

Holographic one-dimensional quantum systems

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Zusammenfassung

Diese Dissertation entwickelt neue nicht-perturbative numerische Methoden sowie exakte analytische Konstruktionen im Kontext niedrigdimensionaler konformer Feldtheorie und Holographie. Die zentralen Ergebnisse sind eine Sechs-Punkt-Erweiterung des numerischen konformen Bootstraps sowie die Bestimmung des holographischen Dualen einer Familie von Anti-de-Sitter-(AdS)-D-Branen im dreidimensionalen AdS-Raum.

Auf der numerischen Seite führt die Arbeit einen Sechs-Punkt-Bootstrap ein, der Positivität der Zerlegungen von Sechs-Punkt-Korrelationsfunktionen in Produkte von Vier-Punkt-Korrelatoren ausnutzt, um rigorose Schranken für Lösungen der Sechs-Punkt-Kreuzungsgleichungen herzuleiten. Das resultierende Optimierungsproblem ist ein semidefinites Programm (SDP) mit unendlichdimensionalen Nebenbedingungsmatrizen. Durch den Einsatz von SDP-Techniken für dünnbesetzte Matrizen wird dieses Problem in eine numerisch handhabbare Form überführt. Als Machbarkeitsnachweis wird die Methode für konform invariante Korrelationsfunktionen, mit einer kinematischen Einschränkung auf den eindimensionalen Fall, implementiert. Dadurch werden unter anderem Korrelationen holographischer Randmoden in wechselwirkenden Quantenfeldtheorien auf AdS_2 nicht-perturbativ zugänglich gemacht.

Auf der analytischen Seite führt die Arbeit eine Familie eindimensionaler holographischer Grenzflächen zwischen symmetrischen Produktorbifaltigkeiten ein. Diese Grenzflächen werden zunächst algebraisch mittels expliziter Randzustände in der gefalteten Theorie analysiert. In diesem Rahmen wird die Partitionsfunktion der Grenzflächenmoden berechnet. Die Ergebnisse befinden sich in exakter Übereinstimmung mit der Annulus-Partitionsfunktion von AdS_2 -Branen im Eberhardt–Gaberdiel–Gopakumar-Grenzfall von Superstringtheorie auf reinem NSNS $\text{AdS}_3 \times S_3 \times \mathbb{T}^4$. Abschließend wird eine alternative Beschreibung der Grenzflächen in Form einer neuartigen Klasse verallgemeinerter Überlagerungen entwickelt, die eine deutlich verbesserte Kontrolle über Grenzflächenkorrelatoren, sowie über Partitionsfunktionen auf Riemann Flächen höheren Geschlechts liefert. Auf Grundlage dieses verbesserten Zugangs wird eine 't Hooft-artige Entwicklung etabliert, die, zu allen Ordnungen in der Kopplungsstärke, strukturell mit der stringtheoretischen Entwicklung von Streuamplituden übereinstimmt.

Abstract

This thesis provides new non-perturbative numerical methods and exact analytic constructions in the context of low-dimensional conformal field theory and holography. The main results are a six-point extension of the numerical conformal bootstrap and the identification of the holographic dual of anti-de Sitter (AdS) D-branes in minimal pure NSNS AdS_3 .

On the numerical side, the thesis introduces a six-point bootstrap framework that exploits positivity in decompositions of six-point correlation functions into products of four-point correlators to derive rigorous bounds on solutions to six-point crossing equations. The resulting optimisation problem involves semi-definite programming (SDP) with infinite-dimensional constraint matrices. By employing sparse SDP techniques, the problem is recast into a numerically tractable form. As a proof of concept, the method is implemented for conformally invariant correlation functions in one dimension, providing e.g. a non-perturbative handle on correlations of holographic boundary modes in interacting quantum field theories defined on AdS_2 .

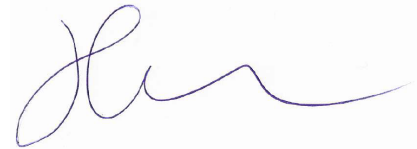
On the analytic side, the thesis introduces a family of one-dimensional holographic interfaces between symmetric product orbifolds. These interfaces are first analysed algebraically via explicit boundary states in the folded theory. In this framework, the partition function of interface-changing operators is computed and exactly matched with the annulus partition function of AdS_2 branes in the Eberhardt–Gaberdiel–Gopakumar tensionless limit of superstring theory on pure NSNS $\text{AdS}_3 \times S_3 \times \mathbb{T}^4$. Finally, an alternative description of the interfaces in terms of a novel class of generalised covering maps is provided, which yields significantly improved control over interface correlators and higher-genus partition functions. With this new handle on the interface correlators, a 't Hooft-like expansion is established which structurally matches string perturbation theory to all orders in the string coupling.

Declaration on Oath

I hereby declare and affirm that this doctoral dissertation is my own work and that I have not used any aids and sources other than those indicated.

If electronic resources based on generative artificial intelligence (gAI) were used in the course of writing this dissertation, I confirm that my own work was the main and value-adding contribution and that complete documentation of all resources used is available in accordance with good scientific practice. I am responsible for any erroneous or distorted content, incorrect references, violations of data protection and copyright law or plagiarism that may have been generated by the gAI.

Hamburg, 11.04.2026

A handwritten signature in blue ink, consisting of a stylized 'H' followed by a cursive 'u' and a long horizontal stroke.

Signature of doctoral candidate

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Această teză de doctorat este dedicată lui Zderie Ionel, a cărui cea de-a 83-a aniversare am sărbătorit-o în ziua finalizării lucrării.

Statement of Originality

Chapters 2 and 7 review known results of the literature. Chapters 3, 4, 5, 8 and 9 are based on the following publications of the author.

- ★ Lining up a positive semi-definite six-point bootstrap [1].
Collaboration with António Antunes, Apratim Kaviraj and Volker Schomerus
- ★ Sparsity in the numerical six-point bootstrap [2].
Single author work
- ★ Five points for the Polyakov bootstrap [3].
Collaboration with António Antunes and Apratim Kaviraj
- ★ Holographic interfaces in symmetric product orbifolds [4].
Collaboration with Yasuaki Hikida, Volker Schomerus and Takashi Tsuda
- ★ Interface correlators in symmetric product orbifolds [5] .
Collaboration with Volker Schomerus and Takashi Tsuda

Ch. 6 is partially based on Refs. [2, 3] and Ch. 10 is partially based on Ref. [5].

As part of his PhD work, the author has also contributed to the following paper:

- ★ Comb channel lightcone bootstrap: triple-twist anomalous dimensions [6].
Collaboration with Apratim Kaviraj, Jeremy Mann, Lorenzo Quintavalle and Volker Schomerus

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Chapter 1

Introduction

– **Multi-point conformal bootstrap & Pure NSNS $\text{AdS}_3/\text{CFT}_2$.**

In many ways, the quest to understand the scattering of nuclei has been one of the most fruitful endeavours in theoretical high-energy physics. Few problems have inspired such a wealth of new ideas and approaches to particle physics as that of the strong force, see Figure 1.1 for a schematic overview of some of these developments. From a modern perspective, the key milestones along this path were the formulation of non-Abelian gauge theory, the discovery of confinement and asymptotic freedom, and, ultimately, the theoretical and experimental establishment of QCD. At the same time, other directions of research, such as S-matrix theory, the bootstrap, and, famously, string theory, failed to provide a theoretical framework consistent with the available experimental data, despite the phenomenological success [7, 8] of Regge theory. Nevertheless, these efforts were far from futile: the wrong approaches to high-energy scattering of hadrons may, in hindsight, provide the right ideas for quantum gravity and the physics of strongly coupled systems.

This chapter is organised into three sections: Section 1.1 gives a non-technical minimal account of the developments that led to the modern understanding of string theory as a framework for quantum gravity and, via the holographic correspondence, an effective description of certain strongly coupled gauge theories. In particular, it sets the stage for the two central objectives that guide the work presented in this thesis and highlights the close connection between them. These objectives are: to increase our computational control over strongly coupled systems and to improve our understanding of the holographic correspondence. With a focus on the first of these two objectives, Section 1.2 introduces new numerical tools for the non-perturbative study of Conformal Field Theories (CFTs) that are presented in Chapters 2, 3, 4, 5, and 6 of the thesis. Shifting the attention to the second objective, Section 1.3 serves as an introduction to new examples of exact instances of holography that are discussed in Chapters 7, 8, 9, and 10. In both cases, the results emphasise the importance of tractable low-dimensional models as, on the one hand, inherently interesting



1.1 From nucleons to gravitons ... and back again

¹The reader is referred to the following references for more on the intertwined history of string theory [9], conformal field theory [10] and the bootstrap [11].

S-matrix theory and the bootstrap. In the 1940s, it was suspected, e.g. by Werner Heisenberg, that the divergences in perturbative QFT could be an indication of a breakdown of our macroscopic notion of space-time at the nuclear scale $\ell_N \sim 10^{-15}\text{m}$. To make progress, Heisenberg proposed to focus, in a theory-agnostic way, on established physical observables and the consistency conditions they must satisfy, independent of the details of the then unknown physics that would emerge at ℓ_N [12]. The observable at the core of this programme was the S-matrix. The key assumption of S-matrix theory is that we can adequately describe the initial and final state of a scattering experiment, given that these consist of approximately decoupled particles that are separated on scales much larger than ℓ_N , without knowing the details of the physics at the nuclear scale. The programme that emerged from these ideas in the 1950s and 1960s was then to “bootstrap” the transition amplitudes between these initial and final states, the entries of the S-matrix, by imposing consistency constraints such as unitarity and crossing symmetry. In the best-case scenario, these considerations would have led to a single S-matrix that, by first-principles reasoning, would be the unique candidate for a description of hadron scattering².

By the mid 1970s it had become clear that perturbative QFT is perfectly able to describe high-energy scattering of hadrons. Fears that the notion of space-time might break down at ℓ_N were by then largely resolved. Moreover, the speculated uniqueness of a consistent S-matrix was understood to be far too ambitious and many UV-complete QFTs had been discovered. Finally, and most importantly, the structure of the consistency conditions for the S-Matrix simply turned out to be too complicated. The progress of the early bootstrappers was significantly constrained and ultimately halted by the lack of suitable numerical techniques and mathematical theory to overcome these technical challenges.

Amplituhedron and the swampland programme. Over the last 20 years, the ideas of the old bootstrap and S-matrix theory have been revived by the modern bootstrap, to which this thesis contributes, the swampland programme, and the amplitudes community.

Today it is still widely believed that, as Heisenberg suspected 80 years ago, space-time breaks down at sufficiently small scales. In our modern understanding, this is not related to the strong force but to gravity, and occurs at a much smaller scale that could potentially be the Planck length $\ell_P \sim 10^{-35}\text{ m}$. This expectation is on conceptually firmer ground than its predecessor. Assuming that general relativity’s identification of geometry with gravity remains valid down to microscopic scales, quantum effects in gravity, which can conservatively be expected to be relevant at the Planck scale, go hand in hand with modifications of classical geometry. In this new

²The author of this thesis is not claiming that the majority of the early bootstrappers ever believed this to be a realistic goal, at least not in this simplified and highly ambitious formulation.

context, and with significantly refined mathematical tools, the modern amplitudes community has returned to the search for new approaches to the S-matrix that may supplement or, though arguably less likely, replace perturbative QFT. For an introduction to the field, see e.g. the SAGEX review articles [13].

Whereas ℓ_N was within the range of experiments that guided the researchers who developed the theory of hadrons, there is little hope of directly probing ℓ_P . However, this does not necessarily imply that imprints of quantum gravity will forever be out of experimental reach. The fact that the Effective Field Theory (EFT) describing observable low-energy physics has to admit a UV-completion to a consistent theory of quantum gravity may, for instance, indirectly determine measurable features. The swampland programme, see e.g. [14] for a review, aims to systematically distinguish seemingly consistent QFTs that, however, fail to admit such a UV-completion, from those that describe low-energy dynamics of a consistent model for quantum gravity. The EFTs of the first type form what in this context is called the “swampland”, whereas the physically more relevant EFTs of the second class form the “landscape”. A main motivation for this endeavour is the hope that, for a theory of gravity, UV-consistency might be so constraining that non-trivial, experimentally testable predictions can be made by demanding that physically observable low-energy dynamics is in the landscape. In this new form, the old dream of the early bootstrappers, that demanding general consistency principles and, in modern language, UV-completeness might single out a unique theory that describes our universe, or at least a strongly constrained class of candidates, continues to influence modern-day research.

It should be added that the swampland programme derives from the old bootstrap in more than one way: in practice, the vast majority of swampland research is dedicated to the low-energy EFTs descending from string theory, which, as we recall in the next paragraph, itself emerged from the old bootstrap.

Dual resonance models and Veneziano amplitude.³ In the beginning of this section, we already mentioned crossing symmetry as a guiding bootstrap principle, and commented on the technical difficulties associated with the complicated structure of crossing constraints. A particularly fruitful direction of research within the old bootstrap, that partially circumvented these issues, was based on demanding scattering amplitudes to not only possess crossing symmetry, but to instead satisfy a stronger constraint named “duality”. To compare these two notions, let us discuss elastic two-to-two scattering of scalar particles with momenta p_i and masses $m_i^2 = -p_i^2$. We choose p_1 and p_2 to be the momenta of the incoming particles and p_3 and p_4 to be outgoing, such that momentum conservation reads $p_1 + p_2 = p_3 + p_4$. Lorentz invariance implies that the scattering amplitude A is a function of the Mandelstam

³This paragraph and the two paragraphs below partially follow the beautiful account of this subject provided by the first chapter of Ref. [15].

variables

$$s = -(p_1 + p_2)^2, \quad t = -(p_2 + p_3)^2 \quad \text{and} \quad u = -(p_1 + p_3)^2 \quad (1.1.1)$$

which are constrained by

$$s + t + u = \sum_{i=1}^4 m_i^2. \quad (1.1.2)$$

In the centre-of-mass frame, s is the square of the total energy and t encodes the scattering angle θ via

$$2t = -s(1 - \cos \theta). \quad (1.1.3)$$

Crossing symmetry is the statement that the amplitude $A(s, t)$ satisfies the identity

$$A(s, t) = A(t, s). \quad (1.1.4)$$

In perturbative QFT, this property is essentially guaranteed by the democratic way in which Feynman rules treat all four external states of the amplitude. At tree level, the contribution of each s -channel exchange of a spin J , mass M_J particle

$$A_J^{(s)} = -\frac{g_J^2(-t)^J}{s - M_J^2} \quad (1.1.5)$$

is, for instance, accompanied by a t -channel exchange contribution

$$A_J^{(t)} = -\frac{g_J^2(-s)^J}{t - M_J^2} \quad (1.1.6)$$

such that the overall sum of all Feynman diagrams, and in fact already the contribution $A_J^{(s)} + A_J^{(t)}$ of each exchanged particle, is manifestly crossing symmetric. The defining feature of dual resonance models is that their scattering amplitudes can be written as a sum over only s -channel poles. In other words, they possess a duality

$$\sum_J A_J^{(s)} = \sum_J A_J^{(t)} \quad (1.1.7)$$

between s -channel and t -channel resonances, which leads to emergent crossing symmetry. Historically, duality was not just an ad hoc way to obtain stronger constraints, but instead a phenomenologically motivated proposal that arose from a bootstrap analysis in Regge theory [16].

In 1968, Gabriele Veneziano discovered that the Euler Beta function

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (1.1.8)$$

can be used to define a crossing-symmetric scattering amplitude with a single (real) linear Regge trajectory $\alpha(s) = \alpha(0) + \alpha' s$ that can be written as the sum

$$B(-\alpha(s), -\alpha(t)) = - \sum_{n=0}^{\infty} \frac{(\alpha(t) + 1)_n}{n!(\alpha(s) - n)} \quad (1.1.9)$$

of s -channel poles [17] corresponding to particles of mass $M^2 = (n - \alpha_0)/\alpha'$ with spin $J \leq n$. A five-point generalisation of the Veneziano amplitude was soon discovered by Bardakci and Ruegg [18] and independently by Virasoro [19], which was in turn swiftly followed by the Koba and Nielsen n -point generalisation [20]. A year later, Leonard Susskind provided a harmonic oscillator construction of these amplitudes, concluding “that the degrees of freedom of the internal state of a hadron are equivalent to those of a violin string or organ pipe” [21]. This was the beginning of string theory.

Fixed t and fixed θ . One reason for the initial phenomenological appeal of dual resonance models was their asymptotics in the large s , fixed t regime. Whereas the contribution of individual t -channel exchanges (1.1.6) scales as s^J in this regime, the Veneziano amplitude has a much softer behaviour than each of its parts. It exhibits $s^{-\alpha(t)}$ Regge asymptotics,

$$B(-\alpha(s), -\alpha(t)) = \Gamma(-\alpha(t))(-\alpha(s))^{\alpha(t)} + \dots, \quad (1.1.10)$$

which is consistent with the intermediate-energy near-beam data measured in the 60s. Explaining such a soft behaviour in standard perturbative QFT, on the other hand, is rather challenging. For a while, dual resonance models were hence popular candidates for a theory of hadrons.

The situation changed in the 70s, when more high-energy fixed-angle data became available. In the large s , fixed θ (or equivalently, large s and t , fixed s/t) regime, the Veneziano amplitude shows an even softer behaviour than eq. (1.1.10), an exponential decay

$$B(-\alpha(s), -\alpha(t)) = e^{-\lambda(\theta)\alpha(s)} + \dots, \quad (1.1.11)$$

at an angle-dependent rate $\lambda(\theta)$. This extremely soft behaviour is incompatible with the experimentally observed fixed-angle asymptotics, which instead are governed by a power law.

Two ground-breaking discoveries of the early 70s placed field theory in a position to explain both regimes of scattering described above. Confinement of quarks [22] explained why free partons were never observed, but also how the phenomenological success of string models could be understood from a field theory perspective: in the soft, near-beam scattering regime, the dynamics of hadronic scattering is dominated

by string-like flux tubes between the confined quarks. Importantly, Wilson’s 1974 paper [22] did not just provide this qualitative picture, but instead laid the foundation of lattice QCD [23, 24], explaining how to quantitatively study confinement by quantising non-Abelian gauge theory on a Euclidean lattice.

How this picture is compatible with the qualitatively different behaviour of high-energy, fixed-angle scattering was, on the other hand, understood by Gross, Politzer and Wilczek through their discovery of asymptotic freedom [25, 26]. The one-loop β -function of QCD with n_f flavours is

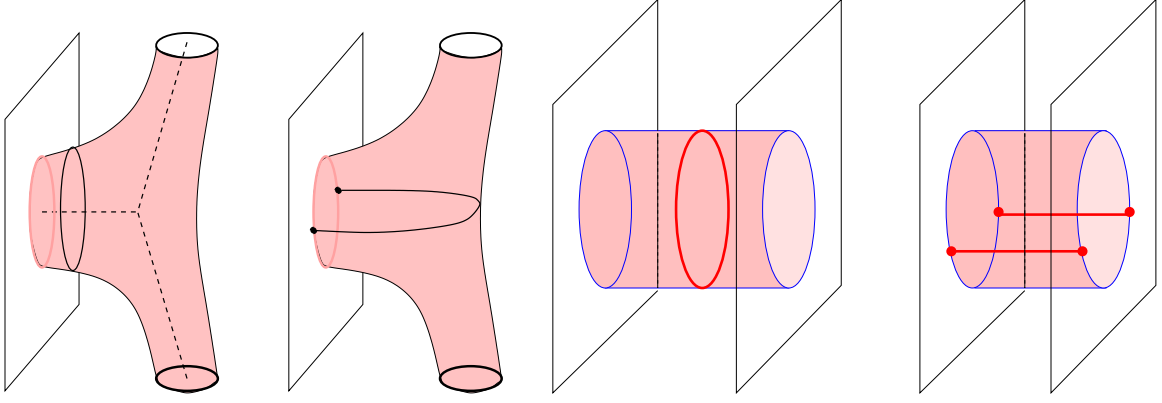
$$\beta(\alpha_s) = -\left(11 - \frac{2n_f}{3}\right) \frac{\alpha_s^2}{2\pi}. \quad (1.1.12)$$

For $n_f \leq 16$, the coupling *decreases* with increasing energy. Thus, while low-energy scattering is dominated by strongly coupled dynamics of confined quarks, the physics relevant for high-energy scattering is that of free partons, in particular leading to the power-law high-energy, fixed-angle behaviour described above. As with confinement and lattice gauge theory, this conceptual insight went hand in hand with a computational breakthrough. The advent of the Wilsonian paradigm [27–29] in the early 1970s led to a clear physical understanding of renormalisation, but the Landau pole encountered in the renormalisation of QED [30] was still suspected to be a generic feature of QFT. Hence, deep concerns that perturbative QFT necessarily breaks down at small scales, indicating fundamental conceptual issues of the framework, remained. The discovery of asymptotic freedom, at once, eased these concerns and laid the foundation for perturbative QCD as a powerful framework to make accurate predictions about the high-energy scattering of hadrons.

As a consequence of these exciting developments in field theory, string theory mostly disappeared from the stage of hadronic physics in the 1970s.

String theory and quantum gravity. While it became more and more clear that non-Abelian gauge theory would supersede dual resonance models as a framework to describe hadronic physics, it was simultaneously realised that the appearance of a massless spin-two state in the spectrum made string theory a natural framework for a microscopic description of gravity [31, 32]. The 25 years that followed this realisation were a particularly fruitful period of research that produced most of the modern formalism of string theory.

By now, numerous textbooks [15, 33–39] have appeared that give a systematic treatment of these developments. Some of the milestones along the path were: the construction of tachyon-free superstring theories via the GSO projection [40], the quantisation of the Polyakov action for bosonic string theory [41], the naming of “type I” and “type II” superstring theory [42], anomaly cancellation in $SO(32)$ type I su-



(a) Open/closed duality for the coupling of strings to a single brane . (b) Open/closed duality for the interaction between branes .

Figure 1.2: Illustrations of open/closed duality taken from the beautifully concise qualitative introduction to holography provided by Ref. [56].

perstring theory and the Green–Schwarz mechanism [43], the discovery of heterotic string theory [44], the beginnings of string field theory [45] and the formulation of cubic bosonic open string field theory [46], Calabi–Yau compactifications [47], the first description of T-duality [48], the construction of an action for supermembranes in 11-dimensional supergravity [49], the proposal of S-duality in string theory [50], the discovery of M-theory [51], the understanding that Dp -branes provide a microscopic description of solitonic p -branes [52], and the counting of microscopic degrees of freedom of extremal black holes [53].

In the late 90s, these discoveries were, in a deep and unexpected way, all related back to non-Abelian gauge theories [54, 55] – the models that had replaced string theory as an approach to hadronic physics more than 30 years earlier.

AdS and CFT. The AdS/CFT correspondence is a duality between gravity in $d + 1$ -dimensional Anti-de Sitter space (AdS) and conformal field theory (CFT) on its d -dimensional asymptotic boundary. Its discovery [54] was closely related to the progress in our understanding of D-brane dynamics made in the 90s.

The key observation is that interactions between strings and branes (Fig. 1.2a) or between branes (Fig. 1.2b) can be interpreted in physically quite distinct ways, either as absorption/emission processes of closed strings, or as the propagation of open strings that end on the branes. This, at first glance perhaps rather inconspicuous, fact has dramatic consequences when combined with a careful construction of stacks of branes.

Indeed, if the distance between the branes in Fig. 1.2b is large compared to the string length (which sets the radius of the blue circles in the illustration), then contributions

from high-energy states are suppressed in the closed string picture. This regime should hence be adequately described by supergravity. On the other hand, the open string picture gives no obvious indication that the dynamics simplify in this limit. Vice versa, bringing the branes close together leads to a regime where the open string dynamics reduces to a gauge theory supported on the world-volume of the branes, whereas the dual closed string picture enters a deeply stringy regime. In this way, the correspondence between open and closed strings hints at a strong–weak duality between theories of gravity and gauge theories.

A prime example of this duality is the correspondence between 4d $\mathcal{N} = 4$ Super Yang–Mills (SYM) and type IIB string theory on $\text{AdS}_5 \times \text{S}_5$. The microscopic brane picture relating these theories via the open/closed duality sketched above is a stack of N parallel D3-branes in 10-dimensional Minkowski space. The gauge theory describing the low-energy dynamics of open strings supported on the 4d world-volume of these branes is $\mathcal{N} = 4$ SYM with gauge group $\text{SU}(N)$. To suppress loop effects in string perturbation theory, we need to restrict our attention to the large- N case, i.e. planar $\mathcal{N} = 4$ SYM. The $\text{AdS}_5 \times \text{S}_5$ geometry enters the setup as the near-horizon limit of a smeared stack of D3-branes in supergravity. At the level of this near-horizon geometry, we no longer resolve individual branes, and in particular there is no direct brane separation parameter as in our discussion of open/closed duality above. This length scale is, in the coarse-grained picture, replaced by the curvature radius R of the AdS_5 geometry (which, for consistency with supersymmetry, has to coincide with the radius of the five-sphere). The concrete manifestation of the strong-weak duality sketched above hence relates R/ℓ_s to the 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$ of planar $\mathcal{N} = 4$ SYM, more precisely

$$\lambda = (R/\ell_s)^4 \quad \text{and} \quad g_s = \lambda/N. \quad (1.1.13)$$

The remarkable conclusion is that strongly coupled planar 4d $\mathcal{N} = 4$ SYM can be solved by studying supergravity on $\text{AdS}_5 \times \text{S}_5$ or, vice versa, perturbative gauge theory allows us to study extremely stringy regimes of quantum gravity.

The upshot is that AdS/CFT produces, at least for some specific models, a large amount of strongly coupled data that is out of reach of standard perturbative methods in QFT and string theory. As such, the correspondence provides an ideal laboratory to test new methods for solving stringy regimes of quantum gravity and strongly coupled CFTs. The arguably most powerful example for comparatively novel tools to attack the latter is the numerical conformal bootstrap.

Modern bootstrap and strong coupling. As discussed above, research on dual resonance models and, later, string theory grew out of the bootstrap. It would, however, be wrong to conclude that string theory in any sense “replaced” the bootstrap.

Instead, the bootstrap has always remained a valuable source of new computational methods that complement perturbation theory at weak coupling and replace it at strong coupling. While the S-matrix bootstrap had lost much of its initial momentum by the end of the 1960s, its ideas resurfaced in the 1970s with the proposal to bootstrap correlation functions in CFT [57, 58]. As we review in Section 2.2.1, the conformal bootstrap was a key ingredient in the analytic solution of many 2d CFTs in the 1980s, 1990s, and early 2000s. A major breakthrough in this field was the 2008 realisation that the bootstrap could be combined with linear programming (or, more generally, semidefinite programming) techniques to yield a non-perturbative handle on higher-dimensional CFTs [59]. The development of the numerical conformal bootstrap from this initial observation to the powerful computational framework available today is reviewed in Section 2.2.2. In the last ten years, these new numerical tools have also been successfully applied to the S-matrix [60, 61], in particular finally bringing the initial dream of bootstrapping hadronic scattering within reach [62–67].

Multi-point conformal bootstrap and Pure NSNS AdS_3 holography. This thesis consists of two parts. The first part develops numerical tools that make new higher-point observables accessible to the bootstrap, which, until recently, focused almost exclusively on four-point correlators. The second part constructs an ensemble of interfaces in certain 2d CFTs that provide a new example in which holography can be derived with a level of detail unprecedented until a recent breakthrough in $\text{AdS}_3/\text{CFT}_2$ holography [68, 69]. Sections 1.2 and 1.3 below provide more technical introductions to these two parts of the thesis.

But even without such details, it is evident that these seemingly unrelated topics are actually two inseparably interwoven threads in a larger fabric of strings and conformal symmetry. In this section, we have seen that this is true in a historical sense, by tracing string theory and the bootstrap back to their common origin in hadronic physics. More importantly, the AdS/CFT correspondence teaches us that this is also true on a deeper conceptual level.

What is the quest to derive holography if not an invitation to confront the challenges of strong coupling?

1.2 Multi-point conformal bootstrap

Over the last two decades⁴, the numerical conformal bootstrap [59] has become an essential asset in the toolbox of the modern theoretical physicist. Like most great tools, it is a versatile instrument that serves its user in various ways.

⁴See [70] for a review of the latest developments and [71] for earlier results.

From a quantitative perspective, the bootstrap offers access to scaling dimensions and OPE coefficients of CFTs as one tool among many, such as, if applicable, Monte Carlo simulations, Hamiltonian truncation, fuzzy sphere regulation, perturbation theory and others. For many theories it outperforms all available alternatives. Famously this is the case, for example, in the 3d Ising model [72–78] as well as in more general small N bosonic $O(N)$ -fixed points [76, 79–82] and $O(N)$ -symmetric Gross-Neveu-Yukawa fixed points [83–85].

From a qualitative perspective, the bootstrap is unique. It allows us to answer questions about the space of all CFTs in a theory-agnostic fashion. This makes it a highly valuable exploratory tool, challenging and inspiring us to think about CFTs in new ways. The purpose of the first part of the thesis is to report on the status of an ongoing effort to significantly enlarge the type of questions that can be addressed via the numerical bootstrap.

Answering new questions typically requires making new sources of information accessible. In this thesis, the new source of information is the set of crossing equations of six-point correlation functions. Studying the crossing equations of higher-point correlators has been fruitful in the analytic bootstrap and, for instance, led to various results on large spin asymptotics of OPE coefficients and multi-twist operators in lightcone bootstrap works such as [6, 86–89]. The first application of numerical techniques to the investigation of higher-point correlators was [90]. In this work and its follow ups [91, 92], numerical solutions to truncated crossing equations of five-point functions were determined in order to obtain approximate conformal data of the 3d Ising CFT. As usual with truncation based approaches to crossing, this five-point bootstrap involved unknown systematic errors that have to carefully be estimated through suitable heuristics. The six-point bootstrap developed simultaneously by the author of this thesis and his collaborators [2, 93] demonstrated that SemiDefinite Programming (SDP) can be used to instead generate exact bounds from higher-point correlators.

The emphasis of the first four chapters is on reporting the current proof-of-concept status of the six-point bootstrap’s practical implementation. The final chapter proposes future steps for the programme to promote numerical six-point bootstrap techniques to a more broadly applicable new tool in the kit of the bootstrapping theorist. Concretely, the logical structure of the text can be summarised as follows.

- **Chapter 2: Objectives and motivation.** We start with a lightning review of conformal field theory, semi-definite programming and the bootstrap. The aim of the review is not to give a fully self contained introduction to these topics, but simply to set the stage by directing the reader’s attention towards the main elements of the discussion. The overall purpose of the chapter is

to formulate sharp objectives for a higher-point extension of the numerical bootstrap, which motivate the work presented in the subsequent chapters.

- **Chapter 3: Existence of six-point bounds.** Based on [93], this chapter formulates optimisation problems whose solutions are rigorous bounds on six-point data. The main result of the chapter is a (constructive) proof of the existence of non-trivial solutions to these problems – which, due to the technically challenging nature of the involved infinite dimensional semi-definite constraints, is a priori not guaranteed. However, the algorithm that is presented should not be viewed as going beyond such a proof of existence. While producing several non-trivial examples of qualitatively new bounds, its applicability is severely limited by three restrictions that are referred to as the **computability problem**, the **convergence problem** and the **input data problem** in this thesis.
- **Chapter 4: Solving the computability problem.** The most problematic shortcoming of the algorithm implemented in Reference [93] is that it could not directly be scaled up to benefit from the power of modern clusters. The numerics of [93] was strictly limited to single core computations. Therefore, while the first implementation of the six-point bootstrap was sufficient to show that in principle computable novel bounds exist, these bounds in practice remained largely inaccessible. This computability problem was solved in [2] by applying sparse semi-definite programming techniques. These allow for a reformulation of the very non-standard SDPs appearing in the six-point bootstrap in terms of equivalent SDPs that can efficiently be solved with the standard machinery of the numerical bootstrap. The chapter based on [2] describes the details of this reformulation as well as new applications that came into reach due to the numerical improvements.
- **Chapter 5: Approaching the convergence problem.** The numerical six-point bootstrap of [93] and [2] is based on constraints on CFT data obtained by acting with a finite set of derivative functionals on the six-point crossing equation. The rate of convergence of six-point bootstrap bounds as functions of the size of this set depends on the scaling dimension Δ_ϕ of the field whose six-point function is considered. While for small Δ_ϕ fast convergence can be expected, larger and larger sets of derivative functionals need to be considered to still obtain reasonable bounds as Δ_ϕ increases. The analogous convergence problem of the four-point bootstrap was solved in dimensions $d \leq 2$ by replacing derivative functionals with so called “analytic functionals”. For $d > 2$ in the four-point case and any dimension in the six-point case, more work is needed to fully address this point. Chapter 5 discusses this problem in more detail and, based on [3], takes some first steps towards its solution in $d = 1$.

- **Chapter 6: Solving the input data problem & other future directions.**

In the currently published work, the six-point bootstrap requires an undesirable amount of theory specific input data. Beyond limiting its applicability to the cases where this data is available, this also undermines its ability to make theory-agnostic statements about the space of all CFTs. The final chapter, sketches a solution to this input data problem. The proposal involves combining the four- and six-point crossing equation in a unified mixed correlator bootstrap problem. Including the four-point crossing equation into the procedure has the potential to both solve the input data problem and significantly broaden the range of questions that can be addressed via the six-point bootstrap. In addition to this particularly promising direction for future research, Chapter 6 also discusses several other aspects of the multi-point bootstrap that should be addressed in future work.

1.3 Tensionless holography

Over the last three decades, AdS_3 backgrounds⁵, especially at pure NSNS flux, played a pivotal role in advancing our understanding of string theory, quantum gravity and holography. Like all great toy models, they have two key complementary features.

On the one hand, AdS_3 preserves a sufficient amount of complexity to still admit highly non-trivial string dynamics, and thus useful insights into the nature of holography as well as the structure of string partition functions and scattering amplitudes. On the other hand, the relation to the $\text{SL}(2; \mathbb{R})_k$ model [96–98], Virasoro symmetry of the boundary CFT_2 [99] and also general simplifications that come with the topological nature of 3d gravity [100], lead to an unusually high level of computational control that makes AdS_3 an ideal testing ground for new ideas.

Concrete examples for new insights gained due to these remarkable properties of AdS_3 include⁶ progress in our understanding of partition function in quantum gravity [101–104], the structure of the α' expansion [105, 106], integrability in string theory and holography [107–109], H_3^+ -Liouville [110, 111] and quantum analytic Langlands correspondence [112, 113], minimal model holography [114–116], black hole physics [117–121] and entanglement in quantum gravity and holography [122, 123]. In 2018 Eberhardt, Gaberdiel and Gopakumar [68, 69] added a derivation of holography to this already impressive list of results, identifying string theory on $\text{AdS}_3 \times \text{S}_3 \times \mathbb{T}^4$ with

⁵See [94] for a review of pure NSNS AdS_3 and [95] for a more general overview of exact approaches to AdS_3 string theory.

⁶All of these are active fields of research, most of which have by now been fruitfully pursued for about two decades. The particular choice of references given above was made with the intention to reflect this fact by, for each field, guiding the reader to some of the foundational early developments and some of the latest contributions.

minimal pure NSNS flux as the exact dual of a grand canonical ensemble of symmetric product orbifolds of the $\mathcal{N} = (4, 4)$ supersymmetric sigma model with target space $\mathbb{T}^4$.

This instance of the duality provides a rare opportunity to uncover the inner mechanisms of AdS/CFT with simultaneous full computational control over both bulk and boundary dynamics. To make the most out of this opportunity, it is important to generalise the setup and enrich the dynamics as much as possible within the constraint of maintaining computational control. Building on previous work on D-branes and open-string dynamics in this context [124, 125], this thesis constructs the holographic dual of a maximally symmetric class of branes, namely the AdS_2 branes introduced in Ref. [126]. With these results, it is now possible to derive holography for open strings on the level of detail that was recently achieved for closed strings in this setup.

The first chapter of the second half of this thesis serves as a general introduction to and review of pure NSNS AdS_3 string theory. This introduction is followed by two chapters which present a novel construction of holographic interfaces in symmetric product orbifolds. The final chapter is devoted to a discussion of future avenues of research. Concretely, the logical structure of the text can be summarised as follows.

- **Chapter 7: Objectives and motivation.** The second part of the thesis, like the first, starts with a short review. The latter begins with general comments on 2d CFTs and Virasoro symmetry, which sets the stage for a more detailed discussion of the $\text{SL}(2; \mathbb{R})_k$ model and its conformal boundary conditions. As we recall in the second section of the chapter, $\text{SL}(2; \mathbb{R})_k$ plays a crucial role in the worldsheet description of strings propagating on AdS_3 geometries. The current algebra of the model ensures a high level of computational control that can be leveraged to treat pure NSNS AdS_3 backgrounds exactly in α' . This provides a formidable opportunity to study stringy regimes of gravity on curved backgrounds. In particular, as we discuss in the last section of the chapter, the worldsheet theory is exactly solvable even in the regime dual to a free gauge theory. Thus, pure NSNS $\text{AdS}_3/\text{CFT}_2$ provides us with the rare chance to understand and derive holography on a fundamental level. The section ends with comments on what is known about the nature of probe branes in this context, and formulates objectives for how to extend this knowledge in the subsequent chapters.
- **Chapter 8: Interface partition functions.** The second chapter on tensionless $\text{AdS}_3/\text{CFT}_2$ holography is based on Reference [4]. In it, a novel family of interfaces in symmetric product orbifolds is constructed. These may be regarded as two-dimensional analogues of Wilson-line operators or Karch-Randall interfaces at the same time. The construction of the interfaces entails the choice

of boundary conditions of the seed theory. For a generic seed theory, the boundary states associated to the interfaces via the folding trick are constructed, and their overlaps are computed. Through modular transformation, this amounts to a determination of the spectrum of interface changing operators. In the specific case of the supersymmetric four-torus, it is shown that the corresponding interfaces of the symmetric product orbifold are dual to AdS_2 branes in the tensionless limit of type IIB superstring theory on $\text{AdS}_3 \times \text{S}_3 \times \mathbb{T}^4$ by matching the known open string partition function of these branes.

- **Chapter 9: Interface correlators.** With the match of the spectrum provided by the computation in Chapter 8, the natural next step is to extend the duality to correlation functions and string scattering amplitudes. For this purpose, the chapter based on Reference [5] introduces a novel class of generalised covering maps, which provides an alternative characterisation of the interfaces constructed in Chapter 8. The new covering-map description enables efficient evaluation of interface correlators via a generalisation of the Lunin-Mathur method. To organise these computations, a generalised Riemann-Hurwitz formula for interface coverings is derived and novel diagrammatic rules that systematically classify these maps are introduced. The new framework in particular leads to a concrete grand-canonical ensemble that has the correct properties to compute correlation functions dual to open string scattering amplitudes. Utilising the generalised Riemann-Hurwitz formula, it is explicitly shown that the correlators of the ensemble structurally match string perturbation theory to all orders in the string coupling.
- **Chapter 10: Open string scattering amplitudes & other future directions.** The work presented in Chapters 8 and 9 made significant progress on the symmetric product orbifold side of open string holography in the minimal pure NSNS AdS_3 setting. At this stage, it is reasonable to return to the AdS_3 side of the duality for further progress. Compared to the available expressions for closed string scattering amplitudes [127–129], little is known about scattering of open and closed strings in the presence of AdS_2 branes. It would be interesting to fill this gap, and in particular provide a detailed analysis of how the interface correlators computed in Chapter 9 are reproduced in the $k = 1$ limit. The final chapter discusses the prospect of pursuing this and several other natural directions of research in the near future.

Part I

Multi-point conformal bootstrap

Chapter 2

The conformal bootstrap

– Objectives and motivation.

Why are the bounds obtained from the numerical conformal bootstrap important? Why do we need to go beyond the results that the community has already achieved using four-point functions? What concrete goals or outcomes can we expect from a six-point bootstrap?

Before we shift the focus to technical aspects of *how* to extract new bounds from six-point functions, let us in this chapter ask *why* we should pursue this problem and *what* we can expect to eventually gain from this. To set the stage for this discussion, Section 2.1 reviews several fundamental aspects of conformal symmetry. This dramatic personae of CFT consists of a collection of individually titled paragraphs, which can be read both independently of each other and of the sequential Section 2.2. The latter has a more linear structure starting with the basic principles of the bootstrap and ending on the concrete expectations that motivate the six-point bootstrap.

2.1 Global conformal symmetry

This section reviews foundations of CFT. We start with a definition of the conformal group and some of its representations, Sec. 2.1.1. The second subsection, Sec. 2.1.2, describes features of CFT Hilbert spaces, such as the operator-state correspondence and unitarity bounds. Finally, Sec. 2.1.3, introduces the central objects studied over the next four chapters: conformally invariant correlation functions and their crossing equations. For an introduction to conformal field theory that is particularly tailored towards the modern conformal bootstrap, see e.g. References [130–132]

2.1.1 The conformal group

The mathematics of the conformal group and the representations of its Lie algebra is a beautiful subject with profound physical implications. Scratching only the very surface of this vast topic, this section introduces the group itself and a few of its most prominent representations. A systematic and more detailed introduction is provided e.g. in the first chapters of Schottenloher's textbook [133] and the references therein.

A conformal map of a Riemannian manifold is a diffeomorphism that preserves the metric up to a, possibly position dependent, rescaling. We are mainly interested in field theories defined on the conformal compactification of Euclidean $\mathbb{R}^d$, which we construct in the first paragraph of the section. Consequently, we refer to its group of conformal maps, described in the second paragraph, simply as *the* conformal group. We end on a brief discussion of some important representations of its Lie algebra.

Conformal compactification of $\mathbb{R}^d$. The conformal compactification of $\mathbb{R}^d$ is most conveniently described by utilising the embedding¹

$$X : \mathbb{R}^d \rightarrow \mathbb{RP}^{1,d+1}, \quad (x_1, \dots, x_d) \mapsto X(x) := \left[\frac{1+x^2}{2} : x_1, \dots, x_d, \frac{1-x^2}{2} \right], \quad (2.1.1)$$

where

$$x^2 := \sum_{i=1}^d x_i^2 \quad (2.1.2)$$

is the squared Euclidean norm of x . The closure of $X(\mathbb{R}^d)$ is the projective null cone

$$\{[X_0 : \dots : X_{d+1}] \in \mathbb{RP}^{1,d+1} \mid X^2 = \sum_{i=1}^{d+1} X_i^2 - X_0^2 = 0\}. \quad (2.1.3)$$

By slight abuse of notation, we shall denote this *conformal compactification of flat d dimensional Euclidean space* simply by $\mathbb{R}^d$. This is justified by the fact that we will exclusively study theories defined on the compactified space.

The conformal group. The conformal group i.e. the group of conformal transformations of $\mathbb{R}^d$ is isomorphic to $SO(1, d+1)$. An explicit isomorphism is the map that identifies the matrix $\Lambda \in SO(1, d+1)$ with the conformal map ψ_Λ defined by

$$\psi_\Lambda([X]) = [\Lambda X]. \quad (2.1.4)$$

The group is generated by translations, rotations, dilatations and special conformal transformations, which are defined as

¹The embedding space formalism [134] is typically attributed to Paul Dirac [135].

- **Translations.** A translation by the vector v i.e. the map $x \mapsto x + v$ is realised by the matrix

$$\Lambda_v = \begin{pmatrix} 1 + \frac{v^2}{2} & v^T & \frac{v^2}{2} \\ v & \mathbb{1}_d & v \\ -\frac{v^2}{2} & -v^T & 1 - \frac{v^2}{2} \end{pmatrix} \quad (2.1.5)$$

- **Rotations.** A rotation $R \in SO(d)$, mapping $x \mapsto Rx$ is realised by the matrix

$$\Lambda_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.1.6)$$

- **Dilatations.** Constant scaling $x \mapsto \lambda x$ by a factor $\lambda \in \mathbb{R}_{>0}$ is realised by the matrix

$$\Lambda_\lambda = \begin{pmatrix} \frac{1+\lambda^2}{2\lambda} & 0 & \frac{1-\lambda^2}{2\lambda} \\ 0 & \mathbb{1}_d & 0 \\ \frac{1-\lambda^2}{2\lambda} & 0 & \frac{1+\lambda^2}{2\lambda} \end{pmatrix} \quad (2.1.7)$$

- **Special conformal transformations.** The special conformal transformation associated to $v \in \mathbb{R}^d$ is obtained by conjugating a translation by v with the inversion $x \mapsto -\frac{x}{x^2}$. It is realised by the matrix

$$\Lambda'_v = \begin{pmatrix} 1 + \frac{v^2}{2} & v^T & -\frac{v^2}{2} \\ v & \mathbb{1}_d & -v \\ \frac{v^2}{2} & v^T & 1 - \frac{v^2}{2} \end{pmatrix} \quad (2.1.8)$$

The conformal algebra. The discussion of the previous paragraph directly suggests a convenient basis for the conformal algebra $\mathfrak{so}(1, d+1)$ consisting of infinitesimal versions of the transformations introduced in eqs. (2.1.5), (2.1.6), (2.1.7) and (2.1.8). We denote the basis elements by

- D , generating the one-parameter family of dilatations.
- P_μ , $\mu = 1, \dots, d$ generating the d translations.
- K_μ , $\mu = 1, \dots, d$ generating the d special conformal transformations.
- $R_{\mu\nu} = -R_{\nu\mu}$, generating the $d(d-1)/2$ rotations.

The non-zero commutation relations involving the dilatation generator are

$$[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu \quad \text{and} \quad [P_\mu, K_\mu] = 2\delta_{\mu\nu}D - 2R_{\mu\nu}. \quad (2.1.9)$$

Furthermore, we have

$$[R_{\mu\nu}, P_\rho] = \delta_{\rho\nu}P_\mu - \delta_{\rho\mu}P_\nu \quad \text{and} \quad [R_{\mu\nu}, K_\rho] = \delta_{\rho\nu}K_\mu - \delta_{\rho\mu}K_\nu \quad (2.1.10)$$

which simply states that translation and special conformal generators transform as vectors, i.e. in the fundamental representation, under rotations. Finally, the rotation generators satisfy the $\mathfrak{so}(d)$ commutation relations

$$[R_{\mu\nu}, R_{\rho\sigma}] = \delta_{\nu\rho}R_{\mu\sigma} - \delta_{\mu\rho}R_{\nu\sigma} + \delta_{\nu\sigma}R_{\rho\mu} - \delta_{\mu\sigma}R_{\rho\nu}. \quad (2.1.11)$$

Lowest weight representations. Every representation V_ρ of $\mathfrak{so}(d)$ induces a family of lowest weight representations $\mathcal{V}_{(\Delta, \rho)}$ of the conformal algebra $\mathfrak{so}(1, d+1)$. The parameter $\Delta \in \mathbb{C}$ is the *conformal weight* of the representation. To construct $\mathcal{V}_{(\Delta, \rho)}$, we first extend V_ρ to a representation of the full conformal algebra by letting all non-rotation generators act trivially. We then consider the tensor product $U(\mathfrak{so}(1, d+1)) \otimes V_\rho^\Delta$ between the universal enveloping algebra of $\mathfrak{so}(1, d+1)$ and the extended representation V_ρ^Δ . Let us use bracket notation for elements of V_ρ and furthermore denote the tensor product of $X \in U(\mathfrak{so}(1, d+1))$ and $|v\rangle \in V_\rho$ simply by

$$X|v\rangle := X \otimes |v\rangle. \quad (2.1.12)$$

The representation $\mathcal{V}_{(\Delta, \rho)}$ is the quotient of the product representation modulo the subspace spanned by elements of the form

$$X(D - \Delta)|v\rangle \quad XK_\mu|v\rangle \quad \text{and} \quad XR_{\mu\nu}|v\rangle - X\rho(R_{\mu\nu})|v\rangle. \quad (2.1.13)$$

If V_ρ is indecomposable then so is $\mathcal{V}_{(\Delta, \rho)}$, independent of the choice of Δ , since acting with translation generators on $V_\rho \subseteq \mathcal{V}_{(\Delta, \rho)}$ generates all of $\mathcal{V}_{(\Delta, \rho)}$. Irreducibility of V_ρ , however, does not generally imply irreducibility of $\mathcal{V}_{(\Delta, \rho)}$. As a simple counter-example, consider the case where $d = 1$, $\Delta = 0$ and $V_\rho = \{|0\rangle\}$ is the trivial representation, which has the set of all vectors of the form $XP|0\rangle$ as non-trivial subrepresentation.

2.1.2 The Hilbert space

Conformal symmetry not only organises the states of CFTs into representations of the conformal algebra, but also leads to a one-to-one map between these states and the local operators of the theory. The ideal setup to understand this identification is *radial quantisation*, which we review in the first paragraph of this section. With this preparation, the *state-operator correspondence* is concisely formulated in the second paragraph. The third paragraph introduces primary operators and their behaviour under conformal transformations, which is then applied in the fourth paragraph to deduce how primaries transform under radial inversion. The latter plays a central role

in the definition of the inner product of the Hilbert space. The final paragraph discusses positivity constraints on this inner product, and a fortiori the CFT spectrum, which are imposed by demanding the existence of a unitary analytic continuation to lorentzian signature.

The vacuum of radial quantisation. In a path integral formulation of euclidean scale invariant QFTs, it is natural to work with a quantisation associated to foliating $\mathbb{R}^d$ by $d - 1$ spheres centered in the origin. States in the Hilbert space of this radial quantisation specify path integral boundary conditions imposed at the unit sphere. We can prepare any state via a suitable path integral over the interior of the unit sphere i.e. the unit ball. In particular, if there is a unique state $|0\rangle$ with vanishing conformal weight i.e. a state annihilated by the dilatation operator D , and if furthermore all other eigenvalues of D have a positive real part, then this vacuum state $|0\rangle$ can be prepared by performing an unconstrained path integral over the unit sphere. This fact is best understood by conformally mapping $\mathbb{R}^d$ to the cylinder $\mathbb{R} \times \mathbb{S}^{d-1}$, an operation that is achieved in polar coordinates by mapping $(r, \Omega) \in \mathbb{R}^d$ to $(\tau, \Omega) \in \mathbb{R} \times \mathbb{S}^{d-1}$. Here, the euclidean time τ is given by

$$\tau = \log r. \quad (2.1.14)$$

Once we have mapped to the cylinder, scaling transformations simply act as translations in euclidean time. Consequently, the dilatation operator is the Hamiltonian of radial quantisation on the cylinder. Hence, the statement that the vacuum of radial quantisation is prepared by a path integral over the unit ball is simply a particular case of the more general fact that propagation over an infinite stretch of euclidean time projects onto the vacuum state of a QFT. Below, we will always assume the existence of a unique, conformally invariant, radial quantisation vacuum state.

Operator-state correspondence. In any QFT, we can prepare a state $|\mathcal{O}\rangle$ on the unit sphere by inserting a local operator $\mathcal{O}$ at the center of the unit ball or, said more algebraically, by acting with $\mathcal{O}(0)$ on the vacuum state. In scale invariant theories, this procedure can be inverted leading to a bijection between local operators and the states of the radial quantisation Hilbert space. Indeed, using scaling transformations, we can collapse the sphere on which a state $|\psi\rangle$ is defined to a point and hence expand $|\psi\rangle$ into local operators inserted at the origin. As an example for this procedure, consider the vacuum state $|0\rangle$. Since this state is prepared by performing an unconstrained path integral over the unit ball, we are left with a trivial puncture after shrinking the sphere down to a point. The insertion of the operator corresponding to the vacuum state therefore has a trivial effect on any correlation function, and we conclude that it is the identity operator $\mathbb{1}$.

Primaries and descendants. The Hilbert space of a CFT is spanned by primary states, states annihilated by the special conformal generators, and their descendants, states obtained by acting on primaries with the momentum generators. A primary operator is an operator that is identified with a primary state via the state-operator-correspondence. Using the fact that the operators P^μ generate translations, i.e.

$$\mathcal{O}(x) = e^{x \cdot P} \mathcal{O}(0) e^{-x \cdot P}, \quad (2.1.15)$$

and the commutation relations of the conformal algebra, one can derive the transformation behaviour

$$[P_\mu, \mathcal{O}(x)] = \partial_\mu \mathcal{O}(x) \quad (2.1.16)$$

$$[D, \mathcal{O}(x)] = (x^\mu \partial_\mu + \Delta_{\mathcal{O}}) \mathcal{O}(x) \quad (2.1.17)$$

$$[R_{\mu\nu}, \mathcal{O}(x)] = (x_\nu \partial_\mu - x_\mu \partial_\nu + \rho(R_{\mu\nu})) \mathcal{O}(x) \quad (2.1.18)$$

$$[K_\mu, \mathcal{O}(x)] = (2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu + 2\Delta_{\mathcal{O}} x_\mu - 2x^\nu \rho(R_{\mu\nu})) \mathcal{O}(x) \quad (2.1.19)$$

for a weight $\Delta_{\mathcal{O}}$ primary operator in the $\mathfrak{so}(d)$ representation ρ . Exponentiating the above infinitesimal conformal transformations gives

$$f \cdot \mathcal{O}(x) = \Omega_f(x)^{\Delta_{\mathcal{O}}} \rho(R_f(x)) \mathcal{O}(f(x)) \quad (2.1.20)$$

for the action of a finite conformal transformation f whose local decomposition into a rescaling $\Omega_f(x) \in \mathbb{R}_{>0}$ and a rotation $R_f(x) \in SO(d)$ is

$$\partial f(x) = \Omega_f(x) R_f(x). \quad (2.1.21)$$

Radial inversion. Considering states in radial quantisation to be prepared by path integrals over the unit sphere leads to a natural inner product: Using the radial inversion

$$x^\mu \rightarrow \bar{x}^\mu := -\frac{x^\mu}{|x|}, \quad (2.1.22)$$

which acts as an operator R on the Hilbert space, we can map the path integral preparing a state $|\psi\rangle$ onto a path integral over the complement of the unit sphere, preparing the state $R|\psi\rangle$. The latter can be glued together with path integrals over the unit sphere to produce path integrals over the entirety of $\mathbb{R}^d$, allowing us to define a pairing $(|\psi\rangle, |\psi'\rangle)$ of $|\psi\rangle$ with an arbitrary state $|\psi'\rangle$. The inner product is obtained from this pairing by applying an additional complex conjugation $\mathfrak{C}|\psi\rangle = |\psi\rangle^*$. That is, $\langle\psi|\psi'\rangle = (\mathfrak{C}|\psi\rangle, |\psi'\rangle)$. Therefore, in radial quantisation, the adjoint of a local operator is given by

$$\mathcal{O}(x)^\dagger = \mathfrak{C} R \mathcal{O}(x) R \mathfrak{C}, \quad (2.1.23)$$

which, by eq. (2.1.20), reduces to

$$\mathcal{O}(x)^\dagger = x^{-2\Delta_{\mathcal{O}}} \mathfrak{C} \mathcal{O}(\bar{x}) \mathfrak{C} \quad (2.1.24)$$

for a primary operator of weight $\Delta_{\mathcal{O}}$.

Hermiticity and reality. Every operator $\mathcal{O}(x)$ can be uniquely decomposed as

$$\mathcal{O}(x) = \mathcal{O}_R(x) + i\mathcal{O}_I(x) \quad (2.1.25)$$

into a sum of real operators $\mathcal{O}_R(x)$ and $\mathcal{O}_I(x)$. A real operator is an operator that after Wick rotation to Lorentzian signature becomes Hermitian. As such, the notion of a reality depends on how we perform the analytic continuation and might in general not coincide with the hermiticity w.r.t. the inner product of the euclidean CFT.

Reflection positivity. We are particularly interested in Euclidean CFTs that upon Wick rotation define unitary Lorentzian theories. A necessary condition for this is reflection positivity, i.e. positivity of the inner product defined via radial inversion. The representations of the conformal algebra that the Hilbert space of a reflection positive CFT can carry are strongly constrained. A simple example for these unitarity bounds is that the dilatation operator must be positive semi-definite: the scaling dimension of any operator must be a non-negative real number. Indeed, for any primary state $|\mathcal{O}\rangle$, we get

$$\langle \mathcal{O}|D|\mathcal{O}\rangle = \frac{1}{2}\langle \mathcal{O}|[K_\mu, P_\mu]|\mathcal{O}\rangle = \frac{1}{2}\|P_\mu|\mathcal{O}\rangle\|^2 \geq 0. \quad (2.1.26)$$

By a slightly refined version of this argument, the bound can be strengthened to conclude that scalar operators in a d dimensional CFT have either vanishing conformal weight or satisfy

$$\Delta \geq \frac{d-2}{2}, \quad (2.1.27)$$

while for spin $J > 1$ operators, the even stronger bound

$$\Delta \geq J + d - 2 \quad (2.1.28)$$

holds. Similar restrictions exist for the quantum numbers of more general mixed-symmetry tensors.

2.1.3 CFT correlators

This section introduces the protagonists of the conformal bootstrap: Local correlators and their crossing equations. The key tool with which we study CFT correlators, the OPE, is discussed in the first paragraph. The OPE allows us to reduce all local correlators on $\mathbb{R}^d$ down to one-point functions. The latter are, as we recall in the second paragraph, constant and, for operators with non-vanishing conformal weight, vanishing. Building on this result we describe two-, three- and finally n -point correlators via the OPE. For $n > 3$ point correlators, the OPE reduction can be

performed in various inequivalent ways, which leads to strong constraints on the CFT data – the crossing equations. The section ends with only a brief comment on this fact, providing a smooth transition to a more detailed discussion of the consequences of crossing in Sec. 2.2.

The OPE. Consider two local operators $\mathcal{O}_1$ and $\mathcal{O}_2$ inserted at positions x_1 and x_2 . If we can find a sphere, centered in a third point x_3 , that contains x_1 and x_2 without including any further operator insertions, then shrinking this sphere down to a point by using scale invariance allows us to expand the product $\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)$ in a basis of local operators inserted at x_3 . As an equation, this Operator Product Expansion (OPE) takes the form

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_{\mathcal{O}_3} \mathbf{C}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}(x_1, x_2)\mathcal{O}_3(x_3), \quad (2.1.29)$$

where the sum runs over a basis of all local operators, both primaries and their descendants. Since by definition any descendant can be generated by acting with derivatives on a primary operator, we can restrict the sum to a basis of primaries, if we furthermore promote the coefficients $\mathbf{C}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$ to differential operators. The resulting equation

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_{\text{primary } \mathcal{O}_3} \mathbf{C}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}(x_1, x_2, \partial_{x_3})\mathcal{O}_3(x_3), \quad (2.1.30)$$

can only make sense if the differential operator $\mathbf{C}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$ is an intertwiner relating the action of conformal generators on $\mathcal{O}_3(x_3)$ to their action on the product $\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)$. Hence the object $\mathbf{C}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$ contains a combination of both general conformal kinematics and theory specific dynamical data. To improve this situation, we can first solve the purely representation theoretical problem of constructing a basis of appropriate intertwiners $\mathcal{D}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}^{(n)}$ and then define the dynamical coefficients $C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}^{(n)}$ by an expansion

$$\mathbf{C}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}(x_1, x_2, \partial_{x_3}) = \sum_n C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}^{(n)} \mathcal{D}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}^{(n)}(x_1, x_2, \partial_{x_3}). \quad (2.1.31)$$

Determining these OPE coefficients $C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}^{(n)}$ is a key aspect of solving a CFT. Together with the spectrum they determine all local correlation functions of a CFT on $\mathbb{R}^d$.

One-point functions. Translation invariance implies that one-point functions $\langle \mathcal{O}(x) \rangle$ must be constant. Therefore, only weight 0 operators can have a non-vanishing one-point function in a scale invariant theory on $\mathbb{R}^d$. Hence, the only non-vanishing one-point function is

$$\langle \mathbb{1} \rangle = 1. \quad (2.1.32)$$

Two-point functions. Since the only non-trivial one-point function is that of the identity operator, the OPE directly implies

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle = \sum_n C_{\mathcal{O}_1 \mathcal{O}_2 \mathbb{1}}^{(n)}(x_1, x_2, \partial_{x_3}) \mathcal{D}_{\mathcal{O}_1 \mathcal{O}_2 \mathbb{1}}^{(n)}(x_1, x_2, \partial_{x_3}) \langle \mathbb{1} \rangle. \quad (2.1.33)$$

Hence, by Schur's Lemma, the two-point function can only be non-vanishing if the irreducible representations of the conformal algebra associated to $\mathcal{O}_1$ and $\mathcal{O}_2$ are dual to each other i.e. $\mathcal{O}_2 \simeq \mathcal{O}_1^*$. Furthermore, Schur's Lemma also implies that the space of relevant intertwiners is one-dimensional. Since derivatives act trivially on $\langle \mathbb{1} \rangle = 1$, the operator $\mathcal{D}_{\mathcal{O}_1 \mathcal{O}_2 \mathbb{1}}$ moreover does not depend on ∂_{x_3} . Using translation and scale invariance, we can finally deduce that it generally takes the form

$$\mathcal{D}_{\mathcal{O}_1 \mathcal{O}_2 \mathbb{1}}(x_1, x_2, \partial_{x_3}) = \frac{T_{\mathcal{O}_1}(\hat{x}_{12})}{x_{12}^{2\Delta}}, \quad (2.1.34)$$

where

$$x_{ij} := x_i - x_j \quad \text{and} \quad \hat{x} := \frac{x}{|x|}. \quad (2.1.35)$$

The precise form of the tensor structure $T_{\mathcal{O}_1}$ depends on the transformation behaviour of $\mathcal{O}_1$ under rotations. In the case of scalar operators, we can simply set $T_{\mathcal{O}_1} \equiv 1$. In conclusion, the two-point function can be written as

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle = C_{\mathcal{O}_1 \mathcal{O}_2 \mathbb{1}} \frac{T_{\mathcal{O}_1}(\hat{x}_{12})}{x_{12}^{2\Delta}}. \quad (2.1.36)$$

We will mostly work with operators normalised such that

$$C_{\mathcal{O}_1 \mathcal{O}_2 \mathbb{1}} = \delta_{\mathcal{O}_1, \mathcal{O}_2^*}. \quad (2.1.37)$$

An important exception to this general rule are conserved currents and the stress tensor, whose normalisation we instead fix via their Ward identities.

Three-point functions. Due to the diagonality of the two-point function, three-point correlators are morally identical with OPE coefficients. Indeed, the OPE implies

$$C_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3}(x_1, x_2, \partial_{x_3}) = \sum_n C_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3^*}^{(n)} \mathcal{D}_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3^*}^{(n)}(x_1, x_2, \partial_{x_2}) \frac{T_{\mathcal{O}_3}(\hat{x}_{23})}{x_{23}^{2\Delta_3}} \quad (2.1.38)$$

for operators normalised as in eq. (2.1.37). Scale and translation invariance allow us to deduce that

$$\mathcal{D}_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3^*}^{(n)}(x_1, x_2, \partial_{x_2}) \frac{T_{\mathcal{O}_3}(\hat{x}_{23})}{x_{23}^{-2\Delta_3}} = \frac{T_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3}^n(\hat{x}_{12}, \hat{x}_{23}, \hat{x}_{13})}{x_{12}^{\Delta_1 + \Delta_2 - \Delta_3} x_{13}^{\Delta_1 + \Delta_3 - \Delta_2} x_{23}^{\Delta_2 + \Delta_3 - \Delta_1}} \quad (2.1.39)$$

where $T_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3}^n$ is an invariant tensor in the $\mathfrak{so}(d)$ representation $\rho_1 \otimes \rho_2 \otimes \rho_3$.

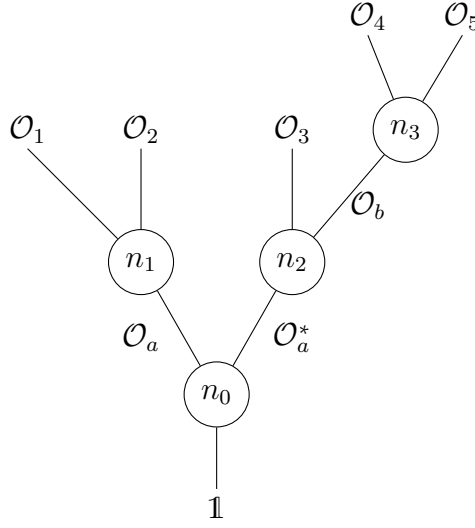


Figure 2.1: This figure shows the tree corresponding to the five-point conformal block $\mathcal{D}_{\mathcal{O}_4\mathcal{O}_5\mathcal{O}_b}(x_4, x_5, \partial_{x_4})\mathcal{D}_{\mathcal{O}_3\mathcal{O}_b\mathcal{O}_a^*}(x_3, x_4, \partial_{x_3})\mathcal{D}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_a}(x_1, x_2, \partial_{x_2})\mathcal{D}_{\mathcal{O}_a\mathcal{O}_a^*\mathbb{1}}(x_2, x_3)(\mathbb{1})$.

Defining conformal blocks. By applying successive OPEs, we can reduce any correlation function of local operators to a sum of kinematically fixed special functions weighted by OPE coefficients. The data specifying these conformal blocks can be efficiently represented via a rooted tree that specifies which intertwiners need to be applied in which order to the one-point function of the identity operator to generate the block.

