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Extensions of Fusion Categories via Fixed-Point-Free Automorphisms

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Abstract.

In this thesis, we will use the structure of quadratic forms and results from group theory to build modular fusion categories. For this, we are using *discriminant forms* with *isometries* that have the property of being *fixed-point-free*.

We first determine conjugacy classes of isometries in the orthogonal group of a regular quadratic module over an Artinian local ring. We describe the conjugacy classes whose isometries are fixed-point-free.

In particular, our classification includes orthogonal groups over the finite ring $\mathbb{Z}/p^l$, where p is a prime and $l \in \mathbb{Z}_{\geq 1}$ a positive integer. We show how to apply our results to discriminant forms. We utilise our result to construct braided $\mathbb{Z}/N$-crossed extensions of hyperbolic pointed modular fusion categories whose $\mathbb{Z}/N$-equivariantisations are Drinfeld centres of pointed fusion categories. We calculate the modular data and fusion rules of these $\mathbb{Z}/N$-equivariantisations.

Furthermore, we define *coherent* Gr-*bicategories*, which are *bi*categories that resemble groups. We show that any coherent Gr-bicategory is equivalent to a special skeletal coherent Gr-bicategory. This should be seen as a setup for future work on extensions of fusion categories.

Zusammenfassung.

In dieser Arbeit benutzen wir die Struktur von quadratischen Formen und Resultate der Gruppentheorie, um modulare Fusionskategorien zu konstruieren. Dafür benutzen wir *Diskriminantenformen* mit *Isometrien*, welche *fixpunktfrei* sind.

Zunächst bestimmen wir Konjugationsklassen von Isometrien in orthogonalen Gruppen von regulären quadratischen Räumen über einem artinschen lokalen Ring. Wir beschreiben diejenigen Konjugationsklassen, deren Isometrien fixpunktfrei sind. Insbesondere schließt unsere Klassifikation orthogonale Gruppen über dem endlichen Ring $\mathbb{Z}/p^l$ ein, wobei p eine Primzahl und $l \geq 1$ eine positive ganze Zahl sind. Wir zeigen, wie sich diese Ergebnisse auf Diskriminantenformen anwenden lassen. Wir benutzen unsere Resultate, um verzopfte $\mathbb{Z}/N$-gekreuzte Erweiterungen von hyperbolischen punktierten modularen Fusionskategorien zu konstruieren, deren $\mathbb{Z}/N$-Equivariantisierungen Drinfeld-Zentren von punktierten Fusionskategorien sind. Wir berechnen die modularen Daten und Fusionsregeln dieser $\mathbb{Z}/N$-Equivariantisierungen.

Weiterhin definieren wir *kohärente* Gr-*Bikategorien*, welche Gruppen für monoidale Bikategorien ähneln. Wir zeigen, dass jede kohärente Gr-Bikategorie äquivalent zu besonderen skeletalen kohärenten Gr-Bikategorien ist. Dies sollte als Vorbereitung der weiteren Untersuchung von Erweiterungen von Fusionskategorien gesehen werden.

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Contents

Introduction

A Brief History

Let us begin with a brief overview of the history of the ideas which form the basis of this work.

Part I: Quadratic Forms and Their Isometries.

In 1936, Ernst Witt constructed a ring structure on equivalence classes of quadratic forms over fields [Wit36]. In the 1960s, quadratic forms over general rings were considered by Manfred Knebusch in [Kne69].

Meanwhile, isometries of quadratic forms over fields *up to conjugacy* were already studied in the 1930s by John Williamson, e.g. in [Wil39], and extended by Gordon Wall in 1962 [Wal63b]. John Milnor and Igor Tsikunov determined conjugacy classes of isometries of quadratic forms over fields at the end of the 1960s; see [Mil69] and [Cik66]. Similar methods were also used by Tonny Springer and Robert Steinberg [Spr22] and [SS70] for classical groups over (finite) fields. Isometries of quadratic forms over general rings are far less studied. In the 1960s, Wilhelm Klingenberg and Manfred Knebusch studied them by decomposing them into reflections in [Kli61] and [Kne69], respectively.

We will follow the ideas of Milnor et al. to classify conjugacy classes of isometries over local rings with finite residue fields in the first part of this thesis. Then we apply our results to *fixed-point-free* isometries. Fixed-point-free automorphisms of groups implicitly appeared in the 1930s in the work of Hans Zassenhaus [Zas35]. Later, they became a valuable tool in group theory. For example, John Thompson used them in [Tho59] to show that finite groups with a fixed-point-free automorphism of prime order are nilpotent.

At the end of the 1970s, Vyacheslav Nikulin introduced *discriminant forms* to study quadratic forms and lattices over the rational integers $\mathbb{Z}$ in [Nik80]. We will use them in our next part.

Part II: Modular Fusion Categories and Fixed-Point-Free Isometries.

In the 1970s and 1980s, tensor and *fusion categories* were studied in many areas, for example by Neantro Saavedra–Rivano [Saa72], Pierre Deligne and James Milne [DM82], and Gregory Moore and Nathan Seiberg [MS89]. The development of their general theory began in the 2000s; see the work of Pavel Etingof, Dimitri Nikshych and Victor Ostrik [ENO05].

Edward Witten suggested in 1988 how to use ideas from quantum field theory to derive invariants of low-dimensional manifolds [Wit88], which he connected to the celebrated polynomials of Vaughan Jones [Jon85] in [Wit89]. In 1991, Nikolai Reshetikhin and Vladimir Turaev reobtained these invariants in [RT90]. It was in the study of invariants for 3-manifolds that Turaev introduced *modular fusion categories* in [Tur92].

Vladimir Drinfeld defined *quasi-Hopf algebras* in 1989 [Dri89] to study quantum groups. Shortly after, Robbert Dijkgraaf, Vincent Pasquier, and Phillipe Roche constructed non-trivial examples of quasi-Hopf algebras using finite groups and 3-cocycles. The categories of representations of these quasi-Hopf algebras are modular fusion categories. In fact, we will use them in our second part.

Turaev introduced *braided* Π-*crossed fusion categories* (for a finite group Π) in the unpublished preprint [Tur00] around 2000, again in the context of 3-dimensional topological field theories: see also [Tur10]. These are fusion categories with a categorical Π-action and a twisted braiding. Given a braided Π-crossed fusion category, one can construct a braided fusion category of elements fixed by the Π-action containing a *Tannakian* category, namely $\mathsf{Rep}(\Pi)$. This process is called *equivariantisation*. Conversely, given a braided category containing a Tannakian subcategory, one can construct braided crossed categories. This was used by Michael Müger [Müg00] and Alain Bruguières [Bru00] to construct modular fusion categories.

Under some regularity assumptions, categories of modules of vertex operator algebras also yield modular fusion categories; see the work of Yi-Zhi Huang [Hua08]. For a vertex operator algebra V with a finite group of automorphisms $\Phi \subseteq \mathrm{Aut}(V)$, there is a fixed-point vertex algebra V^Φ. It turns out that (under some finiteness assumptions on the category of modules of V^Φ), the category of modules twisted by Φ is canonically a braided Φ-crossed fusion category such that its *equivariantisation* is equivalent to the category of modules for V^{Φ}; see the recent work of Robert McRae [McR21]. This correspondence was the starting point of the present work.

In the second part, we will construct examples of modular fusion categories by building braided crossed extensions of pointed modular fusion categories.

Part III: Monoidal Bicategories and Coherent Gr-*bicategories.*

Already in 1926, Otto Schreier considered in [Sch26] extensions of groups. In the 1930s, it became one of the most important tools in the study of finite groups, with mathematicians like Reinhold Baer, Hans Zassenhaus, and Marshall Hall working on it.

Samuel Eilenberg and Saunders Mac Lane developed the connection between group extensions and group homology in the early 1940s [EM42].

In 1947, Eilenberg and Mac Lane suggested in [EM45] that it could be helpful to study *categories*. This quickly led to a lot of research all around the world; see e.g. [Mac88, Section 17]. For example, in 1961, Beno Eckmann and Peter Hilton studied group objects in categories [EH62].

Monoidal categories, i.e. categories with a tensor product functor, were introduced by Jean Bénabou in [Bén63] and Saunders Mac Lane [Mac63] in 1963.

In 1972, Hoàng Xuân Sính completed her thesis under the supervision of Alexandre Grothendieck. Her thesis was concerned with a categorification of the concept of a group: a monoidal category whose objects are *up to isomorphism* invertible with respect to the tensor product functor; see e.g. [Sín78].

In 1967, Jean Bénabou defined *bicategories*. A bicategory with only one object can be identified with a monoidal category.

Bicategories with a tensor product (although defined only later) and the investigation of algebraic homotopy types led André Joyal and Ross Street in 1986 to consider a *braiding* on a monoidal category [JS93]. A braiding consists of a natural isomorphism

$$c_{x,y} \colon x \otimes y \to y \otimes x$$

for objects x and y, satisfying an analogue of the *Yang-Baxter equation*.

In particular, they proved in [JS93] using results of Eilenberg and Mac Lane (see e.g. [EM54]) that quadratic maps are invariants of braided categorified groups. This is the underlying reason connecting our three parts.

In 1995, Robert Gordon, John Power, and Ross Street introduced tricategories together with their strictifications, Gray-enriched categories [GPS95]. They also discuss monoidal bicategories as tricategories with exactly one object; see also [DS97].

A categorification of the work of Sính, where we work with monoidal bicategories, is the topic of the last part of this work.

Detailed Outline and Main Results

Part I: Quadratic Forms and Their Isometries

In the first part, we will work with quadratic forms and classify certain isometries over them. We will put a special emphasis on local rings which are *Henselian*, i.e. those for which *Hensel's lemma* holds:

Definition 1.1 (*Henselian local ring, compare to [Mil80, 1 §4]*).
Let A be a local ring with residue field k. Then A is called *Henselian* if for every polynomial $Q \in A[T]$ such that its reduction over k factors $\overline{Q} = R_0 S_0$ with $R_0, S_0 \in k[T]$ where R_0 is monic and R_0, S_0 are relatively prime in $k[T]$, there exist polynomials $R, S \in A[T]$ such that Q factors as $Q = RS$ with R monic and $\overline{R} = R_0, \overline{S} = S_0$.

Here, we denote the reduction of a polynomial $Q \in A[T]$ to its residue field k by $\overline{Q} \in k[T]$. This lemma and its consequences will allow us to lift many properties of the residue field k to A. Note that Galois theory for Henselian rings is closely related to the Galois theory of their residue fields; see Proposition 2.4.

For example, Artinian local rings are Henselian. In particular, the finite ring $\mathbb{Z}/p^l$ for a prime p and a positive integer $l \in \mathbb{Z}_{\geq 1}$ is Henselian.

In Chapter 1, our first main result concerns the existence of minimal polynomials over Artinian local rings. Over a local ring A which is not a field, the polynomial ring $A[T]$ is not a principal ideal domain. Therefore, minimal polynomials of endomorphisms do not always exist. By Cohen's structure theorem of complete local rings, we can show that Artinian local rings are quotients of regular complete rings; see Corollary 1.13.

For complete local rings, we show that solutions to matrix equations over the residue field can be lifted to solutions over the ring.

PROPOSITION 1.10 (*lifting solutions to matrix equations*).
Let A be a complete local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $\mathfrak{a} \subset A$ be an ideal in A such that A is complete in the $\mathfrak{a}$-adic topology.

Let $Q \in A[T]$ be a separable polynomial. Let $X_0 \in M_n(A/\mathfrak{a})$ be a matrix over the quotient ring $A/\mathfrak{a}$ such that

$$Q(X_0) \equiv 0 \pmod{\mathfrak{a}}.$$

Then there exists an $X \in M_n(A)$ such that

$$Q(X) = 0, \quad X \equiv X_0 \pmod{\mathfrak{a}}.$$

Since matrices over regular local rings admit minimal polynomials, this will allow us to prove our first main theorem.

THEOREM 1.16 (*existence of minimal polynomial*).
Let A be an Artinian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of characteristic p. Let M be a finitely generated projective module over A and let $\phi\colon M \to M$ be an automorphism of order N such that $p \nmid N$. Then ϕ admits a minimal polynomial $\mathrm{minpoly}_\phi$ *which divides the polynomial $T^N - 1$ in $A[T]$.*

In Chapter 2, we introduce some notation for quadratic and symmetric bilinear forms over general rings. In particular, we recall:

DEFINITION 2.3 (*quadratic module, [Knu91, Definition 5.3.5]*).
Let A be a ring. A *quadratic module* over A is a tuple (M, q) consisting of a finitely generated projective module M over A with a map $q\colon M \to A$ satisfying

(i.) $q(ax) = a^2 q(x)$ for all $x \in M$ and $a \in A$,

(ii.) the map $\lambda_q\colon M \times M \to A$ defined by

$$\lambda_q(x, y) := q(x + y) - q(x) - q(y)$$

is a symmetric bilinear form.

A quadratic module (M, q) is called *regular* if the symmetric form (M, λ_q) is regular. An *isometry* of the quadratic module (M, q) is an A-module automorphism $\phi\colon M \to M$ such that $q \circ \phi = q$. The group of isometries is denoted by $O_A(M, q)$ and called the *orthogonal group* of (M, q) over A.

The isometry classes of quadratic modules modulo the *hyperbolic* regular quadratic forms form the famous *Witt ring*. We denote the Witt ring of a ring A by $\mathrm{Wq}(A)$ and the Witt class of a regular quadratic module (M, q) over A by $[(M, q)] \in \mathrm{Wq}(A)$.

We recall and prove some general facts about the Witt ring for Henselian rings with finite residue field. For example, Knebusch showed that regular quadratic modules over local rings also admit the famous Witt cancellation property; see Lemma 2.13. We also note that for a Henselian local ring, its Witt ring is isomorphic to the Witt ring of its residue field.

The Witt ring of finite fields is well-known:

$$\mathrm{Wq}(\mathbb{F}_q) = \begin{cases} \mathbb{Z}/2[(\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2], & q \equiv 1 \pmod 4, \\ \mathbb{Z}/4, & q \equiv 3 \pmod 4. \end{cases}$$

In Chapter 3, we are in the position to determine conjugacy classes of isometries in orthogonal groups over Artinian local rings with finite residue fields.

We use that the reduction map for a local ring A with regular quadratic module (M, q) over A

$$O_A(M, q) \to O_k(\bar{M}, \bar{q})$$

is surjective, which was also shown by Knebusch in [Kne69].

To classify the conjugacy classes of isometries in the orthogonal group, we will follow the general proof strategy outlined by Shinoda in [Shi80]. It is based on the work of [Mil69] and [SS70]; see also [Wal63b].

The study of conjugacy classes of isometries over a finite field naturally leads to their minimal polynomials. This is precisely the reason why we established the existence of minimal polynomials in Chapter 1.

Recall that for a ring A and a polynomial $P = \sum_{i=0}^{d} a_i T^i \in A[T]$ such that $a_0 \in A^\times$, the *reciprocal polynomial* P^* of P is defined by

$$P^* = a_0^{-1} \sum_{i=0}^{d} a_{d-i} T^i.$$

Let A be an Artinian local ring with maximal ideal $\mathfrak{m}$ and residue field k of odd characteristic p. Let (M, q) be a regular quadratic module over A with an isometry $\phi\colon (M, q) \to (M, q)$ of finite order N. The minimal polynomial of ϕ is self-reciprocal; see Lemma 3.14, i.e. we have

$$\mathrm{minpoly}_{\phi}^{*} = \mathrm{minpoly}_{\phi}.$$

Since A is also Henselian, we can decompose the minimal polynomial $\mathrm{minpoly}_{\phi}$ as

$$\mathrm{minpoly}_{\phi} = P_1 \dots P_m Q_1 Q_1^{*} \dots Q_n Q_n^{*}$$

where

(i.) $\{P_i\}$ are pairwise strongly relatively coprime polynomials over A such that
   - $*$ P_i is monic and self-reciprocal,
   - $*$ the reduction $\overline{P}_i \in k[T]$ is irreducible with order $\mathrm{ord}(\overline{P}_i)$ coprime to p;

(ii.) $\{Q_j Q_j^{*}\}$ are pairwise strongly relatively coprime polynomials over A such that
   - $*$ Q_j is monic and non-self-reciprocal,
   - $*$ the reduction $\overline{Q}_j \in k[T]$ is irreducible with order $\mathrm{ord}(\overline{Q}_j)$ coprime to p;

see Lemma 3.18.

Using this decomposition, we can decompose M into annihilated submodules corresponding to the factors of the minimal polynomial. For a factor $R \mid \mathrm{minpoly}_{\phi}$, its annihilated submodule is defined by

$$M_{R,\phi} = \{x \in M\colon R(\phi)[x] = 0\}.$$

Namely, we obtain:

Lemma 3.25 (*decomposing using the minimal polynomial*).
Let A be an Artinian local ring with finite residue field k of odd characteristic p.

Let (M, q) be a regular quadratic module over A. Then an isometry ϕ of finite order N coprime to p has a separable minimal polynomial $\mathrm{minpoly}_\phi$.

Let

$$\mathrm{minpoly}_\phi = P_1 \cdot \dots \cdot P_m Q_1 Q_1^* \dots Q_n Q_n^* \in A[T]$$

be the factorisation as above, see Lemma 3.18. Then M admits the orthogonal decomposition

$$M = \left(\perp_{i=1}^m M_{i,\phi}^{(I)} \right) \perp \left(\perp_{j=1}^n M_{j,\phi}^{(II)} \right)$$

where $M_{i,\phi}^{(I)} := M_{P_i,\phi}$ and $M_{j,\phi}^{(II)} := M_{Q_j,\phi} \oplus M_{Q_j^*,\phi}$ from Lemma 3.19.

The conjugacy class of ϕ in the orthogonal group $O(M, q)$ is completely determined by $\mathrm{minpoly}_\phi$ *together with the isometry classes of the regular quadratic modules $\{M_{i,\phi}^{(I)}\}$, $\{M_{j,\phi}^{(II)}\}$ for all i and j.*

Note that the orthogonal decomposition of M implies that in the Witt ring $\mathrm{Wq}(A)$ over A, we have the equality

$$[(M, q)] = \sum_{i=1}^m [M_{i,\phi}^{(I)}] + \sum_{j=1}^n [M_{j,\phi}^{(II)}].$$

We will reduce the question of whether two isometries are conjugate to each other to the equivalence of Hermitian forms, and then apply the following theorem:

Theorem 3.11 (*equivalence of Hermitian forms over Henselian rings*).
Let A be a Henselian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic. Let E be a connected, separable A-algebra with involution $\sigma \in \mathrm{Aut}_A(E)$. Let M be a finitely generated, projective module over E and $s, t \colon M \times M \to E$ regular ϵ-Hermitian σ-sesquilinear forms.

If the reduction of the involution $\overline{\sigma} \in \mathrm{Aut}_k(\overline{E})$ is non-trivial, then s and t are isometric.

Using Galois theory for Henselian local rings with finite residue fields, we can determine the Witt classes above; see Lemma 3.33.

This culminates in the following theorem.

Theorem 3.34 (*parametrisation of conjugacy classes*).
Let A be an Artinian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic p. Let (M, q) be a regular quadratic module over A. Then conjugacy classes of isometries over (M, q) of order coprime to p can be parametrised by

$$[(h_+, \mu_+), (h_-, \mu_-), (P_1, \mu_1^I), \dots, (P_m, \mu_m^I), (Q_1 Q_1^*, \mu_1^{II}), \dots, (Q_n Q_n^*, \mu_n^{II})]$$

where

(i.) *h_+ is a regular symmetric bilinear module over A of rank μ_+;*

(ii.) *h_- is a regular symmetric bilinear module over A of rank μ_-;*

(iii.) *$\{\mu_i^I\}$ are positive integers and $\{P_i\}$ are pairwise strongly relatively coprime polynomials over A such that*
   - *P_i is monic and self-reciprocal with $\deg(P_i) \geq 2$,*
   - *the reduction $\overline{P}_i \in k[T]$ is irreducible with order $\mathrm{ord}(\overline{P}_i)$ coprime to p;*

(iv.) *$\{\mu_j^{II}\}$ are positive integers and $\{Q_j Q_j^*\}$ are pairwise strongly relatively coprime polynomials over A such that*
   - *Q_j is monic and non-self-reciprocal,*
   - *the reduction $\overline{Q}_j \in k[T]$ is irreducible with order $\mathrm{ord}(\overline{Q}_j)$ coprime to p;*

up to reordering of the polynomials P_i and $Q_j Q_j^*$, respectively. The above data is required to satisfy

$$\mathrm{rank}_A\ M = \mu_+ + \mu_- + \sum_i \mu_i^I \deg P_i + \sum_j 2\mu_j^{II} \deg Q_j$$

and the equality in the Witt ring $\mathrm{Wq}(A)$

$$[M] = \sum_i \mu_i^I \Lambda + [h_+] + [h_-].$$

Here $\Lambda \in \mathrm{Wq}(A)$ denotes the Witt class with even dimension and non-trivial Arf invariant; see Equation (2.2). The symmetric regular forms $h_\pm$ correspond to the polynomials $T \mp 1$ in Lemma 3.25.

We emphasise that irreducible polynomials of finite fields are well-studied. For the convenience of the reader, we include a table of the examples in small rank in Chapter 3.

Here is a small extract:

	1	2	3
3	2_1	4_1	
		8_1	13_2, 26_2
5	2_1	3_1, 6_1	
	4_1	8_1, 12_1, 24_2	31_5, 62_5, 124_{10}
7	2_1	4_1, 8_2	
	3_1, 6_1	12_1, 16_2, 24_2, 48_4	9_1, 18_1, 19_3, 38_3, 57_6, 114_6, 171_{18}, 342_{18}

The columns correspond to the degree of the polynomial; the rows to the order of the finite field $\mathbb{F}_p$. For a given prime, we list the self-reciprocal irreducible polynomials in the first line and the non-self-reciprocal irreducible polynomials in the second line. We list the order of the polynomials with the number of polynomials with a given order as the index. For example, there are two non-self-reciprocal irreducible polynomials of order 24 of degree 2 over $\mathbb{F}_7$. We also discuss how to construct a regular quadratic module with an isometry whose conjugacy class has a given invariant; see Remark 3.35.

In Chapter 4, we apply our classification to fixed-point-free isometries, i.e. isometries $\phi\colon (M,q) \to (M,q)$ such that $\phi(x) = x$ implies $x = 0$ for all $x \in M$. Actions by fixed-point-free automorphisms have a controlled orbit structure (all non-trivial orbits have the same length), and they will be instrumental when we construct groups and modular fusion categories in Chapter 6.

Theorem 4.9 (*conjugacy classes of fixed-point-free isometries*).
Let A be a finite local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic p. Let (M,q) be a regular quadratic module over A. Then conjugacy classes of fixed-point-free isometries of order $N > 2$ with $p \nmid N$ can be parametrised as follows

$$[(P_1, \mu_1^I), \dots, (P_m, \mu_m^I), (Q_1Q_1^*, \mu_1^{II}), \dots, (Q_nQ_n^*, \mu_n^{II})]$$

where

(i.) $\{\mu_i^I\}$ *are positive integers and* $\{P_i\}$ *are pairwise strongly relatively coprime polynomials over* A *such that*

- $*$ P_i *is monic and self-reciprocal with* $\deg(P_i) \geq 2$*,*
- $*$ *the reduction* $\overline{P}_i \in k[T]$ *is irreducible with order* $\operatorname{ord}(\overline{P}_i) = N$*;*

(ii.) $\{\mu_j^{II}\}$ *are positive integers and* $\{Q_j Q_j^*\}$ *are pairwise strongly relatively coprime polynomials over* A *such that*

- $*$ Q_j *is monic and non-self-reciprocal,*
- $*$ *the reduction* $\overline{Q}_j \in k[T]$ *is irreducible with order* $\operatorname{ord}(\overline{Q}_j) = N$*;*

up to reordering the polynomials P_i *and* $Q_j Q_j^*$*, respectively. This data is required to satisfy*

$$\operatorname{rank}_A M = \sum_i \mu_i^I \deg P_i + \sum_j 2\mu_j^{II} \deg Q_j$$

and the equality in the Witt ring $\mathrm{Wq}(A)$

$$[M] = \sum_i \mu_i^I \Lambda.$$

We also describe how to construct a regular quadratic module with a fixed-point-free action; see Lemma 4.8.

Part II: Modular Fusion Categories and Fixed-Point-Free Isometries

In the second part, we will construct modular fusion categories using the quadratic forms with fixed-point-free isometries from Part I.

We choose a modular fusion category D and construct a braided $\mathbb{Z}/N$-crossed extension C of D. The $\mathbb{Z}/N$-*equivariantisation* $(\mathsf{C})^{\mathbb{Z}/N}$ of C will be a modular fusion category again.

We start with a *pointed* modular fusion category, i.e. where every simple object is invertible. In particular, the *modular* pointed fusion categories correspond to discriminant forms.

Definition 5.1 (*discriminant form*).
Let Γ be a finite abelian group and $q\colon \Gamma \to \mathbb{Q}/\mathbb{Z}$ a map such that

(i.) $q(ag) = a^2 q(g)$ for all $a \in \mathbb{Z}$ and $g \in \Gamma$, (ii.) the map $\lambda_q \colon \Gamma \times \Gamma \to \mathbb{Q}/\mathbb{Z}$ defined by

$$\lambda_q(g,h) := q(g+h) - q(g) - q(h)$$

is a regular symmetric bilinear form.

Then, the tuple (Γ, q) is called *discriminant form.*

An *isometry* of a discriminant form (Γ, q) is a group automorphism $\phi \colon \Gamma \to \Gamma$ such that $q \circ \phi = q$. The group of all isometries of a discriminant form (Γ, q) will be denoted by $O(\Gamma, q)$; it is called the *orthogonal group* of (Γ, q).

For a discriminant form (Γ, q), there is an equivalence class of braided *modular* fusion categories $\mathsf{V}(\Gamma, q)$ such that the group of simple objects is the group Γ and whose braiding is related to the quadratic form q.

We recall some basic facts which allow us to reduce the study of discriminant forms to quadratic forms over finite local rings from Part I.

First, discriminant forms split orthogonally into their p-groups; see Lemma 5.6. A discriminant form whose underlying group is a p-group furthermore splits orthogonally into its homogeneous subgroups; see Lemma 5.7. By a theorem due to Maschke and Schur, this splitting can be shown to be invariant under groups of automorphisms; see Corollary 5.12.

Rather than working with a general discriminant form (Γ, q) and its isometries, we can thus work with a discriminant form whose underlying group is the homogeneous group $(\mathbb{Z}/p^l)^r$ for a prime p and positive integers $r, l \in \mathbb{Z}_{\geq 1}$. By changing the coefficients from $\mathbb{Q}/\mathbb{Z}$ to $\mathbb{Z}/p^l$ (compare to Lemma 5.9), we can thus apply our results from Part I for the ring $\mathbb{Z}/p^l$.

In the following, we are working with a cyclic group $\mathbb{Z}/N$ which acts by fixed-point-free isometries ϕ^i on a discriminant form (Γ, q) for $i = 1, \dots, N-1$, i.e. we have an action

$$\begin{aligned} \tau \colon \mathbb{Z}/N &\to O(\Gamma, q) \\ i &\mapsto \phi^i. \end{aligned}$$

Since all isometries ϕ^i for $i = 1, \dots, N$ are fixed-point-free, we get that all cohomology groups $H^n(\mathbb{Z}/N, \Gamma)$ for this action are trivial; see Lemma 6.4. A useful consequence of this fact is that (up to an isomorphism) there is only one realisation of τ as an action of $\mathbb{Z}/N$ on a modular fusion category in $\mathsf{V}(\Gamma, q)$ (at least when Γ is odd); see Lemma 6.6. We will therefore choose the *strict* action whose tensor structures are all trivial; see [Gal17].

Recall from above that fixed-point-free isometries of quadratic modules over $\mathbb{Z}/p^l$ were classified in Theorem 4.9. We apply our results in Chapter 6 where we use an action of a cyclic group by fixed-point-free automorphisms on a hyperbolic discriminant form; see Definition 6.7.

We call them hyperbolic since they canonically determine splittings of the discriminant form in invariant Lagrangian subspaces. We concentrated on hyperbolic actions because the $\mathbb{Z}/N$-equivariantisations of crossed $\mathbb{Z}/N$-braided extensions of discriminant forms with such an action turn out to be Drinfeld centres; see Corollary 6.29.

Using sequential equivariantisation, we can easily build braided $\mathbb{Z}/N$-crossed extensions of discriminant forms with such an action. In particular, we get the following.

Corollary 6.23 (*braided $\mathbb{Z}/N$-crossed extensions for fixed-point-free actions*). *Let H be an odd finite abelian group with a fixed-point-free action τ of $\mathbb{Z}/N$ on H. We define the semi-direct product $\Pi := H \rtimes_\tau \mathbb{Z}/N$. Let $\operatorname{cls}\omega \in H^3(\mathbb{Z}/N, k^\times)$ be a normalised 3-cocycle with inflation $\Omega = \operatorname{Infl}_\Pi^{\mathbb{Z}/N}(\omega)$.*

Then, there is a braided $\mathbb{Z}/N$-crossed extension $(\mathsf{vect}_\Pi^\Omega)^H$ of $(\mathsf{vect}_H)^H$ whose $\mathbb{Z}/N$-equivariantisation is equivalent (as braided fusion categories) to

$$_{D_{\omega^{-1}}(\Pi)}\mathsf{Mod} \simeq Z(\mathsf{vect}_\Pi^\Omega) \simeq (\mathsf{vect}_\Pi^\Omega)^\Pi \simeq ((\mathsf{vect}_\Pi^\Omega)^H)^{\mathbb{Z}/N}.$$

Furthermore, $(\mathsf{vect}_H)^H$ is the hyperbolic modular fusion category $\mathsf{vect}_{H\oplus X_H}^{q_{\mathrm{hyp}}}$ from Example 6.19 and $\mathbb{Z}/N$ acts as $\operatorname{Hyp}\tau$ on $\mathsf{vect}_{H\oplus X_H}^{q_{\mathrm{hyp}}}$.

We can directly calculate that the simple objects of the fusion category $(\mathsf{vect}_\Pi^\Omega)^H$ constructed in Corollary 6.23. We get:

(i.) $|H|^2$ simple objects of quantum dimension 1 indexed by $H \times X_H$. These are denoted by $X_{y,\phi}$ for $y \in H$ and $\phi \in X_H$.

(ii.) $N-1$ simple objects of quantum dimension $|H|$ indexed by the non-trivial $(\mathbb{Z}/N) \setminus \{0\}$. These are denoted by M_i for $i \in (\mathbb{Z}/N) \setminus \{0\}$.

The fusion rules are given by

$$\begin{aligned} X_{y_1,\phi_1} \otimes X_{y_2,\phi_2} &= X_{y_1+y_2,\phi_1\phi_2} \\ X_{y_1,\phi_1} \otimes M_i = M_i &= M_i \otimes X_{y_2,\phi_2} \\ M_i \otimes M_j &= \begin{cases} \bigoplus_{y,\phi} X_{y,\phi}, & i+j=0, \\ |H| M_{i+j} & i+j \neq 0, \end{cases} \end{aligned}$$

where $y_1, y_2 \in H$, $\phi_1, \phi_2 \in X_H$ and $i, j = 1, \dots, N_1$.

Fusion categories with fusion rules as above are called $\mathbb{Z}/N$-*Tambara-Yamagami* in [GR25, Definition 5.1]. Let us emphasise that we know more than just the fusion rules.

We describe the representation theory of the group $\Pi = H \rtimes_\tau \mathbb{Z}/N$ in Corollary 6.23 in Section 6.2. Quite remarkably, every braided $\mathbb{Z}/N$-crossed extension of the hyperbolic discriminant form with compatible fixed-point-free action contains this Tannakian subcategory $\mathsf{Rep}_k(\Pi)$ as Lagrangian subcategory; see Lemma 6.28.

We obtain the following classification of braided $\mathbb{Z}/N$-crossed extensions of hyperbolic discriminant forms with hyperbolic action.

THEOREM 6.31 (*classification of hyperbolic $\mathbb{Z}/N$-crossed extensions*). *Let (Γ, q) be an odd discriminant form with a hyperbolic fixed-point-free action*

$$\tau\colon \mathbb{Z}/N \to O(\Gamma, q).$$

We realise τ as strict action on vect_Γ^q, i.e.

$$\tau^{\otimes,\mathrm{str}}\colon \mathbb{Z}/N^{\otimes} \to \mathrm{Aut}_{\mathrm{br}}(\mathsf{vect}_\Gamma^q).$$

The pointed braided category vect_Γ^q with action $\tau^{\otimes,\mathrm{str}}$ has N inequivalent braided $\mathbb{Z}/N$-crossed extensions.

We split $\Gamma = U \oplus V$ such that U is Lagrangian and τ invariant. Representatives for these N equivalence classes are given by the equivariantisations $(\mathsf{vect}^\Omega_{U\rtimes\mathbb{Z}/N})^U$ for the inflations

$$\Omega = \mathrm{Infl}^{\mathbb{Z}/N}_{U\rtimes\mathbb{Z}/N}(\omega)$$

for $\omega \in H^3(\mathbb{Z}/N, k^\times)$ whose $\mathbb{Z}/N$-equivariantisations are given by the Drinfeld centre $Z(\mathsf{vect}^\Omega_{U\rtimes\mathbb{Z}/N})$.

In particular, the above extensions include the modular fusion categories of dimension p^2q^2 for which Mignard and Schauenburg showed that they are not determined (up to braided equivalence) by their modular data; see [MS21] and Remark 6.33.

We show, using results on categorical Morita equivalence for pointed categories, that for two choices of splittings in invariant maximally isotropic submodules, the resulting equivariantisations will be braided equivalent in Lemma 6.36.

The Drinfeld centre of a pointed fusion category is braided equivalent to the category of representations of the quasi-Hopf algebra introduced by Dijkgraaf, Pasquier, and Roche. We will call this quasi-Hopf algebra twisted quantum double; see e.g. [GMN07]. In the end, we will describe the representation theory of these quasi-Hopf algebras for semi-direct products in some detail. Namely, we define:

DEFINITION 7.1 (*quasi-Hopf algebra* $\mathrm{DPR}(H, \tau, \omega)$).
Let H be an abelian group with a fixed-point-free action

$$\tau\colon \mathbb{Z}/N \to \mathrm{Aut}(H).$$

Then, we can consider the semi-direct product

$$\Pi := H \rtimes_\tau \mathbb{Z}/N.$$

Let $\omega \in H^3(\mathbb{Z}/N, k^\times)$ be a normalised 3-cocycle with inflation

$$\Omega^{\mathrm{inv}} = \mathrm{Infl}_{\Pi}^{\mathbb{Z}/N}(\omega^{-1}) \in H^3(\Pi, k^\times).$$

We will denote the quasi-Hopf algebra $D_{\Omega^{\mathrm{inv}}}(\Pi)$ by $\mathrm{DPR}(H, \tau, \omega)$.

In Chapter 7, we give a complete description of the irreducible representations of this quasi-Hopf algebra $\mathrm{DPR}(H, \tau, \omega)$.

PROPOSITION 7.3 (*irreducible representations of* $\mathrm{DPR}(H, \tau, \omega)$).
Let H be a finite abelian group with fixed-point-free action $\tau\colon \mathbb{Z}/N \to \mathrm{Aut}(H)$ and a normalised 3-cocycle $\omega \in H^3(\mathbb{Z}/N, k^\times)$. Then the irreducible representations of $\mathrm{DPR}(H, \tau, \omega)$ can be parametrised by:

(i.) N irreducible representations of quantum dimension 1 indexed by $X_{\mathbb{Z}/N}$. These are denoted by $(0, \alpha)$ for an $\alpha \in X_{\mathbb{Z}/N}$.

(ii.) $(|H|-1)/N$ *irreducible representations of quantum dimension* N *indexed by* $(X_H/(\mathbb{Z}/N)) \setminus \{0\}$*. These are denoted by* $(0,\chi)$ *for a representative* χ *of a non-trivial orbit of* X_H *under the action of* $\mathbb{Z}/N$*.*

(iii.) $|H|(|H|-1)/N$ *irreducible representations of quantum dimension* N *indexed by* $(H/(\mathbb{Z}/N) \setminus \{0\}) \times X_H$*. These are denoted by* (x,ϕ) *for a representative* x *of a non-trivial orbit of* H *under the action of* $\mathbb{Z}/N$ *and* $\phi \in X_H$*.*

(iv.) $N(N-1)$ *irreducible representations of quantum dimension* $|H|$ *indexed by* $((\mathbb{Z}/N) \setminus \{0\}) \times X_{\mathbb{Z}/N}$*. These are denoted by* (i,η) *for an* $i \in (\mathbb{Z}/N) \setminus \{0\}$ *and* $\eta \in \mathbb{Z}/N$*.*

We calculate the modular data (Proposition 7.4) and charge conjugates (duals) (Proposition 7.5) for the category of representations of $\mathrm{DPR}(H,\tau,\omega)$.

We conclude with the following proposition: Here $\langle *, * \rangle_\Pi$ denotes the scalar product of characters of groups. The 1-dimensional and N-dimensional representations θ_α and θ_χ for the group Π where $\alpha \in X_{\mathbb{Z}/N}$ and χ is a representative of an orbit in $X_H/(\mathbb{Z}/N)$ are introduced in Section 6.2

Proposition 7.6 (*fusion rules of* $\mathrm{DPR}(H,\tau,\omega)$).
Let H *be a finite abelian group with fixed-point-free action* τ *of the cyclic group* $\mathbb{Z}/N$ *and* $\omega \in H^3(\mathbb{Z}/N, k^\times)$ *a normalised* 3*-cocycle. Then the fusion coefficients of the irreducible representations of* $\mathrm{DPR}(H,\tau,\omega)$ *are given by*

$$N_{(0,\alpha),(i,\beta)}^{(i,\gamma)} = \delta_{\gamma,\alpha\beta} \quad N_{(0,\chi),(i,\beta)}^{(i,\gamma)} = 1 \quad N_{(x,\phi),(i,\alpha)}^{(i,\gamma)} = 1$$

$$\begin{gathered} N_{(i,\alpha),(j,\beta)}^{(k,\gamma)} = \frac{1}{N}\delta_{i+j,k}\big(|H|-1+\delta_{\alpha\beta,\gamma c(i,j)}\big) \\ N_{(0,\chi),(0,\kappa)}^{(0,\rho)} = \langle \theta_\chi \theta_\kappa, \theta_\rho \rangle_\Pi \\ N_{(0,\chi),(x,\phi)}^{(y,\psi)} = \delta_{x,y} \langle \mathrm{Red}_H^\Pi(\theta_\chi)\phi, \psi \rangle_H \end{gathered}$$

$$\left.\begin{aligned} N_{(x,\phi),(y,\psi)}^{(0,\alpha)} &= \langle (\phi^{i_0})(\psi^{j_0}), \mathrm{Red}_H^\Pi(\theta_\alpha) \rangle_H \\ N_{(x,\phi),(y,\psi)}^{(0,\chi)} &= \langle (\phi^{i_0})(\psi^{j_0}), \mathrm{Red}_H^\Pi(\theta_\chi) \rangle_H \end{aligned}\right\} \text{ if } {}^{i_0}x + {}^{j_0}y = 0 \text{ for some } i_0, j_0 \in \mathbb{Z}/N$$

$$N_{(x,\phi),(y,\psi)}^{(z,\xi)} = \langle (\phi^{i_0})(\psi^{j_0}), \xi \rangle_H \quad \Big\} \text{ if } {}^{i_0}x + {}^{j_0}y = z \text{ for some } i_0, j_0 \in \mathbb{Z}/N$$

for all $\alpha, \beta, \gamma \in X_{\mathbb{Z}/N}$*,* $i, j, k \in (\mathbb{Z}/N) \setminus \{0\}$*,* χ, κ, ρ *representatives of the non-trivial orbits* $(X_H/(\mathbb{Z}/N)) \setminus \{0\}$*,* x, y, z *representatives of the non-trivial orbits* $(H/(\mathbb{Z}/N)) \setminus \{0\}$*, and* $\phi, \psi, \xi \in X_H$*.*

Here, $c\colon \mathbb{Z}/N \times \mathbb{Z}/N \to X_{\mathbb{Z}/N}$ is a 2-cocycle. We refer to Proposition 7.6 for its definition; it depends on the 3-cocycle ω.

The fact that we assumed the action to be fixed-point-free is very important, as otherwise we would not be able to obtain the fusion rules with the closed form above.

Part III: Monoidal Bicategories and Coherent Gr*-Bicategories*

In the last part, we turn our attention to something slightly different. In Chapter 8, we give a complete definition of monoidal bicategories in Definition 8.13. Monoidal bicategories can be defined as one-object tricategories. The axioms of a monoidal bicategory are quite involved, especially on the level of 2-cells. For example, the pentagon axiom of a monoidal category gets replaced by the existence of a modification, the *pentagonator*, with component 2-cells

$$\begin{tikzcd}
& {[w \boxtimes (x \boxtimes y)] \boxtimes z} \arrow[rr, "A_{w,x\boxtimes y,z}"] & & {w \boxtimes [(x \boxtimes y) \boxtimes z]} \arrow[dr, "\mathbf{1}_w \boxtimes A_{x,y,z}"] & \\
{[(w \boxtimes x) \boxtimes y] \boxtimes z} \arrow[ur, "A_{w,x,y} \boxtimes \mathbf{1}_z"] \arrow[drr, "A_{w\boxtimes x,y,z}"'] \arrow[rrrr, phantom, "\Downarrow \pi_{w,x,y,z}"] & & & & {w \boxtimes [x \boxtimes (y \boxtimes z)]} \\
& & (w \boxtimes x) \boxtimes (y \boxtimes z) \arrow[urr, "A_{w,x,y\boxtimes z}"'] & &
\end{tikzcd}$$

which need to satisfy a non-abelian 4-cocycle condition [GPS95] with 14 vertices (the 14 ways to parenthesise the product of five objects). This equation A fills an entire page. This is the main reason why a complete definition is usually omitted in the literature; see e.g. [Bae23].

We also state the full definition of morphisms between monoidal bicategories in Definition 8.22. A complete definition is necessary since the skeletalisation of a monoidal 2-category (with strict composition) is a monoidal bicategory rather than a monoidal 2-category; see Proposition 9.15.

We state and prove some results that allow us to *transport* bicategorical and monoidal bicategorical structure along biequivalences. For example, we prove in

Lemma 8.7 and Corollary 8.8 that every bicategory is biequivalent to a bicategory with strict identities.

We already mentioned Sính's work on Gr-categories, a categorification of the concept of a group; see e.g. [Bae23]. We will work with (and generalise) *coherent* Gr-categories as in [BL04, Section 3]. These are monoidal categories with invertible arrows where every object x has a weak inverse object x^- such that there exist isomorphisms

$$i_x \colon \mathbb{1} \to x \otimes x^-, \quad e_x \colon x^- \otimes x \to \mathbb{1},$$

and the two triangle identities hold; i.e. the following diagrams commute

$$\begin{tikzcd}
x^- \otimes \mathbb{1} \arrow[r, "\mathrm{id}_{x^-} \otimes i_x"] \arrow[rrdd, "r_{x^-}"'] & x^- \otimes (x \otimes x^-) \arrow[r, "(a_{x^-,x,x^-})^{-1}"] & (x^- \otimes x) \otimes x^- \arrow[d, "e_x \otimes \mathrm{id}_{x^-}"] \\
 & & \mathbb{1} \otimes x^- \arrow[d, "l_{x^-}"] \\
 & & x^-
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \otimes x \arrow[r, "i_x \otimes \mathrm{id}_x"] \arrow[rrdd, "l_x"'] & (x \boxtimes x^-) \otimes x \arrow[r, "a_{x,x^-,x}"] & x \otimes (x^- \otimes x) \arrow[d, "\mathrm{id}_x \otimes e_x"] \\
 & & x \otimes \mathbb{1} \arrow[d, "r_x"] \\
 & & x
\end{tikzcd}$$

We briefly recall facts about coherent Gr-categories as presented in [BL04, Section 3] in Section 9.1.

In Chapter 9, we define coherent Gr-bicategories in Definition 9.2. These are an appropriate categorification of the concept of a group for monoidal *bi*categories. For every object x in a coherent Gr-bicategory there is a biadjoint biequivalence with component 1-cells

$$I_x \colon \mathbb{1} \to x \boxtimes x^-, \quad E_x \colon x^- \boxtimes x \to \mathbb{1},$$

with 2-cells corresponding to the triangle identities; see Definition 9.2.

We show, by transporting monoidal structure along equivalences, that we can always work with *skeletal* coherent Gr-bicategories with controlled unitor and coherence structure in Proposition 9.18.

In Chapter 10, we conclude by calculating the group-theoretic data underlying morphisms of coherent monoidal Gr-bicategories, following the ideas of Baez and Lauda [BL04, Section 8.3] for monoidal Gr-categories. This should be useful for the study of extensions of fusion categories in the future.

In the end, we briefly outline some natural next steps in our outlook.

I.

Quadratic Forms and Their Isometries

In this part, we are concerned with quadratic forms over local rings. Our primary motivation is that in Part II, we will use *discriminant forms* and their *isometries* to build modular fusion categories. A discriminant form (Γ, q) consists of a finite abelian group Γ with a regular quadratic map $q\colon \Gamma \to \mathbb{Q}/\mathbb{Z}$. Later, we will see how to get from a discriminant form to regular quadratic modules over the local ring $\mathbb{Z}/p^l$ for a prime p and positive integer $l \in \mathbb{Z}_{\geq 1}$.

We will see that many of our results apply to all *Henselian* local rings, i.e. local rings for which Hensel's lemma holds. For example, Artinian local rings are Henselian. Thus, this includes the ring $\mathbb{Z}/p^l$ above.

Chapter 1: Minimal Polynomials for Automorphisms.

In Chapter 1, we will establish the existence of minimal polynomials of certain automorphisms over Artinian local rings. In Theorem 3.34, we prove that over Artinian local rings with finite residue field, automorphisms of order coprime to the characteristic of the residue field admit minimal polynomials. Our proof relies on the following: By Cohen's structure theorem of complete local rings, we will show that Artinian rings are quotients of regular complete rings; see Corollary 1.13. For complete local rings, we show that solutions to matrix equations over the residue field can be lifted to solutions over the ring; see Proposition 1.10. Since matrices over regular local rings admit minimal polynomials, this will allow us to prove the existence of minimal polynomials of certain automorphisms over Artinian local rings.

Chapter 2: Hermitian and Quadratic Forms over Henselian Rings.

In Chapter 2, we recall the definitions of quadratic and symmetric bilinear forms over general rings; see Definition 2.3. We note some general facts about the Witt ring for Henselian rings with finite residue field in Section 2.1. For example, Knebusch showed that regular quadratic modules over local rings also admit the famous Witt cancellation property; see Lemma 2.13. We also note that for a Henselian local ring, its Witt ring is isomorphic to the Witt ring of its residue field; see Lemma 2.14.

Chapter 3: Conjugacy Classes in Orthogonal Groups.

In Chapter 3, we will determine conjugacy classes of isometries in orthogonal groups over Artinian local rings with finite residue fields. For quadratic forms over finite fields, this problem is discussed in [SS70], [Shi80], [Mil69], and [Wal63b].

In Theorem 3.34, we give a complete set of invariants for conjugacy classes of quadratic modules over an Artinian ring with finite residue fields whose order are coprime to the characteristic of the residue field. The proof takes up much of Chapter 3. We will see that we can closely follow the calculations of [Shi80] for finite fields. The reason for this again is that for an Henselian local ring with finite residue field, we can lift many properties of its residue field to the ring. We only need the assumption that the ring is Artinian to ensure the existence of the minimal polynomials; see Chapter 1. We proceed as follows. Let us fix an Artinian local ring A with finite residue field k of odd characteristic. Let (M, q) be a regular quadratic module over A.

(i.) If two isometries admit a minimal polynomial and are similar, they have the same minimal polynomial; see Lemma 3.12. Throughout Chapter 3 and 4, we will thus assume that all isometries admit minimal polynomials (which we can only guarantee if they have finite order coprime to the characteristic of the residue field).

(ii.) The minimal polynomial of an isometry of a regular quadratic module over a ring A is self-reciprocal; see Lemma 3.14.

(iii.) Since the ring A is Henselian, we can decompose the minimal polynomial $\mathrm{minpoly}_\phi$ as

$$\mathrm{minpoly}_\phi = P_1 \dots P_m Q_1 Q_1^* \dots Q_m^*$$

with self-reciprocal polynomials $P_i \in A[T]$ and non-self-reciprocal polynomials $Q_j \in A[T]$ such that the reductions $\overline{P}_i$ and $\overline{Q}_j$ are all strongly relatively coprime; see Lemma 3.18.

(iv.) Using the above decomposition of the minimal polynomial, we find an orthogonal decomposition of (M, q); see Lemma 3.25. With this decomposition, we can simplify further.

(v.) For two isometries with the same self-reciprocal polynomial irreducible over the residue field, we can show that they are conjugate by reducing the statement that they are conjugate to a statement about Hermitian forms; see Lemma 3.21 and 3.22.

(vi.) We prove in Theorem 3.11 that over an Henselian ring with finite residue field of odd characteristic, two regular Hermitian forms with non-trivial involution are isometric. Note that this is a crucial property of finite fields which enters Milnor's proof [Mil69]; see also [Shi80].

(vii.) In Section 3.4, we use the Galois theory for Henselian local rings with finite residue fields, to determine the Witt class of a regular quadratic module with an isometry whose minimal polynomial is self-reciprocal and irreducible over the residue field; see Lemma 3.33.

Since irreducible polynomials over finite fields of fixed degree and order are known, we can easily list examples in small rank; see the table in Chapter 3.

Chapter 4: Fixed-Point-Free Isometries.

In our fourth Chapter, we describe the conjugacy classes whose isometries are fixed-point-free, i.e. isometries $\phi\colon (M,q) \to (M,q)$ such that $\phi(x) = x$ implies $x = 0$ for all $x \in M$. We will use this classification in Chapter 6. We obtain in Theorem 4.9 that they can be parametrised by irreducible polynomials of fixed order over finite fields.

1. Minimal Polynomials for Automorphisms

A ring will be associative, commutative, and have a unit. Throughout this part, we assume that all rings are Noetherian. Let A be a ring and $A^\times$ be the multiplicative group consisting of the units of A. For an element $x \in A$, we denote its reduction in the residue field by $\overline{x}$. For a field k, we denote its characteristic by $\operatorname{char}(k)$.

The Local Ring $\mathbb{Z}/p^l$

Let p be a prime number and $l \in \mathbb{Z}_{\geq 1}$ be a positive integer. Then the quotient ring $A = \mathbb{Z}/p^l$ is a local ring with maximal ideal $\mathfrak{m} = (p)$ and the finite residue field $k = \mathbb{F}_p$.

As a finite ring, A is in particular Artinian. The maximal ideal (p) is nilpotent, so A is separated and complete in the $\mathfrak{m}$-adic topology, i.e. $A \cong \hat{A}_{\mathfrak{m}}$, where we denoted the completion of A in the $\mathfrak{m}$-adic topology by $\hat{A}_{\mathfrak{m}}$.

In particular, Artinian local rings are *Henselian.*

Definition 1.1 (*Henselian[1] local ring, compare to [Mil80, 1 §3]).*
Let A be a local ring with residue field k. Then A is called *Henselian* if for every polynomial $Q \in A[T]$ such that its reduction over k factors $\overline{Q} = R_0 S_0$ with $R_0, S_0 \in k[T]$ where R_0 is monic and R_0, S_0 are relatively prime in $k[T]$, there exist polynomials $R, S \in A[T]$ such that Q factors as $Q = RS$ with R monic and $\overline{R} = R_0, \overline{S} = S_0$.

Here we denote the reduction of coefficients of a polynomial $Q \in A[T]$ to its residue field by $\overline{Q} \in k[T]$. The polynomials R and S are uniquely determined with this property; see [Mil80, I Remark 4.1]. In fact, any separated and complete

[1] To our knowledge, Azumaya named this property after Kurt Hensel in [Azu51]. This property was actually already used by Theodor Schönemann in 1846; see [Sch46].

ring is Henselian; see [For17, Theorem 10.1.10].[2] A very detailed account of Henselian rings can be found in [Ray70].

First, we note that for Henselian local rings, we can lift solutions of polynomial equations, in particular, quadratic equations.

LEMMA 1.2 (*compare to [Azu51, Lemma 4]).*
Let A be a Henselian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let Q be a polynomial in $A[T]$ and $\overline{Q}$ be its reduction in $k[T]$. Suppose that $\overline{Q}$ has a non-multiple root $\overline{a}$ in k. Then Q has exactly one root a in A which is a representative of $\overline{a}$.

Since we are only working with Noetherian rings, we don't need to assume that Q is monic; see the proof of [For17, Lemma 10.1.9]. Note that local rings are called *completely primary* in [Azu51].

As a direct consequence of the above lemma, we obtain:

COROLLARY 1.3.
Let A be a Henselian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Then an element $x \in A^\times$ is a perfect square if and only if its reduction $\overline{x} \in k^\times$ is a perfect square.

