{aligned}
2|(\mathcal{L}^{n-1}, \tilde{\mathrm{D}}\rho)|(D \cap (F \times E)) &= \mathrm{P}(A^\star, F \times \mathbb{R} \times E \backslash \mathrm{J}) \\
&= \mathrm{P}(A, F \times \mathbb{R} \times E \backslash \mathrm{J}) \\
&= |(\mathcal{L}^{n-1}, \tilde{\mathrm{D}}(\hat{c} + \rho))|(D \cap (F \times E)) + |(\mathcal{L}^{n-1}, \tilde{\mathrm{D}}(\hat{c} - \rho))|(D \cap (F \times E))
\end{aligned} \tag{3.35}$$

for all Borel sets $F \in \mathcal{B}(\mathbb{R}^{n-2})$ and $E \in \mathcal{B}(I)$, where we used the notation from Theorem 3.48, i.e., we define $\rho(x,t) = \sqrt{((\mathrm{rad}^-_A(t))^2 - |x|^2)_+}$, $\mathrm{J} := \mathrm{J}_{\mathrm{rad}_A} \cup \mathrm{J}_{\mathrm{cen}_A}$, $\hat{c}(x,t) := c(t)$ for $(x,t) \in \mathbb{R}^{n-2} \times I$, and $D := \{\rho > 0\}$. The equality transfers in general to all Borel sets $B \in \mathcal{B}(D_I)$ instead of $D \cap (F \times E)$. Hence, Proposition 2.52 and Theorem 2.16 give

$$2\sqrt{1 + |\mathrm{D}^a \rho|^2}\mathcal{L}^{n-1}(B) = \sqrt{1 + |\mathrm{D}^a \hat{c} + \mathrm{D}^a \rho|^2}\mathcal{L}^{n-1}(B) + \sqrt{1 + |\mathrm{D}^a \hat{c} - \mathrm{D}^a \rho|^2}\mathcal{L}^{n-1}(B) \tag{3.36}$$

for Borel sets $B \in \mathcal{B}(D_I)$. It follows

$$2\sqrt{1 + |\mathrm{D}^a \rho|^2} = \sqrt{1 + |\mathrm{D}^a \hat{c} + \mathrm{D}^a \rho|^2} + \sqrt{1 + |\mathrm{D}^a \hat{c} - \mathrm{D}^a \rho|^2} \qquad \mathcal{L}^{n-1}\text{-a.e. on } D_I\,.$$

Now, the strict convexity of $t \mapsto \sqrt{1 + |t|^2}$ on $\mathbb{R}$ applied to the convex combination $\mathrm{D}^a \rho = \frac{1}{2}(\mathrm{D}^a \hat{c} + \mathrm{D}^a \rho) + \frac{1}{2}(\mathrm{D}^a \hat{c} - \mathrm{D}^a \rho)$ leads to

$$|\mathrm{D}^a \rho| = |\mathrm{D}^a \hat{c} + \mathrm{D}^a \rho| = |\mathrm{D}^a \hat{c} - \mathrm{D}^a \rho| \qquad \mathcal{L}^{n-1}\text{-a.e. on } D_I\,,$$

and thus, $\mathrm{D}^a\hat{c} \equiv 0$ on D_I. Finally, by the definition of c, we obtain $\mathrm{D}^a\hat{c} = e_{n-1}\mathrm{D}^a c$ as functions, such that $\mathrm{D}^a c = 0$ on I follows.
From (3.35), Proposition 2.52 and Theorem 2.16, we conclude

$$2|\mathrm{D}^c\rho|(B) = |\mathrm{D}^c\hat{c} + \mathrm{D}^c\rho|(B) + |\mathrm{D}^c\hat{c} - \mathrm{D}^c\rho|(B) \tag{3.37}$$

for Borel sets $B \in \mathcal{B}(D_I)$. In particular, an application of Lemma 2.62 gives

$$\mathrm{D}^c\hat{c} = g\,\mathrm{D}^c\rho \qquad \text{on } D_I\,, \tag{3.38}$$

for a Borel function $g : \mathbb{R}^{n-1} \to [-1,1]$.
Since we chose $\varepsilon > 0$ such that $I \subseteq \{\mathrm{rad}_A^- > \varepsilon\}$, we define for all $\delta \in (0,\varepsilon)$ the C^1 Lipschitz function

$$f(x,s) := \sqrt{s^2 - |x|^2} \qquad \text{on } \mathrm{B}_\delta^{n-2} \times \mathbb{R}\backslash[-\varepsilon,\varepsilon]\,.$$

Then $\rho(x,t) = (f \circ r)(x,t)$ belongs to $\mathrm{BV}_{\mathrm{loc}}(\mathrm{B}_\delta^{n-2} \times I)$, where $r(x,t) := (x, \mathrm{rad}_A(t))$, and it holds

$$\mathrm{D}\rho = \mathrm{D}(f \circ r) = \nabla f(r^*)\tilde{\mathrm{D}}r\mathcal{L}^{n-1} + (f(r^+) - f(r^-))\nu_r\mathcal{H}^{n-2}\llcorner \mathrm{J}_r$$

according to Proposition 2.20, where r^* denotes the mean-value representative of r. By Theorem 2.16, we conclude $\mathrm{D}^c\rho = \nabla f(r^*)\mathrm{D}^c r$. Since Lemma 2.4 gives

$$\mathrm{D}r = \begin{bmatrix} \mathcal{L}^{n-1} & 0 & \dots & 0 \\ 0 & \mathcal{L}^{n-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{L}^{n-2} \otimes \mathrm{D}\mathrm{rad}_A \end{bmatrix} \qquad \text{on } \mathbb{R}^{n-2} \times \{\mathrm{rad}_A^- > 0\}$$

and $r^\pm(x,s) = (x, \mathrm{rad}_A^\pm(t))$ for all $x \in \mathbb{R}^{n-2}$ and $s \in \{\mathrm{rad}_A^- > 0\}$, we find

$$\mathrm{D}^c\rho = \frac{(0, r_{n-1})}{\rho}\mathcal{L}^{n-2} \otimes \mathrm{D}^c\mathrm{rad}_A \qquad \text{on } \mathrm{B}_\delta^{n-2} \times I\,.$$

For all Borel sets $L \subseteq \mathrm{B}_\delta^{n-2}$ and $K \subseteq I$, we can exploit the decreasing behavior of $s \mapsto \frac{s}{f(x,s)}$ for all $x \in \mathrm{B}_\delta^{n-2}$ to obtain

$$|\mathrm{D}^c\rho|(L \times K) \leqslant \frac{\varepsilon}{\sqrt{\varepsilon^2 - \delta^2}}|L|\,|\mathrm{D}^c\mathrm{rad}_A|(K)\,. \tag{3.39}$$

Furthermore, another application of Lemma 2.4 and Theorem 2.16 yields

$$\mathrm{D}\hat{c} = e_{n-1}\mathcal{L}^{n-2} \otimes \mathrm{D}c \qquad \text{and} \qquad \mathrm{D}^c\hat{c} = e_{n-1}\mathcal{L}^{n-2} \otimes \mathrm{D}^c c \qquad \text{on } \mathrm{B}_\delta^{n-2} \times I\,. \tag{3.40}$$

Together, (3.38), (3.39) and (3.40) lead to

$$|L|\,|\mathrm{D}^c c|(K) = |\mathrm{D}^c\hat{c}|(L \times K) \leqslant |\mathrm{D}^c\rho|(L \times K) \leqslant \frac{\varepsilon}{\sqrt{\varepsilon^2 - \delta^2}}\,|L|\,|\mathrm{D}^c\mathrm{rad}_A|(K) \tag{3.41}$$

for all $L \in \mathcal{B}(\mathrm{B}_\delta^{n-1})$ and $K \in \mathcal{B}(I)$. Hence, $|\mathrm{D}^c c|(K) \leqslant \frac{\varepsilon}{\sqrt{\varepsilon^2-\delta^2}}|\mathrm{D}^c\mathrm{rad}_A|(K)$ holds for all $K \in \mathcal{B}(I)$. Since $\delta \in (0,\varepsilon)$ was arbitrary, statement i) holds in the special case $\mathrm{cen}_A = ce_{n-1}$.
In order to transfer this result to general center functions, we observe that, by Remark 3.2 and the Steiner inequality of Theorem 3.5, every component $(\mathrm{cen}_A)_i$, $i \in \{1,\dots,n-1\}$, satisfies

$$|\mathrm{D}^c(\mathrm{cen}_A)_i| \leqslant |\mathrm{D}^c\mathrm{rad}_A| \qquad \text{on } I\,.$$

In particular, it holds $|\mathrm{D}^c\mathrm{cen}_A| \ll |\mathrm{D}^c\mathrm{rad}_A|$ on I, which implies the existence of some density function $f : I \to \mathbb{R}^{n-1}$ with $\mathrm{D}^c\mathrm{cen}_A = f|\mathrm{D}^c\mathrm{rad}_A|$ on I. Now, for $\nu \in \partial\mathrm{B}_1^{n-1}$, we consider some rotation $R \in \mathcal{O}(n-1)$ with $R^T e_{n-1} = \nu$. In fact, R is chosen such that the $(n-1)$-th column is ν. We extend this rotation to a matrix $\overline{R}$ in $\mathcal{O}(n)$ with n-th row and column e_n. Since A is a stack of balls, we obtain

$$\overline{R}A = \bigcup_{t \in \mathbb{R}} \mathrm{B}^{n-1}_{\mathrm{rad}_A(t)}(R\mathrm{cen}_A(t)) \times \{t\} \qquad \text{and} \qquad (\overline{R}A)^\star = A^\star$$

up to negligible sets. The invariance of the perimeter under rotation yields

$$\mathrm{P}\big((\overline{R}A)^{\star}, \mathbb{R}^{n-1} \times I\big) = \mathrm{P}\big(\overline{R}A, \mathbb{R}^{n-1} \times I\big)\,.$$

According to the previous analysis, it holds

$$|\nu \cdot f|\,|\mathrm{D}^c \mathrm{rad}_A| = |(Rf|\mathrm{D}^c \mathrm{rad}_A|)_{n-1}| = |(R\mathrm{D}^c \mathrm{cen}_A)_{n-1}| = |\mathrm{D}^c (R\mathrm{cen}_A)_{n-1}| \leqslant |\mathrm{D}^c \mathrm{rad}_A| \qquad \text{on } I\,.$$

In particular, for a countable dense subset $\Sigma \subseteq \partial \mathrm{B}_1^{n-1}$, there exists a $|\mathrm{D}^c \mathrm{rad}_A|$-null set $N \subseteq I$ with $|\nu \cdot f| \leqslant 1$ on $I \backslash N$ for all $\nu \in \Sigma$. As for every $\kappa \in \mathbb{R}^{n-1}$ with $|\kappa| > 1$ there exists $\nu \in \Sigma$ with $|\nu \cdot \kappa| > 1$, it follows $|f| \leqslant 1$ on $I \backslash N$ and therefore $|\mathrm{D}^c \mathrm{cen}_A| \leqslant |\mathrm{D}^c \mathrm{rad}_A|$ on I.

Proof of statement ii):
For the second implication, we again assume that $I \Subset \{\mathrm{rad}_A^- > 0\}$ is an open interval such that $I \subseteq \{\mathrm{rad}^- > \varepsilon\}$ for some $\varepsilon \in (0, 1)$.
As before, we consider the jumps $\mathrm{J} := \mathrm{J}_{\mathrm{cen}_A} \cup \mathrm{J}_{\mathrm{rad}_A}$ and obtain

$$\left|\mathrm{B}^{n-1}_{\mathrm{rad}_A(t+)}(\mathrm{cen}_A(t+)) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(t-)}(\mathrm{cen}_A(t-))\right| = \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A^+(t)}(e_1|\mathrm{D}\mathrm{cen}_A|(\{t\})) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(t)}\right| = \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A^+(t)} \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A^-(t)}\right|$$

for all $t \in \mathrm{J}$, since assumption (3.33) implies $|\mathrm{D}\mathrm{cen}_A|(\{t\}) \leqslant |\mathrm{D}\mathrm{rad}_A|(\{t\}) = \mathrm{rad}_A^+(t) - \mathrm{rad}_A^-(t)$, which gives $\mathrm{B}^{n-1}_{\mathrm{rad}_A^-(t)} \subseteq \mathrm{B}^{n-1}_{\mathrm{rad}_A^+(t)}(e_1|\mathrm{D}\mathrm{cen}_A|(\{t\}))$. Hence, Lemma 3.35 yields

$$\mathrm{P}(A, \mathbb{R}^{n-1} \times \mathrm{J}) = \mathrm{P}(A^{\star}, \mathbb{R}^{n-1} \times \mathrm{J})\,.$$

It is left to show the claim for sets A which satisfy the additional requirement that J is a finite set. Indeed, if this claim is given, Remark 3.21 gives a sequence $(A_k)_{k \in \mathbb{N}}$ of stacks of balls such that the radius and the center functions of A_k have only finitely many jumps and such that it holds $|\mathrm{D}^j \mathrm{cen}_{A_k}| \leqslant |\mathrm{D}^j \mathrm{rad}_{A_k}|$ on I and $\lim_{k\to\infty} \mathrm{P}(A_k^{\star}, \mathbb{R}^{n-1} \times I) = \mathrm{P}(A^{\star}, \mathbb{R}^{n-1} \times I)$. Then, the lower semicontinuity of the perimeter implies

$$\mathrm{P}(A, \mathbb{R}^{n-1} \times I) \leqslant \liminf_{k\to\infty} \mathrm{P}(A_k, \mathbb{R}^{n-1} \times I) = \liminf_{k\to\infty} \mathrm{P}(A_k^{\star}, \mathbb{R}^{n-1} \times I) = \mathrm{P}(A^{\star}, \mathbb{R}^{n-1} \times I)\,,$$

and with the Schwarz inequality of Theorem 3.5, equality follows.
To carry out the proof for radius and center functions with finitely many jumps, we observe that the open set $I \backslash \mathrm{J}$ is a finite union of open intervals. Since we have already shown equality on J, we can further restrict the analysis to sets with continuous radius and center functions on the interval $I = (a, b)$ with $a < b$ in $\mathbb{R}$. By considering a slightly smaller interval, we can also assume cen_A and rad_A to be uniformly continuous on $\overline{I}$. Furthermore, we can reduce the following argument to the case where rad_A can be decomposed into the positive functions rad_A^a and rad_A^c from Theorem 2.28. Indeed, according to Lemma 2.30, we find for all $t \in I$ an interval $I_t^\delta := (t - \delta, t + \delta) \subseteq I$ such that we can decompose rad_A into $\mathrm{rad}_A^j \equiv 0$ and the positive functions rad_A^a and rad_A^c that satisfy the properties of Theorem 2.28 on I_t^δ. If we find

$$\mathrm{P}(A^{\star}, \mathbb{R}^{n-1} \times B) = \mathrm{P}(A, \mathbb{R}^{n-1} \times B)\,,$$

for $B = I_t^\delta$, this equality transfers to all Borel subsets B of I_t^δ according to Theorem 3.5 ii). Since we can exhaust I by countably many neighborhoods of the form I_t^δ, we can conclude (3.32) via the σ-additivity of the perimeter with respect to the second component.
Hence, it is left to show (3.32) in the case where $I \subseteq \{\mathrm{rad}_A^- > \varepsilon\}$ is an open interval such that rad_A and cen_A are uniformly continuous on $\overline{I}$ and there exists a decomposition of rad_A into the positive and uniformly continuous functions rad_A^a and rad_A^c.
To this end, we will approximate the Cantor parts of rad_A and cen_A suitably by jump functions on I. Considering the continuous representative of cen_A, we find by Theorem 2.24

$$\mathrm{pVar}(\mathrm{cen}_A, I) = |\mathrm{D}\mathrm{cen}_A|(I) = |\mathrm{D}^c \mathrm{cen}_A|(I) \qquad \text{and} \qquad \mathrm{pVar}(\mathrm{rad}_A^c, I) = |\mathrm{D}\mathrm{rad}_A^c|(I) = |\mathrm{D}^c \mathrm{rad}_A|(I)\,. \tag{3.42}$$

Thus, by the definition of pointwise variation, we find for every $\delta \in (0, \varepsilon)$ a partition $a = t_1 < t_2 < \cdots < t_k = b$ of I such that, for the functions

$$c_\delta := \sum_{i=1}^{k-1} \mathrm{cen}_A(t_i) \mathbb{1}_{[t_i, t_{i+1})} \qquad \text{and} \qquad r_\delta := \sum_{i=1}^{k-1} \mathrm{rad}_A^c(t_i) \mathbb{1}_{[t_i, t_{i+1})}\,,$$

the following properties are satisfied:

$$\begin{aligned} &|\mathrm{cen}_A(t_{i+1}) - \mathrm{cen}_A(t_i)| < \delta\,, &&|\mathrm{cen}_A - c_\delta| < \delta \quad \text{on } I\,, &&|\mathrm{Dcen}_A|(I) - \delta \leqslant |\mathrm{D}c_\delta|(I) \leqslant |\mathrm{Dcen}_A|(I)\,,\\ &|\mathrm{rad}^c_A(t_{i+1}) - \mathrm{rad}^c_A(t_i)| < \delta\,, &&|\mathrm{rad}^c_A - r_\delta| < \delta \quad \text{on } I\,, &&|\mathrm{D}^c\mathrm{rad}_A|(I) - \delta \leqslant |\mathrm{D}r_\delta|(I) \leqslant |\mathrm{Drad}^c_A|(I) \end{aligned}$$

for all $i \in \{1, \dots, k-1\}$. By the uniform continuity of rad^a_A on $\overline{I}$, we can assume

$$\max\left\{|(\mathrm{rad}^a_A)^m(x) - (\mathrm{rad}^a_A)^m(y)| : \, x, y \in [t_i, t_{i+1}],\, m \in \{1, \dots, n-1\}\right\} < \delta \quad \text{for all } i \in \{1, \dots, k-1\}\,. \tag{3.43}$$

In the following, we estimate the jumps of c_δ which exceed the jumps of r_δ. To this end, let

$$s_i := |\mathrm{D}c_\delta|(\{t_i\}) - |\mathrm{D}r_\delta|(\{t_i\}) \quad \text{for } i \in \{2, \dots, k-1\} \qquad \text{and} \qquad \mathrm{J}_+ := \{i \in \{2, \dots, k-1\} : \, s_i > 0\}.$$

Since it holds that

$$|\mathrm{D}c_\delta|(\{t_i\}) = |\mathrm{cen}_A(t_i) - \mathrm{cen}_A(t_{i-1})| = |\mathrm{Dcen}_A((t_{i-1}, t_i))| \leqslant |\mathrm{Dcen}_A|((t_{i-1}, t_i))\,,$$

and similarly for r_δ and rad^c_A, we can exploit the assumption $|\mathrm{Dcen}_A| \leqslant |\mathrm{D}^c\mathrm{rad}_A|$ on I to estimate

$$\begin{aligned} \sum_{t_i \in \mathrm{J}_+} s_i &= \sum_{t_i \in \mathrm{J}_+} |\mathrm{D}c_\delta|(\{t_i\}) - |\mathrm{D}r_\delta|(\{t_i\}) \\ &\leqslant \sum_{t_i \in \mathrm{J}_+} |\mathrm{Dcen}_A|((t_{i-1}, t_i)) - |\mathrm{D}r_\delta|(\{t_i\}) \\ &\leqslant \sum_{t_i \in \mathrm{J}_+} |\mathrm{Drad}^c_A|((t_{i-1}, t_i)) - |\mathrm{D}r_\delta|(\{t_i\}) \\ &\leqslant |\mathrm{Drad}^c_A|(I) - |\mathrm{D}r_\delta|(I) \leqslant \delta\,. \end{aligned} \tag{3.44}$$

We now define the set

$$A_\delta := \bigcup_{t \in I} \mathrm{B}^{n-1}_{\mathrm{rad}^a_A(t) + r_\delta(t)}(c_\delta(t)) \times \{t\}\,,$$

which is the stack of balls with radius function $\mathrm{rad}^a_A + r_\delta$ and center function c_δ. Since c_δ and r_δ are pure jump functions with only finitely many jumps, we can verify

$$\mathrm{P}(A_\delta, \mathbb{R}^{n-1} \times I) = \mathrm{P}(A_\delta, \mathbb{R}^{n-1} \times (I \backslash \{t_2, \dots, t_{k-1}\})) + \sum_{i=2}^{k-1} \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A(t_i)}(\mathrm{cen}_A(t_i)) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(t_{i-1})}(\mathrm{cen}_A(t_{i-1}))\right| < \infty$$

by Lemma 3.35. As $A_\delta \cap \mathbb{R}^{n-1} \times (t_i, t_{i+1})$ is a shift of $A^\star_\delta \cap \mathbb{R}^{n-1} \times (t_i, t_{i+1})$, we have that

$$\mathrm{P}(A_\delta, \mathbb{R}^{n-1} \times (t_i, t_{i+1})) = \mathrm{P}(A^\star_\delta, \mathbb{R}^{n-1} \times (t_i, t_{i+1})) \tag{3.45}$$

holds true for all $i \in \{1, \dots, k-1\}$. This implies

$$\begin{aligned} \mathrm{P}(A_\delta, \mathbb{R}^{n-1} \times I) - \mathrm{P}(A^\star_\delta, \mathbb{R}^{n-1} \times I) &= \mathrm{P}(A_\delta, \mathbb{R}^{n-1} \times \{t_2, \dots, t_{i-1}\}) - \mathrm{P}(A^\star_\delta, \mathbb{R}^{n-1} \times \{t_2, \dots, t_{i-1}\}) \\ &= \sum_{i=2}^{k-1} \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A(t_i)}(\mathrm{cen}_A(t_i)) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(t_{i-1})}(\mathrm{cen}_A(t_{i-1}))\right| - \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A(t_i)} \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(t_{i-1})}\right| \\ &= \sum_{i \in \mathrm{J}_+} \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A(t_i)}(\mathrm{cen}_A(t_i)) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(t_{i-1})}(\mathrm{cen}_A(t_{i-1}))\right| - \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A(t_i)} \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(t_{i-1})}\right| \\ &= \sum_{i \in \mathrm{J}_+} \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A(t_i)}(|\mathrm{cen}_A(t_i) - \mathrm{cen}_A(t_{i-1})|e_1) \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(t_{i-1})}\right| - \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A(t_i)} \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_A(t_{i-1})}\right|. \end{aligned}$$

To closer investigate the final sum, we assume w.l.o.g. $\mathrm{rad}_A(t_i) < \mathrm{rad}_A(t_{i-1})$. Since $|\mathrm{cen}_A(t_i) - \mathrm{cen}_A(t_{i-1})| < \delta < \varepsilon < \mathrm{rad}^-_A(t_i)$ for all $i \in \mathrm{J}_+$, we find

$$\begin{aligned} \mathrm{P}(A_\delta, \mathbb{R}^{n-1} \times \{t_i\}) - \mathrm{P}(A^\star_\delta, \mathbb{R}^{n-1} \times \{t_i\}) &= 2\left|\mathrm{B}^{n-1}_{\mathrm{rad}_A(t_i)}(|\mathrm{cen}_A(t_i) - \mathrm{cen}_A(t_{i-1})|e_1) \backslash \mathrm{B}^{n-1}_{\mathrm{rad}_A(t_{i-1})}\right| \\ &\leqslant \left|\mathrm{B}^{n-1}_{\mathrm{rad}_A(t_{i-1}) + s_i} \backslash \mathrm{B}^{n-1}_{\mathrm{rad}_A(t_{i-1})}\right| \\ &= \omega_{n-1}\left[(\mathrm{rad}_A(t_{i-1}) + s_i)^{n-1} - \mathrm{rad}_A(t_{i-1})^{n-1}\right] \\ &\leqslant K(n)\left(\|\mathrm{rad}_A\|^{n-2}_{\mathrm{L}^\infty(I)} + 1\right) s_i \end{aligned} \tag{3.46}$$

for some constant $K(n) > 0$, which only depends on n, where we applied the Lipschitz continuity of $s \mapsto s^{n-1}$ on bounded sets $[0, R]$ with Lipschitz constant $(n-1)R^{n-2}$ for all $R > 0$.

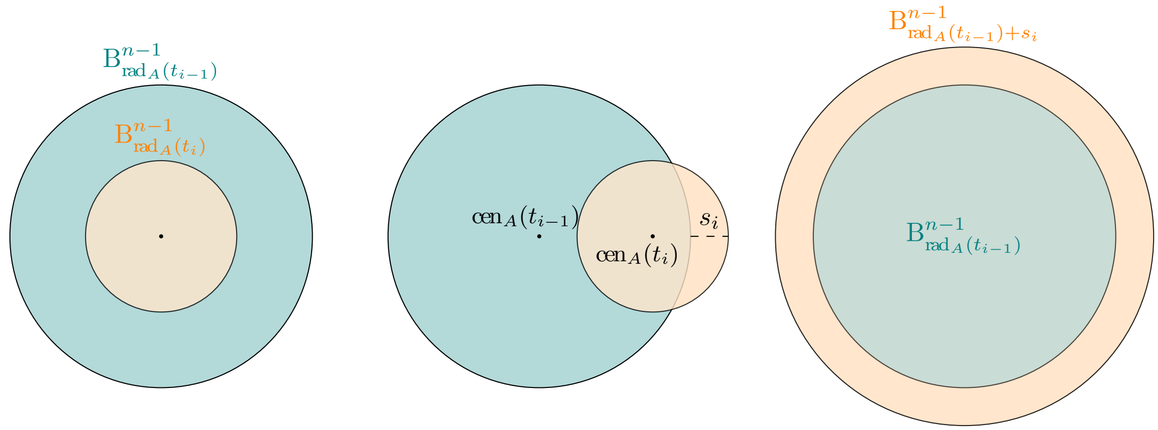


Figure 3.7: Illustration of estimate (3.46)

Hence, (3.44), (3.45) and (3.46) give

$$\mathrm{P}(A_\delta, \mathbb{R}^{n-1} \times I) - \mathrm{P}(A_\delta^\star, \mathbb{R}^{n-1} \times I) \leqslant K(n) \left(\|\mathrm{rad}_A\|^{n-2}_{\mathrm{L}^\infty(I)} + 1 \right) \delta \,,$$

which yields $|\mathrm{P}(A_\delta, \mathbb{R}^{n-1} \times \{t_i\}) - \mathrm{P}(A_\delta^\star, \mathbb{R}^{n-1} \times \{t_i\})| \to 0$ as $\delta \searrow 0$.
Since r_δ and c_δ approximate the radius and the center function uniformly in I, Fubini's theorem ensures that A_δ converges to A in $\mathrm{L}^1(\mathbb{R}^{n-1} \times I)$, compare the proof of Proposition 3.20. The lower semicontinuity of the perimeter and the perimeter convergence from the previous step yield

$$\mathrm{P}(A, \mathbb{R}^{n-1} \times I) \leqslant \liminf_{\delta \searrow 0} \mathrm{P}(A_\delta, \mathbb{R}^{n-1} \times I) = \liminf_{\delta \searrow 0} \mathrm{P}(A_\delta^\star, \mathbb{R}^{n-1} \times I) \,.$$

By the Schwarz inequality, is remains to show that $\liminf_{\delta \searrow 0} \mathrm{P}(A_\delta^\star, \mathbb{R}^{n-1} \times I) \leqslant \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times I)$.
To this end, we apply Lemma 3.45 to find

$$\begin{aligned}
\infty &> \mathrm{P}(A_\delta^\star, \mathbb{R}^{n-1} \times I) \\
&= \omega_{n-1} \left| \left((n-1)(\mathrm{rad}_A^a + r_\delta)^{n-2} \mathcal{L}^1, \mathrm{D}^a((\mathrm{rad}_A^a + r_\delta)^{n-1}) \right) \right| (I) + \omega_{n-1} |\mathrm{D}^s((\mathrm{rad}_A^a + r_\delta)^{n-1})|(I) \\
&= \omega_{n-1}(n-1)(\mathrm{rad}_A^a + r_\delta)^{n-2} \left| \left(\mathcal{L}^1, \mathrm{D}^a \mathrm{rad}_A \right) \right| (I) + \omega_{n-1} |\mathrm{D}^s((\mathrm{rad}_A^a + r_\delta)^{n-1})|(I)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times I) &= \omega_{n-1} \left| \left((n-1)\mathrm{rad}_A^{n-2} \mathcal{L}^1, \mathrm{D}^a(\mathrm{rad}_A^{n-1}) \right) \right| (I) + \omega_{n-1} |\mathrm{D}^s(\mathrm{rad}_A^{n-1})|(I) \\
&= \omega_{n-1}(n-1)\mathrm{rad}_A^{n-2} \left| \left(\mathcal{L}^1, \mathrm{D}^a \mathrm{rad}_A \right) \right| (I) + \omega_{n-1} |\mathrm{D}^s(\mathrm{rad}_A^{n-1})|(I) \,.
\end{aligned}$$

The uniform convergence of $\mathrm{rad}_A^a + r_\delta$ to rad_A on I and the finiteness of $\left| \left(\mathcal{L}^1, \mathrm{D}^a \mathrm{rad}_A \right) \right| (I)$ imply

$$\lim_{\delta \searrow 0} (\mathrm{rad}_A^a + r_\delta)^{n-2} \left| \left(\mathcal{L}^1, \mathrm{D}^a \mathrm{rad}_A \right) \right| (I) = \mathrm{rad}_A^{n-2} \left| \left(\mathcal{L}^1, \mathrm{D}^a \mathrm{rad}_A \right) \right| (I) \,.$$

In order to verify

$$\limsup_{\delta \searrow 0} \mathrm{P}(A_\delta^\star, \mathbb{R}^{n-1} \times I) \leqslant \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times I) \,,$$

we must show

$$\limsup_{\delta \searrow 0} |\mathrm{D}^s((\mathrm{rad}_A^a + r_\delta)^{n-1})|(I) \leqslant |\mathrm{D}^c(\mathrm{rad}_A^{n-1})|(I) \,. \tag{3.47}$$

To this end, we first record that $(\mathrm{rad}_A^a)^m$ is in $\mathrm{W}^{1,1}(I) \cap \mathrm{L}^\infty(I)$ and that $(\mathrm{rad}_A^c)^m$ is in $\mathrm{BV}(I) \cap \mathrm{L}^\infty(I)$ with pure Cantor derivative for all $m \in \{1, \ldots, n-1\}$. We obtain

$$\begin{aligned}
\mathrm{D}^c((\mathrm{rad}_A)^{n-1}) &= \mathrm{D}^c((\mathrm{rad}_A^a + \mathrm{rad}_A^c)^{n-1}) \\
&= \mathrm{D}^c\left[\sum_{m=0}^{n-1}\binom{n-1}{m}(\mathrm{rad}_A^a)^{n-1-m}(\mathrm{rad}_A^c)^m\right] \\
&= \sum_{m=1}^{n-1}\binom{n-1}{m}(\mathrm{rad}_A^a)^{n-1-m}\mathrm{D}(\mathrm{rad}_A^c)^m
\end{aligned} \tag{3.48}$$

by Lemma 2.27. Analogously, we find

$$\begin{aligned}
\mathrm{D}^s((\mathrm{rad}_A^a + r_\delta)^{n-1}) &= \mathrm{D}^s\left[\sum_{m=0}^{n-1}\binom{n-1}{m}(\mathrm{rad}_A^a)^{n-1-m}(r_\delta)^m\right] \\
&= \sum_{m=1}^{n-1}\binom{n-1}{m}(\mathrm{rad}_A^a)^{n-1-m}\mathrm{D}(r_\delta)^m\,.
\end{aligned} \tag{3.49}$$

By the definition of r_δ, the increasing behavior of $t \mapsto t^m$ on $[0,\infty)$ for all $m \in \mathbb{N}$, the positivity of rad_A^a and the formula (3.49), we obtain

$$\begin{aligned}
\left|\mathrm{D}^s((\mathrm{rad}_A^a + r_\delta)^{n-1})\right|(I) &= \sum_{i=1}^{k-2}\left|\sum_{m=1}^{n-1}\binom{n-1}{m}(\mathrm{rad}_A^a(t_{i+1}))^{n-1-m}\left((\mathrm{rad}_A^c)^m(t_{i+1}) - (\mathrm{rad}_A^c)^m(t_i)\right)\right| \\
&\leqslant \sum_{i=1}^{k-1}\sum_{m=1}^{n-1}\binom{n-1}{m}(\mathrm{rad}_A^a(t_{i+1}))^{n-1-m}\left|(\mathrm{rad}_A^c)^m(t_{i+1}) - (\mathrm{rad}_A^c)^m(t_i)\right|
\end{aligned}$$

Now, $|(\mathrm{rad}_A^c)^m(t_{i+1}) - (\mathrm{rad}_A^c)^m(t_i)| \leqslant |\mathrm{D}(\mathrm{rad}_A^c)^m|(t_i, t_{i+1})$ holds for all $m \in \{1, \ldots, n-1\}$ and all $i \in \{1, \ldots, k-1\}$ according to Theorem 2.24, which, together with Proposition 2.20 and the positivity of rad_A^c on I, leads to

$$\begin{aligned}
\left|\mathrm{D}^s((\mathrm{rad}_A^a + r_\delta)^{n-1})\right|(I) &\leqslant \sum_{i=1}^{k-1}\sum_{m=1}^{n-1}\binom{n-1}{m}(\mathrm{rad}_A^a(t_{i+1}))^{n-1-m}\left|\mathrm{D}(\mathrm{rad}_A^c)^m\right|(t_i, t_{i+1}) \\
&= \sum_{i=1}^{k-1}\sum_{m=1}^{n-1}\binom{n-1}{m}(\mathrm{rad}_A^a(t_{i+1}))^{n-1-m}m\left[(\mathrm{rad}_A^c)^{m-1}\left|\mathrm{D}\mathrm{rad}_A^c\right|\right](t_i, t_{i+1})\,.
\end{aligned}$$

We further exploit (3.43) and the positivity of rad_A^a and rad_A^c, and apply Proposition 2.20 to verify

$$\begin{aligned}
\left|\mathrm{D}^s((\mathrm{rad}_A^a + r_\delta)^{n-1})\right|(I) &\leqslant \sum_{i=1}^{k-1}\left[\sum_{m=1}^{n-1}\binom{n-1}{m}\left[(\mathrm{rad}_A^a)^{n-1-m}m(\mathrm{rad}_A^c)^{m-1}\right]\right]\left|\mathrm{D}\mathrm{rad}_A^c\right|(t_i, t_{i+1}) \\
&\quad + \delta\sum_{i=1}^{k-1}\sum_{m=1}^{n-1}\binom{n-1}{m}\left[m(\mathrm{rad}_A^c)^{m-1}\left|\mathrm{D}\mathrm{rad}_A^c\right|\right](t_i, t_{i+1}) \\
&\leqslant \sum_{i=1}^{k-1}\left|\sum_{m=1}^{n-1}\binom{n-1}{m}(\mathrm{rad}_A^a)^{n-1-m}m(\mathrm{rad}_A^c)^{m-1}\mathrm{D}\mathrm{rad}_A^c\right|(t_i, t_{i+1}) + c\,\delta \\
&= \sum_{i=1}^{k-1}\left|\sum_{m=1}^{n-1}\binom{n-1}{m}(\mathrm{rad}_A^a)^{n-1-m}\mathrm{D}(\mathrm{rad}_A^c)^m\right|(t_i, t_{i+1}) + c\,\delta\,,
\end{aligned}$$

where c is the constant $\sum_{m=1}^{n-1}\binom{n-1}{m}m\|\mathrm{rad}_A^c\|_{\mathrm{L}^\infty(I)}^{m-1}\left|\mathrm{D}\mathrm{rad}_A^c\right|(I)$. Together with (3.48), we obtain

$$\begin{aligned}
\left|\mathrm{D}^s((\mathrm{rad}_A^a + r_\delta)^{n-1})\right|(I) &\leqslant \sum_{i=1}^{k-1}\left|\mathrm{D}^c(\mathrm{rad}_A)^{n-1}\right|(t_i, t_{i+1}) + c\,\delta \\
&= \left|\mathrm{D}^c((\mathrm{rad}_A)^{n-1})\right|(I) + c\,\delta\,,
\end{aligned}$$

which confirms (3.47) and therefore proves the claim. □

3.2.3 The vanishing radius part

This section is dedicated to the characterization of the Schwarz inequality on the vanishing radius part $\{\mathrm{rad}_A^- = 0\}$. We will see that, on this part, the perimeter of the symmetrization $A^\star$ coincides with the perimeter of A, regardless of the behavior of the center function. The proof of the following proposition is based on the proof of the corresponding result for the Steiner symmetrization in [14, Proposition 3.8].

Proposition 3.55. *Let $A \subseteq \mathbb{R}^n$ be a set of finite volume and $O \subseteq \mathbb{R}$ open. We assume that A is a stack of balls in O, that A has finite perimeter in $\mathbb{R}^{n-1} \times (\{\mathrm{rad}_A^- > 0\} \cap O)$ and that its symmetrization $A^\star$ has finite perimeter in $\mathbb{R}^{n-1} \times O$. Then it holds that*

$$\mathrm{P}(A, \mathbb{R}^{n-1} \times I) = \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times I) = \sum_{j \in \{\mathrm{rad}_A^+ > 0\} \cap I} \omega_{n-1} (\mathrm{rad}_A^+(j))^{n-1}$$

for all $I \in \mathcal{B}(\{\mathrm{rad}_A^- = 0\} \cap O)$.

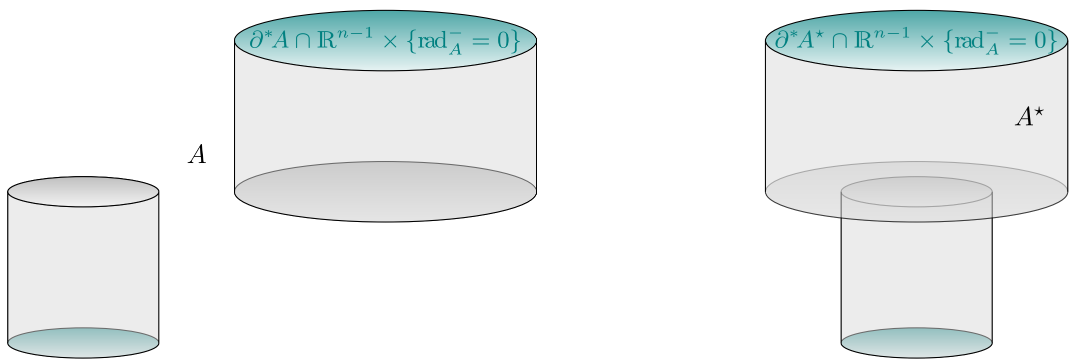


Figure 3.8: Illustration of Proposition 3.55

Remark 3.56. With this result, and with Lemma 3.45 at hand, we can confirm [7, Proposition 3.5] in the case $k = n-1$, which states the equivalence of the properties

i) $\mathcal{H}^{n-1}(\{z \in \partial^* A^\star : \nu_{A^\star}(z) = \pm e_n\}) = 0$

ii) $\mathrm{rad}_A^{n-1} \in \mathrm{W}^{1,1}(\mathbb{R})$

iii) $\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times I) = 0$ for all negligible sets $I \in \mathcal{B}(\mathbb{R})$

for all sets $A \subseteq \mathbb{R}^n$ of finite perimeter and finite volume. In particular, the horizontal boundary parts of $A^\star$ (and, according to [7, Proposition 3.6], therefore of A) are created by the singular parts of the radius function.

To show the equivalence, we record that Fubini's theorem and $|A^\star| < \infty$ give $\mathrm{rad}_A \in \mathrm{L}^{n-1}(\mathbb{R})$. Furthermore, rad_A^{n-1} is in $\mathrm{BV}(\mathbb{R})$ with $|\mathrm{D}\mathrm{rad}_A^{n-1}|(\mathbb{R}) < \infty$, according to Propositions 3.14 and 3.18. Applying Lemma 3.45, Proposition 3.55 and Theorem 2.16 gives that iii) implies $|\mathrm{D}^s \mathrm{rad}_A|(\{\mathrm{rad}_A^- > 0\}) = 0$ and that rad_A is continuous on $\mathbb{R}$. In particular, the equalities $\mathrm{vol}_A^+ = \mathrm{vol}_A^-$ on $\mathbb{R}$ and $0 = |\mathrm{D}^c \mathrm{vol}_A|(\{\mathrm{rad}_A^- > 0\}) = |\mathrm{D}^c \mathrm{vol}_A|(\mathbb{R})$ follow from Lemma 2.21. So we find $\omega_{n-1} \mathrm{rad}_A^{n-1} = \mathrm{vol}_A \in \mathrm{W}^{1,1}(\mathbb{R})$.

On the other hand, $\mathrm{vol}_A \in \mathrm{W}^{1,1}(\mathbb{R})$ implies $|\mathrm{D}^s \mathrm{rad}_A| = 0$ and $\mathrm{rad}_A^+ = \mathrm{rad}_A^-$ on $\mathbb{R}$, which, by Lemma 3.45 and Proposition 3.55, implies iii).

Property i) implies ii) and iii) by Theorem 3.8.

To show that properties iii) and ii) imply i), we use the same strategy as in [7] and notice that, according to Theorem 3.7, there exists a null set $N \subseteq \mathbb{R}$ such that for all $t \in \mathbb{R} \backslash N$, it holds $\mathcal{H}^{n-2}(\partial^*(A_t^\star) \triangle (\partial^* A^\star)_t) = 0$

and $\nu_{A^\star}(x,t) \notin \{-e_n, e_n\}$ for $\mathcal{H}^{n-2}$-a.e. $x \in \partial^*(A_t^\star) \cap (\partial^* A^\star)_t$. In the two-dimensional case, we even have $\nu_{A^\star}(x,t) \notin \{-e_n, e_n\}$ for all $t \in \mathbb{R}\backslash N$ and all $x \in \partial^*(A_t^\star) = (\partial^* A^\star)_t$, such that iii) gives

$$\mathcal{H}^{n-1}\left(\{z \in \partial^* A^\star : \nu_{A^\star}(z) = \pm e_n\}\right) \leqslant \mathcal{H}^{n-1}\left(\partial^* A^\star \cap (\mathbb{R}^{n-1} \times N)\right) = \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times N) = 0\,.$$

If $n > 2$, it holds $\partial^*(A_t^\star) = \partial \mathrm{B}^{n-1}_{\mathrm{rad}_A(t)}$ for all $t \in \mathbb{R}\backslash N$, where we interpret the latter expression as the empty set if $\mathrm{rad}_A(t) = 0$. By $\mathcal{H}^{n-2}(\partial^*(A_t^\star) \triangle (\partial^* A^\star)_t) = 0$, the continuity of rad_A and the symmetry of $A^\star$, we find $\partial \mathrm{B}^{n-1}_{\mathrm{rad}_A(t)} \triangle (\partial^* A^\star)_t \subseteq \{0\}$ for all $t \in \mathbb{R}\backslash N$. As in the two-dimensional case, we conclude

$$\mathcal{H}^{n-1}\left(\{z \in \partial^* A^\star : \nu_{A^\star}(z) = \pm e_n\}\right) \leqslant \mathcal{H}^{n-1}\left(\partial^* A^\star \cap (\mathbb{R}^{n-1} \times N)\right) + \mathcal{H}^{n-1}(\{0\} \times \mathbb{R}) = \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times N) = 0$$

with property iii).

Proof of Proposition 3.55. It suffices to show the claim for the case $O = \mathbb{R}$, since the assumptions of the theorem transfer to $A \cap (\mathbb{R}^{n-1} \times O')$ for all open intervals $O' \Subset O$ such that the general case follows by an exhaustion argument. We treat the special case $I = \{\mathrm{rad}_A^- = 0\}$ first. Throughout this proof, we mainly work with the volume function $\mathrm{vol}_A = \mathrm{vol}_{A^\star} = \omega_{n-1} \mathrm{rad}_A^{n-1}$ instead of the radius function as, by Propositions 3.18 and 3.14, this is a function in $\mathrm{BV}(\mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R})$. Hence, the set $\{\mathrm{vol}_A^+ > 0\} \cap I = \{\mathrm{rad}_A^+ > 0\} \cap I \subseteq \{\mathrm{rad}_A^- = 0 < \mathrm{rad}_A^+\}$ is at most countable. Applying Theorem 2.41 on vol_A provides

$$\int_0^\infty \mathrm{P}(\{\mathrm{vol}_A > t\})\,\mathrm{d}t = |\mathrm{D}\mathrm{vol}_A|(\mathbb{R}) < \infty\,, \tag{3.50}$$

hence we find a null set $N \subseteq [0, \infty)$ with $\mathrm{P}(\{\mathrm{vol}_A > t\}) < \infty$ for all $t \in [0, \infty)\backslash N$. Next, we argue that for all $\delta > 0$, there exists $\varepsilon \in (0, \delta)\backslash N$ with

$$\mathrm{P}(\{\mathrm{vol}_A > \varepsilon\}) \leqslant \frac{\delta}{\varepsilon}.$$

To this end, assume that there exists $\delta > 0$ such that for all $\varepsilon \in (0, \delta)\backslash N$, it holds $\mathrm{P}(\{\mathrm{vol}_A > \varepsilon\}) > \frac{\delta}{\varepsilon}$. This leads to

$$\int_0^\infty \mathrm{P}(\{\mathrm{vol}_A > t\})\,\mathrm{d}t > \int_0^\delta \frac{\delta}{t}\,\mathrm{d}t = \infty\,,$$

in contradiction to (3.50). Thus, we find a decreasing null sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that

$$(\varepsilon_k \mathrm{P}(\{\mathrm{vol}_A > \varepsilon_k\}))_{k\in\mathbb{N}} \subseteq [0, \infty)$$

is a null sequence as well. In particular, for all $k \in \mathbb{N}$, the set $\{\mathrm{vol}_A > \varepsilon_k\}$ is a union of finitely many disjoint intervals except for a null set, according to Lemma 2.39. Hence, for all $t \in \partial^*\{\mathrm{vol}_A > \varepsilon_k\}$, there exists $\delta_t > 0$ such that either $(t - \delta_t, t) \subseteq \{\mathrm{vol}_A > \varepsilon_k\}^1$ and $(t, t + \delta_t) \subseteq \{\mathrm{vol}_A \leqslant \varepsilon_k\}^+$ or such that $(\delta_t, t + \delta_t) \subseteq \{\mathrm{vol}_A > \varepsilon_k\}^1$ and $(t - \delta_t, t) \subseteq \{\mathrm{vol}_A \leqslant \varepsilon_k\}^+$. The upper semicontinuity of vol_A^+ implies $\mathrm{vol}_A^+(t) \geqslant \varepsilon_k$, which leads to the existence of a one-sided limit of cen_A at t by Remark 3.51 for every good representative of cen_A. In fact, cen_A is bounded in $(t - \delta_t, t + \delta_t) \cap \{\mathrm{vol}_A > \varepsilon_k\}^1$. With this, we argue that

$$A_k := A \cap (\mathbb{R}^{n-1} \times \{\mathrm{vol}_A > \varepsilon_k\})$$

is a set of finite perimeter, since

$$\partial^* A_k \subseteq (\partial^* A \cap (\mathbb{R}^{n-1} \times \{\mathrm{vol}_A^- > 0\})) \cup (\mathrm{B}^{n-1}_{R + \|\mathrm{rad}_A\|_{\mathrm{L}^\infty(\mathbb{R})}} \times \partial^*\{\mathrm{vol}_A > \varepsilon_k\})$$

holds for all $R > \sup\left\{|\mathrm{cen}_A(s)| : s \in \bigcup_{t \in \partial^*\{\mathrm{vol}_A > \varepsilon_k\}} (t - \delta_t, t + \delta_t) \cap \{\mathrm{vol}_A > \varepsilon_k\}^1\right\}$. By Fubini's theorem, we further obtain

$$|(A \triangle A_k) \cap (\mathbb{R}^{n-1} \times K)| = \int_{K \backslash \{\mathrm{vol}_A > \varepsilon_k\}} |A_t|\,\mathrm{d}t \leqslant \varepsilon_k |K| \to 0 \qquad \text{as } k \to \infty$$

for all compact sets $K \subseteq \mathbb{R}$. Therefore, we have $A_k \to A$ as $k \to \infty$ in $\mathrm{L}^1_{\mathrm{loc}}$, and the lower semicontinuity of the perimeter gives

$$\mathrm{P}(A) \leqslant \liminf_{k\to\infty} \mathrm{P}(A_k) = \liminf_{k\to\infty} \mathrm{P}(A_k, \mathbb{R}^{n-1} \times \partial^*\{\mathrm{vol}_A > \varepsilon_k\}) + \mathrm{P}(A, \mathbb{R}^{n-1} \times \{\mathrm{vol}_A > \varepsilon_k\}^1)\,, \tag{3.51}$$

where the last equality follows from Lemma 2.49 and the definition of A_k as intersection of sets. In particular, A is a set of finite perimeter. For all $t \in \{\mathrm{vol}_A > \varepsilon_k\}^1$, we find $\mathrm{vol}_A^-(t) \geqslant \varepsilon_k$, since $\mathrm{vol}_A^-(t) \in \{\mathrm{vol}_A^-(t+), \mathrm{vol}_A^-(t-)\}$ and since $\{\mathrm{vol}_A > \varepsilon_k\}^1$ is a finite union of open intervals with pairwise positive distance. Taking into account the openness of $\{\mathrm{vol}_A^- > \varepsilon_k\}$, we find

$$\{\mathrm{vol}_A^- > 0\} = \bigcup_{k=1}^{\infty} \{\mathrm{vol}_A^- > \varepsilon_k\} \subseteq \bigcup_{k=1}^{\infty} \{\mathrm{vol}_A > \varepsilon_k\}^1 \subseteq \bigcup_{k=1}^{\infty} \{\mathrm{vol}_A^- \geqslant \varepsilon_k\} \subseteq \{\mathrm{vol}_A^- > 0\} .$$

With this, (3.51) reads

$$\mathrm{P}(A) \leqslant \liminf_{k\to\infty} \mathrm{P}(A_k, \mathbb{R}^{n-1} \times \partial^*\{\mathrm{vol}_A > \varepsilon_k\}) + \mathrm{P}(A, \mathbb{R}^{n-1} \times \{\mathrm{vol}_A^- > 0\}) ,$$

which can be rewritten as

$$\mathrm{P}(A, \mathbb{R}^{n-1} \times \{\mathrm{vol}_A^- = 0\}) \leqslant \liminf_{k\to\infty} \mathrm{P}(A_k, \mathbb{R}^{n-1} \times \partial^*\{\mathrm{vol}_A > \varepsilon_k\}) . \tag{3.52}$$

To further analyze the right-hand side, we record that, for every $t \in \partial^*\{\mathrm{vol}_A > \varepsilon_k\} \cap \{\mathrm{vol}_A^- > 0\}$, there exists some $\delta > 0$ such that $\mathrm{vol}_{A_k}^-(s) = 0$ holds for either $\mathcal{L}^1$-a.e. $s \in (t-\delta, t)$ or $\mathcal{L}^1$-a.e. $s \in (t, t+\delta)$ by the previous analysis. For the remaining argument, we assume w.l.o.g. that $\mathrm{vol}_{A_k}(t-) = \mathrm{vol}_{A_k}^-(t)$ and conclude that $\mathrm{vol}_{A_k}^- = 0$ on $(t-\delta, t)$ and $\mathrm{vol}_{A_k}^- > \varepsilon_k$ on $(t, t+\delta)$ for some $\delta > 0$. According to the previous observations, the limit $\mathrm{cen}_A(t+) = \mathrm{cen}_{A_k}(t+)$ exists for every good representative of cen_A, and by Lemma 3.35, we find $\mathrm{Tr}_{H_t^+} A_k = \mathbb{1}_{\mathrm{B}^{n-1}_{\mathrm{rad}_A^-(t+)}(\mathrm{cen}_A(t+))}$ and $\mathrm{Tr}_{H_t^-} A_k = 0$. This gives

$$\begin{aligned} \mathrm{P}(A_k, \mathbb{R}^{n-1} \times \partial^*\{\mathrm{vol}_A > \varepsilon_k\}) &= \sum_{t\in\partial^*\{\mathrm{vol}_A>\varepsilon_k\}} \mathrm{P}(A_k, H_t) \\ &= \sum_{t\in\partial^*\{\mathrm{vol}_A>\varepsilon_k\}} |\mathrm{B}^{n-1}_{\mathrm{rad}_A(t+)}(\mathrm{cen}_A(t+))| \\ &= \sum_{t\in\partial^*\{\mathrm{vol}_A>\varepsilon_k\}} \mathrm{vol}_A^+(t) . \end{aligned} \tag{3.53}$$

For every $t \in \mathbb{R}$ with $\mathrm{vol}_A^-(t) = 0$ and $\mathrm{vol}_A^+(t) > \varepsilon_k$, we find $t \in \partial^*\{\mathrm{vol}_A > \varepsilon_k\}$, which yields

$$\{\mathrm{vol}_A^- = 0\} \cap \mathrm{J}_{\mathrm{vol}_A} \subseteq \bigcup_{k=1}^{\infty} \{\mathrm{vol}_A^- = 0\} \cap \partial^*\{\mathrm{vol}_A > \varepsilon_k\} \subseteq \bigcup_{k=1}^{\infty} \{\mathrm{vol}_A^- = 0\} \cap \{\mathrm{vol}_A^+ \geqslant \varepsilon_k\} = \{\mathrm{vol}_A^- = 0\} \cap \mathrm{J}_{\mathrm{vol}_A} .$$

With this, we conclude

$$\lim_{k\to\infty} \sum_{t\in\{\mathrm{vol}_A^-=0\}\cap\partial^*\{\mathrm{vol}_A>\varepsilon_k\}} \mathrm{vol}_A^+(t) = \sum_{t\in\{\mathrm{vol}_A^-=0\}\cap \mathrm{J}_{\mathrm{vol}_A}} \mathrm{vol}_A^+(t) . \tag{3.54}$$

Furthermore, the structure of $\{\mathrm{vol}_A > \varepsilon_k\}$ gives

$$\partial^*\{\mathrm{vol}_A > \varepsilon_k\} \cap \{\mathrm{vol}_A^- > 0\} = \partial^*\{\mathrm{vol}_A > \varepsilon_k\} \cap \{\mathrm{vol}_A^+ \geqslant \varepsilon_k\} \cap \{\mathrm{vol}_A^- > 0\} \subseteq \{0 < \mathrm{vol}_A^- \leqslant \varepsilon_k\} , \tag{3.55}$$

which leads to

$$\lim_{k\to\infty} \sum_{t\in E_k} \mathrm{vol}_A^+(t) - \mathrm{vol}_A^-(t) = \lim_{k\to\infty} |\mathrm{D}\mathrm{vol}_A|(E_k) = 0 \tag{3.56}$$

for

$$E_k := \partial^*\{\mathrm{vol}_A > \varepsilon_k\} \cap \{\mathrm{vol}_A^- > 0\} \cap \mathrm{J}_{\mathrm{vol}_A}$$

as $|\mathrm{D}\mathrm{vol}_A|(\mathbb{R}) < \infty$. We further denote

$$F_k := \big(\partial^*\{\mathrm{vol}_A > \varepsilon_k\} \cap \{\mathrm{vol}_A^- > 0\}\big) \setminus \mathrm{J}_{\mathrm{vol}_A} ,$$

and estimate

$$\begin{aligned}\sum_{t\in\partial^*\{\mathrm{vol}_A>\varepsilon_k\}\cap\{\mathrm{vol}_A^->0\}} \mathrm{vol}_A^+(t) &= \sum_{t\in E_k}\mathrm{vol}_A^+(t)+\sum_{t\in F_k}\mathrm{vol}_A^+(t)\\ &\leqslant \sum_{t\in E_k}[\mathrm{vol}_A^+(t)-\mathrm{vol}_A^-(t)+\varepsilon_k]+\sum_{t\in F_k}\varepsilon_k\\ &\leqslant \sum_{t\in E_k}[\mathrm{vol}_A^+(t)-\mathrm{vol}_A^-(t)]+\sum_{t\in\partial^*\{\mathrm{vol}_A>\varepsilon_k\}\cap\{\mathrm{vol}_A^->0\}}\varepsilon_k\\ &\leqslant \sum_{t\in E_k}[\mathrm{vol}_A^+(t)-\mathrm{vol}_A^-(t)]+\mathrm{P}(\{\mathrm{vol}_A>\varepsilon_k\})\varepsilon_k\,,\end{aligned}$$

where we applied (3.55) and the fact that $\mathrm{vol}_A^+=\mathrm{vol}_A^-$ holds on F_k. With this, along with the choice of $(\varepsilon_k)_{k\in\mathbb{N}}$, (3.52), (3.53), (3.54) and (3.56), we obtain

$$\mathrm{P}(A,\mathbb{R}^{n-1}\times\{\mathrm{vol}_A^-=0\})\leqslant\sum_{t\in\{\mathrm{vol}_A^-=0\}\cap\mathrm{J}_{\mathrm{vol}_A}}\mathrm{vol}_A^+(t)\,.$$

Taking into account Lemma 3.35 and Theorem 3.5, we finally conclude

$$\begin{aligned}\mathrm{P}(A,\mathbb{R}^{n-1}\times\{\mathrm{vol}_A^-=0\}) &\leqslant \sum_{t\in\{\mathrm{vol}_A^-=0\}\cap\mathrm{J}_{\mathrm{vol}_A}}\mathrm{vol}_A^+(t)\\ &=\mathrm{P}(A^\star,\mathbb{R}^{n-1}\times(\{\mathrm{vol}_A^-=0\}\cap\mathrm{J}_{\mathrm{vol}_A}))\\ &\leqslant\mathrm{P}(A^\star,\mathbb{R}^{n-1}\times\{\mathrm{vol}_A^-=0\})\\ &\leqslant\mathrm{P}(A,\mathbb{R}^{n-1}\times\{\mathrm{vol}_A^-=0\})\,.\end{aligned}$$

Hence, we obtain equality, and by the relation between vol_A and rad_A, the claim is proven for the special case $I=\{\mathrm{rad}_A^-=0\}$.
It remains to show the claim for $I\in\mathcal{B}(\{\mathrm{vol}_A^-=0\})$. According to the previously shown equality, it holds $\mathrm{P}(A^\star,\mathbb{R}^{n-1}\times(\{\mathrm{vol}_A^-=0\}\backslash\mathrm{J}_{\mathrm{vol}_A}))=0$. By Lemma 3.35, Remark 3.36 and Theorem 3.5, we have

$$\sum_{t\in I}\mathrm{vol}_A^+(t)=\mathrm{P}(A^\star,\mathbb{R}^{n-1}\times I)\leqslant\mathrm{P}(A,\mathbb{R}^{n-1}\times I)\qquad\text{and}$$
$$\sum_{t\in\{\mathrm{rad}_A^-=0\}\backslash I}\mathrm{vol}_A^+(t)=\mathrm{P}(A^\star,\mathbb{R}^{n-1}\times\{\mathrm{rad}_A^-=0\}\backslash I)\leqslant\mathrm{P}(A,\mathbb{R}^{n-1}\times\{\mathrm{rad}_A^-=0\}\backslash I)\,.$$

In fact, these inequalities turn out to be equalities according to the special cases already proven, which concludes the proof. □

3.2.4 The main theorem

We are now able to bring the previous results together and state the main theorem of this section.

Theorem 3.57 (Characterization of the equality cases in the Schwarz inequality)**.** *Let $A\subseteq\mathbb{R}^n$ be a set of finite perimeter and finite volume and let $I\subseteq\mathbb{R}$ open. Then the following are equivalent:*

i) $\mathrm{P}(A^\star,\mathbb{R}^{n-1}\times I)=\mathrm{P}(A,\mathbb{R}^{n-1}\times I)$

ii) *The set A is a stack of balls in I and $|\mathrm{Dcen}_A|\leqslant|\mathrm{D}^s\mathrm{rad}_A|$ holds on $I\cap\{\mathrm{rad}_A^->0\}$.*

Proof. According to the Schwarz inequality of Theorem 3.5, statement i) is equivalent to

$$\mathrm{P}(A^\star,\mathbb{R}^{n-1}\times(I\cap\{\mathrm{rad}_A^->0\}))=\mathrm{P}(A,\mathbb{R}^{n-1}\times(I\cap\{\mathrm{rad}_A^->0\}))\tag{3.57}$$

and

$$\mathrm{P}(A^\star,\mathbb{R}^{n-1}\times(I\cap\{\mathrm{rad}_A^-=0\}))=\mathrm{P}(A,\mathbb{R}^{n-1}\times(I\cap\{\mathrm{rad}_A^-=0\}))\,.\tag{3.58}$$

Since (3.58) is always satisfied according to Proposition 3.55, it is left to show the equivalence of ii) and (3.57), which is exactly the statement of Proposition 3.53, where we use that $\{\mathrm{rad}_A^->0\}$ is an open set. □

From the main theorem, we can reproduce the rigidity result for Schwarz symmetrizations of [26, Theorem 1.3], see also [7, Theorem 1.2].

Corollary 3.58 (Rigidity of the Schwarz inequality)**.** *Let $A \subseteq \mathbb{R}^n$ be a set of finite volume and finite perimeter. Then the following statements are equivalent:*

i) *The set* $\{\mathrm{rad}_A^- > 0\}$ *is an interval and* $\mathrm{rad}_A \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\})$.

ii) *For every set* $\tilde{A} \subseteq \mathbb{R}^n$ *of finite perimeter with* $\mathrm{rad}_{\tilde{A}} = \mathrm{rad}_A$ *on* $\mathbb{R}$ *that attains equality in the Schwarz perimeter inequality, i.e.,*
$$\mathrm{P}(A^\star) = \mathrm{P}(\tilde{A}),$$
there exists some $x \in \mathbb{R}^{n-1}$ *with* $\tilde{A} = A^\star + (x, 0)$.

Proof. For proving i) $\implies$ ii), we consider the set $\tilde{A}$ satisfying equality in the Schwarz inequality, and apply Theorem 3.57. It follows that $\tilde{A}$ is a stack of balls with $|\mathrm{Dcen}_{\tilde{A}}| \leqslant |\mathrm{D}^s\mathrm{rad}_A| = 0$ on $\{\mathrm{rad}_A^- > 0\}$. Since the latter set is an interval, Lemma 2.13 gives that $\mathrm{cen}_{\tilde{A}}$ is constant on $\{\mathrm{rad}_A^- > 0\}$. Now, $|\tilde{A} \cap \mathbb{R}^{n-1} \times \{\mathrm{rad}_A^- = 0\}| = 0$ holds by Fubini's theorem. It follows that $\tilde{A} = A^\star + (\mathrm{cen}_{\tilde{A}}, 0)$ holds up to a negligible set, where we identified $\mathrm{cen}_{\tilde{A}}$ with the corresponding constant.
In order to prove the other implication, we assume that statement ii) holds true and that either $\{\mathrm{rad}_A^- > 0\}$ is not an interval or $\mathrm{rad}_A \notin \mathrm{W}^{1,1}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\})$.
In the first case, there exists $t \in \{\mathrm{rad}_A^- = 0\}$ with $|A \cap (\mathbb{R}^{n-1} \times (-\infty, t))| \neq 0$ and $|A \cap (\mathbb{R}^{n-1} \times (t, \infty))| \neq 0$. Then we define the set
$$\tilde{A} := (A + (x_1, 0))_{(-\infty,t)} \cup (A + (x_2, 0))_{(t,\infty)} \qquad \text{for any } x_1, x_2 \in \mathbb{R}^{n-1}.$$
It holds that $\mathrm{rad}_{\tilde{A}} = \mathrm{rad}_A$ and $\mathrm{P}(\tilde{A}) = \mathrm{P}(A)$, according to Proposition 3.55 and the invariance of the perimeter with respect to translations. Since x_1 and x_2 were arbitrary, we found a contradiction to ii).
In the case where the radius function is not a Sobolev function, we can apply Lemma 3.34 to conclude $\mathrm{rad}_A \in \mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\})$. If $|\mathrm{D}^j\mathrm{rad}_A| \neq 0$, we can define $\mathrm{cen} := (\mathrm{rad}_A^+(j) - \mathrm{rad}_A^-(j))\mathbb{1}_{(-\infty,j)}e_1$ for some $j \in \mathrm{J}_{\mathrm{rad}_A} \cap \{\mathrm{rad}_A^- > 0\}$ and obtain a contradiction by Lemma 3.35 for $\tilde{A}$ defined as the stack of balls with radius function rad_A and center function cen, i.e.,
$$\tilde{A} := \bigcup_{t\in\mathbb{R}} \mathrm{B}^{n-1}_{\mathrm{rad}_A(t)}(\mathrm{cen}(t)) \times \{t\}.$$
In the case where rad_A is continuous, i.e., the singular part only consists of the Cantor part, we define the non-constant and purely singular BV function $c := \lambda\mathrm{rad}_A^c$ for some $\lambda \in (0, 1]$ on $\{\mathrm{rad}_A^- > 0\}$ and define the set $\tilde{A}$ as the stack of balls with radius function rad_A and center function $c\,e_1$, i.e.,
$$\tilde{A} := \bigcup_{t\in\mathbb{R}} \mathrm{B}^{n-1}_{\mathrm{rad}_A(t)}(c(t)e_1) \times \{t\}.$$
We record $\tilde{A}^\star = A^\star$ by definition and $|A| = |\tilde{A}|$ by Fubini's theorem. Now, we have to ensure that $\tilde{A}$ is a set of finite perimeter.
To this end, we consider the continuous representatives of rad_A and c and record that $C := (\mathrm{rad}_A, c)$ is a 2-dimensional curve of finite variation on all open intervals $I \Subset \{\mathrm{rad}_A^- > 0\}$. Hence, we can parametrize C by arc length into a curve $\hat{C}$ on some bounded interval $\hat{I} \Subset \mathbb{R}$ such that the image of C on I coincides with the image of $\hat{C}$ on $\hat{I}$. In particular, $|\mathrm{D}\hat{C}|((a, b)) = \mathrm{pVar}(\hat{C}, (a, b)) = (b - a)$ holds for all $(a, b) \subseteq \hat{I}$, which implies that $\hat{C}$ is a Lipschitz function with Lipschitz constant 1 on $\hat{I}$ and with $|\mathrm{D}\hat{C}|(\hat{I}) = |\hat{I}|$. We further define the function
$$g : \hat{I} \times \partial\mathrm{B}^{n-1}_1 \to \mathbb{R}^n,\ (t, x) \mapsto \hat{C}_2(t)e_1 + (\hat{C}_1(t)x, t),$$
which satisfies $g(\hat{I} \times \partial\mathrm{B}^{n-1}_1) = (\partial\tilde{A})_I$ by the continuity of rad_A and $\mathrm{cen}_{\tilde{A}} = \lambda\mathrm{rad}_A^c e_1$. Since $\mathrm{rad}_A \in \mathrm{L}^\infty(\mathbb{R})$ by

Lemma 3.34, we find

$$\begin{aligned}|g(t,x)-g(s,y)| &\leqslant |\hat{C}_2(t)-\hat{C}_2(s)|+|\hat{C}_1(t)x-\hat{C}_1(s)y|+|t-s| \\ &\leqslant 2|t-s|+|\hat{C}_1(t)x-\hat{C}_1(s)x|+|\hat{C}_1(s)x-\hat{C}_1(s)y| \\ &\leqslant 2|t-s|+|t-s|+\sup_{\hat{I}}|\hat{C}_1|\,|x-y| \\ &\leqslant \big(3+\|\mathrm{rad}_A\|_{\mathrm{L}^\infty(\mathbb{R})}\big)|(t,x)-(s,y)|\end{aligned}$$

for all (s,y) and (t,x) in $\hat{I}\times\partial\mathrm{B}_1^{n-1}$. With this and the area formula, we conclude

$$\mathrm{P}(\tilde{A},\mathbb{R}^{n-1}\times I)\leqslant\mathcal{H}^{n-1}((\partial\tilde{A})_I)=\mathcal{H}^{n-1}(g(\hat{I}\times\partial\mathrm{B}_1^{n-1}))\leqslant(3+\|\mathrm{rad}_A\|_{\mathrm{L}^\infty(\mathbb{R})})^{n-1}\mathcal{H}^{n-1}(\hat{I}\times\partial\mathrm{B}_1^{n-1})<\infty\,.$$

By the boundedness of rad_A and $\mathrm{cen}_{\tilde{A}}$, Lemma 3.35 implies that the set $\tilde{A}_I$ is of finite perimeter and finite volume. Furthermore, A is a stack of balls such that the properties of Theorem 3.57 ii) are satisfied, which leads to

$$\mathrm{P}\big(\tilde{A},\mathbb{R}^{n-1}\times I\big)=\mathrm{P}\Big(\tilde{A}_I,\mathbb{R}^{n-1}\times I\Big)=\mathrm{P}\Big(\big(\tilde{A}_I\big)^\star,\mathbb{R}^{n-1}\times I\Big)=\mathrm{P}\big(A^\star,\mathbb{R}^{n-1}\times I\big)\,.$$

Since $I\Subset\{\mathrm{rad}_A^->0\}$ was chosen arbitrarily, we conclude that $\tilde{A}$ is a set of finite perimeter in all of $\mathbb{R}^{n-1}\times\{\mathrm{rad}_A^->0\}$ with

$$\mathrm{P}\big(\tilde{A},\mathbb{R}^{n-1}\times\{\mathrm{rad}_A^->0\}\big)=\mathrm{P}\big(A^\star,\mathbb{R}^{n-1}\times\{\mathrm{rad}_A^->0\}\big)<\infty\,. \tag{3.59}$$

With this, we are able to apply Proposition 3.55, which gives

$$\mathrm{P}(\tilde{A},\mathbb{R}^{n-1}\times\{\mathrm{rad}_A^-=0\})=\mathrm{P}(\tilde{A}^\star,\mathbb{R}^{n-1}\times\{\mathrm{rad}_A^-=0\})=\mathrm{P}(A^\star,\mathbb{R}^{n-1}\times\{\mathrm{rad}_A^-=0\})<\infty\,. \tag{3.60}$$

Together, (3.59), (3.60) and the fact that $\mathrm{cen}_{\tilde{A}}$ is not constant are a contradiction to ii), which proves the claim. □

3.3 Generalized Schwarz inequality for functionals with measure data

The main goal of this section is to generalize the Schwarz perimeter inequality and to analyze the equality cases for functionals of this generalized form.

3.3.1 The measure data functional

In the following, we introduce the measure data functional $\mathcal{P}$ and the assumptions which we will impose.

Notation. Let μ_-,μ_+ be two non-negative Radon measures on $\mathbb{R}^n$, such that either $\mu_-(\mathbb{R}^n)$ or $\mu_+(\mathbb{R}^n)$ is finite. We then define the functional

$$\mathcal{P}_{\mu_-,\mu_+}[A]:=\mathrm{P}(A)+\mu_+(A^1)-\mu_-(A^+)$$

for sets $A\subseteq\mathbb{R}^n$ of finite perimeter. In most cases, we will abbreviate

$$\mathcal{P}:=\mathcal{P}_{\mu_-,\mu_+}\,,\qquad\mathcal{P}_-:=\mathcal{P}_{\mu_-,0}\,,\qquad\text{and}\qquad\mathcal{P}_+:=\mathcal{P}_{0,\mu_+}\,.$$

In order to ensure lower semicontinuity of the functional $\mathcal{P}$, we will often require μ_- and μ_+ to satisfy some isoperimetric conditions from Section 2.3.4. Furthermore, we assume that the measures are mutually singular and that they satisfy the monotonicity condition

$$\mu_+(B^\star)-\mu_-(B^\star)\leqslant\mu_+(B)-\mu_-(B)\qquad\text{for all bounded sets }B\in\mathcal{B}(\mathbb{R}^n)\,. \tag{3.61}$$

3.3.2 Measure characterization

To generalize the Schwarz inequality to the functional $\mathcal{P}_{\mu_-,\mu_+}$, it is essential to understand the behavior of the corresponding measures μ_- and μ_+ on $(A^1)^\star \triangle (A^\star)^1$ and $(A^+)^\star \triangle (A^\star)^+$, respectively. To this end, we decompose the measures into a discrete and a regular part via the disintegration theory introduced in Section 2.3.1.
We begin with showing that (3.61) can be reformulated as follows.

Lemma 3.59. *Let μ_-, μ_+ be two mutually singular and non-negative Radon measures on $\mathbb{R}^n$. Then* (3.61) *is equivalent to*

$$\mu_+(B^\star) \leqslant \mu_+(B) \qquad \text{and} \qquad \mu_-(B) \leqslant \mu_-(B^\star) \qquad \text{for all } B \in \mathcal{B}(\mathbb{R}^n)\,. \tag{3.62}$$

Proof. Clearly (3.62) implies (3.61). In order to show the other implication, we first consider $R > 0$ and define $\nu := \mu_+ - \mu_- \llcorner \mathrm{B}_R$. Then (3.61) holds for all $B \in \mathcal{B}(\mathrm{B}_R)$ and there exists a Borel set $N \subseteq \mathrm{B}_R$ such that $\mu_- \llcorner \mathrm{B}_R = \mu_- \llcorner N$ and $\mu_+(N) = 0$. With (3.61), the estimate

$$-\mu_-(N^\star) \leqslant \nu(N^\star) \leqslant \nu(N) = -\mu_-(N)$$

follows, which implies $\mu_-(N) \leqslant \mu_-(N^\star) = \mu_-(N^\star \cap N) \leqslant \mu_-(N)$ and $\mu_+(N^\star) = 0$. Hence, we obtain $\mu_-(N \triangle N^\star) = 0 = |\nu|(N \triangle N^\star)$. For $B \in \mathcal{B}(\mathrm{B}_R)$, we conclude

$$-\mu_-(B^\star) = -\mu_-(B^\star \cap N) = -\mu_-(B^\star \cap N^\star) \leqslant -\mu_-((B \cap N)^\star) = \nu((B \cap N)^\star) \leqslant \mu(B \cap N) = -\mu_-(B)$$

according to (3.61), and the previous inequality confirms $\mu_-(B) \leqslant \mu_-(B^\star)$. Exploiting the finiteness of μ_- on N, the inequality

$$\mu_+(B^\star) - \mu_-(N) = \mu_+(B^\star \cup N^\star) - \mu_-(N^\star) \leqslant \nu((B \cup N)^\star) \leqslant \nu(B \cup N) = \mu_+(B) - \mu_-(N)$$

implies $\mu_+(B^\star) \leqslant \mu_+(B)$. Therefore, we proved (3.62) for all bounded sets $B \in \mathcal{B}(\mathbb{R}^n)$. In order to transfer this to unbounded sets $B \in \mathcal{B}(\mathbb{R}^n)$, we first notice $\bigcup_{R>0}(B \cap \mathrm{B}_R)^\star = B^\star$. Hence, the continuity of measures gives

$$\mu_+(B^\star) = \lim_{R\to\infty} \mu_+((B \cap \mathrm{B}_R)^\star) \leqslant \lim_{R\to\infty} \mu_+(B \cap \mathrm{B}_R) = \mu_+(B)$$

and

$$\mu_-(B^\star) = \lim_{R\to\infty} \mu_-((B \cap \mathrm{B}_R)^\star) \geqslant \lim_{R\to\infty} \mu_-(B \cap \mathrm{B}_R) = \mu_-(B)\,,$$

which completes the proof. □

Proposition 3.60 (Disintegration of measures that satisfy the monotonicity condition)**.** *Let μ be a non-negative Radon measure in $\mathbb{R}^n$. With the notation introduced in Section 2.3.1, the following holds true. If either $\mu(B) \leqslant \mu(B^\star)$ for all $B \in \mathcal{B}(\mathbb{R}^n)$ or $\mu(B^\star) \leqslant \mu(B)$ for all $B \in \mathcal{B}(\mathbb{R}^n)$, there exists a non-negative Borel function $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n; \mathcal{L}^{n-1} \otimes \mu^{\mathrm{V}})$ with $\mu = f(\mathcal{L}^{n-1} \otimes \mu^{\mathrm{V}})$ for μ^{V} as in Definition 2.68.*

Proof. By the Radon-Nikodym theorem, we have to show $\mu \ll \mathcal{L}^{n-1} \otimes \mu^{\mathrm{V}}$. Since $\pi_{n\#}\mu \ll \mu^{\mathrm{V}}$, it suffices to show $\mu \llcorner \mathrm{B}_R \ll \mathcal{L}^{n-1} \otimes \pi_{n\#}(\mu \llcorner \mathrm{B}_R)$ for all $R > 0$. If we consider the monotonicity condition $\mu(B) \leqslant \mu(B^\star)$, we can w.l.o.g. assume $\mu = \mu \llcorner \mathrm{B}_R$ for some $R > 0$. In the case of $\mu(B^\star) \leqslant \mu(B)$, the corresponding monotonicity condition for $\mu \llcorner \mathrm{B}_R$ holds true only for Borel sets $B \subseteq \mathrm{B}_R$.
However, we first assume that $\mu(B) \leqslant \mu(B^\star)$ holds true for all Borel sets B in $\mathbb{R}^n$ and that μ is a finite measure supported in B_R. In this case, we can apply Lemma 2.66 and conclude that, for $\pi_{n\#}\mu$-a.e. $t \in \mathbb{R}$, there exists a Radon measure μ^{H}_t on $\mathbb{R}^{n-1}$ such that the map $\mathbb{R} \to \mathrm{RM}(\mathbb{R}^{n-1}, [0,\infty])$, $t \mapsto \mu^{\mathrm{H}}_t$ is $(\pi_n)_\#\mu$-measurable, and that

$$\mu(B) = \int_{\mathbb{R}} \mu^{\mathrm{H}}_t(B_t)\,\mathrm{d}(\pi_{n\#}\mu)(t) \qquad \text{holds for all } B \in \mathcal{B}(\mathbb{R}^n)\,,$$

as well as (2.15) for all $f \in \mathrm{L}^1(\mathbb{R}^n; \mu)$.
Since $\mathrm{C}^0_{\mathrm{cpt}}$ is separable on open sets, we find for each $k \in \mathbb{N}$ a countable and dense subset $\{\varphi_{k,1}, \varphi_{k,2}, \dots\}$ of

$$\left\{ \varphi \in \mathrm{C}^0_{\mathrm{cpt}}(\mathbb{R}^{n-1} \backslash \{0\}, [0,1]) : |\mathrm{spt}\varphi| < \omega_{n-1}\frac{1}{k^{n-1}} \right\}.$$

Then we define

$$g_{k,i}(t) := \int_{\mathbb{R}^{n-1}} \varphi_{k,i}\,\mathrm{d}\mu_t^{\mathrm{H}}$$

and

$$S(t) := \limsup_{k\to\infty} \sup_{i\in\mathbb{N}} g_{k,i}(t) = \limsup_{k\to\infty} \sup\left\{\mu_t^{\mathrm{H}}(B) :\ B \in \mathcal{B}(\mathbb{R}^{n-1}\backslash\{0\}),\, |B| < \omega_{n-1}\frac{1}{k^{n-1}}\right\}$$

for $t \in \mathbb{R}$, where the latter equality follows from the Riesz-Markov representation theorem. We can split μ_t^{H} into its absolutely continuous part μ_t^a with respect to $\mathcal{L}^{n-1}$ and singular part μ_t^s. In particular, there exists a density $g : \mathbb{R}^{n-1} \to [0,\infty)$ with $\mu^a = g\mathcal{L}^{n-1}$. By the assumptions on μ, we even find $g \in \mathrm{L}^1(\mathbb{R}^{n-1})$. Furthermore, there exists a negligible set $N \subseteq \mathbb{R}^{n-1}$ with $\mu_t^{\mathrm{H}} \llcorner N = \mu_t^s$ such that we can rewrite

$$S(t) = \limsup_{k\to\infty} \sup\left\{\mu_t^a(B) + \mu_t^s(N\backslash\{0\}) :\ B \in \mathcal{B}(\mathbb{R}^{n-1}\backslash\{0\}),\, |B| < \omega_{n-1}\frac{1}{k^{n-1}}\right\} = \mu_t^s(\mathbb{R}^{n-1}\backslash\{0\})\,.$$

By the $\pi_{n\#}\mu$-measurability of $t \mapsto \mu_t^{\mathrm{H}}$ and Lemma 2.65, the functions S and $g_{k,i}$ are $\pi_{n\#}\mu$-measurable.
In the following, we show $\pi_{n\#}\mu(\{S > 0\}) = 0$ by contradiction. If we assume $\pi_{n\#}\mu(\{S > 0\}) > 0$, we find some $\delta > 0$ such that the $\pi_{n\#}\mu$-measurable set $\{S > \delta\}$ has $\pi_{n\#}\mu$-measure equal to some $\lambda > 0$. Hence, we can apply Lusin's theorem on $\mathbb{R}$, obtaining compact sets $E_{k,i} \subseteq \{S > \delta\}$ with $\pi_{n\#}\mu(\{S > \delta\}\backslash E_{k,i}) \leqslant \frac{\lambda}{2^{k+i+1}}$ and $g_{k,i}|_{E_{k,i}}$ continuous for all $i,k \in \mathbb{N}$. We then define the compact set $E := \bigcap_{k,i\in\mathbb{N}} E_{k,i}$ that satisfies

$$\pi_{n\#}\mu(E) = \pi_{n\#}\mu(\{S > \delta\}) - \pi_{n\#}\mu(\{S > \delta\}\backslash E) \geqslant \lambda - \frac{\lambda}{2}\left(\sum_{k=1}^{\infty}\frac{1}{2^k}\right)^2 = \frac{\lambda}{2}\,. \tag{3.63}$$

Since $E \subseteq \{S > \delta\}$, we can fix $\varepsilon > 0$ and find for each $t \in E$ some $k = k(t) \in \mathbb{N}_{\geqslant k_\varepsilon}$ and $i = i(t) \in \mathbb{N}$ with $g_{k,i}(t) > \delta$, where k_ε is chosen such that $\frac{1}{k_\varepsilon} \leqslant \varepsilon$. Note that the dependencies might depend on the exact choice of ε, k and i. By the continuity of $g_{k,i}$ on E, there exists some $\gamma = \gamma(t) > 0$ with $g_{k,i}(s) > \delta$ for all $s \in I_\gamma(t) \cap E$, where we define $I_\gamma(t) := [t-\gamma, t+\gamma]$. Considering the fine cover $\mathcal{C} := \{I_{\frac{1}{\kappa}\gamma(t)}(t) :\ \kappa \in \mathbb{N},\, t \in E\}$ of E, we can apply the covering theorem of Vitali and Besicovitch, which can be found, for instance, in [3, Theorem 2.19]. With this, we find countably many disjoint intervals $(I_j)_{j\in\mathbb{N}} \subseteq \mathcal{C}$ with

$$\pi_{n\#}\mu\left(E\backslash\bigcup_{j=1}^{\infty} I_j\right) = 0\,.$$

By the preliminary assumptions imposed on μ and the compactness of E, there exists $\ell \in \mathbb{N}$ with

$$\pi_{n\#}\mu\left(E\backslash\bigcup_{j=1}^{\ell} I_j\right) < \varepsilon\,. \tag{3.64}$$

In the following, we denote $I_j(t) = I_j$ if there exists some $\kappa \in \mathbb{N}$ such that $I_j = I_{\frac{1}{\kappa}\gamma(t)}(t)$, indicating the middle of I_j. With this, we define the following Borel function

$$\varphi :\ \mathbb{R}^{n-1} \times \mathbb{R} \to [0,1],\ (x,s) \mapsto \begin{cases} \varphi_{k(t),i(t)}(x), & \text{if } s \in I_j(t) \text{ for some } t \in \mathbb{R},\ j \in \{1,\dots,\ell\} \\ 0, & \text{else.} \end{cases}$$

Exploiting the choice of $\varphi_{k,i}$, we find $\mathrm{spt}\varphi \Subset (\mathbb{R}^{n-1}\backslash\{0\}) \times \mathbb{R}$. In particular, there exists $r \in (0,\varepsilon)$ such that $(\mathrm{B}_r^{n-1} \times \mathbb{R}) \cap \mathrm{spt}\varphi = \emptyset$. Moreover, by the definition of k_ε, it holds

$$\left|\left\{x \in \mathbb{R}^{n-1} :\ \varphi(x,s) > 0\right\}\right| \leqslant \left|\mathrm{spt}\varphi_{k(t),i(t)}\right| \leqslant \omega_{n-1}\varepsilon^{n-1}, \qquad \text{for all } s \in I_j(t),\ j \in \{1,\dots,\ell\}\,.$$

It follows

$$\left(\left\{(x,s) \in \mathbb{R}^{n-1} \times \mathbb{R} :\ \varphi(x,s) > 0\right\} \cup (\mathrm{B}_r^{n-1} \times \mathbb{R})\right)^\star \subseteq \mathrm{B}_{2\varepsilon}^{n-1} \times \mathbb{R}\,. \tag{3.65}$$

By (3.64), we have

$$\pi_{n\#}\mu\,(E) \leqslant \pi_{n\#}\mu\left(E \cap \bigcup_{j=1}^{\ell} I_j\right) + \pi_{n\#}\mu\left(E \backslash \bigcup_{j=1}^{\ell} I_j\right) \leqslant \sum_{j=1}^{\ell} \pi_{n\#}\mu\,(E \cap I_j) + \varepsilon\,.$$

Since $g_{k(t),i(t)} > \delta$ on $E \cap I_j(t)$, the estimate

$$\pi_{n\#}\mu\,(E) \leqslant \frac{1}{\delta}\sum_{j=1}^{\ell}\int_{I_j} g_{k(t),i(t)}(s)\,\mathrm{d}\pi_{n\#}\mu(s) + \varepsilon = \frac{1}{\delta}\sum_{j=1}^{\ell}\int_{I_j}\int_{\mathbb{R}^{n-1}} \varphi_{k(t),i(t)}\,\mathrm{d}\mu_t^{\mathrm{H}}\,\mathrm{d}\pi_{n\#}\mu(s) + \varepsilon$$

is verified. Thus, the definition of φ and (2.15) yield

$$\pi_{n\#}\mu\,(E) \leqslant \frac{1}{\delta}\int_{\mathbb{R}^n} \varphi\,\mathrm{d}\mu(x) + \varepsilon\,.$$

As $0 \leqslant \varphi \leqslant 1$, we conclude

$$\begin{aligned}
\pi_{n\#}\mu\,(E) &\leqslant \frac{1}{\delta}\mu(\{\varphi > 0\}) + \varepsilon \\
&\leqslant \frac{1}{\delta}\mu(\{\varphi > 0\} \cup (\mathrm{B}_r^{n-1} \times \mathbb{R})) - \frac{1}{\delta}\mu(\{0\} \times \mathbb{R}) + \varepsilon \\
&\leqslant \frac{1}{\delta}\mu(\mathrm{B}_{2\varepsilon}^{n-1} \times \mathbb{R}) - \frac{1}{\delta}\mu(\{0\} \times \mathbb{R}) + \varepsilon \to 0 \qquad \text{as } \varepsilon \searrow 0\,,
\end{aligned}$$

where we applied (3.65) and the imposed monotonicity assumption on μ. Since we achieved a contradiction to the initial observation (3.63), we find that it holds $S(t) = \mu_t^s(\mathbb{R}^{n-1}\backslash\{0\}) = 0$ for $\pi_{n\#}\mu$-a.e. $t \in \mathbb{R}$. In particular, we have $\mu \ll (\mathcal{L}^{n-1} + \delta_0) \otimes \pi_{n\#}\mu$. With the monotonicity condition $\mu(\{0\} \times \mathbb{R}) \leqslant \mu((\{0\} \times \mathbb{R})^\star) = \mu(\emptyset) = 0$, we confirm the claim in this case.