In such a graph, the root corresponds to the identity one-point function, while the leaves are labelled with the external operators i.e. the operators to whose correlation function the block contributes. All edges connecting internal vertices are labelled with exchanged operators i.e. intermediate operators produced in the OPE. Furthermore, all internal vertices have valency three and are labelled by intertwiners of the representations associated to their adjacent edges. At each internal vertex, we should think of the operators labeling the two edges that are further up the tree as producing the third, further down operator in their OPE. Figure 2.1 shows a concrete example of a five-point block.

The graphical representation in terms of a directed tree carries redundant information that does not describe the block per se, but the particular way in which we constructed it. The orientation of the tree reflects a choice among canonically isomorphic ways to interpret the intertwiners at the vertices. By using dualisation, we can cycle through all the equivalent options of viewing $\mathcal{D}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}^n$ as either an intertwiner mapping from $\mathcal{O}_3$ to $\mathcal{O}_1 \otimes \mathcal{O}_2$, an invariant tensor in $\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3^*$ etc. In particular, the vertex connected to the root of the tree is completely redundant and can just be seen as the identity map via dualisation. To accommodate for these

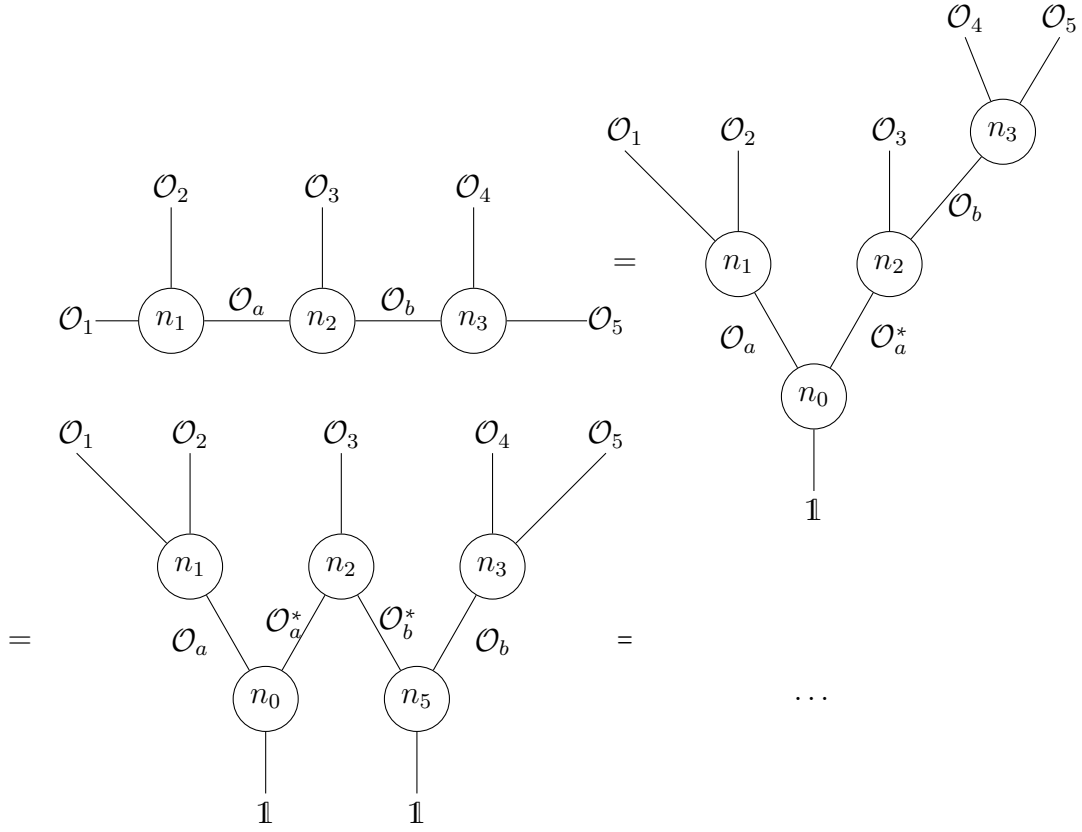


Figure 2.2: The reduced and two redundant representations of a five-point block.

facts, we should both drop the root and the direction of the tree, to obtain a reduced graphical representation of the block. As a concrete example, the reduced version of Figure 2.1 is shown in Figure 2.2.

Computing conformal blocks. The literature on the problem of computing conformal blocks is vast, and only a relatively small amount of conformal block technology is directly relevant for the multi-point bootstrap discussed in the chapters below.

- *Casimir equations.* The definition of conformal blocks in terms of the action of intertwiners given above, directly makes manifest that the Casimirs of exchanged operators can be measured by acting with suitable conformal generators on the external operators. Therefore conformal blocks can be characterised as eigenfunctions of certain differential operators. Supplemented with appropriate boundary conditions, that in particular input the a priori arbitrary choice of tensor structures, the resulting Casimir equations fully fix the blocks. In some cases, the differential equations can be solved explicitly, leading to expressions for conformal blocks in terms of hypergeometric functions, see in particular the

pioneering work of Dolan and Osborne [136–138]. In an extension of this PDE approach to conformal blocks, References [139–142] introduced additional operators commuting with the Casimirs which lead to a canonical basis of tensor structures, see also the three theses [143–145].

- *Projectors.* Another way to compute conformal blocks starts by choosing, for each primary operator $\mathcal{O}$, an orthonormal basis $(\mathcal{O}^{(i)})_{i \in I}$ of descendants. This allows us to write down the projector onto the conformal multiplet generated by $\mathcal{O}$ as²

$$\pi_{\mathcal{O}} = \sum_{i \in I} |\mathcal{O}^{(i)}\rangle \langle \mathcal{O}^{(i)}|. \quad (2.1.40)$$

The identity map of the CFT Hilbert space can be written as a sum over all such projectors. Inserting this decomposition allows us to decompose an arbitrary n -point function into a product of three-point functions. Upon expanding these three-point functions into a basis of tensor structures, this decomposition into three-point functions becomes a conformal block expansion of the original n -point correlator. In this way, the projector $\pi_{\mathcal{O}}$ can be used to compute conformal blocks. Note that, in principle, we are not forced to fully decompose a correlator into three-point functions and hence conformal blocks, but can also use the projectors to form other decompositions of higher- into lower-point functions. This observation is crucial for the six-point bootstrap that we develop below.

- *Zamolodchikov recursion relations.* The numerically most efficient³ way to evaluate conformal blocks is to use Zamolodchikov recursion relations⁴. The implementation of Zamolodchikov recursion relations in the efficient package `blocks_3d` [154] is a key part of the backbone of the modern numerical bootstrap.

The crossing equation. The conformal block decomposition of correlation functions is far from unique. A given correlator can be decomposed into various different channels, see Figure 2.3. To give a concrete example, let us focus on the special case of Figure 2.3, where $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}_3 = \mathcal{O}_4 = \phi$ is a real scalar field. The OPE of a real scalar field with itself only contains real symmetric traceless tensors (STTs).

²A for many purposes more useful representation of the projectors is that in terms of shadow integrals, see Refs. [146–149] for the historical origin, and e.g. Refs. [150, 151] for modern developments.

³This is true for applications in which it is crucial to evaluate families of blocks for a large range of different exchanged dimensions. Evaluating just an individual block to high precision might more efficiently be achieved using other methods.

⁴First proposed by Alexei Zamolodchikov in the context of 2d CFTs [152, 153]

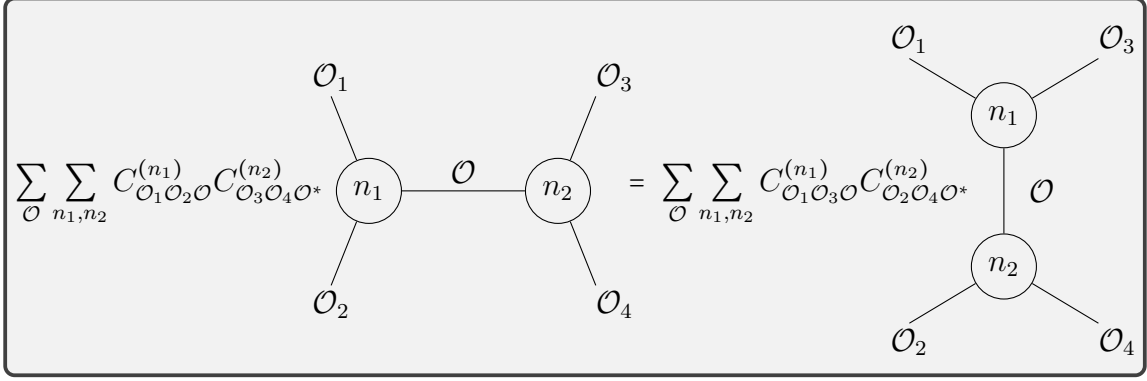


Figure 2.3: Crossing equation for the four-point function $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle$

These are labeled by two quantum numbers, their scaling dimension Δ and their spin J . Moreover, the space of conformally invariant three-point structures of the type $\langle \phi \phi \mathcal{O}_{\Delta, J} \rangle$ is one dimensional. Hence, the tensor structure labels can be dropped for four-point blocks of scalar fields and the crossing equation takes the simple form

$$\sum_{\Delta, J} C_{\Delta, J} G_{\Delta, J}(x_1, x_2, x_3, x_4) = \sum_{\Delta, J} C_{\Delta, J} G_{\Delta, J}(x_2, x_3, x_4, x_1), \quad (2.1.41)$$

where $C_{\Delta, J} = C_{\phi \phi \mathcal{O}_{\Delta, J}}^2$. The conformal block $G_{\Delta, J}$ can be reduced to a function $g_{\Delta, J}$ of two cross-ratios:

$$G_{\Delta, J}(x_1, x_2, x_3, x_4) = \frac{g_{\Delta, J}(z, \bar{z})}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}. \quad (2.1.42)$$

Here, the cross-ratios z and $\bar{z}$ are defined via the two equations

$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \quad \text{and} \quad (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \quad (2.1.43)$$

The reduction to cross-ratios allows us to reformulate the crossing equation as

$$\sum C_{\Delta, J} F_{\Delta, J}(z, \bar{z}) = 0, \quad (2.1.44)$$

where $F_{\Delta, J}(z, \bar{z})$ is the crossing vector

$$F_{\Delta, J}(z, \bar{z}) = g_{\Delta, J}(z, \bar{z}) - \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})} \right)^{\Delta_\phi} g_{\Delta, J}(1-z, 1-\bar{z}). \quad (2.1.45)$$

In even dimensions, $g_{\Delta, J}$ can be written down explicitly in terms of Gauss hypergeometric functions. For instance,

$$g_{\Delta, J}^{d=4}(z, \bar{z}) = \frac{1}{(-2)^J} \frac{z\bar{z}}{z-\bar{z}} (g_{\Delta+J}^{d=1}(z) g_{\Delta-J-2}^{d=1}(\bar{z}) - g_{\Delta+J}^{d=1}(\bar{z}) g_{\Delta-J-2}^{d=1}(z)), \quad (2.1.46)$$

where $g_{\Delta}^{d=1}$ is the $\text{SL}(2; \mathbb{R})$ block

$$g_{2h}^{d=1}(z) = z^h {}_2F_1(h, h, 2h, z). \quad (2.1.47)$$

Crossing equations like (2.1.44) impose highly non-trivial constraints on CFT data. The agenda of the conformal bootstrap is to systematically exploit these constraints in order to solve CFTs.

2.2 The numerical bootstrap

The numerical bootstrap non-perturbatively solves CFTs by exploiting the constraints on their correlation functions imposed by unitarity and the consistency of the OPE, defining individual theories by symmetries and special features of their spectrum rather than via an explicit Lagrangian. In plain words, the conformal bootstrap is the solution theory of crossing equations. Since exact solutions are usually out of reach, various techniques to construct approximations to crossing symmetric correlation functions were developed by the bootstrap community. These primal approaches to crossing are reviewed in Section 2.2.1.

Primal techniques suffer from various issues. On the one hand, they typically come with unknown errors that need to heuristically be estimated. On the other hand, they do not allow for a systematic study of the whole solution space but only produce a potentially biased selection of special points in it. In the semi-definite programming based dual approach to crossing, these issues are solved by shifting the focus to rigorous bounds on the space of all solutions. The dual approach to crossing is reviewed in Section 2.2.2. Prior to the work presented in this thesis, dual approaches were only available for four-point crossing equations. The final section of this chapter, Section 2.2.3, motivates the need to push existing techniques beyond the status quo and unlock the constraining power and CFT data of higher-point functions.

2.2.1 The primal approach to crossing

The most natural first impulse when encountering an interesting system of equations is to construct simple examples of explicit solutions. If successful, this almost always provides valuable insights that can guide the development of a general solution theory and methods to study more generic situations where a fully explicit solution is out of reach. The crossing equation is no exception to this general rule. In this section, we trace back the bootstrap to its origin, the explicit solution of crossing for analytically tractable 2d CFTs in the 80s and 90s. We then discuss various modern primal approaches to crossing, which produce approximate CFT data by numerically solving truncations of the equations in the generic case where the problem becomes analytically intractable.

Minimal models. After its formulation in the 70s [57, 58], the bootstrap dream became reality with the solution of the minimal models in 1984 [155–157]. In these 2d CFTs, Virasoro symmetry combines infinitely many global conformal primaries into larger conformal multiplets generated by Virasoro primaries. Rational theories, such as the minimal models of Belavin, Polyakov and Zamolodchikov, only contain finitely many of the latter. For example, the 2d Ising CFT, which is the $\mathcal{M}(4, 3)$ minimal model, has three Virasoro primaries, the identity $\mathbb{1}$, the energy operator ϵ and the spin σ . The conformal weights of these finitely many Virasoro primaries are completely fixed by the representation-theoretical construction of minimal models as

$$h_{\mathbb{1}} = \bar{h}_{\mathbb{1}} = 0, \quad h_{\sigma} = \bar{h}_{\sigma} = \frac{1}{16} \quad \text{and} \quad h_{\epsilon} = \bar{h}_{\epsilon} = \frac{1}{2}. \quad (2.2.1)$$

Furthermore, representation theory puts additional constraints on the OPE of minimal models that allow us to deduce that only a restricted subset of (the already finite set) of all three-point correlators of Virasoro primaries is non-vanishing. In the case of the Ising model, these fusion rules are

$$\begin{aligned} [\mathbb{1}] \times [\mathbb{1}] &= [\mathbb{1}] & [\mathbb{1}] \times [\sigma] &= [\sigma] & [\mathbb{1}] \times [\epsilon] &= [\epsilon] \\ [\epsilon] \times [\epsilon] &= [\mathbb{1}] & [\sigma] \times [\sigma] &= [\mathbb{1}] + [\epsilon] & [\sigma] \times [\epsilon] &= [\sigma]. \end{aligned} \quad (2.2.2)$$

In this context, the crossing equations include only finitely many unknowns, the OPE coefficients of the Virasoro primaries. For the Ising model, the fusion rules (2.2.2) in fact tell us that there is only a single non-trivial OPE coefficient, $C_{\sigma\sigma\epsilon}$. Since they are equations of finitely many unknowns, the crossing relations of minimal models can straightforwardly be solved to high numerical precision, and with some effort even lead to exact analytic expressions, such as $C_{\sigma\sigma\epsilon} = \frac{1}{2}$. In this way, the bootstrap exactly solves minimal models.

Beyond rational CFTs. The minimal models were an inspiring first success of the bootstrap, which however relied heavily on the drastic simplifications that come with rationality. In the 90s and early 2000s another important milestone was reached when with Liouville theory [158, 159], and its conformal boundaries [160], the first non-rational theories were successfully bootstrapped. However, these analytic results again relied on 2d specific infinite conformal symmetry and on exploiting simplifications in the OPE that occur for degenerate representations of the Virasoro algebra (“the Teshner trick”). Techniques to directly solve the crossing equation in theories that only possess global conformal symmetry were not systematically developed until the dual approach to crossing, which will be discussed in Section 2.2.2, led to a revival of the bootstrap idea in the 2010s.

Gliozzi’s method. In general, crossing equations are analytically intractable functional eqs. with infinitely many unknowns. As a concrete example, let us again focus

on the four-point crossing equation (2.1.44) for a correlation function of identical real scalars, which by singling out the known identity contribution can be formulated as

$$F_{0,0}(z, \bar{z}) + \sum_{\Delta, J} C_{\Delta, J} F_{\Delta, J}(z, \bar{z}) = 0. \quad (2.2.3)$$

This is a non-linear equation for the infinitely many pairs of unknown scaling dimensions and spins (Δ, J) , which however is linear in the squared OPE coefficients $C_{\Delta, J} = C_{\phi\phi\mathcal{O}_{\Delta, J}}^2$. The kinematically fixed crossing vector $F_{\Delta, J}(z, \bar{z})$ is given in eq. (2.1.45). Dividing by $F_{0,0}$ and Taylor expanding around $z = \bar{z} = \frac{1}{2}$, we can rewrite this functional equation as a system of one inhomogeneous equation

$$-\sum_{\Delta, J} C_{\Delta, J} \frac{F_{\Delta, J}(z, \bar{z})}{F_{0,0}(z, \bar{z})} \Big|_{z=\bar{z}=1/2} = 1 \quad (2.2.4)$$

and infinitely many homogeneous equations

$$-\sum_{\Delta, J} C_{\Delta, J} \partial_z^n \partial_{\bar{z}}^m \frac{F_{\Delta, J}(z, \bar{z})}{F_{0,0}(z, \bar{z})} \Big|_{z=\bar{z}=1/2} = 0, \quad \text{for all } (n, m) \neq (0, 0). \quad (2.2.5)$$

In 2013, Gliozzi [161] proposed to numerically approach this problem by replacing the infinite set of homogeneous constraints with the two-parameter family of truncated equations

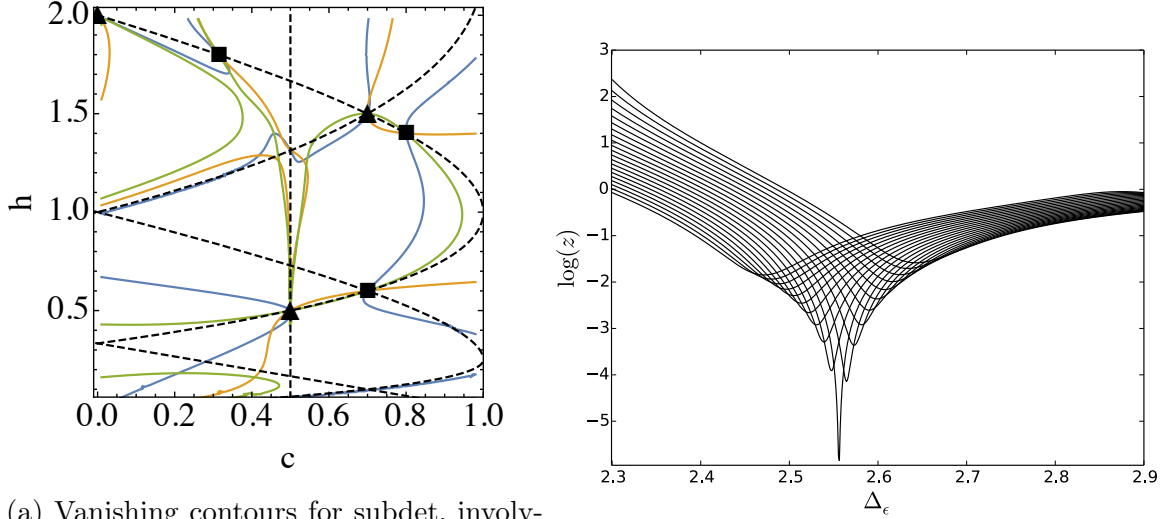
$$\sum_{i=1}^N C_{\Delta_i, J_i} \alpha^{(j)}(\Delta_i, J_i) = 0, \quad \text{for } j = 1, \dots, M. \quad (2.2.6)$$

Here, $\alpha^{(1)}, \dots, \alpha^{(M)}$ are M of the infinitely many homogeneous equations (2.2.5) and $i = 1, \dots, N$ labels the operators of the truncated CFT spectrum. The strategy proposed by Gliozzi is to focus on the overconstrained situation $M > N$ and then construct approximate solutions to crossing in two steps.

1. Determine an approximate spectrum by demanding that all $N \times N$ minors of the $M \times N$ matrix $(\alpha^{(j)}(\Delta_i, J_i))_{j,i}$ vanish, see Figure 2.4a.⁵
2. Solve the truncated homogeneous equations (2.2.6) and the truncated inhomogeneous equation for the OPE coefficients, using the approximate spectrum from step 1 as an input.

There is no a priori reason to expect that the approximate solutions to crossing generated this way should give us meaningful estimates for the spectrum and OPE coefficients of every (or even any) CFT. A direct comparison to known CFT data however provides good evidence that Gliozzi's method is a feasible approach to various

⁵Later versions of the method used the less constrained and numerically more advantageous approach of minimising the smallest singular value of the matrix [162, 163], see Figure 2.4b.



(a) Vanishing contours for subdet. involving Virasoro blocks of a scalar ϕ with chiral weight h . Minimal models with fusion rule $[\phi] \times [\phi] = [1]$ (triangles) and bootstrap of 4d percolation. The result $[\phi] \times [\phi] = [1] + [\phi]$ (squares) sit at intersections.

(b) Minimal singular value curves for various values of Δ_σ , as function of Δ_ϵ in a fusion rule $[\phi] \times [\phi] = [1]$ (triangles) and bootstrap of 4d percolation. The result $[\phi] \times [\phi] = [1] + [\phi]$ (squares) sit at intersections. $\Delta_\epsilon \approx 2.557$, $\Delta_\sigma \approx 0.997$ is close to the four-loop prediction of [164].

Figure 2.4: Examples of Gliozzi bootstrap, taken from [162] (l.h.s.) and [163] (r.h.s.).

CFTs, including the 3d Ising CFT [161], various conformal defects and boundaries [165, 166], and non-unitary CFTs such as the Lee-Yang edge singularity [161], self avoiding random walks [167] and critical percolation [163]. The application to the last three models is particularly important since the more reliable dual approach to crossing, that is free from the uncontrolled errors that come with truncation, is only available for unitary CFTs.

The five-point bootstrap. The main bottle-neck restricting how far Gliozzi's method can be pushed numerically is clearly our limited ability to solve the complicated system of non-linear equations for the scaling dimensions in the first step of the procedure. Once the spectrum is known, the second step, i.e. solving the resulting linear equations for the OPE coefficients, is relatively straightforward. In this sense, the ideal setup for truncation methods is a scenario where the spectrum is already known and step one can be skipped. This situation arises naturally for any CFT whose four-point functions can be bootstrapped to high precision with the dual approach to crossing.

Indeed, bootstrapping the four-point function of e.g. a scalar field ϕ can in the ideal case give us full control over the $\phi \times \phi$ OPE, i.e. both the spectrum of operators produced by fusing ϕ with itself and the associated OPE coefficients. This data

already fixes all exchanges that can occur in the $\langle\phi\phi\phi\phi\rangle$ five-point function. The (truncated) crossing equations of the latter are hence linear equations⁶ whose only unknowns are the OPE coefficients $C_{\mathcal{O}_1\phi\mathcal{O}_2}$ of ϕ with two operator $\mathcal{O}_i \in \phi \times \phi$ from the self OPE of ϕ . This idea was implemented in a series of papers [90–92] that successfully computed previously unknown 3d Ising OPE coefficients with reliable heuristics for the error on this data originating from the truncation. The upshot of this work is that

primal multi-point numerical bootstrap is a powerful complementary tool that helps us to make the most out of the data obtained from dual four-point bootstrap.

By solving six-point crossing in the snow-flake channel, similar reasoning could conceivably produce estimates for unknown OPE coefficients $C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$ of three generic spinning operators of e.g. the 3d Ising model. While somewhat orthogonal to the dual six-point bootstrap developed below, it would be interesting to, along those lines, extend the primal five-point bootstrap to a primal six-point bootstrap.

2.2.2 The dual approach to crossing

The bootstrap renaissance of the 2010s was made possible by three important shifts of paradigm away from the initial programme formulated by Polyakov [57]. First, the modern bootstrap promotes the space of all unitary CFTs to the central object of its analysis, aiming for a full classification rather than sporadically solving special examples. Second, it achieves this goal not by approximately determining the set of solutions to crossing, as in the primal approach, but by rigorously determining its complement, hence deriving rigid bounds on CFT data. Finally, it relies on numerical tools to derive these bounds instead of analysing crossing eqs. analytically.

Rigorous bounds. The key realisation that is necessary to conclude that rigorous bounds on CFT data can be numerically computed is that certificates for non-existence of solutions to crossing can be produced with finite optimisation problems. Thus, while the solution space itself is not directly approachable without truncations that render the problem finite, such as the Gliozzi method discussed in Section 2.2.1, its complement can be charted exactly without uncontrolled relaxations. To make this idea more explicit, let us return to the crossing equation (2.2.3) of a scalar four-point function $\langle\phi\phi\phi\phi\rangle$. However, instead of truncating $\phi \times \phi$ to a finite set of exchanged operators, let us this time make the weaker assumption $\phi \times \phi \subseteq \mathcal{S}$ for some still infinite set $\mathcal{S}$. At the same time, let us again focus on a finite subset of the equations obtained by Taylor expanding the crossing equation around $z = \bar{z} = 1/2$,

⁶A similar situation arises when we are interested in thermal correlation functions of CFTs for which a large amount of spectral data is available. [168–170]

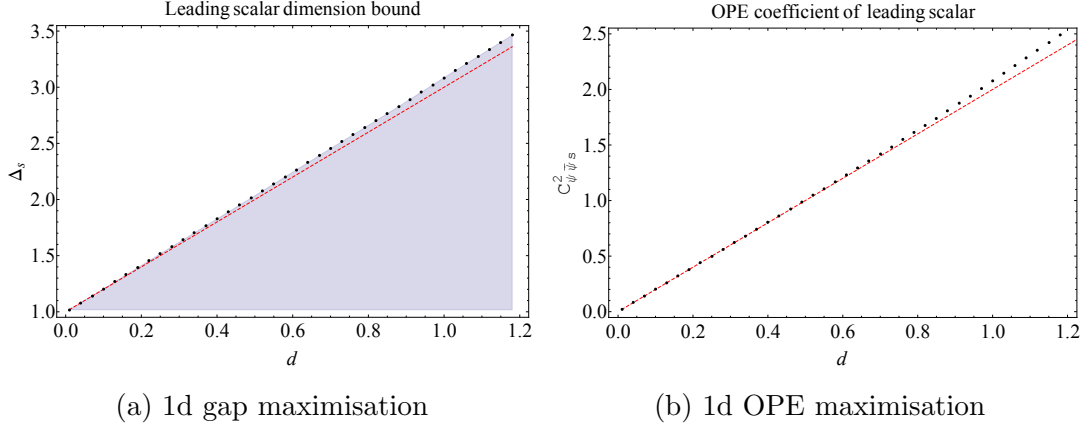


Figure 2.5: Examples of bootstrap bounds in 1d, numerical bounds are shown as black dots, the GFF CFT data is plotted in red. Images taken from Reference [171].

e.g.

$$F_{0,0}^{(n,m)} + \sum_{(\Delta,J) \in \mathcal{S}} C_{\Delta,J} F_{\Delta,J}^{(n,m)} = 0, \quad \text{for all } n+m \leq \Lambda, \quad (2.2.7)$$

where we introduced the notation

$$F_{\Delta,J}^{(n,m)} := \partial_z^n \partial_{\bar{z}}^m F_{\Delta,J}(z, \bar{z})|_{z=\bar{z}=1/2}. \quad (2.2.8)$$

Crucially, the OPE coefficients $C_{\phi\phi\mathcal{O}}$ that enter eq. (2.2.7) are real numbers in a unitary CFT. Hence their squares $C_{\Delta,J} = C_{\phi\phi\mathcal{O}_{\Delta,J}}^2$ are non-negative. Thus, if we manage to find coefficients $c_{(n,m)}$ such that

$$\sum_{n+m \leq \Lambda} c_{(n,m)} F_{0,0}^{(n,m)} = 1 \quad \text{and} \quad \sum_{n+m \leq \Lambda} c_{(n,m)} F_{\Delta,J}^{(n,m)} \geq 0 \quad \forall (\Delta, J) \in \mathcal{S}, \quad (2.2.9)$$

then we obtain the contradiction

$$0 = \sum_{n+m \leq \Lambda} c_{(n,m)} F_{0,0}^{(n,m)} + \sum_{(\Delta,J) \in \mathcal{S}} C_{\Delta,J} \sum_{n+m \leq \Lambda} c_{(n,m)} F_{\Delta,J}^{(n,m)} \geq \sum_{n+m \leq \Lambda} c_{(n,m)} F_{0,0}^{(n,m)} = 1. \quad (2.2.10)$$

In this case, we can exclude the existence of a unitary CFT that satisfies $\phi \times \phi \subseteq \mathcal{S}$. Choosing different types of sets $\mathcal{S}$ allows us to answer a range of different questions about the space of unitary CFTs via the bootstrap.

To give a simple and concrete example, let us focus on the one-dimensional case, where primary operators are only labelled by their scaling dimension (and the crossing equation furthermore depends on only a single cross-ratio z). A natural question to ask in this context is: Given the scaling dimension Δ_ϕ of an operator charged under a $\mathbb{Z}_2$ symmetry, what is the biggest possible value $\Delta_{\phi^2}^*(\Delta_\phi)$ for the dimension Δ_{ϕ^2} of the lightest non-trivial operator appearing in the $\phi \times \phi$ OPE? To answer this

question via the bootstrap, our choice for the superset $\mathcal{S}$ of $\phi \times \phi$ should be the union of the identity contribution $\{0\}$ and the interval $[\Delta_{\phi^2}, \infty[$. At each derivative order Λ , there is a minimal value $\Delta_{\phi^2}^*(\Delta_\phi)_\Lambda$ for which we can derive a contradiction to crossing as in eq. (2.2.10). Truncating the analysis to a finite set of constraints gives up some constraining power. Thus the bound obtained at a given Λ might not be optimal. Crucially, this however does not change the fact that it is a rigorous bound. Moreover, increasing the number of constraints that we consider can only improve the bounds. Hence, the procedure is monotonic in Λ i.e.

$$\Delta_{\phi^2}^*(\Delta_\phi) \leq \Delta_{\phi^2}^*(\Delta_\phi)_\Lambda \leq \Delta_{\phi^2}^*(\Delta_\phi)_{\tilde{\Lambda}} \quad \forall \Lambda > \tilde{\Lambda}. \quad (2.2.11)$$

We can thus systematically compute successively stronger bounds, which in the particular example at hand converge to

$$\Delta_{\phi^2}^*(\Delta_\phi) \leq \lim_{\Lambda \rightarrow \infty} \Delta_{\phi^2}^*(\Delta_\phi)_\Lambda = 2\Delta_\phi + 1. \quad (2.2.12)$$

On the other hand, the Generalised Free Fermion (GFF) is a concrete family of theories with $\Delta_{\phi^2} = 2\Delta_\phi + 1$, which also implies that $2\Delta_\phi + 1$ is a lower bound on $\Delta_{\phi^2}^*(\Delta_\phi)$. Thus,

$$\Delta_{\phi^2}^*(\Delta_\phi) = 2\Delta_\phi + 1. \quad (2.2.13)$$

This was conjectured in [171] based on the numerical data shown in Fig. 2.5a and later proven by analytically constructing the optimal (infinite derivative order) bound [172].

Via a slight generalisation of the setup, the bootstrap can also provide bounds on the OPE coefficients. Indeed, if we manage to find coefficients $c_{n,m}$ such that

$$\sum_{n+m \leq \Lambda} c_{(n,m)} F_{\Delta_*, J_*}^{(n,m)} = 1 \quad \text{and} \quad \sum_{n+m \leq \Lambda} c_{(n,m)} F_{\Delta, J}^{(n,m)} \geq 0 \quad \forall (\Delta, J) \in \mathcal{S} \setminus \{(0,0)\}, \quad (2.2.14)$$

then this implies the upper bound

$$C_{\Delta_*, J_*} = - \sum_{n+m \leq \Lambda} c_{(n,m)} F_{0,0}^{(n,m)} - \sum_{n+m \leq \Lambda} c_{(n,m)} C_{\Delta, J} F_{\Delta, J}^{(n,m)} \leq - \sum_{n+m \leq \Lambda} c_{(n,m)} F_{0,0}^{(n,m)}. \quad (2.2.15)$$

Likewise, flipping the sign of the contribution of (Δ_*, J_*) and hence imposing

$$\sum_{n+m \leq \Lambda} c_{(n,m)} F_{\Delta_*, J_*}^{(n,m)} = -1, \quad (2.2.16)$$

we can derive the lower bound

$$C_{\Delta_*, J_*} = \sum_{n+m \leq \Lambda} c_{(n,m)} F_{0,0}^{(n,m)} + \sum_{n+m \leq \Lambda} c_{(n,m)} C_{\Delta, J} F_{\Delta, J}^{(n,m)} \geq \sum_{n+m \leq \Lambda} c_{(n,m)} F_{0,0}^{(n,m)}. \quad (2.2.17)$$

As an example, consider the task of bounding the OPE coefficient $C_{\phi\phi\phi^2}$ for a given value of Δ_ϕ and Δ_{ϕ^2} in the gap maximisation problem discussed above. Deriving a lower bound is, in this particular situation, not possible since normalising the ϕ^2 contribution to -1 is inconsistent with demanding positivity on the range $\Delta \geq \Delta_{\phi^2}$. An upper bound, however, can be obtained. For $\Delta_{\phi^2} = 2\Delta_\phi + 1$ it is again saturated by the GFF value.

Semi-definite programming. In the previous paragraph we have seen that special linear combinations of four-point crossing equations can provide rigorous bounds on CFT data. We however have not addressed the question of how to find these linear combinations. Numerically, the task of constructing an (optimal) bound is a Semi-Definite Programme (SDP).

Recall that an SDP is a convex optimisation problem in N variables $y \in \mathbb{R}^N$ constrained by a system of linear equations as well as semi-definite constraints

$$\sum_{\alpha=1}^N y_\alpha M_i^\alpha \geq 0 \quad \forall i \in I, \quad (2.2.18)$$

with the objective to maximise the scalar product $c \cdot y$ of y with some target vector $c \in \mathbb{R}^N$. Here, for each i in the index set I , the set $\{M_i^1, \dots, M_i^N\}$ is a collection of real symmetric matrices. If the constraints have no solution, we formally set the objective to minus infinity. SDP is not a one-size-fits-all problem and a large number of solvers based on various algorithms that each have their own benefits and restrictions is available. The most important solvers in the context of conformal bootstrap are the general purpose algorithm SDPA [173], which was especially used in the pre 2015/16 early stages of the bootstrap, and the more recent SDPB [75], which was specifically designed to solve the extremely ill conditioned SDPs relevant for bootstrap. The advantages and disadvantages of both solvers play a central role in Chapter 3, which presents an SDPA implementation of the six-point bootstrap, and Chapter 4, in which SDPB is used instead. For more details, we refer the reader to these chapters.

The gap maximisation problems discussed below eq. (2.2.9) are SDPs involving only constraint matrices of size one⁷. In the notation introduced above, they correspond to

$$I \leftrightarrow \mathcal{S}, \quad y_\alpha \leftrightarrow c_{(n,m)} \quad M_i^\alpha \leftrightarrow (F_{\Delta,J}^{(n,m)}) \quad (2.2.19)$$

further constrained by the linear equation

$$\sum_{n+m \leq \Lambda} c_{(n,m)} F_{0,0}^{(n,m)} = 1. \quad (2.2.20)$$

⁷This special case of SDP is also known as linear programming.

The specific value of the objective is irrelevant to us in this case and we can pick any c we like. The relevant question is only whether a feasible y exists, i.e. whether the objective is finite. In OPE maximisation and minimisation problems on the other hand, the optimal objective value is precisely the optimal bound on the OPE coefficient and c is a list of actions of derivative functionals on the identity crossing vector (with an appropriately chosen sign).

An important issue with the SDP (2.2.19) is that the index set $\mathcal{S}$ of the semi-definite constraints is infinite. In this form, the problem cannot directly be implemented on a finite computer. The two main strategies to deal with this obstruction are

1. Discretisation. In the first bootstrap studies, the set $\mathcal{S}$ of constraints was simply truncated to a finite subset $\tilde{\mathcal{S}}$. If $\tilde{\mathcal{S}}$ is not chosen sufficiently large, this can lead to an overestimate, which will converge to the weaker true bound as $\tilde{\mathcal{S}}$ is increased. Nonetheless it is still possible to obtain fully rigorous bounds from discretisation. This however requires the additional step of proving on a case by case basis that the solution obtained for some large but finite $\tilde{\mathcal{S}}$ already satisfies the full set of constraints associated to $\mathcal{S}$.
2. Polynomial approximation. In Reference [74] a more refined version of the bootstrap was suggested, which instead of discretising the spectrum approximates the constraints by polynomials of Δ . The question of whether a polynomial is positive on a certain interval can be answered without truncations by constructing sum of square decompositions using semi-definite programming, see Section 3.C for more details. This approach essentially leads to a reformulation of the continuum of 1×1 positivity constraints related by choosing different values for the exchanged scaling dimension in eq. (2.2.19) in terms of positivity constraints on larger matrices, whose entries however no longer depend on Δ .

In later chapters, we will encounter the same issue for the six-point bootstrap. The first implementation, discussed in Chapter 3 will take a discretisation approach. The improved procedure presented in Chapter 4 then adapts polynomial SDP, with the special feature that in the six-point case this step does not require any approximations. Instead, the six-point SDP is, in a suitable formulation, exactly polynomial.

Mixed correlator bootstrap. More involved SDPs, whose constraints have a non-trivial matrix structure even before introducing polynomial approximations, arise in the mixed correlator bootstrap i.e. the simultaneous study of systems of multiple crossing equations. Reference [74], which introduced polynomial SDP to the bootstrap, was also the first mixed correlator study, focussing particularly on the $\{\langle\sigma\sigma\sigma\sigma\rangle, \langle\sigma\sigma\epsilon\epsilon\rangle, \langle\epsilon\epsilon\epsilon\epsilon\rangle\}$ system. The basic strategy is to combine the crossing

equations

$$F_{[\epsilon,\epsilon,\epsilon,\epsilon],0,0}^{(n_1,m_1)} + \sum_{(\Delta,J) \in \mathcal{S}_{\text{even}}} C_{[\epsilon,\epsilon],\Delta,J} F_{[\epsilon,\epsilon,\epsilon,\epsilon],\Delta,J}^{(n_1,m_1)} C_{[\epsilon,\epsilon],\Delta,J} = 0 \quad (2.2.21)$$

$$F_{[\sigma,\sigma,\sigma,\sigma],0,0}^{(n_2,m_2)} + \sum_{(\Delta,J) \in \mathcal{S}_{\text{even}}} C_{[\sigma,\sigma],\Delta,J} F_{[\sigma,\sigma,\sigma,\sigma],\Delta,J}^{(n_2,m_2)} C_{[\sigma,\sigma],\Delta,J} = 0 \quad (2.2.22)$$

of the $\langle\sigma\sigma\sigma\sigma\rangle$ and $\langle\epsilon\epsilon\epsilon\epsilon\rangle$ four-point functions and the crossing equation

$$\begin{aligned} F_{[\epsilon,\epsilon,\sigma,\sigma],0,0}^{(n_3,m_3)} + \sum_{(\Delta,J) \in \mathcal{S}_{\text{even}}} C_{[\epsilon,\epsilon],\Delta,J} G_{[\epsilon,\epsilon,\sigma,\sigma],\Delta,J}^{(n_3,m_3)} C_{[\sigma,\sigma],\Delta,J} \\ + \sum_{(\Delta,J) \in \mathcal{S}_{\text{odd}}} C_{[\epsilon,\sigma],\Delta,J} \tilde{G}_{[\epsilon,\epsilon,\sigma,\sigma],\Delta,J}^{(n_3,m_3)} C_{[\epsilon,\sigma],\Delta,J} = 0 \end{aligned} \quad (2.2.23)$$

of $\langle\epsilon\epsilon\sigma\sigma\rangle$ into the equation

$$\begin{aligned} 0 = & F_{[\epsilon,\epsilon,\epsilon,\epsilon],0,0}^{(n_1,m_1)} + F_{[\sigma,\sigma,\sigma,\sigma],0,0}^{(n_2,m_2)} + F_{[\epsilon,\epsilon,\sigma,\sigma],0,0}^{(n_3,m_3)} + \sum_{(\Delta,J) \in \mathcal{S}_{\text{odd}}} C_{[\epsilon,\sigma],\Delta,J} \tilde{G}_{[\epsilon,\epsilon,\sigma,\sigma],\Delta,J}^{(n_3,m_3)} C_{[\epsilon,\sigma],\Delta,J} \\ & + \sum_{(\Delta,J) \in \mathcal{S}_{\text{even}}} (C_{[\epsilon,\epsilon],\Delta,J} C_{[\sigma,\sigma],\Delta,J}) \begin{pmatrix} F_{[\epsilon,\epsilon,\epsilon,\epsilon],0,0}^{(n_1,m_1)} & G_{[\epsilon,\epsilon,\sigma,\sigma],\Delta,J}^{(n_3,m_3)} \\ G_{[\epsilon,\epsilon,\sigma,\sigma],\Delta,J}^{(n_3,m_3)} & F_{[\sigma,\sigma,\sigma,\sigma],\Delta,J}^{(n_2,m_2)} \end{pmatrix} \begin{pmatrix} C_{[\epsilon,\epsilon],\Delta,J} \\ C_{[\sigma,\sigma],\Delta,J} \end{pmatrix}. \end{aligned} \quad (2.2.24)$$

We leave the indices n_i and m_i independent of each other, and allow for an independent scaling of the three components of this equation with arbitrary constants, such that no information of the initial system of three equations is lost. Bounds can be derived by making the 2×2 matrix appearing in the second line of the equation and linear combinations of derivatives of $\tilde{G}$ positive independently. Clearly, this procedure has the advantage of making accessible the $\sigma \times \epsilon$ OPE data, which is out of reach for the positivity based approach to four-point functions of identical operators. Perhaps surprisingly, the inclusion of $\langle\sigma\sigma\epsilon\epsilon\rangle$ can furthermore even strengthen bounds on the $\sigma \times \sigma$ and $\epsilon \times \epsilon$ data. At first glance, this might be counter intuitive since the inclusion of the off diagonal contributions to the 2×2 matrix can only make it harder to find a feasible solution to the semi-definite constraints: Nontrivial off-diagonal matrix elements strengthen the constraints imposed on the diagonal elements and hence should weaken the bounds on solutions to $\langle\sigma\sigma\sigma\sigma\rangle$ and $\langle\epsilon\epsilon\epsilon\epsilon\rangle$ crossing. However, this effect is compensated by the extra freedom which the inclusion of the off-diagonal terms introduces in the linear normalisation constraint. Indeed, while a pure $\{\langle\sigma\sigma\sigma\sigma\rangle, \langle\epsilon\epsilon\epsilon\epsilon\rangle\}$ bootstrap would impose the normalisation constraint

$$1 = \sum_{n_1+m_1 \leq \Lambda} c_{(n_1,m_1)}^{(1)} F_{[\epsilon,\epsilon,\epsilon,\epsilon],0,0}^{(n_1,m_1)} + \sum_{n_2+m_2 \leq \Lambda} c_{(n_2,m_2)}^{(2)} F_{[\sigma,\sigma,\sigma,\sigma],0,0}^{(n_2,m_2)} \quad (2.2.25)$$

which in particular requires the identity contribution from at least one of the two four-point functions to be positive, the $\{\langle\sigma\sigma\sigma\sigma\rangle, \langle\sigma\sigma\epsilon\epsilon\rangle, \langle\epsilon\epsilon\epsilon\epsilon\rangle\}$ bootstrap imposes

$$1 = \sum_{n_1+m_1 \leq \Lambda} c_{(n_1,m_1)}^{(1)} F_{[\epsilon,\epsilon,\epsilon,\epsilon],0,0}^{(n_1,m_1)} + \sum_{n_2+m_2 \leq \Lambda} c_{(n_2,m_2)}^{(2)} F_{[\sigma,\sigma,\sigma,\sigma],0,0}^{(n_2,m_2)} + \sum_{n_3+m_3 \leq \Lambda} c_{(n_3,m_3)}^{(3)} F_{[\epsilon,\epsilon,\sigma,\sigma],0,0}^{(n_3,m_3)}, \quad (2.2.26)$$

which for instance also allows for a negative contribution from both $\langle\sigma\sigma\sigma\sigma\rangle$ and $\langle\epsilon\epsilon\epsilon\epsilon\rangle$.

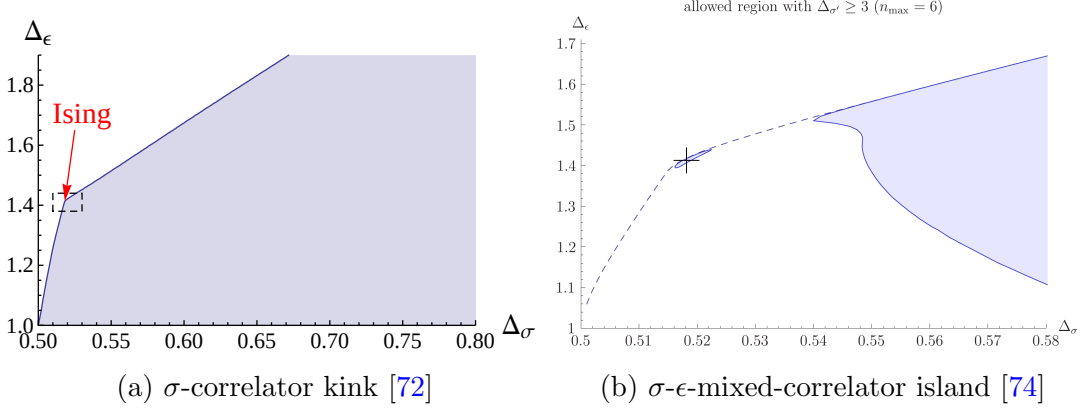


Figure 2.6: The 3d Ising kink and island.

Solving the 3d Ising model. The mixed correlator four-point bootstrap made a vast amount of previously unknown CFT data accessible and taught us invaluable lessons about the space of unitary conformal field theories. The general flavour of these results is best exhibited by focusing on one specific representative bootstrap target. Arguably, the canonical choice for this is the 3d Ising model. For a more general overview of bootstrap bounds on other CFTs, the reader is referred to the reviews [70, 71].

The bounds, shown in Fig. 2.5, that we discussed above, were essentially linear functions without any special features. Compared to other bootstrap bounds, this behaviour is rather the exception than the rule. In general the bounds of the conformal bootstrap tend to have a lot of structure and sharp features which distinguish special points in theory space. These often, but not always, correspond to physically interesting models. The most iconic example for this observation is the Ising kink shown in Fig. 2.6a. This bound was obtained by bootstrapping the four-point correlator $\langle \sigma \sigma \sigma \sigma \rangle$ of the lowest lying $\mathbb{Z}_2$ odd scalar in a 3d CFT. The location of its prominent kink agrees well with Monte Carlo estimates of 3d Ising CFT data. This suggests that determining the position of the kink in the plot, provides a new way to compute the scaling dimensions Δ_σ and Δ_ϵ . The conclusion that the bootstrap can therefore not only teach us general lessons about the qualitative shape of theory space, but instead could be a new computational tool to non-perturbatively solve CFTs had dramatic impact on the CFT community and significantly boosted the development of this programme.

Importantly, the bootstrap managed to keep the promise that the Ising kink is not just a curious observation but rather a foothold that can be expanded to a systematic method to solve the theory. With the implementation of the mixed-correlator σ - ϵ bootstrap, Reference [74] managed to isolate the 3d Ising model on the small island shown in Fig. 2.6b. With this significant step, the bootstrap was promoted from a

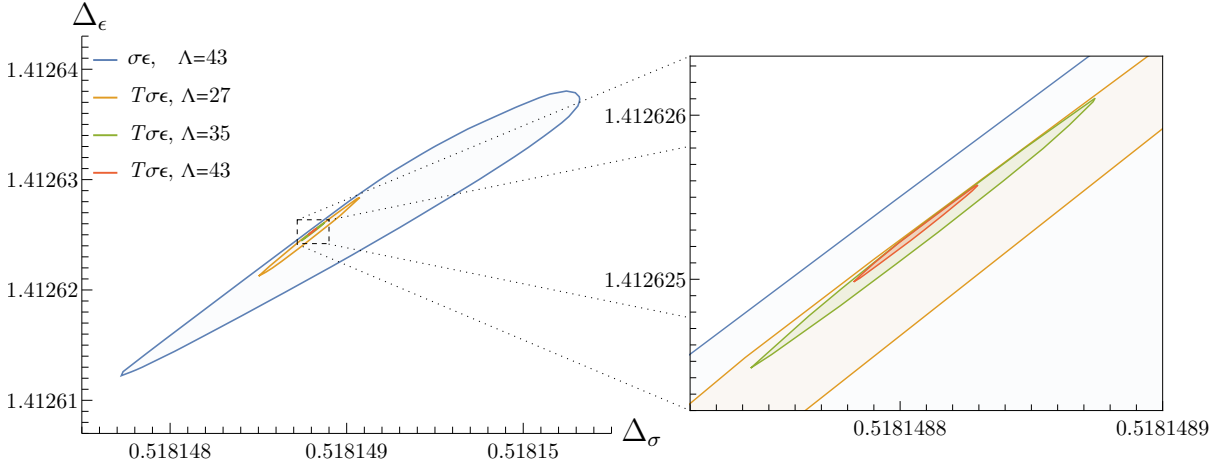


Figure 2.7: The σ - ϵ - T -mixed-correlator island of [78], currently most precise and extensive source of 3d Ising CFT data.

heuristic method to estimate 3d Ising data by determining the position of special features in plots like Fig. 2.6a to a method to compute this data with rigorous error bars. Note that Figure 2.6b, in addition to the Ising island, also contains a larger allowed region to the right. This region is typically referred to as “the continent”. Shrinking the continent to be able to conclude that the island contains all 3d CFTs with only one relevant neutral scalar and one relevant scalar charged under a global $\mathbb{Z}_2$ symmetry (or showing that this is not possible by identifying a currently unknown CFT in the continent) is an important open problem. If successful, and complemented by a proof that the 3d Ising island is really just a 3d Ising point whose finite size is a numerical artifact, the exclusion of the continent would provide a full bootstrap explanation of 3d Ising universality. For further comments on this uniqueness problem see Ref. [11].

In the years that followed the publication of [74], the numerical method was systematically improved to allow for higher derivative order bootstrap and larger mixed-correlator systems. Consequently, the size of the Ising island is a strictly monotonically decreasing function of time, which by now has shrunk to the impressive status shown in Figure 2.7. The lesson from 3d Ising seems to be that, given enough computational power to arbitrarily increase the size of mixed-correlator systems, the bootstrap might be able to access any CFT data to any desired degree of precision. So are we essentially done? In a world of bounded resources, where the four-point approach will always be limited to a finite fraction of the full constraining power of crossing, the answer to this question is clearly: no.

2.2.3 The six-point bootstrap

In Section 2.2.1, we have seen that solving truncations of the crossing equations of five-point functions gives access to an, in principle, infinite amount of OPE coefficients that is not contained in any finite system of four-point functions. What can be gained if we not only use higher point functions to access the CFT data of infinitely many four-point functions, but also the constraining power of their crossing equations via semi-definite programming?

We end this chapter with two paragraphs that provide complementary qualitative answers to the question above, before then beginning to quantitatively answer with concrete six-point bootstrap bounds in Chapter 3.

New observables for the bootstrap. Through the crossing relations of six-point functions, one can naturally answer questions on triple OPEs just as the bootstrap analysis of four-point functions allows us to answer questions on the fusion of pairs of operators. Let us make this more precise by considering the example of correlation functions of a scalar field ϕ in an arbitrary (unitary) CFT. The OPE of two copies of ϕ takes the form

$$\phi(x_1)\phi(x_2) \sim \sum_{\mathcal{O} \in \phi \times \phi} F_{\phi\phi\mathcal{O}}(x_1, x_2) \mathcal{O}(x_2), \quad (2.2.27)$$

where the function $F_{\phi\phi\mathcal{O}}(x_1, x_2)$ is kinematically fixed up to an overall normalisation constant, namely the OPE coefficient $C_{\phi\phi\mathcal{O}}$. By studying crossing of the $\langle\phi\phi\phi\phi\rangle$ four-point function, the numerical bootstrap can answer a large range of questions on the fields $\mathcal{O} \in \phi \times \phi$, putting bounds on the scaling dimensions $\Delta_{\mathcal{O}}$ and the OPE coefficients $C_{\phi\phi\mathcal{O}}$. Consider now the triple OPE

$$\phi(x_1)\phi(x_2)\phi(x_3) \sim \sum_{\mathcal{O} \in \phi \times \phi \times \phi} F_{\phi\phi\phi\mathcal{O}}(x_1, x_2, x_3) \mathcal{O}(x_3). \quad (2.2.28)$$

In this case, conformal symmetry of course still constrains $F_{\phi\phi\phi\mathcal{O}}(x_1, x_2, x_3)$. However, the kinematic constraints are not strong enough to completely fix the functional form of $F_{\phi\phi\phi\mathcal{O}}$. Just as crossing symmetry of the four-point function is a dynamical input that allows the numerical bootstrap to answer questions about the $\phi \times \phi$ OPE, crossing equations of the six-point function $\langle\phi\phi\phi\phi\phi\phi\rangle$ enable the bootstrap to answer questions on the $\phi \times \phi \times \phi$ triple OPE, putting bounds on the dimensions $\Delta_{\mathcal{O}}$ of the operators $\mathcal{O} \in \phi \times \phi \times \phi$ as well as on the values of the function $F_{\phi\phi\phi\mathcal{O}}$ and its derivatives ⁸.

⁸Note that, in principle, all data encoded in the triple OPE should also be accessible through four-point functions. More concretely, it is conceivable that the bounds that we can obtain through the six-point bootstrap could in some way also be determined by studying an (infinitely) large mixed correlator system of four-point functions involving correlation functions of the type $\langle\mathcal{O}\phi\phi\mathcal{O}'\rangle$ with $\mathcal{O}, \mathcal{O}' \in \phi \times \phi$. However, it seems unclear how such an alternative approach would be implemented in practice.

In this way, the six-point bootstrap allows us to explore qualitatively new features of theory space that involve the OPE of pairs and triples of operators and the interplay between these. Concrete examples of qualitatively new bootstrap problems that cannot directly be probed with finite mixed-correlator four-point systems are solved with the six-point bootstrap in both Chapter 3 and Chapter 4.

Better bounds for old problems. In Section 2.2.2 we discussed how the numerical bootstrap of the 3d Ising model is at its core a quest of making increasing subsets of the constraints which crossing imposes on CFT data serviceable to the computation of bounds. However, no matter how much we improve our algorithms, there will always be some upper limit on how many four-point crossing equations we can include in our mixed-correlator studies. It is hence imperative to carefully consider all available options and make an informed decision which constraints to include to make the most of the available computational resources. In particular, it may in some cases be beneficial to directly include infinitely many four-point crossing equations encoded in a single higher-point correlator instead of enlarging the bootstrap SDP by a finite number of additional four-point correlators. As Chapter 6 discusses in more detail, this idea could so far not be adequately explored since an appropriate way to combine the four- and six-point correlator into an extended mixed correlator system was missing. However, with the proposal made in Chapter 6, it will be possible to finally approach this problem in the near future and systematically investigate how much the inclusion of the six-point function can strengthen previously known bounds of the four-point bootstrap.

Chapter 3

Lining up a positive semi-definite six-point bootstrap

– Existence of six-point bounds.

We ended the previous chapter on the conclusion that extending the traditional mixed correlator four-point bootstrap to a multi-point set-up constitutes a qualitatively novel and promising leap ahead. A main argument in that discussion was that multi-point bootstrap naturally combines the constraints from infinitely many four-point crossing equations. While this is true for any number $N > 4$ of external points, the positivity features that are vital for a controlled SDP based numerical bootstrap require N to be even, making $N = 6$ the simplest example. Indeed, as we schematically depict in Figure 3.1 for the relevant case of $N = 6$, the insertion of conformal projectors allows us to decompose any $2k$ -point function into a manifestly positive sum of squares of $(k + 1)$ -point correlators.

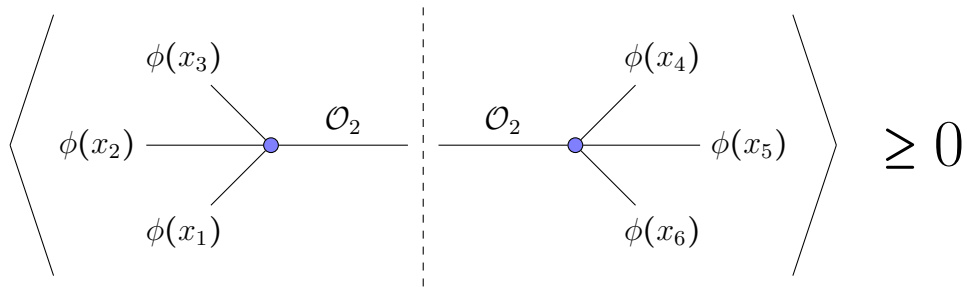


Figure 3.1: Schematic representation of reflection positivity in the decomposition of a six-point function into four-point correlators. The sets $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, x_6\}$ of insertion points are assumed to be related by a reflection.

This chapter takes the first step towards concretely implementing the numerical multi-point bootstrap programme. It contains a formulation of the SDP dual to six-

point crossing and a detailed description of an implementation that produces first bounds. To make progress, we take advantage of the technical simplifications that come with the restriction to one dimensional kinematics. While this step eliminates some of the constraining power that considering crossing in general kinematics would offer, most of the richness and structure of the higher-dimensional case is preserved: As we will show below, we are still solving the bootstrap for an infinite number of four-point functions! Furthermore, inherently one-dimensional conformal theories are interesting physical models on their own. They arise e.g. in the description of conformal boundaries in CFT_2 , “holographic” modes of QFT in AdS_2 [60] and line defects of general higher dimensional theories. In all these examples, the six-point bootstrap presented in this chapter can provide qualitative new results and potentially improve the old four-point bounds which notoriously struggle to isolate theories in one dimension, bringing more structure to 1d bounds such as 2.5.

Outline. Before we end this introduction, let us summarise the main results of the chapter and line out its overall structure. Section 3.1, reviews the kinematics and OPE structure of one-dimensional six-point functions and shows how the comb-channel crossing equation naturally leads to an SDP. We move on to find a reformulation of this SDP, which we dub the *descendant-space* SDP, that only requires differential relations between conformal blocks instead of their explicit evaluations as kinematical input. Instead of performing a full decomposition into three-point coefficients, the descendant-space SDP emerges from expressing the six-point function in terms of products of four-point functions (as in Figure 3.1). This approach turns out to be crucial in establishing a robust numerical algorithm. Section 3.1 also discusses important necessary conditions for the positivity of six-point bootstrap functionals, related to the asymptotics at large scaling dimension. Imposing these additional constraints is a vital aspect of the numerical procedure which guarantees the convergence of the algorithm used in this chapter.

Section 3.2 applies the six-point bootstrap to the toy problem of maximising the gap in a correlation function where no identity operator is exchanged in the OPE of two external scalars. After some simple but illustrative analytical bounds, we develop a numerical truncation scheme which can be fed to **SDPA**, and then show how to rigorously confirm the positivity of the functionals obtained numerically. Our numerical results for the gap maximisation without identity are presented in Subsection 3.2.2. These corroborate and reinforce our analytical expectation: The obtained numerical bound appears to be saturated by a simple crossing symmetric function for which the gap is given by $9/5$ of the external dimension, see Figure 3.7.

With the numerical machinery in place, the final Section 3.3 studies a simple but more realistic class of examples in which the identity field does appear in the OPE of two external scalars ϕ . The section starts by lining out how to constrain triple-twist

data in a rather general setup, assuming that we have solved or bootstrapped the four-point function of the scalar ϕ . This programme is then carried out concretely in two examples. First, we take as an input the four-point function of a generalised free fermion (GFF) of dimension Δ_ψ and bound the gap of the six-point function. We find a bound which approaches $3\Delta_\psi + 3$: the leading triple-twist operator for the GFF theory, see Figure 3.9. Then, we consider the same problem with the four-point function of a generalised free boson (GFB) as input. Surprisingly, the resulting bound on the conformal dimension of the leading triple-twist operator seems to approach a value strictly above $3\Delta_\phi$, the leading triple-twist operator of GFB (Figure 3.10). We explain this result in terms of a perturbative deformation of GFB: A Φ^6 contact interaction in AdS_2 , which can increase the six-point gap without altering the four-point function. A similar interpretation for the GFF result suggests a sign constraint on the leading fermionic sextic coupling. The section ends with a discussion on OPE-coefficient maximisation as well as bounds on full four-point correlators. As a concrete example, we provide a numerical bound on the GFF four-point function of three ψ insertions and the lowest lying triple-twist operator (Figure 3.11).

Some technical aspects of the numerical implementation as well as auxiliary analytical results are relegated to several appendices.

3.1 Six-point Crossing Equation in the Comb-Channel

The goal of this section is to set up an infinite dimensional semi-definite program (SDP) dual to a crossing equation of a six-point function in the comb-channel expansion. We begin by reviewing the kinematics and basic properties of six-point functions and their conformal block expansions in Subsection 3.1.1. An associated crossing equation is then spelled out in Subsection 3.1.2 and it is analysed by rewriting it in terms of an infinite matrix whose rows and columns are labelled by the primaries exchanged in the OPE of two external operators. While this perspective is close to the established multi-correlator four-point bootstrap, we shall argue that it is preferable to pass to a different, yet equivalent, formulation. As explained in Subsection 3.1.3, the rows and columns of the matrices occurring in our second formulation are labelled by descendants of the operators exchanged in the OPE of three external operators. In Subsection 3.1.4, we establish asymptotic properties of these new matrices which will be of key importance in the construction of positive definite combinations in the subsequent sections. In the final Subsection 3.1.5, we present some conceptual remarks that provide a broader perspective on our new approach to six-point bootstrap. While Subsections 3.1.1 to 3.1.4 lay out the foundation for the rest of this work, Subsection 3.1.5 can safely be skipped on a first reading.

3.1.1 Kinematics and conventions

The protagonist of this paper is a six-point function of identical scalar operators $\phi(x_i)$ with scaling dimension Δ_ϕ . All fields are inserted at points $x_i \in \mathbb{R}$ on the line with the ordering prescription $x_i < x_{i+1}$. Their six-point function is denoted by

$$\mathcal{G}(x_1, \dots, x_6) \equiv \langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \phi(x_5) \phi(x_6) \rangle. \quad (3.1.1)$$

In one dimension, four-point functions are parameterised by a single cross-ratio, and each extra point adds a single subsequent degree of freedom. Thus, the correlator $\mathcal{G}$ is effectively a function of three variables, which we choose to be

$$\chi_1 = \frac{x_{12}x_{34}}{x_{13}x_{24}}, \quad \chi_2 = \frac{x_{23}x_{45}}{x_{24}x_{35}}, \quad \chi_3 = \frac{x_{34}x_{56}}{x_{35}x_{46}}, \quad (3.1.2)$$

and $x_{ij} = x_i - x_j$. With the specified ordering of operators on the line, the cross-ratios satisfy $0 < \chi_i < 1$. For covariance under conformal transformations, the correlator must be the product of a conformally invariant function of the cross-ratios with an appropriate dimensionful prefactor $\mathcal{L}$, referred to as the leg factor,

$$\mathcal{G}(x_1, \dots, x_6) = \mathcal{L}(x_i) \mathcal{G}(\chi_1, \chi_2, \chi_3). \quad (3.1.3)$$

By abuse of notation, we use the same letter $\mathcal{G}$ to denote both the entire correlator and the associated invariant function of cross-ratios. For the leg factor we adopt the same choice as in [174],

$$\mathcal{L}(x_i) = \chi_2^{-\Delta_\phi} x_{12}^{-2\Delta_\phi} x_{34}^{-2\Delta_\phi} x_{56}^{-2\Delta_\phi}. \quad (3.1.4)$$

Six-point correlators admit different conformal block decompositions which depend on the order in which the OPEs are performed. There are two channel topologies, snowflake and comb, which each give rise to several possible channels. For example, taking the OPEs in disjoint pairs, as indicated by the brackets on the left hand side of equation (3.1.5), leads to the snowflake expansion

$$\langle (\phi_1 \phi_2) (\phi_3 \phi_4) (\phi_5 \phi_6) \rangle = \mathcal{L}(x_i) \sum_{\mathcal{O}_a, \mathcal{O}_b, \mathcal{O}_c} C_{\phi\phi\mathcal{O}_a} C_{\phi\phi\mathcal{O}_b} C_{\phi\phi\mathcal{O}_c} C_{\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c} G_{\Delta_a, \Delta_b, \Delta_c}^{12,34,56}(\chi_i). \quad (3.1.5)$$

Here, we used the shorthand notation $\phi_i \equiv \phi(x_i)$. On the right hand side, we introduced the OPE coefficients C , and the conformal blocks $G_{\Delta_a, \Delta_b, \Delta_c}(\chi_i)$, whose topology and channel is identified by the superscript, in this case the 12, 34, 56 snowflake-channel. We note that the theory dependent data in eq. (3.1.5) contains the three-point functions $\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_c \rangle$ for any triple of operators that can appear in the OPE of two external scalars ϕ along with three OPE coefficients that appear in the block expansion of the four-point function already. This makes the snowflake-channel an interesting one, especially for higher dimensional theories where it gives

access to OPE coefficients of three fields in STT representations, as discussed in the introduction. On the other hand, the snowflake-channel does not have the positivity features that are needed for the numerical conformal bootstrap.