Proof. If x is a perfect square, then so is $\overline{x}$. Conversely, assume that $\overline{x} = \overline{x}_0^2$ for an $x_0 \in k^\times$. Then the polynomial $Q = T^2 - x$ has the reduction

$$T^2 - \overline{x}_0^2 = (T - \overline{x}_0)(T + \overline{x}_0).$$

Thus, by Lemma 1.2, there exist a unique element $m \in \mathfrak{m}$ such that

$$0 = Q(\overline{x}_0 + m) = (\overline{x}_0 + m)^2 - x,$$

i.e. x is a perfect square. ▪

Finally, we remark here that since the ring is local, every projective module over it is free.

Note that for a local ring A with maximal ideal $\mathfrak{m}$ that is not Noetherian, the $\mathfrak{m}$-adic topology is not necessarily separated. Since our rings are usually even Artinian local, this will not be a problem for us; hopefully, this avoids some confusion with the literature.

[2] For non-Noetherian rings, Hensel's lemma takes a slightly different form; see e.g. [For17, Theorem 10.1.4].

Minimal Polynomials over Artinian Local Rings

In the first part, we will primarily work with Artinian local rings and repeatedly use their completeness. Our first goal is to show that certain automorphisms over Artinian local rings admit *minimal polynomials*.

Let M be a finitely generated projective module over a ring A and let $\phi\colon M \to M$ be an A-endomorphism. Then, the kernel of the ring map

$$\begin{aligned} \theta\colon A[T] &\to \operatorname{End}_A(M), \\ T &\mapsto \phi \end{aligned}$$

is an ideal. If it is a principal ideal, it has a monic generator, the *minimal polynomial* $\operatorname{minpoly}_\phi$; see [FL14, Lemma 4.1].

If A is not a field, the existence of minimal polynomials is not guaranteed since $A[T]$ is not a principal ideal domain. For a discussion of this problem, we refer to [FL14].

We will recall the following characterisation of separable polynomials. Let A be a ring and E an A-algebra with a finite basis of d elements. Recall that the *discriminant* of a tuple $(x_1, \dots, x_d)$ is defined as

$$\operatorname{discr}(E/A)[(x_1, \dots, x_d)] := \det(\operatorname{trace}_{E/A}(x_i x_j));$$

see e.g. [Bou74, III §9 Definition 3]. For a polynomial $P \in A[T]$ of degree d, the discriminant of P is the discriminant of the A-algebra $A[T]/(P)$ with basis $(1, T, \dots, T^{d-1})$; see [Bou90, IV §6 Definition 2].

Lemma 1.4 (*characterisation of separable polynomials).*
Let A be a ring. Let $Q \in A[T]$ be a polynomial and $E = A[T]/(Q)$. We denote the image of T in E by t. The following are equivalent:

(i.) The algebra E is étale over A.

(ii.) The element $Q'(t)$ is invertible in E.

(iii.) The discriminant $\operatorname{discr}(Q)$ *of Q is invertible in A.*

Here, Q' denotes the formal derivative of the polynomial Q. The polynomial Q satisfying the above conditions is called *separable*. Details can be found in e.g. [FL14, Proposition 2.8], [For17, Chapter 6.1], or [DI71].

We quickly record the following helpful fact, which follows directly from the calculation of the discriminant of a product of polynomials.

LEMMA 1.5.
Let A be a local ring with maximal ideal $\mathfrak{m}$. Let $Q \in A[T]$ be a separable polynomial. If $R \in A[T]$ is a polynomial dividing Q, R is also separable.

Proof. As Q is separable, we have $\operatorname{discr}(Q) \in A^\times$. Since A is (Noetherian and) local, this is equivalent to $\operatorname{discr}(Q) \notin \mathfrak{m}$. As R divides Q, we also have that $\operatorname{discr}(R) \mid \operatorname{discr}(Q)$; see e.g. [Bou90, IV §6 Corollary 1 of Proposition 11]. Therefore, $\operatorname{discr}(R) \notin \mathfrak{m}$. ▪

Furthermore, we get since discriminants commute with change of rings; see e.g. [Bou90, IV §6 Proposition 11]:

LEMMA 1.6.
Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $Q \in A[T]$ be a polynomial and $\overline{Q}$ be its reduction in $k[T]$. Then Q is separable if and only if $\overline{Q}$ is separable.

We will use the following criterion for minimal polynomials later. Note that for endomorphisms of a finitely generated projective module, the characteristic polynomial always exists.

LEMMA 1.7.
Let A be a local ring. Let M be a finitely generated projective module over A and let $\phi\colon M \to M$ be an A-endomorphism. Let $Q \in A[T]$ be a separable polynomial dividing the characteristic polynomial of ϕ such that $Q(\phi) = 0$, then Q is the minimal polynomial.

A proof can be found in [FL14, Proposition 4.2]; note that local rings are connected.

Lifting Matrix Equations

Now, we will follow the ideas of [Dav68]. We will show the existence of a minimal polynomial by lifting matrix equations and using the completeness of our local ring. We recall a fact about matrix equations which we will use repeatedly. It is also stated in [Dav68] (before Lemma 1).

LEMMA 1.8.
Let k be a field and $Q \in k[T]$ be a polynomial. Let $X \in M_n(k)$ be a matrix such that $Q(X) = 0$. Then, if Q is separable over k,

$$\det(Q'(X)) \neq 0.$$

Proof. Let L be a field such that Q splits in L. We get for every eigenvalue λ of X

$$Q(\lambda) = 0, \quad Q'(\lambda) \neq 0.$$

Thus $\det(Q'(X)) \neq 0$. ▪

We will see that we can solve matrix equations over complete rings by lifting solutions over the residue field using the following two lemmas.

LEMMA 1.9 (*[Eis95, Lemma 7.14]).*
Let A be a ring with ideals $\mathfrak{a}, \mathfrak{b} \subset A$. If there exist positive integers $i, j \in \mathbb{Z}_{\geq 1}$ such that

$$\mathfrak{a}^i \subseteq \mathfrak{b}, \quad \mathfrak{b}^j \subseteq \mathfrak{a},$$

then there is a natural isomorphism $\hat{A}_{\mathfrak{a}} \cong \hat{A}_{\mathfrak{b}}$.

In particular, we immediately get that if A is Artinian local with maximal ideal $\mathfrak{m}$ and $\mathfrak{a} \subseteq \mathfrak{m}$ is an ideal in A, then A is also complete (and separated) in the $\mathfrak{a}$-adic topology.

The following lemma is based on the first section of [Dav68]. The proof presented there is for the p-adic integers $\mathbb{Z}_p$. We will apply this proposition to Noetherian complete local regular rings coming from Cohen's structure theorem of complete local rings.

PROPOSITION 1.10 (*lifting solutions to matrix equations).*
Let A be a complete local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $\mathfrak{a} \subset A$ be an ideal in A such that A is complete in the $\mathfrak{a}$-adic topology.

Let $Q \in A[T]$ be a separable polynomial. Let $X_0 \in M_n(A/\mathfrak{a})$ be a matrix over the quotient ring $A/\mathfrak{a}$ such that

$$Q(X_0) \equiv 0 \pmod{\mathfrak{a}}.$$

Then there exists an $X \in M_n(A)$ *such that*

$$Q(X) = 0, \quad X \equiv X_0 \pmod{\mathfrak{a}}.$$

Proof. We write the polynomial Q as $Q = \sum_{i=0}^{d} a_i T^i$.

Note that $Q'(X_0) \pmod{\mathfrak{a}}$ viewed as a matrix over the quotient ring $A/\mathfrak{a}$ is invertible if and only if its reduction $\overline{Q}'(\overline{X}_0)$ over k is invertible (follows by considering determinants). Since Q is separable, so is its reduction $\overline{Q}$ modulo $\mathfrak{m}$. Therefore, $\overline{Q}'(\overline{X}_0)$ is invertible by Lemma 1.8.

Now, we can apply the same construction as in the proof of [Dav68, Lemma 1], namely, we define

$$X_1 := X_0 - Q(X_0)(Q'(X_0))^{-1} \pmod{\mathfrak{a}^2}.$$

Then, we get

$$\begin{aligned} Q[X_1] &= Q[X_0 - Q(X_0)(Q'(X_0))^{-1}] \\ &= \sum_{i=0}^{d} a_i [X_0 - Q(X_0)(Q'(X_0))^{-1}]^i \\ &\equiv a_0 + \sum_{i=1}^{d} a_i (X_0^i - i X_0^{i-1} Q(X_0)(Q'(X_0))^{-1}) \pmod{\mathfrak{a}^2} \\ &= Q(X_0) - Q(X_0) = 0. \end{aligned}$$

Similarly, for $i \geq 1$, we define

$$X_{i+1} := X_i - Q(X_i)(Q'(X_i))^{-1}.$$

This is well-defined since $Q'(X_i) \equiv \overline{Q}'(\overline{X}_0) \pmod{\mathfrak{m}(A/\mathfrak{a}^i)}$ is invertible for all $i \in \mathbb{Z}_{\geq 1}$. We get that

$$X_{i+1} \equiv X_i \pmod{\mathfrak{a}^i}, \quad Q(X_{i+1}) \equiv 0 \pmod{\mathfrak{a}^{i+1}}.$$

Thus, all coefficient sequences $(X_i)_{ab} \in A/\mathfrak{a}^i$ for fixed a, b converge to some $x_{ab} \in \hat{A}_{\mathfrak{a}} \cong A$. The matrix $X = (x_{ab})$ satisfies the above requirements. ▪

The above lemma allows us to construct matrices with the same order coprime to the characteristic of the residue field, which are solutions to a particular polynomial equation.

Example 1.11.
For the ring $A = \mathbb{Z}/(7^3)$ and the ideal (7) in A, the matrix

$$\begin{bmatrix} 0 & 6 \\ 1 & 4 \end{bmatrix}$$

has the minimal polynomial $Q := T^2 + 3T + 1 \in \mathbb{F}_7[T]$. Applying the above algorithm, we get matrices

$$X_1 := \begin{bmatrix} 28 & 34 \\ 22 & 18 \end{bmatrix}, \quad X_2 := \begin{bmatrix} 175 & 34 \\ 22 & 165 \end{bmatrix}$$

such that $Q(X_1) \equiv 0 \pmod{7^2}$ and $Q(X_2) = 0$, where we viewed Q as having coefficients in A.

Now, we will prove that automorphisms of finite order not dividing the characteristic of the residue field admit minimal polynomials. Using Cohen's structure theorem for complete local rings, we reduce the statement to regular local rings. We follow [Mat80, Section 28]; for a nice proof using formal smoothness we refer to [Gro64, Théorème (19.8.8)]. Let A be a complete local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $p = \mathrm{char}(k)$ be the characteristic of the residue field k. A *coefficient ring* $R \subset A$ is a subring such that

(i.) R is a Noetherian complete local ring with maximal ideal $R \cap \mathfrak{m}$,

(ii.) the residue field of R is isomorphic to k (by the canonical map),

(iii.) $R \cap \mathfrak{m} = pR$.

Proposition 1.12 (*Cohen's structure theorem).*
Let A be a complete local ring. Then A has a coefficient ring R.

In fact, we will use the following corollary.

Corollary 1.13.
Let A be a complete (separated) local ring with maximal ideal $\mathfrak{m}$ and residue field k such that $\mathfrak{m} = (x_1, \dots, x_n)$ is finitely generated. Then A has a coefficient ring R and the map

$$\begin{aligned} R[[T_1, \dots, T_n]] &\to A \\ T_i &\mapsto x_i \end{aligned}$$

is a surjective map of local rings.

We refer to [Mat80, Corollary 28.3]; a proof can be found in [Sta25, Section 0323].

Remark 1.14.
The ring $\mathbb{Z}/p^l$ has characteristic p^l while its residue field $\mathbb{F}_p$ has characteristic p. The coefficient ring is the ring of p-adic integers $\mathbb{Z}_p$. For the general theory, we refer to [Ser79, Section II §6].

By [AB59, Theorem 5], regular local rings are unique factorisation domains and unique factorisation domains are integrally closed (see e.g. [AM69, Example 5.0]; the same argument applies for all unique factorisation domains). By the [Fri04, Corollary] every square matrix with coefficients in an integrally closed domain admits a minimal polynomial with coefficients in that domain. Thus, we get:

Lemma 1.15.
Let A be a regular local ring. Let $X \in M_n(A)$ be a matrix. Then X admits a minimal polynomial $\mathrm{minpoly}_X \in A[T]$.

Theorem 1.16 (*existence of minimal polynomial).*
Let A be an Artinian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of characteristic p. Let M be a finitely generated projective module over A and let $\phi\colon M \to M$ be an automorphism of finite order N such that $p \nmid N$. Then ϕ admits a minimal polynomial $\mathrm{minpoly}_\phi$ *which divides the polynomial $T^N - 1$ in $A[T]$.*

Proof. First, the polynomial $T^N - 1$ is separable over k since $p \nmid N$. Note that since A is local, M is free, so we can represent ϕ as a matrix $X_0 \in M_n(A)$ with $X_0^N - 1 = 0$. By Corollary 1.13, we get that there is a regular local ring D with a surjective local map $\pi\colon D \to A$. We denote the kernel of this map by $\mathfrak{a}$ and the maximal ideal of D by $\mathfrak{n}$. Then since $\mathfrak{m}$ is nilpotent in A, there exists a positive integer $i \in \mathbb{Z}_{\geq 1}$ such that $\mathfrak{m}^i = 0$. Then the maximal ideal of D, satisfies $\mathfrak{n}^i \subset \mathfrak{a}$ since $\mathfrak{a}$ is the kernel of π. Thus, we get by Lemma 1.9, that D is complete in the $\mathfrak{a}$-adic topology. Now, we can apply Proposition 1.10 and find a matrix $X \in M_n(D)$ which satisfies

$$X^N - 1 = 0 \text{ and } X \equiv X_0 \pmod{\mathfrak{a}}.$$

Since matrices over regular local rings have minimal polynomials by Lemma 1.15, X has a minimal polynomial $\mathrm{minpoly}_X \in D[T]$ by [Fri04] which divides $T^N - 1$ since $X^N - 1 = 0$. Since $T^N - 1$ is separable, so is $\mathrm{minpoly}_X$ by Lemma 1.6.

By reducing the coefficients, we get a monic, separable polynomial

$$Q := \pi(\mathrm{minpoly}_X) \in A[T]$$

which satisfies $Q(X_0) = 0$.

Since determinants commute with change of rings, we get the equality of characteristic polynomials $\mathrm{charpoly}_{X_0} = \pi(\mathrm{charpoly}_X)$ and therefore $Q \mid \mathrm{charpoly}_{X_0}$. Thus, we get the statement with Lemma 1.7. ▪

Note that we only used the fact that A is Artinian in showing that D is complete in the $\mathfrak{a}$-adic topology. An Artinian local ring with finite residue field is finite.

2. Hermitian and Quadratic Forms over Henselian Rings

In this section, we will recall basic definitions and conventions for Hermitian and quadratic forms. Quadratic forms are well-studied mathematical objects. As we already mentioned, the algebraic theory of quadratic forms is quite old and was studied by Ernst Witt in 1936 [Wit36]. Quadratic forms over rings which are not necessarily fields were studied later. A good overview can be found in [Kne70], where Manfred Knebusch even discusses quadratic forms over schemes. Many of the properties of quadratic forms over fields can be generalised to quadratic forms over local rings; see e.g. [Kne69]. For example, Witt's cancellation theorem still holds; see Lemma 2.13. General facts about quadratic forms can be found in the textbooks [OMe63], [Sch09a], and [Shi80]. For quadratic forms over local rings, we refer to [Bae78]. Finally, for finite fields, we use the textbooks [Tay92] and [Lam73].

We will work with quadratic forms over *Henselian* local rings; see Section 1. In the end, we are primarily interested in the finite ring $\mathbb{Z}/p^l$ for a prime number p and a positive integer $l \in \mathbb{Z}_{\geq 1}$, but as we will see, the majority of the results generalise to all Henselian local rings. Note that in Chapter 4, we will work over Artinian local rings again.

Recall that an *involution* on a ring A is a ring endomorphism σ of A of order ≤ 2. If E is a commutative ring extension of A, an involution of E with respect to A is an involution of E which restricts to the identity on A, i.e. $\sigma|_A = \mathrm{id}_A$.

Definition 2.1 (*sesquilinear form, [Knu91, I Section 2.2]).*
Let A be a ring with involution σ and M an A-module. A *σ-sesquilinear form* is a map $s\colon M \times M \to A$ which satisfies the following conditions:

For all $x, y, z \in M$ and $a \in A$,

$$s(x+y,z) = s(x,z) + s(y,z), \qquad s(x,y+z) = s(x,y) + s(x,z),$$
$$s(ax,y) = as(x,y), \qquad s(x,ay) = \sigma(a)s(x,y).$$

The pair (M, s) is called *σ-sesquilinear module* over A.

An *isometry* of sesquilinear forms (M_1, s_1) and (M_2, s_2) is an A-linear isomorphism $\phi\colon M_1 \to M_2$ such that

$$s_2(\phi(x_1), \phi(y_1)) = s_1(x_1, y_1)$$

for all $x_1, y_1 \in M_1$.

For an A-module M, we denote the *dual A-module* by

$$M^\times := \mathrm{Hom}_A(M, A).$$

A sesquilinear form s on M induces an A-linear map

$$\mathrm{Adj}[s]\colon M \mapsto M^\times, \quad \mathrm{Adj}[s](x)(y) = s(x, y).$$

We call $\mathrm{Adj}[s]$ the *adjoint* of s. The sesquilinear form s is called *regular* (or *non-singular*) if its adjoint $\mathrm{Adj}[s]$ is an isomorphism.

DEFINITION 2.2 (*ϵ-Hermitian form, [Knu91, I Section 2.3]).*
Let A be a ring with involution σ and $\epsilon \in A$ with $\epsilon\sigma(\epsilon) = 1$. Then a σ-sesquilinear form $\lambda\colon M \times M \to A$ on an A-module M is called *ϵ-Hermitian* if

$$\lambda(x, y) = \epsilon\sigma(\lambda(y, x))$$

for all $x, y \in M$.

In this case, the pair (M, λ) is called *ϵ-Hermitian σ-sesquilinear module.* We introduce Hermitian forms because they are useful for classifications; see [Jac40]. Of course, a 1-Hermitian id_A-sesquilinear form is called *symmetric* bilinear form.

We denote the category of ϵ-Hermitian modules over A by $\mathrm{Herm}^\epsilon(A)$.

DEFINITION 2.3 (*quadratic module, [Knu91, I Definition 5.3.5]).*
Let A be a ring. A *quadratic module* over A is a tuple (M, q) consisting of a finitely generated projective module M over A with a map $q\colon M \to A$ satisfying

(i.) $q(ax) = a^2 q(x)$ for all $x \in M$ and $a \in A$,

(ii.) the map $\lambda_q\colon M \times M \to A$ defined by

$$\lambda_q(x, y) := q(x + y) - q(x) - q(y)$$

is a symmetric bilinear form.

If 2 is a unit in A, we can identify quadratic forms and symmetric bilinear forms by associating to a symmetric bilinear form λ the quadratic form q_λ defined by $q_\lambda(x) = \lambda(x,x)$ and associating to a quadratic form q the symmetric bilinear form $\frac{1}{2}\lambda_q$. The associated bilinear form is sometimes called *polar form*.

A quadratic module (M,q) is called *regular* (or *non-singular*) if the symmetric bilinear form λ_q is regular.

An *isometry* between quadratic modules (M_1,q_1) and (M_2,q_2) is an A-linear isomorphism $\phi\colon M_1 \to M_2$ such that $q_2 \circ \phi = q_1$. For a quadratic module (M,q), we denote the group of all isometries from (M,q) to (M,q) by $O_A(M,q)$. This group is called *orthogonal group* of (M,q) over A.

We will limit ourselves to local rings for which every finitely generated projective module is a finite-dimensional free module.

We will see that the conjugacy classes of the isometries we will classify impose restrictions in the Witt class of the quadratic forms. We denote the Witt ring of regular quadratic forms (of constant rank) over A by $\mathrm{Wq}(A)$, compare to [Bae78, I §2.4]. For a regular module (M,q) over A, we will denote its Witt class by $[(M,q)]$. We will denote the isometry class of (M,q) by $\{(M,q)\}$ (i.e. its class in the Grothendieck-Witt ring; we will not work with this ring though).

For a projective module U over A, we denote the *hyperbolic* space associated to N by $\mathrm{Hyp}\,(U)$; see [Bae78, i §4].

Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. For a quadratic module (M,q) over A, we can always reduce (M,q) modulo $\mathfrak{m}$ to obtain a quadratic space $(\overline{M},\overline{q})$ over k. In fact, (M,q) is regular if and only if its reduction $(\overline{M},\overline{q})$ is regular; see e.g. [Bae78, I Proposition 1.4].

We show in the next section that the most important properties of quadratic forms over *Henselian* local rings can be studied by looking at the reduced quadratic form over the residue field. We will work with local rings with *finite* residue fields of odd characteristic. So, we quickly recall the structure of the Witt ring of $\mathbb{F}_q$ for an odd q; see e.g. [Lam73, Corollary 3.6]

$$\mathrm{Wq}(\mathbb{F}_q) = \begin{cases} \mathbb{Z}/2[(\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2], & q \equiv 1 \pmod 4, \\ \mathbb{Z}/4, & q \equiv 3 \pmod 4. \end{cases}$$

We have the two ring maps

$$\begin{aligned} d\colon\ & \mathrm{Wq}(\mathbb{F}_q) \to \mathbb{Z}/2, \\ & \mathrm{cls}\,(M,q) \mapsto \mathrm{rank}_A\ M \,(\mathrm{mod}\ 2) \end{aligned} \tag{2.1}$$

$$\begin{aligned} \mathrm{Arf}\colon\ & \mathrm{Wq}(\mathbb{F}_q) \to (\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2 \\ & \mathrm{cls}\,(M,q) \mapsto (-1)^{\mathrm{rank}_A\ M(\mathrm{rank}_A\ M-1)/2}\,\mathrm{discr}((M,q)). \end{aligned} \tag{2.2}$$

We will refer to the latter invariant as the *Arf invariant*. Here the discriminant is the determinant of the matrix $\lambda_q(x_i,x_j)$ for a basis (x_i) of M; it is determined up to $(\mathbb{F}_q^\times)^2$. Together, these two maps completely describe $\mathrm{Wq}(\mathbb{F}_q)$. Similar to [Shi80], we denote by Λ the element of $\mathrm{Wq}(\mathbb{F}_q)$ with $d(\Lambda) = 0$ and $(\mathrm{Arf}(\Lambda)/\mathbb{F}_q) = -1$. Here $(\,*\,/\mathbb{F}_q)$ denotes the Legendre symbol for the finite field $\mathbb{F}_q$; see e.g. [Shi80, Section 1.14]. For the finite field $\mathbb{F}_p$, with p a prime, it coincides with the Kronecker symbol.

In the next section, we show that for a Henselian local ring A with residue field k, the reduction map induces an isomorphism of Witt rings $\mathrm{Wq}(A) \cong \mathrm{Wq}(k)$; see Lemma 2.14. Consequently, many properties of quadratic forms over finite fields hold for Henselian local rings with finite residue fields as well.

§.2.1. Quadratic Forms over Henselian Rings

In the last section, we used the lifting properties in complete local rings to ensure the existence of minimal polynomials of certain automorphisms. In this section, we collect some properties of quadratic forms over Henselian rings. Recall that complete and separated rings are Henselian. Later, we will be concerned with complete rings with finite residue fields of odd characteristic. Our primary example is $\mathbb{Z}/p^l$ for an odd prime p and some integer $l \in \mathbb{Z}_{\geq 1}$.

In the proof of our classification theorem of conjugacy classes, we will use the Fundamental Theorem of Galois Theory for Henselian local rings below. The Galois theory of these rings is more or less controlled by the Galois theory of their residue fields. Recall that a *connected* ring is a ring with no non-trivial idempotent elements (i.e. not equal to 1 and 0). In the proof, we use some general facts about Henselian local rings from [For17]. For a ring A, we denote the set of idempotent elements by $\mathrm{idemp}(A)$.

Proposition 2.4 (*Fundamental Theorem of Galois Theory).*
Let A be a Henselian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let E be a connected, separable A-algebra, finitely generated as an A-module such that for $G = \mathrm{Aut}_A(E)$, $E^G = A$. Then G is finite, and there is an injective order-inverting correspondence of subgroups of G and subrings B of E which contain A and are separable over A.

In particular, all the rings involved can be assumed to be local and Henselian.

Proof. The first part of the proposition is the Fundamental Theorem of Galois Theory and holds for any connected ring; see [For17, Theorem 12.5.4] or [DI71, II §1 Theorem 1.1].

We only need to prove that all the rings involved can be assumed to be Henselian local.

First note that since E is connected, $|\mathrm{idemp}(E)| = 2$. By [For17, Proposition 10.1.6], we have an isomorphism of sets of idempotents

$$\mathrm{idemp}(E) \to \mathrm{idemp}(E \otimes_A k).$$

Thus, we get that $E \otimes_A k$ is connected. Since E is étale, we also get that $E \otimes_A k$ is a commutative separable k-algebra. Thus, we get that $E \otimes_A k$ is a separable field extension of k. This implies that E is local by [For17, Corollary 2.6.2]. Therefore, we also get that E is Henselian by [Sta25, Lemma 04GH].

If we have a subring B of E containing A and separable over A, the ring B is connected, so, by the above argument, it is local and Henselian. ▪

In the following, we quickly recall some basic facts relating quadratic forms over a ring with quadratic forms over its residue field; see e.g. [Bae78].

We need the fact that quadratic forms of rank at least 2 over a Henselian ring with maximal ideal $\mathfrak{m}$ and *finite* residue field of odd characteristic are universal, i.e. represent every unit in A. Therefore, we briefly recall some facts about hyperbolic forms.

Lemma 2.5 (*e.g. [Lam73, 1 Theorem 3.4]).*
Let k be a finite field of odd characteristic. Then a quadratic module (M, q) over k is isotropic, i.e. it contains an element $x \in M$ with $q(x) = 0$, if and only if it contains a hyperbolic plane.

Lemma 2.6 (*e.g. [Tay92, Theorem 11.2]).*
Let k be a finite field of odd characteristic. Let (M,q) be a regular quadratic space over k. Then if M is of dimension at least 3, q is isotropic.

Lemma 2.7 (*compare to [Bae78, V Lemma 1.4]).*
Let A be a Henselian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic. Let (M,q) be a regular quadratic module over A. Then, the reduction $(\overline{M},\overline{q})$ over k is isotropic if and only if (M,q) is isotropic.

We call a quadratic module (M,q) over a ring A *universal* if the quadratic form q represents every element of $A^\times$.

Lemma 2.8 (*compare to [Bae78, V Lemma 1.4]).*
Let A be a Henselian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic. Let (M,q) be a regular quadratic module over A. Then, if (M,q) is isotropic, it is universal.

Proof. Let $a \in A^\times$. Since (M,q) is isotropic, so is its reduction $(\overline{M},\overline{q})$ over k. By Lemma 2.5, $(\overline{M},\overline{q})$ contains a hyperbolic plane $(\overline{e},\overline{f})$. We find elements $e,f \in M$, whose residue classes modulo $\mathfrak{m}$ are $\overline{e},\overline{f}$, respectively. Then the polynomial

$$P := q(e)T^2 + \lambda(e,f)T + q(f) - a \in A[T]$$

satisfies

$$\overline{P} = T - \overline{a} \in k[T]$$

and $\overline{P}' = 1$. So, by Lemma 1.2, there exists an $x \in A$ such that $P(x) = 0$. Thus,

$$q(xe+f) = q(e)x^2 + \lambda(e,f)x + q(f) = P(x) + a = a.$$

▪

For a local ring A with residue field of *odd* characteristic, every regular quadratic module admits an orthogonal basis; see e.g. [Bae78, I Proposition 3.4] or [Sch85, Theorem 1.6.4]. For units $a_1,\dots,a_m \in A^\times$, we denote a quadratic form of rank m with orthogonal basis $\{x_1,\dots,x_n\}$ by $\langle a_1,\dots,a_m\rangle$ where a_i is the value of the quadratic form at x_i for all i, see e.g. [Bae78, I Proposition 3.4].

Lemma 2.9.
Let A be a Henselian ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic. Let (M, q) be a regular quadratic module over A. Then $a \in A^\times$ is represented by q if and only if $q \perp \langle -a \rangle$ is isotropic.

Proof. If a is represented by q, i.e. $a = q(x)$ for some $x \in M$, then $q(x) - a1^2 = 0$, so $q \perp \langle -a \rangle$ is isotropic.

Conversely, let $q \perp \langle -a \rangle$ be isotropic. Then there exist $x \in M$ and $b \in A$ such that

$$0 = q(x) - ab^2.$$

If $b \in A^\times$, then $a = q(b^{-1}m)$, i.e. a is represented by q. On the other hand, if b is not invertible, then $b \in \mathfrak{m}$. Then $\overline{q}$ is isotropic. By Lemma 2.7 and 2.8, q is universal, and, in particular, a is represented by q. ▪

For fields, the corresponding lemma can be found e.g. in [Lam05, Corollary 3.5].

Similarly, the next lemma is also well-known over finite fields; see e.g. [Tay92, Theorem 11.2].

Lemma 2.10.
Let A be a Henselian ring with maximal ideal $\mathfrak{m}$ and finite residue field of odd characteristic. Let (M, q) be a regular quadratic module over A of dimension $\dim_A(M) \geq 2$. *Then (M, q) is universal.*

Proof. Let $a \in A^\times$. Then $M \perp \langle -a \rangle$ is of rank at least 3, i.e. its reduction is isotropic by Lemma 2.6. Thus, q is universal by Lemma 2.9. ▪

Proposition 2.11 (*[Bae78, II Theorem 1.6]*).
Let A be a local ring. Then any quadratic separable A-algebra has the form

$$B = {}^{A[T]}/_{(T^2 - T - b)}$$

for an $b \in A$ with $1 + 4b^2 \in A^\times$.

We denote the image of T in B by t. Then, the (canonical) involution of B keeping A fixed is given by

$$\sigma \colon B \to B, \quad t \mapsto 1 - t.$$

Note that

$$\sigma(t^2) = 1 - t + b = \sigma(t + b).$$

We define the *norm map* $n\colon B \to A$ by $n(x) := x\sigma(x)$ for all $x \in B$. The quadratic module (B, n) is called *norm space*.

LEMMA 2.12.
Let A be a Henselian local ring with maximal ideal $\mathfrak{m}$ and finite residue field of odd characteristic. Then the norm space (B, n) over A is regular and universal.

Proof. The regularity of (B, n) follows from the regularity of the reduction. The universality follows from Lemma 2.10. ▪

LEMMA 2.13 (*Witt cancellation of local rings).*
Let A be a local ring, and let E, F, and G be regular quadratic modules over A. Then, if there exists an isometry $E \perp F \cong E \perp G$, this implies $F \cong G$.

We refer to [Bae78, IV Corollary 4.3] or [Kne69].

Using the fact that every regular quadratic module over a local ring with residue field of *odd* characteristic admits an orthogonal basis, see [Bae78, I Proposition 3.4] or [Sch85, Theorem 1.6.4], we can calculate the Witt ring if the ring is also Henselian.

LEMMA 2.14 (*see e.g. [Sch85, 6 Theorem 2.4]).*
Let A be a Henselian local ring with finite residue field k of odd characteristic. Then the reduction induces an isomorphism of Witt rings

$$\mathrm{Wq}(A) \to \mathrm{Wq}(k).$$

We reproduce the proof here for the convenience of the reader.

Proof. We proceed by induction on the rank of the quadratic modules. Consider two quadratic modules $\langle a_1, \dots, a_m \rangle$ and $\langle b_1, \dots, b_m \rangle$ of rank m, and suppose there is an isometry $\langle \overline{a}_1, \dots, \overline{a}_m \rangle \cong \langle \overline{b}_1, \dots, \overline{b}_m \rangle$ of quadratic forms over k. Thus there exist $x_i \in A$ such that

$$\overline{a}_1 \overline{x}_1^2 + \dots + \overline{a}_m \overline{x}_m^2 = \overline{b}_1$$

at least one of $\overline{x}_i$ is non-zero and thus $x_i \in A^{\times}$. Without loss of generality, we suppose $x_1 \in A^{\times}$. Then the polynomial

$$T^2 - a_1^{-1}(b_1 - a_2 x_2^2 - \cdots - a_m x_m^2)$$

has a (non-multiple) root in $x \in A$ by Lemma 1.2. Thus, the (free module of rank 1) Ax is a regular submodule of $\langle a_1, \dots, a_m \rangle$ since x is non-zero. We get an isometry

$$\langle a_1, \dots, a_m \rangle \cong \langle b_1, c_2, \dots, c_m \rangle.$$

By Witt's cancellation theorem 2.13, we obtain $\langle b_2, \dots, b_m \rangle \cong \langle c_2, \dots, c_m \rangle$. ▪

Remark 2.15.
The above lemma holds for 2-Henselian local rings (called *quadratisch Henselsch* in [Kne69]).

3. Conjugacy Classes in Orthogonal Groups

Let A be a Henselian local ring with maximal ideal $\mathfrak{m}$ and residue field k of odd characteristic. Let (M, q) be a regular quadratic module over A. In the previous chapter, we recalled that the Witt ring over A is isomorphic to the Witt ring over k. Now, we will study isometries of (M, q), i.e. A-linear automorphisms $\phi\colon M \to M$ such that $q \circ \phi = \phi$. The canonical reduction map on orthogonal groups

$$O_A(M, q) \to O_k(\overline{M}, \overline{q})$$

is surjective. For a proof using reflections, we refer to [Kli61] or [Kne69]. This fact was the starting point of this chapter, as isometries of quadratic forms over finite fields are well understood; see e.g. [Tay92] and [Shi80]. For example, one can explicitly construct representatives for conjugacy classes in orthogonal groups over finite fields with `Magma` [BCP97]; see the notes [Tay21].

Conjugacy classes of isometries of quadratic forms over finite fields $\mathbb{F}_q$ were considered in [Wal63b], and later in [SS70] and [Mil69]. We shall follow the general proof strategy of [SS70] as presented in [Shi80].

We will always assume that our automorphisms are of finite order, coprime to the characteristic of the residue field, since this ensures that they admit (separable) minimal polynomials.

The following fact underlies our discussion.

Lemma 3.1 (*[Knu91, II Lemma 4.5.3. 1)]*).
Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k, M, N be projective finitely generated A-modules with an A-module homomorphism $\phi\colon M \to N$. Then ϕ is an isomorphism if and only if the reduction $\overline{\phi}$ is an isomorphism.

We will determine the conjugacy classes of isometries of finite order for which we can guarantee the existence of a minimal polynomial by first reducing it to determining isometries of certain regular Hermitian forms. For their definition, we briefly introduce involution traces as in [Knu91].

§.3.1. Involution Traces and Self-Reciprocal Polynomials

Definition 3.2 (*involution trace, [Knu91, I Proposition 7.2.4]).*
Let A be a ring and B be a commutative A-algebra that is projective and finitely generated as an A-module with an involution $\sigma\colon B \to B$ restricting to the identity on A. Then an A-linear map $\tau\colon B \to A$ is called *involution trace* if it satisfies

(i.) $\tau \circ \sigma = \tau$,

(ii.) the symmetric bilinear form $\lambda_{B/A}\colon B \times B \to A$ defined by

$$\lambda_{B/A}(x, y) := \tau(\sigma(x)y)$$

is regular.

For the trivial involution σ, they are sometimes called *Frobenius extensions*; see e.g. [Bae78, Section I 2.8]

The following example of an involution trace underlies the results in [Mil69].

Example 3.3 (*[Sta25, Lemma 0BIL]).*
Let k be a field and F a finite separable extension of k. Then the trace map $\mathrm{trace}_{F/k}$ is an involution trace for the trivial involution if and only if it is not the zero map. If F is a Galois extension of k and $\sigma\colon F \to F$ is a non-trivial involution in the Galois group $\mathrm{Gal}(F/k)$, then the trace map $\mathrm{trace}_{F/k}$ is an involution trace for σ if and only if it is not the zero map.

We will see that separable polynomials which are self-reciprocal naturally yield involution traces. Let us quickly recall their definition.

Definition 3.4 (*reciprocal polynomial, see e.g. [Coh87, Section 2]).*
Let A be a ring and $P \in A[T]$ be a polynomial of degree d

$$P = \sum_{i=0}^{d} a_i T^i$$

such that $a_0 \in A^\times$. Then the *reciprocal polynomial* P^* of P is defined by

$$P^* := a_0^{-1} T^d P(T^{-1}) = a_0^{-1} \sum_{i=0}^{d} a_{d-i} T^i.$$

A monic polynomial $Q \in A[T]$ is called *self-reciprocal* if $Q = Q^*$.

EXAMPLE 3.5.
For the ring $A = \mathbb{Z}/(7^3)$, the reciprocal polynomial of

$$Q = T^3 + 64T^2 + 32T + 342$$

is given by

$$Q^* = T^3 + 311T^2 + 279T + 342.$$

Note that over the residue field $k = \mathbb{F}_7$, the reduction $\overline{Q}$ of Q is the irreducible polynomial

$$\overline{Q} = T^3 + T^2 + 4T + 6$$

with a reciprocal polynomial

$$\overline{Q}^* = T^3 + 3T^2 + 6T + 6,$$

which is also irreducible. For example, the polynomial

$$P = T^2 + 108T + 1$$

is a self-reciprocal polynomial over A.

PROPOSITION 3.6.
Let A be a local ring with maximal ideal $\mathfrak{m}$ and finite residue field k. Let $P \in A[T]$ be a self-reciprocal separable polynomial whose reduction $\overline{P} \in k[T]$ over k is irreducible. Then $E := A[T]/(P)$ is a commutative A-algebra, projective and finitely generated as an A-module with an involution $\sigma\colon E \to E$ given by $t \mapsto t^{-1}$. Then the trace map $\operatorname{trace}_{E/A}$ is an involution trace.

Proof. The trace map is A-linear. By Lemma 3.8 and Example 3.3, the trace form $(x, y) \mapsto \operatorname{trace}_{E/A}(\sigma(x)y)$ is regular. ▪

The polynomial needs to be self-reciprocal, as otherwise σ is not a well-defined involution on the algebra E.

§.3.2. Hermitian Forms over Henselian Local Rings

To classify conjugacy classes we will use Hermitian forms. Note that the famous results that over a finite field, all regular Hermitian forms of the same rank are equivalent, is used in Shinoda's determination of conjugacy classes; see [Shi80, Section 1.12]. We will prove a corresponding generalisation using the following result:

PROPOSITION 3.7 (*[Knu91, I Proposition 6.2.4]).*
Let (M, λ) be an ϵ-Hermitian space over a field k. Then (M, λ) has an orthogonal basis in the following cases:

(i.) The involution of k is non-trivial.

(ii.) The involution of k is trivial, the bilinear form is symmetric ($\epsilon = 1$) and $\mathrm{char}(k) \neq 2$.

Note that for a ring A with involution σ, the involution maps maximal ideals to maximal ideals, hence for a local ring, the involution of A induces an involution on the residue field (compare to [Knu91, II Section 4.2.5]).

LEMMA 3.8.
Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k with involution σ. Let (E, s) be a σ-sesquilinear module over A.

Then the adjoint $\mathrm{Adj}[s]\colon E \to E^\times$ *is an isomorphism if and only if the adjoint* $\mathrm{Adj}[\overline{s}]\colon \overline{E} \to \overline{E}^\times$ *is an isomorphism.*

Proof. Note that the quotient map $A \to k$ is a map of rings with involution; see above. By Lemma 3.1, the A-linear map $\mathrm{Adj}[s]$ is an isomorphism if and only if $\mathrm{Adj}[\overline{s}]$ is an isomorphism. ▪

LEMMA 3.9 (*lifting orthogonal decompositions over local rings).*
Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k of odd characteristic with involution σ. Let (M, λ) be a regular ϵ-Hermitian σ-sesquilinear module over A. Then an orthogonal decomposition

$$\overline{M} = \overline{E} \perp \overline{F}$$

over k where $\overline{E}$ is a (free) regular subspace of $\overline{M}$, can be lifted to M, i.e. there exists an orthogonal decomposition $M = E \perp F$ of M with regular ϵ-Hermitian σ-sesquilinear modules E and F such that

$$E/\mathfrak{m}E = \overline{E}, \quad F/\mathfrak{m}F = \overline{F}.$$

For the symmetric case, we refer to [Kne69, Lemma 1.1.4] or [Bae78, I Corollary 3.3]. The statement for Hermitian forms is a special case of [Knu91, II Theorem 4.6.1] (note that since 2 is invertible, every Hermitian module is even).

Corollary 3.10.
Let A be a local ring with residue field k of odd characteristic with an involution. Let (M, λ) be a regular ϵ-Hermitian module over A. If either the reduced involution on k is non-trivial or $\epsilon = 1$, then (M, λ) admits an orthogonal basis.

Proof. By Proposition 3.7, the reduction $\overline{M}$ over k has an orthogonal basis. By the Lifting Lemma 3.9, an orthogonal basis of a ϵ-Hermitian module over k can be lifted to an orthogonal basis over A. ▪

For modules with symmetric bilinear form, the above is stated in [Sch85, Theorem 1.6.4].

The proof idea for the following theorem is based on [Spr22, Section 9 (3)]. Let A be a ring with an étale algebra E over A. Note that by [FL14, Proposition 2.9], a module over E that is finitely generated and projective as a module over A, is also projective over E. This will be implicitly used in the proof of the following theorem.

Theorem 3.11 (*equivalence of Hermitian forms over Henselian rings).*
Let A be a Henselian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic. Let E be a connected, separable A-algebra with involution $\sigma \in \operatorname{Aut}_A(E)$. Let M be a finitely generated, projective module over E and $s, t\colon M \times M \to E$ regular ϵ-Hermitian σ-sesquilinear forms.

If the reduction of the involution $\overline{\sigma} \in \operatorname{Aut}_k(\overline{E})$ is non-trivial, then s and t are isometric.

Proof. Note that both (M, s) and (M, t) admit an orthogonal basis by Corollary 3.10, i.e. there exist bases (e_i), (e_i') of M such that

$$s(e_i, e_j) = \eta_i \delta_{i,j}, \quad t(e_i', e_j') = \xi_i \delta_{i,j}$$

for some $\eta_i, \xi_i \in E$. Note that since s and t are σ-sesquilinear,

$$\eta_i = \sigma(\eta_i), \quad \xi_i = \sigma(\xi_i)$$

for all i. Furthermore, if the involution is non-trivial, it has by Proposition 2.4 a fixed subring E^σ of E, such that E is a quadratic separable E^σ-algebra with Galois group $\{\mathrm{id}, \sigma\}$. Since the norm map for E/E^σ is a regular quadratic form over a Henselian ring with finite residue field of odd characteristic, it is universal by Lemma 2.12.

Since E is regular, we have $\eta_i, \xi_i \in E^\times$ for all i. Thus, by the universality of the norm space, there exist $\alpha_i, \beta_i \in E$ such that

$$\sigma(\alpha_i)\alpha_i = \eta_i^{-1}, \quad \sigma(\beta_i)\beta_i = \xi_i^{-1}.$$

Thus, we obtain

$$s(\alpha_i e_i, \alpha_j e_j) = \delta_{i,j}, \quad t(\beta_i e_i', \beta_j e_j') = \delta_{i,j}.$$

We conclude that (M, s) and (M, t) are isometric. ▪

If the reduction $\overline{\sigma}$ is trivial and $\epsilon = 1$, then we also obtain that (M, s) and (M, t) are isometric if for every i there exists a bijective map between the square classes of η_i and ξ_i modulo $(E^\times)^2$.

§.3.3. Conjugacy Classes in Orthogonal Groups

In this section, we generalise many ideas from the first chapter of [Shi80] to Henselian local rings with finite residue fields of odd characteristic. In the end, we specialise to Artinian local rings with finite residue fields, for which we established the existence of minimal polynomials of automorphisms of finite order coprime to the characteristic; see Theorem 1.16. We will use the minimal polynomial to classify isometries of quadratic modules up to conjugacy.

Let A be an Artinian local ring with finite residue field k of characteristic p. In this section, we will use the existence of minimal polynomials of isometries of finite order coprime to the characteristic p to classify them up to conjugacy.

Let A be a ring and M be a finitely generated projective A-module with two automorphisms $\phi, \psi \colon M \to M$. We will say that ϕ and ψ are *similar* if they are conjugate in the general linear group $\mathrm{GL}_n(A)$, i.e. if there is an automorphism $\tau\colon M \to M$ such that $\tau \circ \phi = \psi \circ \tau$. In fact, this implies that $\mathrm{minpoly}_\phi = \mathrm{minpoly}_\psi$ by the following fact.

LEMMA 3.12.
Let A be a ring and M be a finitely generated projective A-module with two automorphisms $\phi, \psi\colon M \to M$. Suppose ϕ and ψ admit minimal polynomials $\mathrm{minpoly}_\phi$ *and* $\mathrm{minpoly}_\psi$*, respectively. Then, if ϕ and ψ are similar, their minimal polynomials are equal.*

Proof. Let $x \in M$. Then, the equalities

$$\mathrm{minpoly}_\phi(\psi)[\tau(x)] = \tau[\mathrm{minpoly}_\phi(x)] = 0,$$
$$\tau(\mathrm{minpoly}_\psi(\phi)[x]) = \mathrm{minpoly}_\psi(\psi)[\tau(x)] = 0$$

imply

$$\mathrm{minpoly}_\phi \mid \mathrm{minpoly}_\psi, \quad \mathrm{minpoly}_\psi \mid \mathrm{minpoly}_\phi .$$

∎

Of course, if two isometries of a regular quadratic module are conjugate in the orthogonal group of that regular quadratic module, they are also similar.

Using Nakayama's lemma, we can show that automorphisms with the same minimal polynomial, which is *irreducible* over the residue field, are similar.

LEMMA 3.13.
Let A be a local ring with maximal ideal $\mathfrak{m}$ and finite residue field k. Let M be a projective finitely generated module over A and $\phi, \psi\colon M \to M$ two A-linear automorphisms with monic separable irreducible minimal polynomial

$$Q := \mathrm{minpoly}_\phi = \mathrm{minpoly}_\psi$$

of degree d. Then ϕ and ψ are similar.

Proof. We will show that M_ϕ and M_ψ both are the direct sum of m cyclic modules over $A[T]$ with the same annihilator and there exist bases of the free module M such that ϕ and ψ represent the same matrix; therefore, they are conjugate; see

[AW92, Corollary 3.25]. As before, we denote the reduction of the module M modulo $\mathfrak{m}$ by $\overline{M}$. Let $E = A[T]/(Q)$. By [Bou90, VII §5 Proposition 4], there exist elements $\overline{x}_1, \ldots, \overline{x}_m$ in $\overline{M}$ such that

$$\overline{M}_{\overline{\phi}} = k[T].\overline{x}_1 \oplus \cdots \oplus k[T].\overline{x}_m$$

as a $k[T]$-module with

$$k[T] \neq \mathrm{Ann}_{k[T]}(\overline{x}_1) \supseteq \mathrm{Ann}_{k[T]}(\overline{x}_2) \supseteq \\ \cdots \supseteq \mathrm{Ann}_{k[T]}(\overline{x}_m) = \mathrm{Ann}_{k[T]}(\overline{M}_{\overline{\phi}}) = (\overline{Q}).$$

Note that since $\overline{Q}$ is irreducible this already implies that

$$\mathrm{Ann}_{k[T]}(\overline{x}_1) = \mathrm{Ann}_{k[T]}(\overline{x}_2) = \cdots = \mathrm{Ann}_{k[T]}(\overline{x}_m) = (\overline{Q}).$$

Choose an $x_i \in M$ with reduction $\overline{x_i}$ modulo $\mathfrak{m}$. Then, we have the inclusion

$$\mathrm{Ann}_{A[T]}(x_i) \supseteq (Q).$$

By Nakayama's lemma, we have the E-linear isomorphism $E \to Ex_i$, thus $\mathrm{Ann}_{A[T]}(x_i) = (Q)$ and M is the sum of m invertible (projective of rank 1) modules over E. The sum is direct, again by Nakayama's lemma. The same argument applies to ψ. ▪

Note that above, we have implicitly shown that there exists bases of M such that ϕ and ψ take the form $C(Q) \oplus \cdots \oplus C(Q)$ (m copies) for the companion matrix $C(Q)$ of the monic polynomial $Q = \sum_{i=0}^{d} a_i T^i$

$$C(Q) = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & 0 & \ldots & 0 & -a_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & -a_{d-1} \end{bmatrix} \in M_d(A).$$

The proof was inspired by [FL14, Proposition 5.3].

Let A be an Artinian local ring with finite residue field of odd characteristic p. Let (M, q) be a regular quadratic module over A, i.e. M is a finitely generated projective A-module and $q\colon M \to A$ is a quadratic form such that the associated bilinear form $\lambda_q\colon M \times M \to A$ is regular. Since A is local, M is free, and we denote its dimension by $n := \dim_A(M)$. Let $\phi\colon (M,q) \to (M,q)$ be an isometry of order N such that $p \nmid N$. By Theorem 1.16, ϕ has a separable minimal polynomial $\mathrm{minpoly}_\phi \in A[T]$. By the next lemma, $\mathrm{minpoly}_\phi$ is *self-reciprocal.*

The following lemma is an analogue of [Mil69, Lemma 1.2] for general rings.

LEMMA 3.14.
Let A be a ring and (M,q) be a regular quadratic module over A with an isometry $\phi\colon (M,q) \to (M,q)$. If ϕ has a minimal polynomial $\mathrm{minpoly}_\phi \in A[T]$*, it is self-reciprocal, i.e.*

$$\mathrm{minpoly}_\phi^* = \mathrm{minpoly}_\phi .$$

Proof. We denote the degree of the minimal polynomial $\mathrm{minpoly}_\phi$ by d. For any polynomial $Q \in A[T]$, we get by the bilinearity of the associated bilinear form λ_q

$$\lambda_q(Q(\phi)x, y) = \lambda_q(x, Q(\phi^{-1})y)$$

for all $x, y \in M$. If we apply this equality to the minimal polynomial $\mathrm{minpoly}_\phi$, we get that $\mathrm{minpoly}_\phi(\phi^{-1})$ is the zero map since λ_q is regular. Thus, we get that

$$\mathrm{minpoly}_\phi(T) \mid T^d\, \mathrm{minpoly}_\phi(T^{-1}).$$

Since they are monic and of the same degree d, they are equal. ▪

The proof is inspired by the proof of [Mil69, Lemma 3.1]; see also the notes [Tay21].

We recall from [Bou98, Section III §4] that two elements a, b of a ring A are called *strongly relatively coprime* if the principal ideals (a) and (b) are relatively prime, i.e. if

$$(a) + (b) = A.$$

In other words, there exist elements $r, s \in A$ such that $ra + sb = 1$.

PROPOSITION 3.15 (*[Bou98, III §4 Proposition 1]).*
Let A be a ring and P and Q be two relatively prime polynomials in $A[T]$ where

P is monic. Then every polynomial X in $A[T]$ can be written uniquely in the form

$$X = RP + SQ$$

where $R, S \in A[T]$ and $\deg S < \deg P$. If further $\deg X \leq d$ for some $d \in \mathbb{Z}_{\geq 0}$ and $\deg S \leq d - \deg P$, then $\deg R \leq d - \deg P$.

Lemma 3.16 (*compare to [Bou98, III §4 Proposition 2]).*
Let A be a local ring with finite residue field k. Let $P, Q \in A[T]$ be monic polynomials. Then, P and Q are strongly relatively coprime in $A[T]$ if and only if their reductions $\overline{P}$ and $\overline{Q}$ in $k[T]$ are relatively prime.

Proof. Suppose P and Q are strongly relatively coprime in $A[T]$. Then there exist polynomials R and S in $A[T]$ such that $1 = RP + SQ$. Thus, we can conclude

$$1 = \overline{R}\,\overline{P} + \overline{S}\,\overline{Q}$$

in $k[T]$.

Conversely, let X be a polynomial in $A[T]$ then we get by the above lemma

$$\overline{X} = \overline{R}\,\overline{P} + \overline{S}\,\overline{Q}$$

for some polynomials $R, S \in A[T]$. Therefore $A[T] = (P, Q) + \ker(f)A[T]$. Now, $A[T]/(P, Q)$ is a finitely generated A-module since every polynomial X is modulo P congruent to a polynomial of degree less than $\deg P$. Since

$$\mathfrak{m}(A[T]/(P, Q)) = k[T]/(\overline{P}, \overline{Q}),$$

we get by Nakayama's lemma that $A[T] = (P, Q)$. ▪

Note that for an A-module M, we define for an element $a \in A$, the *a-annihilated* module by $M_a := \{m \in M \colon am = 0\}$.

Lemma 3.17.
Let A be a ring and let M be a torsion A-module with $\operatorname{Ann}_A(M) = (a)$. Let $a = bc$ be a decomposition of a in pairwise strongly relatively coprime elements in A. Then M admits a direct sum decomposition into annihilated submodules

$$M = M_b \oplus M_c.$$

Proof. There exist elements r and s in A such that

$$1 = rb + sc.$$

Thus, for every $m \in M$ we get

$$m = rbm + scm = m_1 + m_2$$

with $m_1 := rbm$ and $m_2 := scm$. Note that

$$cm_1 = 0 = bm_2.$$

In other words, $M_b + M_c = M$.

Now, let $n \in M_b \cap M_c$, i.e. $bn = 0 = cn$. Then

$$\mathrm{Ann}_A(n) = A,$$

i.e. $n = 0$. ▪

Now, we apply the above to separable minimal polynomials over Henselian local rings.