Now, we show the claim for a non-negative Radon measure μ that satisfies $\mu(B^\star) \leqslant \mu(B)$ for all $B \in \mathcal{B}(\mathbb{R}^n)$. To this end, we again consider the restriction $\mu \llcorner \mathrm{B}_R$, which, for simplicity, we denote by μ. In fact, we consider a finite Radon measure μ that satisfies $\mu(B^\star) \leqslant \mu(B)$ for all $B \in \mathcal{B}(\mathrm{B}_R)$ and argue similar as above with the slight difference of carrying out a sort of complementary symmetrization. For every measurable set $F \subseteq \mathrm{B}_{\frac{R}{2}}^{n-1}$, there exists some radius $r = r(R, |F|) \in \left(0, \frac{R}{2}\right)$ with $|F| = |\mathrm{B}_R^{n-1}\backslash\mathrm{B}_{R-r}^{n-1}|$. It holds

$$\left[\left(\left(\mathrm{B}_{\frac{R}{2}}^{n-1}\backslash F\right) \cup \left(\mathrm{B}_R^{n-1}\backslash\mathrm{B}_{R-r}^{n-1}\right)\right) \times I\right]^\star = \mathrm{B}_{\frac{R}{2}}^{n-1} \times I$$

for all measurable sets $I \subseteq (-R, R)$. The monotonicity condition on μ then yields

$$\mu\left(\mathrm{B}_{\frac{R}{2}}^{n-1} \times I\right) \leqslant \mu\left(\left(\left(\mathrm{B}_{\frac{R}{2}}^{n-1}\backslash F\right) \cup \left(\mathrm{B}_R^{n-1}\backslash\mathrm{B}_{R-r}^{n-1}\right)\right) \times I\right) \leqslant \mu\left(\left(\mathrm{B}_{\frac{R}{2}}^{n-1}\backslash F\right) \times I\right) + \mu\left(\left(\mathrm{B}_R^{n-1}\backslash\mathrm{B}_{R-r}^{n-1}\right) \times I\right)\,,$$

which implies

$$\mu(F \times I) \leqslant \mu\left(\left(\mathrm{B}_R^{n-1}\backslash\mathrm{B}_{R-r}^{n-1}\right) \times I\right)\,. \tag{3.66}$$

As before, we choose countable dense subsets $\{\varphi_{k,1}, \varphi_{k,2}, \dots\}$ of

$$\left\{\varphi \in \mathrm{C}_{\mathrm{cpt}}^0\left(\mathrm{B}_{\frac{R}{2}}^{n-1}, [0,1]\right) : |\mathrm{spt}\varphi| \leqslant \omega_{n-1}\frac{1}{k^{n-1}}\right\}\,.$$

Then the functions $g_{k,i}$ and S are as previously defined, and we obtain $S(t) = \mu_t^s\left(\mathrm{B}_{\frac{R}{2}}^{n-1}\right)$, where μ_t^s again denotes the singular part of μ_t^{H}. Analogously, while assuming $|\{S > 0\}| > 0$, we can repeat the contradiction argument from before and define the set E with $\pi_{n\#}\mu(E) > 0$, while ensuring $E \subseteq (-R, R)$ in this case, and show

$$\pi_{n\#}\mu\,(E) \leqslant \frac{1}{\delta}\mu(\{\varphi > 0\}) + \varepsilon\,.$$

The estimate (3.66) applied for $F = \{\varphi_{k(t),i(t)} > 0\}$ and $I = I_j(t)$ for all $j \in \{1, \ldots, \ell\}$, where $|F| \leqslant \omega_{n-1}\varepsilon^{n-1}$ is satisfied, gives then

$$\pi_{n\#}\mu\,(E) \leqslant \frac{1}{\delta}\mu\left(\mathrm{B}_R^{n-1}\backslash\mathrm{B}^{n-1}_{R-r(R,\omega_{n-1}\varepsilon^{n-1})} \times (-R,R)\right) + \varepsilon\,.$$

Since $r(R,\omega_{n-1}\varepsilon^{n-1}) \to R$ as $\varepsilon \searrow 0$, we conclude the contradiction $\pi_{n\#}\mu\,(E) = 0$, as before. It follows $\mu \llcorner \mathrm{B}_{\frac{R}{2}} \ll \mathcal{L}^{n-1} \otimes \pi_{n\#}\mu$, which transfers to $\mu \ll \mathcal{L}^{n-1} \otimes \mu^{\mathrm{V}}$ without restriction. □

Remark 3.61. The proof of the previous theorem works analogously if we merely impose $\mu(A^+) \leqslant \mu((A^+)^\star)$ for all measurable sets $A \subseteq \mathbb{R}^n$. Indeed, the same construction of φ even gives

$$\begin{aligned}(\{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} :\, \varphi(x,t) > 0\} \cup (\mathrm{B}_r^{n-1} \times \mathbb{R}))^\star &\subseteq ((\{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} :\, \varphi(x,t) > 0\} \cup (\mathrm{B}_r^{n-1} \times \mathbb{R}))^+)^\star \\ &\subseteq \mathrm{B}_{2\varepsilon}^{n-1} \times \mathbb{R}\end{aligned}$$

instead of (3.65). We can then exploit the compactness of E and conclude $\mu \ll (\mathcal{L}^{n-1} + \delta_0) \otimes \pi_{n\#}\mu$ in the same way.
In this case, we cannot remove the additional Dirac measure as before, since $(\{0\} \times \mathbb{R})^+ = \emptyset$ in all dimensions. In order to control the additional Dirac portion on the n-th axis, we consider a Borel set $B = A^+$, and compute

$$\mu\,(B \cup (\{0\} \times [-R,R])) \leqslant \lim_{\varepsilon \searrow 0} \mu\left(\left[B \cup ([-\varepsilon,\varepsilon]^{n-1} \times [-R,R])\right]^\star\right) \leqslant \mu\left(B^{\overline{\star}} \cup (\{0\} \times [-R,R])\right)$$

for all $R > 0$, where we take the closed symmetrization $B^{\overline{\star}}$ of B defined in Definition 3.67. Thus, we obtain

$$\mu\,(B\backslash(\{0\} \times [-R,R])) \leqslant \mu\left(B^{\overline{\star}}\backslash(\{0\} \times [-R,R])\right).$$

Letting $R \to \infty$ yields that $\tilde{\mu} := \mu \llcorner \mathbb{R}^{n-1}\backslash\{0\} \times \mathbb{R}$ satisfies the monotonicity condition with the slight difference that we take the closed symmetrization. However, by the previous analysis, μ does not record the difference between $B^{\overline{\star}}$ and $B^\star$ away from $\{0\} \times \mathbb{R}$ since

$$\mu\left(B^{\overline{\star}}\backslash(B^\star \cup (\{0\} \times \mathbb{R})\right) = \int_{\mathbb{R}} \int_{\partial \mathrm{B}^{n-1}_{\mathrm{rad}_B(t)}} f(x,t)\,\mathrm{d}x\,\mathrm{d}\mu^{\mathrm{V}}(t) = 0\,.$$

Hence, neglecting the Dirac portion of μ on the middle axis still gives a measure which satisfies the monotonicity condition.
A similar analysis shows that if $\mu((A^1)^\star) \leqslant \mu(A^1)$ holds for all measurable sets $A \subseteq \mathbb{R}^n$, then $\mu \ll \mathcal{L}^{n-1} \otimes \mu^{\mathrm{V}}$. Indeed, instead of (3.66) we can show $\mu(F^1 \times I) \leqslant \mu\left((\mathrm{B}_R^{n-1}\backslash\overline{\mathrm{B}_{R-r}^{n-1}}) \times I\right)$ first for all for all open, and then also for all Borel sets $I \subseteq (-R,R)$, where F and r are as in the proof. This follows as before together with the observation $\left((\mathrm{B}^{n-1}_{\frac{R}{2}}\backslash F) \times I\right)^1 = \left(\mathrm{B}^{n-1}_{\frac{R}{2}}\backslash F^+\right) \times I$ if I is an open interval and with the regularity of Radon measures. Taking into account the openness of $\{\varphi_{k(t),i(t)} > 0\}$, we then find $\mu \ll \mathcal{L}^{n-1} \otimes \pi_{n\#}\mu$ as in the proof of the previous proposition.

Proposition 3.62 (Characterization of the monotonicity condition)**.** *Let μ be a non-negative Radon measure in $\mathbb{R}^n$ and $\mu = f(\mathcal{L}^{n-1} \otimes \mu^{\mathrm{V}})$ for some μ^{V}-measurable and non-negative Borel function $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n; \mathcal{L}^{n-1} \otimes \mu^{\mathrm{V}})$. Then the following properties hold true.*

i) *The condition $\mu(B) \leqslant \mu(B^\star)$ for all $B \in \mathcal{B}(\mathbb{R}^n)$ is equivalent to the existence of a μ^{V}-negligible set $N \subseteq \mathbb{R}$ with*

$$f(y,t) \leqslant f(x,t) \qquad \text{for } \mathcal{L}^{n-1}\text{-a.e. } x,y \in \mathbb{R}^{n-1} \text{ with } |x| \leqslant |y| \text{ and all } t \in \mathbb{R}\backslash N\,.$$

ii) *The condition $\mu(B^\star) \leqslant \mu(B)$ for all $B \in \mathcal{B}(\mathbb{R}^n)$ is equivalent to the existence of a μ^{V}-negligible set $N \subseteq \mathbb{R}$ with*

$$f(x,t) \leqslant f(y,t) \qquad \text{for } \mathcal{L}^{n-1}\text{-a.e. } x,y \in \mathbb{R}^{n-1} \text{ with } |x| \leqslant |y| \text{ and all } t \in \mathbb{R}\backslash N\,.$$

Proof. We will only prove the first statement, since the second one can be shown analogously. First, we assume that the monotonicity condition on μ is satisfied, fix $s \geqslant 0$, and notice $(\{f>s\}^\star)_t = \mathrm{B}^{n-1}_{\mathrm{rad}_{\{f>s\}}(t)}$ for all $t \in \mathbb{R}$. Exploiting that $|(A^\star)_t| = |A_t|$ holds for all measurable sets $A \subseteq \mathbb{R}^n$ and all $t \in \mathbb{R}$ by definition, we obtain

$$\int_{(\{f>s\}^\star)_t} f(x,t)\,\mathrm{d}x \leqslant \int_{\{f>s\}_t} f(x,t)\,\mathrm{d}x \tag{3.67}$$

for all $t \in \mathbb{R}$. We compute with Fubini's theorem

$$\int_{\mathbb{R}} \int_{\{f>s\}_t} f(x,t)\,\mathrm{d}x\,\mathrm{d}\mu^{\mathrm{V}}(t) = \int_{\{f>s\}} f(x,t)\,\mathrm{d}(\mathcal{L}^{n-1} \otimes \mu^{\mathrm{V}})(x,t) = \mu(\{f>s\})\,.$$

Applying the monotonicity condition of μ on the Borel set $\{f>s\}$ yields

$$\int_{\mathbb{R}} \int_{\{f>s\}_t} f(x,t)\,\mathrm{d}x\,\mathrm{d}\mu^{\mathrm{V}}(t) \leqslant \mu(\{f>s\}^\star) = \int_{\{f>s\}^\star} f(x,t)\,\mathrm{d}(\mathcal{L}^{n-1} \otimes \mu^{\mathrm{V}})(x,t) = \int_{\mathbb{R}} \int_{(\{f>s\}^\star)_t} f(x,t)\,\mathrm{d}x\,\mathrm{d}\mu^{\mathrm{V}}(t)\,.$$

Together with (3.67), this verifies

$$\int_{\{f>s\}_t} f(x,t)\,\mathrm{d}x = \int_{(\{f>s\}^\star)_t} f(x,t)\,\mathrm{d}x \qquad \text{for } \mu^{\mathrm{V}}\text{-a.e. } t \in \mathbb{R}.$$

In fact, for every $s \geqslant 0$, we find a corresponding μ^{V}-null set $N_s \subseteq \mathbb{R}$ such that it holds $\{f>s\}_t = \mathrm{B}^{n-1}_{\mathrm{rad}_{\{f>s\}}(t)}$ up to negligible sets for all $t \in N_s$. We choose a dense subset $Q := \{q_1, q_2, \dots\}$ of $[0,\infty)$ and define the μ^{V}-null set $N := \bigcup_{k=1}^{\infty} N_{q_k}$. For $t \in \mathbb{R}\backslash N$, we obtain

$$E := \bigcup_{k=1}^{\infty} \mathrm{B}^{n-1}_{\mathrm{rad}_{\{f>q_k\}}(t)} \backslash S = \mathbb{R}^{n-1} \backslash \{f=0\}_t$$

up to negligible sets, where $S := \bigcup_{k=1}^{\infty} \{f>q_k\}_t \triangle \mathrm{B}^{n-1}_{\mathrm{rad}_{\{f>q_k\}}(t)}$. For $x, y \in E$ with $|x| \leqslant |y|$, we now assume $f(x) < f(y)$ and argue by contradiction. The density of Q yields the existence of some $q \in Q$ with $f(x) < q < f(y)$, and therefore, $y \in \{f>q\}_t \cap \mathrm{B}^{n-1}_{\mathrm{rad}_{\{f>q\}}(t)}$ and $x \notin \mathrm{B}^{n-1}_{\mathrm{rad}_{\{f>q\}}(t)}$, in contradiction to the assumption. The equality $\{f=0\}_t = \mathbb{R}^{n-1} \backslash \bigcup_{k=1}^{\infty} \mathrm{B}^{n-1}_{\mathrm{rad}_{\{f>q_k\}}(t)}$ up to null sets provides the previous argument for almost every $x, y \in \mathbb{R}^{n-1}$.
On the other hand, if f satisfies the monotonicity condition and $B \in \mathrm{B}(\mathbb{R}^n)$, then we exploit once again $|B_t| = |(B^\star)_t|$ and compute

$$\mu(B) = \int_{\mathbb{R}} \int_{B_t} f(x,t)\,\mathrm{d}x\,\mathrm{d}\mu^{\mathrm{V}}(t) \leqslant \int_{\mathbb{R}} \int_{(B^\star)_t} f(x,t)\,\mathrm{d}x\,\mathrm{d}\mu^{\mathrm{V}}(t) = \mu(B^\star)\,,$$

which shows the claim. □

With this characterization of the monotonicity condition at hand, we are able to decompose both measures, μ_- and μ_+ into the discrete and the regular part with respect to the Schwarz symmetrization. We then analyze the difference of $(A^\star)^+$ and $(A^+)^\star$, respectively the difference of $(A^\star)^1$ and $(A^1)^\star$, observing that the crucial parts are those with horizontal mass.

Definition 3.63 (Discrete and regular measure parts)**.** Let μ be a non-negative Radon measure on $\mathbb{R}^n$. We call the sets

$$\mathrm{dis}_\mu := \{t \in \mathbb{R} : \mu \llcorner H_t \neq 0\} \qquad \text{and} \qquad \mathrm{reg}_\mu := \mathbb{R} \backslash \mathrm{dis}_\mu$$

the **discrete** and the **regular set of μ**. Furthermore, we define the **discrete** and the **regular part of μ** by

$$\mu_{\mathrm{dis}} := \sum_{t \in \mathrm{dis}_\mu} \mu \llcorner H_t \qquad \text{and} \qquad \mu_{\mathrm{reg}} := \mu \llcorner \mathbb{R}^{n-1} \times \mathrm{reg}_\mu\,,$$

respectively.

Remark 3.64. i) Both the discrete and regular parts of μ are non-negative Radon measures. Since μ is by assumption a Radon measure, the discrete set of μ is at most countable. Moreover, it holds that $\mu = \mu_{\text{dis}} + \mu_{\text{reg}}$.

ii) The relevant properties which we impose on the measure μ, i.e., the isoperimetric and the monotonicity conditions, transfer to the discrete and regular part. The isoperimetric condition especially transfers with the same constant. The monotonicity condition transfers due to the definition of Schwarz symmetrization giving $(B \cap (\mathbb{R}^{n-1} \times S))^\star = B^\star \cap (\mathbb{R}^{n-1} \times S)$ for all measurable sets $B \subseteq \mathbb{R}^n$ and $S \subseteq \mathbb{R}$.

iii) If μ is equal to $f(\mathcal{L}^{n-1} \otimes \mu^{\mathrm{V}})$ for some Borel function $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n; \mathcal{L}^{n-1} \otimes \mu^{\mathrm{V}})$, then $\mu \llcorner H_t \neq 0$ for some $t \in \mathbb{R}$ if and only if $\mu^{\mathrm{V}}(\{t\}) > 0$. In particular, μ_{dis} corresponds to the Dirac portions of μ^{V} and equals $f(\mathcal{L}^{n-1} \otimes (\mu^{\mathrm{V}} \llcorner \mathrm{dis}_\mu))$. It then holds that $\mu_{\text{reg}} = f(\mathcal{L}^{n-1} \otimes (\mu^{\mathrm{V}} \llcorner \mathrm{reg}_\mu))$ vanishes on every set $\mathbb{R}^{n-1} \times T$ with $T \subseteq \mathbb{R}$ countable.

Despite the fact that the isoperimetric condition implies absolute continuity with respect to the Hausdorff measure, we show in the following two lemmas that this follows also from the monotonicity condition.

Lemma 3.65 (Dimension of the regular part)**.** *Let μ be a non-negative Radon measure on $\mathbb{R}^n$ satisfying either $\mu(B^\star) \leqslant \mu(B)$ or $\mu(B) \leqslant \mu(B^\star)$ for all Borel sets $B \subseteq \mathbb{R}^n$. Then μ_{reg} vanishes on every $\mathcal{H}^{n-1}$-σ-finite Borel set.*

Proof. It is enough to show $\mu_{\text{reg}}(B) = 0$ for every Borel set $B \subseteq \mathbb{R}^n$ with $\mathcal{H}^{n-1}(B) < \infty$. With Proposition 3.60 and Remark 3.64, we find a non-negative Borel function f such that $\mu_{\text{reg}} = f(\mathcal{L}^{n-1} \otimes (\mu^{\mathrm{V}} \llcorner \mathrm{reg}_\mu))$. The finiteness of $\mathcal{H}^{n-1}(B)$ yields an at most countable set $T := \{t \in \mathbb{R} : \mathcal{H}^{n-1}(B_t \times \{t\}) > 0\}$. Moreover, since $\mathcal{H}^{n-1}(B_t \times \{t\}) = |B_t|$ for all $t \in \mathbb{R}$ and $\mu^{\mathrm{V}}(T \cap \mathrm{reg}_\mu) = 0$, it holds that

$$\mu_{\text{reg}}(B) = \int_{\mathrm{reg}_\mu} \int_{B_t} f(x,t)\,\mathrm{d}x\,\mathrm{d}\mu^{\mathrm{V}}(t) = \int_{\mathrm{reg}_\mu \cap T} \int_{B_t} f(x,t)\,\mathrm{d}x\,\mathrm{d}\mu^{\mathrm{V}}(t) = 0$$

as claimed. □

Lemma 3.66 (Absolute continuity with respect to the Hausdorff measure)**.** *Let μ be a non-negative Radon measure on $\mathbb{R}^n$ satisfying either $\mu(B^\star) \leqslant \mu(B)$ or $\mu(B) \leqslant \mu(B^\star)$ for all Borel sets $B \subseteq \mathbb{R}^n$. Then it holds $\mu \ll \mathcal{H}^{n-1}$.*

Proof. We consider a Borel set B with vanishing $(n-1)$-Hausdorff measure. It holds that $|B_t| = \mathcal{H}^{n-1}(B_t \times \{t\}) = 0$ for all $t \in \mathbb{R}$. Therefore, an application Proposition 3.60 yields

$$\mu(B) = \int_{\mathbb{R}} \int_{B_t} f(x,t)\,\mathrm{d}x\,\mathrm{d}\mu^{\mathrm{V}}(t) = 0$$

as claimed. □

It is left to show that μ_{reg} is indeed the part of μ which behaves well with respect to the choice of representation of the Schwarz symmetrization. To this end, we show a subset relation with the following notion of closed Schwarz symmetrization.

Definition 3.67 (Closed Schwarz symmetrization)**.** For $A \in \mathcal{M}^n$, the **closed Schwarz symmetrization of $\boldsymbol{A}$** is defined as the set

$$A^{\overline{\star}} := \{x \in \mathbb{R}^n : |\overline{x}| \leqslant \mathrm{rad}_A(x_n)\}\ .$$

Lemma 3.68 (Subset relation of symmetrized representatives)**.** *Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter and finite volume. Then it holds*

$$(A^\star)^+ \subseteq (A^+)^{\overline{\star}} \qquad \textit{and} \qquad (A^1)^\star \subseteq (A^\star)^1\ .$$

For the proof, we use the identification of measure-theoretic interior and exterior via more general sets, rather than balls. We however only need the following equality via cylinders.

Lemma 3.69. *Let $A \in \mathcal{M}^n$, then the following equalities hold true*

i) $A^0 = \left\{x \in \mathbb{R}^n : \lim_{\rho\searrow 0} \frac{|A\cap C_\rho(x)|}{2\omega_{n-1}\rho^n} = 0\right\}$,

ii) $A^1 = \left\{x \in \mathbb{R}^n : \lim_{\rho\searrow 0} \frac{|A\cap C_\rho(x)|}{2\omega_{n-1}\rho^n} = 1\right\}$,

iii) $\partial^e A = \mathbb{R}^n \backslash (A^0 \cup A^1)$.

Proof. We only show the first two equalities, since then the last one follows by definition. For all $\rho > 0$ and $x \in \mathbb{R}^n$, it holds $\mathrm{B}_\rho(x) \subseteq C_\rho(x) \subseteq \mathrm{B}_{\sqrt{2}\rho}(x)$. Hence,

$$\frac{|A \cap \mathrm{B}_\rho(x)|}{\omega_n \rho^n} \frac{\omega_n}{2\omega_{n-1}} \leqslant \frac{|A \cap C_\rho(x)|}{2\omega_{n-1}\rho^n} \leqslant \frac{|A \cap \mathrm{B}_{\sqrt{2}\rho}(x)|}{\omega_n(\sqrt{2}\rho)^n} \frac{\sqrt{2}^n \omega_n}{2\omega_{n-1}} ,$$

which shows the first equality by definition. The second follows from $(\mathbb{R}^n \backslash A)^0 = A^1$. □

Proof of Lemma 3.68. We first consider the case of bounded sets $A \subseteq \mathbb{R}^n$ such that $A \Subset \mathrm{B}_R$ for some $R > 0$. Let $t \in \mathbb{R}$, In what follows, we consider a big portion of A close to the t-slice of A^1. Let ε be an arbitrary number in $(0, 1)$. According to Lemma 3.69, it holds $(A^1)_t \subseteq \left\{x \in \mathbb{R}^{n-1} : \liminf_{\rho\searrow 0} \frac{|A\cap C_\rho((x,t))|}{2\omega_{n-1}\rho^n} > 1-\varepsilon\right\} \subseteq (A^+)_t$. In particular, we find $\rho_0 = \rho_0(\varepsilon) > 0$ with

$$\left|\left\{x \in \mathbb{R}^{n-1} : \frac{|A \cap C_\rho((x,t))|}{2\omega_{n-1}\rho^n} > 1-\varepsilon\right\}\right| \geqslant \mathrm{vol}_{A^1}(t) - \varepsilon$$

for all $\rho \in (0, \rho_0)$. Since we assumed that A is bounded in the first $n-1$ components, we can apply the covering theorem of Besicovitch (see [3, Theorem 2.18]) to find some universal constant $m \in \mathbb{N}$, which only depends on n, and a countable set $\{x_1, x_2, \dots\}$ of points in $\left\{x \in \mathbb{R}^{n-1} : \frac{|A\cap C_\rho((x,t))|}{2\omega_{n-1}\rho^n} > 1-\varepsilon\right\}$ such that

$$\left\{x \in \mathbb{R}^{n-1} : \frac{|A \cap C_\rho((x,t))|}{2\omega_{n-1}\rho^n} > 1-\varepsilon\right\} \subseteq \bigcup_{k=1}^{\infty} \mathrm{B}_\rho^{n-1}(x_k) =: F$$

and such that for every $x \in \mathbb{R}^{n-1}$, there exist at most m different points $x_{k_1}, \dots, x_{k_m}$ with $x \in \mathrm{B}_\rho^{n-1}(x_{k_i})$ for all $i \in \{1, \dots, m\}$. Using again the boundedness assumption on A, we find that the cover F is also bounded and has finite volume. Now, the identification $\bigcup_{k=1}^\infty C_\rho((x_k, t)) = F \times (t-\rho, t+\rho) =: F_\rho$ yields

$$|F_\rho \backslash A| \leqslant \sum_{k=1}^{\infty} |C_\rho((x_k, t)) \backslash A| .$$

We notice that every point in F_ρ lies in at most m different cylinders $C_\rho((x_{k_i}, t))$ by construction. Taking into account the choice of ρ_0, we estimate

$$|F_\rho \backslash A| \leqslant \sum_{k=1}^{\infty} \varepsilon |C_\rho((x_k, t))| \leqslant \varepsilon m |F_\rho| = \varepsilon 2m|F|\rho . \tag{3.68}$$

For the final part of the reasoning, we define the sets

$$\begin{aligned} I &:= \{s \in (t-\rho, t+\rho) : |A_s \cap F| \geqslant (1-\sqrt{m\varepsilon})|F|\} \\ J &:= (t-\rho, t+\rho) \backslash I . \end{aligned}$$

We record $|I| \geqslant (1-\sqrt{m\varepsilon})2\rho$ since otherwise, the definition yields $|I| < (1-\sqrt{m\varepsilon})2\rho$ and $|J| = 2\rho - |I| >$

$\sqrt{m\varepsilon}2\rho$, and the estimate (3.68) results in the following contradiction:

$$\begin{aligned} |F_\rho| - m\varepsilon|F_\rho| \leqslant |F_\rho| - |F_\rho\backslash A| = |A\cap F_\rho| \\ &= \int_{t-\rho}^{t+\rho} |A_s \cap F| \,\mathrm{d}s \\ &= \int_I |A_s\cap F|\,\mathrm{d}s + \int_J |A_s \cap F|\,\mathrm{d}s \\ &< \int_I |F| - |F\backslash A_s|\,\mathrm{d}s + |J|(1-\sqrt{m\varepsilon})|F| \\ &= (|I|+|J|-|J|\sqrt{m\varepsilon})|F| - \int_I |F\backslash A_s|\,\mathrm{d}s \\ &< 2\rho(1-m\varepsilon)|F| - \int_I |F\backslash A_s|\,\mathrm{d}s\,, \end{aligned}$$

where we applied Fubini's theorem in the second step. The estimate

$$|A_s| \geqslant |A_s\cap F| \geqslant (1-\sqrt{m\varepsilon})|F| \geqslant (1-\sqrt{m\varepsilon})(\mathrm{vol}_{A^1}(t)-\varepsilon) \qquad \text{for all } s\in I$$

now follows by the definition of F and I. Thus,

$$\mathrm{B}_r^{n-1} \subseteq (A^\star)_s \qquad \text{for } r = r(\varepsilon) = \left(\frac{(1-\sqrt{m\varepsilon})(\mathrm{vol}_{A^1}(t)-\varepsilon)}{\omega_{n-1}}\right)^{\frac{1}{n-1}} \qquad \text{for all } s\in I\,.$$

Together with the volume estimate for I, this yields

$$\frac{|A^\star\cap C_\rho((x,t))|}{2\omega_{n-1}\rho^n} \geqslant (1-\sqrt{m\varepsilon})\frac{|\mathrm{B}_r^{n-1}\cap\mathrm{B}_\rho^{n-1}(x)|}{|\mathrm{B}_\rho^{n-1}|} \to 1-\sqrt{m\varepsilon} \text{ as } \rho\searrow 0 \qquad \text{for all } x\in\mathrm{B}_r^{n-1}\,.$$

Since ε was chosen arbitrarily small and $r\to\mathrm{rad}_{A^1}(t)$ as $\varepsilon\searrow 0$, we find $((A^1)^\star)_t\subseteq((A^\star)^1)_t$. Since t is also arbitrary, we showed the first claim for sets which are bounded.
In order to transfer the result to unbounded sets, we compute

$$(A^1)^\star = \bigcup_{R\in\mathbb{N}}((\mathrm{B}_R\cap A)^1)^\star \subseteq \bigcup_{R\in\mathbb{N}}((\mathrm{B}_R\cap A)^\star)^1 \subseteq (A^\star)^1\,.$$

In the remaining proof, we show $(A^\star)^+\subseteq(A^+)^{\overline{\star}}$ for bounded sets $A\Subset\mathrm{B}_R$. By the definition of closed Schwarz symmetrization, it suffices to show $|((A^\star)^+)_t|\leqslant|(A^+)_t|$ for all $t\in(-R,R)$. We record that $(\mathrm{B}_R^{n-1}\times(-R,R))\backslash A$ is a set of finite perimeter and finite volume. We denote by s the radius function for the complement of A in $\mathrm{B}_R^{n-1}\times(-R,R)$, i.e., $s(t):=\mathrm{rad}_{(\mathrm{B}_R^{n-1}\times(-R,R))\backslash A}(t)=\left(R^{n-1}-\mathrm{rad}_A(t)^{n-1}\right)^{\frac{1}{n-1}}$. According to Lemma 3.34, we find representatives of s and rad_A such that the one-sided limits exist at each point in $\mathbb{R}$. Furthermore, we observe $s^\pm=\left(R^{n-1}-(\mathrm{rad}_A^\mp)^{n-1}\right)^{\frac{1}{n-1}}$ on $(-R,R)$. Applying Lemma 3.37 gives

$$\begin{aligned} |(((\mathrm{B}_R^{n-1}\times(-R,R))\backslash A^\star)^1)_t| &= |((\mathrm{B}_R^{n-1}\times(-R,R))\backslash((A^\star)^+))_t| \\ &= \left|\mathrm{B}_R^{n-1}\backslash\mathrm{B}^{n-1}_{\mathrm{rad}_A^+(t)}\right| \\ &= \left|\mathrm{B}^{n-1}_{s^-(t)}\right| \\ &= \left|((((\mathrm{B}_R^{n-1}\times(-R,R))\backslash A)^\star)^1)_t\right|\,, \end{aligned}$$

and with the shown subset relation, we conclude

$$\begin{aligned} |((A^\star)^+)_t| &= |\mathrm{B}_R^{n-1}\backslash(\mathrm{B}_R^{n-1}\backslash((A^\star)^+)_t)| \\ &= |\mathrm{B}_R^{n-1}\backslash(((\mathrm{B}_R^{n-1}\times(-R,R)\backslash A)^\star)^1)_t| \\ &\leqslant |\mathrm{B}_R^{n-1}\backslash((\mathrm{B}_R^{n-1}\times(-R,R)\backslash A)^1)_t| \\ &= |(A^+)_t| \end{aligned}$$

for all $t \in \mathbb{R}$. Hence follows the claim in the bounded case.
For proving the result for unbounded sets, we apply Lemma 3.41 and consider a sequence $(R_k)_{k\in\mathbb{N}}$ of positive numbers with $R_k \to \infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty} \left|\left(((A\cap \mathrm{B}_{R_k})^\star)^+\right)_t\right| = \left|\left((A^\star)^+\right)_t\right| \qquad \text{for all } t \in \mathbb{R}\,.$$

With the corresponding estimate for bounded sets, we obtain

$$\left|\left((A^\star)^+\right)_t\right| = \lim_{k\to\infty} \left|\left(((A\cap \mathrm{B}_{R_k})^\star)^+\right)_t\right| \leqslant \lim_{k\to\infty} \left|\left((A\cap \mathrm{B}_{R_k})^+\right)_t\right| \leqslant \left|(A^+)_t\right|$$

for all $t \in \mathbb{R}$, which proves the claim. □

Lemma 3.70 (Behavior of the regular part)**.** *Let μ be a non-negative Radon measure on $\mathbb{R}^n$ satisfying either $\mu(B^\star) \leqslant \mu(B)$ or $\mu(B) \leqslant \mu(B^\star)$ for all Borel sets $B \subseteq \mathbb{R}^n$. For all sets $A \subseteq \mathbb{R}^n$ of finite perimeter and finite volume, it holds that*

$$\mu_{\mathrm{reg}}((A^+)^\star) = \mu_{\mathrm{reg}}((A^\star)^+) = \mu_{\mathrm{reg}}((A^\star)^1) = \mu_{\mathrm{reg}}((A^1)^\star)\,.$$

Proof. We first notice that, according to the Schwarz inequality, the difference $(A^\star)^+\triangle(A^\star)^1 = \partial^{\mathrm{e}}(A^\star)$ is an $\mathcal{H}^{n-1}$-finite set. Therefore, Lemma 3.65 implies $\mu_{\mathrm{reg}}((A^\star)^+) = \mu_{\mathrm{reg}}((A^\star)^1)$. According to Lemma 3.68, we know $\mu_{\mathrm{reg}}((A^\star)^+) \leqslant \mu_{\mathrm{reg}}((A^+)^{\bar\star})$ and $\mu_{\mathrm{reg}}((A^1)^\star) \leqslant \mu_{\mathrm{reg}}((A^\star)^1)$. Also, the set $(A^+)^{\bar\star}\setminus(A^+)^\star$ is in general negligible with respect to μ by Proposition 3.60. Indeed, for f as in the statement, it holds

$$\mu\left((A^+)^{\bar\star}\setminus(A^+)^\star\right) = \int_{\mathbb{R}}\int_{\partial \mathrm{B}^{n-1}_{\mathrm{rad}_{A^+}(t)}} f(x,t)\,\mathrm{d}x\,\mathrm{d}\mu^{\mathrm{V}}(t) = 0\,.$$

Thus, it remains to show $\mu_{\mathrm{reg}}\left((A^+)^\star\right) \leqslant \mu_{\mathrm{reg}}\left((A^1)^\star\right)$. To this end, we exploit the finiteness of ∂^*A under $\mathcal{H}^{n-1}$ to obtain $|(\partial^*A)_t| = 0$ for all but countably many $t \in \mathbb{R}$. Since the Schwarz symmetrization does not see $\mathcal{H}^{n-1}$-negligible sets, the structure theorem of Federer implies $(A^+)^\star = (A^1\cup\partial^*A)^\star \subseteq (A^1)^\star \cup (\mathbb{R}^{n-1}\times T)$ for the countable set $T \coloneqq \{t\in\mathbb{R} : |(\partial^*A)_t| > 0\}$. According to Remark 3.64, the regular part μ_{reg} vanishes on $\mathbb{R}^{n-1}\times T$. Hence, the remaining estimate is shown. □

3.3.3 The generalized Schwarz inequality

We are now able to prove the Schwarz inequality for measure data functionals. Since we have already shown that the regular part of the corresponding measure is indeed well-behaved, we first concentrate on the discrete part, which lives on horizontal hyperplanes. The assumptions on the measures yield suitable compensation of the increase of the measures while changing the representatives of the symmetrized sets by the decrease of the perimeter.

Proposition 3.71 (The generalized Schwarz inequality on hyperplanes)**.** *Let μ_-, μ_+ be two mutually singular and non-negative Radon measures on $\mathbb{R}^n$ with $\mu_-(\mathbb{R}^n\setminus H_t) + \mu_+(\mathbb{R}^n\setminus H_t) = 0$ for some $t \in \mathbb{R}$. If both μ_- and μ_+ satisfy the small-volume isoperimetric condition with constant* 1 *and the monotonicity condition* (3.61)*, then*

$$\mathrm{P}(A^\star, H_t) + \mu_+((A^\star)^1) - \mu_-((A^\star)^+) \leqslant \mathrm{P}(A, H_t) + \mu_+(A^1) - \mu_-(A^+)$$

holds for all sets $A \subseteq \mathbb{R}^n$ of finite volume and finite perimeter in a neighborhood of H_t.

The proof is divided four steps, relaxing the assumptions on A iteratively. First, we will show that smooth bounded sets with no boundary part on H_t are well-behaved, thanks to Lemma 3.68, compare Figure 1.3. We then transfer this result to bounded sets of finite perimeter without boundary portion on H_t via Theorem 2.40 and the lower semicontinuity result, which we gain from the isoperimetric conditions for the measures μ_+ and μ_-. The most crucial step is then to relax the additional assumptions on A, only assuming boundedness. We therefore exploit Proposition 2.61, constructing approximating sets of A, which satisfy the assumptions of the previous step and shift the crucial boundary part into its interior or exterior, respectively. The final step transfers the result to unbounded sets by a standard exhaustion argument.

Proof. It suffices to consider sets $A \subseteq \mathbb{R}^n$ of finite volume and finite perimeter in $\mathbb{R}^n$. We then find at most countably many $\varepsilon > 0$ with $\mathrm{P}(A, \mathbb{R}^{n-1} \times \{t-\varepsilon, t+\varepsilon\}) > 0$. In fact, we find a null sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ with $\mathrm{P}(A, \mathbb{R}^{n-1} \times \{t-\varepsilon_k, t+\varepsilon_k\}) = 0$. If we show

$$\mathrm{P}(A^\star, \mathbb{R}^{n-1}\times(t-\varepsilon_k, t+\varepsilon_k))+\mu_+((A^\star)^1)-\mu_-((A^\star)^+) \leqslant \mathrm{P}(A, \mathbb{R}^{n-1}\times(t-\varepsilon_k, t+\varepsilon_k))+\mu_+(A^1)-\mu_-(A^+) \quad (3.69)$$

for all $k \in \mathbb{N}$, then the regularity of Radon measures implies the claim. Thus, we fix any $\varepsilon > 0$ which satisfies $\mathrm{P}(A, \mathbb{R}^{n-1} \times \{t-\varepsilon, t+\varepsilon\}) = 0$ and define $I := (t-\varepsilon, t+\varepsilon)$. The proof of (3.69) is divided into four steps, considering different levels of regularity of A.

Step 1: Show the claim for smooth and bounded sets A which satisfy $\mathcal{H}^{n-1}(\partial A \cap H_t) = 0$:
In this case, we record $|(A^1 \triangle A^+)_t| = |(\partial A)_t| = 0$, which, by the definition of the (closed) Schwarz symmetrization and Lemma 3.68, gives

$$(A^1)^\star_t = (A^+)^\star_t = (A^+)^{\overline{\star}}_t \supseteq (A^\star)^+_t \supseteq (A^\star)^1_t \supseteq (A^1)^\star_t$$

up to negligible sets. Equality follows in the subset relation above. Now, the monotonicity condition (3.62) follows from Lemma 3.59 and, together with Lemma 3.66, this gives

$$\mu_-(A^+) \leqslant \mu_-((A^+)^\star) = \mu_-((A^\star)^+) \qquad \text{and} \qquad \mu_+(A^1) \geqslant \mu_+((A^1)^\star) = \mu_+((A^\star)^1)\,.$$

Together with the standard Schwarz inequality, Theorem 3.5, we obtain (3.69).

Step 2: Show the claim for bounded sets A which satisfy $\mathrm{P}(A, H_t) = 0$:
According to Theorem 2.40, we find a sequence of bounded and smooth sets $(\tilde{A}_k)_{k\in\mathbb{N}}$ which converge to A in L^1 and which satisfy $\lim_{k\to\infty} \mathrm{P}(\tilde{A}_k, B) = \mathrm{P}(A, B)$ for all $B \in \mathcal{B}(\mathbb{R}^n)$ with $\mathrm{P}(A, \partial B) = 0$. We can further shift every $\tilde{A}_k$ vertically, such that $\mathrm{P}(\tilde{A}_k, H_t) = 0$. Indeed, since $\mathrm{P}(\tilde{A}_k + (0,\delta), H_t) = \mathrm{P}(\tilde{A}_k, H_{t-\delta}) = 0$ holds for all but countably many $\delta \in \mathbb{R}$, there exists a null sequence δ_k with $\mathrm{P}(A_k, H_t) = 0$ and $|\mathrm{P}(A_k, H_t^\pm) - \mathrm{P}(\tilde{A}_k, H_t^\pm)| < \frac{1}{k}$ for $A_k := \tilde{A}_k + (0, \delta_k)$. It is then possible to verify the same suitable properties for the sequence $(A_k)_{k\in\mathbb{N}}$. In particular, we have $A_k \to A$ in L^1 as $k \to \infty$ and

$$\lim_{k\to\infty} \mathrm{P}(A_k, H_t^\pm) = \lim_{k\to\infty} \mathrm{P}(\tilde{A}_k, H_{t-\delta_k}^\pm) = \mathrm{P}(A, H_t^\pm)\,.$$

In fact, the convergence $\mathbb{1}_{A_k} \to \mathbb{1}_A$ is strict in $\mathrm{BV}(H_t^\pm)$ as $k \to \infty$. Applying Proposition 2.55 provides the convergence of $\mathrm{Tr}_{H_t^\pm} \mathbb{1}_{A_k}$ to $\mathrm{Tr}_{H_t^\pm} \mathbb{1}_A$ in $\mathrm{L}^1(H_t, \mathcal{H}^{n-1})$ as $k \to \infty$. Since

$$\mathbb{1}_{A^1 \cap H_t} \leqslant \mathrm{Tr}_{H_t^\pm} \mathbb{1}_A \leqslant \mathbb{1}_{A^+ \cap H_t}$$

is always satisfied up to an $\mathcal{H}^{n-1}$-negligible set, we can apply $\mathrm{P}(A, H_t) = 0$ and $\mathrm{P}(A_k, H_t) = 0$ and obtain equality above when A is replaced by A_k. Hence, it holds that

$$0 = \lim_{k\to\infty} \|\mathrm{Tr}_{H_t^\pm} \mathbb{1}_{A_k} - \mathrm{Tr}_{H_t^\pm} \mathbb{1}_A\|_{\mathrm{L}^1(H_t, \mathcal{H}^{n-1})} = \lim_{k\to\infty} |(A^+ \triangle A_k^+)_t| = \lim_{k\to\infty} |(A^1 \triangle A_k^1)_t|\,.$$

Exploiting once again the absolute continuity of $\mu_\pm$ with respect to the $\mathcal{H}^{n-1} \llcorner H_t$, it follows

$$\lim_{k\to\infty} \mu_-(A_k^+) = \mu_-(A^+) \qquad \text{and} \qquad \lim_{k\to\infty} \mu_+(A_k^1) = \mu_+(A^1)\,.$$

Applying Step 1 yields

$$\begin{aligned}\limsup_{k\to\infty} \mathrm{P}(A_k^\star, \mathbb{R}^{n-1} \times I) + \mu_+((A_k^\star)^1) - \mu_-((A_k^\star)^+) &\leqslant \limsup_{k\to\infty} \mathrm{P}(A_k, \mathbb{R}^{n-1} \times I) + \mu_+(A_k^1) - \mu_-(A_k^+) \\ &= \mathrm{P}(A, \mathbb{R}^{n-1} \times I) + \mu_+(A^1) - \mu_-(A^+)\,.\end{aligned}$$

To complete Step 2, it is left to show the lower semicontinuity of

$$\mathcal{P}(\,\cdot\,; I) : \mathcal{M}^n \to [-\infty, \infty],\; F \mapsto \mathrm{P}(F, \mathbb{R}^{n-1} \times I) + \mu_+(F^1) - \mu_-(F^+)$$

with respect to L^1 convergence, since $|A^\star \triangle A_k^\star| \leqslant |A \triangle A_k| \to 0$ as $k \to \infty$ is satisfied by Remark 3.2. Indeed, by Proposition 2.79 and Lemma 2.80, the measure $\mathrm{P}(\mathbb{R}^{n-1} \times I, \,\cdot\,) + \mu_-(\,\cdot\,) = \mathcal{H}^{n-1} \llcorner \mathbb{R}^{n-1} \times \{t-\varepsilon, t+\varepsilon\} + \mu_-$

satisfies the small-volume isoperimetric condition on $\mathbb{R}^n$ with constant 1. Moreover, $\mu_+(\mathbb{R}^{n-1} \times (\mathbb{R}\backslash I)) + \mu_-(\mathbb{R}^{n-1} \times (\mathbb{R}\backslash I)) = 0$, hence Theorem 2.82 and the boundedness of A give the lower semicontinuity of $\mathcal{P}(\,\cdot\,; I)$ and therefore, Step 2 is completed.

Step 3: Show the claim for bounded sets A:
By the absolute continuity of $\mu_\pm$ with respect to $\mathcal{H}^{n-1} \llcorner H_t = \mathcal{L}^{n-1} \otimes \delta_t$ and Proposition 3.62, we find Borel functions $f_\pm \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n; \mathcal{L}^{n-1} \otimes \delta_t)$ such that $\mu_\pm = f_\pm(\mathcal{L}^{n-1} \otimes \delta_t)$ and both functions, $f_+(\,\cdot\,, t)$ and $f_-(\,\cdot\,, t)$ are radial and $f_-(\,\cdot\,, t)$ is additionally decreasing, whereas $f_+(\,\cdot\,, t)$ is increasing. In particular, we find some $R \in [0, \infty]$ with $\mu_- = \mu_- \llcorner \mathrm{B}^{n-1}_R \times \{t\}$ and $\mu_+ = \mu_+ \llcorner (\mathbb{R}^{n-1} \backslash \mathrm{B}^{n-1}_R) \times \{t\}$, where we exploited that μ_- and μ_+ are mutually singular.
By Lemma 2.49, it holds that $\partial^* A \cap H_t \subseteq \{\nu_A = \pm e_n\}$, which leads to the following decomposition of the trace via Proposition 2.61. In particular, applying the statement twice, we find for each $k \in \mathbb{N}$ some sets $C^\pm_k \in \mathcal{B}(\mathbb{R}^{n-1} \times I)$ satisfying

i) $|C^\pm_k| \leqslant \frac{1}{k}$, $\mathrm{P}(C^\pm_k) < \infty$,

ii) $\mathrm{Tr}_{H^+_t} \mathbb{1}_{C^+_k} = \mathbb{1}_{\{\nu_A = e_n\} \cap \mathrm{B}^{n-1}_R \times \{t\}}$, $\mathrm{Tr}_{H^-_t} \mathbb{1}_{C^+_k} = \mathbb{1}_{\{\nu_A = -e_n\} \cap \mathrm{B}^{n-1}_R \times \{t\}}$,

iii) $\mathrm{Tr}_{H^+_t} \mathbb{1}_{C^-_k} = \mathbb{1}_{\{\nu_A = -e_n\} \cap (\mathbb{R}^{n-1} \backslash \mathrm{B}^{n-1}_R) \times \{t\}}$, $\mathrm{Tr}_{H^-_t} \mathbb{1}_{C^-_k} = \mathbb{1}_{\{\nu_A = e_n\} \cap (\mathbb{R}^{n-1} \backslash \mathrm{B}^{n-1}_R) \times \{t\}}$, and

iv) $\mathrm{P}(C^+_k, \mathbb{R}^n \backslash H_t) + \mathrm{P}(C^-_k, \mathbb{R}^n \backslash H_t) < \mathcal{H}^{n-1}(\partial^* A \cap H_t) + \frac{1}{k}$.

The sets C^+_k form small sets, which have complementary trace of A in H_t, whereas C^-_k has the same trace of A in H_t. Moreover, they are chosen such that the construction $A_k := (A \cup C^+_k) \backslash C^-_k$ for all $k \in \mathbb{N}$ neglects the portion of $\partial^* A$ in $\mathbb{R}^{n-1} \backslash \mathrm{B}^{n-1}_R \times \{t\}$ and includes $\partial^* A$ in $\mathrm{B}^{n-1}_R \times \{t\}$ in its measure-theoretic interior, see Figure 3.9.

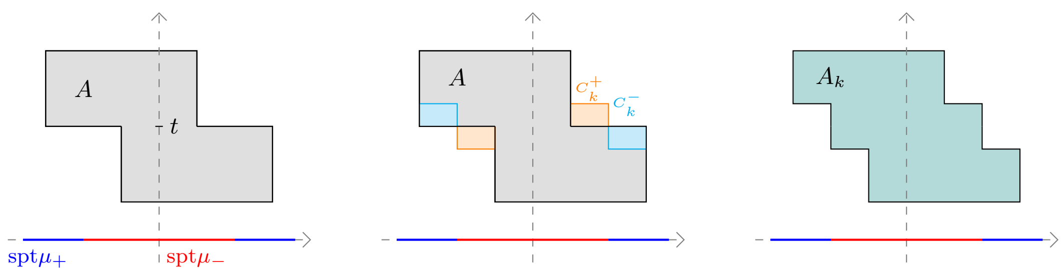


Figure 3.9: Construction of the auxiliary approximate sets A_k

Formally, we use i) to observe that each A_k is a set of finite perimeter with $A_k \to A$ in L^1 as $k \to \infty$ and show in the following that

$$A^1_k \cap H_t = A^+_k \cap H_t = (A^1 \cap (\mathbb{R}^{n-1} \backslash \mathrm{B}^{n-1}_R \times \{t\})) \cup (A^+ \cap (\mathrm{B}^{n-1}_R \times \{t\})) \tag{3.70}$$

holds up to $\mathcal{H}^{n-1}$-negligible sets.
Since Federer's structure theorem ensures $A^1 \cup A^0 \cup A^{\frac{1}{2}} = \mathbb{R}^n$ except for $\mathcal{H}^{n-1}$-null sets, and since we know that $A^{\frac{1}{2}} \cap H_t \subseteq \{\nu_A = \pm e_n\}$ up to $\mathcal{H}^{n-1}$-negligible sets, it suffices to show the following properties:

v) If $x \in A^1 \cap H_t$, then $x \in A^1_k$,

vi) if $x \in A^0 \cap H_t$, then $x \in A^0_k$,

vii) if $x \in A^{\frac{1}{2}} \cap \{\nu_A = \pm e_n\} \cap (\mathrm{B}^{n-1}_R \times \{t\})$, then $x \in A^1_k$,

viii) if $x \in A^{\frac{1}{2}} \cap \{\nu_A = \pm e_n\} \cap (\mathbb{R}^{n-1} \backslash \mathrm{B}^{n-1}_R \times \{t\})$, then $x \in A^0_k$,

where we understand these statements for $\mathcal{H}^{n-1}$-a.e. x in the corresponding sets.
To show v) and vi), it suffices to record that the trace assumptions ii) and iii) on $C^\pm_k$ imply $x \in (C^+_k)^0 \cap (C^-_k)^0$ in both cases. For x as in vii), we assume w.l.o.g. $\nu_A(x) = e_n$ and obtain by Lemma 2.49 and Proposition

2.59 that x is in $(A\cap H_t^-)^{\frac{1}{2}}$. Moreover, the choice of C_k^+ yields $x \in (C_k^+)^{\frac{1}{2}} \cap (C_k^+ \cap H_t^+)^{\frac{1}{2}} \cap (C_k^-)^0$. The density of A_k at x is hence estimated by

$$\liminf_{r\searrow 0} \frac{|A_k \cap \mathrm{B}_r(x)|}{|\mathrm{B}_r|} \geqslant \liminf_{r\searrow 0} \frac{|A\cap H_t^- \cap \mathrm{B}_r(x)|}{|\mathrm{B}_r|} + \frac{|C_k^+ \cap H_t^+ \cap \mathrm{B}_r(x)|}{|\mathrm{B}_r|} = \frac{1}{2} + \frac{1}{2} = 1$$

and therefore, we find $x \in A_k^1$.
For x as in viii), we again assume w.l.o.g. $\nu_A(x) = e_n$ and obtain similarly as before that x is in

$$(A\cap H_t^-)^{\frac{1}{2}} \cap (C_k^-)^{\frac{1}{2}} \cap (C_k^- \cap H_t^-)^{\frac{1}{2}} \cap (C_k^+)^0 \cap (A \triangle H_t^-)^0 \,.$$

Thus, we conclude

$$\begin{aligned}\limsup_{r\searrow 0} \frac{|A_k \cap \mathrm{B}_r(x)|}{|\mathrm{B}_r|} &\leqslant \limsup_{r\searrow 0} \frac{|A \cap \mathrm{B}_r(x)|}{|\mathrm{B}_r|} + \frac{|C_k^+ \cap \mathrm{B}_r(x)|}{|\mathrm{B}_r|} - \frac{|C_k^- \cap A \cap \mathrm{B}_r(x)|}{|\mathrm{B}_r|} \\ &= \limsup_{r\searrow 0} \frac{1}{2} - \frac{|C_k^- \cap H_t^- \cap \mathrm{B}_r(x)|}{|\mathrm{B}_r|} = 0\end{aligned}$$

and with this, we find $x \in A_k^0$.
We have shown that $\mathrm{P}(A_k, H_t) = \mathcal{H}^{n-1}(\partial^* A_k \cap H_t) = 0$, such that Lemma 2.38 implies

$$\mathrm{P}(A_k, \mathbb{R}^{n-1}\times I) = \mathrm{P}(A_k, \mathbb{R}^{n-1}\times I\backslash\{t\}) \leqslant \mathrm{P}(A, \mathbb{R}^{n-1}\times I\backslash\{t\}) + \mathrm{P}(C_k^-, \mathbb{R}^{n-1}\times I\backslash\{t\}) + \mathrm{P}(C_k^+, \mathbb{R}^{n-1}\times I\backslash\{t\})\,.$$

By property iv), we conclude

$$\mathrm{P}(A_k, \mathbb{R}^{n-1}\times I) \leqslant \mathrm{P}(A, \mathbb{R}^{n-1}\times I) + \frac{1}{k}\,.$$

Furthermore, (3.70) and the choice of B_R^{n-1} imply

$$\mu_-(A_k^+) = \mu_-(A^+) \qquad \text{and} \qquad \mu_+(A_k^1) = \mu_+(A^1) \qquad \text{for all } k\in\mathbb{N}\,.$$

All together, we find

$$\begin{aligned}\mathrm{P}(A^\star, \mathbb{R}^{n-1}\times I) + \mu_+((A^\star)^1) - \mu_-((A^\star)^+) &\leqslant \liminf_{k\to\infty} \mathrm{P}(A_k^\star, \mathbb{R}^{n-1}\times I) + \mu_+((A_k^\star)^1) - \mu_-((A_k^\star)^+) \\ &\leqslant \limsup_{k\to\infty} \mathrm{P}(A_k, \mathbb{R}^{n-1}\times I) + \mu_+(A_k^1) - \mu_-(A_k^+) \\ &= \mathrm{P}(A, \mathbb{R}^{n-1}\times I) + \mu_+(A^1) - \mu_-(A^+)\,,\end{aligned}$$

where we applied the result and the lower semicontinuity of $\mathcal{P}(\,\cdot\,, I)$ from Step 2.

Step 4: Show the claim for unbounded sets A:
If A is an unbounded set of finite perimeter and finite volume in $\mathbb{R}^n$, we apply Remark 3.2 and find that $(\mathrm{B}_R \cap A)^\star$ converges to $A^\star$ in L^1 as $R\to\infty$. Hence, we can use the lower semicontinuity of $\mathcal{P}(\,\cdot\,; I)$, Step 3 and the regularity of Radon measures to obtain

$$\begin{aligned}&\mathrm{P}(A^\star, \mathbb{R}^{n-1}\times I) + \mu_+((A^\star)^1) - \mu_-((A^\star)^+) \\ &\qquad\leqslant \liminf_{k\to\infty} \mathrm{P}((\mathrm{B}_{R_k}\cap A)^\star, \mathbb{R}^{n-1}\times I) + \mu_+(((\mathrm{B}_{R_k}\cap A)^\star)^1) - \mu_-(((\mathrm{B}_{R_k}\cap A)^\star)^+) \\ &\qquad\leqslant \limsup_{k\to\infty} \mathrm{P}(\mathrm{B}_{R_k}\cap A, \mathbb{R}^{n-1}\times I) + \mu_+((\mathrm{B}_{R_k}\cap A)^1) - \mu_-((\mathrm{B}_{R_k}\cap A)^+) \\ &\qquad= \limsup_{k\to\infty} \mathrm{P}(\mathrm{B}_{R_k}\cap A, \mathbb{R}^{n-1}\times I) + \mu_+(A^1) - \mu_-(A^+)\end{aligned} \tag{3.71}$$

for all sequences $(R_k)_{k\in\mathbb{N}}$ of positive numbers that satisfy $R_k\to\infty$ as $k\to\infty$. Since A has finite perimeter and finite volume, we can consider a specific sequence $(R_k)_{k\in\mathbb{N}}$ of positive numbers that satisfies $R_k\to\infty$

as $k \to \infty$ and $\mathcal{H}^{n-1}(A^+ \cap (\mathbb{R}^{n-1} \times I) \cap \partial \mathrm{B}_{R_k}) < \frac{1}{k}$. Indeed, the existence of such a sequence falls out of the formula for radial integration applied on $\mathbb{1}_{A^+ \cap (\mathbb{R}^{n-1} \times I)}$, which gives

$$\infty > |A \cap (\mathbb{R}^{n-1} \times I)| = \int_0^\infty \mathcal{H}^{n-1}(A^+ \cap (\mathbb{R}^{n-1} \times I) \cap \partial \mathrm{B}_R) \,\mathrm{d}R\,.$$

For such a sequence, we obtain

$$\mathrm{P}(\mathrm{B}_{R_k} \cap A, \mathbb{R}^{n-1} \times I) \leqslant \mathrm{P}(A, \mathrm{B}_{R_k} \cap (\mathbb{R}^{n-1} \times I)) + \mathcal{H}^{n-1}(A^+ \cap (\mathbb{R}^{n-1} \times I) \cap \partial \mathrm{B}_{R_k}) \to \mathrm{P}(A, \mathbb{R}^{n-1} \times I) \quad (3.72)$$

as $k \to \infty$ by Lemma 2.49. Finally, (3.71) and (3.72) yield

$$\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times I) + \mu_+((A^\star)^1) - \mu_-((A^\star)^+) \leqslant \mathrm{P}(A, \mathbb{R}^{n-1} \times I) + \mu_+(A^1) - \mu_-(A^+)\,,$$

which shows the claim in full generality. □

Example 3.72. We cannot lift this statement to the case of non-mutually singular measures μ_-, μ_+. Indeed, take for instance the set $A := [0,1]^2 \cup [-1,0]^2$ in $\mathbb{R}^2$, $\mu_- = \mu_+ = 2\mathcal{H}^{n-1} \llcorner H_0$. According to the definition of Schwarz symmetrization and Lemma 2.80, both measures satisfy (3.61) and the small-volume isoperimetric condition with constant 1. Since $A^\star = \big((-0.5, 0.5) \times [-1,1]\big) \cup \big((-1,1) \times \{0\}\big)$, it holds that $\mu_-((A^\star)^+) = \mu_-((A^\star)^1)$. Together with $\mu_-(A^+) = 4$ and $\mu_+(A^1) = 0$, we obtain

$$\mathcal{P}(A^\star) = \mathrm{P}(A^\star) = 6 > 4 = \mathrm{P}(A) - 4 = \mathcal{P}(A)\,.$$

In fact, the compensation effect of the perimeter and the measures μ_+ and μ_-, which we exploit in the proof, is no longer fulfilled.

Theorem 3.73 (Generalized Schwarz inequality)**.** *Let μ_-, μ_+ be two mutually singular and non-negative Radon measures on $\mathbb{R}^n$ satisfying* (3.61) *and the small-volume isoperimetric condition with constant* 1*. Then*

$$\mathcal{P}(A^\star) \leqslant \mathcal{P}(A)$$

holds true for all sets $A \subseteq \mathbb{R}^n$ of finite perimeter and finite volume.

Proof. We first denote the Borel sets $\mathrm{dis} := \mathrm{dis}_{\mu_+} \cup \mathrm{dis}_{\mu_-}$ and $\mathrm{reg} := \mathrm{reg}_{\mu_+} \cap \mathrm{reg}_{\mu_-}$. By Lemma 3.59, the condition (3.62) applies for μ_- and μ_+ separately, such that the Schwarz inequality and Lemma 3.70 yield

$$\begin{aligned}
\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times \mathrm{reg}) &+ (\mu_+)_{\mathrm{reg}}((A^\star)^1) - (\mu_-)_{\mathrm{reg}}((A^\star)^+) \\
&= \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times \mathrm{reg}) + (\mu_+)_{\mathrm{reg}}((A^1)^\star) - (\mu_-)_{\mathrm{reg}}((A^+)^\star) \\
&\leqslant \mathrm{P}(A, \mathbb{R}^{n-1} \times \mathrm{reg}) + (\mu_+)_{\mathrm{reg}}(A^1) - (\mu_-)_{\mathrm{reg}}(A^+)\,.
\end{aligned}$$

Since the set dis is at most countable, and Proposition 3.71 applies for each $t \in \mathbb{R}$ with $\mu_\pm \llcorner H_t$, we conclude the generalized Schwarz inequality for $\mathcal{P}$. □

In the following, we present another approach of the proof of Theorem 3.73 for functionals of the reduced form $\mathcal{P}_-$ and $\mathcal{P}_+$, avoiding the extensive characterization of the corresponding measures of Section 3.3.2. The proof uses various approximation results, which eventually reduce the argument to polyhedral sets without flat boundary parts, compare the proof of Theorem 3.5, which was based on [62, Theorem 19.11]. It seems difficult to apply this approach to the general functional $\mathcal{P}$, since the argument is based on an outer approximation argument for $\mathcal{P}_-$ and an inner one for $\mathcal{P}_+$, compare Step 3 of the proof of Proposition 3.71

Theorem 3.74 (Generalized Schwarz inequality for the reduced functional)**.** *Let μ_-, μ_+ be two non-negative Radon measures on $\mathbb{R}^n$ satisfying* (3.61) *and the small-volume isoperimetric condition with constant* 1*. Then*

$$\mathcal{P}_-(A^\star) \leqslant \mathcal{P}_-(A) \qquad \textit{and} \qquad \mathcal{P}_+(A^\star) \leqslant \mathcal{P}_+(A) \tag{3.73}$$

hold true for all sets $A \subseteq \mathbb{R}^n$ of finite perimeter and finite volume.

We first confirm (3.73) for simple polyhedral sets without horizontal boundary parts, before transferring the result iteratively to smooth sets, sets without pathological boundary parts (i.e. without points of density 0 on the boundary), and finally to general measurable sets by approximation results from Section 3.1.3.

Proof. The proof in divided into several steps.

Step 1: Show (3.73) for bounded open polyhedral sets A with $\{\nu_A = \pm e_n\} = \emptyset$:
Since A is polyhedral, the boundary of A is Lipschitz. Thus, $A^1 = A$, $A^+ = \overline{A}$ and $\partial A = \partial^* A$ up to an $\mathcal{H}^{n-1}$-null set. Hence, we may write ∂A instead of $\partial^* A$ throughout this step. Applying the definition of Schwarz symmetrization provides

$$(A^+)^\star = \bigcup_{t\in\mathbb{R}} \mathrm{B}^{n-1}_{\mathrm{rad}_{\overline{A}}(t)} \times \{t\} \qquad \text{and} \qquad A^\star = \bigcup_{t\in\mathbb{R}} \mathrm{B}^{n-1}_{\mathrm{rad}_A(t)} \times \{t\}\,.$$

Step 1.i): Show $\mathrm{rad}_{\overline{A}} = \mathrm{rad}_A$ is continuous on $\mathbb{R}$:
The volume function vol_A, and therefore rad_A, is lower semicontinuous, since Fubini's theorem, Remark 2.18 and Fatou's lemma imply

$$\begin{aligned}
\mathrm{vol}_A(t) = |A_t| &= \int_{\mathbb{R}^{n-1}} \mathbb{1}_{A_t}(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^{n-1}} \mathbb{1}_A((x,t))\,\mathrm{d}x \\
&= \int_{\mathbb{R}^{n-1}} (\mathbb{1}_A)^-((x,t))\,\mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^{n-1}} \liminf_{s\to t} (\mathbb{1}_A)^-((x,s))\,\mathrm{d}x \\
&\leqslant \liminf_{s\to t} \int_{\mathbb{R}^{n-1}} (\mathbb{1}_A)^-((x,s))\,\mathrm{d}x = \liminf_{s\to t} \mathrm{vol}_A(t)\,.
\end{aligned}$$

The same argument shows that vol_{A^+} is upper semicontinuous, which transfers to $\mathrm{rad}_{\overline{A}}$. By Theorem 3.8, it holds that

$$\begin{aligned}
|(A^+)_t| - |A_t| &= |(\partial A)_t| \\
&= \mathcal{H}^{n-1}((\partial A)_t \times \{t\}) \\
&= \mathcal{H}^{n-1}(\partial A \cap H_t) \\
&= \int_{\partial A} \mathbb{1}_{H_t}\,\mathrm{d}\mathcal{H}^{n-1} \\
&= \int_{\{\nu_A = \pm e_n\}} \mathbb{1}_{H_t}\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\mathbb{R}} \int_{\partial(A_s)} \frac{\mathbb{1}_{H_t}(z,s)}{|\overline{\nu_A}(z,s)|}\,\mathrm{d}\mathcal{H}^{n-2}(z)\,\mathrm{d}s \\
&= \mathcal{H}^{n-1}(\{\nu_A = \pm e_n\} \cap H_t) \\
&= 0
\end{aligned}$$

for all $t \in \mathbb{R}$. Since A is assumed to be open, we have $\mathrm{rad}_{\overline{A}} \geqslant \mathrm{rad}_A$ and, by the previous observation, even $\mathrm{rad}_{\overline{A}} = \mathrm{rad}_A$ on $\mathbb{R}$. Thus, $\mathrm{rad}_A = \mathrm{rad}_{\overline{A}}$ is continuous on $\mathbb{R}$.

Step 1.ii): Show that rad_A is locally Lipschitz continuous on $\{\mathrm{rad}_A > 0\}$:
According to Theorem 3.8, vol_A is in $\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R})$ with

$$\mathrm{vol}'_A(t) = -\int_{(\partial^* A)_t} \frac{(\nu_A)_n(z,t)}{|\overline{\nu_A}(z,t)|}\,\mathrm{d}\mathcal{H}^{n-2}(z) \tag{3.74}$$

for a.e. $t \in \mathbb{R}$. Since A is assumed to be bounded, it even is in $\mathrm{W}^{1,1}(\mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R})$. We now show Lipschitz continuity of vol_A in order to transfer the property to rad_A on $\{\mathrm{rad}_A > 0\}$ with the previous step. To this end, we show the uniform boundedness of vol'_A via (3.74). We claim that $\mathcal{H}^{n-2}((\partial^* A)_t)$ is bounded uniformly for a.e. $t \in \mathbb{R}$. Indeed, by Theorem 3.7

$$\mathcal{H}^{n-2}((\partial^* A)_t) = \mathcal{H}^{n-2}(\partial^*(A_t)) = \mathrm{P}(A_t)$$

holds for a.e. $t \in \mathbb{R}$. Since $\{\nu_A = \pm e_n\} = \emptyset$, we can represent ∂A by finitely many rotated images of affine functions $f_1, \ldots, f_k$, defined on open and bounded Lipschitz sets $F_1, \ldots, F_k$ in $\mathbb{R}^{n-1}$ with values in $\mathbb{R}^n$. In particular, we find $R > 0$ such that $\partial A \cap (\mathbb{R}^{n-1} \times I) \subseteq \bigcup_{i=1}^k T_i f_i(\mathrm{B}_R^{n-2} \times I)$ holds for all $I \subset \mathbb{R}$, where $T_1, \ldots, T_k$ are a suitable rotations. For all compact sets $I \subseteq \mathbb{R}$, Lemma 3.42 and a standard estimate of the Hausdorff measure in relation to Lipschitz functions give

$$\begin{aligned}
\int_I \mathrm{P}(A_t)\,\mathrm{d}t \leqslant \mathrm{P}(A, \mathbb{R}^{n-1} \times I) &= \mathcal{H}^{n-1}(\partial A \cap \mathbb{R}^{n-1} \times I) \\
&\leqslant \mathcal{H}^{n-1}\left(\bigcup_{i=1}^k T_i f_i(\mathrm{B}_R^{n-1} \times I)\right) \\
&\leqslant \sum_{i=1}^k \mathcal{H}^{n-1}\left(T_i f_i(\mathrm{B}_R^{n-1} \times I)\right) \\
&\leqslant \sum_{i=1}^k \mathrm{Lip}(f_i)^{n-1} |\mathrm{B}_R^{n-1} \times I| \\
&\leqslant \mathrm{C}|I|\,.
\end{aligned}$$

Thus, $\mathrm{P}(A_t) = \mathcal{H}^{n-2}(\partial^* A)$ is bounded by the constant C for a.e. $t \in \mathbb{R}$. Furthermore, ν_A can only attain finitely many values such that there exists $\delta > 0$ with $|\nu_A \pm e_n| > \delta$ on ∂A, which implies vol'_A is in $\mathrm{L}^\infty(\mathbb{R})$ by (3.74). By the definition of the radius function, it follows that rad_A is locally Lipschitz continuous on $\{\mathrm{rad}_A > 0\}$.

Step 1.iii): Show that for all $t \in \{\mathrm{rad}_A > 0\}$ there exists $\varepsilon_t > 0$ such that $\partial A^\star \cap \mathbb{R}^{n-1} \times (t - \varepsilon_t, t + \varepsilon_t)$ is locally the rotated graph of a Lipschitz function:
Let $t \in \{\mathrm{rad}_A > 0\}$. Then there exists $\varepsilon_t > 0$ such that $\mathrm{rad}_A > \varepsilon_t$ on the open interval $(t - \varepsilon_t, t + \varepsilon_t)$. We show that $\partial A^\star \cap (\mathbb{R}^{n-1} \times (t - \varepsilon_t, t + \varepsilon_t))$ is Lipschitz near (x, t) for all $x \in \partial \mathrm{B}^{n-1}_{\mathrm{rad}_A(t)}$. Since $A^\star$ is a body of revolution, we may assume w.l.o.g. that $x = \mathrm{rad}_A(t) e_{n-1} \in \mathbb{R}^{n-1}$. In this case, $A^\star$ is the rotated subgraph $\mathrm{Sub}_{n-1}[\rho]$ of $\rho : B^{n-1}_{\varepsilon_t}(\overline{x}, t) \to \mathbb{R}$; $(y', s) \mapsto \sqrt{\mathrm{rad}(s)^2 - |y'|^2}$ near (x, t) (this is well-defined since $\mathrm{rad}_A > \varepsilon_t$ and $\overline{x} = 0$). The Lipschitz continuity of rad_A transfers to ρ such that $\partial A^\star$ is Lipschitz near (x, t) with

$$\partial A^\star \cap (\mathbb{R}^{n-1} \times (t - \varepsilon_t, t + \varepsilon_t)) = \bigcup_{s \in (t - \varepsilon_t, t + \varepsilon_t)} \partial \mathrm{B}^{n-1}_{\mathrm{rad}_A(s)} \times \{s\}$$

up to $\mathcal{H}^{n-1}$-negligible sets for all $t \in \{\mathrm{rad}_A > 0\}$. In particular, this regularity implies that

$$N_t := \left[\left(A^\star \cap (A^\star)^0\right) \cup \left((\mathbb{R}^n \backslash A^\star) \cap (\mathbb{R}^n \backslash A^\star)^0\right)\right] \cap (\mathbb{R}^{n-1} \times (t - \varepsilon_t, t + \varepsilon_t))$$

is an $\mathcal{H}^{n-1}$-null set.

Step 1.iv): Conclusion:
First, we notice that A has at most finitely many connected components, since it is a bounded polyhedral set. Together with the continuity of rad_A, it follows that $\{\mathrm{rad}_A > 0\}$ is a union of finitely many disjoint open intervals $((a_i, b_i))_{i=1,\ldots,k}$. According to the previous step, we find for each $t \in (a_i, b_i)$ some $\varepsilon_t > 0$ with $\partial A^\star \cap (\mathbb{R}^{n-1} \times (t - \varepsilon_t, t + \varepsilon_t))$ Lipschitz. Exploiting again the continuity of rad_A and the fact that we can write $\{\mathrm{rad}_A > 0\}$ as a countable union of such subintervals with center t_j, it follows

$$\left[(A^+)^\star \backslash (A^\star)^+\right] \cup \left[(A^\star)^1 \backslash A^\star\right] \subseteq \left[(A^\star)^0 \cap A^\star\right] \cup \left[(\mathbb{R}^n \backslash A^\star) \cap (\mathbb{R}^n \backslash A^\star)^0\right] \subseteq \bigcup_{j=1}^\infty N_{t_j} \cup \bigcup_{i=1}^k \{0\} \times \{a_i, b_i\}\,,$$

where we applied Lemma 3.37 and where N_{t_j} are the $\mathcal{H}^{n-1}$-negligible sets defined as in the previous step. In particular, we have

$$\mathcal{H}^{n-1}((A^+)^\star \backslash (A^\star)^+) = 0 \qquad \text{and} \qquad \mathcal{H}^{n-1}((A^\star)^1 \backslash A^\star) = 0\,.$$

Finally, Theorem 3.5, the monotonicity conditions imposed on μ_- and μ_+, and Lemma 3.66 give

$$\mathcal{P}_-(A^\star) = \mathrm{P}(A^\star) - \mu_-((A^\star)^+) \leqslant \mathrm{P}(A^\star) - \mu_-((A^+)^\star) \leqslant \mathrm{P}(A) - \mu_-(A^+) = \mathcal{P}_-(A)$$

and

$$\mathcal{P}_+(A^\star) = \mathrm{P}(A^\star) + \mu_+((A^\star)^1) \leqslant \mathrm{P}(A^\star) + \mu_+((A^1)^\star) \leqslant \mathrm{P}(A) + \mu_+(A^1) = \mathcal{P}_+(A)\,.$$

Since the remaining proof is similar for $\mathcal{P}_+$, we from now on mainly consider proving the Schwarz inequality only for $\mathcal{P}_-$.

Step 2: Show (3.73) for bounded smooth sets:
By Remark 3.27, we can approximate A by smooth and bounded sets $(\tilde{A}_k)_{k\in\mathbb{N}}$ that compactly contain A with respect to the perimeter, the L^1 convergence and such that $\partial\tilde{A}_k \to \partial A$ with respect to the Hausdorff distance as $k \to \infty$. By Theorem 3.25, we can further approximate each $\tilde{A}_k$ in L^1 and with respect to the perimeter by polyhedral sets $(A_i^k)_{i\in\mathbb{N}}$ such that $\partial A_i^k \to \partial\tilde{A}_k$ with respect to the Hausdorff distance as $i \to \infty$. By applying a diagonal argument, we find a sequence $(A_k)_{k\in\mathbb{N}}$ of bounded polyhedral sets with $A \Subset A_k$ for all $k \in \mathbb{N}$ that approximates A in L^1 and with respect to the perimeter. Clearly, there exist at most finitely many hyperplanes H in $\mathbb{R}^n$ such that $\partial A_k \cap H$ is not an $\mathcal{H}^{n-1}$-null set. Hence, for each $k \in \mathbb{N}$, there exists a rotation T with $A \Subset TA_k$ and $\mathcal{H}^{n-1}(TA_k \cap \{\mathbb{R}^{n-1} \times \{t\} : t \in \mathbb{R}\}) = 0$ for all $t \in \mathbb{R}$. Since the rotations can be chosen without effecting the converging properties, we can further assume $\{\nu_{A_k} = \pm e_n\} = \emptyset$ for all $k \in \mathbb{N}$. Taking into account Step 1, the lower semicontinuity of the functional $\mathcal{P}_-$ from Theorem 2.82 and the positivity of μ_-, we conclude

$$\mathcal{P}_-(A^\star) \leqslant \liminf_{k\to\infty} \mathrm{P}(A_k^\star) - \mu_-((A_k^\star)^+) \leqslant \liminf_{k\to\infty} \mathrm{P}(A_k) - \mu_-(A_k^+) \leqslant \mathrm{P}(A) - \mu_-(A^+) = \mathcal{P}_-(A)\,.$$

An analogous argument with inner approximation by Theorem 3.26 applies to $\mathcal{P}_+$. Hence, we showed (3.73) for arbitrary bounded smooth sets.

Step 3: Show (3.73) for bounded sets A of finite perimeter with $\mathcal{H}^{n-1}(A^0 \cap \partial A) = 0$:
By Remark 3.29 we can approximate A with $\mathcal{H}^{n-1}(A^0 \cap \partial A) = 0$ by bounded smooth sets A_k in L^1 and with respect to the perimeter such that $A \Subset A_k$ for all $k \in \mathbb{N}$. As in Step 2, we obtain (3.73) for the functional $\mathcal{P}_-$ for all bounded sets of finite perimeter with $\mathcal{H}^{n-1}(A^0 \cap \partial A) = 0$.
With Theorem 3.28, we can approximate bounded sets A with $\mathcal{H}^{n-1}(A^1 \cap \partial A) = 0$ by bounded smooth sets A_k in L^1 and with respect to the perimeter such that $A_k \Subset A$ for all $k \in \mathbb{N}$. Then the same argument as before gives the Schwarz inequality for $\mathcal{P}_+$ on A.

Step 4: Show (3.73) for bounded sets of finite perimeter:
According to Theorem 3.30, we can approximate general bounded sets of finite perimeter A by subsets A_k of finite perimeter with $\mathcal{H}^{n-1}(\partial A_k \cap A_k^0) = 0$ for all $k \in \mathbb{N}$ and $\mathrm{P}(A\backslash A_k) \to 0$. In particular, we have $A_k^\star \to A^\star$ in L^1 such that the lower semicontinuity of $\mathcal{P}_-$ from Theorem 2.82 and the results of Step 3 imply

$$\mathcal{P}_-(A^\star) = \mathrm{P}(A^\star) - \mu_-((A^\star)^+) \leqslant \liminf_{k\to\infty} \mathrm{P}(A_k^\star) - \mu_-((A_k^\star)^+) \leqslant \liminf_{k\to\infty} \mathrm{P}(A_k) - \mu_-(A_k^+)\,. \tag{3.75}$$

Since μ_- satisfies the small-volume isoperimetric condition with constant 1, we have

$$0 \leqslant \mu_-(A^+) - \mu_-(A_k^+) \leqslant \mu_-((A\backslash A_k)^+) \leqslant \mathrm{P}(A\backslash A_k) \to 0 \qquad \text{as } k \to \infty\,. \tag{3.76}$$

Furthermore, by Lemma 2.38, it holds $\mathrm{P}(A) - \mathrm{P}(A_k) \leqslant \mathrm{P}(A\backslash A_k) \to 0$ as $k \to \infty$, which, together with (3.75) and (3.76) gives $\mathcal{P}_-(A^\star) \leqslant \mathcal{P}_-(A)$.
We obtain the corresponding inequality for the functional $\mathcal{P}_+$ by applying Remark 3.31 instead of Theorem 3.30.
Hence, we showed (3.73) for general bounded sets of finite perimeter.

Step 5: Show (3.73) for unbounded sets of finite perimeter and finite volume:
As in the proof of Proposition 3.71, we derive

$$\mathcal{P}_-(A^\star) \leqslant \limsup_{R\to\infty} \mathrm{P}(B_R \cap A) - \mu_-(A^+)\,.$$

Applying Lemma 2.50 yields $\mathcal{P}_-(A^\star) \leqslant \mathcal{P}_-(A)$. The argumentation for $\mathcal{P}_+$ instead of $\mathcal{P}_-$ is analogous. $\square$

3.3.4 Equality case analysis

In this section, we present characterizations of the equality cases of the generalized Schwarz inequality, similar to Theorem 3.57. To this end, it is necessary to ensure that the measures μ_- and μ_+ do not fully compensate for $\mathrm{P}(A) - \mathrm{P}(A^\star)$. In fact, the constant in the small-volume isoperimetric condition must be strictly smaller than 1, see Example 3.75, or the corresponding hyperplanes must be excluded from the equality characterization.