In order to develop our positive semi-definite six-point bootstrap program, we shall therefore focus on the comb-topology. One particular conformal block expansion with comb-topology is given by

$$\langle ((\phi_1\phi_2)\phi_3)(\phi_4(\phi_5\phi_6)) \rangle = \mathcal{L}(x_i) \sum_{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3} C_{\phi\phi\mathcal{O}_1} C_{\mathcal{O}_1\phi\mathcal{O}_2} C_{\mathcal{O}_2\phi\mathcal{O}_3} C_{\phi\phi\mathcal{O}_3} G_{\Delta_1, \Delta_2, \Delta_3}^{123, 456}(\chi_i), \quad (3.1.6)$$

where we distinguish the comb from the snowflake through the superscript with a single comma separating two ordered triplets 123, 456. To specify a channel, one has to fix the order in which the OPEs in each triplet are performed, as we did on the left hand side of the previous equation. In our notation for blocks this is achieved by paying attention to the order of the fields in each of the triplets. For the expansion at hand, the superscript 123, 456 encodes the OPE channel $((12)3)(4(56))$ whereas a superscript 132, 456 would label the channel $((13)2)(4(56))$, to give just one example.

Let us note that comb-channel expansion (3.1.6) includes two pairs of OPE coefficients. The OPE coefficients of the first pair arise from the OPE of two external scalars. These coefficients can be studied within the four-point bootstrap. In addition, we now have a pair of new OPE coefficients in which (only) one field is an external scalar. While this is not as generic as the new OPE coefficient we saw in the snowflake expansion above, the pairwise structure of OPE coefficients in the comb topology is the feature that enables us to set up a positive semi-definite bootstrap.

There is one important aspect of the new OPE coefficients in the comb-channel that we must pay some attention to, namely their dependence on ordering. The OPE of two identical scalars ϕ only contains parity even fields and hence the order of operators is irrelevant for the OPE coefficients $C_{\phi\phi\mathcal{O}_a}$ that appear in the snowflake and the comb-channel. The same is actually true for the OPE coefficients $C_{\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c}$ that appear with the central vertex in the snowflake-channel for six identical external fields. Things are different for the OPE coefficients $C_{\mathcal{O}_1\phi\mathcal{O}_2}$ with a single external field ϕ . In this case both parity even and parity odd fields can appear in the operator product. Parity is the one-dimensional avatar of spin and leads to subtle minus signs in correlators, see for example [175]. On the level of OPE coefficients, these signs arise upon reordering of the fields as

$$C_{123} = (-1)^{\sigma_{123}} C_{321}, \quad \sigma_{123} \equiv P_1 + P_2 + P_3 \pmod{2}. \quad (3.1.7)$$

Here, P_i denotes the parity of the i^{th} operator with the convention that $P = 0$ for even and $P = 1$ for odd operators.

Having discussed at length the OPE coefficients in the expansions (3.1.5) and (3.1.6), we also briefly want to comment on the conformal blocks. While in general space-time dimensions the physically relevant case where multiple spinning operators are exchanged is still not understood in full generality,¹ in one dimension there are drastic simplifications due to the kinematics and the absence of spin. Indeed, [174] provided closed form expressions in the form of multi-variable hypergeometric series expansions for all 1d conformal blocks of comb-topology. In the six-point case with identical external operators, we have

$$G_{\Delta_i}^{123,456}(\chi_i) = \chi_1^{\Delta_1} \chi_2^{\Delta_2} \chi_3^{\Delta_3} F_K \left[\begin{matrix} \Delta_1, \Delta_1 + \Delta_2 - \Delta_\phi, \Delta_2 + \Delta_3 - \Delta_\phi, \Delta_3 \\ 2\Delta_1, 2\Delta_2, 2\Delta_3 \end{matrix}; \chi_1, \chi_2, \chi_3 \right], \quad (3.1.8)$$

where the comb-function F_K is defined through the triple series

$$F_K \left[\begin{matrix} a_1, b_1, b_2, a_2 \\ c_1, c_2, c_3 \end{matrix}; \chi_1, \chi_2, \chi_3 \right] \equiv \sum_{n_i=0}^{\infty} \frac{(a_1)_{n_1} (b_1)_{n_1+n_2} (b_2)_{n_2+n_3} (a_2)_{n_3}}{(c_1)_{n_1} (c_2)_{n_2} (c_3)_{n_3}} \frac{\chi_1^{n_1}}{n_1!} \frac{\chi_2^{n_2}}{n_2!} \frac{\chi_3^{n_3}}{n_3!}. \quad (3.1.9)$$

These functions have power law behaviour in each of the χ_i variables, making the identification of the dimensions of exchanged operators $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ straightforward.

3.1.2 The crossing equation and positive semi-definiteness

This subsection describes the comb-topology crossing equations and derives an SDP dual to them. More concretely, we shall argue that crossing symmetry allows us to construct families of symmetric bilinear forms $B_{\Delta_2}^k$ (defined in eq. (3.1.21)) with respect to which certain vectors of OPE coefficients (defined in eq. (3.1.14)) are null. This statement is concisely captured by the main equation of this subsection, namely eq. (3.1.23). It also has an obvious SDP dual: The task of constructing positive definite linear combinations of said bilinear forms to rule out the existence of solutions to the crossing equations.

In one dimension, it is natural to consider OPE channels that share the same cyclic ordering of points to derive crossing equations. All permutations that preserve cyclic ordering are generated single cycle permutation that shifts $x_i \rightarrow x_{i+1}$ with the periodic identification $x_7 = x_1$. The associated crossing equation that relates comb-channel expansions $123, 456 = ((12)3)(4(56))$ and $234, 561 = ((23)4)(5(61))$ is depicted graphically in Figure 3.2. The corresponding conformal blocks are related by

$$\mathcal{L}(x_i) G^{234,561}(\chi_j(x_i)) = \mathcal{L}(x_{i+1}) G^{123,456}(\chi_j(x_{i+1})). \quad (3.1.10)$$

Here, $\mathcal{L}(x_i)$ is the leg factor we defined by eq. (3.1.4) and the superscripts in our notation of the comb-channel blocks follow the conventions we introduced after eq. (3.1.6).

¹See however [139–142, 176–180] for recent progress in this direction.

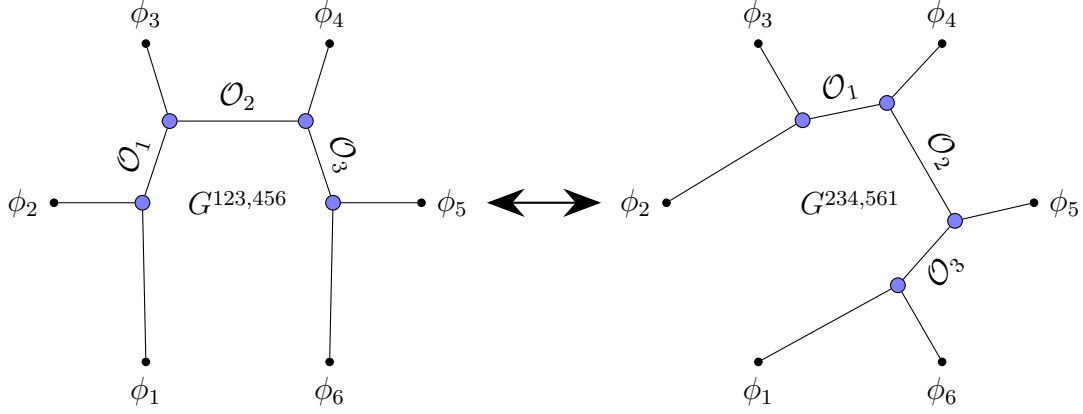


Figure 3.2: The OPE diagram for the crossing of six-point comb-channel expansions.

Upon the cyclic shift $x_i \rightarrow x_{i+1}$, the cross ratios we introduced in eq. (3.1.2) behave as

$$\chi_j(x_{i+1}) = \chi_{j+1}(x_i), \quad (3.1.11)$$

for $j = 1, 2, 3$, provided we introduce χ_4 as

$$\chi_4(x_i) \equiv \chi_3(x_{i+1}) = \frac{\chi_1 + \chi_2 + \chi_3 - \chi_1\chi_3 - 1}{\chi_1 + \chi_2 - 1}. \quad (3.1.12)$$

In order to write down the crossing equation we depicted in Figure (3.2), we shall introduce a shorthand for the ratio of leg factors on the two sides of the crossing equation,

$$Q(\chi_j) \equiv \frac{\mathcal{L}(x_{i+1})}{\mathcal{L}(x_i)} = \left(\frac{\chi_1^2 \chi_2 \chi_3}{(\chi_1 + \chi_2 + \chi_3 - \chi_1\chi_3 - 1)^2} \right)^{\Delta_\phi}. \quad (3.1.13)$$

Note that, by construction of the leg factors, the ratio $Q(\chi_i)$ depends on the insertion points x_i only through the cross ratios χ_j , but we will henceforth omit this dependence. It will also be convenient to abbreviate products of OPE coefficients as

$$C_{\mathcal{O}_2}^{\mathcal{O}_j} \equiv C_{\phi\phi\mathcal{O}_j} C_{\mathcal{O}_j\phi\mathcal{O}_2}. \quad (3.1.14)$$

In applications to our crossing equation, the index j assumes the values $j = 1, 3$. After this preparation, the crossing equation under consideration can be written in the form

$$\sum_{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3} C_{\mathcal{O}_2}^{\mathcal{O}_1} G_{\Delta_1, \Delta_2, \Delta_3}^{123, 456}(\chi_1, \chi_2, \chi_3) C_{\mathcal{O}_2}^{\mathcal{O}_3} = \sum_{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3} Q C_{\mathcal{O}_2}^{\mathcal{O}_1} G_{\Delta_1, \Delta_2, \Delta_3}^{234, 561}(\chi_2, \chi_3, \chi_4) C_{\mathcal{O}_2}^{\mathcal{O}_3}. \quad (3.1.15)$$

The weights Δ_j that label the conformal blocks in eq. (3.1.15) are the conformal weights of the operators $\mathcal{O}_j$ that are summed over. Before we rewrite the crossing equation in the standard form of the numerical conformal bootstrap, we shall

introduce the so-called *crossing vector* as

$$\tilde{F}_{\Delta_1, \Delta_2, \Delta_3}(\chi_1, \chi_2, \chi_3) = G_{\Delta_1, \Delta_2, \Delta_3}^{123, 456}(\chi_1, \chi_2, \chi_3) - Q G_{\Delta_1, \Delta_2, \Delta_3}^{123, 456}(\chi_2, \chi_3, \chi_4) . \quad (3.1.16)$$

This allows us to bring eq. (3.1.15) into the following very concise form

$$\sum_{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3} C_{\mathcal{O}_2}^{\mathcal{O}_1} \tilde{F}_{\Delta_1, \Delta_2, \Delta_3}(\chi_1, \chi_2, \chi_3) C_{\mathcal{O}_2}^{\mathcal{O}_3} = 0 . \quad (3.1.17)$$

Writing the crossing equation in this way makes manifest that it can be seen as a sum of matrices labelled by Δ_2 , which are contracted with two copies of the vector $C_{\mathcal{O}_2} \equiv (C_{\mathcal{O}_2}^{\mathcal{O}})_{\mathcal{O}}$ whose components are labelled by primaries $\mathcal{O}$ that appear in the OPE of two external scalars ϕ . Since we are contracting the crossing vector $\tilde{F}_{\Delta_1, \Delta_2, \Delta_3}(\chi_1, \chi_2, \chi_3)$ with a symmetric tensor, we can symmetrise with respect to the indices Δ_1 and Δ_3 to deduce

$$\sum_{\mathcal{O}_2} \left(\sum_{\mathcal{O}_1 \mathcal{O}_3} C_{\mathcal{O}_2}^{\mathcal{O}_1} F_{\Delta_1 \Delta_2 \Delta_3}(\chi_1, \chi_2, \chi_3) C_{\mathcal{O}_2}^{\mathcal{O}_3} \right) = 0 . \quad (3.1.18)$$

In passing from equation (3.1.17) to the new crossing equation (3.1.18), we have introduced the symmetrised crossing vector F as

$$F_{\Delta_1, \Delta_2, \Delta_3} \equiv \tilde{F}_{\Delta_1, \Delta_2, \Delta_3} + \tilde{F}_{\Delta_3, \Delta_2, \Delta_1} . \quad (3.1.19)$$

The symmetrization procedure we applied in order to get to the crossing equation (3.1.18) can also be understood as the result of adding a second crossing equation to the original one that was given in eq. (3.1.15). The second equation we need to add is obtained from the first one by exchanging Δ_1 and Δ_3 . Hence, our symmetrisation boils down to adding the crossing equation for the comb-channels 654, 123 and 543, 216.

Following the usual lore of the numerical conformal bootstrap, we do not directly study eq. (3.1.18) as a functional equation for arbitrary χ_i . Instead, we choose some finite set of linear functionals $\{\alpha_k\}_{k=1}^{N_\alpha}$ acting on the crossing vector F . These functionals then allow us to focus on the finite subset

$$\sum_{\mathcal{O}_2} \left(\sum_{\mathcal{O}_1 \mathcal{O}_3} C_{\mathcal{O}_2}^{\mathcal{O}_1} \alpha_k(F_{\Delta_1 \Delta_2 \Delta_3}) C_{\mathcal{O}_2}^{\mathcal{O}_3} \right) = 0 , \quad k = 1, \dots, N_\alpha , \quad (3.1.20)$$

of the a priori infinitely many constraints imposed by the crossing equation (3.1.18). An obvious candidate for the choice of functionals α is to apply differential operators $\mathcal{D}_k$ in the three cross-ratios and to evaluate the result for some fixed kinematics². The canonical prescription for the evaluation is the crossing symmetric point which is $\chi_j = 1/3$ in our case. We shall adopt these choices throughout the rest of the work.

²See however [181, 182] for recent usage of other classes of functionals in the numerical bootstrap.

Thanks to the symmetrisation procedure above, we can now define a family of *symmetric* bilinear forms $\{B_{\Delta_2}^k\}_{k=1}^{N_\alpha}$. The arguments of these forms are vectors $C = (C^\mathcal{O})_\mathcal{O}$ whose components are labelled by the primary fields $\mathcal{O}$ that appear in the OPE of two external scalars ϕ . Given any two such vectors C_L and C_R , the bilinear forms are introduced as

$$B_{\Delta_2}^k(C_L, C_R) := \sum_{\mathcal{O}_1 \mathcal{O}_3} C_L^{\mathcal{O}_1} \alpha_k(F_{\Delta_1 \Delta_2 \Delta_3}) C_R^{\mathcal{O}_3} \quad (3.1.21)$$

for all Δ_2 and $k = 1, \dots, N_\alpha$. In other words, the bilinear forms $B_{\Delta_2}^k$ can simply be thought of as matrices whose rows and columns are labelled by the operators exchanged in the $\phi \times \phi$ OPE, namely

$$(B_{\Delta_2}^k)_{\Delta_1, \Delta_3} = \alpha_k(F_{\Delta_1 \Delta_2 \Delta_3}). \quad (3.1.22)$$

With this definition, our crossing equation (3.1.20) finally takes the advocated form

$$\sum_{\mathcal{O}_2} B_{\Delta_2}^k(C_{\mathcal{O}_2}, C_{\mathcal{O}_2}) = 0, \quad k = 1, \dots, N_\alpha. \quad (3.1.23)$$

Equation (3.1.23) is the starting point of the SDP based six-point bootstrap: As in the four-point bootstrap, the existence of positive definite linear combinations of the bilinear forms B^k puts rigorous constraints on the spectra of CFTs. For example, if one can find numerical coefficients $c_k, k = 1, \dots, N_\alpha$, such that

$$\sum_k c_k B_{\Delta_2}^k > 0 \quad \text{for all } \Delta_2 \geq \Delta_2^*, \quad (3.1.24)$$

one can conclude that the leading operator in the $\mathcal{O}_2$ OPE exchange must have conformal weight $\Delta_2 < \Delta_2^*$. This follows by contracting with the vector $C_{\mathcal{O}_2}$ of eq. (3.1.14) and using equation (3.1.23). Put differently, the existence of numerical coefficients c_k that satisfy the condition (3.1.24) provides an upper bound of Δ_2^* on the leading operator that is exchanged in the middle leg.

The proposition for a numerical six-point bootstrap discussed so far is close to the usual four-point bootstrap formulated directly in terms of OPE coefficients and conformal blocks. However, the numerical search for solutions of eq. (3.1.24) in the framework of this subsection is quite challenging: If one naively discretises these equations by considering some finite list of possible values for the weights Δ_1, Δ_2 and Δ_3 , one obtains large dense matrices that lack a particularly useful asymptotic behaviour which would control the convergence of the discretisation procedure. Finding an alternative formulation of eq. (3.1.23) which is more suitable for numerics, is the main goal of the next subsection.

3.1.3 The descendant-space formulation of six-point crossing

We now refine the ideas of the previous subsections and introduce a new set of bilinear forms $b_{\Delta_2}^k$, see eq. (3.1.30), that will become central for the subsequent

numerical analysis. The transition from the bilinear forms $B_{\Delta_2}^k$ we introduced in the previous Subsection 3.1.2 to the new forms $b_{\Delta_2}^k$ reduces the complexity of the corresponding SDPs and it makes the constraints that we impose physically more transparent. Ultimately though, our new formulation turns out to be equivalent to that of the previous section.

The transition to our second formulation starts with the following representation of the comb channel six-point blocks (3.1.8),

$$G_{\Delta_1, \Delta_2, \Delta_3}(\chi_1, \chi_2, \chi_3) = \sum_n F_n(\Delta_1, \chi_1) \frac{\chi_2^{n+\Delta_2}}{(2\Delta_2)_n n!} F_n(\Delta_3, \chi_3), \quad (3.1.25)$$

where we have absorbed sums over n_1 and n_3 in eq. (3.1.9) into the definition of the functions

$$F_n(\Delta_i, \chi_i) := (\Delta_i + \Delta_2 - \Delta_\phi)_n \chi_i^{\Delta_i} {}_2F_1(\Delta_i, \Delta_i + \Delta_2 - \Delta_\phi + n, 2\Delta_i, \chi_i). \quad (3.1.26)$$

Up to normalisation, this is the four-point conformal block for the exchange of $\mathcal{O}_i$ in a correlator of three operators with dimension Δ_ϕ and one (primary) operator with dimension $\Delta_2 + n$. It is tempting to physically identify the operator of dimension $\Delta_2 + n$ with a descendant of $\mathcal{O}_2$. Section 3.1.5 clarifies the exact relation to the correlator $\langle \phi \phi \phi \mathcal{O}_2^{(n)} \rangle$, where $\mathcal{O}_2^{(n)}$ is a level n descendant of $\mathcal{O}_2$.

The functions F_n possess a very helpful feature: Derivatives with respect to the cross ratio simply act by shifting the index n . Indeed, it is easy to see that

$$\chi_1 \partial_{\chi_1} F_n(\Delta_1, \chi_1) = F_{n+1}(\Delta_1, \chi_1) - (\Delta_2 + n - \Delta_\phi) F_n(\Delta_1, \chi_1), \quad (3.1.27)$$

where crucially the coefficients in front of the F_n functions on the r.h.s. do not depend on Δ_1 . Of course, there is a similar formula with $1 \rightarrow 3$. One can view eq. (3.1.27) simply as a mathematical consequence of the hypergeometric nature of comb-channel blocks. However, the interpretation of F_n in terms of correlators of descendants alluded to above allows us to assign some physical meaning to eq. (3.1.27) in Section 3.1.5.

For our program, eq. (3.1.27) is very useful. Recall that the linear functionals α we introduced in the previous subsection involve acting with some partial differential operators in the three cross ratios. Now we see that the action of a such a differential operator on the comb blocks preserves the bilinear structure of equation (3.1.25). Concretely, acting with a differential operator $\mathcal{D}$ of order Λ results in

$$\mathcal{D} G_{\Delta_1, \Delta_2, \Delta_3}(\chi_1, \chi_2, \chi_3) = \sum_{n, m} F_n(\Delta_1, \chi_1) (M[\mathcal{D}])_{nm} F_m(\Delta_3, \chi_3), \quad (3.1.28)$$

where $M[\mathcal{D}]$ is a bandmatrix whose bandwidth is bounded from above by the order Λ of $\mathcal{D}$ with respect to χ_1 and χ_3 , and whose entries are functions of *only* Δ_2 and

the cross-ratios. Consequently, if the N_α functionals α_k with which we act on the crossing equation are given by evaluating partial derivatives with respect to the three cross-ratios at the crossing symmetric point

$$\chi_1 = \chi_2 = \chi_3 = 1/3, \quad (3.1.29)$$

then the bilinear form corresponding to a functional α_k of derivative order Λ_k has the form

$$B_{\Delta_2}^k(C_L, C_R) = \sum_{n,m} F_n(C_L) M_{nm}^k(\Delta_2) F_m(C_R) =: b_{\Delta_2}^k(F(C_L), F(C_R)), \quad (3.1.30)$$

where the bandwidth of M^k is bounded above by Λ_k . The symbol $F(C)$ denotes the sequence $(F_n(C))_{n \in \mathbb{N}}$ defined by

$$F_n(C) := \sum_{\mathcal{O}} F_n(\Delta_{\mathcal{O}}, 1/3) C^{\mathcal{O}}. \quad (3.1.31)$$

In other words, the bilinear form $B_{\Delta_2}^k$ is the pullback of a bilinear form $b_{\Delta_2}^k$ defined on some real sequence space with respect to the linear map F . Hence, $B_{\Delta_2}^k$ is positive semi-definite if and only if $b_{\Delta_2}^k$ is positive semi-definite on the image of F .

Based on numerical tests, we consider it highly plausible that the map F is surjective and that therefore the positive semi-definiteness of $b_{\Delta_2}^k$ is equivalent to that of $B_{\Delta_2}^k$. This strongly supports the claimed equivalence between the two pictures of crossing that we have presented. Note, however, that independently of the surjectivity of the linear map F , a positive semi-definite linear combination of the $b_{\Delta_2}^k$ always corresponds to a positive semi-definite combination of the $B_{\Delta_2}^k$. Thus, establishing surjectivity more rigorously would only improve our a posteriori understanding for why our methods are effective, but it is not necessary to justify their validity.

The main practical consequence of this discussion is that, instead of constructing positive semi-definite linear combinations of the bilinear forms $B_{\Delta_2}^k$, we can equally well work with the $b_{\Delta_2}^k$. Hence, our second formulation of the SDP problem for six-point crossing amounts to finding numerical coefficients c_k that satisfy

$$\sum_k c_k b_{\Delta_2}^k > 0 \quad \text{for all} \quad \Delta_2 \geq \Delta_2^*. \quad (3.1.32)$$

We shall refer to this formulation as *descendant-space* SDP problem and to the objects b^k it involves as *descendant-space bilinear forms*. This is to be contrasted with the original *primary-space* SDP problem (3.1.24) which is based on the *primary-space bilinear forms* B^k .

It turns out that the descendant-space formulation of the SDP problem has a number of important advantages over the primary-space formulation. Let us briefly list them here.

1. One immediate qualitative difference between descendant- and primary-space is that in the descendant-space we do not need to evaluate the six-point block themselves, only establish differential relations between them. This is conceptually quite different from what is done in the standard four-point bootstrap, and suggests that a higher-dimensional generalisation might be possible, even without full control over the blocks.
2. A second difference is that the linear map (3.1.31) sends the continuous parameter $\Delta_{\mathcal{O}}$ to the discrete label n . Any attempt to numerically construct positive semi-definite linear combinations of the bilinear forms that we consider necessarily has to address their infinite dimensional nature, in practice by some truncation to finite submatrices. In primary-space, this truncation requires both cutting-off the matrices at a maximal scaling dimension Δ_{cut} and a choice of discretisation of the interval $[0, \Delta_{cut}]$. In contrast to this, the discrete nature of the descendant-space label n allows for a truncation of the bilinear forms to finite matrices simply by picking a maximal value for their indices.³
3. The third and most important advantage of the descendant-space approach is that the bilinear forms b^k it works with are banded. This suggests that they are much easier to handle numerically than the dense matrices B^k encountered in primary-space.

These advantages, however, could not be profited of, without firm control over the convergence of the truncation procedure that we necessarily have to implement. Thus, we devote the next subsection to the asymptotic structure of the banded matrices occurring in descendant-space.

3.1.4 Large dimension asymptotics and the Toeplitz arc

This subsection is organised in three logical steps that lead from simple asymptotic properties of the descendant-space bilinear forms to an algorithm that stabilises the convergence of truncation schemes for SDPs involving these bilinear forms. We begin by making observations about the asymptotics of descendant-space bilinear forms for large Δ_2 as well as far down their diagonal. Concretely, these are captured by eqs. (3.1.35) and (3.1.36). This sets the stage for a concise characterisation of the asymptotics of general large Δ_2 and n regimes with fixed ratio Δ_2/n . It is in this characterisation, that a curve of Toeplitz matrices enters that is referred to in the subsection title. Let us recall that a matrix $\mathcal{T}$ is called Toeplitz if it is of the form

$$\mathcal{T}_{ij} = \sum_{n=-\infty}^{\infty} \delta_{i-j}^n a_n, \quad (3.1.33)$$

³In both cases, we still truncate and discretise with respect to the Δ_2 variable.

where $(a_n)_{n \in \mathbb{Z}}$ is a bounded family of complex numbers. In the final step, we establish that positive semi-definiteness of the matrices constituting the Toeplitz arc is equivalent to positivity of certain polynomials. This is the central result of this subsection and is concisely formulated in eq. (3.1.52).

Let us start by using the elementary fact that positive semi-definiteness is a basis independent property so that we are free to conjugate the bilinear forms under consideration by an arbitrary invertible matrix. Concretely, positive semi-definiteness of some linear combination of bilinear forms $\{b_{\Delta_2}^k\}_{k=1}^{N_\alpha}$ associated, in the sense of Section 3.1.3, to partial derivatives of order Λ_k , is most conveniently shown after conjugation with the matrix

$$N_{ij} \equiv \sum_{n=0}^{\infty} \delta_i^n \delta_j^n \sqrt{\frac{3^n (2\Delta_2)_n n!}{(\Delta_2 + n)^\Lambda}} \quad \text{where} \quad \Lambda \equiv \max_{k=1, \dots, N_\alpha} \Lambda_k. \quad (3.1.34)$$

It follows directly from eq. (3.1.27) that the matrix $\hat{M}^k$ obtained by conjugating the M^k defined in eq. (3.1.30) with N , has the following asymptotic properties

$$\hat{M}_{nm}^k(\Delta_2) \sim \Delta_2^{\Lambda_k - \Lambda - \frac{1}{2}|n-m|} \quad \text{for } \Delta_2 \gg 1 \quad \text{and } n \text{ fixed}, \quad (3.1.35)$$

$$\hat{M}_{nm}^k(\Delta_2) \sim n^{\Lambda_k - \Lambda} \quad \text{for } n \gg 1 \quad \text{and } \Delta_2 \text{ fixed}. \quad (3.1.36)$$

To see this statement in action, let us consider as an example the functional associated to the first derivative ∂_{χ_1} . Acting with this functional on the crossing vector of eq. (3.1.19) produces the combination

$$\partial_{\chi_1} F_{\Delta_1, \Delta_2, \Delta_3} = -18\Delta_\phi G_{\Delta_1, \Delta_2, \Delta_3} + \partial_{\chi_1} G_{\Delta_1, \Delta_2, \Delta_3} + \partial_{\chi_3} G_{\Delta_1, \Delta_2, \Delta_3} \quad (3.1.37)$$

of derivatives of the comb-channel block evaluated at the crossing symmetric point. Using eqs. (3.1.25) and (3.1.27), this can be rewritten as

$$\partial_{\chi_1} F_{\Delta_1, \Delta_2, \Delta_3} = \sum_{n, m=0}^{\infty} F_n \left(\Delta_1, \frac{1}{3} \right) \frac{\delta_{n-1}^m + \delta_{n+1}^m - 2\delta_n^m (\Delta_2 + n + 2\Delta_\phi)}{3^{n-1} (2\Delta_2)_n n!} F_m \left(\Delta_1, \frac{1}{3} \right). \quad (3.1.38)$$

Hence, the matrix associated to this functional by eq. (3.1.30) is

$$M_{nm}^{\partial_{\chi_1}} = \frac{\delta_{n-1}^m + \delta_{n+1}^m - 2\delta_n^m (\Delta_2 + n + 2\Delta_\phi)}{3^{n-1} (2\Delta_2)_n n!}. \quad (3.1.39)$$

After conjugation by N , defined here with $\Lambda = 1$, we end up with

$$\hat{M}_{nm}^{\partial_{\chi_1}} = 3\sqrt{3} \sqrt{\frac{n+1}{2\Delta_2 + n + 1}} (\delta_{n-1}^m + \delta_{n+1}^m) - \frac{6(\Delta_2 + n + 2\Delta_\phi)}{2\Delta_2 + n} \delta_n^m \quad (3.1.40)$$

which has the anticipated asymptotic properties (3.1.35) and (3.1.36) for $\Lambda - \Lambda_k = 0$.

As announced in the beginning of this subsection, let us now move on to the more general case where both Δ_2 and n are taken to infinity with the ratio Δ_2/n fixed. To describe this regime, we associate to each derivative functional α_k the following one-parameter family of real symmetric Toeplitz matrices:

$$\mathcal{T}_{ij}^k(\theta) \equiv \lim_{n \rightarrow \infty} \hat{M}_{n+i, n+j}^k(n \tan(\theta)), \quad \theta \in [0, \pi/2[. \quad (3.1.41)$$

Clearly, eqs. (3.1.35) and (3.1.36) imply that $\mathcal{T}^k$ vanishes for $\Lambda_k < \Lambda$ and that $\mathcal{T}^k$ approaches a multiple of the identity matrix as θ approaches $\pi/2$. For $\Lambda_k = \Lambda$ and generic values of θ , however, one obtains a banded Toeplitz matrix with bandwidth Λ_k . In the example of the single derivative with respect to the cross ratio χ_1 , this Toeplitz matrix takes the form

$$\mathcal{T}_{ij}^{\partial\chi_1}(\theta) = \sqrt{\frac{3}{1+2\tan(\theta)}}(\delta_{j-1}^i + \delta_{j+1}^i) - \frac{6(1+\tan(\theta))}{1+2\tan(\theta)}\delta_j^i. \quad (3.1.42)$$

A necessary condition for the positive semi-definiteness of a linear combination of functionals for all values of Δ_2 larger than some arbitrary finite value Δ_* is that the corresponding linear combination of asymptotic Toeplitz matrices is positive semi-definite for all values of θ , i.e. our SDP condition (3.1.32) implies that

$$\mathcal{T}(\theta) \equiv \sum_{k=1}^{N_\alpha} c_k \mathcal{T}^k(\theta) \geq 0 \quad \text{for all } \theta \in [0, \pi/2[. \quad (3.1.43)$$

What makes this necessary criterion for positive semi-definiteness especially useful is that it is easy to check. To do so, we rely on the following basic result of the theory of Toeplitz operators (see for instance section 1.5 of [183]).

Theorem (Spectra of Toeplitz operators.). *Let $\mathcal{T}$ be the Toeplitz operator, i.e. an infinite matrix of the form (3.1.33). Define the symbol of $\mathcal{T}$ as the curve*

$$\gamma_{\mathcal{T}} : [0, 2\pi] \rightarrow \mathbb{C}, t \mapsto \sum_{n=-\infty}^{\infty} a_n e^{int}. \quad (3.1.44)$$

Then, $\mathcal{T}$ is a Fredholm operator on $\ell^p(\mathbb{C})$ (for $1 \leq p \leq \infty$) iff $0 \notin \gamma_{\mathcal{T}}([0, 2\pi])$. In that case, the Fredholm Index of $\mathcal{T}$ is minus the winding number of $\gamma_{\mathcal{T}}$ around 0. Thus, the essential spectrum of $\mathcal{T}$ is $\gamma_{\mathcal{T}}([0, 2\pi])$. The spectrum of $\mathcal{T}$ is the set of points around which $\gamma_{\mathcal{T}}$ has a non trivial winding.

This theorem implies that the Toeplitz matrix $\mathcal{T}(\theta)$ from equation (3.1.41) is positive definite iff its symbol

$$\gamma_{\mathcal{T}(\theta)}(t) = \mathcal{T}_{00}(\theta) + 2 \sum_{n=1}^{\Lambda} \mathcal{T}_{0n}(\theta) \cos(nt) \quad (3.1.45)$$

is positive for all $t \in [0, 2\pi]$. It is actually possible to rephrase this as a positivity condition on a polynomial in two variables. In a first step we shall employ the transformation

$$\cos(t) = \frac{x^2 - 1}{1 + x^2}, \quad (3.1.46)$$

to turn the trigonometric dependence in t into a rational dependence on the new variable x . Using the Chebyshev polynomials T_n the positivity condition on the symbol (3.1.45) can be rewritten as

$$\gamma_{\mathcal{T}(\theta)}(x) \equiv \mathcal{T}_{00}(\theta) + 2 \sum_{n=1}^{\Lambda} \mathcal{T}_{0n}(\theta) T_n\left(\frac{x^2 - 1}{1 + x^2}\right) > 0, \quad \text{for all } x \in \mathbb{R}, \quad (3.1.47)$$

which in turn is equivalent to the positivity of the polynomial

$$p_{\mathcal{T}(\theta)}(x) \equiv (1 + x^2)^{\Lambda} \gamma_{\mathcal{T}(\theta)}(x). \quad (3.1.48)$$

Clearly, $p_{\mathcal{T}}(x)$ is linear in $\mathcal{T}$, so

$$p_{\mathcal{T}(\theta)}(x) = \sum_{k=1}^{N_{\alpha}} c_k p_{\mathcal{T}^k(\theta)}(x). \quad (3.1.49)$$

Now we also want to turn the dependence on the variable θ into a rational function. To this end, we can perform the reparametrisation

$$\tan(\theta) = \frac{(1 + y^2)^2 - 1}{2}. \quad (3.1.50)$$

This indeed guarantees that the dependence of the functions $p_{\mathcal{T}^k(\theta)}(x)$ on the position θ along the arc becomes rational too. By multiplying with an appropriate positive factor, one ends up with a polynomial dependence in the variable y . We denote by $P_k(x, y)$ the resulting two variable polynomial associated to a functional α_k .

To illustrate the construction of the polynomial P , let us consider once again our example of the functional that is given by the derivative with respect to χ_1 . From the explicit expression for the arc of asymptotic Toeplitz matrices associated to ∂_{χ_1} that we spelled out in eq. (3.1.42), we find

$$p_{\mathcal{T}^{\partial_{\chi_1}}(\theta)}(x) = 6\sqrt{3}(x^2 - 1) \sqrt{\frac{1}{2 \tan(\theta) + 1}} - \frac{6(x^2 + 1)(\tan(\theta) + 1)}{2 \tan(\theta) + 1}. \quad (3.1.51)$$

Applying transformation (3.1.50) and multiplying by $(1 + y^2)^2$, we obtain the following polynomial

$$P_{\partial_{\chi_1}}(x, y) = -3x^2y^4 + (6\sqrt{3} - 6)x^2y^2 + (6\sqrt{3} - 6)x^2 - 3y^4 - (6 + 6\sqrt{3})y^2 - 6\sqrt{3} - 6.$$

Thereby, we have now understood how to rephrase the positivity condition (3.1.44) as a positivity condition for the associated polynomial $P(x, y)$. Since we argued before that the positivity condition (3.1.44) was necessary for positive semi-definiteness of some linear combination of functionals, we now conclude that our SDP condition (3.1.32) implies that

$$P(x, y) \equiv \sum_{k=1}^{N_\alpha} c_k P^k(x, y) \geq 0 \quad \text{for all} \quad (x, y) \in \mathbb{R}^2 \quad (3.1.52)$$

i.e. in order to construct positive semi-definite linear combinations of the infinite dimensional bilinear forms $\{b^k\}_{k=1}^{N_\alpha}$, one has to construct non-negative linear combinations of the associated two-variable polynomials P^k . A sufficient (but famously not necessary condition) for this is that the polynomial is a sum of squares (SOS). The task of producing a linear combination of a set of polynomials which is an SOS, is equivalent to an SDP and can thus conveniently be incorporated in an SDP based numerical bootstrap algorithm (see Appendix 3.C for details).

3.1.5 Comments on the descendant-space formulation

Our discussion above and in particular the passage from primary-space to descendant-space SDP appeared rather technical since we based the reformulation on the particular property (3.1.25) of six-point comb channel blocks in $d = 1$ dimension. Our main aim in this final subsection is to show that the rewriting we proposed above does actually not depend on the concrete blocks but rather on features of the crossing equation we consider. More concretely, we shall highlight that the descendant-space SDP is based on a crossing between two distinct decompositions of a six-point function into four-point correlators instead of the typical OPE decomposition into three-point functions.

As usual, we begin our discussion of the crossing equation with two different insertions of the identity operator inside the six-point correlator, i.e.

$$\sum_{\mathcal{O}, \underline{n}, \underline{m}} \langle \phi_6 \phi_5 \phi_4 \mathcal{O}_{\underline{n}} \rangle \mathcal{N}^{\underline{n}, \underline{m}} \langle \mathcal{O}_{\underline{m}}^\dagger \phi_3 \phi_2 \phi_1 \rangle = \sum_{\mathcal{O}, \underline{n}, \underline{m}} \langle \phi_4 \phi_3 \phi_2 \mathcal{O}_{\underline{n}} \rangle \mathcal{N}^{\underline{n}, \underline{m}} \langle \mathcal{O}_{\underline{m}}^\dagger \phi_6 \phi_5 \phi_1 \rangle. \quad (3.1.53)$$

Here, the sum extends over all primaries $\mathcal{O}$ of the theory and over a basis of descendants that are labelled by some index $\underline{n}$. In a one-dimensional theory, $\underline{n} = n = 0, 1, 2, \dots$ is just a non-negative integer that counts the number of derivatives that are needed in order to construct $\mathcal{O}_n$ from the corresponding primary. With this choice of basis in the space of descendants, it is easy to compute the normalisation matrix $\mathcal{N}$. Concretely,

$$\mathcal{O}_n = P^n \mathcal{O} \quad \text{gives} \quad \mathcal{N}^{nm} = \frac{\delta_{nm}}{(2\Delta_{\mathcal{O}})_n n!}, \quad (3.1.54)$$

with $P = \partial_x$, the momentum generator. Here, we assumed that the primary fields are normalised canonically. The decomposition formula (3.1.53) focuses on a symmetric split of the fields into two groups of three fields. This is similar to the symmetric split of four-point functions. In the case of four fields, such a symmetric split is actually the only non-trivial one. For higher point correlations, however, there exists other non-trivial decompositions (i.e. the decompositions into a three- and a five-point function). One advantage of the decomposition into four-point correlators is that it contains new data beyond that of the $\phi \times \phi$ OPE. More importantly, the symmetric decomposition leads to a crossing equation that has a clear SDP dual, namely the descendant-space SDP introduced in Section 3.1.3.

To establish this, let us further analyse the quantities that appear in the crossing equation (3.1.53). Through Ward identities, four-point functions that involve the descendants $\mathcal{O}_n$ of a primary field $\mathcal{O}$ can be reduced to derivatives of the underlying correlators of $\mathcal{O}$. If we were analysing the decomposition of four-point correlators into three-point functions, the application of Ward identities would allow us to sum the contributions of the descendants into conformal blocks. The corresponding SDPs hence do not have to deal with matrices labelled by descendants as in eq. (3.1.53), but rather with kinematically fixed functions multiplied by an unknown number, which is the square of an OPE coefficient. In the case of six-point functions, the theory dependent four-point functions of primaries are certainly not just numbers but rather functions of some cross ratios. These functions are the unknowns in eq. (3.1.53) and contain all dynamical data. We show in Appendix 3.A how to directly derive by application of Ward identities and conformal transformations that

$$\sum_{n=0}^{\infty} \frac{\langle \phi_6 \phi_5 \phi_4 \mathcal{O}_n \rangle \langle \mathcal{O}_n^\dagger \phi_3 \phi_2 \phi_1 \rangle}{(2\Delta_{\mathcal{O}})_n n!} = \sum_{n=0}^{\infty} \frac{(\bar{\chi}_1 \chi_2 \bar{\chi}_3)^{\Delta_{\mathcal{O}}+n-\Delta_{\phi}} \partial_{\bar{\chi}_1}^n f(\bar{\chi}_1) \partial_{\bar{\chi}_3}^n f(\bar{\chi}_3)}{(x_{12}^2 x_{34}^2 x_{56}^2)^{\Delta_{\phi}} (2\Delta_{\mathcal{O}})_n n!}, \quad (3.1.55)$$

where

$$\bar{\chi} \equiv \frac{1}{\chi} \quad f(\bar{\chi}) \equiv \langle \phi(\infty) \mathcal{O}(\bar{\chi}) \phi(1) \phi(0) \rangle. \quad (3.1.56)$$

Eq. (3.1.55) implies that in terms of

$$F_n(\chi) \equiv (-1)^n (\bar{\chi})^{\Delta_{\mathcal{O}}+n-\Delta_{\phi}} \partial_{\bar{\chi}}^n f(\bar{\chi}), \quad (3.1.57)$$

which satisfies the recurrence relation

$$\chi \partial_{\chi} F_n(\chi) = F_{n+1}(\chi) - (\Delta_{\mathcal{O}} + n - \Delta_{\phi}) F_n(\chi) \quad (3.1.58)$$

we can write

$$\sum_{n=0}^{\infty} \frac{\langle \phi_6 \phi_5 \phi_4 P^n \mathcal{O} \rangle \langle \mathcal{O}^\dagger K^n \phi_3 \phi_2 \phi_1 \rangle}{(2\Delta_{\mathcal{O}})_n n!} = \mathcal{L}(x_i) \sum_{n=0}^{\infty} F_n(\chi_1) \frac{\chi_2^{\Delta_{\mathcal{O}}+n}}{(2\Delta_{\mathcal{O}})_n n!} F_n(\chi_3), \quad (3.1.59)$$

with

$$\mathcal{L}(x_i) = (x_{12}^2 x_{34}^2 x_{56}^2 \chi_2)^{-\Delta_\phi}. \quad (3.1.60)$$

Comparing this to Section 3.1.3, it is clear that eq. (3.1.59) is exactly a decomposition of the six-point correlator in the form of eq. (3.1.25), while the relation eq. (3.1.58) coincides with the relation eq. (3.1.27) on the level of individual blocks. However, the derivation of the descendant-space SDP dual to six-point crossing in Section 3.1.3 only used this decomposition and the recurrence relation. Hence, applying eq. (3.1.59) on both sides of (3.1.53) and repeating the steps of Section 3.1.3 reproduces the descendant-space SDP.

Even though it just reproduces the SDP formulated in Section 3.1.3, we consider this alternative perspective as an important complement to the derivation given in Section 3.1.3. The reason for this is threefold.

Firstly, the derivation establishes that our ability to formulate the descendant-space SDP is not just a consequence of coincidental special properties of the comb-channel blocks. On the contrary, it makes manifest that exact knowledge of the comb-channel blocks is not crucial and highlights the actually necessary physical input.

Secondly, it thus shows which conceptual aspects of our approach could potentially be useful in other settings beyond the 1d conformal six-point correlators we consider here. For instance, one could in principle follow the same steps as in Appendix 3.A to find a higher dimensional formulation of numerical six-point bootstrap, that does not require the evaluation of six-point blocks. In attempting to pursue this, however, one has to overcome two technical hurdles. Clearly, the structure of the projectors onto a conformal multiplet becomes more complicated in higher dimensions i.e. it is less obvious what the optimal choice for a basis of descendants would be in $d > 2$. Relatedly, it is not clear what the correct choice of functionals should be if one would like to maintain the band-structure that makes the descendant-space matrices so amenable to numerical implementation in 1d. Recall that the band structure arises in 1d because acting with a differential operator of degree Λ on a level n descendant, independently of n can only produce a linear combination of $\Lambda + 1$ descendants (namely those at levels $n, n+1, \dots, n+\Lambda$). In higher d however, the action of a generic degree Λ differential operator on a level n descendant can produce a linear combination of up to $(n + \Lambda)^d - (\max(0, n - 1))^d$ states (since at level k there are k^d states). Hence, one should expect that a careful choice of basis both for the descendants and for the functionals will be important for the generalisation to higher space-time dimension.

Finally, we would like to point out that the perspective taken in this subsection also makes it clearer how to extract the four-point crossing content of the six-point crossing equation. We provide some comments on this in Appendix 3.B.

3.2 First Bounds: Gap Maximisation without Identity

Having established an SDP formulation for six-point crossing (3.1.32), as well as the necessary asymptotic conditions (3.1.52), we now study a simple bootstrap problem where we can compare numerical results to concrete analytical expectations. We dub this first problem “gap maximisation without identity”.

Gap maximisation without identity amounts to finding the largest possible value for the conformal dimensions Δ_2 of the leading operator $\mathcal{O}_2$ in the triple operator product $(\phi \times \phi) \times \phi \sim \mathcal{O}_2$ of the external field ϕ , subject to crossing symmetry and unitarity. Of course, the typical scenario is that the $\phi \times \phi$ OPE contains the identity field. In this case we would expect that the leading operator in the triple OPE is simply given by $\mathcal{O}_2 = \phi$. Hence, it might seem most natural to assume that ϕ is exchanged in the middle channel and to study the gap above it, and we will indeed do so in Section 3.3. On the other hand, nothing stops us from considering the technically somewhat simpler situation where the leading operator $\mathcal{O}_1$ in the double OPE $\phi \times \phi$ is nontrivial and where the OPE of this operator $\mathcal{O}_1$ with ϕ produces some leading contribution $\mathcal{O}_2 \neq \phi$. In this setup, it makes sense to directly study bounds on the leading exchange $\mathcal{O}_2$.⁴

As is standard in optimisation problems, it is useful to formulate both a primal and a dual approach to gap maximisation without identity. Concretely, these take the following form

Gap Maximisation without identity (primal formulation): Find the maximal value Δ_{opt}^P for which there exists a family $K^{\mathcal{O}_1 \mathcal{O}_3}(\Delta_2) = C_{\mathcal{O}_2}^{\mathcal{O}_1} C_{\mathcal{O}_2}^{\mathcal{O}_3}$ of positive semi-definite kernels parametrised by $\Delta_2 \in \mathbb{R}_{>0}$ such that

$$G(\chi_i) = \sum_{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3} C_{\mathcal{O}_2}^{\mathcal{O}_1} C_{\mathcal{O}_2}^{\mathcal{O}_3} G_{\Delta_1 \Delta_2 \Delta_3}^{123,456}(\chi_i) \quad (3.2.1)$$

is a crossing symmetric function and $K(\Delta_2) = 0$ for all $\Delta_2 < \Delta_{\text{opt}}^P$.

Gap Maximisation without identity (dual formulation): Find the minimal value Δ_{opt}^D for which there exists a functional α such that the corresponding family b_{Δ_2} of bilinear forms parameterised by $\Delta_2 \in \mathbb{R}_{>0}$ is positive semi-definite for all $\Delta_2 > \Delta_{\text{opt}}^D$.

⁴Some intuition can be gained by considering the four-point analogue of this problem. What is the upper bound on the dimension $\Delta_{\mathcal{O}}$ in a four-point function $\langle \phi \phi \phi \phi \rangle$, where $\phi \times \phi \sim \mathcal{O} + \dots$? Running the standard numerical four-point bootstrap indicates that $\Delta_{\mathcal{O}, \text{max}} = 4\Delta_{\phi}/3$ [184]. This can be identified as a solution to crossing without an identity exchange which obstructs the existence of bounds on certain OPE coefficients in 1d CFT [185, 186]. It can be written as the correlator of a $:\varphi^3:$ type correlator in GFF where each constituent field is Wick contracted with exactly one field in the other 3 points, in complete analogy to the six-point solution we will describe below.

In practice, we address the primal problem by constructing an exact solution to crossing by hand and reading off the value Δ_*^P of the leading term $\mathcal{O}_2$ in the middle channel. This value may not be $\Delta_{\text{opt.}}^P$, but it clearly sets a lower bound $\Delta_*^P \leq \Delta_{\text{opt.}}^P$. Similarly the solutions to the dual problem that we construct via SDP do not necessarily give $\Delta_{\text{opt.}}^D$, but rather an upper bound $\Delta_*^D \geq \Delta_{\text{opt.}}^D$, i.e.

$$\Delta_*^P \leq \Delta_{\text{opt.}}^P \leq \Delta_{\text{opt.}}^D \leq \Delta_*^D . \quad (3.2.2)$$

If the determined values Δ_*^P and Δ_*^D are close together,⁵ they are good approximations to the solution of the primary and the dual problem. Since we mostly discuss Δ_*^D and only refer to Δ_*^P in relatively small portions of the text, we usually use the short notation

$$\Delta_* \equiv \Delta_*^D . \quad (3.2.3)$$

In the first subsection, we shall construct an exact solution to crossing and read off the corresponding value Δ_*^P . Then, we turn to the dual problem in Subsection 3.2. After discussing a simple analytical bound that arises from first order derivative functionals, we shall describe how to write the dual problem in a numerically treatable way that can be fed to a standard SDP solver, in our case SDPA-GMP [187–189], see Subsection 3.2.2. In Subsection 3.2.2, we elaborate on how to thoroughly check positivity of the numerically constructed functionals. The final Subsection 3.2.2 discusses the concrete numerical results and their comparison to the analytical expectation. In particular we shall see that the values Δ_*^P and Δ_*^D we determine from the exact solution and the numerical study of four-derivative functionals are quite close already.

3.2.1 Exact solution to crossing - primal approach

Before discussing the numerical dual approach to gap maximisation without identity, we present the best candidate for an optimal solution to crossing that we managed to explicitly construct. This primal candidate is meant to serve as a reference point for the dual results discussed in the subsequent subsections.

The correlator in question appears in the direct sum of 15 decoupled copies of GFF, with corresponding scalar fields $\{\varphi_i\}_{i=1}^{15}$ of equal scaling dimension Δ_φ . In this simple CFT, consider the composite operators

$$\phi_{a,b,c,d,e} = \varphi_a \varphi_b \varphi_c \varphi_d \varphi_e, \quad (3.2.4)$$

⁵If there was a large duality gap $\Delta_{\text{opt.}}^D - \Delta_{\text{opt.}}^P$, it would of course be impossible to find Δ_*^D and Δ_*^P that are close to each other. It is however a standard result of finite dimensional SDP that $\Delta_{\text{opt.}}^D = \Delta_{\text{opt.}}^P$. The results presented in this work a posteriori suggest that the duality gap is also small or even vanishing for the infinite dimensional SDP that we study.

with $\{a, b, c, d, e\}$ pairwise distinct. These operators all have the same dimension $\Delta_\phi = 5\Delta_\varphi$. The correlator

$$\langle \phi_{1,2,3,4,5} \phi_{1,6,7,8,9} \phi_{2,6,10,11,12} \phi_{3,7,10,13,14} \phi_{4,8,11,13,15} \phi_{5,9,12,14,15} \rangle = \mathcal{L}(x_i) \mathcal{G}(\chi_j) \quad (3.2.5)$$

can be computed easily by applying Wick contractions between all pairs of external operators. The pattern of these contractions shown in Figure 3.3. After splitting

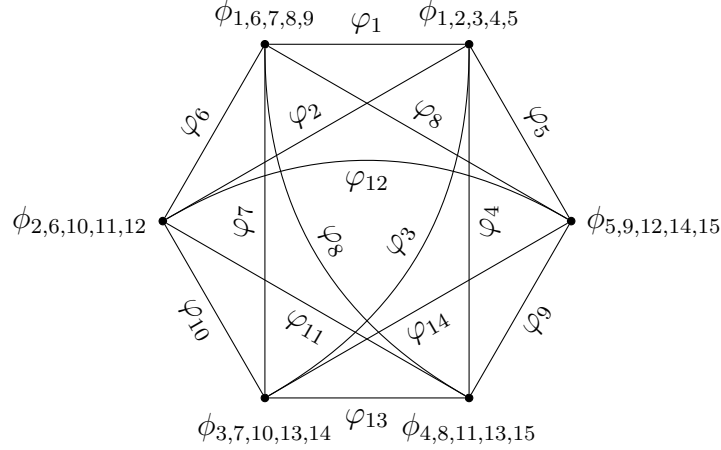


Figure 3.3: Wick contractions leading to the correlator (3.2.5).

off the leg factor $\mathcal{L}$ defined in eq. (3.1.4) and expressing the remaining function $\mathcal{G}$ in terms of the cross ratios (3.1.2), we obtain

$$\mathcal{G}(\chi_j) = \frac{\chi_1^{8\Delta_\varphi} \chi_2^{9\Delta_\varphi} \chi_3^{8\Delta_\varphi}}{\left((1-\chi_1)(1-\chi_2)(1-\chi_3)(1-\chi_1-\chi_2)(1-\chi_2-\chi_3)(1-\chi_1-\chi_2-\chi_3+\chi_1\chi_3) \right)^{2\Delta_\varphi}}. \quad (3.2.6)$$

This expression can now be expanded in a power series. From the exponent of the leading term in each of the cross ratios χ_j , we can read off the conformal dimension $\Delta_{\mathcal{O}_j^*}$ of the leading primary $\mathcal{O}_j^*$ exchanged in the j th internal edge of the comb. Concretely we find

$$\Delta_{\mathcal{O}_1^*} = 8\Delta_\phi/5 \quad \text{and} \quad \Delta_{\mathcal{O}_2^*} = 9\Delta_\phi/5. \quad (3.2.7)$$

A more careful expansion in terms of the conformal blocks (3.1.8) allows us to read off the OPE coefficients, which we then check to satisfy unitarity, as described in Appendix 3.D. We hence conclude that

$$\Delta_{\text{opt.}}^P \geq \Delta_*^P = 9\Delta_\phi/5. \quad (3.2.8)$$

There is another way to arrive at this lower bound for the primal approach, in terms of $\mathbb{Z}_n$ chiral twist fields σ_n . Let us recall that for a parent theory of central charge c the conformal dimension of such a chiral twist field is given by

$$\Delta_{\sigma_n} = \frac{c}{4} \left(\frac{1}{n} - \frac{1}{n^2} \right). \quad (3.2.9)$$

In order to obtain a non-vanishing six-point function without identity exchange in the two outer channels, we consider the case of a $\mathbb{Z}_6$ orbifold and set $\phi = \sigma_6$. The operator product of two such twist fields contains a twist field $\mathcal{O}_1^* = \sigma_3$ of order three while the leading contribution to the intermediate exchange in the middle channel is given by a twist field $\mathcal{O}_2^* = \sigma_2$. Application of the general formula for the conformal dimension of twist fields gives

$$\Delta_\phi = \frac{5}{36} \frac{c}{4}, \quad \Delta_{\mathcal{O}_1^*} = \frac{2}{9} \frac{c}{4} = \frac{8}{5} \Delta_\phi, \quad \Delta_{\mathcal{O}_2^*} = \frac{1}{4} \frac{c}{4} = \frac{9}{5} \Delta_\phi. \quad (3.2.10)$$

The results from this short analysis confirm the conclusion (3.2.8) of the first elementary analysis in terms of generalised free fields, but they do not improve on the lower bound we obtain for the value of Δ_{opt}^P . This concludes our brief discussion of exact solutions to the six-point crossing equation and the primal approach to gap maximisation without identity.

3.2.2 Analytical and numerical bounds - dual approach

In this section, we construct functionals which provide a rigorous upper bound on the value of Δ_{opt}^D . Taking advantage of the “no identity” assumption, we construct a simple one-derivative functional which provides an upper bound that can be easily checked analytically. Then, we consider the semi-definite problem (3.1.32), which we discretise by introducing a grid and a cutoff for the variable Δ_2 . In addition, the asymptotic constraints (3.1.52) are included. This puts the problem in a format amenable to the semi-definite solver SDPA-GMP.⁶

An analytical toy bound

Let us consider a simple functional composed of first order derivatives ∂_{χ_1} and ∂_{χ_2} . Since we have already computed the derivative of the crossing vector F with respect to the cross ratio χ_1 in eq. (3.1.37), it only remains to state a formula for the derivative with respect to χ_2 . Acting on the crossing vector (3.1.19) with ∂_{χ_2} and evaluating at the crossing symmetric point gives

$$\partial_{\chi_2} F_{\Delta_1, \Delta_2, \Delta_3} = -21 \Delta_\phi G_{\Delta_1, \Delta_2, \Delta_3} + \frac{1}{2} (\partial_{\chi_1} + \partial_{\chi_3}) G_{\Delta_1, \Delta_2, \Delta_3} + \partial_{\chi_2} G_{\Delta_1, \Delta_2, \Delta_3}. \quad (3.2.11)$$

⁶For our truncated scheme, where we have many large matrices with numerical entries, we found SDPA-GMP to be faster than SDPB [75]. We note that for most computations performed in this paper SDPA-QD would be sufficient. However, we found it convenient to use the extra-flexibility of the arbitrary precision solver SDPA-GMP throughout. We also note that it is possible to formulate the problem in a way where the Δ_2 dependence is treated as a polynomial, the type of problem SDPB was designed to tackle. We will address this point in Chapter 4.

We can combine this formula with our expression (3.1.37) for the χ_1 -derivative of the crossing vector to obtain

$$\left(\partial_{\chi_2} - \frac{1}{2}\partial_{\chi_1}\right)F_{\Delta_1, \Delta_2, \Delta_3} = -12\Delta_\phi G_{\Delta_1, \Delta_2, \Delta_3} + \partial_{\chi_2} G_{\Delta_1, \Delta_2, \Delta_3}. \quad (3.2.12)$$

Inserting our representation (3.1.25) for the comb channel six-point blocks G , we can further evaluate this expression,

$$\left(\partial_{\chi_2} - \frac{1}{2}\partial_{\chi_1}\right)F_{\Delta_1, \Delta_2, \Delta_3} = \sum_{n=0}^{\infty} \frac{F_n(\Delta_1, 1/3)(n + \Delta_2 - 4\Delta_\phi)F_n(\Delta_3, 1/3)}{3^{n+h_2-1}(2h_2)_n n!}. \quad (3.2.13)$$

From this equation, we may read off the matrix M that characterises the bilinear form, see eq. (3.1.28). In descendant-space the matrix that is associated to $\partial_{\chi_2} - 1/2\partial_{\chi_1}$ is diagonal and, for $\Delta_2 > 4\Delta_\phi$, the diagonal has only positive entries. Hence, we can directly deduce an upper bound of $\Delta_{\text{opt.}}^D \leq 4\Delta_\phi$. If we allow ourselves to also consider non-diagonal matrices in descendant-space, we can actually push this simple analytical bound even down to $\Delta_{\text{opt.}}^D \leq 2\Delta_\phi$, still using only the two linearly independent first derivatives ∂_{χ_1} and ∂_{χ_2} , though now the linear combination $\partial_{\chi_2} - \partial_{\chi_1}$. To see this, let us restrict to the special case $\Delta_\phi = 1$. In eq. (3.1.40), we gave an expression for the normalised descendant-space matrix of the functional corresponding to the derivative ∂_{χ_1} . Analogously, one obtains

$$\hat{M}_{nm}^{\partial_{\chi_2} - \partial_{\chi_1}} = \frac{6(\Delta_2 + n + 2)}{2\Delta_2 + n} \delta_n^m - \frac{18}{2\Delta_2 + n} - \frac{3}{2} \sqrt{3} \sqrt{\frac{n+1}{2\Delta_2 + n + 1}} (\delta_{n-1}^m + \delta_{n+1}^m) \quad (3.2.14)$$

for the normalised descendant-space functional corresponding to $\partial_{\chi_2} - \partial_{\chi_1}$. We can show that this matrix is positive semi-definite by constructing auxiliary matrices with only finitely many entries. This will also serve as a warm-up for a similar method to check the positivity of functionals with arbitrary derivative order, to be discussed in Section 3.2.2. For the simple functional at hand, it is enough to define the matrix

$$M_{ij}^{(n)} \equiv \begin{cases} \frac{1}{2}(1 + \delta_{n,0}) \hat{M}_{ij}^{\partial_{\chi_2} - \partial_{\chi_1}} & i, j = n + 1 \\ \hat{M}_{ij}^{\partial_{\chi_2} - \partial_{\chi_1}} & n + 1 < \max(i, j) \wedge \min(i, j) \leq n + 99 \\ \frac{1}{2} \hat{M}_{ij}^{\partial_{\chi_2} - \partial_{\chi_1}} & i = j = n + 100 \\ 0 & \text{else} \end{cases}, \quad (3.2.15)$$

or more visually:

$$M_{ij}^{(n)} = \left(\begin{array}{c|cccccc} \hline \frac{1}{2}(1 + \delta_{n,0})\hat{M}_{n+1,n+1} & \hat{M}_{n+1,n+2} & 0 & \dots & 0 \\ \hline \hat{M}_{n+2,n+1} & \hat{M}_{n+2,n+2} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \hat{M}_{n+99,n+99} & \hat{M}_{n+99,n+100} \\ 0 & \dots & 0 & \hat{M}_{n+100,n+99} & \frac{1}{2}\hat{M}_{n+100,n+100} \\ \hline \end{array} \right), \quad (3.2.16)$$

where we suppressed the superscript $\partial_{\chi_2} - \partial_{\chi_1}$ for simplicity of notations, and the empty blocks correspond to entries which take the value 0. By construction, we can recover the infinite dimensional matrix $\hat{M}$ from $\hat{M}^{(n)}$ as

$$\hat{M}_{ij}^{\partial_{\chi_2} - \partial_{\chi_1}} = \sum_{n=0}^{\infty} M_{ij}^{(99n)}. \quad (3.2.17)$$

To show that $\hat{M}^{\partial_{\chi_2} - \partial_{\chi_1}}$ is positive semi-definite, it is sufficient to establish positive semi-definiteness for each summand on the r.h.s. of eq. (3.2.17). This is accomplished in two steps. First we shall analyse the matrix $M^{(0)}$ and then in a second step we show that $M^{(n)}$ for any Δ_2 is positive semi-definite for $n \geq 99$.

Addressing $M^{(0)}$, let us start by noting that the limit $\Delta_2 \rightarrow \infty$ of the matrix $M^{(0)}$ is a diagonal matrix whose smallest non-zero eigenvalue $\lambda_{\min} = \lambda_{\min}(\infty)$ is given by $\lambda_{\min} = 1.5$, as can be read off directly from eq. (3.2.14). With this knowledge of the asymptotics, we may demonstrate the positive semi-definiteness of $M_{ij}^{(0)}$ for all $\Delta_2 > 2$ by considering the smallest non-zero eigenvalue as a function of Δ_2 . One can compute and plot this minimal eigenvalue $\lambda_{\min}(\Delta_2)$ in the regime $\Delta_2 \geq 2$ starting from $\Delta_2 = 2$ up until one reaches values of Δ_2 where the minimal eigenvalue of $M^{(0)}$ reaches its large Δ_2 asymptotics value $\lambda_{\min}(\Delta_2) \sim 1.5$. This plot can be seen in Figure 3.4. It shows very clearly that the smallest eigenvalue of $M^{(0)}$ is indeed positive and hence that $M_{ij}^{(0)}$ is a positive semi-definite matrix for all values $\Delta_2 > 2$.