LEMMA 3.18.

Let A be a Henselian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic p. Let M be a projective finitely generated module over A and $\phi\colon M \to M$ be an A-linear automorphism of finite order N such that $p \nmid N$ with minimal polynomial $\mathrm{minpoly}_\phi$*. Then there exists a unique factorisation (up to relabelling the polynomials P_i and Q_j)*

$$\mathrm{minpoly}_\phi = P_1 \cdot \dots \cdot P_m Q_1 Q_1^* \dots Q_n Q_n^* \in A[T],$$

where

(i.) $\{P_i\}$ are pairwise strongly relatively coprime polynomials over A such that

- *P_i is monic and self-reciprocal,*
- *the reduction $\overline{P}_i \in k[T]$ is irreducible with order $\mathrm{ord}(\overline{P}_i)$ coprime to p;*

(ii.) $\{Q_j Q_j^*\}$ are pairwise strongly relatively coprime polynomials over A such that

* *Q_j is monic and non-self-reciprocal,*
* *the reduction $\overline{Q}_j \in k[T]$ is irreducible with order $\operatorname{ord}(\overline{Q}_j)$ coprime to p.*

Then, the module M admits the decomposition into annihilated, free modules

$$M = \bigoplus_{i=1}^{m} M_{P_i} \oplus \bigoplus_{j=1}^{n} M_{Q_j}$$

such that the restrictions of ϕ

$$\phi|_{M_{P_i}}, \quad \phi|_{M_{Q_j}}$$

have minimal polynomials P_i and Q_j, respectively.

Proof. Note that the reduction $\overline{\mathrm{minpoly}_\phi}$ of $\mathrm{minpoly}_\phi$ to $k[T]$ can be decomposed in strongly relatively coprime irreducible factors

$$\overline{\mathrm{minpoly}_\phi} = \overline{P}_1 \cdot \dots \cdot \overline{P}_m \overline{Q}_1 \overline{Q}_1^* \dots \overline{Q}_n \overline{Q}_n^*.$$

Since A is Henselian and $\mathrm{minpoly}_\phi$ is separable, we can lift the above decomposition to A. Note that since $\mathrm{minpoly}_\phi$ is self-reciprocal, $Q_i \mid \mathrm{minpoly}_\phi$ also implies $Q_i^* \mid \mathrm{minpoly}_\phi$.

Since $\mathrm{Ann}_{A[T]}(M) = (\mathrm{minpoly}_\phi)$, the module M admits the decomposition

$$M = \bigoplus_{i=1}^{m} M_{P_i} \oplus \bigoplus_{j=1}^{n} M_{Q_j}.$$

Note that, since the above decomposition is direct, and direct summands of free modules are projective, the direct summands are also free (since projective implies free for local rings). ▪

Lemma 3.19 (*compare to [Mil69, Lemma 3.1]*).
Let A be a Henselian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic p. Let (M, q) be a regular quadratic module over A and $\phi\colon (M,q) \to (M,q)$ be an isometry of finite order N such that $p \nmid N$ with minimal polynomial $\mathrm{minpoly}_\phi$. *Let*

$$\mathrm{minpoly}_\phi = P_1 \cdot \dots \cdot P_m Q_1 Q_1^* \dots Q_n Q_n^* \in A[T]$$

be the factorisation from Lemma 3.18 where *(i.)* $\{P_i\}$ *are pairwise strongly relatively coprime polynomials over* A *such that*

- $*$ P_i *is monic and self-reciprocal,*
- $*$ *the reduction* $\overline{P}_i \in k[T]$ *is irreducible with order* $\operatorname{ord}(\overline{P}_i)$ *coprime to* p*;*

(ii.) $\{Q_j Q_j^*\}$ *are pairwise strongly relatively coprime polynomials over* A *such that*

- $*$ Q_j *is monic and non-self-reciprocal,*
- $*$ *the reduction* $\overline{Q}_j \in k[T]$ *is irreducible with order* $\operatorname{ord}(\overline{Q}_j)$ *coprime to* p.

Then, the decomposition from Lemma 3.18

$$M = \bigoplus_{i=1}^{m} M_{P_i} \oplus \bigoplus_{j=1}^{n} M_{Q_j}$$

yields the orthogonal decomposition in regular quadratic modules

$$M = \left(\perp_{i=1}^{m} M_i^{(I)} \right) \perp \left(\perp_{j=1}^{n} M_j^{(II)} \right)$$

where $M_i^{(I)} := M_{P_i}$ *and* $M_j^{(II)} := M_{Q_j} \oplus M_{Q_j^*}$.

Proof. We view M as an $A[T]$-module where T acts as ϕ on M. For all $x, y \in M$, we get

$$\lambda_q(x, Ty) = \lambda_q(x, \phi(y)) = \lambda_q(\phi^{-1}x, y) = \lambda_q(T^{-1}x, y).$$

Thus, for $y \in M_{P_i}$ for $P_i \in A[T]$ as in the factorisation from Lemma 3.18,

$$0 = \lambda_q(x, P_i(T)y) = \lambda_q(P_i(T^{-1})x, y),$$

so M_{P_i} is orthogonal to $P(T^{-1})M$. For every polynomial $R \in A[T]$ strongly relatively coprime to P_i, the map

$$M_R \to M_R, \quad x \mapsto T^{-d} P_i(T) x$$

defines an isomorphism, hence $M_{P_i} \perp M_R$.

Similarly, we get that M_{Q_j} is totally isotropic for all $Q_j \in A[T]$ in the factorisation above. We can orthogonally split off $M_Q \oplus M_Q^*$. ▪

Let A be a Henselian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic. Let (M,q) be a regular quadratic module over A and $\phi\colon (M,q) \to (M,q)$ be an isometry of order coprime to the characteristic $\mathrm{char}(k)$ with minimal polynomial $\mathrm{minpoly}_\phi$. With the lemma above, we reduced the problem of determining conjugacy classes to the following two cases:

Type I: $\mathrm{minpoly}_\phi = P \in A[T]$ is self-reciprocal and the reduction $\overline{P}$ is irreducible over $k[T]$.

Type II: $\mathrm{minpoly}_\phi = QQ^* \in A[T]$ with $Q \neq Q^*$ and the reduction $\overline{Q}$ is irreducible over $k[T]$.

Type I: Isometries with Self-Reciprocal Minimal Polynomial Irreducible over the Residue Field

Remark 3.20.
Let k be a finite field of odd characteristic. Let $P \in k[T]$ be a monic, irreducible, self-reciprocal polynomial over k. If $\deg P > 1$, then the degree of P is even and $P(0) = 1$; see e.g. [Shi80, Lemma 1.7.3].

Let A be a local ring and (M,q) be a regular quadratic module over A. We consider an isometry $\phi\colon (M,q) \to (M,q)$ with a self-reciprocal irreducible minimal polynomial $P \in A[T]$. The subalgebra of A-module endomorphisms of M generated by ϕ is isomorphic to the algebra $E_\phi = E := A[T]/(P)$. We denote the image of T in E by t. Since the polynomial P is self-reciprocal, we get the involution

$$\sigma\colon E \to E, \quad t \mapsto t^{-1}.$$

Furthermore, the algebra E is separable and free over A with basis $\{1, t, \dots, t^{d-1}\}$. By Proposition 3.6, the étale algebra E admits an involution trace

$$\mathrm{trace}_{E/A}\colon E \to A$$

for the involution σ.

We denote M considered as a E_ϕ-module by M_ϕ, i.e. an element $\sum_{i=0}^{d-1} a_i t^i \in E_\phi$ acts as on an element $x \in M$ by

$$\sum_{i=0}^{d-1} a_i \phi^i(x).$$

LEMMA 3.21 (*compare to [Mil69, Lemma 1.1]).*
Let A be a local ring and (M, q) be a regular quadratic module over A.
Let $\phi\colon (M, q) \to (M, q)$ be an isometry with self-reciprocal irreducible minimal polynomial $P \in A[T]$ and étale algebra $E = A[T]/(P)$ over A with involution $\sigma\colon E \to E,\ t \mapsto t^{-1}$ and involution trace $\mathrm{trace}_{E/A}\colon E \to A$. There exists a unique regular 1-Hermitian σ-sesquilinear form

$$s = s_\phi\colon M_\phi \times M_\phi \to E$$

such that, for all $u \in E$ and all $x, y \in M$,

$$\lambda_q(ux, y) = \mathrm{trace}_{E/A}(us(x, y)).$$

Proof. Since $\mathrm{trace}_{E/A}$ is an involution trace for σ by Proposition 3.6, we get for any $x, y \in M$ an A-linear map

$$E \to A, \quad u \mapsto \lambda_q(ux, y)$$

i.e. there exists a unique element $s(x, y) \in E$ such that for all $u \in E$

$$\mathrm{trace}_{E/A}(us(x, y)) = \lambda_q(ux, y).$$

Now, the map s defined by

$$s\colon M \times M \to E, \quad (x, y) \mapsto s(x, y)$$

is a 1-Hermitian form for the involution σ:

(i.) The biadditivity follows from the A-linearity of $\mathrm{trace}_{E/A}$ and the biadditivity of λ_q.

(ii.) The equality

$$\mathrm{trace}_{E/A}(us(vx, y)) = \lambda_q(uvx, y) = \mathrm{trace}_{E/A}(uvs(x, y))$$

for all $u \in E$ implies $s(vx, y) = vs(x, y)$ for all $v \in E$ and $x, y \in M$.

(iii.) Similarly, the equality

$$\begin{aligned}
\mathrm{trace}_{E/A}(us(x, y)) &= \lambda_q(ux, y) \\
&= \lambda_q(y, ux) \\
&= \lambda_q(\sigma(u)y, x) \\
&= \mathrm{trace}_{E/A}(\sigma(u)s(y, x)) \\
&= \mathrm{trace}_{E/A}(u\sigma(s(y, x)))
\end{aligned}$$

for all $u \in E$ implies $s(x,y) = \sigma(s(y,x))$ for all $x, y \in M$.

▪

The following lemma is an analogue of [SS70, Proposition IV.2.7] for local rings; compare to [Mil69, Lemma 1.1].

LEMMA 3.22.
Let A be a local ring and (M,q) be a regular quadratic module over A. Let $\phi\colon (M,q) \to (M,q)$ be an isometry with self-reciprocal irreducible minimal polynomial $P \in A[T]$ and étale algebra $E = A[T]/(P)$ over A with involution $\sigma\colon E \to E,\ t \mapsto t^{-1}$ and involution trace $\operatorname{trace}_{E/A}\colon E \to A$. Let (M_ϕ, s_ϕ) be the regular 1-Hermitian σ-sesquilinear from Lemma 3.21 with

$$\lambda_q(ux, y) = \operatorname{trace}_{E/A}(us(x,y))$$

for all $u \in E$ and $x, y \in M$.

There is a bijection between isometries $\tau\colon (M,q) \to (M,q)$ such that $\psi \circ \tau = \tau \circ \phi$ and the set of isometries $f\colon (M_\phi, s_\phi) \to (M_\psi, s_\psi)$, i.e. E-linear isomorphism $f\colon M_\phi \to M_\psi$ such that $s_\psi(f(x), f(y)) = s_\phi(x,y)$ for all $x, y \in M$.

Proof. Let $\tau\colon (M,q) \to (M,q)$ be an isometry such that $\psi \circ \tau = \tau \circ \phi$. Then τ is a E-linear map from M_ϕ to M_ψ since for all elements $u = \sum_{i=0}^{d-1} a_i t^i \in E_\phi$

$$\tau(ux) = \tau(\sum_{i=0}^{d-1} a_i \phi^i x) = \sum_{i=0}^{d-1} a_i \psi^i (\tau x).$$

Then, we get for all $u = \sum_{i=0}^{d-1} a_i t^i \in E$ and $x, y \in M$

$$\begin{aligned}
\operatorname{trace}_{E/A}(us_\psi(\tau x, \tau y)) &= \lambda_q(\sum_{i=0}^{d-1} a_i \psi^i \tau x, \tau y) \\
&= \lambda_q(\tau \sum_{i=0}^{d-1} a_i \phi^i x, \tau y) \\
&= \lambda_q(\sum_{i=0}^{d-1} a_i \phi^i x, y) \\
&= \operatorname{trace}_{E/A}(us_\phi(x,y)).
\end{aligned}$$

Conversely, if we get a E-linear map $f\colon M_\phi \to M_\psi$ such that for all $x, y \in M_\phi$

$$s_\psi(f(x), f(y)) = s_\phi(x,y),$$

we get for all $x \in M$

$$f(\phi(x)) = f(tx) = tf(x) = \psi(f(x)),$$

so $f \circ \phi = \psi \circ f$. Furthermore, f is an isometry since

$$\begin{aligned} \lambda_q(fx, fy) &= \operatorname{trace}_{E/A}(s_\psi(fx, fy)) \\ &= \operatorname{trace}_{E/A}(s_\phi(x, y)) \\ &= \lambda_q(x, y). \end{aligned}$$

▪

Lemma 3.23.
Let A be a Henselian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic. Let (M, q) be a regular quadratic module over A and $\phi, \psi \colon (M, q) \to (M, q)$ be two isometries with the same minimal polynomial

$$P = \operatorname{minpoly}_\phi = \operatorname{minpoly}_\psi$$

for a separable self-reciprocal polynomial $P \in A[T]$ such that $\overline{P}$ is irreducible in $k[T]$. Then ϕ and ψ are conjugate, i.e. there exists an $\tau \colon (M, q) \to (M, q)$ such that $\tau \circ \phi = \psi \circ \tau$.

Proof. We denote $E := A[T]/(P)$ with involution $\sigma \colon E \to E,\ t \mapsto t^{-1}$. Since $\overline{P}$ is irreducible, E is local and Henselian.

There exists an A-linear automorphism $\alpha \colon M \to M$ such that $\alpha \circ \phi = \psi \circ \alpha$ by Lemma 3.13. Thus, we get an $A[T]$-linear automorphism $\alpha \colon M_\phi \to M_\psi$. The pullback $(M_\phi, \alpha^* s_\psi)$ is a regular 1-Hermitian σ-sesquilinear form such that

$$\alpha \colon (M_\phi, \alpha^* s_\psi) \to (M_\psi, s_\psi)$$

defines an isometry.

Now, (M_ϕ, s_ϕ) and $(M_\phi, \alpha^* s_\psi)$ are two regular 1-Hermitian σ-sesquilinear forms and thus isometric by Theorem 3.11, i.e. there exists an isometry

$$f \colon (M_\phi, s_\phi) \to (M_\phi, \alpha^* s_\psi).$$

Thus, we get the isometry $\alpha \circ f \colon (M_\phi, s_\phi) \to (M_\psi, s_\psi)$, so ϕ and ψ are conjugate as isometries over (M, q) by Lemma 3.22. ▪

Type II: Isometries with Non-Self-Reciprocal Minimal Polynomial Irreducible over the Residue Field

This case is analogous to Case 3 in [Mil69, Section 3].

Lemma 3.24.
Let A be a Henselian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic. Let (M, q) be a regular quadratic module over A and $\phi, \psi \colon (M, q) \to (M, q)$ be two isometries with minimal polynomial

$$QQ^* = \mathrm{minpoly}_\phi = \mathrm{minpoly}_\psi$$

for a separable $Q \in A[T]$ which is not self-reciprocal such that $\overline{Q}$ is irreducible in $k[T]$. Then ϕ and ψ are conjugate, i.e. there exists an $\tau \colon (M, q) \to (M, q)$ such that $\tau \circ \phi = \psi \circ \tau$.

Proof. First note that since $\overline{Q}$ is irreducible, Q and Q^* are strongly relatively prime in A, so as an $A[T]$-module M splits as

$$M = M_Q \oplus M_{Q^*}.$$

Now, M_Q and M_{Q^*} are ϕ invariant. The restriction ϕ_1 to M_Q has minimal polynomial Q since $\mathrm{minpoly}_{\phi_1} \mid Q$ and Q is separable. Note that (M, q) has a totally isotropic submodule M_Q with $(M_Q)^\perp = M_Q$. Thus, M is hyperbolic and, by [Bae78, I Theorem 4.6], we can find a dual basis, i.e. $\{x_i\} \subset M_Q$ and $\{y^i\} \subset M_{Q^*}$ such that for all i, j

$$\lambda_q(x_i, y^j) = \delta_{i,j}.$$

In this basis, ϕ can be written as

$$\phi(x_i) = \sum_{j=1}^m X^\phi_{i,j} x_j, \quad \phi^{-1}(x_i) = \sum_{j=1}^m Y^\phi_{i,j} x_j$$

for a matrix $X^\phi \in M_m(A)$ with inverse $Y^\phi := (X^\phi)^{-1} \in M_m(A)$. Thus, we get by

$$\lambda_q(x_i, \phi(y_j)) = \lambda_q(\phi^{-1}(x_i), y_j) = Y^\phi_{i,j}$$

that

$$\phi(y_j) = \sum_{i=1}^{m} Y_{i,j}^{\phi} y_i.$$

In the dual basis $(\{x_i\}, \{y_j\})$ of M, ϕ and ψ take thus the forms

$$\begin{bmatrix} X^{\phi} & 0 \\ 0 & ((X^{\phi})^{-1})^t \end{bmatrix}, \quad \begin{bmatrix} X^{\psi} & 0 \\ 0 & ((X^{\psi})^{-1})^t \end{bmatrix}.$$

Since the restrictions $\phi|_{M_Q}$ and $\psi|_{M_Q}$ to M_Q of ϕ and ψ have the same irreducible minimal polynomial Q, they are conjugate by Lemma 3.13, hence there exists an $R \in M_m(A)$ with $RX^{\phi} = X^{\psi}R$. Then

$$\begin{bmatrix} R & 0 \\ 0 & (R^t)^{-1} \end{bmatrix}$$

defines an isometry $\tau\colon (M, q) \to (M, q)$ such that $\tau \circ \phi = \psi \circ \tau$. ▪

We also record for later use that if ϕ is an isometry with minimal polynomial of the above form, the corresponding Witt class of (M, q) is trivial because it is hyperbolic.

Combining the lemmas above, we can prove the following lemma which is closely related to [Mil69, Section 3].

Lemma 3.25 (*decomposing using the minimal polynomial).*
Let A be an Artinian local ring with finite residue field k of odd characteristic p. Let (M, q) be a regular quadratic module over A. Then an isometry ϕ of finite order N coprime to p has a separable minimal polynomial $\mathrm{minpoly}_{\phi}$.

Let

$$\mathrm{minpoly}_{\phi} = P_1 \cdot \dots \cdot P_m Q_1 Q_1^* \dots Q_n Q_n^* \in A[T]$$

be the factorisation from Lemma 3.18 where

(i.) $\{P_i\}$ are pairwise strongly relatively coprime polynomials over A such that

- *P_i is monic and self-reciprocal,*
- *the reduction $\overline{P}_i \in k[T]$ is irreducible with order* $\mathrm{ord}(\overline{P}_i)$ *coprime to p;*

(ii.) $\{Q_jQ_j^*\}$ *are pairwise strongly relatively coprime polynomials over* A *such that*

- $*$ Q_j *is monic and non-self-reciprocal,*
- $*$ *the reduction* $\overline{Q}_j \in k[T]$ *is irreducible with order* $\mathrm{ord}(\overline{Q}_j)$ *coprime to* p.

Then M *admits the orthogonal decomposition*

$$M = \left(\perp_{i=1}^{m} M_{i,\phi}^{(I)} \right) \perp \left(\perp_{j=1}^{n} M_{j,\phi}^{(II)} \right)$$

where $M_{i,\phi}^{(I)} := M_{P_i,\phi}$ *and* $M_{j,\phi}^{(II)} := M_{Q_j,\phi} \oplus M_{Q_j^*,\phi}$ *from Lemma 3.19.*

The conjugacy class of ϕ *in the orthogonal group* $O(M,q)$ *is completely determined by* $\mathrm{minpoly}_\phi$ *together with the isometry classes of the regular quadratic modules* $\{M_{i,\phi}^{(I)}\}$, $\{M_{j,\phi}^{(II)}\}$ *for all* i *and* j.

Proof. Let $\phi, \psi \colon (M,q) \to (M,q)$ be two isometries that are conjugate, i.e. there exists an $\tau \colon (M,q) \to (M,q)$ such that $\tau \circ \phi = \psi \circ \tau$. Then, $\mathrm{minpoly}_\phi = \mathrm{minpoly}_\psi$ by Lemma 3.12. For a polynomial $R \in A[T]$ dividing the minimal polynomial $\mathrm{minpoly}_\phi$, the isometry τ maps $M_{R,\phi}$ to $M_{R,\psi}$ and τ^{-1} maps $M_{R,\psi}$ to $M_{R,\phi}$.

Conversely, let $\psi, \psi \colon (M,q) \to (M,q)$ be two isometries such that

$$\mathrm{minpoly}_\phi = \mathrm{minpoly}_\psi$$

and there exist isometries

$$\tau_i \colon M_{i,\phi}^{(I)} \to M_{i,\psi}^{(I)} \quad \tau_j \colon M_{j,\phi}^{(II)} \to M_{j,\psi}^{(II)}$$

for all i and j. Then,

$$\phi|_{M_{i,\phi}^{(I)}}, \quad \tau_i^{-1} \circ \phi|_{M_{i,\phi}^{(I)}} \circ \tau_i, \qquad \phi|_{M_{j,\phi}^{(II)}}, \quad \tau_j^{-1} \circ \phi|_{M_{j,\phi}^{(II)}} \circ \tau_j,$$

are isometries of $M_{i,\phi}^{(I)}$ and $M_{j,\phi}^{(II)}$ with minimal polynomials P_i and $Q_jQ_j^*$, respectively. Thus, we get the statement by Lemma 3.23 and 3.24. ▪

Recall that the Witt class and the rank of a regular quadratic module determine its isometry class. Determining the Witt class of the above regular quadratic modules will be our goal in the next section.

§.3.4. Calculation in the Witt Ring

In the previous section, we determined conjugacy classes in the orthogonal group of a regular quadratic module (M, q) over an Artinian local ring by decomposing the minimal polynomial and then obtaining an orthogonal decomposition of (M, q). This decomposition gave us an equality in the Witt ring. In this section, we will show that the Witt class of a quadratic module with an isometry with an irreducible minimal polynomial is equal to the Witt class of an involution trace. It can be expressed as a discriminant of the corresponding ring extension.

There is one small caveat, related to [Shi80, Section 1 Case 1]

Remark 3.26.
Let A be an Artinian local ring with finite residue field k of odd characteristic p. Let (M, q) be a regular quadratic module over A. Then an isometry ϕ of finite order N coprime to p has a separable minimal polynomial $\mathrm{minpoly}_{\phi}$.

Let $P \in A[T]$ be a self-reciprocal polynomial of degree 1 such that its reduction $\overline{P}$ is irreducible. For $P = T - a_0 \in A[T]$ with $a_0 \in A^{\times}$, we immediately obtain $a_0 \in \{1, -1\}$ (since $2 \in A^{\times}$ and A is local).

If the minimal polynomial of $P = T \mp 1$ divides the minimal polynomial, there are in general no restrictions on the Witt class of $[M_P]$, as in the notation of Lemma 3.25.

We will see that the Witt class depends only on the discriminant of a field extension. Hence, we recall the following three lemmas.

Lemma 3.27 (*[Bou90, IV §6 Proposition 7 and IV §6 Proposition 11 (ii)-(iii)]*). *Let A be a ring and $P \in A[T]$ be a monic polynomial of degree d. We define $E := A[T]/(P)$. Then the discriminant of P is given by*

$$\mathrm{discr}(P) = \mathrm{discr}(P)[(1, T, \dots, T^{d-1})] = (-1)^{d(d-1)/2}\, \mathrm{Norm}_{E/A} P'.$$

If $P = (T - \alpha_1) \dots (T - \alpha_d)$, then

$$\mathrm{discr}(P) = \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

Here, we use the norm map $\mathrm{Norm}_{E/A} \colon E \to A$ of the A-algebra E; see [Bou74, III §9 Definition 2] or [Sam70, Defintion 2.6.1].

Let K be a field. Then a finite field extension L of K is in particular a K-algebra with a basis of $[L\colon K]$ elements. Hence the discriminant $\operatorname{discr}(L/K)$ is well-defined.

Lemma 3.28 (*[Sam70, Proposition 2.7.3]*).
Let K be a finite field and L an extension of K of degree m, and let $\sigma_1, \dots, \sigma_m$ be m-distinct K-embeddings into an algebraically closed field Ω containing K. Then, if $(x_1, \dots, x_m)$ is a base for L over K, we can calculate the discriminant as

$$\operatorname{discr}(L/K)[x_1, \dots, x_m] = [\det(\sigma_i(x_j))]^2.$$

Lemma 3.29.
Let K be a finite field and L a finite extension of K. Then, for all $x \in L$, the $\operatorname{Norm}_{L/K}(x)$ is a perfect square if and only if x is a perfect square.

Proof. The norm of a finite field extension is multiplicative and a surjective map $\operatorname{Norm}_{L/K}\colon L^\times \to K^\times$; see e.g. [LN83, Theorem 2.28]. ▪

We will use the fundamental theorem of Galois theory and calculate a discriminant using the following (computational) lemma.

Lemma 3.30.
Let $K \subseteq L \subseteq M$ be a tower of finite fields with $m = [L\colon K]$ and $n = [M\colon L]$. Let $(x_1, \dots, x_m)$ be a K-basis for L and $(y_1, \dots, y_n)$ be a L-basis for M. Then $(x_1y_1, x_1y_2, \dots x_my_n)$ is a K-basis for M and

$$\begin{aligned}&\operatorname{discr}(M/K)[x_1y_1, \dots, x_my_n]\\ &\qquad = \operatorname{Norm}_{L/K}[\operatorname{discr}(M/L)[y_1, \dots, y_m]]\big(\operatorname{discr}(L/K)[x_1, \dots, x_m]\big)^n.\end{aligned}$$

The following proof of this proposition is due to Joseph Rabinoff.[1]

Proof. Let Ω be a algebraically closed field containing M and let $\sigma_1, \dots, \sigma_m$ be the m distinct K-embeddings of L into Ω. For every σ_i there exist n distinct $\tau_{i,1}, \dots, \tau_{i,n}$ K-embeddings of M into Ω restricting on L to σ_i. We write $\hat{\sigma}_i\colon \Omega \to \Omega$ for automorphisms such that $\hat{\sigma}_i|_L = \sigma_i$ for all $i = 1, \dots, m$. Then,

[1] We were unable to find a textbook reference, only [Rab15]. For the convenience of the reader, we reproduce it here.

$\hat{\sigma}_i^{-1} \circ \tau_{i,j} \colon M \to \Omega$ for $j = 1, \dots, n$ are the n distinct L-embeddings of M into Ω, compare to [Bou90, V §4 Corollary of Theorem 2]. Now, we can write

$$\tau_{i,j}(x_k y_l) = \sigma_i(x_k)\tau_{i,j}(y_l) = \sum_{(a,b)} \tau_{i,j}(y_b)\delta_{a,i}\delta_{l,b}\sigma_a(x_k)$$

for all $i, k = 1, \dots, m$ and $j, l = 1, \dots, n$. The matrix $(\tau_{i,j}(y_b)\delta_{a,i})_{(i,j),(a,b)}$ is a block diagonal matrix whose determinant is given by

$$\det(\tau_{i,j}(y_b)\delta_{a,i}) = \prod_i \det(\tau_{i,j} y_b)_{j,b}.$$

By taking squares, we get

$$\begin{aligned}
\det(\tau_{i,j}(y_b))^2_{j,b} &= \det(\sum_j \tau_{i,j}(y_b y_c))_{b,c} \\
&= \det(\sum_j \hat{\sigma}_i \circ \hat{\sigma}_i^{-1} \circ \tau_{i,j}(y_b y_c))_{b,c} \\
&= \hat{\sigma}_i \det(\sum_j \hat{\sigma}_i^{-1} \circ \tau_{i,j}(y_b y_c))_{b,c} \\
&= \sigma_i \operatorname{discr}(M/L)(y_1, \dots, y_n),
\end{aligned}$$

where in the last step we used Lemma 3.28 and that $\operatorname{discr}(M/L)(y_1, \dots, y_n) \in L$. Note that the matrix $(\delta_{l,b}\sigma_a(x_k))_{(a,b),(k,l)}$ is the Kronecker product of the matrices $(\sigma_i(x_a))_{i,a=1}^m$ and $(\delta_{j,b})_{j,b=1}^n$. Thus, its determinant is given by

$$\det(\delta_{l,b}\sigma_a(x_k))_{(a,b),(k,l)} = [\det(\sigma_i(x_a))]^n.$$

On the other hand, the square of the determinant of $(\tau_{i,j}(x_k y_l))_{(i,j),(k,l)}$ equals the discriminant $\operatorname{discr}(M/K)(x_1 y_1, \dots, x_m y_n)$. ▪

In particular, we get:

Corollary 3.31.
Let K be a field of characteristic not 2. Let N be a cyclic Galois extension of K of even order d. Let σ be the field automorphism of K of order 2 and N^σ the corresponding fixed field. Then we get in $K^\times/(K^\times)^2$ the equality

$$\operatorname{discr}(N/K) \equiv \operatorname{Norm}_{N^\sigma/K}(\operatorname{discr}(N/K^\sigma)) \pmod{(K^\times)^2}.$$

If t generates a normal basis for N, we get

$$\operatorname{discr}(N/K) \equiv \operatorname{Norm}_{N^\sigma/K}((t - \sigma t)^2) \pmod{(K^\times)^2}.$$

Proof. We get the statement by Lemma 3.27 and Proposition 3.30 since $(t, \sigma t)$ is a N^σ-basis of N. ▪

Recall that we can identify the Witt ring $\mathrm{Wq}(A)$ of A with the Witt ring $\mathrm{Wq}(k)$ of its residue field k.

Lemma 3.32.
Let A be a Henselian local ring with finite residue field k of odd characteristic. Let $P \in A[T]$ be a self-reciprocal polynomial of degree at least 2 such that $\overline{P}$ is irreducible over k. Let $E = A[T]/(P)$ be the étale algebra over A with involution $\sigma\colon E \to E,\ t \mapsto t^{-1}$ and involution trace $\mathrm{trace}_{E/A}\colon E \to A$.

The Witt class of the quadratic module $(E, \lambda_{\mathrm{trace}_{E/A}})$ is given by Λ, i.e. it has even dimension and non-trivial Arf invariant:

$$d(\mathrm{cls}\, \lambda_{E/A}) \equiv 0, \quad (\mathrm{Arf}(\mathrm{cls}\, \lambda_{E/A})/k) = -1,$$

where $(\,*\,/k)$ denotes the Legendre symbol of the field k.

Proof. We denote a generator of $\mathrm{Gal}(E/A)$ by α. We calculate the determinant

$$\begin{aligned}\det(\mathrm{trace}_{E/A}(t^i\sigma(t^j))) &= \det(\mathrm{trace}_{E/A}(t^{i-j}))\\ &= \prod_{k<l} (\alpha^l(t) - \alpha^k(t))(\alpha^l(t^{-1}) - \alpha^k(t^{-1})),\end{aligned}$$

where we used that the matrix $\big(\mathrm{trace}_{E/A}(t^{i-j})\big)_{i,j} = \big(\sum_{k=0}^{d-1} \alpha^k(t^{i-j})\big)_{i,j}$ can be written as the product of the matrices

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ t & \alpha(t) & \dots & \alpha^{d-1}(t) \\ t^2 & \alpha(t)^2 & \dots & \alpha^{d-1}(t)^2 \\ & & \dots & \\ t^{d-1} & \alpha(t)^{d-1} & \dots & \alpha^{d-1}(t)^{d-1} \end{bmatrix} \begin{bmatrix} 1 & t^{-1} & \dots & (t^{-1})^{d-1} \\ 1 & \alpha(t^{-1}) & \dots & \alpha(t^{-1})^{d-1} \\ 1 & \alpha^2(t^{-1}) & \dots & \alpha^2(^{(d-1)})^{d-1} \\ & & \dots & \\ 1 & \alpha^{d-1}(t^{-1}) & \dots & \alpha^{d-1}(t^{-1})^{d-1}. \end{bmatrix}$$

Since

$$\prod_{k<l} (\alpha^l(t^{-1}) - \alpha^k(t^{-1})) = \prod_{k<l} (\alpha^k(t) - \alpha^l(t))(\alpha^k(t)\alpha^l(t))^{-1}$$

and

$$\prod_{k<l} (\alpha^k(t)\alpha^l(t)) = \prod_{i=0}^{d-1} \alpha^i(t^{d-1}) = \mathrm{Norm}_{E/A}(t)^{d-1},$$

we can conclude

$$\begin{aligned}\det(\mathrm{trace}_{E/A}(t^i\sigma(t^j))) &= \det(\mathrm{trace}_{E/A}(t^{i-j})) \\ &= (-1)^{d(d-1)/2}\,\mathrm{Norm}_{E/A}(t^{-(d-1)})\prod_{k<l}\left(\alpha^l(t)-\alpha^k(t)\right)^2.\end{aligned}$$

Since $E = A[T]/(P)$ and $\mathrm{Norm}_{E/A}(t) = \det(C(P))$, we get

$$\mathrm{Norm}_{E/A}(t) = P(0) = 1.$$

Thus, we get

$$(-1)^{d(d-1)/2}\det(\mathrm{trace}_{E/A}(t^i\sigma(t^j))) = \mathrm{discr}(E/A)(1, t, \alpha(t), \dots, \alpha^{d-1}(t)).$$

By Lemma 1.3, the square class of the discriminant is determined by the square class on the corresponding residue fields. By Corollary 3.31, the square class of the discriminant can be calculated as

$$\mathrm{discr}(\overline{E}/k)(1, \bar{t}, \overline{\alpha}(\bar{t}), \dots, \overline{\alpha}^{d-1}(\bar{t})) \equiv \mathrm{Norm}_{k^{\overline{\sigma}}/k}(\bar{t} - \overline{\sigma}\bar{t})^2 \pmod{(k^\times)^2}.$$

Since $\bar{t} - \sigma\bar{t}$ is not in the fixed field $k^{\overline{\sigma}}$ of $\overline{\sigma}$, we get the statement by Lemma 3.29. ▪

The following is an analogue of [Shi80, Lemma 1.17] for Henselian local rings with finite residue field of odd characteristic.

Lemma 3.33.
Let A be a Henselian local ring with residue field k of odd characteristic and (M, q) be a regular quadratic module over A. Let $\phi\colon (M, q) \to (M, q)$ be an isometry with self-reciprocal minimal polynomial P of degree at least 2 such that $\overline{P}$ is irreducible over k. We define $E := A[T]/(P)$. Then

$$[(M, q)] = (\mathrm{rank}_E\ M)\Lambda.$$

Proof. By Lemma 3.21, there is a regular 1-Hermitian module (M_ϕ, s_ϕ) over E such that

$$\lambda_q(x, y) = \mathrm{trace}_{E/A}(s_\phi(x, y))$$

for all $x, y \in M$. We define $n := \mathrm{rank}_E\ M$.

By Proposition 3.7, (M, s_ϕ) has an orthonormal basis, so let $\{x_k\}_{k=1}^n$ be an orthonormal basis of the E-module M_ϕ. Then, we get that the set $\{t^i x_k\}$ forms an A-basis of M and

$$\begin{aligned}\lambda_q(t^i x_k, t^j x_l) &= \mathrm{trace}_{E/A}(t^i \sigma(t^j) s_\phi(x_k, x_l)) \\ &= \mathrm{trace}_{E/A}(t^{i-j})\delta_{k,l}.\end{aligned}$$

Thus, we get the statement from the previous lemma. ▪

§.3.5. The Main Theorem

Combining the above subsections, we have thus proven the following theorem.

THEOREM 3.34 (*parametrisation of conjugacy classes*).
Let A be an Artinian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic p. Let (M, q) be a regular quadratic module over A. Then conjugacy classes of isometries over (M, q) of order coprime to p can be parametrised by

$$[(h_+, \mu_+), (h_-, \mu_-), (P_1, \mu_1^I), \dots, (P_m, \mu_m^I), (Q_1 Q_1^*, \mu_1^{II}), \dots, (Q_n Q_n^*, \mu_n^{II})]$$

where

(i.) h_+ is a regular symmetric bilinear module over A of rank μ_+;

(ii.) h_- is a regular symmetric bilinear module over A of rank μ_-;

(iii.) $\{\mu_i^I\}$ are positive integers and $\{P_i\}$ are pairwise strongly relatively coprime polynomials over A such that

- *P_i is monic and self-reciprocal with $\deg(P_i) \geq 2$,*
- *the reduction $\overline{P}_i \in k[T]$ is irreducible with order $\mathrm{ord}(\overline{P}_i)$ coprime to p;*

(iv.) $\{\mu_j^{II}\}$ are positive integers and $\{Q_j Q_j^*\}$ are pairwise strongly relatively coprime polynomials over A such that

- *Q_j is monic and non-self-reciprocal,*
- *the reduction $\overline{Q}_j \in k[T]$ is irreducible with order $\mathrm{ord}(\overline{Q}_j)$ coprime to p;*

up to reordering of the polynomials P_i *and* $Q_jQ_j^*$, *respectively. The above data is required to satisfy*

$$\operatorname{rank}_A M = \mu_+ + \mu_- + \sum_i \mu_i^I \deg P_i + \sum_j 2\mu_j^{II} \deg Q_j$$

and the equality in the Witt ring $\mathrm{Wq}(A)$

$$[M] = \sum_i \mu_i^I \Lambda + [h_+] + [h_-].$$

For an isometry ϕ in a conjugacy class with the above invariants, the ranks μ_+ and μ_- are the multiplicity of $T \mp 1$ in the characteristic polynomial of ϕ. Similarly, the numbers μ_i^I and μ_j^{II} are the multiplicities of P_i and Q_j in the characteristic polynomial of ϕ.

Theorem 3.34 should be seen as a generalisation of [Shi80, Theorem 1.20] to Henselian local rings with finite residue field of odd characteristic. Note however that we only considered isometries whose order is coprime to the characteristic.

Note that the order of a polynomial $P \in k[T]$ over a finite field k has a well-defined *order* which is the least positive integer e such that it divides the polynomial $T^e - 1$; see [LN83, Definition 3.2]. The proof above is constructive, namely, for every set of invariants as above, we can construct a regular quadratic module with an isometry with these invariants.

Remark 3.35.
Let A be an Artinian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic p. If we are given a tuple

$$[(h_+, \mu_+), (h_-, \mu_-), (P_1, \mu_1^I), \dots, (P_m, \mu_m^I), (Q_1Q_1^*, \mu_1^{II}), \dots, (Q_nQ_n^*, \mu_n^{II})]$$

as above we can build a regular quadratic module (M, q) with an isometry with the given invariants as follows:

For a self-reciprocal polynomial $P_i \in A[T]$ of order coprime to p and of degree d_i^I at least 2 such that its reductions $\overline{P_i}$ is irreducible over $k[T]$, we define $E_{P_i} := A[T]/(P_i)$. Then, M_{P_i} is the orthogonal sum of μ_i^I copies of the symmetric module $M_{E_{P_i}}$ corresponding to the involution trace $\operatorname{trace}_{E_{P_i}/A}$.

For a polynomial $Q_j \in A[T]$ with $Q_j \neq Q_j^*$ with order coprime to p and of degree d_j^{II} such that the reductions $\overline{Q}_j$ are irreducible in $k[T]$, we define $\hat{M}_{Q_j}$

as the orthogonal sum of μ_j^{II} copies of the hyperbolic space $\mathrm{Hyp}\,(A[T]/(Q_j))$ of dimension $2d_j$; here we consider $A[T]/(Q_j)$ as a (finitely generated projective) A-module.

Then, we can define

$$M := h_+ \perp h_- \perp (\perp_i M_{P_i}) \perp (\perp_j \hat{M}_{Q_j}).$$

Now, we can define an isometry $\phi\colon (M,q) \to (M,q)$ which acts as follows:

(i.) On h_+, ϕ acts as the identity,

(ii.) On h_-, ϕ acts as the inversion -1,

(iii.) On M_{P_i}, ϕ acts on each copy of $M_{E_{P_i}}$ as the companion matrix $C(P_i)$ of P_i,

(iv.) On $\hat{M}_{Q_j}$, ϕ acts on each copy of $A[T]/(Q_j)$ as the companion matrix $C(Q_j)$ of Q_j; this already fixes how ϕ acts on the hyperbolic space $\mathrm{Hyp}\,(A[T]/(Q_j))$; see Lemma 3.24.

Thus ϕ restricted to M_{P_i} and M_{Q_j} has the minimal polynomial P_i and $Q_jQ_j^*$, respectively.

Example 3.36.
Using the above lemma, it is straightforward to write down examples of isometries of a given order N.

For example, for the prime $p = 7$, the irreducible self-reciprocal polynomials of degree 2 are

$$1+T^2, \quad 1+3T+T^2, \quad 1+4T+T^2$$

which are of order 4, 8, and 8, respectively.

Then, we can construct quadratic spaces M_P as in Remark 3.35 for P a polynomial above:

$$\begin{array}{r|lll}
P & 1+T^2 & 1+3T+T^2 & 1+4T+T^2 \\
N & 4 & 8 & 8 \\
\lambda & \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} & \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} & \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \\
C(P) & \begin{bmatrix} 0 & 6 \\ 1 & 4 \end{bmatrix} & \begin{bmatrix} 0 & 6 \\ 1 & 3 \end{bmatrix} & \begin{bmatrix} 0 & 6 \\ 1 & 3 \end{bmatrix}
\end{array}$$

We listed the polynomial, its order, the symmetric bilinear form, and the companion matrix, respectively. The symmetric bilinear form for the involution trace is written in the basis $\{1, t\}$, where t is the image of T in $E = A[T]/(P)$ for P a polynomial above. One can directly check that, as quadratic spaces, they are all isometric, since they have a non-trivial Arf invariant and the same rank.

Furthermore, we can use results on self-reciprocal irreducible polynomials to calculate the number of conjugacy classes for a given rank and degree. We recall some facts.

For a finite field k and a polynomial $Q \in k[T]$ of degree d, we define its *associated* self-reciprocal polynomial by

$$Q^{\mathrm{ass}}(T) := T^d Q(T + T^{-1}) \in k[T].$$

It is a polynomial of degree $2d$.

Lemma 3.37 (*[Coh87, Lemma 2.1]*).
Let k be a finite field.

(i.) If P is an irreducible, self-reciprocal polynomial in $k[T]$. Then $P = T \pm 1$ or P is the associate of another polynomial.

(ii.) If Q is an irreducible polynomial in $k[T]$, its associate Q^{ass} is reducible if and only if $Q^{\mathrm{ass}} = RR^*$ for some irreducible polynomial R where either $R = T \pm 1$ or R is not self-reciprocal.

Lemma 3.38 (*[Coh87, Lemma 2.3]*).
Let Q be an irreducible polynomial in $\mathbb{F}_q[T]$ of degree $2d$. Then Q is self-reciprocal if and only if its order $\operatorname{ord}(Q)$ *divides $q^d + 1$.*

Proposition 3.39 (*[Coh87, Theorem 2.4]*).
Let Q be an irreducible polynomial in $\mathbb{F}_q[T]$ of degree d and level N.

(i.) If $N = 1$, then $Q(T) = T - 1$ and if $N = 2$ then q is odd and $Q(T) = T + 1$.

(ii.) If $N > 2$, then $N = \operatorname{lvl}(Q) = \operatorname{ord}(Q^{\mathrm{ass}})$.

(iii.) d is the least positive integer satisfying $q^d \equiv \pm 1 \pmod{N}$

(iv.) If Q^{ass} is reducible, then d is the least integer satisfying $q^d \equiv 1 \pmod{N}$. If Q^{ass} is irreducible then $N > 2$ and d is the least integer satisfying $q^d \equiv -1 \pmod{N}$.

For $N > 2$ with $p \nmid N$, there are $\phi(N)/2d$ irreducible polynomials P over $\mathbb{F}_q$ of degree d with level N.

	1	2	3	4
3	2_1	4_1		$5_1, 10_1$
		8_1	$13_2, 26_2$	$16_1, 20_1, 40_2, 80_4$
5	2_1	$3_1, 6_1$		$13_3, 26_3$
	4_1	$8_1, 12_1, 24_2$	$31_5, 62_5, 124_{10}$	$16_1, 39_3, 48_2, 52_3, 78_3, 104_6, 156_6, 208_{12}, 312_{12}, 624_{24}$
7	2_1	$4_1, 8_2$		$5_1, 10_1, 25_5, 50_5$
	$3_1, 6_1$	$12_1, 16_2, 24_2, 48_4$	$9_1, 18_1, 19_3, 38_3, 57_6, 114_6, 171_{18}, 342_{18}$	$15_1, 20_1, 30_1, 32_2, 40_2, 60_2, 75_5, 80_4, 96_4, 100_5, 120_4, 150_5, 160_8, 200_{10}, 240_8, 300_{10}, 400_{20}, 480_{16}, 600_{20}, 800_{40}, 1200_{40}, 2400_{80}$
11	2_1	$3_1, 4_1, 6_1, 12_2$		$61_{15}, 122_{15}$
	$5_2, 10_2$	$8_1, 15_2, 20_2, 24_2, 30_2, 40_4, 60_4, 120_8$	$7_1, 14_1, 19_3, 35_4, 38_3, 70_4, 95_{12}, 133_{18}, 190_{12}, 266_{18}, 665_{72}, 1330_{72}$	$16_1, 48_2, 80_4, 183_{15}, 240_8, 244_{15}, 305_{30}, 366_{15}, 488_{30}, 610_{30}, 732_{30}, 915_{60}, 976_{60}, 1220_{60}, 1464_{60}, 1830_{60}, 2440_{120}, 2928_{120}, 3660_{120}, 4880_{240}, 7320_{240}, 14640_{480}$
13	2_1	$7_3, 14_3$		$5_1, 10_1, 17_4, 34_4, 85_{16}, 170_{16}$
	$3_1, 4_1, 6_1, 12_2$	$8_1, 21_3, 24_2, 28_3, 42_3, 56_6, 84_6, 168_{12}$	$9_1, 18_1, 36_2, 61_{10}, 122_{10}, 183_{20}, 244_{20}, 366_{20}, 549_{60}, 732_{40}, 1098_{60}, 2196_{120}$	$15_1, 16_1, 20_1, 30_1, 35_3, 40_2, 48_2, 51_4, 60_2, 68_4, 70_3, 80_4, 102_4, 105_6, 112_6, 119_{12}, 120_4, 136_8, 140_6, 204_8, 210_6, 238_{12}, 240_8, 255_{16}, 272_{16}, 280_{12}, 336_{12}, 340_{16}, 357_{24}, 408_{16}, 420_{12}, 476_{24}, 510_{16}, 560_{24}, 595_{48}, 680_{32}, 714_{24}, 816_{32}, 840_{24}, 952_{48}, 1020_{32}, 1190_{48}, 1360_{64}, 1428_{48}, 1680_{48}, 1785_{96}, 1904_{96}, 2040_{64}, 2380_{96}, 2856_{96}, 3570_{96}, 4080_{128}, 4760_{192}, 5712_{192}, 7140_{192}, 9520_{384}, 14280_{384}, 28560_{768}$

The columns correspond to the degree of the polynomial; the rows to the order of the finite field $\mathbb{F}_p$. For a given prime, we list the self-reciprocal irreducible polynomials in the first and the non-self-reciprocal irreducible polynomials in the second line. We list the order of the polynomials with the number of polynomials with given order as index. For example, there are 20 non-self-reciprocal polynomials of order 366 and rank 3 over $\mathbb{F}_{13}$.

4. Fixed-Point-Free Isometries

In Chapter 6, we will use fixed-point-free isometries of discriminant forms. Fixed-point-free automorphisms of groups are well-studied; see e.g. [Rob96, Section 10.5] or [Gor80, Chapter 10]. A very explicit treatment can be found in [May98]. They appear in the study of near fields or Frobenius groups.

In this chapter, we will apply our main theorem above to classify all automorphisms which generate a fixed-point-free group of isometries for quadratic forms over a finite local ring A with maximal ideal $\mathfrak{m}$ and residue field k of finite order coprime to the characteristic $\mathrm{char}(k)$. We will see that all of them correspond to lifts of fixed-point-free automorphisms over the finite residue field k. We will need the finiteness assumption on A to guarantee that the lift has the same order. Even though every Artinian local ring with finite residue is a finite local ring, we will explicitly speak of finite local rings here to make clear that this is important.

In the end, we want to apply our classification of fixed-point-free isometries to the finite local ring $\mathbb{Z}/p^l$ for an odd prime $p \in \mathbb{Z}$ and a positive integer $l \in \mathbb{Z}_{\geq 1}$.

Definition 4.1 (*fixed-point-free automorphism).*
Let A be a local ring and M be a finitely generated projective A-module. Let $\phi\colon M \to M$ be an automorphism of finite order N. Then ϕ is called *fixed-point-free* if $\phi(x) = x$ for $x \in M$ implies $x = 0$.

We immediately obtain many properties of fixed-point-free automorphisms of groups, see e.g. [Rob96, Proposition 10.5.1], here stated for modules over rings.

Lemma 4.2.
Let A be a local ring and M be a finitely generated projective A-module. Let $\phi\colon M \to M$ be a fixed-point-free automorphism of finite order N.

(i.) If i is coprime to N, ϕ^i is fixed-point-free.

(ii.) For all $x \in M$, we have

$$x + \phi(x) + \dots + \phi^{N-1}(x) = 0.$$

In particular, if the order N of the automorphism ϕ is prime, the non-trivial power ϕ^i of ϕ is fixed-point-free for all $i = 1, \dots, N-1$. In this work, we will be interested in automorphisms for which all non-trivial powers are fixed-point-free, as they have the nice property that they define an action of $\mathbb{Z}/N$ such that all non-trivial orbits have the same length N; see Lemma 6.2.

In particular, groups of fixed-point-free automorphisms have the property that they restrict nicely to invariant subspaces.

LEMMA 4.3 (*compare to [May98, Proposition 3.3]*).
Let A be a local ring and M be a finitely generated projective A-module. Let Φ be a finite group of fixed-point-free automorphisms and $\tilde{M}$ a non-trivial Φ-invariant submodule of M. Then the group of automorphisms restricted to $\tilde{M}$ is isomorphic to Φ.

LEMMA 4.4 (*compare to [Rob96, Proposition 10.5.1]*).
Let A be a local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of characteristic p. Let M be a finitely generated projective A-module. Let $\phi\colon M \to M$ be an automorphism of finite order. Then ϕ is fixed-point-free if and only if the endomorphism of M defined by

$$W_\phi\colon M \to M, \quad x \mapsto x - \phi(x)$$

is an automorphism.

Proof. Assume that $x - \phi(x) = y - \phi(y)$ for some $x, y \in M$. Then ϕ fixes $x - y$, so $x = y$. Thus, W_ϕ is injective. Thus, the reduction modulo $\mathfrak{m}$ is an automorphism and, by Lemma 3.1, so is W_ϕ. ▪

We thus obtain:

COROLLARY 4.5.
Let A be a local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of characteristic p. Let M be a finitely generated projective A-module. Let $\phi\colon M \to M$ be an automorphism of finite order. Then ϕ is fixed-point-free if and only if its reduction ϕ over k is fixed-point-free.

Now, we use the fact that the reduction map is surjective and for *finite rings*, its kernel is a p-group where p is the characteristic of the residue field.

LEMMA 4.6 (*see e.g. [Pom73]).*
Let A be a finite local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of characteristic p. Then the reduction map

$$\mathrm{GL}_n(A) \to \mathrm{GL}_n(k)$$

is a surjective group homomorphism, whose kernel is a (solvable) p-group.

We thus can conclude that an automorphism $f \in \mathrm{GL}_n(A)$ of order coprime to p has the same order as its reduction $\overline{f} \in \mathrm{GL}_n(k)$.

REMARK 4.7.
In fact, this way Pomfret proved that conjugacy classes in the general linear group over a finite local ring A can be identified with the conjugacy classes in the general linear group of its residue field; see [Pom73, Theorem 1].

Now, the only thing left to do is to show that irreducible polynomials generate fixed-point-free actions.

LEMMA 4.8.
Let A be a finite local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic p. Let $P \in A[T]$ be a minimal polynomial such that its reduction $\overline{P} \in k[T]$ is irreducible with order $N := \mathrm{ord}(P)$, $p \nmid N$. Define $E := A[T]/(P)$ and denote the image of T in E by t. Then multiplication by t^i is a fixed-point-free automorphisms of E of order N for all $i = 1, \dots, N-1$.

Proof. By Lemma 4.6 and 4.5, it suffices to show that $\overline{t}^i$ acts fixed-point-free with order N on $\overline{E} = k[T]/(\overline{P})$ for all $i = 1, \dots, N-1$.

Let $\overline{x} \in \overline{E}$ be a fixed point of $\overline{t}^i$ for some $i = 1, \dots, N-1$. We pick an $\overline{S} \in k[T]$ such that its image in $\overline{E}$ equals x. Then

$$\overline{t}^i \overline{x} = \overline{x}$$

implies

$$\overline{P} \mid (\overline{T}^i - 1)\overline{S}.$$

Since the order of $\overline{P}$ is N and $\overline{P}$ is irreducible, we get that $\overline{P}$ divides $\overline{S}$, i.e. $\overline{x} = 0$. ▪

For example, the isometries in Example 3.36 are fixed-point-free.