Example 3.75. We consider the set $A := \mathrm{B}_1^{n-1}(-2e_1) \times (0,1) \cup \mathrm{B}_1^{n-1}(2e_1) \times (-1,0]$ and the measure $\mu = -\mu_- = -2\mathcal{H}^{n-1} \llcorner H_0$.

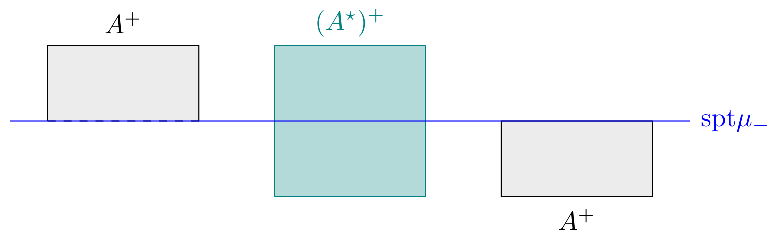


Figure 3.10: Illustration of an equality case for $\mathcal{P}$ without full rigidity

It holds that $A^\star = \mathrm{B}_1 \times (-1,1)$ and that μ_- satisfies the small-volume isoperimetric condition with constant 1 by Lemma 2.80. Moreover,

$$\mathcal{P}(A) = \mathrm{P}(A) - \mu_-(A^+) = \mathrm{P}(A) - 4\omega_{n-1} = \mathrm{P}(A^\star) - 2\omega_{n-1} = \mathrm{P}(A^\star) - \mu_-((A^\star)^+) = \mathcal{P}(A^\star)\,,$$

although rigidity is not satisfied in this case.

Lemma 3.76 (Equality cases of the Schwarz inequality for functionals with measure data on the regular part)**.** *Let μ_+ and μ_- be two mutually singular and non-negative Radon measures on $\mathbb{R}^n$ which both satisfy* (3.61) *and the small-volume isoperimetric condition in $\mathbb{R}^n$ with constant* 1*. If a set $A \subseteq \mathbb{R}^n$ of finite perimeter and finite volume satisfies $\mathcal{P}(A^\star) = \mathcal{P}(A)$, then $\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times \mathrm{reg}) = \mathrm{P}(A, \mathbb{R}^{n-1} \times \mathrm{reg})$ for $\mathrm{reg} := \mathrm{reg}_{\mu_+} \cap \mathrm{reg}_{\mu_-}$. In particular, Theorem 3.57 can be applied for all open sets $I \subseteq \mathrm{reg}$.*

Proof. Remark 3.64, Lemmas 3.70 and 3.59, and the Schwarz inequality yield

$$\begin{aligned} &\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times \mathrm{reg}) + (\mu_+)_{\mathrm{reg}}((A^\star)^1) - (\mu_-)_{\mathrm{reg}}((A^\star)^+) \\ &\qquad \leqslant \mathrm{P}(A^\star, \mathbb{R}^{n-1} \times \mathrm{reg}) + (\mu_+)_{\mathrm{reg}}(A^1) - (\mu_-)_{\mathrm{reg}}(A^+) \\ &\qquad \leqslant \mathrm{P}(A, \mathbb{R}^{n-1} \times \mathrm{reg}) + (\mu_+)_{\mathrm{reg}}(A^1) - (\mu_-)_{\mathrm{reg}}(A^+)\,, \end{aligned} \tag{3.77}$$

and by Proposition 3.71, we also obtain

$$\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times \mathrm{dis}) + (\mu_+)_{\mathrm{dis}}((A^\star)^1) - (\mu_-)_{\mathrm{dis}}((A^\star)^+) \leqslant \mathrm{P}(A, \mathbb{R}^{n-1} \times \mathrm{dis}) + (\mu_+)_{\mathrm{dis}}(A^1) - (\mu_-)_{\mathrm{dis}}(A^+)$$

for $\mathrm{dis} := \mathrm{dis}_{\mu_-} \cup \mathrm{dis}_{\mu_+}$. By Theorem 3.73, we obtain equality in every step of (3.77) and conclude $\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times \mathrm{reg}) = \mathrm{P}(A, \mathbb{R}^{n-1} \times \mathrm{reg})$. ☐

Notation. In the following, we introduce for a Radon measure μ on $\mathbb{R}^n$ the set

$$\mathrm{dis}_\mu^{\geqslant 2} := \left\{ t \in \mathbb{R} : \mathcal{H}^{n-1}\left(\left\{ \frac{\mathrm{d}|\mu| \llcorner H_t}{\mathrm{d}\mathcal{H}^{n-1} \llcorner H_t} \geqslant 2 \right\} \right) > 0 \right\}.$$

Theorem 3.77 (Equality characterization of the Schwarz inequality on the discrete set)**.** *Let μ_+ and μ_- be two mutually singular and non-negative Radon measures on $\mathbb{R}^n$ which both satisfy* (3.61) *and the small-volume isoperimetric condition in $\mathbb{R}^n$ with constant* 1*. If a set $A \subseteq \mathbb{R}^n$ of finite perimeter and finite volume satisfies $\mathcal{P}(A^\star) = \mathcal{P}(A)$, then*

$$\mathrm{P}\Big(A^\star, \mathbb{R}^{n-1} \times \Big(\mathbb{R} \backslash \mathrm{dis}^{\geqslant 2}_{\mu_+ + \mu_-}\Big)\Big) = \mathrm{P}\Big(A, \mathbb{R}^{n-1} \times \Big(\mathbb{R} \backslash \mathrm{dis}^{\geqslant 2}_{\mu_+ + \mu_-}\Big)\Big)\,.$$

In particular, Theorem 3.57 can be applied for all open sets $I \subseteq \mathrm{dis}^{\geqslant 2}_{\mu_+ + \mu_-}$.

Proof. According to Lemma 3.76, we directly obtain $\mathrm{P}(A^\star, \mathbb{R}^{n-1}\times \mathrm{reg}) = \mathrm{P}(A, \mathbb{R}^{n-1}\times \mathrm{reg})$ if $\mathcal{P}(A^\star) = \mathcal{P}(A)$. Since $\mathrm{reg} = \mathbb{R}$ up to a countable set, Theorem 3.5 gives that A is a stack of balls. Hence, we can apply Proposition 3.55, which yields $\mathrm{P}(A^\star, \mathbb{R}^{n-1} \times \{\mathrm{rad}_A^- = 0\}) = \mathrm{P}(A, \mathbb{R}^{n-1} \times \{\mathrm{rad}_A^- = 0\})$. It is therefore left to show

$$\mathrm{P}(A^\star, H_t) = \mathrm{P}(A, H_t)$$

for every t in the countable set $\mathrm{dis} \cap \{\mathrm{rad}_A^- > 0\}\backslash \mathrm{dis}^{\geqslant 2}_{\mu_+ + \mu_-}$. Applying Theorem 3.48 provides $\mathrm{rad}_A \in \mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\})$ and $\mathrm{cen}_A \in \mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}_A^- > 0\}, \mathbb{R}^{n-1})$. With these preliminary thoughts at hand, we fix $t \in \mathrm{dis} \cap \{\mathrm{rad}_A^- > 0\}\backslash \mathrm{dis}^{\geqslant 2}_{\mu_+ + \mu_-}$ and define the Borel functions $f_+ := \frac{\mathrm{d}(\mu_+)\llcorner H_t}{\mathrm{d}\mathcal{H}^{n-1}\llcorner H_t}$ and $f_- := \frac{\mathrm{d}(\mu_-)\llcorner H_t}{\mathrm{d}\mathcal{H}^{n-1}\llcorner H_t}$, which satisfy $f_+ + f_- < 2$ for $\mathcal{H}^{n-1}$-a.e. point in H_t. For the remaining proof, we assume w.l.o.g. that $r^+(t) = r(t+)$ and $r^-(t) = r(t-)$, and abbreviate $\mathrm{B}_+ := \mathrm{B}^{n-1}_{\mathrm{rad}_A(t+)}(\mathrm{cen}_A(t+))$ and $\mathrm{B}_- := \mathrm{B}^{n-1}_{\mathrm{rad}_A(t-)}(\mathrm{cen}_A(t-))$. In order to determine the perimeter and the measure of $A^\star$ and A on H_t with respect to μ_+ and μ_-, we apply the structure theorems of De Giorgi and Federer in combination with Lemma 3.37 to derive

$$\mathrm{P}(A^\star, H_t) = \mathcal{H}^{n-1}(\partial^*(A^\star) \cap H_t) = \big|((A^\star)^+)_t \backslash ((A^\star)^1)_t\big| = \big|\mathrm{B}^{n-1}_{\mathrm{rad}_A(t+)}\big| - \big|\mathrm{B}^{n-1}_{\mathrm{rad}_A(t-)}\big| \tag{3.78}$$

and

$$\mu_-\big((A^\star)^+\big) = \mu_-\big(\mathrm{B}^{n-1}_{\mathrm{rad}_A(t+)} \times \{t\}\big) \qquad \text{and} \qquad \mu_+\big((A^\star)^1\big) = \mu_+\big(\mathrm{B}^{n-1}_{\mathrm{rad}_A(t-)} \times \{t\}\big)\,. \tag{3.79}$$

Another application of the structure theorem of Federer and Lemma 3.35 yields

$$(\partial^* A)_t = \mathrm{B}_+ \triangle \mathrm{B}_- \qquad \text{and} \qquad (A^1)_t = \mathrm{B}_+ \cap \mathrm{B}_-$$

up to $\mathcal{H}^{n-1}$-negligible sets. In particular, the structure theorem of De Giorgi and Lemma 3.66, and Proposition 2.79, respectively, give

$$\mathrm{P}(A, H_t) = |(\partial^* A)_t| = \big|\mathrm{B}_+ \triangle \mathrm{B}_-\big| \tag{3.80}$$

and

$$\mu_-\big(A^+\big) = \mu_-\big((\mathrm{B}_+ \cup \mathrm{B}_-) \times \{t\}\big) \qquad \text{and} \qquad \mu_+\big(A^1\big) = \mu_+\big((\mathrm{B}_+ \cap \mathrm{B}_-) \times \{t\}\big)\,. \tag{3.81}$$

In the following, we argue by contraposition and assume $\mathrm{P}(A, H_t) > \mathrm{P}(A^\star, H_t)$. The formulas (3.78) and (3.80) yield

$$\begin{aligned}
\mathrm{P}(A, H_t) - \mathrm{P}(A^\star, H_t) &= |\mathrm{B}_- \backslash \mathrm{B}_+| + |\mathrm{B}_+ \backslash \mathrm{B}_-| - \big|\mathrm{B}^{n-1}_{\mathrm{rad}_A(t+)} \backslash \mathrm{B}^{n-1}_{\mathrm{rad}_A(t-)}\big| \\
&= |\mathrm{B}_- \backslash \mathrm{B}_+| + |\mathrm{B}_+| - |\mathrm{B}_+ \cap \mathrm{B}_-| - |\mathrm{B}_+| + |\mathrm{B}_-| \\
&= 2|\mathrm{B}_- \backslash \mathrm{B}_+|\,.
\end{aligned} \tag{3.82}$$

From the assumption $\mathcal{P}(A^\star) = \mathcal{P}(A)$, Lemmas 3.70 and Proposition 3.71, we obtain equality of $\mathcal{P}$ on every horizontal hyperplane, i.e.,

$$\mathrm{P}(A^\star, H_t) + \mu_+((A^\star)^1) - \mu_-((A^\star)^+) = \mathrm{P}(A, H_t) + \mu_+(A^1) - \mu_-(A^+)\,,$$

which, together with (3.79) and (3.81) and (3.82), implies

$$\begin{aligned}
0 < 2|\mathrm{B}_- \backslash \mathrm{B}_+| &= \mu_+\big((\mathrm{B}^{n-1}_{\mathrm{rad}_A(t-)}) \times \{t\}\big) - \mu_-\big((\mathrm{B}^{n-1}_{\mathrm{rad}_A(t+)}) \times \{t\}\big) \\
&\quad - \mu_+\big((\mathrm{B}_+ \cap \mathrm{B}_-) \times \{t\}\big) + \mu_-\big((\mathrm{B}_+ \cup \mathrm{B}_-) \times \{t\}\big)\,.
\end{aligned}$$

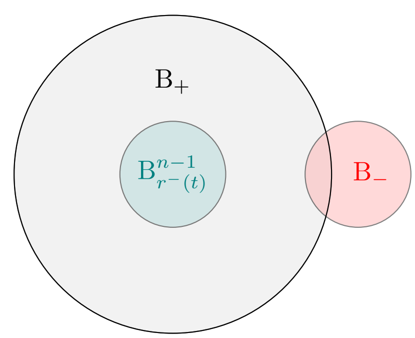


Figure 3.11: Illustration of the overlap of B_- and B_+

Then, Lemma 3.59 and the definition of $f_\pm$ yield

$$\begin{aligned}
0 < 2|\mathrm{B}_- \backslash \mathrm{B}_+| &\leqslant \mu_+\big(\mathrm{B}_- \times \{t\}\big) - \mu_-\big(\mathrm{B}_+ \times \{t\}\big) - \mu_+\big((\mathrm{B}_+ \cap \mathrm{B}_-) \times \{t\}\big) + \mu_-\big((\mathrm{B}_+ \cup \mathrm{B}_-) \times \{t\}\big) \\
&= \mu_+\big((\mathrm{B}_- \backslash \mathrm{B}_+) \times \{t\}\big) + \mu_-\big((\mathrm{B}_- \backslash \mathrm{B}_+) \times \{t\}\big) \\
&= \int_{(\mathrm{B}_- \backslash \mathrm{B}_+) \times \{t\}} f_+ + f_- \,\mathrm{d}\mathcal{H}^{n-1}\,.
\end{aligned}$$

Finally, the assumption $f_+ + f_- < 2$ on H_t gives the contradiction, and hence $\mathrm{P}(A^\star, H_t) = \mathrm{P}(A, H_t)$ holds. $\square$

A complete characterization of the equality cases in the Schwarz inequality for measure data functionals requires distinguishing certain cases, where A has weight on H_t such that $f_- + f_+ = 2$, following the notation of the previous proof. We can avoid this by assuming that the isoperimetric inequality is satisfied only for some constant $C < 1$.

Corollary 3.78 (Characterization of the equality cases in the Schwarz inequality for functionals with measure data)**.** *Let μ_+ and μ_- be two mutually singular and non-negative Radon measures on $\mathbb{R}^n$ which both satisfy* (3.61) *and the small-volume isoperimetric condition in $\mathbb{R}^n$ with constant $C < 1$. Then*

$$\mathcal{P}(A^\star) = \mathcal{P}(A) \qquad \textit{implies} \qquad \mathrm{P}(A^\star) = \mathrm{P}(A)$$

for all sets $A \subseteq \mathbb{R}^n$ of finite perimeter and finite volume. In this case, A is a stack of balls with $|\mathrm{D}\mathrm{cen}_A| \leqslant |\mathrm{D}^s \mathrm{rad}_A|$ on $\{\mathrm{rad}_A^- > 0\}$.

The other direction is not true in general, since the measure $\mu = \mu_+ - \mu_-$ might still decrease under symmetrization.

Example 3.79. Let $A = [-2, -1] \times [-1, 1]$, $\mu_- = \mathcal{H}^1 \llcorner [-1, 1] \times \{0\}$ and $\mu_+ = 0$. Then the small-volume isoperimetric condition is satisfied with constant $\frac{1}{2}$ by Lemma 2.80. It further holds that $\mathrm{P}(A^\star) = \mathrm{P}(A)$ but we have strict inequality in

$$\mathcal{P}(A^\star) = \mathrm{P}(A) - 1 < \mathrm{P}(A) = \mathcal{P}(A)\,.$$

Proof of Corollary 3.78. According to Theorem 3.77, it suffices to show $\mathrm{dis}^{\geqslant 2}_{\mu_+ + \mu_-} = \emptyset$ if μ_+ and μ_- both satisfy the small-volume isoperimetric condition with constant strictly smaller than 1. To this end, we consider $t \in \mathrm{dis}_{\mu_+ + \mu_-}$ and denote the Borel functions

$$f_+ := \frac{\mathrm{d}(\mu_+) \llcorner H_t}{\mathrm{d}\mathcal{H}^{n-1} \llcorner H_t} \qquad \text{and} \qquad f_- := \frac{\mathrm{d}(\mu_-) \llcorner H_t}{\mathrm{d}\mathcal{H}^{n-1} \llcorner H_t}\,.$$

Both functions f_- and f_+ are in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{n-1} \times \{t\}; \mathcal{H}^{n-1} \llcorner H_t)$, and $\mu_+ = f_+ \mathcal{H}^{n-1} \llcorner H_t$ as well as $\mu_- = f_- \mathcal{H}^{n-1} \llcorner H_t$ hold true on H_t. The assumptions on μ_+ and μ_- and Lemma 2.81 lead to $f_-(x) + f_+(x) \leqslant 2C < 2$ for $\mathcal{H}^{n-1}$-a.e. $x \in H_t$. Hence, we found $\mathrm{dis}^{\geqslant 2}_{\mu_+ + \mu_-} = \emptyset$ and therefore proved the claim. □

Combining Proposition 3.55 and Corollaries 3.58 and 3.78 gives the following criterion of rigidity.

Corollary 3.80 (Rigidity criterion)**.** *Let μ_+ and μ_- be two mutually singular and non-negative Radon measures on $\mathbb{R}^n$ which both satisfy* (3.61) *and the small-volume isoperimetric condition in $\mathbb{R}^n$ with constant $C < 1$. Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter and finite volume for which $\{\mathrm{rad}_A^- > 0\}$ is an open interval and $\mathrm{rad}_A^- \in \mathrm{W}^{1,1}(\{\mathrm{rad}_A^- > 0\})$. Then rigidity is satisfied for A with respect to the functional $\mathcal{P}$. In particular, the equality $\mathcal{P}(F^\star) = \mathcal{P}(F)$ for a set $F \subseteq \mathbb{R}^n$ with $\mathrm{rad}_F = \mathrm{rad}_A$ on $\mathbb{R}$ implies the existence of some $x \in \mathbb{R}^{n-1}$ with $F = A^\star + (x, 0)$.*

Rigidity can hold true for a set which does not satisfy the assumptions given in the previous corollary.

Example 3.81. Let $r < R$ in $(0, \infty)$ and $A := \big(\mathrm{B}^2_R \times (0, 2)\big) \cup \big(\mathrm{B}^2_r \times (-2, 0)\big)$.
The set A equals its symmetrization and is indeed the unique minimizer of the functional $\mathcal{P}$ among all sets that have the volume function vol_A, where

$$\mu_+ = 0 \qquad \text{and} \qquad \mu_- := \mathcal{H}^{n-1} \llcorner \mathrm{B}^2_R \times \{1\} + \mathcal{H}^{n-1} \llcorner \mathrm{B}^2_r \times \{-1\}\,.$$

In fact, A satisfies an extreme form of rigidity with respect to the functional $\mathcal{P}$, since the measure μ_- holds A in place. Nonetheless, rad_A is not a Sobolev function in this situation. Similarly, we can take the set $A := \mathrm{B}^2_R \times (1, 2) \cup \mathrm{B}^2_r \times (-2, -1)$ and still obtain the extreme rigidity condition with $\{\mathrm{rad}_A^- > 0\}$ not being an interval. If we define

$$\mu_+ = 0 \qquad \text{and} \qquad \mu_- := 2\mathcal{H}^{n-1} \llcorner \mathrm{B}^2_R \times \{1\} + 2\mathcal{H}^{n-1} \llcorner \mathrm{B}^2_r \times \{-1\}\,,$$

we further obtain rigidity for A without μ_- satisfying the small-volume isoperimetric condition with constant strictly smaller than 1, according to Lemmas 2.80 and 2.81.

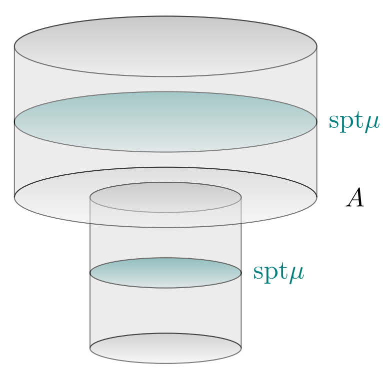


Figure 3.12: Illustration of A and μ

3.3.5 Model cases for the generalized Schwarz inequality

This section is devoted to the application of the so far accumulated results to model cases in the form of obstacle problems. Due to the extensive computation, we restrict the analysis to functionals of the form $\mathcal{P}_-$. We consider two different planar problems, one with squares as obstacles and non-constant density function of μ_- with respect to the Hausdorff measure, and another with discs as obstacles and constant density function. The last case is the three-dimensional version of the latter. The general form of the problems is as follows. For two Borel sets $I, O \subseteq \mathbb{R}^n$, with $I \subseteq O$ up to null sets, we consider the admissible set

$$\mathcal{A}_I^O := \left\{A \in \mathcal{B}(\mathbb{R}^n) : I \subseteq A \subseteq O \text{ up to negligible sets, } |A| + \mathrm{P}(A) < \infty\right\}.$$

The corresponding measure is then defined on some hypersurface of the form H_t so that $\mathrm{dis}_\mu = \{t\}$. Depending on the shape of the obstacles and the location of t, the corresponding minimization problem takes different forms, see Figures 3.13, 3.15 and 3.16.

We first consider the planar obstacle problems, starting with the square obstacles.

Notation. Let $0 < r < s$ and $t \in \mathbb{R}$. We denote the convex hull of $[-r, -r]^2$ and $\{(-s, t), (s, t)\}$ by $K(r, s, t)$. Moreover, if $|t| < r$, we denote by $s_f^r(t)$ the unique value s in $[r, \infty)$ which satisfies

$$f(s) = \frac{1}{2}\left[\frac{s-r}{\sqrt{(r-t)^2 + (s-r)^2}} + \frac{s-r}{\sqrt{(r+t)^2 + (s-r)^2}}\right] \tag{3.83}$$

for some non-negative function $f \not\equiv 0$ in $\mathrm{L}^1(\mathbb{R}, [0,1]) \cap \mathrm{C}^0(\mathbb{R}, [0,1])$, which is radial decreasing (i.e., $f(x) \geqslant f(y)$ for all $x, y \in \mathbb{R}$ with $|x| \leqslant |y|$).

In the case of $|t| \geqslant r$ and $2f(0) > 1 - \frac{r}{\sqrt{r^2 + (|t|-r)^2}}$, we denote by $s_f^r(t)$ the unique value s in $(0, \infty)$ which satisfies

$$2f(s) = \begin{cases} 1 + \frac{s-r}{\sqrt{(s-r)^2 + (|t|+r)^2}} & \text{if } s > r\,, \\ 1 & \text{if } s = r\,, \\ 1 + \frac{s-r}{\sqrt{(s-r)^2 + (|t|-r)^2}} & \text{if } s < r\,. \end{cases} \tag{3.84}$$

Remark 3.82. The uniqueness of the value $s_f^r(t)$ can be obtained by the strict monotonicity of the functions on the right-hand side of (3.83) and (3.84), and the monotonicity assumption on f.

Theorem 3.83 (The planar obstacle problem with squares as obstacles)**.** *Let $r < R$ in $(0, \infty)$ and let $f \in \mathrm{L}^1(\mathbb{R}, [0,1]) \cap \mathrm{C}^0(\mathbb{R}, [0,1])$ be radial decreasing. For $t \in (-R, R)$, we define the measure $\mu_- = 2f_t\mathcal{H}^1 \llcorner \mathbb{R} \times \{t\}$ with the density function*

$$f_t(s, \tau) := \begin{cases} 0 & \text{if } \tau \neq t\,, \\ f(s) & \text{if } \tau = t\,, \end{cases} \qquad \text{for } (s, \tau) \in \mathbb{R}^2\,.$$

Then there exists a unique minimizer $A_{\min}$ of the functional $\mathcal{P}_-$ in the admissible class $\mathcal{A}_I^O$ with the obstacles $I := [-r, r]^2$ and $O := [-R, R]^2$. In particular, if either $|t| > r$ and $2f(0) > 1 - \frac{r}{\sqrt{r^2 + (|t|-r)^2}}$ or $|t| \leqslant r$ and $f(r) > 0$, then

$$A_{\min} = K\big(r, \min\{s_f^r(t), R\}, t\big)\,.$$

If $|t| > r$ and $2f(0) \leqslant 1 - \frac{r}{\sqrt{r^2 + (|t|-r)^2}}$ or $|t| \leqslant r$ and $f(r) = 0$, then $A_{\min}$ equals the inner obstacle I.

We first show the following auxiliary result about one-dimensional BV minimizers.

Lemma 3.84. *Let $a, b, t, s, r, R \in \mathbb{R}$ with $a < t < b$ and $u \in \mathrm{BV}((a, b))$.*

i) *If $u(a+) = r$ and $u(b-) = R$, then the estimate*

$$|(\mathcal{L}^1, \mathrm{D}u_0)|((a, b)) \leqslant |(\mathcal{L}^1, \mathrm{D}u)|((a, b))$$

holds true, where

$$u_0(x) := r + \frac{R - r}{b - a}(x - a) \qquad \text{if} \qquad x \in (a, b)\,.$$

Moreover, equality in the previous inequality holds true if and only if $u = u_0$.

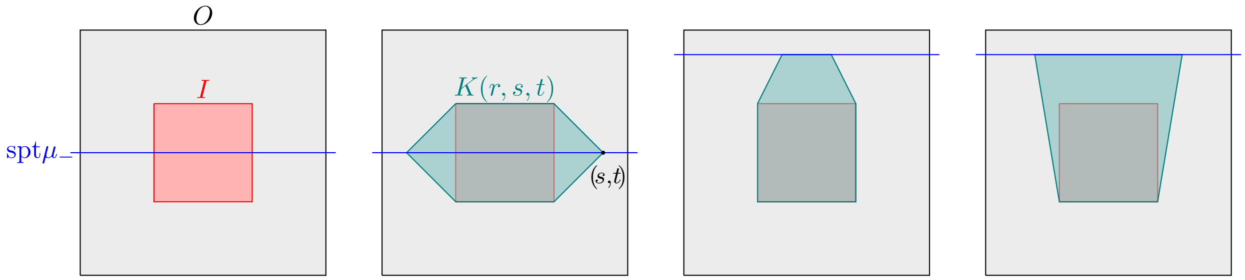


Figure 3.13: Obstacle problem with rectangles in $\mathbb{R}^2$ and different values of t

ii) *If* $u(a+) = r$, $u(b-) = R$ *and* $u^+(t) = s$. *Then the estimate*

$$|(\mathcal{L}^1, \mathrm{D}u_0)|((a,b)) \leqslant |(\mathcal{L}^1, \mathrm{D}u)|((a,b))$$

holds true, where

$$u_0(x) := \begin{cases} r + \frac{s-r}{t-a}(x-a) & \text{if } x \in (a,t) \\ R + \frac{R-s}{b-t}(x-b) & \text{if } x \in (t,b)\,. \end{cases}$$

Moreover, equality in the previous inequality holds true if and only if $u = u_0$.

Proof. We will only prove ii), since i) can be shown analogously. Let μ be a non-negative Radon measure on I with $|(\mathcal{L}^1, \mathrm{D}u)| \ll \mu$ on $\mathcal{B}(I)$. Then either $u(t-) = s$ or $u(t+) = s$ by Theorem 2.24. We continue the proof by assuming $u(t-) = s$ and denote the two-dimensional Borel function $\frac{\mathrm{d}(\mathcal{L}^1, \mathrm{D}u)}{\mathrm{d}\mu}$ by f. The estimate

$$|(\mathcal{L}^1, \mathrm{D}u)|((a,t)) = \int_{(a,t)} |f| \,\mathrm{d}\mu \geqslant \int_{(a,t)} \nu \cdot f \,\mathrm{d}\mu = \int_{(a,t)} \nu_1 f_1 \,\mathrm{d}\mu + \int_{(a,t)} \nu_2 f_2 \,\mathrm{d}\mu = \nu_1 \mathcal{L}^1((a,t)) + \nu_2 \mathrm{D}u((a,t))$$

follows from the Cauchy-Schwarz inequality for all unit vectors $\nu \in \mathbb{R}^2$. We now specify $\nu = \frac{(t-a,s-r)}{|(t-a,s-r)|}$ and obtain $\mathcal{L}^1((a,t)) = t-a$ and $\mathrm{D}u((a,t)) = s-r$ by Theorem 2.28, exploiting the assumptions imposed on u. Hence, the previous estimate becomes

$$|(\mathcal{L}^1, \mathrm{D}u)|((a,t)) \geqslant |(t-a, s-r)| = \int_{(a,t)} \sqrt{1 + |u_0'|^2}\,\mathrm{d}x\,.$$

Having equality in the Cauchy-Schwarz inequality above is equivalent to the existence of some $\lambda \in \mathbb{R}\backslash\{0\}$ with $\nu = \lambda f$ holding μ-a.e. on (a,t). This can be written as

$$\mathcal{L}^1 = \gamma(t-a)\mu \qquad \text{and} \qquad \mathrm{D}u = \gamma(s-r)\mu \qquad \text{for some } \gamma \in \mathbb{R}\backslash\{0\}\,,$$

which gives $\mathrm{D}u = \frac{s-r}{t-a}\mathcal{L}^1 = \mathrm{D}u_0$. Another application of Theorem 2.24 together with $u(a-) = u_0(a-)$ yields $u = u_0$ on (a,t). We continue with considering the interval $[t,b)$ and first obtain $\mathrm{D}u([t,b))$ to equal $u(b-) - u(t-) = R - s$. Again, we can show

$$|(\mathcal{L}^1, \mathrm{D}u)|([t,b)) \geqslant \nu_1 \mathcal{L}^1([t,b)) + \nu_2 \mathrm{D}u([t,b))$$

for each unit vector $v \in \mathbb{R}^2$. This time, we specify $\nu = \frac{(b-t,R-s)}{|(b-t,R-s)|}$ and obtain

$$|(\mathcal{L}^1, \mathrm{D}u)|([t,b)) \geqslant \int_{[t,b)} \sqrt{1 + |u_0'|^2}\,\mathrm{d}x\,.$$

The equality case characterization can be concluded analogously. □

Proof of Theorem 3.83. We start with some observations on μ_-. According to Lemma 2.80, μ_- satisfies the small-volume isoperimetric condition with constant 1. Moreover, the given assumptions on f imply that μ_-

satisfies the monotonicity condition $\mu_-(B) \leqslant \mu_-(B^\star)$ for all Borel sets $B \subseteq \mathbb{R}^2$. The finiteness of μ_- on O^+ ensures the existence of a minimizer $A = A_{\min}$ in $\mathcal{A}_I^O$ by Theorem 2.83.

We first consider the case $|t| \leqslant r$ in the following. If $f(r) = 0$, the monotonicity of f implies $f = 0$ on $[-R, -r] \cup [r, R]$, such that A minimizes the perimeter in $\mathcal{A}_I^O$. Since I is convex, Lemma 2.50 implies that in this case, the inner obstacle minimizes $\mathcal{P}_-$.

Thus, we assume in the following that $f(r) > 0$. By Theorem 3.73, it holds that $\mathcal{P}_-(A^\star) \leqslant \mathcal{P}_-(A)$ and by definition of the Schwarz inequality, we have $A^\star \in \mathcal{A}_I^O$ such that $A^\star$ is a minimizer of the functional as well. The symmetrization has the advantage that we can represent it as body of revolution $\mathrm{R}^2_{\mathrm{rad}_A}$ with $\mathrm{rad}_A \in \mathrm{BV}(\mathbb{R})$ and with $r \leqslant \mathrm{rad}_A$ a.e. on $(-r, r)$ and $\mathrm{rad}_A \leqslant R$ a.e. on $(-R, R)$, compare Proposition 3.14. By Lemma 3.37, we have $((A^\star)^+)_t = \mathrm{rad}_A^+(t)$, hence the definition of μ_- yields

$$\mu_-\left((A^\star)^+\right) = \mu_-\left(K(r, \mathrm{rad}_A^+(t), t)^+\right) = \mu_-\left((A^\star \cap K(r, \mathrm{rad}_A^+(t), t))^+\right) .$$

Since $K(r, \mathrm{rad}_A^+(t), t)$ is convex and $A^\star \cap K(r, \mathrm{rad}_A^+(t), t)$ is in $\mathcal{A}_I^O$, Lemma 2.50 implies

$$\begin{aligned}
\mathcal{P}_-(A^\star) &= \mathrm{P}(A^\star) - \mu_-((A^\star)^+) \\
&= \mathrm{P}(A^\star) - \mu_-((A^\star \cap K(r, \mathrm{rad}_A^+(t), t))^+) \\
&\geqslant \mathrm{P}(A^\star \cap K(r, \mathrm{rad}_A^+(t), t)) - \mu_-((A^\star \cap K(r, \mathrm{rad}_A^+(t), t))^+) \\
&= \mathcal{P}_-(A^\star \cap K(r, \mathrm{rad}_A^+(t), t)) ,
\end{aligned}$$

with strict inequality if $|A^\star \backslash K(r, \mathrm{rad}_A^+(t), t)| > 0$. Now, we exploit the choice of $A^\star$ as a minimizer of $\mathcal{P}_-$, which implies $\mathcal{P}_-(A^\star \cap K(r, \mathrm{rad}_A^+(t), t)) \geqslant \mathcal{P}_-(A^\star)$, and it therefore holds that $|A^\star \backslash K(r, \mathrm{rad}_A^+(t), t)| = 0$. In fact, we have $\mathrm{R}^2_{\mathrm{rad}_A} = A^\star \subseteq K(r, \mathrm{rad}_A^+(t), t) = \mathrm{R}^2_\gamma$ up to null sets, where

$$\gamma(x) := \begin{cases} r + \frac{\mathrm{rad}_A^+(t) - r}{t + r}(x + r) & \text{if } x \in (-r, t) \\ r + \frac{r - \mathrm{rad}_A^+(t)}{r - t}(x - r) & \text{if } x \in (t, r) \\ 0 & \text{else} \end{cases} \qquad \text{for } x \in \mathbb{R} .$$

In particular, $\mathrm{rad}_A \leqslant \gamma$ holds a.e. on $\mathbb{R}$.

Every radius function $w \in \mathrm{BV}(\mathbb{R})$ that satisfies $r \leqslant w$ on $(-r, r)$, $w \leqslant \gamma$ on $\mathbb{R}$ and $w^+(t) = \mathrm{rad}_A^+(t)$ generates a body of revolution in the class of admissible sets such that we can restate the minimization problem in the following way:

$$\mathrm{R}^2_{\mathrm{rad}_A} \quad \text{minimizes} \quad \mathcal{P}_- \quad \text{among} \quad \left\{\mathrm{R}^2_w : w \in \mathrm{BV}(\mathbb{R}),\, r\mathbb{1}_{(-r,r)} \leqslant w \leqslant \gamma \text{ a.e. on } \mathbb{R},\, w^+(t) = \mathrm{rad}_A^+(t)\right\} . \tag{3.85}$$

With Propositions 3.18 and 3.14, we rewrite the functional for such sets as

$$\mathcal{P}_-(\mathrm{R}^2_w) = \mathrm{P}(\mathrm{R}^2_w) - 2 f_t \mathcal{H}^1((\mathrm{R}^2_w)^+ \cap H_t) = 2\big|(\mathcal{L}^1, \mathrm{D}w)\big|(\mathbb{R}) - 2\int_{-w^+(t)}^{w^+(t)} f \,\mathrm{d}x .$$

Hence (3.85) yields the non-parametric minimization problem in the form

$$\mathrm{rad}_A \quad \text{minimizes} \quad \left\{\big|(\mathcal{L}^1, \mathrm{D}w)\big|(\mathbb{R}) : w \in \mathrm{BV}(\mathbb{R}),\, w^+(t) = \mathrm{rad}_A^+(t),\, r\mathbb{1}_{(-r,r)} \leqslant w \leqslant \gamma \text{ a.e. on } \mathbb{R}\right\} .$$

Since the assumption $r\mathbb{1}_{(-r,r)} \leqslant w \leqslant \gamma$ implies

$$\big|(\mathcal{L}^1, \mathrm{D}w)\big|(\mathbb{R}\backslash(-r, r)) = 2r = \big|(\mathcal{L}^1, \mathrm{D}\mathrm{rad}_A)\big|(\mathbb{R}\backslash(-r, r)) ,$$

we can apply Lemma 3.84, obtaining $\mathrm{rad}_A = \gamma$ on $\mathbb{R}$. It is left to show the equality $\mathrm{rad}_A^+(t) = \min\{s_f^r(t), R\}$, i.e., $A^\star = K(r, \min\{s_f^r(t), R\}, t)$. For fixed $s \in [r, R]$, we compute

$$\mathcal{P}_-(K(r, s, t)) = 4r + 2\left[\sqrt{(r - t)^2 + (s - r)^2} + \sqrt{(r + t)^2 + (s - r)^2}\right] - 4\int_0^s f \,\mathrm{d}x ,$$

compare Figure 3.13. The latter can be verified as convex in s and to have a critical point at $s = s_f^r(t)$. Moreover, $\mathcal{P}_-(K(r,s,t))$ is strictly decreasing if $s < s_f^r(t)$ and strictly increasing if $s > s_f^r(t)$, since $f(r) > 0$. Hence, $A^\star$ equals the set $K(r, \min\{s_f^r(t), R\}, t)$. Applying Theorem 3.77 provides

$$\mathrm{P}(A^\star, \mathbb{R}^2 \backslash H_t) = \mathrm{P}(A, \mathbb{R}^2 \backslash H_t)\,.$$

Then, Theorem 3.57 implies

$$A \cap \mathbb{R} \times (t,\infty) = A^\star \cap \mathbb{R} \times (t,\infty) + (x,0) \qquad \text{and} \qquad A \cap \mathbb{R} \times (-\infty,t) = A^\star \cap \mathbb{R} \times (-\infty,t) + (y,0) \qquad \text{for } x,y \in \mathbb{R}\,.$$

The inner obstacle condition gives $[-r,r] \subseteq (A^\star)_z + (y,0)$ and $[-r,r] \subseteq (A^\star)_z + (x,0)$ for all $z \in (-r,r)$, which yields $x = y = 0$ and therefore, $A = A^\star = K(r, \min\{s_f^r(t), R\}, t)$.
We continue with the case $|t| \geqslant r$. We can argue as in the first part of the proof, showing $A^\star = A = K(r, \mathrm{rad}_A^+(t), t)$ if $\mathrm{rad}_A^+(t) > 0$ or $A = I$ if $\mathrm{rad}_A^+(t) = 0$. In order to specify the cases, we compute

$$\mathcal{P}_-[K(r,s,t)] = \begin{cases} 2s + 2r + 2\sqrt{(s-r)^2 + (|t|+r)^2} - 4\int_0^s f \,\mathrm{d}x & \text{if } s > r\,, \\ 6r + 2|t| - 4\int_0^r f \,\mathrm{d}x & \text{if } s = r\,, \\ 2s + 2r + 4r + 2\sqrt{(s-r)^2 + (|t|-r)^2} - 4\int_0^s f \,\mathrm{d}x & \text{if } s \in [0,r)\,, \end{cases}$$

compare again Figure 3.13. We record that the right hand side is convex in s by the monotonicity of f. If $2f(0) > 1 - \frac{r}{\sqrt{r^2 + (|t|-r)^2}}$, then $s = s_f^r(t)$ is a critical point of the function above, and for $s < s_f^r(t)$, the functional $\mathcal{P}_-[K(r,s,t)]$ is strictly decreasing, whereas $\mathcal{P}_-[K(r,s,t)]$ is strictly increasing for $s > s_f^r(t)$. This yields in fact $\mathrm{rad}_A^+(t) = \min\{s_f^r(t), R\}$, as above. If $2f(0) \leqslant 1 - \frac{r}{\sqrt{r^2+(|t|-r)^2}}$, then $\mathcal{P}_-[K(r,s,t)]$ is strictly increasing for all $s \geqslant 0$, giving $\mathrm{rad}_A^+(t) = 0$ and $A = I$ by Lemma 2.50. □

In the following, we deepen the planar model case analysis and consider discs as obstacles. The argument for the shape of the minimizing set essentially carries over, but the convex hull sets get more subtle such that a simplification ensures the reasoning not getting too extensive. In fact, we assume that f is constant from now on.

Notation. Let $r, s > 0$ and $t \in \mathbb{R}$. The two-dimensional convex hull of $\mathrm{B}_r^2 \cup \{(-s,t),(s,t)\}$ is denoted by $C(r,s,t)$.
Moreover, if $(s,t) \notin \mathrm{B}_r^2$ and $|t| < r$, then we set

$$\rho_u^s = \rho_u^s(r,t) := \arccos\left(\mathrm{sgn}(s-b)\frac{\sqrt{r^2-b^2}}{r}\right) \in [0,\pi) \quad \text{with} \quad b := \frac{r}{s^2+t^2}\left(rs - t\sqrt{s^2+t^2-r^2}\right) \in (0,r]\,, \tag{3.86}$$

$$\rho_\ell^s = \rho_\ell^s(r,t) := \arccos\left(\mathrm{sgn}(s-a)\frac{\sqrt{r^2-a^2}}{r}\right) \in [0,\pi) \quad \text{with} \quad a := \frac{r}{s^2+t^2}\left(rs + t\sqrt{s^2+t^2-r^2}\right) \in (0,r]\,. \tag{3.87}$$

If $(s,t) \notin \mathrm{B}_r^2$ with $|t| \geqslant r$, we define

$$\rho^s = \rho^s(r,t) := \arccos\left(\mathrm{sgn}(s-c)\frac{\sqrt{r^2-c^2}}{r}\right) \in [0,\pi) \quad \text{with} \quad c := \frac{r}{s^2+t^2}\left(rs + |t|\sqrt{s^2+t^2-r^2}\right) \in (0,r]\,. \tag{3.88}$$

The interpretation of the quantities is as follows: If $|t| < r$, then $\partial C(r,s,t)$ consists of two circular arcs and two symmetric corners at $(\pm s, t)$, each created by lines extending the circle smoothly. The upper lines meet $(0,t) + (1,0)\mathbb{R}$ with the angle ρ_u^s and the lower lines meet the affine subspace with the angle ρ_ℓ^s. The vectors $(b, \mathrm{sgn}(s-b)\sqrt{r^2-b^2})$ and $(a, -\mathrm{sgn}(s-a)\sqrt{r^2-a^2})$ are the unique points with positive first coordinate, where the disc and the upper and lower lines meet, respectively.
In the case $|t| \geqslant r$, there is only one line (reflected by the middle axis) extending the circle and meeting $(0,t) + (1,0)\mathbb{R}$ with angle ρ^s.

Remark 3.85. In order to suitably define $s_f^r(t)$ in this case, we need the following properties:

i) The set $C(r,s,t)$ has C^1 boundary except for the points $\{(-s,t),(s,t)\}$ for all $r, s > 0$, $t \in \mathbb{R}$.

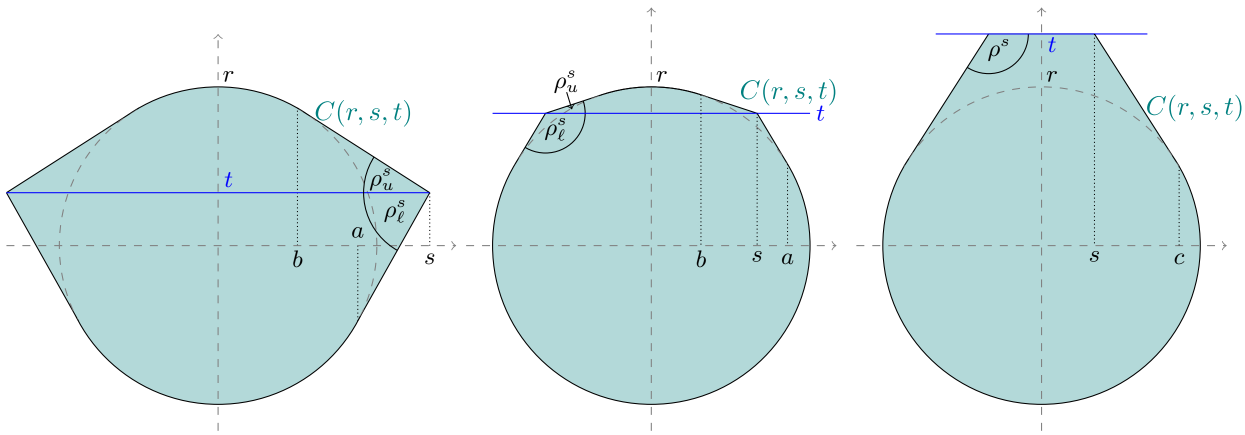


Figure 3.14: Illustration of the set $C(r,s,t)$ for different t and s

ii) If $|t| < r$, the functions $\cos(\rho_u^s)$, $\cos(\rho_\ell^s)$ are defined for $s \in [\sqrt{r^2-t^2}, \infty)$. Furthermore, both functions are continuous on the corresponding domain, and are even C^1 on $(\sqrt{r^2-t^2}, \infty)\backslash\{r\}$. If $|t| \geqslant r$, the same is true for $\cos(\rho^s)$, defined for $s \in [0, \infty)$.

Indeed, $b = s$ is satisfied if and only if $s = r$ and $t < 0$ or $s = \sqrt{r^2-t^2}$ and, similarly, $a = s$ holds if and only if $s = r$ and $t > 0$ or $s = \sqrt{r^2-t^2}$. In particular, if $t \in (-r, r)$, then $a = b = s$ is equivalent to $s = \sqrt{r^2-t^2}$, and in this case, we have $\cos(\rho_u^s) + \cos(\rho_\ell^s) = 0$.

Furthermore, $c = s$ is equivalent to $s = r$ independently of the sign of t.

iii) All three of the functions $\cos(\rho_u^s)$, $\cos(\rho_\ell^s)$, and $\cos(\rho^s)$ are strictly increasing with respect to s.

To see this, we consider first the case $|t| < r$ and assume w.l.o.g. that $t \geqslant 0$. Then we can compute

$$\begin{aligned}\frac{\mathrm{d}b}{\mathrm{d}s} &= -\frac{2sr}{(s^2+t^2)^2}\left(rs - t\sqrt{s^2+t^2-r^2}\right) + \frac{r}{s^2+t^2}\left(r - st\frac{1}{\sqrt{s^2+t^2-r^2}}\right) \\ &= \frac{r}{(s^2+t^2)^2\sqrt{s^2+t^2-r^2}}\left[rt^2\sqrt{s^2+t^2-r^2} - str^2 + \left(t\sqrt{s^2+t^2-r^2} - rs\right)s\sqrt{s^2+t^2-r^2}\right]\end{aligned}$$

for all $s \in (\sqrt{r^2-t^2}, \infty)\backslash\{r\}$. We exploit the fact that $t < r$ implies $\sqrt{s^2+t^2-r^2} < s$, which gives

$$\frac{\mathrm{d}b}{\mathrm{d}s} < \frac{r}{(s^2+t^2)^2\sqrt{s^2+t^2-r^2}}\left[rst(t-r) + (t-r)\,s^2\sqrt{s^2+t^2-r^2}\right] < 0\,.$$

Hence, b is strictly decreasing on $(\sqrt{r^2-t^2}, \infty)$. Furthermore, as $b = s$ if $s = \sqrt{r^2-t^2}$, we have $\operatorname{sgn}(s-b) \geqslant 0$ for all $s \geqslant (\sqrt{r^2-t^2}, \infty)$, which implies that $\cos(\rho_u^s) = \operatorname{sgn}(s-b)\frac{\sqrt{r^2-b^2}}{r}$ is increasing with respect to s.

Still in the case $t \in (0, r)$, the derivative

$$\frac{\mathrm{d}a}{\mathrm{d}s} = \frac{r}{(s^2+t^2)^2\sqrt{s^2+t^2-r^2}}\left[str^2 + \left(r(t^2-s^2) - st\sqrt{s^2+t^2-r^2}\right)\sqrt{s^2+t^2-r^2}\right], \tag{3.89}$$

together with the fact that $\sqrt{s^2+t^2-r^2} < t$ if $s < r$, gives

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}s} &> \frac{r}{(s^2+t^2)^2\sqrt{s^2+t^2-r^2}}\left[str^2 + \left((r-s)t^2 - rs^2\right)\sqrt{s^2+t^2-r^2}\right] \\ &> \frac{r}{(s^2+t^2)^2\sqrt{s^2+t^2-r^2}}\left[str^2 - rs^2\sqrt{s^2+t^2-r^2}\right] \\ &> \frac{r}{(s^2+t^2)^2\sqrt{s^2+t^2-r^2}}\left[str(r-s)\right] > 0\,.\end{aligned}$$

If $s > r$, (3.89) and $\sqrt{s^2+t^2-r^2} > t$ yield

$$\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}s} &< \frac{r}{(s^2+t^2)^2\sqrt{s^2+t^2-r^2}}\left[str^2 + \big((r-s)t^2 - rs^2\big)\sqrt{s^2+t^2-r^2}\right]\\
&< \frac{r}{(s^2+t^2)^2\sqrt{s^2+t^2-r^2}}\left[str^2 - rs^2t + (r-s)t^2\sqrt{s^2+t^2-r^2}\right]\\
&= \frac{r}{(s^2+t^2)^2\sqrt{s^2+t^2-r^2}}(r-s)\left[str + t^2\sqrt{s^2+t^2-r^2}\right] < 0\,.
\end{aligned}$$

Hence, a is strictly increasing on $(\sqrt{r^2-t^2}, r)$ and strictly decreasing on (r,∞). According to the analysis in ii) it holds that $\operatorname{sgn}(s-a) = 1$ if and only if $s > r$ and $\operatorname{sgn}(s-a) = -1$ if and only if $s < r$. Thus, $\cos(\rho_\ell^s) = \operatorname{sgn}(s-a)\frac{\sqrt{r^2-a^2}}{r}$ is strictly increasing with respect to s.

For c, we argue analogously that it is strictly increasing on $(0,r)$ and strictly decreasing on (r,∞), which implies that $\cos(\rho^s)$ is strictly increasing in s by ii).

iv) For $f = f(s) \in \{\cos(\rho_u^s), \cos(\rho_\ell^s), \cos(\rho^s)\}$, it holds that $f(s) \to 1$ as $s \to \infty$.

v) It holds that $\cos(\rho^0) = -\frac{r}{|t|}$.

Notation. Let $t \in \mathbb{R}$, $R > r > 0$ and $\theta \in (0,1)$. If $|t| < r$, then we denote by $s_\theta^r(t)$ the unique value $s \in (\sqrt{r^2-t^2}, \infty)$ with $\cos\rho_\ell^s + \cos\rho_u^s = 2\theta$. If $|t| \geqslant r$ and $\theta \geqslant \frac{1}{2} - \frac{r}{2|t|}$, the notion $s_\theta^r(t)$ represents the unique value in $s \in (0,\infty)$ with $\cos\rho^s = 2\theta - 1$. In some contexts, we will abbreviate $C_\theta := C(r, \min\{s_\theta^r(t), \sqrt{R^2-t^2}\}, t)$. We furthermore define the function

$$f_{r,\theta,t}(s) := 2s(1-2\theta) + 2\int_{-r}^{-\operatorname{sgn}(s-c)\sqrt{r^2-c^2}} \frac{r}{\sqrt{r^2-x^2}}\,\mathrm{d}x + 2\sqrt{(s-c)^2 + \big(|t| + \operatorname{sgn}(s-c)\sqrt{r^2-c^2}\big)^2} - 2\pi r$$

for $s \in [0,\infty)$, where c is defined as in (3.88).

Theorem 3.86 (The planar obstacle problem with discs as obstacles)**.** *Let $r < R$ in $(0,\infty]$ and $\theta \in (0,1)$. For $t \in (-R,R)$, we define the measure $\mu_- = 2\theta\mathcal{H}^1 \llcorner \mathbb{R}\times\{t\}$. Then there exists a minimizer $A_{\min}$ of the functional $\mathcal{P}_-$ in the admissible class $\mathcal{A}_I^O$ with the obstacles $I := \mathrm{B}_r^2$ and $O := \mathrm{B}_R^2$. In particular,*

i) *if $|t| < r$, then $A_{\min} = C_\theta$ and the minimizer is unique,*

ii) *if $|t| \geqslant r$ and $\theta \leqslant \frac{1}{2} - \frac{r}{2|t|}$, then $A_{\min} = \mathrm{B}_r^2$ and the minimizer is unique,*

iii) *if $|t| \geqslant r$ and $\theta > \frac{1}{2} - \frac{r}{2|t|}$, then $A_{\min} \in \{C_\theta, \mathrm{B}_r^2\}$. In fact, if*

$$f_{r,\theta,t}\left(\min\left\{s_\theta^r(t), \sqrt{R^2-t^2}\right\}\right)\begin{cases} <0, & \text{then } C_\theta \text{ uniquely minimizes } \mathcal{P}_-\,,\\ =0, & \text{then both } C_\theta \text{ and } \mathrm{B}_r \text{ minimize } \mathcal{P}_-\,,\\ >0, & \text{then } \mathrm{B}_r \text{ uniquely minimizes } \mathcal{P}_-\,.\end{cases}$$

If $\theta = 1$ and $R < \infty$, then the preceding applies with C_θ replaced by $C\big(r, \sqrt{R^2-t^2}, t\big)$. If $\theta = 1$ and $R = \infty$, no minimizer exists in any case. Moreover, if $\theta = 0$, then B_r^2 is the unique minimizer of $\mathcal{P}_-$ in $\mathcal{A}_I^O$.

Proof. We first notice that, according to the isoperimetric inequality, the inner obstacle B_r^2 is the unique minimizer of $\mathcal{P}_-$ if $\theta = 0$. In the general case, we observe that the measure μ_- satisfies the small-volume isoperimetric inequality with constant θ according to Lemma 2.80. By the definition of Schwarz symmetrization, we also confirm that μ_- satisfies the monotonicity condition (3.62). Furthermore, if $R < \infty$ or $\theta < 1$, Theorem 2.83 ensures the existence of a minimizer A of $\mathcal{P}_-$. By Theorem 3.73 and the symmetry of the obstacles, we can further conclude that $A^\star$ is a minimizer as well. In the following, we consider the case $\theta \in (0,1]$ and $R < \infty$ and distinguish between the absolute values of t.

Case $|t| < r$: Since $A^\star \in \mathcal{A}_I^O$, Lemma 3.37 implies $((A^\star)^+)_t = (-\mathrm{rad}_A^+(t), \mathrm{rad}_A^+(t))$ up to $\mathcal{L}^1$-null sets with $\mathrm{rad}_A^+(t) \in [\sqrt{r^2-t^2}, \sqrt{R^2-t^2}]$. We then consider the convex set $C := C(r, \mathrm{rad}_A^+(y), t)$ and obtain $\mu_-(A^\star) =$

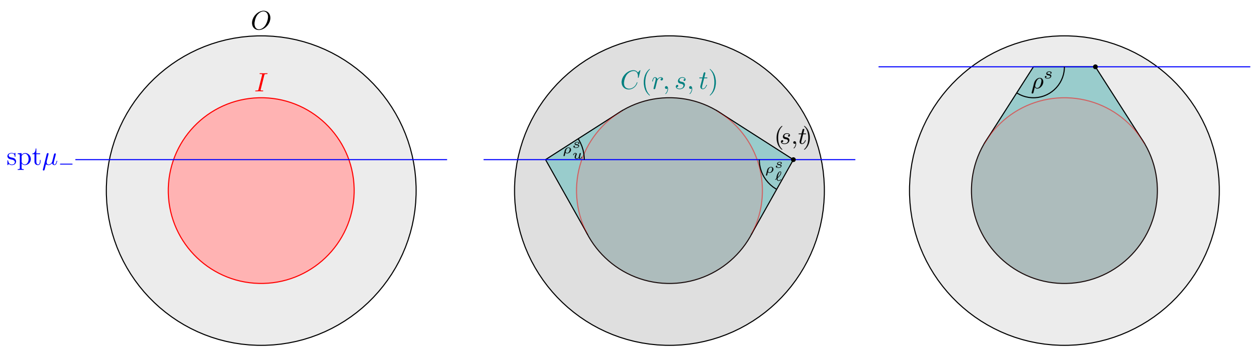


Figure 3.15: Obstacle problem with discs in $\mathbb{R}^2$ and different t

$\mu_-(C)$ by definition. Together with Lemma 2.50, this implies $A^\star \subseteq C$. By the symmetry of C, we can rewrite the set as $\mathrm{R}^2_{\mathrm{rad}_C}$ and observe $\mathrm{rad}_{\mathrm{B}^2_r} \leqslant \mathrm{rad}_A \leqslant \mathrm{rad}_C$ a.e. on $\mathbb{R}$ with $\mathrm{rad}_A = \mathrm{rad}_C = 0$ on $\mathbb{R}\backslash(-r,r)$. Propositions 3.18 and 3.14 ensure that the radius function rad_A is in $\mathrm{BV}(\mathbb{R})$. In fact, for every $w \in \mathrm{BV}(\mathbb{R})$ satisfying $\mathrm{rad}_{\mathrm{B}^2_r} \leqslant w \leqslant \mathrm{rad}_C$ a.e. on $\mathbb{R}$, we obtain $\mathrm{R}^2_w \in \mathcal{A}^O_I$ with

$$\mathcal{P}_-(\mathrm{R}^2_w) = 2|(\mathcal{L}^1, \mathrm{D}w)|(\mathbb{R}) - 4\theta|w^+(t)|\,.$$

Similar to the proof of Theorem 3.83, we reduce the minimization problem to the following non-parametric one:

$$\mathrm{rad}_A \qquad \text{minimizes} \qquad \left\{|(\mathcal{L}^1, \mathrm{D}w)|(\mathbb{R}) \,:\, w \in \mathrm{BV}(\mathbb{R}),\, \mathrm{rad}_{\mathrm{B}^2_r} \leqslant w \leqslant \mathrm{rad}_{\mathrm{B}^2_R},\, w^+(t) = \mathrm{rad}^+_A(t)\right\}.$$

In the following, we define

$$b := \frac{r}{\mathrm{rad}^+_A(t)^2 + t^2}\left(r\,\mathrm{rad}^+_A(t) - t\sqrt{\mathrm{rad}^+_A(t)^2 + t^2 - r^2}\right) \in (0,r]$$
$$a := \frac{r}{\mathrm{rad}^+_A(t)^2 + t^2}\left(r\,\mathrm{rad}^+_A(t) + t\sqrt{\mathrm{rad}^+_A(t)^2 + t^2 - r^2}\right) \in (0,r]\,.$$

We obtain $-\mathrm{sgn}(\mathrm{rad}^+_A(t) - a)\sqrt{r^2-a^2} < t < \mathrm{sgn}(\mathrm{rad}^+_A(t) - b)\sqrt{r^2-b^2}$ by the monotonicity analysis from Remark 3.85. Furthermore, a and b are the unique positive values such that the lower and upper contact points of $\partial\mathrm{B}^2_r$ and ∂C are $(\pm b, \mathrm{sgn}(\mathrm{rad}^+_A(t)-b)\sqrt{r^2-b^2})$ and $(\pm a, -\mathrm{sgn}(\mathrm{rad}^+_A(t)-a)\sqrt{r^2-a^2})$, respectively. An application of Lemma 3.84 then gives $A^\star = C$. In fact, $\mathrm{rad}^+_A = \mathrm{rad}^-_A$ is C^1 and equals rad_C on $\mathbb{R}\backslash(-\mathrm{sgn}(\mathrm{rad}^+_A(t)-a)\sqrt{r^2-a^2}, \mathrm{sgn}(\mathrm{rad}^+_A(t)-b)\sqrt{r^2-b^2})$. At this point, we can argue for $A = A^\star = C$ as in the proof of Theorem 3.83, where we exploit the symmetry of the obstacles, Theorem 3.77 and Theorem 3.57.
In order to fully show the claim in this case, we have to prove $\mathrm{rad}^+_A(t) = \min\{s^r_\theta(t), \sqrt{R^2-t^2}\}$. To this end, we first assume $\sqrt{r^2-t^2} < \mathrm{rad}^+_A(t) < \sqrt{R^2-t^2}$ and compute the Euler equation. From the previous argument, we know that rad_A minimizes $\left\{|(\mathcal{L}^1, \mathrm{D}w)|(\mathbb{R}) - 2\theta w^+(t) \,:\, w \in \mathrm{BV}(\mathbb{R}),\, \mathrm{rad}_{\mathrm{B}^2_r} \leqslant w \leqslant \mathrm{rad}_{\mathrm{B}^2_R}\right\}$, which gives

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0} \left|\left(\mathcal{L}^1, \mathrm{D}\mathrm{rad}_A + \tau\mathrm{D}\varphi\right)\right|\Big(\big(-\mathrm{sgn}(\mathrm{rad}^+_A(t)-a)\sqrt{r^2-a^2}, \mathrm{sgn}(\mathrm{rad}^+_A(t)-b)\sqrt{r^2-b^2}\big)\Big) - 2\theta(\mathrm{rad}_A + \tau\varphi)^+(t)\\ &= \int_{-\mathrm{sgn}(\mathrm{rad}^+_A(t)-a)\sqrt{r^2-a^2}}^{\mathrm{sgn}(\mathrm{rad}^+_A(t)-b)\sqrt{r^2-b^2}} \frac{\mathrm{rad}'_A\,\varphi'}{\sqrt{1+(\mathrm{rad}'_A)^2}}\,\mathrm{d}x - 2\theta\varphi(t)\end{aligned}$$

for all $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}\big(\big(-\mathrm{sgn}(\mathrm{rad}^+_A(t)-a)\sqrt{r^2-a^2}, \mathrm{sgn}(\mathrm{rad}^+_A(t)-b)\sqrt{r^2-b^2}\big)\big)$, where we used the fact hat $A = C$ implies $\mathrm{rad}_A \in \mathrm{W}^{1,1}(\mathbb{R})$. This implies that rad'_A is constant on $(-\mathrm{sgn}(\mathrm{rad}^+_A(t)-a)\sqrt{r^2-a^2}, t)$ and on $(t, \mathrm{sgn}(\mathrm{rad}^+_A(t)-b)\sqrt{r^2-b^2})$, compare Figure 3.14. In the following, we denote the values of the derivative of the radius function on those intervals by rad_a and rad_b. We continue the previous computation with

$$\begin{aligned} 0 &= \int_{-\operatorname{sgn}(\mathrm{rad}_A^+(t)-a)\sqrt{r^2-a^2}}^{t} \frac{\mathrm{rad}_a}{\sqrt{1+(\mathrm{rad}_a)^2}}\varphi'\,\mathrm{d}x + \int_t^{\operatorname{sgn}(\mathrm{rad}_A^+(t)-b)\sqrt{r^2-b^2}} \frac{\mathrm{rad}_b}{\sqrt{1+(\mathrm{rad}_b)^2}}\varphi'\,\mathrm{d}x - 2\theta\varphi(t) \\ &= \left(\frac{\mathrm{rad}_a}{\sqrt{1+(\mathrm{rad}_a)^2}} - \frac{\mathrm{rad}_b}{\sqrt{1+(\mathrm{rad}_b)^2}} - 2\theta\right)\varphi(t) \end{aligned}$$

for all $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}\big(\big(-\operatorname{sgn}(\mathrm{rad}_A^+(t)-a)\sqrt{r^2-a^2}, \operatorname{sgn}(\mathrm{rad}_A^+(t)-b)\sqrt{r^2-b^2}\big)\big)$. Hence, it holds

$$\frac{\mathrm{rad}_a}{\sqrt{1+(\mathrm{rad}_a)^2}} - \frac{\mathrm{rad}_b}{\sqrt{1+(\mathrm{rad}_b)^2}} = 2\theta\,.$$

In order to connect this with the introduced notation, we notice that

$$\mathrm{rad}_a = \frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=-\operatorname{sgn}(\mathrm{rad}_A^+(t)-a)\sqrt{r^2-a^2}} \sqrt{r^2-x^2} = \operatorname{sgn}(\mathrm{rad}_A^+(t)-a)\frac{\sqrt{r^2-a^2}}{a}$$

and

$$\mathrm{rad}_b = \frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=\operatorname{sgn}(\mathrm{rad}_A^+(t)-b)\sqrt{r^2-b^2}} \sqrt{r^2-x^2} = -\operatorname{sgn}(\mathrm{rad}_A^+(t)-b)\frac{\sqrt{r^2-b^2}}{b}$$

results in

$$\operatorname{sgn}(\mathrm{rad}_A^+(t)-a)\frac{\sqrt{r^2-a^2}}{r} + \operatorname{sgn}(\mathrm{rad}_A^+(t)-b)\frac{\sqrt{r^2-b^2}}{r} = 2\theta\,.$$

In fact, we have

$$\cos\rho_\ell^{\mathrm{rad}_A^+(t)} + \cos\rho_u^{\mathrm{rad}_A^+(t)} = 2\theta\,,$$

which implies $\mathrm{rad}_A^+(t) = s_\theta^r(t)$.
We now consider the case $\mathrm{rad}_A^+(t) = \sqrt{R^2-t^2}$ and only take variations from the left side into account, i.e., for $0 \geqslant \varphi \in \mathrm{C}^1_{\mathrm{cpt}}\big(\big(-\operatorname{sgn}(\mathrm{rad}_A^+(t)-a)\sqrt{r^2-a^2}, \sqrt{r^2-b^2}\big)\big)$, we determine

$$0 \leqslant \left(\frac{\mathrm{rad}_a}{\sqrt{1+(\mathrm{rad}_a)^2}} - \frac{\mathrm{rad}_b}{\sqrt{1+(\mathrm{rad}_b)^2}} - 2\theta\right)\varphi(t)$$

and

$$\cos\rho_\ell^{\mathrm{rad}_A^+(t)} + \cos\rho_u^{\mathrm{rad}_A^+(t)} \leqslant 2\theta$$

as before. By the monotonicity of the left term from Remark 3.85, we conclude $\mathrm{rad}_A^+(t) = \min\{s_\theta^r(t), \sqrt{R^2-t^2}\}$ also in this case.
For $\mathrm{rad}_A^+(t) = \sqrt{r^2-t^2}$, we can argue $A = \mathrm{B}_r^2$ by Lemma 2.50 and obtain $\mathrm{rad}_A(x) = \sqrt{(r^2-x^2)_+}$. Hence, for each $0 \leqslant \varphi \in \mathrm{C}^1_{\mathrm{cpt}}((-r,r))$, it holds that

$$0 \leqslant \int_{-r}^{r} \frac{\mathrm{rad}_A'\,\varphi'}{\sqrt{1+(\mathrm{rad}_A')^2}}\,\mathrm{d}x - 2\theta\varphi(t) = \frac{1}{r}\int_{-r}^{r}\varphi\,\mathrm{d}x - 2\theta\varphi(t)\,.$$

We now choose $0 \leqslant \varphi \in \mathrm{C}^1_{\mathrm{cpt}}(t-\varepsilon, t+\varepsilon)$ for some small $\varepsilon > 0$ such that φ has a maximum at t with positive value, which leads to $0 < \frac{2}{r}\varepsilon\varphi(t) - 2\theta\varphi(t)$. Hence, we have $0 < \frac{1}{r}\varepsilon - \theta$, which is a contradiction to $\theta > 0$ as $\varepsilon \searrow 0$.

Case $|t| \geqslant r$: As before, we record that $A^\star$ is also a minimizer and first consider the case $\mathrm{rad}_A^+(t) \in (0, \sqrt{R^2-t^2})$. Moreover, we have $A \subseteq C := C(r, \mathrm{rad}_A^+(t), t)$ again and conclude analogously that

$$\mathrm{rad}_A \qquad \text{minimizes} \qquad \Big\{|(\mathcal{L}^1, \mathrm{D}w)|(\mathbb{R}) \,:\, w \in \mathrm{BV}(\mathbb{R}),\ \mathrm{rad}_{\mathrm{B}_r^2} \leqslant w \leqslant \mathrm{rad}_{\mathrm{B}_R^2},\ w^+(t) = \mathrm{rad}_A^+(t)\Big\}\,.$$

We define the unique positive value

$$c := \frac{r}{(\mathrm{rad}_A^+(t))^2 + t^2}\left(r\,\mathrm{rad}_A^+(t) + |t|\sqrt{(\mathrm{rad}_A^+(t))^2 + t^2 - r^2}\right),$$

such that $(\pm c, -\mathrm{sgn}(\mathrm{rad}_A^+(t) - c)\sqrt{r^2 - c^2})$ are the contact points of ∂C and B_r^2, compare Figure 3.14. The relation $A^\star \subseteq C$ implies $\mathrm{rad}_A(t-) = \mathrm{rad}_A^+(t)$ if $t > 0$ and $\mathrm{rad}_A(t+) = \mathrm{rad}_A^+(t)$ if $t < 0$. In both cases, Lemma 3.84 implies $A^\star = C$ and therefore rad_A is in $\mathrm{W}^{1,1}(\mathbb{R}\backslash\{t\})$ with only one jump at t from $\mathrm{rad}_A^+(t)$ to 0, as illustrated in Figure 3.14. As before, we obtain $A = A^\star = C$. It is left to determine the exact value of $\mathrm{rad}_A^+(t)$ in this case.
To this end, we assume w.l.o.g. $t > 0$. Computing the first variation yields

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0} \left|(\mathcal{L}^1, \mathrm{D}\mathrm{rad}_A + \tau \mathrm{D}\varphi)\right|\left(\left(-\mathrm{sgn}(\mathrm{rad}_A^+(t) - c)\sqrt{r^2 - c^2}, t\right]\right) - 2\theta(\mathrm{rad}_A + \tau\varphi)^+(t)\\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0} \left|(\mathcal{L}^1, \mathrm{D}\mathrm{rad}_A + \tau \mathrm{D}\varphi)\right|\left(\left(-\mathrm{sgn}(\mathrm{rad}_A^+(t) - c)\sqrt{r^2 - c^2}, t\right)\right) + (1-2\theta)(\mathrm{rad}_A + \tau\varphi)^+(t)\\
&= \int_{-\mathrm{sgn}(\mathrm{rad}_A^+(t)-c)\sqrt{r^2-c^2}}^{t} \frac{\mathrm{rad}_A'\,\varphi'}{\sqrt{1+(\mathrm{rad}_A')^2}}\,\mathrm{d}x + (1-2\theta)\varphi(t-)
\end{aligned}$$

for $\varphi := \psi \mathbb{1}_{(-\mathrm{sgn}(\mathrm{rad}_A^+(t)-c)\sqrt{r^2-c^2}, t)}$ for all $\psi \in \mathrm{C}^1\left(\left(-\mathrm{sgn}(\mathrm{rad}_A^+(t)-c)\sqrt{r^2-c^2}, \frac{R+t}{2}\right)\right)$. Again, we notice that rad_A' is constantly $\mathrm{rad}_c = \mathrm{sgn}(\mathrm{rad}_A^+(t) - c)\frac{\sqrt{r^2-c^2}}{c}$ on $(-\mathrm{sgn}(\mathrm{rad}_A^+(t) - c)\sqrt{r^2 - c^2}, t)$, hence

$$\cos\rho^{\mathrm{rad}_A^+(t)} = \frac{\mathrm{rad}_c}{\sqrt{1+\mathrm{rad}_c^2}} = 2\theta - 1$$

follows. By the monotonicity of $\cos\rho^s$, it follows $\mathrm{rad}_A^+(t) = s_\theta^r(t)$.
We continue with considering the limit cases. If $\mathrm{rad}_A^+(t) = 0$, Lemma 2.50 implies $A = \mathrm{B}_r^2$. If $\mathrm{rad}_A^+(t) = \sqrt{R^2 - t^2}$, taking the one-sided variation yields with the same reasoning

$$\cos\rho^{\mathrm{rad}_A^+(t)} \leqslant 2\theta - 1\,,$$

hence follows $s_\theta^r(t) \geqslant \sqrt{R^2 - t^2} = \mathrm{rad}_A^+(t)$ in this case.
We have proven that the only competitors are C_θ and B_r^2. If, moreover, θ is in $\left[0, \frac{1}{2} - \frac{r}{2|t|}\right]$ and $\mathrm{rad}_A^+(t) > 0$, then Remark 3.85 implies

$$\cos\rho^{\mathrm{rad}_A^+(t)} > \cos\rho^0 = -\frac{r}{|t|} \geqslant 2\theta - 1\,,$$

in contradiction to the previous analysis. In the remaining case, one has to check whether the functional $\mathcal{P}_-$ is minimized by C_θ or B_r^2, i.e., the relation of

$$\begin{aligned}
&\mathcal{P}_-(\mathrm{B}_r^2) = 2\pi r \qquad \text{and}\\
&\mathcal{P}_-(C_\theta) = 2s(1-2\theta) + 2\int_{-r}^{-\mathrm{sgn}(s-c)\sqrt{r^2-c^2}} \frac{r}{\sqrt{r^2-x^2}}\,\mathrm{d}x + \sqrt{(s-c)^2 + \left(|t| - \mathrm{sgn}(r-s)\sqrt{r^2-c^2}\right)^2}\,,
\end{aligned}$$

where $s = \min\{s_\theta^r(t), \sqrt{R^2 - t^2}\}$, which gives the claim in this case.
It remains to consider the case $R = \infty$ and $\theta \in (0, 1]$. If $\theta < 1$, we can argue the claim analogous to the case $R < \infty$. If $\theta = 1$ and $|t| < r$, the existence theory for minimizers fails, and we conclude from the other cases that

$$\mathcal{P}_-(C(r, s_1, t)) > \mathcal{P}_-(C(r, s_2, t)) \qquad \text{for all } s_1 < s_2 \text{ in } \left(\sqrt{r^2 - t^2}, \infty\right) \tag{3.90}$$

if $|t| < r$. In fact, for every minimizer A of the problem, Lemma 2.50, Proposition 3.14 and Theorem 3.73 imply that $A^\star$ is a minimizer in $\left(-\|\mathrm{rad}_A\|_{\mathrm{L}^\infty(\mathbb{R})}, \|\mathrm{rad}_A\|_{\mathrm{L}^\infty(\mathbb{R})}\right) \times (-r, r)$. With the previous analysis, we find $A^\star = C(r, s, t)$ for some $s \in \left(\sqrt{r^2 - t^2}, \infty\right)$, which is a contradiction to (3.90).
If $\theta = 1$ and $|t| \geqslant r$, we record that $f_{r,1,t}(s)$ is decreasing for large s and that $f_{r,1,t}(s) \to -2\pi r$ as $s \to \infty$. In particular, $\mathcal{P}_-(C(r, s_1, t)) > \mathcal{P}_-(C(r, s_2, t))$ holds for large $s_1 < s_2$ and with this, we can argue the non-existence of a minimizer as before. □

We proceed with a three-dimensional obstacle problem with balls as obstacles. Due to the higher dimension, the argument becomes even more subtle, which motivates a restriction of the analysis to the case of $|t| < r$ and only state the general shape of the minimizers. As usual, we start by introducing some notation.

Theorem 3.87 (The three-dimensional obstacle problem with balls as obstacles)**.** *Let $r < R$ in $(0,\infty]$, $t \in (-r,r)$, $\theta \in [0,1]$, $I = \mathrm{B}^3_r$, $O = \mathrm{B}^3_R$ and $\mu_- = 2\theta\mathcal{H}^2 \llcorner \mathbb{R}^2 \times \{t\}$. In the special case $\theta = 1$ and $R = \infty$, no minimizer of $\mathcal{P}_-$ exist in $\mathcal{A}_I^O$. If $\theta = 0$, then the ball B^3_r is the unique minimizer of the obstacle problem. In any other case, every minimizer $A_{\min}$ of $\mathcal{P}_-$ in $\mathcal{A}_I^O$ satisfies the following properties:*
There exist some $s \in \left(\sqrt{r^2-t^2}, \sqrt{R^2-t^2}\right]$, $a \in (-r,t)$, $b \in (t,r)$, $\alpha_1, \alpha_2 \in (0,\infty)$ and $\beta_1, \beta_2 \in \mathbb{R}$ such that

$$A_{\min} = \mathrm{R}^3_w\,, \qquad \textit{where} \qquad w(x) := \begin{cases} \sqrt{(r^2-x^2)_+}\,, & \textit{if } x \in \mathbb{R}\backslash(a,b) \\ \alpha_1 \cosh\left(\frac{x-\beta_1}{\alpha_1}\right)\,, & \textit{if } x \in (a,t] \\ \alpha_2 \cosh\left(\frac{x-\beta_2}{\alpha_2}\right)\,, & \textit{if } x \in (t,b) \end{cases}$$

and the following properties hold true:

i) $s = \alpha_1 \cosh\left(\frac{t-\beta_1}{\alpha_1}\right) = \alpha_2 \cosh\left(\frac{t-\beta_2}{\alpha_2}\right)$,

ii) $2\theta s = \alpha_1 \sinh\left(\frac{t-\beta_1}{\alpha_1}\right) - \alpha_2 \sinh\left(\frac{t-\beta_2}{\alpha_2}\right)$ *or* $s = \sqrt{R^2-t^2}$,

iii) $\partial\mathrm{R}^3_w$ *is* C^1 *away from* $\mathbb{R}^2 \times \{t\}$.

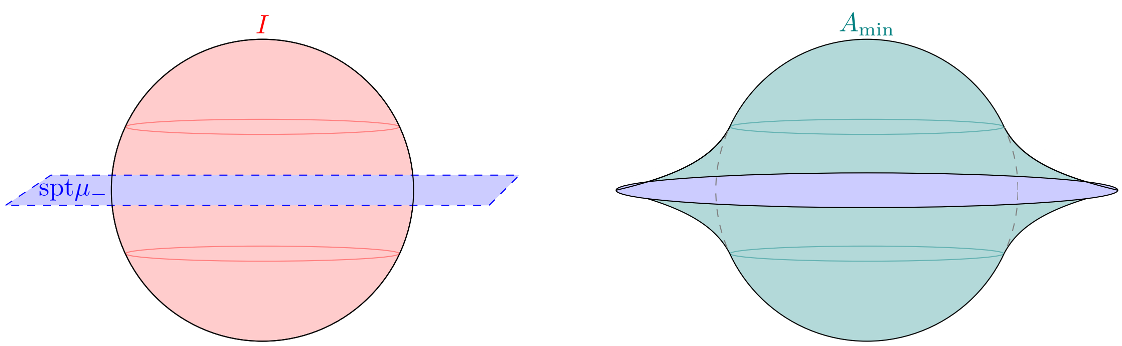


Figure 3.16: Obstacle problem with balls in $\mathbb{R}^3$

Remark 3.88. With respect to the assumptions and notation from the previous theorem, we have $\theta \in (0,1)$ for all $s, a, b, \alpha_1, \alpha_2, \beta_1, \beta_2$ such that i), ii) and iii) are satisfied. Furthermore, $\theta \to 1$ corresponds to $\frac{t-\beta_1}{\alpha_1} \to \infty$ and $\frac{t-\beta_2}{\alpha_2} \to \infty$.
To show this, we assume w.l.o.g. $t \geqslant 0$ and rearrange the formulas in i) and ii) to obtain

$$\theta = \frac{\alpha_1 \sinh\left(\frac{t-\beta_1}{\alpha_1}\right) - \alpha_2 \sinh\left(\frac{t-\beta_2}{\alpha_2}\right)}{2s} = \frac{\sinh\left(\frac{t-\beta_1}{\alpha_1}\right)}{2\sqrt{1+\left(\sinh\left(\frac{t-\beta_1}{\alpha_1}\right)\right)^2}} - \frac{\sinh\left(\frac{t-\beta_2}{\alpha_2}\right)}{2\sqrt{1+\left(\sinh\left(\frac{t-\beta_2}{\alpha_2}\right)\right)^2}}\,. \tag{3.91}$$

Taking the behavior of sinh into account, the right term is increasing with respect to $\frac{t-\beta_1}{\alpha_1}$ and decreasing with respect to $\frac{t-\beta_2}{\alpha_2}$. The strict decreasing behavior of $x \mapsto \sqrt{r^2-x^2}$ on $[0,r]$ and property iii) give

$$\alpha_2 \cosh\left(\frac{b-\beta_2}{\alpha_2}\right) = \sqrt{r^2-b^2} < \sqrt{r^2-t^2} < s = \alpha_2 \cosh\left(\frac{t-\beta_2}{\alpha_2}\right)\,.$$

Now, the fact that $\alpha \cosh\left(\frac{x-\beta}{\alpha}\right)$ is strictly decreasing on $(-\infty,\beta]$ and strictly increasing on $[\beta,\infty)$ for all $\alpha > 0$ and $\beta \in \mathbb{R}$ implies $\beta_2 > t$.

In the following, we make a case distinction with respect to s.
If $\sqrt{r^2-a^2} \leqslant s$, we can show $\beta_1 < t$ as before. Hence, in this case, it holds that $\frac{t-\beta_2}{\alpha_2} < 0 < \frac{t-\beta_1}{\alpha_1}$, leading to $\theta \in (0,1)$ according to (3.91) and the behavior of sinh.
If $s < \sqrt{r^2-a^2}$ and $\beta_1 \leqslant t$, the conclusion follows as in the previous case.
In the remaining case $s < \sqrt{r^2-a^2}$ and $\beta_1 \geqslant t > a$, property iii) implies

$$0 > \frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=a} \alpha_1 \cosh\left(\frac{x-\beta_1}{\alpha_1}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=a} \sqrt{r^2-x^2} = \frac{-a}{\sqrt{r^2-a^2}}$$

and therefore, $a > 0$. The regularity assumptions at a give

$$\sqrt{r^2-a^2} = \alpha_1 \cosh\left(\frac{a-\beta_1}{\alpha_1}\right) = \alpha_1\sqrt{1+\left(\sinh\left(\frac{a-\beta_1}{\alpha_1}\right)\right)^2} = \alpha_1\sqrt{1+\frac{a^2}{r^2-a^2}}\,,$$

which can be reformulated as $\alpha_1 = r - \frac{a^2}{r}$. Analogously, we can show $\alpha_2 = r - \frac{b^2}{r}$, which, together with $0 < a < b$, implies $\alpha_2 < \alpha_1$. Property i) yields

$$\cosh\left(\frac{t-\beta_1}{\alpha_1}\right) < \cosh\left(\frac{t-\beta_2}{\alpha_2}\right)$$

and exploiting $\beta_2 > t$ and $\beta_1 \geqslant t$ leads to $\frac{t-\beta_2}{\alpha_2} < \frac{t-\beta_1}{\alpha_1}$. We obtain

$$\sinh\left(\frac{t-\beta_2}{\alpha_2}\right) < \sinh\left(\frac{t-\beta_1}{\alpha_1}\right).$$

Finally $\theta \in (0,1)$ follows from formula (5.7) and the corresponding monotonicity analysis of the right-hand side in this case as well.

In order to keep the proof of Theorem 3.87 organized, we present the following lemma, which gives the first variation of the corresponding non-parametric obstacle problem in the most relevant cases.