Before we consider general $M_{ij}^{(n)}$ with $n \geq 99$, let us briefly comment on the asymptotic Toeplitz arc of $\hat{M}^{\partial_{\chi_2} - \partial_{\chi_1}}$ in terms of the matrices $M^{(n)}$. In other words, let us consider the spectrum of the matrices

$$M^{(\infty)}(r) := \lim_{n \rightarrow \infty} M^{(n)}(\Delta_2 = nr). \quad (3.2.18)$$

Clearly, it is a necessary condition for the positivity property of $\hat{M}^{\partial_{\chi_2} - \partial_{\chi_1}}$ in question that $M^{(\infty)}(r)$ is positive semi-definite for all $r > 0$. Note that $r \rightarrow \infty$ is just the large Δ_2 limit, where we expect the smallest non vanishing eigenvalue to take the value $\lambda_{\min}(\infty) = 1.5$. On the other hand, $r = 0$ corresponds to sending n to infinity at finite Δ_2 . The smallest non vanishing eigenvalue of the $r = 0$ matrix is $\lambda_{\min}(r = 0) = 0.75$. Having analysed these two extremal values of r , it now remains to plot the minimal

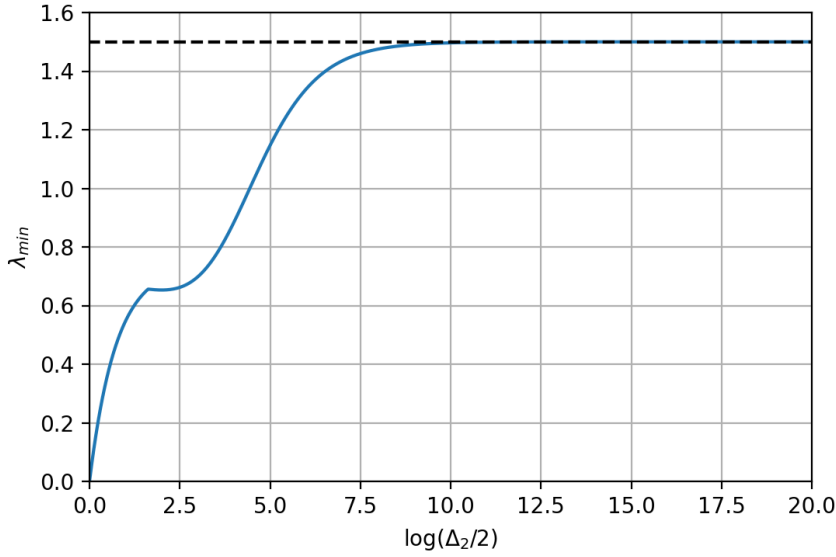


Figure 3.4: In blue, we plot the minimal non zero eigenvalue $\lambda_{\min}$ as a function of Δ_2 on a logarithmic scale. The black dashed line corresponds to the asymptotic large Δ_2 value.

eigenvalue of $M^{(\infty)}$ as a function of r . This plot can be seen in Figure 3.2. It shows clearly that the minimal eigenvalue of the matrix $M^{(\infty)}(r)$ is positive for all values of r

Let us finally comment on the spectrum of $M^{(n)}$ for finite n . Though we only need to show $M^{(n)}(\Delta_2) \geq 0$ for $n \geq 100$ and $\Delta_2 > 2$, we choose to consider all $\Delta_2 > 0$ and $n \geq 10$, as this is slightly more instructive. The minimal eigenvalue of $M^{(n)}(nr)$ for small n is a function of r about which we only know the asymptotic value $\lambda_{\min}(\infty) = 1.5$ for sufficiently large Δ_2 . Thus, in order to demonstrate positivity of the minimal eigenvalue, we need to compute $\lambda_{\min}(r)$ for $r \geq 0$ up until values of r at which the minimal eigenvalue reaches its asymptotic value. As n increases, however, the r -profile for the minimal eigenvalue of $M^{(n)}(nr)$ approaches the function plotted in Figure 3.5. Thus, only a finite range of r and n has to be plotted explicitly to conclude that $M^{(n)}(\Delta_2)$ is positive semi-definite for all $\Delta_2 > 0$ and $n \geq 10$. This plot can be found in Figure 3.6, which finishes our argument for the positivity of the functional that is associated with $\partial_{\chi_2} - \partial_{\chi_1}$, for all values $\Delta_2 \geq 2$ and with the conformal weight of the external field set to $\Delta_\phi = 1$.

Producing the input for SDPA

We now want to automatise the task of producing positive functionals by implementing the discretised version of the SDP (3.1.32) alluded to above. In particular, we

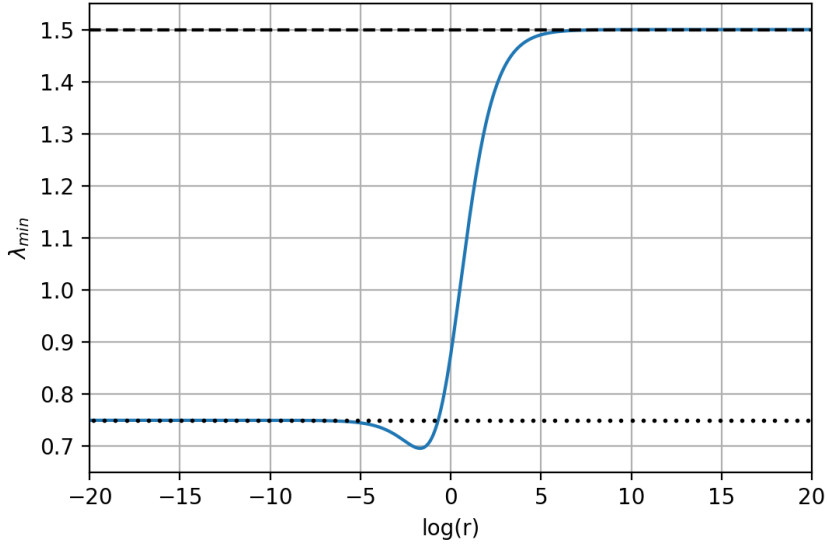


Figure 3.5: In blue, we plot the minimal non zero eigenvalue of $M^{(\infty)}$ as a function of the ratio r between n and Δ_2 on a logarithmic scale. The black dotted line corresponds to the asymptotic $n \rightarrow \infty$ limit, while the black dashed line represents the $\Delta_2 \rightarrow \infty$ regime.

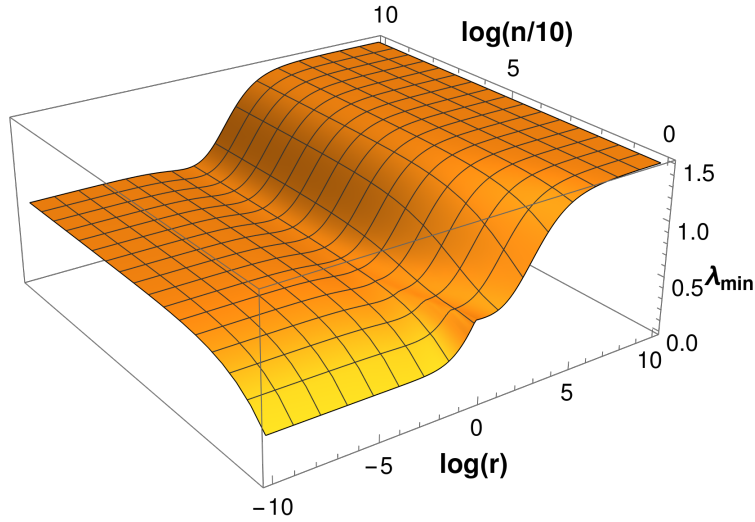


Figure 3.6: Here, we illustrate the surface of minimal non-vanishing eigenvalues of the matrices $M^{(n)}(\Delta_2 = nr)$ as functions of r and n .

will describe the explicit input that we feed into SDPA in order to solve the SDP dual to gap maximisation without identity. The `Mathematica13.2` notebook that

we wrote for this purpose can be found as an ancillary file in this arXiv submission⁷ It implements the following six steps:

1. Read in the external parameters. In the first section of the notebook, we input key parameters, like the highest derivative order to be considered (Λ), the total number of functionals (`Nfunc`), the gap that we would like to exclude (`gap`), the truncation size (`DTrunc`), the discrete range of Δ_2 values on which we explicitly impose positivity (`h2ran`) and a choice of an overall sign (`sign`).

2. Generate a basis of independent derivative functionals. Next, we generate a basis of linearly independent derivative functionals with a total number of derivatives less or equal to Λ . Concretely, we apply partial derivatives to the function defined in eq. (3.1.19) and evaluate them at the crossing symmetric point, without explicitly evaluating the conformal blocks G . We then treat different derivatives of G as linearly independent. Thus each derivative functional corresponds to a specific linear combination of these linearly independent functions and we can determine a basis of this space directly by Gaussian elimination.

3. Transform to the descendant-space basis. Now, we compute the normalised descendant-space matrices associated to the selected derivative functionals. As explained in Section 3.1.3, to achieve this, one simply has to act with the derivatives on the blocks written in the form of eq. (3.1.25) and then apply eq. (3.1.27). Once we have computed formulas for arbitrary entries of the descendant-space matrices this way, we explicitly evaluate the upper left $\text{DTrunc} \times \text{DTrunc}$ block of each functional.

4. Compute the asymptotic arc polynomials. Then, we use the expressions generated in step 3 in order to compute the asymptotic arc polynomials as described in Section 3.1.4.

5. Generate an SDP to construct SOS combinations of the asymptotic arc polynomials. The final piece of input data that we need to compute for SDPA is the set of matrices that realise the SDP formulation of the task of constructing SOS linear combinations of the asymptotic arc polynomials. We achieve this through the minimalistic approach described in Appendix 3.C, and use the result in the final step.

6. Formulate the gap maximisation as an SDP and export SDPA input file.

⁷We use SPDA-GMP 7.1.3 with the following parameters (with minor modifications depending on the specific SDP at hand): Large `maxIteration` (e.g. 5000) to ensure that the solver does not stop before convergence, `epsilonStar` and `epsilonDash` varying between 10^{-10} to 10^{-20} depending on the specific input SDP, Large `lambdaStar` (10^5) and order one `omegaStar` (2), `lowerBound`= 0 and `upperBound`= 0 so that SDPA terminates once it knows that a certain assumption on the gap can be excluded or cannot be excluded, `betaStar`=0.01, `betaBar`=0.02, `gammaStar`=0.9, `precision`=128 (some runs also with 256).

SDPA solves the following semi-definite optimisation problem: Given $N + 1$ (block-diagonal) matrices $\{\mathbf{F}_i\}_{i=0}^N$ and an objective $\{c_i\}_{i=1}^N$, find coefficients $\{x_i\}_{i=1}^N$ such that

$$\sum_{i=1}^N c_i x_i \text{ is minimised and } \sum_{i=1}^N x_i \mathbf{F}_i - \mathbf{F}_0 \geq 0. \quad (3.2.19)$$

In its formulation (3.1.32), however, the dual to gap maximisation without identity is not directly in this form. Instead it is rather a family of Linear Matrix Inequalities (LMI)⁸ which express the constraint that the linear combination of functionals under consideration should be positive semi-definite above the chosen gap. The existence (or not) of solutions to these LMI allows us to exclude the gap (or not). Of course, we could simply transform the LMI to an optimisation problem that can be solved by SDPA by choosing an arbitrary objective function (say for instance a vanishing objective). But as suggested in the SDPA manual [173], it is preferable to instead add an identity matrix $(\mathbf{F}_{N+1})_{ij} = \delta_{ij}$ and consider the optimisation problem of finding $\{x_i\}_{i=1}^{N+1}$ such that

$$x_{N+1} \text{ is minimised and } \sum_{i=1}^{N+1} x_i \mathbf{F}_i - \mathbf{F}_0 \geq 0. \quad (3.2.20)$$

If this problem has a solution with negative objective, we are guaranteed that

$$\mathbf{F} \equiv \sum_{i=1}^N x_i \mathbf{F}_i - \mathbf{F}_0 \geq 0 \quad (3.2.21)$$

and moreover, $0 < -x_{N+1}$ is a lower bound on the smallest eigenvalue of $\mathbf{F}$.

Apart from the identity that is added to realise the LMI, there are three qualitatively different matrices $\mathbf{F}_i$ that we feed into SDPA.

First of all we have the matrices $\mathbf{F}_1, \dots, \mathbf{F}_{\text{Nfunc}-1}$ corresponding to $\text{Nfunc} - 1$ of the functionals under consideration. Each of these is a large block diagonal matrix whose blocks are: For each value $h_2 > \text{gap}$ in h2ran , a $\text{DTrunc} \times \text{DTrunc}$ matrix corresponding to the upper left corner of the full descendant-space matrix, evaluated at $\Delta_2 = h_2$; and the coefficient matrix of the associated asymptotic arc polynomial obtained through the SOS SDP described in step 5.

The role of $\mathbf{F}_0$ is played by the matrix corresponding to the one derivative functional described in eqs. (3.2.12) and (3.2.13), which is diagonal in the descendant space basis. As the coefficient of $\mathbf{F}_0$ is not variable, we in principle need to consider also the case where $\mathbf{F}_0$ is multiplied by a sign in order to have access to the most general linear combination (at high enough derivative order the choice of the sign does not seem to matter anymore). This sign is determined through the parameter **sign**

⁸A LMI is the problem of finding x_i for given matrices $\{\mathbf{F}_i\}_{i=0}^N$ s.t. $\sum_{i=1}^N x_i \mathbf{F}_i - \mathbf{F}_0 \geq 0$

defined in the first section of the notebook. Up to the sign, $\mathbf{F}_0$ is on the same footing as the other $\mathbf{F}_i$.

In addition to $\mathbf{F}_0, \dots, \mathbf{F}_{\text{Nfunc}-1}$, we also have to include $\mathbf{F}_i$ corresponding to redundancies of the SOS SDP (see Appendix 3.C). These have vanishing $\text{DTrunc} \times \text{DTrunc}$ blocks for all h_2 in h2ran above gap and a block that is one of the matrices $\mathbf{K}[[j]]$ satisfying eq. (3.C.4).

Checking the output of SDPA

A priori, the functionals that we determine numerically are not guaranteed to fulfil the full set of positivity constraints that are needed to establish bounds on CFT data. After all, we only ask SDPA to construct functionals that are positive semi-definite asymptotically in the sense of Section 3.1.4 and for a specific truncation in the matrix size and discretisation of the Δ_2 -range, h2ran .

There are several criteria which we use to rigorously establish the full positivity of the functionals. Every bound claimed here has been checked using some combination of these.

The simplest criterion which can sometimes be used to rigorously establish the positivity of a given functional is the Gershgorin circle theorem:

Theorem (Gershgorin). *Let M be a $n \times n$ matrix with complex entries and $r_i := \sum_{j \neq i} |M_{ij}|$. Then the spectrum of M is a subset of the union*

$$\bigcup_{i=1}^n B_{r_i}(M_{ii}) \quad (3.2.22)$$

of all Gershgorin circles $B_{r_i}(M_{ii}) := \{z \in \mathbb{C} : |z - M_{ii}| \leq r_i\}$.

Since we have concrete analytic expressions for the entries of the infinite-dimensional matrix corresponding to a given functional, we can simply compute the distance of all of its Gershgorin circles from the negative real axis and then take a minimum. If this minimum is strictly bigger than zero, one has established positive semi-definiteness of the bilinear form in question. This simple criterion works well for functionals involving only a small number of derivatives, as one would expect: In the simplest case of just first order derivatives, for example, the asymptotic Toeplitz matrix corresponding to the functional is just tridiagonal. But, as a consequence of Theorem 3.1.4, the Gerschgorin criterion is not only sufficient for positive semi-definiteness but even necessary in the case of tridiagonal Toeplitz matrices.

For functionals involving a larger number of derivatives, the Gerschgorin criterion is however too restrictive. The remainder of this section aims at describing the more

generally applicable higher derivative extension of the ideas described in Section 3.2.2, where we demonstrated the positivity of a particular first derivative functional.

In order to show positive semi-definiteness of an infinite band matrix M of bandwidth Λ , we introduce an auxiliary $n_c \times n_c$ matrix C , which we refer to as the *cutting matrix*. We demand that $n_c \geq \Lambda + 1$. To this matrix, we associate the matrix $\bar{C}$ defined by

$$\bar{C}_{ij} \equiv 1 - C_{ij}. \quad (3.2.23)$$

Next, we choose two integers $n_l > n_c$ and $n_h > 2n_c$. These choices let us define matrices $M^{(0,C)}$ and $\{M^{(n,C)}\}_{n>0}$ as

$$M_{ij}^{(0,C)} \equiv \begin{cases} M_{ij} & \min(i, j) \leq n_l - n_c \\ M_{ij} \bar{C}_{i-(n_l-n_c), j-(n_l-n_c)} & n_l - n_c < i, j \leq n_l \\ 0 & \text{else} \end{cases} \quad (3.2.24)$$

and

$$M_{ij}^{(n,C)} \equiv \begin{cases} M_{ij} C_{i-n, j-n} & n < i, j \leq n + n_c \\ M_{ij} & n + n_c < \max(i, j) \wedge \min(i, j) \leq n + n_h - n_c \\ M_{ij} \bar{C}_{i-(n+n_h-n_c), j-(n+n_h-n_c)} & n + n_h - n_c < i, j \leq n + n_h \\ 0 & \text{else} \end{cases}. \quad (3.2.25)$$

Note that we already discussed a special case with $n_c = 1$ in eq. (3.2.15). With these more general definitions, we conclude that

$$M_{ij} = M_{ij}^{(0,C)} + \sum_{\ell=0}^{\infty} M_{ij}^{(\ell(n_h-n_c)+n_l-n_c, n_c)}. \quad (3.2.26)$$

Thus, if we are able to show for some specific choice of the cutting matrix C and of integers n_h, n_l that $M^{(0,C)}$ and $M^{(n,C)}$ are positive semi-definite for all $n \geq n_l - n_c$ then we can conclude that M is positive semi-definite. In principle, a suitable cutting matrix C tailored specifically to the matrix M at hand, could be constructed using SDP. However, we found it sufficient to simply always take an Ansatz

$$C_{ij} = a + (1 - 2a) \frac{i + j - 2}{2n_c - 2}, \quad (3.2.27)$$

for some constant a together with sufficiently large integers n_c, n_l and n_h .

In practice we perform these checks in an automated iterative procedure consisting of the following four steps:

1. Use **SDPA** to construct a functional that is positive on a certain grid **h2ran** of Δ_2 values up to a certain cutoff **DTrunc**. Then initiate testing by proceeding to step 2.

2. Refine the grid by complementing it with the midpoints of all neighbouring grid-elements. If the functional is negative somewhere on the refined grid, replace `h2ran` by the refined grid and return to step 1. Otherwise proceed with step 3.
3. Increase `DTrunc` by some fixed value and check whether the functional stays positive on the refined grid for the new value of `DTrunc`. If so, continue with the final step. Otherwise, return to step 1 with the new value of `DTrunc`.
4. Generate plots of the minimal eigenvalues of the cut matrices, similar to those in Section 3.2.2, that allow the user to visually verify full positivity.

Ideally, one would avoid this verification process altogether by treating both Δ_2 and n as polynomial variables, and subsequently imposing polynomial inequalities. While this seems natural and straightforward for the Δ_2 dependence, some extra thought is required for the n dependence, which controls the size of the matrices we work with. As we will see in Chapter 4, a version of the cut decomposition described here can be used to not only check positivity after running an SDP solver, but rather to impose it systematically with an improved, reformulated SDP.

Discussion of numerical results

Having established the methods to obtain linear combinations of functionals whose positivity we can thoroughly check, we now present the resulting numerical bounds for the problem of gap maximisation without identity. In Figure 3.7, we plot the maximum allowed value for $\Delta_2 = \Delta_*$, the dimension of the leading operator in the $\mathcal{O}_2$ channel, as a function of the external dimension Δ_ϕ . We present results for several values of $0 < \Delta_\phi \leq 5$, with a more refined grid in the range $0 < \Delta_\phi \leq 1$. The bound is roughly linear, and completely smooth.

Remarkably, even at the lowest derivative order $\Lambda = 2$ (the lightest shade of green in the plot) we find percent-level agreement with the primal solution $\Delta_2 = 9\Delta_\phi/5$ discussed in section 3.2.1. In fact, even at this derivative order all the points with $0 < \Delta_\phi \leq 1$ give a bound strictly stronger than $\Delta_2 < 2\Delta_\phi$, which leads us to conjecture that $\Delta_2 = 9\Delta_\phi/5$ actually saturates the bound on the gap maximisation without identity problem. For small Δ_ϕ the bound is actually so close to the conjectured optimum $\Delta_*^P = 9\Delta_\phi/5$ that the difference between the green and the black lines cannot be resolved in Figure 3.7. Therefore, we plot the difference between the obtained bounds and Δ_*^P in the range $0 < \Delta_\phi < 1$ in the separate Figure 3.8.

Subsequent increases of the derivative order Λ lead only to a slight (but finite) improvement to the bound, which is unsurprising, given how close the $\Lambda = 2$ answer already is to the primal solution we constructed. This good agreement at low

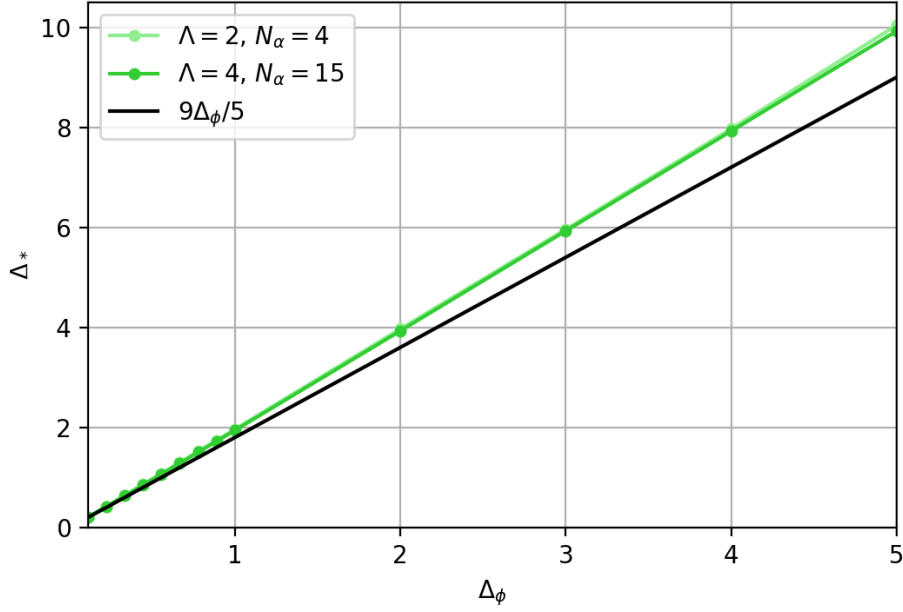


Figure 3.7: Bound on the dimension of the leading exchange on the $\mathcal{O}_2$ channel: Δ_* as a function of the external dimension Δ_ϕ . In green, we plot the upper bounds derived from the numerical SDP approach, with darker colours corresponding to higher derivative order Λ . In black, we plot the primal solution constructed in Section 3.2.1, which agrees remarkably well with the numerical bound for small values of Δ_ϕ .

derivative order and slow improvement with Λ seems to simply be an artefact of the “no-identity” problem, as we will find a more familiar behaviour in the GFF-type problems considered in the next section.

3.3 Bounds on Theories with GF Four-Point Functions

With all the numerical machinery in place, we now tackle more realistic bootstrap problems in which the operator product of the external field ϕ with itself contains the identity operator. The appearance of the identity implies that in the OPE limit the $\mathcal{O}_2$ channel is dominated by ϕ , i.e. the external operator. Including also the first subleading contributions we have

$$(\phi \times \phi) \times \phi \sim (1 + \mathcal{O}_1^* + \dots) \times \phi \sim \phi + \mathcal{O}_2^* + \dots$$

where we denoted the first nontrivial operator in the intermediate channels by $\mathcal{O}_j^*$. Given this natural setup, an immediate quantity to consider is the gap above ϕ in the middle channel, i.e. the conformal dimension Δ_* of the leading non-trivial field $\mathcal{O}_2^*$. To obtain a bound on this quantity, we should solve the following positivity

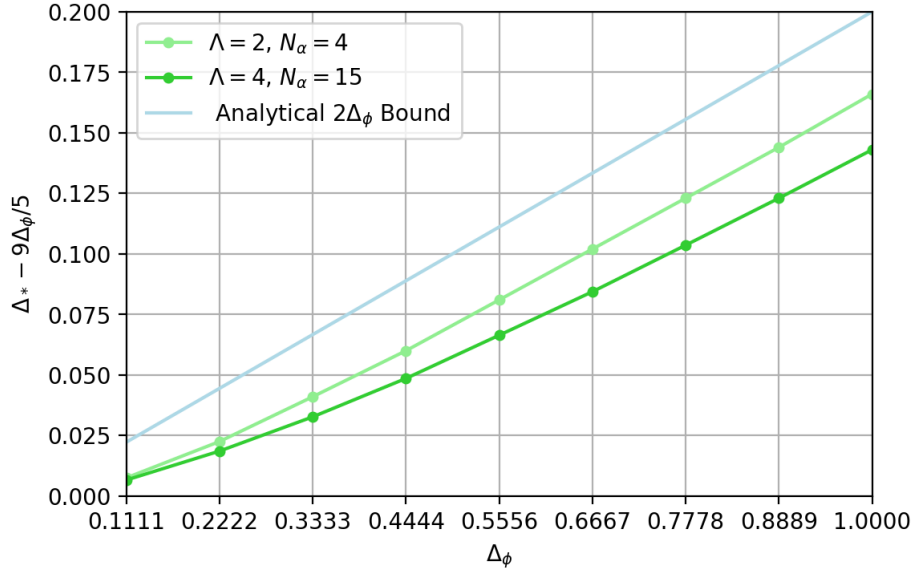


Figure 3.8: Difference between the small Δ_ϕ data-points plotted in Figure 3.7 and the best primal solution to gap maximisation without identity that we were able to construct. The thin blue line represents the analytical toy bound of Section 3.2.2.

conditions⁹

$$\sum_k c_k B_{\Delta_\phi}^k > 0, \quad \sum_k c_k B_{\Delta_2}^k > 0 \quad \text{for all } \Delta_2 \geq \Delta_*. \quad (3.3.1)$$

It is important to note, however, that the six-point function $\mathcal{G}$ now contains contributions from the intermediate exchange of $\mathcal{O}_2 = \phi$ that are qualitatively different from the remaining terms in the block expansion. Let us collect all the terms in the conformal blocks expansion (3.1.6) that involve six-point blocks with $\Delta_2 = \Delta_\phi$ in a new function

$$\mathcal{G}_{4\text{-pt}}(\chi_1, \chi_2, \chi_3) \equiv \sum_{\mathcal{O}_1, \mathcal{O}_3} C_\phi^{\mathcal{O}_1} C_\phi^{\mathcal{O}_3} G_{\Delta_1, \Delta_\phi, \Delta_3}^{123,456} = \sum_{\mathcal{O}_1, \mathcal{O}_3} C_{\phi\phi\mathcal{O}_1}^2 C_{\phi\phi\mathcal{O}_3}^2 G_{\Delta_1, \Delta_\phi, \Delta_3}^{123,456}. \quad (3.3.2)$$

We stress that this function depends on all three six-point cross ratios. The reason we denoted it with a subscript ‘4-pt’ is that it only involves CFT data contained in the four-point function of ϕ . One should also observe that the product of OPE coefficients appearing in this special contribution to the full six-point function is manifestly positive.

These observations have important consequences for the choice of strategy that

⁹We prefer to formulate equation (3.3.1) in terms of the primary space bilinear-forms B , as this seems to be more natural for the point that is made here. We could have also chosen to use the descendant-space formulation and replace B by b .

should be implemented to ensure

$$\alpha(\mathcal{G}_{4\text{-pt}}) = \sum_k c_k \alpha_k(\mathcal{G}_{4\text{-pt}}) \geq 0. \quad (3.3.3)$$

First of all, note that the transformation to descendant-space discussed in Section 3.1.3 preserves the positivity property of the primary-space OPE vector $C_{\phi\phi\mathcal{O}}^2$. Thus, both in primary- and descendant-space, ensuring eq. (3.3.3) does not require positive semi-definiteness of the bilinear form B_{Δ_ϕ} but only co-positivity i.e. the property $B_{\Delta_\phi}(C, C) \geq 0$ for all OPE vectors C that are componentwise positive (see e.g. [190] for a review on co-positive programming). For instance, every matrix with only positive entries is manifestly co-positive, and of course every positive semi-definite matrix is co-positive. However, solving a co-positive program is generally NP-hard. Nonetheless, it might be sufficient to relax co-positivity to componentwise positivity or positive semi-definiteness of the bilinear forms to construct valid bootstrap bounds, which of course cannot be stronger than in the co-positive case.

We discuss this question to more detail in Chapter 6 and for now take the perspective that one has solved or bootstrapped the four-point function of ϕ , and hence knows all non-vanishing OPE coefficients $C_{\phi\phi\mathcal{O}}^2$. In this scenario, we can just compute $\mathcal{G}_{4\text{-pt}}$ and directly impose eq. (3.3.3) as a constraint on the choice of the functionals. A simple case where we are in this situation is for generalised free theories, which we will proceed to analyse in the subsections below.

3.3.1 Bounds on the triple-twist gap for GFF

Let us first consider the generalised free fermion theory with a real fermionic field ψ . In this case, the spectrum in the $\psi \times \psi$ OPE contains operators of conformal dimension $\Delta_n^F = 2\Delta_\psi + 2n + 1$ with $n \in \mathbb{Z}_{\geq 0}$, and the corresponding OPE coefficients are given in eq. (3.D.7).

We can then easily reconstruct the function $\mathcal{G}_{4\text{-pt}}$ we defined in eq. (3.3.2) and which now takes the following explicit form

$$\begin{aligned} \mathcal{G}_{4\text{-pt}}^{\text{GFF}}(\chi_i)/\chi_2^{\Delta_\psi} = & 1 - \chi_1^{2\Delta_\psi} + \left(\frac{\chi_1}{1-\chi_1}\right)^{2\Delta_\psi} - \chi_3^{2\Delta_\psi} + \left(\frac{\chi_3}{1-\chi_3}\right)^{2\Delta_\psi} + \left(\frac{\chi_1\chi_3}{1-\chi_2}\right)^{2\Delta_\psi} \\ & - \left(\frac{\chi_1\chi_3}{1-\chi_1-\chi_2}\right)^{2\Delta_\psi} - \left(\frac{\chi_1\chi_3}{1-\chi_3-\chi_2}\right)^{2\Delta_\psi} + \left(\frac{\chi_1\chi_3}{1-\chi_1-\chi_2-\chi_3+\chi_1\chi_3}\right)^{2\Delta_\psi}. \end{aligned} \quad (3.3.4)$$

This piece of the full correlator can also be identified directly with a Wick contraction computation of the six-point function, keeping track of the usual fermionic minus signs. Each of the nine terms above can be identified with a Wick contraction where exactly one of the propagators crosses from the set of points $\{x_1, x_2, x_3\}$ to the set $\{x_4, x_5, x_6\}$. We can finally use eq. (3.3.4) to add a constraint to our SDP according

to eq. (3.3.3). Our results for the bound on the dimension Δ_* of the first operator $\mathcal{O}_2^*$ above the operator ψ in the $\mathcal{O}_2$ channel are shown in Figure 3.9.

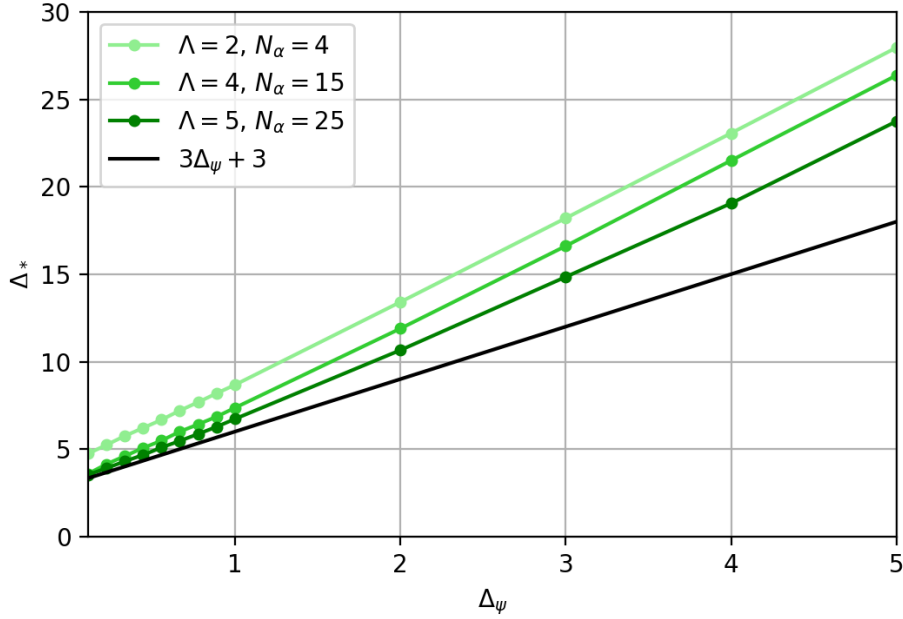


Figure 3.9: Bound on the dimension of the subleading exchange on the $\mathcal{O}_2$ channel: Δ_* as a function of the external dimension Δ_ψ . In green, we plot the upper bounds derived from the numerical SDP approach, with darker colours corresponding to higher derivative order Λ . In black, we plot the expected gap from the leading triple-twist operator in GFF theory, which agrees remarkably well with the numerical bound for low values of Δ_ψ .

As in Section 3.2.2, we present results for several values of $0 < \Delta_\psi \leq 5$, with a more refined grid in the range $0 < \Delta_\psi \leq 1$. At fixed derivative order Λ , the bound is roughly linear in the interval $0 < \Delta_\psi \leq 1$ but starts bending upward for larger values of Δ_ψ . Nonetheless, the bound is completely smooth, and the non-linearity seems to simply be caused by a slower convergence at larger values of Δ_ψ . In this case, as Λ increases, and the tone of green gets darker, there is a more substantial improvement of the bounds when compared to the “no-identity” case. For reference, we note that at the highest derivative order $\Lambda = 5$ each value of Δ_ψ took a few days on a single core of a 2.8 GHz AMD EPYC 7402 24-core processor, and is roughly 50 to 100 times slower than the smaller value $\Lambda = 4$. The majority of this time is taken by solving the actual SDP. We expect that making full use of the sparsity structure of the matrix, this can be sped up by several orders of magnitude.

Returning to the value of the bound itself, we see that it is always above the black line. The latter corresponds to the value $\Delta_*^P = 3\Delta_\psi + 3$ which is the gap for the $\mathcal{O}_2$ exchange in the GFF six-point function. The spectrum of non-trivial fields that are

exchanged in the $\mathcal{O}_2$ channel of GFF can be read off easily from the following function in which we subtract all contributions from the exchange of ψ and its descendants,

$$\begin{aligned}
(\mathcal{G}^{\text{GFF}} - \mathcal{G}_{4\text{-pt}}^{\text{GFF}})/\chi_2^{\Delta_\psi} = & \left(\frac{\chi_1\chi_2\chi_3}{(1-\chi_1-\chi_2)(1-\chi_3)} \right)^{2\Delta_\psi} + \left(\frac{\chi_1\chi_2\chi_3}{(1-\chi_3-\chi_2)(1-\chi_1)} \right)^{2\Delta_\psi} \\
& - \left(\frac{\chi_1\chi_2\chi_3}{(1-\chi_1)(1-\chi_2)(1-\chi_3)} \right)^{2\Delta_\psi} - \left(\frac{\chi_1\chi_2\chi_3}{(1-\chi_1-\chi_2)(1-\chi_2-\chi_3)} \right)^{2\Delta_\psi} \\
& - \left(\frac{\chi_1\chi_2\chi_3}{1-\chi_1-\chi_2-\chi_3+\chi_1\chi_3} \right)^{2\Delta_\psi} + \left(\frac{\chi_1\chi_2\chi_3}{(1-\chi_1-\chi_2-\chi_3+\chi_1\chi_3)(1-\chi_2)} \right)^{2\Delta_\psi}.
\end{aligned} \tag{3.3.5}$$

Expanding this function in conformal blocks (3.1.8), and matching the Taylor series at small χ_i allows us to read off the spectrum of primaries in all channels, see Appendix 3.D for more details. In particular the first non-trivial operator $\mathcal{O}_2^*$ indeed has dimension $\Delta_*^P = 3\Delta_\psi + 3$. This was to be expected also from the Fock space realisation of GFF. Note that in GFF the OPE of the single field ψ and the two-field/double-twist operators $\mathcal{O}_1$, only contains ψ itself and three-field/triple-twist operators. Because of fermionic statistics, simple powers of ψ and its derivatives vanish and one has to insert derivatives of different orders to obtain a non-zero operator. In the case of three-field/triple-twist, the lowest non-trivial operator is obviously given by

$$\mathcal{O}_2^* = \psi(\partial\psi)(\partial^2\psi). \tag{3.3.6}$$

Here normal ordering is understood. The conformal dimension of this operator is indeed given by $\Delta(\mathcal{O}_2^*) = 3\Delta_\psi + 3$, in agreement with the result we obtained by expanding the function (3.3.5) in the cross ratio χ_2 . More details on how to count multi-twist operators in GFF can be found in Appendix 3.E.

While the numerics have yet to fully converge, we conjecture that the bound on the dimension of $\mathcal{O}_2^*$ is saturated by $\Delta_*^P = 3\Delta_\psi + 3$, the GFF value. At first glance our conjecture that the optimal value Δ_{opt}^D of the dual SDP problem coincides with the free field result, i.e. $\Delta_{\text{opt}}^D = \Delta_*^P$ might seem tautological, since we assume the CFT data of GFF in the $\psi \times \psi$ OPE. However, the bound on the dimension Δ_{opt}^D in the dual SDP problem is answering a non-trivial question: Assuming that the four-point function of ψ matches GFF, is it true that the six-point function of ψ must agree with GFF as well?

In order to build some more intuition into how this could indeed fail to be the case, let us make use of the description of GFF in terms of a free massive Majorana fermion Ψ in AdS_2 . If we turn on a sextic coupling g_6 in AdS , as shown in the schematic

EFT action¹⁰

$$S[\Psi] = \int_{AdS_2} \sqrt{g} d^2x \left(\bar{\Psi}(\not{\nabla} - m)\Psi + g_6(\nabla\Psi)^6 + \dots \right), \quad (3.3.7)$$

while requiring quartic couplings to be zero, we change the triple-twist dimensions at leading order in g_6 without touching the four-point function of ψ , due to the contribution of the six-point contact diagram. In this language, the conjecture we formulated in the previous paragraph can be reformulated as follows: Within the family of theories described by the action (3.3.7), the $g_6 = 0$ point (GFF) is extremal, saturating the bound.¹¹

The reformulation in terms of AdS also implies that our analysis could lead to bounds on the sign of the irrelevant sextic coupling, since linearity of the anomalous dimensions in g_6 could either push us above or below the extremal value. This would be the six-point generalisation of the bounds on the sign of the $(\partial\phi)^4$ couplings obtained with the four-point bootstrap in [185]. While such sign constraints on irrelevant quartic couplings are well understood from many other points of view [197–200], to the best of our knowledge, sign constraint for a higher-point coupling would be new.¹² See however [201] for recent progress.

3.3.2 Bounds on the triple-twist gap for GFB

It is straightforward to modify the above analysis to the case of generalised free bosons. For this theory, the OPE of the external field ϕ with itself contains operators of dimensions $\Delta_n^B = 2\Delta_\psi + 2n$ with $n \in \mathbb{Z}_{\geq 0}$, and the corresponding OPE coefficients are given in eq. (3.D.11). Once again, we impose the positivity condition on the Δ_ϕ block by requiring eq. (3.3.3) on

$$\begin{aligned} \mathcal{G}_{4\text{-pt}}^{\text{GFB}}(\chi_i)/\chi_2^{\Delta_\phi} = & 1 + \chi_1^{2\Delta_\phi} + \left(\frac{\chi_1}{1-\chi_1}\right)^{2\Delta_\phi} + \chi_3^{2\Delta_\phi} + \left(\frac{\chi_3}{1-\chi_3}\right)^{2\Delta_\phi} + \left(\frac{\chi_1\chi_3}{1-\chi_2}\right)^{2\Delta_\phi} \\ & + \left(\frac{\chi_1\chi_3}{1-\chi_1-\chi_2}\right)^{2\Delta_\phi} + \left(\frac{\chi_1\chi_3}{1-\chi_3-\chi_2}\right)^{2\Delta_\phi} + \left(\frac{\chi_1\chi_3}{1-\chi_1-\chi_2-\chi_3+\chi_1\chi_3}\right)^{2\Delta_\phi}, \end{aligned} \quad (3.3.8)$$

which is reconstructed from the bosonic four-point data, and can of course be obtained by the same Wick contractions as the GFF correlator above. With this, we present the results of the bound Δ_* on the dimension Δ_2^* of the first operator $\mathcal{O}_2^*$ above ϕ in the $\mathcal{O}_2$ channel in Figure 3.10. Once again, we choose several values of

¹⁰While the distribution of derivatives in the sextic coupling is schematic, it is indeed true that the leading interaction for a single Majorana fermion has six derivatives. Furthermore, note that a single coupling g_6 is irrelevant and presumably incompatible with Regge boundedness and therefore the action has to be understood in the EFT sense.

¹¹It would be interesting to prove extremality of the GFF six-point correlator using higher-point analytic functionals extending the work of [172, 191–196]

¹²We thank Sasha Zhiboedov for a discussion on this point.

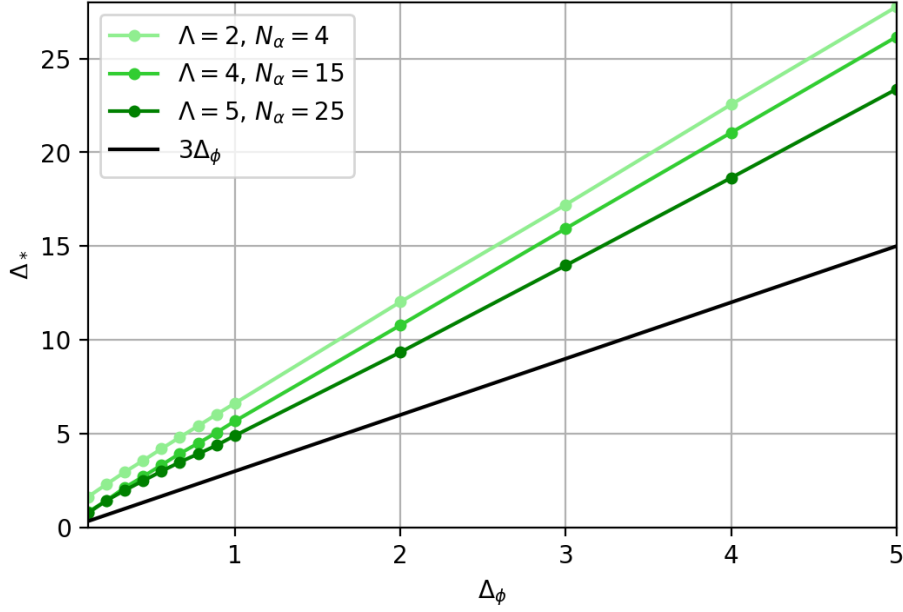


Figure 3.10: Bound on the dimension of the subleading exchange on the $\mathcal{O}_2$ channel: Δ_* as a function of the external dimension Δ_ϕ . In green, we plot the upper bounds derived from the numerical SDP approach, with darker colours corresponding to higher derivative order Λ . In black, we plot the expected gap from the leading triple-twist operator in GFB theory, which appears to be strictly below the numerical bound, even as Λ is increased.

$0 < \Delta_\phi \leq 5$, with a more refined grid in the range $0 < \Delta_\phi \leq 1$. At fixed derivative order Λ , the bound is roughly linear in the interval $0 < \Delta_\phi \leq 1$ and seems to remain so for larger values of Δ_ϕ . The values of the bound are always well below their GFF counterparts. However, as Λ increases, and the tone of green gets darker, there is a slower improvement of the bounds when compared to the GFF case. We see that the bound is always above the black line, corresponding to the value $3\Delta_\phi$ which is the gap in the $\mathcal{O}_2$ sector of the GFB six-point function

$$\begin{aligned}
 (\mathcal{G}^{\text{GFB}} - \mathcal{G}_{4\text{-pt}}^{\text{GFB}})/\chi_2^{\Delta_\phi} &= \left(\frac{\chi_1 \chi_2 \chi_3}{(1 - \chi_1 - \chi_2)(1 - \chi_3)} \right)^{2\Delta_\phi} + \left(\frac{\chi_1 \chi_2 \chi_3}{(1 - \chi_3 - \chi_2)(1 - \chi_1)} \right)^{2\Delta_\phi} \\
 &+ \left(\frac{\chi_1 \chi_2 \chi_3}{(1 - \chi_1)(1 - \chi_2)(1 - \chi_3)} \right)^{2\Delta_\phi} + \left(\frac{\chi_1 \chi_2 \chi_3}{(1 - \chi_1 - \chi_2)(1 - \chi_2 - \chi_3)} \right)^{2\Delta_\phi} \\
 &+ \left(\frac{\chi_1 \chi_2 \chi_3}{1 - \chi_1 - \chi_2 - \chi_3 + \chi_1 \chi_3} \right)^{2\Delta_\phi} + \left(\frac{\chi_1 \chi_2 \chi_3}{(1 - \chi_1 - \chi_2 - \chi_3 + \chi_1 \chi_3)(1 - \chi_2)} \right)^{2\Delta_\phi},
 \end{aligned} \tag{3.3.9}$$

which we expand in conformal blocks in Appendix 3.D. Somewhat surprisingly, however, in this case it seems that the bound does not converge all the way to the ϕ^3 operator of GFB, meaning the extremal theory whose four-point function agrees with

GFB might not be GFB! Once again the AdS_2 perspective sheds some light on why this might be the case. Consider the following action in AdS

$$S = \int_{\text{AdS}_2} \sqrt{g} d^2x \left(\frac{1}{2} \nabla^\mu \Phi \nabla_\mu \Phi + \frac{1}{2} m^2 \Phi^2 + g_6 \Phi^6 + \dots \right), \quad (3.3.10)$$

where Φ is the bulk scalar dual to ϕ on the boundary and we initially fine tuned quartic couplings to zero. As before, to leading order in g_6 , the four-point function of ϕ is unaltered, but there is a six-point contact diagram leading to anomalous dimensions of triple-twists.¹³ Unlike for fermions, though, the Φ^6 coupling is relevant, and hence we don't expect bounds on the sign of its coupling. In particular, we should be able to increase the dimension of ϕ^3 above the GFB value consistently, at least perturbatively. In practice, the bound should be a finite distance away from the GFB value, and a detailed explanation could involve finite coupling dynamics (perhaps related to instability of the potential), or an altogether unphysical solution, so it is hard to estimate the actual upper bound from this analysis. We emphasise, however, that very similar phenomena were observed in [185, 193], for the mixed four-point system of ϕ and ϕ^2 , where certain bounds are not saturated by GFB, even though one imposes by hand part of the GFB CFT data.

3.3.3 Comments on bounding three- and four-point functions

One of the main appeals of the six-point bootstrap in higher dimensions is the access to new OPE coefficients that are not probed by the scalar four point bootstrap such as OPE coefficients involving two spinning operators. In the 3d Ising model, for example, the six-point correlators can access the OPE coefficient for $TT\epsilon$ in the six-point function of ϵ or the OPE coefficient for $T\sigma S$, where S is a $\mathbb{Z}_2$ odd spinning operator, in the six-point function of σ . Hence, it would certainly be of interest to bound (products of) OPE coefficients $C_{\mathcal{O}_1}^{\mathcal{O}_2}$ within the numerical six-point bootstrap.

A very basic (but non-generic) example is the OPE coefficient $C_{\phi^2}^{\phi^3}$ in GFB. In this case, the only operator $\mathcal{O}_1$ with a non-vanishing OPE coefficient $C_{\phi\mathcal{O}_1\phi^3}$ is $\mathcal{O}_1 = \mathcal{O}_1^* = \phi^2$, i.e. the OPE coefficients for the GFB obey

$$C_{\phi\mathcal{O}_1\mathcal{O}_2^*} = 0 \quad \text{for all} \quad \mathcal{O}_1 \neq \mathcal{O}_1^*. \quad (3.3.11)$$

Indeed, for the GFB this OPE coefficient vanishes for all other double-twist operators with more derivatives, as one may infer from eq. (3.3.9) by expanding in conformal

¹³It is not hard to see, based on the spectral or Mellin representation of the six-point contact diagram that the comb-channel block expansion of this object contains only blocks of the form $G_{4\Delta_\phi+n_1, 3\Delta_\phi+n_2, 2\Delta_\phi+n_3}$ as well as $G_{2\Delta_\phi+n_1, 3\Delta_\phi+n_2, 4\Delta_\phi+n_3}$ and $\partial_{n_2} G_{2\Delta_\phi+n_1, 3\Delta_\phi+n_2, 2\Delta_\phi+n_3}$ which indeed corresponds to having anomalous dimension for triple-twists but no anomalous dimensions for double-twists, at leading order.

blocks (see Appendix 3.D). We can now separate out the contribution from $\mathcal{O}_2^* = \phi^3$ exchange in the middle channel and rewrite the six-point crossing equation as

$$(C_{\mathcal{O}_1^*}^{\mathcal{O}_2^*})^2 \alpha(F_{2\Delta_\phi, 3\Delta_\phi, 2\Delta_\phi}) = -\alpha(\mathcal{G}_{4\text{-pt}}) - \sum_{\Delta_2 > \Delta_2^*} \left(\sum_{\mathcal{O}_1, \mathcal{O}_3} C_{\mathcal{O}_1}^{\mathcal{O}_2} \alpha(F_{\Delta_1, \Delta_2, \Delta_3}) C_{\mathcal{O}_3}^{\mathcal{O}_2} \right). \quad (3.3.12)$$

Here, we have also applied a functional α to both sides of the equation. Let us agree to normalise the functional α such that it $\alpha(F_{2\Delta_\phi, 3\Delta_\phi, 2\Delta_\phi}) = 1$. If we now require positive semi-definiteness above the gap, as usual, we can maximise $\alpha(\mathcal{G}_{4\text{-pt}})$ to obtain the bound

$$(C_{\mathcal{O}_1^*}^{\mathcal{O}_2^*})^2 \leq -\max \alpha(\mathcal{G}_{4\text{-pt}}). \quad (3.3.13)$$

Recall that the vector $C_{\mathcal{O}_1^*}^{\mathcal{O}_2^*}$ is itself a product of two OPE coefficients so that we have now bounded a product of four OPE coefficients. But it is easy to obtain a bound on the interesting factor $C_{\mathcal{O}_1^* \phi \mathcal{O}_2^*}^2$ by taking the resulting bound (3.3.13) and dividing it by a bound on $C_{\phi \phi \mathcal{O}_1^*}^2$ which can be obtained from the usual four-point bootstrap. Note that to normalise on $F_{2\Delta_\phi, 3\Delta_\phi, 2\Delta_\phi}$ we are working in the primary-space basis, which makes it easy to isolate the contribution of the specific OPE coefficient we are interested in. However, the positivity conditions above the gap, are most conveniently implemented in the descendant space basis, as we have argued.

In a more generic situation, however, property (3.3.11) is not satisfied and instead infinitely many operators $\mathcal{O}_1$ will have a non-vanishing OPE coefficient $C_{\phi \mathcal{O}_1 \mathcal{O}_2^*}$. Hence, accessing individual OPE coefficients will be more challenging. Furthermore, directly extracting bounds on OPE coefficients from six-point crossing might also be considered somewhat artificial. As we have argued in Section 3.1.5, from the six-point bootstrap perspective it is natural to view the comb-channel OPE as a sum of scalar products of four-point and not three-point correlators whose positivity constraints CFT data. Consequently, the most straightforward quantity to bound in this framework is the full four-point correlator and not an individual OPE coefficient.¹⁴

For example, let us discuss how to produce an upper bound on the four-point function of three external scalars ϕ with the order p descendant of the primary $\mathcal{O}_2^*$, i.e. the correlator $\langle \phi \phi \phi P^n \mathcal{O}_2^* \rangle$. More precisely, we will consider the closely related (see Section 3.1.5) quantity

$$F_{\mathcal{O}_2^*}^{(p)} \equiv \sum_{\mathcal{O}} C_{\mathcal{O}_2^*}^{\mathcal{O}} F_p(\Delta). \quad (3.3.14)$$

Here we have used the functions F_n that were introduced in eq. (3.1.26) and we evaluated both sides at the crossing symmetric point $\chi_j = 1/3$ (we actually suppressed

¹⁴See also [194, 202] for recent work on bounding the four-point correlators directly from the four-point bootstrap.

the χ_j -dependence entirely). Instead of thinking of $F_{\mathcal{O}_2^*}^{(p)}$ in terms of a descendant four-point correlator evaluated at $\chi = 1/3$, we can equivalently think of it as a derivative of the four-point function with the primary $\mathcal{O}_2^*$. Hence we have access to all derivatives of the four-point correlator $\langle \phi \phi \phi \mathcal{O}_2^* \rangle$ at $\chi_j = 1/3$ and can reconstruct its Taylor expansion around that point. To obtain the bounds, we simply construct functionals $\alpha = \sum_{k=1}^{N_\alpha} c_k \alpha_k$ such that, in the notation of Section 3.1.3,

$$\left(\sum_{k=1}^{N_\alpha} c_k M_{nm}^k(\Delta_2) - \delta_{\Delta_2^*}^{\Delta_2^*} \delta_n^p \delta_m^p \right)_{n,m} > 0, \quad \text{for all } \Delta_2 \geq \Delta_2^*, \quad (3.3.15)$$

and maximise $\alpha(\mathcal{G}_{4\text{-pt}})$, i.e. the action of the functional α on the function $\mathcal{G}_{4\text{-pt}}$ we defined in eq. (3.3.2). The crossing equation then implies

$$0 = \alpha(\mathcal{G}_{4\text{-pt}}) + \left(F_{\mathcal{O}_2^*}^{(p)} \right)^2 + \sum_{\Delta_2 \geq \Delta_2^*} \left(\sum_{n,m} F_{\mathcal{O}_2}^{(n)} \left(\sum_{k=1}^{N_\alpha} c_k M_{nm}^k(\Delta_2) - \delta_{\Delta_2^*}^{\Delta_2^*} \delta_n^p \delta_m^p \right) F_{\mathcal{O}_2}^{(m)} \right), \quad (3.3.16)$$

and hence

$$\left(F_{\mathcal{O}_2^*}^{(p)} \right)^2 \leq -\alpha(\mathcal{G}_{4\text{-pt}}). \quad (3.3.17)$$

Just as for the bounds on OPE coefficients in the usual four-point bootstrap, changing the minus sign in eq. (3.3.15) to a plus allows one to analogously also construct lower bounds on $\left(F_{\mathcal{O}_2^*}^{(p)} \right)^2$. Note that to get non-trivial lower bounds, the assumption of a gap below Δ_2^* is not sufficient and we need to also impose an additional gap above Δ_2^* .

As an application, let us consider an upper bound on $F_{\mathcal{O}_2^*}^{(0)}$ for the case of GFF, where $\Delta_2^* = 3\Delta_\phi + 3$ corresponds to the leading triple-twist and $\mathcal{G}_{4\text{-pt}}$ was given in eq. (3.3.4). Figure 3.11 shows this upper bound for $\Lambda = 4$ and compares it to the exact result. We would like to stress that the scale of the plot is logarithmic – the bootstrap bound is thus nearly saturated by GFF over several orders of magnitude. As expected, the bound worsens significantly as Δ_ψ increases, which can already be noticed in Figure 3.11. We expect that a higher number of functionals would be necessary in order to extend the range of Δ_ψ where a reasonable agreement with the exact result is possible.

We leave a more systematic exploration of six-point bounds on four-point functions to Chapter 4, where a drastic improvement of the numerical setup will provide us with the computational power to appropriately address this problem.

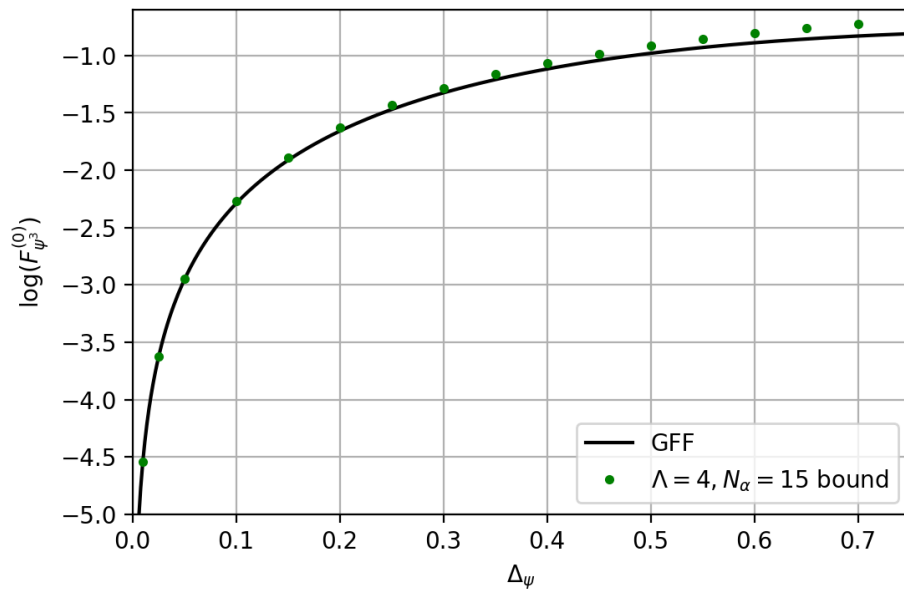


Figure 3.11: Bound on the $\langle \psi\psi\psi\mathcal{O}_2^* \rangle$ GFF four-point function evaluated at $\chi = 1/3$. The green dots represent a numerical upper bound obtained at $\Lambda = 4$. The black curve corresponds to the exact GFF result.

Appendix

3.A Decomposition of six-point into four-point correlators

The purpose of this appendix is to establish the identity

$$\sum_{n=0}^{\infty} \frac{\langle \phi_6 \phi_5 \phi_4 P^n \mathcal{O} \rangle \langle \mathcal{O}^\dagger K^n \phi_3 \phi_2 \phi_1 \rangle}{(2\Delta_{\mathcal{O}})_n n!} = \sum_{n=0}^{\infty} \frac{(\bar{\chi}_1 \chi_2 \bar{\chi}_3)^{\Delta_{\mathcal{O}}+n-\Delta_{\phi}} \partial_{\bar{\chi}_1}^n f(\bar{\chi}_1) \partial_{\bar{\chi}_3}^n f(\bar{\chi}_3)}{(x_{12}^2 x_{34}^2 x_{56}^2)^{\Delta_{\phi}} (2\Delta_{\mathcal{O}})_n n!}, \quad (3.A.1)$$

where

$$\bar{\chi} \equiv \frac{1}{\chi} \quad f(\bar{\chi}) \equiv \langle \phi(\infty) \mathcal{O}(\bar{\chi}) \phi(1) \phi(0) \rangle. \quad (3.A.2)$$

Three conformal transformations play a central role in the derivation. These are

$$\mu(z) \equiv \frac{z - x_3}{z - x_4} x_4 \quad \partial_z \mu(z) = \frac{x_{34}}{(z - x_4)^2} x_4 < 0 \quad (3.A.3)$$

$$\bar{\chi}_1(z) \equiv \frac{(z - x_2)x_{13}}{(z - x_3)x_{12}} \quad \partial_z \bar{\chi}_1(z) = \frac{x_{13}x_{23}}{(z - x_3)^2 x_{12}} < 0 \quad (3.A.4)$$

$$\bar{\chi}_3(z) \equiv \frac{(z - x_5)x_{46}}{(z - x_4)x_{56}} \quad \partial_z \bar{\chi}_3(z) = \frac{x_{45}x_{46}}{(z - x_5)^3 x_{65}} > 0. \quad (3.A.5)$$

As indicated above, $\mu(z)$ and $\bar{\chi}_1(z)$ are orientation reversing maps whereas $\bar{\chi}_3(z)$ preserves the orientation $x_i < x_{i+1}$. Note that the coincidence of our notation for the conformal maps $\bar{\chi}_1(z)$ and $\bar{\chi}_3(z)$ and the cross-ratios $\bar{\chi}_1$ and $\bar{\chi}_3$ is justified by

$$\bar{\chi}_1(x_4) = \bar{\chi}_1 \quad \bar{\chi}_3(x_3) = \bar{\chi}_3. \quad (3.A.6)$$

Before beginning the actual computation, let us qualitatively outline the strategy. Roughly speaking, we would like to insert $\mathcal{O}$ and $\mathcal{O}^\dagger$ into the six-point correlator at x_3 and x_4 respectively, so that we have a product of two four-point functions that depend on $\{x_6, x_5, x_4, x_3\}$ and $\{x_4, x_3, x_2, x_1\}$ only. Applying a conformal transformation on these two four-point functions, we could then further simplify to a product of two functions that depend only on χ_3 and χ_1 respectively multiplied by some simple prefactor. There are two obstructions to this plan. Firstly, in order to use the simple projector

$$\mathcal{P}_{\mathcal{O}} = \sum_{n=0}^{\infty} \frac{|P^n \mathcal{O}\rangle \langle \mathcal{O}^\dagger K^n|}{(2\Delta_{\mathcal{O}})_n n!}, \quad (3.A.7)$$

we should insert $\mathcal{O}$ and $\mathcal{O}^\dagger$ at 0 and ∞ not at x_3 and x_4 . To tackle this obstruction, we apply the conformal involution $\mu(z)$ that exchanges $x_3 \leftrightarrow 0$, $x_4 \leftrightarrow \infty$ before inserting the projector $\mathcal{P}_{\mathcal{O}}$ and would then like to apply it a second time after inserting the projector to undo the first transformation. However, when we try to apply $\mu(z)$ after inserting the projector, we encounter the second obstruction: After inserting the projector, we no longer have a correlator of primaries, but a sum of products of correlators of descendants, which do not transform as nicely under conformal mappings. To overcome this second problem, we should first apply conformal Ward identities to express the descendant correlators in terms of primary correlators and only afterwards apply $\mu(z)$ a second time.