We conclude with the following classification following from Theorem 3.34 and Lemma 4.8 with Remark 3.35. We exclude $N = 1$ and $N = 2$ as there is only one irreducible polynomial of order 1 and 2 (namely $T \mp 1$, respectively).

THEOREM 4.9 (*conjugacy classes of fixed-point-free isometries).*
Let A be a finite local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic p. Let (M, q) be a regular quadratic module over A. Then conjugacy classes of fixed-point-free isometries of order $N > 2$ with $p \nmid N$ can be parametrised as follows

$$[(P_1, \mu_1^I), \dots, (P_m, \mu_m^I), (Q_1 Q_1^*, \mu_1^{II}), \dots, (Q_n Q_n^*, \mu_n^{II})]$$

where

(i.) $\{\mu_i^I\}$ *are positive integers and* $\{P_i\}$ *are pairwise strongly relatively coprime polynomials over A such that*

- P_i *is monic and self-reciprocal with* $\deg(P_i) \geq 2$,
- *the reduction* $\overline{P}_i \in k[T]$ *is irreducible with order* $\operatorname{ord}(\overline{P}_i) = N$;

(ii.) $\{\mu_j^{II}\}$ *are positive integers and* $\{Q_j Q_j^*\}$ *are pairwise strongly relatively coprime polynomials over A such that*

- Q_j *is monic and non-self-reciprocal,*
- *the reduction* $\overline{Q}_j \in k[T]$ *is irreducible with order* $\operatorname{ord}(\overline{Q}_j) = N$;

up to reordering the polynomials P_i and $Q_j Q_j^*$, respectively. This data is required to satisfy

$$\operatorname{rank}_A M = \sum_i \mu_i^I \deg P_i + \sum_j 2\mu_j^{II} \deg Q_j$$

and the equality in the Witt ring $\mathrm{Wq}(A)$

$$[M] = \sum_i \mu_i^I \Lambda.$$

We will apply this theorem in Chapter 6. Note that here all polynomials are of order N as the restriction of the isometry to invariant subspaces needs to be of the same order by Lemma 4.3.

REMARK 4.10.
We did not determine the conjugacy class of the group of automorphisms generated by an isometry with the above invariants. For fixed-point-free automorphisms on abelian groups, this is discussed in [May98, Chapter 6]. For our applications in Part II, this question does not appear to be relevant.

Let us end this part with an example.

EXAMPLE 4.11.
We construct an example with order 57 on a regular quadratic module over the ring $\mathbb{Z}/(7^3) = A$ with residue field $\mathbb{F}_7$. We find that the least positive integer d with $p^d \equiv \pm 1 \pmod{57}$ is given by $d = 3$ with $p^d = 1 \pmod{57}$. Then there are $\phi(57)/2d = 6$ irreducible non-self-reciprocal polynomials over $\mathbb{F}_7$ of degree 3. For example, we can pick

$$\overline{Q} = T^3 + T^2 + 4T + 6 \in \mathbb{F}_7$$

with reciprocal polynomial

$$\overline{Q}^* = T^3 + 3T^2 + 6T + 6 \in \mathbb{F}_7.$$

Then, we can use Hensel's lemma to lift $\overline{Q}$ to a polynomial $Q \in A[T]$ of order 57, i.e. dividing $T^{57} - 1 \in A[T]$. The lift is given by

$$Q = T^3 + 64T^2 + 32T + 342.$$

Thus, we can construct $\hat{M}_Q$ as in Remark 3.35: The isometry acts as on the dual basis of the hyperbolic space $\hat{M}_Q$ as

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 311 & 0 & 0 & 0 \\ 0 & 1 & 279 & 0 & 0 & 0 \\ 0 & 0 & 0 & 32 & 1 & 0 \\ 0 & 0 & 0 & 64 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

with the companion matrix $C(Q)$ and its inverted transpose $(C(Q)^{-1})^t$ on the diagonal.

II.

Modular Fusion Categories and Fixed-Point-Free Isometries

Now, we will construct modular fusion categories. We will use our results from Part I.

Chapter 5: Pointed Modular Fusion Categories.

We first recall the correspondence between pointed modular fusion categories and discriminant forms. In Chapter 5, we will see that *modular* pointed fusion categories correspond to discriminant forms. We recall the definition of discriminant forms in Definition 5.1. For any discriminant form (Γ, q), there is an equivalence class of braided modular fusion categories whose group of simple objects are given by Γ and whose braiding is related to the quadratic form q.

By decomposing the discriminant form and changing the coefficients (compare to Lemma 5.9), we will see that we can apply results from Part I for the ring $\mathbb{Z}/p^l$ for a prime p and a positive integer $l \in \mathbb{Z}_{\geq 1}$.

Chapter 6: Building Extensions via Fixed-Point-Free Isometries.

Then, in Chapter 6, we are working with a cyclic group $\mathbb{Z}/N$ which acts by fixed-point-free isometries on a discriminant form (Γ, q). Since the isometries are fixed-point-free, we get that all cohomology groups $H^n(\mathbb{Z}/N, \Gamma)$ are trivial; see Lemma 6.4. For example, this implies that there is a unique lifting of the action of $\mathbb{Z}/N$ by isometries of (Γ, q) to a categorical action of $\mathbb{Z}/N$ on a modular fusion category corresponding to (Γ, q); see Lemma 6.6. This will slightly simplify our exposition.

In Part I, we determined conjugacy classes in orthogonal groups whose isometries are fixed-point-free in Theorem 4.9. We will use this classification to build an action of a cyclic group on a hyperbolic discriminant form; see Definition 6.7.

We focused on hyperbolic actions since we will see that the equivariantisations of crossed braided extensions of discriminant forms with such an action turn out to be Drinfeld centres.

Using sequential equivariantisation, we can easily build braided $\mathbb{Z}/N$-crossed extensions for hyperbolic discriminant forms; see Corollary 6.23. We directly calculate the fusion rings of these extensions; see Remark 6.24.

Fusion categories with fusion rules of this form appeared in [GR25, Definition 5.1], where they are called $\mathbb{Z}/N$-*Tambara-Yamagami*. For $N = 3$ such categories already appeared in [JL09, Definition 1.3].

We describe the representation theory of the group $\Pi = H \rtimes \mathbb{Z}/N$ in Section 6.2.

Quite remarkably, every braided $\mathbb{Z}/N$-crossed extension of the hyperbolic discriminant form with compatible fixed-point-free action contains this Tannakian subcategory as Lagrangian subcategory; see Lemma 6.28.

We obtain a classification of braided $\mathbb{Z}/N$-crossed extensions of hyperbolic discriminant forms with hyperbolic action (up to equivalence) in Theorem 6.31.

We remark that, in particular, the modular fusion categories of dimension p^2q^2 constructed by Mignard and Schauenburg in [MS21] can be realised as equivariantisations of the above extensions; see Remark 6.33.

We show, using results on categorical Morita equivalence for pointed categories, that for two choices of hyperbolic actions, the resulting equivariantisations will be braided equivalent in Lemma 6.36.

Fusion rings with fixed-point-free automorphisms have also appeared in [Sch25]. Pointed modular fusion categories with fixed-point-free actions were also studied in [GR25].

Chapter 7: Representation Theory of Twisted Quantum Doubles.

We can describe the $\mathbb{Z}/N$-equivariantisations of the extensions from Theorem 6.31 as the category of representations of a quasi-Hopf algebra introduced by Dijkgraaf, Pasquier, and Roche. In Chapter 7, we will describe the representation theory of these quasi-Hopf algebras for semi-direct products in some detail. Namely, we define a quasi-Hopf algebra using the fixed-point-free action; see Definition 7.1. Then, we give a complete description of the irreducible representations of this quasi-Hopf algebra (Proposition 7.3), the modular data (Proposition 7.4), charge conjugates (duals) (Proposition 7.5), and fusion rules of these modular fusion categories (Proposition 7.6).

Here, we use that the action of $\mathbb{Z}/N$ is by fixed-point-free automorphisms and thus every non-trivial orbit has the same length N.

Notation and Conventions

For this chapter, we work with fusion categories over the complex numbers $k = \mathbb{C}$; see [ENO05]. For braided fusion categories, we refer to [Dri+10].

We recall some notation. For a good overview, we refer to [ENO11].

We denote the set of isomorphism classes of simple objects of C by $O(\mathsf{C})$.

Note that a fusion category C always contains the fusion subcategory C_{pt} generated by invertible objects, and a tensor functor sends invertible objects to invertible objects.

A fusion category is called *pseudo-unitary* if the categorical dimension coincides with its Frobenius-Perron dimension. In this case, C admits a spherical structure such that categorical dimensions coincide with their Frobenius-Perron dimensions; see [ENO05, Proposition 8.23]. In this part, all fusion categories will be weakly integral (their Frobenius-Perron dimension is an integer), so by [ENO05, Proposition 8.24] they are always pseudo-unitary. We denote the Frobenius-Perron dimension of an object x of C by $\mathrm{FPdim}(x)$; see [ENO05, Section 8].

For a fusion subcategory K of a braided fusion category C with braiding c, we will consider the centraliser $C_{\mathsf{C}}(\mathsf{K})$ of K in C consisting of all objects x of C such that

$$c_{y,x} \circ c_{x,y} = \mathrm{id}_{x \otimes y}$$

for all objects y in K. A fusion subcategory K is *symmetric* if and only if $\mathsf{K} \subseteq C_{\mathsf{C}}(\mathsf{K})$.

A symmetric fusion category E is called *Tannakian* if it is equivalent (as braided fusion categories) to the representation category of a finite group Π; see [Del02] and [Del90].

A braided fusion category C with twist θ is called *ribbon* (or premodular) if θ preserve duals, i.e. $\theta_{x^\times} = (\theta_x)^\times$ for all objects x in C. A ribbon fusion category admits an S-matrix, defined as a trace over the double braiding. A modular fusion category is a ribbon fusion category with an invertible S-matrix.

We denote the categorical (or quantum) dimensions of an object x in a ribbon category C by $\dim(x)$. The categorical dimension of C is defined by

$$\dim(\mathsf{C}) = \sum_{\mathrm{cls}\, x \in O(\mathsf{C})} \dim(x)^2.$$

In a modular fusion category C, we have Müger's Centraliser Theorem [Müg03, Theorem 3.2]

$$\dim(\mathsf{K})\dim(C_{\mathsf{C}}(\mathsf{K})) = \dim(\mathsf{C}).$$

Let C be a ribbon (or premodular) fusion category with braiding c and twist θ. Let E be a fusion subcategory of C.

(i.) The fusion subcategory E is called *isotropic* if the twist θ restricts to the identity on E.

(ii.) An isotropic subcategory E is called *Lagrangian* if

$$\mathsf{E} = C_{\mathsf{C}}(\mathsf{E}).$$

Note that an isotropic subcategory E of C is always symmetric, i.e. $\mathsf{E} \subseteq C_{\mathsf{C}}(\mathsf{E})$. If C contains a Lagrangian subcategory, it is called *hyperbolic*.

We will freely use module categories over and algebras in (braided) fusion categories. For details, we refer to e.g. [Nai07].

For a (indecomposable) right module category M over the fusion category C, we denote the category of (right) C-module endofunctors of M by $\mathsf{C}^{\times}_{\mathsf{M}} = \mathrm{Fun}(\mathsf{M},\mathsf{M})_{\mathsf{C}}$. We will refer to this category as the dual category of C with respect to M.

Two fusion categories C and D are called *categorically Morita equivalent* if there exists an indecomposable right module category M over C such that the dual category and D are equivalent as fusion categories. Equivalently, two fusion categories are categorically Morita equivalent if and only if their Drinfeld centres are equivalent (as braided fusion categories); see e.g. [ENO11, Theorem 3.1].

A fusion category is called *group-theoretical* if it is equivalent to the dual of a pointed fusion category with respect to an indecomposable module category.

We will also use group actions on fusion categories and braided fusion categories. For details, we refer to [Gal17].

5. Pointed Modular Fusion Categories

In the following, we will implicitly use the classification of braided categorical groups; see [JS93].

A fusion category is called *pointed* if every simple object is invertible. Every pointed fusion category is equivalent to the fusion category vect_Π^Ω, i.e. the category of finite-dimensional vector spaces graded by the finite group Π and with associativity constraint given by a (normalised) 3-cocycle $\Omega \in H^3(\Pi, k^\times)$; see e.g. [Nai07].[1]

Discriminant Forms and Metric Groups

There is a bijection between equivalence classes of braided pointed fusion categories and *pre-metric groups*, i.e. finite abelian groups Γ with a quadratic map $q\colon \Gamma \to k^\times$; see [Dri+10, Proposition 2.41].

Since the order of every element in the image of a quadratic map $q\colon \Gamma \to k^\times$ divides $|\Gamma|^2$, we can identify a quadratic map $q\colon \Gamma \to k^\times$ uniquely with a quadratic map $q^+\colon \Gamma \to \mathbb{Q}/\mathbb{Z}$. We will do so from now on.

Definition 5.1 (*discriminant form).*
Let Γ be a finite abelian group and $q\colon \Gamma \to \mathbb{Q}/\mathbb{Z}$ a map such that

(i.) $q(ag) = a^2 q(g)$ for all $a \in \mathbb{Z}$ and $g \in \Gamma$,

(ii.) the map $\lambda_q\colon \Gamma \times \Gamma \to \mathbb{Q}/\mathbb{Z}$ defined by

$$\lambda_q(g,h) := q(g+h) - q(g) - q(h)$$

is a regular symmetric bilinear form.

Then, the tuple (Γ, q) is called *discriminant form.*

[1] Note however that a pointed fusion category of course is not a Gr-category, but they are intricately related; see [Dri+10] and [JS93, Remark 3.2].

Here *regularity* means that the map

$$\begin{aligned}\mathrm{Adj}[\lambda]\colon \Gamma &\to \mathrm{Hom}(\Gamma, \mathbb{Q}/\mathbb{Z}),\\ g &\mapsto \lambda(g, *)\end{aligned}$$

is an isomorphism for all $g \in \Gamma$. We use the name 'discriminant forms' since Nikulin used them in [Nik80]. However, our discussion is independent of lattice realisation(s). Discriminant forms also appear in [Bor00]. They are called *finite quadratic modules* in [Str13, Definition 2.5].

We will call $g, h \in \Gamma$ orthogonal if $\lambda_q(g, h) = 0$. For a subgroup $H \leq \Gamma$, we denote its *orthogonal complement* (or *annihilator*) by $H^\perp$. A subgroup $H \leq \Gamma$ is called *isotropic* if q restricts to $0 + \mathbb{Z}$ on H. An isotropic subgroup H is called *Lagrangian* if $H = H^\perp$. An isometry of discriminant forms (Γ_1, q_1) and (Γ_2, q_2) is a group isomorphism $\phi\colon \Gamma_1 \to \Gamma_2$ such that $q_2 \circ \phi = q_1$. The group of all isometries of a discriminant form (Γ, q) will be denoted by $O(\Gamma, q)$ and is called the *orthogonal group* of (Γ, q).

Modular Fusion Categories Corresponding to Metric Groups

For a discriminant form (Γ, q), there is an equivalence class of braided *modular* fusion categories $\mathsf{V}(\Gamma, q)$ such that the group of simple objects is Γ and the trace of the braiding is given by q. For details, we refer to [Dri+10, Section 2.11.5], [Dri+07, Section 5] and [GJ16]; see also the recent preprint [Gal25]. The spherical structure that we choose on $\mathsf{V}(\Gamma, q)$ is the unique positive one; see [Dri+10, Remark 2.26].

Remark 5.2.
For a discriminant form (Γ, q) where Γ is a finite abelian group of *odd order*, we can always choose D in $\mathsf{V}(\Gamma, q)$ such that D is equal to vect_Γ as fusion categories and the braiding on vect_Γ is a bicharacter whose trace is given by the quadratic form q; see [Dri+10, Appendix D] or [Gal25, Corollary 5.5]. We denote this modular fusion category by vect_Γ^q.

For a modular fusion category in $\mathsf{V}(\Gamma, q)$, isotropic (resp. Lagrangian) subcategories correspond to isotropic (resp. Lagrangian) subgroups of (Γ, q).

Remark 5.3.
Every braided autoequivalence of a modular fusion category in $\mathsf{V}(\Gamma, q)$ associated to a discriminant form (Γ, q) naturally determines an isometry of (Γ, q).

We will use the fact that there is an isomorphism between the group of isomorphism classes of braided autoequivalences of a modular fusion category D in $\mathsf{V}(\Gamma, q)$, denoted by $\mathrm{Aut}_{\mathrm{br}}(\mathsf{D})^0$, and the orthogonal group $O(\Gamma, q)$

$$\mathrm{Aut}_{\mathrm{br}}(\mathsf{D})^0 \cong O(\Gamma, q),$$

see e.g. [DN13, Section 2C] or [JS93, Remark 3.3].

In the following, we will often treat an isometry ϕ of (Γ, q) as a braided autoequivalence of D whose underlying isomorphism on simple objects is given by ϕ.

If the discriminant form is odd, we can, by Remark 5.2, always work with vect_Γ^q. Then, we can choose the *trivial* tensor structure; see also [Cui+16, Section 5.1.1].

Furthermore, we note the following fact, which can be proven using basic results on abelian cohomology.

Lemma 5.4 (*see [Gal25, Corollary 5.5]*).
Let (Γ, q) be a discriminant form of odd order. Then, the normalised 3-cocycle $\mathrm{cls}\,\Omega \in H^3(\Gamma, k^\times)$ determined by the associativity constraint of D in $\mathsf{V}(\Gamma, q)$ is cohomologically trivial.

Recall that the category of braided autoequivalences of a braided category is a coherent Gr-category. We now apply this fact to the modular fusion categories in $\mathsf{V}(\Gamma, q)$.

Remark 5.5.
Let (Γ, q) be a discriminant form of odd order. By Lemma 5.4, we can choose a modular fusion category D in $\mathsf{V}(\Gamma, q)$ whose underlying fusion category is skeletal and strict with simple objects indexed by the elements of Γ; see also e.g. [Nai07, Section 2.2]. By [Dri+10, Lemma 3.31], there is an isomorphism between the group of simple objects and tensor automorphisms of the identity functor, i.e. in our case

$$\begin{aligned} \alpha \colon \Gamma &\to \mathrm{Aut}_\otimes(1_{\mathsf{D}}) \\ x &\mapsto \eta^x \colon 1_{\mathsf{D}} \to 1_{\mathsf{D}} \end{aligned}$$

defined by

$$\eta^x_y : y \xrightarrow{\lambda_q(x,y)} y$$

where λ_q is the associated symmetric form on (Γ, q).

We know that, for every isomorphism class of braided autoequivalences $\operatorname{cls} F$ of D, there is a representative F^{str} with strict tensor structure; see Remark 5.3. This way we obtain a *special* coherent Gr-category $\mathrm{Aut}_{\mathrm{br}}(\mathsf{D})^{\mathrm{str}}$ equivalent to $\mathrm{Aut}_{\mathrm{br}}(\mathsf{D})$; see [BL04, Definition 8.3.2] for the definition. For a strict functor F^{str} with underlying isometry f, the strict functor with underlying isometry f^{-1} is a strict inverse $(F^{\mathrm{str}})^{-1}$ of F^{str}. Now, the action of F^{str} on an element $\eta \in \mathrm{Aut}_{\otimes}(1_{\mathsf{D}})$ was defined as

$$(F^{\mathrm{str}} * \eta) * (F^{\mathrm{str}})^{-1}$$

with strict inverse $(F^{\mathrm{str}})^{-1}$ of F^{str}.

We get by evaluating the action on η^x for some $x \in \Gamma$

$$\begin{aligned}
[(F^{\mathrm{str}} * \eta^x) * (F^{\mathrm{str}})^{-1}]_y &= F^{\mathrm{str}}(\eta^x_{f^{-1}(y)}) \\
&= \lambda_q(x, f^{-1}(y)) \operatorname{id}_y \\
&= \lambda_q(f(x), y) \operatorname{id}_y = \eta^{f(x)}_y
\end{aligned}$$

where $f\colon (\Gamma, q) \to (\Gamma, q)$ is the isometry underlying the functor F^{str}.

Under the above isomorphism α, an element $f \in O(\Gamma, q)$ of the orthogonal group $O(\Gamma, q)$ thus acts on an element $x \in \Gamma$ by $f(x)$.

We will discuss coherent Gr-categories later in Part III; see Chapter 9.1.

Decomposition of Discriminant Forms

We will now show that discriminant forms and their isometries can be understood by studying quadratic forms over $\mathbb{Z}/p^l$ for prime numbers p and positive integers $l \in \mathbb{Z}_{\geq 1}$. This will allow us to apply results from Part I.

Lemma 5.6 (*see e.g. [Str13]).*
Let (Γ, q) be a discriminant form. Then (Γ, q) splits orthogonally into its p-subgroups.

Lemma 5.7 (*splitting lemma, [Wal63a, Lemma 8]).*
Let (Γ, q) *be a discriminant form such that the underlying finite abelian group* Γ *is a p-group for a prime p. Then* (Γ, q) *splits orthogonally into homogeneous subgroups*

$$(\Gamma, q) = \perp_{l\in\mathbb{Z}_{\geq 1}} (\Gamma^{(p^l)}, q^{(p^l)})$$

where $(\Gamma^{(p^l)}, q^{(p^l)})$ *is an discriminant form with* $\Gamma^{(p^l)} = (\mathbb{Z}/p^l)^{r_l}$ *for some* $r_l \in \mathbb{Z}_{\geq 0}$*.*

The homogeneous group $\Gamma^{(p^l)}$ is also called *homocyclic*. Of course, $r_l = 0$ for all but finitely many $l \in \mathbb{Z}_{\geq 1}$. Note that Wall has a slightly different notion of quadratic forms. The result also appears in [Str13].

Remark 5.8.
This decomposition is closely related to the *Jordan decomposition* of discriminant forms, see e.g. [Sch09a, Section 2], which is based on the diagonalisation of symmetric forms. Note that the sign of the Jordan decomposition corresponds (up to a normalisation) to the Arf invariant in our notation.

Lemma 5.9 (*changing coefficients, see e.g. [Str13, Lemma 2.7 and 2.10]).*
Let p *be an odd prime and* $l \geq 1$ *be a positive integer. Then there is a bijection between discriminant forms* (Γ, q) *whose underlying group is homogeneous of exponent* p^l *and regular quadratic forms* $(M, \hat{q})$ *over the ring* $A = \mathbb{Z}/p^l$ *where we denote* Γ *viewed as an* A*-module by* M*.*

Let us define the ring isomorphism

$$\begin{aligned}\kappa_{p^l} \colon ((1/p^l)\mathbb{Z})/\mathbb{Z} &\to \mathbb{Z}/p^l,\\ \frac{x}{p^l} \,(\mathrm{mod}\ 1) &\mapsto x \,(\mathrm{mod}\ p^l).\end{aligned}$$

Then, the isomorphism $q \mapsto \hat{q}$ *is given by*

$$\hat{q}(x) = \kappa_{p^l}(2q(x)).$$

A regular quadratic map on a finite abelian group Γ of odd order thus decomposes as quadratic forms (in the sense of Part I) which are of the form $(M, \hat{q}) \colon (\mathbb{Z}/p^l)^r \to \mathbb{Z}/p^l$ for odd primes p and positive integers $l, r \in \mathbb{Z}_{\geq 1}$. Note that since p is odd and q is a quadratic map, we get for all $x \in \Gamma$

$$q(p^l x) = p^{2l} q(x) = 0 \,(\mathrm{mod}\ 1),$$

i.e. the denominator of $q(x)$ is odd for all $x \in \Gamma$.

Remark 5.10.
There are Lagrangian subgroups of discriminant forms which do not correspond to maximally isotropic subspaces of the associated quadratic form above, simply because they do not split off as a direct summand. Consider, for example, the characteristic subgroup $p(\mathbb{Z}/(p^2))$ in a discriminant form with underlying abelian group $\mathbb{Z}/(p^2)$. However, every maximally isotropic subspace naturally yields a Lagrangian subgroup.

In fact, a well-known theorem by Maschke and Schur will allow us to reduce the study of isometries of a discriminant form (Γ, q) to isometries of the homocyclic components of Γ.

Proposition 5.11 (*Maschke, Schur; see e.g. [Hup25, Theorem I.17.6], [Gas52]).*
Let Π be a finite group and let M be a Π-module. Let $\sigma\colon M \to M$ be a endomorphism of Π-modules such that $\sigma(|\Pi|x) = x$ for all $x \in M$. Let $M = M_1 \oplus \tilde{M}_2$ be a direct sum decomposition as abelian groups such that M_1 is a Π-submodule of M. Then there exists a composition of $M = M_1 \oplus M_2$ as Π-modules.

Corollary 5.12.
Let H be a finite abelian p-group and Φ be a group of automorphisms of order not divisible by p. Then there is a decomposition in Φ-invariant subgroups

$$H = \bigoplus_{l \in \mathbb{Z}_{\geq 1}} H_l$$

such that $H_l = (\mathbb{Z}/p^l)^{r_l}$ for some $r_l \in \mathbb{Z}_{\geq 0}$.

This statement appears as [Hup25, Exercise 68-69]. A proof can be found in [May98, Proposition 6.5]; for an elementary proof of this fact, we refer to [Tau49, Lemma 2.1].

As we are just changing the coefficients of the quadratic map, there is a bijection between isometries of the discriminant form (Γ, q) and the isometries of $(M, \hat{q})$. This is very helpful for us as it will allow us to reduce the study of isometries of (odd) discriminant forms to isometries of quadratic forms over rings $\mathbb{Z}/p^l$ for odd primes p and positive integers $l \geq 1$.

Remark 5.13.
We note that a similar decomposition based on the classification results of

[Wal63a, Section §6] (later completed in [Mir84]) appeared in [Dur77, Section 3]; see also [Müg03, Theorem 4.7]. Since we need a decomposition invariant under a group of automorphisms, we needed a slight modification using the above proposition due to Maschke and Schur.

6. Building Extensions via Fixed-Point-Free Isometries

In this chapter, we discuss fixed-point-free automorphisms of groups and discriminant forms. For an introduction to fixed-point-free automorphisms of finite groups, we refer to [Rob96, Section 10.5] or [Gor80, Chapter 10]. A very explicit treatment can be found in [May98].

DEFINITION 6.1 (*fixed-point-free automorphism).*
Let Π be a group. Then an automorphism of groups $\phi\colon \Pi \to \Pi$ is called *fixed-point-free* if $\phi(x) = x$ for $x \in \Pi$ already implies $x = 1$.

A subgroup Φ of $\mathrm{Aut}(\Pi)$ is called *fixed-point-free* if every non-trivial element of Φ is fixed-point-free. We call an action of a group Ψ on Π by automorphisms *fixed-point-free* if every non-trivial element of Ψ acts by a fixed-point-free automorphism on Π.

LEMMA 6.2 (*see e.g. [May98, Proposition 3.1]).*
Let Π be a finite group with a fixed-point-free action τ of $\mathbb{Z}/N$ on Π. Then the orbit of every non-trivial element $x \in \Pi$ is of length N.

Proof. Assume there is an element $x \in \Pi$ such that its orbit under τ is not of length N. Then there exist $i, j \in \mathbb{Z}/N$ with $\tau[i](x) = \tau[j](x)$ but $i \not\equiv j$. Thus, x is fixed by the non-trivial element $\tau[i-j](x)$. We get $x = 1$. ▪

In particular, the order $|\Pi|$ is coprime to N.

For the cyclic group of order N with generator α, we define the *norm element* by

$$\nu := 1 + \alpha + \cdots + \alpha^{N-1}.$$

LEMMA 6.3 (*[Wei94, Theorem 6.2.2]).*
Let M be a module of the cyclic group $\mathbb{Z}/N$ with generator α. Then the cohomology

groups are given by

$$H^n(\mathbb{Z}/N, M) = \begin{cases} M^{\mathbb{Z}/N}, & n = 0, \\ \{x \in M \colon \nu x = 0\}/(\alpha - 1)M, & n \text{ odd}, \\ M^{\mathbb{Z}/N}/NM, & n > 0 \text{ and } n \text{ even}. \end{cases}$$

One possible reason to study fixed-point-free automorphisms is proven in the following lemma. We apply it to fixed-point-free isometries and pointed modular fusion categories in Lemma 6.6 and Example 6.16.

Lemma 6.4.
Let Γ be a finite abelian group with a fixed-point-free action τ of $\mathbb{Z}/N$. Then all cohomology group $H^n(\mathbb{Z}/N, \Gamma)$ are trivial.

Proof. Let us define $\alpha := \tau[1]$. By definition, Γ^{Π} is trivial since α is fixed-point-free. Since α is a fixed-point-free automorphism, we have for all elements $x \in \Gamma$ that the norm element ν sends x to 0, i.e.

$$\nu x = x + \alpha(x) \cdots + \alpha^{N-1}(x) = 0.$$

Since $1 - \alpha$ is a permutation of Γ, it is in particular surjective, hence we get that

$$(1 - \alpha)\Gamma = \Gamma.$$

By the above formula in Lemma 6.3, every cohomology group is trivial. ▪

Let (Γ, q) be a discriminant form and D a modular fusion category in $\mathsf{V}(\Gamma, q)$. We already noted that the group of isomorphism classes of braided autoequivalences $\mathrm{Aut}_{\mathrm{br}}(\mathsf{D})^0$ is isomorphic to the orthogonal group $O(\Gamma, q)$; see Remark 5.3.

Definition 6.5 (*realisable action).*
Let (Γ, q) be a discriminant form and D a modular fusion category in $\mathsf{V}(\Gamma, q)$. Let Π be a finite group. A group homomorphism

$$\tau \colon \Pi \to O(\mathsf{D})$$

is called *realisable* if there exists a monoidal functor

$$T\colon \Pi^{\otimes} \to \mathrm{Aut}_{\mathrm{br}}(\mathsf{D}), \quad g \mapsto T_g$$

such that the underlying isometry of T_g is given by $\tau(g)$ for all $g \in \Pi$.

Realisable actions on not necessarily braided tensor categories are discussed in e.g. [Gal11, Theorem 5.5].

Recall that since $\mathrm{Aut}_{\mathrm{br}}(\mathsf{D})$ is a coherent Gr-category, $O(\Gamma, q)$ acts on the group $\mathrm{Aut}_{\otimes}(1_{\mathsf{D}})$; see Remark 5.5. Then [Gal11, Theorem 5.5] asserts that τ is realisable if and only if an obstruction in the cohomology group $H^3(\Pi, \mathrm{Aut}_{\otimes}(1_{\mathsf{D}}))$ vanishes. The set of equivalence classes of realisations is isomorphic to $H^2(\Pi, \mathrm{Aut}_{\otimes}(1_{\mathsf{D}}))$. Here τ induced a change of groups; see [Mac95, Chapter IV.2].

Following [Gal17, Section 3.1] (see Example 5.5. (2)), we call an action T of a finite group Π on a braided fusion category strict if both the tensor structure of T and of all component functors T_g are identities for all $g \in \Pi$.

Lemma 6.6.
Let (Γ, q) be an odd discriminant form with fixed-point-free action

$$\tau\colon \mathbb{Z}/N \to O(\Gamma, q).$$

Let vect_Γ^q be the modular fusion category from Remark 5.2. Then we can realise τ as a strict action on vect_Γ^q. Furthermore, every realisation of τ is isomorphic (as tensor functors) to the strict one.

Proof. We already noted that every braided autoequivalence of vect_Γ^q is isomorphic to a strict one in Remark 5.3.

By Remark 5.5, the action of the group $O(\Gamma, q)$ on Γ is given by

$$\phi \triangleright x = \phi(x)$$

for all $\phi \in O(\Gamma, q)$ and $x \in \Gamma$. Thus, we get by the Lemma 6.4 above all cohomology groups $H^*(\mathbb{Z}/N, \Gamma)$ vanish. Therefore, the strict action is realisable, and every action is isomorphic to it by [Gal11, Theorem 5.5]. ▪

In the following, we associate to an action of $\mathbb{Z}/N$ by fixed-point-free isometries on (Γ, q)

$$\tau\colon \mathbb{Z}/N \to O(\Gamma, q)$$

the strict action by braided autoequivalences on the modular fusion category vect_Γ^q in $\mathsf{V}(\Gamma, q)$

$$\tau^{\otimes,\mathrm{str}} \colon \mathbb{Z}/N^{\otimes} \to \mathrm{Aut}_{\mathrm{br}}(\mathsf{vect}_\Gamma^q).$$

We recall a result from Part I:

THEOREM 4.9 (*conjugacy classes of fixed-point-free isometries*).
Let A be a finite local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic p. Let (M, q) be a regular quadratic module over A. Then conjugacy classes of fixed-point-free isometries of order $N > 2$ with $p \nmid N$ can be parametrised as follows

$$[(P_1, \mu_1^I), \dots, (P_m, \mu_m^I), (Q_1 Q_1^*, \mu_1^{II}), \dots, (Q_n Q_n^*, \mu_n^{II})]$$

where

(i.) $\{\mu_i^I\}$ are positive integers and $\{P_i\}$ are pairwise strongly relatively coprime polynomials over A such that

- *P_i is monic and self-reciprocal with $\deg(P_i) \geq 2$,*
- *the reduction $\overline{P}_i \in k[T]$ is irreducible with order $\mathrm{ord}(\overline{P}_i) = N$;*

(ii.) $\{\mu_j^{II}\}$ are positive integers and $\{Q_j Q_j^*\}$ are pairwise strongly relatively coprime polynomials over A such that

- *Q_j is monic and non-self-reciprocal,*
- *the reduction $\overline{Q}_j \in k[T]$ is irreducible with order $\mathrm{ord}(\overline{Q}_j) = N$;*

up to reordering the polynomials P_i and $Q_j Q_j^*$, respectively. This data is required to satisfy

$$\mathrm{rank}_A\, M = \sum_i \mu_i^I \deg P_i + \sum_j 2\mu_j^{II} \deg Q_j$$

and the equality in the Witt ring $\mathrm{Wq}(A)$

$$[M] = \sum_i \mu_i^I \Lambda.$$

Note that the order $|M|$ of M is finite and a perfect square.

Definition 6.7 (*hyperbolic action*).
Let (Γ, q) be a discriminant form and let $\tau\colon \mathbb{Z}/N \to O(\Gamma, q)$ be a fixed-point-free action on (Γ, q). Using Lemma 5.6, 5.7, and Corollary 5.12, we can decompose

$$(\Gamma, q) = \perp_{p||\Gamma|} \perp_{l\in\mathbb{Z}_{\geq 1}} (\Gamma^{(p^l)}, q^{(p^l)})$$

where $\Gamma^{(p^l)} = (\mathbb{Z}/p^l)^{r_l}$ for some $r_l \in \mathbb{Z}_{\geq 0}$ and $(\Gamma^{(p^l)}, q^{(p^l)})$ is τ-invariant for all $l \in \mathbb{Z}_{\geq 1}$. We denote the minimal polynomial of the isometry $\tau[i]$ over the non-trivial homocyclic components $\Gamma^{(p^l)}$ by $Q_{p,l,i} \in (\mathbb{Z}/p^l)[T]$. We will call the action τ *hyperbolic* if the reduction $\overline{Q_{p,l,i}} \in \mathbb{F}_p[T]$ has no irreducible self-reciprocal factor for all $i = 1, \dots, N$, and primes p with positive integers $l \in \mathbb{Z}$ such that $r_l > 0$.

Of course, in general, we can also construct fixed-point-free actions which are not hyperbolic. We only assume this because we will see later that they have an extension theory (as braided fusion categories) that is remarkably well-behaved. Note that this excludes isometries of order 1 and 2 since their minimal polynomials are self-reciprocal. For the construction of braided $\mathbb{Z}/2$-crossed extensions of braided fusion categories, we refer to [GLM24a] and [GLM24b].

Let us describe why we called the actions above hyperbolic (and why this definition is well-defined).

Remark 6.8.
Let (Γ, q) be an odd discriminant form with a hyperbolic action $\tau\colon \mathbb{Z}/N \to O(\Gamma, q)$. By Lemma 5.6 and 5.7, we can decompose

$$(\Gamma, q) = \perp_{p||\Gamma|} \perp_{l\in\mathbb{Z}_{\geq 1}} (\Gamma^{(p^l)}, q^{(p^l)})$$

such that $\Gamma^{(p^l)}$ is trivial or an homogeneous p-group of exponent p^l for $l \in \mathbb{Z}_{\geq 1}$. Of course, only a finite number of these groups are non-trivial. By changing the coefficients with Lemma 5.9, the restriction of τ to $(\Gamma^{(p^l)}, q^{(p^l)})$ yields an action of $\mathbb{Z}/N$ by isometries on a regular quadratic module $(M^{(p^l)}, \hat{q}^{(p^l)})$ over $\mathbb{Z}/p^l$. Let us denote the generator of this action of $\mathbb{Z}/N$ on $(M^{(p^l)}, \hat{q}^{(p^l)})$ by $\phi^{(p^l)}$. By Theorem 4.9, there thus exists a finite set of tuples $\{(Q_j^{(p^l)}(Q_j^{(p^l)})^*, \mu_j^{(p^l)})\}_{j=1}^{m_{p^l}}$, where $\mu_j^{(p^l)} \in \mathbb{Z}_{\geq 1}$ and $\{Q_j^{(p^l)}(Q_j^{(p^l)})^*\}$ are pairwise strongly relatively coprime polynomials over $\mathbb{Z}/p^l$ such that

* $Q_j^{(p^l)}$ is monic and non-self-reciprocal,

* the reduction $\overline{Q}_j^{(p^l)} \in \mathbb{F}_p[T]$ is irreducible with order $\mathrm{ord}(\overline{Q}_j^{(p^l)}) = N$.

and the regular quadratic module $(M^{(p^l)}, \hat{q}^{(p^l)})$ decomposes as

$$(M^{(p^l)}, \hat{q}^{(p^l)}) = \perp_j \hat{M}_j^{(p^l)}.$$

Here, the regular quadratic module $\hat{M}_j^{(p^l)}$ is hyperbolic over $\mathbb{Z}/p^l$ and has dimension $2\mu_j^{(p^l)} \deg(Q_j^{(p^l)})$. The minimal polynomial of $\phi^{(p^l)}$ is given by the product

$$\mathrm{minpoly}_{\phi^{(p^l)}} = (Q_1^{(p^l)})(Q_1^{(p^l)})^* \dots (Q_{n_{p^l}}^{(p^l)})(Q_{n_{p^l}}^{(p^l)})^*.$$

In particular, $(M^{(p^l)}, \hat{q}^{(p^l)})$ is a hyperbolic regular quadratic module. Note that this already gives us (in general, many different) decompositions of $(M^{(p^l)}, \hat{q}^{(p^l)})$ in maximally isotropic subspaces which are all τ-invariant. Choosing one splitting for every non-trivial subgroup $(\Gamma^{(p^l)}, q^{(p^l)})$ will give us a splitting of Γ in τ-invariant Lagrangian subgroups.

We will see in Lemma 6.36 that for the equivariantisations of the extensions we will construct, the choice of splitting will not matter.

Remark 6.9.
For an odd prime p and a positive integer l, we constructed in Remark 3.35 an explicit regular quadratic module with an isometry whose conjugacy class has given invariants. Using Lemma 5.7, 5.9, and Corollary 5.12, they can be used to construct fixed-point-free actions on discriminant forms.

Let us construct an example order $N = 12$.

Example 6.10.
We choose the monic non-self-reciprocal polynomials

* $Q_1 := T + 2 \in (\mathbb{Z}/13)[T]$ with rank $\mu_1 = 2$,
* $Q_2 := T^2 + 30 \in (\mathbb{Z}/7^2)[T]$ with rank $\mu_2 = 1$.

Then, we have a fixed-point-free action of $\mathbb{Z}/12$ on the discriminant form with symmetric form on $(\mathbb{Z}/13)^4$ and $(\mathbb{Z}/7^2)^4$

$$\begin{bmatrix} 0 & \frac{7}{13} & 0 & 0 \\ \frac{7}{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{7}{13} \\ 0 & 0 & \frac{7}{13} & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & \frac{25}{7^2} & 0 \\ 0 & 0 & 0 & \frac{25}{7^2} \\ \frac{25}{7^2} & 0 & 0 & 0 \\ 0 & \frac{25}{7^2} & 0 & 0 \end{bmatrix}$$

generated by the automorphism acting as

$$\begin{bmatrix} 11 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \quad \begin{bmatrix} 0 & 19 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 31 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

§.6.1. Braided Crossed Fusion Categories and (De-)Equivariantisation

Let Π be a finite group and D a braided fusion category. Then an action is a tensor functor $T\colon \Pi^{\otimes} \to \mathrm{Aut}_{\otimes}(\mathsf{D})$.

Later, we will use (de)equivariantisation to pass from a braided Π-crossed to a braided fusion category containing a Tannakian category $\mathsf{Rep}_k(\Pi)$ and vice versa.

A braided Π-crossed category C consists of an action T of Π on C, a grading

$$\mathsf{C} = \bigoplus_{g \in \Pi} \mathsf{C}_g$$

compatible with the action, i.e. $T_g(\mathsf{C}_h) \subseteq \mathsf{C}_{ghg^{-1}}$, and Π-braiding isomorphisms

$$c_{x,y}\colon x \otimes y \to T_g(y) \otimes x$$

for $g \in \Pi$, objects x in C_g and y in C satisfying various compatibility axioms; see e.g. [GN21, Definition 3.1] for the diagrams for a (not necessarily strict) fusion category C.

In particular, the action of the group Π on C_e is an action by braided auto-equivalences, i.e. T_g restricts on C_e to a *braided* auto-equivalence for every $g \in \Pi$.

In this case, we can define a fusion category $(\mathsf{C})^{\Pi}$, called the Π-*equivariantisation* of C, of Π-equivariant objects in C, i.e. tuples (x, u) with an object x of C with a family of isomorphisms $u_g\colon T_g(x) \to x$ in C such that

$$u_g \circ T_g(u_h) = u_{gh} \circ T^2_{g,h}$$

where $T^2\colon T_g \circ T_h \to T_{gh}$ is the tensor structure of the categorical action.

There is a procedure (in a certain sense) inverse to equivariantisation called *de-equivariantisation*: A braided fusion category D with a full Tannakian subcategory $\mathsf{E} \simeq \mathsf{Rep}_k(\Pi)$ admits a commutative étale algebra in E and thus in D

(up to isomorphism, it is the algebra of functions from Π to k). The category of modules $\mathsf{D} \rtimes \mathsf{E}$ of this algebra in D has the structure of a braided Π-crossed fusion category. For the details, we refer to [Dri+10, Section 4.4.4].

Note that we have the equalities

$$\mathrm{FPdim}((\mathsf{C})^{\Pi}) = |\Pi| \,\mathrm{FPdim}(\mathsf{C}), \quad \mathrm{FPdim}(\mathsf{D} \rtimes \mathsf{Rep}_k(\Pi)) = \frac{\mathrm{FPdim}(\mathsf{D})}{|\Pi|}, \tag{6.1}$$

see e.g. [Dri+10, Proposition 4.26].

For a discussion of braided Π-crossed equivalences and strictification of braided Π-crossed categories, we refer to [Gal17].

For a fixed braided fusion category D with a fixed *braided* action T of a finite group Π on D, a braided Π-crossed category C such that $\mathsf{C}_e = \mathsf{D}$ and the action of Π on C restricts on D to T is called Π-*crossed extension* of D.[1]

Remark 6.11.
By [ENO05, Proposition 8.20], an extension of an weakly integral braided fusion category is weakly integral. By Equation (6.1), the equivariantisation of this extension is also weakly integral. Hence, both the extension and its equivarianti-sation are in particular pseudo-unitary.

We are mainly interested in extensions of *modular* fusion categories, i.e. in braided Π-crossed categories C such that C_e is a modular fusion category. They have the following very useful property.

Lemma 6.12.
Let Π be a finite group. For a braided Π-crossed extension C of a modular fusion category D, the rank of the component categories C_g is equal to the size of the stabiliser of the action of g on the set of isomorphism classes of simple objects of D for all $g \in \Pi$, i.e.

$$|O(\mathsf{C}_g)| = |\{\mathrm{cls}\, x \in O(\mathsf{D}) \colon T_g(x) \cong x\}|,$$

where we denoted the action of Π on D defined by C by

$$T \colon \Pi^{\otimes} \to \mathrm{Aut}_{\mathrm{br}}(\mathsf{D}).$$

[1] There is a notion of extension where one requires only an equivalence of D and C_e (rather than equality); see e.g. [Jon+22, Example 2.11]. Still, we shall follow [Cui+16].

For a proof, we refer to [Bis20, Proposition 1.1].

The above lemma has an interesting consequence for extensions of pointed modular fusion categories with an action by fixed-point-free isometries.

Remark 6.13.
Let (Γ, q) be a discriminant form. Then any modular fusion category in $\mathsf{V}(\Gamma, q)$ is equivalent (as fusion categories) to $\mathsf{vect}_\Gamma^\Omega$ for some normalised 3-cocycle Ω with $\operatorname{cls}\Omega \in H^3(\Gamma, k^\times)$.

Suppose there exists a braided Π-crossed extension C of a modular fusion category D in $\mathsf{V}(\Gamma, q)$ with an action by braided tensor functors τ such that at least one of the isometries $\tau(g)$ is fixed-point-free for some $g \in \Pi$. Then, by the above lemma, the component category C_g has up to isomorphism exactly one simple object. This is equivalent to the existence of a fibre functor for $\mathsf{vect}_\Gamma^\Omega$; see e.g. [Ost03, Section 2.3]. Therefore, Ω is cohomologically trivial $0 = \operatorname{cls}\Omega$; see also [JL09, Lemma 2.6].

Note that if Γ is odd, the 3-cocycle corresponding to the associator is cohomologically trivial by Lemma 5.4.

Remark 6.14.
For a finite group Π, the Π-equivariantisation of a faithfully graded Π-crossed extension C of a modular fusion category D is modular; see [Tur10, Appendix S, Theorem 4.1].

There are several related notions of *modular* Π-crossed fusion categories; see [Tur10, Chapter VII, 1.1] or the preprint [Kir04, Definition 10.1].

In the next remark, we briefly recall some results on the extension theory of braided fusion categories as summarised in [Cui+16, Section 3.4]. It is based on [ENO10] and [DN21]; see also [DN13] and [Jon+22]. We will see that our results are independent of this theory. However, it was a major inspiration for this work, especially Part III.

Remark 6.15.
Let D be a braided fusion category and Π be a finite group. Then braided Π-crossed extensions of D can be equivalently described as 3-homomorphisms from Π as 3-group to the *Picard* 3-group over the braided fusion category D. This is

based on the observation that for a braided Π-crossed extension $\mathsf{C} = \bigoplus_{g\in\Pi} \mathsf{C}_g$ of D, the component categories C_g are invertible (one-sided) D-bimodule categories. A group homomorphism $\tau\colon \Pi \to \mathrm{Aut}_{\mathrm{br}}(\mathsf{D})$ can be lifted to a monoidal functor if an obstruction $O_3(\tau) \in H^3(\Pi, \mathrm{Aut}_\otimes(1_{\mathsf{D}}))$ vanishes. If the obstruction vanishes, the possible liftings form a torsor over $\alpha \in H^2(\Pi, \mathrm{Aut}_\otimes(1_{\mathsf{D}}))$. A monoidal functor $\tau^\otimes\colon \Pi^\otimes \to \mathrm{Aut}_{\mathrm{br}}(\mathsf{D})$ can be lifted to a 3-homomorphism if a obstruction $O_4(\tau^\otimes, \alpha)$ vanishes. Equivalences classes of crossed Π-extensions form a torsor over the cokernel of a map

$$H^1(\Pi, \mathrm{Aut}_\otimes(1_{\mathsf{D}})) \to H^3(\Pi, k^\times);$$

see [DN21, Corollary 2.25]. If D is a modular fusion category, we can identify by [Dri+10, Lemma 3.31] the group $\mathrm{Aut}_\otimes(1_{\mathsf{D}})$ with the group of simple invertible objects $O(\mathsf{D}_{\mathrm{pt}})$; see also [ENO10].

Example 6.16 (*braided crossed extensions of pointed braided categories with a fixed-point-free isometry).*
We now consider a pointed modular fusion category $\mathsf{V}(\Gamma, q)$ with odd discriminant form (Γ, q) with a fixed-point-free action τ of $\mathbb{Z}/N$. Let vect_Γ^q be the modular fusion category in $\mathsf{V}(\Gamma, q)$ from Remark 5.2. By Lemma 6.6, we can realise τ as strict action $\tau^{\otimes,\mathrm{str}}$ on D. Now, we can apply the obstruction theory in Remark 6.15 above.

Recall that the cohomology groups $H^*(\mathbb{Z}/N, \Gamma)$ are all trivial; see Lemma 6.4. For the cyclic group $\mathbb{Z}/N$, the cohomology group $H^4(\mathbb{Z}/N, k^\times) = 0$ is also trivial. Thus, the equivalence classes of braided $\mathbb{Z}/N$-crossed extensions form a torsor over the cyclic group $H^3(\mathbb{Z}/N, k^\times) \cong \mathbb{Z}/N$.

We will build braided crossed categories by starting with a pointed fusion category and considering the equivariantisation of a subgroup.

Example 6.17 (*Pointed fusion categories as braided crossed fusion categories).*
Let C be a pointed fusion category. Then the simple objects form a finite group Π and $\mathsf{C} \simeq \mathsf{vect}_\Pi^\Omega$ for some normalised 3-cocycle $\Omega \in H^3(\Pi, k^\times)$. Then there is a unique (up to a unique isomorphism) structure of a braided Π-crossed fusion category on C; see [Dri+10, Proposition 4.61], [GN21, Proposition 3.6] or [Jon+22, Corollary 7.7 and Remark 7.8]. The Π-equivariantisation of this braided Π-crossed fusion category is braided equivalent to the Drinfeld centre of C.

The Drinfeld centre of a pointed braided fusion category is braided equivalent to the category of representations of the well-known quasi-Hopf algebra introduced by Dijkgraaf, Pasquier, and Roche in [DPR90], i.e.

$$Z(\mathsf{C}) \simeq {}_{D_{\Omega^{-1}}(\Pi)}\mathsf{Mod},$$

see e.g. [Nai10, Lemma 6.3].

Remark 6.18.
There is a correspondence between *skeletal* pointed braided fusion category with underlying finite group Π and the third abelian cohomology group $H^3_{\mathrm{ab}}(\Pi, k^\times)$. For details, we refer to [JS93, section 3].

For a finite abelian group H, we denote its group of irreducible characters by X_H.

Example 6.19.
Let H be a finite abelian group. Then the H-equivariantisation of the *symmetric* pointed vect_H category (i.e. corresponding to the pre-metric group $(H, 1)$ with trivial quadratic form) is the modular pointed category with underlying metric group $(H \oplus X_H, q_{\mathrm{hyp}})$ where q_{hyp} is the hyperbolic quadratic form

$$\begin{aligned} q_{\mathrm{hyp}} \colon H \oplus X_H &\to k^\times, \\ (x, \phi) &\mapsto \phi(x). \end{aligned}$$

Note that it is hyperbolic since it contains the Lagrangian subgroups H and X_H. We denote the finite abelian group $H \oplus X_H$ with the hyperbolic quadratic form above by $\mathrm{Hyp}\, H$.

An action τ of $\mathbb{Z}/N$ on H induces an action $\mathrm{Hyp}\,\tau$ on $\mathrm{Hyp}\, H$: If we write $\tau[i](x) := {}^i x$ for $i \in \mathbb{Z}/N$, then $i \in \mathbb{Z}/N$ acts on a character $\phi \in X_H$ as

$${}^i\phi(x) = \phi({}^{-i}x)$$

for all $x \in H$. Then, we have

$$q_{\mathrm{hyp}}({}^i x, {}^i\phi) = ({}^i\phi)({}^i x) = \phi(x) = q_{\mathrm{hyp}}(x, \phi)$$

for all $x \in H$, $\phi \in X_H$ and $i \in \mathbb{Z}/N$.

We will consider semi-direct product groups constructed from a fixed-point free-action in the sequel. Recall that for a group Π with normal subgroup H, we have the *inflation* map on cohomology groups

$$\mathrm{Infl}_{\Pi}^{\Pi/H}(\,*\,)\colon H^n(\Pi/H,k^\times)\to H^n(\Pi,k^\times),$$

see [Mac95, Chapter XI 9].

Definition 6.20.
Let H be an odd finite abelian group with a fixed-point-free action τ of $\mathbb{Z}/N$. We define the semi-direct product $\Pi := H \rtimes_\tau \mathbb{Z}/N$.

Let $\omega \in H^3(\mathbb{Z}/N,k^\times)$ be a normalised 3-cocycle with inflation $\Omega = \mathrm{Infl}_{\Pi}^{\mathbb{Z}/N}(\omega)$. We will denote the Π-crossed category $\mathsf{vect}_{\Pi}^{\Omega}$ by $\mathsf{Cr}(H,\tau,\omega)$.

Explicitly, the 3-cocycle Ω takes the values

$$\Omega[(x_1,i_1),(x_2,i_2),(x_3,i_3)] = \omega(i_1,i_2,i_3)$$

for all $(x_1,i_1),(x_2,i_2),(x_3,i_3)\in\Pi$.