Lemma 3.89 (First variations of the functional in the proof of Theorem 3.87)**.** *Let $r_1 < r_2$, $t \in (r_1, r_2)$, $\theta \in [0,1]$, $a \in (r_1, t]$ and $b \in [t, r_2)$ with $a < b$. We consider ψ_1 and ψ_2 in $\mathrm{L}^\infty((r_1,r_2)) \cap \mathrm{C}^1((r_1,r_2))$ and a function $u \in \mathrm{C}^1((r_1,r_2)\backslash\{t\}) \cap \mathrm{BV}_{\mathrm{loc}}((r_1,r_2))$ that satisfies $\psi_1 \leqslant u \leqslant \psi_2$ on (r_1,r_2). If u minimizes*

$$\mathcal{A}^{\mathrm{rev}}_{3,(r_1,r_2)}[w] - \theta(w^+(t))^2 \qquad \text{among } w \in \{v \in \mathrm{BV}_{\mathrm{loc}}((r_1,r_2)),\ \psi_1 \leqslant v \leqslant \psi_2\}\,, \tag{3.92}$$

with value in $\mathbb{R}$, then the following properties hold true.

i) *If $\psi_1 < u < \psi_2$ on (a,b) and $u^-(t) = u^+(t)$, then*

$$0 = \int_a^b \varphi\sqrt{1+(u')^2} + \varphi'\frac{u\,u'}{\sqrt{1+(u')^2}}\,\mathrm{d}x - 2\theta\varphi(t)u(t)$$

holds for all $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}((a,b))$.

ii) *If $\psi_1 < u < \psi_2$ on (a,b) and $u(t-) = u^-(t) < u^+(t) = u(t+)$, then*

$$0 = \int_a^b \varphi\sqrt{1+(u')^2} + \varphi'\frac{u\,u'}{\sqrt{1+(u')^2}}\,\mathrm{d}x + u(t+)\varphi(t+) - u(t-)\varphi(t-) - 2\theta\varphi(t+)u(t+)$$

holds for all $\varphi \in \mathrm{C}^1((a,b)\backslash\{t\}) \cap \mathrm{BV}((a,b))$ with $\mathrm{spt}(\varphi) \subseteq (a,b)$.

iii) *If $\psi_1 < u < \psi_2$ on $(a,t) \neq \emptyset$ and $\psi_1(t) < u(t-) = u^-(t) < u^+(t) \leqslant \psi_2(t)$, then*

$$0 = \int_a^t \varphi\sqrt{1+(u')^2} + \varphi'\frac{u\,u'}{\sqrt{1+(u')^2}}\,\mathrm{d}x - u(t-)\varphi(t-)$$

holds for all $\varphi \in \mathrm{C}^1((a,r_2)\backslash\{t\}) \cap \mathrm{BV}((a,r_2))$ with $\mathrm{spt}(\varphi) \subseteq (a,t]$.

iv) *If $u=\psi_1$ on (r_1,t) and $u(t-)=u^-(t)<u^+(t)$, then*

$$0 \leqslant \int_{r_1}^{t} \varphi\sqrt{1+(\psi_1')^2}+\varphi'\frac{\psi_1\,\psi_1'}{\sqrt{1+(\psi_1')^2}}\,\mathrm{d}x-\psi_1(t)\varphi(t-)$$

holds for all non-negative $\varphi\in \mathrm{C}^1((r_1,r_2)\backslash\{t\})\cap \mathrm{BV}((r_1,r_2))$ with $\mathrm{spt}(\varphi)\subseteq(r_1,t]$.

v) *If $u=\psi_1$ on (r_1,r_2), then*

$$0 \leqslant \int_{r_1}^{r_2} \varphi\sqrt{1+(\psi_1')^2}+\varphi'\left(\frac{\psi_1\,\psi_1'}{\sqrt{1+(\psi_1')^2}}\right)\mathrm{d}x-2\theta\varphi(t)\psi_1(t)$$

holds for all non-negative $\varphi\in \mathrm{C}^1_{\mathrm{cpt}}((r_1,r_2))$.

Proof. We first notice that, according to Proposition 3.14, u minimizes

$$\left|\left(w\mathcal{L}^1,\frac{1}{2}\mathrm{D}(w^2)\right)\right|(-r,r)=\int_{-r}^{r} w\sqrt{1+(w')^2}\,\mathrm{d}x+\frac{(w^+(t))^2-(w^-(t))^2}{2}-\theta(w^+(t))^2$$

among all $w\in\left\{v\in \mathrm{C}^1((-r,r)\backslash\{t\})\cap \mathrm{BV}_{\mathrm{loc}}((-r,r)),\ \psi_1\leqslant v\leqslant\psi_2\right\}$, where we consider a good representative of w. By the assumptions on u, it is easy to check that, for every interval $K\subseteq(-r,r)$ with $t\in K$, u also minimizes

$$\int_K w\sqrt{1+(w')^2}\,\mathrm{d}x+\frac{(w^+(t))^2-(w^-(t))^2}{2}-\theta(w^+(t))^2 \tag{3.93}$$

among $w=u+\varphi$ for all $\varphi\in \mathrm{C}^1(K\backslash\{t\})\cap \mathrm{BV}(K)$ with $\mathrm{spt}(\varphi)\Subset K$ and $\psi_1\leqslant w\leqslant\psi_2$ on K. With this at hand, we turn to the individual statements of the Lemma.
We first consider u and φ as in i) and record that

$$(u+\tau\varphi)^+(t)=(u+\tau\varphi)^-(t)=u(t)+\tau\varphi(t)$$

holds for all $\tau\in\mathbb{R}$ by the continuity of the functions. Since $\psi_1<u<\psi_2$ on (a,b), we can consider the two-sided variation of (3.93) with $K=(a,b)$ and compute

$$\begin{aligned}
0&=\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0}\int_a^b(u+\tau\varphi)\sqrt{1+(u'+\tau\varphi')^2}\,\mathrm{d}x+\frac{((u+\tau\varphi)^+(t))^2-((u+\tau\varphi)^-(t))^2}{2}-\theta((u+\tau\varphi)^+(t))^2\\
&=\int_a^b\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0}(u+\tau\varphi)\sqrt{1+(u'+\tau\varphi')^2}\,\mathrm{d}x-2\theta\varphi(t)u(t)\\
&=\int_a^b\varphi\sqrt{1+(u')^2}+\varphi'\frac{u\,u'}{\sqrt{1+(u')^2}}\,\mathrm{d}x-2\theta\varphi(t)u(t)\,.
\end{aligned}$$

This proves the first case.
We now take u and φ as in ii) and observe that there exists some $\varepsilon>0$ such that

$$(u+\tau\varphi)^+(t)=u(t+)+\tau\varphi(t+)\qquad\text{and}\qquad(u+\tau\varphi)^-(t)=u(t-)+\tau\varphi(t-)$$

holds for all $\tau\in(-\varepsilon,\varepsilon)$. With this, we find

$$\begin{aligned}
0&=\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0}\int_a^b(u+\tau\varphi)\sqrt{1+(u'+\tau\varphi')^2}\,\mathrm{d}x+\frac{((u+\tau\varphi)^+(t))^2-((u+\tau\varphi)^-(t))^2}{2}-\theta((u+\tau\varphi)^+(t))^2\\
&=\int_a^b\varphi\sqrt{1+(u')^2}+\varphi'\frac{u\,u'}{\sqrt{1+(u')^2}}\,\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0}\left[\frac{(u(t+)+\tau\varphi(t+))^2-(u(t-)+\tau\varphi(t-))^2}{2}-\theta(u(t+)+\tau\varphi(t+))^2\right]\\
&=\int_a^b\varphi\sqrt{1+(u')^2}+\varphi'\frac{u\,u'}{\sqrt{1+(u')^2}}\,\mathrm{d}x+u(t+)\varphi(t+)-u(t-)\varphi(t-)-2\theta\varphi(t+)u(t+)\,.
\end{aligned}$$

To show iii), we first observe that, according to the assumptions on u and φ, there exists $\varepsilon > 0$ such that $\psi_1 \leqslant u + \tau\varphi \leqslant \psi_2$ holds on (r_1, r_2) for all $\tau \in (-\varepsilon, \varepsilon)$. With this at hand, we repeat the argument of the case ii), where (3.93) is applied to $K = (a, t]$ this time.
The proof of iv) and v) is similar to the proof of i), where we consider one-sided variations rather than two-sided. □

Similar to the planar case, we need the following additional result from [65, Teorema 2] to ensure that the minimizer is manageable.

Theorem 3.90 (Regularity of perimeter minimizers of obstacle problems)**.** *Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter and let $I \subseteq \Omega$ an open set with C^1 boundary in Ω. If A is a local perimeter minimizer in Ω among sets in $\mathcal{A}_I^{\mathbb{R}^n}$, i.e., if for all measurable sets $F \in \mathcal{A}_I^{\mathbb{R}^n}$ with $A \triangle F \Subset \Omega$, it holds $\mathrm{P}(A, \Omega) \leqslant \mathrm{P}(F, \Omega)$, then there exists a neighborhood of $\partial I \cap \Omega$ such that $\partial^* A$ is of class C^1 in this neighborhood.*

Proof of Theorem 3.87. As in the proof of Theorem 3.86, we argue that $\theta = 0$ implies that B_r^3 is the unique minimizer of $\mathcal{P}_-$ in $\mathcal{A}_I^O$, and that μ_- satisfies the small-volume isoperimetric condition with constant 1 and the monotonicity condition (3.62). Furthermore, Theorem 2.83 implies the existence of a minimizer $A \in \mathcal{A}_I^O$ of $\mathcal{P}_-$ if $R < \infty$ or $\theta < 1$.

We continue the analysis first for the case $R < \infty$. By the symmetry of the obstacles and Theorem 3.73, the symmetrization $A^\star$ is in the admissible class, and also minimizes the functional $\mathcal{P}_-$. Furthermore, Lemma 3.37 implies $((A^\star)^+)_t = \mathrm{B}^2_{\mathrm{rad}_A^+(t)}$ up to an $\mathcal{L}^2$-null set with $\mathrm{rad}_A^+(t) \in \left[\sqrt{r^2 - t^2}, \sqrt{R^2 - t^2}\right]$.

We argue that $\partial(A^\star)\backslash\left(\overline{\mathrm{B}_r^3} \cup H_t\right)$ is analytic and that $\partial(A^\star)\backslash H_t$ is C^1, i.e., property iii) is satisfied.
Indeed, $A^\star$ not only minimizes $\mathcal{P}_-$ but also the localized version of the inner obstacle problem, i.e., $A^\star$ locally minimizes the perimeter in $\Omega := \mathrm{B}_R^3 \cap (\mathbb{R}^2 \times (t, \infty))$ with inner obstacle B_r^3. In particular, for every measurable set $F \subseteq \mathbb{R}^3$ with $F \triangle A^\star \Subset \Omega$ and $\mathrm{B}_r^3 \subseteq F$, we find $\mu_-(F^+) = \mu_-((A^\star)^+)$ and therefore $\mathrm{P}(A^\star, \Omega) \leqslant \mathrm{P}(F, \Omega)$ follows from $\mathcal{P}_-(A^\star) \leqslant \mathcal{P}_-(F)$. Theorem 3.90 then implies that $A^\star$ has C^1 boundary in a neighborhood of $\partial\mathrm{B}_r^3 \cap (\mathbb{R}^2 \times (t, \infty))$, which is contained in Ω. Analogously, we argue that $A^\star$ has C^1 boundary in a neighborhood of $\partial\mathrm{B}_r^3 \cap (\mathbb{R}^2 \times (-\infty, t))$. We further obtain that $A^\star$ is locally perimeter minimizing away from the obstacles and $\mathbb{R}^2 \times \{t\}$. Applying the well-known regularity theorem for local perimeter minimizers, i.e., the later Theorem 4.3, leads to $\partial(A^\star)$ being analytic in $\mathrm{B}_R^3\backslash\left(\overline{\mathrm{B}_r^3} \cup H_t\right)$. According to Lemma 2.50, $A^\star$ is a subset of the set C, defined as the convex hull of $\mathrm{B}_r^3 \cup \left(\mathrm{B}^2_{\mathrm{rad}_A^+(t)} \times \{t\}\right)$, and we find that $\mathrm{rad}_{\mathrm{B}_r^3} \leqslant \mathrm{rad}_A \leqslant \mathrm{rad}_C$ holds a.e. on $\mathbb{R}$. Hence, there exists at most one contact point of rad_A^+ and $\mathrm{rad}_{\mathrm{B}_R^3}$ in $(-R, R)$ (at t if $\mathrm{rad}_A^+(t) = \sqrt{R^2 - t^2}$), which implies that $\partial(A^\star)$ is even analytic in $\mathbb{R}^3\backslash\left(\overline{\mathrm{B}_r^3} \cup H_t\right)$ by the previous analysis.

We have $\mathrm{rad}_A = 0$ on $\mathbb{R}\backslash(-r, r)$, $\mathrm{rad}_A = \mathrm{rad}_{\mathrm{B}_r^3}$ near $\{-r, r\}$ and $\mathrm{rad}_C \in \mathrm{BV}_{\mathrm{loc}}((-r, r))$. Together with Propositions 3.14 and 3.18, we conclude $\mathrm{rad}_A \in \mathrm{BV}_{\mathrm{loc}}((-r, r))$ and

$$A^\star = \mathrm{R}^3_{\mathrm{rad}_A} \quad \text{minimizes} \quad \mathcal{P}_- \quad \text{in} \quad \left\{\mathrm{R}^3_w \,:\, w \in \mathrm{BV}_{\mathrm{loc}}((-r, r)),\, \mathrm{rad}_{\mathrm{B}_r^3} \leqslant w \leqslant \mathrm{rad}_{\mathrm{B}_R^3}\right\}.$$

Since every function w in the admissible class is also in $\mathrm{L}^\infty((-r, r))$, we can apply Proposition 3.18 and Lemma 3.37 to write

$$\mathcal{P}_-(\mathrm{R}^3_w) = \mathrm{P}(\mathrm{R}^3_w) - 2\theta\mathcal{H}^2(((\mathrm{R}^3_w)^+)_t) = 2\pi\mathcal{A}^{\mathrm{rev}}_{3,(-r,r)}[w] - 2\theta\pi(w^+(t))^2\,.$$

Hence, the non-parametric formulation of the obstacle problem is as follows

$$\mathrm{rad}_A \quad \text{minimizes} \quad \mathcal{A}^{\mathrm{rev}}_{3,(-r,r)}[w] - \theta(w^+(t))^2 \quad \text{in} \quad \left\{w \in \mathrm{BV}_{\mathrm{loc}}((-r, r)) \,:\, \mathrm{rad}_{\mathrm{B}_r^3} \leqslant w \leqslant \mathrm{rad}_{\mathrm{B}_R^2}\right\}. \tag{3.94}$$

In the sequel, we show that we can find unique contact points of $\partial(A^\star)$ and $\partial\mathrm{B}_r^2$.
Since $\partial(A^\star)$ is analytic in $\mathbb{R}^3\backslash\left(\overline{\mathrm{B}_r^3} \cup H_t\right)$, we find $\tau_1 \leqslant \tau_2$ in $[t, r)$ with $\mathrm{rad}_A = \mathrm{rad}_{\mathrm{B}_r^3}$ on $[\tau_2, r)$, $\mathrm{rad}_A^+(\tau_1+) = \mathrm{rad}_{\mathrm{B}_r^3}(\tau_1)$ and $\mathrm{rad}_A \neq \mathrm{rad}_{\mathrm{B}_r^3}$ on (t, τ_1), where the latter might be empty if $\tau_1 = t$. Hence, by exploiting the regularity of $\partial(A^\star)$ on $\mathbb{R}^3\backslash H_t$, it holds that $F := \left(A^\star \cap (\mathbb{R}^2 \times (\tau_1, \tau_2))\right) \cup \mathrm{B}_r^3$ locally minimizes the perimeter in $\mathcal{A}^{\mathrm{B}_R^3}_{\mathrm{B}_r^3}$, which, by the isoperimetric inequality, gives $F = \mathrm{B}_r^3$. In particular, we can choose $\tau_1 = \tau_2$ in the first

place. By repeating the same argument for $\tau_1 \leqslant \tau_2$ in $(-r,t]$, we conclude the existence of unique contact points $-r < a \leqslant t \leqslant b < r$ such that $\mathrm{rad}^+_A(x) > \mathrm{rad}_{\mathrm{B}^3_r}(x)$ for all $x \in (a,b)$ and $\mathrm{rad}^+_A(x) = \mathrm{rad}_{\mathrm{B}^3_r}(x)$ for all $x \in (-r,r)\backslash(a,b)$. Note that if $\mathrm{rad}^+_A(t) = \sqrt{r^2-t^2}$, then we obtain $a=b=t$.

We now consider the case $a<t$ and argue that $\mathrm{rad}_A(x) = \alpha_1 \cosh\left(\frac{x-\beta_1}{\alpha_1}\right)$ holds on (a,t) for some $\alpha_1 > 0$ and $\beta_1 \in \mathbb{R}$.
Since $A^\star$ is C^1 in $\mathbb{R}^3\backslash H_t$ and $A \subseteq C$, we find some neighborhood N of $\partial\mathrm{B}^3_r \cap (\mathbb{R}^2 \times (a,b))$ such that $\nu_A \neq \pm e_1$ on $\partial A \cap (N\backslash H_t)$. Hence, rad_A is C^1 on $(a, a+\varepsilon)$ for some $\varepsilon > 0$. Since $\partial(A^\star)$ is analytic in $\mathbb{R}^3\backslash\big(\overline{\mathrm{B}^3_r} \cup H_t\big)$, rad_A is even smooth on $(a,a+\varepsilon)$. Furthermore, rad_A locally minimizes $\mathcal{A}^{\mathrm{rev}}_{3,(a,a+\varepsilon)}$ among $\mathrm{rad}_A + \varphi$ with $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}((a,a+\varepsilon))$. As in the proof of Lemma 3.89 i), we can show that

$$0 = \int_a^{a+\varepsilon} \varphi\sqrt{1+(\mathrm{rad}_A')^2} + \varphi' \frac{\mathrm{rad}_A \,\mathrm{rad}_A'}{\sqrt{1+(\mathrm{rad}_A')^2}} \,\mathrm{d}x$$

holds for all $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}((a,a+\varepsilon))$. Exploiting the derived regularity of the radius function, the integration by parts formula and the fundamental lemma of calculus of variation, we obtain

$$0 = \sqrt{1+(\mathrm{rad}_A')^2} - \left(\frac{\mathrm{rad}_A\,\mathrm{rad}_A'}{\sqrt{1+(\mathrm{rad}_A')^2}}\right)' = \sqrt{1+(\mathrm{rad}_A')^2} - \frac{(\mathrm{rad}_A')^2}{\sqrt{1+(\mathrm{rad}_A')^2}} - \frac{\mathrm{rad}_A''\,\mathrm{rad}_A}{\sqrt{1+(\mathrm{rad}_A')^2}^3} \qquad \text{on } (a,a+\varepsilon)\,.$$

Rearranging the terms yields

$$\mathrm{rad}_A''\,\mathrm{rad}_A = 1 + (\mathrm{rad}_A')^2 \qquad \text{on } (a,a+\varepsilon)\,.$$

Since $\mathrm{rad}_A > 0$ on $(-r,r)$, we can conclude

$$\frac{\mathrm{rad}_A''}{1+(\mathrm{rad}_A')^2}\mathrm{rad}_A' = \frac{1}{\mathrm{rad}_A}\mathrm{rad}_A' \qquad \text{on } (a,a+\varepsilon)\,.$$

Thus, the separation of variables method leads to

$$\frac{\mathrm{rad}_A}{\alpha_1} = \sqrt{1+(\mathrm{rad}_A')^2} = \cosh\sinh^{-1}(\mathrm{rad}_A') \qquad \text{on } (a,a+\varepsilon) \qquad \text{for some } \alpha_1 > 0\,.$$

Rewriting this to

$$1 = \frac{\mathrm{rad}_A'(x)}{\sinh\cosh^{-1}\left(\frac{\mathrm{rad}_A(x)}{\alpha_1}\right)} = \frac{\mathrm{d}}{\mathrm{d}x}\alpha_1\cosh^{-1}\left(\frac{\mathrm{rad}_A(x)}{\alpha_1}\right)$$

gives $\mathrm{rad}_A(x) = \alpha_1\cosh\left(\frac{x-\beta_1}{\alpha_1}\right)$ for some $\alpha_1 > 0$ and $\beta_1 \in \mathbb{R}$ for all $x \in (a,a+\varepsilon)$. This, however, implies that rad_A is smooth in a neighborhood of $a+\varepsilon$ by a similar argument as before which exploits the regularity analysis of $\partial(A^\star)$ in $\mathbb{R}^3\backslash\big(\overline{\mathrm{B}^3_r}\cup H_t\big)$. Hence, the supremum of values $\hat{a} \in (a,t)$, such that rad_A is smooth on $(a,\hat{a})$ and not smooth in any neighborhood of $\hat{a}$, is equal to t, and the previous argumentation yields $\mathrm{rad}_A(x) = \alpha_1\cosh\left(\frac{x-\beta_1}{\alpha_1}\right)$ for all $x \in (a,t)$.

Analogously, we can conclude $\mathrm{rad}_A(x) = \alpha_2\cosh\left(\frac{x-\beta_2}{\alpha_2}\right)$ on (t,b) for some $\alpha_2 > 0$ and $\beta_2 \in \mathbb{R}$ if $b > t$. In particular, $\mathrm{rad}_A \in \mathrm{C}^1((-r,r)\backslash\{t\})$ with only one possible jump at t.

With the general structure of the radius function at hand, we can show $\mathrm{rad}^+_A(t) = \mathrm{rad}^-_A(t) > \sqrt{r^2-t^2}$ via the following case distinction and the corresponding contradiction arguments.
Case 1: $\sqrt{r^2-t^2} < \mathrm{rad}^-_A(t) < \mathrm{rad}^+_A(t) \leqslant \sqrt{R^2-t^2}$:
In this case, we have $a<t<b$ and $\mathrm{rad}^+_A < \mathrm{rad}_{\mathrm{B}^2_R}$ on $(a,b)\backslash\{t\}$ by the regularity analysis of rad_A. We assume w.l.o.g. $\mathrm{rad}^+_A(t) = \mathrm{rad}_A(t+)$ and $\mathrm{rad}^-_A(t) = \mathrm{rad}_A(t-)$. In this framework, Lemma 3.89 iii) yields

$$0 = \int_a^t \varphi\sqrt{1+(\mathrm{rad}_A')^2} + \varphi'\frac{\mathrm{rad}_A\,\mathrm{rad}_A'}{\sqrt{1+(\mathrm{rad}_A')^2}}\,\mathrm{d}x - \mathrm{rad}_A(t-)\varphi(t-)$$

for all $\varphi \in \mathrm{C}^1((a,r)\backslash\{t\}) \cap \mathrm{BV}((a,r))$ with $\mathrm{spt}(\varphi) \subseteq (a,t]$. The derived structure of rad_A and the integration by parts formula give

$$\begin{aligned} 0 &= \int_a^t \varphi \frac{\mathrm{rad}_A}{\alpha_1} + \varphi' \alpha_1 \mathrm{rad}_A' \,\mathrm{d}x - \mathrm{rad}_A(t-)\varphi(t-) \\ &= \alpha_1 \mathrm{rad}_A'(t)\varphi(t-) - \mathrm{rad}_A(t-)\varphi(t-) \\ &= \varphi(t-)\alpha_1 \left[\sinh\left(\frac{t-\beta_1}{\alpha_1}\right) - \cosh\left(\frac{t-\beta_1}{\alpha_1}\right)\right]. \end{aligned}$$

Since $\varphi(t-) \in \mathbb{R}$ can be chosen arbitrarily, it follows

$$\sinh\left(\frac{t-\beta_1}{\alpha_1}\right) = \cosh\left(\frac{t-\beta_1}{\alpha_1}\right),$$

which is a contradiction to $|\sinh| < \cosh$ on $\mathbb{R}$.

Case 2: $\sqrt{r^2-t^2} = \mathrm{rad}_A^-(t) < \mathrm{rad}_A^+(t) \leqslant \sqrt{R^2-t^2}$:
As in the previous case, we assume w.l.o.g. $\mathrm{rad}_A^-(t) = \mathrm{rad}_A(t-)$. We have $\mathrm{rad}_A = \mathrm{rad}_{\mathrm{B}_r^3}$ on $(-r,t)$ such that Lemma 3.89 iv) yields

$$\begin{aligned} 0 &\leqslant \int_{-r}^t \varphi\sqrt{1+(\mathrm{rad}_{\mathrm{B}_r^3}')^2} + \varphi' \frac{\mathrm{rad}_{\mathrm{B}_r^3}\mathrm{rad}_{\mathrm{B}_r^3}'}{\sqrt{1+(\mathrm{rad}_{\mathrm{B}_r^3}')^2}} \,\mathrm{d}x - \varphi(t-)\mathrm{rad}_{\mathrm{B}_r^3}^-(t) \\ &= \int_{-r}^t \varphi(x)\frac{r}{\sqrt{r^2-x^2}} - \varphi'(x)\frac{x}{r}\sqrt{r^2-x^2}\,\mathrm{d}x - \varphi(t-)\sqrt{r^2-t^2} \end{aligned}$$

for non-negative test functions $\varphi \in \mathrm{C}^1((-r,r)\backslash\{t\}) \cap \mathrm{BV}((-r,r))$ with $\mathrm{spt}(\varphi) \subseteq (-r,t]$. The integration by parts formula leads to

$$0 \leqslant \frac{2}{r}\int_{-r}^t \varphi(x)\sqrt{r^2-x^2}\,\mathrm{d}x - \varphi(t-)\left(\frac{t}{r}+1\right)\sqrt{r^2-t^2}\,.$$

Since for every $\varepsilon > 0$ we can find a non-negative test function $\varphi \in \mathrm{C}^1((a,b)\backslash\{t\}) \cap \mathrm{BV}((a,b))$ with $\varphi(t-) = 1$ and $\frac{2}{r}\int_{-r}^t \varphi(x)\sqrt{r^2-x^2}\,\mathrm{d}x < \varepsilon$, the previous inequality becomes $0 \leqslant -\left(\frac{t}{r}+1\right)\sqrt{r^2-t^2}$, which is a contradiction to the choice of t in $(-r,r)$.

Case 3: $\mathrm{rad}_A = \mathrm{rad}_{\mathrm{B}_r^3}$ on $(-r,r)$:
Here, Lemma 3.89 v) and the integration by parts formula yield

$$0 \leqslant \frac{2}{r}\int_{-r}^r \varphi(x)\sqrt{r^2-x^2}\,\mathrm{d}x - 2\theta\varphi(t)\sqrt{r^2-t^2}$$

for non-negative test functions $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}((-r,r))$. We again conclude $0 \leqslant -2\theta\sqrt{r^2-t^2}$, in contradiction to the choice of θ and t.

Hence, we have shown $\sqrt{r^2-t^2} < \mathrm{rad}_A^+(t) = \mathrm{rad}_A^-(t)$ and therefore verified property i).
It is left to show the property ii). To this end, we assume $\mathrm{rad}_A(t) < \sqrt{R^2-t^2}$ and determine the Euler equation once more by Lemma 3.89 i), which gives

$$0 = \int_a^b \varphi\sqrt{1+(\mathrm{rad}_A')^2} + \varphi'\frac{\mathrm{rad}_A\,\mathrm{rad}_A'}{\sqrt{1+(\mathrm{rad}_A')^2}}\,\mathrm{d}x - 2\theta\varphi(t)\mathrm{rad}_A(t)$$

for all $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}((a,b))$. The structure of rad_A and the integration by parts formula lead to

$$\begin{aligned} 0 &= \int_a^t \varphi\frac{\mathrm{rad}_A}{\alpha_1} + \varphi'\alpha_1\mathrm{rad}_A'\,\mathrm{d}x + \int_t^b \varphi\frac{\mathrm{rad}_A}{\alpha_2} + \varphi'\alpha_2\mathrm{rad}_A'\,\mathrm{d}x - 2\theta\varphi(t)\mathrm{rad}_A(t) \\ &= \alpha_1\mathrm{rad}_A'(t)\varphi(t) - \alpha_2\mathrm{rad}_A'(t)\varphi(t) - 2\theta\varphi(t)\mathrm{rad}_A(t)\,. \end{aligned}$$

It follows

$$0 = \alpha_1 \sinh\left(\frac{t-\beta_1}{\alpha_1}\right) - \alpha_2 \sinh\left(\frac{t-\beta_2}{\alpha_2}\right) - 2\theta \mathrm{rad}_A(t)$$

and therefore, ii) is satisfied.

Finally, the symmetry of the obstacles, Theorem 3.77 and Theorem 3.57 imply $A^\star = A$, which proves the claim in the case $R < \infty$.

If $R = \infty$ and $\theta < 1$, the existence theory from Theorem 2.83 still applies and by Remark 3.88, we find some $s \in (0, \infty)$ which satisfies ii). The same argument as before leads to the claimed shape of A, in particular, A is a bounded set of finite perimeter and therefore admissible.

In the case $R = \infty$ and $\theta = 1$, the existence theory fails. In particular, the previous analysis yields

$$\mathcal{P}_-(A(s-1)) > \mathcal{P}_-(A(s)) \tag{3.95}$$

for all $s \geqslant \sqrt{r^2 - t^2} + 1$, where $A(s)$ is a minimizer of the corresponding obstacle problem from the previous analysis with $\theta = 1$, outer obstacle $O = \mathrm{B}^3_{\sqrt{s^2-t^2}}$, and with $\mathrm{rad}_{A(s)}(t) = s$. According to Lemma 2.50, Proposition 3.14 and Theorem 3.73, $A^\star$ is in $\mathrm{B}^2_{\|\mathrm{rad}_A\|_{\mathrm{L}^\infty(\mathbb{R})}} \times (-r, r)$ for every minimizer A of the unbounded obstacle problem. Hence, $A^\star$ is a bounded set and therefore, $A^\star$ minimizes $\mathcal{P}_-$ in the admissible class $\mathcal{A}^{\mathrm{B}^3_R}_{\mathrm{B}^2_r}$ for $R > 0$ large enough. In particular, the previous analysis applies and gives the existence of some $s > 0$ such that $A(s)$ is a minimizer of the unbounded obstacle problem as well, which contradicts (3.95) and Remark 3.88. $\square$

Chapter 4

Variational mean curvatures

In this chapter, we introduce variational mean curvatures, give interesting and useful results and discuss the connection to the main topics of this thesis.

4.1 Motivation, definition, and regularity theorem

Variational mean curvatures generalize the classical mean curvature (which is defined on C^2 manifolds) to (boundaries of) sets of finite perimeter. They were first introduced in [63] and [64] together with the following motivation.
Let $A \subseteq \mathbb{R}^n$ be a C^2 set and assume there exists a continuous function $H \in \mathrm{C}(\mathbb{R}^n) \cap \mathrm{L}^1(\mathbb{R}^n)$ such that A minimizes the Massari functional

$$\mathcal{F}_H : \mathcal{M}^n \to (-\infty, \infty]\,,\; F \mapsto \mathrm{P}(F) - \int_F H \,\mathrm{d}x\,, \tag{4.1}$$

then $\frac{1}{n-1} H|_{\partial A}$ is the mean curvature of ∂A.
Indeed, according to the regularity of A, we can represent ∂A near any point of ∂A as subgraph of some $u \in \mathrm{C}^2(\mathrm{B}_r^{n-1})$ with values in $(-r, r)$, possibly rotating and translating A. By Proposition 2.52, the function u minimizes

$$\int_{\mathrm{B}_r^{n-1}} \sqrt{1 + |\nabla v|^2}\,\mathrm{d}x - \int_{\mathrm{Sub}[v,\mathrm{B}_r^{n-1}]} H \,\mathrm{d}z = \int_{\mathrm{B}_r^{n-1}} \sqrt{1 + |\nabla v|^2} - \int_{\mathrm{B}_r^{n-1}} \int_{-\infty}^{v(x)} H(x,t)\,\mathrm{d}t\,\mathrm{d}x$$

among functions $v = u + \varphi$ with values in $(-r, r)$ and $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}(\mathrm{B}_r^{n-1})$. With this, we can compute the Euler equation explicitly, using also the continuity of H. The equation yields

$$0 = \int_{\mathrm{B}_r^{n-1}} \frac{\nabla\varphi(x) \cdot \nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} - \varphi\, H(x, u(x))\,\mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{C}^1_{\mathrm{cpt}}(\mathrm{B}_r^{n-1})\,.$$

Since u is C^2, we can rewrite the outer unit normal ν_A of A at $(x, u(x)) \in \mathrm{B}_r^{n-1} \times (-r, r)$ as $\frac{(\nabla u(x), -1)}{\sqrt{1+|\nabla u(x)|^2}}$ and obtain

$$\operatorname{div} \nu_A = \operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|}} = -H(\,\cdot\,, u(\,\cdot\,)) \qquad \text{on } \mathrm{B}_r^{n-1}\,,$$

where we identify $\nu_A(x)$ with $\nu_A(x, u(x))$.
The definition of variational mean curvatures merely omits the regularity assumptions on A and H.

Definition 4.1 (Variational mean curvatures)**.** Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter in Ω. The function $H \in \mathrm{L}^1(\Omega)$ is called a **(local) variational mean curvature of $\boldsymbol{A}$ in $\boldsymbol{\Omega}$**, if A locally minimizes the functional

$$\mathcal{F}_H^\Omega : \mathcal{M}^n \to (-\infty, \infty],\; F \mapsto \mathrm{P}(F, \Omega) - \int_{F\cap\Omega} H \;\mathrm{d}x\,,$$

i.e., if $\mathcal{F}_H^\Omega(A) \leqslant \mathcal{F}_H^\Omega(F)$ for all $F \in \mathcal{M}^n$ with $F \triangle A \Subset \Omega$.

If $\Omega = \mathbb{R}^n$ and A globally minimizes the functional $\mathcal{F}_H = \mathcal{F}_H^{\mathbb{R}^n}$ from (4.1), i.e., if

$$\mathcal{F}_H(A) \leqslant \mathcal{F}_H(F) \qquad \text{for all } F \in \mathcal{M}^n,$$

then we call $H \in \mathrm{L}^1(\mathbb{R}^n)$ a **(global) variational mean curvature of A**.

In contrast to the classical mean curvature, the variational mean curvature is defined on the ambient space and, as it is an L^1 function, there is no suitable restriction onto lower dimensional sets.
The following remark collects some well-known results about variational mean curvatures. The existence result is the main statement of [9], ii) follows from the definition and iii) can be found, for instance, in [50, Remark 2.2].

Remark 4.2. Let A be a set of finite perimeter in $\mathbb{R}^n$.

i) There exists a global variational mean curvature of A.

ii) If $H \in \mathrm{L}^1(\mathbb{R}^n)$ is a global variational mean curvature of A, then $H|_\Omega$ is a local variational mean curvature of A in Ω.

iii) For every global variational mean curvature $H \in \mathrm{L}^1(\mathbb{R}^n)$, and for every $\Phi \in \mathrm{L}^1(\mathbb{R}^n)$ with $\Phi \geqslant 0$ a.e. on A and $\Phi \leqslant 0$ a.e. on $\mathbb{R}^n \backslash A$, the function $H + \Phi$ is also a global variational mean curvature of A.

Hence, for all sets $A \subseteq \mathbb{R}^n$ of finite perimeter, there exist infinitely many variational mean curvatures, which may not include much information about A. In [63, 64], the author proved a remarkable theorem connecting the L^p-integrability of variational mean curvatures of sets A and the $\mathrm{C}^{1,\alpha}$ regularity of the reduced boundary $\partial^* A$. The theorem was restated in the survey [50, Theorem 3.6], where it was conjectured in Remark 3.6 that the optimal Hölder exponent α_{opt} converges to 1 as $p \to \infty$, improving the original exponent, which has limit $\frac{1}{4}$. In [81, Theorem 3.3], this is confirmed by showing that the theorem holds for all $\alpha < \alpha_{\mathrm{opt}} = \frac{p-n}{p+1}$, and counterexamples to $\alpha > \alpha_{\mathrm{opt}}$ have been established, leaving open whether the limit exponent α_{opt} is attained or not.
Before we present the regularity theorem for sets with variational mean curvature in L^p, we give the well-known regularity theorem for local perimeter minimizers, which was first stated in [20, Teorema VI] (see also [21, p. 262]) and refined (with respect to the dimension of the singularity set) in [66, Teorema 6.5], [84, Theorem 6.1.1, Theorem 6.2.1] and [41, Theorem 1].

Theorem 4.3 (De Giorgi-Miranda-Simons-Federer regularity theorem)**.** *Let A be a set with vanishing local variational mean curvature in Ω, i.e., it holds*

$$\mathrm{P}(A, \Omega) \leqslant \mathrm{P}(F, \Omega) \qquad \textit{for every measurable set } F \subseteq \mathbb{R}^n \textit{ with } F \triangle A \Subset \Omega\,.$$

Then $\partial^* A \cap \Omega$ is a relatively open $(n-1)$-dimensional analytic manifold in $\partial A^1 \cap \Omega$ and $\mathcal{H}^s((\partial A^1 \backslash \partial^* A) \cap \Omega) = 0$ holds for all $s \in (n-8, n]$.

The aforementioned regularity theorem from [64, Teorema 3.1, Teorema 3.2] lifts the previous result to sets with well-behaved curvatures, while losing a substantial part of the regularity. We directly state the version with the refined exponent from [81, Theorem 3.3].

Theorem 4.4 (Massari's regularity theorem)**.** *For a set $A \subseteq \mathbb{R}^n$ of finite perimeter in Ω and with local variational mean curvature $H \in \mathrm{L}^p(\Omega)$ in Ω for $p \in (n, \infty]$, the following properties hold true.*

i) *The set $\partial^* A \cap \Omega$ is relatively open in $\partial A^1 \cap \Omega$.*

ii) *The set $\partial^* A \cap \Omega$ is an $(n-1)$-dimensional $\mathrm{C}^{1,\alpha}$ manifold for all $\alpha \in \left(0, \frac{p-n}{p+1}\right)$ if $p < \infty$ and for all $\alpha \in (0,1)$ if $p = \infty$.*

iii) *It holds $\mathcal{H}^s((\partial A^1 \backslash \partial^* A) \cap \Omega) = 0$ for all $s \in (n-8, n]$.*

It can be shown that the dimension of the singularities as well as the bound on p are optimal. Indeed, there exist sets with variational mean curvature in L^p for $p \leqslant n$ having singularities, see [50, Example 2.2] for the case $p < n$ and [51, Section 2] for the case $p = n$. Moreover, the Simons cone is the standard example of a set with vanishing mean curvature and a singularity point in $\mathbb{R}^8$, which shows optimality for the Hausdorff dimension $n - 8$ of the singularity set, see [11, Theorem A].
In the 2-dimensional case, having a variational mean curvature in L^∞ implies $\mathrm{C}^{1,1}$ regularity of the boundary of the corresponding set according to [81, Proposition 3.6], but [50, Remark 3.6] gives an example of a three-dimensional set, which is of class $\mathrm{C}^{1,\alpha}$ for all $\alpha \in (0,1)$, but not of class $\mathrm{C}^{1,1}$, and which has a variational mean curvature in L^∞. In Chapter 6, we show that the theorem remains valid for $\alpha = \frac{p-n}{p+1}$. Considering the counterexamples [81, Theorem 4.2, Theorem 4.3] for the case $\alpha > \frac{p-n}{p+1}$, Massari's regularity theorem is then optimal in all its parameters.

4.2 Constructions of variational mean curvatures

As Massari's regularity theorem suggests, finding variational mean curvatures with good integrability can reveal relevant regularity properties. Therefore, we present two ways of constructing variational mean curvatures for sets. The first was derived in [8] as an improved version of the construction in [9]. The Barozzi curvature turns out to be optimal with respect to its integrability, and can therefore be used to present sharp counterexamples similar to [81, Theorem 4.3]. Since the construction works with sets of constant mean curvatures, the exact shape of the curvature remains unknown up to some special cases, see Section 4.3.
The second construction is a callback to the motivation of variational mean curvatures, showing that the divergence of (an extension of) the unit normal can sometimes result in a variational mean curvature. The advantage of this divergence curvature is that the resulting functions are explicit compared to the Barozzi curvature. We give a version of this construction based on the generalized normal trace, which, to our knowledge, was never before stated and which is more general than the known variants.

4.2.1 Barozzi's curvature

In the following, we present the construction from [8, Section 2] and the corresponding properties.

Definition 4.5 (Sets of constant curvature)**.** Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter and $f_A \in \mathrm{L}^1(A)$ an a.e. strictly positive function on A. For $\lambda \in [0,\infty)$, we call any minimizer of

$$\mathcal{F}_\lambda^{f_A} : \{F \in \mathcal{M}^n : F \subseteq A\} \to (-\infty, \infty],\, F \mapsto \mathrm{P}(F) - \lambda \int_F f_A \,\mathrm{d}x$$

a **set of $\boldsymbol{f_A}$-constant mean curvature $\boldsymbol{\lambda}$ in $\boldsymbol{A}$** and denote it by $\mathcal{C}_{A,\lambda}^{f_A}$.
If A is additionally a set of finite volume, we abbreviate $\mathcal{F}_\lambda := \mathcal{F}_\lambda^1$ and call it **a constant mean curvature functional**. Moreover, we write $\mathcal{C}_{A,\lambda}$ instead of $\mathcal{C}_{A,\lambda}^1$ and call this a **set of constant (variational) mean curvature $\boldsymbol{\lambda}$ in $\boldsymbol{A}$**.

Remark 4.6. Let $A \subseteq F \subseteq \mathbb{R}^n$ be two sets of finite perimeter and $f \in \mathrm{L}^1(F)$ an a.e. strictly positive function on F. Then, for all $\lambda > 0$, $\mathcal{C}_{A,\lambda}^f \cup \mathcal{C}_{F,\lambda}^f$ is a minimizer of $\mathcal{F}_\lambda^f$ among subsets of F of finite perimeter.
Indeed, by the previous definition and Lemma 2.38, it holds

$$\begin{aligned}
\mathcal{F}_\lambda^f\big(\mathcal{C}_{A,\lambda}^f \cup \mathcal{C}_{F,\lambda}^f\big) - \mathcal{F}_\lambda^f\big(\mathcal{C}_{F,\lambda}^f\big) &= \mathrm{P}\big(\mathcal{C}_{A,\lambda}^f \cup \mathcal{C}_{F,\lambda}^f\big) - \mathrm{P}\big(\mathcal{C}_{F,\lambda}^f\big) - \lambda \int_{\mathcal{C}_{A,\lambda}^f \setminus \mathcal{C}_{F,\lambda}^f} f \,\mathrm{d}x \\
&\leqslant \mathrm{P}\big(\mathcal{C}_{A,\lambda}^f\big) - \mathrm{P}\big(\mathcal{C}_{A,\lambda}^f \cap \mathcal{C}_{F,\lambda}^f\big) - \lambda \int_{\mathcal{C}_{A,\lambda}^f \setminus \mathcal{C}_{F,\lambda}^f} f \,\mathrm{d}x \\
&= \mathcal{F}_\lambda^f\big(\mathcal{C}_{A,\lambda}^f\big) - \mathcal{F}_\lambda^f\big(\mathcal{C}_{A,\lambda}^f \cap \mathcal{C}_{F,\lambda}^f\big) \\
&\leqslant 0\,.
\end{aligned}$$

In order to define the Barozzi curvature, we need to ensure that it is well-defined, regardless of the choice of the minimizers $\mathcal{C}_{A,\lambda}^{f_A}$.

Lemma 4.7. *For all choices of A, f_A and λ as in Definition 4.5, there exist sets of f_A-constant mean curvature λ in A. Furthermore, for all $0<\lambda<\gamma$, it holds*

$$\left|\mathcal{C}^{f_A}_{A,\lambda}\backslash\mathcal{C}^{f_A}_{A,\gamma}\right| = 0 \qquad \textit{and} \qquad \left|\bigcap_{\lambda\in\mathbb{Q}_{>0}} A\backslash\mathcal{C}^{f_A}_{A,\lambda}\right| = 0$$

for all choices of $\mathcal{C}^{f_A}_{A,\lambda}$ and $\mathcal{C}^{f_A}_{A,\gamma}$.

The first statement of the following remark can be found in [8, Remark 2.4].

Remark 4.8. We consider A, f_A and λ as in Definition 4.5.

i) The sets $\mathcal{C}^{f_A,+}_{A,\lambda}$ and $\mathcal{C}^{f_A,-}_{A,\lambda}$, defined by

$$\mathcal{C}^{f_A,+}_{A,\lambda} := \bigcap_{k=1}^{\infty}\mathcal{C}^{f_A}_{A,\gamma_k^+} \qquad \text{and} \qquad \mathcal{C}^{f_A,-}_{A,\lambda} := \bigcup_{k=1}^{\infty}\mathcal{C}^{f_A}_{A,\gamma_k^-}$$

for any sequences $(\gamma_k^{\pm})_{k\in\mathbb{N}}\subseteq\mathbb{R}$ that satisfy $\gamma_k^-\nearrow\lambda$ and $\gamma_k^+\searrow\lambda$ as $k\to\infty$, are well-defined except for null sets according to Lemma 4.7, and they minimize $\mathcal{F}^{f_A}_{\lambda}$ among measurable subsets of A.

Indeed, since $\mathcal{C}^{f_A}_{A,\gamma_k^{\pm}}$ converges to $\mathcal{C}^{f_A,\pm}_{A,\lambda}$ in $\mathrm{L}^1_{\mathrm{loc}}$ as $k\to\infty$, we can use the lower semicontinuity of the perimeter and estimate

$$\mathcal{F}^{f_A}_{\lambda}\big(\mathcal{C}^{f_A,\pm}_{A,\lambda}\big) \leqslant \liminf_{k\to\infty}\mathrm{P}\Big(\mathcal{C}^{f_A}_{A,\gamma_k^{\pm}}\Big) - \gamma_k^{\pm}\int_{\mathcal{C}^{f_A}_{A,\gamma_k^{\pm}}} f_A\,\mathrm{d}x \leqslant \liminf_{k\to\infty}\mathrm{P}(F) - \gamma_k^{\pm}\int_F f_A\,\mathrm{d}x = \mathcal{F}^{f_A}_{\lambda}(F)$$

for all measurable sets $F\subseteq A$.

ii) The set $\mathcal{C}^{f_A,+}_{A,\lambda}$ is sometimes called the maximal minimizer of $\mathcal{F}^{f_A}_{\lambda}$ in A and $\mathcal{C}^{f_A,-}_{A,\lambda}$ is called the minimal minimizer of $\mathcal{F}^{f_A}_{\lambda}$ in A, compare [60]. By Lemma 4.7, it holds $\mathcal{C}^{f_A,-}_{A,\lambda}\subseteq\mathcal{C}^{f_A}_{A,\lambda}\subseteq\mathcal{C}^{f_A,+}_{A,\lambda}$ up to null sets for every choice of $\mathcal{C}^{f_A}_{A,\lambda}$ and for all $\lambda>0$.

iii) Since $\lambda\mapsto\int_{\mathcal{C}^{f_A}_{A,\lambda}} f_A\,\mathrm{d}x$ is increasing and bounded, it is continuous except for countably many values $\lambda>0$. Hence, $\mathcal{C}^{f_A,+}_{A,\lambda}=\mathcal{C}^{f_A}_{A,\gamma}=\mathcal{C}^{f_A,-}_{A,\lambda}$ holds true up to a null set for all but countably many $\lambda>0$.

Definition 4.9 (Barozzi curvature)**.** Let $A\subseteq\mathbb{R}^n$ be a set of finite perimeter. We call a measurable function $H:\mathbb{R}^n\to\mathbb{R}$ a **Barozzi curvature of A** if there exists a function $f_A\in\mathrm{L}^1(\mathbb{R}^n)$, which is a.e. strictly positive on A, and a.e. strictly negative on $\mathbb{R}^n\backslash A$, and there exists a fixed choice of $\mathcal{C}^{f_A}_{A,\lambda}$ and $\mathcal{C}^{-f_A}_{\mathbb{R}^n\backslash A,\lambda}$ for all $\lambda\in\mathbb{Q}_{>0}$, such that $H=H^{f_A}_A\big((\mathcal{C}^{f_A}_{A,\lambda},\mathcal{C}^{-f_A}_{\mathbb{R}^n\backslash A,\lambda})_{\lambda\in\mathbb{Q}_{>0}}\big)$, where

$$H^{f_A}_A\big((\mathcal{C}^{f_A}_{A,\lambda},\mathcal{C}^{-f_A}_{\mathbb{R}^n\backslash A,\lambda})_{\lambda\in\mathbb{Q}_{>0}}\big):\mathbb{R}^n\to\mathbb{R}\,,\; x\mapsto\begin{cases}\inf\Big\{\lambda f_A(x):\, x\in\mathcal{C}^{f_A}_{A,\lambda},\,\lambda\in\mathbb{Q}_{>0}\Big\} & \text{if } x\in A\,,\\ \sup\Big\{\lambda f_A(x):\, x\in\mathcal{C}^{-f_A}_{\mathbb{R}^n\backslash A,\lambda},\,\lambda\in\mathbb{Q}_{>0}\Big\} & \text{if } x\in\mathbb{R}^n\backslash A\,.\end{cases}$$

If A is additionally of finite volume, then we write H_A for any Barozzi curvature $H^{f_A}_A\big((\mathcal{C}^{f_A}_{A,\lambda},\mathcal{C}^{-f_A}_{\mathbb{R}^n\backslash A,\lambda})_{\lambda\in\mathbb{Q}_{>0}}\big)$ with the function $f_A=\mathbb{1}_A+f\mathbb{1}_{\mathbb{R}^n\backslash A}$ for some $f\in\mathrm{L}^1(\mathbb{R}^n\backslash A)$ that is a.e. negative on $\mathbb{R}^n\backslash A$.

Remark 4.10. Let A and f_A be as in the previous definition.

i) According to Lemma 4.7 and Remark 4.8, a Barozzi curvature is measurable and a.e. invariant under the exact choice of $\mathcal{C}^{f_A}_{A,\lambda}$ and $\mathcal{C}^{-f_A}_{\mathbb{R}^n\backslash A,\lambda}$, respectively. Instead of $H^{f_A}((\mathcal{C}^{f_A}_{A,\lambda},\mathcal{C}^{-f_A}_{\mathbb{R}^n\backslash A,\lambda})_{\lambda\in\mathbb{Q}_{>0}})$, we therefore only write $H^{f_A}_A$. In particular, H_A is well-defined on A if A has finite volume.

ii) According to the definition, it holds $H^{f_A}_A=-H^{-f_A}_{\mathbb{R}^n\backslash A}$ on $\mathbb{R}^n$.

iii) The function $H_A^{f_A}$ is a.e. non-negative on A and a.e. non-positive on $\mathbb{R}^n\backslash A$.

iv) The Barozzi curvature depends on the choice of f_A, see the next simple example.

Example 4.11. Let A be the two-dimensional unit ball B_1^2. For a measurable set $F \subseteq A$, we choose $s \in [0,1]$ such that $|F| = \pi s^2$ holds true. The estimates

$$\mathrm{P}(F) - \lambda|F| \geqslant \mathrm{P}(\mathrm{B}_s^2) - \lambda|\mathrm{B}_s^2| = (2-\lambda s)\pi s \quad \text{and} \quad \mathrm{P}(F) - \lambda\int_F \frac{1}{|x|}\,\mathrm{d}x \geqslant \mathrm{P}(\mathrm{B}_s^2) - \lambda\int_{\mathrm{B}_s^2}\frac{1}{|x|}\,\mathrm{d}x = (1-\lambda)2\pi s$$

follow for all $\lambda > 0$ by symmetry and by the isoperimetric inequality. A monotonicity analysis of the right-hand sides of the inequalities give that $\mathcal{C}_{A,\lambda}$ is the empty set if $\lambda \in (0,2)$ and equal to A if $\lambda > 2$ and that $\mathcal{C}_{A,\lambda}^{\frac{1}{|\cdot|}}$ is the empty set if $\lambda \in (0,1)$ and equal to A if $\lambda > 1$. By definition, it follows $H_A \equiv 2$ on A and $H_A^f(x) = \frac{1}{|x|}$ for a.e. $x \in A$, where f is a function in $\mathrm{L}^1(\mathbb{R}^2)$ which is a.e. negative on $\mathbb{R}^2\backslash\mathrm{B}_1^2$ and equal to $\frac{1}{|\cdot|}$ on B_1^2.

In [8, Remark 2.1, Theorem 2.1, Theorem 3.1, Theorem 3.2], the following properties are shown.

Theorem 4.12 (Properties of the Barozzi curvature)**.** *Let A be a set of finite perimeter in $\mathbb{R}^n$.*

i) *Every Barozzi curvature of A is a global variational mean curvature of A in $\mathrm{L}^1(\mathbb{R}^n)$.*

ii) *For all global variational mean curvatures $G \in \mathrm{L}^1(\mathbb{R}^n)$ of A, it holds*

$$\mathrm{P}(A) = \|H\|_{\mathrm{L}^1(A)} \leqslant \|G\|_{\mathrm{L}^1(A)} \qquad \text{and} \qquad \mathrm{P}(A) = \|H\|_{\mathrm{L}^1(\mathbb{R}^n\backslash A)} \leqslant \|G\|_{\mathrm{L}^1(\mathbb{R}^n\backslash A)}\,,$$

where H is a Barozzi curvature of A.

iii) *If A has additionally finite volume, $H_A \in \mathrm{L}^p(A)$ for some $p \in (1,\infty)$, and $H_A \notin \mathrm{L}^q(A)$ for all $q > p$, then*

$$G \notin \mathrm{L}^q(A) \quad \text{for all } q > p \qquad \text{and} \qquad \|H_A\|_{\mathrm{L}^p(A)} < \|G\|_{\mathrm{L}^p(A)}$$

hold true for all variational mean curvatures $G \in \mathrm{L}^1(\mathbb{R}^n)$ with $\|H_A - G\|_{\mathrm{L}^1(A)} > 0$.

For sets A of finite perimeter and finite volume, the Barozzi curvature H_A has the best possible integrability among all variational mean curvatures. Hence, in view of Theorem 4.4, it indicates the best regularity results for the reduced boundary of A. Indeed, in [81], this is exploited in order to obtain optimality of α_{opt} in every dimension. In Chapter 5, we will use this L^p-optimality to show that any $\mathrm{C}^{1,\alpha}$ set admits a variational mean curvature in L^p, along with corresponding counterexamples. In order to connect the global Barozzi curvature result with the regularity result for local curvatures, we need the following lemmas, which can be found in [81, Lemma 2.5].

Lemma 4.13 (Curvature composition)**.** *Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter. For all local variational mean curvatures H_1, H_2 of A in Ω, the composition $H := H_1\,\mathbb{1}_{A\cap\Omega} + H_2\,\mathbb{1}_{\Omega\backslash A}$ is a local variational mean curvature of A in Ω.*

Remark 4.14. From the previous lemma, Remark 4.2 and Theorem 4.12, we conclude that, if $A \subseteq \mathbb{R}^n$ is a set of finite perimeter and $f \in \mathrm{L}^1(\mathbb{R}^n)$ is a function which is a.e. strictly positive on A and a.e. strictly negative on $\mathbb{R}^n\backslash A$, then the function

$$H(x) := \begin{cases} H^f_{A\cap\Omega}(x) & \text{if } x \in A\cap\Omega\,, \\ -H^{-f}_{\Omega\backslash A}(x) & \text{if } x \in \Omega\backslash A \end{cases}$$

is a local variational mean curvature of A in Ω.

Lemma 4.15 (Local to global curvatures)**.** *Let $A \subseteq \mathbb{R}^n$ be a bounded set of finite perimeter with $A \Subset \Omega$. Moreover, let $H \in \mathrm{L}^1(\Omega)$ be a local variational mean curvature of A in Ω and $G \in \mathrm{L}^1(\mathbb{R}^n)$ a global variational mean curvature of A. Then the function*

$$H\mathbb{1}_A + G\mathbb{1}_{\mathbb{R}^n\backslash A}$$

is a global variational mean curvature of A.

4.2.2 Divergence curvature

We now give some brief results about normal traces, which we use to define the divergence curvature. The first existence result in this direction is [5, Theorem 1.2]. The slightly more general result which appears in our analysis can be found in the unpublished paper [6, Theorem 3.5]. In [73, Section 2], a compact introduction is given. Further references in this direction are [15, 17, 16, 2].

Definition 4.16 (Normal traces)**.** Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter in Ω. Assume that $V \in \mathrm{L}^1(\Omega, \mathbb{R}^n)$ has distributional divergence $\operatorname{div} V$ in $\mathrm{L}^1(\Omega)$. The **normal trace of $\boldsymbol{V}$ on $\boldsymbol{\partial^* A} \cap \boldsymbol{\Omega}$** is defined as the distribution

$$\mathrm{nTr}_A V : \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega) \to \mathbb{R},\ \varphi \mapsto \int_A \varphi \operatorname{div} V + V \cdot \nabla\varphi \,\mathrm{d}x\,.$$

The following result is can be found in [73, Lemma 2.24, Definition 2.25].

Proposition 4.17 (Normal trace)**.** *Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter in Ω and let $V \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n)$ with distributional divergence* div V *in* $\mathrm{L}^1(\Omega)$*. Then there exists a function in* $\mathrm{L}^\infty(\Omega \cap \partial^* A; \mathcal{H}^{n-1})$*, which we denote by $V \bullet \nu_A$, such that it holds*

$$\|V \bullet \nu_A\|_{\mathrm{L}^\infty(\partial^* A \cap \Omega; \mathcal{H}^{n-1})} \leqslant \|V\|_{\mathrm{L}^\infty(\Omega, \mathbb{R}^n)} \quad \textit{and} \quad \langle \mathrm{nTr}_A V, \varphi\rangle = \int_{\partial^* A} V \bullet \nu_A \,\varphi \,\mathrm{d}\mathcal{H}^{n-1}(x) \quad \textit{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)\,.$$

Definition 4.18 (Generalized normal trace)**.** In the situation of Proposition 4.17, the density $V \bullet \nu_A$ is called the **generalized normal trace of $\boldsymbol{V}$ on $\boldsymbol{\partial^* A} \cap \boldsymbol{\Omega}$**.

Remark 4.19. If the set A from the previous proposition is of finite volume and if $\Omega = \mathbb{R}^n$, then a suitable approximation of φ implies

$$\int_A \varphi \operatorname{div} V + V \cdot \nabla\varphi \,\mathrm{d}x = \int_{\partial^* A} V \bullet \nu_A \,\varphi \,\mathrm{d}\mathcal{H}^{n-1}(x) \qquad \text{for all } \varphi \in \mathrm{C}^\infty(\mathbb{R}^n) \text{ that satisfies } \sup_{\mathbb{R}^n} |\varphi| + |\nabla\varphi| < \infty\,.$$

Definition 4.20 (Divergence curvature)**.** Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter in Ω and let $V \in \mathrm{L}^1(\Omega, \mathbb{R}^n)$. We call $H \in \mathrm{L}^1(\Omega)$ a **$\boldsymbol{V}$-divergence curvature of $\boldsymbol{A}$ in $\boldsymbol{\Omega}$** if the following properties are satisfied:

i) $V \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n)$ with $\|V\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \leqslant 1$,

ii) V has distributional divergence $\operatorname{div} V$ in $\mathrm{L}^1(\Omega)$ such that $V \bullet \nu_A = 1$ on $\partial^* A \cap \Omega$ except for $\mathcal{H}^{n-1}$-null sets,

iii) $H = \operatorname{div} V$ in Ω.

We call $H \in \mathrm{L}^1(\Omega)$ a **divergence curvature of $\boldsymbol{A}$ in $\boldsymbol{\Omega}$** if there exists $V \in \mathrm{L}^1(\Omega, \mathbb{R}^n)$ such that H is a V-divergence curvature of A in Ω.

Lemma 4.21. *Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter. If $V \in \mathrm{L}^1(\Omega, \mathbb{R}^n)$ satisfies either*

i) *there exist a relatively closed $\mathcal{H}^{n-1}$-null set $\Gamma \subseteq \Omega$ such that $V \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^n) \cap \mathrm{C}(\Omega \backslash \Gamma, \mathbb{R}^n)$ is an extension of ν_A in Ω with $\sup_\Omega |V| \leqslant 1$,*

or

ii) *A is a Lipschitz set in Ω and $V \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^n)$ is a subunit extension of ν_A in Ω, i.e., it holds $\|V\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \leqslant 1$ and $\mathrm{Tr}_A V = \nu_A$ on $\partial A \cap \Omega$,*

then $\operatorname{div} V$ *is a V-divergence curvature of A in Ω.*

Proof. To argue for i), we first assume that V is additionally in $\mathrm{C}^\infty(\Omega, \mathbb{R}^n)$ and apply the product rule for BV functions, Lemma 2.3, on each component, showing that $\mathbb{1}_A V$ is in $\mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and that

$$\operatorname{div}(\mathbb{1}_A V) = \mathbb{1}_A \operatorname{div} V \,\mathcal{L}^n|_\Omega - V \cdot \nu_A \,\mathcal{H}^{n-1} \llcorner (\partial^* A \cap \Omega)$$

is satisfied by Theorem 2.45. With this, we find

$$\langle \mathrm{n}\mathrm{Tr}_A V, \varphi\rangle = \int_\Omega \varphi\, \mathbb{1}_A \operatorname{div} V + \mathbb{1}_A V\cdot\nabla\varphi \,\mathrm{d}x = \int_\Omega \varphi \operatorname{div}(\mathbb{1}_A V) + \mathbb{1}_A V\cdot\nabla\varphi\,\mathrm{d}x + \int_{\partial^* A} V\cdot\nu_A\,\varphi\,\mathrm{d}\mathcal{H}^{n-1}(x)\,,$$

for all $\varphi\in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$. By the definition of the weak derivative measure, the previous equality leads to $V\bullet\nu_A = V\cdot\nu_A = |\nu_A|^2 = 1$ on $\partial^*A\cap\Omega$. For general $V\in \mathrm{W}^{1,1}(\Omega,\mathbb{R}^n)\cap \mathrm{C}(\Omega\backslash\Gamma,\mathbb{R}^n)$, we find for each $\delta>0$ some open neighborhood N_δ of Γ with $\mathcal{H}^{n-1}(N_\delta)<\delta$ by the definition of the Hausdorff measure. Now, we consider the mollifications V_ε in $\mathrm{C}^\infty([\Omega]_\varepsilon)$, which converge to V uniformly in $[\Omega]_\varepsilon\backslash N_\delta$ as $\varepsilon\searrow 0$. Furthermore, $\operatorname{div} V_\varepsilon\to\operatorname{div} V$ in $\mathrm{L}^1([\Omega]_\varepsilon)$ and $V_\varepsilon\to V$ in $\mathrm{L}^1([\Omega]_\varepsilon,\mathbb{R}^n)$ as $\varepsilon\searrow 0$. With this and the first part of this argument, we conclude

$$\int_\Omega \varphi\,\mathbb{1}_A\operatorname{div} V + \mathbb{1}_A V\cdot\nabla\varphi\,\mathrm{d}x = \int_{\partial^*A\backslash N_\delta} V\cdot\nu_A\,\varphi\,\mathrm{d}\mathcal{H}^{n-1}(x) + \lim_{\varepsilon\searrow 0}\int_{\partial^*A\cap N_\delta} V_\varepsilon\cdot\nu_A\,\varphi\,\mathrm{d}\mathcal{H}^{n-1}(x)\,.$$

The latter term is estimated by

$$\left|\int_{\partial^*A\cap N_\delta} V_\varepsilon\cdot\nu_A\,\varphi\,\mathrm{d}\mathcal{H}^{n-1}(x)\right| \leqslant \mathcal{H}^{n-1}(N_\delta)\|V_\varepsilon\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}\|\varphi\|_{\mathrm{L}^\infty(\Omega)} \leqslant \delta\|\varphi\|_{\mathrm{L}^\infty(\Omega)}\,,$$

which leads to $V\bullet\nu_A = V\cdot\nu_A = 1$ on $\partial^*A\cap\Omega$ except for $\mathcal{H}^{n-1}$-null sets.
The case ii) can be shown straightforwardly by the trace definition for Sobolev functions. □

In [10, p. 152-153], [51, Lemma 1.3], [50, Proposition 4.1], and [81, Proposition 4.1], slightly different versions of the following statement have been presented. These results are covered by our statement according to Lemma 4.21.

Proposition 4.22 (Divergence curvature). *For a set $A\subseteq\mathbb{R}^n$ of finite perimeter in Ω, any divergence curvature of A in Ω is a local variational mean curvature of A in Ω.*

The following proof is based on [81, Proposition 4.1].

Proof. Let $\operatorname{div} V$ be a V-divergence curvature of A in Ω as in Definition 4.20. We further consider a test set $F\subseteq\mathbb{R}^n$ with finite perimeter in Ω and $F\triangle A\Subset\Omega$ and choose a bounded open set $\Omega'\Subset\Omega$ with $F\triangle A\Subset\Omega'$. According to the assumptions on V and Proposition 4.17, there exists a function $V\bullet\nu_F$ in $\mathrm{L}^\infty(\partial^*F\cap\Omega;\mathcal{H}^{n-1})$ with $\|V\bullet\nu_F\|_{\mathrm{L}^\infty(\partial^*F\cap\Omega;\mathcal{H}^{n-1})}\leqslant 1$ such that $\mathrm{n}\mathrm{Tr}_F V = V\bullet\nu_F\,\mathcal{H}^{n-1}\llcorner(\partial^*F\cap\Omega)$. Clearly, we have equality $V\bullet\nu_A = V\bullet\nu_F$ on $\partial^*F\cap(\Omega\backslash\Omega') = \partial^*A\cap(\Omega\backslash\Omega')$. Now, for $\varphi\in\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,[0,1])$ with $\varphi=1$ on Ω' it holds

$$\begin{aligned}
\mathcal{F}^\Omega_{\operatorname{div} V}(A) &= \mathrm{P}(A,\Omega\backslash\Omega') - \int_{A\cap(\Omega\backslash\Omega')}(1-\varphi)\operatorname{div} V\,\mathrm{d}x + \mathrm{P}(A,\Omega') - \int_{A\cap\Omega}\varphi\operatorname{div} V\,\mathrm{d}x\\
&= \mathrm{P}(F,\Omega\backslash\Omega') - \int_{F\cap(\Omega\backslash\Omega')}(1-\varphi)\operatorname{div} V - \nabla\varphi\cdot V\,\mathrm{d}x + \mathcal{H}^{n-1}(\partial^*A\cap\Omega') - \int_{\partial^*A} V\bullet\nu_A\,\varphi\,\mathrm{d}\mathcal{H}^{n-1}(x)\\
&= \mathrm{P}(F,\Omega\backslash\Omega') - \int_{F\cap(\Omega\backslash\Omega')}(1-\varphi)\operatorname{div} V - \nabla\varphi\cdot V\,\mathrm{d}x - \int_{\partial^*F\cap(\Omega\backslash\Omega')} V\bullet\nu_F\,\varphi\,\mathrm{d}\mathcal{H}^{n-1}(x)\\
&\quad + \mathcal{H}^{n-1}(\partial^*A\cap\Omega') - \int_{\partial^*A\cap\Omega'} V\bullet\nu_A\,\mathrm{d}\mathcal{H}^{n-1}(x)
\end{aligned}$$

by the definition of normal trace. The two latter terms are the only ones which depend on A, and they can be estimated as

$$\mathcal{H}^{n-1}(\partial^*A\cap\Omega') - \int_{\partial^*A\cap\Omega'} V\bullet\nu_A\,\mathrm{d}\mathcal{H}^{n-1}(x) = \int_{\partial^*A\cap\Omega'} 1 - V\bullet\nu_A\mathrm{d}\mathcal{H}^{n-1}(x) = 0 \leqslant \int_{\partial^*F\cap\Omega'} 1 - V\bullet\nu_F\mathrm{d}\mathcal{H}^{n-1}(x)$$

by exploiting $|V\bullet\nu_F|\leqslant 1$ on Ω and $V\bullet\nu_A = 1$ on $\partial^*A\cap\Omega$. Similar arguments as before yield

$$\mathcal{F}^\Omega_{\operatorname{div} V}(A) \leqslant \mathrm{P}(F,\Omega) - \int_{F\cap(\Omega\backslash\Omega')}(1-\varphi)\operatorname{div} V - \nabla\varphi\cdot V\,\mathrm{d}x - \int_{\partial^*F\cap\Omega} V\bullet\nu_F\,\varphi\,\mathrm{d}\mathcal{H}^{n-1}(x) = \mathcal{F}^\Omega_{\operatorname{div} V}(F)\,,$$

which concludes the claim. □

Remarkably, the reversed statement is also true to some extent. The proof is part of the argument of [73, Theorem 8.2].

Proposition 4.23. *Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter and $H \in \mathrm{L}^1(\mathbb{R}^n)$ a Barozzi curvature of A. Then there exists a vector field $V \in \mathrm{L}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $\|V\|_{\mathrm{L}^\infty(\mathbb{R}^n,\mathbb{R}^n)} \leqslant 1$, $V \bullet \nu_A = 1$ on $\partial^* A$, and with distributional divergence H.*

4.3 Sets of constant variational mean curvature

In order to apply the results of Section 4.2.1 to general $\mathrm{C}^{1,\alpha}$ sets, we collect some information about sets of constant curvature, as they are the basis of the Barozzi curvature definition.
We first recall [60, Theorem 2.3], characterizing sets of constant curvature in certain planar situations, which is based on the analysis of planar Cheeger sets in [58].

Definition 4.24 (No necks)**.** Let $A \subseteq \mathbb{R}^n$ and $r > 0$. We call A a set **with no necks of radius r** if for all balls $\mathrm{B}_r(x_1)$, $\mathrm{B}_r(x_2) \subseteq A$ with x_1, $x_2 \in A$, there exists a curve $c : [0,1] \to A$ with $c(0) = x_1$ and $c(1) = x_2$ such that $\mathrm{B}_r(c(t)) \subseteq A$ for all $t \in [0,1]$. If there exists no such curve, then A is called set with **necks of radius r**.

Remark 4.25. A set $A \subseteq \mathbb{R}^n$ has no necks of radius $r > 0$ if and only if $A \backslash (\partial A)_r$ is path-connected.
This can be easily seen from the fact that a ball of radius r is in A if and only if the center is in $A \backslash (\partial A)_r$.

Theorem 4.26 (Rolling ball)**.** *Let $A \subseteq \mathbb{R}^2$ be a Jordan domain (i.e., a region, which is bounded by an injective curve from $\partial \mathrm{B}_1^2$ to $\mathbb{R}^2$) such that ∂A is a null set. There exists $\kappa > 0$ such that if A has no neck of radius $\frac{1}{\lambda}$ for some $\lambda > \kappa$, then*

$$\bigcup \left\{ \mathrm{B}_{\frac{1}{\lambda}}(x) : x \in A,\, \mathrm{B}_{\frac{1}{\lambda}}(x) \subseteq A \right\}$$

is a set of constant variational mean curvature λ in A, i.e., it minimizes $\mathcal{F}_\lambda$ among measurable subsets of A.

Remark 4.27. The value κ can be identified as the Cheeger constant of A, that is, the minimizer of the fraction $\frac{\mathrm{P}(F)}{|F|}$ among subsets F of A of finite perimeter. It is the largest value in $[0, \infty)$ such that $\emptyset$ minimizes $\mathcal{F}_\kappa$.

There is no such identification for higher dimensional sets. However, it can be shown that the union of all balls in A of a certain radius is a subset of each $\mathcal{C}_{A,\lambda}$, see the following result from [86, Lemma 2.4].

Proposition 4.28 (Minimizers contain balls)**.** *Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter and finite volume and let $r > 0$. For all $\lambda \geqslant \frac{n}{r}$ and all balls $\mathrm{B}_r(x) \subseteq A$ with $x \in A$, the set $\mathcal{C}_{A,\lambda} \cup \mathrm{B}_r(x)$ is a set of constant variational mean curvature λ in A. If $\lambda > \frac{n}{r}$, then $\mathrm{B}_r(x) \subseteq \mathcal{C}_{A,\lambda}$.*

With the previous proposition at hand, we are able to take a closer look at Massari's regularity theorem. In particular, apart from the optimality analysis of the parameters and a reverse formulation of the theorem, one could also wonder whether there exist weaker results for $p \leqslant n$. For $p < n$ the answer is still fully open. In the special case $p = n$, De Giorgi conjectured that Massari's regularity theorem still holds true when replacing the $\mathrm{C}^{1,\alpha}$ regularity with bilipschitz regularity (i.e., that the boundary of a set with local variational mean curvature in L^n is locally parametrizable by a bilipschitz map, defined on an $(n-1)$-dimensional ball except for singularities of dimension at most $n - 8$). In [51, Theorem 1.1, Theorem 1.2], weak density results in this direction have been achieved, which were improved by [69, Theorem 0.1], where it is shown that, if $n \leqslant 7$, sets with variational mean curvatures in L^n are $\mathrm{C}^{0,\alpha}$ regular for all $\alpha \in (0,1)$. Finally, in [4, Theorem 4.10, Theorem 5.2], this result has been lifted to all dimensions with possible singularities of dimension at most $n - 8$, and, for $n = 2$, the De Giorgi conjecture has been confirmed.
Similar to [81, Theorem 4.3], we can draw a link between $\mathrm{C}^{0,\alpha}$ sets and having variational mean curvature in L^p.

Example 4.29. Let $n \geqslant 2$, $\alpha \in (0,1)$ and define the set

$$A := \{x \in \mathbb{R}^{n-1} \times [0,1] \,:\, |\overline{x}|^\alpha < |x_n|\} \cup \mathrm{B}_{\sqrt{1+\alpha^{-2}}}\left(\left(0, 1 + \frac{1}{\alpha}\right)\right).$$

Then $\partial^* A = \partial A\backslash\{0\}$ is of class $\mathrm{C}^{0,\alpha}$ and not $\mathrm{C}^{0,\beta}$ for all $\beta > \alpha$. Furthermore, for all balls $B \subseteq \mathbb{R}^n$ with $A \Subset B$, there exists a local variational mean curvature H of A in B in $\mathrm{L}^p(B)$ for all $p \in [1, n)$ such that $\alpha > 1 + \frac{p-n}{p+1}$.

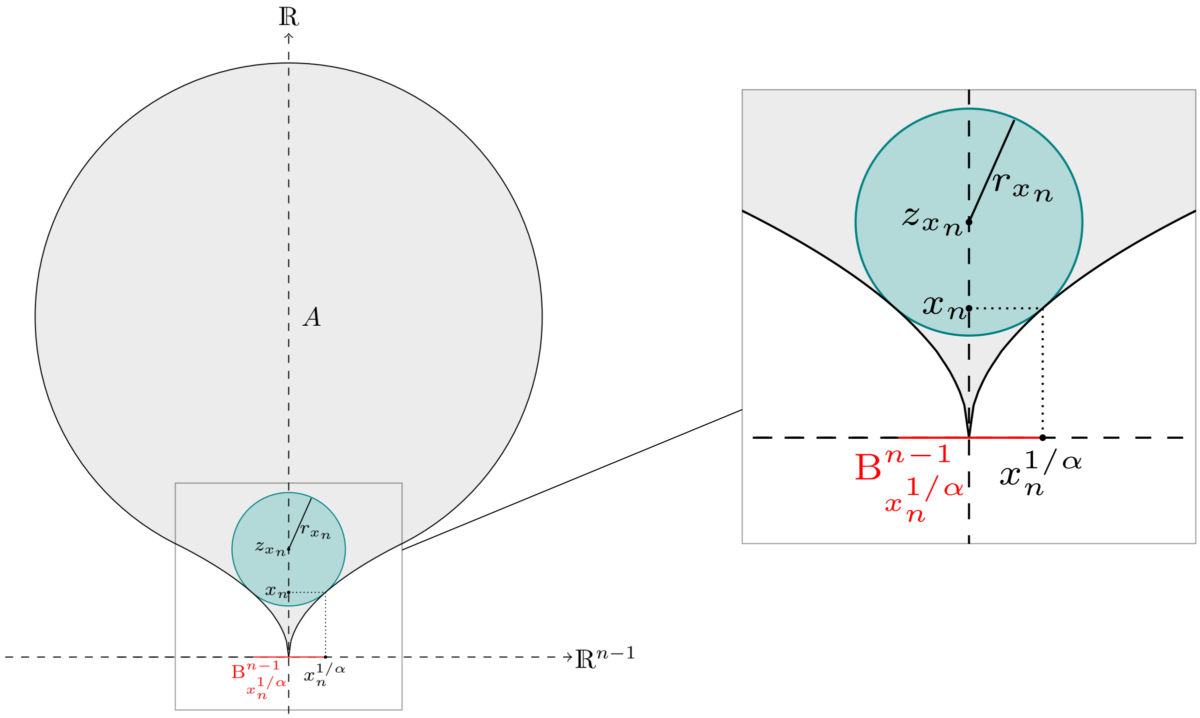


Figure 4.1: Counterexample to Massari-type regularity for Hölder continuous sets

According to Lemma 4.15, it suffices to investigate the integrability of H_A on A and $H_{B\backslash A}$ on B. By Proposition 4.28 and the construction of A and B, $H_{B\backslash A}$ is bounded, such that it is left to consider H_A in A. In order to estimate Barozzi's optimal variational mean curvature, we notice that for $x \in A$, the unique ball that is fully included in A and that has contact points with ∂A at height x_n has radius

$$r_{x_n} = \sqrt{\left(\frac{1}{\alpha} x_n^{\frac{2-\alpha}{\alpha}}\right)^2 + x_n^{\frac{2}{\alpha}}} \geqslant \frac{1}{\alpha} x_n^{\frac{2-\alpha}{\alpha}} \tag{4.2}$$

and center

$$z_{x_n} = \left(0, x_n + \frac{1}{\alpha} x_n^{\frac{2-\alpha}{\alpha}}\right)$$

for all $x_n \in (0, \varepsilon)$ with small $\varepsilon > 0$ fixed, compare Figure 4.1. Hence, Proposition 4.28 gives that for a.e. $x_n \in (0, \varepsilon)$, it holds $H_A \leqslant \frac{n}{r_{x_n}}$ on $A_{x_n} = \mathrm{B}^{n-1}_{x_n^{1/\alpha}}$. We can then apply Fubini's theorem and the estimate (4.2) to obtain

$$\int_{A\cap[0,\varepsilon]^n} |H_A|^p \,\mathrm{d}x \leqslant \int_0^\varepsilon \left|\mathrm{B}^{n-1}_{x_n^{1/\alpha}}\right| \left(\frac{n}{r_{x_n}}\right)^p \mathrm{d}x_n \leqslant \mathrm{c} \int_0^1 x_n^{\frac{n-1+p(\alpha-2)}{\alpha}} \,\mathrm{d}x_n,$$

for $\mathrm{c} := \omega_{n-1} \left(\frac{n}{\alpha}\right)^p$. The latter term is finite if and only if $\alpha > 1 + \frac{p-n}{p+1}$.

Since a ball of radius $r > 0$ in $\mathbb{R}^n$ has constant mean curvature $\frac{n-1}{r}$, the question arises of whether Proposition 4.28 even holds true when replacing $\frac{n}{r}$ with $\frac{n-1}{r}$. The following result applies the strategy of the proof of [86, Lemma 2.4], replacing the constant variational mean curvature $\frac{n}{r}$ by a divergence curvature.

Lemma 4.30. *Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter and finite volume and let $r > 0$. For all $\lambda > \frac{n-1}{r}$, there exists $\delta = \delta(r, \lambda) > 0$ such that for all balls $\mathrm{B}_r(x)$ in A with $\omega_n r^n - |\mathrm{B}_r(x) \cap \mathcal{C}_{A,\lambda}| < \delta$, the ball $\mathrm{B}_r(x)$ is a subset of $\mathcal{C}_{A,\lambda}$ except for null sets.*

Proof. In order to prove the claim, we first compute for $s \in [0, r]$ and $x \in \mathbb{R}^n$, the difference

$$\mathrm{P}(\mathrm{B}_r(x)) - \mathrm{P}(\mathrm{B}_s(x)) = n\omega_n(r^{n-1} - s^{n-1}) = n\omega_n \int_s^r (n-1)t^{n-2}\,\mathrm{d}t = \int_{\mathrm{B}_r(x)\setminus\mathrm{B}_s(x)} \frac{n-1}{|x-y|}\,\mathrm{d}y\,. \tag{4.3}$$

Since λ is strictly greater than $\frac{n-1}{r}$, we can fix $\delta > 0$ small enough such that $|\mathrm{B}_r| - |F \cap \mathrm{B}_r(x)| < \delta$ implies

$$s_F := \left(\frac{1}{\omega_n}|F \cap \mathrm{B}_r(x)|\right)^{\frac{1}{n}} \in \left(\frac{n-1}{\lambda}, r\right] \qquad \text{for all } x \in \mathbb{R}^n \text{ and } F \subseteq A \text{ in } \mathcal{M}^n\,.$$

In particular, s_F is chosen such that any ball of this radius has volume $|F \cap \mathrm{B}_r(x)|$. Now, Lemma 2.38 gives

$$\begin{aligned}
\mathcal{F}_\lambda(F \cup \mathrm{B}_r(x)) - \mathcal{F}_\lambda(F) &= \mathrm{P}(\mathrm{B}_r(x) \cup F) - \mathrm{P}(F) - \lambda|\mathrm{B}_r(x)\setminus F| \\
&= \mathrm{P}(\mathrm{B}_r(x) \cup F) - \mathrm{P}(F) - \lambda|\mathrm{B}_r(x)\setminus \mathrm{B}_{s_F}(x)| \\
&\leqslant \mathrm{P}(\mathrm{B}_r(x)) - \mathrm{P}(\mathrm{B}_r(x) \cap F) - \lambda|\mathrm{B}_r(x)\setminus \mathrm{B}_{s_F}(x)|
\end{aligned}$$

for all $F \subseteq A$ of finite perimeter. Taking into account the isoperimetric inequality and (4.3), it follows

$$\begin{aligned}
\mathcal{F}_\lambda(F \cup \mathrm{B}_r(x)) - \mathcal{F}_\lambda(F) &\leqslant \mathrm{P}(\mathrm{B}_r(x)) - \mathrm{P}(\mathrm{B}_{s_F}(x)) - \lambda|\mathrm{B}_r(x)\setminus \mathrm{B}_{s_F}(x)| \\
&= \int_{\mathrm{B}_r(x)\setminus\mathrm{B}_{s_F}(x)} \frac{n-1}{|x-y|}\,\mathrm{d}y - \lambda|\mathrm{B}_r(x)\setminus \mathrm{B}_{s_F}(x)| \\
&\leqslant \left(\frac{n-1}{s_F} - \lambda\right)|\mathrm{B}_r(x)\setminus \mathrm{B}_{s_F}(x)|.
\end{aligned}$$

The last term is either negative, or $\mathrm{B}_r(x)$ is a subset of F already, i.e., it holds $s_F = r$. Thus, replacing F by a minimizer $\mathcal{C}_{A,\lambda}$ of $\mathcal{F}_\lambda$, we conclude the corresponding subset relation. ☐

Proposition 4.31 (Improved sets of constant curvature contain balls)**.** *Let $A \subseteq \mathbb{R}^n$ be a set of finite perimeter and finite volume, $r > 0$ and $\lambda > \frac{n-1}{r}$. If A has no necks of radius r and if $\mathcal{C}_{A,\lambda}$ already contains at least one ball of radius r, then $\mathcal{C}_{A,\lambda}$ contains all balls of radius r in A.*

Proof. We first choose δ as in Lemma 4.30. Since $\mathrm{B}_r(x) \subseteq A$, there exists for all $y \in A$ a curve $c : [0,1] \to A$ connecting x and y such that $\mathrm{B}_r(c(t)) \subseteq A$ for all $t \in (0,1)$. Exploiting the continuity of c, we find for each $t \in [0,1]$ and $\varepsilon > 0$ some $\gamma = \gamma(t,\varepsilon) > 0$ such that for all $s_1, s_2 \in (t-\gamma, t+\gamma)$, it holds $|c(s_1) - c(s_2)| < \varepsilon$. By the compactness of $c([0,1])$, we find a finite cover of $[0,1]$ by these intervals. Making $\varepsilon > 0$ small enough such that $|\mathrm{B}_r| - |\mathrm{B}_r(z_1) \cap \mathrm{B}_r(z_2)| \leqslant \delta$ holds for all z_1, $z_2 \in \mathbb{R}^n$ with $|z_1 - z_2| \leqslant \varepsilon$, we can apply Lemma 4.30 finitely many times to ensure $\mathrm{B}_r(y) \subseteq \mathcal{C}_{A,\lambda}$. ☐

Remark 4.32.

i) We can lift the previous proposition up to the case $\lambda = \frac{n-1}{r}$ under some additional assumptions. In particular, if we additionally assume that A has no necks of radius r and that there exists a ball of radius r that is contained in the maximal minimizer $\mathcal{C}^+_{A,\lambda}$ (defined in Remark 4.8), then the set $\mathcal{C}^+_{A,\lambda}$ contains all balls in A of radius r.

ii) Under some additional assumptions, we can also ensure that all balls in A of radius $r = \frac{n-1}{\lambda}$ are in $\mathcal{C}^-_{A,\lambda}$. In particular, if there exists a decreasing null sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ and $x \in A$ with $\mathrm{B}_{r+\varepsilon_1}(x) \subseteq \mathcal{C}^-_{A,\lambda}$, if A has no necks of radius $r + \varepsilon_k$ for all $k \in \mathbb{N}$, and if for each ball $\mathrm{B}_r(y) \subseteq A$, we find some $y_k \in A$ with $\mathrm{B}_{r+\varepsilon_k}(y_k) \subseteq A$ such that $|\mathrm{B}_r(y) \triangle \mathrm{B}_{r+\varepsilon_k}(y_k)| \to 0$ as $k \to \infty$, then

$$\bigcup\{\mathrm{B}_{r+\varepsilon_k}(y) : \, y \in A,\, \mathrm{B}_{r+\varepsilon_k}(y) \subseteq A\} \subseteq \mathcal{C}^-_{A,\lambda} \qquad \text{for all } k \in \mathbb{N}\,.$$

In fact, the imposed assumptions on A give

$$\bigcup\{\mathrm{B}_r(y) : \, y \in A,\, \mathrm{B}_r(y) \subseteq A\} = \bigcup_{k=1}^{\infty}\bigcup\{\mathrm{B}_{r+\varepsilon_k}(y) : \, y \in A,\, \mathrm{B}_{r+\varepsilon_k}(y) \subseteq A\} \subseteq \mathcal{C}^-_{A,\lambda}\,.$$

iii) If A from the proposition is additionally open, we can choose r small enough such that we find a ball of radius $r = \frac{n}{\lambda}$ in the set A. According to Proposition 4.28, this ball is contained in $\mathcal{C}_{A,\lambda}$. Hence, Proposition 4.31 applies and also ensures that the balls $\mathrm{B}_{\tilde{r}}$ are in $\mathcal{C}_{A,\lambda}$ for $\tilde{r} = \frac{n-1}{n} r = \frac{n-1}{\lambda}$ (under the certain assumptions on A as in i) and ii)).

iv) It can be shown that the no-neck assumption, and the assumption that a suitable ball in $\mathcal{C}_{A,\lambda}$ is found, is necessary, see [59, Example 4.1, Example 4.5] and [55, Section 6].