Let us now perform the steps sketched above. First, we apply $\mu(z)$ to the six-point correlator, obtaining

$$\langle \phi_6 \phi_5 \phi_4 \phi_3 \phi_2 \phi_1 \rangle = \Omega \langle \phi(\mu(x_1)) \phi(\mu(x_2)) \phi(0) \phi(\infty) \phi(\mu(x_5)) \phi(\mu(x_6)) \rangle, \quad (3.A.8)$$

where

$$\Omega = \left| \partial_z \mu(x_1) \partial_z \mu(x_2) \partial_z \mu(x_3) \lim_{x \rightarrow x_4} \frac{\partial_z \mu(x)}{\mu(x)^2} \partial_z \mu(x_5) \partial_z \mu(x_6) \right|^{\Delta_\phi}. \quad (3.A.9)$$

Thus, with the projectors inserted, we have

$$\sum_{n=0}^{\infty} \frac{\langle \phi_6 \phi_5 \phi_4 P^n \mathcal{O} \rangle \langle \mathcal{O}^\dagger K^n \phi_3 \phi_2 \phi_1 \rangle}{(2\Delta_{\mathcal{O}})_n n!} \quad (3.A.10)$$

$$= \Omega \sum_{n=0}^{\infty} \frac{\langle \phi(\infty) \phi(\mu(x_5)) \phi(\mu(x_6)) P^n \mathcal{O} \rangle \langle \mathcal{O}^\dagger K^n \phi(\mu(x_1)) \phi(\mu(x_2)) \phi(0) \rangle}{(2\Delta_{\mathcal{O}})_n n!}. \quad (3.A.11)$$

Now it is time to use Ward identities and trade $P^n \mathcal{O}$ and $\mathcal{O}^\dagger K^n$ for

$$P^n |\mathcal{O}\rangle = \partial_{z_3}^n \mathcal{O}(z_3) |0\rangle|_{z_3=0}, \quad (3.A.12)$$

$$\langle \mathcal{O} | K^n = \langle 0 | (P^n \mathcal{O})^\dagger = \langle 0 | \partial_{z_1}^n \mathcal{O}(z_1)^\dagger |_{z_1=0} = \partial_{z_1}^n z_1^{-2\Delta_{\mathcal{O}}} \langle 0 | \mathcal{O}(1/z_1) |_{z_1=0}, \quad (3.A.13)$$

where for ease of notation, we will drop the $z_1 = 0$ and $z_3 = 0$ subscripts for the rest of the computation. Inserting these identities, we can apply $\mu(z)$ to deduce that

$$\langle \phi(\infty) \phi(\mu(x_5)) \phi(\mu(x_6)) P^n \mathcal{O} \rangle = \partial_{z_3}^n \langle \phi(\infty) \phi(\mu(x_5)) \phi(\mu(x_6)) \mathcal{O}(z_3) \rangle \quad (3.A.14)$$

$$= \partial_{z_3}^n \Omega_{456} \langle \phi_6 \phi_5 \phi_4 \mathcal{O}(\mu(z_3)) \rangle \quad (3.A.15)$$

and

$$\langle \mathcal{O}^\dagger K^n \phi(\mu(x_1)) \phi(\mu(x_2)) \phi(0) \rangle = \partial_{z_1}^n z_1^{-2\Delta_{\mathcal{O}}} \Omega_{123} \langle \mathcal{O}(1/z_1) \phi_3 \phi_2 \phi_1 \rangle, \quad (3.A.16)$$

where

$$\Omega_{456} = \left| \lim_{x \rightarrow x_4} \frac{\partial_z \mu(x)}{\mu(x)^2} \partial_z \mu(x_5) \partial_z \mu(x_6) \right|^{-\Delta_\phi} |\partial_z \mu(z_3)|^{\Delta_{\mathcal{O}}} \quad (3.A.17)$$

and

$$\Omega_{123} = |\partial_z \mu(x_1) \partial_z \mu(x_2) \partial_z \mu(x_3)|^{-\Delta_\phi} |\partial_z \mu(1/z_1)|^{\Delta_\phi}. \quad (3.A.18)$$

Next, we apply $\bar{\chi}_1(z)$ and $\bar{\chi}_3(z)$ to obtain

$$\partial_{z_3}^n \Omega_{456} \langle \phi_6 \phi_5 \phi_4 \mathcal{O}(\mu(z_3)) \rangle = \partial_{z_3}^n \Omega_{456} \Omega_{\bar{\chi}_3} \langle \phi(\infty) \mathcal{O}(\bar{\chi}_3(\mu(z_3))) \phi(1) \phi(0) \rangle \quad (3.A.19)$$

$$= \partial_{z_3}^n \Omega_{456} \Omega_{\bar{\chi}_3} f(\bar{\chi}_3(\mu(z_3))) \quad (3.A.20)$$

and likewise

$$\partial_{z_1}^n z_1^{-2\Delta_\phi} \Omega_{123} \langle \mathcal{O}(1/z_1) \phi_3 \phi_2 \phi_1 \rangle = \partial_{z_1}^n z_1^{-2\Delta_\phi} \Omega_{123} \Omega_{\bar{\chi}_1} f(\bar{\chi}_1(\mu(1/z_1))), \quad (3.A.21)$$

where

$$\Omega_{\bar{\chi}_3} = \left| \partial_z \bar{\chi}_3(x_6) \partial_z \bar{\chi}_3(x_5) \lim_{x \rightarrow x_4} \frac{\partial_z \bar{\chi}_3(x)}{\bar{\chi}_3(x)} \right|^{\Delta_\phi} |(\partial_z \bar{\chi}_3)(\mu(z_3))|^{\Delta_\phi} = \frac{|(\partial_z \bar{\chi}_3)(\mu(z_3))|^{\Delta_\phi}}{|x_{45} x_{56} x_{64}|^{\Delta_\phi}} \quad (3.A.22)$$

and likewise

$$\Omega_{\bar{\chi}_1} = \frac{|(\partial_z \bar{\chi}_1)(\mu(1/z_1))|^{\Delta_\phi}}{|x_{12} x_{23} x_{31}|^{\Delta_\phi}}. \quad (3.A.23)$$

Note that the products $\Omega_{123} \Omega_{\bar{\chi}_1} z_1^{-2\Delta_\phi}$ and $\Omega_{456} \Omega_{\bar{\chi}_3}$ are independent of z_1 and z_3 since

$$\Omega_{123} \Omega_{\bar{\chi}_1} z_1^{-2\Delta_\phi} = \frac{|\partial_{z_1} \bar{\chi}_1(\mu(1/z_1))|^{\Delta_\phi}}{|\partial_z \mu(x_1) \partial_z \mu(x_2) \partial_z \mu(x_3) x_{12} x_{23} x_{31}|^{\Delta_\phi}} \quad (3.A.24)$$

and

$$\Omega_{456} \Omega_{\bar{\chi}_3} = \frac{|\partial_{z_3} \bar{\chi}_3(\mu(z_3))|^{\Delta_\phi}}{\left| \lim_{x \rightarrow x_4} \frac{\partial_z \mu(x)}{\mu(x)^2} \partial_z \mu(x_5) \partial_z \mu(x_6) x_{45} x_{56} x_{64} \right|^{\Delta_\phi}} \quad (3.A.25)$$

but $\bar{\chi}_1(\mu(1/z_1))$ and $\bar{\chi}_3(\mu(z_3))$ just depend linearly on z_1 and z_3

$$\bar{\chi}_1(\mu(1/z_1)) = \bar{\chi}_1 + x_4 \frac{x_{13} x_{32}}{x_{12} x_{34}} z_1 \quad \bar{\chi}_3(\mu(z_3)) = \bar{\chi}_3 + \frac{1}{x_4} \frac{x_{45} x_{46}}{x_{43} x_{56}} z_3. \quad (3.A.26)$$

Apart from ensuring that $\Omega_{123} \Omega_{\bar{\chi}_1}$ and $\Omega_{456} \Omega_{\bar{\chi}_3}$ are independent of z_1 and z_3 , eq. (3.A.26) also implies that

$$\partial_{z_1}^n f(\bar{\chi}_1(\mu(1/z_1)))|_{z_1=0} = \left(x_4 \frac{x_{13} x_{32}}{x_{12} x_{34}} \right)^n \partial_{\bar{\chi}_1}^n f(\bar{\chi}_1) = \left| x_4 \frac{x_{13} x_{32}}{x_{12} x_{34}} \right|^n (-1)^n \partial_{\bar{\chi}_1}^n f(\bar{\chi}_1) \quad (3.A.27)$$

and

$$\partial_{z_3}^n f(\bar{\chi}_3(\mu(z_3)))|_{z_3=0} = \left(\frac{1}{x_4} \frac{x_{45} x_{46}}{x_{43} x_{56}} \right)^n \partial_{\bar{\chi}_3}^n f(\bar{\chi}_3) = \left| \frac{1}{x_4} \frac{x_{45} x_{46}}{x_{43} x_{56}} \right|^n (-1)^n \partial_{\bar{\chi}_3}^n f(\bar{\chi}_3). \quad (3.A.28)$$

Thus, we conclude that

$$\langle \phi(\infty)\phi(\mu(x_5))\phi(\mu(x_6))P^n\mathcal{O} \rangle = \frac{\left| x_4 \frac{x_{13}x_{32}}{x_{12}x_{34}} \right|^{\Delta_{\mathcal{O}}+n} (-1)^n \partial_{\bar{\chi}_1}^n f(\bar{\chi}_1)}{|\partial_z \mu(x_1)\partial_z \mu(x_2)\partial_z \mu(x_3)x_{12}x_{23}x_{31}|^{\Delta_\phi}} \quad (3.A.29)$$

and

$$\langle \mathcal{O}^\dagger K^n \phi(\mu(x_1))\phi(\mu(x_2))\phi(0) \rangle = \frac{\left| \frac{1}{x_4} \frac{x_{45}x_{46}}{x_{43}x_{56}} \right|^{\Delta_{\mathcal{O}}+n} (-1)^n \partial_{\bar{\chi}_3}^n f(\bar{\chi}_3)}{\left| \lim_{x \rightarrow x_4} \frac{\partial_z \mu(x)}{\mu(x)^2} \partial_z \mu(x_5)\partial_z \mu(x_6)x_{45}x_{56}x_{64} \right|^{\Delta_\phi}}, \quad (3.A.30)$$

Inserting these two expressions into eq. (3.A.10) finally leads to

$$\sum_{n=0}^{\infty} \frac{\langle \phi_6\phi_5\phi_4 P^n \mathcal{O} \rangle \langle \mathcal{O}^\dagger K^n \phi_3\phi_2\phi_1 \rangle}{(2\Delta_{\mathcal{O}})_n n!} = \sum_{n=0}^{\infty} \frac{\left| \frac{x_{13}x_{32}}{x_{12}x_{34}} \frac{x_{45}x_{46}}{x_{43}x_{56}} \right|^{\Delta_{\mathcal{O}}+n} \partial_{\bar{\chi}_1}^n f(\bar{\chi}_1) \partial_{\bar{\chi}_3}^n f(\bar{\chi}_3)}{|x_{12}x_{23}x_{31}x_{45}x_{56}x_{64}|^{\Delta_\phi} (2\Delta_{\mathcal{O}})_n n!} \quad (3.A.31)$$

$$= \sum_{n=0}^{\infty} \frac{(\bar{\chi}_1 \chi_2 \bar{\chi}_3)^{\Delta+n-\Delta_\phi} \partial_{\bar{\chi}_1}^n f(\bar{\chi}_1) \partial_{\bar{\chi}_3}^n f(\bar{\chi}_3)}{(x_{12}^2 x_{34}^2 x_{56}^2)^{\Delta_\phi} (2\Delta_{\mathcal{O}})_n n!}, \quad (3.A.32)$$

which is what we wanted to show.

3.B Comments on n -point crossing equations

The aim of this appendix is to elaborate further on the relation between the six-point bootstrap and mixed correlator systems of four-point crossing equations. En passant, the discussion will naturally lead to some motivation for extending our programme from six- to eight-point correlators.

Our starting point is the search for a mixed four-point correlator interpretation of the crossing equation

$$\sum_{\mathcal{O}} \sum_b \langle \phi_1\phi_2\phi_3 \mathcal{O}^b \rangle \langle \mathcal{O}^b \phi_4\phi_5\phi_6 \rangle = \sum_{\mathcal{O}} \sum_b \langle \phi_2\phi_3\phi_4 \mathcal{O}^b \rangle \langle \mathcal{O}^b \phi_5\phi_6\phi_1 \rangle, \quad (3.B.1)$$

for decompositions of a six-point function into products of four-point functions. Note that, as opposed to the main text (eq. (3.1.53)), we find it more convenient to work with an orthonormal basis here, so that $\mathcal{N}_{ab} = \delta_{ab}$.

Inserting the identity in a carefully chosen way, we can directly see that this crossing equation is exactly equivalent to the full system of all s- to t-channel four-point crossing equations for correlators of two copies of ϕ and two arbitrary operators that appear in the $\phi \times \phi$ OPE.

Indeed, eq. (3.B.1) is equivalent to¹⁵

$$\begin{aligned} & \sum_{\mathcal{O}_1, \mathcal{O}_3} \sum_{a,b,c} \langle \phi_2 \phi_3 \mathcal{O}_1^a \rangle \langle \mathcal{O}_1^a \phi_1 \mathcal{O}^b \rangle \langle \mathcal{O}^b \phi_4 \mathcal{O}_3^c \rangle \langle \mathcal{O}_3^c \phi_5 \phi_6 \rangle \\ &= \sum_{\mathcal{O}_1, \mathcal{O}_3} \sum_{a,b,c} \langle \phi_2 \phi_3 \mathcal{O}_1^a \rangle \langle \mathcal{O}_1^a \phi_4 \mathcal{O}^b \rangle \langle \mathcal{O}^b \phi_1 \mathcal{O}_3^c \rangle \langle \mathcal{O}_3^c \phi_5 \phi_6 \rangle, \end{aligned} \quad (3.B.2)$$

which due to linear independence of the occurring three-point structures implies that

$$\sum_{\mathcal{O}} \sum_b \langle \mathcal{O}_1^a \phi_1 \mathcal{O}^b \rangle \langle \mathcal{O}^b \phi_4 \mathcal{O}_3^c \rangle = \sum_{\mathcal{O}} \sum_b \langle \mathcal{O}_1^a \phi_4 \mathcal{O}^b \rangle \langle \mathcal{O}^b \phi_1 \mathcal{O}_3^c \rangle \quad \forall \mathcal{O}_1, \mathcal{O}_3 \subseteq \phi \times \phi. \quad (3.B.3)$$

Pictorially, we have said nothing more than

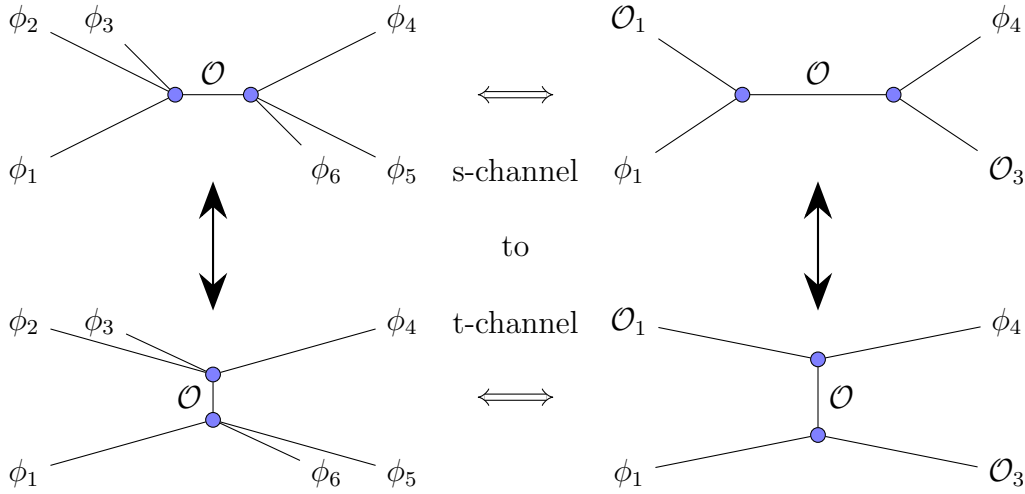


Figure 3.B.1: Visualising the equivalence between crossing of four-point decompositions of six-point correlators and infinite systems of s- to t-channel four-point crossing equations.

It is reassuring that this perspective now gives us the precise answer to the questions raised at the end of Section 3.1.2: We exactly capture all s- to t-channel crossings, involving two external ϕ and two arbitrary operators $\mathcal{O}_1, \mathcal{O}_3$ in the $\phi \times \phi$ OPE. But can we do better? How could a multi-point bootstrap access crossing of all possible operators in the $\phi \times \phi$ OPE?

Figure 3.B.1 already suggests the answer to these questions. We can go to eight-point functions. By the same arguments that led to eq. (3.B.3), one arrives at the

¹⁵Let us stress that this seemingly obvious claim is not trivial: There is a surjectivity statement of a map from a vector-space of real numbers labelled by operators occurring in the $\phi \times \phi$ -OPE to a vector-space of real numbers labelled by descendants underlying it. Concretely, this map is induced by conformal blocks for $\langle \phi \phi \phi \mathcal{O}^a \rangle$ four-point correlators. In the 1d case, we have discussed it as the map from primary-space to descendant-space.

conclusion that imposing the following crossing equation

$$\sum_{\mathcal{O}} \sum_b \langle \phi_1 \phi_2 \phi_3 \phi_4 \mathcal{O}^b \rangle \langle \mathcal{O}^b \phi_5 \phi_6 \phi_7 \phi_8 \rangle = \sum_{\mathcal{O}} \sum_b \langle \phi_1 \phi_2 \phi_7 \phi_8 \mathcal{O}^b \rangle \langle \mathcal{O}^b \phi_3 \phi_4 \phi_5 \phi_6 \rangle. \quad (3.B.4)$$

is directly related to imposing s- to t-channel crossing for the entirety of all operators in the $\phi \times \phi$ OPE. This is visualised in the figure below.

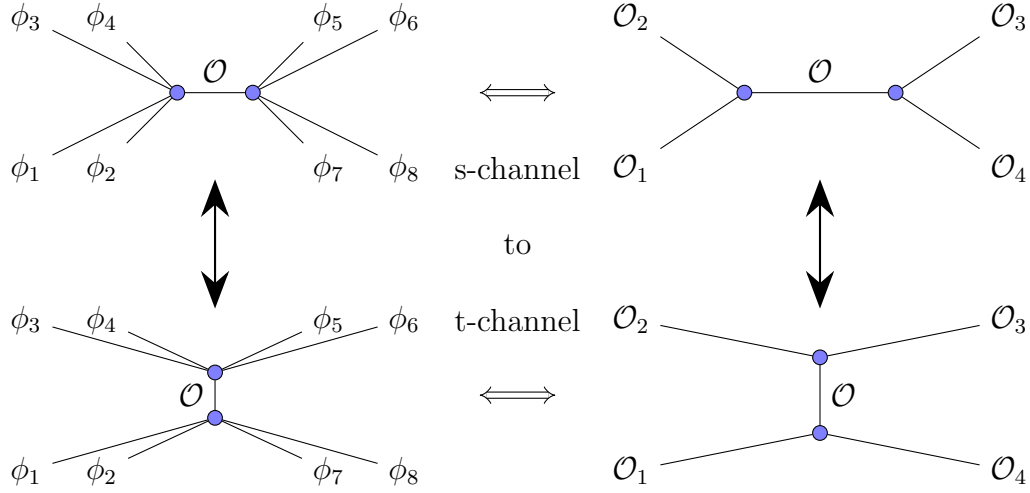


Figure 3.B.2: Equivalence between eight-point crossing and the full system of all four-point crossing equations, s- and t-channel.

3.C SDP implementation: polynomial SOS

The notebook `polynomial_sos.nb` attached to the arXiv submission contains the algorithm that we use to recast the problem of constructing SOS linear combinations of two variable polynomials as an SDP. At the current stage of the numerical six-point bootstrap, we found that solving this SDP is not the bottleneck holding us back from higher precision results and therefore decided to stick to the minimalistic implementation described below. Nevertheless, due to the notoriously bad scaling of SDP formulations of polynomial SOS optimisation problems with the degree of the polynomials, it might become necessary at high derivative order to implement some more sophisticated algorithm. For a short review of directions for optimisation see [203].

The input of the notebook `polynomial_sos.nb` is a list P of polynomials of the form

$$\sum_{i=0}^{2n} \sum_{j=0}^{2m} c_{ij} x^i y^j, \quad (3.C.1)$$

for some reals coefficients c . Corresponding to these polynomials, we consider a list

$$\mathbf{v} = \{x^i y^j | 0 \leq i \leq n, 0 \leq j \leq m\} \quad (3.C.2)$$

of monomials. Now, for each polynomial $\mathbf{p} \in \mathbf{P}$, we construct a $(n+1)(m+1) \times (n+1)(m+1)$ matrix $\mathbf{Q}[\mathbf{p}]$ such that

$$\mathbf{v}^T \mathbf{Q}[\mathbf{p}] \mathbf{v} = \mathbf{p}. \quad (3.C.3)$$

Next, we construct a basis of the kernel of the conjugation by $\mathbf{v}$ i.e. a basis $\mathbf{K}$ of matrices $\mathbf{K}[[j]]$ s.t.

$$\mathbf{v}^T \mathbf{K}[[j]] \mathbf{v} = 0. \quad (3.C.4)$$

We then input the problem of finding positive semi-definite linear combinations of the matrices $\mathbf{Q}[\mathbf{P}[[i]]]$ and the kernel matrices $\mathbf{K}$ into SDPA. If we manage to find such a positive semi-definite linear combination

$$\mathbf{M} \equiv \sum_i a_i \mathbf{Q}[\mathbf{P}[[i]]] + \sum_j b_j \mathbf{K}[[j]] \quad (3.C.5)$$

with some real coefficients a and b , then the polynomial

$$\mathbf{v}^T \mathbf{M} \mathbf{v} = \sum_i a_i \mathbf{v}^T \mathbf{Q}[\mathbf{P}[[i]]] \mathbf{v} = \sum_i a_i \mathbf{P}[[i]] \quad (3.C.6)$$

is manifestly a sum of squares.

3.D Selected conformal block expansions

In this appendix, we provide explicit expressions for the conformal block expansions of some of the correlators which appeared in the main text. In practice, we consider a given correlator and observe its leading power law dependence on the cross-ratios χ_i . We then write an Ansatz with a finite sum of conformal blocks (3.1.8) which we can be Taylor expanded and matched to the correlator's Taylor expansion, up to an order determined by the number of blocks included. In the cases we consider, this is simple to do since the scaling dimensions are of the form $\Delta_i = a_i \Delta_\phi + n_i$, for some integer or rational numbers a_i and some integers n_i . The Ansatz can then be written as a sum over the n_i which are chosen consistently with the order at which we truncate the Taylor expansion.

No-identity correlator Let us first consider the “no-identity” correlator (3.2.6), repeated here for convenience:

$$\mathcal{G} = \frac{\chi_1^{8\Delta_\phi} \chi_2^{9\Delta_\phi} \chi_3^{8\Delta_\phi}}{\left((1-\chi_1)(1-\chi_2)(1-\chi_3)(1-\chi_1-\chi_2)(1-\chi_2-\chi_3)(1-\chi_1-\chi_2-\chi_3+\chi_1\chi_3)\right)^{2\Delta_\phi}}. \quad (3.D.1)$$

By Taylor expanding it becomes clear that the dimensions of the exchanged operators have to be of the form $\Delta_1 = 8\Delta_\phi + n_1$ and $\Delta_2 = 9\Delta_\phi + n_2$. We can then propose an Ansatz

$$\mathcal{G} = \sum_{n_1, n_2, n_3=0}^{\infty} P_{n_1, n_2, n_3} G_{8\Delta_\phi + n_1, 9\Delta_\phi + n_2, 8\Delta_\phi + n_3} \quad (3.D.2)$$

where $P_{n_1, n_2, n_3} = P_{n_3, n_2, n_1}$, and we recall that the external operator has dimension $5\Delta_\phi$ in these conventions. Truncating the Taylor expansion to the appropriate order we can read off the first few non-vanishing OPE coefficients

$$\begin{aligned} P_{000} &= 1, \quad P_{002} = \frac{6\Delta_\phi^2}{1 + 16\Delta_\phi}, \quad P_{004} = \frac{3\Delta_\phi^2(2\Delta_\phi + 1)(6\Delta_\phi + 1)}{2(16\Delta_\phi + 3)(16\Delta_\phi + 5)}, \\ P_{020} &= \frac{8\Delta_\phi^2}{1 + 18\Delta_\phi}, \quad P_{022} = \frac{4\Delta_\phi^2(4\Delta_\phi(3\Delta_\phi + 7) + 1)}{(16\Delta_\phi + 1)(18\Delta_\phi + 1)}, \quad P_{030} = \frac{8\Delta_\phi^3}{3(9\Delta_\phi + 1)(9\Delta_\phi + 2)}, \\ P_{202} &= \frac{36\Delta_\phi^4}{(16\Delta_\phi + 1)^2}, \quad P_{040} = \frac{2\Delta_\phi^2(2\Delta_\phi + 1)(4\Delta_\phi + 1)}{3(3\Delta_\phi + 1)(18\Delta_\phi + 5)}, \quad \dots \end{aligned} \quad (3.D.3)$$

We note that n_1 and n_3 can only take even values, while n_2 can take all even and odd values except 1. This is consistent with what one would naively expect from parity and a counting of primaries vs. descendants. Unitarity is also easy to check: For fixed n_2 , we found that the matrices $(P_{n, n_2, m})_{nm}$ that we have computed are positive semi-definite.

Generalised free fermion We now consider the non-trivial piece of the GFF six-point function (3.3.5), repeated below

$$\begin{aligned} (\mathcal{G}^{\text{GFF}} - \mathcal{G}_{4\text{-pt}}^{\text{GFF}})/\chi_2^{\Delta_\psi} &= \left(\frac{\chi_1\chi_2\chi_3}{(1-\chi_1-\chi_2)(1-\chi_3)} \right)^{2\Delta_\psi} + \left(\frac{\chi_1\chi_2\chi_3}{(1-\chi_3-\chi_2)(1-\chi_1)} \right)^{2\Delta_\psi} \\ &- \left(\frac{\chi_1\chi_2\chi_3}{(1-\chi_1)(1-\chi_2)(1-\chi_3)} \right)^{2\Delta_\psi} - \left(\frac{\chi_1\chi_2\chi_3}{(1-\chi_1-\chi_2)(1-\chi_2-\chi_3)} \right)^{2\Delta_\psi} \\ &- \left(\frac{\chi_1\chi_2\chi_3}{1-\chi_1-\chi_2-\chi_3+\chi_1\chi_3} \right)^{2\Delta_\psi} + \left(\frac{\chi_1\chi_2\chi_3}{(1-\chi_1-\chi_2-\chi_3+\chi_1\chi_3)(1-\chi_2)} \right)^{2\Delta_\psi}. \end{aligned} \quad (3.D.4)$$

This is the piece of the correlator with triple-twist exchange in the $\mathcal{O}_2$ sector as can be seen from the behaviour as $\chi_2 \rightarrow 0$. Using the Ansatz

$$\mathcal{G}_{3t}^{\text{GFF}} = \sum_{n_1, n_2, n_3=0}^{\infty} P_{n_1, n_2, n_3} G_{2\Delta_\psi + n_1, 3\Delta_\psi + n_2, 2\Delta_\psi + n_3}, \quad (3.D.5)$$

where the subscript 3t denotes the triple-twist contribution, allows us to extract the first non-vanishing OPE coefficients

$$\begin{aligned}
P_{131} &= 2\Delta_\psi^2(1 + 2\Delta_\psi), & P_{133} &= -\frac{2\Delta_\psi^2(1 + \Delta_\psi)(1 + 2\Delta_\psi)}{3 + 4\Delta_\psi}, \\
P_{333} &= \frac{2\Delta_\psi^2(1 + \Delta_\psi)^2(1 + 2\Delta_\psi)}{(3 + 4\Delta_\psi)^2}, & P_{151} &= \frac{\Delta_\psi^2(\Delta_\psi + 1)(2\Delta_\psi + 1)^2(2\Delta_\psi + 3)}{3(6\Delta_\psi + 7)}, \\
P_{161} &= \frac{2\Delta_\psi^3(\Delta_\psi + 1)(\Delta_\psi + 2)(2\Delta_\psi + 1)^2(2\Delta_\psi + 3)}{45(3\Delta_\psi + 4)(3\Delta_\psi + 5)}, & \dots & .
\end{aligned} \tag{3.D.6}$$

The OPE coefficients pass all the unitarity checks mentioned above. Furthermore one observes that n_1 and n_3 take only odd values as expected from the fermionic double-trace spectrum. One also immediately sees that the leading triple-trace operator has dimension $3\Delta_\psi + 3$ as discussed in the main text. Additionally we see that there is no state with $n_2 = 4$ which is compatible with a counting of triple-twist primaries in GFF discussed in Appendix 3.E.

Using the double trace OPE coefficients

$$C_{\psi\psi\Delta_n}^2 = \frac{2\Gamma^2(\Delta_n)\Gamma(\Delta_n + 2\Delta_\psi - 1)}{\Gamma^2(2\Delta_\psi)\Gamma(2\Delta_n - 1)\Gamma(\Delta_n - 2\Delta_\psi + 1)} \tag{3.D.7}$$

one can extract squares of triple-trace OPE coefficients from the coefficients P_{n_1, n_2, n_3} given in Eq. (3.D.6)

Generalised free boson Finally we take the triple-twist piece of the GFB correlator (3.3.9) given by

$$\begin{aligned}
(\mathcal{G}^{\text{GFB}} - \mathcal{G}_{4\text{-pt}}^{\text{GFB}})/\chi_2^{\Delta_\phi} &= \left(\frac{\chi_1\chi_2\chi_3}{(1 - \chi_1 - \chi_2)(1 - \chi_3)}\right)^{2\Delta_\phi} + \left(\frac{\chi_1\chi_2\chi_3}{(1 - \chi_3 - \chi_2)(1 - \chi_1)}\right)^{2\Delta_\phi} \\
&+ \left(\frac{\chi_1\chi_2\chi_3}{(1 - \chi_1)(1 - \chi_2)(1 - \chi_3)}\right)^{2\Delta_\phi} + \left(\frac{\chi_1\chi_2\chi_3}{(1 - \chi_1 - \chi_2)(1 - \chi_2 - \chi_3)}\right)^{2\Delta_\phi} \\
&+ \left(\frac{\chi_1\chi_2\chi_3}{1 - \chi_1 - \chi_2 - \chi_3 + \chi_1\chi_3}\right)^{2\Delta_\phi} + \left(\frac{\chi_1\chi_2\chi_3}{(1 - \chi_1 - \chi_2 - \chi_3 + \chi_1\chi_3)(1 - \chi_2)}\right)^{2\Delta_\phi}. \tag{3.D.8}
\end{aligned}$$

Once again we can use the Ansatz

$$\mathcal{G}_{3t}^{\text{GFB}} = \sum_{n_1, n_2, n_3=0}^{\infty} P_{n_1, n_2, n_3} G_{2\Delta_\phi + n_1, 3\Delta_\phi + n_2, 2\Delta_\phi + n_3}, \tag{3.D.9}$$

which allows us to extract the leading OPE coefficients

$$\begin{aligned}
P_{0,0,0} &= 6, & P_{0,2,0} &= \frac{8\Delta_\phi^2 (2\Delta_\phi + 1)}{6\Delta_\phi + 1}, & P_{0,2,2} &= \frac{4\Delta_\phi^2 (2\Delta_\phi + 1)}{4\Delta_\phi + 1}, \\
P_{0,3,0} &= \frac{8\Delta_\phi^3 (\Delta_\phi + 1) (2\Delta_\phi + 1)}{3(3\Delta_\phi + 1)(3\Delta_\phi + 2)}, & P_{0,3,2} &= -\frac{8\Delta_\phi^3 (\Delta_\phi + 1) (2\Delta_\phi + 1)}{(3\Delta_\phi + 2)(4\Delta_\phi + 1)}, \\
P_{0,4,0} &= \frac{2\Delta_\phi^2 (2\Delta_\phi + 1)^2 (2\Delta_\phi + 3)}{9(6\Delta_\phi + 5)}, & P_{0,4,2} &= \frac{4\Delta_\phi^3 (2\Delta_\phi + 1)^2 (2\Delta_\phi + 3)}{3(4\Delta_\phi + 1)(6\Delta_\phi + 5)}, \quad \dots
\end{aligned} \tag{3.D.10}$$

which yet again passes all the unitarity checks, is consistent with a double-trace spectrum with only even integer shifts and with the counting of triple-twist primary operators.

Finally note that these expansion coefficients fit nicely with the closed OPE-coefficient formulas: The OPE coefficient between two ϕ and a double trace operator $[\phi\phi]_n$ of dimension Δ_n^B is

$$C_{[\phi\phi]_n \phi\phi}^2 = \frac{2(2\Delta_\phi)_{2n}^2}{(4\Delta_\phi + 2n - 1)_n (2n)!}. \tag{3.D.11}$$

The squared OPE coefficient $C_{\phi[\phi\phi]_n[\phi\phi\phi]_m}^2$ between ϕ the double twist $[\phi\phi]_n$ and the sum of all m -derivative triple-twist operators $[\phi\phi\phi]_m$ vanishes if $m < 2n$ and is given by

$$\begin{aligned}
C_{\phi[\phi\phi]_n[\phi\phi\phi]_m}^2 &= \frac{(2\Delta)_{m-2n} (4n + 4\Delta)_{m-2n}}{(m-2n)! (m+2n+6\Delta-1)_{m-2n}} \left(1 + \right. \\
&\quad \left. \frac{2(-1)^m (4\Delta + 4n - 1) (2\Delta)_m}{(2n)! (2n + 4\Delta - 1)_{m-2n+1}} {}_4F_3 \left[\begin{matrix} -2n, 1-2\Delta-2n, 2\Delta+2n, 4\Delta+2n-1 \\ 1-2\Delta-m, 2\Delta, 4\Delta+m \end{matrix}; 1 \right] \right),
\end{aligned} \tag{3.D.12}$$

for $m \geq 2n$. In terms of these OPE coefficients, we have

$$P_{2n_1, 2n_2, 2n_3}^2 = C_{[\phi\phi]_{n_1} \phi\phi}^2 C_{[\phi\phi\phi]_{n_2} [\phi\phi]_{n_1} \phi}^2 C_{[\phi\phi\phi]_{n_2} [\phi\phi]_{n_3} \phi}^2 C_{[\phi\phi]_{n_3} \phi\phi}^2. \tag{3.D.13}$$

3.E Counting multi-twist operators in GFF Theory

A straightforward way to obtain the spectrum of multi-twist operators in GFF is to use its Fock space description and compute the thermal partition function [204].

We define the partition function

$$Z(\beta) = \text{Tr} e^{-\beta\Delta}, \tag{3.E.1}$$

which is guaranteed by conformal representation theory to admit the decomposition

$$Z(\beta) = 1 + \sum_{\Delta} N_{\Delta} \chi_{\Delta}(\beta), \tag{3.E.2}$$

where we have the conformal character χ_Δ ¹⁶

$$\chi_\Delta(\beta) = \frac{q^\Delta}{1-q}, \quad q \equiv e^{-\beta}, \quad (3.E.3)$$

and the degeneracies N_Δ . We can use the fermionic Fock space to directly compute $Z(\beta)$. For that we define the partial partition function Z_n built out of constituents with exactly n insertions of the momentum generator P . The Pauli exclusion principle leads to

$$Z_n = 1 + q^{\Delta_\psi + n}. \quad (3.E.4)$$

The full partition function is then easily computed through

$$Z(\beta) = \prod_{n=0}^{\infty} Z_n = (-q^{\Delta_\psi}, q)_\infty, \quad (3.E.5)$$

where we used the q -deformed Pochhammer symbol

$$(a, q)_n = \prod_{k=0}^{n-1} (1 - aq^k). \quad (3.E.6)$$

It is straightforward to expand the explicit partition function in powers of the form $q^{n\Delta_\psi + m}$ and match this to the character expansion, reading off the degeneracies

$$N_{2\Delta_\psi + n} = \frac{1 + (-1)^{n+1}}{2}, \quad (3.E.7)$$

which correspond to the usual non-degenerate fermionic double-twists with an odd number of derivatives. For the triple-twists we find in terms of the floor function

$$N_{3\Delta_\psi + n} = \lfloor (n-1)/2 \rfloor - \lfloor (n-1)/3 \rfloor. \quad (3.E.8)$$

In particular, we see that the first non-vanishing primary has dimension $3\Delta_\psi + 3$, and that there is no primary of dimension $3\Delta_\psi + 4$, in agreement with the explicit conformal block expansion. For completeness we add the quadruple-twist degeneracies

$$N_{4\Delta_\psi + n} = \lfloor ((n-3)^2 + 6)/12 \rfloor - \lfloor (n-3)/4 \rfloor \lfloor (n-1)/4 \rfloor. \quad (3.E.9)$$

¹⁶Hopefully, there is no confusion with the cross-ratios χ_i

Chapter 4

Sparsity in the six-point bootstrap

– Solving the computability problem.

In the previous chapter, we have demonstrated that interesting qualitatively novel bounds can be computed by analysing six-point crossing via semi-definite programming. However, this first implementation of the six-point bootstrap suffers from various technical difficulties that limit its range of applicability. On the one hand, it has to resort to a discretisation of the spectrum of exchanged operators and cannot rely on polynomial SDP techniques that have been the standard in four-point bootstrap since [74, 75]. On the other hand, SDPB – the highly optimised solver designed specifically to address the needs of the bootstrap community – is not well suited to solve the SDPs encountered in [93], which in the previous chapter forced us to resort to the general purpose solver SDPA [173]. Support for the latter has however seized ten years ago and, crucially, no arbitrary precision, parallelisable version of SDPA is available. This fact is a major bottle neck for SDPA based approaches to bootstrap, making it virtually impossible to scale programs up and benefit from the computational power of modern clusters.

This chapter introduces a conceptual improvement of the six-point bootstrap which allows us to overcome the two mayor obstacles that have been described in the previous paragraph. Using sparse SDP techniques, the banded structure of the infinite matrices that occur in the six-point bootstrap of [93] can be efficiently exploited. The final result is that 1d six-point bootstrap problems reduce to SDPs that are of the same type as those encountered in finite mixed correlator systems of 2d four-point bootstrap. Since SDPB was specifically designed to excel at solving SDPs of that type, it can thus directly be applied to the six-point bootstrap. This both removes the need of discretisations and makes large scale parallelisation applicable. These technical improvements extend the possibilities for six-point bootstrap applications, which is demonstrated through two examples in this chapter.

As a first example, Section 4.3.1 revisits the “gap maximisation without identity”

problem that was introduced in Section 3.2. In the previous formulation of the six-point bootstrap it was even challenging to obtain the 15 functional (four derivative) bounds shown in Figures 3.7 and 3.8. With the refined method presented in the current chapter, the bound was determined with up to 53 functionals (seven derivatives), while at the same time also scanning over a much finer grid of external scaling dimensions, see Figure 4.3.1.

The main application of numerical six-point bootstrap studied in this chapter is described in Sections 4.3.2, 4.3.3 and 4.3.4. Concretely, a novel two parameter family of extremal solutions to crossing interpolating between the Generalised Free Boson (GFB) and Generalised Free Fermion (GFF) is found using the new numerical tools. A similar, OPE maximising, interpolation between GFB and GFF has been studied before, using the four-point bootstrap, for instance in [181]. Close to GFB, the four-point extremal solution was found to match the $\lambda_4\Phi^4$ deformation of the massive free boson in AdS_2 . Likewise, we show that, around the GFB point, the plane tangent to the interpolating surface obtained from the six-point bootstrap is spanned by the $\lambda_4\Phi^4$ and $\lambda_6\Phi^6$ deformation.

Outline. First, Section 4.1 provides a concise review of the six-point crossing equations, derivative functionals for them and the numerical six-point bootstrap. Section 4.2 then describes in detail the main technical improvements of this chapter. After the review Section 4.2.1, which discusses sparse SDP for banded matrices in general terms, Section 4.2.2 specialises to the particular SDPs that arise in the numerical six-point bootstrap. As already outlined in the previous paragraphs, Section 4.3 describes two applications of the improved six-point bootstrap. The ancillary files available on the arXiv page of the original publication [2] on which this chapter is based should allow the reader to reproduce the bounds presented in Section 4.3. Appendix 4.A contains instructions on how to use the corresponding code.

4.1 Semidefinite programming and six-point crossing

This section briefly reviews the semidefinite programming based numerical six-point bootstrap of Reference [93]. It starts in Section 4.1.1 with a discussion of the crossing equation, which is followed by a description of derivative functionals that act on that equation in Section 4.1.2. Finally, the dual SDP formulation of the six-point crossing equation and some aspects of its numerical implementation are discussed in Section 4.1.3.

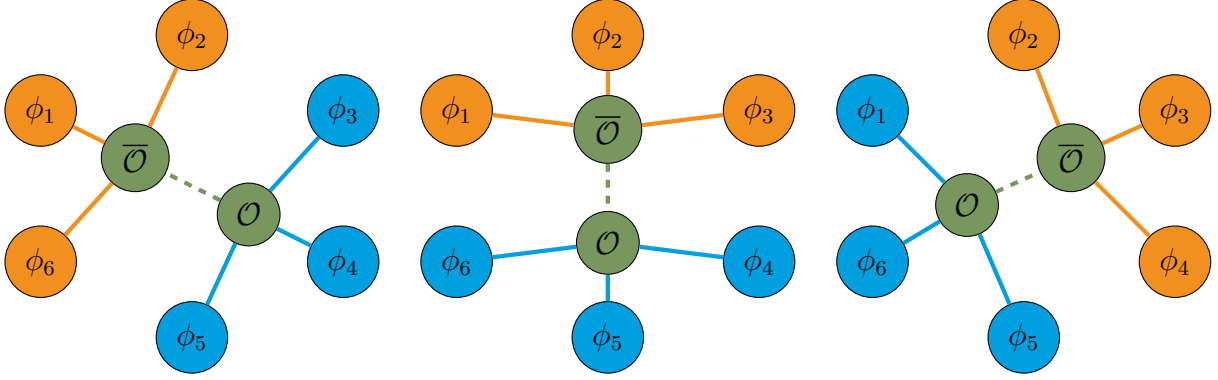


Figure 4.1.1: The three different decompositions of a six-point correlator into products of four-point functions.

4.1.1 The crossing equation

Throughout this chapter, we study conformally invariant six-point functions of identical scalars $\phi_i := \phi(x_i)$ inserted on points $x_i < x_{i+1}$ on the line. Such a correlator can always be reduced to a function of three cross-ratios χ_i by extracting a suitable kinematic prefactor $\mathcal{L}(x_i)$. More concretely, let us use $x_{ij} := x_i - x_j$ and $x_{i+6} := x_i$ to define

$$\chi_i := \frac{x_{i,i+1} x_{i+2,i+3}}{x_{i,i+2} x_{i+1,i+3}} \quad \text{and} \quad \mathcal{L}(x_i) := (x_{12}^2 x_{34}^2 x_{56}^2 \chi_2)^{-\Delta_\phi} \quad (4.1.1)$$

such that the reduction of the correlator to a function G of cross-ratios becomes

$$\langle \phi_6 \phi_5 \phi_4 \phi_3 \phi_2 \phi_1 \rangle = \mathcal{L}(x_i) G(\chi_1, \chi_2, \chi_3). \quad (4.1.2)$$

By inserting projectors onto conformal multiplets, the six-point function can be written as a sum of products of lower-point correlators. In particular, reducing it to products of the kinematically fixed three-point functions results in the conformal block decomposition of the correlator. For our purposes, the relevant decomposition is that into four-point functions. As usual in the conformal bootstrap, we leverage the fact that there are multiple inequivalent decompositions of the same correlator, leading to a power full set of functional equations – the crossing equations. The three different possibilities for decomposing the correlator (4.1.2) into products of four-point functions are illustrated in Figure 4.1.1.

In equations, the central decomposition in Figure 4.1.1 is

$$\langle \phi_6 \phi_5 \phi_4 \phi_3 \phi_2 \phi_1 \rangle = \sum_{\mathcal{O}} \sum_{n=0}^{\infty} \frac{\langle \phi_6 \phi_5 \phi_4 P^n \mathcal{O} \rangle \langle \mathcal{O}^\dagger K^n \phi_3 \phi_2 \phi_1 \rangle}{(2\Delta_{\mathcal{O}})_n n!}, \quad (4.1.3)$$

which was shown in Appendix A of [93] to reduce to

$$G(\chi_1, \chi_2, \chi_3) = \sum_{\mathcal{O}} \sum_{n=0}^{\infty} F_n^{\mathcal{O}}(\chi_1) \frac{\chi_2^{\Delta_{\mathcal{O}}+n}}{(2\Delta_{\mathcal{O}})_n n!} F_n^{\mathcal{O}}(\chi_3), \quad (4.1.4)$$

where, for $0 < \chi < 1$,

$$F_n^{\mathcal{O}}(\chi) := (-1)^n (\bar{\chi})^{\Delta_{\mathcal{O}}+n-\Delta_{\phi}} \partial_{\bar{\chi}}^n f^{\mathcal{O}}(\bar{\chi}) \quad (4.1.5)$$

is defined in terms of

$$\bar{\chi} := \chi^{-1} \quad \text{and} \quad f^{\mathcal{O}}(\bar{\chi}) := \langle \phi(\infty) \mathcal{O}(\bar{\chi}) \phi(1) \phi(0) \rangle. \quad (4.1.6)$$

The other decompositions can be obtained by shifting $x_i \rightarrow x_{i\pm 1}$. Therefore, the functions $F_n^{\mathcal{O}}$ obey the functional equation¹

$$\sum_{\mathcal{O}} \sum_{n=0}^{\infty} \left[F_n^{\mathcal{O}}(\chi_1) \frac{\chi_2^{\Delta_{\mathcal{O}}+n}}{(2\Delta_{\mathcal{O}})_n n!} F_n^{\mathcal{O}}(\chi_3) - \frac{\mathcal{L}(x_{i+1})}{\mathcal{L}(x_i)} F_n^{\mathcal{O}}(\chi_2) \frac{\chi_3^{\Delta_{\mathcal{O}}+n}}{(2\Delta_{\mathcal{O}})_n n!} F_n^{\mathcal{O}}(\chi_4) \right] = 0 \quad (4.1.8)$$

whose solutions are bootstrapped in this chapter.

4.1.2 The functionals

As usual in the bootstrap, we need to act with certain functionals on the crossing equation (4.1.8) to extract equations that do not depend on undetermined cross-ratios anymore. Concretely, we act with combinations $\partial_{\chi_1}^a \partial_{\chi_2}^b \partial_{\chi_3}^c$ of derivatives on eq. (4.1.8).

Before we do so observe that the functions $F_n^{\mathcal{O}}$ defined in eq. (4.1.5) satisfy

$$\chi \partial_{\chi} F_n^{\mathcal{O}}(\chi) = F_{n+1}^{\mathcal{O}}(\chi) - (\Delta_{\mathcal{O}} + n - \Delta_{\phi}) F_n^{\mathcal{O}}(\chi). \quad (4.1.9)$$

Therefore, the a^{th} derivative $\partial_{\chi}^a F_n^{\mathcal{O}}(\chi)$ can be written as

$$\partial_{\chi}^a F_n^{\mathcal{O}}(\chi) = \sum_m D_{nm}^{(a)}(\Delta_{\mathcal{O}}, \chi) F_m^{\mathcal{O}}(\chi), \quad (4.1.10)$$

where $D^{(a)}(\Delta_{\mathcal{O}}, \chi)$ is a banded upper triangular matrix with bandwidth a whose entries depend on χ and $\Delta_{\mathcal{O}}$.

Thus, the expression obtained by acting with $\partial_{\chi_1}^a \partial_{\chi_2}^b \partial_{\chi_3}^c$ on eq. (4.1.8) takes the form

$$\sum_{\mathcal{O}} \sum_{m,n=0}^{\infty} \left[F_m^{\mathcal{O}}(\chi_1) M_{mn}^{(a,b,c)}(\Delta_{\mathcal{O}}, \chi_i) F_n^{\mathcal{O}}(\chi_3) - F_m^{\mathcal{O}}(\chi_2) \tilde{M}_{mn}^{(a,b,c)}(\Delta_{\mathcal{O}}, \chi_i) F_n^{\mathcal{O}}(\chi_4) \right] = 0, \quad (4.1.11)$$

¹Note that χ_4 and the ratio $\frac{\mathcal{L}(x_{i+1})}{\mathcal{L}(x_i)}$ can be expressed in terms of χ_1 , χ_2 and χ_3 as

$$\chi_4 = \frac{\chi_1 + \chi_2 + \chi_3 - \chi_1 \chi_3 - 1}{\chi_1 + \chi_2 - 1} \quad \text{and} \quad \frac{\mathcal{L}(x_{i+1})}{\mathcal{L}(x_i)} = \left(\frac{\chi_1^2 \chi_2 \chi_3}{(\chi_1 + \chi_2 + \chi_3 - \chi_1 \chi_3 - 1)^2} \right)^{\Delta_{\phi}}. \quad (4.1.7)$$

Thus, eq. (4.1.8) is a functional equation depending on the three cross-ratios only.

where $M_{mn}^{(a,b,c)}(\Delta_{\mathcal{O}}, \chi_i)$ and $\tilde{M}_{mn}^{(a,b,c)}(\Delta_{\mathcal{O}}, \chi_i)$ are banded matrices whose bandwidth is bounded above by $\Lambda := a + b + c$ and whose entries depend on $\Delta_{\mathcal{O}}$ and all three cross-ratios.

Finally, we would like to enforce $\chi_1 = \chi_2 = \chi_3 = \chi_4 =: \chi$, which implies $\chi = \frac{1}{3}$ or $\chi = 1$. The constraint $0 < \chi_i < 1$ leads us to exclude $\chi = 1$. The preferred kinematics at which we evaluate the derivative functionals is therefore $\chi_i = \frac{1}{3}$. Evaluating eq. (4.1.11) at $\chi_i = \frac{1}{3}$ yields the sum rule

$$\sum_{\mathcal{O}} \sum_{m,n=0}^{\infty} F_m^{\mathcal{O}}(\frac{1}{3}) M_{mn}^{(a,b,c)}(\Delta_{\mathcal{O}}) F_n^{\mathcal{O}}(\frac{1}{3}) = 0 \quad (4.1.12)$$

with some banded matrix $M_{mn}^{(a,b,c)}(\Delta_{\mathcal{O}}) = M_{mn}^{(a,b,c)}(\Delta_{\mathcal{O}}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}) - \tilde{M}_{mn}^{(a,b,c)}(\Delta_{\mathcal{O}}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ of bandwidth $\Lambda \leq a + b + c$. It is straight forward to write down analytic expressions for the matrices $M_{mn}^{(a,b,c)}(\Delta_{\mathcal{O}})$. They can be found in the `GenerateFunctionals.nb` file (see Appendix 4.A.1) provided with the arXiv submission associated to this chapter.

4.1.3 The bootstrap

It is straight forward to formulate an SDPs dual to the sum rules (4.1.12): For each $\Lambda \in \mathbb{Z}_{>0}$ let $\mathcal{F}_{\Lambda}$ be a basis of $\text{span}(\{M^{(a,b,c)} | \text{with } a + b + c \leq \Lambda\})$. A spectrum $\mathcal{S}$ of operators occurring in the triple OPE $\phi \times \phi \times \phi$ is inconsistent with the crossing equation (4.1.8) if there are coefficients c_{α} such that

$$\sum_{\alpha \in \mathcal{F}_{\Lambda}} c_{\alpha} \alpha(\Delta_{\mathcal{O}}) \geq 0 \text{ for all } \mathcal{O} \in \mathcal{S} \quad \text{and} \quad \sum_{\alpha \in \mathcal{F}_{\Lambda}} c_{\alpha} \alpha(\Delta_{\mathcal{O}}) > 0 \text{ for at least one } \mathcal{O} \in \mathcal{S}. \quad (4.1.13)$$

Using essentially standard reasoning of the conformal bootstrap, conditions on the coefficients c_{α} that lead to bounds on scaling dimensions in $\phi \times \phi \times \phi$ and bounds on the four-point functions $F_n^{\mathcal{O}}$ were formulated in Reference [93].

The key challenge is then to efficiently search for c_{α} that have the desired properties. The problem to determine these coefficients is an SDP with a one-parameter family of infinite dimensional semidefinite constraints. To solve it numerically, one has to reduce this problem to an SDP with finitely many finite dimensional semidefinite constraints.

This has been done in Reference [93] by applying the following four steps.

1. Truncate the infinite-dimensional matrices in $\mathcal{F}_{\Lambda}$ to finite $N \times N$ matrices.
2. Truncate the continuum of semidefinite constraints labelled by $\Delta_{\mathcal{O}}$ to a finite grid.

3. Solve the truncated SDP for the coefficients c_α .
4. If the c_α obtained this way satisfy the full untruncated constraints, stop.
Else, increase N and consider a denser grid of $\Delta_{\mathcal{O}}$.

For a successful implementation of this scheme in Reference [93], the banded sparsity structure of the derivative functionals reviewed in Section 4.1.2 was essential in two ways.

1. For the convergence of the truncation scheme it was crucial that positivity of the functionals at large $\Delta_{\mathcal{O}}$ and far down the diagonal was imposed by additional semidefinite constraints that supplement the truncated SDP. These asymptotic considerations heavily relied on the banded structure of the functionals.
2. The method applied to determine if a given functional satisfies the full untruncated infinite dimensional constraints consisted of manually constructing sparse matrix decompositions, which reduced the problem to numerically checking positive semidefiniteness for a two-parameter family of finite dimensional matrices.

While the truncation scheme of [93] did successfully produce bounds, the computational effort that was necessary to do so does not compare well with the amount of information contained in them. The truncations, the checks performed after obtaining a candidate functional and the potential necessity to iterate the procedure several times with increasing N and finer grids of scaling dimensions, comes with an undesirably large computational cost. Even more severely, as explained in [93], the need to consider large N of order 10^2 to 10^3 makes it necessary to use SDPA instead of SDPB. As already mentioned in the introduction, this is highly problematic since no arbitrary precision parallelisable version of SDPA is available, which implies that bootstrap computations based on SDPA cannot be scaled up to efficiently benefit from the resources provided by large computer clusters.

The procedure that is described in the next section solves these problems by making extensive use of sparse matrix decompositions similar to those that were applied manually in the “checking” step of the truncation scheme of [93]. This simultaneously eliminates the need for any truncations and reduces the six-point bootstrap to SDPs that are very similar to those encountered in the four-point bootstrap. In particular, by using the new formulation, SDPB’s capability to efficiently solve such four-point bootstrap problems can directly be applied to the six-point bootstrap.

4.2 New numerics for the six-point bootstrap

This section presents the main technical result of the chapter – a new, sparse SDP based formulation of the numerical six-point bootstrap. To this end, Section 4.2.1 briefly reviews how an SDP for large banded² matrices can be reformulated as a system of coupled SDPs for smaller dense matrices. After the general review, Section 4.2.2 specialises to the specific SDP arising in the six-point bootstrap. Besides improving efficiency of the bootstrap algorithms, we show that exploiting the banded structure makes it straight forward to use polynomial SDP, thereby overcoming the discretisations necessary in [93].

4.2.1 Banded SDP

This section reviews sparsity exploiting decompositions in SDP with banded matrices. We start by setting up such an SDP and some useful notation. Then, the SDP is reformulated in terms of two coupled SDPs of smaller dense matrices by decomposing each of the constraint matrices into an upper left and a lower right part. After this warm-up, a more general decomposition into multiple coupled SDPs is stated. The section ends with a simple example that illustrates the application of the general result.

Setup and notation. For any objective $c \in \mathbb{R}^N$ and any collection of $N + 1$ symmetric $m \times m$ matrices $\{M^\alpha\}_{\alpha=0}^N$, we can use SDP to solve the following optimisation problem.

$$\text{Maximise } c \cdot y \text{ over } y \in \mathbb{R}^N \text{ such that } M^0 + \sum_{\alpha=1}^N y_\alpha M^\alpha \geq 0. \quad (4.2.1)$$

In this section, we are interested in the case where $\{M^\alpha\}_{\alpha=0}^N$ are banded matrices with bandwidth Λ . That is,

$$|i - j| > \Lambda \Rightarrow M_{ij}^\alpha = 0. \quad (4.2.2)$$

For a discussion of such a sparse SDP, it is useful to introduce $s \times t$ matrices $E_{s,t}^{m_0}$ with

$$(E_{s,t}^{m_0})_{ij} = \delta_{i-j}^{m_0}, \quad (4.2.3)$$

i.e. $E_{s,t}^{m_0}$ is a matrix that is 1 on a diagonal specified by m_0 and 0 otherwise. Note that

$$(E_{s,t}^{m_0})^T = E_{t,s}^{-m_0}. \quad (4.2.4)$$

Throughout this section, Sym_k denotes the set of symmetric $k \times k$ matrices.

²The general theory of sparse SDP is discussed for example in the excellent review [205].

Splitting the SDP in two. The matrices defined in eq. (4.2.3) can be used to conveniently describe projections of matrices onto submatrices. For instance, the upper left $a \times a$ submatrix of the $m \times m$ matrix M^α can be written as

$$A^\alpha := E_{a,m}^0 M^\alpha E_{m,a}^0. \quad (4.2.5)$$

Moreover, $E_{s,t}^{m_0}$ can also be used to embed small matrices into larger matrices. For instance, $E_{m,a}^0 A^\alpha E_{a,m}^0$ is a $m \times m$ matrix that coincides with M^α in the upper left $a \times a$ submatrix, but vanishes otherwise. In particular, if we project onto the lower right $b \times b$ submatrix

$$B^\alpha := E_{b,m}^{b-m} (M^\alpha - E_{m,a}^0 A^\alpha E_{a,m}^0) E_{m,b}^{m-b}. \quad (4.2.6)$$

of $M^\alpha - E_{m,a}^0 A^\alpha E_{a,m}^0$ then we can recover M^α as

$$E_{m,a}^0 A^\alpha E_{a,m}^0 + E_{m,b}^{m-b} B^\alpha E_{b,m}^{b-m} = M^\alpha \quad (4.2.7)$$

for suitably large a and b . More concretely, since M^α has bandwidth Λ , it is enough for A and B to overlap on a $\Lambda \times \Lambda$ submatrix of M^α . Because the parts of M^α captured by A^α and B^α overlap in the lower right and upper left $(a+b-m) \times (a+b-m)$ corners respectively, this amounts to the constraint $a+b \geq m + \Lambda$.

The decomposition is of course not unique. For any $X \in \text{Sym}_{a+b-m}$, we can define

$$A_X^\alpha = A^\alpha + E_{a,a+b-m}^{m-b} X E_{a+b-m,a}^{b-m}, \quad B_X^\alpha = B^\alpha - E_{b,a+b-m}^0 X E_{a+b-m,b}^0 \quad (4.2.8)$$

and still have the decomposition property

$$E_{m,a}^0 A_X^\alpha E_{a,m}^0 + E_{m,b}^{m-b} B_X^\alpha E_{b,m}^{b-m} = M^\alpha. \quad (4.2.9)$$

With this ambiguity in mind, let us consider an optimisation problem with an extended set of variables $y \in \mathbb{R}^N$ and $X \in \text{Sym}_{a+b-m}$, namely

$$\text{Maximise } c \cdot y \text{ such that } A_X^0 + \sum_{\alpha=1}^N y_\alpha A^\alpha \geq 0 \text{ and } B_X^0 + \sum_{\alpha=1}^N y_\alpha B^\alpha \geq 0. \quad (4.2.10)$$

Clearly, every y that solves the constraints of this SDP also solves the constraint of our original problem (4.2.1). A less trivial, but very useful fact is that the converse holds too [205]. That is, for every positive semidefinite banded matrix

$$M = M^0 + \sum_{\alpha=1}^N y_\alpha M^\alpha, \quad (4.2.11)$$

there exists a symmetric matrix X such that

$$A^0 + \sum_{\alpha=1}^N y_\alpha A^\alpha \geq -E_{a,a+b-m}^{m-b} X E_{a+b-m,a}^{b-m} \quad (4.2.12)$$

and

$$B^0 + \sum_{\alpha=1}^N y_{\alpha} B^{\alpha} \geq E_{b,a+b-m}^0 X E_{a+b-m,b}^0. \quad (4.2.13)$$

Therefore, the two SDPs (4.2.1) and (4.2.10) are in fact equivalent. Applying this equivalence iteratively until we arrive at $m - \Lambda$ coupled SDPs of $(\Lambda + 1) \times (\Lambda + 1)$ matrices allows us to deduce the following key result.

Dense decomposition of banded SDPs. *Let $\{M^{\alpha}\}_{\alpha=0}^N$ be symmetric banded $m \times m$ matrices with bandwidth Λ and $c \in \mathbb{R}^N$. Then, the task of maximising $c \cdot y$ over $y \in \mathbb{R}^N$ with the positive semidefinite constraint*

$$M^0 + \sum_{\alpha=1}^N y_{\alpha} M^{\alpha} \geq 0 \quad (4.2.14)$$

is equivalent to the task of maximising $c \cdot y$ over $(y, X_0, \dots, X_{m-\Lambda-1}) \in \mathbb{R}^N \times (\text{Sym}_{\Lambda+1})^{m-\Lambda}$ with the constraint

$$X_n + M_n^0 + \sum_{\alpha=1}^N y_{\alpha} M_n^{\alpha} \geq 0 \quad \sum_{n=0}^{m-\Lambda-1} E_{m,\Lambda+1}^n X_n E_{\Lambda+1,m}^{-n} = 0, \quad (4.2.15)$$

where $M_n^{\alpha} \in \text{Sym}_{\Lambda+1}$ are some arbitrary fixed matrices with the property

$$\sum_{n=0}^{m-\Lambda-1} E_{m,\Lambda+1}^n M_n^{\alpha} E_{\Lambda+1,m}^{-n} = M^{\alpha}. \quad (4.2.16)$$

Example. As a simple example to illustrate the decomposition discussed in the previous section, consider the problem of determining the maximal y such that

$$M^0 + yM^1 := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + y \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \geq 0. \quad (4.2.17)$$

This SDP is simple enough to just write down the spectrum of $M^0 + yM^1$, namely

$$\text{spec}(M^0 + yM^1) = \left\{ y + 1, -\sqrt{2y^2 - 2y + 1} + y + 1, \sqrt{2y^2 - 2y + 1} + y + 1 \right\}, \quad (4.2.18)$$

and simplify $\text{spec}(M^0 + yM^1) \subseteq \mathbb{R}_{\geq 0}$ to conclude that the feasible range of y is

$$0 \leq y \leq 4. \quad (4.2.19)$$

Alternatively, we can use the decomposition approach discussed in the previous section and conclude that the SDP formulated in eq. (4.2.17) is equivalent to the problem of finding the maximal y such that there is a pair of symmetric 2×2 matrices X_0, X_1 with

$$X_0 + M_0^0 + yM_0^1 \geq 0 \quad X_1 + M_1^0 + yM_1^1 \geq 0 \quad E_{3,2}^0 X_0 E_{2,3}^0 + E_{3,2}^1 X_1 E_{2,3}^{-1} = 0, \quad (4.2.20)$$

where

$$M_0^0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_1^0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_0^1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad M_1^1 = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}. \quad (4.2.21)$$

The equality constraint is easily solved. Indeed,

$$0 = E_{3,2}^0 X_0 E_{2,3}^0 + E_{3,2}^1 X_1 E_{2,3}^{-1} = \begin{pmatrix} (X_0)_{11} & (X_0)_{12} & 0 \\ (X_0)_{21} & (X_0)_{22} + (X_1)_{11} & (X_1)_{12} \\ 0 & (X_1)_{21} & (X_1)_{22} \end{pmatrix} \quad (4.2.22)$$

implies

$$X_0 = \begin{pmatrix} 0 & 0 \\ 0 & -x \end{pmatrix} \quad \text{and} \quad X_1 = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}. \quad (4.2.23)$$

Thus, our original single variable, 3×3 matrix SDP has reduced to the system

$$\begin{pmatrix} 1+y & 1-y \\ 1-y & 1+y-x \end{pmatrix} \geq 0 \quad \text{and} \quad \begin{pmatrix} x & -y \\ -y & y+1 \end{pmatrix} \geq 0 \quad (4.2.24)$$

of two coupled two variable 2×2 matrix SDPs. Demanding a non negative trace and determinant for the matrices in eq. (4.2.24) leads to quadratic inequalities which simplify to

$$0 \leq y \leq 4 \quad \text{and} \quad \frac{y^2}{y+1} \leq x \leq \frac{4y}{y+1}. \quad (4.2.25)$$

Projecting to y , we recover the feasible interval $0 \leq y \leq 4$ given in eq. (4.2.19).

4.2.2 Improved six-point bootstrap

We now apply the sparse SDP techniques reviewed in Section 4.2.1 to the numerical six-point bootstrap. For this, we first lay out what choices have to be made in the course of the implementation. Then, we detail the specific choices that have been made for this work.

Overview. As reviewed in Secs. 4.1.2 and 4.1.3, the six-point bootstrap involves constructing linear combinations of infinite dimensional symmetric matrices with bandwidth Λ , whose entries are rational functions of a conformal weight Δ . Positive semidefiniteness is imposed on the matrices for certain ranges of Δ , e.g. $\Delta > \Delta_*$, that is for all conformal weights above a gap Δ_* . Concretely, the matrices $M^{(a,b,c)}(\Delta)$ defining the sum rules (4.1.12) take the form

$$M^\alpha(\Delta) = \sum_{n=0}^{\infty} E_{\infty, \Lambda+1}^n \tilde{M}^\alpha(\Delta, n) E_{\Lambda+1, \infty}^{-n} \quad \text{with} \quad \tilde{M}^\alpha(\Delta, n) = \frac{M^\alpha(\Delta, n)}{3^n (2\Delta)_n n!}, \quad (4.2.26)$$

where the matrices E are defined as in eq. (4.2.3) and the entries of $M(\Delta, n) \in \text{Sym}_{\Lambda+1}$ are polynomials in Δ and n of degree

$$\deg_{\Delta} M_{jk}(\Delta) \leq \Lambda + (2 - j - k) \quad \deg_n M_{jk}(\Delta, n) \leq \Lambda + (2 - j - k). \quad (4.2.27)$$

In particular,

$$j + k > \Lambda + 2 \Rightarrow M_{jk}(\Delta, n) = 0. \quad (4.2.28)$$

In the spirit of the approach to sparse SDPs laid out in Section 4.2.1, we reformulate the infinite dimensional semidefinite constraints involving sparse matrices of the type (4.2.26) as infinite systems of semidefinite constraints for dense $(\Lambda + 1) \times (\Lambda + 1)$ matrices. Eq. (4.2.26) is already a decomposition of the form (4.2.16). Let us directly work with this particular choice of breaking the infinite banded matrices up into families of $\text{Sym}_{\Lambda+1}$ matrices. There are two important choices that we need to additionally make. These are

1. *Choice of X matrices.* As in the example (Sec. 4.2.1) we would like to choose a convenient parametrisation of the constraints on the X matrices in eq. (4.2.15).
2. *Choice of C matrices.* After we have decomposed the sparse SDP into a coupled family of SDPs for smaller matrices, we can choose a different matrix to conjugate each of the coupled SDPs.

The remainder of this section gives a detailed description of these two choices.

X matrices. To describe the construction of X matrices, the following four paragraphs approach the SDPs relevant for numerical six-point bootstrap with four steps of increasing complexity. Concretely, we consider

1. Banded SDPs with matrices whose entries are independent real numbers.
2. Banded SDPs with matrices whose entries are independent polynomials in Δ .
3. Banded SDPs with matrices whose entries are not independent of each other but instead polynomials in two variables, namely Δ and the position n along the diagonal.
4. The specific SDPs relevant for six-point bootstrap, where matrix elements are, up to an exponential prefactor, rational functions of Δ and n as indicated in eq. (4.2.26).

X matrices, step 1: Constant matrices. As in the example given at the end of Section 4.2.1, we would like to find a set of sequences of matrices $\{(X_n^\beta)_{n=0}^\infty\}$ that is a basis of solutions to the constraint

$$\sum_{n=0}^{\infty} E_{m,\Lambda+1}^n X_n E_{\Lambda+1,m}^{-n} = 0 \quad (4.2.29)$$

arising in the decomposition of banded SDPs i.e. in eq. (4.2.15).

One choice of such a basis is given by the sequences $X^{(n_0,I,J)}$ labelled by natural numbers n_0, I, J with $2 \leq I \leq J \leq \Lambda + 1$ and consisting of matrices $X_n^{(n_0,I,J)}$ with entries

$$(X_n^{(n_0,I,J)})_{ij} = \delta_{n_0}^n \delta_I^{(i)} \delta_J^{(j)} - \delta_{n_0+I-1}^n \delta_1^{(i)} \delta_{1+J-I}^{(j)} \quad (4.2.30)$$

It is simple to verify that $X^{(n_0,I,J)}$ indeed satisfies the constraints:

$$\sum_{n=0}^{\infty} (E_{m,\Lambda+1}^n X_n^{(n_0,I,J)} E_{\Lambda+1,m}^{-n})_{ij} = \sum_{n=0}^{\infty} \sum_{k,l=1}^{\Lambda+1} \delta_{i-k}^n (X_n^{(n_0,I,J)})_{kl} \delta_{l-j}^{-n} \quad (4.2.31)$$

$$= \sum_{n=0}^{\infty} (X_n^{(n_0,I,J)})_{i-n,j-n} = \delta_I^{(i-n_0)} \delta_J^{(j-n_0)} - \delta_1^{(i-n_0-I+1)} \delta_{1+J-I}^{(j-n_0-I+1)} = 0. \quad (4.2.32)$$

Using this basis, we can rephrase the maximisation of $b \cdot y$ over y constrained by

$$M^0 + \sum_{\alpha=1}^N y_\alpha M^\alpha \geq 0 \quad (4.2.33)$$

with infinite dimensional banded matrices M^α as a maximisation of $b \cdot y$ over two variables x and y constrained by the system

$$M_n^0 + \sum_{p=0}^{\min(\Lambda,n)} \sum_{J=2}^{\Lambda+1} \sum_{I=2}^J x_{(n-p,I,J)} X_n^{(n-p,I,J)} + \sum_{\alpha=1}^N y_\alpha M_n^\alpha \geq 0 \quad \forall n \in \mathbb{N}_0 \quad (4.2.34)$$

of coupled semidefinite constraints involving only $(\Lambda + 1) \times (\Lambda + 1)$ matrices.