Remark 6.21 (*see e.g. [Mac95, Corollary XI.10.3]).*
Let H be a finite group with an action τ by the cyclic group $\mathbb{Z}/N$. We define the semi-direct product $\Pi := H \rtimes_\tau \mathbb{Z}/N$. If the order of the group H is coprime to N, there is a split exact sequence

$$0\to H^n(\mathbb{Z}/N,k^\times)\xrightarrow{\mathrm{Infl}_{\Pi}^{\mathbb{Z}/N}(\,*\,)} H^n(\Pi,k^\times)\xrightarrow{\mathrm{Res}_{\mathbb{Z}/N}^{\Pi}(\,*\,)} H^n(H,k^\times)^{\Pi}\to 0$$

In particular, a 3-cocycle $\Omega\in H^3(\Pi,k^\times)$ whose restriction to H is cohomologically trivial is cohomologous to an inflation.

Note that if $\mathbb{Z}/N$ acts on H by fixed-point-free automorphisms, the orders of the groups H and $\mathbb{Z}/N$ are coprime, so the above remark applies.

In the above setting, we will consider the equivariantisation of H, a normal subgroup of Π, which has a canonical structure of a braided $\mathbb{Z}/N$-crossed fusion category, by the following Lemma.

Lemma 6.22 (*sequential equivariantisation, [Cui+16, Section 4]).*
Let $\Pi = H\rtimes M$ be a semi-direct product of finite groups H and M. Let

C *be a braided* Π*-crossed extension of the braided fusion category* D*. Then* $\mathsf{C}(H) := \bigoplus_{h\in H} C_h$ *admits the structure of a braided* H*-crossed extension of* D *and the* H*-equivariantisation* $(\mathsf{C})^H$ *admits the structure of a braided* M*-crossed extension of the* H*-equivariantisation* $(\mathsf{C}(H))^H$ *such that there is an equivalence (of braided fusion categories)*

$$((\mathsf{C})^H)^M \simeq (\mathsf{C})^\Pi.$$

Similar arguments can be found in [GJ25, Section 3] or [Dri+10, Equation 75].

By applying the above lemma to the pointed braided Π-crossed categories $\mathsf{Cr}(H, \tau, \omega)$, we get:

Corollary 6.23 (*braided* $\mathbb{Z}/N$*-crossed extensions for fixed-point-free actions). Let* H *be an odd finite abelian group with a fixed-point-free action* τ *of* $\mathbb{Z}/N$ *on* H*. We define the semi-direct product* $\Pi := H \rtimes_\tau \mathbb{Z}/N$*. Let* $\operatorname{cls}\omega \in H^3(\mathbb{Z}/N, k^\times)$ *be a normalised* 3*-cocycle with inflation* $\Omega = \operatorname{Infl}_\Pi^{\mathbb{Z}/N}(\omega)$*.*

Then, there is a braided $\mathbb{Z}/N$*-crossed extension* $(\mathsf{Cr}(H, \tau, \omega))^H$ *of* $(\mathsf{vect}_H)^H$ *whose* $\mathbb{Z}/N$*-equivariantisation is equivalent (as braided fusion categories) to*

$$_{D_{\omega^{-1}}(\Pi)}\mathsf{Mod} \simeq Z(\mathsf{vect}_\Pi^\Omega) \simeq (\mathsf{Cr}(H, \tau, \omega))^\Pi \simeq ((\mathsf{vect}_\Pi^\Omega)^H)^{\mathbb{Z}/N}.$$

Furthermore, $(\mathsf{vect}_H)^H$ *is the hyperbolic modular fusion category* $\mathsf{vect}_{H\oplus X_H}^{q_{\mathrm{hyp}}}$ *from Example 6.19 and* $\mathbb{Z}/N$ *acts on* $\mathsf{vect}_{H\oplus X_H}^{q_{\mathrm{hyp}}}$ *as* $\operatorname{Hyp}\tau$*.*

To verify that $\mathbb{Z}/N$ indeed acts as $\operatorname{Hyp}\tau$ on $\mathsf{vect}_{H\oplus X_H}^{q_{\mathrm{hyp}}}$, we refer to the explicit action defined in the proof of [Cui+16, Lemma 2]. The calculation is straightforward.

Let us describe the fusion category constructed above; see e.g. [BN13].

Remark 6.24.
The group H acts by left conjugation on Π (and thus on vect_Π^Ω), i.e.

$$(x, 0)(y, i)(-x, 0) = (x - {}^i x + y, i)$$

for all $x \in H$ and $(y, i) \in \Pi$. The tensor structure of this action is trivial since Ω is an inflation. We can therefore directly calculate that the simple objects of the fusion category $\mathsf{C} = (\mathsf{Cr}(H, \tau, \omega))^H$ constructed in Corollary 6.23 are given by:

(i.) $|H|^2$ simple objects of quantum dimension 1 indexed by $H \times X_H$. For $y \in H$ and $\phi \in X_H$, the underlying object in vect_Π^Ω is $(y, 0)$ with equivariant structure given by ϕ, i.e.

$$^{(x,0)}(y,0) \xrightarrow{\phi(x)} (y,0).$$

We denote this simple object of C by $X_{y,\phi}$.

(ii.) $N-1$ simple objects of quantum dimension $|H|$ indexed by $(\mathbb{Z}/N) \setminus \{0\}$. For $i \in (\mathbb{Z}/N) \setminus \{0\}$, the underlying object in vect_Π^Ω is $\oplus_{z \in H}(z, i)$ with the obvious equivariant structure. We denote this simple object of C by M_i.

The fusion rules are easily computed. They are given by

$$\begin{aligned} X_{y_1,\phi_1} \otimes X_{y_2,\phi_2} &= X_{y_1+y_2,\phi_1\phi_2} \\ X_{y_1,\phi_1} \otimes M_i = M_i &= M_i \otimes X_{y_2,\phi_2} \\ M_i \otimes M_j &= \begin{cases} \bigoplus_{y,\phi} X_{y,\phi}, & i+j=0, \\ |H| M_{i+j} & i+j \neq 0. \end{cases} \end{aligned}$$

for $y_1, y_2 \in H$, $\phi_1, \phi_2 \in X_H$ and $i, j = 1, \dots, N-1$.

We also remark that the associators of the fusion category C can also easily be calculated since we know the associator of the underlying fusion category braided Π-crossed fusion category $\mathsf{Cr}(H, \tau, \omega)$. Namely, the only non-trivial associator is given by

$$(M_i \otimes M_j) \otimes M_k \xrightarrow{\omega(i,j,k)} M_i \otimes (M_j \otimes M_k).$$

for all $i, j, k = 1, \dots, N-1$.

In fact, the results in [Cui+16, Section 4] allow for a calculation of a braided $\mathbb{Z}/N$-crossed structure of $(\mathsf{vect}_\Pi^\Omega)^H$.

REMARK 6.25.
Let C be a braided Π-crossed extension of a pointed modular fusion category in $\mathsf{V}(\Gamma, q)$ with action of Π on $\mathsf{V}(\Gamma, q)$. If Π acts by a group of fixed-point-free isometries on (Γ, q), then we will always get fusion rules similar to the ones above:

By Lemma 6.12, there exists exactly one simple object, up to isomorphism, in C_g for $g \in \Pi$. Then, the fusion rules take the above form; see [GR25, Theorem

5.1]. Since for $N = 2$, these recover the fusion rules for the Tambara-Yamagami fusion categories [TY98], they are called Π-*Tambara-Yamagami* in [GR25, Definition 5.1]. For $\Pi = \mathbb{Z}/3$ such categories have appeared in [JL09, Definition 1.3].

Remark 6.26.
For the trivial group H, the identity is a fixed-point-free automorphism. Thus, Corollary 6.23 applies. We get for every normalised 3-cocycle $\omega \in H^3(\mathbb{Z}/N, k^\times)$, a braided $\mathbb{Z}/N$-crossed extension $\mathsf{vect}^{\omega}_{\mathbb{Z}/N}$ of vect_k whose equivariantisation is braided equivalent to a Drinfeld centre. This is related to the Dijkgraaf-Witten conjecture; for details, we refer to [GR25, Section 3].

§.6.2. Representation Theory of Semi-Direct Products with Fixed-Point-Free Action

We briefly give an overview of the representation theory of semi-direct products with an action by fixed-point-free automorphism using the *method of little groups* by Wigner and Mackey; see e.g. [Ser77, Section 8.2].

Recall that for an abelian group, all irreducible representations are of dimension 1 and form a group which we will called X_H.

Lemma 6.27.
Let H be a finite abelian group with a fixed-point-free action τ of $\mathbb{Z}/N$. Then the action of $\mathbb{Z}/N$ on on the group X_H given by

$$(\phi^i)(a) = \phi({}^i a)$$

for $i \in \mathbb{Z}/N$, $\phi \in X_H$ and $a \in H$ is fixed-point-free, i.e. for $i \neq 0$, $\phi^i = \phi$ for some $\phi \in X_H$ implies $\phi \equiv 1$.

Proof. Let $i \in (\mathbb{Z}/N) \setminus \{1\}$ such that $\phi^i = \phi$ for some $\phi \in X_H$. We denote the order of i by $n > 1$. Since i is non-trivial, it acts fixed-point-free on H. Since ϕ is constant on the orbits of i, we get that

$$1 = \phi(0) = \phi(a)\phi({}^i a) \ldots \phi({}^{i^{n-1}} a) = \phi(a)^n$$

for all $a \in H$. Thus, the order of ϕ divides n. As N and $|H|$ are coprime, this implies $\phi \equiv 1$. ▪

We get for a fixed-point-free action τ of $\mathbb{Z}/N$ on H, there are exactly two classes of irreducible representations of $\Pi = H \rtimes_\tau \mathbb{Z}/N$: First, there are irreducible representations θ_α for $\alpha \in X_{\mathbb{Z}/N}$ with characters

$$\theta_\alpha[(a,i)] = \alpha(i)$$

for all $(a,i) \in \Pi$.
Secondly, pick representatives χ for the non-trivial orbits of $X_H/(\mathbb{Z}/N)$. Then, there are irreducible representations θ_χ with characters

$$\begin{aligned} \theta_\chi[(a,i)] &= 0 \\ \theta_\chi[(a,0)] &= \sum_{i\in\mathbb{Z}/N} ({}^i\chi)(a) \end{aligned}$$

for $i \in (\mathbb{Z}/N) \setminus \{0\}$ and $a \in H$. There are $(|X_H|-1)/N = (|H|-1)/N$ representations of the form θ_χ.

The degree of θ_α is 1 while the degree of θ_χ is N. These are all the irreducible representations as

$$N \cdot 1 + \frac{|H|-1}{N} N^2 = |H|N = |\Pi|.$$

In the next section, we will see that every braided $\mathbb{Z}/N$-extension of a hyperbolic discriminant form Γ with Lagrangian subgroup U and fixed-point-free action induced by an action on U contains this Tannakian category $\mathsf{Rep}_k(U \rtimes \mathbb{Z}/N)$ as a Lagrangian subcategory.

§.6.3. Braided Crossed Extensions of Hyperbolic Discriminant Forms

In the following, we will fix an odd discriminant group (Γ, q) with a hyperbolic fixed-point-free action τ of a cyclic group $\mathbb{Z}/N$

$$\tau\colon \mathbb{Z}/N \to O(\Gamma, q)$$

which extends to an action of braided autoequivalences the modular fusion category vect_Γ^q

$$\tau^{\otimes,\mathrm{str}}\colon \mathbb{Z}/N^{\otimes} \to \mathrm{Aut}_{\mathrm{br}}(\mathsf{vect}_\Gamma^q)$$

by Remark 5.3. Since we work with *odd* discriminant forms in this section, we can always choose the category of graded vector spaces vect_Γ^q with appropriate braiding in $\mathsf{V}(\Gamma, q)$; see Remark 5.2.

LEMMA 6.28.
Let (Γ, q) be an odd discriminant form with a hyperbolic fixed-point-free action τ of $\mathbb{Z}/N$. We realise τ as strict action on vect_Γ^q, i.e.

$$\tau^{\otimes,\mathrm{str}} \colon \mathbb{Z}/N^{\otimes} \to \mathrm{Aut}_{\mathrm{br}}(\mathsf{vect}_\Gamma^q).$$

We split $\Gamma = U \oplus V$ such that U is Lagrangian and τ invariant. Let C be a braided $\mathbb{Z}/N$-crossed extension of vect_Γ^q with action $\tau^{\otimes,\mathrm{str}}$. Then its $\mathbb{Z}/N$-equivariantisation $(\mathsf{C})^{\mathbb{Z}/N}$ contains the Lagrangian subcategory $(\mathsf{vect}_V)^{\mathbb{Z}/N}$.

Proof. First note that $(\mathsf{vect}_V)^{\mathbb{Z}/N}$ is a subcategory of $(\mathsf{vect}_\Gamma)^{\mathbb{Z}/N}$ which is a subcategory of $(\mathsf{C})^{\mathbb{Z}/N}$. Thus, we only need to show that $(\mathsf{vect}_V)^{\mathbb{Z}/N}$ is Lagrangian in C: Note that it is isotropic and symmetric since V is totally isotropic in Γ. By Müger's Centraliser Theorem [Müg03, Theorem 3.2], it is therefore Lagrangian since it has dimension $N|\Gamma|^{\frac{1}{2}}$. ▪

It was shown in [Dri+07, Section 4] that modular fusion categories containing a Lagrangian subcategory are equivalent to Drinfeld centres of pointed fusion categories. In fact, they even show how to reconstruct a Drinfeld centre from a Lagrangian subcategory; see [Dri+07, Theorem 4.5]. They explicitly use Deligne's characterisation of symmetric fusion categories; see [Dri+07, Theorem 2.6], [Del02], and [DM82].

COROLLARY 6.29.
Let (Γ, q) be an odd discriminant form with a hyperbolic fixed-point-free action τ of $\mathbb{Z}/N$. We realise τ as strict action on vect_Γ^q, i.e.

$$\tau^{\otimes,\mathrm{str}} \colon \mathbb{Z}/N^{\otimes} \to \mathrm{Aut}_{\mathrm{br}}(\mathsf{vect}_\Gamma^q).$$

We split $\Gamma = U \oplus V$ such that U is Lagrangian and τ invariant.

Let C be a braided $\mathbb{Z}/N$-crossed extension of vect_Γ^q with action $\tau^{\otimes,\mathrm{str}}$. Then its equivariantisation is equivalent (as a braided fusion category) to the Drinfeld centre of $Z(\mathsf{vect}^{\Omega}_{U \rtimes \mathbb{Z}/N})$ for some $\Omega = \mathrm{Infl}^{\mathbb{Z}/N}_{U \rtimes \mathbb{Z}/N}(\omega)$ for an $\omega \in H^3(\mathbb{Z}/N, k^\times)$.

Proof. By Lemma 6.28, the equivariantisation $(\mathsf{C})^{\mathbb{Z}/N}$ contains the Lagrangian subcategory $(\mathsf{vect}_V)^{\mathbb{Z}/N}$. Note that $(\mathsf{vect}_V)^{\mathbb{Z}/N} \simeq \mathsf{Rep}_k(U \rtimes \mathbb{Z}/N)$; see e.g. [Nik08, Example 2.7]. Thus, we can now apply the proof of [Dri+07, Theorem 4.5] and the equivariantisation is braided equivalent to the Drinfeld centre $Z(\mathsf{vect}^{\omega}_{U \rtimes \mathbb{Z}/N})$ for some 3-cocycle $\omega \in H^3(U \rtimes \mathbb{Z}/N, k^\times)$.

Note that they choose the commutative étale algebra A in $\mathsf{Rep}_k(U \rtimes \mathbb{Z}/N)$ (corresponding to the algebra of functions) of dimension $|U|N$ and show that the category of (right) modules over A in $(\mathsf{C})^{\mathbb{Z}/N}$ is pointed with simple objects indexed by $U \rtimes \mathbb{Z}/N$. Since by Remark 6.13, the 3-cocycle on vect^q_Γ is trivial, the restriction of Ω to U is cohomologically trivial. Thus Ω is cohomologous to an inflation by Remark 6.21. ▪

Lemma 6.30.
Let (Γ, q) be an odd discriminant form with a hyperbolic fixed-point-free action τ of $\mathbb{Z}/N$. We realise τ as strict action on vect^q_Γ, i.e.

$$\tau^{\otimes,\mathrm{str}} \colon \mathbb{Z}/N^{\otimes} \to \mathrm{Aut}_{\mathrm{br}}(\mathsf{vect}^q_\Gamma).$$

We split $\Gamma = U \oplus V$ such that U is Lagrangian and τ invariant. We define the semi-direct product by $\Pi := U \rtimes_\tau \mathbb{Z}/N$. Then the braided $\mathbb{Z}/N$-crossed extensions $(\mathsf{vect}^{\Omega}_{U \rtimes_\tau \mathbb{Z}/N})^U$ for $\omega \in H^3(\mathbb{Z}/N, k^\times)$ a normalised 3-cocycle with inflation $\Omega = \mathrm{Infl}^{\mathbb{Z}/N}_{\Pi}(\omega)$ are inequivalent.

Proof. Let $\omega_1, \omega_2 \in H^3(\mathbb{Z}/N, k^\times)$ be normalised 3-cocycles with inflations to Π, $\Omega_w \in \mathrm{Infl}^{\mathbb{Z}/N}_{\Pi}(\omega_w)$ for $w = 1, 2$. Assume we have an equivalence of braided $\mathbb{Z}/N$-crossed extensions

$$B \colon (\mathsf{vect}^{\Omega_1}_{U \rtimes_\tau \mathbb{Z}/N})^U \to (\mathsf{vect}^{\Omega_2}_{U \rtimes_\tau \mathbb{Z}/N})^U.$$

Then $B|_{\mathsf{vect}^U_U} = \mathrm{id}_{\mathsf{vect}^U_U}$ so in particular, it maps $\mathsf{Rep}_k(U) \simeq \mathsf{vect}^U_k$ to itself. We get the (braided U-crossed) tensor equivalence

$$\mathsf{vect}^{\Omega_1}_{U \rtimes_\tau \mathbb{Z}/N} \to \mathsf{vect}^{\Omega_2}_{U \rtimes_\tau \mathbb{Z}/N},$$

thus $\mathrm{cls}\,\Omega_1 = \mathrm{cls}\,\Omega_2$. ▪

Theorem 6.31 (*classification of hyperbolic $\mathbb{Z}/N$-crossed extensions).*
Let (Γ, q) be an odd discriminant form with a hyperbolic fixed-point-free action

$$\tau \colon \mathbb{Z}/N \to O(\Gamma, q).$$

We realise τ as strict action on vect_Γ^q, i.e.

$$\tau^{\otimes,\mathrm{str}}\colon \mathbb{Z}/N^{\otimes} \to \mathrm{Aut}_{\mathrm{br}}(\mathsf{vect}_\Gamma^q).$$

The pointed braided category vect_Γ^q with action $\tau^{\otimes,\mathrm{str}}$ has N inequivalent braided $\mathbb{Z}/N$-crossed extensions.

We split $\Gamma = U \oplus V$ such that U is Lagrangian and τ invariant. Representatives for these N equivalence classes are given by the equivariantisations $(\mathsf{vect}^\Omega_{U\rtimes\mathbb{Z}/N})^U$ for the inflations

$$\Omega = \mathrm{Infl}^{\mathbb{Z}/N}_{U\rtimes\mathbb{Z}/N}(\omega)$$

for $\omega \in H^3(\mathbb{Z}/N, k^\times)$ whose $\mathbb{Z}/N$-equivariantisations are given by the Drinfeld centre $Z(\mathsf{vect}^\Omega_{U\rtimes\mathbb{Z}/N})$.

Remark 6.32.
Recall from Remark 6.16 that the extension theory of braided fusion categories predicts N inequivalent braided $\mathbb{Z}/N$-crossed extensions of vect_Γ^q with action $\tau^{\otimes,\mathrm{str}}$. Our result above is thus in perfect agreement with this expectation, but our proof is independent of this result.

Remark 6.33.
Let p, q be odd primes such that $p \mid q-1$. Then there exists an element $\alpha \in \mathbb{F}_q^\times$ with order p, i.e. $\alpha^p = 1$. Then $\phi\colon \mathbb{Z}/q \to \mathbb{Z}/q,\ x \mapsto \alpha x$ is an automorphism of $\mathbb{Z}/q$ of order p. Thus, we have a fixed-point-free action $\mathrm{Hyp}\,\phi$ of $\mathbb{Z}/p$ on the hyperbolic quadratic form

$$q_{\mathrm{hyp}}\colon \mathbb{Z}/q \oplus X_{\mathbb{Z}/q} \to k^\times$$

with Lagrangian subgroup $\mathbb{Z}/q$; see Example 6.19.

Therefore, Theorem 6.31 tells us that there are p inequivalent braided $\mathbb{Z}/p$-crossed extensions of $\mathsf{vect}^{q_{\mathrm{hyp}}}_{\mathbb{Z}/q\oplus X_{\mathbb{Z}/q}}$ given by the $\mathbb{Z}/q$-equivariantisations with inflations

$$(\mathsf{vect}^\Omega_{\mathbb{Z}/q\rtimes\mathbb{Z}/p})^{\mathbb{Z}/q} \text{ with } \Omega = \mathrm{Infl}^{\mathbb{Z}/p}_{\mathbb{Z}/q\rtimes\mathbb{Z}/p}(\omega), \quad \omega \in H^3(\mathbb{Z}/p, k^\times)$$

whose $\mathbb{Z}/p$-equivariantisations are given by the Drinfeld centre $Z(\mathsf{vect}^\Omega_{\mathbb{Z}/q\rtimes\mathbb{Z}/p})$. These are exactly the Drinfeld centres appearing in [MS21, Corollary 4.2]: For $p > 3$, the modular data does not distinguish them up to braided equivalence.

In particular, we proved in Theorem 6.31 that $\mathbb{Z}/N$-equivariantisations of braided $\mathbb{Z}/N$-crossed extensions of pointed braided fusion categories with cyclic hyperbolic fixed-point-free action are described (up to braided equivalence) by the representation theory of a certain quasi-Hopf algebra. In the next chapter, we describe these modular fusion categories. We end this chapter by showing in the next section that two different choices of invariant Lagrangian subgroups in Theorem 6.31 yield braided equivalent equivariantisations.

§.6.4. Categorical Morita Equivalence of Pointed Fusion Categories

We constructed braided $\mathbb{Z}/N$-crossed extensions of pointed braided fusion categories with a *hyperbolic* action. Their equivariantisations are given by Drinfeld centres of pointed categories. As we already mentioned, Drinfeld centres of pointed categories are equivalent as braided fusion categories if and only if the pointed fusion categories are categorically Morita equivalent. Therefore, we briefly recall the explicit group-theoretic description of categorically Morita equivalent pointed fusion categories; see [Uri17] and [Nai07].

Recall that for a finite group Π with normal abelian subgroup $H \leq \Pi$, we can assign a double complex

$$C^{p,q} := C^p(H\backslash\Pi, C^q(\Pi, \mathrm{Coind}_H^\Pi(k^\times)))$$

with horizontal differential $\delta_{H\backslash\Pi}$ and vertical differential δ_Π; see [Uri17, Section 1B1]. The cohomology of the total complex is canonically isomorphic to the cohomology of Π with values in $k^\times$. By considering the right filtration, we obtain the well-known Lyndon-Hochschild-Serre spectral sequence.

Proposition 6.34 (*[Uri17, Theorem 3.9]*).
Let Π_1 and Π_2 be finite groups with normalised 3-cocycles $\Omega_1 \in H^3(\Pi_1, k^\times)$ and $\Omega_2 \in H^3(\Pi_2, k^\times)$. Then the pointed fusion categories $\mathsf{vect}_{\Pi_1}^{\Omega_1}$ and $\mathsf{vect}_{\Pi_2}^{\Omega_2}$ are categorically Morita equivalent if and only if

(i.) there exists a normal abelian subgroup $H_1 \leq \Pi_1$ with isomorphisms of groups

$$a\colon H_1 \rtimes_{\epsilon_1} (H_1\backslash\Pi_1) \to \Pi_1 \quad b\colon (H_1\backslash\Pi_1) \ltimes_{\epsilon_2} X_{H_1} \to \Pi_2$$

with 2-cocycles

$$\epsilon_1 \in Z^2(H_1\backslash\Pi_1, H_1) \text{ and } \epsilon_2 \in Z^2(H_1\backslash\Pi_1, X_{H_1}),$$

(ii.) *there exists a map* $f \in C^3(H_1\backslash\Pi_1, k^\times)$ *such that* $\delta_{H_1\backslash\Pi_1} f = \epsilon_2 \wedge \epsilon_1$,

(iii.) *the 3-cocycles defined by*

$$\omega_1[(x_1, z_1), (x_2, z_2), (x_3, z_3)] := \epsilon_2(z_1, z_2)(x_3) f(z_1, z_2, z_3)$$
$$\omega_2[(z_1, \phi_1), (z_2, \phi_2), (z_3, \phi_3)] := f(z_1, z_2, z_3)\phi_1(\epsilon_1(z_2, z_3))$$

satisfy

$$\operatorname{cls}\omega_1 = \operatorname{cls} a^*\Omega_1 \qquad \operatorname{cls}\omega_2 = \operatorname{cls} b^*\Omega_2$$

in $H^3(H_1 \rtimes_{\epsilon_1} H_1\backslash\Pi_1, k^\times)$ *and* $H^3(H_1\backslash\Pi_1 \ltimes_{\epsilon_2} X_{H_1}, k^\times)$, *respectively.*

For trivial 2-cocycles ϵ_i above, the groups $H_1 \rtimes_{\epsilon_1} H_1\backslash\Pi_1$ and $H_1\backslash\Pi_1 \ltimes_{\epsilon_2} X_{H_1}$ are semi-direct products. For details, we refer to [Uri17, Section 1A].

Proposition 6.35 (*see [Tau55, Theorem 3.3]).*
Let H_1 *and* H_2 *be two finite groups of order coprime to* N *with actions*

$$\tau_i \colon \mathbb{Z}/N \to \operatorname{Aut}(H_i).$$

Then the semi-direct products

$$\Pi_1 := H_1 \rtimes_{\tau_1} \mathbb{Z}/N, \quad \Pi_2 := H_2 \rtimes_{\tau_2} \mathbb{Z}/N$$

are isomorphic if and only if there exist automorphisms $\chi \in \operatorname{Aut}(\mathbb{Z}/N)$ *and an isomorphism* $\omega \colon H_2 \to H_1$ *such that*

$$\tau_2(\chi(i)) = \omega^{-1}\tau_1(i)\omega$$

for all $i \in \mathbb{Z}/N$.

Since H_i and $\mathbb{Z}/N$ are groups of coprime order, then H_i and $\mathbb{Z}/N$ are characteristic in their semi-direct products Π_i. Note that this Proposition 6.35 together with 6.34 shows that there is a connection between the conjugacy class of the action on H and the equivariantisations of the extensions which we constructed in Theorem 6.31.

In general, we have the symmetry:

Lemma 6.36.
Let H be an odd finite abelian group with a fixed-point-free action τ of $\mathbb{Z}/N$. We define the semi-direct product $\Pi = H \rtimes_\tau \mathbb{Z}/N$ with action $\tau\colon \mathbb{Z}/N \to \mathrm{Aut}(H)$. Let $\omega \in H^3(\mathbb{Z}/N, k^\times)$ be a normalised 3-cocycle and $\Omega = \mathrm{Infl}^{\Pi}_{\mathbb{Z}/N}(\omega)$. We decompose as before

$$H = \bigoplus_{p,l} \bigoplus_{\iota \in I_{(p,l)}} H^{\iota}_{(p,l)}.$$

Let H' be the odd finite abelian group

$$H' := \bigoplus_{q,m} \bigoplus_{\kappa \in J_{(q,m)}} (H')^{\kappa}_{(q,m)}$$

such that we have

- *an isomorphism $\Xi\colon I_{(p,l)} \to J_{(p,l)}$,*
- *a partition $I_{(p,l)} = I^{(+)}_{(p,l)} \cup I^{(-)}_{(p,l)}$,*

such that for $\iota^+ \in I^{(+)}_{(p,l)}, \iota^- \in I^{(-)}_{(p,l)}$

$$H^{\iota^+}_{(p,l)} = (H')^{(\Xi(\iota^+))}_{(p,l)} \qquad X_{H^{\iota^-}_{(p,l)}} = (H')^{(\Xi(\iota^-))}_{(p,l)}.$$

We define $\Pi' := H' \rtimes_{\tau'} \mathbb{Z}/N$ with

$$\tau'\colon \mathbb{Z}/N \to \mathrm{Aut}(H'),$$
$$\tau'[i]|_{H^{\iota^+}_{(p,l)}} = \tau[i] \qquad \tau'[i]|_{H^{\iota^-}_{(p,l)}} = \tau[-i]$$

for $\iota^+ \in I^{(+)}_{(p,l)}, \iota^- \in I^{(-)}_{(p,l)}$. Finally, denote the inflation $\mathrm{Infl}^{\Pi'}_{\mathbb{Z}/N}(\omega)$ of ω to Π' by Ω'. Then (Π, Ω) and (Π', Ω') are categorically Morita equivalent.

The proof follows directly from Proposition 6.34 by considering

$$H_1 := \bigoplus_{p,l} \bigoplus_{\iota^- \in I^-_{(p,l)}} H^{\iota^-}_{(p,l)}.$$

and $f = \mathrm{Infl}^{H_1 \backslash H \rtimes \mathbb{Z}/N}_{\mathbb{Z}/N}(\omega)$ where the 2-cocycles ϵ_1 and ϵ_2 are also trivial. This should not come as a surprise: it follows from the choice to split our hyperbolic quadratic form.

7. Representation Theory of Twisted Quantum Doubles

In this section, we review the representation theory of the quasi-Hopf algebra $D_\Omega(\Pi)$ associated to a group Π with a 3-cocycle Ω on Π in [DPR90]. Later, we specialise to groups built from a fixed-point-free action by $\mathbb{Z}/N$ on a finite abelian group H.

We will follow the notation of [DPR90] and [CGR00, Section 2.2 and 5.2].

For general facts about modular data, we always refer to [Gan05].

Let Π be a finite group. For $y \in \Pi$, we denote left and right conjugation of $x \in \Pi$ with y by ${}^y x := yxy^{-1}$ and $x^y := y^{-1}xy$, respectively.

Let R be a set of representatives of each conjugacy class of Π. For an element $x \in \Pi$, we denote its centraliser by $C_\Pi(x)$.

Let $\Omega \in H^3(\Pi, k^\times)$ be a normalised 3-cocycle. We will assume that $\Omega(x, y, z)$ is a root of unity for all $x, y, z \in \Pi$; see [CGR00, Section 5.2]. For $a, x, y \in \Pi$, we define

$$\beta_a(x, y) := \frac{\Omega(a, x, y)\Omega(x, y, a^{xy})}{\Omega(x, a^x, y)}$$

Note that in [DPR90], they are denoted by the letter θ.

The irreducible representations of $D_\Omega(\Pi)$ are indexed by pairs $(a, \hat{\phi})$ where $a \in R$ is a representative of a conjugacy class of Π and $\hat{\phi}$ the character of a irreducible projective β_a-representation of the centraliser $C_\Pi(a)$.

General facts about twisted group algebras can be found in [Thé95].

We will be concerned with groups Π for which the 2-cocycles β_a are coboundaries for all $a \in \Pi$. They are called *cohomologically trivial* in [CGR00]. In this case, there exist 1-cochains $\epsilon_a \colon C_\Pi(a) \to k^\times$ with $\epsilon_a(1) = 1$ and

$$\begin{aligned}
\beta_a(x, y) &= \epsilon_a(x)\epsilon_a(y)\epsilon_a(x + y)^{-1} \\
\epsilon_{a^y}(x^y) &= \frac{\beta_a(y, x^y)}{\beta_a(x, y)}\epsilon_a(x)
\end{aligned}$$

for all $x, y \in C_\Pi(a)$. As Ω is normalised, we get that $\beta_1 \equiv 1$, so we can assume without loss of generality that $\epsilon_1 \equiv 1$. Furthermore, we can and will assume that ϵ_a satisfies $\epsilon_a^* = \epsilon_a^{-1}$ for all $a \in \Pi$, see [CGR00, Section 5.2], where $(\ )^*$ denotes complex conjugation.

For the projective β_a-representation of $C_\Pi(a)$ with character $\hat{\phi}$, we denote by $\phi = \epsilon_a^{-1}\hat{\phi}$ the character of the corresponding linear representation; we identify $(a, \hat{\phi})$ with (a, ϕ).

In this case, the modular matrices become

$$S_{(a,\phi),(b,\psi)} = \frac{1}{|C_\Pi(a)||C_\Pi(b)|} \sum_{x \in \Pi(a,b)} \phi^*({}^x b)\psi^*(a^x)\sigma^*(a|{}^x b) \tag{7.1}$$

$$T_{(a,\phi),(b,\psi)} = \delta_{a,b}\delta_{\phi,\psi}\frac{\phi(a)}{\phi(1)}\epsilon_a(a) \tag{7.2}$$

where

$$\Pi(a,b) := \{x \in \Pi \colon a({}^x b) = ({}^x a)b\}, \quad \sigma(a|b) = \epsilon_a(b)\epsilon_b(a)$$

for all representatives of conjugacy classes $a, b \in R$ with $\hat{\phi} \in \beta_a - \mathrm{Irr}(C_\Pi(a))$ and $\hat{\psi} \in \beta_b - \mathrm{Irr}(C_\Pi(b))$.

Note that Coste et al. also state the modular data in the most general case in [CGR00, Equation 5.23]; compare also to [DW90, Equation 6.48].

With the S-matrix above, we can calculate the fusion coefficients $N_{(a,\phi),(b,\psi)}^{(c,\chi)}$ using *Verlinde's formula*

$$N_{(a,\phi),(b,\psi)}^{(c,\chi)} = \sum_{(d,\kappa)} \frac{S_{(a,\phi),(d,\kappa)}S_{(b,\psi)}S^*_{(c,\chi),(d,\kappa)}}{S_{0,(d,\kappa)}}$$

for three representatives of conjugacy classes $a, b, c \in R$ with $\hat{\phi} \in \beta_a - \mathrm{Irr}(C_\Pi(a))$, $\hat{\psi} \in \beta_b - \mathrm{Irr}(C_\Pi(b))$, and $\hat{\chi} \in \beta_c - \mathrm{Irr}(C_\Pi(c))$.

For $a, x, y \in \Pi$, we define

$$\gamma_a(x,y) := \frac{\Omega(x,y,a)\Omega(a,x^a,y^a)}{\Omega(x,a,y^a)}.$$

In [GM10, Section 3], the authors calculated the fusion coefficients for (generalised) twisted quantum doubles

$$\begin{aligned} N_{(a,\hat{\phi}),(b,\hat{\psi})}^{(c,\hat{\chi})} = \frac{1}{|\Pi|} \sum_{c^z \in K_c} \sum_{\substack{a^x \in K_a, b^y \in K_b \\ (a^x b^y = c^z)}} \sum_{g \in C_\Pi(a^x) \cap C_\Pi(b^y)} \hat{\phi}({}^x g)\hat{\psi}({}^y g)\hat{\chi}^*({}^z g) \\ \gamma_g(a^x, b^y)\frac{\beta_{a^x}(g,x^{-1})\beta_{b^y}(g,y^{-1})\beta^*_{c^z}(g,z^{-1})}{\beta_a(x^{-1},{}^x g)\beta_b(y^{-1},{}^y g)\beta^*_c(z^{-1},{}^z g)} \end{aligned} \tag{7.3}$$

for $K_c \subseteq K_a K_b$ for three representatives of conjugacy classes $a, b, c \in R$ and $\hat{\phi} \in \beta_a - \mathrm{Irr}(C_\Pi(a))$, $\hat{\psi} \in \beta_b - \mathrm{Irr}(C_\Pi(b))$ and $\hat{\chi} \in \beta_c - \mathrm{Irr}(C_\Pi(c))$.

Note that a similar inner product for right modules appears in [Gof12, Section 5]. Several identities involving the cocycles above are proven in [ACM04].

Twisted Quantum Doubles of Semi-Direct Products

Definition 7.1 (*quasi-Hopf algebra* $\mathrm{DPR}(H, \tau, \omega)$).
Let H be an abelian group with a fixed-point-free action

$$\tau\colon \mathbb{Z}/N \to \mathrm{Aut}(H).$$

Then, we can consider the semi-direct product

$$\Pi := H \rtimes_\tau \mathbb{Z}/N.$$

Let $\omega \in H^3(\mathbb{Z}/N, k^\times)$ be a normalised 3-cocycle with inflation

$$\Omega^{\mathrm{inv}} = \mathrm{Infl}_\Pi^{\mathbb{Z}/N}(\omega^{-1}) \in H^3(\Pi, k^\times).$$

We will denote the quasi-Hopf algebra $D_{\Omega^{\mathrm{inv}}}(\Pi)$ by $\mathrm{DPR}(H, \tau, \omega)$.

We sometimes will identify H with its image $\{(x, 0)\colon x \in H\}$ in Π. Similarly, we identify $\mathbb{Z}/N$ with $\{(0, i)\colon i \in \mathbb{Z}/N\}$ in Π.

Note that since the orders of H and $\mathbb{Z}/N$ are coprime, the second cohomology group $H^2(\mathbb{Z}/N, H)$ is trivial. Thus, this is in fact the unique extension of H by $\mathbb{Z}/N$ with this action.

We note that we choose the following representatives for conjugacy classes with corresponding centralisers

a	K_a	$\lvert K_a\rvert$	$C_\Pi(a)$	$\lvert C_\Pi(a)\rvert$
$(0,0)$	0	1	Π	$\lvert H\rvert N$
$(x,0)$	$\{({}^i x, 0)\colon i \in \mathbb{Z}/N\}$	N	H	$\lvert H\rvert$
$(0,i)$	$\{(a,i) : a \in H\}$	$\lvert H\rvert$	$\mathbb{Z}/N$	N

where x is a representatives for the non-trivial orbit of $\mathbb{Z}/N$ acting on H and $i \in (\mathbb{Z}/N) \setminus \{0\}$.

Then, we directly get that $\beta_a(x, y) = 1$ if a, x or y is an element of H. Therefore, in this case, the twisting is cohomologically trivial.

We recall the representation theory of the semi-direct product Π from Section 6.2.

Remark 7.2.
Let H be a finite abelian group with fixed-point-free action τ of the cyclic group $\mathbb{Z}/N$. The group Π has

* N irreducible representations θ_α of degree 1 indexed by $X_{\mathbb{Z}/N}$. For $\alpha \in X_{\mathbb{Z}/N}$, the character of the corresponding irreducible representation is given by

$$\theta_\alpha[(a,i)] = \alpha(i)$$

for all $(a,i) \in \Pi$.

* $(|H|-1)/N$ representations θ_χ of degree N indexed by $X_H/(\mathbb{Z}/N) \setminus \{0\}$. For a representatives $\chi \in X_H/(\mathbb{Z}/N) \setminus \{0\}$, the character of the corresponding irreducible representation is given by

$$\begin{aligned} \theta_\chi[(a,i)] &= 0 \\ \theta_\chi[(a,0)] &= \sum_{i \in \mathbb{Z}/N} ({}^i\chi)(a) \end{aligned}$$

for $i \in (\mathbb{Z}/N) \setminus \{0\}$ and $a \in H$.

We recalled in Chapter 7 that irreducible representations of the quasi-Hopf algebra $D_\Omega(\Pi)$ can be parametrised by pairs $(a, \hat{\phi})$ of a representative of a conjugacy class $a \in R$ and an irreducible representation of the centraliser $C_\Pi(a)$. With the above choice of representatives for the conjugacy classes of Π, we thus directly obtain the following parametrisation.

Proposition 7.3 (*irreducible representations of* $\mathrm{DPR}(H,\tau,\omega)$).
Let H be a finite abelian group with fixed-point-free action $\tau\colon \mathbb{Z}/N \to \mathrm{Aut}(H)$ and a normalised 3-cocycle $\omega \in H^3(\mathbb{Z}/N, k^\times)$. Then the irreducible representations of $\mathrm{DPR}(H,\tau,\omega)$ can be parametrised by:

(i.) N irreducible representations of quantum dimension 1 indexed by $X_{\mathbb{Z}/N}$.

(ii.) $(|H|-1)/N$ irreducible representations of quantum dimension N indexed by $(X_H/(\mathbb{Z}/N)) \setminus \{0\}$

(iii.) $|H|(|H|-1)/N$ *irreducible representations of quantum dimension* N *indexed by* $(H/(\mathbb{Z}/N) \setminus \{0\}) \times X_H$.

(iv.) $N(N-1)$ *irreducible representations of quantum dimension* $|H|$ *indexed by* $((\mathbb{Z}/N) \setminus \{0\}) \times X_{\mathbb{Z}/N}$.

The above parametrisation follows directly since the centralisers of the choices of representatives for the conjugacy classes are given by the following:

a	$C_\Pi(a)$	$\beta_a - \mathrm{Irr}(C_\Pi(a))$	qdim
$(0,0)$	Π	θ_α	1
$(0,0)$	Π	θ_χ	N
$(x,0)$	H	ϕ	N
$(0,i)$	$\mathbb{Z}/N$	η	$\|H\|$

We will denote them by $(0,\alpha)$, $(0,\chi)$, (x,ϕ) and (i,η), respectively.

Note that, as expected, we get by computing the sums of the squares of the quantum dimensions of the irreducible representations

$$N \cdot 1 + \frac{|H|-1}{N} N^2 + \frac{|H|-1}{N} |H| N^2 + (N-1)N|H|^2 = N^2|H|^2.$$

Now, we can calculate the modular matrices of the twisted group doubles $\mathrm{DPR}(H,\tau,\omega)$.

Proposition 7.4 (*modular data of* $\mathrm{DPR}(H,\tau,\omega)$*).*
Let H *be a finite abelian group with fixed-point-free action* $\tau\colon \mathbb{Z}/N \to \mathrm{Aut}(H)$ *and* $\omega \in H^3(\mathbb{Z}/N, k^\times)$ *a normalised* 3*-cocycle. Then the modular matrices of*

$\mathrm{DPR}(H,\tau,\omega)$ *are given by*

$$\begin{aligned} S_{(0,\alpha),(0,\beta)} &= \frac{1}{N|H|} \quad S_{(0,\alpha),(0,\chi)} = \frac{1}{|H|} \\ S_{(0,\alpha),(x,\phi)} &= \frac{1}{|H|} \qquad S_{(0,\alpha),(j,\beta)} = \frac{1}{N}\alpha^*(i) \\ S_{(0,\chi),(0,\kappa)} = \frac{N}{|H|} \quad & S_{(0,\chi),(x,\phi)} = \frac{\theta_\chi(x,0)}{|H|} \quad S_{(0,\chi),(i,\alpha)} = 0 \\ S_{(x,\phi),(y,\psi)} &= \frac{1}{|H|}\sum_{k\in\mathbb{Z}/N}\phi^*({}^k y)\psi^*({}^{-k}x) \quad S_{(x,\phi),(i,\alpha)} = 0 \\ S_{(i,\alpha),(j,\beta)} &= \frac{1}{N}\alpha^*(j)\beta^*(i)\sigma^*(i|j). \end{aligned} \tag{7.4}$$

$$\begin{aligned} T_{(0,\alpha),(0,\beta)} &= \delta_{\alpha,\beta} & T_{(0,\chi),(0,\kappa)} &= \delta_{\chi,\kappa} \\ T_{(x,\phi),(y,\psi)} &= \delta_{x,y}\delta_{\phi,\psi}\phi(x) & T_{(i,\alpha),(j,\beta)} &= \delta_{i,j}\delta_{\alpha,\beta}\alpha(i)\epsilon_i(i) \end{aligned} \tag{7.5}$$

where we used the parametrisation of irreducible representations from Proposition 7.3, namely

- $\alpha,\beta \in X_{\mathbb{Z}/N}$*;*
- *representatives* χ,κ *of the non-trivial orbits* $X_H/(\mathbb{Z}/N)$*;*
- *representatives* x,y *of the non-trivial orbits* $H/(\mathbb{Z}/N)$ *and* $\phi,\psi\in X_H$*;*
- $i,j\in(\mathbb{Z}/N)\setminus\{0\}$ *and* $\alpha,\beta\in X_{\mathbb{Z}/N}$.

Proof. We calculate the modular matrices using the formula (7.1). For this note, the following:

$\Pi(\,*\,,\,*\,)$	$(0,0)$	$(x,0)$	$(0,i)$
$(0,0)$	Π	Π	Π
$(y,0)$	Π	Π	$\emptyset$
$(0,j)$	Π	$\emptyset$	$\mathbb{Z}/N$

∎

Note that here the Kronecker delta $\delta_{x,y}$ is 1 if x and y are conjugate, i.e. lie in the same orbit.

By inspecting the S-matrices above, we can directly read off the charge conjugates of the irreducible modules. For detail on charge conjugates in general, we refer to [CGR00] and [CG94].

Proposition 7.5 (*charge conjugates of* $\mathrm{DPR}(H, \tau, \omega)$).
Let H be a finite abelian group with fixed-point-free action $\tau\colon \mathbb{Z}/N \to \mathrm{Aut}(H)$ and $\omega \in H^3(\mathbb{Z}/N, k^\times)$ a normalised 3-cocycle. Then the charge conjugates of the irreducible representations are given by

C	$(0,\alpha)$	$(0,\chi)$	(x,ϕ)	(i,α)
	$(0,\alpha^*)$	$(0,\chi^*)$	$(-x,\phi^*)$	$(-i,\alpha^*\zeta_i)$

where χ^{inv} is the representative of an orbit of the action $\mathbb{Z}/N$ on X_H containing χ^* and similarly x^{inv} is the representative of an orbit of the action $\mathbb{Z}/N$ on H containing $-x$. The character $\zeta_i\colon \mathbb{Z}/N \to k^\times$ is defined by

$$\zeta_i(j) := \sigma^*(i|j)\sigma^*(-i|j)$$

for all $i, j \in \mathbb{Z}/N$.

We will show later that ζ_i is a well-defined character; compare to [Dij+89, Equation 4.28].

Now, we are ready to calculate the fusion rules for all irreducible representations.

The fusion rules of the irreducible representations indexed with representations of the group Π or H will depend on how the tensor product of these representations decomposes. Therefore, the scalar product $\langle *, * \rangle_\Pi$ of characters of groups naturally appears; see e.g. [Ser77, Section 2.3]. We denote the reduction of a representation τ of the group Π to the subgroup H by $\mathrm{Red}^\Pi_H(\tau)$; see [Ser77, Section 7.2].

Here, we again use the representations θ_α and θ_χ of Π of degree 1 and N, respectively; see Remark 7.2.

Proposition 7.6 (*fusion rules of* $\mathrm{DPR}(H, \tau, \omega)$).
Let H be a finite abelian group with fixed-point-free action τ of the cyclic group $\mathbb{Z}/N$ and $\omega \in H^3(\mathbb{Z}/N, k^\times)$ a normalised 3-cocycle. Then the fusion coefficients of the irreducible representations of $\mathrm{DPR}(H, \tau, \omega)$ are given by

$$N^{(i,\gamma)}_{(0,\alpha),(i,\beta)} = \delta_{\gamma,\alpha\beta} \quad N^{(i,\gamma)}_{(0,\chi),(i,\beta)} = 1 \quad N^{(i,\gamma)}_{(x,\phi),(i,\alpha)} = 1$$

$$N_{(i,\alpha),(j,\beta)}^{(k,\gamma)} = \frac{1}{N}\delta_{i+j,k}\big(|H|-1+\delta_{\alpha\beta,\gamma c(i,j)}\big)$$
$$N_{(0,\chi),(0,\kappa)}^{(0,\rho)} = \langle\theta_\chi\theta_\kappa,\theta_\rho\rangle_\Pi$$
$$N_{(0,\chi),(x,\phi)}^{(y,\psi)} = \delta_{x,y}\langle \mathrm{Red}_H^\Pi(\theta_\chi)\phi,\psi\rangle_H$$

$$\left.\begin{aligned} N_{(x,\phi),(y,\psi)}^{(0,\alpha)} &= \langle(\phi^{i_0})(\psi^{j_0}),\mathrm{Red}_H^\Pi(\theta_\alpha)\rangle_H \\ N_{(x,\phi),(y,\psi)}^{(0,\chi)} &= \langle(\phi^{i_0})(\psi^{j_0}),\mathrm{Red}_H^\Pi(\theta_\chi)\rangle_H \end{aligned}\right\} \textit{ if } {}^{i_0}x+{}^{j_0}y=0 \textit{ for some } i_0,j_0\in\mathbb{Z}/N$$
$$\left. N_{(x,\phi),(y,\psi)}^{(z,\xi)} = \langle(\phi^{i_0})(\psi^{j_0}),\xi\rangle_H \right\} \textit{ if } {}^{i_0}x+{}^{j_0}y=z \textit{ for some } i_0,j_0\in\mathbb{Z}/N$$

for all $\alpha,\beta,\gamma\in X_{\mathbb{Z}/N}$, $i,j,k\in(\mathbb{Z}/N)\setminus\{0\}$, χ,κ,ρ representatives of the non-trivial orbits $(X_H/(\mathbb{Z}/N))\setminus\{0\}$, x,y,z representatives of the non-trivial orbits $(H/(\mathbb{Z}/N))\setminus\{0\}$, and $\phi,\psi,\xi\in X_H$. Here, we defined

$$c(i,j)(l) := \frac{\sigma(i+j|l)}{\sigma(i|l)\sigma(j|l)}.$$

Proof. With the S-matrix as in (7.4), let us calculate a fusion coefficient using Verlinde's formula

$$\begin{aligned} N_{(i,\alpha),(j,\beta)}^{(k,\gamma)} &= \sum_{\lambda\in X_{\mathbb{Z}/N}} \frac{S_{(i,\alpha),(0,\lambda)}S_{(j,\beta),(0,\lambda)}S^*_{(k,\gamma),(0,\lambda)}}{S_{(0,1),(0,\lambda)}} \\ &\quad + \sum_{l=1}^{N-1}\sum_{\lambda\in X_{\mathbb{Z}/N}} \frac{S_{(i,\alpha),(l,\lambda)}S_{(j,\beta),(l,\lambda)}S^*_{(k,\gamma),(l,\lambda)}}{S_{(0,1),(l,\lambda)}} \\ &= \frac{|H|}{N^2}\sum_\lambda \lambda(i)\lambda(j)\lambda^*(k) \\ &\quad + \frac{1}{N^2}\sum_l\sum_\lambda \alpha^*(l)\lambda^*(i)\sigma^*(i|l)\beta^*(l)\lambda^*(j)\sigma^*(j|l)\gamma(l)\lambda(k)\sigma(k|l) \\ &= \frac{1}{N}\delta_{i+j,k}\big(|H|+\frac{1}{N}\sum_l \alpha^*(l)\beta^*(l)\gamma(l)\frac{\sigma(i+j|l)}{\sigma(i|l)\sigma(j|l)}\big) \\ &= \frac{1}{N}\delta_{i+j,k}\big(|H|-1+\delta_{\alpha\beta,\gamma c(i,j)}\big) \end{aligned}$$

For fixed $i,j\in\mathbb{Z}/N$, $c(i,j)$ is a (irreducible) character of $\mathbb{Z}/N$ since

$$\frac{c(i,j)(k)c(i,j)(l)}{c(i,j)(k+l)} = \frac{\beta_{i+j}(k,l)}{\beta_i(k,l)\beta_j(k,l)}\frac{\beta_{k+l}(i,j)}{\beta_k(i,j)\beta_l(i,j)} = 1,$$

for all $i, j, k, l \in \mathbb{Z}/N$; see e.g. [DPR90, Equation 3.2.9]. Therefore, the map c defines a 2-cocycle valued in $X_{\mathbb{Z}/N}$ since

$$c(i,j)(l)c(i+j,k)(l) = \frac{\sigma(i+j+k|l)}{\sigma(i|l)\sigma(j|l)\sigma(k|l)} = c(i,j+k)(l)c(j,k)(l)$$

for all $i, j, k, l \in \mathbb{Z}/N$.

Using Verlinde's formula, we also get the first line (7.6).

Note that to obtain the fusion coefficients involving the irreducibles of quantum dimension N, we need to take into account how H decomposes into $\mathbb{Z}/N$-orbits. We get the remaining fusion coefficients using equation (7.3). We directly obtain

$$N^{0,\rho}_{(0,\chi),(0,\kappa)} = \langle \theta_\chi \theta_\kappa, \theta_\rho \rangle_\Pi.$$

Note that the reduction $\mathrm{Red}^\Pi_H(\theta_\chi)$ of θ_χ to H decomposes into the orbit of χ under the action of $\mathbb{Z}/N$, so we get

$$N^{(y,\psi)}_{(0,\chi),(x,\phi)} = \delta_{x,y} \langle \mathrm{Red}^\Pi_H(\theta_\chi)\phi, \psi \rangle_H$$

$$\begin{aligned} N^{(0,\alpha)}_{(x,\phi),(y,\psi)} &= \frac{1}{N|H|} \sum_{\substack{i,j\in\mathbb{Z}/N \\ x^i+y^j=0}} \sum_{a\in H} \phi({}^i a)\psi({}^j a) \\ &= \frac{1}{N} \sum_{\substack{i,j\in\mathbb{Z}/N \\ x^i+y^j=0}} \langle \phi^i \psi^j, 1 \rangle_H \\ &= \langle (\phi^{i_0})(\psi^{j_0}), 1 \rangle_H \end{aligned}$$

for $x^{i_0} + y^{j_0} = 0$ (independent of α). Thus, in that case, we get N irreducible representations of quantum dimension 1.

$$\begin{aligned} N^{(0,\chi)}_{(x,\phi),(y,\psi)} &= \frac{1}{N|H|} \sum_{\substack{i,j\in\mathbb{Z}/N \\ x^i+y^j=0}} \sum_{a\in H} \phi({}^i a)\psi({}^j a)\theta^*_\chi(a) \\ &= \frac{1}{N} \sum_{\substack{i,j\in\mathbb{Z}/N \\ x^i+y^j=0}} \langle \phi^i \psi^j, \mathrm{Red}^\Pi_H(\theta_\chi) \rangle_H \\ &= \langle (\phi^{i_0})(\psi^{j_0}), \mathrm{Red}^\Pi_H(\theta_\chi) \rangle_H \end{aligned}$$

for $x^{i_0}+y^{j_0}=0$. If $(\phi^{i_0})(\psi^{j_0})\not\equiv 1$, there exists an χ such that $(\phi^{i_0})(\psi^{j_0})$ is contained in the orbit of χ under $\mathbb{Z}/N$. Thus, in this case, we get one irreducible module of quantum dimension N.