Example 4.33. In order to see why the assumptions of Remark 4.32 ii) are necessary, we define the two-dimensional set

$$A := \mathrm{B}_1 \cup [0,2] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$$

and consider

$$A(a) := \mathrm{B}_1 \cup \bigcup_{t \in (0, 1+a)} \mathrm{B}_{\frac{1}{4}}((t,0))$$

for all $a \in \left[0, \frac{3}{4}\right]$, see Figure 4.2.

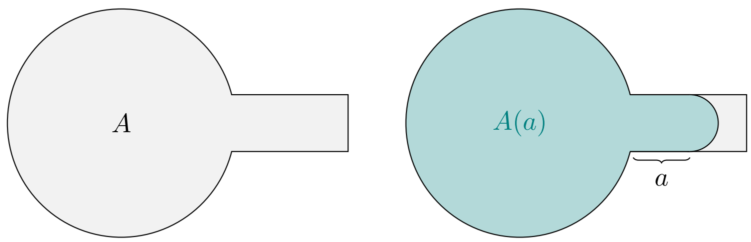


Figure 4.2: Counterexample of the limit case of the improved minimizers contain balls lemma

We record that, according to Theorem 4.26, the union of balls $A\left(\frac{3}{4}\right)$ of radius $\frac{1}{4}$ in A minimizes $\mathcal{F}_4$ among all measurable subsets of A, where we used the fact that $\mathrm{B}_1 \subseteq \mathcal{C}_{A,\lambda}$ for all $\lambda > 2$ by Proposition 4.28, which gives $4 > \kappa$. Furthermore, since

$$\mathcal{F}_4(A(a)) = \mathrm{P}(A(a)) - 4|A(a)| = \mathrm{P}(A(b)) + 2(b-a) - 4|A(b)| - 4\left((b-a)2\frac{1}{4}\right) = \mathcal{F}_4(A(b))$$

holds for all $a \in \left[0, \frac{3}{4}\right]$ and all $b \in \left[a, \frac{3}{4}\right]$, we cannot conclude that every set of constant mean curvature 4 in A contains all balls of radius $\frac{1}{4}$ in A.

It might be important to obtain some results for sets of constant mean curvatures in unbounded sets as well, hence the following statement for the complement of bounded sets is given.

Lemma 4.34. *Let $E \subseteq \mathbb{R}^n$ be a bounded set of finite perimeter. There exists a function $f \in \mathrm{L}^1(\mathbb{R}^n)$, which is a.e. positive on $\mathbb{R}^n \backslash E$ such that for all $\lambda > 1$, Lemma 4.30 and Proposition 4.31 apply for $\mathbb{R}^n \backslash E$ instead of A and $\mathcal{C}^f_{\mathbb{R}^n \backslash E, \lambda}$ instead of $\mathcal{C}_{A,\lambda}$.*

Proof. It suffices to show that Lemma 4.30 applies in this case, as the proof of Proposition 4.31 remains the same.

First, we show that for suitable balls $B \supseteq E$, the set $\mathcal{C}^f_{\mathbb{R}^n \backslash E, \lambda}$ contains the complement of B. To this end, we choose $R > 0$ large enough such that $\operatorname{dist}(E, \partial \mathrm{B}_R) > 2r$. According to Remark 4.2, there exists a global variational mean curvature $H \in \mathrm{L}^1(\mathbb{R}^n)$ of $\mathbb{R}^n \backslash \mathrm{B}_R$, which is a.e. positive on $\mathbb{R}^n \backslash \mathrm{B}_R$. We define $f := 2H \mathbb{1}_{\mathbb{R}^n \backslash \mathrm{B}_R} + \mathbb{1}_{\mathrm{B}_R \backslash E}$. For all test sets $F \subseteq \mathbb{R}^n \backslash \mathrm{B}_R$ of finite perimeter, we estimate

$$\mathrm{P}(\mathrm{B}_R) - \mathrm{P}(F) \leqslant \int_{(\mathbb{R}^n \backslash \mathrm{B}_R) \backslash F} H \,\mathrm{d}x\,. \tag{4.4}$$

We now show, that for all $\lambda \geqslant 1$, the set $\mathbb{R}^n \backslash \mathrm{B}_R$ is in $\mathcal{C}^f_{\mathbb{R}^n \backslash E, \lambda}$. Indeed, according to the construction and Lemma 2.38, it holds

$$\mathcal{F}^f_\lambda(\mathbb{R}^n \backslash \mathrm{B}_R) - \mathcal{F}^f_\lambda(F) = \mathrm{P}(\mathbb{R}^n \backslash \mathrm{B}_R) - \mathrm{P}(F) - \lambda \int_{(\mathbb{R}^n \backslash \mathrm{B}_R) \backslash F} f \,\mathrm{d}x \leqslant \int_{(\mathbb{R}^n \backslash \mathrm{B}_R) \backslash F} H - \lambda f \,\mathrm{d}x$$

for all $F \subseteq \mathbb{R}^n \backslash \mathrm{B}_R$ of finite perimeter. The last term is either strictly negative, or $F = \mathbb{R}^n \backslash \mathrm{B}_R$. Clearly, for F replaced by $\mathcal{C}^f_{\mathbb{R}^n \backslash \mathrm{B}_R, \lambda}$, the latter holds true and the claim follows by Remark 4.6 for all $\lambda > 1$.

By the choice of R, it is left to show $\mathrm{B}_r(y) \subseteq \mathcal{C}^f_{\mathbb{R}^n \backslash E, \lambda}$ for all $y \in \mathbb{R}^n \backslash E$ such that $\mathrm{B}_r(y) \subseteq \mathrm{B}_R \backslash E$. Analogously to the proof of Lemma 4.30, we consider $F \subseteq \mathbb{R}^n \backslash E$ of finite perimeter and define $\delta > 0$ and $s_F \in \left(\frac{n-1}{\lambda}, r\right]$ as in the aforementioned proof such that $|\mathrm{B}_r| - |\mathrm{B}_r(y) \backslash F| < \delta$ implies

$$\begin{aligned} \mathcal{F}^f_\lambda(F \cup \mathrm{B}_r(y)) - \mathcal{F}^f_\lambda(F) &\leqslant \mathrm{P}(F \cup \mathrm{B}_r(y)) - \mathrm{P}(F) - \lambda \int_{\mathrm{B}_r(y) \backslash F} f \,\mathrm{d}x \\ &\leqslant \mathrm{P}(\mathrm{B}_r(y)) - \mathrm{P}(F \cap \mathrm{B}_r(y)) - \lambda |\mathrm{B}_r(y) \backslash F| \\ &\leqslant \left(\frac{n-1}{s_F} - \lambda \right) |\mathrm{B}_r(y) \backslash F| \,. \end{aligned}$$

It follows $\mathrm{B}_r(y) \subseteq \mathcal{C}^f_{\mathbb{R}^n \backslash E, \lambda}$ and by the choice of B, we conclude the claim. ☐

There are regularity statements for sets of constant variational mean curvature in A. The particular case in which we are interested is the behavior near ∂A if A is regular. The following theorem has been stated in many versions throughout the literature. The statements i) and ii) were originally shown for perimeter minimizers in A with volume constraints in [52, Theorem 2], which can be transferred to minimizers of $\mathcal{F}_\lambda$ in A straightforwardly. Then, iii) follows from the first observation of this chapter. Statement iv) is [35, Teor. 1.1].

Theorem 4.35 (Regularity of sets of constant curvature)**.** *Let $A \subseteq \mathbb{R}^n$ be a bounded set of class C^1 and let $\lambda > 0$. The set $\mathcal{C}_{A,\lambda}$ of constant mean curvature λ in A has a representative that satisfies the following properties.*

i) *The set $\partial^* \mathcal{C}_{A,\lambda} \cap A$ is an $(n-1)$-dimensional analytic manifold.*

ii) *The Hausdorff dimension of the singularity set $(\partial \mathcal{C}_{A,\lambda} \backslash \partial^* \mathcal{C}_{A,\lambda}) \cap A$ is at most $n - 8$.*

iii) *The set $\partial^* \mathcal{C}_{A,\lambda} \cap A$ has classical constant mean curvature $\frac{\lambda}{n-1}$.*

iv) *There exists a neighborhood N of ∂A such that $N \cap \partial \mathcal{C}_{A,\lambda}$ is of class C^1.*

Chapter 5

Variational mean curvatures of $\mathrm{C}^{1,\alpha}$ sets

In this chapter, we prove that all sets A of class $\mathrm{C}^{1,\alpha}$ with $\alpha \in (0,1]$ have a variational mean curvature H in L^p for p depending on α. We first consider an approach via the divergence curvature, introduced in Section 4.2.2. This will not provide the optimal result concerning the dependency of the corresponding Hölder and the integrability exponent. Thus, we make use of the optimality of Barozzi's curvature from Section 4.2.1. According to the construction, we need to consider sets $\mathcal{C}_{A,\lambda}$ of constant variational mean curvature λ in A, which are strongly connected to the union of balls of radius $\frac{n-1}{\lambda}$ in A, see Section 4.3. Hence, we will first investigate those unions in order to analyze Barozzi curvatures suitably. Later in Section 5.3, we will give counterexamples to underpin the optimality of the statement. The main result, however, is only stated for either convex or planar sets, and leaves open the limit case.

5.1 Approach via trace theory

We start the analysis by giving a rather simple trace theory approach, using the divergence curvature. Apart from the case $\alpha = 1$, this does not lead to the optimal integrability exponent. However, this method affirms the idea of requiring good properties of curvature of $\mathrm{C}^{1,\alpha}$ sets.

5.1.1 Preliminaries

In the following, we collect some helpful results from trace theory for Sobolev functions, which can be found, for instance, in [61, Chapter 18] and [23, Chapter 3.2].

Definition 5.1 (Trace space for Sobolev functions)**.** Let $A \subseteq \mathbb{R}^n$ be an open C^1 set such that ∂A is compact. Let also $p \in [1,\infty)$ and $s \in (0,1)$. For a function $u \in \mathrm{L}^p(\partial A, \mathcal{H}^{n-1})$, we define the **$\mathrm{W}^{s,p}(\partial A)$-norm** of u by

$$\|u\|_{\mathrm{W}^{s,p}(\partial A)} := \|u\|_{\mathrm{L}^p(\partial A,\mathcal{H}^{n-1})} + \left[\int_{\partial A}\int_{\partial A} \frac{|u(x)-u(y)|^p}{|x-y|^{n-1+sp}}\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}\mathcal{H}^{n-1}(x)\right]^{\frac{1}{p}}$$

and the **fractional Sobolev space $\mathrm{W}^{s,p}(\partial A)$** as the set of functions in $\mathrm{L}^p(\partial A, \mathcal{H}^{n-1})$ with finite $\mathrm{W}^{s,p}(\partial A)$-norm.

Remark 5.2. The space $(\mathrm{W}^{s,p}(\partial A), \|\cdot\|_{\mathrm{W}^{s,p}(\partial A)})$ is a Banach space, where s, p and A are defined as in the previous definition.

Lemma 5.3. *For an open set $A \subseteq \mathbb{R}^n$ of class C^1 with compact boundary, $\alpha \in (0,1)$ and $p \in \left[1, \frac{1}{1-\alpha}\right)$, it holds that $\mathrm{C}^{0,\alpha}(\partial A) \subseteq \mathrm{W}^{1-\frac{1}{p},p}(\partial A)$.*

Proof. For $u \in \mathrm{C}^{0,\alpha}(\partial A)$ the compactness of ∂A implies $u \in \mathrm{L}^p(\partial A)$. Furthermore, we have

$$\int_{\partial A}\int_{\partial A} \frac{|u(x)-u(y)|^p}{|x-y|^{n-2+p}}\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}\mathcal{H}^{n-1}(x) \leqslant \int_{\partial A}\int_{\partial A} C|x-y|^{\alpha p-n+2-p}\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}\mathcal{H}^{n-1}(x)$$

for some constant $C>0$. Since ∂A is bounded, we can ensure that the right side of the integral is finite if $\alpha p - n + 2 - p > -(n-1)$, which is exactly the case for $p < \frac{1}{1-\alpha}$. □

The following theorem identifies fractional Sobolev functions and traces and it can be found in [61, Theorem 18.34] and in [23, Proposition 3.31].

Theorem 5.4 (Characterization of trace functions)**.** *Let $A \subseteq \mathbb{R}^n$ be an open set of class C^1 with compact boundary set and let $p \in (1,\infty)$. It holds that*

$$\mathrm{W}^{1-\frac{1}{p},p}(\partial A, \mathbb{R}^N) = \mathrm{Tr}_A \mathrm{W}^{1,p}(A,\mathbb{R}^N)\,.$$

In order to transfer this concept to variational mean curvatures of $\mathrm{C}^{1,\alpha}$ functions, we make use of the divergence curvature, introduced in Section 4.2.2 via the following results from [81, Lemma 3.1, Lemma 3.2].

Lemma 5.5 (Hölder continuity of the unit normal)**.** *Let $\alpha \in (0,1)$, $u \in \mathrm{C}^1(\overline{A})$ with $\|\nabla u\|_{\mathrm{L}^\infty(A,\mathbb{R}^n)} < \infty$. Then it holds that*

$$u \in \mathrm{C}^{1,\alpha}(A) \qquad \text{if and only if} \qquad \nu_{\mathrm{Sub}\,u} \in \mathrm{C}^{0,\alpha}(\mathrm{Graph}\,u, \mathbb{R}^{n+1})\,.$$

Lemma 5.6 (Good graph representation of $\mathrm{C}^{1,\alpha}$ sets)**.** *Let $f \in \mathrm{C}^{1,\alpha}(A)$ with $\alpha \in (0,1]$, $\sup_A |\nabla f| < \infty$ and Hölder constant C. For all $x_0 \in \mathrm{Graph}\,f$, we find a constant $R>0$ such that for all $x \in \mathrm{C}_R(x_0) \cap \mathrm{Graph}\,f$, there exists a rotation T with $Tx = x$ such that $\mathrm{C}_R(x) \cap T\mathrm{Graph}\,f$ is the graph of a $\mathrm{C}^{1,\alpha}$ function on $\mathrm{B}_R(\overline{x})$ which has vanishing gradient at $\overline{x}$ and has Hölder constant in $[0,KC]$ for some constant $K>0$ only depending on α.*

Remark 5.7. By following the proofs of [81, Lemma 3.1, Lemma 3.2], it is easy to show that it holds $K \leqslant 16$ in every case.

5.1.2 Integrability of variational mean curvatures for $\mathrm{C}^{1,\alpha}$ sets

We are now prepared to show the existence of a variational mean curvature for every $\mathrm{C}^{1,\alpha}$ function.

Theorem 5.8 (Weak integrability for variational mean curvatures of $\mathrm{C}^{1,\alpha}$ sets)**.** *Let A be a bounded open $\mathrm{C}^{1,\alpha}$ set for some $\alpha \in (0,1]$. If $\alpha < 1$ and $p \in \left[1, \frac{1}{1-\alpha}\right)$, then there exists a local variational mean curvature of A in $\mathbb{R}^n$ which is in $\mathrm{L}^p(\mathbb{R}^n)$. If $\alpha = 1$, then A has a local variational mean curvature in $\mathbb{R}^n$ which is in $\mathrm{L}^\infty(\mathbb{R}^n)$.*

Remark 5.9. We can refine the result from the previous theorem as follows: For $\alpha \in (0,1)$ and A as in the theorem, there exists a local variational mean curvature of A in B, which is in $\mathrm{L}^p(B)$ for all $p \in \left[1, \frac{1}{1-\alpha}\right)$ and for all balls $B \subseteq \mathbb{R}^n$ with $A \Subset B$.
In the theorem, the variational mean curvature depends on the choice of p, however, we can apply Theorem 4.12 to ensure that the Barozzi curvature has optimal integrability on A among all global variational mean curvatures of A. In particular, if A is a bounded open $\mathrm{C}^{1,\alpha}$ set with $\alpha < 1$, then the previous statement implies $H_A \in \mathrm{L}^p(A)$ for all $p \in \left[1, \frac{1}{1-\alpha}\right)$, since we can convert every local variational mean curvature of A to a global one, while only changing the values outside A by Lemma 4.15. Furthermore, for every open ball $B \subseteq \mathbb{R}^n$ with $A \Subset B$, the set $B\backslash\overline{A}$ is of class $\mathrm{C}^{1,\alpha}$ such that the same argument implies $H_{B\backslash A} \in \mathrm{L}^p(B\backslash A)$ for all $p \in \left[1, \frac{1}{1-\alpha}\right)$. Finally, Remark 4.14 yields the existence of a local variational mean curvature of A in $\mathrm{L}^p(B)$ for all $p \in \left[1, \frac{1}{1-\alpha}\right)$.

Proof. We first consider $\alpha < 1$. Since ∂A is compact and can be locally represented by subgraphs of $\mathrm{C}^{1,\alpha}$ functions, we can apply the Lemmas 5.5, 5.3 and Theorem 5.4 to record

$$\nu_A \in \mathrm{C}^{0,\alpha}(\partial A, \mathbb{R}^n) \subseteq \mathrm{W}^{1-\frac{1}{p},p}(\partial A, \mathbb{R}^n) = \mathrm{Tr}_A \mathrm{W}^{1,p}(A, \mathbb{R}^n) = \mathrm{Tr}_{\mathbb{R}^n \backslash A} \mathrm{W}^{1,p}(\mathbb{R}^n \backslash A, \mathbb{R}^n)\,.$$

for $p \in \left[1, \frac{1}{1-\alpha}\right)$. Hence, we find Sobolev vector fields $v_1 \in \mathrm{W}^{1,p}(A, \mathbb{R}^n)$ and $v_2 \in \mathrm{W}^{1,p}(\mathbb{R}^n \backslash A, \mathbb{R}^n)$ both having trace ν_A on ∂A. Since ∂A is C^1, the function v, defined as v_1 on A and as v_2 on $\mathbb{R}^n \backslash A$, is in $\mathrm{W}^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ and has inner and outer trace ν_A on ∂A. We further define the truncation

$$T : \mathbb{R}^n \to \mathbb{R}^n,\, x \mapsto \begin{cases} \frac{x}{|x|}, & \text{if } |x| > 1\,, \\ x, & \text{else,} \end{cases}$$

which is a Lipschitz continuous vector field. By the chain rule for Sobolev functions, it holds that $V := T \circ v$ is in $\mathrm{W}^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$. In the following, we show that $\mathrm{Tr}_A V = \nu_A = \mathrm{Tr}_{\mathbb{R}^n \backslash A} V$ holds on ∂A. The Lipschitz continuity of T gives

$$|V(x) - \nu_A(y)| = |T(v(x)) - T(\nu_A(y))| \leqslant \mathrm{Lip}(T)|v(x) - \nu_A(y)|$$

for all $x \in \mathbb{R}^n$ and $y \in \partial A$. Since v has inner and outer trace ν_A on ∂A, the same follows for V by definition. Now, Lemma 4.21 and Proposition 4.22 show that $\operatorname{div} V \in \mathrm{L}^p(\mathbb{R}^n)$ is a local variational mean curvature of A in $\mathbb{R}^n$.

For $\alpha = 1$, we find $\nu_A \in \mathrm{C}^{1,1}(\partial A)$ as above. In this case, Kirszbraun's theorem ensures the existence of a Lipschitz continuous extension v of ν_A defined on $\mathbb{R}^n$. Then V defined as before is in $\mathrm{W}^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ and together, Lemma 4.21 and Proposition 4.22 imply again that $\operatorname{div} V \in \mathrm{L}^\infty(\mathbb{R}^n)$ is a local variational mean curvature of A in $\mathbb{R}^n$. □

5.2 Approach via Barozzi's curvature

5.2.1 Preliminary and auxiliary results

We now collect the results required to prove a stronger version of Theorem 5.8.

Definition 5.10 (Free rolling)**.** A set $A \subseteq \mathbb{R}^n$ satisfies the **free rolling condition for the radius $r > 0$** if $A \backslash (\partial A)_r$ is path-connected and for every $x \in \partial A$, there exists a closed ball B of radius r in A with $x \in B$.

Remark 5.11.

i) If $A \subseteq \mathbb{R}^n$ is a closed set such that $A \backslash (\partial A)_r$ is path-connected, then A satisfies the free rolling condition if and only if for every $x \in A$ there exists a closed ball $B \subseteq A$ of radius r with $x \in B$.

ii) By Remark 4.25, $A \backslash (\partial A)_r$ being path-connected is equivalent to A having no necks of radius r. Hence, every set that satisfies the free rolling condition has no neck of the corresponding radius.

iii) Not every set that has no neck of radius $r > 0$ satisfies the free rolling condition for the radius r. A simple instance is the square.

The following theorem can be found in a stronger version in [90, Theorem 1].

Theorem 5.12 (Free rolling characterization)**.** *Let $A \subseteq \mathbb{R}^n$ be a path-connected, non-empty and compact set and let $r_0 > 0$. Then the following statements are equivalent:*

i) *A and $\overline{\mathbb{R}^n \backslash A}$ both satisfy the free rolling conditions for all radii $r \in (0, r_0)$,*

ii) *∂A is an $(n-1)$-dimensional C^1 manifold and $\nu_A \in \mathrm{C}^{0,1}(\partial A, \mathbb{R}^n)$ has Lipschitz constant $\frac{1}{r_0}$.*

The following Lemma is a technical inequality, which we will apply later in Section 5.2.3.

Lemma 5.13. *For $0 \leqslant t \leqslant r \leqslant R$ the following inequality is satisfied*

$$\sqrt{R^2 - t^2} - R - \left(\sqrt{r^2 - t^2} - r\right) \leqslant R t^2 \frac{R - r}{(r^2 - t^2)^{\frac{3}{2}}}\,. \tag{5.1}$$

Proof. We first notice that $\theta \mapsto \frac{s\theta}{(s^2-\theta^2)^{\frac{3}{2}}}$ is increasing on $(0,s)$ for all $s \geqslant 0$. Hence, we conclude

$$\begin{aligned}\sqrt{R^2-t^2} - R - \left(\sqrt{r^2-t^2} - r\right) &= \int_r^R \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \frac{\mathrm{d}}{\mathrm{d}\theta} \sqrt{s^2-\theta^2}\,\mathrm{d}\theta\,\mathrm{d}s \\ &= \int_r^R \int_0^t \frac{s\theta}{(s^2-\theta^2)^{\frac{3}{2}}}\,\mathrm{d}\theta\,\mathrm{d}s \\ &\leqslant \int_r^R \int_0^t \frac{s\,t}{(s^2-t^2)^{\frac{3}{2}}}\,\mathrm{d}\theta\,\mathrm{d}s \\ &\leqslant \int_r^R \frac{R\,t^2}{(r^2-t^2)^{\frac{3}{2}}}\,\mathrm{d}s = Rt^2 \frac{R-r}{(r^2-t^2)^{\frac{3}{2}}}\,,\end{aligned}$$

which proves the claim. □

Remark 5.14. We later apply the inequality from the previous lemma for values which additionally satisfy $t \leqslant \frac{3}{5}r$ and $R \leqslant \frac{3}{2}r$. In this case, we can verify $\sqrt{r^2-t^2} \geqslant \frac{4}{5}r$, such that (5.1) turns into

$$\sqrt{R^2-t^2} - R - \left(\sqrt{r^2-t^2} - r\right) \leqslant \frac{3}{2}\left(\frac{5}{4}\right)^3 t^2 \frac{R-r}{r^2} \leqslant 3t^2 \frac{R-r}{r^2}\,. \tag{5.2}$$

Definition 5.15. Let $A \subseteq \mathbb{R}^n$ and let $\lambda > 0$. We call the open set

$$A_{\mathrm{B},\lambda} := \bigcup \left\{ \mathrm{B}_{\frac{n-1}{\lambda}}(x) : x \in \mathbb{R}^n,\, \mathrm{B}_{\frac{n-1}{\lambda}}(x) \subseteq A \right\}$$

the **union of balls with curvature $\boldsymbol{\lambda}$ in $\boldsymbol{A}$**.

Lemma 5.16 (Convergence of the union of balls with respect to the Hausdorff distance)**.** *Let $A \subseteq \mathbb{R}^n$ be a measurable, open and bounded set. Then*

$$\partial A_{\mathrm{B},\lambda} \to \partial A \qquad \textit{with respect to the Hausdorff distance as } \lambda \to \infty\,.$$

Proof. Let $\varepsilon > 0$. We divide the proof in two parts.

Step 1: Show $\sup_{x \in \partial A_{\mathrm{B},\lambda_1}} \operatorname{dist}(x, \partial A) < \varepsilon$ for some $\lambda_1 > 0$:
For $\lambda_1 > 2\frac{n-1}{\varepsilon}$, we obtain

$$A \backslash \overline{(\partial A)_\varepsilon} \Subset \bigcup \left\{ \mathrm{B}_{\frac{\varepsilon}{2}}(x) :\, x \in A\backslash(\partial A)_\varepsilon \right\} \subseteq A_{\mathrm{B},\lambda_1} \subseteq A\,.$$

This implies $\partial A_{\mathrm{B},\lambda_1} \subseteq (\partial A)_\varepsilon$ and $\sup_{x \in \partial A_{\mathrm{B},\lambda_1}} \operatorname{dist}(x, \partial A) < \varepsilon$.

Step 2: Show $\sup_{x \in \partial A} \operatorname{dist}(x, \partial A_{\mathrm{B},\lambda_2}) < \varepsilon$ for some λ_2:
We first exploit the openness and boundedness of A to obtain that $A_{\mathrm{B},\lambda} \to A$ in L^1 as $\lambda \to \infty$. The compactness of ∂A implies the existence of finitely many $x_1, \dots, x_k \in \partial A$ such that $\partial A \subseteq B := \bigcup_{i=1}^k \mathrm{B}_{\frac{\varepsilon}{2}}(x_i)$. Furthermore, the openness of A ensures $|\mathrm{B}_{\frac{\varepsilon}{2}}(x_i) \cap A| > 0$ for all $i \in \{1, \dots, k\}$. Taking into account the first observation of this step, it holds $|\mathrm{B}_{\frac{\varepsilon}{2}}(x_i) \cap A_{\mathrm{B},\lambda}| \to |\mathrm{B}_{\frac{\varepsilon}{2}}(x_i) \cap A| > 0$ such that for each $i \in \{1, \dots, k\}$ we find $\lambda_{x_i} > 0$ with $|\mathrm{B}_{\frac{\varepsilon}{2}}(x_i) \cap A_{\mathrm{B},\lambda_{x_i}}| > 0$. For $\lambda_2 := \max\{\lambda_{x_1}, \dots, \lambda_{x_k}\}$, we even find $|\mathrm{B}_{\frac{\varepsilon}{2}}(x_i) \cap A_{\mathrm{B},\lambda_2}| > 0$ for all $i \in \{1, \dots, k\}$. Since $A_{\mathrm{B},\lambda_2} \subseteq A$, there exists some $y_i \in \mathrm{B}_{\frac{\varepsilon}{2}}(x_i) \cap \partial A_{\mathrm{B},\lambda_2}$ for all $i \in \mathbb{N}$, which implies

$$\sup_{x \in \partial A} \operatorname{dist}(x, \partial A_{\mathrm{B},\lambda_2}) \leqslant \sup_{x \in B} \operatorname{dist}(x, \partial A_{\mathrm{B},\lambda_2}) < \varepsilon\,.$$

Taking into account $A_{\mathrm{B},\lambda} \subseteq A_{\mathrm{B},\gamma} \subseteq A$ for all $0 < \lambda \leqslant \gamma$, we have $\operatorname{dist}_{\mathcal{H}}(\partial A, \partial A_{\mathrm{B},\lambda}) < \varepsilon$ for all $\lambda > \max\{\lambda_1, \lambda_2\}$. □

5.2.2 Properties of the union of balls in two dimensions

The main purpose of this section is to prove the following crucial properties for the union of balls of curvature $\lambda \gg 0$ in a two-dimensional C^1-set.

Proposition 5.17 (Properties of the union of balls inside planar C^1 sets)**.** *Let $A \subseteq \mathbb{R}^2$ be an open, bounded and connected set of class C^1. Then there exists $\lambda^* > 0$ such that the following properties are satisfied for all $\lambda > \lambda^*$.*

i) The set $A_{\mathrm{B},\lambda}$ is of class C^1,

ii) it holds $\mathrm{P}(A_{\mathrm{B},\lambda}) \leqslant \mathrm{P}(A)$,

iii) for all $x \in \partial A_{\mathrm{B},\lambda} \cap A$, there exists $y \in A$ such that there exists a connected component of $\partial \mathrm{B}_{\frac{1}{\lambda}}(y) \backslash \partial A$, which contains x and this component is a subset of $\partial A_{\mathrm{B},\lambda} \cap A$.

We will use this result later in Sections 5.2.3 and 5.2.4 to conclude properties of the Barozzi curvature in the two-dimensional case. In higher dimensions, no result of this kind is known so far.
The proof of Proposition 5.17 is divided into several subresults.

Lemma 5.18. *Let $A \subseteq \mathbb{R}^2$ be an open and connected set of class C^1 with compact boundary. Then there exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$, the set A has no necks of radius $\frac{1}{\lambda}$.*

Proof. We first assume that ∂A is connected and consider a (possibly unbounded) set A of class C^1 with compact boundary.
To show the claim, we first choose a suitable cover of ∂A. According to the compactness and the regularity of ∂A, we find a finite set $\{x_1, \dots, x_k\} \subseteq \partial A$ of boundary points such that for each $i \in \{1, \dots, k\}$, there exists some $\delta_i > 0$, a rotation $T \in \mathcal{O}(2)$, an open neighborhood of the form $N_i := x_i + T[(-\delta_i, \delta_i) \times (-3\delta_i, 3\delta_i)]$, and a C^1 function $f_i : (-\delta_i, \delta_i) \to (-\delta_i, \delta_i)$ with $f_i(0) = f_i'(0) = 0$ such that $A \cap N_i$ is the (possibly rotated and translated) supergraph of f_i and $\partial A \Subset N := \bigcup_{i=1}^k N_i$. Furthermore, we can even assume that f_i is C^1 in a neighborhood of $[-\delta_i, \delta_i]$ for all $i \in \{1, \dots, k\}$. To be able to refer to adjacent neighborhoods, we assume that $\overline{(\partial A) \cap N_i^+} \subseteq \partial A$ is covered by N_i and N_{i+1} and that $\overline{(\partial A) \cap N_i^-} \subseteq \partial A$ is covered by N_{i-1} and N_i for all $i \in \{1, \dots, k-1\}$, where $N_i^+ := x_i + T[(0, \delta_i) \times (-3\delta_i, 3\delta_i)]$ and $N_i^- := x_i + T[(-\delta_i, 0) \times (-3\delta_i, 3\delta_i)]$.
In what follows, we fix $\delta_0 > 0$ small enough such that $\delta_0 < \min\{\delta_1, \dots, \delta_k\}$ and such that $(\partial A)_{2\delta_0} \Subset N$. We further assume that $\delta \in (0, \delta_0)$ is bounded from above by the Lebesgue number of the open cover $N_1, \dots, N_k$ of the compact set $\overline{(\partial A)_{2\delta_0}}$. Moreover, by the assumptions on N_i, we can make δ small enough such that for all $i \in \{1, \dots, k\}$, there exists a ball of radius $\frac{\delta}{2}$ in the set $A \cap (\partial A)_{2\delta_0} \cap N_i \cap N_{i+1}$.
We now consider two balls $\mathrm{B}_r(y)$ and $\mathrm{B}_r(z)$ of some radius $r \in \left(0, \frac{\min\{\delta,1\}}{2}\right)$ in A and show the existence of a curve that connects y and z such that every ball with radius r and center on the curve is contained in A.
Since $\operatorname{dist}(\partial N, (\partial A)_{2\delta_0}) > 0$, it suffices to consider balls $\mathrm{B}_r(y) \subseteq N \cap (\partial A)_{2\delta_0} \cap A$ and $\mathrm{B}_r(z) \subseteq N \cap (\partial A)_{2\delta_0} \cap A$ and show that they satisfy the previously given condition. Indeed, if $\mathrm{B}_r(y)$ is not a subset of $A \cap N$, then we can consider the nearest point projection $p \in \partial A$ of y on ∂A and shift $\mathrm{B}_r(y)$ along $\nu_A(p)$ until it is fully contained in $A \cap (\partial A)_{2\delta_0} \cap N$, which is possible by the choice of r. This shift can be realized by a curve that satisfies the previous conditions and since we can reparametrize and compose two curves C_1, C_2 on $[0,1]$ with $C_1(1) = C_2(0)$ that satisfy $\mathrm{B}_r(C_i(t)) \subseteq A$ for all $t \in [0,1]$ and $i \in \{1,2\}$ to a curve C parametrized over $[0,1]$ that connects $C_1(0)$ and $C_2(1)$ and satisfies $\mathrm{B}_r(C(t)) \subseteq A$ for all $t \in [0,1]$, it suffices to consider $\mathrm{B}_r(y) \subseteq N \cap (\partial A)_{2\delta_0} \cap A$ and $\mathrm{B}_r(z) \subseteq N \cap (\partial A)_{2\delta_0} \cap A$ in what follows.
Since $2r$ is smaller than the Lebesgue number of the open cover $N_1, \dots, N_k$ of $\overline{(\partial A)_{2\delta_0}}$, we observe that $\mathrm{B}_r(z) \subseteq N_i$ and $\mathrm{B}_r(y) \subseteq N_j$ for some $i, j \in \{1, \dots, k\}$. We now assume w.l.o.g. $i \leqslant j$ and make a case distinction.
We first treat the case $i = j$ and show that $A \cap N_i$ has no necks of any radius for all $i \in \{1, \dots, k\}$. To this end, we assume that it holds $N_i = (-\delta_i, \delta_i) \times (-3\delta_i, 3\delta_i)$ and construct the curve C which first connects z with $(z_1, 2\delta_i)$, then $(z_1, 2\delta_i)$ with $(y_1, 2\delta_i)$, and finally $(y_1, 2\delta_i)$ with y via three straight lines. Reparametrizing the curve, we can ensure that C is defined on $[0,1]$ and by $\delta < \delta_i$, we find $\mathrm{B}_r(C(t)) \subseteq A \cap N_i$ for all $t \in [0,1]$.
In the case $i + 1 = j$, we make use of the case $i = j$ and of the fact that for all $i \in \{1, \dots, k\}$ there exists a ball with center a and radius r inside $A \cap (\partial A)_{2\delta_0} \cap N_i \cap N_{i+1}$, to find a curve C_1 parametrized over $\left[0, \frac{1}{2}\right]$

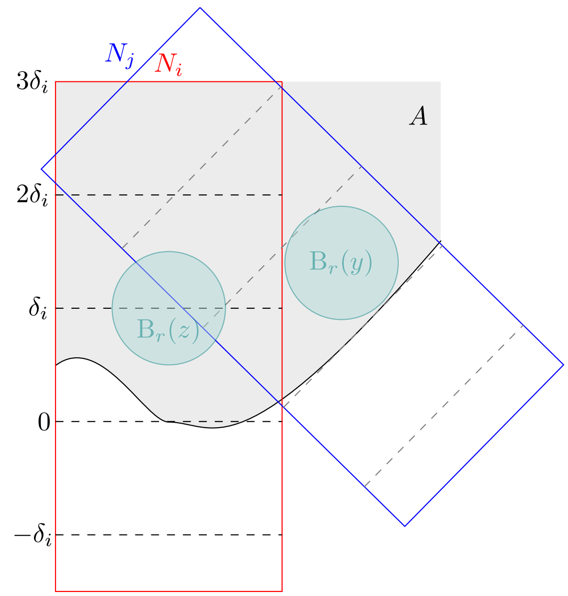


Figure 5.1: No small necks in C^1 sets

which connects z and a such that $\mathrm{B}_r(C_1(t)) \subseteq A \cap N_i$ for all $t \in \left[0, \frac{1}{2}\right]$. With the same argumentation, we find a curve C_2 parametrized over $\left[\frac{1}{2}, 1\right]$ which connects a and y and satisfies $\mathrm{B}_r(C_2(t)) \subseteq A \cap N_{i+1}$ for all $t \in \left[\frac{1}{2}, 1\right]$. Hence, the curve

$$C(t) := \begin{cases} C_1(t) & \text{if } t \in \left[0, \frac{1}{2}\right] , \\ C_2(t) & \text{if } t \in \left(\frac{1}{2}, 1\right] , \end{cases}$$

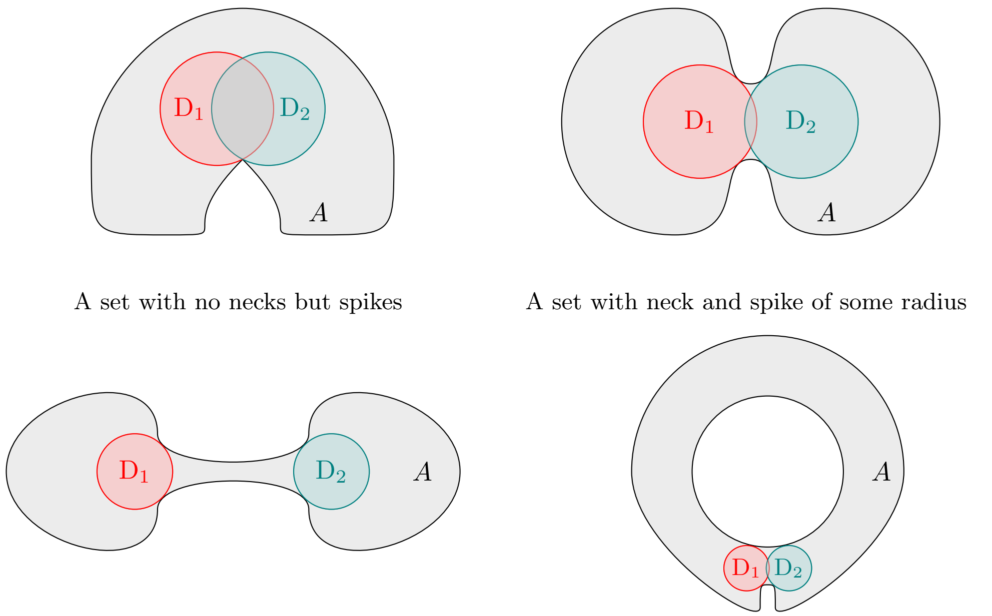
connects z and y and satisfies $\mathrm{B}_r(C(t)) \subseteq A$ for all $t \in [0,1]$.
The case $i+1 < j$ follows from repeating the previous argument iteratively.
With this, the claim is proven for sets A such that ∂A has only one connected component, where we choose $\lambda^* \geqslant \frac{2}{\min\{\delta,1\}}$.
If the boundary of A is not connected, we can conclude from its regularity that there exists at most finitely many connected components of ∂A. Indeed, if we have infinitely many connected components $\Gamma_1, \Gamma_2, \ldots$ of ∂A, then we could construct a sequence $(x_k)_{k\in\mathbb{N}} \subseteq \partial A$ with $x_k \in \Gamma_k$ for all $k \in \mathbb{N}$. Since ∂A is compact, we find a converging subsequence with limit $x \in \partial A$, which we also denote by $(x_k)_{k\in\mathbb{N}}$. By construction, we find infinitely many subsets of distinct connected components in every neighborhood of x, in contradiction to the regularity of ∂A. Thus, we may assume that $\Gamma_1, \ldots, \Gamma_k$ are the connected components of ∂A.
Since A is connected, we find open and connected C^1 sets $A_1, \ldots, A_k$ with $\partial A_i = \Gamma_i$ for all $i \in \{1, \ldots, k\}$ and such that $A = A_1 \backslash \bigcup_{i=2}^k \overline{A_i}$. In the following, we denote $F_1 = A_1$ and $F_i = \mathbb{R}^2 \backslash \overline{A_i}$ for all $i \in \{2, \ldots, k\}$. As shown in the first part of the proof, for every $i \in \{1, \ldots, k\}$ there exists $\lambda_i^* > 0$ such that F_i has no necks of radius $\frac{1}{\lambda}$ for all $\lambda > \lambda_i^*$. We further define

$$\gamma := \min\left\{\operatorname{dist}(\Gamma_i, \Gamma_j) : i \neq j \in \{1, \ldots, k\}\right\} > 0 \qquad \text{and} \qquad \lambda^* > \max\left\{\frac{2}{\gamma}, \lambda_1^*, \ldots, \lambda_k^*\right\} .$$

We now obtain by Remark 4.25 that $F_i \backslash (\partial F_i)_{\frac{1}{\lambda}}$ has at most one path-connected component for all $i \in \{1, \ldots, k\}$ and for all $\lambda > \lambda^*$. Since $\frac{1}{\lambda} < \frac{\gamma}{2}$, we have $\operatorname{dist}\big((\partial F_i)_{\frac{1}{\lambda}}, (\partial F_j)_{\frac{1}{\lambda}}\big) > 0$ for all $i \neq j$ in $\{1, \ldots, k\}$ such that $A \backslash (\partial A)_{\frac{1}{\lambda}} = \bigcap_{i=1}^k F_i \backslash (\partial F_i)_{\frac{1}{\lambda}}$ is path-connected as well for all $\lambda > \lambda^*$. Together with Remark 4.25, we have shown the claim. $\square$

We introduce a new notion, which is similar to the no-neck notion, but prevents bottlenecks which formally fulfill the no-neck condition by allowing balls to approach from the other side, see Figure 5.2.

Definition 5.19 (Spike)**.** A two-dimensional set $A \subseteq \mathbb{R}^2$ is said to have **no spike of reach $r > 0$** if for all open balls $D_1 \neq D_2$ of radius r in A, we find another open ball D_3 of radius r in A with $\overline{D}_1 \cap \overline{D}_2 \subseteq D_3$.



A set with no necks but spikes

A set with neck and spike of some radius

A set with neck and no spike of some radius

A set with no neck and spike of some radius

Figure 5.2: Connection between spikes and necks

Lemma 5.20. *Let $A \subseteq \mathbb{R}^2$ be a bounded open set of class C^1. Then there exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$, the set A has no spikes of reach $\frac{1}{\lambda}$.*

Proof. If the claim is not true, we could find a bounded open C^1 set $A \subseteq \mathbb{R}^2$ such that for all $k \in \mathbb{N}$, we find a radius $r_k \in \left(0, \frac{1}{k}\right)$, and two different balls D_{k_1}, D_{k_2} of radius r_k in A such that there exists no open ball of radius r_k in A which contains $\overline{D}_{k_1} \cap \overline{D}_{k_2}$. We choose $x_k \in \overline{D}_{k_1} \cap \overline{D}_{k_2}$ arbitrarily, constructing a sequence in the compact set $\overline{A}$. Thus, there exists $x \in \overline{A}$ such that $x_k \to x$ as $k \to \infty$ (where we possibly restrict to a subsequence, which we denote by $(x_k)_{k \in \mathbb{N}}$ again).

Since $\mathrm{B}_{3r_k}(x_k)$ is not contained in A for all $k \in \mathbb{N}$, we conclude $\operatorname{dist}(x_k, \partial A) \leqslant 3r_k$, such that it follows

$$\operatorname{dist}(x, \partial A) \leqslant \lim_{k\to\infty} |x - x_k| + \operatorname{dist}(x_k, \partial A) \leqslant \lim_{k\to\infty} |x - x_k| + \frac{3}{k} = 0\,.$$

In particular, $x \in \partial A$ and we find $\varepsilon, \delta > 0$ and some C^1 function $f : (-\varepsilon, \varepsilon) \to (-\delta, \delta)$ such that $A - x$ is the rotated supergraph of f in $(-\varepsilon, \varepsilon) \times (-2\delta, 2\delta)$. In the remaining proof, we assume w.l.o.g. $x = 0$. Taking k large enough such that $2 \max\left\{|x - x_k|, \frac{1}{k}\right\} < \min\{\varepsilon, \delta\}$, it holds that $D_{k_1} \cup D_{k_2} \Subset N$ and, in particular, $\overline{D}_{k_1} \cap \overline{D}_{k_2} \Subset N$ for $N := A \cap T\left[(-\varepsilon, \varepsilon) \times (-\delta, \delta)\right]$, where we applied the first observation and chose the corresponding rotation T. Since ∂A is C^1, it holds that $\rho := \operatorname{dist}(\overline{D}_{k_1} \cap \overline{D}_{k_2}, \partial A) > 0$. We assume w.l.o.g. that $T = I$ and find that

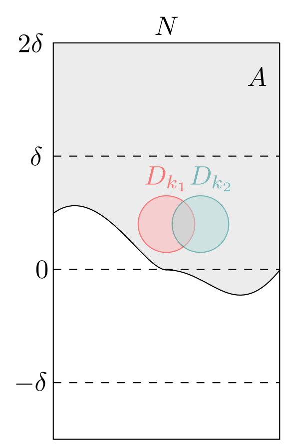


Figure 5.3: No spikes in C^1 sets

$$(D_{k_1} \cap D_{k_2})_\rho \cup \{y \in \mathbb{R} \times (-2\delta, 2\delta) : \, y_1 = z_1,\, y_2 \geqslant z_2 \text{ for some } z \in D_{k_1} \cup D_{k_2}\}$$

is a subset of A. This implies the existence of a suitable ball of radius r_k in A which contains $\overline{D}_{k_1} \cap \overline{D}_{k_2}$, in contradiction to the choice of D_{k_1} and D_{k_2}. □

Lemma 5.21. *Let $A \subseteq \mathbb{R}^2$ be an open set and let $\lambda > 0$. For all $x \in \partial A_{\mathrm{B},\lambda}$ there exists a ball with radius $\frac{1}{\lambda}$ in A which contains x in the boundary. If A has no spike of reach $\frac{1}{\lambda}$ for some $\lambda > 0$ and $x \in \partial A_{\mathrm{B},\lambda}$, then there exists at most one ball B with radius $\frac{1}{\lambda}$ in A with $x \in \partial B$.*

Proof. For $x \in \partial A_{\mathrm{B},\lambda} \setminus \bigcup \left\{ \partial \mathrm{B}_{\frac{1}{\lambda}}(y) : \mathrm{B}_{\frac{1}{\lambda}}(y) \subseteq A \right\}$ we record that $\mathrm{B}_\varepsilon(x) \cap A_{\mathrm{B},\lambda} \neq \emptyset \neq \mathrm{B}_\varepsilon(x) \setminus A_{\mathrm{B},\lambda}$ holds for all $\varepsilon > 0$. By the definition of $A_{\mathrm{B},\lambda}$, we find for each $\varepsilon > 0$ some $y_\varepsilon \in A$, such that $\mathrm{B}_{\frac{1}{\lambda}}(y_\varepsilon) \subseteq A$ and $\mathrm{B}_\varepsilon(x) \cap \mathrm{B}_{\frac{1}{\lambda}}(y_\varepsilon) \neq \emptyset \neq \mathrm{B}_\varepsilon(x) \setminus \mathrm{B}_{\frac{1}{\lambda}}(y_\varepsilon)$. Since $|x - y_\varepsilon| \leqslant \varepsilon + \frac{1}{\lambda}$, there exists a converging subsequence of $(y_\varepsilon)_{\varepsilon > 0}$ (which we also denote by $(y_\varepsilon)_{\varepsilon > 0}$) with limit y in A as $\varepsilon \searrow 0$. By construction, it holds that $\mathrm{B}_{\frac{1}{\lambda}}(y) \subseteq \bigcup_{\varepsilon > 0} \mathrm{B}_{\frac{1}{\lambda}}(y_\varepsilon) \subseteq A_{\mathrm{B},\lambda}$. For any choice of $x_\varepsilon \in \partial \mathrm{B}_{\frac{1}{\lambda}}(y_\varepsilon) \cap \mathrm{B}_\varepsilon(x)$ we find

$$\operatorname{dist}\big(x, \partial \mathrm{B}_{\frac{1}{\lambda}}(y)\big) \leqslant \lim_{\varepsilon \searrow 0} |x - x_\varepsilon| + \operatorname{dist}(x_\varepsilon, \partial \mathrm{B}_{\frac{1}{\lambda}}(y)) = 0\,.$$

Hence $x \in \partial \mathrm{B}_{\frac{1}{\lambda}}(y)$, in contradiction to the choice of x, and therefore existence is proven.
To show uniqueness, we assume $x \in \partial A_{\mathrm{B},\lambda}$ and $x \in \partial \mathrm{B}_{\frac{1}{\lambda}}(y) \cap \partial \mathrm{B}_{\frac{1}{\lambda}}(z)$ for $z \neq y \in A$, which contradicts the assumption of A having no spike of reach $\frac{1}{\lambda}$, and the definition of $A_{\mathrm{B},\lambda}$. □

Lemma 5.22. *Let $A \subseteq \mathbb{R}^2$ be an open set and let $\lambda > 0$. If $\partial A_{\mathrm{B},\lambda} \cap A \cap \partial \mathrm{B}_{\frac{1}{\lambda}}(y) \neq \emptyset$ for some $\mathrm{B}_{\frac{1}{\lambda}}(y) \subseteq A$, then $\partial A \cap \partial \mathrm{B}_{\frac{1}{\lambda}}(y)$ has at least two elements.*

Proof. Let $x \in \partial A_{\mathrm{B},\lambda} \cap A \cap \partial \mathrm{B}_{\frac{1}{\lambda}}(y)$ for some ball $\mathrm{B}_{\frac{1}{\lambda}}(y) \subseteq A$.
If $\partial A \cap \partial \mathrm{B}_{\frac{1}{\lambda}}(y) = \emptyset$, then $\overline{\mathrm{B}}_{\frac{1}{\lambda}}(y) \Subset A$ such that we find slight translations of the ball containing x, which gives $x \in A_{\mathrm{B},\lambda}$.
Hence, it is left to exclude that $\partial A \cap \partial \mathrm{B}_{\frac{1}{\lambda}}(y)$ only contains one element. To this end, we assume $\partial A \cap \partial \mathrm{B}_{\frac{1}{\lambda}}(y) = \{z\}$ for some $z \in \mathbb{R}^2$. Let $r := \frac{1}{\lambda}$ and assume w.l.o.g. $y = 0$ and $z = (0, -r)$. We now consider $0 < \varepsilon < \min\left\{ \frac{r}{2}, \sqrt{2 - \sqrt{2}}\, r, \operatorname{dist}(x, \partial A) \right\}$ and obtain $\mathrm{B}_\varepsilon(z) \subseteq \mathbb{R} \times (-\infty, 0)$. According to the assumptions, it holds that $\mathrm{B}_r \setminus \mathrm{B}_\varepsilon(z) \Subset A$. Hence, there exists $\delta \in (0, \varepsilon)$ such that

$$N := \big(\mathrm{B}_{r+\delta} \setminus \overline{\mathrm{B}}_\varepsilon(z)\big) \cup \mathrm{B}_r \subseteq A\,.$$

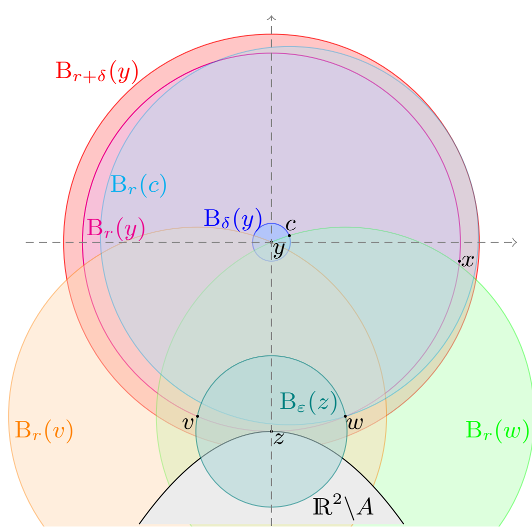


Figure 5.4: Contradiction to having one contact point only

We continue with a case distinction with respect to the sign of x_2.

Case: $x_2 > 0$:
We show that x is in the ball $\mathrm{B}_r((0, \min\{x_2, \delta\}))$. This ball is a subset of N and therefore of A since $\varepsilon < \frac{r}{2}$. If $x_2 < \delta$, it holds that $|(0, \min\{x_2, \delta\}) - x| = |x_1| < r$. If $\delta \leqslant x_2$, we compute

$$|(0, \min\{x_2, \delta\}) - x| = \sqrt{x_1^2 + (x_2 - \delta)^2} < |x| = r\,,$$

such that the claim follows in both cases, as a contradiction is produced.

Case: $x_2 \leqslant 0$:
To show $x \in A_{\mathrm{B},\lambda}$ in this case, we first observe that $\partial \mathrm{B}_\varepsilon(z) \cap \partial \mathrm{B}_r$ contains two points which we denote by v and w such that $v_1 < 0 < w_1$. We assume w.l.o.g. that $x_1 > 0$, i.e., x is contained in the arc of $\partial \mathrm{B}_r$ between w and $(r, 0)$. Since $\varepsilon < \frac{r}{2}$, we find $\mathrm{B}_\varepsilon(z) \subseteq \mathrm{B}_r(v)$ and $\mathrm{B}_\varepsilon(z) \subseteq \mathrm{B}_r(w)$. Moreover, for c being the unique point in $\partial \mathrm{B}_\delta \cap \partial \mathrm{B}_r(w) \backslash \mathrm{B}_r(v)$, it holds that $c_1 > 0$. With $w_1 = \varepsilon\sqrt{1 - \left(\frac{\varepsilon}{2r}\right)^2}$, we can compute that the tangent of $\partial \mathrm{B}_r$ at w has slope $\sqrt{\frac{4\varepsilon^2 r^2 - \varepsilon^4}{4r^4 - 4\varepsilon^2 r^2 + \varepsilon^4}}$, which is strictly smaller than 1 for $\varepsilon < \sqrt{2 - \sqrt{2}}\, r$. The tangent of $\partial \mathrm{B}_r(w)$ has the same slope at 0, leading to $c_2 < c_1$ by convexity of the ball $\mathrm{B}_r(w)$. It follows

$$|c - (r, 0)|^2 = (r - c_1)^2 + c_2^2 < (r - c_2)^2 + c_2^2 = r^2 + 2c_2(c_2 - r) \leqslant r^2 + 2c_2(\delta - r) \leqslant r^2.$$

In particular, $(r, 0) \in \mathrm{B}_r(c)$. Since $|c - (-r, 0)| > r$ holds by the positivity of c_1, this shows that the arc of B_r between w and $(r, 0)$, which contains x by assumption, is contained in $\mathrm{B}_r(c)$. Furthermore, by construction we have $w \in \partial \mathrm{B}_r(c)$. Since $\varepsilon < r$, we can conclude $\mathrm{B}_r(c) \cap \mathrm{B}_\varepsilon(z) \subseteq \mathrm{B}_r(x)$ by elementary geometric considerations. Hence, we have $x \in \mathrm{B}_r(c) \subseteq N \subseteq A$, which contradicts the assumption $x \in \partial A_{\mathrm{B},\lambda}$. □

Lemma 5.23. *Let $A \subseteq \mathbb{R}^2$ be a bounded and connected open set of class C^1 such that ∂A is the union of its connected components $\Gamma_1, \ldots, \Gamma_k$. Moreover, let $\lambda > 0$ be large enough such that $r := \frac{1}{\lambda}$ is strictly smaller than $\frac{1}{2}\min\{\mathrm{dist}(\Gamma_i, \Gamma_j) : i \neq j \in \{1, \ldots, k\}\}$. If A has no necks of radius r and no spikes of reach r, then the following properties are satisfied.*

i) *For all $x \in \partial A_{\mathrm{B},\lambda} \cap A$, there exists a ball $D \subseteq \mathbb{R}^2$ of radius r such that the relatively open connected component of $\partial D \backslash \partial A$ containing x is a connected component of $\partial A_{\mathrm{B},\lambda} \backslash \partial A$.*

ii) *The set $A_{\mathrm{B},\lambda}$ is C^1.*

iii) *It holds that $\mathrm{P}(A_{\mathrm{B},\lambda}) \leqslant \mathrm{P}(A)$.*

Remark 5.24. For bounded sets $A \subseteq \mathbb{R}^n$ and $\lambda > 0$, it holds $\mathrm{P}(\mathcal{C}_{A,\lambda}) \leqslant \mathrm{P}(A) - \lambda|A \backslash \mathcal{C}_{A,\lambda}| \leqslant \mathrm{P}(A)$ by definition of $\mathcal{C}_{A,\lambda}$. However, since our domains are not necessarily Jordan domains, we cannot directly apply Theorem 4.26 to conclude $A_{\mathrm{B},\lambda} = \mathcal{C}_{A,\lambda}$.

Proof. We will first prove property i). Let $x \in \partial A_{\mathrm{B},\lambda} \cap A$. According to Lemmas 5.21 and 5.22, there exists a unique ball $\mathrm{B}_r(y)$ in A with $x \in \partial \mathrm{B}_r(y)$, and $\partial \mathrm{B}_r(y) \cap \partial A$ contains at least two elements. Since $\partial \mathrm{B}_r(y) \cap \partial A$ is compact, the connected component Γ of $\partial \mathrm{B}_r(y) \backslash \partial A$ containing x is relatively open in $\partial \mathrm{B}_r(y)$ and it is a subset of A by construction. It remains to show $\Gamma \subseteq \partial A_{\mathrm{B},\lambda}$.
To this end, we denote by $v, w \in \partial A$ the geometric boundary of Γ and assume w.l.o.g. $y = 0$, $v = (-r, 0)$ and $x_2 < 0$. According to the choice of λ and r, the points v and w belong to one connected component Γ_i of ∂A, where $i \in \{1, \ldots, k\}$. It holds that $w_2 \leqslant 0$, since otherwise, we could shift $\mathrm{B}_r(y)$ in any direction ν with $\nu \in (\partial \mathrm{B}_r \backslash \overline{\Gamma})$ and find $x \in A_{\mathrm{B},\lambda}$ similar to the proof of Lemma 5.22.
In order to argue by contradiction we choose $\tilde{x} \in \Gamma$ arbitrarily and assume $\tilde{x} \in A_{\mathrm{B},\lambda}$. Then we find another ball $\mathrm{B}_r(\tilde{y})$ in A containing $\tilde{x}$. Hence, by the no neck assumption, there exists a curve $C : [0, 1] \to A$ with $C(0) = \tilde{y}$ and $C(1) = y$ and $\mathrm{B}_r(C(t)) \subseteq A$ for all $t \in [0, 1]$. Furthermore, by the choice of x, it holds that $x \notin \mathrm{B}_r(C(t))$ for all $t \in [0, 1]$. Since $\mathrm{dist}(x, \{v, w\}) < 2r$, this means that the curve C does not cross the line between v and x nor the line between w and x. In particular, we showed that $C([0, 1])$ and the straight lines connecting $\tilde{y}$ and $\tilde{x}$ and $\tilde{x}$ and y are fully contained in A and together enclose either v or w, leading to a contradiction of the observation $v, w \in \Gamma_i$.

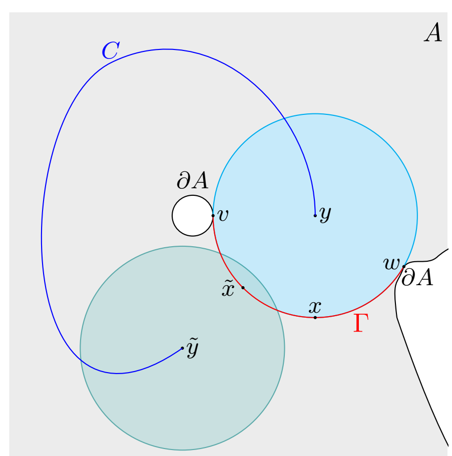


Figure 5.5: Contradiction argument

In order to prove ii), we make a case distinction.
If $x \in \partial A_{\mathrm{B},\lambda} \cap A$ then there exists Γ as in i) including x, which gives that $\partial A_{\mathrm{B},\lambda}$ is C^1 at x.

If $x \in \partial A_{\mathrm{B},\lambda} \cap \partial A$, we notice that there exists some unique ball D of radius r with $x \in \partial D$. If moreover ∂A equals $\partial A_{\mathrm{B},\lambda}$ in a neighborhood of x, the regularity of A transfers to $A_{\mathrm{B},\lambda}$.
The other case is more delicate, however, we ensure in the following that $\partial A_{\mathrm{B},\lambda}$ has a graph representation close to x by Lemma 5.21 and the regularity of ∂A. To this end, we assume w.l.o.g. that there exists a supergraph representation of A near $x = 0$ in the neighborhood $(-\varepsilon, \varepsilon) \times (-\varepsilon, r)$ via the C^1 function $f : (-\varepsilon, \varepsilon) \to (-\varepsilon, r)$ with $f(0) = 0 = f'(0)$ for some $\varepsilon \in \left(0, \frac{r}{4}\right)$, and that $D = \mathrm{B}_r((0,r))$. The assumptions give $f(s) \leqslant r - \sqrt{r^2 - s^2}$ with equality at $s = 0$. As $f'(0) = 0$, we can additionally assume that f only attains values in $\left(-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}\right)$. In particular, for all $(s,t) \in \overline{A_{\mathrm{B},\lambda}} \cap \big((-\varepsilon,\varepsilon) \times (-\varepsilon, r)\big)$, it holds that $t > -\frac{\varepsilon}{4}$.
With this setup, we now show that there exists $\delta > 0$ such that for every $s \in \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$ the value $t := \min\{\tau \in [-\varepsilon, r] : (s,\tau) \in \partial A_{\mathrm{B},\lambda}\}$ satisfies $(s, t + \tilde{\delta}) \in A_{\mathrm{B},\lambda}$ for all $\tilde{\delta} \in (0, \delta]$.

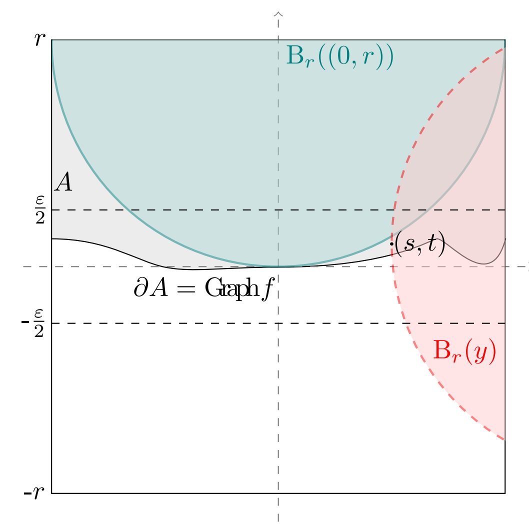


Figure 5.6: Contradiction for $\partial A_{\mathrm{B},\lambda}$ having a graph parametrization

By Lemma 5.21, we find another ball $\mathrm{B}_r(y)$ in A with $(s,t) \in \partial \mathrm{B}_r(y)$ and by the choice of f and ε we have

$$\overline{\mathrm{B}_r(y)} \cap \big((-\varepsilon,\varepsilon) \times \mathbb{R}\big) \subseteq (-\varepsilon, \varepsilon) \times \left(-\frac{\varepsilon}{4}, \infty\right). \tag{5.3}$$

It remains to show that $(s, t+\gamma)$ is contained in the lower half of $\partial \mathrm{B}_r(y)$, i.e., that $t \leqslant y_2 - \gamma$ holds for some $\gamma > 0$ independent of the exact choice of s. To this end, we set $\gamma := \frac{\varepsilon}{4}$ and make another case distinction with respect to the value of y_2.

Case: $y_2 \leqslant -\frac{\varepsilon}{2}$:
If $t > y_2 - \gamma$ holds in this case, then we have $(s, 2y_2 - t) \in \partial \mathrm{B}_r(y) \subseteq \overline{A}$ with $2y_2 - t < y_2 + \gamma \leqslant -\frac{\varepsilon}{4}$, in contradiction to (5.3).

Case: $y_2 > -\frac{\varepsilon}{2}$:
To treat this case, we again assume $t > y_2 - \gamma$ and record that since (s,t) is between $\operatorname{Graph} f$ and the lower part of $\partial \mathrm{B}_r((0,r))$, we find $-\frac{\varepsilon}{4} < t \leqslant r - \sqrt{r^2 - s^2} \leqslant s \leqslant \frac{\varepsilon}{2}$. Hence, we have $y_2 < t + \gamma < \frac{3}{4}\varepsilon$ and with this we can conclude $|y_1| > 2\varepsilon$: Indeed, if $|y_1| \leqslant 2\varepsilon$, we find $0 \in (y_1 - r, y_1 + r)$ with $(0, \tau) \in \partial \mathrm{B}_r(y)$, where

$$\tau = y_2 - \sqrt{r^2 - y_1^2} < \frac{3}{4}\varepsilon - \sqrt{r^2 - (2\varepsilon)^2} < \frac{3}{4}\varepsilon - 2\varepsilon < -\frac{\varepsilon}{4}$$

follows from the choice of ε and is a contradiction to (5.3).
From now on, we assume w.l.o.g. $y_1 > 0$ (in fact, $y_1 > 2\varepsilon > s + \varepsilon$ by the previous analysis) and apply $\varepsilon < y_1 - s \leqslant r$ to estimate

$$y_2 - \sqrt{r^2 - \left(y_1 - \left(s + \frac{\varepsilon}{2}\right)\right)^2} < \frac{3}{4}\varepsilon - \sqrt{r^2 - \left(r - \frac{\varepsilon}{2}\right)^2} = \frac{3}{4}\varepsilon - \sqrt{r\varepsilon - \frac{\varepsilon^2}{4}} < \frac{3}{4}\varepsilon - \sqrt{4 - \frac{1}{4}}\,\varepsilon < -\frac{\varepsilon}{4}.$$

This is a contradiction to (5.3). We conclude the existence of δ as claimed.
From this, we finally derive that $\partial A_{\mathrm{B},\lambda}$ indeed has a graph representation in some neighborhood of the form $(-\tilde{\varepsilon}, \tilde{\varepsilon}) \times \left(-\frac{\delta}{2}, \frac{\delta}{2}\right)$, where $\tilde{\varepsilon} > 0$ is chosen small enough such that $|f| < \frac{\delta}{2}$ and $r^2 - \sqrt{r^2 - (\,\cdot\,)^2} < \frac{\delta}{2}$ on $(-\tilde{\varepsilon}, \tilde{\varepsilon})$. The existence of this graph representation g of $\partial A_{\mathrm{B},\lambda}$ between f and $r - \sqrt{r^2 - (\,\cdot\,)^2}$ enforces it to be differentiable at $x_1 = 0$ with $g'(0) = f'(0) = 0$. With the arbitrariness of x and the first step of this proof, we conclude that g is differentiable on all of $(-\tilde{\varepsilon}, \tilde{\varepsilon})$.
We now estimate the derivative of g close to 0 in order to show that g' is continuous at 0. To this end, we fix $\hat{\varepsilon} \in \left(0, \min\left\{\frac{1}{2}, \frac{r}{2}\right\}\right)$ and $\hat{\delta} \in (0, \min\{\tilde{\varepsilon}, \hat{\varepsilon}\})$ such that $\left|r - \sqrt{r^2 - z^2}\right| + |f(z)| + |f'(z)| < \hat{\varepsilon}$ for all $z \in (-\hat{\delta}, \hat{\delta})$.
In order to bound $|g'|$ on $(-\hat{\delta}, \hat{\delta})$, we make another case distinction and consider w.l.o.g. $s \in (0, \hat{\delta})$.

If $g(s) = f(s)$, then the previous analysis implies $|g'(s)| = |f'(s)| \leqslant \hat{\varepsilon}$.
If $g(s) > f(s)$, then $g(\tilde{s}) = y_1 - \sqrt{r^2 - (y_2 - \tilde{s})^2}$ for some $y \in \mathrm{B}_{2r}$ and all $\tilde{s}$ in a neighborhood of s according to property i). The same property implies that the arc of $\partial \mathrm{B}_r(y) \backslash \partial A$ which contains $(s, g(s))$ is a connected component of $\partial A_{\mathrm{B},\lambda} \backslash \partial A$. Since we chose $\hat{\delta}$ such that the line from $(s, g(s))$ to $(s, s - \sqrt{r^2 - s^2})$ is contained in $A_{\mathrm{B},\lambda}$, there exist points v in $[0, s)$ and $w \in (s, \hat{\delta})$ with $g(\,\cdot\,) = y_1 - \sqrt{r^2 - (y_2 - \,\cdot\,)^2}$ on $[v, w)$, $g(v) = f(v)$ and $g'(v) = f'(v)$. Since $|g'(v)| = |f'(v)| \leqslant \hat{\varepsilon}$ implies $|y_2 - v| \leqslant \frac{r\hat{\varepsilon}}{\sqrt{1+\hat{\varepsilon}^2}} < r\hat{\varepsilon} < r - \hat{\delta}$, we can estimate $|g'| \leqslant \frac{r\hat{\varepsilon}+\hat{\delta}}{\sqrt{r^2-(r\hat{\varepsilon}+\hat{\delta})^2}}$ on (v, w) and therefore at s. By letting $\hat{\varepsilon}$ and $\hat{\delta}$ converge to 0, we conclude that g' is continuous at 0.
It is left to show iii). To this end, we consider the contraction

$$K_y : \mathbb{R}^2 \backslash \mathrm{B}_r(y) \to \partial \mathrm{B}_r(y),\ z \mapsto r\frac{z-y}{|z-y|} + y$$

for $y \in \mathbb{R}^2$. We notice that there are at most countably many connected components $\Xi_1, \Xi_2, \dots$ of $\partial A_{\mathrm{B},\lambda} \backslash \partial A$, which are all relatively open arcs of balls $\mathrm{B}_r(y_1), \mathrm{B}_r(y_2), \dots$ according to i). We further argue that we can find a connected component Σ_i of $\partial A \backslash \overline{\Xi}_i$ such that $\Sigma_i \cup (\partial \mathrm{B}_r(y_i) \backslash \Xi_i)$ is a C^1 submanifold.
Indeed, by the shape of Ξ_i we have that $\overline{\Xi}_i \backslash \Xi_i$ contains exactly two points $a \neq b$, which are both elements the boundary of A. Since $|a - b| \leqslant 2r$, the choice of λ implies that both a and b are contained in one connected component Λ of ∂A. Hence, $\Lambda \backslash \overline{\Xi}_i$ has two connected components Λ_1 and Λ_2 with relative boundary $\{a, b\}$. Since a and b are C^1 contact points of $\partial \mathrm{B}_r(y_i)$ and Λ (as $\mathrm{B}_r(y_i) \subseteq A$), we find that $\Lambda_j \cup (\partial \mathrm{B}_r(y_i) \backslash \Xi_i)$ is a C^1 submanifold for one $j \in \{1, 2\}$ and we denote $\Sigma_i := \Lambda_j$. Having Σ_i leads to $\Xi_i \subseteq K_{y_i}\Sigma_i$ such that it follows

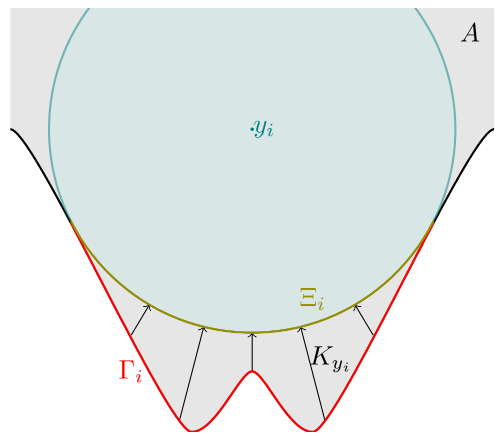


Figure 5.7: Projection argument

$$\begin{aligned}
\mathrm{P}(A_{\mathrm{B},\lambda}) &= \mathcal{H}^1(A_{\mathrm{B},\lambda}) \\
&= \mathcal{H}^1(\partial A_{\mathrm{B},\lambda} \cap \partial A) + \sum_{i=1}^{\infty} \mathcal{H}^1(\Xi_i) \\
&\leqslant \mathcal{H}^1(\partial A_{\mathrm{B},\lambda} \cap \partial A) + \sum_{i=1}^{\infty} \mathcal{H}^1(K_{y_i}\Sigma_i) \\
&\leqslant \mathcal{H}^1(\partial A_{\mathrm{B},\lambda} \cap \partial A) + \sum_{i=1}^{\infty} \mathcal{H}^1(\Sigma_i) \\
&= \mathcal{H}^1(\partial A)\,.
\end{aligned}$$

Together with the regularity of A, we conclude $\mathrm{P}(A_{\mathrm{B},\lambda}) \leqslant \mathrm{P}(A)$. □

Proof of Proposition 5.17. According to Lemmas 5.18 and 5.20, we find $\lambda^* > 0$ such that for all $\lambda > \lambda^*$, the set A has no necks of radius $\frac{1}{\lambda}$ and no spikes of reach $\frac{1}{\lambda}$. Furthermore, since A is C^1 regular, its boundary has at most finitely many connected components. Hence, we can make λ^* large enough such that $\frac{1}{\lambda^*} < \frac{1}{2}\mathrm{dist}(\Gamma, \widetilde{\Gamma})$ holds for all connected components $\Gamma \neq \widetilde{\Gamma}$ of ∂A. For $\lambda > \lambda^*$, Lemma 5.23 implies the claim. □

5.2.3 Behavior of the union of balls in $\mathrm{C}^{1,\alpha}$ sets

In the following, we argue that the union of balls $A_{\mathrm{B},\lambda}$ in $\mathrm{C}^{1,\alpha}$ sets A can be controlled with respect to λ. This enables us to estimate the derivative of $\lambda \mapsto |A_{\mathrm{B},\lambda}|$ in the proof of Theorem 5.33. Note that we assume that A is connected, but its boundary may have different connected components, which is necessary in order to construct the corresponding Barozzi curvature on the complement of A as well.

Proposition 5.25 (Control of rolling balls in $\mathrm{C}^{1,\alpha}$ sets)**.** *Let $f \in \mathrm{C}^{1,\alpha}(\mathbb{R}^{n-1})$ for some $\alpha \in (0,1)$ with Hölder constant $C > 0$ of ∇f. Then there exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$ and $\varepsilon \in \left(0, \frac{1}{3}\lambda\right]$ it holds that*

$$\mathrm{B}_r(0) \subseteq \operatorname{Supp} f \cap [-3r, 3r]^n \quad \textit{and} \quad (0, -r) \in \operatorname{Graph} f \qquad \textit{imply} \qquad \mathrm{B}_{r_\varepsilon}((0, r_\varepsilon - r + \delta)) \subseteq \operatorname{Supp} f \cap [-3r, 3r]^n$$

for all $\delta \in \left[\frac{9}{2}(3Cr)^{\frac{2}{1-\alpha}}\varepsilon, r\right]$, where $r := \frac{n-1}{\lambda}$ and $r_\varepsilon := \frac{n-1}{\lambda - \varepsilon}$.

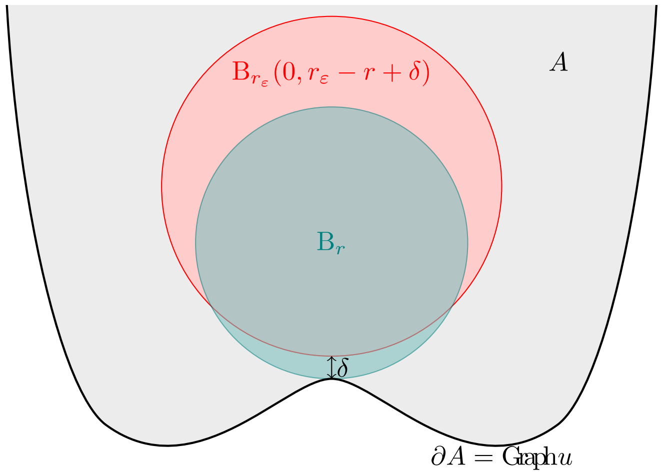


Figure 5.8: Control of balls in $\mathrm{C}^{1,\alpha}$ sets

Proof. In order to approach the proof, we first estimate

$$r_\varepsilon - r = (n-1)\left(\frac{\varepsilon}{\lambda(\lambda - \varepsilon)}\right) \leqslant \frac{3}{2}(n-1)\frac{\varepsilon}{\lambda^2} \leqslant \frac{3}{2(n-1)}\varepsilon r^2. \tag{5.4}$$

By the choice of ε, this leads to $r_\varepsilon - r \leqslant \frac{1}{2}r$. Hence, $r_\varepsilon \leqslant \frac{3}{2}r$ and $r_\varepsilon - r + \delta + r_\varepsilon \leqslant 3r$ for all $\delta \leqslant r$, which ensures $\mathrm{B}_{r_\varepsilon}((0, r_\varepsilon - r + \delta)) \subseteq [-3r, 3r]^n$. The assumptions from the proposition translate to

$$f(x) \leqslant -\sqrt{r^2 - |x|^2} \qquad \text{on } [-r, r] \qquad \text{and} \qquad f(0) = -r\,.$$

We have to show

$$f(x) \leqslant r_\varepsilon - r + \delta - \sqrt{r_\varepsilon^2 - |x|^2} \qquad \text{for all } x \in [-r_\varepsilon, r_\varepsilon]$$

for any $\delta \in \left[\frac{9}{2}(3Cr)^{\frac{2}{1-\alpha}}\varepsilon, r\right]$.

To this end, we make λ^* large enough to ensure $(3Cr)^{\frac{1}{1-\alpha}} \leqslant \frac{3}{5}r$. We distinguish two cases.

Case: $|x| \leqslant (3Cr)^{\frac{1}{1-\alpha}}$:

We can apply Lemma 5.13 in the specific case of Remark 5.14 since the choice of λ^* ensures $|x| \leqslant \frac{3}{5}r$ in the current case. Together with (5.4), we estimate

$$\sqrt{r_\varepsilon^2 - |x|^2} - \sqrt{r^2 - |x|^2} - r_\varepsilon + r \leqslant 3|x|^2\frac{r_\varepsilon - r}{r^2} \leqslant \frac{9}{2(n-1)}|x|^2\varepsilon \leqslant \frac{9}{2}|x|^2\varepsilon \leqslant \frac{9}{2}(3Cr)^{\frac{2}{1-\alpha}}\varepsilon \leqslant \delta\,.$$

Now, for δ as above, the previous estimate, together with the initial assumption, yield

$$f(x) \leqslant -\sqrt{r^2 - |x|^2} \leqslant r_\varepsilon - r + \delta - \sqrt{r_\varepsilon^2 - |x|^2}\,,$$

showing the claim for this case.

Case: $(3Cr)^{\frac{1}{1-\alpha}} \leqslant |x| \leqslant r_\varepsilon$:

Since $f(x) \leqslant -\sqrt{r^2 - |x|^2}$ for all $x \in (-r, r)$ with equality for $x = 0$, it follows $\nabla f(x) = 0$. Thus, the Hölder condition on ∇f implies

$$f(x) + r = f(x) - f(0) \leqslant |f(x) - f(0)| \leqslant \sup_{\mathrm{B}^{n-1}_{|x|}} |\nabla f||x| \leqslant \sup_{\mathrm{B}^{n-1}_{|x|}} |\nabla f - \nabla f(0)||x| \leqslant C|x|^{1+\alpha}\,. \tag{5.5}$$

Furthermore, we can reformulate the lower bound of the case distinction to $C \leqslant \frac{|x|^{1-\alpha}}{3r}$, which yields $C|x|^{1+\alpha} \leqslant \frac{|x|^2}{3r}$. Now, since $r_\varepsilon + \sqrt{r_\varepsilon^2 - |x|^2} \leqslant 2r_\varepsilon \leqslant 3r$, we conclude

$$C|x|^{1+\alpha} \leqslant \frac{|x|^2}{r_\varepsilon + \sqrt{r_\varepsilon^2 - |x|^2}} = r_\varepsilon - \sqrt{r_\varepsilon^2 - |x|^2}\,. \tag{5.6}$$

The combination of (5.5) and (5.6) leads to

$$f(x) \leqslant C|x|^{1+\alpha} - r \leqslant r_\varepsilon - r - \sqrt{r_\varepsilon^2 - |x|^2}\,,$$

which proves the claim. □

The previous proposition shows that if the boundary of a suitably small ball in some $\mathrm{C}^{1,\alpha}$ set A has at least one contact point x with ∂A, then a slightly larger ball which is shifted in the direction $-\nu_A(x)$ is in A as well. The specific δ-shift dependence will later lead to the optimal relation of $|A_{\mathrm{B},\lambda-\varepsilon}|$, $|A_{\mathrm{B},\lambda+\varepsilon}|$ and $|A_{\mathrm{B},\lambda}|$. As we have seen in the previous section for the two-dimensional case, if a ball has multiple contact points with ∂A, we find some arc of the ball contained in A. In this case, one can improve the previous proposition in the sense that one can shift the ball not only in the normal direction of the contact point, but also in the normal direction of every point of the aforementioned arc.
The proof of this result exploits the advantages of considering sets in only two dimensions, which cannot be extended to higher dimensions. It is for this reason that we eventually we concentrate on convex sets in higher dimensions.