X matrices, step 2: Polynomial in Δ matrices. It is straight forward to generalise eq. (4.2.30) to deal with polynomial SDPs of the form

$$\text{maximise } b \cdot y \text{ over } y \text{ such that } M^0(\Delta) + \sum_{\alpha=1}^N y_\alpha M^\alpha(\Delta) \geq 0 \quad \forall \Delta \in \mathcal{S}, \quad (4.2.35)$$

where $\mathcal{S}$ is some subset of $\mathbb{R}$. We simply need to multiply the $X^{(n_0,I,J)}$ with some suitable set of polynomials. Concretely, if we define

$$X^{(n_0,I,J,d_\Delta)} := X^{(n_0,I,J)} \Delta^{d_\Delta}, \quad (4.2.36)$$

then the SDP in eq. (4.2.35) is equivalent to

$$\begin{aligned} & \text{maximise } b \cdot y \text{ over } (x, y) \text{ such that} \\ & M_n^0(\Delta) + \sum_{p=0}^{\min(\Lambda, n)} \sum_{J=2}^{\Lambda+1} \sum_{I=2}^J \sum_{d_\Delta=0}^{\deg_\Delta(n-p, I, J)} x_{(n-p, I, J, d_\Delta)} X_n^{(n-p, I, J, d_\Delta)}(\Delta) \\ & + \sum_{\alpha=1}^N y_\alpha M_n^\alpha(\Delta) \geq 0 \quad \forall n \in \mathbb{N}_0, \Delta \in \mathcal{S}, \end{aligned} \quad (4.2.37)$$

where

$$\deg_\Delta(n_0, I, J) := \max_{0 \leq \alpha \leq N} (\max[\deg_\Delta(M_{n_0+I, n_0+J}^\alpha(\Delta)), \deg_\Delta(M_{n_0+1, n_0+1+J-I}^\alpha(\Delta))]). \quad (4.2.38)$$

X matrices, step 3: Matrix entries polynomial in Δ and n . With an infinite number of undetermined coefficients x_{n, I, J, d_Δ} , the SDP formulated in eq. (4.2.37) seems to be somewhat abstract and not particularly useful for concrete numerical computations. If however, the matrices $M_n^\alpha(\Delta)$ with different n are not independent of each other, but instead have entries that are for instance polynomials $M_{ij}^\alpha(\Delta, n)$ in n , we can use eq. (4.2.37) to formulate a finite dimensional version of the a priori infinite dimensional SDP (4.2.35).

To do so, we define

$$X_{ij}^{(I, J, d_\Delta, d_n)}(\Delta, n) := \Delta^{d_\Delta} ((n+I-1)^{d_n} \delta_I^{(i)} \delta_J^{(j)} - n^{d_n} \delta_1^{(i)} \delta_{1+J-I}^{(j)} H(n-I+1)), \quad (4.2.39)$$

where H is the Heaviside step function

$$H(x) := \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}. \quad (4.2.40)$$

Again, it is easy to check that the constraints are satisfied. Indeed,

$$\sum_{n=0}^{\infty} (E_{m, \Lambda+1}^n X^{(I, J, d_\Delta, d_n)}(\Delta, n) E_{\Lambda+1, m}^{-n})_{ij} = \sum_{n=0}^{\infty} X_{i-n, j-n}^{(I, J, d_\Delta, d_n)}(\Delta, n) \quad (4.2.41)$$

$$= \Delta^{d_\Delta} \left(\sum_{n=0}^{\infty} (n+I-1)^{d_n} \delta_I^{(i-n)} \delta_J^{(j-n)} - \sum_{n=0}^{\infty} n^{d_n} \delta_1^{(i-n)} \delta_{1+J-I}^{(j-n)} H(n-I+1) \right) \quad (4.2.42)$$

$$= \Delta^{d_\Delta} \left(\sum_{n=0}^{\infty} (n+I-1)^{d_n} \delta_I^{(i-n)} \delta_J^{(j-n)} - \sum_{n=0}^{\infty} (n+I-1)^{d_n} \delta_1^{(i-n-I+1)} \delta_{1+J-I}^{(j-n-I+1)} \right) = 0. \quad (4.2.43)$$

Using the matrices $X^{(I, J, d_\Delta, d_n)}(\Delta, n)$, we can formulate the SDP

$$\begin{aligned} & \text{maximise } b \cdot y \text{ over } (x, y) \text{ such that} \\ & M^0(\Delta, n) + \sum_{J=2}^{\Lambda+1} \sum_{I=2}^J \sum_{d_\Delta=0}^{\deg_\Delta(I, J)} \sum_{d_n=0}^{\deg_n(I, J)} x_{(I, J, d_\Delta, d_n)} X^{(I, J, d_\Delta, d_n)}(\Delta, n) \\ & + \sum_{\alpha=1}^N y_\alpha M^\alpha(\Delta, n) \geq 0 \quad \forall n \in \mathbb{N}_0, \Delta \in \mathcal{S}, \end{aligned} \quad (4.2.44)$$

where for both $x = n$ and $x = \Delta$, we define

$$\deg_x(I, J) := \max_{0 \leq \alpha \leq N} (\max[\deg_x(M_{I,J}^\alpha(\Delta, n)), \deg_x(M_{1,1+J-I}^\alpha(\Delta, n))]). \quad (4.2.45)$$

In general, this finite SDP has stronger constraints than the original SDP (4.2.35). Thus, the optimal solution to (4.2.44) only gives a lower bound for the maximal value of $b \cdot y$ in (4.2.35). For bootstrap applications this means that, using (4.2.44), we will still find rigorous bounds on CFT data – however possibly not the optimal bounds. In practice this is not particularly concerning. After all, restricting to a finite number of derivative functionals, we cannot expect optimal bounds in any case.

It is however important to note that we can combine the $X^{(I,J,d_\Delta,d_n)}$ matrices with any choice of finitely many $X^{(n_0,I,J,d_\Delta)}$. The SDPs constructed in that way in principle can be made as close to being equivalent to eq. (4.2.35) as we like.

X-matrices step 4: SDPs of the six-point bootstrap. As already stated earlier in this section, the matrices relevant for six-point bootstrap do not have entries that are polynomial in Δ and n but rather take the form indicated in equation (4.2.26). Thus, we cannot directly work with the matrices $X^{(I,J,d_\Delta,d_n)}$ and $X^{(n_0,I,J,d_\Delta)}$. The modified versions that we use instead for numerical bootstrap are

$$\tilde{X}_{ij}^{(I,J,d_\Delta,d_n)}(\Delta, n) = \frac{\Delta^{d_\Delta}}{3^n} \left[\frac{(n+I-1)^{d_n} \delta_I^{(i)} \delta_J^{(j)}}{3^{I-1} (2\Delta)_{n+I-1} (n+I-1)!} - \frac{n^{d_n} \delta_1^{(i)} \delta_{1+J-I}^{(j)} H(n-I+1)}{(2\Delta)_n n!} \right] \quad (4.2.46)$$

and $(\tilde{X}_n^{(n_0,I,J,d_\Delta)}(\Delta))_{ij} = \frac{\Delta^{d_\Delta}}{3^n (2\Delta)_n n!} \left(\delta_{n_0}^n \delta_I^{(i)} \delta_J^{(j)} - \delta_{n_0+I-1}^n \delta_1^{(i)} \delta_{1+J-I}^{(j)} \right). \quad (4.2.47)$

With these, we map the SDP (4.2.35) with matrices $M^\alpha(\Delta)$ as in (4.2.26) into the SDP³

³The choice to only include $\tilde{X}_n^{(n_0,I,J,d_\Delta)}(\Delta)$ matrices with n_0 up to $\Lambda - 1$ in eq. (4.2.48) is somewhat arbitrary. As noted towards the end of the description of step 3, one could in principle include a larger number of these matrices. However, the bounds presented in this chapter were empirically found not to be sensitive to this choice i.e. including more $\tilde{X}_n^{(n_0,I,J,d_\Delta)}(\Delta)$ matrices than those with n_0 up to $\Lambda - 1$ has an effect that is orders of magnitude smaller than increasing Λ . This aligns well with the observation that in the cases where Reference [93] provided optimal fixed Λ bounds to compare with, they agreed with the ones computed with the choice in eq. (4.2.48), implying that there is no room for improvement.

maximise $b \cdot y$ over (x, y) such that

$$\begin{aligned} \tilde{M}^0(\Delta, n) + \sum_{\alpha=1}^N y_{\alpha} \tilde{M}^{\alpha}(\Delta, n) + \sum_{J=2}^{\Lambda+1} \sum_{I=2}^J \sum_{d_n=0}^{\Lambda+I-J} \sum_{d_{\Delta}=0}^{\Lambda+I-J-d_n} x_{(I,J,d_{\Delta},d_n)} \tilde{X}^{(I,J,d_{\Delta},d_n)}(\Delta, n) \\ + \sum_{n_0=0}^{\Lambda-1} \sum_{J=2}^{\Lambda+1} \sum_{I=2}^{\min(J, \Lambda-n_0)} \sum_{d_{\Delta}=0}^{\Lambda+I-J} x_{(n_0,I,J,d_{\Delta})} \tilde{X}_n^{(n_0,I,J,d_{\Delta})}(\Delta) \geq 0 \quad \forall n \in \mathbb{N}_0, \Delta \in \mathcal{S}. \end{aligned} \quad (4.2.48)$$

As a final comment on the decomposition, let us state the empirical observation that not all of the X matrices have an equal impact on the bootstrap bounds. In particular, some of them do not contribute to the optimal functionals at all. Therefore, we usually only work with a subset of the general matrices described here.

C matrices. In order to efficiently solve the SDP formulated in eq. (4.2.48), one has to use the freedom to conjugate the semidefinite constraints by some sequences of invertible matrices $C_n(\Delta)$. In doing so there are two goals

1. One would like to make sure that after conjugation all semidefinite constraints only involve matrices that depend polynomially on Δ , preferably with a low degree. This allows us to treat the continuous parameter Δ of the constraints in eq. (4.2.48) with standard polynomial SDP techniques without truncation.
2. Additionally, one would like the matrix entries of $C_n(\Delta)$ to depend on n in precisely such a way that, after conjugation, the diagonal entries of the constraint matrices are constant to leading order in $n \gg 1$. This allows us to deal with the infinite range of the n label in eq. (4.2.48) by imposing positivity in the $n \rightarrow \infty$ limit and a finite range of n up to some truncation value⁴.

To achieve the first goal, conjugate by the diagonal matrices C_n^{Δ} with

$$(C_n)_{ii} = \begin{cases} \sqrt{3^n (2\Delta)_{n+\lfloor \frac{\Lambda+1}{2} \rfloor} \left(n + \lfloor \frac{\Lambda+1}{2} \rfloor\right)!}, & 0 \leq i-1 \leq \lfloor \frac{\Lambda+1}{2} \rfloor \\ \sqrt{\frac{3^n (n+\lfloor \frac{\Lambda+1}{2} \rfloor)!}{(2\Delta)_{n+\lfloor \frac{\Lambda+1}{2} \rfloor}} (2\Delta)_{n+i-1}}, & \lfloor \frac{\Lambda+1}{2} \rfloor < i-1 \leq \Lambda \end{cases} \quad (4.2.49)$$

To achieve the second goal, additionally conjugate matrices with $n > \Lambda$ by the diagonal matrices $\hat{C}_n$ defined as

⁴The truncation in n is completely analogous to the standard truncation of the spin parameter in higher dimensional four-point bootstrap. In this work, only fully converged bounds for which the truncation parameter was chosen large enough to ensure that the functionals are positive for all n are presented.

$$(\hat{C}_n)_{ii} = n^{-\deg_n(i)} \quad \text{where} \quad \deg_n(i) := \max_{\alpha} \deg_n(C_n M^{\alpha}(n\Delta, n) C_n)_{ii}. \quad (4.2.50)$$

4.3 Extremal flow from GFF to GFB

This section uses the improved numerical six-point bootstrap to study extremal correlators that interpolate between the Generalised Free Fermion (GFF) and the Generalised Free Boson (GFB).

A similar four-point bootstrap analysis was performed in Reference [181]. There, a one-parameter family of OPE maximising solutions to four-point crossing that interpolates between GFB and GFF was studied using the numerical bootstrap. Close to GFB, it was related to the deformation of the massive free boson in AdS_2 by a quartic interaction. Here, this one-parameter family of extremal correlators is extended to a two-parameter family incorporating also the sextic interaction. The logic of this six-point extension follows similar bootstrap problems studied in Reference [93] where the CFT data contained in the four-point function of four identical generalised free bosons or fermions was used as an input to study the constraints that six-point crossing puts on additional CFT data occurring in the respective six-point functions.

As a warm-up, Section 4.3.1 briefly showcases the consequences of the improvements that were presented in Section 4.2 by revisiting the “gap maximisation without identity” problem of [93]. Section 4.3.2 then turns to the extremal four-point correlators interpolating between GFB and GFF. In Section 4.3.3, a six-point bootstrap problem that produces bounds on the triple twist data along the four-point deformation is introduced and the results of its implementation are discussed. Finally, Section 4.3.4 relates these results to perturbative computations in AdS_2 .

4.3.1 Warm-up: Gap maximisation without identity revisited

The first application of numerical six-point bootstrap studied in Reference [93] was “gap maximisation without identity” i.e. the problem of maximising the dimension of the leading operator in the $\phi \times \phi \times \phi$ triple OPE.

Gap maximisation without identity. For fixed Δ_{ϕ} , determine the maximal Δ_* such that there is a crossing symmetric six-point function $\langle \phi \phi \phi \phi \phi \phi \rangle$ consistent with unitarity and the assumption that

$$\phi \times \phi \times \phi = \phi_* + \dots, \quad (4.3.1)$$

where the operators that have been omitted have a scaling dimension $\Delta > \Delta_*$.

In Reference [93] an explicit candidate for the extremal solution to gap maximisation without identity was proposed. The proposed correlator has

$$\frac{\Delta_*}{\Delta_\phi} = \frac{9}{5}. \quad (4.3.2)$$

The qualifier “without identity” is motivated by the fact that the identity operator neither appears in the $\phi \times \phi \times \phi$ nor in the $\phi \times \phi$ OPE of this extremal solution (the former would mean $\Delta_* = 0$, while the latter would imply $\Delta_* = \Delta_\phi$). Some initial numerical data supporting the claim that the correlator associated with eq. (4.3.2) is the optimal solution to gap maximisation without identity was already presented in Reference [93].

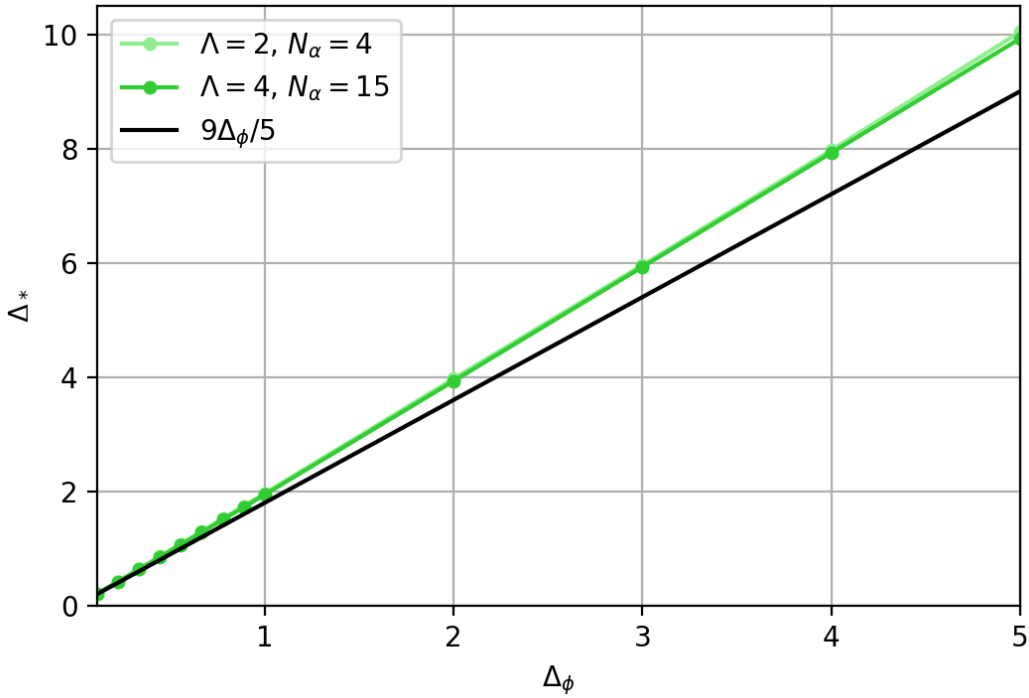


Figure 4.3.1: Allowed region for gap maximisation without identity.

The methods introduced in Section 4.2 enable a significant increase of the number of derivative functionals: While the numerical data of the previous publication only contained bounds with up to $\Lambda = 4$ derivatives (i.e. $N_\alpha = 15$ functionals), Figure 4.3.1 now shows the bounds for values of Λ up to 7 i.e. $N_\alpha = 53$. Apart from increasing Λ , the new numerical implementation also helped to increase the range of external dimensions Δ_ϕ that could be scanned over. Thanks to these two improvements, a clear picture arises: For small Δ_ϕ the bound rapidly converges to the expected gap of $\frac{9}{5}\Delta_\phi$. For any fixed Λ , the bound then weakens and monotonically tends to $2\Delta_\phi$ at large Δ_ϕ . As Λ is increased the region where this transition happens is pushed to larger and larger Δ_ϕ .

Appendix 4.A.2 describes some code shared in the ancillary files of this publication that reproduces the bound presented in Figure 4.3.1.

4.3.2 Double twist data from four-point bootstrap

Let us now state the four-point bootstrap problem whose six-point generalisation is the main target of this section and briefly discuss its solution.

Three-point maximisation. For fixed Δ_ϕ and Δ_{ϕ^2} , maximise the OPE coefficient $C_{\phi\phi\phi^2}$ such that $\langle\phi\phi\phi\phi\rangle$ is consistent with crossing, unitarity and the assumption that

$$\phi \times \phi = \mathbf{1} + \phi^2 + \dots, \quad (4.3.3)$$

where the operators that have been omitted have a scaling dimension $\Delta > \Delta_{\phi^2}$.

The $\Delta_{\phi^2} = 2\Delta_\phi$ extremal solution to the three-point maximisation problem is the GFB four-point function. For $\Delta_{\phi^2} = 2\Delta_\phi + 1$, one recovers the GFF [181]. Scaling dimensions and OPE coefficients of the one-parameter family of solutions to crossing interpolating between these two end points are most conveniently inferred by numerically computing solutions to truncated crossing equations that approximate the GFF (or GFB) and applying Newton's method to gradually deform these solutions by varying Δ_{ϕ^2} . While it was observed in Reference [181] that this is most elegantly done using analytic functionals, it is sufficient for our purposes to simply compute the truncated spectra for up to $\Lambda_4 = 20$ derivative functionals combined with extrapolation to $\Lambda_4 = \infty$.

Since [181] already gave a detailed description of the spectrum of the extremal correlators, we refrain from further discussing the well-known four-point data and instead directly move on to the new information that can be extracted from the six-point bootstrap. The reader interested in more details on the four-point data is invited to inspect the mathematica notebook `GFF_to_GFB.nb` which we used to generate this data and which is provided in the ancillary files associated to this publication.

4.3.3 Triple twist data from six-point bootstrap

Analogously to the three-point maximisation problem formulated in the previous section, one can study an extended six-point bootstrap problem that maximises four-point functions.

Four-point maximisation. For fixed Δ_ϕ , Δ_{ϕ^2} and Δ_{ϕ^3} , maximise the value of the four-point function $F_{\phi^3} = \langle \phi(\infty)\phi(1)\phi^3(1/3)\phi(0) \rangle$ with the constraint that $\langle \phi\phi\phi\phi \rangle$ is consistent with crossing, unitarity and the assumption that

$$\langle \phi\phi\phi\phi \rangle = F_\phi^{\Delta_{\phi^2}} \quad \text{and} \quad \phi \times \phi \times \phi = \phi + \phi^3 + \dots, \quad (4.3.4)$$

where the operators that have been omitted have a scaling dimension $\Delta > \Delta_{\phi^3}$.

Here, $F_\phi^{\Delta_{\phi^2}}$ denotes the solution to three-point maximisation for fixed Δ_{ϕ^2} . Upper bounds on the solutions to four-point maximisation can be constructed through the procedure outlined in Section 4.3. of Reference [93]. In fact, that paper already presented numerical data strongly suggesting that for $(\Delta_{\phi^2}, \Delta_{\phi^3}) = (2\Delta_\phi + 1, 3\Delta_\phi + 3)$, where the four-point input $F_\phi^{2\Delta_\phi+1}$ is just the GFF four-point correlator, the four-point maximisation problem is solved by the GFF six-point function. We change the procedure used to obtain this result in two ways here.

1. We use the improved six-point bootstrap described in Section 4.2.
2. As input data, we use the ϕ -exchange contribution to the six-point function computed using the extremal solutions to the three-point maximisation problem i.e. if

$$F_\phi^{\Delta_{\phi^2}} = \sum_{\mathcal{O} \in (\phi \times \phi)_{\Delta_{\phi^2}}} C_{\phi\phi\mathcal{O}}^2(\Delta_{\phi^2}) G_{\mathcal{O}}, \quad (4.3.5)$$

then the input for the six-point bootstrap is

$$\sum_{\mathcal{O}_1, \mathcal{O}_2 \in (\phi \times \phi)_{\Delta_{\phi^2}}} C_{\phi\phi\mathcal{O}_1}^2(\Delta_{\phi^2}) C_{\phi\phi\mathcal{O}_2}^2(\Delta_{\phi^2}) G_{\mathcal{O}_1\phi\mathcal{O}_2}, \quad (4.3.6)$$

where $G_{\mathcal{O}_1\phi\mathcal{O}_2}$ is the $(\mathcal{O}_1\phi\mathcal{O}_2)$ -exchange comb-channel six-point block [174, 178].

For fast, precise evaluation of derivatives comb-channel conformal blocks that enters in eq. (4.3.6), we use a $\mathbb{C}$ programme that together with a python and mathematica wrapper is available in the ancillary files provided with this publication (see also Appendix 4.A.4).

The numerical upper bound for the cases $\Delta_\phi = 0.01$ and $\Delta_\phi = 0.1$ is shown in figures 4.3.2 and 4.3.3 respectively. For $(\Delta_{\phi^2}, \Delta_{\phi^3}) = (2\Delta_\phi, 3\Delta_\phi)$, the bound is well saturated by the GFB correlator. Likewise, it is saturated by the GFF for $(\Delta_{\phi^2}, \Delta_{\phi^3}) = (2\Delta_\phi + 1, 3\Delta_\phi + 3)$. As a function of Δ_{ϕ^3} at fixed Δ_{ϕ^2} , the bound converges rapidly close to the maximal allowed value of Δ_{ϕ^3} . As Δ_{ϕ^3} is decreased the bound monotonically increases on the GFB slice $\Delta_{\phi^2} = 2\Delta_\phi$. Closer to the GFF slice $\Delta_{\phi^2} = 2\Delta_\phi + 1$, however, it first drops to a local minimum before finally rising again. This qualitative behaviour does not greatly vary as the value of Δ_ϕ is changed.

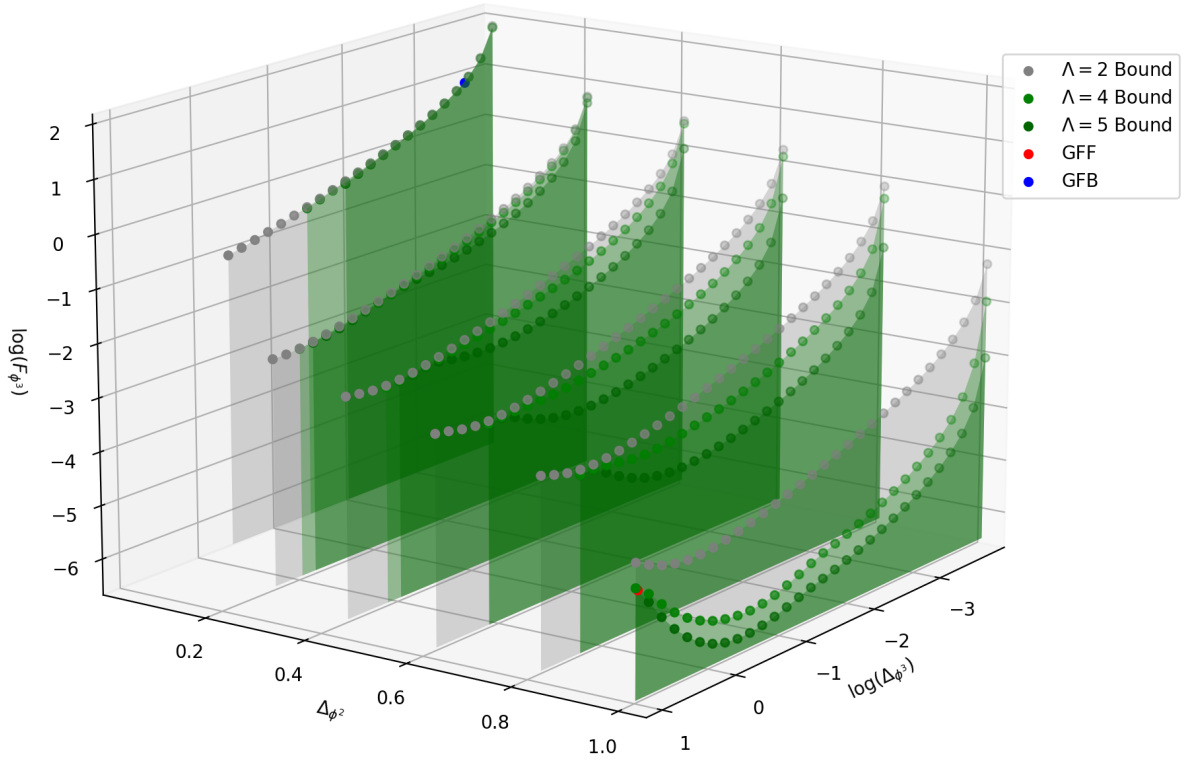


Figure 4.3.2: Upper bound on solutions of the four-point maximisation problem for $\Delta_\phi = 0.01$.

4.3.4 Matching with perturbation theory in AdS_2

As pointed out in Section 4.3.3, the GFB/GFF correlators (dual to a massive free boson/fermion in AdS_2) saturate the six-point bootstrap bounds and are therefore extremal correlators. A natural expectation is that deformations which preserve extremality, i.e. generate the extremal surface, correspond to interactions that do not introduce new states in the OPE, but instead just modify the dimensions and OPE coefficients of states that are already present. At leading order in a quartic or sextic deformation of the GFB, it is easy to see that no new states are introduced. Therefore, we focus on these examples and relate them to the bounds obtained in Section 4.3.3.

Φ^6 interaction. Let us start from the UV action

$$S = \frac{1}{2} \int d^2x \sqrt{g} \left(\partial_\mu \Phi \partial^\mu \Phi + m^2 : \Phi^2 : + \frac{\lambda_6}{6!} : \Phi^6 : \right), \quad (4.3.7)$$

where $\lambda_6 = 0$ describes a free boson Φ of mass $m^2 = \Delta_\phi(\Delta_\phi - 1)$ dual to a GFB ϕ of dimension Δ_ϕ . Normal ordering in the sextic interaction by which we deform the boson ensures that no effective quartic interaction is induced at tree level. Therefore,

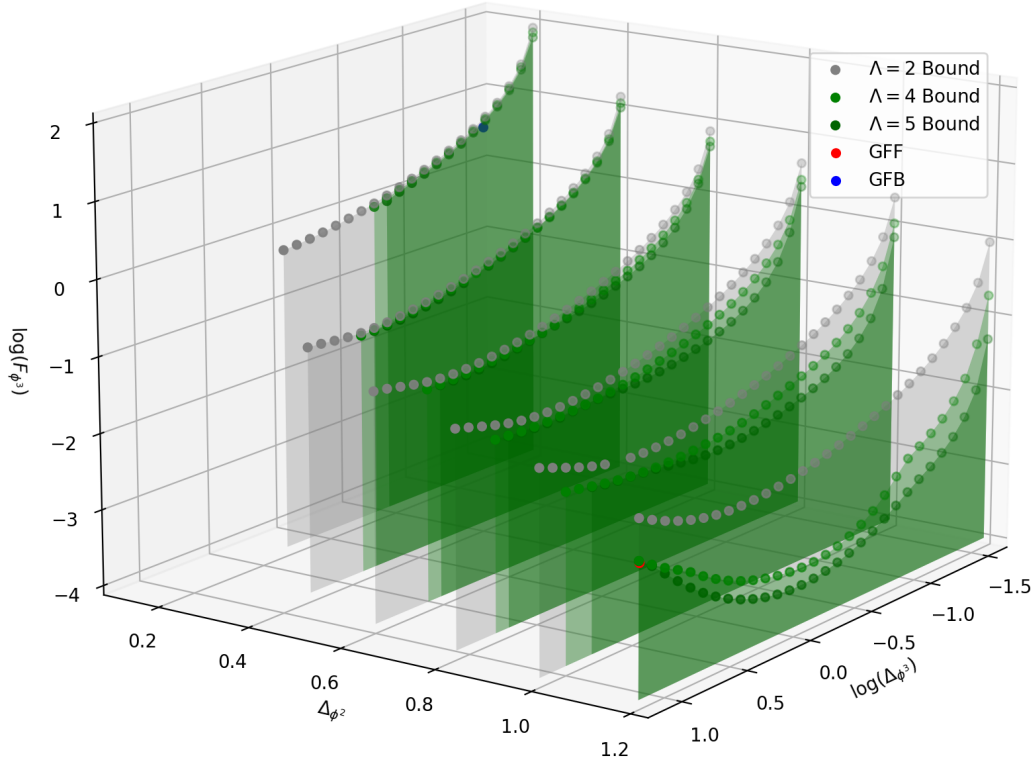


Figure 4.3.3: Upper bound on solutions of the four-point maximisation problem for $\Delta_\phi = 0.1$.

the boundary four-point function is unaltered by the deformation at leading order. That is,

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle_{\text{GFB}} + O(\lambda_6^2) \quad (4.3.8)$$

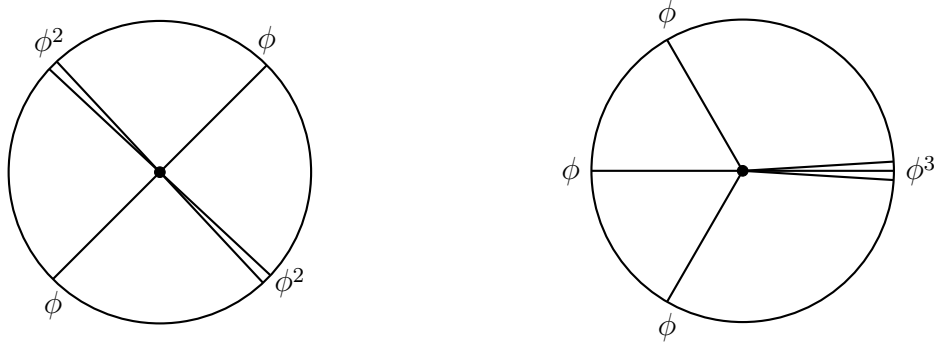
which means that, at this order, the sextic interaction takes us away from the GFB point in Figures 4.3.2 and 4.3.3 by moving only in the Δ_{ϕ^3} direction while keeping Δ_{ϕ^2} fixed. To make contact with the bounds of Section 4.3.3, we not only need to compute tree-level corrections to the four-point function $\langle \phi\phi\phi\phi^3 \rangle$, but also the anomalous dimension $\gamma_{\phi^3} = \Delta_{\phi^3} - 3\Delta_\phi$ of ϕ^3 . The latter quantifies the rate at which we move away from the GFB point in the $(\Delta_{\phi^2}, \Delta_{\phi^3})$ plane.

To determine γ_{ϕ^3} , consider the correlator

$$\langle \phi(x_1)\phi^2(x_2)\phi(x_3)\phi^2(x_4) \rangle = \text{GFB} - \lambda_6 N_\phi^6 D_{\Delta_\phi, 2\Delta_\phi, \Delta_\phi, 2\Delta_\phi}(x_i) + O(\lambda_6^2) \quad (4.3.9)$$

where $D_{\Delta_\phi, 2\Delta_\phi, \Delta_\phi, 2\Delta_\phi}(x_i)$ denotes a D-function, i.e. a quartic contact Witten-diagram and N_ϕ is the normalisation of the bulk-to-boundary propagator associated to the diagram. The diagram is drawn on the l.h.s. of Figure 4.3.4. Concerning D functions, we use the convention

$$D_{\Delta_1, \Delta_2, \Delta_3, \Delta_4}(x_i) = \int d^2x \sqrt{g} K_{\Delta_1}(x, x_1) K_{\Delta_2}(x, x_2) K_{\Delta_3}(x, x_3) K_{\Delta_4}(x, x_4), \quad (4.3.10)$$



(a) Sextic contact diagram used to determine the anomalous dimension γ_{ϕ^3} of the triple trace operator ϕ^3 . (b) Sextic contact diagram leading to a correction to the correlator $\langle \phi\phi\phi\phi^3 \rangle$. The diagram is naively divergent.

Figure 4.3.4: Tree level Witten diagrams occurring in $\lambda_6\Phi^6$ perturbation theory.

where $K_{\Delta_i}(x, x_i)$ is the bulk-to boundary propagator from the point x in the bulk to the point x_i on the boundary. In Poincaré coordinates for AdS_2 , the bulk-to-boundary propagator reads

$$K_{\Delta_i}(x, x_i) = \left(\frac{y}{y^2 + (x - x_i)^2} \right)^{\Delta_i}. \quad (4.3.11)$$

Crucially, eq. (4.3.11) makes the fact that the product of two bulk-to-boundary propagators of dimension Δ_ϕ is proportional to a single propagator of dimension $2\Delta_\phi$ manifest. This justifies the subscripts of the D-function in eq. (4.3.9).

Reading off the correction to CFT data from the conformal block expansion of the contact diagram is a standard computation described for example in [206]. For generic external dimensions, the s -channel block expansion of the contact diagram contains only double trace towers of the channel pairs, i.e.

$$D_{\Delta_1, \Delta_2, \Delta_3, \Delta_4} = \sum_{n=0}^{\infty} a_n^{12} G_{\Delta_1 + \Delta_2 + 2n}^{12, 34} + \sum_{n=0}^{\infty} a_n^{34} G_{\Delta_3 + \Delta_4 + 2n}^{12, 34}, \quad (4.3.12)$$

with the product of OPE coefficients a_n^{12} given by

$$a_n^{12} = \frac{\sqrt{\pi}(-1)^{-n} \Gamma(n + \Delta_1) \Gamma(n + \Delta_2) \Gamma\left(n + \Delta_1 + \Delta_2 - \frac{1}{2}\right) \Gamma\left(\frac{2n + \Delta_1 + \Delta_2 + \Delta_3 - \Delta_4}{2}\right)}{2n! \Gamma(\Delta_1) \Gamma(\Delta_2) \Gamma(\Delta_3) \Gamma(\Delta_4)} \frac{\Gamma\left(\frac{2n + \Delta_1 + \Delta_2 - \Delta_3 + \Delta_4}{2}\right) \Gamma\left(\frac{-2n - \Delta_1 - \Delta_2 + \Delta_3 + \Delta_4}{2}\right) \Gamma\left(\frac{2n + \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 - 1}{2}\right)}{\Gamma\left(2n + \Delta_1 + \Delta_2 - \frac{1}{2}\right) \Gamma(2n + \Delta_1 + \Delta_2)}. \quad (4.3.13)$$

Considering the first order piece of eq. (4.3.9) and extracting a dimensionful prefactor

$$\langle \phi(x_1) \phi^2(x_2) \phi(x_3) \phi^2(x_4) \rangle^{(1)} = \frac{1}{x_{12}^{3\Delta_\phi} x_{34}^{3\Delta_\phi}} \left(\frac{x_{13}}{x_{24}} \right)^{\Delta_\phi} f_{\phi\phi^2\phi\phi^2}(z), \quad (4.3.14)$$

one has the conformal block expansion

$$f_{\phi\phi^2\phi^2}(z) = \sum_{n=0}^{\infty} (C_{\phi\phi^2(\phi^3)_n}^2)^{(1)} G_{(\phi^3)_n}^{\phi\phi^2\phi^2}(z) + \sum_{n=0}^{\infty} (C_{\phi\phi^2(\phi^3)_n}^2)^{(0)} \gamma_{(\phi^3)_n}^{(1)} \partial_{\Delta} G_{(\phi^3)_n}^{\phi\phi^2\phi^2}(z), \quad (4.3.15)$$

which is obtained from the generic expression in the case where $\Delta_1 + \Delta_2 = \Delta_3 + \Delta_4$. In this case the individual coefficients a_n^{12} and a_n^{34} in eq. (4.3.12) diverge but the sum of the two towers of double trace operators is finite. Through the computation of this finite limit, derivatives of conformal blocks enter on the r.h.s. of eq. (4.3.15). The anomalous dimension of ϕ^3 can be read off as the coefficient which multiplies the derivative of the block corresponding to the non-degenerate state at $n = 0$. Dividing by the free OPE coefficient $(C_{\phi\phi^2(\phi^3)_n}^2)^{(0)} = 3$ and by a factor of 2 ensuring unit normalisation of the ϕ^2 operators, one finds

$$\gamma_{\phi^3} = -\lambda_6 N_{\phi}^6 \left(-\frac{\sqrt{\pi} \Gamma(3\Delta_{\phi} - \frac{1}{2})}{6 \Gamma(3\Delta_{\phi})} \right). \quad (4.3.16)$$

We can also read off the OPE coefficient $C_{\phi\phi^2\phi^3}^{(1)}$ by noting that the square OPE coefficient appearing in the correlator is $(C_{\phi\phi^2\phi^3}^2)^{(1)} = 2C_{\phi\phi^2\phi^3}^{(0)} C_{\phi\phi^2\phi^3}^{(1)}$. We find

$$C_{\phi\phi^2\phi^3}^{(1)} = \lambda_6 N_{\phi}^6 \frac{\sqrt{\frac{\pi}{3}} \Gamma(3\Delta_{\phi} - \frac{1}{2}) \left[H_{\Delta_{\phi}-1} + H_{2\Delta_{\phi}-1} + H_{3\Delta_{\phi}+\frac{3}{2}} - 2H_{6\Delta_{\phi}-2} + \log(4) \right]}{4\Gamma(3\Delta_{\phi})}, \quad (4.3.17)$$

where H_{α} is an analytically continued harmonic number.

Having computed the anomalous dimension, let us turn to the four-point function $f_{\phi\phi\phi\phi^3}(z = 1/3)$, which is the quantity that was bounded in Section 4.3.3 by the numerical bootstrap. The tree level correction to this correlator is again given by a contact diagram

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi^3(x_4) \rangle^{(1)} = -\lambda_6 N_{\phi}^6 D_{\Delta_{\phi}\Delta_{\phi}\Delta_{\phi}3\Delta_{\phi}}(x_i). \quad (4.3.18)$$

However this D-function, illustrated on the r.h.s. of Fig. 4.3.4, is actually divergent. The divergence is due to the extremality condition $\Delta_{\phi} + \Delta_{\phi} + \Delta_{\phi} = 3\Delta_{\phi}$, and is a familiar phenomenon in holographic three point-functions. Physically, the extremality condition should be avoided by the fact that the external operator ϕ^3 acquires an anomalous dimension. Therefore, we regulate the divergence by working in terms of physical CFT data. Let us first remove a kinematic prefactor and define a function $f_{\phi\phi\phi\phi^3}^{(1)}$ of the cross-ratio through

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi^3(x_4) \rangle^{(1)} = \frac{1}{x_{12}^{2\Delta_{\phi}} x_{34}^{\Delta_{\phi}+\Delta_{\phi^3}}} \left(\frac{x_{13}}{x_{14}} \right)^{\Delta_{\phi^3}-\Delta_{\phi}} f_{\phi\phi\phi\phi^3}^{(1)}(z). \quad (4.3.19)$$

The decomposition of $f_{\phi\phi\phi\phi^3}^{(1)}$ into conformal blocks takes the form

$$\begin{aligned} f_{\phi\phi\phi\phi^3}^{(1)}(z) = & \sum_{n=0}^{\infty} C_{\phi\phi(\phi^4)_n}^{(1)} C_{\phi\phi^3(\phi^4)_n}^{(0)} G_{(\phi^4)_n}^{\phi\phi\phi\phi^3} + \sum_{n=0}^{\infty} C_{\phi\phi(\phi^2)_n}^{(0)} C_{\phi\phi^3(\phi^2)_n}^{(1)} G_{(\phi^2)_n}^{\phi\phi\phi\phi^3} \\ & + C_{\phi\phi\phi^2}^{(0)} C_{\phi\phi^3\phi^2}^{(0)} \gamma_{\phi^3}^{(1)} \partial_{\phi^3} G_{\phi^2}^{\phi\phi\phi\phi^3}. \end{aligned} \quad (4.3.20)$$

Clearly, the anomalous dimension of ϕ^3 , which should be unit normalised by diving by $\sqrt{6}$, only affects the exchange of the operator ϕ^2 . This is consistent with the fact that the free piece of the correlator actually contains only a single conformal block

$$f_{\phi\phi\phi\phi^3}^{(0)}(z) = \sqrt{6} G_{\phi^2}^{\phi\phi\phi\phi^3}(z). \quad (4.3.21)$$

Since therefore only the coefficient of the ϕ^2 block is divergent when using the standard expansion of the contact diagram, we can extract all the coefficients for exchanges of $(\phi^4)_n$ and $(\phi^2)_n$ with $n \geq 1$ from the standard expansion of the D-function. To reconstruct the ϕ^2 exchange, we use the expressions (4.3.16) and (4.3.17) for γ_{ϕ^3} and $C_{\phi\phi^2\phi^3}^{(1)}$ that were extracted from the finite $\langle\phi\phi^2\phi\phi^2\rangle$ correlator. Since the OPE converges quickly at $z = 1/3$, it is sufficient to sum only a few blocks to obtain an answer that has converged to several decimal places.

The result can be matched to the derivative of the numerical bound $F_{\phi^3}(\Delta_\phi, \Delta_{\phi^2}, \Delta_{\phi^3})$ w.r.t. Δ_{ϕ^3} . As Figure 4.3.5 shows, we indeed find

$$\partial_{\Delta_{\phi^3}} F_{\phi^3}|_{GFB} = \frac{\partial_{\lambda_6} F_{\phi^3}}{\partial_{\lambda_6} \Delta_{\phi^3}} = \frac{f_{\phi\phi\phi\phi^3}^{(1)}}{\gamma_{\phi^3}^{(1)}} \quad (4.3.22)$$

to high precision at small Δ_ϕ .

As already pointed out in Section 4.3.1, the bounds obtained using six-point derivative functionals behave qualitatively similar to those obtained using derivative functionals in the four-point bootstrap, in that the convergence of the bound as a function of the number of functionals significantly slows down as the external dimension Δ_ϕ is increased. To see the effect of this slow down also for the problem studied here, the range of Δ_ϕ shown in Figure 4.3.5 also includes a region where the numerical bound is not yet well converged. In that region, the numerical answer and the analytic prediction do of course not agree, it is however reassuring that increasing Λ from 4 to 5 reduces the mismatch. This suggests that eventually, the numerical data should match the analytic curve also for large values of Δ_ϕ if a sufficiently high value of Λ is chosen.

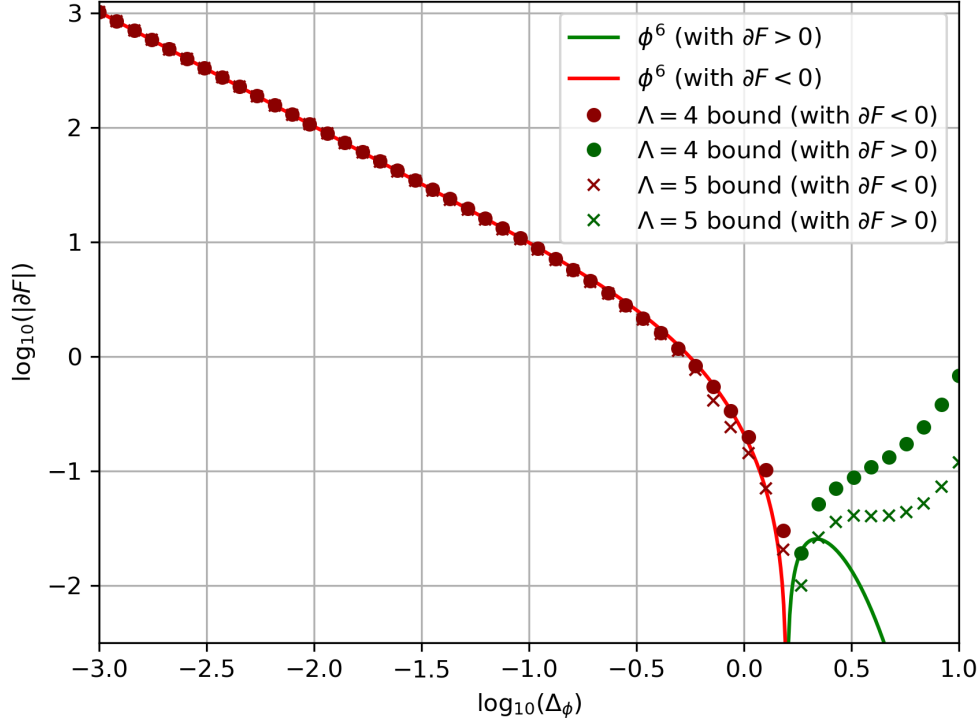


Figure 4.3.5: Derivative of the four-point function w.r.t. the Φ^6 coupling and numerical derivative of the bound at GFB as a function of Δ_ϕ .

Φ^4 interaction. The quartic coupling case is similar but has some important differences. We once again normal order the interaction and write the action as

$$S = \frac{1}{2} \int d^2x \sqrt{g} \left(\partial_\mu \Phi \partial^\mu \Phi + m^2 : \Phi^2 : + \frac{\lambda_4}{4!} : \Phi^4 : \right). \quad (4.3.23)$$

This of course means that the four-point function of ϕ gets modified by a contact diagram

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle = \text{GFF} - \lambda_4 N_\phi^4 D_{\Delta_\phi \Delta_\phi \Delta_\phi \Delta_\phi}(x_i) + O(\lambda_4^2), \quad (4.3.24)$$

which in particular induces an anomalous dimension for ϕ^2

$$\gamma_{\phi^2} = -\lambda_4 N_\phi^4 \left(-\frac{\sqrt{\pi} \Gamma(2\Delta_\phi - \frac{1}{2})}{2\Gamma(2\Delta_\phi)} \right), \quad (4.3.25)$$

and a correction to the OPE coefficient

$$C_{\phi\phi\phi^2}^{(1)} = (-\lambda_4 N_\phi^4) \frac{-\sqrt{\pi} \Gamma(2\Delta_\phi - \frac{1}{2}) \left(2H_{\Delta_\phi-1} + H_{2\Delta_\phi-\frac{3}{2}} - 2H_{4\Delta_\phi-2} + \log(4) \right)}{2\Gamma(2\Delta_\phi)}. \quad (4.3.26)$$

To obtain the anomalous dimension of ϕ^3 we could once again study the correlator $\langle \phi\phi^2\phi\phi^2 \rangle$, but we will instead consider a shortcut. Directly computing the two-point

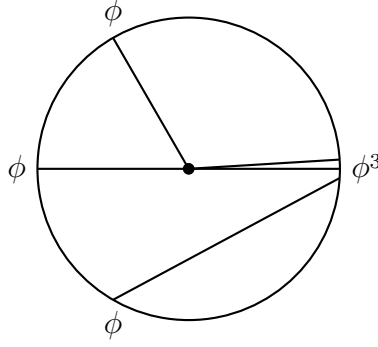


Figure 4.3.6: Factorised diagram contributing to the $\langle \phi \phi \phi \phi^3 \rangle$ correlator with a quartic vertex. The sub-diagram corresponding to the three-point function is divergent.

function of ϕ^3 perturbatively makes it clear that the resulting integral is the same as in the ϕ^2 case. Counting diagrams leads to a simple combinatorial factor

$$\gamma_{\phi^3} = 3\gamma_{\phi^2}. \quad (4.3.27)$$

Finally, we need to compute $\langle \phi \phi \phi \phi^3 \rangle$. Since the ϕ^3 operator is normal ordered, there is no contribution to the correlator that connects all four-points. Instead, the correlator factorises into a product of two- and three-point functions, as illustrated in Figure 4.3.6. However these three-point functions are extremal (due to the property $\Delta_\phi + \Delta_\phi = 2\Delta_\phi$) and are therefore divergent. We once again regulate by working in terms of physical data. The divergent three-point integral computes the correction to the three-point function. Therefore, we simply write down the three-point function in terms of OPE coefficients and scaling dimensions and expand them to first order. These are known from the expansion of the finite four-point function $\langle \phi \phi \phi \phi \rangle$ that we wrote above. We then have

$$\langle \phi^2(x_1)\phi(x_2)\phi(x_3) \rangle^{(1)} = \frac{C_{\phi\phi\phi^2}^1 - \sqrt{2}\gamma_{\phi^2}(\log(x_{12}x_{13}x_{23}^{-1}))}{x_{12}^{2\Delta_\phi} x_{13}^{2\Delta_\phi}}. \quad (4.3.28)$$

Adding all the permutations of the above diagram and extracting the prefactor yields a correlator that is easy to write as an explicit function of z

$$f_{\phi\phi\phi\phi^3}^{(1)}(z) = -\lambda_4 N_\phi^4 \left(\sqrt{3} z^{2\Delta_\phi} (3C_{\phi\phi\phi^2}^{(1)} + \sqrt{2}\gamma_{\phi^2} \log(z(1-z))) \right), \quad (4.3.29)$$

which can be straightforwardly evaluated at $z = 1/3$.

Figure 4.3.7 visualises numerical data indicating that the Φ^4 deformation is tangential to bound in the same way as the Φ^6 deformation. As opposed to Figure 4.3.5, this plot is restricted to the small Δ_ϕ range where the bound is well converged at $\Lambda = 4$. Similar comments to those made about convergence above Figure 4.3.5 also hold here.

Note that, since now both Δ_{ϕ^2} and Δ_{ϕ^3} are changed by the deformation, a combination of the two derivatives has to be taken and we find

$$3\partial_{\Delta_{\phi^3}}F_{\phi^3}|_{GFB} + \partial_{\Delta_{\phi^2}}F_{\phi^3}|_{GFB} = \frac{f_{\phi\phi\phi\phi^3}^{(1)}}{\gamma_{\phi^2}^{(1)}}. \quad (4.3.30)$$

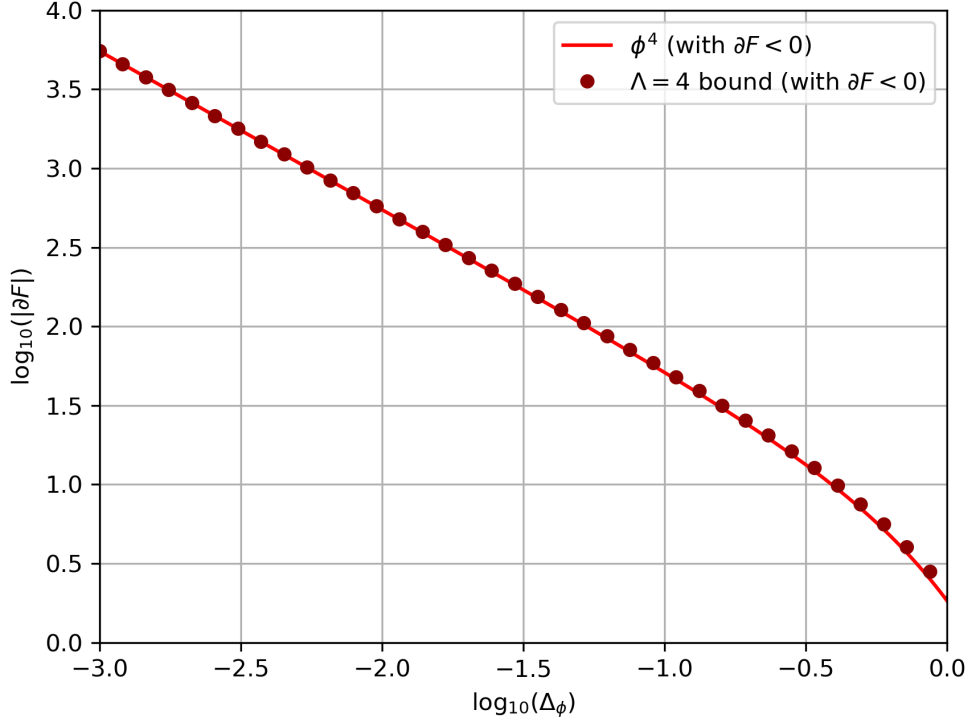


Figure 4.3.7: Derivative of the four-point function w.r.t. the Φ^4 coupling and numerical derivative of the bound at GFB as a function of Δ_ϕ .

Appendix

4.A Code

This appendix collects remarks on the code that is made available with this publication. The focus is on increasing usability of the code and not on a detailed description of its precise structure.

4.A.1 `GenerateFunctionals.nb`

The mathematica notebook `GenerateFunctionals.nb` can be used to generate and store all derivative functionals used in this work as well as their decomposition according to the procedure described in Section 4.2. Upon running the notebook, it generates the file `Functionals_A2.txt` in the folder in which the notebook is stored. The file contains the functionals with up to 2 derivatives. The parameter Λ can be changed in the first section of the notebook.

4.A.2 `Generate_JSON_no1.py`

The python script `Generate_JSON_no1.py` generates the file `pmp.JSON` in the working folder, which serves as the input that has to be provided to `SDPB` in order to compute the bound discussed in Section 4.3.1. Several parameters need to be provided to `Generate_JSON_no1.py`. These are

- `fundir`: The folder in which the functionals generated by `GenerateFunctionals.nb` (see App. 4.A.1) are stored. In the example below, we assume that the functionals are stored in a folder “`functionals`” inside of the folder in which `Generate_JSON_no1.py` is called.
- `Lambda`: Derivative order Λ .
- `nTrunc`: Truncation in the n parameter introduced through the decomposition described in Section 4.2.
- `gap`: Value of the gap to be tested.

- **h**: External scaling dimension Δ_ϕ .
- **prec**: Internal working precision.
- **extra_eval**: A string of integers separated by commas, that indicates which particular values of n should be considered beyond $n \leq \mathbf{nTrunc}$. Entering -1 as part of **extra_eval** dictates that positivity should be imposed asymptotically for $n \rightarrow \infty$.

Using the `pmp.JSON` file generated by the example call

```
python3 generate_JSON_no1.py --fundir=functionals --Lambda=2 --nTrunc=20
--gap=0.187 --h=0.1 --prec=512 --extra_eval=100,-1
```

as input to SDPB should result in SDPB terminating with a dual feasible solution, indicating that for $\Delta_\phi = 0.1$ a gap of 0.187 can be excluded with $\Lambda = 2$ functionals. If on the other hand, the **gap** parameter is changed to 0.186, SDPB will not be able to find a dual feasible solution and e.g. terminate with the final status **maxComplementarity exceeded**. To improve the numerical bound and get closer to 0.18, higher derivative functionals need to be considered.

4.A.3 GFF_to_GFB_Flow.nb

In section two of the notebook, the GFF spectrum and OPE data is computed by solving the truncated crossing equation for a scalar field ϕ of dimension $\Delta_\phi = 0.1$ with $\Lambda_{\max} = 10$ operators above the identity exchange. The parameters Δ_ϕ and $\Lambda_{\max}$ can be adapted in the first lines of the second section.

After running the second section of the notebook, one can run the third section, which uses the GFF data that has been computed as initial data to flow to GFB. In the first two subsections i.e. 3.1 and 3.2, the flow is performed for all values of Λ up to $\Lambda_{\max}$ and a folder called **h=0.1** (or **h=...** if Δ_ϕ is changed) is created in which the resulting data is stored. Finally, an extrapolation to $\Lambda = \infty$ is performed. Running the notebook with the default parameters will take several hours.

In addition to $\Lambda_{\max}$, parameters Λ_6 and **dynamiclibPATH** are specified in the first lines of the third section of the notebook. The variable **dynamiclibPATH** needs to be set to the absolute path of the **arbitrary_precision_blocks.so** shared object that is described in Appendix 4.A.4. Λ_6 on the other hand specifies which derivative order of the six-point bootstrap the notebook should generate the input data corresponding to the ϕ exchange in the central exchange of the six-point comb for. Upon running Section 3.3 of the notebook, six-point comb channel blocks and their derivatives are computed using **arbitrary_precision_blocks.so** and multiplied with the OPE data of the extremal flow. The result of the computation is stored inside of the **h=0.1** folder in the form of txt files that serve as an input to the six-point bootstrap.

4.A.4 arbitrary_precision_blocks.c

For the purpose of this publication, a short mathematica script that evaluates comb-channel blocks to a few digits precision by summing the series definition would have been completely sufficient. However, since we already had code available that efficiently evaluates the blocks to high precision, namely the `arbitrary_precision_blocks.c` of the ancillary files of this publication, we decided to use that code instead. The mathematica wrapper `arbitrary_precision_blocks.nb` can be used to call our C implementation of the blocks in mathematica. To do so, the user has to compile the C code into a shared object that has to be called `arbitrary_precision_blocks.so`.

4.A.5 Generate_JSON_GFF_to_GFB.py

`Generate_JSON_GFF_to_GFB.py` is for the interpolation between GFF and GFB discussed in Section 4.3 what the script `Generate_JSON_no1.py` described in Appendix 4.A.2 is for gap maximisation without identity. Hence, most of the parameters that need to be passed to it coincide with those discussed in Appendix 4.A.2. Therefore, let us here only highlight one example call of `Generate_JSON_GFF_to_GFB.py` and then explain the three new parameters that occur in it.

```
python3 generate_JSON_GFF_to_GFB.py --h=0.1 --Lambda=2 --nTrunc=30 --prec=512
--share=input_data --triple=2.1 --D1=0.6 --extra_eval=100,-1
```

To successfully run the script with the arguments chosen above, the working folder must contain a folder called `input_data`. The `input_data` folder in turn must contain a subfolder called `functionals` and a subfolder called `objectives`. Inside of `input_data/functionals`, one has to place the file `Functionals_Λ2.txt` generated by `Generate_Functionals.nb` (see App. 4.A.1). Inside of `input_data/objectives`, the file `Λ2_Δφ0.1_Δ10.6.txt` generated by `GFF_to_GFB_Flow.nb` (see App. 4.A.3) has to be placed.

Finally, let us comment on the arguments `--triple` and `--D1`.

- **D1:** Dimension Δ_{ϕ^2} of the lowest lying double twist operator (note that only values for D1 can be used for which the necessary input data has been computed using `GFF_to_GFB_Flow.nb`).
- **triple:** Dimension Δ_{ϕ^3} of the lowest lying double twist operator (in principle arbitrary positive real number).

With the input data generated by the example call of `Generate_JSON_GFF_to_GFB.py` given above, running `SDPB` should result in convergence to a primal-dual optimal solution putting an upper bound of 0.110 on the square of the four-point function F_{ϕ^3} of the four-point maximisation problem for $\Delta_{\phi} = 0.1$, $\Delta_{\phi^2} = 0.6$ and $\Delta_{\phi^3} = 2.1$.

Chapter 5

Multi-point analytic functionals

– Approaching the large Δ problem.

After the numerical work of the previous chapters, we now change gears and focus on analytical properties of higher-point functionals, approaching a solution to the large Δ convergence problem for the multi-point bootstrap.

To formulate this problem, recall that the numerical six-point bootstrap of the previous two chapters derives bounds by solving SDPs dual to a somewhat arbitrarily selected finite set of sum rules for the CFT data. More specifically, we chose to work with the constraints obtained by evaluating all degree less than or equal to Λ derivatives of a six-point crossing equation at the crossing symmetric point $\chi_i = 1/3$. Every numerical bound $\mathcal{B}$ obtained in this way depends on at least two parameters: The scaling dimension Δ_ϕ of the operator whose six-point correlator we study, and the upper limit Λ on the degree of the derivative functionals. As we increase Λ , the bound monotonously improves and ultimately we are interested in the optimal bound

$$\mathcal{B}_*(\Delta_\phi) = \lim_{\Lambda \rightarrow \infty} \mathcal{B}(\Delta_\phi, \Lambda). \quad (5.0.1)$$

However, as Λ is increasing, the size of the SDP that needs to be solved to compute the bounds is growing as well. How large the parameter Λ can be made is hence limited by the computation time that we can dedicate to the problem. As we observed in the previous two chapters, the rate of convergence of the limit (5.0.1) rapidly slows down as Δ_ϕ is increased, making it challenging to infer the value of $\mathcal{B}_*(\Delta_\phi)$ for large Δ_ϕ .

This large Δ convergence problem is not a new feature of the multi-point bootstrap, but a well known issue already on the level of four-point functions. A standard approach established in that context is to guess an ansatz for the Λ dependence of the function $\mathcal{B}(\Delta_\phi, \Lambda)$ and fit the free parameters of the ansatz to a large range of bounds computed at various Λ values. This approach was, for instance, impressively

implemented in Ref. [60], which reliably performed a double extrapolation first to infinite Λ and then to infinite Δ_ϕ , reproducing flat space 2d S-matrices with 1d extremal CFT correlators. But it does not generally solve the problem, and e.g. reached its limits when the authors of Ref. [60] tried to apply the same approach in higher dimensions [207].

More conceptual issues of the extrapolation approach are that, even when applicable, it is both numerically costly and based on heuristic arguments that pollute potentially rigorous bootstrap bounds with uncontrolled errors. These problems are, even in the four-point setting, only partially solved for theories in generic space time dimensions, see e.g. [11] for a concise discussion of the current status. For one-dimensional conformal systems, the situation is much clearer, and the ‘analytic functionals’ of [172, 191, 208] provide both a practical solution to the problem and a systematic understanding of the issue. This solution of the large Δ convergence problem of the numerical bootstrap was concretely implemented e.g. in References [181, 182, 202, 209].

The analytic functionals can be seen as integral transforms acting on crossing vectors. The construction of the corresponding integral kernels requires a careful analysis of the analytic structure of conformal blocks and correlation functions. Beyond the case of 4-point identical scalar correlators, this set-up is notoriously difficult to work with and to generalise¹. Fortunately this complication can be bypassed with an alternative method, the Polyakov bootstrap. The idea here is to express a CFT correlator as an expansion in Polyakov blocks, which are crossing symmetric combinations of tree-level Witten diagrams. The equality of this description with the usual conformal block expansion furnishes a set of conditions on the OPE data that is equivalent to analytic functional sum-rules. The Polyakov block is also a natural replacement of a conformal block in Mellin space, where it perfectly mimics all the analytic and crossing properties of a CFT correlation function. In fact, the Polyakov block expansion can be rigorously established from crossing symmetric dispersion relations for CFTs in general spacetime dimensions [215, 216]. See [217–220] for related works.