Similarly,

$$\begin{aligned} N^{(z,\xi)}_{(x,\phi),(y,\psi)} &= \frac{1}{N|H|}\sum_{k\in\mathbb{Z}/N}\sum_{\substack{i,j\in\mathbb{Z}/N\\ x^i+y^j=z^k}}\sum_{a\in H}\phi({}^i a)\psi({}^j a)\xi^*({}^k a)\\ &= \frac{1}{N}\sum_{k}\sum_{\substack{i,j\in\mathbb{Z}/N\\ x^i+y^j=z^k}}\langle\phi^i\psi^j,\xi^k\rangle_H\\ &= \langle(\phi^{i_0})(\psi^{j_0}),\xi\rangle_H \end{aligned}$$

for $x^{i_0}+y^{j_0}=z$. Note that the set $\{x^{i_0}+y^{j_0}\colon i_0,j_0\in\mathbb{Z}/N\}$ decomposes either in N orbits of length N (if it does not contain 0) or in $N-1$ orbits of length N (if it does). By the above, the quantum dimensions are thus indeed preserved. ▪

In other words, in the tensor product of the irreducible representations (i,α) and (j,β) every $(i+j,\gamma)$ but one appears $(|H|-1)/N$-times. Note that this is an integer, since we assumed a fixed-point-free action.

In the above proof, we have shown that the characters $\zeta_i=c(i,-i)$ defined above are well-defined; see Proposition 7.5.

Example 7.7 (*[CGR00, Section 6.1]*).
We can pick an explicit representative for a normalised 3-cocycle in $H^3(\mathbb{Z}/N,k^\times)$. Namely, for $l\in\mathbb{Z}/N$, we can choose

$$\omega_l(i,j,k)=\xi_{N^2}^{li(j+k-\lfloor j+k\rfloor_N)},$$

where we denoted the reduction modulo N by $\lfloor * \rfloor_N$ and ξ_{N^2} is a primitive N^2-root of unity. Then, we get

$$\beta_i(j,k)=\omega_l(i,j,k)^{-1}\quad\text{and}\quad\sigma(i|j)=\xi_{N^2}^{-2lij}.$$

III.

Monoidal Bicategories and Coherent Gr-bicategories

In the last part of this thesis, we will introduce coherent Gr-bicategories.

Chapter 8: Monoidal Bicategories and Their Morphisms.

Recall that a bicategory with exactly one object is a monoidal category; see e.g. [CG07]. Similarly, a monoidal bicategory is a tricategory with exactly one object. In Chapter 8, we start by giving a complete definition of monoidal bicategories in Definition 8.13. We also define morphisms between monoidal bicategories in Definition 8.22. As the tricategorical axioms are technical and pose typesetting difficulties, they are rarely spelt out; a notable exception is [Sch09b, Appendix C.1]. We will need the complete definition of a morphism between monoidal bicategories later when we work with coherent Gr-categories.

We state and prove some results that allow us to *transport* bicategorical and monoidal bicategorical structure along biequivalences. For example, we prove in Lemma 8.7 and Corollary 8.8 that every bicategory is biequivalent to a bicategory with strict identities.

When we are working with bicategories, we are usually following [Bén67], [Str80], or [Str72]; also note the very extensive textbook [JY21]. The definitions of monoidal bicategories and their morphisms are based on [Gur13]; in contrast to [GPS95].

Chapter 9: Coherent Gr-*bicategories.*

In Chapter 9, we will work with (and generalise) *coherent* Gr-categories as in [BL04, Section 3]. We briefly recall facts about coherent Gr-categories as presented in [BL04, Section 3] in Section 9.1.

We define coherent Gr-bicategories in Definition 9.2. For every object x in a coherent Gr-bicategory there is a biadjoint biequivalence with component 1-cells

$$I_x \colon \mathbb{1} \to x \boxtimes x^- \quad E_x \colon x^- \boxtimes x \to \mathbb{1}$$

with 2-cells exhibiting the triangle identities.

We show, by transporting monoidal structure along equivalences, that we can always work with *skeletal* coherent Gr-bicategories with controlled unitor and coherence structure in Proposition 9.18.

This chapter should be seen as a categorification of the results [BL04] on coherent Gr-categories to coherent Gr-*bi*categories. It was motivated by and is reliant on the results in [Gur12].

Chapter 9: Morphisms between Coherent Gr*-bicategories.*

In Chapter 10, we conclude by sketching the group-theoretic data induced by coherent monoidal Gr-bicategory, following the ideas of Baez and Lauda [BL04, Section 8.3] for monoidal Gr-categories. This can be viewed as an explicit calculation of the obstructions to lifting a group map to a morphism of (special) coherent Gr-categories.

In our Outlook, we will sketch how we think this theory should be applied to construct extensions of fusion and tensor categories; see e.g. [DN21].

8. Monoidal Bicategories and Their Morphisms

Notation and Conventions

Bicategorical Conventions

For our conventions on bicategories, we will follow [Bén67], [Str80] or [Str72]. Note, however, that we write composition of arrows $f\colon x_1 \to x_2$, $g\colon x_2 \to x_3$ as $g \circ f\colon x_1 \to x_3$ in contrast to [Bén67]. We will read pasting diagrams as left-bracketed.

We denote the identity 1- and 2-cells of a bicategory B by 1.

For bicategories A, B a *morphism* $S\colon \mathsf{A} \to \mathsf{B}$ of bicategories comes with an invertible natural 2-cell

$$(S^2)_{f,g}\colon S(g) \overset{\mathsf{B}}{\circ} S(f) \to S(g \overset{\mathsf{A}}{\circ} f).$$

for every composable pair of arrows $f\colon x_1 \to x_2$, $g\colon x_2 \to x_3$ in A, and, for all objects x in A, an invertible natural 2-cell

$$S^0_x\colon 1^B_{Sx} \to S(1^{\mathsf{A}}_x)$$

satisfying some natural compatibility axioms; see e.g. [JY21, Definition 4.1.2]. We will call S^2 the (strong lax) *functoriality constraint* and S^0 the (strong lax) *unitality constraint*.

For two morphisms $S, T\colon \mathsf{A} \to \mathsf{B}$ of bicategories, an (oplax) *transformation* $\alpha\colon S \to T$ consists of

* for every object x in A, an invertible 1-cell $\alpha_x\colon S(x) \to T(x)$,
* for every 1-cell $f\colon x_1 \to x_2$ in A, an invertible 2-cell

$$\begin{tikzcd} S(x_1) \arrow[r, "S(f)"] \arrow[d, "\alpha_{x_1}"'] \arrow[dr, phantom, "\swArrow\alpha_f"] & S(x_2) \arrow[d, "\alpha_{x_2}"] \\ T(x_1) \arrow[r, "T(f)"'] & T(x_2) \end{tikzcd},$$

which satisfy for all objects x in A

$$
\begin{tikzcd}
S(x) \arrow[r, bend left=40, "1_{S(x)}", ""{name=U, below}] \arrow[r, bend right=40, "S(1_x)"', ""{name=D}] \arrow[d, "\alpha_x"'] \arrow[dr, phantom, "\Swarrow\alpha_{1_x}"] & S(x) \arrow[d, "\alpha_x"] \\
T(x) \arrow[r, "T(1_x)"'] & T(x)
\arrow[Rightarrow, from=U, to=D, "S^0_x"]
\end{tikzcd}
=
\begin{tikzcd}
S(x) \arrow[r, "1_{S(x)}"] \arrow[d, "\alpha_x"'] \arrow[dr, "\alpha_x" description] & S(x) \arrow[d, "\alpha_x"] \\
T(x) \arrow[r, bend left=40, "1_{T(x)}", ""{name=U2, below}] \arrow[r, bend right=40, "T(1_x)"', ""{name=D2}] & T(x)
\arrow[Rightarrow, from=U2, to=D2, "T^0_x"]
\end{tikzcd}
$$

and, for every pair of composable 1-cells $f\colon x_1 \to x_2$, $g\colon x_2 \to x_3$ in A

$$
\begin{tikzcd}
& S(x_2) \arrow[dr, "S(g)"] \arrow[d, phantom, "\Downarrow S^2_{f,g}"] & \\
S(x_1) \arrow[ur, "S(f)"] \arrow[rr, "S(g\circ f)" description] \arrow[d, "\alpha_{x_1}"'] & {} \arrow[d, phantom, "\Swarrow\alpha_{g\circ f}"] & S(x_3) \arrow[d, "\alpha_{x_3}"] \\
T(x_1) \arrow[rr, "T(g\circ f)"'] & {} & T(x_3)
\end{tikzcd}
=
\begin{tikzcd}
& S(x_2) \arrow[dr, "S(g)"] \arrow[dd, "\alpha_{x_2}"] & \\
S(x_1) \arrow[ur, "S(f)"] \arrow[dd, "\alpha_{x_1}"'] \arrow[r, phantom, "\Swarrow\alpha_f"] & {} \arrow[r, phantom, "\Swarrow\alpha_g"] & S(x_3) \arrow[dd, "\alpha_{x_3}"] \\
& T(x_2) \arrow[dr, "T(g)"] \arrow[d, phantom, "\Downarrow T^2_{f,g}"] & \\
T(x_1) \arrow[ur, "T(f)"] \arrow[rr, "T(g\circ f)"'] & {} & T(x_3)
\end{tikzcd}
.
$$

For two transformations $\alpha, \beta\colon S \to T$, an (oplax) *modification* $s\colon \alpha \to \beta$ is a family of invertible 2-cells with components

$$s_x\colon \alpha_x \to \beta_x$$

for every object x in A such that

$$
\begin{tikzcd}
S(x_1) \arrow[r, "S(f)"] \arrow[d, bend right=40, "\beta_{x_1}"', ""{name=B}] \arrow[d, bend left=40, "\alpha_{x_1}", ""{name=A}] \arrow[dr, phantom, "\Swarrow\alpha_f"] & S(x_2) \arrow[d, "\alpha_{x_2}"] \\
S(x_2) \arrow[r, "T(f)"'] & T(x_2)
\arrow[Rightarrow, from=A, to=B, "s_{x_1}"]
\end{tikzcd}
=
\begin{tikzcd}
S(x_1) \arrow[r, "S(f)"] \arrow[d, "\beta_{x_1}"'] \arrow[dr, phantom, "\Swarrow\beta_f"] & S(x_2) \arrow[d, bend right=40, "\beta_{x_2}"', ""{name=B2}] \arrow[d, bend left=40, "\alpha_{x_2}", ""{name=A2}] \\
T(x_1) \arrow[r, "T(f)"'] & T(x_2)
\arrow[Rightarrow, from=A2, to=B2, "s_{x_2}"]
\end{tikzcd}
.
$$

Note that in [Str72], these transformations are called *right lax*.

We denote the bicategory of morphisms, transformations and modifications for two bicategories A and B by $\mathsf{Bicat}(\mathsf{A}, \mathsf{B})$; see e.g. [JY21, Corollary 4.4.13]. Note that in [JY21, Definition 4.1.2] morphisms are called *pseudofunctors*.

Example 8.1.
A bicategory with exactly one object is a monoidal category, compare to [JY21, Example 2.1.19] or [CG07, Theorem 4.1].

We quickly recall different notions of dual bicategories. Later, we will use the categorifications of the following two examples.

Example 8.2.
For a bicategory B, we denote the opposite bicategory by B^{op}, compare to [JY21, Definition 2.6.2]. Its hom categories are given by

$$\mathrm{Hom}_{\mathsf{B}^{\mathrm{op}}}(x,y) = \mathrm{Hom}_{\mathsf{B}}(y,x)$$

for every pair of objects x, y in B with the composition of 1-cells defined accordingly as

$$g \overset{\mathrm{op}}{\circ} f := f \circ g$$

for 1-cells $f\colon x_1 \to^{\mathrm{op}} x_2$, $g\colon x_2 \to^{\mathrm{op}} x_3$.

Example 8.3.
For a one-object bicategory, i.e. a monoidal category C, we get a monoidal category with reversed monoidal product when we consider the opposite bicategory as above, compare to [JY21, Example 1.2.9]. We will denote this monoidal category by $\mathsf{C}^{\otimes\,\mathrm{rev}}$.

Example 8.4.
For a bicategory B, we denote the co-bicategory by B^{co}, compare to [JY21, Definition 2.6.3]. Its hom categories are given by the opposite categories

$$\mathrm{Hom}_{\mathsf{B}^{\mathrm{co}}}(x,y) = \mathrm{Hom}_{\mathsf{B}}(x,y)^{\mathrm{op}}$$

for every pair of objects x, y in B.

Example 8.5.
For a one-object bicategory, i.e. a monoidal category C, we get a monoidal category with opposite monoidal product when we consider the co-bicategory as above, compare to [JY21, Example 1.2.10]. We will denote this monoidal category by $\mathsf{C}^{\mathrm{op}\,\otimes}$.

Adjoints and Mates in Bicategories

Throughout this part, we work with internal adjoints in bicategories. For example, we will see that in a coherent Gr-bicategory every 1-cell will be part of an adjoint equivalence. Thus, we introduce some notation.

Internal Adjunctions and Biadjoint Biequivalences

Let B be a bicategory. An *internal adjunction* $f \dashv f^\bullet$ in B consists of

(i.) Two 1-cells $f\colon x_1 \to x_2$ and $f^\bullet\colon x_2 \to x_1$.

(ii.) Two 2-cells $\epsilon\colon f \circ f^\bullet \Rightarrow 1_{x_2}$ and $\eta\colon 1_{x_1} \Rightarrow f^\bullet \circ f$ such that

$$
\begin{tikzcd}
x_1 \arrow[r, "1_{x_1}"] \arrow[d, "f"'] \arrow[dr, phantom, near start, "\eta \swArrow"] & x_1 \arrow[d, "f"] \\
x_2 \arrow[ur, "f^\bullet"] \arrow[r, "1_{x_2}"'] \arrow[ur, phantom, near end, shift right=4, "\epsilon \swArrow"] & x_2
\end{tikzcd}
=
\begin{tikzcd}
x_1 \arrow[r, "1_{x_1}"] \arrow[d, "f"'] \arrow[dr, phantom, "\swArrow 1_f"] & x_1 \arrow[d, "f"] \\
x_2 \arrow[r, "1_{x_2}"'] & x_2
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
x_2 \arrow[r, "f^\bullet"] \arrow[d, "1_{x_2}"'] \arrow[dr, phantom, near start, "\epsilon \swArrow"] & x_1 \arrow[d, "1_{x_1}"] \arrow[dl, "f"'] \\
x_2 \arrow[r, "f^\bullet"'] \arrow[ur, phantom, near end, shift right=4, "\eta \swArrow"] & x_1
\end{tikzcd}
=
\begin{tikzcd}
x_2 \arrow[r, "f^\bullet"] \arrow[d, "1_{x_2}"'] \arrow[dr, phantom, "\swArrow 1_{f^\bullet}"] & x_1 \arrow[d, "1_{x_1}"] \\
x_2 \arrow[r, "f^\bullet"'] & x_1
\end{tikzcd} .
$$

In the above diagrams, we have omitted the appropriate associators and unitors. For details, we refer to [JY21, Example 3.4.20]. The 1-cell $f^\bullet$ is the *right adjoint* of the 1-cell f; f is the left-adjoint of $f^\bullet$. Similarly, we will denote for a 1-cell $g\colon x_2 \to x_1$ a left adjoint by ${}^\bullet g\colon x_1 \to x_2$. An internal adjunction is called *internal equivalence* if ϵ and η are invertible. Internal equivalences are ambidextrous adjunctions, i.e. in that case $f^\bullet$ is both left and right adjoint[1] to f. We will repeatedly use the fact that morphisms of bicategories preserve internal adjunctions.

A morphism $S\colon \mathsf{A} \to \mathsf{B}$ of bicategories is a *biequivalence* if there is a biadjoint biequivalence $S \dashv_{\mathrm{bieq}} T$, compare to [Gur12, Theorem 3.2]. This consists

[1]Note, however, that the unit and counit of the adjunctions $f \dashv f^\bullet$ and $f^\bullet \dashv f$ will, in general, be different.

of a biadjunction $S \dashv_{\mathrm{bi}} T$ with counit $\epsilon\colon S \circ T \to 1_{\mathsf{B}}$ and unit $\eta\colon 1_{\mathsf{A}} \to T \circ S$ with modifications Φ and Ψ witnessing the triangle identities and chosen adjoint equivalences $\epsilon \dashv_{\mathrm{eq}} \epsilon^\bullet$ and $\eta \dashv_{\mathrm{eq}} \eta^\bullet$ in the bicategory of morphisms $\mathsf{Bicat}(\mathsf{B}, \mathsf{B})$ and $\mathsf{Bicat}(\mathsf{A}, \mathsf{A})$, respectively. We refer to [Gur12, Definition 2.1] for the details.

Mates

In our calculations, we will use the calculus of mates for adjunctions. We summarise our conventions.

Let $(f_1, (f_1)^\bullet, \eta_1, \epsilon_1)$ and $(f_2, (f_2)^\bullet, \eta_2, \epsilon_2)$ be two internal adjunctions in a bicategory B. Then for a 2-cell

$$\begin{tikzcd} x_1 \arrow[r, "h"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\swarrow\alpha"] & x_3 \arrow[d, "f_2"] \\ x_2 \arrow[r, "i"'] & x_4 \end{tikzcd},$$

we define its *mate* as

$$\begin{tikzcd} x_2 \arrow[r, "f_1^\bullet"] \arrow[d, "i"'] \arrow[dr, phantom, "\swarrow\alpha^\bullet"] & x_1 \arrow[d, "h"] \\ x_4 \arrow[r, "f_2^\bullet"'] & x_3 \end{tikzcd} := \begin{tikzcd} x_2 \arrow[r, "f_1^\bullet"] \arrow[dr, "1_{x_2}"'] & x_1 \arrow[r, "h"] \arrow[d, "f_1"] \arrow[dr, phantom, "\swarrow\alpha"] & x_3 \arrow[d, "f_2"] \arrow[dr, "1_{x_3}"] & \\ & x_2 \arrow[r, "i"'] & x_4 \arrow[r, "f_2^\bullet"'] & x_3 \end{tikzcd}.$$

Similarly, for a 2-cell

$$\begin{tikzcd} x_2 \arrow[r, "g_1"'] \arrow[d, "i"'] \arrow[dr, phantom, "\swarrow\beta"] & x_1 \arrow[d, "h"] \\ x_4 \arrow[r, "g_2"] & x_3 \end{tikzcd}$$

its *mate* is defined as

$$
\begin{tikzcd}
x_1 \arrow[r, "h"] \arrow[d, "{}^\bullet(g_1)"'] \arrow[dr, phantom, "\swarrow {}^\bullet(\beta)"] & x_3 \arrow[d, "{}^\bullet(g_2)"] \\
x_2 \arrow[r, "i"'] & x_4
\end{tikzcd}
:=
\begin{tikzcd}
x_1 \arrow[d, "{}^\bullet(g_1)"'] \arrow[dr, "1_{x_1}"] & \\
x_2 \arrow[r, "g_1"'] \arrow[d, "i"'] \arrow[ur, phantom, "\swarrow \eta_2"] \arrow[dr, phantom, "\swarrow \beta"] & x_1 \arrow[d, "h"] \\
x_4 \arrow[r, "g_2"] \arrow[dr, "1_{x_4}"'] & x_3 \arrow[d, "{}^\bullet(g_2)"] \arrow[dl, phantom, "\swarrow \epsilon_2"] \\
& x_4
\end{tikzcd}
$$

We will repeatedly use the mates of squares with only one adjoint edge. We give them the following names

$$
\begin{tikzcd}
x_1 \arrow[r, "h"] \arrow[d, "j"] \arrow[dr, phantom, "\swarrow (\alpha)^\sharp"] & x_3 \\
x_2 \arrow[r, "i"'] & x_4 \arrow[u, "f^\bullet"']
\end{tikzcd}
:=
\begin{tikzcd}
x_1 \arrow[r, "h"] \arrow[d, "j"] \arrow[dr, phantom, "\swarrow \alpha"] & x_3 \arrow[d, "f"] \arrow[dr, "1_{x_3}"] & \\
x_2 \arrow[r, "i"'] & x_4 \arrow[r, "f^\bullet"'] \arrow[ur, phantom, "\swarrow \eta"] & x_3
\end{tikzcd}
$$

$$
\begin{tikzcd}
x_1 \arrow[r, "h"] \arrow[d, "{}^\bullet(g)"'] \arrow[dr, phantom, "\swarrow {}^\flat(\beta)"] & x_3 \\
x_2 \arrow[r, "i"'] & x_4 \arrow[u, "k"']
\end{tikzcd}
:=
\begin{tikzcd}
x_1 \arrow[d, "{}^\bullet(g)"'] \arrow[dr, "1_{x_1}"] & \\
x_2 \arrow[r, "g"'] \arrow[d, "i"'] \arrow[ur, phantom, "\swarrow \eta"] \arrow[dr, phantom, "\swarrow \beta"] & x_1 \arrow[d, "h"] \\
x_4 \arrow[r, "k"] & x_3
\end{tikzcd}
$$

REMARK 8.6.
Taking mates, in fact, defines a bijection at the level of two cells. Given two internal adjunctions $(f_1, (f_1)^\bullet, \eta_1, \epsilon_1)$ and $(f_2, (f_2)^\bullet, \eta_2, \epsilon_2)$ in a bicategory B with 1-cells $f_1 \colon x_1 \to x_2$ and $f_2 \colon x_3 \to x_4$, we have an isomorphism

$$
(\,*\,)^\bullet \colon \mathsf{B}(x_1, x_4)(f_2 \circ h, f_1 \circ i) \to \mathsf{B}(x_2, x_3)(h \circ f_1^\bullet, i \circ f_2^\bullet)
$$

$$
\begin{tikzcd}
x_1 \arrow[r, "h"] \arrow[d, "f_1"] \arrow[dr, phantom, "\swarrow \alpha"] & x_3 \arrow[d, "f_2"] \\
x_2 \arrow[r, "i"'] & x_4
\end{tikzcd}
\mapsto
\begin{tikzcd}
x_2 \arrow[r, "f_1^\bullet"] \arrow[d, "i"'] \arrow[dr, phantom, "\swarrow \alpha^\bullet"] & x_1 \arrow[d, "h"] \\
x_4 \arrow[r, "f_2^\bullet"'] & x_3
\end{tikzcd}
.
$$

with inverse given by ${}^{\bullet}(\,*\,)$. For a proof, we refer to [JY21, Lemma 6.1.13]; see also [KS74, Proposition 2.1].

§8.1. Identity Normalised Bicategories

In the sequel, we will use many results that allow us to transport bicategorical structure from one bicategory to another. Later, we will use a similar strategy for tricategorical structure. The following lemma is the first example of such a result. We will use this result to show that every bicategory is biequivalent to a bicategory with strict identities. For monoidal categories, it was stated in [Sch01, Lemma 3.1].

LEMMA 8.7.
Let $(\mathsf{B}, 1, c, a, l, r)$ be a bicategory. Assume we have the following data:

(i.) *For every object x of B, an arrow $\dot{1}_x \colon x \to x$.*

(ii.) *For objects x_1, x_2, x_3 in B, a map*

$$\dot{c}\colon \operatorname{ob}(\mathsf{B}(x_2,x_3)) \times \operatorname{ob}(\mathsf{B}(x_1,x_2)) \to \operatorname{ob}(\mathsf{B}(x_1,x_3)).$$

We denote $\dot{c}(g,f)$ by $g \circledcirc f$ for composable arrows $f\colon x_1 \to x_2$, $g\colon x_2 \to x_3$.

(iii.) *For a pair of composable arrows $f\colon x_1 \to x_2$, $g\colon x_2 \to x_3$, an invertible 2-cell*

$$(D^2)_{f,g}\colon g \circledcirc f \to g \circ f.$$

(iv.) *For every object x in B, an invertible 2-cell*

$$(D^0)_x\colon \dot{1}_x \to 1_x.$$

Then there is a structure of a bicategory on B such that

$$(\mathrm{id}, D^2, D^0)\colon (\mathsf{B}, \dot{1}, \dot{c}, \dot{a}, \dot{l}, \dot{r}) \to (\mathsf{B}, 1, c, a, l, r)$$

is a morphism of bicategories.

Proof. For composable arrows $f_1, f_2\colon x_1 \to x_2$ and $g_1, g_2\colon x_2 \to x_3$ with 2-cells $\alpha\colon f_1 \Rightarrow f_2$ and $\beta\colon g_1 \Rightarrow g_2$, we define the horizontal composition of 2-cells for the composition $\dot{c}$ by

$$\beta \dot{*} \alpha\colon g_1 \odot f_1 \Rightarrow g_2 \odot f_2 := \begin{tikzcd} x_1 \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[rr, bend left=60, "g_1 \odot f_1"] \arrow[rr, bend right=60, "g_2 \odot f_2"'] & x_2 \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\Downarrow\beta"] \arrow[u, phantom, "\Downarrow (D^2)_{f_1,g_1}"] \arrow[d, phantom, "\Uparrow (D^2)_{f_2,g_2}"] & x_3 \end{tikzcd}$$

Note that $1_g \dot{*} 1_f = 1_{g \odot f}$.

We define the associativity constraint $\dot{a}$ for three composable 1-cells f, g, and h by

$$\begin{tikzcd} (h \odot g) \odot f \arrow[r, "\dot{a}_{h,g,f}"] \arrow[d, "D^2_{f,h \odot g}"'] & h \odot (g \odot f) \arrow[d, "D^2_{g \odot f,h}"] \\ (h \odot g) \circ f \arrow[d, "D^2_{g,h} * 1_f"'] & h \circ (g \odot f) \arrow[d, "1_h * D^2_{f,g}"] \\ (h \circ g) \circ f \arrow[r, "a_{h,g,f}"] & h \circ (g \circ f) \end{tikzcd}$$

The left and right unitors $\dot{l}$ and $\dot{r}$ for a 1-cell are defined by

$$\begin{tikzcd} x_1 \arrow[r, "f"] \arrow[rr, bend left=60, "\mathrm{i}_{x_2} \odot f"] \arrow[rr, bend right=60, "f"'] & x_2 \arrow[r, bend left, "\mathrm{i}_{x_2}"] \arrow[r, bend right, "1_{x_2}"'] \arrow[r, phantom, "\Downarrow D^0_{x_2}"] \arrow[u, phantom, "\Downarrow D^2_{f,\mathrm{i}_{x_2}}"] \arrow[d, phantom, "\Uparrow D^2_{f,1_{x_2}}"] & x_2 \end{tikzcd} \qquad \begin{tikzcd} x_1 \arrow[r, bend left, "\mathrm{i}_{x_1}"] \arrow[r, bend right, "1_{x_1}"'] \arrow[r, phantom, "\Downarrow D^0_{x_1}"] \arrow[rr, bend left=60, "f \odot \mathrm{i}_{x_1}"] \arrow[rr, bend right=60, "f"'] & x_1 \arrow[r, "f"] \arrow[u, phantom, "\Downarrow D^2_{\mathrm{i}_{x_1},f}"] \arrow[d, phantom, "\Uparrow D^2_{1_{x_1},f}"] & x_2 \end{tikzcd} .$$

With those definitions, we defined the structure of a bicategory since the following two diagrams are commutative

$$
\begin{tikzcd}
 & & (k\odot h)\odot(g\odot f) \arrow[d, "D^2_{g\odot f,k\odot h}"] \arrow[rr, rounded corners, to path={-| (\tikztotarget)}, "\dot a_{k,h,g\odot f}"] & & \\
 & & (k\odot h)\circ(g\odot f) \arrow[dl, "\mathsf{1}_{k\odot h}*D^2_{f,g}"'] \arrow[dr, "D^2_{h,k}*\mathsf{1}_{g\odot f}"] & & \\
((k\odot h)\odot g)\odot f \arrow[uurr, rounded corners, to path={|- (\tikztotarget)}, "\dot a_{k\odot h,g,f}"] \arrow[d, "D^2_{f,(k\odot h)\odot g}"] \arrow[ddddd, bend right=40, "\dot a_{k,h,g}\dot{*}\mathsf{1}_f"'] & (k\odot h)\circ(g\circ f) \arrow[dr, "D^2_{h,k}*\mathsf{1}_{g\circ f}"] & & (k\circ h)\circ(g\odot f) \arrow[dl, "\mathsf{1}_{k\circ h}*D^2_{f,g}"'] \arrow[d, "a_{k,h,g\odot f}"] & k\odot(h\odot(g\odot f)) \arrow[d, "D^2_{h\odot(g\odot f),k}"] \\
((k\odot h)\odot g)\circ f \arrow[r, "D^2_{g,k\odot h}*\mathsf{1}_f"] \arrow[ddd, "\dot a_{k,h,g}*\mathsf{1}_f"'] & ((k\odot h)\circ g)\circ f \arrow[u, "a_{k\odot h,g,f}"] \arrow[d, "(D^2_{h,k}*\mathsf{1}_g)*\mathsf{1}_f"'] & (k\circ h)\circ(g\circ f) \arrow[dr, "a_{k,h,g\circ f}"] & k\circ(h\circ(g\odot f)) \arrow[d, "\mathsf{1}_k*(\mathsf{1}_h*D^2_{f,g})"] & k\circ(h\odot(g\odot f)) \arrow[l, "\mathsf{1}_k*D^2_{g\odot f,h}"] \\
 & ((k\circ h)\circ g)\circ f \arrow[ur, "a_{k\circ h,g,f}"] \arrow[d, "a_{k,h,g}*\mathsf{1}_f"'] & & k\circ(h\circ(g\circ f)) & \\
 & (k\circ(h\circ g))\circ f \arrow[rr, "a_{k,h\circ g,f}"'] & & k\circ((h\circ g)\circ f) \arrow[u, "\mathsf{1}_k*a_{h,g,f}"'] & \\
(k\odot(h\odot g))\circ f \arrow[r, "D^2_{h\odot h,k}*\mathsf{1}_f"'] & (k\circ(h\odot g))\circ f \arrow[u, "(\mathsf{1}_k*D^2_{g,h})*\mathsf{1}_f"'] \arrow[rr, "a_{k,h\odot g,f}"'] & & k\circ((h\odot g)\circ f) \arrow[u, "\mathsf{1}_k*(D^2_{g,h}*\mathsf{1}_f)"] & k\circ((h\odot g)\odot f) \arrow[l, "\mathsf{1}_k*D^2_{f,h\odot g}"'] \arrow[uuu, "\mathsf{1}_k*\dot a_{h,g,f}"'] \\
(k\odot(h\odot g))\odot f \arrow[u, "D^2_{f,(k\odot(h\odot g))}"] \arrow[rrrr, "\dot a_{k,h\odot g,f}"'] & & & & k\odot((h\odot g)\odot f) \arrow[u, "D^2_{(h\odot g)\odot f,k}"'] \arrow[uuuuu, bend right=40, "\mathsf{1}_k\dot{*}\dot a_{h,g,f}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
(g\odot \mathrm{i}_{x_2})\odot f \arrow[rr, "\dot a_{g,\mathrm{i}_{x_2},f}"] \arrow[d, "D^2_{f,g\odot \mathrm{i}_{x_2}}"'] \arrow[ddddr, bend right=60, "\dot r_g\dot{*}\mathsf{1}_f"'] & & g\odot(\mathrm{i}_{x_2}\odot f) \arrow[d, "D^2_{\mathrm{i}_{x_2}\odot f,g}"] \arrow[ddddl, bend left=60, "\mathsf{1}_g\dot{*}\dot l_f"] \\
(g\odot \mathrm{i}_{x_2})\circ f \arrow[d, "D^2_{\mathrm{i}_{x_2},g}*\mathsf{1}_f"'] \arrow[dddr, bend right=45, "\dot r_g*\mathsf{1}_f"'] & & g\circ(\mathrm{i}_{x_2}\odot f) \arrow[d, "\mathsf{1}_g*D^2_{f,\mathrm{i}_{x_2}}"] \arrow[dddl, bend left=45, "\mathsf{1}_g*\dot l_f"] \\
(g\circ \mathrm{i}_{x_2})\circ f \arrow[rr, "a_{g,\mathrm{i}_{x_2},f}"] \arrow[d, "(\mathsf{1}_g*D^0_{x_2})*\mathsf{1}_f"'] & & g\circ(\mathrm{i}_{x_2}\circ f) \arrow[d, "\mathsf{1}_g*(D^0_{x_2}*\mathsf{1}_f)"] \\
(g\circ \mathsf{1}_{x_2})\circ f \arrow[rr, "a_{g,\mathsf{1}_{x_2},f}"] \arrow[dr, "r_g*\mathsf{1}_f"'] & & g\circ(\mathsf{1}_{x_2}\circ f) \arrow[dl, "\mathsf{1}_g*l_f"] \\
 & g\odot f &
\end{tikzcd}.
$$

∎

Corollary 8.8.
Let $(\mathsf{B}, \mathsf{1}, c, a, l, r)$ *be a bicategory. Then, there is a bicategory* $\mathsf{B}^{\mathrm{str}\,\mathsf{1}}$ *whose left and*

right unit constraints l and r are identities with a biequivalence $\mathrm{str}^1\colon \mathsf{B}^{\mathrm{str}\,1} \to \mathsf{B}$*, whose underlying map on objects is the identity.*

Proof. By choosing, for 1-cells f, g in B

$$\dot{c}(g,f) := \begin{cases} g, & f = 1, \\ f, & g = 1, \\ g \circ f, & \text{else}, \end{cases} \text{ and } D^2_{f,g} := \begin{cases} l_g^{-1}, & f = 1, \\ r_f^{-1}, & g = 1, \\ 1_{g\circ f}, & \text{else}. \end{cases}$$

with $\dot{1}_x := 1_x$ and $(D^0)_x\colon 1_x \to 1_x$, we get the statement by Lemma 8.7. ▪

The corresponding statement for monoidal categories is [Sch01, Theorem 3.2].

Lemma 8.9.
Let $(\mathsf{B}, 1, c, a, l, r)$ *be a bicategory. Then in* $\mathsf{B}^{\mathrm{str}\,1}$ *with associativity constraint* $a^{\mathrm{str}\,1}$ *we have for composable 1-cells* $f\colon x_1 \to x_2$*,* $g\colon x_2 \to x_3$ *the equalities*

$$a^{\mathrm{str}\,1}_{g,f,1_{x_1}} = 1_{g\circ f}, \quad a^{\mathrm{str}\,1}_{g,1_{x_2},f} = 1_{g\circ f}, \quad a^{\mathrm{str}\,1}_{1_{x_3},g,f} = 1_{g\circ f}$$

Proof. We get the statement immediately since in any bicategory the following diagrams commute

$$\begin{tikzcd} (g\circ 1_{x_2})\circ f \arrow[rr, "a^{\mathrm{str}1}_{g,1_{x_2},f}"] \arrow[dr, "r_g * 1_f"'] & & g\circ(1_{x_2}\circ f) \arrow[dl, "1_g * l_f"] \\ & g\circ f & \end{tikzcd} ,$$

$$\begin{tikzcd} (1_{x_3}\circ g)\circ f \arrow[rr, "a^{\mathrm{str}1}_{1_{x_3},g,f}"] \arrow[dr, "l_g * 1_f"'] & & 1_{x_3}\circ(g\circ f) \arrow[dl, "l_{g\circ f}"] \\ & g\circ f & \end{tikzcd} ,$$

$$\begin{tikzcd} (g\circ f)\circ 1_{x_1} \arrow[rr, "a^{\mathrm{str}1}_{g,f,1_{x_1}}"] \arrow[dr, "r_{g\circ f}"'] & & g\circ(f\circ 1_{x_1}) \arrow[dl, "1_g * r_f"] \\ & g\circ f & \end{tikzcd} ;$$

see e.g. [JY21, Proposition 2.2.4]. ▪

We will call a bicategory whose left and right unit constraints are identities *identity-normalised*.

By the unitality of morphisms between bicategories, see e.g. [JY21, Equation 4.1.4], we can thus conclude the following.

LEMMA 8.10.
Let $S\colon \mathsf{A} \to \mathsf{B}$ be a morphism of bicategories such that A and B have trivial left and right identity constraints and such that S^0 is trivial for every object of A. Then $S^2_{f,g}$ is an identity if at least one of the 1-cells f and g is an identity.

Similarly, we also get the following.

LEMMA 8.11.
Let $S\colon \mathsf{A} \to \mathsf{B}$ be a morphism of bicategories. Then there exists a morphism of bicategories $S^{\mathrm{str}\,1}\colon \mathsf{A}^{\mathrm{str}\,1} \to \mathsf{B}^{\mathrm{str}\,1}$ such that the following diagram commutes up to an invertible 2-cell

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "S"] \arrow[d, "\mathrm{str}^1"'] \arrow[dr, phantom, "\swarrow\alpha"] & \mathsf{B} \arrow[d, "\mathrm{str}^1"] \\
\mathsf{A}^{\mathrm{str}\,1} \arrow[r, "S^{\mathrm{str}\,1}"'] & \mathsf{B}^{\mathrm{str}\,1}
\end{tikzcd}$$

whose unit constraints $(S^{\mathrm{str}\,1})^0$ is the identity for all objects and such that $(S^{\mathrm{str}\,1})^2_{f,g} = 1$ if at least one of the 1-cells f and g is an identity.

A morphism between identity-normalised bicategories of the above form is called *identity-normalised*.

LEMMA 8.12.
Let A and B be two identity-normalised bicategories and $S, T\colon \mathsf{A} \to \mathsf{B}$ identity-normalised morphisms. Then a transformation $\alpha\colon S \to T$ has the naturality constraints $\alpha_{1_x} = 1_{\alpha_x}$ for every object x of A.

Proof. Follows directly from the oplax unity axiom. ▪

§8.2. Definition of a Monoidal Bicategory

In this section, we will define *monoidal bicategories*. A monoidal bicategory B comes with a morphism of bicategories $\boxtimes\colon \mathsf{B} \times \mathsf{B} \to \mathsf{B}$ that we will call the *tensor product*. We will see them as a categorification of a *monoidal category* $(\mathsf{C}, \otimes)$. For a monoidal category $(\mathsf{C}, \otimes)$, there are natural isomorphisms $a_{x,y,z} \colon (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ for which a pentagonal diagram commutes; see [Mac63] and [Bén63].

For a monoidal bicategory, $(\mathsf{B}, \boxtimes)$, we will have natural adjoint equivalences $A_{x,y,z} \colon (x \boxtimes y) \boxtimes z \to x \boxtimes (y \boxtimes z)$ and a modification π replacing the pentagon axiom. This data will be quite involved. By applying a coherence theorem (see [GPS95]), it is possible to show that every monoidal bicategory is equivalent (in an appropriate sense) to a Gray monoid, whose underlying bicategory is actually a strict 2-category and whose tensor product is *cubical*; see Remark 8.21. They have been studied quite extensively; see e.g. [DS97]. With the additional structure of a braiding they were used in e.g. [KV94], [Cra98], and [BN96].

In many applications, such as the extension theory for fusion categories [ENO10] and finite tensor categories [DN21], or the classification of algebraic homotopy types [Ber99], people often work with Gray-monoids to simplify the exposition. However, examples naturally appear as monoidal bicategories. In fact, even if we start with a Gray-monoid, once we take the *skeleton* of the underlying 2-category, i.e. choose representatives of equivalence classes of objects and representatives of isomorphism classes of 1-cells, we are naturally led to the structure of a monoidal bicategory rather than a Gray-monoid. This is the reason why we state the definition of a monoidal bicategory in its full generality.

In the following chapter, we will furthermore equip a monoidal bicategory with a coherent Gr-structure and show that we can extend many of the known results for coherent Gr-categories.

Monoidal Bicategories

A monoidal bicategory can be defined as a one-object tricategory. Our definition is based on [Gur13]. We picked specific adjoint equivalences in contrast to e.g. [GPS95].

Definition 8.13 (*monoidal bicategory*).
A *monoidal bicategory* consists of the following data:

(i.) A bicategory $(\mathsf{B}, 1, c, a, l, r)$.

(ii.) A morphism $(\boxtimes, \boxtimes^2, \boxtimes^0)\colon \mathsf{B} \times \mathsf{B} \to \mathsf{B}$ with component 2-cells

$$\begin{tikzcd} x \boxtimes y \arrow[r, bend left, "\mathbf{1}_{x\boxtimes y}"] \arrow[r, bend right, "\mathbf{1}_x \boxtimes \mathbf{1}_y"'] \arrow[r, phantom, "\Downarrow \boxtimes^0_{x,y}"] & x \boxtimes y \end{tikzcd}$$

for objects x and y in B, and

$$\begin{tikzcd} x_1 \boxtimes y_1 \arrow[r, "f_1 \boxtimes g_1"] \arrow[dr, "(f_2 \circ f_1)\boxtimes(g_2 \circ g_1)"'] & x_2 \boxtimes y_2 \arrow[d, "f_2 \boxtimes g_2"] \arrow[dl, phantom, "\Downarrow \boxtimes^2_{(f_1,g_1),(f_2,g_2)}"] \\ & x_3 \boxtimes y_3 \end{tikzcd}$$

for 1-cells $f_1\colon x_1 \to x_2$, $f_2\colon x_2 \to x_3$, $g_1\colon y_1 \to y_2$, $g_2\colon y_2 \to y_3$.

(iii.) A morphism $(\mathbb{1}, \mathbb{1}^2, \mathbb{1}^0)\colon \mathbf{1} \to \mathsf{B}$. By abuse of notation, we identify $\mathbb{1}$ with the image of the object in $\mathbf{1}$.

(iv.) An adjoint equivalence

$$A\colon\ \boxtimes \circ(\boxtimes \times \mathrm{id}_{\mathsf{B}}) \to \boxtimes \circ(\mathrm{id}_{\mathsf{B}} \times \boxtimes),$$

called the *associator*, with right adjoint $A^\bullet$ with invertible components

$$\begin{tikzcd} (x_1 \boxtimes y_1) \boxtimes z_1 \arrow[r, "(f\boxtimes g)\boxtimes h"] \arrow[d, "A_{x_1,y_1,z_1}"'] \arrow[dr, phantom, "\Downarrow A_{f,g,h}"] & (x_2 \boxtimes y_2) \boxtimes z_2 \arrow[d, "A_{x_2,y_2,z_2}"] \\ x_1 \boxtimes (y_1 \boxtimes z_1) \arrow[r, "f\boxtimes(g\boxtimes h)"'] & x_2 \boxtimes (y_2 \boxtimes z_2) \end{tikzcd}$$

for $f\colon x_1 \to x_2$, $g\colon y_1 \to y_2$ and $h\colon z_1 \to z_2$.

(v.) Adjoint equivalences L and R, called the *left* and *right unitor*,

$$L\colon\ \boxtimes \circ(\mathbb{1} \times 1) \to 1$$
$$R\colon\ \boxtimes \circ(1 \times \mathbb{1}) \to 1$$

with right adjoints $L^\bullet$ and $R^\bullet$, respectively, with invertible components

$$\begin{tikzcd}
\mathbb{1} \boxtimes x_1 \arrow[r, "1_{\mathbb{1}} \boxtimes f"] \arrow[d, "L_{x_1}"'] \arrow[dr, phantom, "\swarrow L_f"] & \mathbb{1} \boxtimes x_2 \arrow[d, "L_{x_2}"] \\
x_1 \arrow[r, "f"'] & x_2
\end{tikzcd}
\qquad
\begin{tikzcd}
x_1 \boxtimes \mathbb{1} \arrow[r, "f \boxtimes 1_{\mathbb{1}}"] \arrow[d, "R_{x_1}"'] \arrow[dr, phantom, "\swarrow R_f"] & x_2 \boxtimes \mathbb{1} \arrow[d, "R_{x_2}"] \\
x_1 \arrow[r, "f"'] & x_2
\end{tikzcd}$$

for $f\colon x_1 \to x_2$, where we denote $\mathbb{1} := \mathbb{1}(*)$ for the unique object $*$ in $\mathbf{1}$.

(vi.) A modification π, called the *pentagonator*, with components

$$\begin{tikzcd}
& {[w \boxtimes (x \boxtimes y)] \boxtimes z} \arrow[rr, "A_{w,x\boxtimes y,z}"] & & {w \boxtimes [(x \boxtimes y) \boxtimes z]} \arrow[dr, "1_w \boxtimes A_{x,y,z}"] & \\
{[(w \boxtimes x) \boxtimes y] \boxtimes z} \arrow[ur, "A_{w,x,y} \boxtimes 1_z"] \arrow[drr, "A_{w\boxtimes x,y,z}"'] & & \Downarrow \pi_{w,x,y,z} & & {w \boxtimes [x \boxtimes (y \boxtimes z)]} \\
& & (w \boxtimes x) \boxtimes (y \boxtimes z) \arrow[urr, "A_{w,x,y\boxtimes z}"'] & &
\end{tikzcd} .$$

(vii.) Modifications μ, λ, and ρ, called the *middle*, *left* and *right* 2*-unitor* respectively, with components

$$\begin{tikzcd}
(x \boxtimes \mathbb{1}) \boxtimes y \arrow[rr, "A_{x,\mathbb{1},y}"] \arrow[dr, "R_x \boxtimes 1_y"'] & \Downarrow \mu_{x,y} & x \boxtimes (\mathbb{1} \boxtimes y) \arrow[dl, "1_x \boxtimes L_y"] \\
& x \boxtimes y &
\end{tikzcd} ,
\quad
\begin{tikzcd}
(\mathbb{1} \boxtimes x) \boxtimes y \arrow[rr, "L_x \boxtimes 1_y"] \arrow[dr, "A_{\mathbb{1},x,y}"'] & \Downarrow \lambda_{x,y} & x \boxtimes y \\
& \mathbb{1} \boxtimes (x \boxtimes y) \arrow[ur, "L_{x\boxtimes y}"'] &
\end{tikzcd} ,$$

$$\begin{tikzcd}
(x \boxtimes y) \boxtimes \mathbb{1} \arrow[rr, "R_{x\boxtimes y}"] \arrow[dr, "A_{x,y,\mathbb{1}}"'] & \Downarrow \rho_{x,y} & x \boxtimes y \\
& x \boxtimes (y \boxtimes \mathbb{1}) \arrow[ur, "1_x \boxtimes R_y"'] &
\end{tikzcd} .$$

We summarise our colouring scheme

$\boxtimes^0$	$\boxtimes^2$	A	μ	λ	ρ	π

.

This data is required to satisfy:

(A.) For all objects v, w, x, y, z in B, we have the equality

$$
\begin{tikzcd}
\{[v \boxtimes (w \boxtimes x)] \boxtimes y\} \boxtimes z \arrow[rr, "A_{v,w\boxtimes x,y} \boxtimes 1_z"] \arrow[d, "A_{v\boxtimes(w\boxtimes x),y,z}"'] & & \{v \boxtimes [(w \boxtimes x) \boxtimes y]\} \boxtimes z \arrow[r, "\{1_v \boxtimes A_{w,x,y}\}\boxtimes 1_z"] \arrow[d, "A_{v,(w\boxtimes x)\boxtimes y,z}"] & \{v \boxtimes [w \boxtimes (x \boxtimes y)]\} \boxtimes z \arrow[d, "A_{v,w\boxtimes(x\boxtimes y),z}"] \\
{[v \boxtimes (w \boxtimes x)] \boxtimes (y \boxtimes z)} \arrow[drr, "A_{v,w\boxtimes x,y\boxtimes z}"] \arrow[dd, bend right, "{}^\bullet(A_{v,w,x})\boxtimes(1_y \boxtimes 1_z)"'] \arrow[dd, bend left, "{}^\bullet(A_{v,w,x})\boxtimes 1_{y\boxtimes z}"] & & v \boxtimes \{[(w \boxtimes x) \boxtimes y] \boxtimes z\} \arrow[r, "1_v \boxtimes\{A_{w,x,y}\boxtimes 1_z\}"'] \arrow[d, "1_v \boxtimes A_{w\boxtimes x,y,z}"] & v \boxtimes \{[w \boxtimes (x \boxtimes y)] \boxtimes z\} \arrow[dd, "1_v \boxtimes A_{w,x\boxtimes y,z}"] \\
& & v \boxtimes \{(w \boxtimes x) \boxtimes (y \boxtimes z)\} \arrow[d, "1_v \boxtimes A_{w,x,y\boxtimes z}"] & \\
{[(v \boxtimes w) \boxtimes x] \boxtimes (y \boxtimes z)} \arrow[r, "A_{v\boxtimes w,x,y\boxtimes z}"'] & (v \boxtimes w) \boxtimes [x \boxtimes (y \boxtimes z)] \arrow[r, "A_{v,w,x\boxtimes(y\boxtimes z)}"'] & v \boxtimes \{w \boxtimes [x \boxtimes (y \boxtimes z)]\} \arrow[r, "1_v \boxtimes\{1_w \boxtimes A^\bullet_{x,y,z}\}"'] & v \boxtimes \{w \boxtimes [(x \boxtimes y) \boxtimes z]\}
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
\{[v \boxtimes (w \boxtimes x)] \boxtimes y\} \boxtimes z \arrow[rr, "A_{v,w\boxtimes x,y} \boxtimes 1_z"] \arrow[d, "A_{v\boxtimes(w\boxtimes x),y,z}"'] \arrow[dr, "\{{}^\bullet(A_{v,w,x})\boxtimes 1_y\}\boxtimes 1_z"] & & \{v \boxtimes [(w \boxtimes x) \boxtimes y]\} \boxtimes z \arrow[r, "\{1_v \boxtimes A_{w,x,y}\}\boxtimes 1_z"] & \{v \boxtimes [w \boxtimes (x \boxtimes y)]\} \boxtimes z \arrow[dd, "A_{v,w\boxtimes(x\boxtimes y),z}"] \\
{[v \boxtimes (w \boxtimes x)] \boxtimes (y \boxtimes z)} \arrow[dd, "{}^\bullet(A_{v,w,x})\boxtimes\{1_y \boxtimes 1_z\}"'] & \{[(v \boxtimes w) \boxtimes x] \boxtimes y\} \boxtimes z \arrow[r, "A_{v\boxtimes w,x,y}\boxtimes 1_z"'] \arrow[ddl, "A_{(v\boxtimes w)\boxtimes x,y,z}"] & \{(v \boxtimes w) \boxtimes (x \boxtimes y)\} \boxtimes z \arrow[ur, "A_{v,w,x\boxtimes y}\boxtimes 1_z"'] \arrow[dl, "A_{v\boxtimes w,x\boxtimes y,z}"] & \\
& (v \boxtimes w) \boxtimes [(x \boxtimes y) \boxtimes z] \arrow[drr, "A_{v,w,(x\boxtimes y)\boxtimes z}"] & & v \boxtimes \{[w \boxtimes (x \boxtimes y)] \boxtimes z\} \arrow[d, "1_v \boxtimes A_{w,x\boxtimes y,z}"] \\
{[(v \boxtimes w) \boxtimes x] \boxtimes (y \boxtimes z)} \arrow[r, "A_{v\boxtimes w,x,y\boxtimes z}"'] & (v \boxtimes w) \boxtimes [x \boxtimes (y \boxtimes z)] \arrow[r, "A_{v,w,x\boxtimes(y\boxtimes z)}"'] \arrow[u, bend left, "1_{v\boxtimes w}\boxtimes A^\bullet_{x,y,z}"] \arrow[u, bend right, "(1_v \boxtimes 1_w)\boxtimes A^\bullet_{x,y,z}"'] & v \boxtimes \{w \boxtimes [x \boxtimes (y \boxtimes z)]\} \arrow[r, "1_v \boxtimes\{1_w \boxtimes A^\bullet_{x,y,z}\}"'] & v \boxtimes \{w \boxtimes [(x \boxtimes y) \boxtimes z]\}
\end{tikzcd}
.
$$

(B.) For all objects x, y, z in B, we have the following two equalities

$$
\begin{tikzcd}
(x \boxtimes y) \boxtimes (\mathbb{1} \boxtimes z) \arrow[rrr, "\mathbf{1}_{x\boxtimes y}\boxtimes L_z"] \arrow[dr, "{}^\bullet(A_{x\boxtimes y,\mathbb{1},z})"'] \arrow[ddd, "A_{x,y,\mathbb{1}\boxtimes z}"'] & & & (x \boxtimes y) \boxtimes z \arrow[ddd, "A_{x,y,z}"] \\
& {[(x \boxtimes y) \boxtimes \mathbb{1}] \boxtimes z} \arrow[urr, "R_{x\boxtimes y}\boxtimes \mathbf{1}_z"'] \arrow[dr, "A_{x,y,\mathbb{1}}\boxtimes \mathbf{1}_z"'] & & \\
& & {[x \boxtimes (y \boxtimes \mathbb{1})] \boxtimes z} \arrow[uur, "{[\mathbf{1}_x\boxtimes R_y]\boxtimes \mathbf{1}_z}"'] \arrow[d, "A_{x,y\boxtimes\mathbb{1},z}"] & \\
x \boxtimes [y \boxtimes (\mathbb{1} \boxtimes z)] \arrow[rr, "\mathbf{1}_x\boxtimes{}^\bullet(A_{y,\mathbb{1},z})"'] & & x \boxtimes [(y \boxtimes \mathbb{1}) \boxtimes z] \arrow[r, "{\mathbf{1}_x\boxtimes[R_y\boxtimes \mathbf{1}_z]}"'] & x \boxtimes (y \boxtimes z)
\end{tikzcd}
$$

$$=$$

$$
\begin{tikzcd}
(x \boxtimes y) \boxtimes (\mathbb{1} \boxtimes z) \arrow[rr, bend left, "\mathbf{1}_{x\boxtimes z}\boxtimes L_z"] \arrow[rr, bend right, "(\mathbf{1}_x\boxtimes \mathbf{1}_y)\boxtimes L_z"'] \arrow[d, "A_{x,y,\mathbb{1}\boxtimes z}"'] & & (x \boxtimes y) \boxtimes z \arrow[d, "A_{x,y,z}"] \\
x \boxtimes [y \boxtimes (\mathbb{1} \boxtimes z)] \arrow[rr, "\mathbf{1}_x\boxtimes(\mathbf{1}_y\boxtimes L_z)"] \arrow[dr, "\mathbf{1}_x\boxtimes{}^\bullet(A_{y,\mathbb{1},z})"'] & & x \boxtimes (y \boxtimes z) \\
& x \boxtimes [(y \boxtimes \mathbb{1}) \boxtimes z] \arrow[ur, "{\mathbf{1}_x\boxtimes[R_y\boxtimes \mathbf{1}_z]}"'] &
\end{tikzcd}
.
$$

$$
\begin{tikzcd}
{[(x \boxtimes \mathbb{1}) \boxtimes y] \boxtimes z} \arrow[r, "A_{x,\mathbb{1},y} \boxtimes \mathbf{1}_z"] \arrow[dddd, "A_{x\boxtimes\mathbb{1},y,z}"'] & {[x \boxtimes (\mathbb{1} \boxtimes y)] \boxtimes z} \arrow[r, "{[\mathbf{1}_x \boxtimes L_y] \boxtimes \mathbf{1}_z}"] \arrow[d, "A_{x,\mathbb{1}\boxtimes y,z}"] & (x \boxtimes y) \boxtimes z \arrow[dddd, "A_{x,y,z}"] \\
& x \boxtimes [(\mathbb{1} \boxtimes y) \boxtimes z] \arrow[dd, "\mathbf{1}_x \boxtimes A_{\mathbb{1},y,z}"'] \arrow[dddr, "{\mathbf{1}_x \boxtimes [L_y \boxtimes \mathbf{1}_z]}"] & \\
& & \\
& x \boxtimes [\mathbb{1} \boxtimes (y \boxtimes z)] \arrow[ddr, "\mathbf{1}_x \boxtimes L_{y\boxtimes z}"] & \\
(x \boxtimes \mathbb{1}) \boxtimes (y \boxtimes z) \arrow[uur, "A_{x,\mathbb{1},y\boxtimes z}"] \arrow[rr, bend left=10, "R_x \boxtimes \mathbf{1}_{y\boxtimes z}"] \arrow[rr, bend right=10, "R_x \boxtimes (\mathbf{1}_y \boxtimes \mathbf{1}_z)"'] & & x \boxtimes (y \boxtimes z)
\end{tikzcd}
$$

$$=$$

$$
\begin{tikzcd}
{[(x \boxtimes \mathbb{1}) \boxtimes y] \boxtimes z} \arrow[r, "A_{x,\mathbb{1},y} \boxtimes \mathbf{1}_z"] \arrow[dd, "A_{x\boxtimes\mathbb{1},y,z}"'] \arrow[dr, "{[R_x \boxtimes \mathbf{1}_y] \boxtimes \mathbf{1}_z}"'] & {[x \boxtimes (\mathbb{1} \boxtimes y)] \boxtimes z} \arrow[d, "{[\mathbf{1}_x \boxtimes L_y] \boxtimes \mathbf{1}_z}"] \\
& (x \boxtimes y) \boxtimes z \arrow[d, "A_{x,y,z}"] \\
(x \boxtimes \mathbb{1}) \boxtimes (y \boxtimes z) \arrow[r, "R_x \boxtimes (\mathbf{1}_y \boxtimes \mathbf{1}_z)"'] & x \boxtimes (y \boxtimes z)
\end{tikzcd}
\quad .
$$

Remark 8.14.