Proposition 5.26 (Control of rolling balls in two-dimensional $\mathrm{C}^{1,\alpha}$ sets)**.** *Let $f \in \mathrm{C}^{1,\alpha}(\mathbb{R})$ for some $\alpha \in (0,1)$ with Hölder constant $C > 0$ of ∇f. Then there exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$ and $\varepsilon \in \left(0, \frac{1}{3}\lambda\right]$ it holds that*

$$\mathrm{B}_r(0) \subseteq \mathrm{Sup} f \cap [-3r, 3r]^2 \quad \textit{and} \quad \left(a, -\sqrt{r^2 - a^2}\right), \left(b, -\sqrt{r^2 - b^2}\right) \in \mathrm{Graph} f \cap \partial \mathrm{B}_r(0)\,,$$

for some $-r \leqslant a < 0 < b \leqslant r$, imply

$$\mathrm{B}_{r_\varepsilon}((0, r_\varepsilon - r + \delta)) \subseteq \mathrm{Sup} f \cap [-3r, 3r]^2 \qquad \textit{for all } \delta \in \left[\frac{9}{2}\left(3(32C)r\right)^{\frac{2}{1-\alpha}} \varepsilon, r\right]\,,$$

where $r := \frac{1}{\lambda}$ and $r_\varepsilon := \frac{1}{\lambda - \varepsilon}$.

Proof. In the following, we consider λ^* large enough such that for all $\lambda > \lambda^*$ it holds that

$$r^{\frac{2\alpha}{1-\alpha}} \leqslant \frac{3}{4}\left(2^\alpha C\right)^{-\frac{2}{1-\alpha}}\,. \tag{5.7}$$

Since the graph of f and the boundary of $\mathrm{B}_r(0)$ are both C^1, it holds that $a, b \in (-r, r)$ and we can define the C^1 function

$$g : \mathbb{R} \to [-3r, 3r],\; x \mapsto \begin{cases} f(x) & \text{if } x \notin (a,b)\,, \\ -\sqrt{r^2 - x^2} & \text{if } x \in (a,b)\,. \end{cases}$$

We now show that g is in $\mathrm{C}^{1,\alpha}(\mathbb{R})$ and that the Hölder constant of ∇g is bounded from above by $32C$. To this end, we assume w.l.o.g. $|a| \leqslant |b|$. Since $f(a) > f(0) < f(b)$, the function f attains a local minimum for some $t \in (a,b)$. Hence, as a critical point, we verified $f'(t) = 0$. Since f and g have the same derivative at b, we find

$$\frac{b}{\sqrt{r^2 - b^2}} = |f'(b)| = |f'(b) - f'(t)| \leqslant C|b - t|^\alpha \leqslant C|b - a|^\alpha \leqslant C2^\alpha b^\alpha\,.$$

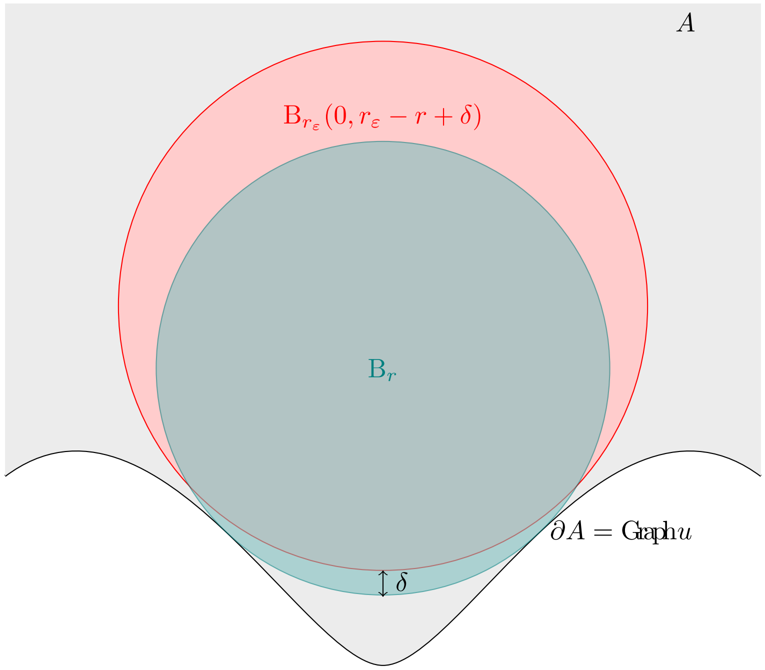


Figure 5.9: Control of balls in two-dimensional $\mathrm{C}^{1,\alpha}$ sets

Rewriting the previous estimate yields

$$b^{1-\alpha} \leqslant C2^\alpha \sqrt{r^2 - b^2} < C2^\alpha r\,. \tag{5.8}$$

Hence, the distance between the contact points and $(0, -r)$ decreases as $r \searrow 0$. Applying (5.7) and (5.8), we obtain the estimate

$$|g''(x)| = \frac{r^2}{(r^2 - x^2)^{\frac{3}{2}}} \leqslant \frac{r^2}{(r^2 - b^2)^{\frac{3}{2}}} \leqslant \frac{r^2}{\left(r^2 - (C2^\alpha r)^{\frac{2}{1-\alpha}}\right)^{\frac{3}{2}}} \leqslant \frac{r^2}{\left(r^2 - \frac{3}{4}r^2\right)^{\frac{3}{2}}} = \frac{8}{r}$$

for all $x \in [a, b]$. Thus, g' is Lipschitz continuous on (a, b) with Hölder constant $\frac{8}{r}$. In particular, for $x, y \in [a, b]$ we have $|x - y| \leqslant 2b$ which leads to

$$|g'(x) - g'(y)| \leqslant \frac{8}{r}|x-y|^{1-\alpha}|x-y|^\alpha \leqslant \frac{8}{r}2^{1-\alpha}b^{1-\alpha}|x-y|^\alpha \leqslant 16C|x-y|^\alpha\,,$$

where we used the estimate (5.8) in the last step. If $x, y \in [-3r, 3r]\backslash[a, b]$, the Hölder continuity of f transfers to g. For $x \in [a, b]$ and $y > b$, it holds that

$$\begin{aligned}
|g'(y) - g'(x)| &\leqslant |g'(y) - g'(b)| + |g'(b) - g'(x)| \\
&= |f'(y) - f'(b)| + |g'(b) - g'(x)| \\
&\leqslant C(y-b)^\alpha + 16C(b-x)^\alpha \\
&\leqslant 16C\big((y-b)^\alpha + (b-x)^\alpha\big) \\
&= 32C(y-x)^\alpha\,.
\end{aligned}$$

The same estimate can be achieved for $x < a$ and $y \in [a, b]$. Hence, $g \in \mathrm{C}^{1,\alpha}([-3r, 3r])$ with Hölder constant $32C$ of ∇g. We now apply Proposition 5.25 on g instead of f, which gives

$$\mathrm{B}_{r_\varepsilon}((0, r_\varepsilon - r + \delta)) \subseteq \operatorname{Sup} g \cap [-3r, 3r]^2 \subseteq \operatorname{Sup} f \cap [-3r, 3r]^2\,. \qquad \square$$

Proposition 5.27. *Let $A \subseteq \mathbb{R}^2$ be an open, bounded and connected $\mathrm{C}^{1,\alpha}$ set for some $\alpha \in (0,1)$. Then there exist constants $C > 0$ and $\lambda^* > 0$ such that for all $\lambda > \lambda^*$ and $\varepsilon \in \left(0, \frac{1}{3}\lambda\right]$, it holds that*

$$A_{\mathrm{B},\lambda} \backslash A_{\mathrm{B},\lambda-\varepsilon} \subseteq \overline{(\partial A_{\mathrm{B},\lambda})_\delta} \qquad \textit{and} \qquad A_{\mathrm{B},\lambda} \backslash A_{\mathrm{B},\lambda-\varepsilon} \subseteq \overline{(\partial A_{\mathrm{B},\lambda-\varepsilon})_\delta}\,, \qquad \textit{where } \delta := C\lambda^{-\frac{2}{1-\alpha}}\varepsilon\,.$$

The constant $C > 0$ only depends on α and A.

Remark 5.28. Proposition 5.27 remains true if A is not connected.
Indeed, if A is a bounded open $\mathrm{C}^{1,\alpha}$ set which is not connected, the regularity and boundedness imply that there exist finitely many connected components $A_1, \ldots, A_k$ of A which are each open and bounded $\mathrm{C}^{1,\alpha}$ sets that satisfy $\operatorname{dist}(A_i, A_j) > 0$ for all $i \neq j$. Hence, $A_{\mathrm{B},\lambda}$ is a disjoint union of $(A_i)_{\mathrm{B},\lambda}$ with $i \in \{1, \ldots, k\}$ for all $\lambda > 0$ and $\operatorname{dist}((A_i)_{\mathrm{B},\lambda}, (A_j)_{\mathrm{B},\lambda}) > 0$ for all $i \neq j$. Since Proposition 5.27 applies for all A_i, it applies for A.

Proof. We first set λ^* large enough such that Propositions 5.17 and 5.26 apply.
By the boundedness and regularity of A, we can cover ∂A by finitely many open rectangles $N_1, \ldots, N_k$ such that for each $i \in \{1, \ldots, k\}$, the set $A \cap N_i$ is the rotated and shifted supergraph of some $\mathrm{C}^{1,\alpha}$ function f_i with Hölder constant $C_i \geqslant 0$ of ∇f_i. Furthermore, we can make all N_i slightly smaller, while maintaining the covering property, but ensuring that each f_i is indeed $\mathrm{C}^{1,\alpha}$ on some open neighborhood of $\overline{N_i}$ and that the Lipschitz constant of f_i on N_i is bounded by 1 (in order to restrict the domain and still maintain the graph representation property). As in the proof of Lemma 5.18, we find $\gamma_0 > 0$ such that $\overline{(\partial A)_{\gamma_0}}$ is covered by $N_1, \ldots, N_k$ as well. We then make $\gamma \in (0, \gamma_0)$ smaller than the Lebesgue number of this covering such that for each $x \in \partial A$, there exists $i \in \{1, \ldots, k\}$ with $\mathrm{B}_\gamma(x) \subseteq N_i$. Applying Lemma 5.6 and Remark 5.7 ensures the existence of some $R \in (0, \gamma)$ such that for each $x \in \partial A$, we find some rotation $T_x \in \mathcal{O}(2)$ and some $\mathrm{C}^{1,\alpha}$ function f_x on $[-R, R]$ such that $(T_x(A - x)) \cap [-R, R]^2$ is the supergraph of f_x on $[-R, R]$. The Hölder constant of ∇f_x then has the upper bound $C_{\max} := 16 \max\{C_i,\, i \in \{1, \ldots, k\}\}$ and it holds that $f_x(0) = f_x'(0) = 0$, i.e., $T_x(-\nu_A(x)) = e_2$ for the exterior unit normal $\nu_A(x)$ of A at x. Moreover, the Lipschitz constant of each f_x is still bounded by 1 for all $x \in \partial A$.
We now consider $r_0 < \frac{1}{3}R$ small enough such that $C_{\max} 2^{1+\alpha} r^{1+\alpha} \leqslant r$ for all $r \in (0, r_0)$, and set $\lambda^* > \frac{1}{r_0}$.
In order to prove the claim for this λ^*, we abbreviate $r = \frac{1}{\lambda}$ and $r_\varepsilon = \frac{1}{\lambda - \varepsilon}$. For $z \in A_{\mathrm{B},\lambda} \backslash A_{\mathrm{B},\lambda-\varepsilon}$, we obtain $\operatorname{dist}(z, \partial A) < r_\varepsilon$ according to the definition of $A_{\mathrm{B},\lambda-\varepsilon}$ as the union of all balls of radius r_ε in A. By the compactness of $\partial A_{\mathrm{B},\lambda}$, there exists $p \in \partial A_{\mathrm{B},\lambda}$ with $\operatorname{dist}(z, \partial A_{\mathrm{B},\lambda}) = |z - p|$.
The C^1 regularity of $\partial A_{\mathrm{B},\lambda}$ from Proposition 5.17 ensures that

$$z = p - t\nu_{A_{\mathrm{B},\lambda}}(p) \tag{5.9}$$

holds for some $t \in (0, r_\varepsilon) \subseteq \left(0, \frac{2}{3}r\right)$. The same proposition yields the existence of some ball $\mathrm{B}_r(y) \subseteq A$ such that $p \in \partial \mathrm{B}_r(y)$ and of two contact points a, b of $\partial \mathrm{B}_r(y)$ and ∂A such that the arc of $\partial \mathrm{B}_r(y) \backslash A_{\mathrm{B},\lambda}$ contains p and is either a subset of $A \cap \partial A_{\mathrm{B},\lambda}$ or $\partial A \cap \partial A_{\mathrm{B},\lambda}$. In the latter case, the proof reduces to the case $a = p = b$.
In the general case we consider the parametrization of ∂A near a via f_a and assume w.l.o.g. $A \cap [-R, R]^2 = \operatorname{Sup}[f_a]$ and $a_1 \leqslant b_1$. According to the intermediate value theorem, there exists $x_1 \in [a_1, b_1]$ with $\nu_A(x) = \nu_{\mathrm{B}_r(y)}(p)$, where $x_2 = f_a(x_1)$, such that there exists a $\mathrm{C}^{1,\alpha}$ supergraph representation f_x of A in some rotated and shifted rectangle of the form $[-3r, 3r]^2$. In particular, it holds $(T_x(A - x)) \cap [-3r, 3r]^2 = \operatorname{Sup}[f_x]$.

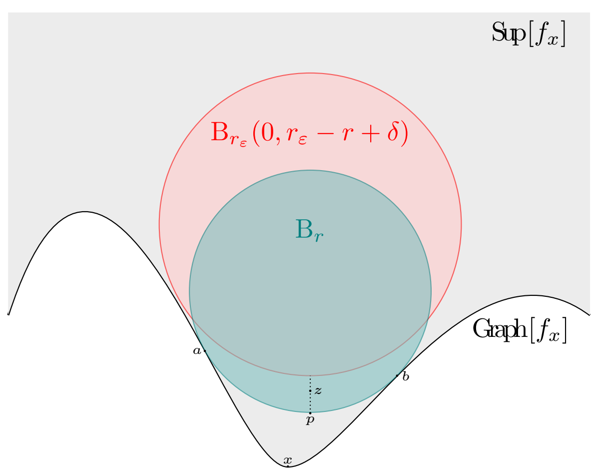


Figure 5.10: Projection argument

We have to ensure that $\mathrm{B}_r(T_x(y - x))$ is contained in $\operatorname{Sup}[f_x] \cap [-3r, 3r]^2$ in order to apply Proposition 5.26. Since $\mathrm{B}_r(y) \subseteq A$, it remains to show $\mathrm{B}_r(T_x(y - x)) \subseteq [-3r, 3r]^2$. To this end, we exploit that it holds

$f_x'(0) = f_x(0) = 0$ and estimate

$$|f_x(s) - f_x(t)| \leqslant \sup_{[s,t]} |\nabla f_x|\,|s-t| = \sup_{[s,t]} |\nabla f_x - \nabla f_x(0)|\,|s-t| \leqslant C_{\max}|s-t|^{1+\alpha} \tag{5.10}$$

for all $s < t$ in $[-3r, 3r]$ such that $0 \in [s,t]$. Since $|a_1 - x_1| \leqslant |a_1 - b_1| \leqslant 2r$, it follows $|s_a| \leqslant 2r$ for $s_a := (T(a-x))_1$. This, the choice of r_0, and (5.10) yield

$$|f_x(s_a)| = |f_x(s_a) - f_x(0)| \leqslant C_{\max}(2r)^{1+\alpha} \leqslant r\,.$$

Now, as $a \in \partial \mathrm{B}_r(y)$, we find $(s_a, f(s_a)) \in \partial \mathrm{B}_r(T_x(y-x))$ and by the previous estimate we have $\mathrm{B}_r(T_x(y-x)) \subseteq \mathbb{R} \times [-3r, 3r]$. By construction, there exists $(0,s)$ in $\mathrm{B}_r(T_x(y-x))$ for some $s \in [-3r, 3r]$, which finally leads to $\mathrm{B}_r(T_x(y-x)) \subseteq [-3r, 3r]^2$.
We now set $C := \frac{9}{2}(3(32C_{\max}))^{\frac{2}{1-\alpha}}$ and $\delta := Cr^{\frac{2}{1-\alpha}}\varepsilon$.
An application of Proposition 5.26 gives $\mathrm{B}_{r_\varepsilon}(T_x(y-x) + (0, r_\varepsilon - r + \delta)) \subseteq \mathrm{Sp}[f_x] \cap [-3r, 3r]^2$. Since $T_x(-\nu_{\mathrm{B}_r(y)}(p)) = T_x(-\nu_A(x)) = e_2$, we obtain $\mathrm{B}_{r_\varepsilon}(y - (r_\varepsilon - r + \delta)\nu_{\mathrm{B}_r(y)}(p)) \subseteq A$. Taking into account $\nu_{\mathrm{B}_r(y)}(p) = \nu_{A_{\mathrm{B},\lambda}}(p)$, this yields $p - \tau\nu_{A_{\mathrm{B},\lambda}}(p) \in A_{\mathrm{B},\lambda-\varepsilon}$ for all $\tau \in (\delta, 2r_\varepsilon + \delta)$. Since (5.9) holds for some $t \in (0, r_\varepsilon)$, we conclude $|z - p| \leqslant \delta$ and $\mathrm{dist}(z, A_{\mathrm{B},\lambda-\varepsilon}) \leqslant \delta$. This finally yields $z \in \overline{(\partial A_{\mathrm{B},\lambda})_\delta}$ and $z \in \overline{(\partial A_{\mathrm{B},\lambda-\varepsilon})_\delta}$. □

In order to prove the corresponding statement for convex higher dimensional sets, we need the following lemma, which is the higher dimensional version of Proposition 5.17 for convex sets.

Lemma 5.29. *Let $A \subseteq \mathbb{R}^n$ be an open, bounded and convex $\mathrm{C}^{1,\alpha}$ set for $\alpha \in (0,1)$. Then there exists some $\lambda^* > 0$ such that for all $\lambda > \lambda^*$, it holds that*

i) $A_{\mathrm{B},\lambda}$ *is convex,*

ii) $\mathrm{P}(A_{\mathrm{B},\lambda}) \leqslant \mathrm{P}(A)$,

iii) *for all $x \in \partial A_{\mathrm{B},\lambda}$, we find some unique ball $B \subseteq A$ of radius $\frac{n-1}{\lambda}$ with $x \in \partial B$,*

iv) $A_{\mathrm{B},\lambda}$ *is a C^1 set.*

Proof. Since A is convex, the convex hull of two balls B_1, B_2 of radius $\frac{n-1}{\lambda}$ in A, is also in A, and therefore in $A_{\mathrm{B},\lambda}$. Hence, for two points $x, y \in A_{\mathrm{B},\lambda}$, we find balls B_1 and B_2 in A with radius as above and with $x \in B_1$ and $y \in B_2$, such that the convex combination is also in $A_{\mathrm{B},\lambda}$. It follows that $A_{\mathrm{B},\lambda}$ is a convex set.
With i) at hand, property ii) follows from Lemma 2.50.
To show iii), we consider $x \in \partial A_{\mathrm{B},\lambda}$. By definition, we find for each $\varepsilon > 0$ some ball $\mathrm{B}_{\frac{n-1}{\lambda}}(y_\varepsilon) \subseteq A$ with $\mathrm{B}_\varepsilon(x) \cap \mathrm{B}_{\frac{n-1}{\lambda}}(y_\varepsilon) \neq \emptyset$. As $y_\varepsilon \in \overline{A}$ for all $\varepsilon > 0$, there exists a subsequence $(y_{\varepsilon_k})_{k\in\mathbb{N}}$ and some $y \in \overline{A}$ with $y_{\varepsilon_k} \to y$ and $\varepsilon_k \to 0$ as $k \to \infty$. Furthermore, for each $k \in \mathbb{N}$, we find some $x_k \in \mathrm{B}_\varepsilon(x) \cap \mathrm{B}_{\frac{n-1}{\lambda}}(y_{\varepsilon_k})$. The openness of $A_{\mathrm{B},\lambda}$ gives $\mathrm{B}_{\frac{n-1}{\lambda}}(y) \subseteq A_{\mathrm{B},\lambda}$ and

$$\mathrm{dist}(x, \mathrm{B}_r(y)) \leqslant |x - x_k| + \mathrm{dist}(x_k, \mathrm{B}_r(y)) \leqslant |x - x_k| + |y - y_{\varepsilon_k}| \to 0 \qquad \text{as } k \to \infty\,.$$

Hence, $x \in \partial \mathrm{B}_{\frac{n-1}{\lambda}}(y)$. We now assume that x is in the boundary of two balls B_1, B_2 in A of radius $\frac{n-1}{\lambda}$ with $x \in \partial B_1 \cap \partial B_2$. Since the convex hull of B_1 and B_2 is a subset of $A_{\mathrm{B},\lambda}$, it either holds $x \in A_{\mathrm{B},\lambda}$ or $B_1 = B_2$, which proves the claimed uniqueness.
To prove iv), we want to apply the free rolling theorem. To this end, we show that $\overline{A_{\mathrm{B},\lambda}}$ as well as $\mathbb{R}^n \backslash A_{\mathrm{B},\lambda}$ both satisfy the free rolling condition with respect to the radius $\frac{n-1}{\lambda}$. We record that the boundedness and the convexity of $A_{\mathrm{B},\lambda}$ imply that both $\overline{A_{\mathrm{B},\lambda}}$ and $\mathbb{R}^n \backslash A_{\mathrm{B},\lambda}$ have no neck of any radius, which is equivalent to the fact that $\overline{A_{\mathrm{B},\lambda}} \backslash (\partial A_{\mathrm{B},\lambda})_\rho$ and $(\mathbb{R}^n \backslash A_{\mathrm{B},\lambda}) \backslash (\partial A_{\mathrm{B},\lambda})_\rho$ are path-connected for all $\rho > 0$ according to Remark 4.25. Furthermore, the convexity of $A_{\mathrm{B},\lambda}$ yields that for every $x \in \partial A_{\mathrm{B},\lambda}$, we find a closed half-space $H \subseteq \mathbb{R}^n \backslash A_{\mathrm{B},\lambda}$ with $x \in \partial H$ and thus find some closed ball $B \subseteq H$ of any radius with $x \in B$. The free rolling condition is also satisfied for $\overline{A_{\mathrm{B},\lambda}}$ by iii). In particular, i) from Theorem 5.12 is satisfied such that ii) follows and ensures that $A_{\mathrm{B},\lambda}$ is a set of class C^1.* □

*In fact, the Lipschitz continuity of the unit normal even implies that $A_{\mathrm{B},\lambda}$ is $\mathrm{C}^{1,a}$ for all $a \in (0,1)$, according to Lemma 5.5.

Proposition 5.30. *Let $A \subseteq \mathbb{R}^n$ be an open, bounded and convex $\mathrm{C}^{1,\alpha}$ set for $\alpha \in (0,1)$. Then there exists some $\lambda^* > 0$ such that for all $\lambda > \lambda^*$ and all $\varepsilon \in \big(0, \frac{1}{3}\lambda\big]$, it holds that*

$$A_{\mathrm{B},\lambda} \backslash A_{\mathrm{B},\lambda-\varepsilon} \subseteq \overline{(\partial A_{\mathrm{B},\lambda})_\delta} \qquad \textit{and} \qquad A_{\mathrm{B},\lambda} \backslash A_{\mathrm{B},\lambda-\varepsilon} \subseteq \overline{(\partial A_{\mathrm{B},\lambda-\varepsilon})_\delta}\,, \qquad \textit{where } \delta := C\lambda^{-\frac{2}{1-\alpha}}\varepsilon$$

for some $C > 0$ depending only on α and A.

Remark 5.31. Proposition 5.30 remains true if A is a finite union of bounded and convex sets $A_1, \dots, A_k$ of class $\mathrm{C}^{1,\alpha}$ such that $\operatorname{dist}(A_i, A_j) > 0$ for all $i \neq j$, compare Remark 5.28.

Proof. The proof is divided into three steps.

Step 1: Choice of λ^*:
As in the proof of Proposition 5.27, we pick $r_0 > 0$ and a constant $\tilde{C} > 0$ such that for all $x \in \partial A$, we find some $\mathrm{C}^{1,\alpha}$ function f_x on $[-2r_0, 4r_0]^{n-1}$ with $\operatorname{Im}(f_x) \subseteq (-2r_0, 4r_0)$, $|\nabla f_x(0)| = 0 = f(0)$ and

$$\big(T_x(A-x)\big) \cap [-2r_0, 4r_0]^n = \operatorname{Sup} f_x \cap [-2r_0, 4r_0]^n$$

for a suitable rotation $T_x \in \mathcal{O}(n)$. We then make λ^* large enough such that Proposition 5.25 applies and such that $\lambda^* > \frac{n-1}{r_0}$ holds true.
Step 2: Show that for all $\lambda > \lambda^*$, all $\varepsilon \in \big(0, \frac{1}{3}\lambda\big]$ and all $x \in \partial A_{\mathrm{B},\lambda-\varepsilon}$, there exists a point $p \in \partial A_{\mathrm{B},\lambda}$ which satisfies $p = x + t\nu_{A_{\mathrm{B},\lambda-\varepsilon}}(x)$ for $t = |x-p| = \operatorname{dist}(p, \partial A_{\mathrm{B},\lambda-\varepsilon}) \leqslant \delta$:
Let $x \in \partial A_{\mathrm{B},\lambda-\varepsilon}$. If x is also in the boundary of A, the relation $A_{\mathrm{B},\lambda-\varepsilon} \subseteq A_{\mathrm{B},\lambda} \subseteq A$ gives $x \in \partial A_{\mathrm{B},\lambda}$, which implies the claim of this step for $p = x$ trivially.
If x is in the interior of A, we use that $A_{\mathrm{B},\lambda-\varepsilon} \subseteq A_{\mathrm{B},\lambda}$ and that $A_{\mathrm{B},\lambda}$ is a convex set of class C^1 by Lemma 5.29 to find the unique point $p \in \partial A_{\mathrm{B},\lambda}$ with $p = x + t\nu_{A_{\mathrm{B},\lambda-\varepsilon}}(x)$ for some $t \geqslant 0$. The convexity of $A_{\mathrm{B},\lambda-\varepsilon}$ implies $A_{\mathrm{B},\lambda-\varepsilon} \subseteq H^-_{0,\nu_{A_{\mathrm{B},\lambda-\varepsilon}}(x)} + x$, which yields that x is the nearest point projection of p in $H_{0,\nu_{A_{\mathrm{B},\lambda-\varepsilon}}(x)} + x$, and therefore also in $\partial A_{\mathrm{B},\lambda}$. Hence, it holds $t = \operatorname{dist}(p, \partial A_{\mathrm{B},\lambda-\varepsilon})$.
It is left to show $\operatorname{dist}(p, \partial A_{\mathrm{B},\lambda-\varepsilon}) \leqslant \delta$ for δ as in Proposition 5.25.
To this end, we consider the case $\operatorname{dist}(p, \partial A_{\mathrm{B},\lambda-\varepsilon}) = \operatorname{dist}(p, A_{\mathrm{B},\lambda-\varepsilon}) > 0$ and apply Lemma 5.29 in order to find some unique ball $\mathrm{B}_{\frac{n-1}{\lambda}}(y) \subseteq A$ with $p \in \partial \mathrm{B}_{\frac{n-1}{\lambda}}(y)$. According to the definition of $A_{\mathrm{B},\lambda}$, there exists a contact point $z \in \partial \mathrm{B}_{\frac{n-1}{\lambda}}(y) \cap \partial A$. Now, we can apply Step 1 to find a suitable supergraph representation near z, which enables an application of Proposition 5.25 to find

$$\mathrm{B}_{\frac{n-1}{\lambda}}\left(y - \delta\nu_{\mathrm{B}_{\frac{n-1}{\lambda}}(y)}(z)\right) \subseteq \mathrm{B}_{\frac{n-1}{\lambda-\varepsilon}}\left(y - \left(\frac{n-1}{\lambda-\varepsilon} - \frac{n-1}{\lambda} + \delta\right)\nu_{\mathrm{B}_{\frac{n-1}{\lambda}}(y)}(z)\right) \subseteq A\,.$$

Hence, we conclude $\mathrm{B}_{\frac{n-1}{\lambda-\varepsilon}}\big(y - \big(\frac{n-1}{\lambda-\varepsilon} - \frac{n-1}{\lambda} + \delta\big)\nu_{\mathrm{B}_{\frac{n-1}{\lambda}}(y)}(z)\big) \subseteq A_{\mathrm{B},\lambda-\varepsilon}$ and

$$\operatorname{dist}(p, A_{\mathrm{B},\lambda-\varepsilon}) \leqslant \operatorname{dist}\left(p, \mathrm{B}_{\frac{n-1}{\lambda}}\left(y - \delta\nu_{\mathrm{B}_{\frac{n-1}{\lambda}}(y)}(z)\right)\right) \leqslant \operatorname{dist}_{\mathcal{H}}\left(\mathrm{B}_{\frac{n-1}{\lambda}}(y), \mathrm{B}_{\frac{n-1}{\lambda}}\left(y - \delta\nu_{\mathrm{B}_{\frac{n-1}{\lambda}}(y)}(z)\right)\right) = \delta\,.$$

Step 3: Conclusion:
Let $z \in A_{\mathrm{B},\lambda} \backslash A_{\mathrm{B},\lambda-\varepsilon}$. By Lemma 5.29, we find some $x \in \partial A_{\mathrm{B},\lambda-\varepsilon}$ as the nearest point projection of z in $\partial A_{\mathrm{B},\lambda-\varepsilon}$. In particular, $z - x = \operatorname{dist}(z, \partial A_{\mathrm{B},\lambda-\varepsilon}))\nu_{A_{\mathrm{B},\lambda-\varepsilon}}(x)$. We find $p \in \partial A_{\mathrm{B},\lambda}$ as in Step 2 such that z is on the straight line from x to p. Hence, we obtain

$$\max\{\operatorname{dist}(x, \partial A_{\mathrm{B},\lambda}), \operatorname{dist}(x, \partial A_{\mathrm{B},\lambda-\varepsilon})\} \leqslant |z-p| \leqslant \delta$$

such that $z \in \overline{(\partial A_{\mathrm{B},\lambda})_\delta} \cap \overline{(\partial A_{\mathrm{B},\lambda-\varepsilon})_\delta}$ follows. □

Remark 5.32. Note that the previous proof only applies the convexity of the union of balls with curvature λ from Lemma 5.29 during Step 2 for the choice of the existence of the point p. This can be overcome, if, instead of assuming A to be convex, we directly assume the existence of λ^* such that $A_{\mathrm{B},\lambda}$ is C^1 for all $\lambda > \lambda^*$ and such that for each $x \in \partial A_{\mathrm{B},\lambda-\varepsilon}$, there exists a point $p \in \partial A_{\mathrm{B},\lambda}$ with $p = x + \operatorname{dist}(p, \partial A_{\mathrm{B},\lambda-\varepsilon})\nu_{A_{\mathrm{B},\lambda-\varepsilon}}(x)$ for all $\varepsilon \in \big(0, \frac{1}{3}\lambda\big]$.

For the following argument to work as well, we have to additionally assume that A has no necks of radius in $\big(0, \frac{n-1}{\lambda}\big]$, $\mathrm{P}(A_{\mathrm{B},\lambda}) \leqslant \mathrm{P}(A)$ (or that $(\mathrm{P}(A_{\mathrm{B},\lambda}))_{\lambda>\lambda^*}$ is uniformly bounded for some $\lambda^* > 0$) and that the same assumptions hold true for $B\backslash A$ instead of A, for some ball $B \subseteq \mathbb{R}^n$ which compactly contains A. In fact, one still has to show that, for all $x \in \partial A_{\mathrm{B},\lambda}$, there exists a (not necessarily unique) ball $\mathrm{B}_{\frac{n-1}{\lambda}}(y)$ in A with $x \in \partial \mathrm{B}_{\frac{n-1}{\lambda}}(y)$, which works as in the two-dimensional case in the proof of Lemma 5.21.

5.2.4 Integrability of variational mean curvatures for $\mathrm{C}^{1,\alpha}$ sets

In this section, we state and prove the main result of this chapter.

Theorem 5.33 (Integrability of variational mean curvatures of $\mathrm{C}^{1,\alpha}$ sets)**.** *Let $A \subseteq \mathbb{R}^n$ be a bounded open $\mathrm{C}^{1,\alpha}$ set with $\alpha \in (0,1)$. Moreover, assume either that A is convex or that $n = 2$ and A is connected. Then, for every ball $B \subseteq \mathbb{R}^n$ with $A \Subset B$, there exists a local variational mean curvature H of A in B with $H \in \mathrm{L}^p(B)$ for all $p \in \big[0, \frac{1+\alpha}{1-\alpha}\big)$.*

Proof. We derive this main statement by using the Barozzi curvature, which gives the best integrability according to Theorem 4.12. To this end, we recall that by $\mathcal{C}_{A,\lambda}$ we denote the set of constant mean curvature $\lambda > 0$ in A according to Definition 4.5, i.e., the minimizer of $\mathrm{P}(\,\cdot\,) - \lambda|\,\cdot\,|$ among measurable subsets of A. According to Remark 4.14, it is left to show $H \in \mathrm{L}^p(B)$, where

$$H(x) := \begin{cases} H_A(x) & \text{if } x \in A\,, \\ -H_{B\backslash A}(x) & \text{if } x \in B\backslash A \end{cases}$$

and H_A and $H_{B\backslash A}$ are the Barozzi curvatures from Definition 4.9. We first show $H \in \mathrm{L}^p(A)$. By definition, it holds $H(x) = \inf\{\lambda \in \mathbb{Q}_{>0} : x \in \mathcal{C}_{A,\lambda}\} \geqslant 0$ for all $x \in A$, which leads to $A\backslash\overline{\mathcal{C}_{A,\lambda}} \subseteq \{H_A > \lambda\} \subseteq A\backslash\mathcal{C}_{A,\lambda}$ for all $\lambda > 0$. With this and Fubini's theorem, we estimate

$$\begin{aligned} \int_A H_A(x)^p \,\mathrm{d}x &= \int_{A\backslash\overline{\mathcal{C}_{A,\lambda^*}}} H_A(x)^p \,\mathrm{d}x + \int_{\overline{\mathcal{C}_{A,\lambda^*}}} H_A(x)^p \,\mathrm{d}x \\ &\leqslant \int_{A\backslash\overline{\mathcal{C}_{A,\lambda^*}}} H_A(x)^p \,\mathrm{d}x + \int_{\overline{\mathcal{C}_{A,\lambda^*}}} (\lambda^*)^p \,\mathrm{d}x \\ &\leqslant p \int_{A\backslash\overline{\mathcal{C}_{A,\lambda^*}}} \int_{\lambda^*}^{H_A(x)} \lambda^{p-1} \,\mathrm{d}\lambda \,\mathrm{d}x + |A|(\lambda^*)^p \\ &\leqslant p \int_A \int_{\lambda^*}^{\infty} \lambda^{p-1} \mathbb{1}_{(0,H_A(x))}(\lambda) \,\mathrm{d}\lambda \,\mathrm{d}x + |A|(\lambda^*)^p \\ &\leqslant p \int_{\lambda^*}^{\infty} \lambda^{p-1} \int_A \mathbb{1}_{(0,H_A(x))}(\lambda) \,\mathrm{d}x \,\mathrm{d}\lambda + |A|(\lambda^*)^p \\ &\leqslant p \int_{\lambda^*}^{\infty} \lambda^{p-1} |A\backslash\mathcal{C}_{A,\lambda}| \,\mathrm{d}\lambda + |A|(\lambda^*)^p \end{aligned} \tag{5.11}$$

for some $\lambda^* > 0$, which will be determined later. We want to bound $|A\backslash\mathcal{C}_{A,\lambda}|$ from above suitably.
According to Propositions 4.28, 4.31, Remark 4.32 and Lemma 5.18 in the two-dimensional case, and the fact that A is convex in the higher dimensional cases, we find $\lambda^* > 0$ large enough such that it holds

$$A_{\mathrm{B},\lambda} \subseteq \mathcal{C}_{A,\lambda} \qquad \text{for all } \lambda > \lambda^*\,. \tag{5.12}$$

Now, Propositions 5.27 and 5.30 imply that we can enlarge λ^* such that

$$A_{\mathrm{B},\lambda}\backslash A_{\mathrm{B},\lambda-\varepsilon} \subseteq \overline{(\partial A_{\mathrm{B},\lambda})_\delta} \qquad \text{and} \qquad A_{\mathrm{B},\lambda+\varepsilon}\backslash A_{\mathrm{B},\lambda} \subseteq \overline{(\partial A_{\mathrm{B},\lambda})_\delta} \tag{5.13}$$

are satisfied for all $\lambda > \lambda^*$, $\varepsilon \in \big(0, \frac{1}{3}\lambda\big]$ and for $\delta = \delta(\varepsilon) = C\lambda^{-\frac{2}{1-\alpha}}\varepsilon$. Indeed, to formally derive the second formula of (5.13), we apply the corresponding propositions to $\tilde{\lambda} := \lambda + \varepsilon$. Since $\varepsilon \in \Big(0, \frac{1}{3}\tilde{\lambda}\Big]$, we find $A_{\mathrm{B},\tilde{\lambda}}\backslash A_{\mathrm{B},\tilde{\lambda}-\varepsilon} \subseteq \overline{(\partial A_{\mathrm{B},\lambda})_{\tilde{\delta}}}$ for $\tilde{\delta} = C(\tilde{\lambda})^{-\frac{2}{1-\alpha}}\varepsilon$. Moreover, it holds $\tilde{\delta} < \delta$, such that the second formula in (5.13) follows.

We make λ^* large enough such that

$$A_{\mathrm{B},\lambda} \text{ is of class } \mathrm{C}^1 \qquad\qquad \text{and} \qquad\qquad \mathrm{P}(A_{\mathrm{B},\lambda}) \leqslant \mathrm{P}(A)\,,$$

which is possible due to Proposition 5.17 and Lemma 5.29, respectively. In combination with Remark 2.77, this gives

$$\lim_{\rho\searrow 0} \frac{|(\partial A_{\mathrm{B},\lambda})_\rho|}{2\rho} = \mathcal{H}^{n-1}(\partial A_{\mathrm{B},\lambda}) = \mathrm{P}(A_{\mathrm{B},\lambda}) \leqslant \mathrm{P}(A)\,.$$

Hence, we can fix $\varepsilon_0 = \varepsilon_0(\lambda)$ small enough, such that we have

$$\left|\overline{(\partial A_{\mathrm{B},\lambda})_{\delta(\varepsilon)}}\right| \leqslant |(\partial A_{\mathrm{B},\lambda})_{2\delta(\varepsilon)}| \leqslant 6\delta(\varepsilon)\mathrm{P}(A) \qquad \text{for all } \varepsilon \in (0,\varepsilon_0)\,. \tag{5.14}$$

For such ε and $\lambda > \lambda^*$, we estimate

$$|A_{\mathrm{B},\lambda}| - |A_{\mathrm{B},\lambda-\varepsilon}| = |A_{\mathrm{B},\lambda}\backslash A_{\mathrm{B},\lambda-\varepsilon}| \leqslant \tilde{C}\lambda^{-\frac{2}{1-\alpha}}\varepsilon \qquad \text{and} \qquad |A_{\mathrm{B},\lambda+\varepsilon}| - |A_{\mathrm{B},\lambda}| = |A_{\mathrm{B},\lambda+\varepsilon}\backslash A_{\mathrm{B},\lambda}| \leqslant \tilde{C}\lambda^{-\frac{2}{1-\alpha}}\varepsilon\,,$$

where we applied (5.13), (5.14) and set $\tilde{C} := 6C\mathrm{P}(A)$. To conclude that $\lambda \mapsto |A_{\mathrm{B},\lambda}|$ is locally Lipschitz continuous on (λ^*,∞), we consider $s < t$ in (λ^*,∞). We then cover $[s,t]$ by k-many open intervals of the form $(\lambda_i - \varepsilon_i, \lambda_i + \varepsilon_i)$ with $s = \lambda_1 < \lambda_2 < \cdots < \lambda_k = t$ and with $\varepsilon_i > 0$ such that

$$I_j := [\lambda_j, \lambda_j + \varepsilon_j) \cap (\lambda_{j+1} - \varepsilon_{j+1}, \lambda_{j+1}] \neq \emptyset \quad \text{and} \quad \max\left\{|A_{\mathrm{B},\lambda_i}| - |A_{\mathrm{B},\lambda_i-\varepsilon}|, |A_{\mathrm{B},\lambda_i+\varepsilon}| - |A_{\mathrm{B},\lambda_i}|\right\} \leqslant \tilde{C}\lambda_i^{-\frac{2}{1-\alpha}}\varepsilon$$

holds for all $\varepsilon \in (0,\varepsilon_i)$, $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,k-1\}$. With $s_0 := s$, $s_k := t$, and $s_i \in I_i$ for all $i \in \{1,\dots,k-1\}$ we estimate

$$|A_{\mathrm{B},t}| - |A_{\mathrm{B},s}| = \sum_{i=1}^{k} |A_{\mathrm{B},s_i}| - |A_{\mathrm{B},\lambda_i}| + |A_{\mathrm{B},\lambda_i}| - |A_{\mathrm{B},s_{i-1}}| \leqslant \tilde{C}\sum_{i=1}^{k} \lambda_i^{-\frac{2}{1-\alpha}}(s_i - \lambda_i + \lambda_i - s_{i-1}) \leqslant \tilde{C}s^{-\frac{2}{1-\alpha}}|t-s|\,.$$

Hence, $\lambda \mapsto |A_{\mathrm{B},\lambda}|$ is locally Lipschitz continuous on (λ^*,∞), i.e., it is in $\mathrm{W}^{1,\infty}_{\mathrm{loc}}((\lambda^*,\infty))$ with

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}|A_{\mathrm{B},\lambda}| \leqslant \tilde{C}\lambda^{-\frac{2}{1-\alpha}} \qquad \text{for a.e. } \lambda \in (\lambda^*,\infty)\,.$$

Now, the fundamental theorem of calculus for Sobolev functions implies

$$|A\backslash A_{\mathrm{B},\lambda}| = |A| - |A_{\mathrm{B},\lambda}| = \int_\lambda^\infty \frac{\mathrm{d}}{\mathrm{d}\gamma}|A_{\mathrm{B},\gamma}|\,\mathrm{d}\gamma \leqslant \tilde{C}\int_\lambda^\infty \gamma^{-\frac{2}{1-\alpha}}\,\mathrm{d}\gamma = \tilde{C}\lambda^{-\frac{1+\alpha}{1-\alpha}}\,,$$

where we exploited the fact that $|A_{\mathrm{B},\gamma}| \to |A|$ as $\gamma \to \infty$ and possibly change the constant $\tilde{C}$ during the steps. This, together with (5.11) and (5.12), implies

$$\int_A H_A(x)^p\,\mathrm{d}x \leqslant p\int_{\lambda^*}^\infty \lambda^{p-1}|A\backslash A_{\mathrm{B},\lambda}|\,\mathrm{d}\lambda + |A|(\lambda^*)^p \leqslant \tilde{C}\int_{\lambda^*}^\infty \lambda^{p-1-\frac{1+\alpha}{1-\alpha}}\,\mathrm{d}\lambda + |A|(\lambda^*)^p\,,$$

and the latter is finite if and only if $p < \frac{1+\alpha}{1-\alpha}$. It is left to show $H_{B\backslash A} \in \mathrm{L}^p(B\backslash A)$. In the convex case, we find some $\gamma > 0$ such that $\mathcal{C}_{B\backslash A,\gamma} = B\backslash A$, since A is convex and $A \Subset B$. Hence, according to Proposition 4.28, $H_{B\backslash A}$ is in this case bounded on $B\backslash A$. Finally, in the planar case, $B\backslash A$ is equivalent to an open and connected $\mathrm{C}^{1,\alpha}$ set, such that the previous reasoning applies to $H_{B\backslash A}$, resulting in $H \in \mathrm{L}^p(B)$. □

Remark 5.34. i) According to Remarks 5.28 and 5.31, Theorem 5.33 remains true (with analogous proof) if A is either a finite union of convex $\mathrm{C}^{1,\alpha}$ sets with positive distance, or a two-dimensional bounded and open $\mathrm{C}^{1,\alpha}$ set.

ii) The previous theorem remains true for $A \subseteq \mathbb{R}^n$, $n \geqslant 3$ and balls B such that the assumptions mentioned in Remark 5.32 still apply on A and $B\backslash A$.

iii) If only A satisfies the assumptions of Remark 5.32 (and not necessarily $B\backslash A$), then there exists a local variational mean curvature of A in $\mathbb{R}^n$ in $\mathrm{L}^p(A)$ for all $p \in \left[1, \frac{1+\alpha}{1-\alpha}\right)$.

iv) The general higher dimensional case is more delicate to handle since we cannot guarantee that the union of balls with curvature λ in A are of class C^1. Controlling the shift of those balls is more subtle and requires extra attention since a corresponding result to Proposition 5.26 has yet to be proven, where the shift direction at non-C^1 points $x \in \partial A_{\mathrm{B},\lambda}$ is in the convex cone of the inner unit normals $-\nu_{\mathrm{B}}(x)$ of balls B in A with radius $\frac{n-1}{\lambda}$ such that $x \in \partial\mathrm{B}$. A result of this kind is in progress.

5.3 Sharp counterexamples

In this section, we construct sharp counterexamples which show that the choice $p = \frac{1+\alpha}{1-\alpha}$ is optimal to some extent. In particular, in the two-dimensional case, we find bounded and connected $\mathrm{C}^{1,\alpha}$ sets, such that the Barozzi curvature of such a set is not in $\mathrm{L}^p(A)$ for all $p > \frac{1+\alpha}{1-\alpha}$. According to Theorem 4.12, it follows that those sets do not have any variational mean curvatures in $\mathrm{L}^p(A)$ for any $p > \frac{1+\alpha}{1-\alpha}$. Still, the limit case $p = \frac{1+\alpha}{1-\alpha}$ remains open. In the higher dimensional case, we also construct bounded and connected $\mathrm{C}^{1,\alpha}$ sets such that the Barozzi curvature has similar integrability properties as in the two-dimensional case. Those sets are not convex, but one can show that Theorem 5.33 essentially carries over in this case, compare Remark 5.32.

Theorem 5.35 (Two-dimensional counterexample to $\mathrm{C}^{1,\alpha}$ sets with good curvatures)**.** *For all $\alpha \in (0,1)$ and all $\varepsilon > 0$ there exists a bounded, open and connected $\mathrm{C}^{1,\alpha}$ set $A \subseteq \mathbb{R}^2$ with no local variational mean curvature in $\mathrm{L}^p(B)$ for all $p \geqslant \frac{1+\alpha}{1-\alpha} + \varepsilon$ and all open sets $B \subseteq \mathbb{R}^2$ with $A \Subset B$.*

Proof. In order to construct the set A we assume w.l.o.g. $\varepsilon < \frac{1}{1-\alpha}$ and fix the parameters

$$\ell_k := k^{-\frac{1}{1-(1-\alpha)\varepsilon}}, \qquad L_k := 2\sum_{i=1}^{k} \ell_i, \qquad L := \lim_{k\to\infty} L_k \tag{5.15}$$

for $k \in \mathbb{N}$. Moreover, we define the $\mathrm{C}^{1,\alpha}$ function

$$u_k(t) := \begin{cases} -\left(\ell_k + t\right)^{1+\alpha} & \text{if } t \in \left[-\ell_k, -\frac{1}{2}\ell_k\right), \\ |t|^{1+\alpha} - 2\left(\frac{1}{2}\ell_k\right)^{1+\alpha} & \text{if } t \in \left[-\frac{1}{2}\ell_k, \frac{1}{2}\ell_k\right], \\ -\left(\ell_k - t\right)^{1+\alpha} & \text{if } t \in \left(\frac{1}{2}\ell_k, \ell_k\right], \\ 0 & \text{otherwise,} \end{cases} \tag{5.16}$$

on $\mathbb{R}$. We then define the function

$$u(t) := \sum_{k=1}^{\infty} u_k(t - L_k + \ell_k)$$

on $\mathbb{R}$. In the following, we show that u is a $\mathrm{C}^{1,\alpha}$ function on $\mathbb{R}$. Indeed, the fact that u is in $\mathrm{C}^{1,\alpha}((L_k - 2\ell_k, L_k))$ with uniform Hölder constant $2(1+\alpha)$ of u' on $(L_k - 2\ell_k, L_k)$ for all $k \in \mathbb{N}$ is straightforward.

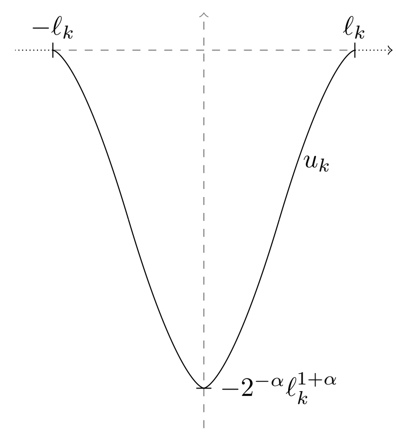


Figure 5.11: The function u_k

With this, we can directly conclude the Hölder continuity of u on $(0, L)$ with uniform Hölder constant $4(1+\alpha)$ of u', since for $s < t$ in $(0, L)$ with $s \in (L_{k-1}, L_k]$ and $t \in (L_i, L_{i+1}]$ for some $i \geqslant k$ in $\mathbb{N}$, we know that $s \leqslant L_k \leqslant L_i < t$ and that $u'(L_k) = u'(L_i) = 0$. Hence, we estimate

$$\left|u'(t) - u'(s)\right| \leqslant \left|u'(t) - u'(L_i)\right| + \left|u'(L_k) - u'(s)\right| \leqslant 2(1+\alpha)\left((t - L_k)^\alpha + (L_i - s)^\alpha\right) \leqslant 4(1+\alpha)(t-s)^\alpha.$$

By the shape of u and the continuity at L, we conclude $u \in \mathrm{C}^{1,\alpha}(\mathbb{R})$ with uniform Hölder constant $4(1+\alpha)$.

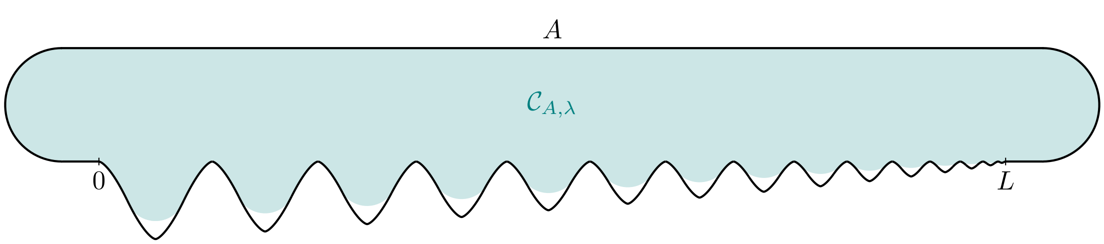


Figure 5.12: Two-dimensional counterexample construction

For a constant $K > 0$ to be fixed later, we define an open, bounded $\mathrm{C}^{1,\alpha}$ Jordan domain A, such that $A \cap [-K, L+K] \times [-2, K] = \mathrm{Sup}\, u \cap [-K, L+K] \times [-2, K]$ and ∂A is locally C^∞ away from $[0, L] \times [-2, 0]$. The set is of finite perimeter by construction as a bounded C^1 set.

We now continue by estimating the Barozzi curvature H_A from Definition 4.9. To this end, we apply Theorem 4.26, obtaining that for λ large enough, the union of balls $A_{\mathrm{B},\lambda}$ with curvature λ in A coincides with the set $\mathcal{C}_{A,\lambda}$ of constant curvature λ in A. We use this and Definition 4.9 to estimate $H_A(x) \geqslant \lambda$ for all $x \in A \backslash A_{\mathrm{B},\lambda}$. Thus, we analyze the set $A_{\mathrm{B},\lambda}$ in the following.

For better readability, we denote the supergraph of $u_k + 2^{-\alpha}\ell_k^{1+\alpha}$ by A_k and analyze first the union of balls in this set, before transferring the results to A. We now consider some ball $\mathrm{B}_{r_{k,s}}(0,t)$ with the lower pole height $s := t - r_{k,s}$ and such that there exists exactly two contact points $\{(x, x^{1+\alpha}), (-x, x^{1+\alpha})\}$ of $\partial \mathrm{B}_{r_k}(0,t)$ and $\mathrm{Graph}[u_k + 2^{-\alpha}\ell_k^{1+\alpha}]$ with $x \in \left(0, \frac{1}{2}\ell_k\right)$.

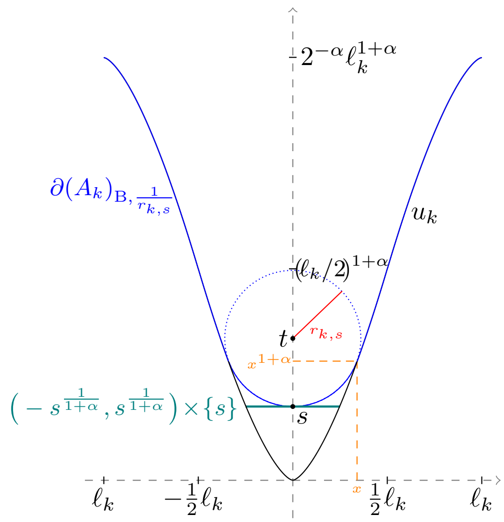


Figure 5.13: Estimate of the Barozzi curvature in the cup A_k

With these conditions, it holds

$$r_{k,s} := \sqrt{\left(\frac{x^{1-\alpha}}{1+\alpha}\right)^2 + x^2}$$

and

$$s = x^{1+\alpha} - \frac{x^{1-\alpha}}{1+\alpha}\left(\sqrt{1 + (1+\alpha)^2 x^{2\alpha}} - 1\right).$$

Since $\sqrt{1+a} \leqslant 1 + \frac{a}{2}$ for all $a \geqslant 0$, we further estimate

$$s \geqslant \frac{1-\alpha}{2} x^{1+\alpha}.$$

In particular, $s \leqslant \frac{1-\alpha}{2}\left(\frac{1}{2}\ell_k\right)^{1+\alpha}$ implies $x \leqslant \frac{1}{2}\ell_k$. Since $x \leqslant \frac{1}{2}\ell_k < \ell_1 = 1$, we further bound

$$r_{k,s} \leqslant \sqrt{(1+\alpha)^{-2} + 1}\, x^{1-\alpha} \leqslant \mathrm{C}\, s^{\frac{1-\alpha}{1+\alpha}}$$

for some constant $\mathrm{C} > 0$. Hence, $r_{k,s} \leqslant \mathrm{C}\ell_k^{1-\alpha} \to 0$ as $k \to \infty$ and therefore, we can make K large enough such that for all $k \in \mathbb{N}$, every ball of the form $\mathrm{B}_{r_{k,s}}(L_k - \ell_k, t - 2^{-\alpha}\ell_k^{1+\alpha})$ as above is in A.

In the following, we estimate the Barozzi curvature on $\left(-s^{\frac{1}{1+\alpha}}, s^{\frac{1}{1+\alpha}}\right) \times \{s\} \subseteq A_k \backslash (A_k)_{\mathrm{B},\frac{1}{r_{k,s}}}$ from below by

$\frac{1}{r_{k,s}}$ for all $s \in \left[0, \frac{1-\alpha}{2}\left(\frac{1}{2}\ell_k\right)^{1+\alpha}\right]$. With the estimates above and Fubini's theorem we compute

$$\begin{aligned}\int_A |H_A(x)|^p \,\mathrm{d}x &\geqslant \sum_{k=1}^{\infty} \int_{A_k \cap \left([-\ell_k,\ell_k]\times\left[0,\frac{1-\alpha}{2}\left(\frac{1}{2}\ell_k\right)^{1+\alpha}\right]\right)} |H_{A_k}(x)|^p \,\mathrm{d}x \\ &\geqslant \sum_{k=1}^{\infty} \int_0^{\frac{1-\alpha}{2}\left(\frac{1}{2}\ell_k\right)^{1+\alpha}} 2s^{\frac{1}{1+\alpha}} \left(\frac{1}{r_{k,s}}\right)^p \mathrm{d}s \\ &\geqslant \frac{2}{C^p} \sum_{k=1}^{\infty} \int_0^{\frac{1-\alpha}{2}\left(\frac{1}{2}\ell_k\right)^{1+\alpha}} s^{\frac{1-p(1-\alpha)}{1+\alpha}} \,\mathrm{d}s\,.\end{aligned}$$

The integrals on the right hand side are unbounded for all $p \geqslant \frac{2+\alpha}{1-\alpha}$. For $p < \frac{2+\alpha}{1-\alpha}$ we have

$$\begin{aligned}\int_A |H_A(x)|^p \,\mathrm{d}x &\geqslant \hat{C} \sum_{k=1}^{\infty} \ell_k^{1-p(1-\alpha)+1+\alpha} \\ &= \hat{C} \sum_{k=1}^{\infty} k^{-\frac{1-p(1-\alpha)+1+\alpha}{1-(1\alpha)\varepsilon}}\,,\end{aligned}$$

where $\hat{C} := 2^{p(1-\alpha)-1-\alpha} \frac{1+\alpha}{C^p(2+\alpha-p(1-\alpha))} \left(\frac{1-\alpha}{2}\right)^{\frac{1-p(1-\alpha)}{1+\alpha}+1}$. The latter term of the previous estimate is finite if and only if $p < \frac{1+\alpha}{1-\alpha} + \varepsilon$ by (5.15). Hence, according to Theorem 4.12, all global variational mean curvatures of A are not in $\mathrm{L}^p(A)$ for $p \geqslant \frac{1+\alpha}{1-\alpha} + \varepsilon$. By Lemma 4.15, we conclude the theorem. $\square$

In the higher dimensional case, Theorem 4.26 does not anymore apply. To still be able to control the Barozzi curvature, we must argue that the set of constant variational mean curvature λ in some specific $\mathrm{C}^{1,\alpha}$ set corresponds to the union of balls with curvature λ in this set.

Proposition 5.36. *Let $n > 2$, $\alpha \in (0,1)$ and*

$$A := \{x \in \mathbb{R}^n : |\overline{x}|^{1+\alpha} < x_n < 1\} \cup \mathrm{B}^n_{\sqrt{1+\frac{1}{(1+\alpha)^2}}}\left(0, 1 + \frac{1}{1+\alpha}\right).$$

Then A is a $\mathrm{C}^{1,\alpha}$ set and it holds that

$$A_{\mathrm{B},\lambda} \qquad \textit{is a minimizer of} \qquad \mathcal{F}_\lambda$$

among measurable subsets of A for all $\lambda > \lambda^* := n\frac{1+\alpha}{\sqrt{(1+\alpha)^2+1}}$.

Remark 5.37. Note that this means that we can set $\mathcal{C}_{A,\lambda} = A_{\mathrm{B},\lambda}$ in the construction of the Barozzi curvature according to Remark 4.10.

Proof. Checking the regularity of A is straightforward. The other statement is proven in several steps.

Step 1: Show $A \neq \mathcal{C}_{A,\lambda} \supseteq A_{\mathrm{B},\lambda}$ for any minimizer $\mathcal{C}_{A,\lambda}$ of $\mathcal{F}_\lambda$:
If A minimizes $\mathcal{F}_\lambda$ from Definition 4.5, we can choose $0 < h \ll 1$ arbitrarily and show that the cut-off set

$$F = F(h) := A \cap (\mathbb{R}^{n-1} \times (h, \infty))$$

is in fact a better minimizer. Indeed, both sets A and F, are bodies of revolution with $\mathrm{rad}_A(t) = t^{\frac{1}{1+\alpha}}$ for

$t \in (0,1)$ and $\mathrm{rad}_F = \mathrm{rad}_A \mathbb{1}_{(h,\infty)}$ on $\mathbb{R}$. Applying Propositions 3.18 and 3.14 gives

$$\begin{aligned}
\frac{1}{(n-1)\omega_{n-1}}\big(\mathcal{F}_\lambda(A) - \mathcal{F}_\lambda(F)\big) &= \frac{1}{(n-1)\omega_{n-1}}\big[\mathrm{P}(A) - \mathrm{P}(F) - \lambda|A\backslash F|\big] \\
&= \mathcal{A}^{\mathrm{rev}}_{n,\mathbb{R}}[\mathrm{rad}_A] - \mathcal{A}^{\mathrm{rev}}_{n,\mathbb{R}}[\mathrm{rad}_F] - \frac{\lambda}{(n-1)\omega_{n-1}}|A\backslash F| \\
&= \int_0^h \mathrm{rad}_A^{n-2}(t)\sqrt{1+(\mathrm{rad}'_A(t))^2}\,\mathrm{d}t - \frac{1}{n-1}\mathrm{rad}_A^{n-1}(h) - \frac{\lambda}{(n-1)\omega_{n-1}}\int_0^h \omega_{n-1}\mathrm{rad}_A^{n-1}(t)\,\mathrm{d}t \qquad (5.17)\\
&= \int_0^h \mathrm{rad}_A^{n-2}(t)\sqrt{1+(\mathrm{rad}'_A(t))^2}\,\mathrm{d}t - \int_0^h \mathrm{rad}'_A(t)\mathrm{rad}_A^{n-2}(t)\,\mathrm{d}t - \frac{\lambda}{n-1}\int_0^h \mathrm{rad}_A^{n-1}(t)\,\mathrm{d}t \\
&= \int_0^h \mathrm{rad}_A^{n-2}(t)\left[\sqrt{1+(\mathrm{rad}'_A(t))^2} - \mathrm{rad}'_A(t) - \frac{\lambda}{n-1}\mathrm{rad}_A(t)\right]\mathrm{d}t\,.
\end{aligned}$$

In the following, we use the identification

$$\sqrt{1+x^2} = x + \frac{1}{2x} + \mathcal{O}(x^{-3}) \qquad \text{as } x \to \infty\,,$$

which holds since

$$\lim_{x\to\infty} x^3\left(\left(x+\frac{1}{2x}\right) - \sqrt{1+x^2}\right) = \lim_{x\to\infty} x^3 \frac{\frac{1}{4x^2}}{x+\frac{1}{2x}+\sqrt{1+x^2}} = \lim_{x\to\infty}\frac{1}{4}\frac{x}{x+\frac{1}{2x}+\sqrt{1+x^2}} = \frac{1}{8}\,.$$

Taking into account $\mathrm{rad}'_A(t) \to \infty$ as $t \searrow 0$, we obtain

$$\begin{aligned}
\frac{1}{(n-1)\omega_{n-1}}\big(\mathcal{F}_\lambda(A) - \mathcal{F}_\lambda(F)\big) &= \int_0^h \mathrm{rad}_A^{n-2}(t)\left[\frac{1}{2\mathrm{rad}'_A(t)} - \frac{\lambda}{n-1}\mathrm{rad}_A(t) + \mathcal{O}(\mathrm{rad}'_A(t)^{-3})\right]\mathrm{d}t \\
&= \int_0^h \mathrm{rad}_A^{n-2}(t)\left[\frac{1+\alpha}{2}t^{\frac{\alpha}{1+\alpha}} - \frac{\lambda}{n-1}t^{\frac{1}{1+\alpha}} + \mathcal{O}(t^{\frac{3\alpha}{1+\alpha}})\right]\mathrm{d}t\,.
\end{aligned}$$

Hence, for h small enough, we can ensure $\frac{1}{(n-1)\omega_{n-1}}\big(\mathcal{F}_\lambda(A) - \mathcal{F}_\lambda(F)\big) > 0$ and therefore $A \neq \mathcal{C}_{A,\lambda}$.
In order to show the subset relation, we first notice that A has no necks of any radius such that Propositions 4.28, 4.31 and Remark 4.32 give $A_{\mathrm{B},\lambda} \subseteq \mathcal{C}_{A,\lambda}$ for all $\lambda > n\sqrt{1+\frac{1}{(1+\alpha)^2}}$. Indeed, the choice of λ in the claim is such that a ball of radius $\frac{n-1}{\lambda}$ is contained in $\mathcal{C}_{A,\lambda}$ by the first proposition, and the second statement then implies that $A_{\mathrm{B},\lambda}$ is a subset of $\mathcal{C}_{A,\lambda}$ while exploiting that there exists no necks of any radius in A.

Step 2: Show that if $\mathcal{C}_{A,\lambda}$ minimizes $\mathcal{F}_\lambda$ among measurable subsets of A, then the Schwarz symmetrization $(\mathcal{C}_{A,\lambda})^\star$ is also a minimizer:
Indeed, by Theorem 3.5 and Remark 3.2, we find

$$\mathcal{F}_\lambda((\mathcal{C}_{A,\lambda})^\star) \leqslant \mathcal{F}_\lambda(\mathcal{C}_{A,\lambda})$$

and since A itself is a body of revolution, $(\mathcal{C}_{A,\lambda})^\star \subseteq A$ remains an admissible test set.

In the following, we assume that $\mathcal{C}_{A,\lambda} = (\mathcal{C}_{A,\lambda})^\star$ is a body of revolution.

Step 3: Show that $\partial\mathcal{C}_{A,\lambda} \cap A$ is analytic with mean curvature in $\frac{\lambda}{n-1}$ up to singularities at most in $\{0\}\times\mathbb{R}$, and the radius function $\mathrm{rad}_{\mathcal{C}_{A,\lambda}}$ is in $\mathrm{W}^{1,1}_{\mathrm{loc}}(\{\mathrm{rad}_{\mathcal{C}_{A,\lambda}} > 0\}) \cap \mathrm{L}^\infty(\mathbb{R}) \cap \mathrm{C}^0(\mathbb{R})$:
According to Theorem 4.35, $\partial\mathcal{C}_{A,\lambda}$ is C^1 near ∂A and is even analytic in A with curvature $\frac{\lambda}{n-1}$ up to a set $S \subseteq \partial\mathcal{C}_{A,\lambda}$ of Hausdorff dimension at most $n-8$. Furthermore, applying Lemma 3.34 provides $\mathrm{rad}_{\mathcal{C}_{A,\lambda}} \in \mathrm{BV}_{\mathrm{loc}}(\{\mathrm{rad}^-_{\mathcal{C}_{A,\lambda}} > 0\}) \cap \mathrm{L}^\infty(\mathbb{R})$. In order to prove that $\mathrm{rad}_{\mathcal{C}_{A,\lambda}}$ is continuous, we argue by contraposition and assume that there exists a jump $t \in \mathbb{R}$ with $\mathrm{rad}^+_{\mathcal{C}_{A,\lambda}}(t) > \mathrm{rad}^-_{\mathcal{C}_{A,\lambda}}(t)$. Then, the symmetry assumption and Lemma 3.37 imply

$$G := \left(\mathrm{B}^{n-1}_{\mathrm{rad}^+_{\mathcal{C}_{A,\lambda}}(t)} \backslash \overline{\mathrm{B}^{n-1}_{\mathrm{rad}^-_{\mathcal{C}_{A,\lambda}}(t)}}\right) \times \{t\} \subseteq \partial\mathcal{C}_{A,\lambda}\,.$$

In fact, for every $x \in G$, the curvature of $\partial\mathcal{C}_{A,\lambda}$ at x is vanishing, which contradicts the dimension of the singularity set S. Similarly, if $\mathrm{rad}_{\mathcal{C}_{A,\lambda}}$ has non-vanishing Cantor part concentrated on $C \subseteq \mathbb{R}\backslash\{0\}$, then

$$\bigcup_{t\in C} \partial\mathrm{B}^{n-1}_{\mathrm{rad}_{\mathcal{C}_{A,\lambda}}(t)} \times \{t\} \subseteq S\,,$$

contradicting the dimension of S, as before. Hence, $\mathrm{rad}_{\mathcal{C}_{A,\lambda}} \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\{\mathrm{rad}^-_{\mathcal{C}_{A,\lambda}} > 0\})$. In the same way, we conclude $S \subseteq \{0\}\times\mathbb{R}$. We want to mention that we cannot conclude $\mathrm{rad}_{\mathcal{C}_{A,\lambda}} \in \mathrm{C}^1(\{\mathrm{rad}_{\mathcal{C}_{A,\lambda}} > 0\})$ in the same way due to the possible existence of other parametrizations (i.e., $\mathrm{rad}'_{\mathcal{C}_{A,\lambda}}$ could be unbounded at some point).

Step 4: Show that $\mathrm{rad}_{\mathcal{C}_{A,\lambda}}$ is not constant with positive value on an open interval:
If the radius function $\mathrm{rad}_{\mathcal{C}_{A,\lambda}}$ would be equal to some $C > 0$ on an open interval $(a,b) \neq \emptyset$, then the curvature analysis of Step 3 implies that $C = \frac{n-1}{\lambda}$. Furthermore, setting $\tilde{a} := \min\{t > 0 : \mathrm{rad}_{\mathcal{C}_{A,\lambda}} = C \text{ on } (\tilde{a},b)\}$, we conclude from Step 3 and the shape of A that $\mathrm{rad}_A(\tilde{a}) > \mathrm{rad}_{\mathcal{C}_{A,\lambda}}(\tilde{a})$ and $\mathrm{rad}_A(b) > \mathrm{rad}_{\mathcal{C}_{A,\lambda}}(b)$. It follows that $\mathrm{rad}_{\mathcal{C}_{A,\lambda}}$ is analytic on an open neighborhood N of $[\tilde{a},b]$. This analyticity on N together with the constancy on $(\tilde{a},b)$ gives $\mathrm{rad}_{\mathcal{C}_{A,\lambda}} \equiv C$ on N, which leads to a contradiction to the choice of $\tilde{a}$.

Step 5: Show that if $0 \notin \mathcal{C}_{A,\lambda}$, then there exists a minimizer $\mathcal{C}$ of $\mathcal{F}_\lambda$ among measurable subsets of A such that $\mathrm{rad}_{\mathcal{C}}$ is increasing on some interval $(t_0,t_1) \Subset (0,1)$ with

$$\mathrm{rad}_{\mathcal{C}}(t_0) = 0\,, \qquad \mathrm{rad}_{\mathcal{C}}(t_1) = \mathrm{rad}_A(t_1) \qquad \text{and} \qquad \mathrm{rad}_{\mathcal{C}} > 0 \qquad \text{on } (t_0,t_1)\,.$$

We define

$$s_0 := \inf\{t \in \mathbb{R} : \mathrm{rad}_{\mathcal{C}_{A,\lambda}}(t) > 0\}\,,$$

which is in $(0,1)$ by Steps 1 and 2 and since $0 \notin \mathcal{C}_{A,\lambda}$. We furthermore define

$$s_1 := \max\big\{t > s_0 : \mathrm{rad}_{\mathcal{C}_{A,\lambda}} < \mathrm{rad}_A \text{ on } (s_0,t)\big\} = \min\{t > s_0 : \mathrm{rad}_{\mathcal{C}_{A,\lambda}}(t) = \mathrm{rad}_A(t)\}\,,$$

which exists according to Steps 1 and 2. In the following, we apply the Steiner symmetrization with respect to the n-th component on $\mathcal{C}_{A,\lambda} \cap (\mathbb{R}^{n-1}\times(0,s_1))$ and its reflection on H_{s_1}. In particular, we define

$$\begin{aligned}
\mathcal{C} &:= \big[\mathcal{C}_{A,\lambda} \cap \mathbb{R}^{n-1}\times[s_1,\infty)\big] \cup S\,, \text{where}\\
\tilde{\mathcal{C}} &:= \big[\mathcal{C}_{A,\lambda} \cap \big(\mathbb{R}^{n-1}\times(0,s_1)\big)\big] - s_1 e_n\,,\\
S &:= \left[\left(\tilde{\mathcal{C}} \cup (-\tilde{\mathcal{C}})\right)^S \cap \big(\mathbb{R}^{n-1}\times(0,-s_1)\big)\right] + s_1 e_n\,.
\end{aligned}$$

With Theorem 3.5, one can check that

$$\mathcal{C}^\star = \mathcal{C} \subseteq A\,, \qquad |\mathcal{C}_{A,\lambda}| = |\mathcal{C}|\,, \qquad \mathrm{P}(\mathcal{C}) \leqslant \mathrm{P}(\mathcal{C}_{A,\lambda})\,.$$

The set $\mathcal{C}$ can be interpreted as the graphification of $\mathcal{C}_{A,\lambda}$ from [47, Theorem 14.8], compare Figure 5.14. Hence, $\mathcal{C}$ is also a minimizer of $\mathcal{F}_\lambda$. By the definition of Steiner symmetrization, it holds that $\mathrm{rad}_{\mathcal{C}}$ is increasing on $(0,s_1)$ and therefore, we redefine

$$t_0 := \inf\{t \in \mathbb{R} : \mathrm{rad}_{\mathcal{C}}(t) > 0\} \geqslant s_0 \qquad \text{and} \qquad t_1 := \max\{t > t_0 : \mathrm{rad}_{\mathcal{C}} < \mathrm{rad}_A \text{ on } (t_0,t)\} = s_1\,,$$

which satisfy the properties from the claim.
In the following, we assume w.l.o.g. $\mathcal{C}_{A,\lambda} = \mathcal{C}$.

Step 6: Conclusion in the case $0 \notin \mathcal{C}_{A,\lambda}$:
According to Steps 3 and 4, we can define the bijective horizontal graph parametrization

$$u : \mathrm{B}^{n-1}_{t_1^{\frac{1}{1+\alpha}}} \to [t_0,t_1]\,,\; x \mapsto \mathrm{rad}^{-1}_{\mathcal{C}_{A,\lambda}}(|x|)$$

of $\Sigma := \partial\mathcal{C}_{A,\lambda} \cap \left(\mathrm{B}^{n-1}_{t_1^{\frac{1}{1+\alpha}}} \times [0,t_1]\right)$. Since $0 \notin \mathcal{C}_{A,\lambda}$, we have $t_0 > 0$, which, in combination with the regularity analysis of Step 3, gives that for every $x \in \Sigma\backslash\{(0,t_0)\}$, at least one of u and v is analytic in a suitable neighborhood of $\overline{x}$ and $(0,x_n) \in \mathbb{R}^{n-1}$, respectively, where

$$v : \mathbb{R}^{n-2}\times\mathbb{R} \to (0,\infty)\,,\; (y,s) \mapsto \sqrt{(\mathrm{rad}_{\mathcal{C}_{A,\lambda}}(s)^2 - |y|^2)_+}\,.$$

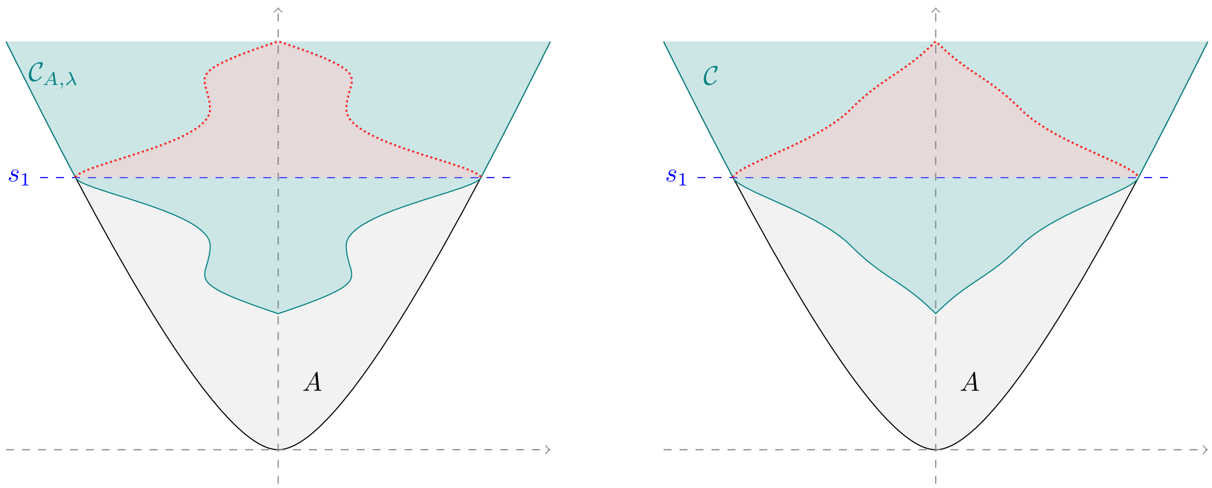


Figure 5.14: Illustration of the Steiner symmetrization argument

Indeed, with suitable rotation and translation, one can show that v gives a graph representation of $\partial\mathcal{C}_{A,\lambda}$ in the corresponding neighborhood.

If v is analytic in some neighborhood of $(0,t)$, then $\mathrm{rad}_{\mathcal{C}_{A,\lambda}}$ is smooth in some neighborhood of t and the mean curvature of the graph of v at $(0,v(0,t),t)$ is the value

$$-\frac{1}{n-1}\mathrm{div}\left(\frac{\nabla v(0,t)}{\sqrt{1+|\nabla v(0,t)|^2}}\right)=\frac{1}{n-1}\left[\frac{n-2}{\mathrm{rad}_{\mathcal{C}_{A,\lambda}}(t)\sqrt{1+\mathrm{rad}'_{\mathcal{C}_{A,\lambda}}(t)^2}}-\frac{\mathrm{rad}''_{\mathcal{C}_{A,\lambda}}(t)}{(1+\mathrm{rad}'_{\mathcal{C}_{A,\lambda}}(t)^2)^{\frac{3}{2}}}\right].$$

Since rotating and shifting does not change the curvature, we find the previous value to be equal to $\frac{\lambda}{n-1}$, which leads to

$$\mathrm{rad}''_{\mathcal{C}_{A,\lambda}}(t)=\frac{n-2}{\mathrm{rad}_{\mathcal{C}_{A,\lambda}}(t)}\left(1+\mathrm{rad}'_{\mathcal{C}_{A,\lambda}}(t)^2\right)-\lambda\left(1+\mathrm{rad}'_{\mathcal{C}_{A,\lambda}}(t)^2\right)^{\frac{3}{2}}$$

for all $(x,t)\in\Sigma$ near which v is analytic. Since $\mathrm{rad}_{\mathcal{C}_{A,\lambda}}$ is increasing, $\mathrm{rad}'_{\mathcal{C}_{A,\lambda}}(t)=0$ implies $\mathrm{rad}''_{\mathcal{C}_{A,\lambda}}(t)=0$, i.e., $\mathrm{rad}_{\mathcal{C}_{A,\lambda}}$ can only have saddles and no extreme points in (t_0,t_1), which leads to

$$\mathrm{rad}_{\mathcal{C}_{A,\lambda}}(t)=\frac{n-2}{\lambda}$$

at those saddle points. Hence, we define

$$t_2:=\min\left\{t_1,\min\left\{\mathrm{rad}_{\mathcal{C}_{A,\lambda}}=\frac{n-2}{\lambda}\right\}\right\}>t_0\,,$$

which leads to $\mathrm{rad}'_{\mathcal{C}_{A,\lambda}}>0$ for each $t\in(t_0,t_2)$ such that v is analytic in some neighborhood of $(0,t)$. Later, we will show that $t_2=t_1$ in the first place. By the definition of u and the inverse function theorem, it follows that u is C^2 on $\mathrm{B}^{n-1}_{t_2^{\frac{1}{1+\alpha}}}\backslash\{0\}$. Furthermore, by defining

$$N:=\mathrm{B}^{n-1}_R\backslash\mathrm{B}^{n-1}_r\times\left(r^{1+\alpha},R^{1+\alpha}\right)$$

for some $0 < r < R < t_2^{\frac{1}{1+\alpha}}$ and applying Proposition 2.52 and Lemma 2.50, we obtain

$$
\begin{aligned}
\mathcal{F}_\lambda^N(\mathcal{C}_{A,\lambda}) &= \mathcal{F}_\lambda^N(\mathrm{Sup}\,u)\\
&= \int_{\mathrm{B}_R^{n-1}\setminus\mathrm{B}_r^{n-1}} \sqrt{1+|\nabla u|^2}\,\mathrm{d}x - \lambda|\mathrm{Sup}\,u\cap N|\\
&= \int_{\mathrm{B}_R^{n-1}\setminus\mathrm{B}_r^{n-1}} \sqrt{1+|\nabla u|^2}\,\mathrm{d}x - \lambda|N| + \lambda|\mathrm{Sub}\,u\cap N|\\
&= \int_{\mathrm{B}_R^{n-1}\setminus\mathrm{B}_r^{n-1}} \sqrt{1+|\nabla u|^2} + \lambda u\,\mathrm{d}x - \lambda(|\mathrm{B}_R^{n-1}\setminus\mathrm{B}_r^{n-1}|r^{1+\alpha} + |N|)\\
&= (n-1)\omega_{n-1}\int_r^R s^{n-2}\left[\sqrt{1+|\overline{u}'|^2} + \lambda\overline{u}\right]\mathrm{d}s - \lambda\left(|\mathrm{B}_R^{n-1}\setminus\mathrm{B}_r^{n-1}|r^{1+\alpha} + |N|\right)\,,
\end{aligned}
$$

with the definition $\overline{u}(|x|) = u(x)$ for all $x \in \mathrm{B}^{n-1}_{t_2^{\frac{1}{1+\alpha}}}$. We continue with computing the Euler equation, which is possible since $\mathrm{Sup}\,u \cap N \Subset A$ (i.e., small variations of u in N still satisfy the outer obstacle condition). For any $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}((r,R))$, it holds that

$$
\begin{aligned}
0 &= \int_r^R \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} s^{n-2}\left[\sqrt{1+|\overline{u}'+t\varphi'|^2} + \lambda(\overline{u}+t\varphi)\right]\mathrm{d}s\\
&= \int_r^R s^{n-2}\left[\frac{\overline{u}'\varphi'}{\sqrt{1+|\overline{u}'|^2}} + \lambda\varphi\right]\mathrm{d}s\\
&= \int_r^R \varphi\left[-\left(s^{n-2}\frac{\overline{u}'}{\sqrt{1+|\overline{u}'|^2}}\right)' + s^{n-2}\lambda\right]\varphi\,\mathrm{d}s\,.
\end{aligned}
$$

Hence, it holds that

$$
\left(\frac{\overline{u}'}{\sqrt{1+|\overline{u}'|^2}}\right)' = -\frac{n-2}{s}\frac{\overline{u}'}{\sqrt{1+|\overline{u}'|^2}} + \lambda \qquad \text{for all } s \in (r,R)\,.
$$

To solve this, we substitute $\psi := \frac{\overline{u}'}{\sqrt{1+|\overline{u}'|^2}}$, and obtain the linear ODE

$$
\psi'(s) = -\frac{n-2}{s}\psi(s) + \lambda\,,
$$

which yields the unique solution

$$
\psi(s) = cs^{2-n} + \frac{\lambda}{n-1}s
$$

for some $c \in \mathbb{R}$. Since $\mathrm{rad}_{\mathcal{C}_{A,\lambda}} \neq 0$ on (t_0,t_2), the horizontal representation $\overline{u}$ has locally bounded nonnegative derivative on $\left(0, t_2^{\frac{1}{1+\alpha}}\right)$. Assuming $c \neq 0$, this would lead to $|\psi| > 1$ near 0. However, this is a contradiction to the construction of ψ, thus, c vanishes and ψ becomes $\frac{\lambda}{n-1}s$. We obtain

$$
\overline{u}'(s) = \frac{\psi(s)}{\sqrt{1-\psi(s)^2}} = \frac{s}{\sqrt{\left(\frac{n-1}{\lambda}\right)^2 - s^2}}\,,
$$

which implies

$$
\overline{u}(s) = C - \sqrt{\left(\frac{n-1}{\lambda}\right)^2 - s^2} \qquad \text{for all } s \in \left(0, t_2^{\frac{1}{1+\alpha}}\right)
$$

for some $C > 0$. We now show that the previous investigation also works for t_1 instead of t_2, which proves the claim since u then describes a ball of radius $\frac{n-1}{\lambda}$ until it touches the boundary of A, which is exactly

$\partial A_{\mathrm{B},\lambda} \cap N$. To this end, we assume $t_2 < t_1$ such that $\mathrm{rad}_{\mathcal{C}_{A,\lambda}}(t_2) = \frac{n-2}{\lambda}$. Then, $\mathrm{rad}_{\mathcal{C}_{A,\lambda}}$ equals the smooth function $t \mapsto \sqrt{\left(\frac{n-1}{2}\right)^2 + (C-t)^2}$ on (t_0, t_2).
Now, if, on the one hand, $\mathrm{rad}_{\mathcal{C}_{A,\lambda}}$ is not smooth in a neighborhood of t_2, then the investigation in the beginning of this step implies that u is smooth on $\mathrm{B}^{n-1}_{t_2^{\frac{1}{1+\alpha}}+\varepsilon}$ for some $\varepsilon > 0$. We can therefore repeat Step 7 on $\mathrm{B}^{n-1}_{t_2^{\frac{1}{1+\alpha}}+\varepsilon} \setminus \{0\}$ and find that $\mathrm{rad}_{\mathcal{C}_{A,\lambda}}$ is in fact smooth at t_2 in contradiction to the assumption.
On the other hand, if $\mathrm{rad}_{\mathcal{C}_{A,\lambda}}$ is smooth in a neighborhood of t_2, then $\mathrm{rad}'_{\mathcal{C}_{A,\lambda}}$ is strictly positive on an interval of the form $(t_0, t_2 + \varepsilon)$ for some $\varepsilon > 0$. In particular, we have $\mathrm{rad}_{\mathcal{C}_{A,\lambda}} \neq \frac{n-2}{\lambda}$ on (t_2, t_1), such that we can repeat Step 7 up to this point even in $\mathrm{B}^{n-1}_{t_1^{\frac{1}{1+\alpha}}} \setminus \{0\}$ and conclude that u is in fact the graph parametrization of the part of the unique ball of radius $\frac{n-1}{\lambda}$ in A which equals $\partial A_{\mathrm{B},\lambda} \cap A$.