In this Chapter, we derive analytic functionals for conformal 5-point functions in one-dimensional kinematics, making use of the Polyakov bootstrap formalism. We begin by assuming the existence of an expansion of the 5-point correlator $\mathcal{G}$ of identical operators in terms of crossing-symmetric Polyakov blocks P which must be compatible with the usual OPE, i.e. with the expansion in terms of ordinary conformal blocks G . Concretely, defining the five point correlator $\mathcal{G}$ as

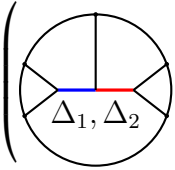
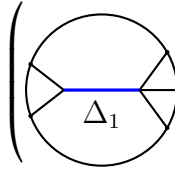
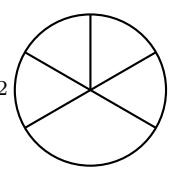
$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi(x_5) \rangle (\mathcal{L}(x_i))^{-1} = \mathcal{G}(\chi_1, \chi_2), \quad (5.0.2)$$

¹For various extensions of analytic functionals see e.g. Ref. [192] (global symmetries), Refs. [210, 211] (higher dimensions), Refs. [212, 213] (boundary CFTs) and Ref. [214] (CFTs in real projective space).

where $\mathcal{L}$ is a scaling covariant prefactor and $\chi_{1,2}$ are two invariant cross-ratios of five points, we must have

$$\mathcal{G}(\chi_1, \chi_2) = \sum_{\Delta_1, \Delta_2} a_{\Delta_1, \Delta_2} G_{\Delta_1, \Delta_2}(\chi_1, \chi_2) = \sum_{\Delta_1, \Delta_2} a_{\Delta_1, \Delta_2} P_{\Delta_1, \Delta_2}(\chi_1, \chi_2), \quad (5.0.3)$$

where a_{Δ_1, Δ_2} is the appropriate product of OPE coefficients. Our first main result is an explicit construction of the Polyakov block. As in the 4-point case, this object can be written as a linear combination of tree-level exchange Witten diagrams, with some subtractions determined by contact interactions. We show that the Polyakov block can be written as

$$P_{\Delta_1, \Delta_2} = \left(\text{Diagram 1} + \dots \right) + c_1 \left(\text{Diagram 2} + \dots \right) + c_2 \left(\text{Diagram 3} \right), \quad (5.0.4)$$




where the c_i coefficients depend on Δ_i and on the external dimension Δ_ϕ , the middle term is understood to be symmetrised under $(\Delta_1 \leftrightarrow \Delta_2)$ and the ellipsis denote all permutations of the external legs of the diagrams. Clearly, along with the exchanges of the form $\{\Delta_1, \Delta_2\}$, the Polyakov blocks contain several other families of exchanges in their OPE, including $\{\Delta_1, 2\Delta_\phi + n\}$, $\{\Delta_1, \Delta_1 + \Delta_\phi + n\}$, $\{2\Delta_\phi + n, 2\Delta_\phi + m\}$ and $\{2\Delta_\phi + n, 3\Delta_\phi + m\}$. Consistency with the OPE leads to the requirement that the exchanges of these spurious families of operators cancel out when summing over all Polyakov blocks. We then find several families of sum-rules for the 5-point function, with the most important ones being

$$\sum_{\Delta_1, \Delta_2} a_{\Delta_1, \Delta_2} \alpha_{n, m}^{(2\Delta_\phi, 2\Delta_\phi)}(\Delta_1, \Delta_2) = 0, \quad \sum_{\Delta_1, \Delta_2} a_{\Delta_1, \Delta_2} \alpha_{n, m}^{(2\Delta_\phi, 3\Delta_\phi)}(\Delta_1, \Delta_2) = 0, \quad (5.0.5)$$

where we introduced the crossing functionals α labeled by the different families of spurious exchanges in the obvious way. These functionals depend on the two variables Δ_1, Δ_2 and capture information about OPE coefficients of the form $C_{\phi, [\phi\phi]_n, [\phi\phi]_m}$ and $C_{\phi, [\phi\phi]_n, [\phi\phi\phi]_m}$; they are therefore a vast generalisation of the usual 4-point functionals. With concrete expressions for these functionals at hand, it is then straightforward to use them in the truncated 5-point bootstrap as a replacement (or complement) to the standard derivative functionals. We find that our functionals have multiple advantages over the familiar derivative basis and constitute the first step towards a solution of the large Δ convergence problem at higher points.

Overview. Section 5.1 reviews the Polyakov bootstrap approach to the 4-point crossing equation, emphasising the extraction of double-trace coefficients in Witten diagrams through a one-dimensional Mellin formalism. In Section 5.2, which is the core of the chapter, we first recall basic facts about conformal 5-point functions, and

then write down an ansatz for the crossing symmetric Polyakov block as a sum of tree-level Witten diagrams. We subsequently derive several families of two-variable sum-rules by requiring consistency with the OPE. We also test these sum-rules in several non-trivial correlators confirming the validity of our results. In Section 5.3, we give a simple application of our functionals, using them to bootstrap the OPE coefficients of a non-trivial solution to crossing through the truncated bootstrap approach. The chapter is complemented by several appendices. Appendix 5.A develops in detail the Mellin techniques used throughout the main text. Appendix 5.B reviews the spectral representation which makes the structure of Witten diagrams more manifest. Appendix 5.C reviews the so-called integrated vertex identities which can be used as a cross-check of the Mellin computations and streamline certain computations. Finally, Appendix 5.D establishes Casimir recursion relations for the 5-point blocks and Witten diagrams which serve as a further cross-check to the other techniques used in the main text, but could also be of independent interest.

5.1 Polyakov Bootstrap review

In this section, we review the necessary ingredients to derive Polyakov Bootstrap sum-rules for 4-point functions, emphasizing the techniques that will be needed in the 5-point case. We first review the basic kinematics of 1d CFTs in Section 5.1.1 and then the principles of Polyakov bootstrap for the simple case of 4-point correlation function of identical scalars in Section 5.1.2, followed by the necessary 1d Mellin amplitude methods in Section 5.1.3.

5.1.1 Four-point kinematics

In general dimensions the 4-point correlator is given by

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{\mathcal{G}(u, v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}, \quad (5.1.1)$$

where $u = \chi\bar{\chi} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = (1 - \chi)(1 - \bar{\chi}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$.

In 1d the two crossratios are related as $\chi = \bar{\chi}$, so we may write $\mathcal{G}(u, v) = \mathcal{G}(\chi)$.

If the $\phi(x_i)\phi(x_j)$ OPE is composed of operators with dimensions denoted by Δ with respective OPE coefficients C_Δ , then the s -channel conformal block expansion of the function $\mathcal{G}(\chi)$ is given by

$$\mathcal{G}(\chi) = \sum_{\Delta} C_\Delta^2 G_\Delta(\chi). \quad (5.1.2)$$

where $G_\Delta(\chi) = \chi^{\Delta_2} F_1(\Delta, \Delta, 2\Delta, \chi)$ denotes $\text{SL}(2, \mathbb{R})$ conformal blocks.

For bosonic operators, the correlation function is crossing symmetric in the exchanges $x_2 \leftrightarrow x_4$ as well as $x_2 \leftrightarrow x_4$. This implies the conditions:

$$\mathcal{G}(\chi) = \left[\frac{\chi}{1-\chi} \right]^{2\Delta_\phi} \mathcal{G}(1-\chi) = \chi^{2\Delta_\phi} \mathcal{G}(1/(1-\chi)) \quad (5.1.3)$$

The r.h.s. of the first and second equalities give the t and u channel conformal block expansions. The bootstrap equation for the s - t channel crossing equation is given by

$$\sum_{\Delta} C_{\Delta}^2 F_{\Delta}(\chi) = 0 \quad (5.1.4)$$

where $F_{\Delta}(\chi) = \chi^{-2\Delta_\phi} G_{\Delta}(\chi) - (1-\chi)^{-2\Delta_\phi} G_{\Delta}(1-\chi)$ is the bootstrap vector.

We will focus on correlation functions that diverge at most with the rate $\chi^{2\Delta_\phi}$ in the limit of large χ , i.e. correlators that satisfy

$$\chi^{-2\Delta_\phi - \epsilon} \mathcal{G}(\chi) \xrightarrow{\chi \rightarrow \infty} 0 \quad \text{for any } \epsilon > 0. \quad (5.1.5)$$

This is called boundedness in the u -channel Regge limit and is a consequence of unitarity [208]. In conventional bootstrap methods (5.1.5) is not assumed for bounding OPE data, but it is expected to hold for a complete solution to crossing symmetry.

Generalised free theory: An important solution to crossing, that we will keep coming back to, is the 4-point function of a generalised free theory (GFF). It is given by the correlation function

$$\mathcal{G}(\chi) = 1 + \chi^{2\Delta_\phi} + \left[\frac{\chi}{1-\chi} \right]^{2\Delta_\phi}. \quad (5.1.6)$$

This corresponds to the CFT at the boundary of a bulk free massive bosonic theory in AdS_2 . The mass is related to scaling dimension by $m^2 = \Delta_\phi(\Delta_\phi - 1)$. Since the theory is free its correlators are given by Wick contractions. From the conformal block decomposition one can see the operator dimensions $\Delta = \Delta_n := 2\Delta_\phi + 2n$ in the OPE with OPE coefficients

$$C_{\Delta_n}^2 = \frac{2\Gamma(4\Delta_\phi + 2n - 1)((2\Delta_\phi)_{2n})^2}{\Gamma(2n + 1)\Gamma(4\Delta_\phi + 4n - 1)}. \quad (5.1.7)$$

Clearly (5.1.6) satisfied both conditions (5.1.3) and (5.1.5). In a bootstrap problem the GFF corresponds to a spectrum with the maximum OPE coefficient of the first non-identity operator whose dimension is fixed at $2\Delta_\phi$.

5.1.2 Polyakov bootstrap

The main idea in Polyakov bootstrap is to replace the conformal blocks in the expansion (5.1.2) with a new object that is manifestly crossing symmetric and Regge bounded. We write this demand as follows

$$\mathcal{G}(\chi) = \sum_{\Delta} C_{\Delta}^2 P_{\Delta}(\chi), \quad (5.1.8)$$

where $P_{\Delta}(\chi)$ is called a Polyakov block that satisfies

$$P_{\Delta}(\chi) = (\chi/(1-\chi))^{2\Delta_{\phi}} P_{\Delta}(1-\chi) = \chi^{2\Delta_{\phi}} P_{\Delta}(1/(1-\chi)) \quad (5.1.9)$$

and also $\chi^{-2\Delta_{\phi}} P_{\Delta}(\chi) = 0$ is finite as $\chi \rightarrow \infty$. These requirements indicate that a Polyakov block is itself a consistent solution of the crossing equation, but it does not have a unique form. A possible definition is in terms of tree level Witten diagrams as follows

$$P_{\Delta}(\chi) = W_{\Delta}^{(s)}(\chi) + W_{\Delta}^{(t)}(\chi) + W_{\Delta}^{(u)}(\chi) + c_{\Delta} W^{\text{con}}(\chi) \quad (5.1.10)$$

Here $W_{\Delta}^{(i)}(\chi)$ denotes a tree level Witten diagram exchanging a bulk operator corresponding to dimension Δ in the $i = s, t, u$ channel, and $W^{\text{con}}(\chi)$ is a 4-point contact diagram. The term c_{Δ} can be chosen in multiple ways. One possible choice is given below (5.1.13) and is discussed further at the end of this subsection.

We discuss the structure of the Witten diagrams in more detail in the next subsection. For now let us note two important properties. Firstly, all the diagrams in (5.1.10) are Regge bounded. Also each of them can be written as a sum over conformal blocks and their derivatives as follows:

$$\begin{aligned} W_{\Delta}^{(i)}(\chi) = & \delta_{i,s} G_{\Delta}(\chi) + \sum_n \left[a_n^{(i)}(\Delta, \Delta_{\phi}) G_{\Delta_n}(\chi) + b_n^{(i)}(\Delta, \Delta_{\phi}) \partial_{\Delta} G_{\Delta_n}(\chi) \right. \\ & \left. + \tilde{a}_n^{(i)}(\Delta, \Delta_{\phi}) G_{\Delta_n+1}(\chi) + \tilde{b}_n^{(i)}(\Delta, \Delta_{\phi}) \partial_{\Delta} G_{\Delta_n+1}(\chi) \right], \end{aligned} \quad (5.1.11)$$

$$W^{\text{con}}(\chi) = \sum_n \left[a_n^{(c)}(\Delta_{\phi}) G_{\Delta_n}(\chi) + b_n^{(c)}(\Delta_{\phi}) \partial_{\Delta} G_{\Delta_n}(\chi) \right]. \quad (5.1.12)$$

We additionally have $\tilde{a}_n^{(s)} = \tilde{b}_n^{(s)} = 0$ and $\tilde{a}_n^{(t)} = -\tilde{a}_n^{(u)}, \tilde{b}_n^{(t)} = -\tilde{b}_n^{(u)}$. Putting these together in (5.1.10) gives the conformal block decomposition of a Polyakov block

$$P_{\Delta}(\chi) = G_{\Delta}(\chi) + \sum_n \left[\alpha_n(\Delta, \Delta_{\phi}) G_{\Delta_n}(\chi) + \beta_n(\Delta, \Delta_{\phi}) \partial_{\Delta} G_{\Delta_n}(\chi) \right], \quad (5.1.13)$$

where $\alpha_n := \sum_i a_n^{(i)} + c_{\Delta} a_n^{(c)}$ and $\beta_n := \sum_i b_n^{(i)} + c_{\Delta} b_n^{(c)}$. The blocks and derivatives of blocks corresponding to Δ_n values are sometimes called ‘spurious blocks’. Finally let us point out that we choose c_{Δ} so that $\beta_0(\Delta, \Delta_{\phi}) = 0$.

The statement of Polyakov bootstrap is that the Polyakov block expansion is the same as conformal block expansion:

$$\sum_{\Delta} C_{\Delta}^2 P_{\Delta}(\chi) = \sum_{\Delta} C_{\Delta}^2 G_{\Delta}(\chi). \quad (5.1.14)$$

This means if we choose the right dimensions in the OPE in a $P_{\Delta}(\chi)$ expansion, the spurious blocks cancel among themselves leaving only the sum over physical block as in r.h.s. of (5.1.14). So the Polyakov bootstrap conditions can be restated as follows:

$$\sum_{\Delta} C_{\Delta}^2 \alpha_n(\Delta, \Delta_{\phi}) = \sum_{\Delta} C_{\Delta}^2 \beta_n(\Delta, \Delta_{\phi}) = 0 \quad (5.1.15)$$

Let us mention here, in words, that there is an alternative way to derive (5.1.15) from the crossing symmetry equation. This is by using analytic functionals which are a set of dual functionals (call α_n and β_n) to a basis of the crossing vectors $F_{\Delta_n}(\chi)$ and $\partial_{\Delta} F_{\Delta_n}(\chi)$ respectively. The fact that any $F_{\Delta}(\chi)$ can be expanded in such a basis is nothing but the consequence of a Polyakov block $P_{\Delta}(\chi)$ being crossing symmetric on its own and its decomposition satisfies eqn. (5.1.4). The exact forms of the functionals is not important here, but their actions on $F_{\Delta}(\chi)$ are precisely $\alpha_n(\Delta, \Delta_{\phi})$ and $\beta_n(\Delta, \Delta_{\phi})$ [172, 191, 208], and acting on the bootstrap equation (5.1.4) they give the sum-rules (5.1.15).

An important feature of the Polyakov block coefficients (or analytic functional actions) is the orthogonality property

$$\begin{aligned} \alpha_n(\Delta_m, \Delta_{\phi}) &= \delta_{mn} & \partial_{\Delta} \alpha_n(\Delta_m, \Delta_{\phi}) &= c_n \delta_{m0} \\ \beta_n(\Delta_m, \Delta_{\phi}) &= 0 & \partial_{\Delta} \beta_n(\Delta_m, \Delta_{\phi}) &= \delta_{mn} + d_n \delta_{m0}. \end{aligned} \quad (5.1.16)$$

These are consequences of the dualities with the basis vectors $F_{\Delta_n}, \partial_{\Delta} F_{\Delta_n}$ mentioned in the paragraph above. All β sum-rules also satisfy $\beta_n(\Delta = 0) = 0$ so they vanish identically on the GFF spectrum. The α 's also trivialise, but $\alpha_n(\Delta = 0) = -C_{\Delta_n}$ so that each α_n sum-rule corresponds to the solution of the n -th GFF OPE coefficient.

Furthermore since the sum-rules have mostly double zeroes, they let us solve any small deformation around GFF also very easily. To be concrete, if we consider a deformation

$$\Delta_{(n)} = \Delta_n + \gamma_n, \quad C_{\Delta_{(n)}}^2 = C_{\Delta_n}^2 + \delta_n. \quad (5.1.17)$$

and assume γ_0 to be a fixed parameter. Then all other γ_n and δ_n 's could be obtained as

$$\delta_n = -\gamma_0 c_n, \quad \gamma_n = -\gamma_0 d_n. \quad (5.1.18)$$

The fact that we chose γ_0 as the deformation parameter goes back to our choice of the Polyakov block. We chose c_{Δ} in (5.1.10) such that there is no β_0 sum-rule, so we

cannot solve γ_0 from the sum-rules. Alternatively we could have chosen to eliminate any other sum-rule. The justification of this is that given a Regge-bounded crossing solution $\mathcal{G}_0(\chi)$ we can define, with an arbitrary parameter λ , another correlator $\mathcal{G}_\lambda(\chi) = \mathcal{G}_0(\chi) + \lambda W^{\text{con}}(\chi)$ which is still Regge-bounded and crossing symmetric.

5.1.3 Mellin approach in one dimension

In this subsection we will review Mellin transforms of CFT correlators, especially Witten diagrams in CFT Mellin space. We will show how the Mellin space is a natural framework to set up the Polyakov bootstrap.

The inverse Mellin transform of a correlation function of four identical scalars is defined by

$$\mathcal{G}(u, v) = \int_{-i\infty}^{i\infty} \frac{u^{\Delta_\phi - s_{12}} v^{-s_{14}}}{(2\pi i)^2} M(s_{12}, s_{14}) \Gamma(s_{12})^2 \Gamma(s_{14})^2 \Gamma(\Delta_\phi - s_{12} - s_{14})^2. \quad (5.1.19)$$

In 1d we simply set $u = \chi^2$ and $v = (1 - \chi)^2$ and continue to work with two Mellin variables.² Here $M(s_{12}, s_{14})$ is the Mellin transform or Mellin amplitude of $\mathcal{G}(\chi)$ which we will denote as $\mathcal{M}[\mathcal{G}](s_{12}, s_{14}) = M(s_{12}, s_{14})$.

Now let us discuss Witten diagrams in AdS_2 . To define them let us consider a free bosonic theory with cubic and quartic interactions as follows

$$S_{\text{AdS}_2} = \int d^2x \sqrt{g} \left[(\partial\Phi)^2 + m_{\Delta_\phi}^2 \Phi^2 + (\partial\mathcal{O})^2 + m_\Delta^2 \mathcal{O}^2 + g_1 \Phi^2 \mathcal{O} + g_2 \Phi^4 \right]. \quad (5.1.20)$$

In this subsection we only consider Witten diagrams with four external Φ 's on the boundary. We get an exchange diagrams using the cubic vertex. It exchanges the scalar bulk operator $\mathcal{O}$ with mass $m^2 = \Delta(\Delta - 1)$, and is proportional to g_1^2 . The contact diagram is obtained from the quartic vertex and is proportional to g_2 .

We will ignore the vertices g_1 and g_2 below and review only the kinematical part. An s -channel exchange diagram has the Mellin transform $\mathcal{M}[W_\Delta^{(s)}] = M_\Delta^{(s)}$ given by [222–224]

$$M_\Delta^{(s)}(s_{12}, s_{14}) = \sum_{m=0}^{\infty} \frac{\mathcal{Q}_m^{(\Delta_\phi)}}{s_{12} + \frac{\Delta}{2} - \Delta_\phi + m}. \quad (5.1.21)$$

The $\mathcal{Q}_m^{(\Delta_\phi)}$ is defined in App. 5.A.1. Notice that it is independent of s_{14} which is a property of the spin 0 exchange diagram in s channel. The Mellin transforms of t and u channel exchange diagrams are readily obtained from $M_\Delta^{(s)}$ by switching $s_{12} \leftrightarrow s_{14}$

²There is no ambiguity since we always compute the Mellin integrals in a way that is compatible with the OPE. This formalism has the advantage of simplifying the form of the Mellin amplitudes, unlike an intrinsically one-dimensional Mellin formalism [221].

and $s_{12} \rightarrow \Delta_\phi - s_{12} - s_{14}$ respectively:

$$M_\Delta^{(t)}(s_{12}, s_{14}) = \sum_{m=0}^{\infty} \frac{\mathcal{Q}_m^{(\Delta_\phi)}}{s_{14} + \frac{\Delta}{2} - \Delta_\phi + m}, \quad (5.1.22)$$

$$M_\Delta^{(u)}(s_{12}, s_{14}) = \sum_{m=0}^{\infty} \frac{\mathcal{Q}_m^{(\Delta_\phi)}}{\frac{\Delta}{2} + m - s_{12} - s_{14}}. \quad (5.1.23)$$

The contact diagram Mellin transform $\mathcal{M}[W^{\text{con}}] = M^{\text{con}}$ is simply a constant. We may normalise it to be unity

$$M^{\text{con}}(s_{12}, s_{14}) = 1. \quad (5.1.24)$$

In 1d, the poles in $M_\Delta^{(s)}$ under the inverse Mellin integral (5.1.19) gives the block $G_\Delta(\chi)$ from the decomposition of s-channel diagram (see (5.1.11)). Taking the poles in s_{12} in the negative direction is the correct thing to do when we want a small χ expansion, as one has to close the integral contour at large negative s_{12} . This naturally corresponds to the s channel conformal block expansion.

The spurious blocks in that decomposition come from the s_{12} poles of the Γ -functions in the integral measure of inverse Mellin transformation. This way the crossed channel and contact diagram Mellin amplitudes also follow the decompositions of (5.1.11) and (5.1.12) respectively. We come back to them in a moment.

We can now write down the Mellin transform of a Polyakov block. For 4-point identical scalar correlator the Polyakov block in Mellin space is as follows

$$P_\Delta(s_{12}, s_{14}) = \sum_{m=0}^{\infty} \left[\frac{\mathcal{Q}_m^{(\Delta_\phi)}}{s_{12} + \frac{\Delta}{2} - \Delta_\phi + m} + \frac{\mathcal{Q}_m^{(\Delta_\phi)}}{s_{14} + \frac{\Delta}{2} - \Delta_\phi + m} + \frac{\mathcal{Q}_m^{(\Delta_\phi)}}{\frac{\Delta}{2} + m - s_{12} - s_{14}} \right] + c_\Delta. \quad (5.1.25)$$

Let us now discuss why $P_\Delta(s_{12}, s_{14})$ is the most natural analogue of a conformal block in Mellin space. First let us point out that it is not possible to write a Mellin amplitude whose poles give only $G_\Delta(\chi)$ since it is not possible to close s_{12} contour with those poles alone. So s channel diagram $M_\Delta^{(s)}$ is the closest and simplest thing to a conformal block in Mellin space although it generates the spurious blocks in addition. The addition of $M_\Delta^{(t)}$ and $M_\Delta^{(u)}$ to it makes it crossing symmetric without adding any new spurious blocks.

The addition of a constant c_Δ in the Polyakov block $P_\Delta(s_{12}, s_{14})$ preserves the crossing symmetry and the spurious block content. It turns out that (5.1.25) is the most general expression we can write that does not grow at large s_{12} . This is the Mellin version of Regge boundedness condition. The general large χ behavior in Mellin space becomes [208]

$$\mathcal{G}(\chi) \stackrel{\chi \rightarrow \infty}{\sim} \chi^{2j-1} \implies \mathcal{M}[\mathcal{G}] = M(s_{12}) \stackrel{s_{12} \rightarrow \infty}{\sim} s_{12}^{2j}. \quad (5.1.26)$$

A bounded correlator in Regge limit requires $j \leq \frac{1}{2}$ or Mellin amplitude growing at most linearly in s_{12} . For the identical external scalar case the only completely crossing symmetric polynomial (contact diagram) with this property is a constant. The exchange diagrams also share the same property as can be checked easily.

As discussed in Section 5.1.2 the constant c_Δ is fixed by eliminating the coefficient of a spurious block. we may of course relax the Regge boundedness condition and require a softer $M \sim s_{12}^{2j}$, $j > \frac{1}{2}$ behavior for large s_{12} . We then have to add more crossing symmetric polynomials. So if there are n independent polynomials we have to fix them by eliminating n sum-rules. This represents the freedom of deforming a crossing solution by n independent contact diagrams.

Polyakov block rules: Let us summarise the rules for writing a 4-point Polyakov block $P_\Delta(s_{12}, s_{14})$ for a correlator $\mathcal{G}(\chi)$ in Mellin space are as follows:

- $P_\Delta(s_{12}, s_{14})$ is completely crossing symmetric in the Mellin variables s_{12}, s_{14} .
- $P_\Delta(s_{12}, s_{14})$ has simple poles in s_{12} that reproduce $G_\Delta(\chi)$ and regular terms (polynomials).
- The regular terms should respect the Regge boundedness of $\mathcal{G}(\chi)$.
- The number n of independent regular terms has to fixed by eliminating n sum-rules.

These rules can be established from the consistency with the analytic functional method discussed in the previous subsection. But a more direct way to derive $P_\Delta(s_{12}, s_{14})$ is from the crossing symmetric dispersion relation (CSDR) shown in [215, 216]. The CSDR is a dispersion relation that manifests the crossing symmetry of a function (Mellin amplitude in our case), with a special dispersive kernel, at the cost of locality or Regge boundedness. One has to further tune it to restore the missing properties - then CSDR reduces to a sum of the general spin of a Polyakov block. The spin 0 sector of this sum, relevant in 1d, corresponds to our $P_\Delta(s_{12}, s_{14})$.

Computation of sum-rules: Finally let us discuss the computation of the coefficients $\alpha(\Delta, \Delta_\phi)$ and $\beta(\Delta, \Delta_\phi)$ from the decomposition (5.1.13) of a Polyakov block in Mellin space.

Let us start from the inverse Mellin transform (5.1.19). If we sum the residues

of $s_{12} = -n$ we get

$$\sum_n \int \frac{ds_{14}}{2\pi i} \chi^{2\Delta_\phi+2n} (1-\chi)^{2s_{14}} \frac{\Gamma(s_{14})^2 \Gamma(n+\Delta_\phi-s_{14})^2}{(n!)^2} (2\log(\chi) M(-n, s_{14}) \\ 2M(-n, s_{14}) (\psi(n+\Delta_\phi+s_{14}) - \psi(n+1)) - \partial_s M(s=-n, s_{14})) . \quad (5.1.27)$$

For simplicity let us focus on the coefficient of $\chi^{2\Delta_\phi} \log(\chi)$ which gives $b_0(\Delta, \Delta_\phi)$ i.e. the coefficient of $\partial_\Delta G_{\Delta=2\Delta_\phi}(\chi)$ in the notation from (5.1.11), (5.1.12). The easiest case is that of the contact diagram because the Mellin amplitude is unity. This gives an s_{14} integral which can be carried out using Barnes' first lemma:

$$\chi^{2\Delta_\phi} \log(\chi) \int_{-i\infty}^{i\infty} \frac{ds_{14}}{2\pi i} 2\Gamma(s_{14})^2 \Gamma(\Delta_\phi - s_{14})^2 = \chi^{2\Delta_\phi} \log(\chi) \frac{2\Gamma(\Delta_\phi)^4}{\Gamma(2\Delta_\phi)} . \quad (5.1.28)$$

For the s -channel diagram the s_{14} integral is the same since $M^{(s)}$ is independent of s_{14} . So we only have the sum over m in (5.1.21) which is straightforward to carry out. For the t -channel it is harder since the $M^{(t)}$ has an s_{14} in the denominator and so the s_{14} integral cannot be evaluated using Barnes lemma. We show both of these cases in detail in App. 5.A.1.

5.2 Five-point Polyakov Bootstrap

In this section we derive the main results of the current chapter, the 5-point Polyakov bootstrap sum-rules. We begin by reviewing 5-point function kinematics in Section 5.2.1 and move on to construct the Polyakov blocks in Section 5.2.2. We systematically extract sum-rules from these blocks in Section 5.2.3 and describe general properties of the associated functionals, as well as provide some consistency checks in Section 5.2.4.

5.2.1 Setup and conventions

To streamline notation, we will be working with 5-point functions of identical operators

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \phi(x_5) \rangle = \mathcal{L}(x_i) \mathcal{G}(\chi_1, \chi_2) , \quad (5.2.1)$$

with the generalisation to external operators with different dimensions being straightforward. Here we defined the conformal covariant prefactor

$$\mathcal{L}(x_i) = \left(\frac{x_{24}}{x_{12}^2 x_{23} x_{34} x_{45}^2} \right)^\Delta , \quad (5.2.2)$$

as well as the two independent conformal cross-ratios

$$\chi_1 = \frac{x_{12} x_{34}}{x_{13} x_{24}} , \quad \chi_2 = \frac{x_{23} x_{45}}{x_{24} x_{35}} , \quad (5.2.3)$$

where we use the notation $x_{ij} \equiv |x_i - x_j|$. These 5-point conformal correlators can be decomposed into 5-point conformal blocks (note that we sum over *pairs* of operators)

$$\mathcal{G}(\chi_1, \chi_2) = \sum_{(\Delta_1, \Delta_2)} C_{\phi\phi\Delta_1} C_{\Delta_1\phi\Delta_2} C_{\phi\phi\Delta_2} G_{\Delta_1, \Delta_2}(\chi_1, \chi_2) \equiv \sum_{(\Delta_1, \Delta_2)} a_{\Delta_1, \Delta_2} G_{\Delta_1, \Delta_2}(\chi_1, \chi_2), \quad (5.2.4)$$

which are known explicitly in terms of a double power series [174]³

$$G_{\Delta_1, \Delta_2}(\chi_1, \chi_2) = \chi_1^{\Delta_1} \chi_2^{\Delta_2} \sum_{n_1, n_2=1}^{\infty} \frac{(\Delta_1)_{n_1} (\Delta_1 + \Delta_2 - \Delta_\phi)_{n_1+n_2} (\Delta_2)_{n_2}}{(2\Delta_1)_{n_1} (2\Delta_2)_{n_2}} \frac{\chi_1^{n_1}}{n_1!} \frac{\chi_2^{n_2}}{n_2!}. \quad (5.2.5)$$

These correlators satisfy several crossing relations which can be generated by a single cyclic permutation $x_i \rightarrow x_{i+1}$ where $x_6 \equiv x_1$. In terms of cross-ratios the crossing equation reads

$$\mathcal{G}(\chi_1, \chi_2) = \left(\frac{\chi_1^2 \chi_2}{(\chi_1 + \chi_2 - 1)^2} \right)^{\Delta_\phi} \mathcal{G}\left(\chi_2, 1 + \frac{\chi_2}{\chi_1 - 1}\right). \quad (5.2.6)$$

Disconnected correlator: To illustrate the points above let us consider the simplest 5-point correlator, the disconnected correlator, which is a sum of products of two- and three- point functions $\langle\phi\phi\rangle\langle\phi\phi\phi\rangle$. Such a correlator naturally appears for singlet operators in $\mathbb{Z}_2$ preserving perturbative theories or for charged operators in perturbative theories with cubic couplings where the $\mathbb{Z}_2$ symmetry is broken. This correlator contains $\binom{5}{2} = 10$ different permutations which contribute to three different sectors in the OPE corresponding to having single- or double-trace operators in the left and right exchanged channels. We can write

$$\mathcal{G}_{\text{discon}}(\chi_1, \chi_2) = C_{\phi\phi\phi} (\mathcal{G}_{\text{sing-Id}}(\chi_1, \chi_2) + \mathcal{G}_{\text{sing-doub}}(\chi_1, \chi_2) + \mathcal{G}_{\text{doub-doub}}(\chi_1, \chi_2)), \quad (5.2.7)$$

with

$$\mathcal{G}_{\text{sing-Id}}(\chi_1, \chi_2) = \chi_1^{\Delta_\phi} + \chi_2^{\Delta_\phi}, \quad (5.2.8)$$

$$\mathcal{G}_{\text{sing-doub}}(\chi_1, \chi_2) = \chi_1^{2\Delta_\phi} \chi_2^{\Delta_\phi} \left(\frac{1}{(1 - \chi_2)^{\Delta_\phi}} + \frac{1}{((1 - \chi_1)(1 - \chi_1 - \chi_2))^{\Delta_\phi}} \right) + (\chi_1 \leftrightarrow \chi_2), \quad (5.2.9)$$

$$\begin{aligned} \mathcal{G}_{\text{doub-doub}}(\chi_1, \chi_2) = \chi_1^{2\Delta_\phi} \chi_2^{2\Delta_\phi} & \left(\frac{1}{((1 - \chi_1)^2(1 - \chi_2))^{\Delta_\phi}} + \frac{1}{(1 - \chi_1 - \chi_2)^{\Delta_\phi}} \right. \\ & \left. + \frac{1}{((1 - \chi_1 - \chi_2)^2)^{\Delta_\phi}} + \frac{1}{((1 - \chi_1)(1 - \chi_2)^2)^{\Delta_\phi}} \right), \end{aligned} \quad (5.2.10)$$

³See also [139, 140, 176, 178] for relevant work on multi-point conformal blocks.

where we isolated different power-laws in the cross-ratio to make manifest the scaling dimensions of the exchanged operators. From this we can read off several families of OPE coefficients. It is easy to see directly from the structure of the OPE that

$$a_{\phi,1} = a_{1,\phi} = C_{\phi\phi\phi}, \quad a_{\phi,[\phi\phi]_{2n}} = a_{[\phi\phi]_{2n},\phi} = C_{\phi\phi\phi} (C_{\phi\phi[\phi\phi]_{2n}}^{\text{GFF}})^2. \quad (5.2.11)$$

Here $C_{\phi\phi[\phi\phi]_{2n}}^{\text{GFF}} = C_{\Delta_n}$ from (5.1.7). Furthermore, it is straightforward to expand $\mathcal{G}_{\text{doub-doub}}$ in blocks and read-off the product of OPE coefficients $a_{[\phi\phi]_{2n},[\phi\phi]_{2m}}$ from which we can infer the non-trivial OPE coefficients $C_{[\phi\phi]_{2n}\phi[\phi\phi]_{2m}}$. The first few are given by

$$a_{2\Delta_\phi,2\Delta_\phi} = 2 C_{\phi\phi\phi}, \quad a_{2\Delta_\phi+2,2\Delta_\phi} = 2 C_{\phi\phi\phi} \frac{\Delta_\phi^2(1+\Delta_\phi)}{1+4\Delta_\phi}, \quad (5.2.12)$$

$$a_{2\Delta_\phi+2,2\Delta_\phi+2} = C_{\phi\phi\phi} \frac{\Delta_\phi^2(1+\Delta_\phi)^2(2+\Delta_\phi(12+\Delta_\phi))}{2(1+4\Delta_\phi)^2}. \quad (5.2.13)$$

5.2.2 Polyakov Blocks

Let us now introduce the Polyakov blocks for 5-point functions. We will do this entirely in Mellin space following our discussions for four points in section 5.1.3.

Consider a correlation function $A(x_{ij})$ of n scalar operators with dimensions and coordinates labelled by Δ_i and x_i respectively ($i, j = 1, \dots, n$). Its inverse Mellin transform is given by

$$A(x_{ij}) = \int_{-i\infty}^{i\infty} [ds_{ij}] \prod_{i<j\leq n} \left(|x_{ij}|^{-s_{ij}} \Gamma(s_{ij}) \right) M(s_{ij}). \quad (5.2.14)$$

The Mellin variables s_{ij} are symmetric i.e. $s_{ij} = s_{ji}$. Furthermore they are constrained as follows

$$\sum_{i(\neq j)} s_{ij} = \Delta_j. \quad (5.2.15)$$

The notation $[ds_{ij}]$ for the differential element indicates that the above condition is being imposed.

For $n = 4$ and with identical scalars $\Delta_i = \Delta_\phi$ the Mellin integral reduces to (5.1.19) where the 4-point conformal prefactor has been factored out and the x_{ij} 's have reorganised into 4-point cross ratios. Notice that in that case one has only two independent Mellin variables corresponding to two cross ratios u, v in general d .

For $n = 5$ one has 10 variables s_{ij} (with $i < j$) among which five are independent. We may choose the independent ones to be $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$. Any other choice

$\{s_{pq}\}$ can be independent if p, q are consecutive pairs in a permutation of $1, 2, 3, 4, 5$. With our choice, (5.2.15) decides the form of the dependent variables as:

$$\begin{aligned} s_{13} &= \frac{\Delta_\phi}{2} - s_{12} - s_{23} + s_{45}, & s_{14} &= \frac{\Delta_\phi}{2} - s_{15} + s_{23} - s_{45}, & s_{24} &= \frac{\Delta_\phi}{2} + s_{15} - s_{23} - s_{34}, \\ s_{25} &= \frac{\Delta_\phi}{2} - s_{12} - s_{15} + s_{34}, & s_{35} &= \frac{\Delta_\phi}{2} + s_{12} - s_{34} - s_{45}. \end{aligned} \quad (5.2.16)$$

The forms of the dependent variables are such that the distances $|x_{ij}|$ rearrange into five cross ratios of the 5-point function in general d [141, 142]. If we further restrict to 1d by rewriting all position dependence in terms of the only two independent cross-ratios χ_1, χ_2 we get

$$\begin{aligned} \mathcal{G}(\chi_1, \chi_2) &= \int [ds_{ij}] \prod_{i < j \leq 5} (\Gamma(s_{ij})) f(\chi_1, \chi_2 | s_{ij}) M(s_{12}, s_{23}, s_{34}, s_{45}, s_{15}) \\ \text{where } f(\chi_1, \chi_2 | s_{ij}) &= \frac{(1 - \chi_1 - \chi_2)^{-2s_{15}} \chi_1^{2\Delta_\phi - 2s_{12}} \chi_2^{2\Delta_\phi - 2s_{45}}}{(1 - \chi_1)^{\Delta_\phi - 2s_{15} + 2s_{23} - 2s_{45}} (1 - \chi_2)^{\Delta_\phi - 2s_{12} - 2s_{15} + 2s_{34}}}. \end{aligned} \quad (5.2.17)$$

Here we have removed the 5-point prefactor, and we show $M(s_{ij})$ as a function of only the independent variables.

Notice that the 12-3-45 OPE channel, which corresponds to the expansion in small χ_1 and χ_2 , is controlled by the poles of s_{12} and s_{45} respectively. This is analogous to the s -channel expansion being controlled by s_{12} poles in the 4-point case. A general channel $ij-k-lm$ corresponds to poles in the pair of variables (s_{ij}, s_{lm}) . A pair with a repeated index (e.g. (s_{12}, s_{23})) cannot correspond to an OPE channel in this sense.

We will bootstrap $\mathcal{G}(\chi_1, \chi_2)$ with its Mellin transform $M(s_{ij})$. This means we will impose the following conditions:

Crossing symmetry: This corresponds to $M(s_{ij})$ remaining invariant under any permutation of $1, 2, 3, 4, 5$: the external operator labels. This can be stated explicitly as

$$M(s_{12}, s_{23}, s_{34}, s_{45}, s_{15}) \stackrel{2 \leftrightarrow 3}{=} M(s_{13}, s_{23}, s_{24}, s_{45}, s_{15}) = \dots (\text{all permutations}). \quad (5.2.18)$$

The first equality indicates that under $2 \leftrightarrow 3$ we should replace the functional dependence of M on s_{12} and s_{34} by $s_{13} = \frac{\Delta_\phi}{2} - s_{12} - s_{23} + s_{45}$ and $s_{24} = \frac{\Delta_\phi}{2} + s_{15} - s_{23} - s_{34}$ respectively.

Regge boundedness: Just like with four points, we restrict our attention to Mellin amplitudes which are polynomially bounded at infinity.⁴ Specifically we require:

$$M(s_{12}, s_{23}, s_{34}, s_{45}, s_{15}) \lesssim s_{12} s_{45}, \quad \text{as } s_{12}, s_{45} \gg 1. \quad (5.2.19)$$

⁴See also [225] for a detailed discussion on the Regge limit of 5-point functions in general d .

The limits on s_{12}, s_{45} are taken in no specific order and s_{23}, s_{34}, s_{15} are kept fixed. The pair (s_{12}, s_{45}) corresponds to the 12-3-45 channel. By crossing symmetry the same bound also exists for other channels i.e. large values of two variables (s_{ij}, s_{lm}) where the indices are not repeated.

Operator product expansion: The OPE guarantees that the correlator $\mathcal{G}(\chi_1, \chi_2)$ obeys the conformal block expansion (5.2.4). In Mellin space we implement the OPE by a Polyakov block expansion

$$M(s_{12}, s_{23}, s_{34}, s_{45}, s_{15}) = \sum_{\Delta_1, \Delta_2} a_{\Delta_1, \Delta_2} P_{\Delta_1, \Delta_2}(s_{12}, s_{23}, s_{34}, s_{45}, s_{15}). \quad (5.2.20)$$

The resulting $M(s_{ij})$ should be crossing symmetric, Regge bounded and compatible with the conformal block expansion in position space where we sum over $\{\Delta_1, \Delta_2\}$. As we show below this is the bootstrap condition on $\{\Delta_1, \Delta_2\}$ and a_{Δ_1, Δ_2} .

We will now lay down the rules for a 5-point Polyakov block, and we will elaborate on them further below.

- $P_{\Delta_1, \Delta_2}(s_{ij})$ is crossing symmetric i.e. invariant under all permutations of the external operators.
- $P_{\Delta_1, \Delta_2}(s_{ij})$ has joint simple poles in two variables (s_{12}, s_{45}) and their crossing symmetric counter-parts. The joint poles should produce only $G_{\Delta_1, \Delta_2}(\chi_1, \chi_2)$, but there can be spurious blocks from the Mellin measure.
- The ‘regular terms’ in $P_{\Delta_1, \Delta_2}(s_{ij})$ correspond to less singular terms i.e. isolated simple poles and polynomials. They should not add any new spurious blocks.
- The regular terms should respect the Regge boundedness of $\mathcal{G}(\chi_1, \chi_2)$. This amounts to satisfying (5.2.19).
- The number n of independent regular terms has to be fixed by eliminating n sum-rules.

These are simple extensions of the four points conditions from Section 5.1.3. The first bullet point simply reflects the crossing property of $M(s_{ij})$.

In the second point we demand that the Polyakov block has the simplest set of poles whose residues give a conformal block. A (non-crossing symmetric) Mellin amplitude that has this property is as follows:

$$M_{\Delta_1, \Delta_2}^{12-3-45}(s_{ij}) = \sum_{n_1, n_2=0}^{\infty} \frac{\mathcal{Q}_{n_1, n_2}(\Delta_1, \Delta_2 | \Delta_\phi)}{(s_{12} - (\Delta_\phi - \frac{\Delta_1}{2} - n_1))(s_{45} - (\Delta_\phi - \frac{\Delta_2}{2} - n_2))}. \quad (5.2.21)$$

where

$$\begin{aligned} \mathcal{Q}_{n_1, n_2}(\Delta_1, \Delta_2 | \Delta_\phi) = & \mathcal{N}_{\Delta_1, \Delta_2} \Gamma\left(\frac{\Delta_1 + \Delta_2 + \Delta_\phi - 1}{2}\right) \left[\frac{\Gamma\left(\frac{\Delta_1 - 1 + 2\Delta_\phi}{2}\right) \Gamma\left(\frac{\Delta_1 - 2\Delta_\phi + 2}{2}\right) \Gamma\left(\frac{\Delta_1 - \Delta_2 - \Delta_\phi + 2}{2}\right)}{2\Gamma(n_1 + 1) \Gamma\left(\frac{2n_1 + 2\Delta_1 + 1}{2}\right)} \right]^{n_1} \times (\Delta_1 \leftrightarrow \Delta_2) \\ & \times {}_3F_2\left(-n_1, -n_2, \frac{\Delta_1 + \Delta_2 + \Delta_\phi - 1}{2}; \frac{\Delta_2 - \Delta_1 + \Delta_\phi - 2n_1}{2}, \frac{\Delta_1 - \Delta_2 + \Delta_\phi - 2n_2}{2}; 1\right). \end{aligned} \quad (5.2.22)$$

The amplitude (5.2.21) is nothing but the Mellin transform of a 5-point double-exchange Witten diagram (see e.g. [224])⁵ of Figure 5.2.1.

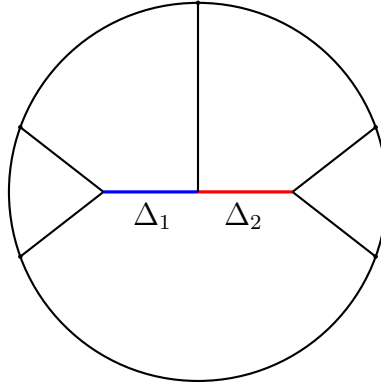


Figure 5.2.1: Double-exchange Witten diagram with a Δ_1 exchange in the 12 OPE and a Δ_2 exchange in the 45 OPE

The residues of the (s_{12}, s_{45}) poles correspond to the powers $\chi^{\Delta_1 + 2n_1} \chi^{\Delta_2 + 2n_2}$. The form of $\mathcal{Q}_{n_1, n_2}$ ensures that carrying out the integrals over s_{23}, s_{34}, s_{15} we recover the exact form of $G_{\Delta_1, \Delta_2}(\chi_1, \chi_2)$. The normalisation $\mathcal{N}_{\Delta_1, \Delta_2}$ is chosen such that the block comes with unit coefficient. The superscript 12-3-45 stands for the channel exchanges in the bulk.

However, there are also spurious blocks, which involve unphysical poles i.e. those from the $\Gamma(s_{ij})$ factors of (5.2.16). The spurious poles can be either in both of the variables (s_{12}, s_{45}) or in just one of them. In the latter case the other variable is at a ‘physical’ pole of (5.2.21), and one obtains blocks of the form $G_{\Delta_1, \Delta_{2\phi, n}}(\chi_1, \chi_2)$ or $G_{\Delta_{2\phi, n}, \Delta_2}(\chi_1, \chi_2)$, with $\Delta_{2\phi, n}$ being the following shorthand notation for the spurious dimension

$$\Delta_{2\phi, n} = 2\Delta_\phi + n. \quad (5.2.23)$$

(In this section we will use the above notation for double twist operator dimensions instead of Δ_n in order to not confuse with ‘physical dimensions’ Δ_1, Δ_2 .) Note that these particular spurious blocks may be present also for some of the crossed versions

⁵See also [226–229] for other studies of higher-point Witten diagrams.

of $M_{\Delta_1, \Delta_2}(s_{ij})$ but not all, e.g. we will get $G_{\Delta_1, \Delta_{2\phi, n}}$ by replacing $(s_{12}, s_{45}) \leftrightarrow (s_{12}, s_{34})$ in (5.2.21) but not with $(s_{12}, s_{45}) \leftrightarrow (s_{13}, s_{25})$. We discuss all the spurious blocks more elaborately in the next subsection.

Consider now the ‘regular’ terms in the third bullet point. These could be simple poles in a single Mellin variable or just a polynomial, but we do not want them to add more spurious blocks than the ones from $M_{\Delta_1, \Delta_2}^{12-3-45}(s_{ij})$. Such an amplitude having simple poles only in s_{12} is given as follows:

$$M_{\Delta}^{12-3-45}(s_{ij}) = \sum_{n=0}^{\infty} \frac{f_n(\Delta|\Delta_{\phi})}{(s_{12} - (\Delta_{\phi} - \frac{\Delta}{2} - n))} \quad (5.2.24)$$

where

$$f_n(\Delta|\Delta_{\phi}) = \frac{\mathcal{N}_{\Delta} \left(\frac{\Delta-3\Delta_{\phi}+4}{2} \right)_{n-1} \left(\frac{\Delta-2\Delta_{\phi}+4}{2} \right)_{n-1}}{n! \left(\Delta + \frac{3}{2} \right)_{n-1}}. \quad (5.2.25)$$

The expression in (5.2.24) is actually the Mellin transform of a 5-point Witten diagram that exchanges a single operator in the bulk corresponding to CFT dimension Δ (see e.g. [222]) as represented in Figure 5.2.2. The residue function $f_n(\Delta|\Delta_{\phi})$ is such

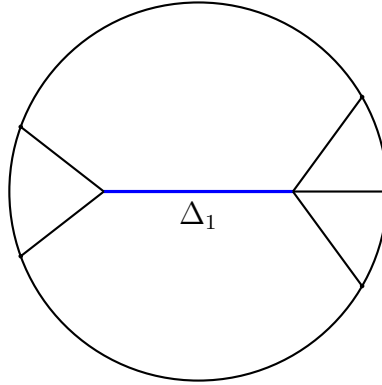


Figure 5.2.2: Single-exchange diagram with a Δ_1 exchange in the 12 OPE channel.

that it leads only to the spurious blocks $G_{\Delta, \Delta_{2\phi, m}}$ and not $G_{\Delta+1, \Delta_{2\phi, m}}$ or $G_{\Delta+2, \Delta_{2\phi, m}}$, etc, and the normalisation $\mathcal{N}_{\Delta}$ is chosen such that $G_{\Delta_{2\phi, 0}}$ has unit coefficient. In fact we can fix f_n by demanding that there is no spurious block $G_{\Delta+p, \Delta_{2\phi, 0}}$ except for $p = 0$.

It is easy to check that $M_{\Delta}(s_{ij})$ is Regge bounded in the sense of (5.2.19). There are also crossed versions of $M_{\Delta}(s_{ij})$ where we replace s_{12} with other variables but none of them can give rise to $G_{\Delta, \Delta_{2\phi, m}}$ spurious blocks. On the other hand, the replacement $s_{12} \leftrightarrow s_{45}$ leads to spurious $G_{\Delta_{2\phi, m}, \Delta}$ blocks.

The other allowed regular terms are polynomials, but the only one that is crossing symmetric and respects the Regge condition (5.2.19) is a constant (note that the

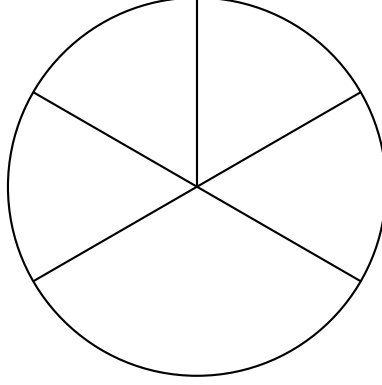


Figure 5.2.3: Five-point contact Witten diagram

crossing symmetric linear polynomial $\sum s_{ij}$ is a constant). We choose it to be unity:

$$M^{(\text{con})}(s_{ij}) = 1. \quad (5.2.26)$$

This is the Mellin transform of a contact Witten diagram with a 5-point vertex without derivatives (see [222]) as given in Figure 5.2.3.

All the Witten diagrams M_{Δ_1, Δ_2} , M_Δ and $M^{(\text{con})}$ can be derived from an AdS_2 scalar field theory with the following action:

$$S_{\text{AdS}_2} = S_{\text{free}}[\Phi, \Delta_\phi] + S_{\text{free}}[\mathcal{O}_1, \Delta_1] + S_{\text{free}}[\mathcal{O}_2, \Delta_2] + \int d^2x \sqrt{g} [g_1 \Phi^2 \mathcal{O}_1 + g_2 \Phi^2 \mathcal{O}_2 + g_3 \Phi \mathcal{O}_1 \mathcal{O}_2 + g_4 \Phi^3 \mathcal{O}_1 + g_5 \Phi^5], \quad (5.2.27)$$

where $S_{\text{free}}[\mathcal{O}, \Delta]$ denotes the free scalar action of an operator $\mathcal{O}$ with mass $m_\Delta^2 = \Delta(\Delta - 1)$. The double exchange diagram is composed with the vertices g_1, g_2, g_3 , the single exchange with g_1, g_4 (with the relabeling $\Delta_1 \rightarrow \Delta$) and the contact diagram with g_5 .

Having introduced all the ingredients let us write down the full structure of the 5-point Polyakov block (ignoring Δ_ϕ dependencies in the constants):

$$P_{\Delta_1, \Delta_2}(s_{ij}) = \sum_{n_1, n_2=0}^{\infty} \frac{\mathcal{Q}_{n_1, n_2}(\Delta_1, \Delta_2) + c_1(\Delta_1, \Delta_2)(f_n(\Delta_1)s_{45} + f_n(\Delta_2)s_{12}) + c_2(\Delta_1, \Delta_2)}{(s_{12} - (\Delta_\phi - \frac{\Delta_1}{2} - n_1))(s_{45} - (\Delta_\phi - \frac{\Delta_2}{2} - n_2))} + \text{all permutations} + (\Delta_1 \leftrightarrow \Delta_2). \quad (5.2.28)$$

There are 30 terms in total, one for each pair (s_{ij}, s_{lm}) with all indices different. Note that we consider (s_{ij}, s_{lm}) to be different from (s_{lm}, s_{ij}) .

The whole expression has only two independent undetermined constants $c_1(\Delta_1, \Delta_2)$

and $c_1(\Delta_1, \Delta_2)$. As indicated in the final bullet point rule for writing this Polyakov block, we will fix c_1 and c_2 by demanding that the overall coefficients of any two spurious blocks are zero. We will specify our choice for these two spurious blocks below.

5.2.3 Sum-rules

Let us write down all the spurious blocks that arise from all possible 5-point diagrams. Since we introduced the double-exchange, single-exchange and contact diagrams in Mellin space, let us denote their inverse transforms as $W_{\Delta_1, \Delta_2}(\chi_a)$, $W_{\Delta}(\chi_a)$ and $W^{\text{con}}(\chi_a)$ respectively. Let us also introduce the following shorthand notations for the different spurious exchanges:

$$\begin{aligned}\Delta_{3\phi, n} &= 3\Delta_{\phi} + n, \\ \Delta_{i, \phi, n} &= \Delta_i + \Delta_{\phi} + n, \quad i = 1, 2.\end{aligned}\tag{5.2.29}$$

With this, the block decomposition of double-exchange diagrams can be written as (denoting $\chi_a := \chi_1, \chi_2$)

$$\begin{aligned}W_{\Delta_1, \Delta_2}^{(i)}(\chi_a) &= \delta_{i, \text{dir}} G_{\Delta_1, \Delta_2}(\chi_a) \\ &- \sum_{n_1, n_2} \left[a_{n_1, n_2}^{2\phi, 2\phi, (i)} G_{\Delta_{2\phi, n_1}, \Delta_{2\phi, n_2}}(\chi_a) + a_{n_1, n_2}^{2\phi, 3\phi, (i)} G_{\Delta_{2\phi, n_1}, \Delta_{3\phi, n_2}}(\chi_a) + a_{n_1, n_2}^{3\phi, 2\phi, (i)} G_{\Delta_{3\phi, n_1}, \Delta_{2\phi, n_2}}(\chi_a) \right] \\ &- \sum_n \left[a_n^{1, 2\phi, (i)} G_{\Delta_1, \Delta_{2\phi, n}}(\chi_a) + a_n^{1, (1, \phi), (i)} G_{\Delta_1, \Delta_{1, \phi, n}}(\chi_a) \right. \\ &\quad \left. + a_n^{2\phi, 2, (i)} G_{\Delta_{2\phi, n}, \Delta_2}(\chi_a) + a_n^{(2, \phi), 2, (i)} G_{\Delta_{2, \phi, n}, \Delta_2}(\chi_a) \right].\end{aligned}\tag{5.2.30}$$

Here the superscript (i) refers to the channel of the exchange and ‘dir’ denotes the direct channel i.e. 12-3-45. Also for the direct channel all $a_{n_1, n_2}^{2\phi, 2\phi}$, $a_{n_1, n_2}^{2\phi, 3\phi}$, $a_{n_1, n_2}^{3\phi, 2\phi}$, $a_n^{1, 2\phi}$, $a_n^{2\phi, 2}$, $a_n^{1, (1, \phi)}$ or $a_n^{(2, \phi), 2}$ are zero for n_1, n_2 or n odd. The same spurious blocks are also absent when we add all the diagrams. One way to understand this is that, as we will see in the next subsection, the Polyakov block trivializes the disconnected correlator and deformations around it. So the disconnected correlator plays the role of GFF for the 5-point case, and it does not contain the blocks that correspond to the odd n ’s, as these do not occur in the OPE of identical scalars.

The coefficients $a_n^{1, 2\phi, (i)}$ and $a_n^{1, (1, \phi), (i)}$ are nonzero only for the channels $i = (12\text{-}3\text{-}45)$, $(12\text{-}4\text{-}35)$, $(12\text{-}5\text{-}34)$, while $a_n^{2\phi, (i), 2}$ and $a_n^{(2, \phi), 2, (i)}$ are nonzero only in the $i = (12\text{-}3\text{-}45)$, $(13\text{-}2\text{-}45)$, $(23\text{-}1\text{-}45)$ channels.

The direct channel block decomposition of single-exchange diagrams is as follows:

$$\begin{aligned}
W_{\Delta_1}^{(i)}(\chi_a) = & - \sum_{n_1, n_2} \left[b_{n_1, n_2}^{2\phi, 2\phi, (i)} G_{\Delta_{2\phi, n_1}, \Delta_{2\phi, n_2}}(\chi_a) + \left(b_{n_1, n_2}^{2\phi, 3\phi, (i)} G_{\Delta_{2\phi, n_1}, \Delta_{3\phi, n_2}}(\chi_a) + (2\phi \leftrightarrow 3\phi) \right) \right] \\
& - \sum_n \left[\delta_{i, (12-345)} \left(b_n^{1, 2\phi} G_{\Delta_1, \Delta_{2\phi, n}}(\chi_a) + b_n^{1, (1, \phi)} G_{\Delta_1, \Delta_{1, \phi, n}}(\chi_a) \right) \right. \\
& \left. \delta_{i, (45-123)} \left(b_n^{2\phi, 1} G_{\Delta_{2\phi, n}, \Delta_1}(\chi_a) + b_n^{(1, \phi), 1} G_{\Delta_{1, \phi, n}, \Delta_1}(\chi_a) \right) \right]. \quad (5.2.31)
\end{aligned}$$

In the above expression (i) refers to all the channels $ij-klm$ of a single operator exchange in the bulk. There are ten such channels. Notice that for $i = (12-345)$ one can have the spurious blocks involving a Δ_1 in the $\phi(x_1)\phi(x_2)$ OPE whereas $i = (45-123)$ involves spurious blocks with a Δ_1 in the $\phi(x_4)\phi(x_5)$ OPE.

The contact diagram decomposes into three types of spurious blocks:

$$W^{\text{con}} = - \sum_{n_1, n_2} \left[c_{n_1, n_2}^{2\phi, 2\phi, \text{con}} G_{\Delta_{2\phi, n_1}, \Delta_{2\phi, n_2}}(\chi_a) + \left(c_{n_1, n_2}^{2\phi, 3\phi, \text{con}} G_{\Delta_{2\phi, n_1}, \Delta_{3\phi, n_2}}(\chi_a) + (2\phi \leftrightarrow 3\phi) \right) \right]. \quad (5.2.32)$$

These three types of diagrams are all the ingredients one has in the Polyakov block, which we defined in Mellin space in (5.2.28). Let us denote its inverse Mellin transform by $P_{\Delta_1, \Delta_2}(\chi_1, \chi_2)$. Its block decomposition is given by

$$\begin{aligned}
P_{\Delta_1, \Delta_2}(\chi_a) = & G_{\Delta_1, \Delta_2}(\chi_a) + G_{\Delta_2, \Delta_1}(\chi_a) \\
& - \sum_{n_1, n_2} \left[\alpha_{n_1, n_2}^{2\phi, 2\phi} G_{\Delta_{2\phi, 2n_1}, \Delta_{2\phi, 2n_2}}(\chi_a) + \left(\alpha_{n_1, n_2}^{2\phi, 3\phi} G_{\Delta_{2\phi, 2n_1}, \Delta_{3\phi, 2n_2}}(\chi_a) + (2\phi \leftrightarrow 3\phi) \right) \right] \\
& - \sum_n \left[\alpha_n^{1, 2\phi} G_{\Delta_1, \Delta_{2\phi, 2n}}(\chi_a) + \alpha_n^{1, (1, \phi)} G_{\Delta_1, \Delta_{1, \phi, n}}(\chi_a) \right. \\
& \left. + \alpha_n^{2\phi, 1} G_{\Delta_{2\phi, 2n}, \Delta_1}(\chi_a) + \alpha_n^{(1, \phi), 1} G_{\Delta_{1, \phi, n}, \Delta_1}(\chi_a) + (\Delta_1 \leftrightarrow \Delta_2) \right]. \quad (5.2.33)
\end{aligned}$$

We will refer to the α coefficients as functionals since they will describe the sum-rules below. Let us write them in terms of the individual Witten diagram coefficients defined before. For convenience we define the sets $I' = \{12-3-45, 12-4-35, 12-5-34\}$, $I = \{\text{all double exchange channels}\}$, and $J = \{\text{all single exchange channels}\}$. Then we have

$$\alpha_n^{1, 2\phi} = \sum_{i \in I'} a_{2n}^{1, 2\phi, (i)} + c_1 b_{2n}^{1, 2\phi, (12-345)}. \quad (5.2.34)$$

$$\alpha_n^{1, (1, \phi)} = \sum_{i \in I'} a_n^{1, (1, \phi), (i)} + c_1 b_n^{1, (1, \phi), (12-345)}. \quad (5.2.35)$$

$$\alpha_n^{2\phi, 2\phi} = \sum_{i \in I} a_{2n}^{2\phi, 2\phi, (i)} + c_1 \sum_{j \in J} b_{2n}^{2\phi, 2\phi, (j)} + c_2 c_{2n}^{2\phi, 2\phi}. \quad (5.2.36)$$

$$\alpha_n^{2\phi, 3\phi} = \sum_{i \in I} a_{2n}^{2\phi, 3\phi, (i)} + c_1 \sum_{j \in J} b_{2n}^{2\phi, 3\phi, (j)} + c_2 c_{2n}^{2\phi, 3\phi}. \quad (5.2.37)$$

The other coefficients like $a_n^{2,2\phi}, a_n^{1,(1,\phi)}$, etc that are related by swapping $x_1, x_2 \leftrightarrow x_4, x_5$ or $\Delta_1 \leftrightarrow \Delta_2$ are defined analogously. Recall that the coefficients c_1 and c_2 were introduced as undetermined constants coming with the ‘regular terms’ in the definition of the Polyakov block (5.2.28). We specify how to fix them at the end of this subsection.

The 5-point Polyakov bootstrap equation is given by

$$\boxed{\sum_{\Delta_1, \Delta_2} a_{\Delta_1, \Delta_2} G_{\Delta_1, \Delta_2}(\chi_1, \chi_2) = \sum_{\substack{\Delta_1, \Delta_2 \\ \text{pairs}}} a_{\Delta_1, \Delta_2} P_{\Delta_1, \Delta_2}(\chi_1, \chi_2)} \quad (5.2.38)$$

This implies that every family of spurious blocks must separately be set to zero, leading to multiple families of sum-rules. Let us discuss them one by one.

The (1, 2 ϕ) sum-rule: This comes from eliminating $G_{\Delta_1, \Delta_{2\phi, 2n}}$ blocks. It is given by

$$\boxed{\sum_{\Delta_2} a_{\Delta_1, \Delta_2} \alpha_n^{1, 2\phi}(\Delta_1, \Delta_2) = 0.} \quad (5.2.39)$$

The (1, 2 ϕ) sum-rules fix an operator $\mathcal{O}_{\Delta_1}(x)$ in the $\phi(x_1)\phi(x_2)$ OPE and sum over all the operators $\mathcal{O}_{\Delta_2}(y)$ in the $\phi(x_4)\phi(x_5)$ OPE. It is clear that the sum-rules coming from the vanishing of $G_{\Delta_{2\phi, 2n}, \Delta_1}$, $G_{\Delta_2, \Delta_{2\phi, 2n}}$ and $G_{\Delta_{2\phi, 2n}, \Delta_2}$ are all equivalent to (5.2.39) since they merely correspond to exchange of external coordinates $(x_1, x_2) \leftrightarrow (x_4, x_5)$ and/or dimensions $\Delta_1 \leftrightarrow \Delta_2$.

The functional $\alpha_n^{1, 2\phi}$ involves coefficients from three double-exchange diagrams and one single exchange diagram. We can think of the ‘1, 2 ϕ ’ conditions as a 4-point Polyakov bootstrap sum-rule, for the correlator

$$\langle \mathcal{O}_{\Delta_1}(x)\phi(x_3)\phi(x_4)\phi(x_5) \rangle = \sum_{\Delta_2} \lambda_{\phi 12} \lambda_{\phi \phi 2} G_{\Delta_2}^{\Delta_1 \Delta_\phi \Delta_\phi \Delta_\phi}(x, x_3, x_4, x_5). \quad (5.2.40)$$

This has three external operator which are identical (ϕ) and one which is different ($\mathcal{O}_{\Delta_1}$). Here $\lambda_{\phi 12}$, $\lambda_{\phi \phi 2}$ denote the OPE coefficients in the $\phi\mathcal{O}_{\Delta_1}$ and $\phi\phi$ OPE’s respectively. Notice that this is crossing symmetric in all three channels. Therefore, its Polyakov blocks will be defined similarly to (5.1.10) i.e. three exchange and a contact Witten diagram, with the difference that the diagrams will have one non-identical external leg of dimension Δ_1 . Unlike (5.1.13), the corresponding spurious blocks in this case will be $G_{\Delta_{2\phi, n}}$ and $G_{\Delta_{1, \phi, n}}$. The resulting sum-rules have been explored in [230]. Our equation (5.2.39) is the same as the one there, specifically the sum-rule for eliminating $G_{\Delta_{2\phi, n}}$.

The $(1, (1, \phi))$ sum-rule: This follows from eliminating the $G_{\Delta_1, \Delta_1, \phi, n}$ blocks. It is given by

$$\boxed{\sum_{\Delta_2} a_{\Delta_1, \Delta_2} \alpha_n^{1, (1, \phi)}(\Delta_1, \Delta_2) = 0.} \quad (5.2.41)$$

Just like the previous sum-rule, here we fix one dimension Δ_1 and sum over the other dimension Δ_2 in the 5-point OPE. As before, the functional $\alpha_n^{1, (1, \phi)}$ involves three double exchange and a single-exchange diagram. So it is equivalent to the Polyakov bootstrap sum-rule for $\langle \mathcal{O}_{\Delta_1} \phi \phi \phi \rangle$, and in this case corresponds to the vanishing of $G_{\Delta_1, \phi, n}$ blocks. Once again the condition (5.2.41) automatically eliminates $G_{\Delta_1, \phi, n, \Delta_1}$, $G_{\Delta_2, \Delta_2, \phi, n}$ and $G_{\Delta_2, \phi, n, \Delta_2}$ by appropriate swapping of coordinates and dimensions.