Alternatively, we could have worked with *weak units* as in [JK07] or [JK13], inspired by [Saa72]. We will see that by an appropriate choice of (the tricategorical) composition, we can assume that the adjoint equivalences L and R are identities; compare to [Eti+15, Remark 2.2.9].

Remark .

In a bicategory, composition of 1-cells is only defined up to 2-cells. In the pasting diagrams above, we omitted all the bicategorical constraints a, l, and r. They can be reconstructed as in [Gur13, Remark 4.5] or in [JY21, Example 3.4.20]. Furthermore, we used mates of 2-cells in the above diagrams (as indicated by the adjoint edges). Up to the bicategorical constraints, we can thus read off the corresponding composition of 2-cells easily. We left the direction of the 2-cells implicit by only colouring the faces. In particular, we also coloured the mate of a 2-cell with the same colour as the corresponding 2-cell.

Example 8.15.
Let M be a monoid (in Set). Then M regarded as a discrete category is a strict monoidal category with the multiplications of M as the monoidal product. We will denote this monoidal category by $M^{\otimes}$.

If we regard this category as a locally discrete bicategory (i.e. with only identity 2-cells), we get a monoidal bicategory in which A, L, and R are identity equivalences and π, μ, λ, and ρ are identity 2-cells. We will denote this monoidal bicategory as $M^{\mathsf{bi}\boxtimes}$.

Example 8.16.
For a bicategory B, the bicategory of endomorphisms $\mathsf{Bicat}(\mathsf{B}, \mathsf{B})$ is a monoidal bicategory, with the composition of morphisms as tensor product. The proof is discussed at length in [JY21, Chapter 11]. In fact, they prove that there is a tricategory with bicategories as objects, morphisms of bicategories as 1-cells, transformations as 2-cells and modifications as 3-cells. Note that the component 1-cells of the associator are identities, while the component 2-cells, in general, are not identities and depend on the composition structure of B; see [JY21, Equation 11.5.6].

By using the same argument as in Example 8.3, i.e. by considering the opposite and co-tricategory of a one-object tricategory, we obtain monoidal structures on a monoidal bicategory.

Lemma 8.17.
Let

$$(\mathsf{B}, \boxtimes, \mathbb{1}, A, L, R, \pi, \mu, \lambda, \rho)$$

be a monoidal bicategory. Then we can define two other structures of a monoidal bicategory:

(i.) The bicategory B *with monoidal product* $\boxtimes^{\mathrm{rev}}$*, such that*

$$x \boxtimes^{\mathrm{rev}} y = y \boxtimes x$$

for every pair of objects x and y in B. The associator is given by

$$A^{\bullet}_{z,y,x} \colon (x \boxtimes^{\mathrm{rev}} y) \boxtimes^{\mathrm{rev}} z \to x \boxtimes^{\mathrm{rev}} (y \boxtimes^{\mathrm{rev}} z)$$

for all objects x, y, z in B.

(ii.) The opposite bicategory B^{op} monoidal product $\boxtimes^{\mathrm{op}}$, such that

$$x \boxtimes^{\mathrm{op}} y = x \boxtimes y$$

for every pair of objects x and y in B. The associator is given by

$$A^{\bullet}_{x,y,z} \colon (x \boxtimes^{\mathrm{op}} y) \boxtimes^{\mathrm{op}} z \to^{\mathrm{op}} x \boxtimes^{\mathrm{op}} (y \boxtimes^{\mathrm{op}} z)$$

for all objects x, y, z in B.

We will call the first monoidal bicategory $\mathsf{B}^{\boxtimes\,\mathrm{rev}}$ and the second one $\mathsf{B}^{\mathrm{op}\,\boxtimes}$. For details on the opposite of a tricategory, we refer to [Gur13, Definition 3.2.1 and Corollary 3.2.2].

Note that

$$(\mathsf{B}^{\mathrm{op}\,\boxtimes})^{\boxtimes\,\mathrm{rev}} = (\mathsf{B}^{\boxtimes\,\mathrm{rev}})^{\mathrm{op}\,\boxtimes}$$

(if we choose the adjoints accordingly). This follows from

$$(A^{\mathrm{op}})^{\bullet}_{z,y,x} = A_{z,y,x} = (A^{\mathrm{rev}})^{\bullet}_{x,y,z}.$$

Lemma 8.18.
Let

$$(\mathsf{B}, \boxtimes, \mathbb{1}, A, L, R, \pi, \mu, \lambda, \rho)$$

be a monoidal bicategory. Then we have an invertible 2-cell

$$L_{\mathbb{1}} \to R_{\mathbb{1}}.$$

Proof. Recall that we have the invertible naturality 2-cells

$$\begin{tikzcd}
(\mathbb{1}\boxtimes\mathbb{1})\boxtimes\mathbb{1} \arrow[r, "R_{\mathbb{1}}\boxtimes 1_{\mathbb{1}}"] \arrow[d, "R_{\mathbb{1}\boxtimes\mathbb{1}}"'] \arrow[dr, phantom, "\swArrow R_{R_{\mathbb{1}}}"] & \mathbb{1}\boxtimes\mathbb{1} \arrow[d, "R_{\mathbb{1}}"] \\
\mathbb{1}\boxtimes\mathbb{1} \arrow[r, "R_{\mathbb{1}}"'] & \mathbb{1}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{1}\boxtimes(\mathbb{1}\boxtimes\mathbb{1}) \arrow[r, "1_{\mathbb{1}}\boxtimes L_{\mathbb{1}}"] \arrow[d, "L_{\mathbb{1}\boxtimes\mathbb{1}}"'] \arrow[dr, phantom, "\swArrow L_{L_{\mathbb{1}}}"] & \mathbb{1}\boxtimes\mathbb{1} \arrow[d, "L_{\mathbb{1}}"] \\
\mathbb{1}\boxtimes\mathbb{1} \arrow[r, "L_{\mathbb{1}}"'] & \mathbb{1}
\end{tikzcd}$$

By post-whiskering the first diagram with $R_{\mathbb{1}}^{\bullet}$ and using the unit of the internal adjunction $R_{\mathbb{1}} \dashv R_{\mathbb{1}}^{\bullet}$, we get an invertible 2-cell $p\colon R_{\mathbb{1}} \boxtimes 1_{\mathbb{1}} \to R_{\mathbb{1}\boxtimes\mathbb{1}}$. Combining with μ and ρ, we obtain an invertible 2-cell

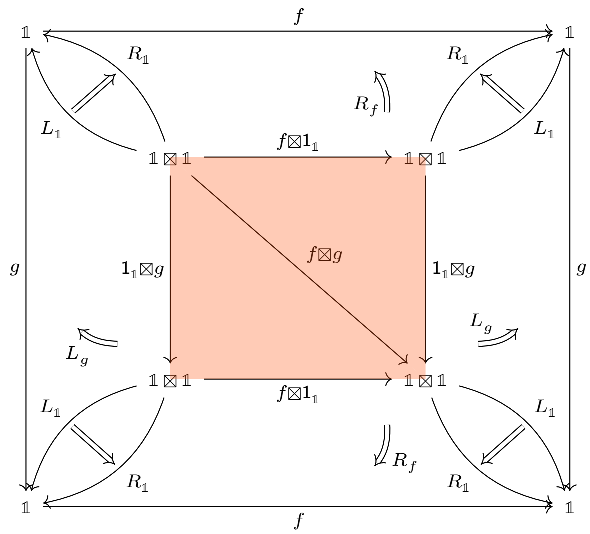
$$(1_{\mathbb{1}} \boxtimes L_{\mathbb{1}}) \circ A_{\mathbb{1},\mathbb{1},\mathbb{1}} \xrightarrow{\mu_{\mathbb{1},\mathbb{1}}} R_{\mathbb{1}} \boxtimes 1_{\mathbb{1}} \xrightarrow{p} R_{\mathbb{1}\boxtimes\mathbb{1}} \xrightarrow{\rho_{\mathbb{1},\mathbb{1}}} (1_{\mathbb{1}} \boxtimes R_{\mathbb{1}}) \circ A_{\mathbb{1},\mathbb{1},\mathbb{1}}.$$

So, by pre-whiskering with the adjoint 1-cell $A^{\bullet}_{\mathbb{1},\mathbb{1},\mathbb{1}}$, we get an invertible 2-cell $q\colon 1_{\mathbb{1}} \boxtimes L_{\mathbb{1}} \to 1_{\mathbb{1}} \boxtimes R_{\mathbb{1}}$. Thus, we get

$$R_{\mathbb{1}} \circ L_{\mathbb{1}\boxtimes\mathbb{1}} \xleftarrow[L_{R_{\mathbb{1}}}]{} L_{\mathbb{1}} \circ (1_{\mathbb{1}} \boxtimes R_{\mathbb{1}}) \xleftarrow[1_{L_{\mathbb{1}}} * q]{} L_{\mathbb{1}} \circ (1_{\mathbb{1}} \boxtimes L_{\mathbb{1}}) \xrightarrow{L_{L_{\mathbb{1}}}} R_{\mathbb{1}} \circ L_{\mathbb{1}\boxtimes\mathbb{1}}$$

so by pre-whiskering with $L^{\bullet}_{\mathbb{1}\boxtimes\mathbb{1}}$, we get the desired invertible 2-cell. ▪

Remark 8.19.
Combining the functoriality of $\boxtimes$ with the adjoint equivalences L and R induces a braiding on the category $\mathrm{Hom}_{\mathsf{B}}(\mathbb{1}, \mathbb{1})$ and the above invertible 2-cell $L_{\mathbb{1}} \to R_{\mathbb{1}}$. The argument boils down to the use of the 2-isomorphisms



We will come back to this point in Remark 8.32. We denote this 2-cell by

$$c_{f,g}\colon g \circ f \to f \circ g.$$

Using the above Remark 8.19, we get the following easy lemma, which we will use extensively later.

LEMMA 8.20.
Consider two 1*-cells* $f, g\colon \mathbb{1} \to \mathbb{1}$ *such that there exists a square*

$$\begin{tikzcd} \mathbb{1} \arrow[r, "f"] \arrow[d, "p"'] & \mathbb{1} \arrow[d, "p"] \\ \mathbb{1} \arrow[r, "g"'] \arrow[ur, phantom, "\swArrow_{\alpha}"] & \mathbb{1} \end{tikzcd}$$

with an invertible 2*-cell* α *and an equivalence* p *with adjoint* $p^\bullet$ *and counit* $\epsilon\colon p \circ p^\bullet \to 1_{\mathbb{1}}$ *and unit* $\eta\colon 1_{\mathbb{1}} \to p^\bullet \circ p$*. Then there exists an invertible* 2*-cell from* f *to* g*.*

Proof. We can calculate directly using the invertible 2-cell c from Remark 8.19

$$f \cong 1_{\mathbb{1}} \circ f \xrightarrow{\eta * 1_f} (p^\bullet \circ p) \circ f \cong p^\bullet \circ (p \circ f) \xrightarrow{1_{p^\bullet} * \alpha} p^\bullet \circ (g \circ p) \xrightarrow{c_{p^\bullet, g \circ p}} (g \circ p) \circ p^\bullet \cong g \circ (p \circ p^\bullet) \xrightarrow{1_g * \epsilon} g \circ 1_{\mathbb{1}} \cong g \ .$$

We did not write out the bicategorical constraints a, l, and r. ▪

REMARK 8.21.
Using the Yoneda lemma for bicategories, see [Str80], every bicategory is biequivalent to a 2-category. However, not every tricategory is triequivalent to a 3-category; see [GPS95, Section 8.9]. Every tricategory is triequivalent to a Gray-category; see [GPS95, Theorem 8.1]. We refer to [GPS95] and [Gur13] for details. The main aim of the results in this chapter is to reproduce statements that are known for monoidal 2-categories. We specifically work in this generality because once we take the skeleton of a monoidal bicategory (even if the underlying bicategory is a 2-category), we obtain a monoidal bicategory whose underlying bicategory is not a 2-category. This will enable us to recover information that would otherwise be lost when applying coherence results as above. For details on monoidal 2-categories, which are also called Gray-*monoids* in the literature, we refer to [DS97]. Day and Street admit that 'Examples naturally occur as monoidal bicategories rather than Gray-monoids. However, the coherence theorem of [GPS95] allows us to transfer our definitions and results.'

§.8.3. Definition of a Monoidal Morphism

Definition 8.22 (*monoidal morphism*).
We consider two monoidal bicategories

$$(\mathsf{A}, \boxtimes^{\mathsf{A}}, \mathbb{1}^{\mathsf{A}}, A^{\mathsf{A}}, L^{\mathsf{A}}, R^{\mathsf{A}}, \pi^{\mathsf{A}}, \mu^{\mathsf{A}}, \lambda^{\mathsf{A}}, \rho^{\mathsf{A}}), (\mathsf{B}, \boxtimes^{\mathsf{B}}, \mathbb{1}^{\mathsf{B}}, A^{\mathsf{B}}, L^{\mathsf{B}}, R^{\mathsf{B}}, \pi^{\mathsf{B}}, \mu^{\mathsf{B}}, \lambda^{\mathsf{B}}, \rho^{\mathsf{B}}).$$

A monoidal morphism

$$(S, \chi, \iota, \omega, \gamma, \delta)\colon \mathsf{A} \to \mathsf{B}$$

from A to B consists of the following data:

(i.) A morphism

$$(S, S^2, S^0)\colon \mathsf{A} \to \mathsf{B}.$$

(ii.) An adjoint equivalence χ with components

$$\begin{tikzcd}
S(x_1) \boxtimes^{\mathsf{B}} S(y_1) \arrow[r, "S(f) \boxtimes^{\mathsf{B}} S(g)"] \arrow[d, "\chi_{x_1,y_1}"'] \arrow[dr, phantom, "\swarrow \chi_{f,g}"] & S(x_2) \boxtimes^{\mathsf{B}} S(y_2) \arrow[d, "\chi_{x_2,y_2}"] \\
S(x_1 \boxtimes^{\mathsf{A}} y_1) \arrow[r, "S(f \boxtimes^{\mathsf{A}} g)"'] & S(x_2 \boxtimes^{\mathsf{A}} y_2)
\end{tikzcd}.$$

(iii.) An adjoint equivalence ι with component

$$\iota\colon \mathbb{1}^{\mathsf{B}} \to S(\mathbb{1}^{\mathsf{A}}).$$

(iv.) An invertible modification ω with components

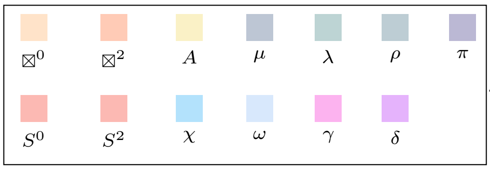
$$\begin{tikzcd}
& S(x \boxtimes^{\mathsf{A}} y) \boxtimes^{\mathsf{B}} S(z) \arrow[r, "\chi_{x \boxtimes y, z}"] & S[(x \boxtimes^{\mathsf{A}} y) \boxtimes^{\mathsf{A}} z] \arrow[dr, "S(A^{\mathsf{A}}_{x,y,z})"] & \\
(S(x) \boxtimes^{\mathsf{B}} S(y)) \boxtimes^{\mathsf{B}} S(z) \arrow[ur, "\chi_{x,y} \boxtimes^{\mathsf{B}} 1^{\mathsf{B}}_{S(z)}"] \arrow[dr, "A^{\mathsf{B}}_{Sx,Sy,Sz}"'] & \arrow[r, phantom, "\Downarrow \omega_{x,y,z}"] & \, & S[x \boxtimes^{\mathsf{A}} (y \boxtimes^{\mathsf{A}} z)] \\
& S(x) \boxtimes^{\mathsf{B}} (S(y) \boxtimes^{\mathsf{B}} S(z)) \arrow[r, "1^{\mathsf{B}}_{S(x)} \boxtimes^{\mathsf{B}} \chi_{y,z}"'] & S(x) \boxtimes^{\mathsf{B}} S(y \boxtimes^{\mathsf{A}} z) \arrow[ur, "\chi_{x, y \boxtimes z}"'] &
\end{tikzcd}.$$

(v.) Two invertible modifications γ and δ with components

$$\begin{tikzcd}
& S(\mathbb{1}) \boxtimes^{\mathsf{B}} S(x) \arrow[r, "\chi_{\mathbb{1},x}"] \arrow[d, phantom, "\Downarrow\gamma_x"] & S(\mathbb{1}^{\mathsf{A}} \boxtimes^{\mathsf{A}} x) \arrow[dr, "S(L^{\mathsf{A}}_x)"] & \\
\mathbb{1}^{\mathsf{B}} \boxtimes^{\mathsf{B}} S(x) \arrow[ur, "\iota \boxtimes^{\mathsf{B}} 1^{\mathsf{B}}_{S(x)}"] \arrow[rrr, "L^{\mathsf{B}}_{S(x)}"'] & & & S(x)
\end{tikzcd},$$

$$\begin{tikzcd}
& S(x) \boxtimes^{\mathsf{B}} S(\mathbb{1}) \arrow[r, "\chi_{x,\mathbb{1}}"] \arrow[d, phantom, "\Downarrow\delta_x"] & S(x \boxtimes^{\mathsf{A}} \mathbb{1}^{\mathsf{A}}) \arrow[dr, "S(R^{\mathsf{A}}_x)"] & \\
S(x) \boxtimes^{\mathsf{B}} \mathbb{1}^{\mathsf{B}} \arrow[ur, "1^{\mathsf{B}}_{S(x)} \boxtimes^{\mathsf{B}} \iota"] \arrow[rrr, "R^{\mathsf{B}}_{S(x)}"'] & & & S(x)
\end{tikzcd}.$$

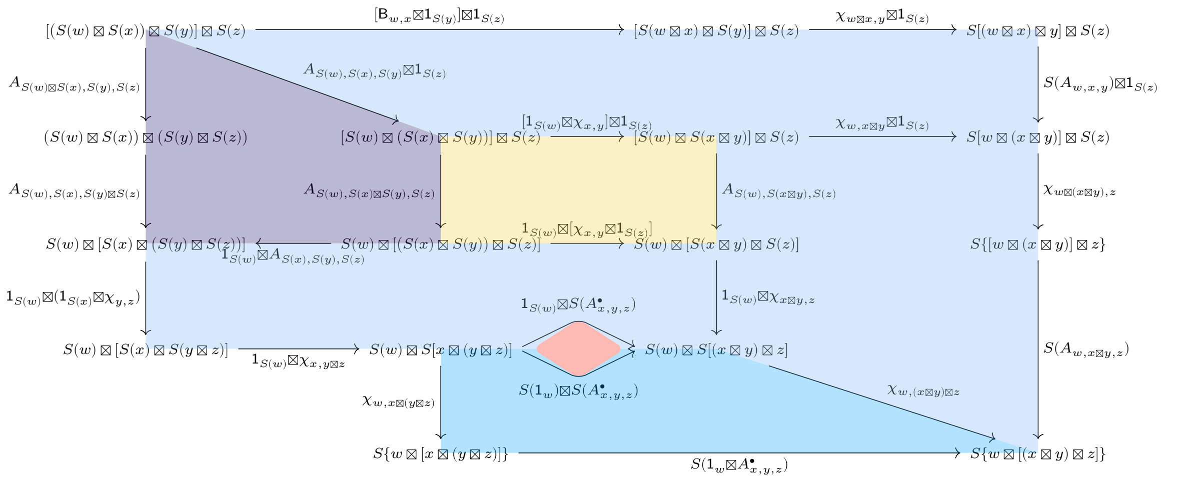
Again, let us summarise our colouring scheme



.

This data is required to satisfy:

(a.) For all objects w, x, y, z in A, we have the equality



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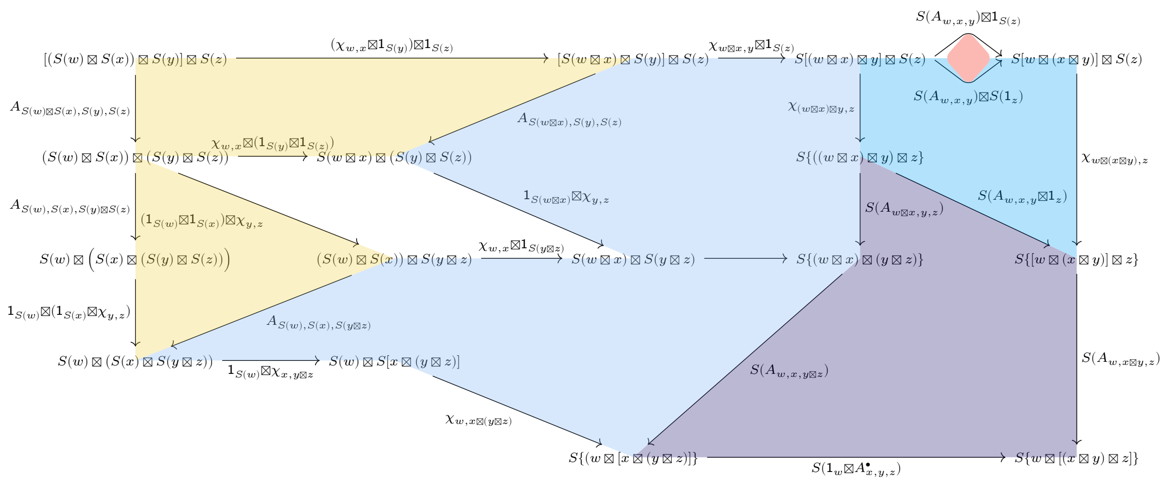


.

(b.) For all objects x, y in A, we have the equality

$$
\begin{tikzcd}
 & & (S(x)\boxtimes S(\mathbb{1}))\boxtimes S(y) \arrow[rr, "A_{S(x),S(\mathbb{1}),S(y)}"] \arrow[d, "\chi_{x,\mathbb{1}}\boxtimes\mathbf{1}_{S(y)}"] & & S(x)\boxtimes(S(\mathbb{1})\boxtimes S(y)) \arrow[dd, "\mathbf{1}_{S(x)}\boxtimes\chi_{\mathbb{1},y}"] \\
(S(x)\boxtimes\mathbb{1})\boxtimes S(y) \arrow[urr, "(\mathbf{1}_{S(x)}\boxtimes\iota)\boxtimes\mathbf{1}_{S(y)}"] \arrow[drr, "R_{S(x)}\boxtimes\mathbf{1}_{S(y)}"'] & & S(x\boxtimes\mathbb{1})\boxtimes S(y) \arrow[r, "\chi_{x\boxtimes\mathbb{1},y}"] & S[(x\boxtimes\mathbb{1})\boxtimes y] \arrow[ddr, "S(A_{x,\mathbb{1},y})"] & \\
 & & S(x)\boxtimes S(y) \arrow[u, bend left, "S(^{\bullet}(R_x))\boxtimes\mathbf{1}_{S(y)}"] \arrow[u, bend right, "S(^{\bullet}(R_x))\boxtimes S(\mathbf{1}_y)"'] \arrow[dr, "\chi_{x,y}"'] & & S(x)\boxtimes S(\mathbb{1}\boxtimes y) \arrow[d, "\chi_{x,\mathbb{1}\boxtimes y}"] \\
 & & & S(x\boxtimes y) \arrow[uu, "S(^{\bullet}(R_x)\boxtimes\mathbf{1}_y)"] \arrow[r, "S(\mathbf{1}_x\boxtimes L_y^{\bullet})"'] & S[x\boxtimes(\mathbb{1}\boxtimes y)]
\end{tikzcd}
$$

$$=$$

$$
\begin{tikzcd}
 & (S(x)\boxtimes S(\mathbb{1}))\boxtimes S(y) \arrow[rr, "A_{S(x),S(\mathbb{1}),S(y)}"] & & S(x)\boxtimes(S(\mathbb{1})\boxtimes S(y)) \arrow[d, "\mathbf{1}_{S(x)}\boxtimes\chi_{\mathbb{1},y}"] \\
(S(x)\boxtimes\mathbb{1})\boxtimes S(y) \arrow[ur, "(\mathbf{1}_{S(x)}\boxtimes\iota)\boxtimes\mathbf{1}_{S(y)}"] \arrow[r, "A_{S(x),\mathbb{1},S(y)}"] \arrow[dr, "R_{S(x)}\boxtimes\mathbf{1}_{S(y)}"'] & S(x)\boxtimes(\mathbb{1}\boxtimes S(y)) \arrow[urr, "\mathbf{1}_{S(x)}\boxtimes(\iota\boxtimes\mathbf{1}_{S(y)})"'] \arrow[d, "\mathbf{1}_{S(y)}\boxtimes L_{S(y)}"] & & \\
 & S(x)\boxtimes S(y) \arrow[rr, bend left, "\mathbf{1}_{S(x)}\boxtimes S(L_y^{\bullet})"] \arrow[rr, bend right, "S(\mathbf{1}_x)\boxtimes S(L_y^{\bullet})"'] \arrow[dr, "\chi_{x,y}"'] & & S(x)\boxtimes S(\mathbb{1}\boxtimes y) \arrow[d, "\chi_{x,\mathbb{1}\boxtimes y}"] \\
 & & S(x\boxtimes y) \arrow[r, "S(\mathbf{1}_x\boxtimes L_y^{\bullet})"'] & S[x\boxtimes(\mathbb{1}\boxtimes y)]
\end{tikzcd}
.
$$

In the first equation, we need to insert

$$
\begin{tikzcd}
(S(w)\boxtimes^{\mathsf{B}} S(x))\boxtimes^{\mathsf{B}}(S(y)\boxtimes^{\mathsf{B}} S(z)) \arrow[rr, bend left, "\chi_{w,x}\boxtimes^{\mathsf{B}}(\mathbf{1}^{\mathsf{B}}_{S(y)}\boxtimes^{\mathsf{B}}\mathbf{1}^{\mathsf{B}}_{S(z)})"] \arrow[rr, bend right, "\chi_{w,x}\boxtimes^{\mathsf{B}}\mathbf{1}^{\mathsf{B}}_{S(y)\boxtimes S(z)}"'] \arrow[d, bend right, "(\mathbf{1}^{\mathsf{B}}_{S(w)}\boxtimes^{\mathsf{B}}\mathbf{1}^{\mathsf{B}}_{S(x)})\boxtimes^{\mathsf{B}}\chi_{y,z}"'] \arrow[d, bend left, "\mathbf{1}^{\mathsf{B}}_{S(w)\boxtimes S(x)}\boxtimes^{\mathsf{B}}\chi_{y,z}"] \arrow[drr, "\chi_{w,x}\boxtimes\chi_{y,z}"] & & S(w\boxtimes x)\boxtimes^{\mathsf{B}}(S(y)\boxtimes^{\mathsf{B}} S(z)) \arrow[d, "\mathbf{1}^{\mathsf{B}}_{S(w\boxtimes x)}\boxtimes^{\mathsf{B}}\chi_{y,z}"] \\
(S(w)\boxtimes^{\mathsf{B}} S(x))\boxtimes^{\mathsf{B}} S(y\boxtimes z) \arrow[rr, "\chi_{w,x}\boxtimes^{\mathsf{B}}\mathbf{1}^{\mathsf{B}}_{S(y\boxtimes z)}"'] & & S(w\boxtimes x)\boxtimes^{\mathsf{B}} S(y\boxtimes z)
\end{tikzcd}
$$

Remark 8.23.
For two monoidal morphisms

$$(S, \chi^S, \iota^S, \omega^S, \gamma^S, \delta^S), (T, \chi^T, \iota^T, \omega^T, \gamma^T, \delta^T) \colon \mathsf{A} \to \mathsf{B}$$

we could similarly define monoidal transformations $\theta \colon S \to T$. As they will not play a major role in this work, we choose to omit them. We refer to [CG11, Definition 3.3]; see also [Gur13] and [GPS95].

Remark 8.24 (*compare to [Gur12, Remark 5.2]).*
A monoidal morphism is called a *monoidal biequivalence* if the underlying morphism is a (biadjoint) biequivalence. We will use the fact that the biadjoint equivalence can be lifted such that both adjoints are in fact *monoidal morphisms.*

Lemma 8.25.
Every morphism (S, S^2, S^0) between bicategories is isomorphic to a morphism (T, T^2, T^0) such that T^0 is the identity for all objects.

Using the above lemma, we will assume that in the following, all morphisms we will consider have trivial identity constraints. The proof is similar to the proof of Lemma 8.7.

We quickly remark that the composite of two monoidal morphisms can be equipped with the structure of a monoidal morphism. Rather than spelling this out in full, we summarise only the data that will be important to us.

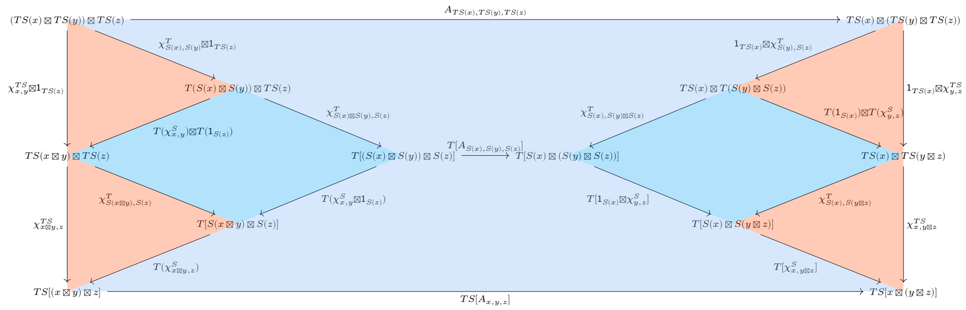
Remark 8.26 (*compare to [Gur06, Section 4.1]).*
For two composable monoidal morphisms

$$(S, \chi^S, \iota^S, \omega^S, \gamma^S, \delta^S) \colon \mathsf{A} \to \mathsf{B}$$
$$(T, \chi^T, \iota^T, \omega^T, \gamma^T, \delta^T) \colon \mathsf{B} \to \mathsf{C}$$

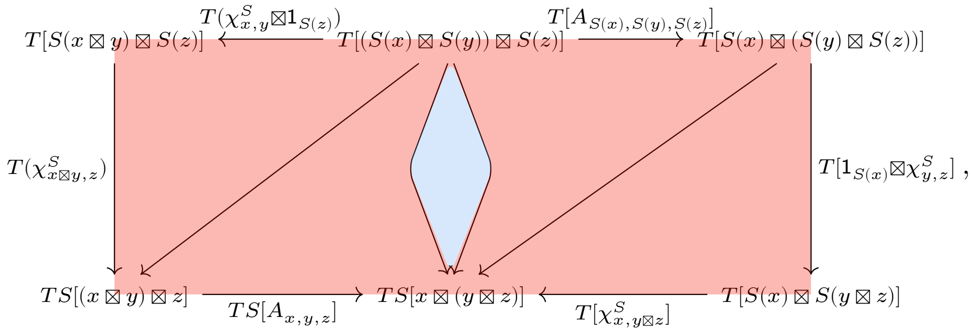
the composite of S and T has the bicategorical composite TS as underlying morphism and we define χ^{TS} component-wise as

$$TS(x) \boxtimes TS(y) \xrightarrow{\chi^T_{S(x),S(y)}} T(S(x) \boxtimes S(y)) \xrightarrow{T(\chi^S_{x,y})} TS(x \boxtimes y)$$

Then, the compatibility cell ω^{TS} is defined as



Here we identified e.g. $1_{TS(z)}$ and $T(1_{S(z)})$ using Lemma 8.25. The lower hexagon above is filled using T^2 and ω^S as



consistent with our left-bracketed conventions for the composition of 1-cells in pasting diagrams.

Let us recall some useful theorems for tricategories, which we state for monoidal bicategories.

PROPOSITION 8.27 (*transport of structure; see [Gur12, Theorem 5.1]).*
Let

$$(\mathsf{B}, \boxtimes, \mathbb{1}, A, L, R, \pi, \mu, \lambda, \rho)$$

be a monoidal bicategory. Let $S \colon \mathsf{A} \to \mathsf{B}$ *be a biadjoint biequivalence. Then there is a structure of a monoidal bicategory on* A *such that* S *underlies a monoidal morphism from* A *to* B*.*

PROPOSITION 8.28 (*change of composition and unit, compare to [Gur13, Theorem 7.23 and 7.24]).*

Let

$$(\mathsf{B}, \boxtimes, \mathbb{1}, A, L, R, \pi, \mu, \lambda, \rho)$$

be a monoidal bicategory. Assume we have the following data:

(i.) *An object* $\dot{\mathbb{1}}$.

(ii.) *An adjoint equivalence*

$$\kappa\colon \dot{\mathbb{1}} \to \mathbb{1}.$$

(iii.) *A morphism*

$$\boxdot\colon \mathsf{B} \times \mathsf{B} \to \mathsf{B}.$$

(iv.) *A transformation*

$$\xi\colon \boxdot \to \boxtimes.$$

Let

$$\boxdot\colon \mathsf{B} \times \mathsf{B} \to \mathsf{B}$$

be a morphism such that there exists a transformation

$$\xi\colon \boxdot \to \boxtimes.$$

Then there is a structure of a monoidal bicategory on B *such that*

$$(1, \xi, \kappa, \omega, \gamma, \delta)\colon (\mathsf{B}, \boxtimes) \to (\mathsf{B}, \boxdot)$$

is a morphism of monoidal bicategories.

Note that this can be seen as a categorification of Lemma 8.7 above.

We will apply the above proposition in a slightly different form. In fact, if we choose a tensor product $x \boxdot y$ of two objects x and y in B with adjoint equivalences

$$\xi_{x,y}\colon x \boxdot y \to x \boxtimes y$$

we can extend $\boxdot$ to a morphism such that ξ defines a transformation from $\boxdot$ to $\boxtimes$. We summarise:

Corollary 8.29.
Let

$$(\mathsf{B}, \boxtimes, \mathbb{1}, A, L, R, \pi, \mu, \lambda, \rho)$$

be a monoidal bicategory. Assume we have the following data:

(i.) *An object* $\dot{\mathbb{1}}$.

(ii.) *A map*

$$\boxdot\colon \operatorname{ob}(\mathsf{B}) \times \operatorname{ob}(B) \to \operatorname{ob}(\mathsf{B}).$$

(iii.) *For every pair of objects* x, y *in* B*, an adjoint equivalence*

$$\xi_{x,y}\colon x \boxdot y \to x \boxtimes y.$$

(iv.) *An adjoint equivalence*

$$\kappa\colon \dot{\mathbb{1}} \to \mathbb{1}.$$

Then there is a structure of a monoidal bicategory on B *such that*

$$(1, \xi, \kappa, \omega, \gamma, \delta)\colon (\mathsf{B}, \boxtimes) \to (\mathsf{B}, \boxdot)$$

is a morphism of monoidal bicategories.

Proof. We extend $\boxdot$ to a morphism from $\mathsf{B} \times \mathsf{B}$ to B. For two 1-cells $f\colon x_1 \to x_2$ and $g\colon y_1 \to y_2$, we define $f \boxdot g$ by the composition

$$x_1 \boxdot y_1 \xrightarrow{\xi_{x_1,y_1}} x_1 \boxtimes y_1 \xrightarrow{f \boxtimes g} x_2 \boxtimes y_2 \xrightarrow{(\xi_{x_2,y_2})^\bullet} x_2 \boxdot y_2$$

To define the morphism constraints, we denote the counit and unit of the adjoint equivalences $\xi_{x,y} \dashv \xi^\bullet_{x,y}$ by

$$\epsilon^{x,y}\colon \xi_{x,y} \circ (\xi_{x,y})^\bullet \to 1_{x \boxtimes y}$$
$$\eta^{x,y}\colon 1_{x \boxdot y} \to (\xi_{x,y})^\bullet \circ \xi_{x,y}.$$

With this definition the oplax structure $\xi_{f,g}$ is given by whiskering with ϵ^{x_2,y_2}. For 2-cells $\alpha\colon f_1 \to f_2$, $\beta\colon g_1 \to g_2$, we define $\alpha \boxdot \beta$ by

$$\begin{tikzcd}
x_1 \boxdot y_1 \arrow[r, "\xi_{x_1,y_1}"] & x_1 \boxtimes y_1 \arrow[r, bend left, "f_1 \boxtimes g_1"] \arrow[r, bend right, "f_2 \boxtimes g_2"'] \arrow[r, phantom, "\Downarrow \alpha\boxtimes\beta"] & x_2 \boxtimes y_2 \arrow[r, "\xi^{\bullet}_{x_2,y_2}"] & x_2 \boxdot y_2
\end{tikzcd}$$

We define $\boxdot_{x,y}$ for two objects x and y in B by

$$\begin{tikzcd}
x \boxdot y \arrow[rr, "1_{x \boxdot y}"] \arrow[d, "\xi_{x,y}"'] \arrow[rr, phantom, shift right=3ex, "\Downarrow \eta^{x,y}"] & & x \boxdot y \\
x \boxtimes y \arrow[rr, bend left, "1_{x \boxtimes z}"] \arrow[rr, bend right, "1_x \boxtimes 1_z"'] \arrow[rr, phantom, "\Downarrow \boxtimes^0_{x,y}"] & & x \boxtimes y \arrow[u, "\xi^{\bullet}_{x,y}"']
\end{tikzcd},$$

and, for 1-cells $f_1 \colon x_1 \to x_2$, $f_2 \colon x_2 \to x_3$, $g_1 \colon y_1 \to y_2$, $g_2 \colon y_2 \to y_3$, we define

$$(f_2 \boxdot g_2) \circ (f_1 \boxdot g_1) \xrightarrow{\boxdot^2_{(f_1,g_1),(f_2,g_2)}} (f_2 \circ f_1) \boxdot (g_2 \circ g_1)$$

by

$$\begin{tikzcd}
& & x_2 \boxdot y_2 \arrow[dr, "\xi_{x_2,y_2}"] \arrow[d, phantom, "\Downarrow \epsilon^{x_2,y_2}"] & & \\
& x_2 \boxtimes y_2 \arrow[ur, "\xi^{\bullet}_{x_2,y_2}"] \arrow[rr, "1_{x_2 \boxtimes y_2}"] & \arrow[d, phantom, "\Downarrow \boxtimes^2_{(f_1,g_1),(f_2,g_2)}"] & x_2 \boxtimes y_2 \arrow[d, "f_2 \boxtimes g_2"] & \\
x_1 \boxdot y_1 \arrow[r, "\xi_{x_1,y_1}"] & x_1 \boxtimes y_1 \arrow[u, "f_1 \boxtimes g_1"] \arrow[rr, "(f_2 \circ f_1) \boxtimes (g_2 \circ g_1)"'] & & x_3 \boxtimes y_3 \arrow[r, "\xi^{\bullet}_{x_3,y_3}"] & x_3 \boxdot y_3
\end{tikzcd}$$

With this definition, $\boxdot$ defines a morphism, i.e. $\boxdot^2$ and $\boxdot^0$ obey lax associativity and unity. The adjoint equivalences ξ define a transformation from $\boxdot$ to $\boxtimes$. ▪

In the above definitions, we again omitted the bicategorical constraints a, l, and r.

In what follows, we will use monoidal bicategories extensively. By proceeding similarly to the proof of Lemma 8.11, we can simplify our discussion slightly.

LEMMA 8.30.
Let

$$(\mathsf{B}, \boxtimes, \mathbb{1}, A, L, R, \pi, \mu, \lambda, \rho)$$

be a monoidal bicategory. Then B *is equivalent as a monoidal bicategory to a monoidal bicategory*

$$(\mathsf{B}^{\mathrm{str}\,1}, \boxtimes^{\mathrm{str}\,1}, \mathbb{1}^{\mathrm{str}\,1}, A^{\mathrm{str}\,1}, L^{\mathrm{str}\,1}, R^{\mathrm{str}\,1}, \pi^{\mathrm{str}\,1}, \mu^{\mathrm{str}\,1}, \lambda^{\mathrm{str}\,1}, \rho^{\mathrm{str}\,1})$$

such that $\boxtimes^{\mathrm{str}\,1}$ *is an identity-normalised morphism.*

Proof. First, we can apply transport of structure to the equivalence str^1 and obtain a tensor product on $\mathsf{B}^{\mathrm{str}\,1}$

$$\boxdot\colon \mathsf{B}^{\mathrm{str}\,1} \times \mathsf{B}^{\mathrm{str}\,1} \to \mathsf{B}^{\mathrm{str}\,1}$$

such that $\mathrm{str}\,1\colon (\mathsf{B}^{\mathrm{str}\,1}, \boxdot) \to (\mathsf{B}, \boxtimes)$ underlies a monoidal morphism. Now, we can use Lemma 8.25 to get a morphism

$$\square\colon \mathsf{B}^{\mathrm{str}\,1} \times \mathsf{B}^{\mathrm{str}\,1} \to \mathsf{B}^{\mathrm{str}\,1}$$

with trivial identity constraints $\square^0$ and a transformation

$$\theta\colon \square \to \boxdot.$$

By applying Proposition 8.28, we get that there exists a monoidal structure on $\mathsf{B}^{\mathrm{str}\,1}$ such that $(\mathsf{B}^{\mathrm{str}\,1}, \square) \to (\mathsf{B}^{\mathrm{str}\,1}, \boxdot)$ is a monoidal morphism. By composing, we get a biequivalence $(\mathsf{B}^{\mathrm{str}\,1}, \square) \to (\mathsf{B}, \boxtimes)$. ▪

We will call a monoidal bicategory $(\mathsf{B}, \boxtimes)$ such that B has trivial unit constraints and the functoriality constraints $\boxtimes^2$ and unitarity constraints $\boxtimes^0$ are of the above form *identity-normalised*. In the following, we will repeatedly assume that our monoidal bicategories are identity-normalised, which, according to the last lemma, is no restriction of generality.

LEMMA 8.31.
Let $(\mathsf{B}, \boxtimes)$ *be an identity-normalised monoidal bicategory with associator A. Then we get that the oplax naturality constraint*

$$A_{1_x, 1_y, 1_z} = 1_{A_{x,y,z}}.$$

Proof. This is a consequence of Lemma 8.12. We only need to check that if $(\mathsf{B}, \boxtimes)$ is identity normalised, then so are $\boxtimes \circ (\boxtimes \times 1_{\mathsf{B}})$ and $\boxtimes \circ (1_{\mathsf{B}} \times \boxtimes)$. This is a direct consequence of the definition of the unity and functoriality constraints of these morphisms, since they are given as the composition of the unity and functoriality constraints of $\boxtimes$, respectively. ▪

Remark 8.32.
Degenerate monoidal bicategories, i.e. monoidal bicategories with exactly one object, naturally correspond to *braided* monoidal categories. However, making this correspondence explicit without appealing to the strictification results for tricategories is very hard, in contrast to e.g. [GPS95, Proposition 8.6]. Many results in this direction are contained in [CG11].

Given a braided monoidal category C, it is, however, comparatively straightforward to construct a degenerate monoidal bicategory $\Sigma\mathsf{C}$ whose tensor product (on 1-cells) is given by the tensor product of C.

9. Coherent Gr-bicategories

§.9.1. Coherent Gr-categories

Here is a good place to recall the classification of Gr-categories as outlined in [JS93] and [BL04], which will guide us in this section.

A monoidal category $(\mathsf{G}, \otimes, \mathbb{1})$ is called *coherent* Gr-*category* if every 1-cell is invertible and it is equipped with a *coherent* structure. Namely, every object x in G is equipped with an adjoint equivalence (x, x^-, i_x, e_x), i.e. with isomorphisms

$$\begin{aligned} i_x &\colon \mathbb{1} \to x \otimes x^-, \\ e_x &\colon x^- \otimes x \to \mathbb{1}, \end{aligned}$$

satisfying the two triangle identities, compare to [BL04, Section 3], i.e. the following two triangles commute

$$\begin{tikzcd} x^- \otimes \mathbb{1} \arrow[r, "\mathrm{id}_{x^-} \otimes i_x"] \arrow[ddrr, "r_{x^-}"'] & x^- \otimes (x \otimes x^-) \arrow[r, "(a_{x^-,x,x^-})^{-1}"] & (x^- \otimes x) \otimes x^- \arrow[d, "e_x \otimes \mathrm{id}_{x^-}"] \\ & & \mathbb{1} \otimes x^- \arrow[d, "l_{x^-}"] \\ & & x^- \end{tikzcd} , \tag{9.1}$$

$$\begin{tikzcd} \mathbb{1} \otimes x \arrow[r, "i_x \otimes \mathrm{id}_x"] \arrow[ddrr, "l_x"'] & (x \boxtimes x^-) \otimes x \arrow[r, "a_{x,x^-,x}"] & x \otimes (x^- \otimes x) \arrow[d, "\mathrm{id}_x \otimes e_x"] \\ & & x \otimes \mathbb{1} \arrow[d, "r_x"] \\ & & x \end{tikzcd} . \tag{9.2}$$

We will call the object x^- *weak inverse* of x. In Section 5, Baez and Lauda go on to show that the choices involved can always be made.[1]

A *morphism* $(F, F^2, F^0)\colon \mathsf{G} \to \mathsf{H}$ of coherent Gr-categories G and H is a tensor functor. By [BL04, Theorem 6.1], there exists for every object x in G a unique isomorphism

$$F_{-1}\colon F(x)^- \to F(x^-)$$

compatible with the tensor and unit structure of F.

In Section 8.3, Baez and Lauda show that coherent Gr-categories can be described up to equivalence by the following data

(i.) a group Π,

(ii.) an abelian group Γ,

(iii.) an action of $\alpha\colon \Pi \to \mathrm{Aut}(\Gamma)$,

(iv.) a (normalised) 3-cocycle $a\colon \Pi^3 \to \Gamma$.

They show that every coherent Gr-category is equivalent (as coherent Gr-category) to a *special* coherent Gr-category which is a *skeletal* coherent Gr-category such that the unit laws are identities, and, for every object x, the chosen inverse x^- is strict, i.e. unit i and counit e are identities as well. For a special Gr-category $(\mathsf{G}, \mathbb{1}, \otimes)$, we get the above data as follows: Since a special coherent Gr-category comes with strict inverses, the set of objects defines our group Π. The automorphisms of the unit object $\mathbb{1}$ form our group Γ. By the Eckmann-Hilton argument, it is abelian. The action of Π on Γ is defined as

$$(\mathrm{id}_x \otimes f) \otimes \mathrm{id}_{x^-}$$

for all objects x of G and morphisms $f\colon \mathbb{1} \to \mathbb{1}$. The associator of the monoidal category G defines the normalised 3-cocycle a.

A morphism of *special* coherent Gr-categories G and H is a morphism of coherent Gr-categories such that the unit constraint F^0 is the identity. For two special coherent Gr-categories defined by the data $(\Pi_1, \Gamma_1, \alpha_1, a_1)$ and $(\Pi_2, \Gamma_2, \alpha_2, a_2)$, a morphism of special coherent Gr-categories defines

[1] Note that x^- is a right dual object to x such that evaluation and coevaluation are isomorphisms.

(i.) a group homomorphism $\phi\colon \Pi_1 \to \Pi_2$,

(ii.) a module map $\psi\colon \Gamma_1 \to \Gamma_2$,

(iii.) a (normalised) 2-cochain k such that $dk = \psi a_1 - a_2 \phi$.

We have seen an example of a coherent Gr-category in Remark 5.5.

EXAMPLE 9.1.
Let C be a monoidal category. Then the category of adjoint equivalences is a coherent Gr-category with composition as monoidal product and the identity functor as unit. Then, by choosing representatives of isomorphism classes of the autoequivalences, we get a special Gr-category whose objects are those isomorphism classes.

An action of a group Π on the monoidal category C is a monoidal functor

$$\Pi^{\otimes} \to \mathrm{Aut}_{\otimes}(\mathsf{C}).$$

Thus, we get that a group homomorphism $\Pi \to \mathrm{Aut}_{\otimes}(\mathsf{C})$ lifts to an action if and only if an obstruction in $H^3(\mathrm{Aut}_{\otimes}(\mathsf{C}), \mathrm{Aut}_{\otimes}(1_{\mathsf{C}}))$ vanishes. For details, we refer to [Gal11, Section 5.2.1].

Note that the functor category $\mathrm{Aut}_{\otimes}(\mathsf{C})$ is strict monoidal (as composition of functors is strict), yet we still have the above obstruction.

§.9.2. Coherent Gr-bicategories

Now, our primary goal will be to categorify the definition and results of the last section. In the definition of a coherent Gr-category, we chose an adjoint pair for every object x and its weak inverse x^{-}. When we work in a monoidal bicategory, we have invertible 2-cells corresponding to the two triangle equations. They need to be compatible with the structure of our monoidal bicategory.

The following definition is based on [Gur12, Definition 6.4].

DEFINITION 9.2 (*coherent* Gr-*bicategory).*
Let

$$(\mathsf{B}, \boxtimes, \mathbb{1}, A, L, R, \pi, \mu, \lambda, \rho)$$

be a monoidal bicategory where every 2-cell is invertible.