Step 7: Conclusion in the case $0 \in \mathcal{C}_{A,\lambda}$:
As in Step 1 we can show that $\mathrm{rad}_{\mathcal{C}_{A,\lambda}}$ is not equal to rad_A on some interval $(0, \varepsilon)$, where $\varepsilon > 0$. Hence, we may assume that $\{\mathrm{rad}_{\mathcal{C}_{A,\lambda}} < \mathrm{rad}_A\} \cap (0, \varepsilon)$ is a countable union of disjoint intervals for all $\varepsilon > 0$. For some $\varepsilon > 0$ to be fixed later, we assume that there exists $a < b$ such that $(a, b) \subseteq \{\mathrm{rad}_{\mathcal{C}_{A,\lambda}} < \mathrm{rad}_A\} \cap (0, \varepsilon)$ and $\{a, b\} \subseteq \{\mathrm{rad}_{\mathcal{C}_{A,\lambda}} = \mathrm{rad}_A\}$. As in Step 4, we can use the gaphification argument for $s_0 = 0$ and $s_1 := \max\{t > 0 : \mathrm{rad}_{\mathcal{C}_{A,\lambda}}(t) < \mathrm{rad}_A(t)\} \subseteq (0, 1)$ to find that we can assume w.l.o.g. $\mathrm{rad}'_{\mathcal{C}_{A,\lambda}} > 0$ on (a, b). We can then repeat Step 6 for $t_0 = a$, $t_1 = b = t_2$, $r = a^{\frac{1}{1+\alpha}}$ and $R = b^{\frac{1}{1+\alpha}}$, where we possibly made ε smaller in order to satisfy $t_1 = t_2$. We again find

$$cs^{2-n} + \frac{\lambda}{n-1} s = \frac{\overline{u}'(s)}{\sqrt{1 + |\overline{u}'(s)|^2}}$$

for some $c \in \mathbb{R}$ and for all $s \in (r, R)$. For $s \in \{r, R\}$, it holds $\overline{u}'(s) = (1 + \alpha)s^\alpha$, such that it follows

$$cs^{2-n} + \frac{\lambda}{n-1} s = \frac{(1+\alpha)s^\alpha}{\sqrt{1 + (1+\alpha)^2 s^{2\alpha}}}$$

and therefore, we have

$$c = \left(\frac{(1+\alpha)s^\alpha}{\sqrt{1 + (1+\alpha)^2 s^{2\alpha}}} - \frac{\lambda}{n-1} s \right) s^{n-2} \tag{5.18}$$

for $s \in \{r, R\}$. We can make ε small enough such that the right hand side of (5.18) is strictly increasing as a function of s on $(0, \varepsilon^{\frac{1}{1+\alpha}})$, which leads a contradiction as $r < R \leqslant \varepsilon^{\frac{1}{1+\alpha}}$. Hence, we find that $\mathrm{rad}_{\mathcal{C}_{A,\lambda}} < \mathrm{rad}_A$ on $(0, \varepsilon)$. With this, we can repeat Step 5 and Step 6 as before. In particular, we find $(t_0, t_1) \subseteq (0, 1)$ as in the claim of Step 5 and conclude $\mathcal{C}_{A,\lambda} = A_{\mathrm{B},\lambda}$ analogously. Thus, $0 \notin \mathcal{C}_{A,\lambda}$, which is a contradiction to the main assumption of this step. □

In the following, we give a version of the counterexample of Theorem 5.35 for higher dimensions, which satisfies the relaxed assumptions mentioned in Remark 5.32 such that Theorem 5.33 still applies according to Remark 5.34.

Theorem 5.38 (Higher dimensional counterexample to $\mathrm{C}^{1,\alpha}$ sets with good curvature)**.** *For all $\alpha \in (0, 1)$ and all $\varepsilon > 0$, there exists a $\mathrm{C}^{1,\alpha}$ set $A \subseteq \mathbb{R}^n$, $n \geqslant 2$, which satisfies the properties from Remark 5.34* iii) *but has no local variational mean curvature in $\mathbb{R}^n$ which is in $\mathrm{L}^p(A)$ for all $p \geqslant \frac{1+\alpha}{1-\alpha} + \varepsilon$.*

Proof. Let $(s_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}$ be a dyadic sequence, in particular, a decreasing nonnegative sequence satisfying

$$1 = s_1 \geqslant s_2 \geqslant \dots\,, \qquad \sum_{k=1}^{\infty} s_k^{n-1-\varepsilon(1-\alpha)} = \infty\,, \qquad \sum_{k=1}^{\infty} s_k^{n-1} < \infty\,,$$

and for every $i \in \mathbb{N}$, there exists some $k_i \in \mathbb{N}_0$ such that $s_i = 2^{-k_i}$. We define the $(n-1)$-dimensional open cubes $\tilde{Q}_k := (-s_k, s_k)^{n-1}$ of side length $2s_k$. According to the assumptions of $(s_k)_{k\in\mathbb{N}}$, the sum of the volumes of $(\tilde{Q}_k)_{k\in\mathbb{N}}$ finite. In particular, there exists an integer ℓ in $(n-1)\mathbb{N}$, which is large enough such that

$$\sum_{k=1}^{\infty} |\tilde{Q}_k| < 2^{\ell}\,.$$

We now show that there exist $x_k \in \mathbb{R}^{n-1}$ for all $k \in \mathbb{N}$ such that the cubes defined by $Q_k := x_k + \tilde{Q}_k$ are mutually disjoint and $\bigcup_{k=1}^{\infty} Q_k \subseteq Q$, where $Q := \left[-2^{\frac{\ell}{n-1}}, 2^{\frac{\ell}{n-1}}\right]^{n-1}$.

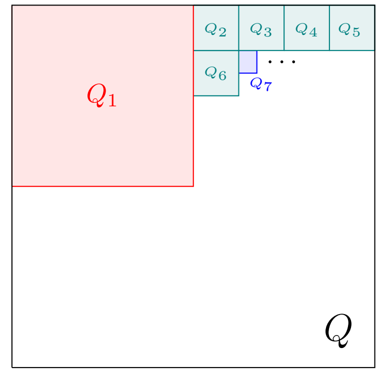


Figure 5.15: Cube construction

To this end, we make a construction as follows. For $k \in \mathbb{N}_0$, let $i_k \in \mathbb{N}_0$ be the unique number with $s_{i_k} = 2^{-k}$ and $s_{i_k+1} > 2^k$ if existent and 0 else. For the following, we assume w.l.o.g. that $i_k > 0$ for all $k \in \mathbb{N}$.
According to the choice of ℓ, it holds that $\sum_{i=1}^{i_0} |\tilde{Q}_i| \leqslant \sum_{i=1}^{\infty} |\tilde{Q}_i| < 2^{\ell} = |Q|$. Since the side length of $Q \cap (0,\infty)^{n-1}$ is even, we find that the disjoint union

$$\bigcup \left\{y + (-1,1)^{n-1} : y \in S_0\right\} \qquad \text{with} \qquad S_0 := \{y \in Q : y_i \in (1+2\mathbb{Z}) \text{ for } i \in \{1,\dots,n-1\}\}$$

equals Q except for a null set. Hence, we find different centers $x_i \in S_0$ with $Q_i \subseteq Q$ for all $i \in \{1,\dots,i_0\}$.
For $k=1$, it holds $\sum_{i=i_0+1}^{i_1} |\tilde{Q}_i| < \left|Q \setminus \bigcup_{i=1}^{i_0} Q_i\right|$, and the choice of the sequence $(s_i)_{i\in\mathbb{N}}$ gives that the disjoint union

$$\bigcup \left\{ y + \left(-\frac{1}{2}, \frac{1}{2}\right)^{n-1} : y \in S_1 \right\} \quad \text{with} \quad S_1 := \left\{ y \in Q \setminus \bigcup_{i=1}^{i_0} Q_i : y_i \in \left(\frac{1}{2} + \mathbb{Z}\right) \text{ for } i \in \{1,\dots,n-1\} \right\}$$

equals $Q \setminus \bigcup_{i=1}^{i_0} Q_i$ except for a null set. Hence, we can find different $x_i \in S_i$ with $Q_i \subseteq Q$ for all $i \in \{i_0+1,\dots,i_1\}$.
Repeating this process iteratively gives countably many mutually disjoint cubes Q_i of side length $2s_i$ in Q for all $i \in \mathbb{N}$, see also Figure 5.15.
We now define the functions u_k on $\mathbb{R}$, similar to (5.16), by

$$u_k(t) := \begin{cases} -\left(s_k + t\right)^{1+\alpha} & \text{if } t \in \left[-s_k, -\frac{1}{2}s_k\right)\,, \\ |t|^{1+\alpha} - 2\left(\frac{1}{2}s_k\right)^{1+\alpha} & \text{if } t \in \left[-\frac{1}{2}s_k, \frac{1}{2}s_k\right]\,, \\ -\left(s_k - t\right)^{1+\alpha} & \text{if } t \in \left(\frac{1}{2}s_k, s_k\right]\,, \\ 0 & \text{otherwise,} \end{cases}$$

and finally set

$$u(x) := \sum_{k=1}^{\infty} u_k(|x - x_k|)\,.$$

Now, u is a C^1 function on $\mathbb{R}^{n-1}$, and, as $u = u_k(|(\,\cdot\,) - x_k|)$ on Q_k, we find that u is $\mathrm{C}^{1,\alpha}$ on every cube $\overline{Q_k}$ and on $\mathbb{R}^{n-1} \setminus \overline{\bigcup_{k\in\mathbb{N}} Q_k}$ with uniform Hölder constant $2(1+\alpha)$ of ∇u. Similar to the proof of Theorem 5.35, we can show that u' is Hölder continuous on $\mathbb{R}^{n-1} \setminus \overline{\bigcup_{k\in\mathbb{N}} Q_k}$ with Hölder constant $4(1+\alpha)$. In order to check that u is a $\mathrm{C}^{1,\alpha}$ function on all of $\mathbb{R}^{n-1}$, we fix $x \in \partial\left(\bigcup_{k\in\mathbb{N}} Q_k\right)$. By the definition of u, we have $u(x) = 0 = \nabla u(x)$. For each value $h > 0$ and direction $\nu \in \partial \mathrm{B}_1^{n-1}$, it either holds $\nabla u(x + h\nu) = 0$ or $x + h\nu \in Q_k$ for some $k \in \mathbb{N}$. In the latter case we can compute

$$|\nabla u(x) - \nabla u(x + h\nu)| = |\nabla u(x + h\nu)| = |u_k'(|x + h\nu - x_k|)|\,.$$

Since $x \notin Q_k$, it holds that $x - x_k \notin \tilde{Q}_k$. Thus, we can exploit that u_k' is Hölder continuous on the closure of $\tilde{Q}_k$ with Hölder constant $2(1+\alpha)$, that $\mathrm{dist}(x + h\nu - x_k, \partial\tilde{Q}_k) \leqslant h$, and that $u_k' = 0$ on $\partial\tilde{Q}_k$, to estimate

$$|\nabla u(x) - \nabla u(x + h\nu)| \leqslant 2(1+\alpha)h^\alpha \,.$$

As in the proof of Theorem 5.35, we choose an open C^∞ set $U \subseteq \mathbb{R}^{n-1} \times [0, \infty)$ such that

$$A := \{x \in 2Q :\, u(x) \leqslant x_n \leqslant 0\} \cup U$$

is an open bounded and connected $\mathrm{C}^{1,\alpha}$ set without necks of any radius in $\big(0, \frac{n-1}{\lambda}\big]$ and which satisfies $A_{\mathrm{B},\lambda} = U \cup (A \cap (\mathrm{Sp}[u])_{\mathrm{B},\lambda})$ for λ large enough. The set U exists since $u \equiv 0$ on $\mathbb{R}^n \backslash Q$.
In the following, we argue that we can locally apply Proposition 5.36. Exploiting the assumptions on U, we can make λ^* large enough such that $A_{\mathrm{B},\lambda} = U \cup (A \cap (\mathrm{Sp}[u])_{\mathrm{B},\lambda})$ and therefore $Q_k^+ := \big(Q_k \times (0, \infty)\big) \cap U \subseteq A_{\mathrm{B},\lambda}$ for all $\lambda > \lambda^*$ and $k \in \mathbb{N}$. Thus, each $Q_k^- \cap \mathcal{C}_{A,\lambda}$ then locally minimizes $\mathcal{F}_\lambda^{Q_k^-}$ in $Q_k^- := (Q_k \times (-\infty, 0)) \cap A$. Now, we can repeat the proof of Proposition 5.36 on Q_k^-, exploiting that $Q_k^+ \subseteq A_{\mathrm{B},\lambda} \subseteq \mathcal{C}_{A,\lambda}$ by Proposition 4.31 and Remark 4.32. As we chose λ^* in Proposition 5.36 merely to ensure $\mathcal{C}_{A,\lambda} \neq \emptyset$, we find $Q_k^- \cap \mathcal{C}_{A,\lambda} = Q_k^- \cap A_{\mathrm{B},\lambda}$ for each $k \in \mathbb{N}$ and for all $\lambda > \lambda^*$.
Since $Q_k^- \cap \mathcal{C}_{A,\lambda}$ is the body of revolution with profile curve u_k, we can use the corresponding estimates from the last part of the proof of Theorem 5.35 to find

$$\int_A |H_A(x)|^p \,\mathrm{d}x \geqslant \omega_{n-1} \sum_{k=1}^\infty \int_0^{\frac{1-\alpha}{2}\left(\frac{1}{2}s_k\right)^{1+\alpha}} s^{\frac{n-1}{1+\alpha}} \left(\frac{1}{r_{k,s}}\right)^p \mathrm{d}s \geqslant \frac{\omega_{n-1}}{C^p} \sum_{k=1}^\infty \int_0^{\frac{1-\alpha}{2}\left(\frac{1}{2}s_k\right)^{1+\alpha}} s^{\frac{n-1-p(1-\alpha)}{1+\alpha}} \,\mathrm{d}s \,.$$

The integrals on the right-hand side are all infinite if $p \geqslant \frac{n+\alpha}{1-\alpha}$. For $p < \frac{n+\alpha}{1-\alpha}$, there exists a constant $\hat{C} > 0$ with

$$\int_A |H_A(x)|^p \,\mathrm{d}x \geqslant \hat{C} \sum_{k=1}^\infty s_k^{n-1-p(1-\alpha)+1+\alpha}$$

such that, by the choice of s_k, $H_A \notin \mathrm{L}^p(A)$ for all $p \geqslant \frac{1+\alpha}{1-\alpha} + \varepsilon$. We conclude that there exists no local variational mean curvature in L^p with Theorem 4.12 and Lemma 4.15.
It is left to show that A satisfies the requirements of Remark 5.32. For that, we choose λ^* as before. According to Theorem 4.35, $\mathcal{C}_{A,\lambda}$ is C^1 near ∂A. By the shape of A and since $\mathcal{C}_{A,\lambda} = A_{\mathrm{B},\lambda}$, each $x \in \partial\mathcal{C}_{A,\lambda}$ is in some (relatively open) spherical cap, which is a subset of $\partial\mathcal{C}_{A,\lambda} \cap A$. Taking into account that $\partial\mathcal{C}_{A,\lambda} \triangle \partial A \subseteq \bigcup_{k\in\mathbb{N}} Q_k \times (-\infty, 0)$, we can make λ^* slightly larger such that for all $\varepsilon \in \big(0, \frac{1}{3}\lambda\big]$ and all $x \in \partial A_{\mathrm{B},\lambda-\varepsilon}$, we find some $p \in \partial A_{\mathrm{B},\lambda}$ with $p = x + \mathrm{dist}(p, \partial A_{\mathrm{B},\lambda-\varepsilon})\nu_{A_{\mathrm{B},\lambda-\varepsilon}}(x)$. Furthermore, by definition, we have

$$\mathrm{P}(\mathcal{C}_{A,\lambda}) - \mathrm{P}(A) \leqslant -\lambda|A \backslash \mathcal{C}_{A,\lambda}| \leqslant 0 \,.$$

Hence, Remark 5.34 iii) applies for A. $\square$

Chapter 6

The limit exponent of Massari's regularity theorem via partial regularity for variational integrals

In this chapter, which is based on [80], we show that Theorem 4.4 applies also in the limit case $\alpha = \frac{p-n}{p+1}$, hence proving the theorem in maximal generality. To achieve this, we transform the parametric problem of minimizing the functional $\mathcal{F}_H^\Omega$ among admissible sets into a non-parametric problem. In the latter, we search for a minimizer of the variational integral

$$\mathbb{F}[w] := \int_V [f(\mathrm{D}w) + g(\,\cdot\,, w)]\,\mathrm{d}x \tag{6.1}$$

among corresponding Sobolev functions $w \in \mathrm{W}^{1,2}_{\mathrm{loc}}(V)$, where V is an open set in $\mathbb{R}^{n-1}$. Depending on the exact properties of f and g, we will continue with some partial regularity results for minimizers of $\mathbb{F}$, which can then be transferred to Massari's regularity theorem. It turns out that we can identify an even more general version of the non-parametric problem, which admits partial regularity results.

6.1 Transfer to the non-parametric problem

In the following, we make the aforementioned transfer of the parametric problem into a non-parametric one precise.

Proposition 6.1 (Transfer of the partial regularity)**.** *Let $p \in (n, \infty)$. For all sets $A \subseteq \mathbb{R}^n$ of finite perimeter in Ω and local variational mean curvature $H \in \mathrm{L}^p(\Omega)$ in Ω, the following properties are satisfied.*

i) *For all $a \in \partial^* A \cap \Omega$, the set A can be represented in a neighborhood of a in Ω as the rotated and translated subgraph of some function $u_a \in \mathrm{C}^1(\overline{V})$ for some open set $V = V(a) \subseteq \mathbb{R}^{n-1}$. Furthermore, the image of u_a is compactly contained in $(0, R)$ and $V \times (0, R)$ is compactly contained in Ω for some $R = R(a) > 0$.*

ii) *The function $\tilde{H} := \mathbb{1}_{A\cap\Omega} H_+ - \mathbb{1}_{\Omega\setminus A} H_-$ is a local variational mean curvature of A in Ω and it is additionally in $\mathrm{L}^p(\Omega)$.*

iii) *For all $a \in \partial^* A$, the corresponding function u_a minimizes the functional*

$$\mathbb{G}[w] := \int_V \left[\sqrt{1 + |\nabla w(x)|^2} - \int_0^{\min\{\max\{w(x),0\},R\}} \tilde{H}(x,t)\mathrm{d}t\right] \mathrm{d}x$$

among functions $w \in \mathrm{W}^{1,2}_{u_a}(V)$.

iv) *All regularity results which apply for minimizers of* $\mathbb{G}$ *in general transfer to* A*, i.e., for all* $\hat{\alpha} \in (0,1]$*, it holds that if* $u_a \in \mathrm{C}^{1,\hat{\alpha}}_{\mathrm{loc}}(V)$ *for all* $a \in \partial^* A$*, then Theorem 4.4 applies for* A *with* $\alpha = \hat{\alpha}$*.*

Remark 6.2. In the zero-order term of the functional $\mathbb{G}$, the upper integral bound $\min\{\max\{w(x),0\},R\}$ is chosen in order to ensure that the variational mean curvature only affects the neighborhood $V \times [0,R]$, where the parametrization u_a describes A suitably.

Proof. Property i) follows directly from Theorem 4.4.
For ii), it suffices to notice that $H \leqslant \tilde{H}$ on $A \cap \Omega$, $\tilde{H} \leqslant H$ on $\Omega \backslash A$ and $|\tilde{H}| \leqslant |H|$ on Ω to conclude $\tilde{H} \in \mathrm{L}^p(\Omega)$ and that $\tilde{H}$ is a variational mean curvature of A in Ω according to Remark 4.2 iii).
In order to prove iii), we assume w.l.o.g. that $A \cap V \times [0,R] = \operatorname{Sub} u_a \cap V \times [0,R]$, hence, neglecting the rotation and translation from i). Now, for any function $\varphi \in \mathrm{C}^1_{\mathrm{cpt}}(V)$ with $w(V) \Subset [0,R]$, where $w := u + \varphi$, we define the test set $A_\varphi := \big(A \backslash (V \times [0,R])\big) \cup \big(\operatorname{Sub} w \cap (V \times [0,R])\big)$, which itself is a set of finite perimeter in Ω and satisfies $A \triangle A_\varphi \Subset \Omega$.

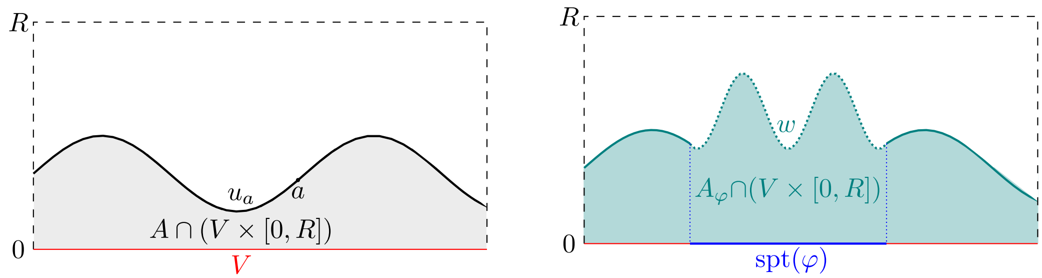


Figure 6.1: Illustration of the argument of the proof of Proposition 6.1

It then follows that $\mathcal{F}^{\Omega}_{\tilde{H}}[A] \leqslant \mathcal{F}^{\Omega}_{\tilde{H}}[A_\varphi]$, by the definition of variational mean curvature. This gives

$$\begin{aligned}
\mathrm{P}(\operatorname{Sub}[u_a], V \times (0,R)) - \int_{\operatorname{Sub}[u_a] \cap (V \times (0,R))} \tilde{H} \,\mathrm{d}z &= \mathrm{P}(A, V \times (0,R)) - \int_{A \cap (V \times (0,R))} \tilde{H} \,\mathrm{d}z \\
&\leqslant \mathrm{P}(A_\varphi, V \times (0,R)) - \int_{A_\varphi \cap (V \times (0,R))} \tilde{H} \,\mathrm{d}z \\
&= \mathrm{P}(\operatorname{Sub}[w], V \times (0,R)) - \int_{\operatorname{Sub}[w] \cap (V \times (0,R))} \tilde{H} \,\mathrm{d}z \,.
\end{aligned}$$

An application of Proposition 2.52 and Fubini's theorem yields $\mathbb{G}[u_a] \leqslant \mathbb{G}[w]$.
We now turn to functions $\eta \in \mathrm{W}^{1,2}_0(V)$ such that $u_a + \eta$ has values in $[\varepsilon, R-\varepsilon]$ a.e. on V for any positive value $\varepsilon \in \big(0, \inf_V \{u_a, R-u_a\}\big)$. Then there exists a sequence $(\eta_k)_{k\in\mathbb{N}}$ of functions in $\mathrm{C}^1_{\mathrm{cpt}}(V)$ which converges to η in $\mathrm{W}^{1,2}(V)$ and a.e. on V. Let now C be any function in $\mathrm{C}^1(\mathbb{R})$ with values only in $\big[\frac{\varepsilon}{3}, R-\frac{\varepsilon}{3}\big]$, with bounded derivative and which equals the identity on $\big[\frac{2\varepsilon}{3}, R-\frac{2\varepsilon}{3}\big]$. We conclude

$$\begin{aligned}
\|u_a + \eta - C(u_a+\eta_k)\|^2_{\mathrm{W}^{1,2}(V)} &= \int_V |u_a + \eta - C[u_a+\eta_k]|^2 + |\nabla u_a + \nabla\eta - \nabla C[u_a+\eta_k]|^2 \,\mathrm{d}x \\
\leqslant \int_V |\eta-\eta_k|^2 &+ |\nabla\eta - \nabla\eta_k + (|\nabla u_a + \nabla\eta| + \sup|C'|\,|\nabla u_a + \nabla\eta_k)|)^2\, \mathbb{1}_{\{u_a+\eta_k \in \mathbb{R}\backslash[\frac{2\varepsilon}{3}, R-\frac{2\varepsilon}{3}]\}} \,\mathrm{d}x \\
&\leqslant \|\eta-\eta_k\|^2_{\mathrm{W}^{1,2}(V)} + \int_{\{|\eta-\eta_k| > \frac{1}{3}\varepsilon\}} (|\nabla u_a + \nabla\eta| + \sup|C'|\,|\nabla u_a + \nabla\eta_k)|)^2 \,\mathrm{d}x \to 0\,,
\end{aligned}$$

as $k \to \infty$, which follows from the assumptions on C and the convergence of $(\eta_k)_{k\in\mathbb{N}}$. Hence, the sequence of functions $\varphi_k := C(u_a+\eta_k) - u_a \in \mathrm{C}^1_{\mathrm{cpt}}(V)$ converges to η in $\mathrm{W}^{1,2}(V)$ and a.e. on V, and it satisfies $(u_a+\varphi_k)(\overline{V}_a) \Subset [0,R]$. Exploiting the first observation, the Lipschitz continuity of the area functional and the Lemma of Fatou yields

$$\begin{aligned}\mathbb{G}[u_a] &\leqslant \limsup_{k\to\infty} \mathbb{G}[u_a+\varphi_k]\\ &\leqslant \limsup_{k\to\infty}\int_V \sqrt{1+|\nabla u_a+\nabla\varphi_k|^2}\,\mathrm{d}x - \int_V \liminf_{k\to\infty}\int_0^{u_a(x)+\varphi_k(x)} \tilde{H}(x,t)\,\mathrm{d}t\,\mathrm{d}x\\ &\leqslant \int_V \sqrt{1+|\nabla u_a+\nabla\eta|^2}\,\mathrm{d}x - \int_V\int_0^{u_a(x)+\eta(x)} \tilde{H}(x,t)\,\mathrm{d}t\,\mathrm{d}x = \mathbb{G}[u_a+\eta]\,.\end{aligned}$$

It remains to show that u_a minimizes the functional among arbitrary functions $w \in \mathrm{W}^{1,2}_{u_a}(V)$. To this end, we choose ε as before, ensuring that $u_a(\overline{V}_a) \subseteq [\varepsilon, R-\varepsilon]$. It can be shown that the truncation

$$\tilde{C}: V_u \to [\varepsilon, R-\varepsilon],\, x \mapsto \min\{\max\{w(x),\varepsilon\}, R-\varepsilon\}$$

is in $\mathrm{W}^{1,2}_{u_a}(V)$ and that $\nabla\tilde{C} = \nabla w \mathbb{1}_{[\varepsilon,R-\varepsilon]}(w)$ holds a.e. on V. Now we exploit property ii), from which it follows that $\tilde{H}$ is non-negative on $V\times[0,\varepsilon]\subseteq A$ and non-positive on $V\times[R-\varepsilon,R]\subseteq\Omega\backslash A$. With this, we conclude

$$\begin{aligned}\mathbb{G}[u_a] \leqslant \mathbb{G}[\tilde{C}] &= \int_V \sqrt{1+|\nabla\tilde{C}(x)|} - \int_0^{\tilde{C}(x)} \tilde{H}(x,t)\,\mathrm{d}t\,\mathrm{d}x\\ &\leqslant \int_V \sqrt{1+|\nabla w(x)|} - \int_0^{\min\{\max\{w(x),0\},R\}} \tilde{H}(x,t)\,\mathrm{d}t\,\mathrm{d}x = \mathbb{G}[w]\,.\end{aligned}$$

This proves property iii) in full generality.
Finally, iv) is a consequence of Theorem 4.4 and the properties i), ii), and iii). □

With the previous proposition in mind, we investigate variational integrals in the generalized version $\mathbb{F}$ of $\mathbb{G}$ in order to achieve the limit case of Massari's regularity theorem. We first notice that the first-order term $f_{\mathbb{G}}(z) := \sqrt{1+|z|^2}$ of $\mathbb{G}$ is smooth and has linear growth. Moreover, the underlying PDE is not uniformly-in-z elliptic, but is still locally-uniformly-in-z elliptic. However, one can easily prove that such a constant exists, whenever one assumes z to be in a bounded set on which the constant strongly depends on. It is convenient that we already know from Proposition 6.1 that u_a is C^1 in $\overline{V}$, and is therefore bounded. For the zero-order term $g_{\mathbb{G}}(x,y) := \int_0^y H(x,t)\mathbb{1}_{[0,R]}(y)\,\mathrm{d}t$, it is apparent that, according to the Hölder inequality, a sort of L^p-Hölder continuity of the form

$$|g_{\mathbb{G}}(x,y) - g_{\mathbb{G}}(x,\tilde{y})| \leqslant \int_{\mathbb{R}} |H(x,t)|\mathbb{1}_{(\tilde{y},y)}\,\mathrm{d}t \leqslant \left(\int_{\mathbb{R}} |H(x,t)|^p\right)^{\frac{1}{p}} |y-\tilde{y}|^{1-\frac{1}{p}}$$

is satisfied for all $\tilde{y} < y \in \mathbb{R}$.

6.2 Preliminaries, notation, assumptions and settings

In the following, we introduce some preliminaries, notation and assumptions which only apply for this part of the thesis.

6.2.1 Generalities

Recall that the transfer of the non-parametric prescribed mean curvature problem to the parametric problem from the previous section reduces the dimension of the framework by 1. We thus introduce the reduced dimension $m := n-1 \in \mathbb{N}$ and the general open and bounded set $V \subseteq \mathbb{R}^m$. Throughout this chapter, we write u^* for the Lebesgue representative of a function $u \in \mathrm{L}^1_{\mathrm{loc}}(V, \mathbb{R}^N)$.

6.2.2 Results on the properties of the zero and first-order terms – Morrey spaces, growth conditions and quasiconvexity

In every considered setting of this chapter, the x-dependency of the zero order term g is regulated by a Morrey function. Morrey functions have certain integrability and growth control, which enable suitable excess bounds for minimizers. For further reference, see [43, Chapter III].

Definition 6.3 (Morrey spaces)**.** Let $U \subseteq \mathbb{R}^m$ be an open set. For $p \in [1, \infty)$ and $\lambda \in [0, \infty)$, the $\mathbf{L^{p,\lambda}}$ **Morrey norm** of some $h \in \mathrm{L}^p(U)$ is defined as

$$\|h\|_{\mathrm{L}^{p,\lambda}(U)} := \left(\sup_{r>0,\, x \in U} \frac{1}{r^\lambda} \int_{U \cap \mathrm{B}_r(x)} |h|^p \mathrm{d}z \right)^{\frac{1}{p}},$$

and the corresponding **Morrey space** is defined as the set

$$\mathrm{L}^{p,\lambda}(U) := \{ h \in \mathrm{L}^p(U) : \|h\|_{\mathrm{L}^{p,\lambda}(U)} < \infty \} .$$

Remark 6.4. For bounded open sets $U \subseteq \mathbb{R}^m$, $p \in [1, \infty)$, the following relations are satisfied.

i) $\mathrm{L}^{p,m}(U) = \mathrm{L}^\infty(U)$ and $\mathrm{L}^{p,\lambda} = \{0\}$ for all $\lambda > m$,

ii) $\mathrm{L}^{p,m-a}(U) \subseteq \mathrm{L}^{q,m-b}(U)$ for all $q \leqslant \min\left\{1, \frac{b}{a}\right\} p$, and all a, b in $[0, m]$,

iii) $\mathrm{L}^p(U) = \mathrm{L}^{p,0}(U) \subseteq \mathrm{L}^{q, m - \frac{m}{p} q}(U)$ for all $q \in [1, p]$.

Moreover, in the situation of ii), the $\mathrm{L}^{p,m-a}$ Morrey norm is controlled by m, p, q, $\operatorname{diam}(U)$ and the $\mathrm{L}^{q,m-b}$ Morrey norm of the corresponding function.
The argument for showing property i) uses the differentiation theorem of Lebesgue, which states that for each $h \in \mathrm{L}^p(U)$ and a.e. $x \in U$, it holds that

$$\lim_{r \searrow 0} \fint_{\mathrm{B}_r(x)} |h(x) - h|^p \mathrm{d}z = 0 \qquad \text{and} \qquad \lim_{r \searrow 0} \fint_{\mathrm{B}_r(x)} |h|^p \mathrm{d}z = |h(x)|^p .$$

The boundedness of U and definition of Morrey norm now implies that h is essentially bounded if and only if $\|h\|_{\mathrm{L}^{p,m}(U)} < \infty$. Furthermore, for $\lambda > m$ it holds that

$$\lim_{r \searrow 0} \frac{1}{r^\lambda} \int_{\mathrm{B}_r(x)} |h|^p \mathrm{d}z = \begin{cases} \infty, & \text{if } h(x) \neq 0 \\ 0, & \text{if } h(x) = 0 . \end{cases}$$

For the second relation, we apply the Hölder inequality with exponents $\frac{p}{q}$ and $\frac{p}{p-q}$, which yields

$$\begin{aligned} \frac{1}{r^{m-b}} \int_{U \cap \mathrm{B}_r(x)} |h|^q \mathrm{d}z &\leqslant \frac{1}{r^{m-b}} |U \cap \mathrm{B}_r(x)|^{1 - \frac{q}{p}} \left(\int_{U \cap \mathrm{B}_r(x)} |h|^p \mathrm{d}z \right)^{\frac{q}{p}} \\ &\leqslant \omega_m^{1-\frac{q}{p}} \left(r^{\frac{p}{q} b - m} \int_{U \cap \mathrm{B}_r(x)} |h|^p \mathrm{d}z \right)^{\frac{q}{p}} \\ &\leqslant \omega_m^{1-\frac{q}{p}} \left(r^{\frac{p}{q} b - a} \|h\|^p_{\mathrm{L}^{p,m-a}(U)} \right)^{\frac{q}{p}} \\ &\leqslant \omega_m^{1-\frac{q}{p}} \|h\|^q_{\mathrm{L}^{p,m-a}(U)} \end{aligned}$$

for all $x \in U$ and $r \leqslant 1$. If $r > 1$, we find $r^{\frac{p}{q} b - a} \leqslant r^m$, and since U is bounded, it suffices to consider radii $r \leqslant \frac{\operatorname{diam}(U)}{2}$, which gives the claimed relation with the additional factor $\left(\frac{\operatorname{diam}(U)}{2}\right)^m$. Property iii) follows from ii) with $a = m$ and $b = \frac{m}{p} r$.

We briefly introduce the quadratic growth condition, which we will impose on f together with quasiconvexity to conclude useful properties. We recommend [36, 37, 46, 44] as references for the quasiconvexity notion, where the latter two also concern the general p-growth condition.

Definition 6.5 (Quadratic growth)**.** We say that a function $h \in \mathrm{C}^0(\mathbb{R}^m)$ has **at most quadratic growth** if it holds that

$$\lim_{z\in\mathbb{R}^m, |z|\to\infty} \frac{|h(z)|}{|z|^2} < \infty\,.$$

Remark 6.6. A continuous function $h : \mathbb{R}^m \to \mathbb{R}$ has at most quadratic growth if and only if there exists a constant $\mathrm{c} > 0$ such that $|h(z)| \leqslant \mathrm{c}(1 + |z|^2)$ holds for all $z \in \mathbb{R}^m$.
To see this, we notice that the backwards implication can be concluded from $\lim_{|z|\to\infty} \frac{\mathrm{c}(1+|z|^2)}{|z|^2} = \mathrm{c}$. For the other implication, due to the local boundedness of h, we conclude $\frac{|h(z)|}{|z|^2} \leqslant \tilde{\mathrm{c}}$ for all $z \in \mathbb{R}^m \setminus \mathrm{B}_1$ and some constant $\tilde{\mathrm{c}} > 0$. As h is also bounded on B_1 by some constant $\hat{\mathrm{c}}$, we find $|h(z)| \leqslant \mathrm{c}(1+|z|^2)$ for $\mathrm{c} = \max\{\tilde{\mathrm{c}}, \hat{\mathrm{c}}\}$.
Convexity of an integrand implies weak lower semicontinuity of the corresponding variational integral. In an attempt to find a weaker notion of the former, which is even equivalent to the latter, Morrey introduced the term quasiconvexity in [68]. In many cases, it turns out that strict quasiconvexity is indeed the correct notion to characterize lower semicontinuous variational integrals, which makes it a crucial constraint for partial regularity theory.

Definition 6.7 ((2-strict) quasiconvexity)**.** Let $h : \mathbb{R}^{N\times m} \to \mathbb{R}$ be a continuous function. Then h is called

i) **quasiconvex** if

$$h(z) \leqslant \fint_{\mathrm{B}_1} h(z + \mathrm{D}\varphi)\,\mathrm{d}x \qquad \text{for all } z \in \mathbb{R}^{N\times m} \text{ and } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathrm{B}_1, \mathbb{R}^N)\,,$$

ii) **2-strictly quasiconvex** if for all $s > 0$, there exists a constant $\mathrm{c}(s) > 0$ such that

$$h(z) + \mathrm{c}(s) \fint_{\mathrm{B}_1} |\mathrm{D}\varphi|^2\,\mathrm{d}x \leqslant \fint_{\mathrm{B}_1} h(z + \mathrm{D}\varphi)\,\mathrm{d}x \qquad \text{for all } z \in \mathrm{B}^{N\times m}_s(0) \text{ and } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathrm{B}_1, \mathbb{R}^N)\,.$$

We call $\mathrm{c}(s)$ the **quasiconvexity constant** of h with respect to the bound s.

In the following, we collect some properties associated with quasiconvexity. Remark 6.8 is well-known and simple to see, thus we provide a proof. Lemmas 6.9 and 6.10 can be found, for instance, in [46, Proposition 5.2, Lemma 5.2], and [1, Lemma II.3]. Lemma 6.11 is proven analogously to [46, Proposition 5.2] with some minor adjustments.

Remark 6.8. i) A 2-strictly quasiconvex function is quasiconvex.

ii) A convex function $h \in \mathrm{C}^0(\mathbb{R}^{N\times m})$ is quasiconvex.

iii) If $h \in \mathrm{C}^0(\mathbb{R}^{N\times m})$ has at most quadratic growth, then quasiconvexity of h is equivalent to

$$h(z) \leqslant \fint_{\mathrm{B}_1} h(z + \mathrm{D}\varphi)\,\mathrm{d}x \qquad \text{for all } z \in \mathbb{R}^{N\times m} \text{ and } \varphi \in \mathrm{W}^{1,2}_0(\mathrm{B}_1, \mathbb{R}^N)\,.$$

iv) If $h \in \mathrm{C}^0(\mathbb{R}^{N\times m})$ has at most quadratic growth, then 2-strict quasiconvexity of h is equivalent to the existence of some constant $\mathrm{c}(s)$ for every $s > 0$ such that it holds

$$h(z) + \mathrm{c}(s) \fint_{\mathrm{B}_1} |\mathrm{D}\varphi|^2\,\mathrm{d}x \leqslant \fint_{\mathrm{B}_1} h(z + \mathrm{D}\varphi)\,\mathrm{d}x \qquad \text{for all } z \in \mathbb{R}^{N\times m} \text{ and } \varphi \in \mathrm{W}^{1,2}_0(\mathrm{B}_1, \mathbb{R}^N)\,.$$

Showing i) is straightforward. For ii), one uses the Hahn-Banach separation theorem to conclude $h(z + \mathrm{D}\varphi) \geqslant h(z) + A \cdot \mathrm{D}\varphi$ for some $A \in \mathbb{R}^{N\times m}$. Integrating this inequality and applying the divergence theorem concludes the claim. The argument for iii) is analogous to the following proof of iv), where we employ that for every $\varphi \in \mathrm{W}^{1,2}_0(\mathrm{B}_1, \mathbb{R}^N)$, we can find a sequence $(\varphi_k)_{k\in\mathbb{N}}$ of functions in $\mathrm{C}^\infty_{\mathrm{cpt}}(\mathrm{B}_1, \mathbb{R}^N)$ which has limit φ in $\mathrm{W}^{1,2}(\mathrm{B}_1)$. By the growth condition on h, we can then find a uniform upper bound in $\mathrm{L}^1(\mathrm{B}_1)$ of $(|h(z+\mathrm{D}\varphi_k)|)_{k\in\mathbb{N}}$. Furthermore, we find a subsequence $(\varphi_{k_\ell})_{\ell\in\mathbb{N}}$ of $(\varphi_k)_{k\in\mathbb{N}}$ such that $\liminf_{k\to\infty} \fint_{\mathrm{B}_1} h(z+\mathrm{D}\varphi_k)\,\mathrm{d}x = \lim_{\ell\to\infty} \fint_{\mathrm{B}_1} h(z+\mathrm{D}\varphi_{k_\ell})\,\mathrm{d}x$ and such that φ_{k_ℓ} converges to φ a.e. on B_1 as $\ell\to\infty$. The dominated convergence theorem and the convergence in $\mathrm{W}^{1,2}(\mathrm{B}_1)$ yield

$$\begin{aligned} h(z) &\leqslant \lim_{\ell\to\infty} \fint_{\mathrm{B}_1} h(z+\mathrm{D}\varphi_{k_\ell})\,\mathrm{d}x - \lim_{\ell\to\infty} \mathrm{c}(s) \fint_{\mathrm{B}_1} |\mathrm{D}\varphi_{k_\ell}|^2\,\mathrm{d}x \\ &= \fint_{\mathrm{B}_1} \lim_{\ell\to\infty} h(z+\mathrm{D}\varphi_{k_\ell})\,\mathrm{d}x - \mathrm{c}(s) \fint_{\mathrm{B}_1} |\mathrm{D}\varphi|^2\,\mathrm{d}x \\ &= \fint_{\mathrm{B}_1} h(z+\mathrm{D}\varphi)\,\mathrm{d}x - \mathrm{c}(s) \fint_{\mathrm{B}_1} |\mathrm{D}\varphi|^2\,\mathrm{d}x \end{aligned}$$

for all $z\in\mathbb{R}^{N\times m}$ and $s>0$, where the last equality follows from the continuity of h.

Lemma 6.9. *For a quasiconvex function $h\in\mathrm{C}^1(\mathbb{R}^m)$ of at most quadratic growth with constant $\tilde{\mathrm{c}}$, there exists a constant c, only depending on $\tilde{\mathrm{c}}$ and m, such that*

$$|\mathrm{D}h(z)| \leqslant \mathrm{c}(1+|z|) \qquad \textit{for all } z\in\mathbb{R}^m\,.$$

Lemma 6.10. *Consider a function $h\in\mathrm{C}^2(\mathbb{R}^m)$, which has at most quadratic growth with constant $\tilde{\mathrm{c}}$ and additionally satisfies*

$$|\mathrm{D}h(z)| \leqslant \hat{\mathrm{c}}(1+|z|) \qquad \textit{for some } \hat{\mathrm{c}}>0 \textit{ and all } z\in\mathbb{R}^m\,.$$

Then, for each $r>0$, there exists a constant $\mathrm{c}>0$, which depends on $\tilde{\mathrm{c}}$, $\hat{\mathrm{c}}$, $r>0$ and $\sup_{\mathrm{B}_{r+1}}|\mathrm{D}^2h|$ such that it holds

$$\frac{|h(z+ty)-h(z)-t\mathrm{D}h(z)\cdot y|}{t^2} \leqslant \mathrm{c}|y|^2 \qquad \textit{and} \qquad \frac{|\mathrm{D}h(z+ty)-\mathrm{D}h(z)|}{t} \leqslant \mathrm{c}|y|$$

for all $z\in\mathrm{B}_r$, $y\in\mathbb{R}^m$ and $t>0$.

Lemma 6.11 (Legendre-Hadamard condition)**.** *Let $h\in\mathrm{C}^2(\mathbb{R}^{N\times m})$ be a 2-strictly quasiconvex function. Then it holds*

$$2\,\mathrm{c}(K)|z|^2|\zeta|^2 \leqslant \mathrm{D}^2h(\xi)\zeta z^T\cdot\zeta z^T \qquad \textit{for all } z\in\mathbb{R}^m,\ \zeta\in\mathbb{R}^N \textit{ and } \xi\in\mathrm{B}_K^{N\times m}\,,$$

where $\mathrm{c}(K)$ is the quasiconvexity constant of h with respect to the bound $K>0$.

Proof. Unsing the 2-strict quasiconvexity of h, we find that for each affine function $u:\mathrm{B}_1\to\mathbb{R}^N$ with $\mathrm{D}u\equiv\xi$, it holds that

$$\int_{\mathrm{B}_1} h(\mathrm{D}u)\,\mathrm{d}x \leqslant \int_{\mathrm{B}_1} h(\mathrm{D}u+t\mathrm{D}\varphi) - \mathrm{c}(K)\,t^2|\mathrm{D}\varphi|^2\,\mathrm{d}x \qquad \text{for all } \varphi\in\mathrm{C}^\infty_{\mathrm{cpt}}(\mathrm{B}_1,\mathbb{R}^N) \text{ and all } t\in\mathbb{R}\,.$$

In fact, the smooth function $\Psi:\mathbb{R}\to\mathbb{R}$, $t\mapsto \int_{\mathrm{B}_1} h(\mathrm{D}u+t\mathrm{D}\varphi)-\mathrm{c}(K)\,t^2|\mathrm{D}\varphi|^2\,\mathrm{d}x$ attains its minimum at $t=0$ with $\Psi'(0)=0$ and $\Psi''(0)\geqslant 0$. The last inequality translates to

$$0 \leqslant \int_{\mathrm{B}_1} \mathrm{D}^2h(\mathrm{D}u)\mathrm{D}\varphi\cdot\mathrm{D}\varphi - 2\,\mathrm{c}(K)\,|\mathrm{D}\varphi|^2\,\mathrm{d}x\,.$$

This yields

$$0 \leqslant \int_{\mathrm{B}_1} \sum_{i,k=1}^n \sum_{j,\ell=1}^N \left[\left(\frac{\mathrm{d}^2}{\mathrm{d}x_{i,j}\,\mathrm{d}x_{k,\ell}}h\right)(\xi)\frac{\mathrm{d}}{\mathrm{d}x_i}\varphi_j\frac{\mathrm{d}}{\mathrm{d}x_k}\varphi_\ell\right] - 2\,\mathrm{c}(K)\sum_{i=1}^n\sum_{j=1}^N\left(\frac{\mathrm{d}}{\mathrm{d}x_i}\varphi_j\right)^2\,\mathrm{d}x\,.$$

Now, we insert the specific functions $\tilde{\varphi}(x) \coloneqq \eta(x)\sin(\tau z\cdot x)\zeta$ and $\hat{\varphi}(x) \coloneqq \eta(x)\cos(\tau z\cdot x)\zeta$ for some $\tau\in\mathbb{R}$ and $\eta\in\mathrm{C}^\infty_{\mathrm{cpt}}(\mathrm{B}_1)$, which have derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x_i}\tilde{\varphi}_j(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x_i}\eta\right)(x)\sin(\tau z\cdot x)\zeta_j + \tau\eta(x)z_i\cos(\tau z\cdot x)\zeta_j$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x_i}\hat{\varphi}_j(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x_i}\eta\right)(x)\cos(\tau z\cdot x)\zeta_j - \tau\eta(x)z_i\sin(\tau z\cdot x)\zeta_j\,,$$

respectively. With the dominated convergence theorem, we obtain

$$\begin{aligned}
0 &\leqslant \lim_{\tau\to\infty}\frac{1}{\tau^2}\int_{\mathrm{B}_1}\mathrm{D}^2h(\mathrm{D}u)\mathrm{D}\tilde{\varphi}\cdot\mathrm{D}\tilde{\varphi} - 2\,\mathrm{c}(K)|\mathrm{D}\tilde{\varphi}|^2\,\mathrm{d}x + \int_{\mathrm{B}_1}\mathrm{D}^2h(\mathrm{D}u)\mathrm{D}\hat{\varphi}\cdot\mathrm{D}\hat{\varphi} - 2\,\mathrm{c}(K)|\mathrm{D}\hat{\varphi}|^2\,\mathrm{d}x\\
&= \lim_{\tau\to\infty}\int_{\mathrm{B}_1}\sum_{i,k=1}^{n}\sum_{j,\ell=1}^{N}\left[\left(\frac{\mathrm{d}^2}{\mathrm{d}x_{i,j}\,\mathrm{d}x_{k,\ell}}h\right)(\xi)\left(\frac{1}{\tau^2}\frac{\mathrm{d}}{\mathrm{d}x_i}\eta\frac{\mathrm{d}}{\mathrm{d}x_k}\eta+\eta^2z_iz_k\right)\zeta_j\zeta_\ell\right]\\
&\quad - 2\,\mathrm{c}(K)\sum_{i=1}^{n}\sum_{j=1}^{N}\left(\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}x_i}\eta\right)^2\zeta_j^2+\eta^2z_i^2\zeta_j^2\,\mathrm{d}x\\
&= \int_{\mathrm{B}_1}\sum_{i,k=1}^{n}\sum_{j,\ell=1}^{N}\left[\left(\frac{\mathrm{d}^2}{\mathrm{d}x_{i,j}\,\mathrm{d}x_{k,\ell}}h\right)(\xi)\eta^2z_iz_k\zeta_j\zeta_\ell\right] - 2\,\mathrm{c}(K)\sum_{i=1}^{n}\sum_{j=1}^{N}\eta^2z_i^2\zeta_j^2\,\mathrm{d}x\\
&= \int_{\mathrm{B}_1}\left[\mathrm{D}^2h(\xi)\zeta z^T\cdot\zeta z^T - 2\,\mathrm{c}(K)|z|^2|\zeta|^2\right]\eta^2\,\mathrm{d}x\,.
\end{aligned}$$

Since η was chosen arbitrarily in $\mathrm{C}^\infty_{\mathrm{cpt}}(\mathrm{B}_1)$, the previous inequality implies the claimed estimate. □

Lemma 6.12 (Concave upper bound)**.** *Let $h : [0,\infty) \to [0,\infty)$ be a bounded function which satisfies $\lim_{t\searrow 0} h(t) = 0 = h(0)$. Then there exists a non-decreasing function h_{conc} in the set*

$$\mathbb{C}_h := \{\varphi : [0,\infty)\to[0,\infty)\,|\,\varphi \textit{ is concave and } \varphi \geqslant h\}$$

of concave upper bounds of h, such that $\lim_{t\searrow 0} h_{\mathrm{conc}}(t) = 0 = h_{\mathrm{conc}}(0)$ and $\sup_{[0,\infty)} h_{\mathrm{conc}} = \sup_{[0,\infty)} h$.

Proof. We first notice that the constant function $S := \sup_{[0,\infty)} h$ is a concave upper bound of h. Hence, we can define the function $h_{\mathrm{conc}}(t) := \inf\{\varphi(t) : \varphi\in\mathbb{C}_h\}$, $t\in[0,\infty)$. Checking that $h_{\mathrm{conc}}\in\mathbb{C}_h$ and that $h \leqslant S$ is straightforward due to the first observation. It follows from the assumptions on h that for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $h([0,\delta]) \subseteq [0,\varepsilon]$. Thus, we can construct a function which is equal to C on $[\delta,\infty]$ and whose graph connects $(0,\varepsilon)$ and (δ, C) by a line. This function is in $\mathbb{C}_h$ and therefore ensures $\lim_{t\searrow 0} h_{\mathrm{conc}}(t) = 0 = h_{\mathrm{conc}}(0)$. It remains to show that h_{conc} is non-decreasing. To this end, we consider $s < t$ in $[0,\infty)$, $\lambda\in(0,1]$ and define $r := \frac{t-(1-\lambda)s}{\lambda}$. The concavity of h_{conc} implies

$$\lambda h_{\mathrm{conc}}(r) + (1-\lambda)h_{\mathrm{conc}}(s) \leqslant h_{\mathrm{conc}}(\lambda r + (1-\lambda)s) = h_{\mathrm{conc}}(t)\,.$$

Taking the limit of the previous inequality as $\lambda\to 0$ gives $h_{\mathrm{conc}}(s) \leqslant h_{\mathrm{conc}}(t)$ as claimed. □

6.2.3 Results on the partial regularity proof strategy – Caccioppoli inequality, *A*-harmonic approximation, excess estimate and Campanato's Hölder space characterization

The approach of partial regularity theory via A-harmonic approximation can be divided into four main steps, which fit to the order of the following results and, in fact, to the whole division of the remaining sections. First, the Caccioppoli inequality controls the excess of minimizers by perturbed mean integrals of the minimizer itself. Then, the approximate A-harmonicity of the minimizer is observed, which gives significant control of its mean. In the third part, the previous steps are used to provide a suitable excess control on small balls. Finally, Campanato's characterization of Hölder continuity together with a suitable choice of the regularity set give partial regularity.

The later stated Caccioppoli inequality substantiates our regularity analysis and is based on the following lemma, which allows for the extrapolation of certain estimates of scalar functions. This can be found, for instance, in [46, Lemma 6.1].

Lemma 6.13 (Iteration lemma)**.** *For $0 \leqslant r < R$, we consider a bounded function $u : [r, R] \to [0, \infty)$. If u satisfies*

$$u(\rho_1) \leqslant \frac{a}{(\rho_2 - \rho_1)^s} + \frac{b}{(\rho_2 - \rho_1)^t} + c + d\, u(\rho_2)$$

for all $\rho_1 < \rho_2$ in $[r, R]$, $s > t > 0$, $d \in [0, 1)$ and $a, b, c \geqslant 0$, then there exists a constant $\mathrm{C} > 0$, which only depends on s and d, such that

$$u(r) \leqslant \mathrm{C} \left(\frac{a}{(R - r)^s} + \frac{b}{(R - r)^t} + c \right).$$

Harmonic functions naturally appear with high regularity. By extending this notion to second order PDEs related to a bilinear form A, one still obtains smooth approximation results for almost-A-harmonic functions together with controlled perturbations. This fact is decisively used in order to receive a suitable excess estimate. For more information, including the proof of the following lemma (in generalized form), we refer to [75, Lemma 6.8].

Definition 6.14 (A-harmonic function)**.** For a bilinear form A on $\mathbb{R}^{N \times m}$ satisfying

$$\mathrm{c}_1 |\zeta|^2 |z|^2 \leqslant A(\zeta z^T, \zeta z^T) \qquad \text{and} \qquad |A| \leqslant \mathrm{c}_2 \tag{6.2}$$

for constants $\mathrm{c}_1, \mathrm{c}_2 > 0$ and all $z \in \mathbb{R}^m$ and $\zeta \in \mathbb{R}^N$, we say that $h \in \mathrm{W}^{1,2}(V, \mathbb{R}^N)$ is an ***A*-harmonic function** on V if it holds that

$$\int_V A(\mathrm{D}h, \mathrm{D}\varphi)\, \mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{W}_0^{1,2}(V, \mathbb{R}^N).$$

Lemma 6.15 (A-harmonic approximation)**.** *Let A be as in Definition 6.14, $\rho > 0$ and $x_0 \in \mathbb{R}^m$. There exists a constant $\mathrm{c} = \mathrm{c}(n, N, c_1, c_2)$ such that for all $\varepsilon > 0$, there exists $\delta > 0$, which depends only on n, N, c_1, c_2 and ε, such that, whenever $v \in \mathrm{W}^{1,2}(\mathrm{B}_\rho(x_0), \mathbb{R}^N)$ and $s \in (0, 1]$ satisfy*

$$\fint_{\mathrm{B}_\rho(x_0)} |\mathrm{D}v|^2\, \mathrm{d}x \leqslant s^2 \qquad \textit{and} \qquad \left| \fint_{\mathrm{B}_\rho(x_0)} A(\mathrm{D}v, \mathrm{D}\varphi)\, \mathrm{d}x \right| \leqslant \delta\, s\, \|\mathrm{D}\varphi\|_{\mathrm{L}^\infty(\mathrm{B}_\rho(x_0), \mathbb{R}^{N \times m})}$$

for all $\varphi \in \mathrm{W}_0^{1,\infty}(\mathrm{B}_\rho(x_0), \mathbb{R}^N)$, then there exists an A-harmonic function $h \in \mathrm{C}^\infty(\mathrm{B}_\rho(x_0), \mathbb{R}^N)$ with

$$\sup_{\mathrm{B}_{\frac{\rho}{2}}(x_0)} |\mathrm{D}h| + \rho \sup_{\mathrm{B}_{\frac{\rho}{2}}(x_0)} |\mathrm{D}^2 h| \leqslant \mathrm{c} \qquad \textit{and} \qquad \fint_{\mathrm{B}_{\frac{\rho}{2}}(x_0)} \left| \frac{v - sh}{\rho} \right| \mathrm{d}x \leqslant s^2 \varepsilon.$$

The following statement provides the optimal connection between integral oscillation controls (i.e. excess estimates) and Hölder continuity. It can be found for instance in [46, Theorem 2.9] in a slightly more general version concerning the corresponding domain.

Theorem 6.16 (Campanato's characterization of Hölder continuity)**.** *Let $V \subseteq \mathbb{R}^m$ be a bounded open Lipschitz set, $p \in [1, \infty)$, $u \in \mathrm{L}^p(V)$ and $\lambda \in (m, m + p]$. Then u^* is in $\mathrm{C}^{0, \frac{\lambda - m}{p}}(V)$ if and only if it is in the (p, λ)-Campanato space on V, i.e., if*

$$\sup_{x \in V, \rho > 0} \rho^{-\lambda} \int_{\mathrm{B}_\rho(x)} |u - (u)_{x,\rho}|^p \mathrm{d}y < \infty.$$

In most references, the characterization is used in the following localized reformulation.

Corollary 6.17 (Integral characterization of Hölder continuity)**.** *For some $p \in [1, \infty)$, $\alpha \in (0, 1]$ and $u \in \mathrm{L}^p(V)$, it holds that $u^* \in \mathrm{C}^{0,\alpha}_{\mathrm{loc}}(V)$ if and only if for all $U \Subset V$, there exists a constant $\mathrm{c} > 0$ such that for all $x \in U$ and $\rho > 0$ with $\mathrm{B}_\rho(x) \subseteq U$, we have*

$$\fint_{\mathrm{B}_\rho(x)} |u - (u)_{x,\rho}|^p \mathrm{d}y \leqslant \mathrm{c} \rho^{\alpha p}.$$

Proof. We first consider $u^* \in \mathrm{C}^{0,\alpha}_{\mathrm{loc}}(V)$. For U as in the statement, we find a bounded open Lipschitz set $\hat{U} \subseteq \mathbb{R}^m$, which satisfies $U \Subset \hat{U} \Subset V$. Applying Theorem 6.16 gives that the claim holds on $\hat{U}$ and, in particular, on U.

For the other implication, we consider $x \in V$ and a bounded open Lipschitz neighborhood $U \Subset V$ of x. Then Theorem 6.16 implies $u^* \in \mathrm{C}^{0,\alpha}(\overline{U})$. □

6.2.4 Overall assumptions and settings

In what follows, we set up general assumptions and settings which are valid for this chapter if not otherwise stated.

Generalities:
Since we first determine regularity of the minimizer locally on balls, we consider some arbitrary ball $\mathrm{B}_{r_0}(x_0)$, which is compactly contained in V and with radius $r_0 \in (0,1]$.
For a number $p \in [1,m)$, we abbreviate the Sobolev exponent $\frac{mp}{m-p}$ by p^*, and if $p \in (1,\infty)$, we denote by p' for the conjugate exponent $\frac{p}{p-1}$. For $p \in [m,\infty)$, we use p^* to denote an arbitrarily large number in (p,∞).
Throughout this chapter, we write c_k with $k \in \mathbb{N}$ when working with specific positive constants, and c when working with general positive constants that might change throughout the computation.

First-order integrand: In each setting, we will impose certain properties on the first-order integrand. In each of these cases, we assume the function f to be in $\mathrm{C}^2(\mathbb{R}^{N\times m})$ and of at most quadratic growth. Furthermore, for each $C > 0$, we find some bounded function $\omega : [0,\infty) \to [0,\infty)$ that satisfies $\lim_{t\searrow 0} \omega(t) = 0 = \omega(0)$ and $|\mathrm{D}^2 f(A) - \mathrm{D}^2 f(B)|^2 \leqslant \omega(|A-B|^2)$ for all $A, B \in \mathbb{R}^{N\times m}$ with $|A| \leqslant C$ and $|B| \leqslant C+1$. We denote by $\mathcal{U}_C$ the non-decreasing concave upper bound of ω, i.e., $\mathcal{U}_C$ satisfies the properties proven in Lemma 6.12 and

$$\left|\mathrm{D}^2 f(A) - \mathrm{D}^2 f(B)\right| \leqslant \sqrt{\mathcal{U}_C(|A-B|^2)} \qquad \text{for all } A, B \in \mathbb{R}^{N\times m} \text{ with } |A| \leqslant C \text{ and } |B| \leqslant C+1\,.$$

Zero-order integrand: We assume $g : V \times \mathbb{R}^N \to \mathbb{R}$ to be a Carathéodory function which satisfies the Morrey-Hölder condition

$$|g(x,y) - g(x,\tilde{y})| \leqslant \Gamma(x)(1+|y|+|\tilde{y}|)^{q-\beta}|y-\tilde{y}|^{\beta} \qquad \text{for all } x \in V,\ y, \tilde{y} \in \mathbb{R}^N \tag{6.3}$$

with respect to the second component and some $\beta \in (0,1]$, $q \in [\beta, 2^*)$.
The function Γ: In general, we assume that Γ is a non-negative function on V. Before we can further set up the conditions for Γ, we introduce the exponents

$$s_\beta := \left(\frac{2^*}{\beta}\right)' = \frac{2^*}{2^*-\beta} \qquad \text{and} \qquad s_q := \left(\frac{2^*}{q}\right)' = \frac{2^*}{2^*-q}$$

for β and q as before. We assume

$$\Gamma \in \mathrm{L}^{s_\beta, m+s_\beta((2-\beta)\alpha-\beta)}(V)\,, \qquad \text{where } \alpha \in (0,1)\,, \tag{6.4}$$

and denote

$$\Gamma_\beta := \|\Gamma\|_{\mathrm{L}^{s_\beta, m-s_\beta\beta+s_\beta(2-\beta)\alpha}(V)}$$

in the case $m \geqslant 3$. If $m \leqslant 2$, we assume* $\alpha \in \left(0, \frac{\beta}{2-\beta}\right] \cap (0,1)$ and $q \in [\beta,\infty)$. Then, we consider 2^* as number which is large enough that s_β is arbitrarily close to 1 for some smallness conditions introduced later.

Minimizers: In all statements of this chapter, we consider $u \in \mathrm{W}^{1,2}(V,\mathbb{R}^N)$ a local minimizer of the functional $\mathbb{F}$ from (6.1), i.e., it holds that $|\mathbb{F}[u]| < \infty$ and, for all $x \in V$, we find some radius $r > 0$ such that the ball $\mathrm{B}_r(x)$ is compactly contained in V and

$$\mathbb{F}_{\mathrm{B}_r(x)}[u] \leqslant \mathbb{F}_{\mathrm{B}_r(x)}[w] \qquad \text{for all } w \in \mathrm{W}^{1,2}_u(\mathrm{B}_r(x),\mathbb{R}^N)\,,\ \text{where } \mathbb{F}_{\mathrm{B}_r(x)}[w] := \int_{\mathrm{B}_r(x)} f(\mathrm{D}w) + g(\,\cdot\,, w)\,\mathrm{d}x\,.$$

Settings: In the following, we present three different settings, which all eventually yield partial regularity. The first setting is the most standard in the partial regularity theory. During the application to Massari's regularity theorem, we a priori know much about the local minimizer u, which leads to a simplified reformulation of the setting.

*In fact, for $\alpha > \frac{\beta}{2-\beta}$, the assumption for Γ reads as $\Gamma \in \{0\}$ by Remark 6.4, such that the functional $\mathbb{F}$ degenerates to only depend on the first-order terms, allowing for an easier access, see, for instance, [36, Theorem 2].

Setting 1. Whenever we refer to Setting 1, we additionally assume that f is 2-strictly quasiconvex. Furthermore, depending on the value of q, we impose

$$\Gamma \in \mathrm{L}^{s_q}(V) \tag{6.5}$$

if $q > \min\{2, m\}$ and fix $\delta \in \left(0, \frac{s_q q - m}{2}\right] \cap (0,1)$. In the other case $q \leqslant \min\{2, m\}$, we consider

$$\Gamma \in \mathrm{L}^{s_q, m - s_q q + 2\delta}(V) \tag{6.6}$$

for some $\delta \in \left(0, \frac{s_q q - m}{2}\right) \cap (0,1)$. Accordingly, we abbreviate $\Gamma_q := \|\Gamma\|_{\mathrm{L}^{s_q, m - s_q q + 2\delta}(V)}$ if $q \leqslant \min\{2, m\}$ and $\Gamma_q := \|\Gamma\|_{\mathrm{L}^{s_q}(V)}$ if $q > \min\{2, m\}$.

Setting 2. Here, we also consider f 2-strictly quasiconvex and additionally assume that the local minimizer u is essentially bounded on V, i.e., $u \in \mathrm{L}^\infty(V, \mathbb{R}^N)$. This restriction allows for the weakening of the conditions on g and Γ, respectively. In particular, instead of $q \in [\beta, 2^*)$, we allow for $q \in [\beta, \infty)$ and we drop the assumptions (6.5) and (6.6).

Setting 3. This setting is applied for the functional $\mathbb{G}$ from Section 6.1. We assume $u \in \mathrm{W}^{1,\infty}(V, \mathbb{R}^N)$ and set $\|u\|_{\mathrm{W}^{1,\infty}} := \|u\|_{\mathrm{W}^{1,\infty}(V,\mathbb{R}^N)}$. As we impose further restrictions on u, we can consider q and Γ as in Setting 2. In order to capture the local convexity of the area functional, we assume that for every $K > 0$, there exists a constant $\mathrm{c}(K) > 0$ such that

$$\mathrm{D}^2 f(z)\xi \cdot \xi \geqslant \mathrm{c}(K)|\xi|^2 \qquad \text{for all } z \in \mathrm{B}_K^{N\times m} \text{ and } \xi \in \mathbb{R}^{N\times m}\,. \tag{6.7}$$

In fact, $\mathrm{D}^2 f$ is positive definite on $\mathbb{R}^{N\times m}$, i.e., f is convex and therefore, by Remark 6.8, quasiconvex.

6.3 Caccioppoli inequality, approximate A-harmonicity and excess estimate

6.3.1 Caccioppoli inequalities

Caccioppoli inequalities are a crucial tool for the partial regularity analysis of minimizers of variational integrals. They are sometimes referred to as reverse Poincaré inequalities, as they estimate the L^2-norm of $\mathrm{D}u$ from above by the L^2 norm of u and some perturbation terms. However, in this case, we obtain some L^{2^*} term in the upper bound, which can be controlled later via the Sobolev inequality.

Lemma 6.18 (Caccioppoli inequality for Setting 1)**.** *In the situation of Setting 1, there exists, for every $C > 0$, a constant*[†] $\mathrm{c} > 0$ *such that*

$$\fint_{\mathrm{B}_{\frac{r_0}{2}}(x_0)} |\mathrm{D}v|^2 \,\mathrm{d}x \leqslant \mathrm{c}\left[\fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^2 \mathrm{d}x + \fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x + \varepsilon\left(\fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}} + \frac{r_0^{2\min\{\alpha,\delta\}}}{\varepsilon^{\frac{\beta}{2-\beta}}}\right]$$

holds for all $\varepsilon \in (0,1]$, $\zeta \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{N\times m}$ with $|\xi| + |\zeta| \leqslant C$, where $v(x) := u(x) - \zeta - \xi x$ for all $x \in V$.

Proof. First, we fix $\rho_1 < \rho_2$ in $\left(\frac{r_0}{2}, r_0\right)$. In order to simplify the notation, we set $F_1 := \mathrm{B}_{\rho_1}(x_0)$ and $F_2 := \mathrm{B}_{\rho_2}(x_0)$ and choose some cut-off function $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^m)$ with vales in $[0,1]$, $\mathrm{spt}(\eta) \Subset F_2$, $\eta = 1$ in F_1, and $|\mathrm{D}\eta| \leqslant \frac{2}{\rho_2 - \rho_1}$. Moreover, we define the functions $\varphi_1 := \eta v$ and $\varphi_2 := (1-\eta)v$ in $\mathrm{W}^{1,2}(\mathrm{B}_{r_0}(x_0), \mathbb{R}^N)$. Since φ_1 has vanishing boundary values on ∂F_2, we can apply Remark 6.8 and the 2-strict quasiconvexity of f with constant c_1 with respect to the bound C to deduce

$$\begin{aligned}
\mathrm{c}_1 \fint_{F_2} |\mathrm{D}\varphi_1|^2 \,\mathrm{d}x &= \mathrm{c}_1 \fint_{\mathrm{B}_1} |\mathrm{D}\varphi_1(\rho_2 y + x_0)|^2 \mathrm{d}y \\
&\leqslant \fint_{\mathrm{B}_1} f(\mathrm{D}\varphi_1(\rho_2 y + x_0) + \xi) - f(\xi)\mathrm{d}y \\
&= \fint_{F_2} f(\mathrm{D}\varphi_1 + \xi) - f(\xi)\,\mathrm{d}x\,.
\end{aligned} \tag{6.8}$$

[†]The constant c depends on $m, N, \beta, q, \Gamma_\beta, \Gamma_q, C$, the function f, and the exact value of 2^* in the case $m \leqslant 2$.

With $v = \varphi_1$ on $F_1 \subseteq F_2$, we conclude

$$\mathrm{c}_1 \int_{F_1} |\mathrm{D}v|^2 \,\mathrm{d}x \leqslant \int_{F_2} f(\mathrm{D}\varphi_1 + \xi) - f(\xi) \,\mathrm{d}x \,. \tag{6.9}$$

By the definition of φ_1, φ_2 and v, it holds that

$$\begin{aligned} \mathbb{F}[u] - \mathbb{F}[u - \varphi_1] &= \int_{F_2} f(\mathrm{D}u) - f(\mathrm{D}u - \mathrm{D}\varphi_1) + g(x,u) - g(x, u+\varphi_1) \,\mathrm{d}x \\ &= \int_{F_2} f(\mathrm{D}u) - f(\mathrm{D}v + \xi - \mathrm{D}(v - \varphi_2)) + g(x,u) - g(x,u+\varphi_1) \,\mathrm{d}x \\ &= \int_{F_2} f(\mathrm{D}u) - f(\mathrm{D}\varphi_2 + \xi) + g(x,u) - g(x,u+\varphi_1) \,\mathrm{d}x \,. \end{aligned}$$

With this, (6.9), and $\varphi_2 \equiv 0$ on F_1, we conclude

$$\begin{aligned} \mathrm{c}_1 \int_{F_1} |\mathrm{D}v|^2 \,\mathrm{d}x \leqslant & \int_{F_2} f(\mathrm{D}u - \mathrm{D}\varphi_2) - f(\xi) \,\mathrm{d}x \\ &= \mathbb{F}[u] - \mathbb{F}[u-\varphi_1] + \int_{F_2} f(\mathrm{D}u - \mathrm{D}\varphi_2) - f(\mathrm{D}u) + f(\mathrm{D}\varphi_2 + \xi) - f(\xi) \,\mathrm{d}x \\ &+ \int_{F_2} g(x, u+\varphi_1) - g(x,u) \,\mathrm{d}x \\ &= \mathbb{F}[u] - \mathbb{F}[u-\varphi_1] + \int_{F_2 \setminus F_1} f(\mathrm{D}u - \mathrm{D}\varphi_2) - f(\mathrm{D}u) + f(\mathrm{D}\varphi_2 + \xi) - f(\xi) \,\mathrm{d}x \\ &+ \int_{F_2} g(x,u+\varphi_1) - g(x,u) \,\mathrm{d}x \,. \end{aligned} \tag{6.10}$$

It is left to estimate the terms on the right-hand side suitably. For the first term, we make use of the fact that u is a local minimizer and that $u - \varphi_1$ is in $\mathrm{W}_0^{1,2}(F_2, \mathbb{R}^N)$, hence is an admissible test function such that

$$\mathbb{F}[u] - \mathbb{F}[u - \varphi_1] \leqslant 0 \tag{6.11}$$

holds. The second term can be controlled by known methods of the partial regularity theory. For the second term, we take Lemma 6.9 into account, in order to apply Lemma 6.10, which leads to

$$\begin{aligned} &\int_{F_2 \setminus F_1} f(\mathrm{D}u - \mathrm{D}\varphi_2) - f(\mathrm{D}u) + f(\mathrm{D}\varphi_2 + \xi) - f(\xi) \,\mathrm{d}x \\ &\quad = \int_{F_2 \setminus F_1} f(\mathrm{D}u - \mathrm{D}\varphi_2) - f(\mathrm{D}u) + \mathrm{D}f(\xi)\mathrm{D}\varphi_2 + f(\mathrm{D}\varphi_2 + \xi) - f(\xi) - \mathrm{D}f(\xi)\mathrm{D}\varphi_2 \,\mathrm{d}x \\ &\quad = \int_{F_2 \setminus F_1} f(\mathrm{D}v + \xi - \mathrm{D}\varphi_2) - f(\xi) - f(\mathrm{D}v + \xi) + \mathrm{D}f(\xi)\mathrm{D}\varphi_2 + f(\mathrm{D}\varphi_2 + \xi) - \mathrm{D}f(\xi)\mathrm{D}\varphi_2 \,\mathrm{d}x \\ &\quad = \int_{F_2 \setminus F_1} f(\mathrm{D}v + \xi - \mathrm{D}\varphi_2) - f(\xi) - \mathrm{D}f(\xi)\big[\mathrm{D}v - \mathrm{D}\varphi_2\big] - \int_0^1 \big[\mathrm{D}f(\xi + t\mathrm{D}v) - \mathrm{D}f(\xi)\big]\mathrm{D}v \,\mathrm{d}t \\ &\quad + f(\mathrm{D}\varphi_2 + \xi) - f(\xi) - \mathrm{D}f(\xi)\mathrm{D}\varphi_2 \,\mathrm{d}x \\ &\quad \leqslant \mathrm{c}_2 \int_{F_2 \setminus F_1} |\mathrm{D}v - \mathrm{D}\varphi_2|^2 + \int_0^1 t|\mathrm{D}v|^2 \,\mathrm{d}t + |\mathrm{D}\varphi_2|^2 \,\mathrm{d}x \\ &\quad \leqslant \mathrm{c}_2 \int_{F_2 \setminus F_1} |\mathrm{D}v - \mathrm{D}\varphi_2|^2 + |\mathrm{D}v|^2 + |\mathrm{D}\varphi_2|^2 \,\mathrm{d}x \end{aligned}$$

for the corresponding constant $\mathrm{c}_2 = \mathrm{c} > 0$ from Lemma 6.10. Taking into account the definition of φ_2, it follows

$$\int_{F_2 \setminus F_1} f(\mathrm{D}u - \mathrm{D}\varphi_2) - f(\mathrm{D}u) + f(\mathrm{D}\varphi_2 + \xi) - f(\xi) \,\mathrm{d}x \leqslant 3\,\mathrm{c}_2 \int_{F_2 \setminus F_1} |\mathrm{D}v|^2 \,\mathrm{d}x + \int_{\mathrm{B}_{r_0}(x_0)} \left| \frac{v}{\rho_2 - \rho_1} \right|^2 \,\mathrm{d}x \,. \tag{6.12}$$

It is left to estimate the last part, which is connected to the zero-order integrand g, and which is therefore the crucial part. To this end, we apply the imposed Morrey-Hölder condition (6.3) and the definition of φ_1, which together give

$$\int_{F_2} g(x, u+\varphi_1) - g(x,u)\,\mathrm{d}x \leqslant \int_{F_2} \Gamma(1+|u|+|u+\varphi_1|)^{q-\beta}|\varphi_1|^\beta\,\mathrm{d}x \leqslant \int_{F_2} \Gamma(1+2|u|+|v|)^{q-\beta}|v|^\beta\,\mathrm{d}x\,. \quad (6.13)$$

Since $|\zeta|+|\xi|$ is bounded by C, we find another constant $\mathrm{c}_3>0$ which satisfies

$$\int_{F_2} \Gamma(1+2|u|+|v|)^{q-\beta}|v|^\beta\,\mathrm{d}x \leqslant \int_{F_2} \Gamma(1+2C+2|v|+|v|)^{q-\beta}|v|^\beta\,\mathrm{d}x \leqslant \mathrm{c}_3 \int_{F_2} \Gamma|v|^q + \Gamma|v|^\beta\,\mathrm{d}x\,. \quad (6.14)$$

To estimate the latter integral, we take a closer look at the terms of the integrand. In fact, for the first term, if $q \leqslant \min\{2,m\}$, we apply Young's inequality with the exponents $\frac{2^*}{q}$ and s_q to obtain

$$\int_{F_2} \Gamma|v|^q\,\mathrm{d}x = \int_{F_2} \left(r_0^q\Gamma\right)\left|\frac{v}{r_0}\right|^q \mathrm{d}x \leqslant \int_{F_2}\left|\frac{v}{r_0}\right|^{2^*}\mathrm{d}x + r_0^{s_q q}\int_{F_2}\Gamma^{s_q}\,\mathrm{d}x \leqslant \int_{F_2}\left|\frac{v}{r_0}\right|^{2^*}\mathrm{d}x + r_0^{s_q q}\int_{\mathrm{B}_{r_0}(x_0)}\Gamma^{s_q}\,\mathrm{d}x\,.$$

Now, by (6.6), we conclude

$$\int_{F_2} \Gamma|v|^q\,\mathrm{d}x \leqslant \int_{F_2}\left|\frac{v}{r_0}\right|^{2^*}\mathrm{d}x + \Gamma_q^{s_q} r_0^{m+2\delta}\,. \quad (6.15)$$

If $q>\min\{2,m\}$, we are in the situation of (6.5). Since $2^*>q$ by assumption, Young's inequality remains applicable. Moreover, since $r_0 \leqslant 1$ and $2\delta \leqslant s_q q - m$ in this case, it follows

$$\int_{F_2} \Gamma|v|^q\,\mathrm{d}x \leqslant \int_{\mathrm{B}_{r_0}(x_0)}\left|\frac{v}{r_0}\right|^{2^*}\mathrm{d}x + \Gamma_q^{s_q} r_0^{m+2\delta} \quad (6.16)$$

similar to before.