Then a *coherent* Gr-*structure* consists of the following data:

(i.) For every 1-cells f, we have chosen an adjoint equivalence with adjoint $f^{\bullet}$.

(ii.) For every object x of B, we have an internal biadjoint biequivalence $x \dashv x^{-}$, i.e. we have

a) an adjoint equivalence with components

$$I_x \colon \mathbb{1} \to x \boxtimes x^{-},$$

b) an adjoint equivalence

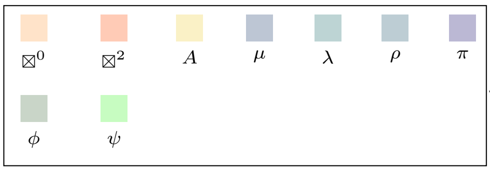
$$E_x \colon x^{-} \boxtimes x \to \mathbb{1},$$

c) invertible 2-cells ϕ_x and ψ_x

$$\begin{tikzcd}
x^{-} \boxtimes \mathbb{1} \arrow[r, "\mathbf{1}_{x^{-}} \boxtimes I_x"] \arrow[ddrr, "R_{x^{-}}"'] & x^{-} \boxtimes (x \boxtimes x^{-}) \arrow[r, "A^{\bullet}_{x^{-},x,x^{-}}"] \arrow[d, phantom, "\Downarrow \phi_x"] & (x^{-} \boxtimes x) \boxtimes x^{-} \arrow[d, "E_x \boxtimes \mathbf{1}_{x^{-}}"] \\
 & \, & \mathbb{1} \boxtimes x^{-} \arrow[d, "L_{x^{-}}"] \\
 & & x^{-}
\end{tikzcd},$$

$$\begin{tikzcd}
\mathbb{1} \boxtimes x \arrow[r, "I_x \boxtimes \mathbf{1}_x"] \arrow[ddrr, "L_x"'] & (x \boxtimes x^{-}) \boxtimes x \arrow[r, "A_{x,x^{-},x}"] \arrow[d, phantom, "\Downarrow \psi_x"] & x \boxtimes (x^{-} \boxtimes x) \arrow[d, "\mathbf{1}_x \boxtimes E_x"] \\
 & \, & x \boxtimes \mathbb{1} \arrow[d, "R_x"] \\
 & & x
\end{tikzcd}.$$

We use the colouring as above, namely



.

The above data is required to satisfy for every object x of B:

(a.)

$$
\begin{tikzcd}
(x^- \boxtimes \mathbb{1}) \boxtimes x \arrow[r, "(\mathbf{1}_{x^-} \boxtimes I_x) \boxtimes \mathbf{1}_x"] \arrow[dddd, "R_{x^-} \boxtimes \mathbf{1}_x"'] \arrow[dr, "A_{x^-, \mathbb{1}, x}"'] & {[x^- \boxtimes (x \boxtimes x^-)] \boxtimes x} \arrow[r, "A_{x^-, x \boxtimes x^-, x}"] & x^- \boxtimes [(x \boxtimes x^-) \boxtimes x] \arrow[r, "\mathbf{1}_{x^-} \boxtimes A_{x, x^-, x}"] & x^- \boxtimes [x \boxtimes (x^- \boxtimes x)] \arrow[d, "A^{\bullet}_{x^-, x, x^- \boxtimes x}"] \arrow[ddl, "\mathbf{1}_{x^-} \boxtimes (\mathbf{1}_x \boxtimes E_x)"] \\
& x^- \boxtimes (\mathbb{1} \boxtimes x) \arrow[ur, "\mathbf{1}_{x^-} \boxtimes (I_x \boxtimes \mathbf{1}_x)"] \arrow[dddl, "\mathbf{1}_{x^-} \boxtimes L_x"] & & (x^- \boxtimes x) \boxtimes (x^- \boxtimes x) \arrow[d, "E_x \boxtimes (\mathbf{1}_{x^-} \boxtimes \mathbf{1}_x)"] \arrow[ddl, "(\mathbf{1}_{x^-} \boxtimes \mathbf{1}_x) \boxtimes E_x"] \\
& & x^- \boxtimes (x \boxtimes \mathbb{1}) \arrow[d, "A^{\bullet}_{x^-, x, \mathbb{1}}"'] \arrow[ddll, "\mathbf{1}_{x^-} \boxtimes R_x"] & \mathbb{1} \boxtimes (x^- \boxtimes x) \arrow[d, "\mathbf{1}_{\mathbb{1}} \boxtimes E_x"] \\
& & (x^- \boxtimes x) \boxtimes \mathbb{1} \arrow[r, "E_x \boxtimes \mathbf{1}_{\mathbb{1}}"'] \arrow[dll, "R_{x^- \boxtimes x}"] & \mathbb{1} \boxtimes \mathbb{1} \arrow[d, "R_{\mathbb{1}}"] \\
x^- \boxtimes x \arrow[rrr, "E_x"'] & & & \mathbb{1}
\end{tikzcd}
$$

$$=$$

$$
\begin{tikzcd}
(x^- \boxtimes \mathbb{1}) \boxtimes x \arrow[r, "(\mathbf{1}_{x^-} \boxtimes I_x) \boxtimes \mathbf{1}_x"] \arrow[dddd, "R_{x^-} \boxtimes \mathbf{1}_x"'] & {[x^- \boxtimes (x \boxtimes x^-)] \boxtimes x} \arrow[r, "A_{x^-, x \boxtimes x^-, x}"] \arrow[d, "A^{\bullet}_{x^-, x, x^-} \boxtimes \mathbf{1}_x"] & x^- \boxtimes [(x \boxtimes x^-) \boxtimes x] \arrow[r, "\mathbf{1}_{x^-} \boxtimes A_{x, x^-, x}"] & x^- \boxtimes [x \boxtimes (x^- \boxtimes x)] \arrow[d, "A^{\bullet}_{x^-, x, x^- \boxtimes x}"] \\
& {[(x^- \boxtimes x) \boxtimes x^-] \boxtimes x} \arrow[rr, "A_{x^- \boxtimes x, x^-, x}"] \arrow[d, "(E_x \boxtimes \mathbf{1}_{x^-}) \boxtimes \mathbf{1}_x"] & & (x^- \boxtimes x) \boxtimes (x^- \boxtimes x) \arrow[d, "E_x \boxtimes (\mathbf{1}_{x^-} \boxtimes \mathbf{1}_x)"] \\
& {[\mathbb{1} \boxtimes x^-] \boxtimes x} \arrow[rr, "A_{\mathbb{1}, x^-, x}"] \arrow[ddl, "L_{x^-} \boxtimes \mathbf{1}_x"] & & \mathbb{1} \boxtimes (x^- \boxtimes x) \arrow[d, "\mathbf{1}_{\mathbb{1}} \boxtimes E_x"] \arrow[ddlll, "L_{x^- \boxtimes x}"] \\
& & & \mathbb{1} \boxtimes \mathbb{1} \arrow[d, "L_{\mathbb{1}}"] \\
x^- \boxtimes x \arrow[rrr, "E_x"] & & & \mathbb{1}
\end{tikzcd}
\quad .
$$

(b.)

$$\begin{tikzcd}
(x \boxtimes x^-) \boxtimes (x \boxtimes x^-) \arrow[rrr, "A^{\bullet}_{x\boxtimes x^-,x,x^-}"] & & & {[(x \boxtimes x^-) \boxtimes x] \boxtimes x^-} \arrow[d, "A_{x,x^-,x}\boxtimes \mathbf{1}_{x^-}"] \\
(x \boxtimes x^-) \boxtimes \mathbb{1} \arrow[u, "(\mathbf{1}_x \boxtimes \mathbf{1}_{x^-})\boxtimes I_x"] & & (\mathbb{1} \boxtimes x) \boxtimes x^- \arrow[ur, "(I_x\boxtimes \mathbf{1}_x)\boxtimes \mathbf{1}_{x^-}"] \arrow[ddr, "L_x\boxtimes \mathbf{1}_{x^-}"] & {[x \boxtimes (x^- \boxtimes x)] \boxtimes x^-} \arrow[d, "{[\mathbf{1}_x\boxtimes E_x]\boxtimes \mathbf{1}_{x^-}}"] \\
\mathbb{1} \boxtimes \mathbb{1} \arrow[u, "I_x\boxtimes \mathbf{1}_{\mathbb{1}}"] \arrow[r, "\mathbf{1}_{\mathbb{1}}\boxtimes I_x"'] \arrow[d, "L_{\mathbb{1}}"'] & \mathbb{1} \boxtimes (x \boxtimes x^-) \arrow[uul, "I_x\boxtimes(\mathbf{1}_x\boxtimes \mathbf{1}_{x^-})"] \arrow[ur, "A^{\bullet}_{\mathbb{1},x,x^-}"] \arrow[drr, "L_{x\boxtimes x^-}"'] & & (x \boxtimes \mathbb{1}) \boxtimes x^- \arrow[d, "R_x\boxtimes \mathbf{1}_{x^-}"] \\
\mathbb{1} \arrow[rrr, "I_x"] & & & x \boxtimes x^-
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
(x \boxtimes x^-) \boxtimes (x \boxtimes x^-) \arrow[rrr, "A^{\bullet}_{x\boxtimes x^-,x,x^-}"] \arrow[dr, "A_{x,x^-,x\boxtimes x^-}"] & & & {[(x \boxtimes x^-) \boxtimes x] \boxtimes x^-} \arrow[d, "A_{x,x^-,x}\boxtimes \mathbf{1}_{x^-}"] \\
 & x \boxtimes [x^- \boxtimes (x \boxtimes x^-)] \arrow[r, "\mathbf{1}_x\boxtimes A^{\bullet}_{x^-,x,x^-}"] & x \boxtimes [(x^- \boxtimes x) \boxtimes x^-] \arrow[d, "{\mathbf{1}_x\boxtimes[E_x\boxtimes \mathbf{1}_{x^-}]}"] & {[x \boxtimes (x^- \boxtimes x)] \boxtimes x^-} \arrow[l, "A_{x,x^-\boxtimes x,x^-}"] \arrow[d, "{[\mathbf{1}_x\boxtimes E_x]\boxtimes \mathbf{1}_{x^-}}"] \\
(x \boxtimes x^-) \boxtimes \mathbb{1} \arrow[uu, "(\mathbf{1}_x\boxtimes \mathbf{1}_{x^-})\boxtimes I_x"] \arrow[r, "A_{x,x^-,\mathbb{1}}"'] \arrow[ddrrr, "R_{x\boxtimes x^-}"'] & x \boxtimes (x^- \boxtimes \mathbb{1}) \arrow[u, "{\mathbf{1}_x\boxtimes[\mathbf{1}_{x^-}\boxtimes I_x]}"] \arrow[ddrr, "\mathbf{1}_x\boxtimes R_{x^-}"] & x \boxtimes (\mathbb{1} \boxtimes x^-) \arrow[ddr, "\mathbf{1}_x\boxtimes L_{x^-}"] & (x \boxtimes \mathbb{1}) \boxtimes x^- \arrow[l, "A_{x,\mathbb{1},x^-}"] \arrow[dd, "R_x\boxtimes \mathbf{1}_{x^-}"] \\
\mathbb{1} \boxtimes \mathbb{1} \arrow[u, "I_x\boxtimes \mathbf{1}_{\mathbb{1}}"] \arrow[d, "R_{\mathbb{1}}"'] & & & \\
\mathbb{1} \arrow[rrr, "I_x"] & & & x \boxtimes x^-
\end{tikzcd}$$

.

The monoidal bicategory B equipped with a coherent Gr-structure is called *coherent* Gr*-category*.

In the above diagrams, we did not colour the naturality 2-cells for L and R. If the monoidal bicategory is a Gray-monoid, the above definition simplifies considerably; see [Gur12].

Example 9.3.
A monoidal category C can be viewed as a discrete monoidal bicategory (with only identity 2-cells) $\mathsf{C}^{\boxtimes}$. Then, a coherent Gr-structure on $\mathsf{C}^{\boxtimes}$ is nothing more than a coherent Gr-structure on the monoidal category C. The existence of the

invertible 2-cells ϕ_x and ψ_x corresponds to the two triangle identities (9.1) and (9.2), respectively. The two equalities (a.) and (b.) follow directly from the definition of the monoidal category C.

The above example is why we refer to the 2-cells ϕ_x and ψ_x as triangulators.

Example 9.4.
Let C be a *braided* monoidal category. Then, we can view C as a monoidal bicategory with one object $\Sigma\mathsf{C}$; see Remark 8.32. A coherent Gr-structure on C naturally yields a coherent Gr-structure on $\Sigma\mathsf{C}$ if we choose the adjoint equivalences I and E to be the identity adjoint equivalence.

Example 9.5.
Let B be a bicategory. Recall that bicategories form a tricategory. For every biautoequivalence $S\colon \mathsf{B} \to \mathsf{B}$ there is by [Gur12, Theorem 3.2], a biadjoint biequivalence $S \dashv_{\mathrm{bi}} T$. This way, we can build a coherent Gr-category whose objects are biequivalences of B. This can be seen as a categorification of [BL04, Example 8.1.3].

Similar arguments apply to monoidal bicategories, as they also form a tricategory; see [CG11, Theorem 3.8].

Remark 9.6.
We chose to define coherent bicategories with right inverses; for the naming conventions, we refer to [DSS20, Definitions 2.1.2-2.1.4]. This differs from the choice made in [Gur12] or [JS93], but agrees with [BL04]. For a coherent Gr-bicategory B, we have an equivalence P_x for every x in B

$$(x^-)^- \simeq \mathbb{1} \boxtimes (x^-)^- \xrightarrow{I_x \boxtimes 1} (x \boxtimes x^-) \boxtimes (x^-)^- \simeq x \boxtimes (x^- \boxtimes (x^-)^-) \xrightarrow{1 \boxtimes E_x} x \boxtimes \mathbb{1} \simeq x,$$

so, we also get equivalences

$$\mathbb{1} \xrightarrow{I_{x^-}} x^- \boxtimes (x^-)^- \xrightarrow{1 \boxtimes P_x} x^- \boxtimes x.$$

Definition 9.7 (*coherent monoidal morphism*).
Let A and B be two coherent Gr-bicategories. Then a coherent morphism from A to B is a monoidal morphism from A to B.

One could expect that the appropriate notion of a morphism between monoidal bicategories with the structure of a coherent Gr-bicategory should come with an additional property, namely that it is compatible with the coherent data. The following proposition explains why this is not necessary. It is a categorification of [BL04, Theorem 6.1].

PROPOSITION 9.8 (*compare to [Gur12, Theorem 6.12]).*
Let A *and* B *be two coherent* Gr-*bicategories and let*

$$(S, \chi, \iota, \omega, \gamma, \delta)\colon \mathsf{A} \to \mathsf{B}$$

be a morphism. Then there exists an equivalence 1-*cell*

$$\zeta\colon S(x)^{-} \to S(x^{-})$$

and invertible 2-*cells for every object* x *in* B

$$\begin{tikzcd}
S(x) \boxtimes S(x)^{-} \arrow[r, "1\boxtimes\zeta"] & S(x) \boxtimes S(x^{-}) \arrow[r, "\chi_{x,x^{-}}"] \arrow[d, phantom, "\Downarrow\alpha_x"] & S(x \boxtimes x^{-}) \\
\mathbb{1} \arrow[u, "I_{S(x)}"] \arrow[rr, "\iota"] & & S(\mathbb{1}) \arrow[u, "S(I_x)"']
\end{tikzcd}$$

$$\begin{tikzcd}
S(x)^{-} \boxtimes S(x) \arrow[r, "\zeta\boxtimes 1"] \arrow[d, "E_{S(x)}"] & S(x^{-}) \boxtimes S(x) \arrow[r, "\chi_{x^{-},x}"] \arrow[d, phantom, "\Downarrow\beta_x"] & S(x^{-} \boxtimes x) \arrow[d, "S(E_x)"] \\
\mathbb{1} \arrow[rr, "\iota"] & & S(\mathbb{1})
\end{tikzcd}$$

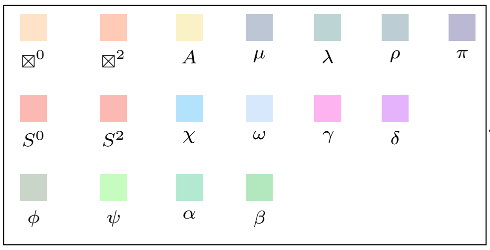
such that

$$
\begin{tikzcd}
S(\mathbb{1}\boxtimes x) \arrow[r, "S(I\boxtimes 1)"] & S[(x\boxtimes x^{-})\boxtimes x] \arrow[r, "S(A)"] & S[x\boxtimes(x^{-}\boxtimes x)] \arrow[r, "S(1\boxtimes E)"] & S(x\boxtimes\mathbb{1}) \\
S(\mathbb{1})\boxtimes S(x) \arrow[u, "\chi"] \arrow[r, "S(I)\boxtimes S(1)"] & S(x\boxtimes x^{-})\boxtimes S(x) \arrow[u, "\chi"'] & S(x)\boxtimes S(x^{-}\boxtimes x) \arrow[u, "\chi"] \arrow[r, "S(1)\boxtimes S(E)"] & S(x)\boxtimes S(\mathbb{1}) \arrow[u, "\chi"'] \\
 & (S(x)\boxtimes S(x^{-}))\boxtimes S(x) \arrow[u, "\chi\boxtimes 1"'] \arrow[r, "A"] & S(x)\boxtimes(S(x^{-})\boxtimes S(x)) \arrow[u, "\chi"] & \\
 & (S(x)\boxtimes S(x)^{-})\boxtimes S(x) \arrow[u, "(1\boxtimes\zeta)\boxtimes 1"'] \arrow[r, "A"'] & S(x)\boxtimes(S(x)^{-}\boxtimes S(x)) \arrow[u, "1\boxtimes(\zeta\boxtimes 1)"] \arrow[r, "1\boxtimes E"'] & S(x)\boxtimes\mathbb{1} \arrow[uu, "1\boxtimes\iota"'] \arrow[dl, "R"] \\
\mathbb{1}\boxtimes S(x) \arrow[uuu, "\iota\boxtimes 1"] \arrow[ur, "I\boxtimes 1"'] \arrow[rr, "L"'] & & S(x) &
\end{tikzcd}
$$

$$=$$

$$
\begin{tikzcd}
S(\mathbb{1}\boxtimes x) \arrow[r, "S(I\boxtimes 1)"] \arrow[dddrr, "S(L)"] & S[(x\boxtimes x^{-})\boxtimes x] \arrow[r, "S(A)"] & S[x\boxtimes(x^{-}\boxtimes x)] \arrow[r, "S(1\boxtimes E)"] & S(x\boxtimes\mathbb{1}) \arrow[dddl, "S(R)"'] \\
S(\mathbb{1})\boxtimes S(x) \arrow[u, "\chi"] & & & S(x)\boxtimes S(\mathbb{1}) \arrow[u, "\chi"'] \\
 & & & S(x)\boxtimes\mathbb{1} \arrow[u, "1\boxtimes\iota"'] \arrow[dl, "R"] \\
\mathbb{1}\boxtimes S(x) \arrow[uu, "\iota\boxtimes 1"] \arrow[rr, "L"'] & & S(x) &
\end{tikzcd}
$$

In the above diagram, we assumed that S^0 is the identity for all objects x. We used the same colouring as above, namely



.

We can transport coherent structure along monoidal equivalences.

Remark 9.9.
Let A and B be two monoidal bicategories such that A is a coherent Gr-bicategory and let $(S, \chi, \iota)\colon \mathsf{A} \to \mathsf{B}$ be a monoidal biequivalence. Then we can define a coherent structure on B by choosing elements $S(x)^-$ with an adjoint equivalence $\zeta\colon S(x)^- \to S(x^-)$ and defining

$$\begin{aligned} I_{S(x)} &:= (1 \boxtimes \zeta)^\bullet \circ (\chi_{x,x^-})^\bullet \circ S(I_x) \circ \iota, \\ E_{S(x)} &:= \iota^\bullet \circ S(E_x) \circ \chi_{x^-,x} \circ (1 \boxtimes \zeta). \end{aligned}$$

Then there exist 2-cells α and β as in Proposition 9.8. By using the fact that the biequivalence S is essentially surjective, we can show that this defines a coherent structure on B.

The proof is a rather long calculation. With the above definition of I and E, one can calculate the triangulator cells ϕ and ψ using the triangulator cells of A and the modifications of S. It can be seen as a categorification of [BL04, Proposition 2.3].

Remark 9.10.
In the literature, Gr-bicategories are sometimes called *Picard bicategories*; see e.g. [Gur12]. We would use this term for symmetric Gr-bicategories; see [Sín78].

Lemma 9.11 (*compare to [GPS95, Proposition 8.6]*).
Let $(\mathsf{B}, \boxtimes, \mathbb{1})$ *be a coherent* Gr*-bicategory with biadjoint biequivalences*

$$(x, x^-, I_x, E_x, \phi_x, \psi_x)$$

for every object x in B. Then the conjugation functors defined by

$$C(x)[\,*\,] := (x \boxtimes *\,) \boxtimes x^-$$

are monoidal morphisms. Furthermore, they restrict to monoidal morphisms of the (degenerate) full monoidal sub-bicategory on the unit object $\mathbb{1}$.

Proof. We assume the statement holds for a coherent Gray-monoid M.

If B is a general coherent Gr-bicategory, we choose a monoidal biequivalence $T\colon \mathsf{B} \to \mathsf{M}$ where M is a Gray monoid. Then by Remark 9.9, we get a coherent structure on M induced by T. Now, for every object x in B, $T(x) \boxtimes * \boxtimes T(x^-)$ are monoidal equivalences by the above. Thus, we get a monoidal equivalence U^x on B such that $T(U^x(y)) = T(x) \boxtimes T(y) \boxtimes T(x^-)$ for all objects y. Then, since T is monoidal, we have an equivalence (the tensor structure of T)

$$f^x_y\colon T(U^x(y)) \xrightarrow{\cong} T[(x \boxtimes y) \boxtimes x^-].$$

Since T is essentially surjective on 1-cells there thus exists a 1-cell

$$g^x_y\colon U^x(y) \to (x \boxtimes y) \boxtimes x^-$$

and an invertible 2-cell $\alpha^x_y\colon T(g^x_y) \to f^x_y$. Thus, we get the component 1-cells of a transformation $U^x \to C(x)$ inducing the structure of a monoidal morphism. ▪

§.9.3. Skeletalisation of (Monoidal) Bicategories

Definition 9.12 (*skeletal bicategory, [Sch09b, Definition 2.19]).*
A bicategory B is 1-*skeletal* if the categories $\mathrm{Hom}_{\mathsf{B}}(x, y)$ are skeletal for every pair of objects x, y. Furthermore, B is called 0-*skeletal* if for two objects x and y, x is equivalent to y if and only if $x = y$ in B. Finally, B is *skeletal* if it is both 0- and 1-skeletal.

A monoidal bicategory is *skeletal* if the underlying bicategory is skeletal.

Proposition 9.13 (*[Sch09b, Lemma 2.20 and 2.22]).*
Every bicategory is biequivalent to a skeletal bicategory.

Note that Schommer-Pries even discusses monoidal bicategories equipped with a symmetry.

Example 9.14.
For a monoid M, the discrete bicategory $M^{\mathsf{bi}\boxtimes}$ is skeletal.

Using the transport of structure theorem, see Theorem 8.27, we obtain the corresponding statement for monoidal bicategories.

Proposition 9.15.
Every monoidal bicategory is biequivalent (as a monoidal bicategory) to a skeletal monoidal bicategory.

We will not prove the statement here and instead refer to the proof of [Sch09b, Lemma 2.22].

Definition 9.16 (*special coherent* Gr-*bicategory*).
A skeletal coherent Gr-bicategory B is called *special* if

(i.) left and right unitors L and R are identity adjoint equivalences,

(ii.) for every object x of B we have biadjoint biequivalences $x \dashv x^-$ such that unit I_x and counit E_x are identity equivalences.

Note that, in general, we cannot demand the triangulator 2-cells to be identities. In a skeletal bicategory requiring the existence of equivalences $\mathbb{1} \to x \boxtimes x^-$ and $x^- \boxtimes x \to \mathbb{1}$ for all objects x already tells us that $x \boxtimes x^-$ is equal to $\mathbb{1}$ as objects of B.

Example 9.17.
Let Π be a finite group. Then $\Pi^{\mathsf{bi}\boxtimes}$ is a special coherent Gr-bicategory.

With our setup, we can now prove the categorification of [BL04, Proposition 8.3.3]. We think it is fair to say that this is the key lemma in their proof of the main theorem [BL04, Theorem 8.3.7].

Proposition 9.18.
Let $(\mathsf{B}, \boxtimes, \mathbb{1}, A, L, R)$ *be a monoidal bicategory equipped with a coherent* Gr-*structure* (I, E, ϕ, ψ). *Then* B *is biequivalent to a special coherent* Gr-*bicategory.*

Proof. First, by Proposition 9.15, we can construct a biequivalence $S\colon \mathsf{B} \to \mathsf{B}^{\mathrm{sk}}$. By transporting the coherent structure along S by Remark 9.9, we already have a skeletal coherent Gr-category equivalent to B. Without loss of generality, we can thus assume that B is skeletal.

Now, we are going to change the tensor product using Corollary 8.29. We choose

$$x \boxdot y := \begin{cases} x, & y = \mathbb{1}, \\ y, & x = \mathbb{1}, \\ \mathbb{1}, & x = y^-, \\ \mathbb{1}, & y = x^-, \\ x \boxtimes y, & \text{else}; \end{cases} \qquad \xi_{x,y} := \begin{cases} R^\bullet_x, & y = \mathbb{1}, \\ L^\bullet_y, & x = \mathbb{1}, \\ E^\bullet_y, & x = y^-, \\ I_x, & x = y^-, \\ \mathbf{1}_{x \boxtimes y}, & \text{else}; \end{cases}$$

where $\xi_{x,y} \colon x \boxdot y \to x \boxtimes y$ for all objects x, y in B. Then, we get a monoidal biequivalence $(\mathsf{B}, \boxtimes) \to (\mathsf{B}, \boxdot)$. We now transport the coherent structure along this biequivalence, as in Remark 9.9. We get that $(\mathsf{B}, \boxdot)$ is a special coherent Gr-bicategory. For example, the existence of the 2-cells

$$\begin{tikzcd} \mathbb{1} \boxdot x \arrow[r, "\xi_{\mathbb{1},x}"] & \mathbb{1} \boxtimes x) \arrow[d, "L_x"] \\ \mathbb{1} \boxdot x \arrow[u, "\mathbf{1}_{\mathbb{1}} \boxdot \mathbf{1}_x"] \arrow[r, "\dot{L}_x"'] & x \end{tikzcd} \qquad \begin{tikzcd} x \boxdot \mathbb{1} \arrow[r, "\xi_{x,\mathbb{1}}"] & x \boxtimes \mathbb{1}) \arrow[d, "R_x"] \\ x \boxdot \mathbb{1} \arrow[u, "\mathbf{1}_x \boxdot \mathbf{1}_{\mathbb{1}}"] \arrow[r, "\dot{R}_x"'] & x \end{tikzcd}$$

implies that $\dot{L}_x$ and $\dot{R}_x$ is the identity for all objects x in B. ▪

Recall that for a monoidal bicategory $(\mathsf{B}, \boxtimes)$, there are two monoidal structures $(\mathsf{B}, \boxtimes^{\mathrm{rev}})$ and $(\mathsf{B}^{\mathrm{op}}, \boxtimes^{\mathrm{op}})$ by Lemma 8.17. By combining them, we get a monoidal structure $(\mathsf{B}^{\mathrm{op}}, \boxtimes^{\mathrm{coop}})$.

Lemma 9.19 (*compare to [Gur12, Proposition 6.8]).*
Let B *be a coherent* Gr*-bicategory. Then there exists a monoidal morphism*

$$(\mathrm{inv}, \mathrm{inv}^2) \colon (\mathsf{B}, \boxtimes) \to (\mathsf{B}^{\mathrm{op}}, \boxtimes^{\mathrm{coop}})$$

extending $x \mapsto x^-$*.*

Note that the components of the tensor structure of inv are internal equivalences

$$\mathrm{inv}^2_{x,y} \colon x^- \boxtimes^{\mathrm{coop}} y^- \to^{\mathrm{op}} (x \boxtimes y)^-$$

i.e. in B they correspond to 1-cells

$$\mathrm{inv}^2_{x,y} \colon (x \boxtimes y)^- \to y^- \boxtimes x^-$$

given by the composition

$$\begin{tikzcd}
(x \boxtimes y)^- \simeq (x \boxtimes y)^- \boxtimes \mathbb{1} \arrow[r, "I_x"] & (x \boxtimes y)^- \boxtimes (x \boxtimes x^-) \simeq (x \boxtimes y)^- \boxtimes ((x \boxtimes \mathbb{1}) \boxtimes x^-) \arrow[d, "I_y"] \\
\mathbb{1} \boxtimes (y^- \boxtimes x^-) \simeq y^- \boxtimes x^- & (x \boxtimes y)^- \boxtimes \{[x \boxtimes (y \boxtimes y^-)] \boxtimes x^-\} \simeq [(x \boxtimes y)^- \boxtimes (x \boxtimes y)] \boxtimes (y^- \boxtimes x^-) \arrow[l, "E_{x\boxtimes y}"]
\end{tikzcd}$$

Remark 9.20.
Similar to [Gur12, Remark 6.9], it is not difficult to show that inv is actually an equivalence with $\mathrm{inv}^{\mathrm{coop}}$ as its inverse.

10. Morphisms between Coherent Gr-bicategories

Let Π be a finite group. We consider Π as a discrete monoidal bicategory $\Pi^{\mathsf{bi}\boxtimes}$. Furthermore, we fix a coherent Gr-bicategory

$$(\mathsf{B}, \boxtimes, \mathbb{1}, A, L, R, \pi, \mu, \lambda, \rho).$$

In this section, we will combine the results of the last chapters to study monoidal morphisms

$$(S, \chi) \colon \Pi^{\mathsf{bi}\boxtimes} \to \mathsf{B}.$$

In this chapter, we will be rather informal. Our primary goal is just to show that one can work with coherent Gr-bicategories. We proceed in several steps.

First, by Corollary 8.8, the Transport of Structure Theorem 8.27 and Remark 9.9, we can, without loss of generality, assume that composition of identities is strict in B. Secondly, by Lemma 8.25, the morphism $\boxtimes$ of bicategories is isomorphic as a morphism to a morphism $\tilde{\boxdot}$ such that the identity constraints $\tilde{\boxdot}^0$ is the identity for every pair of objects x and y. By Proposition 8.28, we can thus assume, without loss of generality, that $\boxtimes^0_{x,y}$ is the identity for all objects x and y of B. We will do so from now on.

For a skeletal monoidal bicategory, identifying isomorphic 1-cells amounts to just forgetting the 2-cells to get a monoidal category $\mathsf{B}^{\leq 1}$.

Note that for the discrete monoidal bicategory $\Pi^{\mathsf{bi}\boxtimes}$, we get $(\Pi^{\mathsf{bi}\boxtimes})^{\leq 1} = \Pi^{\otimes}$. If the skeletal monoidal bicategory B is coherent, then $\mathsf{B}^{\leq 1}$ is a coherent Gr-category. If B is a special coherent bicategory, then $\mathsf{B}^{\leq 1}$ is special coherent Gr-category. By transporting the structure of a monoidal morphism, we can, without loss of generality, assume that the monoidal morphism S is identity-normalised.

Obstruction Theory for Coherent Gr-bicategories

Now, we will use the fact that our bicategory B is skeletal. We will show that the associativity 1-cells A on B define a 3-cocycle of the group of objects $\mathrm{ob}(\mathsf{B})$ with values in the abelian group $\mathrm{ob}(\mathsf{B}(\mathbb{1},\mathbb{1}))$. This exactly corresponds to the calculation in [BL04, Section 8.3] for the special coherent Gr-category $\mathsf{B}^{\leq 1}$.

We define for objects x, y, z in B

$$\bar{A}_{x,y,z} := A_{x,y,z} \boxtimes 1_{(x \boxtimes y \boxtimes z)^-} \colon \mathbb{1} \to \mathbb{1}.$$

Here, we used that B is 0-skeletal.

Note that according to Lemma 8.20, if we have two 1-cells $f, g \colon \mathbb{1} \to \mathbb{1}$ such that

$$\begin{tikzcd} \mathbb{1} \arrow[r, "f"] \arrow[d, "p"'] & \mathbb{1} \arrow[d, "p"] \arrow[dl, phantom, "\swArrow\alpha"] \\ \mathbb{1} \arrow[r, "g"'] & \mathbb{1} \end{tikzcd}$$

with an invertible 2-cell α and an internal equivalence p, then f and g are isomorphic and hence (since B is 1-skeletal), they are, in fact, equal.

Similar arguments apply for all endomorphisms: Note that tensoring with an object x is a biequivalence from the (degenerate) full sub-bicategory $\mathsf{B}[\mathbb{1}]$ on the object $\mathbb{1}$ to the (degenerate) full sub-bicategory $\mathsf{B}[x]$ on the object x. Since $\mathsf{B}[\mathbb{1}]$ is braided, see Remark 8.32, so is $\mathsf{B}[x]$.

For endomorphisms $f \colon x \to x$, $g \colon y \to y$ and $h \colon z \to z$, we have the square

$$\begin{tikzcd} (x \boxtimes y) \boxtimes z \arrow[r, "(f \boxtimes g) \boxtimes h"] \arrow[d, "A_{x,y,z}"'] & (x \boxtimes y) \boxtimes z \arrow[d, "A_{x,y,z}"] \arrow[dl, phantom, "\swArrow A_{f,g,h}"] \\ x \boxtimes (y \boxtimes z) \arrow[r, "f \boxtimes (g \boxtimes h)"'] & x \boxtimes (y \boxtimes z) \end{tikzcd}$$

so, we get

$$(f \boxtimes g) \boxtimes h = f \boxtimes (g \boxtimes h).$$

Note that every equivalence in a skeletal bicategory is an endomorphism.

By Lemma 9.19, we have a monoidal morphism inv whose tensor structure has components[1] for endomorphisms $f\colon x \to x$, $g\colon y \to y$

$$\begin{tikzcd}
y^- \boxtimes x^- & y^- \boxtimes x^- \arrow[l, "\mathrm{inv}(g)\boxtimes\mathrm{inv}(f)"'] \\
(x \boxtimes y)^- \arrow[u, "\mathrm{inv}^2_{x,y}"] \arrow[ur, phantom, "\mathrm{inv}^2_{f,g}\Nearrow"] & (x \boxtimes y)^- \arrow[l, "\mathrm{inv}(f\boxtimes g)"] \arrow[u, "\mathrm{inv}^2_{x,y}"']
\end{tikzcd}$$

so we get

$$\mathrm{inv}\, g \boxtimes \mathrm{inv}\, f = \mathrm{inv}\,(f \boxtimes g).$$

Using the above, we can define an action of the objects of $\mathrm{ob}(\mathsf{B})$ on $\mathrm{ob}(\mathsf{B}(\mathbb{1}, \mathbb{1}))$ by setting for an object x of B and a 1-cell $f\colon \mathbb{1} \to \mathbb{1}$

$$x \triangleright f := 1_x \boxtimes f \boxtimes 1_{x^-}.$$

Thus, we get for objects x, y, z, w in B

$$\begin{aligned}
(A_{w,x,y} \boxtimes 1_z) \boxtimes 1_{(wxyz)^-} &= A_{w,x,y} \boxtimes (1_z \boxtimes 1_{(wxyz)^-}) \\
&= A_{w,x,y} \boxtimes [1_z \boxtimes (1_{z^-} \boxtimes 1_{(wxy)^-})] \\
&= A_{w,x,y} \boxtimes ((1_z \boxtimes 1_{z^-}) \boxtimes 1_{(wxy)^-}) \\
&= A_{w,x,y} \boxtimes (1_{\mathbb{1}} \boxtimes 1_{(wxy)^-}) \\
&= A_{w,x,y} \boxtimes 1_{(wxy)^-}
\end{aligned}$$

where we used that, by the existence of $\boxtimes^0$, we have

$$\boxtimes_{\mathbb{1}} = 1_x \boxtimes 1_{x^-}.$$

Similarly, we get

$$\begin{aligned}
(1_w \boxtimes A_{x,y,z}) \boxtimes 1_{(wxyz)^-} &= (1_w \boxtimes A_{x,y,z}) \boxtimes (1_{(xyz)^-} \boxtimes 1_w) \\
&= [1_w \boxtimes (A_{x,y,z} \boxtimes 1_{(xyz)^-})] \boxtimes 1_{w^-}
\end{aligned}$$

The existence of the modification π thus yields

$$\bar{A}_{w,x,y} + \bar{A}_{w,xy,z} + w \triangleright \bar{A}_{x,y,z} = \bar{A}_{wx,y,z} + \bar{A}_{w,x,yz},$$

[1]Note that they live in $\mathsf{B}^{\mathrm{coop}}$, hence all the cells are reversed.

i.e. that $\bar{A}_{*,*,*}$ defines a 3-cocycle with values in $\mathrm{ob}(\mathsf{B}(\mathbb{1},\mathbb{1}))$.

Now, we come back to our goal of constructing a monoidal morphism from $\Pi^{\mathsf{bi}\boxtimes}$ to B. First, a map of groups

$$S\colon \Pi \to \mathrm{ob}(\mathsf{B}),$$
$$g \mapsto x_g$$

is underlying a monoidal functor

$$(S,\chi,\iota)\colon \Pi^{\otimes} \to \mathsf{B}^{\leq 1}$$

if there exist invertible 1-cells

$$\chi_{g,h}\colon x_g \boxtimes x_h \to x_{gh} \qquad \iota\colon \mathbb{1} \to x_1$$

such that the following diagrams commute

$$\begin{tikzcd}
 & x_g \boxtimes (x_h \boxtimes x_k) \arrow[rr, "\mathbf{1}_{x_g}\boxtimes\chi_{h,k}"] & & x_g \boxtimes x_{hk} \arrow[dr, "\chi_{g,hk}"] & \\
(x_g \boxtimes x_h)\boxtimes x_k \arrow[ur, "A_{x_g,x_h,x_k}"] \arrow[drr, "\chi_{g,h}\boxtimes\mathbf{1}_{x_k}"'] & & & & x_{ghk} \\
 & & x_{gh}\boxtimes x_k \arrow[urr, "\chi_{gh,k}"'] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{1}\boxtimes x_g \arrow[r, "L_{x_g}"] \arrow[d, "\iota\boxtimes\mathbf{1}_{x_g}"'] & x_g \\
x_g \boxtimes x_1 \arrow[ur, "\chi_{1,g}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
x_g\boxtimes\mathbb{1} \arrow[r, "R_{x_g}"] \arrow[d, "\mathbf{1}_{x_g}\boxtimes\iota"'] & x_g \\
x_g \boxtimes x_1 \arrow[ur, "\chi_{g,1}"'] &
\end{tikzcd}.$$

We denote the full monoidal sub-bicategory whose objects are in the image of S by $\mathsf{B}[\Pi]$. The commutativity of the first diagram above implies that on the image $\mathsf{B}[\Pi]$, the pullback of $\bar{A}$ along S is cohomologically trivial.

Remark 10.1.
The existence of this obstruction can be puzzling at first sight. From the perspective of algebraic homotopy types and 2-crossed modules, we would expect

the appearance of a 4-cocycle; see [Con84, Théorème 4.7]. In fact, the first axiom of a monoidal bicategory A will provide us with a 4-cocycle if we can deal with the first obstruction. Since we are interested in morphisms of special Gr-bicategories from a special Gr-Gray-category Π to B, luckily for us, the existence of such a morphism already ensures the vanishing of the (pullback of the) first obstruction.

Now, we will apply Corollary 8.29 with the above data to obtain a coherent Gr-bicategory equivalent to the image $\mathsf{B}[\Pi]$ of S. We choose $\dot{\mathbb{1}} := x_1$,

$$x_g \boxdot x_h := x_{gh} \text{ and } \xi_{g,h} := \chi_{g,h}^{-1}$$

for all $g, h \in G$. We consider these 1-cells as internal biadjoint equivalences with their inverse as (right) adjoint.

With this definition, $\boxdot$ defines a monoidal product on $\mathsf{B}[\Pi]$. If we choose the weak inverses $x_g^{\dot{-}} := x_{g^{-1}}$ for every $g \in \Pi$, we get that $(\mathsf{B}[\Pi], \boxdot)$ is special.

Note that by the definition of $\boxdot^0$ in the proof of Corollary 8.29, we have that $\boxdot^0$ is the identity for every pair of objects.

Let us denote the pentagonator of $(\mathsf{B}[\Pi], \boxdot, \dot{\mathbb{1}})$ by $\dot{\pi}$. The underlying morphism $\boxdot$ is identity-normalised. Since $(\mathsf{B}[\Pi], \boxdot, \dot{\mathbb{1}})$ is a special Gr-bicategory such that the associator 1-cells are identities, we get, by Lemma 8.31, that the associator compatibility cells are all identities. Thus, the first axiom (a.) of the monoidal bicategory $\mathsf{B}[\Pi]$ reduces to

$$\bar{\pi}_{g,h,i,jk} + g \triangleright \bar{\pi}_{h,k,i,j} + \bar{\pi}_{g,hk,i,j} = \bar{\pi}_{gh,i,j,k} + \bar{\pi}_{g,h,ij,k} + \bar{\pi}_{g,h,k,i}$$

where we define the invertible 2-cell

$$\bar{\pi}_{g,h,i,j} := \dot{\pi}_{x_g,x_h,x_i,x_j} \boxtimes 1_{1_{(x_g x_h x_i x_j)^-}} : 1_{\mathbb{1}} \to 1_{\mathbb{1}}.$$

In other words, $\dot{\pi}$ defines a 4-cocycle on $\mathrm{ob}(\mathsf{B}[\Pi])$ with values in the group $\mathrm{ob}(\mathsf{B}[\Pi](\dot{\mathbb{1}}, \dot{\mathbb{1}})(1_{\dot{\mathbb{1}}}, 1_{\dot{\mathbb{1}}}))$.

Now, a monoidal morphism

$$(S, \chi, \iota, \omega, \gamma, \delta) \colon \Pi^{\mathsf{bi}\boxtimes} \to \mathsf{B}$$

can be factored using the monoidal equivalence and $(\mathsf{B}[\Pi], \boxtimes)$ to $(\mathsf{B}[\Pi], \boxdot)$ as a monoidal morphism

$$(S, 1, 1, \dot{\omega}, \dot{\gamma}, \dot{\delta}) \colon \Pi^{\mathsf{bi}\boxtimes} \to (\mathsf{B}, \boxdot).$$

Then, the first axiom of a monoidal functor A corresponds to

$$\bar{\pi}_{g,h,i,j} + \bar{\omega}_{g,h,i} + \bar{\omega}_{g,hi,j} + g \triangleright \bar{\omega}_{h,i,j} = \bar{\omega}_{gh,i,j} + \bar{\omega}_{g,h,ij},$$

where we again defined the invertible 2-cell

$$\bar{\omega}_{i,j,k} := \dot{\omega}_{x_i,x_j,x_k} \boxtimes 1_{1_{(x_i x_j x_k)^-}} : 1_{\mathbb{1}} \to 1_{\mathbb{1}}.$$

Thus, we conclude that the 4-cocycle defined by $\dot{\pi}$ is cohomologically trivial.

Remark 10.2.
In the above discussion, we focused on the first axiom of a monoidal bicategory A. The normalisation axioms B1 and B2 reduce in the above setting to an equation with values in $\mathrm{ob}(\mathsf{B}[\Pi](\dot{\mathbb{1}}, \dot{\mathbb{1}})(1_{\dot{\mathbb{1}}}, 1_{\dot{\mathbb{1}}}))$ involving only the pentagonator and the 2-unitors (all of which are 2-automorphisms of $1_{\dot{\mathbb{1}}}$. We believe that with the right notion of special morphisms between coherent Gr-categories, we can achieve that $\dot{\pi}$ is a normalised 4-cocycle.

The above discussion should serve as proof of concept that this could be useful.

Remark 10.3.
Let B be a bicategory. Then, as discussed in Example 9.5, there is a coherent Gr-bicategory with the biautoequivalences of B as its objects. Then the above invariants can be seen as obstructions to a group action on B.

Remark 10.4.
Our primary motivation to study the obstructions to lifting maps of groups to coherent Gr-bicategories was a similar calculation for semi-strict monoidal 2-categories (Gray-monoids) in [DN21, Section 2.5]. As summarised in [Cui+16, Section 3.4] and Remark 6.15, it can be helpful in constructing extensions of fusion categories to also proceed in two steps: By first, lifting a map of groups to a monoidal functor, which comes with an O_3 obstruction in a third cohomology group. And then, in a second step, lifting a monoidal functor to a functor of 3-groups (Gray-monoids), which comes with an O_4 obstruction in a fourth cohomology group. We think that for a correctly chosen coherent Gr-bicategory, we will be able to reprove their results. We will come back to this in the outlook.

Outlook

Let us remark on some natural next steps.

Part I: Quadratic Forms and Their Isometries.

In Chapter 1, we established the existence of minimal polynomials for automorphisms of modules over Artinian local rings (Theorem 1.16). In the proof, we used the structure of complete local rings. A natural next step would be to investigate if similar ideas apply to other excellent Henselian rings as in [Art69],

In Chapter 2 and 3, we worked with quadratic forms over Henselian local rings with finite residue fields of *odd* characteristic. Thus, of course, the question arises as to what happens in even characteristic. The theory of quadratic forms and symmetric bilinear forms in even characteristic is more involved. Still, by working with form parameters (see [Knu91] and [Bak81]), many of the ideas presented here should generalise to even characteristic.

In Chapter 3, we worked with an Artinian local ring A with maximal ideal $\mathfrak{m}$ and finite residue field k of odd characteristic p. For a regular quadratic module (M, q), we listed in Theorem 3.34 the conjugacy classes of isometries in the orthogonal group of (M, q) of order coprime to the characteristic p. By further investigating the kernel of the (surjective) reduction map,

$$O_A(M, q) \to O_k(\bar{M}, \bar{q})$$

it should be possible to give a complete description of the orthogonal group $O_A(M, q)$ (at least up to conjugacy). Furthermore, we think it is worthwhile to study how the generation of the orthogonal group by reflections corresponds to this; see [Kne69].

In Chapter 4, we described in Theorem 4.9 which of the conjugacy classes, characterised above, contain isometries that are fixed-point-free. We used the fact that we are working with a finite ring, since we needed the kernel of the reduction map to be a finite p-group. Thus, it would be interesting to see what happens if we can drop the finiteness assumption on the ring.

Part II: Modular Fusion Categories and Fixed-Point-Free Isometries.

In Chapter 5, we recalled the correspondence between pointed braided modular fusion categories and discriminant forms. Of course, it would be a natural next step to generalise the ideas in this section to non-semi-simple pointed braided fusion categories as in [BN19]. Note that braided autoequivalences of pointed braided fusion categories still are (extensions of) orthogonal groups, see [BN19, Corollary 7.5].

In Chapter 6, we constructed and classified crossed $\mathbb{Z}/N$-crossed extensions of hyperbolic pointed modular categories with fixed-point-free action in Theorem 6.31. A natural next step would also be to calculate the $\mathbb{Z}/N$-crossed modular data of the constructed extensions; see [Del+24]. We also refer to [Bar+19] for skeletal and to [Kir04] for strict modular crossed categories. It would especially be interesting to see what the crossed modular data corresponds to if we realise the above pointed modular categories as categories of modules over a vertex operator algebra, as in [GR25, Section 5.3].

In the end, let us remark that there are most likely different tools needed to study examples with a not necessarily hyperbolic fixed-point-free action.

Let q be an odd prime. Then, we can consider a finite Galois extension $\mathbb{F}_q \subset \mathbb{F}_{q^2}$. We define the involution σ by

$$\begin{aligned} \sigma\colon \mathbb{F}_{q^2} &\to \mathbb{F}_{q^2}, \\ x &\mapsto x^q. \end{aligned}$$

Then, the norm map

$$\begin{aligned} \mathrm{Norm}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\colon \mathbb{F}_{q^2} &\to \mathbb{F}_q \\ x &\mapsto x\sigma(x) \end{aligned}$$

defines an anisotropic quadratic form.

The orthogonal group of $(\mathbb{F}_{q^2}, \mathrm{Norm}_{\mathbb{F}_{q^2}/\mathbb{F}_q})$ (as quadratic form over $\mathbb{F}_q$) can easily be described; see [Tay92, Theorem 11.4]. In particular, every element $x \in \mathbb{F}_{q^2}$ with $\mathrm{Norm}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(x) = 1$ defines an isometry ϕ_x by multiplication by x.

If we now assume that x has odd prime order p with $p \mid q+1$, then we get that the minimal polynomial $P \in \mathbb{F}_p$ of ϕ_x is an irreducible polynomial of degree 2 and order p. It is self-reciprocal. The isometry ϕ_x is also fixed-point-free, and, since it is of prime order, it induces a fixed-point-free action.

It is shown in [GPR24, Theorem 4.5] that the $\mathbb{Z}/p$-equivariantisation of any braided $\mathbb{Z}/p$-crossed extension of the pointed modular category corresponding to $(\mathbb{F}_{q^2}, \mathrm{Norm}_{\mathbb{F}_{q^2}/\mathbb{F}_q})$ is non-group-theoretical. In fact, they even showed in [GPR24, Theorem 4.6] that every non-group-theoretical modular fusion category of dimension p^2q^2 is of the above form. By [NNW09, Theorem 7.2], it can not be described as the Drinfeld centre of a pointed category.

In Chapter 7, we described the modular categories that we obtain as $\mathbb{Z}/N$-equivariantisations of the extensions we constructed in Theorem 6.31 up to braided equivalence. This was only possible because the quasi-Hopf algebra introduced by Dijkgraaf, Pasquier, and Roche is particularly well-understood. We think that a natural next step would be to investigate the Hopf algebra whose category of representations is equivalent to the braided $\mathbb{Z}/N$-crossed extension. Similar ideas for $\mathbb{Z}/2$ appeared in [Nat10], [Nik08, Remark 5.1 and Proposition 5.2], and [CM17, Section 4]; see also [Mas02].

Part III: Monoidal Bicategories and Coherent Gr*-bicategories.*

In Chapter 8 and 9, we recalled the definition of monoidal bicategories and their morphisms and introduced coherent Gr-bicategories. As we mentioned earlier, our main motivation for working with coherent bicategories stems from the following example. Let C be a *braided* fusion category. It should not be too hard to show that *invertible* module categories, (one-sided) module equivalences and isomorphisms of such equivalences form a coherent Gr-bicategory. In fact, the semistrict (Gray) analogue of this statement already appears in [DN21], [DN13] and [ENO10]; see also [Str04].

Schaumann proved in his thesis [Sch13, Theorem 3.6.1] that there is a well-defined monoidal bicategory with bimodule categories over a fusion category C as its objects, bimodule functors as 1-cells, and bimodule natural transformations as 2-cells. By choosing biadjoint equivalences (which is possible by [Gur12, Theorem 4.5]), it should be possible to build a coherent Gr-bicategory.

This does not depend on the semi-simplicity of the fusion category and can thus be generalised to finite tensor categories, see e.g. [Sch15, Theorem 1.2].

We are interested in these monoidal bicategories as monoidal morphisms into them will (conjecturally) classify extensions of fusion/tensor categories, see [DN21, Theorem 8.5 or 8.13].

For us, the main problem with working with semi-strict monoidal 2-categories rather than coherent monoidal bicategories is that it turns out to be very hard to explicitly construct the classified extensions, as we did in Chapter 6, especially including the associator and braiding of the constructed extension. Of course, this approach has many benefits, for example, the obstruction theory gets simplified considerably, see [DN21, Section 2.5]. If the group by which we extend has small order, this is, in fact, sometimes enough to write down all extensions; see e.g. [VV12, Theorem 3.16] building on the preprint [Lip10] or [JL09, Theorem 1.1].

For *braided* fusion/tensor categories, similar ideas should apply. Namely, by considering invertible *one-sided* bimodule categories [DN13], their bimodule functors, and natural transformations, it should be possible to build a coherent Gr-category such that monoidal morphisms in this Gr-category should classify crossed braided extensions; see also [Jon+22] and [Cui+16].

In Chapter 10, we sketched how to lift a map of groups to a morphism between coherent Gr-bicategories. This was mainly motivated by [DN21, Section 2.5], which the authors later applied to classify extensions of tensor categories; see [DN21, Corollary 8.7 and Remark 8.14]. Once the existence of the coherent Gr-category of invertible (one-sided) bimodule categories over a fusion/tensor category above is established, this section should be applied. We hope that working with coherent Gr-categories will allow for explicit calculations. Our primary goal was to show that even when we work with coherent Gr-bicategories, it is possible to proceed in two steps and 'decouple' the two obstructions.

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Hamburg, 17. Dezember 2025

H. Knötzele

Hannes Knötzele

Eigenanteilserklärung:

The results in this thesis are my own, unless indicated otherwise. The idea of using fixed-point-free isometries of discriminant forms to construct modular fusion categories was suggested to me by my advisor, Sven Möller, and served as the starting point of this thesis.

Some results of this thesis were announced at the conference 'Vertex algebras and related topics' in 2024 at the RIMS, Kyoto.

Joint publications with Sven Möller are planned.