For estimating the second term in (6.14) we use the Hölder inequality with exponents $\frac{2^*}{\beta}$ and $\left(\frac{2^*}{\beta}\right)' = s_\beta$, which yields

$$\int_{F_2}\Gamma|v|^\beta\,\mathrm{d}x \leqslant \left(\int_{F_2}|v|^{2^*}\,\mathrm{d}x\right)^{\frac{\beta}{2^*}}\left(\int_{F_2}\Gamma^{s_\beta}\,\mathrm{d}x\right)^{\frac{1}{s_\beta}}\,. \quad (6.17)$$

We first cover the case $m\in\{1,2\}$. To this end, we extend the right-hand side of (6.17) and then apply Young's inequality with the exponents $\frac{2}{\beta}$ and $\frac{2}{2-\beta}$, i.e.,

$$\begin{aligned}
\int_{F_2}\Gamma|v|^\beta\,\mathrm{d}x &\leqslant \left[\varepsilon^{\frac{\beta}{2}} r_0^{\frac{\beta}{2}m-\frac{\beta}{2^*}m-\beta}\left(\int_{F_2}|v|^{2^*}\,\mathrm{d}x\right)^{\frac{\beta}{2^*}}\right]\left[\varepsilon^{-\frac{\beta}{2}} r_0^{-\frac{\beta}{2}m+\frac{\beta}{2^*}m+\beta}\left(\int_{F_2}\Gamma^{s_\beta}\,\mathrm{d}x\right)^{\frac{1}{s_\beta}}\right] \\
&\leqslant \varepsilon r_0^{m-\frac{2}{2^*}m-2}\left(\int_{\mathrm{B}_{r_0}(x_0)}|v|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} r_0^{-\frac{\beta}{2-\beta}(m-\frac{2}{2^*}m-2)}\left(\int_{\mathrm{B}_{r_0}(x_0)}\Gamma^{s_\beta}\,\mathrm{d}x\right)^{\frac{2}{s_\beta(2-\beta)}} \\
&= \mathrm{c}_4\varepsilon r_0^m\left(\fint_{\mathrm{B}_{r_0}(x_0)}\left|\frac{v}{r_0}\right|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} r_0^{-\frac{\beta}{2-\beta}(m-\frac{2}{2^*}m-2)}\left(\int_{\mathrm{B}_{r_0}(x_0)}\Gamma^{s_\beta}\,\mathrm{d}x\right)^{\frac{2}{s_\beta(2-\beta)}}
\end{aligned}$$

for some $c_4>0$. Employing (6.4) gives

$$\int_{F_2}\Gamma|v|^\beta\,\mathrm{d}x \leqslant \mathrm{c}_4\varepsilon r_0^m\left(\fint_{\mathrm{B}_{r_0}(x_0)}\left|\frac{v}{r_0}\right|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} r_0^{-\frac{\beta}{2-\beta}(m-\frac{2}{2^*}m-2)}\left(\Gamma_\beta\, r_0^{m-s_\beta\beta+s_\beta(2-\beta)\alpha}\right)^{\frac{2}{s_\beta(2-\beta)}}\,.$$

Since $m\leqslant 2$, we find $\frac{2}{2^*}m+2>m$ and furthermore

$$\frac{\beta}{2-\beta}\left(\frac{2}{2^*}m+2-m\right)+\frac{2}{(2-\beta)s_\beta}(m-\beta s_\beta) = m$$

for all positive choices of 2^*, which yields

$$\begin{aligned}\int_{F_2} \Gamma|v|^\beta \,\mathrm{d}x &\leqslant \mathrm{c}_4 \varepsilon r_0^m \left(\fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} r_0^{-\frac{\beta}{2-\beta}(m-\frac{2}{2^*}m-2)} \left(\Gamma_\beta\, r_0^{m-s_\beta\beta+s_\beta(2-\beta)\alpha} \right)^{\frac{2}{s_\beta(2-\beta)}} \\ &= \mathrm{c}_4 \left[\varepsilon r_0^m \left(\fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} r_0^{m+2\alpha} \right].\end{aligned}$$

In order to estimate (6.17) in the case $m \geqslant 3$, we again apply Young's inequality with the same exponents as before, but expanding differently leads to

$$\int_{F_2} \Gamma|v|^\beta \mathrm{d}x \leqslant \left[\varepsilon^{\frac{\beta}{2}} \left(\int_{F_2} |v|^{2^*} \mathrm{d}x \right)^{\frac{\beta}{2^*}} \right] \left[\varepsilon^{-\frac{\beta}{2}} \left(\int_{F_2} \Gamma^{s_\beta} \mathrm{d}x \right)^{\frac{1}{s_\beta}} \right] \leqslant \varepsilon \left(\int_{F_2} |v|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} \left(\int_{F_2} \Gamma^{s_\beta} \mathrm{d}x \right)^{\frac{2}{s_\beta(2-\beta)}}.$$

Exploiting the Morrey condition (6.4) again and noting that $\frac{2}{s_\beta(2-\beta)}(m-\beta s_\beta) = m$ by the definition of 2^*, we obtain

$$\begin{aligned}\int_{F_2} \Gamma|v|^\beta \,\mathrm{d}x &\leqslant \mathrm{c}_4 \varepsilon r_0^{m\frac{2}{2^*}+2} \left(\fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} \left(\Gamma_\beta\, r_0^{m-s_\beta\beta+s_\beta(2-\beta)\alpha} \right)^{\frac{2}{s_\beta(2-\beta)}} \\ &\leqslant c_4 \left[\varepsilon r_0^m \left(\fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} r_0^{m+2\alpha} \right]\end{aligned} \tag{6.18}$$

for some $\mathrm{c}_4 > 0$ as in the case $m \leqslant 2$. Combining the estimates (6.13), (6.14), (6.15), (6.16) and (6.18) provides

$$\int_{F_2} g(x,u+\varphi_1) - g(x,u)\,\mathrm{d}x \leqslant \mathrm{c}_5 \left[\int_{F_2} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x + \varepsilon r_0^m \left(\fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} r_0^{m+2\min\{\alpha,\delta\}} \right]$$

for all choices of q and m. Together with (6.10), (6.11), and (6.12), we conclude

$$\begin{aligned}\int_{F_1} |\mathrm{D}v|^2 \,\mathrm{d}x \leqslant \mathrm{c}_6 \Bigg[&\int_{F_2\setminus F_1} |\mathrm{D}v|^2 \,\mathrm{d}x + \int_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{\rho_2-\rho_1}\right|^2 + \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x \\ &+ \varepsilon r_0^m \left(\fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} r_0^{m+2\min\{\alpha,\delta\}} \Bigg].\end{aligned}$$

In order to ensure the constant c_6 to be in $(0,1)$, we use the Widman's hole filling trick, which adds the term $\mathrm{c}_6 \int_{F_1} |\mathrm{D}v|^2 \,\mathrm{d}x$ on both sides of the equation (i.e., filling the hole on the right-hand side). This leads to

$$\begin{aligned}\int_{F_1} |\mathrm{D}v|^2 \mathrm{d}x \leqslant \mathrm{c}_7 \int_{F_2} |\mathrm{D}v|^2 \,\mathrm{d}x &+ \int_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{\rho_2-\rho_1}\right|^2 \mathrm{d}x + \int_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x \\ &+ \varepsilon r_0^m \left(\fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} r_0^{m+2\min\{\alpha,\delta\}}\end{aligned}$$

with $\mathrm{c}_7 = \frac{\mathrm{c}_6}{1+\mathrm{c}_6} < 1$. Hence, we are able to apply Lemma 6.13 with $R = r_0$ and $r = \frac{r_0}{2}$, which states the existence of some $\mathrm{c} > 0$ such that the claimed statement is satisfied. □

In the following, we prove the Caccioppoli inequality for Setting 2, i.e., a priori bounded minimizers u, and released conditions on Γ. In fact, as we do no longer assume (6.5) or (6.6), additionally the definition of δ is obsolete. Hence, $\min\{\alpha,\delta\}$ is replaced by α in the reformulation.

Lemma 6.19 (Caccioppoli inequality for Setting 2)**.** *In the situation of Setting 2, there exists, for every $C > 0$, a constant*[‡] c > 0 *such that*

$$\fint_{\mathrm{B}_{\frac{r_0}{2}}(x_0)} |\mathrm{D}v|^2 \,\mathrm{d}x \leqslant \mathrm{c}\left[\fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^2 \mathrm{d}x + \varepsilon\left(\fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} r_0^{2\alpha}\right]$$

holds for all $\varepsilon \in (0,1]$, $\zeta \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{N\times m}$ with $|\xi| + |\zeta| \leqslant C$, where $v(x) := u(x) - \zeta - \xi x$ for all $x \in V$.

Proof. As in the proof of Lemma 6.18, we derive the inequality (6.10) and continue with estimating the first and second term as in (6.11) and (6.12), respectively. In order to estimate the third term of (6.10), we first notice that, by definition, the function v is uniformly bounded by a constant $K > 0$ on $\mathrm{B}_{r_0}(x_0)$. In fact, we can estimate $|v|^q \leqslant K^{q-\beta}|v|^\beta$, such that, instead of (6.13) and (6.14), the Morrey-Hölder condition (6.3) yields

$$\int_{F_2} g(x, u + \varphi_1) - g(x,u) \,\mathrm{d}x \leqslant \mathrm{c}_3 \int_{F_2} \Gamma |v|^\beta \,\mathrm{d}x$$

for some constant c_3. We continue with deriving the estimate (6.18) and conclude the claim as in the proof of Lemma 6.18. □

It is left to prove the Caccioppoli inequality in the case of Setting 3, where $u \in \mathrm{W}^{1,\infty}(V)$ is assumed, hence the relaxed conditions on Γ from Setting 2 apply and f only satisfies the local ellipticity condition (6.7).

Lemma 6.20 (Caccioppoli inequality for Setting 3)**.** *In the situation of Setting 3, there exists, for every $C > 0$, a constant*[§] c > 0 *such that*

$$\fint_{\mathrm{B}_{\frac{r_0}{2}}(x_0)} |\mathrm{D}v|^2 \,\mathrm{d}x \leqslant \mathrm{c}\left[\fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^2 \mathrm{d}x + \varepsilon\left(\fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}} + \varepsilon^{-\frac{\beta}{2-\beta}} r_0^{2\alpha}\right]$$

holds for all $\varepsilon \in (0,1]$, $\zeta \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{N\times m}$ with $|\xi| + |\zeta| \leqslant C$, where $v(x) := u(x) - \zeta - \xi x$ for all $x \in V$.

Proof. We argue as in the proof of Lemma 6.18, but instead of deriving equation (6.9) via the 2-strict quasiconvexity of f, we use the local ellipticity condition (6.7) to conclude

$$\begin{aligned}\frac{\mathrm{c}_1}{2}\int_{F_1} |\mathrm{D}v|^2 \,\mathrm{d}x &= \mathrm{c}_1 \int_{F_1}\int_0^1\int_0^1 t\,|\mathrm{D}v|^2 \,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x \\ &\leqslant \int_{F_1}\int_0^1\int_0^1 t\,\mathrm{D}^2 f(\xi + s\,t\,\mathrm{D}v)\mathrm{D}v\cdot\mathrm{D}v \,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x\end{aligned}$$

for the bound $\mathrm{c}_1 = \mathrm{c}(K)$ from (6.7) with respect to the bound

$$K := \sup\{|\xi + t\,\mathrm{D}v(x)| : t \in [0,1], x \in \mathrm{B}_{r_0}(x_0)\} \leqslant \|u\|_{\mathrm{W}^{1,\infty}} + C\,.$$

Now, by definition, it holds that $\varphi_1 = v$ on F_1 and according to the local ellipticity, $\mathrm{D}^2 f(\xi + s\,t\,\mathrm{D}v)\mathrm{D}v\cdot\mathrm{D}v$ remains non-negative on $F_2 \backslash F_1$. It follows

$$\begin{aligned}\frac{\mathrm{c}_1}{2}\int_{F_1} |\mathrm{D}v|^2 \,\mathrm{d}x &\leqslant \int_{F_2}\int_0^1\int_0^1 t\,\mathrm{D}^2 f(\xi + s\,t\,\mathrm{D}\varphi_1)\mathrm{D}\varphi_1\cdot\mathrm{D}\varphi_1 \,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_{F_2}\int_0^1 t\,\mathrm{D}f(\xi + t\,\mathrm{D}\varphi_1)\mathrm{D}\varphi_1 - \mathrm{D}f(\xi)\mathrm{D}\varphi_1 \,\mathrm{d}t\,\mathrm{d}x \\ &= \int_{F_2} f(\xi + \mathrm{D}\varphi_1) - f(\xi) - \mathrm{D}f(\xi)\mathrm{D}\varphi_1 \,\mathrm{d}x \\ &= \int_{F_2} f(\xi + \mathrm{D}\varphi_1) - f(\xi) \,\mathrm{d}x\,,\end{aligned}$$

[‡] The constant c has the same dependencies as in Lemma 6.18, where Γ_q is replaced by $\|u\|_{\mathrm{L}^\infty(\mathrm{B}_{r_0}(x_0),\mathbb{R}^N)}$.

[§] The constant c has the same dependencies as in Lemma 6.19, where $\|u\|_{\mathrm{L}^\infty(\mathrm{B}_{r_0}(x_0),\mathbb{R}^N)}$ is replaced by $\|u\|_{\mathrm{W}^{1,\infty}}$.

where the last equality follows from the divergence theorem and $\varphi_1 \in \mathrm{W}_0^{1,2}(F_2, \mathbb{R}^N)$. With this, we can conclude the claim as in the proof of Lemma 6.19, where the application of Lemma 6.9 is possible since quasiconvexity follows from the local ellipticity condition by Remark 6.8. $\square$

6.3.2 Approximate *A*-harmonicity

The following lemma provides the basis to apply Lemma 6.15 on $A = \mathrm{D}^2 f(\xi)$. We want to mention that $\|u\|^2_{\mathrm{L}^{2^*}}$ can be crucially estimated by the value Φ via the Sobolev inequality later in Section 6.3.3, such that control parameter ε as in the Caccioppoli inequalities can be avoided.

Lemma 6.21 (Approximate A-harmonicity for Setting 1)**.** *In the case of Setting 1, there exists a constant*[¶] $\mathrm{c} > 0$ *for every* $C > 0$ *such that*

$$\left| \fint_{\mathrm{B}_{r_0}(x_0)} \mathrm{D}^2 f(\xi)\, \mathrm{D}v \cdot \mathrm{D}\varphi \,\mathrm{d}x \right| \leqslant \mathrm{c} \left[\fint_{\mathrm{B}_{r_0}(x_0)} \left| \frac{v}{r_0} \right|^{2^*} \mathrm{d}x + \sqrt{\mathcal{U}_C(\Phi)\Phi} + \Phi + r_0^\alpha \right] \|\mathrm{D}\varphi\|_{\mathrm{L}^\infty(\mathrm{B}_{r_0}(x_0), \mathbb{R}^{N\times m})}$$

holds true for all $\varphi \in \mathrm{W}_0^{1,\infty}(\mathrm{B}_{r_0}(x_0), \mathbb{R}^N)$, $\zeta \in \mathbb{R}^N$ *and* $\xi \in \mathbb{R}^{N\times m}$ *with* $|\xi| + |\zeta| \leqslant C$, *where* $v(x)$ *is defined as* $u(x) - \zeta - \xi x$ *for all* $x \in V$ *and* $\Phi := \fint_{\mathrm{B}_{r_0}(x_0)} |\mathrm{D}v|^2 \,\mathrm{d}x$.

Proof. We first prove the claim for functions φ that satisfy $\|\mathrm{D}\varphi\|_{\mathrm{L}^\infty(\mathrm{B}_{r_0}(x_0), \mathbb{R}^{N\times m})} \leqslant 1$. Similar to the proof of Lemma 6.18, we rewrite the left-hand side of the equation as three different terms and estimate each term suitably. We first notice that, since $\varphi \in \mathrm{W}^{1,\infty}(\mathrm{B}_{r_0}(x_0), \mathbb{R}^N)$ with vanishing boundary values, it holds that

$$\|\varphi\|_{\mathrm{L}^\infty(\mathrm{B}_{r_0}(x_0), \mathbb{R}^N)} \leqslant r_0\,, \tag{6.19}$$

and $\int_{\mathrm{B}_{r_0}(x_0)} \mathrm{D}\varphi \,\mathrm{d}x = 0$ since $\mathrm{Tr}_{\mathrm{B}_{r_0}(x_0)}\varphi \equiv 0$ on $\partial \mathrm{B}_{r_0}(x_0)$. With this, we rewrite

$$\begin{aligned} \fint_{\mathrm{B}_{r_0}(x_0)} \mathrm{D}^2 f(\xi)\, \mathrm{D}v \cdot \mathrm{D}\varphi \,\mathrm{d}x &= \fint_{\mathrm{B}_{r_0}(x_0)} \mathrm{D}^2 f(\xi)\, \mathrm{D}v \cdot \mathrm{D}\varphi - \mathrm{D}f(\mathrm{D}u)\cdot \mathrm{D}\varphi + \mathrm{D}f(\xi)\cdot \mathrm{D}\varphi \,\mathrm{d}x \\ &\quad + \fint_{\mathrm{B}_{r_0}(x_0)} \mathrm{D}f(\mathrm{D}u)\cdot\mathrm{D}\varphi + r_0^{-\alpha}\left[f(\mathrm{D}u - r_0^\alpha \mathrm{D}\varphi) - f(\mathrm{D}u)\right] \mathrm{d}x \\ &\quad + r_0^{-\alpha} \fint_{\mathrm{B}_{r_0}(x_0)} f(\mathrm{D}u) - f(\mathrm{D}u - r_0^\alpha \mathrm{D}\varphi) \,\mathrm{d}x\,. \end{aligned} \tag{6.20}$$

To estimate the first term on the right-hand side of the equation, we rewrite

$$\begin{aligned} &\fint_{\mathrm{B}_{r_0}(x_0)} \mathrm{D}^2 f(\xi)\, \mathrm{D}v \cdot \mathrm{D}\varphi - \mathrm{D}f(\mathrm{D}u)\cdot\mathrm{D}\varphi + \mathrm{D}f(\xi)\cdot\mathrm{D}\varphi \,\mathrm{d}x \\ &\qquad = \fint_{\mathrm{B}_{r_0}(x_0)} \int_0^1 \mathrm{D}^2 f(\xi)\, \mathrm{D}v \cdot \mathrm{D}\varphi - \mathrm{D}^2 f(\xi + t(\mathrm{D}u - \xi))(\mathrm{D}u - \xi)\cdot \mathrm{D}\varphi \,\mathrm{d}t\,\mathrm{d}x \\ &\qquad = \fint_{\mathrm{B}_{r_0}(x_0)} \int_0^1 \mathrm{D}^2 f(\xi)\, \mathrm{D}v \cdot \mathrm{D}\varphi - \mathrm{D}^2 f(\xi + t\mathrm{D}v)\mathrm{D}v\cdot \mathrm{D}\varphi \,\mathrm{d}t\,\mathrm{d}x \\ &\qquad = \frac{1}{|\mathrm{B}_{r_0}|} \int_{\mathrm{B}_{r_0}(x_0)\cap\{|\mathrm{D}v|<1\}} \int_0^1 \mathrm{D}^2 f(\xi)\, \mathrm{D}v \cdot \mathrm{D}\varphi - \mathrm{D}^2 f(\xi + t\mathrm{D}v)\mathrm{D}v\cdot \mathrm{D}\varphi \,\mathrm{d}t\,\mathrm{d}x \\ &\qquad\quad + \frac{1}{|\mathrm{B}_{r_0}|} \int_{\mathrm{B}_{r_0}(x_0)\cap\{|\mathrm{D}v|\geqslant 1\}} \mathrm{D}^2 f(\xi)\, \mathrm{D}v \cdot \mathrm{D}\varphi - \mathrm{D}f(\mathrm{D}u)\cdot\mathrm{D}\varphi + \mathrm{D}f(\xi)\cdot\mathrm{D}\varphi \,\mathrm{d}t\,\mathrm{d}x\,. \end{aligned}$$

Since $|\xi|$ is bounded by C, we can apply the definition of $\mathcal{U}_C$, together with the fact that it is non-decreasing, and estimate

$$\frac{1}{|\mathrm{B}_{r_0}|} \int_{\mathrm{B}_{r_0}(x_0)\cap\{|\mathrm{D}v|<1\}} \int_0^1 \mathrm{D}^2 f(\xi)\, \mathrm{D}v \cdot \mathrm{D}\varphi - \mathrm{D}^2 f(\xi + t\mathrm{D}v)\mathrm{D}v\cdot \mathrm{D}\varphi \,\mathrm{d}t\,\mathrm{d}x \leqslant \fint_{\mathrm{B}_{r_0}(x_0)} \sqrt{\mathcal{U}_C(|\mathrm{D}v|^2)}|\mathrm{D}v| \,\mathrm{d}x\,.$$

[¶]The constant c depends on $n, N, \beta, q, \Gamma_\beta, \Gamma_q, f$ and C.

Now applying the Cauchy-Schwarz inequality provides us with

$$\frac{1}{|\mathrm{B}_{r_0}|}\int_{\mathrm{B}_{r_0}(x_0)\cap\{|\mathrm{D}v|<1\}}\int_0^1 \mathrm{D}^2 f(\xi)\,\mathrm{D}v\,\cdot\,\mathrm{D}\varphi - \mathrm{D}^2 f(\xi+t\mathrm{D}v)\mathrm{D}v\cdot\mathrm{D}\varphi\,\mathrm{d}t\,\mathrm{d}x \leqslant \sqrt{\fint_{\mathrm{B}_{r_0}(x_0)} \mathcal{U}_C(|\mathrm{D}v|^2)\,\mathrm{d}x\,\Phi}\,,$$

which, together with Jensen's inequality, yields

$$\frac{1}{|\mathrm{B}_{r_0}|}\int_{\mathrm{B}_{r_0}(x_0)\cap\{|\mathrm{D}v|<1\}}\int_0^1 \mathrm{D}^2 f(\xi)\,\mathrm{D}v\,\cdot\,\mathrm{D}\varphi - \mathrm{D}^2 f(\xi+t\mathrm{D}v)\mathrm{D}v\cdot\mathrm{D}\varphi\,\mathrm{d}t\,\mathrm{d}x \leqslant \sqrt{\mathcal{U}_C(\Phi)\,\Phi}\,.$$

By Lemma 6.9, we obtain

$$\begin{aligned}
&\frac{1}{|\mathrm{B}_{r_0}|}\int_{\mathrm{B}_{r_0}(x_0)\cap\{|\mathrm{D}v|\geqslant 1\}} \mathrm{D}^2 f(\xi)\,\mathrm{D}v\,\cdot\,\mathrm{D}\varphi - \mathrm{D}f(\mathrm{D}u)\cdot\mathrm{D}\varphi + \mathrm{D}f(\xi)\cdot\mathrm{D}\varphi\,\mathrm{d}x\\
&\qquad\leqslant \frac{K}{|\mathrm{B}_{r_0}|}\fint_{\mathrm{B}_{r_0}(x_0)\cap\{|\mathrm{D}v|\geqslant 1\}} \sup_{\mathrm{B}_C^{N\times m}} |\mathrm{D}^2 f||\mathrm{D}v| + \mathrm{c}_1(2+|\mathrm{D}u|+|\xi|)\,\mathrm{d}x\\
&\qquad\leqslant \fint_{\mathrm{B}_{r_0}(x_0)} \sup_{\mathrm{B}_C^{N\times m}} |\mathrm{D}^2 f||\mathrm{D}v|^2 + 2\mathrm{c}_1(1+C)|\mathrm{D}v|^2\,\mathrm{d}x\\
&\qquad\leqslant \mathrm{c}\,\Phi
\end{aligned}$$

with c_1 being the constant from the corresponding Lemma. Together, we have

$$\fint_{\mathrm{B}_{r_0}(x_0)} \mathrm{D}^2 f(\xi)\,\mathrm{D}v\,\cdot\,\mathrm{D}\varphi - \mathrm{D}f(\mathrm{D}u)\cdot\mathrm{D}\varphi + \mathrm{D}f(\xi)\cdot\mathrm{D}\varphi\,\mathrm{d}x \leqslant \mathrm{c}\left(\Phi + \sqrt{\mathcal{U}_C(\Phi)\,\Phi}\right)\,. \tag{6.21}$$

In order to estimate the second term of (6.20), we similarly rewrite

$$\begin{aligned}
&\fint_{\mathrm{B}_{r_0}(x_0)} \mathrm{D}f(\mathrm{D}u)\cdot\mathrm{D}\varphi + r_0^{-\alpha}\left[f(\mathrm{D}u - r_0^{\alpha}\mathrm{D}\varphi) - f(\mathrm{D}u)\right]\,\mathrm{d}x\\
&\qquad= \fint_{\mathrm{B}_{r_0}(x_0)}\fint_0^{r_0^\alpha} \left[\mathrm{D}f(\mathrm{D}u) + \mathrm{D}f(\mathrm{D}u - t\mathrm{D}\varphi)\right]\cdot\mathrm{D}\varphi\,\mathrm{d}t\,\mathrm{d}x\\
&\qquad= \frac{1}{|\mathrm{B}_{r_0}|}\int_{\mathrm{B}_{r_0}(x_0)\cap\{|\mathrm{D}v|<1\}}\fint_0^{r_0^\alpha}\int_0^1 t\,\mathrm{D}^2 f(\mathrm{D}u - s\,t\,\mathrm{D}\varphi)\mathrm{D}\varphi\cdot\mathrm{D}\varphi\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x\\
&\qquad+ \frac{1}{|\mathrm{B}_{r_0}|}\int_{\mathrm{B}_{r_0}(x_0)\cap\{|\mathrm{D}v|\geqslant 1\}}\fint_0^{r_0^\alpha} \left[\mathrm{D}f(\mathrm{D}u) + \mathrm{D}f(\mathrm{D}u - t\mathrm{D}\varphi)\right]\cdot\mathrm{D}\varphi\,\mathrm{d}t\,\mathrm{d}x\,.
\end{aligned}$$

We further estimate

$$\begin{aligned}
&\frac{1}{|\mathrm{B}_{r_0}|}\int_{\mathrm{B}_{r_0}(x_0)\cap\{|\mathrm{D}v|<1\}}\fint_0^{r_0^\alpha}\int_0^1 t\,\mathrm{D}^2 f(\mathrm{D}u - s\,t\,\mathrm{D}\varphi)\mathrm{D}\varphi\cdot\mathrm{D}\varphi\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x\\
&\qquad\qquad\leqslant \sup_{\mathrm{B}_{C+2}^{N\times m}} |\mathrm{D}^2 f|\fint_{\mathrm{B}_{r_0}(x_0)}\fint_0^{r_0^\alpha}\int_0^1 t\,|\mathrm{D}\varphi|^2\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x = \mathrm{c}r_0^\alpha\,.
\end{aligned}$$

As before, we use Lemma 6.9, which yields

$$\begin{aligned}
&\frac{1}{|\mathrm{B}_{r_0}|}\int_{\mathrm{B}_{r_0}(x_0)\cap\{|\mathrm{D}v|\geqslant 1\}}\fint_0^{r_0^\alpha} \left[\mathrm{D}f(\mathrm{D}u) + \mathrm{D}f(\mathrm{D}u - t\mathrm{D}\varphi)\right]\cdot\mathrm{D}\varphi\,\mathrm{d}t\,\mathrm{d}x\\
&\qquad\leqslant \frac{\mathrm{c}_1}{|\mathrm{B}_{r_0}|}\int_{\mathrm{B}_{r_0}(x_0)\cap\{|\mathrm{D}v|\geqslant 1\}}\fint_0^{r_0^\alpha} \left[2 + |\mathrm{D}u| + |\mathrm{D}u - t\mathrm{D}\varphi|\right]\cdot|\mathrm{D}\varphi|\,\mathrm{d}t\,\mathrm{d}x\\
&\qquad\leqslant \mathrm{c}_1\fint_{\mathrm{B}_{r_0}(x_0)\cap\{|\mathrm{D}v|\geqslant 1\}}\fint_0^{r_0^\alpha} 3 + 2C + 2|\mathrm{D}v|\,\mathrm{d}t\,\mathrm{d}x\\
&\qquad\leqslant \mathrm{c}\,\Phi\,.
\end{aligned}$$

Together, we obtain

$$\fint_{\mathrm{B}_{r_0}(x_0)} \mathrm{D}f(\mathrm{D}u)\cdot\mathrm{D}\varphi + r_0^{-\alpha}\left[f(\mathrm{D}u - r_0^{\alpha}\mathrm{D}\varphi) - f(\mathrm{D}u)\right]\mathrm{d}x \leqslant \mathrm{c}(r_0^{\alpha} + \Phi)\,. \tag{6.22}$$

For the third term, we exploit the local minimality of u, i.e., $\mathbb{F}_{\mathrm{B}_{r_0}(x_0)}[u] \leqslant \mathbb{F}_{\mathrm{B}_{r_0}(x_0)}[u - r_0^{\alpha}\varphi]$, which can be rewritten as

$$r_0^{-\alpha}\fint_{\mathrm{B}_{r_0}(x_0)} f(\mathrm{D}u) - f(\mathrm{D}u - r_0^{\alpha}\mathrm{D}\varphi)\,\mathrm{d}x \leqslant r_0^{-\alpha}\fint_{\mathrm{B}_{r_0}(x_0)} g(x, u - r_0^{\alpha}\varphi) - g(x,u)\,\mathrm{d}x\,.$$

Using the Morrey-Hölder conditions (6.3) and (6.19) yields

$$\begin{aligned}
r_0^{-\alpha}\fint_{\mathrm{B}_{r_0}(x_0)} f(\mathrm{D}u) - f(\mathrm{D}u - r_0^{\alpha}\mathrm{D}\varphi)\,\mathrm{d}x &\leqslant r_0^{-\alpha}\fint_{\mathrm{B}_{r_0}(x_0)} (1 + 2|u| + r_0^{\alpha}\varphi)^{q-\beta}(r_0^{\alpha}\varphi)^{\beta}\Gamma\,\mathrm{d}x\\
&\leqslant r_0^{-\alpha}\fint_{\mathrm{B}_{r_0}(x_0)} (1 + 2C + 2|v|)^{q-\beta} r_0^{(1+\alpha)\beta}\Gamma\,\mathrm{d}x\\
&\leqslant r_0^{-\alpha}\fint_{\mathrm{B}_{r_0}(x_0)} 2(1 + C + |v|)^{q-\beta} r_0^{(1+\alpha)\beta}\Gamma\,\mathrm{d}x\\
&= \mathrm{c}r_0^{(1+\alpha)\beta-\alpha}\fint_{\mathrm{B}_{r_0}(x_0)} (1 + |v|^q)\Gamma\,\mathrm{d}x\,.
\end{aligned}$$

Since $r_0 \leqslant 1$ and $\mathrm{L}^{s_\beta, m+s_\beta(2-\beta)\alpha - s_\beta\beta}(V) \subseteq \mathrm{L}^{1,m+(2-\beta)\alpha-\beta}(V)$ according to Remark 6.4, the condition (6.4) gives

$$\fint_{\mathrm{B}_{r_0}(x_0)} \Gamma\,\mathrm{d}x \leqslant \mathrm{c}r_0^{(2-\beta)\alpha-\beta}\,. \tag{6.23}$$

As in the proof of Lemma 6.18, specifically, equations (6.15) and (6.16) on $\mathrm{B}_{r_0}(x_0)$ instead of F_2, we can estimate

$$\fint_{\mathrm{B}_{r_0}(x_0)} |v|^q\Gamma\,\mathrm{d}x \leqslant \fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^{2^*}\mathrm{d}x + \mathrm{c}r_0^{2\delta}\,. \tag{6.24}$$

Together, the previous inequalities give

$$r_0^{-\alpha}\fint_{\mathrm{B}_{r_0}(x_0)} f(\mathrm{D}u) - f(\mathrm{D}u - r_0^{\alpha}\mathrm{D}\varphi)\,\mathrm{d}x \leqslant \mathrm{c}r_0^{(1+\alpha)\beta-\alpha}\fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^{2^*}\mathrm{d}x + \mathrm{c}r_0^{(1+\alpha)\beta-\alpha+2\delta} + \mathrm{c}r_0^{\alpha}\,.$$

Exploiting $\alpha \leqslant \frac{\beta}{2-\beta}$ and $r_0 \leqslant 1$ yields

$$r_0^{-\alpha}\fint_{\mathrm{B}_{r_0}(x_0)} f(\mathrm{D}u) - f(\mathrm{D}u - r_0^{\alpha}\mathrm{D}\varphi)\,\mathrm{d}x \leqslant \mathrm{c}\left[\fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^{2^*}\mathrm{d}x + r_0^{\alpha}\right]. \tag{6.25}$$

We now put together (6.20), (6.21), (6.22) and (6.25) to obtain

$$\fint_{\mathrm{B}_{r_0}(x_0)} \mathrm{D}^2 f(\xi)\,\mathrm{D}v\,\cdot\,\mathrm{D}\varphi\,\mathrm{d}x \leqslant \mathrm{c}\left[\fint_{\mathrm{B}_{r_0}(x_0)} \left|\frac{v}{r_0}\right|^{2^*}\mathrm{d}x + \sqrt{\mathcal{U}_K(\Phi)\Phi} + \Phi + r_0^{\alpha}\right].$$

The lower bound estimation is analogous with the slight difference that, instead of (6.20), the reformulation of the left-hand side is

$$\begin{aligned}
\fint_{\mathrm{B}_{r_0}(x_0)} \mathrm{D}^2 f(\xi)\,\mathrm{D}v\,\cdot\,\mathrm{D}\varphi\,\mathrm{d}x = &\fint_{\mathrm{B}_{r_0}(x_0)} \mathrm{D}^2 f(\xi)\,\mathrm{D}v\,\cdot\,\mathrm{D}\varphi - \mathrm{D}f(\mathrm{D}u)\cdot\mathrm{D}\varphi + \mathrm{D}f(\xi)\cdot\mathrm{D}\varphi\,\mathrm{d}x\\
&+ \fint_{\mathrm{B}_{r_0}(x_0)} \mathrm{D}f(\mathrm{D}u)\cdot\mathrm{D}\varphi + r_0^{-\alpha}\left[f(\mathrm{D}u + r_0^{\alpha}\mathrm{D}\varphi) - f(\mathrm{D}u)\right]\mathrm{d}x\\
&+ r_0^{-\alpha}\fint_{\mathrm{B}_{r_0}(x_0)} f(\mathrm{D}u) - f(\mathrm{D}u + r_0^{\alpha}\mathrm{D}\varphi)\,\mathrm{d}x\,.
\end{aligned}$$

The claimed estimate for general functions $\varphi \in \mathrm{W}^{1,\infty}(\mathrm{B}_{r_0}(x_0), \mathbb{R}^N)$ with $K := \|\mathrm{D}\varphi\|_{\mathrm{L}^\infty(\mathrm{B}_{r_0}(x_0),\mathbb{R}^{N\times m})}$ possibly not bounded by 1 follows by inserting $\frac{\varphi}{K}$ in the derived formula. $\square$

We show the corresponding Lemma for the remaining settings.

Lemma 6.22 (Approximate A-harmonicity for Setting 2 or Setting 3)**.** *In the case of Setting 2 or Setting 3, there exists a constant*‖ $\mathrm{c} > 0$ *for every* $C > 0$ *such that*

$$\left| \fint_{\mathrm{B}_{r_0}(x_0)} \mathrm{D}^2 f(\xi)\, \mathrm{D}v \cdot \mathrm{D}\varphi \,\mathrm{d}x \right| \leqslant \mathrm{c} \left[\sqrt{\mathcal{U}_C(\Phi)\Phi} + \Phi + r_0^{\alpha} \right] \|\mathrm{D}\varphi\|_{\mathrm{L}^\infty(\mathrm{B}_{r_0}(x_0),\mathbb{R}^{N\times m})}$$

holds true for all $\varphi \in \mathrm{W}^{1,\infty}(\mathrm{B}_{r_0}(x_0), \mathbb{R}^N)$, $\zeta \in \mathbb{R}^N$ *and* $\xi \in \mathbb{R}^{N\times m}$ *with* $|\xi| + |\zeta| \leqslant C$, *where* $v(x)$ *is defined as* $u(x) - \zeta - \xi x$ *for all* $x \in V$ *and* $\Phi := \fint_{\mathrm{B}_{r_0}(x_0)} |\mathrm{D}v|^2 \,\mathrm{d}x$.

Proof. The proof is analogous to the proof of Lemma 6.21, where Lemma 6.9 remains applicable in Setting 3 since the local ellipticity of f implies its quasiconvexity. Moreover, in both settings, we replace the estimate (6.24) by

$$\fint_{\mathrm{B}_{r_0}(x_0)} |v|^q \Gamma \,\mathrm{d}x \leqslant (\|u\|_{\mathrm{L}^\infty(V,\mathbb{R}^N)} + C)^q \fint_{\mathrm{B}_{r_0}(x_0)} \Gamma \,\mathrm{d}x\,,$$

which can be estimated from above by $\mathrm{c}r_0^{(1-\beta)\alpha-\beta}$ as in (6.23). Hence, (6.25) turns into

$$r_0^{-\alpha} \fint_{\mathrm{B}_{r_0}(x_0)} f(\mathrm{D}u) - f(\mathrm{D}u - r_0^{\alpha}\mathrm{D}\varphi) \,\mathrm{d}x \leqslant \mathrm{c}r_0^{\alpha}\,,$$

and the proof of the claim follows as in Lemma 6.21. □

6.3.3 Excess estimates

We now have all required tools to derive the crucial estimates for the quadratic excess, which are essential in order to apply Campanato's characterization of Hölder continuity, i.e., Corollary 6.17 for $\mathrm{D}u$.

Definition 6.23 (Quadratic excess for $\mathrm{D}u$)**.** Let $x \in V$ and $r > 0$ such that $\mathrm{B}_r(x) \subseteq V$. Then the **quadratic excess** (of $\mathrm{D}u$ in $\mathrm{B}_r(x)$) is defined as

$$\Phi(x,r) := \fint_{\mathrm{B}_r(x)} |\mathrm{D}u - (\mathrm{D}u)_{x,r}|^2 \,\mathrm{d}x\,.$$

We furthermore abbreviate $\Phi(x_0, r)$ by $\Phi(r)$ and $\Phi(x_0, r_0)$ by Φ.

Remark 6.24. i) For all $\xi \in \mathbb{R}^{N\times m}$, it holds that $\Phi \leqslant \fint_{\mathrm{B}_{r_0}(x_0)} |\mathrm{D}u - \xi|^2 \,\mathrm{d}x$.

ii) For all $\rho \in (0, r_0]$, it holds that $\Phi(\rho) \leqslant \left(\frac{r_0}{\rho}\right)^m \Phi$.

To show i), we define the auxiliary convex function $\Psi : \mathbb{R}^{N\times m} \to [0,\infty)$, $\xi \mapsto \fint_{\mathrm{B}_{r_0}(x_0)} |\mathrm{D}u - \xi|^2 \,\mathrm{d}x$ and notice that there exists at least one local minimum of Ψ. Since

$$\mathrm{D}\Psi(\xi) = 2 \fint_{\mathrm{B}_{r_0}(x_0)} (\mathrm{D}u - \xi) \,\mathrm{d}x\,,$$

the only critical point, and therefore the global minimizer is $(\mathrm{D}u)_{x_0,r_0}$. For proving ii), we rewrite the left-hand side and apply i), which gives

$$\Phi(\rho) = \fint_{\mathrm{B}_\rho(x_0)} |\mathrm{D}u - (\mathrm{D}u)_{x_0,\rho}|^2 \,\mathrm{d}x \leqslant \frac{|\mathrm{B}_{r_0}|}{|\mathrm{B}_\rho|} \fint_{\mathrm{B}_{r_0}(x_0)} |\mathrm{D}u - (\mathrm{D}u)_{x_0,r_0}|^2 \,\mathrm{d}x \leqslant \left(\frac{r_0}{\rho}\right)^m \Phi\,.$$

‖The constant c depends on $m, N, \beta, q, \Gamma_\beta, \|u\|_{\mathrm{L}^\infty(V,\mathbb{R}^N)}, f$ and C.

Lemma 6.25 (First excess estimate). *We consider Setting 1, Setting 2, or Setting 3. For every $\kappa \in (\min\{\delta,\alpha\},1)$ in the case of Setting 1, and $\kappa \in (\alpha,1)$ in the case of Settings 2 and 3, and for every constant $C>0$, there exist** some $\lambda \in (0,1)$, $\varepsilon > 0$ and $\mathrm{c}>0$, such that*

$$r_0 + \Phi \leqslant \varepsilon \qquad \textit{and} \qquad |(\mathrm{D}u)_{x_0,r_0}| + |(u)_{x_0,r_0}| \leqslant C \tag{6.26}$$

imply

$$\Phi(\lambda r_0) \leqslant \lambda^{2\kappa}\Phi + \mathrm{c} r_0^{2\min\{\delta,\alpha\}}$$

in the case of Setting 1 and

$$\Phi(\lambda r_0) \leqslant \lambda^{2\kappa}\Phi + \mathrm{c} r_0^{2\alpha}$$

in the case of the other settings.

Proof. W.l.o.g., we assume $\Phi>0$ and $\delta>\alpha$ if we consider Setting 2 and 3. Moreover, we use the following abbreviations

$$\xi := (\mathrm{D}u)_{x_0,r_0}\,, \qquad \zeta := (u)_{x_0,r_0}\,, \qquad A := \mathrm{D}^2 f(\xi)\,, \qquad v(x) = u(x+x_0) - \zeta - \xi x \quad \text{for all } x \in \mathrm{B}_{r_0}\,.$$

The shift in the argument of v has technical reasons and does not have significant effect since $x \mapsto u(x+x_0)$ still locally minimizes the functional $w \mapsto \int_{V-x_0} f(\mathrm{D}w) + g(x+x_0,w)\,\mathrm{d}x$ suitably, and the corresponding zero-order term still satisfies (6.3).
In order to apply Lemma 6.15 on A, we have to ensure that the corresponding assumptions are satisfied via Lemmas 6.21 and 6.22. The upper bound of $|A|$ in Definition 6.14 is realized by $\sup_{\mathrm{B}_C^{N\times m}} |\mathrm{D}^2 f|$. Since in both Settings 1 and 2 f is assumed to be 2-strictly quasiconvex, the second assumption in (6.2) is ensured by Lemma 6.11. In Setting 3, the boundedness of $|\xi|$ by C and the local ellipticity (6.7) give the required estimate. Since Φ is assumed to be controlled, it remains to show $\left|⨍_{\mathrm{B}_{r_0}} A(\mathrm{D}v,\mathrm{D}\varphi)\,\mathrm{d}x\right| \leqslant \delta_A\, s\, \|\mathrm{D}\varphi\|_{\mathrm{L}^\infty(\mathrm{B}_\rho,\mathbb{R}^{N\times m})}$ for suitable $\delta_A>0$, $s\in(0,1]$ and for all $\varphi \in \mathrm{W}_0^{1,\infty}(\mathrm{B}_{r_0},\mathbb{R}^N)$, in order to fit the premises of Lemma 6.15. To this end, we apply Lemmas 6.21 and 6.22, which yields

$$\left|⨍_{\mathrm{B}_{r_0}} A(\mathrm{D}v,\mathrm{D}\varphi)\,\mathrm{d}x\right| \leqslant \mathrm{c}\left[⨍_{\mathrm{B}_{r_0}} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x + \sqrt{\mathcal{U}_K(\Phi)\Phi} + \Phi + r_0^\alpha\right] \|\mathrm{D}\varphi\|_{\mathrm{L}^\infty(\mathrm{B}_{r_0}(x_0),\mathbb{R}^{N\times m})}\,. \tag{6.27}$$

Since $(v)_{0,r_0} = ⨍_{\mathrm{B}_{r_0}(x_0)} u(x) - \xi x - \zeta\,\mathrm{d}x = \xi ⨍_{\mathrm{B}_{r_0}} x\,\mathrm{d}x = 0$, the Sobolev inequality in the case $m \geqslant 3$ yields

$$⨍_{\mathrm{B}_{r_0}} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x \leqslant \mathrm{c}\left(⨍_{\mathrm{B}_{r_0}} |\mathrm{D}v|^2 \mathrm{d}x\right)^{\frac{2^*}{2}}\,.$$

In the case $m=2$, we find $p\in[1,2)$ with $p^* = 2^*$, then the Sobolev and the Hölder inequalities imply

$$⨍_{\mathrm{B}_{r_0}} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x \leqslant \mathrm{c} r_0^{-2-2^*}\left(\int_{\mathrm{B}_{r_0}} |\mathrm{D}v|^q\,\mathrm{d}x\right)^{\frac{2^*}{p}} \leqslant \mathrm{c} r_0 \left(⨍_{\mathrm{B}_{r_0}} |\mathrm{D}v|^2\,\mathrm{d}x\right)^{\frac{2^*}{2}} = \mathrm{c}\left(⨍_{\mathrm{B}_{r_0}} |\mathrm{D}v|^2\,\mathrm{d}x\right)^{\frac{2^*}{2}}\,.$$

In the case $m=1$, the Sobolev embedding implies $v \in \mathrm{C}^{0,\frac{1}{2}}(\mathrm{B}_{r_0},\mathbb{R}^N)$ with

$$\sup_{x\neq y\in \mathrm{B}_{r_0}} \frac{|v(x)-v(y)|}{\sqrt{|x-y|}} \leqslant \mathrm{c}\|\mathrm{D}v\|_{\mathrm{L}^2(\mathrm{B}_{r_0},\mathbb{R}^{N\times m})}\,.$$

Furthermore, since $(v)_{0,r_0}=0$, we find $y\in \mathrm{B}_{r_0}$ with $v(y)$, which leads to the estimate

$$⨍_{\mathrm{B}_{r_0}} \left|\frac{v}{r_0}\right|^{2^*} \mathrm{d}x \leqslant r_0^{-2^*} \sup_{x\in \mathrm{B}_{r_0}} |v(x)-v(y)|^{2^*} \leqslant \mathrm{c} r_0^{-\frac{2^*}{2}}\left(\int_{\mathrm{B}_{r_0}} |\mathrm{D}v|^2\,\mathrm{d}x\right)^{\frac{2^*}{2}} = \mathrm{c}\left(⨍_{\mathrm{B}_{r_0}} |\mathrm{D}v|^2\,\mathrm{d}x\right)^{\frac{2^*}{2}}\,.$$

**All constants depend on $m,N,\beta,q,\Gamma_\beta,C,\kappa$ and f in all settings. The number ε additionally depends on α and the choice $\mathcal{U}_C$. In the case of Setting 1, all values also depend on Γ_q, which is replaced by $\|u\|_{\mathrm{L}^\infty(V,\mathbb{R}^N)}$ and by $\|u\|_{W^{1,\infty}}$ in the other settings, respectively.

Hence, in all cases, (6.27) turns into

$$\left| \fint_{\mathrm{B}_{r_0}} A(\mathrm{D}v, \mathrm{D}\varphi)\,\mathrm{d}x \right| \leqslant \mathrm{c}_1 \left[\Phi^{\frac{2^*}{2}} + \sqrt{\mathcal{U}_K(\Phi)\Phi} + \Phi + r_0^\alpha \right] \|\mathrm{D}\varphi\|_{\mathrm{L}^\infty(\mathrm{B}_{r_0}(x_0), \mathbb{R}^{N\times m})} . \tag{6.28}$$

In the following, we consider $\varepsilon_A \in (0,1]$ to be fixed later and the corresponding δ_A from Lemma 6.15. Then we define $s := \sqrt{\Phi} + \frac{2\mathrm{c}_1}{\delta_A} r_0^\alpha$, such that (6.28) turns into

$$\left| \fint_{\mathrm{B}_{r_0}} A(\mathrm{D}v, \mathrm{D}\varphi)\,\mathrm{d}x \right| \leqslant \left[\mathrm{c}_1 \left(\sqrt{\mathcal{U}_K(\Phi)} + \sqrt{\Phi} + \Phi^{\frac{2^*-1}{2}} \right) + \frac{1}{2}\delta_A \right] s \, \|\mathrm{D}\varphi\|_{\mathrm{L}^\infty(\mathrm{B}_{r_0}(x_0), \mathbb{R}^{N\times m})} .$$

Due to (6.26), we can make $\varepsilon \in (0, s)$ small enough such that $\mathrm{c}_1 \left(\sqrt{\mathcal{U}_K(\Phi)} + \sqrt{\Phi} + \Phi^{\frac{2^*-1}{2}} \right) \leqslant \frac{1}{2}\delta_A$, which yields

$$\left| \fint_{\mathrm{B}_{r_0}} A(\mathrm{D}v, \mathrm{D}\varphi)\,\mathrm{d}x \right| \leqslant \delta_A s \, \|\mathrm{D}\varphi\|_{\mathrm{L}^\infty(\mathrm{B}_{r_0}(x_0), \mathbb{R}^{N\times m})}$$

for all $\varphi \in \mathrm{W}_0^{1,\infty}(\mathrm{B}_{r_0}, \mathbb{R}^N)$. Thus, we can finally apply Lemma 6.15, which ensures the existence of some A-harmonic function $h \in \mathrm{C}^\infty(\mathrm{B}_{r_0}, \mathbb{R}^N)$ and some constant $\mathrm{c}_2 > 0$, which satisfy

$$\sup_{\mathrm{B}_{\frac{r_0}{2}}} |\mathrm{D}h| + r_0 \sup_{\mathrm{B}_{\frac{r_0}{2}}} |\mathrm{D}^2 h| \leqslant \mathrm{c}_2 \qquad \text{and} \qquad \fint_{\mathrm{B}_{\frac{r_0}{2}}} \left| \frac{v - sh}{r_0} \right| \mathrm{d}x \leqslant s^2 \varepsilon_A . \tag{6.29}$$

Now, for $\lambda \in \left(0, \frac{1}{4}\right]$ and $\varepsilon_A := \lambda^{m+4}$, it holds that

$$\fint_{\mathrm{B}_{2\lambda r_0}} \left| \frac{v - s\mathrm{D}h(0)x - sh(0)}{\lambda r_0} \right|^2 \mathrm{d}x \leqslant \frac{2}{4^m \lambda^{m+2}} \fint_{\mathrm{B}_{\frac{r_0}{2}}} \left| \frac{v - sh}{r_0} \right|^2 \mathrm{d}x + 2s^2 \sup_{x \in \mathrm{B}_{2\lambda r_0}} \left| \frac{h(x) - h(0) - \mathrm{D}h(0)x}{2\lambda r_0} \right|^2 .$$

Since the mean value inequality produces the bound

$$|h(x) - h(0) - \mathrm{D}h(0)x| \leqslant \sup_{\mathrm{B}_{2\lambda r_0}} |\mathrm{D}^2 h| (2\lambda r_0)^2 \leqslant 4\mathrm{c}_2 \lambda^2 r_0$$

for all $x \in \mathrm{B}_{2\lambda r_0}$, by (6.29) we obtain

$$\fint_{\mathrm{B}_{2\lambda r_0}} \left| \frac{v - s\mathrm{D}h(0)x - sh(0)}{\lambda r_0} \right|^2 \mathrm{d}x \leqslant \mathrm{c} \left(\frac{\varepsilon_A}{\lambda^{m+2}} + \lambda^2 \right) s^2 \leqslant \mathrm{c}\lambda^2 s^2 . \tag{6.30}$$

Since (6.29) implies $|h(x) - h(0)| \leqslant \mathrm{c}r_0$ for all $x \in \mathrm{B}_{\frac{r_0}{2}}$, the triangle inequality yields

$$|h(0)|^2 \leqslant \mathrm{c} \left(\fint_{\mathrm{B}_{\frac{r_0}{2}}} |h|^2 \,\mathrm{d}x + r_0^2 \right) \leqslant \frac{\mathrm{c}}{s^2} \left(\fint_{\mathrm{B}_{\frac{r_0}{2}}} |v - sh|^2 \,\mathrm{d}x + \fint_{\mathrm{B}_{\frac{r_0}{2}}} |v|^2 \,\mathrm{d}x \right) + \mathrm{c}r_0^2 .$$

Now, the second inequality of (6.29), $\varepsilon_A \leqslant 1$ and the Poincaré inequality guarantee

$$|h(0)|^2 \leqslant \mathrm{c} \left(\varepsilon_A r_0^2 + r_0^2 \Phi + r_0^2 \right) ,$$

which, together with (6.26) and the choice of ε, gives $\Phi \leqslant \sqrt{\Phi} \leqslant s$, hence,

$$|h(0)|^2 \leqslant \mathrm{c}r_0^2 . \tag{6.31}$$

In fact, we can bound $|\xi + s\mathrm{D}h(0)| + |\zeta + sh(0)|$ by a constant c_3. According to Lemmas 6.18 and 6.19, we can then find another constant $\mathrm{c}_4 > 0$, such that the corresponding estimates are satisfied for all $\varepsilon_C \in (0,1]$

instead of ε, and for the function $\tilde{v} := v - s\mathrm{D}h(0)x - sh(0)$ instead of v. Combined with Remark 6.24, we obtain

$$\Phi(\lambda r_0) \leqslant \fint_{\mathrm{B}_{\lambda r_0}} |\mathrm{D}\tilde{v}|^2 \mathrm{d}x \leqslant \mathrm{c}_4 \left[\fint_{\mathrm{B}_{2\lambda r_0}} \left| \frac{\tilde{v}}{\lambda r_0} \right|^2 \mathrm{d}x + \varepsilon_C \left(\fint_{\mathrm{B}_{2\lambda r_0}} \left| \frac{\tilde{v}}{\lambda r_0} \right|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}} + \fint_{\mathrm{B}_{2\lambda r_0}} \left| \frac{\tilde{v}}{\lambda r_0} \right|^{2^*} \mathrm{d}x + \frac{(\lambda r_0)^{2\min\{\delta,\alpha\}}}{\varepsilon_C^{\frac{\beta}{2-\beta}}} \right] .$$

The definition of $\tilde{v}$ and the triangle inequality yield

$$\begin{aligned} \Phi(\lambda r_0) \leqslant \mathrm{c} \Bigg[& \fint_{\mathrm{B}_{2\lambda r_0}} \left| \frac{\tilde{v}}{\lambda r_0} \right|^2 \mathrm{d}x + \varepsilon_C \left(\lambda^{-m-2^*} \fint_{\mathrm{B}_{r_0}} \left| \frac{v}{r_0} \right|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}} + \lambda^{-m-2^*} \fint_{\mathrm{B}_{r_0}} \left| \frac{v}{r_0} \right|^{2^*} \mathrm{d}x \\ & + \frac{(\lambda r_0)^{2\min\{\delta,\alpha\}}}{\varepsilon_C^{\frac{\beta}{2-\beta}}} + \varepsilon_C s^2 \left(\frac{|h(0)|}{\lambda r_0} + |\mathrm{D}h(0)| \right)^2 + s^{2^*} \left(\frac{|h(0)|}{\lambda r_0} + |\mathrm{D}h(0)| \right)^{2^*} \Bigg] . \end{aligned}$$

The latter can be estimated with (6.26), (6.30), (6.31) and the Sobolev inequality or Poincaré and Hölder inequalities, respectively. This results in

$$\Phi(\lambda r_0) \leqslant \mathrm{c} \left[\lambda^2 s^2 + \varepsilon_C \lambda^{-\frac{2}{2^*}m-2} \Phi + \lambda^{-m-2^*} \Phi^{\frac{2^*}{2}} + \varepsilon_C^{-\frac{\beta}{2-\beta}} (\lambda r_0)^{2\min\{\delta,\alpha\}} + \varepsilon_C \lambda^{-2} s^2 + \lambda^{-2^*} s^{2^*} \right] .$$

With $\Phi \leqslant s^2$, we can sum up some terms and obtain

$$\Phi(\lambda r_0) \leqslant \mathrm{c}_5 \left[\left(\lambda^2 + \varepsilon_C \lambda^{-\frac{2}{2^*}m-2} + \lambda^{-m-2^*} s^{2^*-2} \right) s^2 + \varepsilon_C^{-\frac{\beta}{2-\beta}} (\lambda r_0)^{2\min\{\delta,\alpha\}} \right] .$$

Finally, we fix the parameters. To this end, we choose λ, ε_C and ε in that order small enough such that

$$6\mathrm{c}_5 \lambda^2 \leqslant \lambda^{2\kappa}, \qquad 6\mathrm{c}_5 \varepsilon_C \lambda^{-\frac{2}{2^*}m-2} \leqslant \lambda^{2\kappa} \qquad \text{and} \qquad 6\mathrm{c}_5 s^{2^*-2} \lambda^{-m-2^*} \leqslant \lambda^{2\kappa} .$$

Exploiting the choice of s, in particular, that $s^2 \leqslant 2\left(\Phi + \frac{2\mathrm{c}_1}{\delta_A} r_0^{2\alpha}\right) \leqslant 2\left(\Phi + \frac{2\mathrm{c}_1}{\delta_A} r_0^{2\min\{\delta,\alpha\}}\right)$, we find

$$\Phi(\lambda r_0) \leqslant \frac{1}{2} \lambda^{2\kappa} s^2 + \mathrm{c}_5 \varepsilon_C^{-\frac{\beta}{2-\beta}} (\lambda r_0)^{2\min\{\delta,\alpha\}} \leqslant \lambda^{2\kappa} \Phi + \mathrm{c} r_0^{2\min\{\delta,\alpha\}} .$$

Taking into account the choice of δ in case of Setting 2 and 3, the claim is proven. □

The previous Lemma prepares the following main excess estimate, which uses a standard induction argument.

Lemma 6.26 (Second excess estimate)**.** *We consider Setting 1, 2 or 3. For every* $\kappa \in (\min\{\delta,\alpha\}, 1)$ *and* $C > 0$, *there exist values*[††] $\varepsilon \in (0, 1]$ *and* $\mathrm{c} > 0$ *such that, whenever*

$$r_0 + \Phi \leqslant \varepsilon \qquad \text{and} \qquad |(u)_{x_0,r_0}| + |(\mathrm{D}u)_{x_0,r_0}| \leqslant C \tag{6.32}$$

are satisfied, then it holds that

$$\Phi(\rho) \leqslant \mathrm{c} \left[\left(\frac{\rho}{r_0} \right)^{2\kappa} \Phi + \rho^{2\min\{\alpha,\delta\}} \right] \qquad \text{for all } \rho \in (0, r_0].$$

In the cases of Settings 2 and 3, the statement remains valid with the only difference that $\min\{\delta,\alpha\}$ *is replaced by* α.

[††] The constants both depend on $m, N, \beta, q, \Gamma_\beta, \Gamma_q, \kappa, C$ and f. The value ε additionally depends on α and the exact choice of $\mathcal{U}_C$. If Settings 2 or 3 are considered, Γ_q is replaced by $\|u\|_{\mathrm{L}^\infty(V,\mathbb{R}^N)}$ and by $\|u\|_{\mathrm{W}^{1,\infty}}$, respectively.

Proof. We again consider $\delta > \alpha$ for Settings 2 and 3. Since κ is chosen as in Lemma 6.25, we can find λ, ε_E and c_E accordingly, where C in Lemma 6.25 is replaced by $4C$. Furthermore, we can make $\varepsilon \in (0, \varepsilon_E)$ small enough such that the smallness assumption (6.32) implies

$$r_0 + \mathrm{c}_1 r_0^{2\min\{\alpha,\delta\}} + \Phi \leqslant \varepsilon_E \,, \tag{6.33}$$

$$\frac{\sqrt{\Phi}}{\lambda^{\frac{m}{2}}(1-\lambda^k)} + \frac{\sqrt{\mathrm{c}_1} r_0^{\min\{\alpha,\delta\}}}{\lambda^{\frac{m}{2}}(1-\lambda^{\min\{\alpha,\delta\}})} \leqslant C \,, \tag{6.34}$$

$$\frac{\mathrm{c}_2}{\lambda^m(1-\lambda)}\, r_0 \,(\sqrt{\varepsilon_E} + 2C) \leqslant C \,, \tag{6.35}$$

for the specific constants $\mathrm{c}_1 := \frac{\mathrm{c}_E}{\lambda^{2\min\{\alpha,\delta\}} - \lambda^{2\kappa}}$ and c_2 from the Poincaré inequality for $p = 1$ on balls, which only depends on n. Now, we claim that

$$\Phi(\lambda^k r_0) \leqslant \lambda^{2k\kappa}\Phi + \mathrm{c}_1(\lambda^k r_0)^{2\min\{\alpha,\delta\}} \qquad \text{for all } k \in \mathbb{N}_0 \,. \tag{6.36}$$

Once (6.36) is established, we find for every $\rho \in (0, r_0]$ an integer $k \in \mathbb{N}_0$ such that $\lambda^{k+1} r_0 \leqslant \rho \leqslant \lambda^k r_0$, which, together with (6.33) and (6.36), implies

$$\begin{aligned}
\Phi(\rho) &\leqslant \left(\frac{\lambda^k r_0}{\rho}\right)^m \Phi(\lambda^k r_0) \\
&\leqslant \lambda^{-m}\Phi(\lambda^k r_0) \\
&\leqslant \lambda^{-m}\left(\lambda^{2k\kappa}\Phi + \mathrm{c}_1(\lambda^k r_0)^{2\min\{\alpha,\delta\}}\right) \\
&\leqslant \lambda^{-m-2\kappa}\max\{1,\mathrm{c}_1\}\left(\lambda^{2(k+1)\kappa}\Phi + (\lambda^{k+1} r_0)^{2\min\{\alpha,\delta\}}\right) \\
&\leqslant \mathrm{c}\left(\left(\frac{\rho}{r_0}\right)^{2\kappa}\Phi + \rho^{2\min\{\alpha,\delta\}}\right) .
\end{aligned}$$

Thus, it is left to show (6.36) via induction. To this end, we extend the claim by

$$\lambda^k r_0 + \Phi(\lambda^k r_0) \leqslant \varepsilon_E \,, \tag{6.37}$$

$$|(\mathrm{D}u)_{x_0,\lambda^k r_0}| \leqslant 2C \,, \tag{6.38}$$

$$|(u)_{x_0,\lambda^k r_0}| \leqslant 2C \tag{6.39}$$

for all $k \in \mathbb{N}_0$. For $k = 0$, the inequalities (6.37), (6.38) and (6.39) follow from the smallness assumption (6.32) and the choice of ε. The estimate (6.36) follows straightforwardly in this case. Now, for the main step in this proof, we assume that (6.36)–(6.39) hold true for all $k \in \{0, \dots, k_0\}$. Since with (6.37)–(6.39) the requirements of Lemma 6.25 with $\lambda^{k_0} r_0$ instead of r_0 (and still with $4C$ instead of C) are satisfied, we find

$$\Phi(\lambda^{k_0+1} r_0) \leqslant \lambda^{2\kappa}\Phi(\lambda^{k_0} r_0) + \mathrm{c}_E(\lambda^{k_0} r_0)^{2\min\{\alpha,\delta\}} .$$

Together with (6.36) and the definition of c_1, we conclude

$$\begin{aligned}
\Phi(\lambda^{k_0+1} r_0) &\leqslant \lambda^{2\kappa(k_0+1)}\Phi(r_0) + \mathrm{c}_1(\lambda^{k_0} r_0)^{2\min\{\alpha,\delta\}}\lambda^{2\kappa} + \mathrm{c}_E(\lambda^{k_0} r_0)^{2\min\{\alpha,\delta\}} \\
&\leqslant \lambda^{2\kappa(k_0+1)}\Phi(r_0) + \mathrm{c}_E(\lambda^{k_0} r_0)^{2\min\{\alpha,\delta\}}\left(\frac{\lambda^{2\kappa}}{\lambda^{2\min\{\alpha,\delta\}} - \lambda^{2\kappa}} + 1\right) \\
&= \lambda^{2\kappa(k_0+1)}\Phi(r_0) + \mathrm{c}_1(\lambda^{k_0+1} r_0)^{2\min\{\alpha,\delta\}} ,
\end{aligned}$$

which gives (6.36) for $k = k_0 + 1$. Inequality (6.37) for $k = k_0 + 1$ then follows with $\lambda < 1$ and (6.33). To show (6.38), we apply the Cauchy-Schwarz inequality and equation (6.36) for all $k \in \{0, \dots, k_0\}$ and obtain

$$\begin{aligned}
\left|(\mathrm{D}u)_{x_0,\lambda^{k_0+1}r_0} - (\mathrm{D}u)_{x_0,r_0}\right| &\leqslant \sum_{k=0}^{k_0} \left|(\mathrm{D}u)_{x_0,\lambda^{k+1}r_0} - (\mathrm{D}u)_{x_0,\lambda^k r_0}\right| \\
&\leqslant \sum_{k=0}^{k_0} \sqrt{\fint_{\mathrm{B}_{\lambda^{k+1}r_0}(x_0)} \left|\mathrm{D}u - (\mathrm{D}u)_{x_0,\lambda^k r_0}\right|^2 \mathrm{d}x} \\
&\leqslant \lambda^{-\frac{m}{2}} \sum_{k=0}^{k_0} \sqrt{\Phi(\lambda^k r_0)} \\
&\leqslant \lambda^{-\frac{m}{2}} \sum_{k=0}^{k_0} \lambda^{k\kappa}\sqrt{\Phi} + \sqrt{\mathrm{c}_1}(\lambda^k r_0)^{\min\{\alpha,\delta\}} \\
&\leqslant \frac{\sqrt{\Phi}}{\lambda^{\frac{m}{2}}(1-\lambda^\kappa)} + \frac{\sqrt{\mathrm{c}_1} r_0^{\min\{\alpha,\delta\}}}{\lambda^{\frac{m}{2}}(1-\lambda^{\min\{\alpha,\delta\}})}\,.
\end{aligned}$$

Taking (6.34) and (6.32), we conclude the claimed estimate. Showing (6.39) for $k = k_0 + 1$ takes a similar approach. In particular, applying the Poincaré inequality and (6.38) for $k \in \{0, \dots, k_0\}$ provides

$$\begin{aligned}
\left|(u)_{x_0,\lambda^{k_0+1}r_0} - (u)_{x_0,r_0}\right| &\leqslant \lambda^{-m} \sum_{k=0}^{k_0} |u - (u)_{x_0,\lambda^k r_0}|\,\mathrm{d}x \\
&\leqslant \mathrm{c}_2 \lambda^{-m} \sum_{k=0}^{k_0} \lambda^k r_0 \fint_{\mathrm{B}_{\lambda^k r_0}(x_0)} |\mathrm{D}u|\,\mathrm{d}x \\
&\leqslant \mathrm{c}_2 \lambda^{-m} \sum_{k=0}^{k_0} \lambda^k r_0 \fint_{\mathrm{B}_{\lambda^k r_0}(x_0)} |\mathrm{D}u - (\mathrm{D}u)_{x_0,\lambda^k r_0}| + |(\mathrm{D}u)_{\lambda^k r_0}|\,\mathrm{d}x \\
&\leqslant \mathrm{c}_2 \lambda^{-m} \sum_{k=0}^{k_0} \lambda^k r_0 \left(\sqrt{\Phi(\lambda^k r_0)} + 2C\right) \\
&\leqslant \mathrm{c}_2 \frac{\lambda^{-m}}{1-\lambda} r_0 (\sqrt{\varepsilon} + 2C)\,.
\end{aligned}$$

The claim can be concluded with (6.35) and (6.32). □

6.4 Partial regularity theorems for variational integrals

In this section, we state and prove the partial regularity of the minimizer u in all three settings and transfer these results to more specific frameworks.

Theorem 6.27 (Partial regularity of local minimizers of $\mathbb{F}$)**.** *Let all overall assumptions from Section 6.2.4 hold true. If Settings 1, 2 or 3 apply, then it holds that*

$$|V \backslash V_{\mathrm{reg}}| = 0 \qquad \textit{and} \qquad u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(V_{\mathrm{reg}}, \mathbb{R}^N)\,,$$

where we define the open set $V_{\mathrm{reg}} := \left\{x \in V : \liminf_{r\searrow 0} \Phi(x,r) = 0,\, \limsup_{r\searrow 0} |(\mathrm{D}u)_{x,r}| + |(u)_{x,r}| < \infty\right\}$.

Remark 6.28. The Theorem remains true if we assume $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(V, \mathbb{R}^N)$ and replace $|\mathbb{F}[u]| < \infty$ by $g(\,\cdot\,, u) \in \mathrm{L}^1_{\mathrm{loc}}(V)$ and replace the conditions (6.4), (6.6) and (6.5) with local Morrey spaces instead of the global ones.

During the proof, we first show the weaker property $u \in \mathrm{C}^{1,\min\{\alpha,\delta\}}(V_{\mathrm{reg}}, \mathbb{R}^N)$. In fact, as this implies that u and $\mathrm{D}u$ are essentially bounded, Setting 2 applies. Hence, we are able to further lift this result to $u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(V_{\mathrm{reg}}, \mathbb{R}^N)$.

Proof. We first record that, for every function $w \in \mathrm{L}^p_{\mathrm{loc}}(V,\mathbb{R}^N)$, it holds $\lim_{r\searrow 0} \fint_{\mathrm{B}_r(x)} |w-w(x)|^p \,\mathrm{d}x = 0$ for a.e. $x \in V$. As the latter holds for our minimizer u with $p \in [1,2]$, we conclude $|V\setminus V_{\mathrm{reg}}| = 0$.
For proving the stated regularity of u, we set to be δ any number larger than α in the case of Setting 2 and 3, respectively, such that $\min\{\alpha,\delta\} = \alpha$. With this is mind, we choose $x_0 \in V_{\mathrm{reg}}$. By definition, we can define $C := 1 + \limsup_{r\searrow 0} |(\mathrm{D}u)_{x_0,r}| + |(u)_{x_0,r}| \in \mathbb{R}$ and furthermore choose $\varepsilon \in (0,1]$ from Lemma 6.26 for $\kappa := \frac{1+\min\{\alpha,\delta\}}{2}$ and C as just defined. According to the definitions of V_{reg} and C, there exists $r_0 \in \left(0,\frac{1}{2}\right]$ such that $\mathrm{B}_{2r_0}(x_0)$ is compactly contained in V and such that

$$r_0 + 2^m \Phi(x_0, 2r_0) \leqslant \varepsilon \qquad \text{and} \qquad |(\mathrm{D}u)_{x_0,r_0}| + |(u)_{x_0,r_0}| < C\,. \tag{6.40}$$

In order to be able to apply Lemma 6.26 on a neighborhood of x_0, we exploit the continuity of the function $x \mapsto |(\mathrm{D}u)_{x,r_0}| + |(u)_{x,r_0}|$ to choose some $r_1 \in (0, r_0]$ such that

$$|(\mathrm{D}u)_{x,r_0}| + |(u)_{x,r_0}| \leqslant C \qquad \text{for all } x \in \mathrm{B}_{r_1}(x_0)\,. \tag{6.41}$$

For extending the first estimate of (6.40), we apply Remark 6.24, which yields

$$\Phi(x,r_0) \leqslant \int_{\mathrm{B}_{r_0}(x)} |\mathrm{D}u - (\mathrm{D}u)_{x_0,2r_0}|^2 \,\mathrm{d}x \leqslant 2^m \Phi(x_0, 2r_0)\,.$$

Thus, with (6.40) we have

$$r_0 + \Phi(x,r_0) \leqslant \varepsilon \qquad \text{for all } x \in \mathrm{B}_{r_1}(x_0)\,. \tag{6.42}$$

Together with (6.41), we are in the position to apply Lemma 6.26 to obtain

$$\Phi(x,r) \leqslant \mathrm{c}\left(\left(\frac{r}{r_0}\right)^{1+\min\{\alpha,\delta\}} \Phi(x,r_0) + r^{2\min\{\alpha,\delta\}}\right) \qquad \text{for all } x \in \mathrm{B}_{r_1}(x_0) \text{ and all } r \in (0,r_0]\,. \tag{6.43}$$

Since $\alpha < 1$, we can refine the previous inequality to

$$\Phi(x,r) \leqslant \mathrm{c}\left(\frac{r^{1-\min\{\alpha,\delta\}}}{r_0^{1+\min\{\alpha,\delta\}}}\varepsilon + 1\right) r^{2\min\{\alpha,\delta\}} \leqslant \mathrm{c}\left(r_0^{-2\min\{\alpha,\delta\}}\varepsilon + 1\right) r^{2\min\{\alpha,\delta\}}\,,$$

where we exploited (6.42) once again. Finally, we can apply Corollary 6.17 to obtain that $(\mathrm{D}u)^*$ belongs to $\mathrm{C}^{0,\min\{\alpha,\delta\}}_{\mathrm{loc}}(\mathrm{B}_{r_0}(x_0),\mathbb{R}^{N\times m})$, which implies $u^* \in \mathrm{C}^{1,\min\{\alpha,\delta\}}_{\mathrm{loc}}(\mathrm{B}_{r_0}(x_0),\mathbb{R}^N)$. By the choice of x_0, we conclude that $u^* \in \mathrm{C}^{1,\min\{\alpha,\delta\}}_{\mathrm{loc}}(V_{\mathrm{reg}},\mathbb{R}^N)$, which proves the claim for Settings 2 and 3.
In the exceptional case of Setting 1, we have to rule out the possibility that $u^* \in \mathrm{C}^{1,\delta}_{\mathrm{loc}}(V_{\mathrm{reg}},\mathbb{R}^N)\setminus\mathrm{C}^{1,\alpha}_{\mathrm{loc}}(V_{\mathrm{reg}},\mathbb{R}^N)$. First, we notice that the previous property still implies $u \in \mathrm{L}^\infty_{\mathrm{loc}}(V,\mathbb{R}^N)$. By slightly restricting V to a smaller domain, u satisfies all assumptions of Setting 2, which leads to $u^* \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(V_{\mathrm{reg}},\mathbb{R}^N)$ (first in the restricted domain, and on V by an exhaustion argument), which completes the proof. □

Corollary 6.29 (Partial regularity for variational integrands with L^p-Hölder zero-order integrands)**.** *We consider the Settings 1, 2 or 3 with the slight adjustment that, instead of* (6.4), (6.5) *and* (6.6), *we merely assume* $\Gamma \in \mathrm{L}^p_{\mathrm{loc}}(V)$ *for some* $p \in \left(\frac{m}{\beta},\infty\right]$. *Moreover, we assume* $p \geqslant s_q$ *if* $m \geqslant 2$ *in the case of Setting 1. Then the local minimizer* u *of* $\mathbb{F}$ *satisfies* $u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(V_{\mathrm{reg}},\mathbb{R}^N)$, *where* V_{reg} *is defined as in Theorem 6.27, and*

$$\alpha = \begin{cases} \frac{\beta p - m}{(2-\beta)p}\,, & \text{if } p < \infty \\ \frac{\beta}{2-\beta}\,, & \text{if } p = \infty,\ \beta < 1\,, \end{cases}$$

and any $\alpha \in (0,1)$ *if* $p = \infty$ *and* $\beta = 1$.

Proof. We argue that the original conditions for Γ from Theorem 6.27 are satisfied. To this end, we first consider the case $p < \infty$ and $n \geqslant 3$. The assumptions on p ensure $s_\beta = \frac{2m}{2m-\beta(m-2)} < \frac{m}{\beta} < p$ and then Remark 6.4 implies $\Gamma \in \mathrm{L}^p(V) \subseteq \mathrm{L}^{s_\beta, m - s_\beta \frac{m}{p}}(V) = \mathrm{L}^{s_\beta, m + s_\beta((2-\beta)\alpha - \beta)}(V)$ for $\alpha = \frac{\beta p - m}{(2-\beta)p}$. Similarly, if $n \leqslant 2$,

2^* can be chosen large enough such that $p \geqslant s_\beta$, which is possible due to the choice $p > \frac{m}{\beta} \geqslant 1$. Hence, $\Gamma \in \mathrm{L}^{s_\beta, m+s_\beta((2-\beta)\alpha-\beta)}(V)$ follows in this case as well.
In the case of Setting 1, $p \geqslant s_q$ implies $\Gamma \in \mathrm{L}^p(V) \subseteq \mathrm{L}^{s_q, m-s_q\frac{m}{p}}(V) \subseteq \mathrm{L}^{s_q, m-s_q q+2\delta}(V)$ with Remark 6.4, where we consider $\delta > 0$ small enough (which exists since $p > \frac{m}{\beta} \geqslant \frac{m}{q}$).
In any case, we are in the situation of Theorem 6.27 with $\alpha = \frac{\beta p - m}{(2-\beta)p}$, which leads to the corresponding claim.
If $p = \infty$ and $\beta < 1$, Remark 6.4 says $\mathrm{L}^\infty(V) = \mathrm{L}^{s,m}(V)$ for all $s \in [1, \infty)$. Hence, $\Gamma \in \mathrm{L}^{s, m-s((2-\beta)\alpha-\beta)}(V)$ and $\Gamma \in \mathrm{L}^{s_q, m-s_q q+2\delta}(V)$ for $\alpha = \frac{\beta}{2-\beta}$ and suitable $\delta > 0$. As s was chosen arbitrarily, i.e., it can be replaced with s_β and s_q, respectively, Theorem 6.27 implies $u \in \mathrm{C}^{1,\frac{\beta}{2-\beta}}_{\mathrm{loc}}(V, \mathbb{R}^N)$ in this case.
We turn to the remaining case $p = \infty$ and $\beta = 1$. Since condition (6.3) weakens as β decreases, we can apply the previous reasoning for $\alpha = \frac{\tilde\beta}{2-\tilde\beta}$ with $\tilde\beta \in (0,1)$ being arbitrary close to 1, which results in $u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(V, \mathbb{R}^N)$ for all $\alpha \in (0,1)$. □

The following functional is much more similar to $\mathbb{G}$, but has more general zero-order integrands H, which become more concrete in the very last statement of this subsection.

Corollary 6.30 (Partial regularity for functionals of the form $\mathbb{G}$ with generalized H)**.** *We consider Settings 1, 2 or 3 with $N = 1$, where we define the zero-order term of $\mathbb{F}$ in (6.1) by*

$$g(x,y) := -\int_0^y H(x,t)\,\mathrm{d}t \qquad \textit{for all } x \in V \textit{ and } y \in \mathbb{R}$$

for $H \in \mathrm{L}^1_{\mathrm{loc}}(V \times \mathbb{R})$ satisfying $\int_V \|H(x,\cdot)\|^p_{\mathrm{L}^s(\mathbb{R})}\,\mathrm{d}x < \infty$ for some $s \in (1,\infty]$ and $p \in \left(\frac{ms}{s-1}, \infty\right]$. Still considering a local minimizer $u \in \mathrm{W}^{1,2}(V)$ of $\mathbb{F}$, it holds that $u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(V_{\mathrm{reg}})$, where V_{reg} is defined as in Theorem 6.27 and where

$$\alpha = \begin{cases} \frac{s-1-\frac{ms}{p}}{s+1}, & \textit{if } p,s < \infty \\ 1 - \frac{m}{p}, & \textit{if } p < s = \infty \\ \frac{s-1}{s+1}, & \textit{if } s < p = \infty . \end{cases}$$

In the special case $p = s = \infty$, it holds that $u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(V_{\mathrm{reg}})$ for all $\alpha \in (0,1)$.

Proof. As in the previous proof, we show that the conditions of Corollary 6.29 on g are satisfied. For this, we use the Hölder inequality, which gives the estimate

$$|g(x,y) - g(x,\hat y)| = \left|\int_{\hat y}^y H(x,t)\,\mathrm{d}t\right| \leqslant |y - \hat y|^{\frac{s-1}{s}} \|H(x,\cdot)\|_{\mathrm{L}^s(\mathbb{R})} .$$

Thus, we set $q = \beta = \frac{s-1}{s} \in (0,1]$ (where we identify $\frac{s-1}{s} = 1$ if $s = \infty$) and $\Gamma(x) = \|H(x,\cdot)\|_{\mathrm{L}^s(\mathbb{R})}$, which is in $\mathrm{L}^p(V)$ by assumption. Moreover, $p > \frac{m}{\beta}$ follows from the choice of p, which implies $p > s_\beta = s_q$ as in the previous proof in the case of Setting 1. Now, we are in the situation of Corollary 6.29, which implies $u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(V_{\mathrm{reg}})$ with

$$\alpha = \begin{cases} \frac{\frac{s-1}{s}p - m}{(2-\frac{s-1}{s})p} = \frac{s-1-\frac{ms}{p}}{s+1}, & \text{if } p,s < \infty \\ 1 - \frac{m}{p} & \text{if } p < s = \infty \\ \frac{\frac{s-1}{s}}{2-\frac{s-1}{s}} = \frac{s-1}{s+1}, & \text{if } s < p = \infty , \end{cases}$$

and any $\alpha \in (0,1)$ if $s = p = \infty$. □

We conclude with the final version of the partial regularity result, which we will then apply to Massari's regularity theorem.

Corollary 6.31 (Partial regularity for functionals of the form $\mathbb{G}$)**.** *We consider Settings 1, 2 or 3 with $N = 1$, where we define the zero-order term of $\mathbb{F}$ by*

$$g(x,y) := -\int_0^y H(x,t)\,\mathrm{d}t \qquad \textit{for all } x \in V \textit{ and } y \in \mathbb{R}$$

for $H \in \mathrm{L}^p(V \times \mathbb{R})$, $p \in (m+1,\infty]$. In this case, a local minimizer $u \in \mathrm{W}^{1,2}(V)$ of $\mathbb{F}$ is in the space $\mathrm{C}^{1,\alpha}_{\mathrm{loc}}(V_{\mathrm{reg}})$, where V_{reg} is defined as in Theorem 6.27 and

$$\alpha = \frac{p-(m+1)}{p+1} \quad \text{if } p<\infty \qquad \text{and} \qquad \alpha \in (0,1) \quad \text{arbitrary if } p=\infty\,.$$

Proof. The described situation matches Corollary 6.30 with $p = s$. In particular, $p > \frac{ms}{s-1}$ is equivalent to $p > m+1$ in this case. Since $\frac{s-1-\frac{ms}{p}}{s+1} = \frac{p-(m+1)}{p+1}$, the conclusion follows from the application of the previous result. □

6.5 The limit exponent of Massari's regularity theorem

We now apply the results of this chapter to the original parametric problem.

Theorem 6.32 (The limit exponent of Massari's regularity theorem)**.** *For a set $A \subseteq \mathbb{R}^n$ of finite perimeter in Ω with local variational mean curvature $H \in \mathrm{L}^p(\Omega)$ in Ω for $p \in (n,\infty]$, the following properties hold true:*

- i) *The set $\partial^* A \cap \Omega$ is relatively open in $\partial A^1 \cap \Omega$.*
- ii) *The set $\partial^* A \cap \Omega$ is an $(n-1)$-dimensional $\mathrm{C}^{1,\alpha}$ manifold for $\alpha = \frac{p-n}{p+1}$ if $p<\infty$ and for all $\alpha \in (0,1)$ if $p=\infty$.*
- iii) *It holds $\mathcal{H}^s((\partial A^1 \setminus \partial^* A) \cap \Omega) = 0$ for all $s \in (n-8, n]$, where we identify $\mathcal{H}^s = \mathcal{H}^0$ for $s<0$.*

Proof. According to Theorem 4.4 and Proposition 6.1, it suffices to show that every local minimizer $u \in \mathrm{C}^1(\overline{V})$ of the functional $\mathbb{G}$ is in $\mathrm{C}^{1,\alpha}_{\mathrm{loc}}(V)$, where $\mathbb{G}$ is defined as

$$\mathbb{G}[w] := \int_V \sqrt{1+|\nabla w(x)|^2} - \int_0^{w(x)} \big[\mathbb{1}_{A\cap\Omega}(x) H_+(x,t) - \mathbb{1}_{\Omega\setminus A}(x) H_-(x,t)\big]\, \mathbb{1}_{[0,R]}(t)\,\mathrm{d}t\,\mathrm{d}x$$

for all $w \in \mathrm{W}^{1,2}_u(V)$ and V and R as in Proposition 6.1. To this end, we show that $f_{\mathbb{G}} : \mathbb{R}^{n-1} \to \mathbb{R}$, $z \mapsto \sqrt{1+|z|^2}$ satisfies the assumptions on the first-order integrand f of Setting 3. Clearly, $f_{\mathbb{G}} \in \mathrm{C}^2(\mathbb{R}^{n-1})$ has at most quadratic growth. For positive definiteness of $\mathrm{D}^2 f_{\mathbb{G}}$, we compute

$$\mathrm{D}^2 f_{\mathbb{G}}(z)\,\xi\cdot\xi = \frac{\mathrm{I}\,(1+|z|^2) - z\otimes z}{(1+|z|^2)^{\frac{3}{2}}}\,\xi\cdot\xi = \frac{|\xi|^2(1+|z|^2) - (z\cdot\xi)^2}{(1+|z|^2)^{\frac{3}{2}}} \geqslant \frac{|\xi|^2}{(1+|z|^2)^{\frac{3}{2}}}$$

for all $z,\xi \in \mathbb{R}^{n-1}$, where $z\otimes z$ denotes the dyadic product $[z_i z_j]_{i,j=1,\dots,n}$. This leads to (6.7) with $\mathrm{c}(K) := \frac{1}{(1+K^2)^{\frac{3}{2}}}$. Furthermore, as $u \in \mathrm{C}^1(\overline{V}) \subseteq \mathrm{W}^{1,\infty}(V)$, we are in the situation of Setting 3 with $m = n-1$, and may apply Corollary 6.31. We find $u \in \mathrm{C}^{1,\frac{p-n}{p+1}}_{\mathrm{loc}}(V_{\mathrm{reg}})$ with

$$V_{\mathrm{reg}} = \left\{ x \in V : \liminf_{r\searrow 0} \Phi(x,r) = 0,\ \limsup_{r\searrow 0} |(\mathrm{D}u)_{x,r}| + |(u)_{x,r}| < \infty \right\}.$$

Since u is in $\mathrm{C}^1(V)$, it follows that each $x \in V$ is an (L^2-)Lebesgue point of u and $\mathrm{D}u$. In particular, $V_{\mathrm{reg}} = V$ and the claim follows from Proposition 6.1. □

Remark 6.33. According to [81, Section 4], there exists counterexamples to Marssari's regularity theorem for $\alpha > \frac{p-n}{p+1}$. Hence, Theorem 6.32 is optimal with respect to α and this optimality transfers to Corollary 6.31 by Proposition 6.1.

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