The $(2\phi, 2\phi)$ sum-rule: This corresponds to the vanishing of $G_{\Delta_{2\phi, 2n_1}, \Delta_{2\phi, 2n_2}}$ blocks, and given by

$$\boxed{\sum_{(\Delta_1, \Delta_2)} a_{\Delta_1, \Delta_2} \alpha_{n_1, n_2}^{2\phi, 2\phi}(\Delta_1, \Delta_2) = 0.} \quad (5.2.42)$$

The (Δ_1, Δ_2) indicates the sum should be over all such distinct pairs. The functional $\alpha_{n_1, n_2}^{2\phi, 2\phi}$ involves all the channels of double- and single-exchange diagrams as well as the contact diagram.

This is a very nontrivial sum-rule since we have to sum over both Δ_1 and Δ_2 in the 5-point OPE. This cannot be derived from any individual 4-point function. Rather it can be thought of as a bootstrap condition on an infinite number of correlators $\langle \mathcal{O} \phi \phi \phi \rangle$ or an infinite number of OPEs $\mathcal{O} \times \phi$ for all $\mathcal{O} \in \phi \times \phi$.

The $(2\phi, 3\phi)$ sum-rule: This sum-rule comes from eliminating $G_{\Delta_{2\phi, 2n_1}, \Delta_{3\phi, 2n_2}}$ blocks. It is given by

$$\boxed{\sum_{(\Delta_1, \Delta_2)} a_{\Delta_1, \Delta_2} \alpha_{n_1, n_2}^{2\phi, 3\phi}(\Delta_1, \Delta_2) = 0.} \quad (5.2.43)$$

Like the previous sum-rule the sum is over all pairs (Δ_1, Δ_2) and the functional $\alpha_{n_1, n_2}^{2\phi, 3\phi}$ involves all the 5-point Witten diagrams.

Once again since it involves sum over both Δ_1 and Δ_2 the $(2\phi, 3\phi)$ sum-rule cannot be obtained from a single 4-point function, and is a collective condition on an infinite number of OPEs $\mathcal{O} \times \phi$. Also by the exchange of $x_1, x_2 \leftrightarrow x_4, x_5$ we can see that vanishing of the blocks $G_{\Delta_{3\phi, n_1}, \Delta_{2\phi, n_2}}$ imposes a sum-rule equivalent to (5.2.43).

Fixing c_1 and c_2 :

Finally let us discuss how to fix the constants multiplying the ‘regular terms’ in the Polyakov block. We do this by eliminating two of the sum-rules described above, i.e.

by setting to zero two functionals which depend on c_1 and c_2 . One possible choice is:

$$\alpha_0^{1,2\phi}(\Delta_1, \Delta_2) = 0, \quad (5.2.44)$$

$$\alpha_0^{2\phi,3\phi}(\Delta_1, \Delta_2) = 0. \quad (5.2.45)$$

This is a simple choice since the functional $\alpha_0^{1,2\phi}$ does not depend on c_2 , so (5.2.44) directly fixes c_1 . Note that the $\alpha_n^{1,2\phi}$ family is easy to evaluate since it depends on only four Witten diagrams, and $n = 0$ is the simplest one. Since $\alpha_0^{2\phi,3\phi}$ depends on both c_1, c_2 we can use (5.2.45) to fix c_2 .

There are, of course, other choices with functionals that involve both c_1 and c_2 . E.g. having $\alpha_0^{1,(1,\phi)} = 0$ and $\alpha_0^{2\phi,2\phi} = 0$ would work and is as simple as (5.2.44) and (5.2.45). However, setting $\alpha_0^{2\phi,2\phi} = 0$ and $\alpha_0^{2\phi,3\phi} = 0$, or $\alpha_0^{2\phi,3\phi} = 0$ and $\alpha_2^{2\phi,3\phi} = 0$ would be more cumbersome to implement since these functionals individually involve all the Witten diagrams.

5.2.4 Structure of functionals

We will now discuss the orthogonality properties of the 5-point Polyakov bootstrap functionals. These are analogous to the ones we showed in (5.1.16) for four points. The simple functionals $\alpha^{1,2\phi}$ and $\alpha^{1,(1,\phi)}$ have the following properties

$$\begin{aligned} \alpha_n^{1,2\phi}(\Delta_1, \Delta_2 = \Delta_{2\phi,m}) &= \delta_{mn} - d_n^{(1)} \delta_{m0}, & \alpha_n^{1,2\phi}(\Delta_1, \Delta_2 = \Delta_{1,\phi,m}) &= 0 \\ \alpha_n^{1,(1,\phi)}(\Delta_1, \Delta_2 = \Delta_{2\phi,m}) &= -d_n^{(2)} \delta_{m0}, & \alpha_n^{1,(1,\phi)}(\Delta_1, \Delta_2 = \Delta_{1,\phi,m}) &= \delta_{mn}. \end{aligned} \quad (5.2.46)$$

Since these functionals are equivalent to those from the correlator $\langle \mathcal{O}_{\Delta_1} \phi \phi \phi \rangle$, (5.2.46) are the same orthogonality properties found from the 4-point Polyakov block [230] for that correlator. Note that $d_n^{(1)}$ and $d_n^{(2)}$ are related to the constant c_1 in the Polyakov block and the respective spurious block coefficient of the single-exchange diagram. E.g. $d_n^{(1)} = b_n^{1,2\phi}$ and $d_n^{(2)} = b_n^{1,(1,\phi)}$ in our choice of normalisation and fixing of c_1 .

The orthogonality properties of $\alpha^{2\phi,2\phi}$ and $\alpha^{2\phi,3\phi}$ are more interesting. They are as follows

$$\begin{aligned} \alpha_{n_1, n_2}^{2\phi,2\phi}(\Delta_{2\phi, m_1}, \Delta_{2\phi, m_2}) &= \delta_{n_1 m_1} \delta_{n_2 m_2}, & \alpha_{n_1, n_2}^{2\phi,2\phi}(\Delta_{2\phi, m_1}, \Delta_{3\phi, m_2}) &= d^{(3)} \delta_{0 m_1} \delta_{0 m_2}, \\ \alpha_{n_1, n_2}^{2\phi,3\phi}(\Delta_{2\phi, m_1}, \Delta_{2\phi, m_2}) &= 0, & \alpha_{n_1, n_2}^{2\phi,3\phi}(\Delta_{2\phi, m_1}, \Delta_{3\phi, m_2}) &= \delta_{n_1 m_1} \delta_{n_2 m_2} \\ & & &+ d^{(4)} \delta_{0 m_1} \delta_{0 m_2}. \end{aligned} \quad (5.2.47)$$

The constants $d_n^{(3)}$ and $d_n^{(4)}$ are related to both c_1 and c_2 in the Polyakov block and the respective coefficients of $G_{\Delta_{2\phi, n_1}, \Delta_{2\phi, n_2}}$ and $G_{\Delta_{2\phi, n_1}, \Delta_{3\phi, n_2}}$ of single-exchange and contact diagrams. Seen as functions of two variables (Δ_1, Δ_2) , the functionals have

a rather non-trivial additional structure, as they exhibit co-dimension 1 loci where they vanish. Indeed, it is sufficient for one of the exchanged dimensions to be a double- or triple twist value, for the functional to vanish with an arbitrary value of the other dimension. While we observe this experimentally, we leave a more rigorous understanding of this property for future work.

Since they trivialise on the double-twist and triple-twist dimensions, the functionals are ideal to bootstrap certain special correlation functions. We will now discuss some of these examples.

Bootstrapping the disconnected correlator

Consider the disconnected correlator that we introduced in Section 5.2.1. It has the following classes of operator pairs in the OPE

$$\begin{aligned} (\phi, 1) : \quad \Delta_1 = \Delta_\phi, \quad \Delta_2 = 0 & \quad a_{\Delta_1, \Delta_2} : a_{\phi, 1} = C_{\phi\phi\phi}, \\ (\phi, [\phi\phi]_n) : \quad \Delta_1 = \Delta_\phi, \quad \Delta_2 = \Delta_{2\phi, n} & \quad a_{\Delta_1, \Delta_2} : a_{\phi, [\phi\phi]_n} = C_{\phi\phi\phi} (C_{\phi\phi[\phi\phi]_n}^{\text{GFF}})^2, \\ ([\phi\phi]_{n_1}, [\phi\phi]_{n_2}) : \quad \Delta_1 = \Delta_{2\phi, n_1}, \quad \Delta_2 = \Delta_{2\phi, n_2} & \quad a_{\Delta_1, \Delta_2} : a_{[\phi\phi]_{n_1}, [\phi\phi]_{n_2}}. \end{aligned} \quad (5.2.48)$$

The first few $a_{[\phi\phi]_{n_1}, [\phi\phi]_{n_2}}$ coefficients were given in (5.2.12). We will show how our sum-rules trivialise on this set of OPE data by using two of them explicitly.

Using $(1, 2\phi)$ functionals: First we will use the n -th $(1, 2\phi)$ sum-rule, given by (5.2.39). In that equation one of the exchanged dimensions i.e. Δ_1 is fixed. Let us take $\Delta_1 = \Delta_\phi$. We have

$$C_{\phi\phi\phi} \alpha_n^{1, 2\phi}(\Delta_\phi, 0) + \sum_m a_{\phi, [\phi\phi]_m} \alpha_n^{1, 2\phi}(\Delta_\phi, \Delta_{2\phi, m}) = 0 \quad (5.2.49)$$

The sum in the second term reduces to $a_{\phi, [\phi\phi]_n} - a_{\phi, [\phi\phi]_0} b_n^{1, 2\phi}(\Delta_\phi)$ using (5.2.46). Now the term $\alpha_n^{1, 2\phi}(\Delta_\phi, 0)$ comes from a Polyakov block $P_{\Delta_\phi, 0}$, specifically from double-exchange diagrams $W_{\Delta_\phi, 0}^{(i)}$ and single-exchange $W_{\Delta_\phi}^{(j)}$. In the direct channel one has

$$W_{\Delta_\phi, 0}^{12-3-45}(\chi_1, \chi_2) = \mathcal{L}(x_i)^{-1} C_{\phi\phi\phi}^{-1} \langle \phi(x_1) \phi(x_2) \phi(x_3) \rangle \langle \phi(x_4) \phi(x_5) \rangle. \quad (5.2.50)$$

where the exchange of the ϕ and the identity correspond to the 3-point function and the 2-point function factors, respectively. As discussed previously, the disconnected correlator is a sum of products of 2- and 3-point factors. This means that the double-exchange contribution to the $(1, 2\phi)$ functional (5.2.49) is nothing but the OPE coefficient $-a_{\phi, [\phi\phi]_n}$ itself. For a similar reason the constant $c_1 = -a_{\phi, [\phi\phi]_0}$. In other words we must have

$$C_{\phi\phi\phi} \alpha_n^{1, 2\phi}(\Delta_\phi, 0) = -C_{\phi\phi\phi} (C_{\phi\phi[\phi\phi]_n}^{\text{GFF}})^2 + C_{\phi\phi\phi} (C_{\phi\phi[\phi\phi]_0}^{\text{GFF}})^2 b_n^{1, 2\phi}(\Delta_\phi). \quad (5.2.51)$$

Therefore the equation (5.2.49) is trivially satisfied.

Using $(2\phi, 2\phi)$ functionals: Now consider the $(2\phi, 2\phi)$ sum-rule given by (5.2.42), specialised to the case $n_1 = n_2 = 0$ for simplicity. This sum-rule runs over all the operators in (5.2.48) and we hence have the equation

$$\begin{aligned} C_{\phi\phi\phi} \alpha_{0,0}^{2\phi,2\phi}(\Delta_\phi, 0) + \sum_n a_{\phi, [\phi\phi]_n} \alpha_{0,0}^{2\phi,2\phi}(\Delta_\phi, \Delta_{2\phi,n}) \\ + \sum_{n,m} a_{[\phi\phi]_n, [\phi\phi]_m} \alpha_{0,0}^{2\phi,2\phi}(\Delta_{2\phi,n}, \Delta_{2\phi,m}) = 0. \end{aligned} \quad (5.2.52)$$

In general, the term $\alpha_{0,0}^{2\phi,2\phi}(\Delta_1, \Delta_2)$ is cumbersome to derive. This is because it involves the coefficient of $G_{\Delta_{2\phi,0}, \Delta_{2\phi,0}}$ and $G_{\Delta_{2\phi,0}, \Delta_{3\phi,0}}$ (to fix c_2) from all the Witten diagrams, as well as those corresponding to $G_{\Delta_{2\phi,0}, \Delta_2}$ (to fix c_1) from four diagrams. The evaluation of some of these coefficients, specifically the ones from crossed channel diagrams e.g. 14-2-35 or 14-3-25 channel double-exchanges or 14-235 single-exchange, are quite involved but we chalk them out in Appendix 5.A.2.

However, for the evaluation at dimensions appearing in (5.2.52) most of the difficult coefficients vanish,⁶ so the evaluation of $\alpha_{0,0}^{2\phi,2\phi}(\Delta_1, \Delta_2)$ is easier. One obtains for the various terms in the sum-rule

$$\begin{aligned} \alpha_{0,0}^{2\phi,2\phi}(\Delta_\phi, 0) &= -4, \\ \alpha_{0,0}^{2\phi,2\phi}(\Delta_\phi, \Delta_{2\phi,0}) &= -2, \\ \alpha_{0,0}^{2\phi,2\phi}(\Delta_\phi, \Delta_{2\phi,n}) &= 0, \quad (n \neq 0), \\ \alpha_{0,0}^{2\phi,2\phi}(\Delta_{2\phi,0}, \Delta_{2\phi,0}) &= 4, \\ \alpha_{0,0}^{2\phi,2\phi}(\Delta_{2\phi,n}, \Delta_{2\phi,m}) &= 0, \quad (n, m \neq 0). \end{aligned} \quad (5.2.53)$$

Let us point out that the term $\alpha_{0,0}^{2\phi,2\phi}(\Delta_\phi, 0)$ corresponds to Witten diagrams with an identity exchange. So, as we explained around eqn. (5.2.50), it is obtained from the disconnected correlator itself, specifically from the OPE coefficients of $G_{\Delta_{2\phi,0}, \Delta_{2\phi,0}}$ and $G_{\Delta_{2\phi,0}, \Delta_{3\phi,0}}$.⁷

If we use (5.2.53) along-with the respective OPE coefficients $a_{\phi, [\phi\phi]_0} = 2C_{\phi\phi\phi}$, $a_{[\phi\phi]_0, [\phi\phi]_0} = 2C_{\phi\phi\phi}$ we see that (5.2.52) is satisfied.

⁶This is because coefficients from the ‘difficult’ channels do not have poles in Δ_1, Δ_2 while there are zeroes from the normalisation due to which they vanish. See appendix 5.A.2.

⁷Notice that (5.2.53) slightly contradict the orthogonality conditions (5.2.47) because we get, e.g. $\alpha_{0,0}^{2\phi,2\phi}(\Delta_{2\phi,0}, \Delta_{2\phi,0}) = 4$ instead of 1. The value is different from 1 for special cases: if we have $n_1 = n_2$ in $\alpha_{n_1, n_2}^{2\phi,2\phi}$, or if one or both of the dimensions Δ_1 or Δ_2 is $\Delta_{2\phi,0}$ (the consequence of the latter being that we get a term $\propto c_1$). The present case satisfies both conditions.

Deforming the disconnected correlator

We showed from the orthogonality relations (5.2.46) and (5.2.47) that our sum-rules trivialise when the 5-point OPE has dimension pairs of the type $(\Delta_{2\phi,n_1}, \Delta_{2\phi,n_2})$, $(\Delta_{2\phi,n_1}, \Delta_{3\phi,n_2})$, $(\Delta_1, \Delta_{2\phi,n})$ or $(\Delta_1, \Delta_{1,\phi,n})$. From the examples in subsection 5.2.4 we see that in such cases, one can determine all OPE coefficients from the knowledge of only a few. For instance, in (5.2.49) we can determine $a_{\phi, [\phi\phi]_n}$ from $a_{\phi, [\phi\phi]_0}$ and in (5.2.52) we obtain $a_{[\phi\phi]_0, [\phi\phi]_0}$ in terms of $a_{\phi, [\phi\phi]_0}$.

This feature of the sum-rules remains valid even if the respective operators acquire anomalous dimensions and OPE coefficient corrections. To see this, consider a (somewhat artificial) deformation of the disconnected correlator where the double-twist dimensions are $\Delta = \Delta_{2\phi,n} + \gamma_n$ and the OPE coefficients are modified as follows

$$\begin{aligned} (\phi, [\phi\phi]_n) : \quad & a_{\Delta_1, \Delta_2} \rightarrow a_{\phi, [\phi\phi]_n} + \delta_n^{(1)}, \\ ([\phi\phi]_{n_1}, [\phi\phi]_{n_2}) : \quad & a_{\Delta_1, \Delta_2} \rightarrow a_{[\phi\phi]_{n_1}, [\phi\phi]_{n_2}} + \delta_{n_1, n_2}^{(2)}. \end{aligned} \quad (5.2.54)$$

Then the $(2\phi, 2\phi)$ sum-rule from (5.2.52) gets corrected and the leading order correction terms look like

$$\begin{aligned} \delta_0^{(1)} \alpha_{0,0}^{2\phi, 2\phi}(\Delta_\phi, \Delta_{2\phi,0}) + \sum_n \gamma_n a_{\phi, [\phi\phi]_n} \partial_{\Delta_2} \alpha_{0,0}^{2\phi, 2\phi}(\Delta_\phi, \Delta_2 = \Delta_{2\phi,n}) + \delta_{0,0}^{(2)} \alpha_{0,0}^{2\phi, 2\phi}(\Delta_{2\phi,0}, \Delta_{2\phi,0}) \\ + \sum_{(n,m)} 2\gamma_m a_{[\phi\phi]_n, [\phi\phi]_m} \partial_{\Delta_2} \alpha_{0,0}^{2\phi, 2\phi}(\Delta_{2\phi,n}, \Delta_2 = \Delta_{2\phi,m}) = 0. \end{aligned} \quad (5.2.55)$$

In the last term we assumed $\partial_{\Delta_1} \alpha_{0,0}^{2\phi, 2\phi}(\Delta_1, \Delta_2) = \partial_{\Delta_1} \alpha_{0,0}^{2\phi, 2\phi}(\Delta_2, \Delta_1)$. This equation determines $\delta_{0,0}^{(2)}$ in terms of $\delta_0^{(1)}$ if we assume that all γ_n -s are fixed.

It is not hard to see that if the pairs $(\Delta_{2\phi,n_1}, \Delta_{2\phi,n_2})$, $(\Delta_{2\phi,n_1}, \Delta_{3\phi,n_2})$, $(\Delta_1, \Delta_{2\phi,n})$ or $(\Delta_1, \Delta_{1,\phi,n})$ get modified by some given anomalous dimensions their OPE coefficient corrections would be obtained by $\alpha_{n_1, n_2}^{2\phi, 2\phi}$, $\alpha_{n_1, n_2}^{2\phi, 3\phi}$, $\alpha_n^{1, 2\phi}$ or $\alpha_n^{1, (1, \phi)}$ functionals respectively.⁸ The rationale behind treating modified dimensions as inputs is that we consider them to be fixed at the level of a 4-point function that has previously been determined by bootstrap, or other methods (e.g. the 4-point sum-rules (5.1.15) are suitable to fix γ_n).

Further checks Along with the disconnected correlator and their deformations, we also considered simple correlation functions whose OPE decomposition contains infinite families of operators which are not of double- or triple-twist type, and therefore need an infinite number of contributions to satisfy the sum-rules. Examples of

⁸In fact one can achieve the same result in perturbation theory by repeated actions of only $\alpha_n^{1, 2\phi}$ and $\alpha_n^{1, (1, \phi)}$ for suitable Δ_1 and n .

such correlators can be obtained by considering composite operators such as ϕ^2 and ϕ^4 in generalised free theory, and we verified the associated sum rules extensively. Instead of presenting these explicit (but cumbersome) checks here, we will instead describe how to bootstrap such a correlator in the following section.

5.3 Application: Truncated numerical bootstrap

In this section we apply our functionals to bootstrap the 5-point crossing equations. Since the OPE decomposition does not have manifest positivity properties, we resort to the truncated bootstrap as done in [90–92]. As a target, we consider an exactly known but non-trivial solution to crossing, introduced in Section 5.3.1. An important feature of this solution is that its spectrum is not a perturbative deformation of double-twist or triple-twist towers. We treat the spectrum as an input (that in a realistic situation would be provided by the 4-point bootstrap or other methods) and bootstrap the non-trivial OPE coefficients, first using derivative functionals (Section 5.3.2) and then using the new functionals proposed in this work (Section 5.3.3). Our knowledge of the exact correlator makes this example an ideal setup to compare the two different sets of functionals in a controlled environment. We comment on this comparison at the end of Section 5.3.3.

5.3.1 Bootstrap target: The ‘all-to-all’ correlator

Consider the correlator,

$$G_{\text{all-to-all}}(x_i) = \prod_{i < j}^5 (x_{ij})^{2\Delta_\phi/4}, \quad (5.3.1)$$

which can be obtained by computing the five point function of $:\phi^4:$ operators in a generalised free theory through Wick contractions that connect every operator to every other exactly once. Here, Δ_ϕ denotes the dimension of the external operator.⁹ This function is manifestly crossing symmetric and by pulling out the prefactor, we find the following simple function of the cross-ratios

$$\mathcal{G}_{\text{all-to-all}}(\chi_a) = \left(\frac{\chi_1^3 \chi_2^3}{(1 - \chi_1)(1 - \chi_2)(1 - \chi_1 - \chi_2)} \right)^{\Delta_\phi/2}. \quad (5.3.2)$$

From the above expression it can easily be seen that the OPE is controlled by a tower of operators of scaling dimensions

$$\Delta_{(n)} = \frac{3\Delta_\phi}{2} + 2n, \quad (5.3.3)$$

⁹This dimension is four times that of the constituent field ϕ , but we do not introduce notation for this to avoid confusion.

which as previously advertised are neither double- nor triple-traces of the external operator. They can instead be understood as $:\phi^6:$ -type operators in the $:\phi^4:\times:\phi^4:$ OPE. It is straightforward to expand this correlator in conformal blocks and extract OPE coefficients, which will subsequently be compared with those obtained by truncating the crossing equation with derivative and analytic functionals. Specialising to the case $\Delta_\phi = 1$ as we will below, and introducing by abuse of notation $a_{n,m} \equiv a_{3/2+2n,3/2+2m}$, the first few coefficients are given by

$$a_{0,0} = 1, a_{0,1} = 1/16, a_{0,2} = 1/256, a_{0,3} = 1/4096, a_{1,1} = 7/128, a_{1,2} = 23/4096. \quad (5.3.4)$$

5.3.2 Bootstrap with derivative functionals

A simple way to extract the constraints on CFT data encoded in the crossing equation (5.2.6) is to Taylor expand in χ_1 and χ_2 around the crossing symmetric point

$$\chi_1 = \chi_2 = \chi_s \equiv \frac{1}{2}(3 - \sqrt{5}), \quad (5.3.5)$$

leading to the derivative sum-rules

$$0 = \partial_{\chi_1}^n \partial_{\chi_2}^m \left[\sum_{\Delta_1, \Delta_2} a_{\Delta_1, \Delta_2} F(\chi_1, \chi_2) \right] \Big|_{\chi_1 = \chi_2 = \chi_s}. \quad (5.3.6)$$

labeled by two integers n and m . Here,

$$F_{\Delta_1, \Delta_2}(\chi_1, \chi_2) = \left(G_{\Delta_1, \Delta_2}(\chi_1, \chi_2) - \left(\frac{\chi_1^2 \chi_2}{(\chi_1 + \chi_2 - 1)^2} \right)^{\Delta_\phi} G_{\Delta_1, \Delta_2} \left(\chi_2, 1 + \frac{\chi_2}{\chi_1 - 1} \right) \right). \quad (5.3.7)$$

We use derivative sum-rules to bootstrap the all-to-all correlator in the following way. Considering six randomly chosen derivative sum-rules, we truncate the spectrum to the lowest seven operators and solve the six resulting equations for the six unknowns $a_{0,1}$, $a_{0,2}$, $a_{0,3}$, $a_{1,1}$, $a_{1,2}$ and $a_{2,2}$ with $a_{1,1}$ normalised to 1. The result of this procedure is shown in figure 5.3.1 for 10 iterations selecting out of derivative functionals with $n, m \leq 3$. The above procedure is chosen in order to have a simple meaningful comparison with the analytic functionals and not with the goal of formulating an optimal strategy for truncated bootstrap.

5.3.3 Bootstrap with analytic functionals

As before, we will assume the scaling dimensions to be given. We take them to be as follows:

$$(\Delta_{(p)}, \Delta_{(q)}) := \left(\frac{3}{2} + 2p, \frac{3}{2} + 2q \right), \quad p, q \in \mathbb{Z}_{\geq 0}. \quad (5.3.8)$$

and $\Delta_\phi = 1$. Our goal is to obtain the OPE coefficients $a_{p,q} := a_{\Delta_{(p)}, \Delta_{(q)}}$ with the input of $a_{0,0} = 1$.

The strategy is simple: we will choose a set of operator pairs $(\Delta_{(p)}, \Delta_{(q)})$ corresponding to $p \leq p_{\max}$, $q \leq q_{\max}(p)$. We will choose the same number of sum-rules. For this problem each sum-rule will couple multiple unknowns $a_{p,q}$, so we cannot solve them individually. Our hope is to solve them as a set of simultaneous linear equations and obtain $a_{p,q}$ to a good numerical accuracy.

Let us choose $p_{\max} = 2$, and $q_{\max}(0) = 3, q_{\max}(1) = 2, q_{\max}(2) = 2$. This means we have the following pairs:

$$(\Delta_{(p)}, \Delta_{(q)}) = \left(\frac{3}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{7}{2}\right), \left(\frac{3}{2}, \frac{11}{2}\right), \left(\frac{3}{2}, \frac{15}{2}\right), \left(\frac{7}{2}, \frac{7}{2}\right), \left(\frac{7}{2}, \frac{11}{2}\right), \left(\frac{11}{2}, \frac{11}{2}\right), \quad (5.3.9)$$

$$a_{p,q} = a_{0,0}, \quad a_{0,1}, \quad a_{0,2}, \quad a_{0,3}, \quad a_{1,1}, \quad a_{1,2}, \quad a_{2,2}. \quad (5.3.10)$$

There are six variables, so we need six sum-rules. Let us first choose the functional $\alpha_0^{1,2\phi}$. For this we choose the fixed dimension to be $\Delta_1 = \frac{3}{2}, \frac{7}{2}, \frac{11}{2}$ which gives, from (5.2.39), three distinct equations. Numerically they read:¹⁰

$(1, 2\phi)$, $n = 0$ sum-rules:

$$\Delta_1 = \frac{3}{2} : \quad a_{0,0} 0.06696 - a_{0,1} 1.13115 + a_{0,2} 0.84200 + a_{0,3} 1.40585 = 0, \quad (5.3.11)$$

$$\Delta_1 = \frac{7}{2} : \quad a_{0,1} 1.52816 - a_{1,1} 1.78614 + a_{1,2} 0.34937 = 0, \quad (5.3.12)$$

$$\Delta_1 = \frac{11}{2} : \quad a_{0,2} 1.30190 - a_{1,2} 0.80423 - a_{2,2} 0.18168 = 0. \quad (5.3.13)$$

We will not use $(1, 2\phi)$ sum-rules with $n > 0$ because for a given Δ_1 they would sum over the same series of operators but would require more of them for numerical convergence. E.g. the $(1, 2\phi)$ sum-rule for $n = 3, 4$, etc with $\Delta_1 = \frac{3}{2}$ we get a sum over $a_{0,q}$ that is different from $n = 0$ i.e. (5.3.11), but we would require to truncate q at higher values than what we should have for $n = 0, 1$. This is evident from their orthogonality property - we cannot satisfy the n -th $(1, 2\phi)$ sum-rule with operator pairs $(\Delta_1, \Delta_{2\phi, m})$ with all $m < n$.

We need three more equations. So, we will use the sum-rules $(2\phi, 2\phi)$ and $(2\phi, 3\phi)$. The former will be for $n_1 = n_2 = 0$ and the latter for $n_1 = 0, n_2 = 1$ and $n_1 = 0, n_2 = 2$, hence three equations. The advantage is that each of them sum over all the OPE coefficients, and for low enough n_1, n_2 converge quickly.

¹⁰The numerical values reported below are obtained from the techniques discussed in Appendices 5.A and 5.C. All functional expressions involve infinite sums. We chose suitable truncations so that the results from both techniques match up to a certain accuracy.

$(2\phi, 2\phi)$, $n = 0$ sum-rule:

$$\begin{aligned} -a_{0,0} 0.20280 + a_{0,1} 2.00474 + a_{0,2} 0.38663 + a_{0,3} 0.74856 + a_{1,1} 1.40910 \\ - a_{1,2} 0.76550 + a_{2,2} 1.13883 = 0. \end{aligned} \quad (5.3.14)$$

$(2\phi, 3\phi)$ sum-rules:

$$\begin{aligned} (n_1, n_2) = (0, 1) : \quad a_{0,0} 0.00928 + a_{0,1} 0.62012 - a_{0,2} 0.1216 - a_{0,3} 0.23437 \\ - a_{1,1} 0.99493 + a_{1,2} 0.81915 + a_{2,2} 0.80916 = 0. \end{aligned} \quad (5.3.15)$$

$$\begin{aligned} (n_1, n_2) = (0, 2) : \quad a_{0,0} 0.00099 + a_{0,1} 0.06188 + a_{0,2} 0.37021 - a_{0,3} 0.32135 \\ - a_{1,1} 0.08503 - a_{1,2} 0.05202 - a_{2,2} 0.62975 = 0. \end{aligned} \quad (5.3.16)$$

Solving the equations (5.3.11)-(5.3.16) we obtain

$$\begin{aligned} a_{0,1} = 0.0623, \quad a_{0,2} = 0.0043, \quad a_{0,3} = 0.00030, \\ a_{1,1} = 0.055, \quad a_{1,2} = 0.0065, \quad a_{2,2} = 0.0022. \end{aligned} \quad (5.3.17)$$

Comparison

Finally, let us compare the results (5.3.17) obtained from the analytic functionals with the results obtained by using derivative functionals as described in Section 5.3.2. For the comparison, we have plotted the logarithmic error

$$\log_{10} \left(\frac{|a_{p,q}^{\text{exact}} - a_{p,q}|}{a_{p,q}^{\text{exact}}} \right) \quad (5.3.18)$$

of the numerically determined OPE coefficients in Figure 5.3.1.

We observe that the accuracy of the numerical estimate for the OPE coefficients of light operators varies considerably between different choices of derivative sum-rules with a mean that is comparable to the error of the solution obtained from analytic functionals. For heavier operators, however, the analytic functionals considerably outperform derivative functionals, producing significantly smaller errors than the best choice of derivative sum-rules.

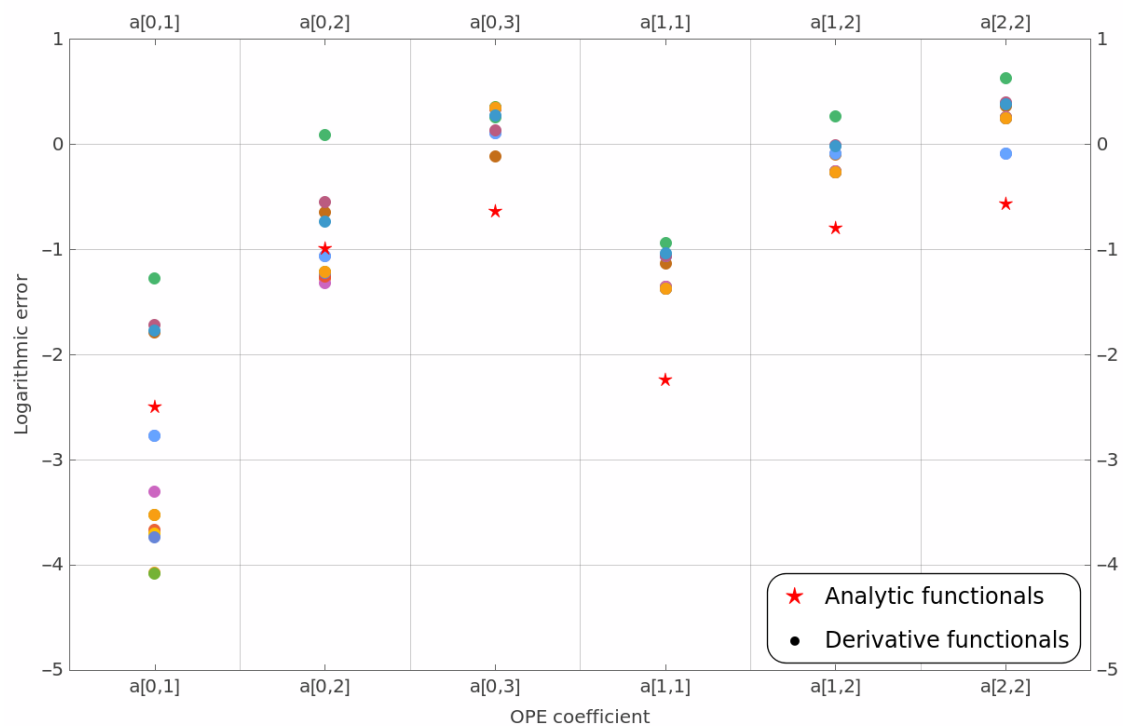


Figure 5.3.1: The absolute value of the relative error of numerical estimates for the OPE coefficients $a_{p,q}$ for derivative and analytic functionals plotted on a logarithmic scale.

Appendix

5.A Details of Mellin space computations

5.A.1 4-point Witten diagrams

In this appendix we will provide some details of the definitions of Witten diagrams in Mellin space and the computations related to them. Recall the 4-point s -channel diagram that is given by

$$M_{\Delta}^{(s)}(s_{12}, s_{14}) = \sum_{m=0}^{\infty} \frac{\mathcal{Q}_m^{(\Delta_{\phi})}}{s_{12} + \frac{\Delta}{2} - \Delta_{\phi} + m}. \quad (5.A.1)$$

Here the numerator is given by

$$\mathcal{Q}_m^{(\Delta_{\phi})} = \frac{\Gamma(\Delta)\Gamma(\Delta + \frac{1}{2})\left(\left(\frac{\Delta}{2} - \Delta_{\phi} + 1\right)_m\right)^2}{\Gamma\left(\frac{\Delta}{2}\right)^4 \Gamma(m+1)\Gamma(\Delta_{\phi} - \frac{\Delta}{2})^2 \Gamma(m + \Delta + \frac{1}{2})}. \quad (5.A.2)$$

The t -channel exchange diagram is given by replacing $s_{12} \leftrightarrow s_{14}$. Let us demonstrate computation of a spurious term $\chi^{2\Delta_{\phi}} \log \chi$ which we initiated for a general amplitude in (5.1.27). For the s -channel diagram the s_{14} integral is the same as the contact diagram (see (5.1.28)) since $M^{(s)}$ is independent of s_{14} , and we carry it out using Barnes first lemma. So we only have the sum over m in (5.A.1) which is straightforward to carry out. We obtain

$$\begin{aligned} & \chi^{2\Delta_{\phi}} \log(\chi) \int_{-i\infty}^{i\infty} \frac{ds_{14}}{2\pi i} 2\Gamma(s_{14})^2 \Gamma(\Delta_{\phi} - s_{14})^2 \sum_{m=0}^{\infty} \frac{\mathcal{Q}_m^{(\Delta_{\phi})}}{m + \frac{\Delta}{2} - \Delta_{\phi}} \\ &= \frac{\chi^{2\Delta_{\phi}} \log(\chi) 4\Gamma(\Delta)\Gamma(\Delta + \frac{1}{2})\Gamma(\Delta_{\phi})^4 \Gamma(2\Delta_{\phi} - \frac{1}{2})}{(\Delta - 2\Delta_{\phi})\Gamma(\frac{\Delta}{2})^4 \Gamma(2\Delta_{\phi})\Gamma(\frac{\Delta-1+2\Delta_{\phi}}{2})\Gamma(\frac{2\Delta_{\phi}-\Delta}{2})^2 \Gamma(\frac{\Delta+1+2\Delta_{\phi}}{2})}. \end{aligned} \quad (5.A.3)$$

Let us now show the evaluation of the analogous coefficient in the t -channel diagram. This is harder since the $M^{(t)}$ has an s_{14} in the denominator and so the s_{14} integral cannot be evaluated using Barnes lemma. To carry out the integral we write ($s_{14} +$

$\frac{\Delta}{2} - \Delta_\phi + m)^{-1} = \int_0^1 \frac{dy}{y} y^{s_{14} + \frac{\Delta}{2} - \Delta_\phi + m}$. So for t -channel diagram we obtain

$$\begin{aligned} & \chi^{2\Delta_\phi} \log(\chi) \sum_{m=0}^{\infty} \int_{-i\infty}^{i\infty} \frac{ds_{14}}{2\pi i} 2 \mathcal{Q}_m^{(\Delta_\phi)} \Gamma(s_{14})^2 \Gamma(\Delta_\phi - s_{14})^2 \int_0^1 \frac{dy}{y} y^{s_{14} + \frac{\Delta}{2} - \Delta_\phi + m} \\ &= \chi^{2\Delta_\phi} \log(\chi) \sum_{m=0}^{\infty} \int_0^1 \frac{dy}{y} y^{\frac{\Delta}{2} - \Delta_\phi + m} 2 \mathcal{Q}_m^{(\Delta_\phi)} \Gamma(\Delta_\phi)^4 {}_2\tilde{F}_1(\Delta_\phi, \Delta_\phi, 2\Delta_\phi, 1 - \frac{1}{y}) \\ &= \chi^{2\Delta_\phi} \log(\chi) \sum_{m=0}^{\infty} 2 \mathcal{Q}_m^{(\Delta_\phi)} \Gamma(\Delta_\phi)^4 \Gamma(\frac{\Delta}{2} + m) {}_3\tilde{F}_2[1, \Delta_\phi, \Delta_\phi; 2\Delta_\phi, \frac{2m+\Delta+2}{2}; 1] \quad (5.A.4) \end{aligned}$$

Here $\tilde{F}$ denotes the regularised hypergeometric functions. To go from the second to third step we used the Pfaff transformation of ${}_2F_1$ to convert the argument $(1 - \frac{1}{y})$ to $1 - y$, then expanded in $(1 - y)$ powers - then we integrated the k -th expansion term over y and summed over k . So the final expression involves an infinite sum. It converges fast and we can evaluate it numerically by truncating.

The u -channel coefficient of $\chi^{2\Delta_\phi} \log(\chi)$ is carried out similarly to t -channel. It also comes out to be identical to that of the t -channel. This shows the property $b_n^{(t)} = b_n^{(u)}$ pointed out below (5.1.12) in the main text.

5.A.2 5-point Witten diagrams

In this appendix we will show how to obtain the coefficients of the various conformal blocks from 5-point Witten diagrams. Let us start with the definition of inverse Mellin transform (5.2.16) which we repeat for convenience of the reader

$$\mathcal{G}(\chi_1, \chi_2) = \int [ds_{ij}] \prod_{i < j \leq 5} (\Gamma(s_{ij})) f(\chi_1, \chi_2 | \{s_{ij}\}) M(s_{12}, s_{23}, s_{34}, s_{45}, s_{15}) \quad (5.A.5)$$

$$\text{where } f(\chi_1, \chi_2 | \{s_{ij}\}) = \frac{(1 - \chi_1 - \chi_2)^{-2s_{15}} \chi_1^{2\Delta_\phi - 2s_{12}} \chi_2^{2\Delta_\phi - 2s_{45}}}{(1 - \chi_1)^{\Delta_\phi - 2s_{15} + 2s_{23} - 2s_{45}} (1 - \chi_2)^{\Delta_\phi - 2s_{12} - 2s_{15} + 2s_{34}}}. \quad (5.A.6)$$

Let us first consider the blocks $G_{\Delta_{2\phi, n_1}, \Delta_{2\phi, n_2}}$ and $G_{\Delta_{2\phi, n_1}, \Delta_{3\phi, n_2}}$, $G_{\Delta_{3\phi, n_1}, \Delta_{2\phi, n_2}}$ which feature in the decomposition of all the Witten diagrams (see (5.2.30), (5.2.31), (5.2.32)).

Coefficients of $G_{\Delta_{2\phi, n_1}, \Delta_{3\phi, n_2}}$

Let us first focus on the blocks $G_{\Delta_{2\phi, n_1}, \Delta_{3\phi, n_2}}$. This involves the power laws $\chi_1^{2\Delta_\phi + m'} \chi_2^{3\Delta_\phi + n'}$ for $m', n' \in \mathbb{Z}_{\geq 0}$. They are obtained from the following residues:

$$s_{12} = -m, s_{45} = -n - m + s_{23} - \frac{\Delta_\phi}{2}, s_{23} = -p. \quad (5.A.7)$$

In choosing the contour we stay careful about the following two things: 1. if we are considering negative integer poles of a $\Gamma(s_{ij})$ then all those poles are enclosed in the contour, and 2. we take all the terms $\chi_1, \chi_2, (1 - \chi_1), (1 - \chi_2), (1 - \chi_1 - \chi_2)$ are less than 1 individually, which would require all s_{ij} contours to be closed on the left. These make it clear that (5.A.7) is the only combination of poles how the required power laws can emerge.

With these poles the kinematic part (5.A.6) becomes

$$\begin{aligned} & \frac{(1 - \chi_1 - \chi_2)^{-2s_{15}} \chi_1^{2\Delta_\phi+2m} \chi_2^{3\Delta_\phi+2m+2n+2p} M(-m, -p, s_{34}, -n - m - p - \frac{\Delta_\phi}{2}, s_{15})}{(1 - \chi_1)^{2\Delta_\phi-2s_{15}+2m+2n} (1 - \chi_2)^{\Delta_\phi+2m-2s_{15}+2s_{34}}} \\ &= \chi_1^{2\Delta_\phi+2m} \chi_2^{3\Delta_\phi+2m+2n+2p} M(-m, -p, s_{34}, -n - m - p - \frac{\Delta_\phi}{2}, s_{15}) \sum_{q,r} \chi_1^q \chi_2^r N_{q,r}(s_{34}, s_{15}). \end{aligned} \quad (5.A.8)$$

Here $N_{q,r}$ are the coefficients appearing in the expansion of the powers of $(1 - \chi_1 - \chi_2)$, $(1 - \chi_1)$ and $(1 - \chi_2)$. Collecting the powers of χ_1 and χ_2 we obtain

$$\chi_1^{2\Delta_\phi+m'} \chi_2^{3\Delta_\phi+n'} \int ds_{34} ds_{15} \prod_{\substack{(ij) \neq \\ 12, 45, 23}} (\Gamma(s_{ij})) \widetilde{M}(s_{34}, s_{15}) \quad (5.A.9)$$

where

$$\widetilde{M}(s_{34}, s_{15}) = \sum_{(q,r)=(0,0)}^{(m',n')} M\left(\frac{q-m'}{2}, m+n+\frac{r-n'}{2}, s_{34}, \frac{r-n'-\Delta_\phi}{2}, s_{15}\right) N_{q,r}(s_{34}, s_{15}). \quad (5.A.10)$$

There are only two integrals to do in (5.A.9), so it is simpler to evaluate than the coefficients for $G_{\Delta_{2\phi, n_1}, \Delta_{2\phi, n_2}}$ (see (5.A.18) below).

Eqn. (5.A.9) is easiest to evaluate when $M(\{s_{ij}\})$ independent of s_{34}, s_{15} . This happens for the following cases:

- Contact diagram with $M^{(\text{con})} = 1$ (see (5.2.26)).
- Single-exchange diagrams $M_{\Delta}^{12-345}, M_{\Delta}^{13-245}, M_{\Delta}^{23-145}, M_{\Delta}^{45-123}$ (see (5.2.24)).
- Double-exchange diagrams $M_{\Delta_1, \Delta_2}^{12-3-45}, M_{\Delta_1, \Delta_2}^{13-2-45}, M_{\Delta_1, \Delta_2}^{23-1-45}$ (see (5.2.21)).

For all these diagrams the integral can be evaluated using first Barnes lemma twice. E.g. for the case of $m' = n' = 0$ one has

$$\begin{aligned} & \chi_1^{2\Delta_\phi} \chi_2^{3\Delta_\phi} M(s_{12} = 0, s_{23} = 0, s_{45} = -\frac{\Delta_\phi}{2}) \Gamma(-\frac{\Delta_\phi}{2}) \int ds_{34} ds_{15} \Gamma(s_{15}) \Gamma(s_{34}) \\ & \quad \times \Gamma(\Delta_\phi - s_{15}) \Gamma(s_{15} - s_{34} + \frac{\Delta_\phi}{2}) \Gamma(\Delta_\phi - s_{34}) \Gamma(s_{34} - s_{15} + \frac{\Delta_\phi}{2}) \\ &= \chi_1^{2\Delta_\phi} \chi_2^{3\Delta_\phi} M(s_{12} = 0, s_{23} = 0, s_{45} = -\frac{\Delta_\phi}{2}) \frac{\Gamma(-\frac{\Delta_\phi}{2}) \Gamma(\Delta_\phi)^3 \Gamma(\frac{3\Delta_\phi}{2})^2}{\Gamma(3\Delta_\phi)}. \end{aligned} \quad (5.A.11)$$

For higher m', n' which involves higher order polynomials, it remains an easy exercise to evaluate the integrals by Barnes lemma, simply by shifting the arguments of the Gamma functions, e.g. writing $s_{15} \Gamma(s_{15}) = \Gamma(s_{15} + 1)$, etc.

A slightly harder case is when $M(\{s_{ij}\})$ has a pole in either s_{34} or s_{15} . This happens for the following cases:

- Single-exchange diagrams $M_{\Delta}^{14-235}, M_{\Delta}^{15-234}, M_{\Delta}^{24-135}, M_{\Delta}^{25-134}, M_{\Delta}^{34-125}, M_{\Delta}^{35-124}$.
- Double-exchange diagrams $M_{\Delta_1, \Delta_2}^{12-5-34}, M_{\Delta_1, \Delta_2}^{12-4-35}, M_{\Delta_1, \Delta_2}^{13-4-25}, M_{\Delta_1, \Delta_2}^{13-5-24}, M_{\Delta_1, \Delta_2}^{23-4-15}, M_{\Delta_1, \Delta_2}^{23-5-14}$.

Consider that there is a pole in $(s_{34} + \frac{\Delta}{2} - \Delta_{\phi} + p)^{-1}$ with residue A_p (the form of A_p depends on the diagram we have), followed by a sum over p . We can carry out the s_{15} integral using Barnes lemma, and the s_{34} integral with the same technique as in t -channel Witten diagram for four points (see (5.A.4)). So, for the $m' = n' = 0$ case, for instance, one has for the coefficient of $\chi_1^{2\Delta_{\phi}} \chi_2^{3\Delta_{\phi}}$

$$\begin{aligned}
& \Gamma\left(-\frac{\Delta_{\phi}}{2}\right) \int ds_{34} ds_{15} \left[\sum_{p=0}^{\infty} \frac{A_p}{s_{34} + \frac{\Delta}{2} - \Delta_{\phi} + p} \right] (\Gamma(s_{15}) \Gamma(s_{34}) \cdots) \\
&= \int ds_{34} \frac{\Gamma(s_{34}) \Gamma\left(-\frac{\Delta_{\phi}}{2}\right) \Gamma(\Delta_{\phi})^2 \Gamma\left(s_{34} + \frac{\Delta_{\phi}}{2}\right) \Gamma(\Delta_{\phi} - s_{34}) \Gamma\left(\frac{3\Delta_{\phi}}{2} - s_{34}\right)}{\Gamma(2\Delta_{\phi})} \\
&\quad \times \left[\sum_{p=0}^{\infty} A_p \int_0^1 \frac{dy}{y} y^{s_{34} + \frac{\Delta}{2} - \Delta_{\phi} + p} \right] \\
&= \sum_{p=0}^{\infty} A_p \Gamma\left(-\frac{\Delta_{\phi}}{2}\right) \Gamma(\Delta_{\phi})^3 \Gamma\left(\frac{3\Delta_{\phi}}{2}\right)^2 \Gamma\left(p + \frac{\Delta}{2}\right) {}_3\tilde{F}_2\left(1, \Delta_{\phi}, \frac{3\Delta_{\phi}}{2}; p + \frac{\Delta}{2} + 1, 3\Delta_{\phi}; 1\right).
\end{aligned} \tag{5.A.12}$$

In the last line we evaluated the s_{34} integral which gives a ${}_2F_1$ function with $(1-y)$ argument - and expanded the ${}_2F_1$ in the powers $(1-y)^q$ to evaluate the y integral, and finally summed over q to obtain the ${}_3\tilde{F}_2$. Once again the higher polynomials are obtained by a simple generalisation of the above treatment.

The hardest case in evaluating (5.A.9) is that where the $M(\{s_{ij}\})$ has poles in both s_{34} and s_{15} . This happens for

- Double-exchange diagrams $M_{\Delta_1, \Delta_2}^{14-3-25}, M_{\Delta_1, \Delta_2}^{15-3-24}, M_{\Delta_1, \Delta_2}^{14-2-35}, M_{\Delta_1, \Delta_2}^{15-2-34}, M_{\Delta_1, \Delta_2}^{24-1-35}, M_{\Delta_1, \Delta_2}^{25-1-34}$.

Consider the following pole structure: $\mathcal{Q}_{kl}(s_{34} + \frac{\Delta_1}{2} - \Delta_{\phi} + k)^{-1}(s_{15} + \frac{\Delta_2}{2} - \Delta_{\phi} + l)^{-1}$ followed by sums over k, l . It is easy to see that this is the $M^{15-2-34}$ diagram. The other diagrams can be brought to the same integral form by shifting the Mellin

variables. Let us carry out the s_{15} integral first. We may convert the s_{15} pole to a y_2 -integral. Then for the $m' = n' = 0$ term we get the coefficient

$$\begin{aligned} & \int ds_{15} \Gamma(s_{15}) \Gamma(\Delta_\phi - s_{15}) \Gamma(s_{15} - s_{34} + \frac{\Delta_\phi}{2}) \Gamma(s_{34} - s_{15} + \frac{\Delta_\phi}{2}) \int_0^1 \frac{dy_2}{y_2} y_2^{s_{15} + \frac{\Delta_2}{2} - \Delta_\phi + l} \\ &= \int_0^1 \frac{dy_2}{y_2} \Gamma(\Delta_\phi)^2 \Gamma(s_{34} + \frac{\Delta_\phi}{2}) \Gamma(\frac{3\Delta_\phi}{2} - s_{34}) {}_2\tilde{F}_1(s_{34} + \frac{\Delta_\phi}{2}, \Delta_\phi; 2\Delta_\phi; 1 - \frac{1}{y_2}) y_2^{\frac{\Delta_2}{2} - \Delta_\phi + l} \\ &= \sum_{p=0}^{\infty} \frac{\Gamma(\Delta_\phi)^2 \Gamma(s_{34} + \frac{\Delta_\phi}{2}) \Gamma(\frac{3\Delta_\phi}{2} - s_{34} + p) (\Delta_\phi)_p \Gamma(l + \frac{\Delta_2}{2})}{\Gamma(2\Delta_\phi + p) \Gamma(l + p + \frac{\Delta_2}{2} + 1)} \end{aligned} \quad (5.A.13)$$

In the last line we expanded the ${}_2\tilde{F}_1$ and then integrated the p -th expansion term over y_2 . The s_{34} integral can then be carried out. If we write the s_{34} pole as a y_1 integral then we get (isolating only the s_{34} dependent terms)

$$\begin{aligned} & \int ds_{34} \Gamma(s_{34}) \Gamma(s_{34} + \frac{\Delta_\phi}{2}) \Gamma(\frac{3\Delta_\phi}{2} - s_{34} + p) \Gamma(\Delta_\phi - s_{34}) \int \frac{dy_1}{y_1} y_1^{s_{34} + \frac{\Delta_1}{2} - \Delta_\phi + k} \\ &= \Gamma(\Delta_\phi) \Gamma(\frac{3\Delta_\phi}{2}) \Gamma(k + \frac{\Delta_1}{2}) \Gamma(p + \frac{3\Delta_\phi}{2}) \Gamma(p + 2\Delta_\phi) \\ & \quad \times {}_3\tilde{F}_2(1, \Delta_\phi, \frac{3\Delta_\phi}{2}; \frac{\Delta_1}{2} + 1 + k, p + 3\Delta_\phi; 1). \end{aligned} \quad (5.A.14)$$

The steps leading to (5.A.14) is similar to what we used for (5.A.12). The final answer hence combines (5.A.13) and (5.A.14) and gives

$$\begin{aligned} & \sum_{p,k,l=0}^{\infty} \frac{\Gamma(\Delta_\phi)^3 (\Delta_\phi)_p \Gamma(l + \frac{\Delta_2}{2}) \Gamma(\frac{3\Delta_\phi}{2}) \Gamma(k + \frac{\Delta_1}{2}) \Gamma(p + \frac{3\Delta_\phi}{2}) \Gamma(p + 2\Delta_\phi)}{\Gamma(2\Delta_\phi + p) \Gamma(l + p + \frac{\Delta_2}{2} + 1)} \\ & \quad \times \mathcal{Q}_{kl}(\Delta_1, \Delta_2) \Gamma(-\frac{\Delta_\phi}{2}) {}_3\tilde{F}_2(1, \Delta_\phi, \frac{3\Delta_\phi}{2}; \frac{\Delta_1}{2} + 1 + k, p + 3\Delta_\phi; 1). \end{aligned} \quad (5.A.15)$$

The coefficient of $\chi_1^{2\Delta_\phi} \chi_2^{3\Delta_\phi}$, from $M^{15-2-34}$ and related diagrams, is hence three nested infinite sums. It can be evaluated numerically by truncation. These nested sums also feature in the higher powers of the same family. We observed that for higher powers the truncation needs to be at higher values for same numerical accuracy.

Coefficients of $G_{\Delta_{2\phi,n_1}, \Delta_{2\phi,n_2}}$

The $G_{\Delta_{2\phi,n_1}, \Delta_{2\phi,n_2}}$ block expands into the power laws $\chi_1^{2\Delta_\phi+m'} \chi_2^{2\Delta_\phi+n'}$ for $m', n' \in \mathbb{Z}_{\geq 0}$. Looking at (5.A.5) it is clear that these powers follow from the residues of the poles

$$s_{12} = -m, \quad s_{45} = -n, \quad \forall m, n \in \mathbb{Z}_{\geq 0}, \quad (5.A.16)$$

from the $\Gamma(s_{12})\Gamma(s_{45})$ in the measure.

The kinematic part (5.A.6) then becomes

$$\begin{aligned} & \frac{(1 - \chi_1 - \chi_2)^{-2s_{15}} \chi_1^{2\Delta_\phi+2m} \chi_2^{2\Delta_\phi+2n} M(-m, s_{23}, s_{34}, -n, s_{15})}{(1 - \chi_1)^{\Delta_\phi-2s_{15}+2s_{23}+2n} (1 - \chi_2)^{\Delta_\phi+2m-2s_{15}+2s_{34}}} \\ &= \chi_1^{2\Delta_\phi+2m} \chi_2^{2\Delta_\phi+2n} M(-m, s_{23}, s_{34}, -n, s_{15}) \sum_{p,q} \chi_1^p \chi_2^q N_{p,q}(s_{23}, s_{34}, s_{15}), \end{aligned} \quad (5.A.17)$$

where we further expanded $(1 - \chi_1 - \chi_2)$, $(1 - \chi_1)$ and $(1 - \chi_2)$ powers in χ_1 and χ_2 . Each coefficient $N_{p,q}$ appearing here is a polynomial in s_{23}, s_{34}, s_{15} and we have to carry out the respective integrals. We can then collect the powers of χ_1 and χ_2 to write

$$\chi_1^{2\Delta_\phi+m'} \chi_2^{2\Delta_\phi+n'} \int ds_{23} ds_{34} ds_{15} \prod_{\substack{(ij) \neq \\ 12, 45}} (\Gamma(s_{ij})) \widetilde{M}(s_{23}, s_{34}, s_{15}) \quad (5.A.18)$$

where

$$\widetilde{M}(s_{23}, s_{34}, s_{15}) = \sum_{(p,q)=(0,0)}^{(m',n')} M\left(\frac{p-m'}{2}, s_{23}, s_{34}, \frac{q-n'}{2}, s_{15}\right) N_{p,q}(s_{23}, s_{34}, s_{15}). \quad (5.A.19)$$

The eqn. (5.A.18) is more involved than (5.A.9) since there are three integrals instead of two. But the difficulty in evaluating (5.A.18) also depends on the form of Mellin amplitude $M(\{s_{ij}\})$ i.e. what diagram we have at hand.

Once again it is easiest when $M\left(\frac{p-m'}{2}, s_{23}, s_{34}, \frac{q-n'}{2}, s_{15}\right)$ has no poles in s_{23}, s_{34}, s_{15} . This is the case when we have

- Contact diagram, $M^{(\text{con})} = 1$.
- Single-exchange diagrams M_{Δ}^{12-345} , M_{Δ}^{45-123} .
- Double-exchange diagram $M_{\Delta_1, \Delta_2}^{12-3-45}$.

For these diagrams the integrals can be evaluated using Barnes lemma three times. The procedure mimics (5.A.11) and we will not repeat it here.

The integral is harder when $M\left(\frac{p-m'}{2}, s_{23}, s_{34}, \frac{q-n'}{2}, s_{15}\right)$ has one pole in s_{23}, s_{34} or s_{15} . This is the case when we have

- Single-exchange diagrams M_{Δ}^{13-245} , M_{Δ}^{23-145} , M_{Δ}^{14-235} , M_{Δ}^{15-234} , M_{Δ}^{24-135} , M_{Δ}^{25-134} , M_{Δ}^{34-125} , M_{Δ}^{35-124} .
- Double-exchange diagram $M_{\Delta_1, \Delta_2}^{13-2-45}$, $M_{\Delta_1, \Delta_2}^{23-1-45}$, $M_{\Delta_1, \Delta_2}^{12-4-35}$, $M_{\Delta_1, \Delta_2}^{12-5-34}$.

In the above cases two of the Mellin integrals can be evaluated using Barnes lemma, and for the one with the pole has to be done using the treatment similar to (5.A.12).

Our integral (5.A.18) is even harder when $M\left(\frac{p-m'}{2}, s_{23}, s_{34}, \frac{q-n'}{2}, s_{15}\right)$ has two simple poles. But now there are two cases to distinguish

- Double-exchange diagrams $M_{\Delta_1, \Delta_2}^{14-2-35}$, $M_{\Delta_1, \Delta_2}^{15-2-34}$, $M_{\Delta_1, \Delta_2}^{24-1-35}$, $M_{\Delta_1, \Delta_2}^{25-1-34}$.

- Double-exchange diagram $M_{\Delta_1, \Delta_2}^{14-3-25}$, $M_{\Delta_1, \Delta_2}^{15-3-24}$.

For the first set of four diagrams, it is possible to carry out one of the three Mellin integrals using Barnes lemma, as the respective variable will have no pole, e.g. with $M_{\Delta_1, \Delta_2}^{15-2-34}$ this will be the s_{23} variable integral. The other two integrals will have a pair of simple poles, and we have to follow the steps shown in (5.A.13) and (5.A.14).

For the remaining two double-exchange diagrams, it is not possible to carry out any of the Mellin integrals using Barnes lemma, as all the variables will feature in one or the other pole. We can still try to evaluate them with steps similar to (5.A.13) and (5.A.14). However the resulting expression is either a set of very poorly convergent nested infinite sums, or infinite sums where with a y -integral (from 0 to 1) at every summand. In both cases it is not feasible to use the expression efficiently for numerical evaluation. So, for these two diagrams, we have to use the method of App. 5.C.

Coefficients of $G_{\Delta_1, \Delta_2, \phi, n}$ and $G_{\Delta_1, \Delta_1, \phi, n}$

We will discuss the coefficients of these blocks together as they both feature in the same four diagrams: $M_{\Delta_1, \Delta_2}^{12-3-45}$, $M_{\Delta_1, \Delta_2}^{12-4-35}$, $M_{\Delta_1, \Delta_2}^{12-5-34}$ and $M_{\Delta_1}^{12-345}$.

The block $G_{\Delta_1, \Delta_2, \phi, n}$ involves the terms $\chi_1^{\Delta_1+m'} \chi_2^{2\Delta_\phi+n'}$. They are obtained from the following poles

$$s_{12} = -\frac{\Delta_1}{2} + \Delta_\phi - m, \quad s_{45} = -n, \quad \forall m, n \in \mathbb{Z}_{\geq 0}. \quad (5.A.20)$$

We will expand the kinematical part in Mellin integral just like (5.A.17) and collect the necessary powers of χ_1 and χ_2 to write

$$\chi_1^{\Delta_1+m'} \chi_2^{2\Delta_\phi+n'} \int ds_{23} ds_{34} ds_{15} \prod_{\substack{(ij) \neq \\ 12, 45}} (\Gamma(s_{ij})) \widetilde{M}(s_{23}, s_{34}, s_{15}) \quad (5.A.21)$$

where

$$\widetilde{M}(s_{23}, s_{34}, s_{15}) = \sum_{(p,q)=(0,0)}^{(m',n')} M\left(-\frac{\Delta}{2} + \Delta_\phi + \frac{p-m'}{2}, s_{23}, s_{34}, \frac{q-n'}{2}, s_{15}\right) N_{p,q}(s_{23}, s_{34}, s_{15}). \quad (5.A.22)$$

The coefficient $N_{p,q}$ is defined similar to (5.A.17). It is easy to evaluate for $M_{\Delta_1, \Delta_2}^{12-3-45}$ and $M_{\Delta_1}^{12-345}$ as there is no pole in s_{23}, s_{34} or s_{45} from $M(\{s_{ij}\})$. We can use Barnes lemma three times to evaluate it. For $M_{\Delta_1, \Delta_2}^{12-4-35}$ and $M_{\Delta_1, \Delta_2}^{12-5-34}$ there is a pole one of the three variables. We have already seen how to handle such integrals in (5.A.12) and we proceed as shown there to evaluate it.

The block $G_{\Delta_1, \Delta_1, \phi, n}$ will follow from the terms $\chi_1^{\Delta_1+m'} \chi_2^{\Delta_1+\Delta_\phi+n'}$. They are obtained from the following poles

$$s_{12} = -\frac{\Delta_1}{2} + \Delta_\phi - m, \quad s_{45} = \frac{\Delta_\phi - \Delta_1}{2} - m - n - p, \quad s_{23} = -p, \quad \forall m, n \in \mathbb{Z}_{\geq 0}. \quad (5.A.23)$$

This results in the powers $\chi_1^{\Delta_1+2m} \chi_2^{\Delta_1+\Delta_\phi+2m+2n+2p}$. The kinematical part in Mellin integral should be expanded as in (5.A.8). Collecting the appropriate powers of χ_1, χ_2 one gets

$$\chi_1^{\Delta_1+m'} \chi_2^{\Delta_1+\Delta_\phi+n'} \int ds_{34} ds_{15} \prod_{\substack{(ij)^\# \\ 12, 45}} (\Gamma(s_{ij})) \widetilde{M}(s_{34}, s_{15}) \quad (5.A.24)$$

where

$$\widetilde{M}(s_{34}, s_{15}) = \sum_{(q,r)=(0,0)}^{(m',n')} M\left(\Delta_\phi - \frac{\Delta}{2} + \frac{q-m'}{2}, m+n + \frac{r-n'}{2}, s_{34}, \frac{r-n'+\Delta_\phi-\Delta_1}{2}, s_{15}\right) N_{q,r}(s_{34}, s_{15}). \quad (5.A.25)$$

Here the definition of $N_{q,r}$ is analogous to (5.A.8). Once again the integral is simple for $M_{\Delta_1, \Delta_2}^{12-3-45}$ and $M_{\Delta_1}^{12-345}$ (no pole in s_{34} or s_{15}) and a bit harder for $M_{\Delta_1, \Delta_2}^{12-4-35}$ and $M_{\Delta_1, \Delta_2}^{12-5-34}$ (one pole). As appropriate, we follow the steps of (5.A.11) and (5.A.12) respectively.

5.B Spectral decomposition of five-point Witten diagrams

In this appendix, we review the spectral approach to Witten diagrams which is very useful to derive the conformal block expansion of these objects, particularly in certain OPE channels which do not require complicated crossing moves [227, 231]. This will serve as a useful crosscheck to the Mellin techniques, which, while more versatile, often lead to more complicated expressions.

This formalism attempts to use the conformal structure of three-point integrals to trade complicated AdS position space integrals by spectral integrals that can be evaluated in the end of the computation by picking residues. There are two key identities in this process: the spectral representation of the scalar bulk-to-bulk propagator

$$G_{BB}(\zeta) = \int d\nu \rho_\Delta(\nu) \Omega_\nu(\zeta), \quad (5.B.1)$$

which is key for exchange diagrams, and the spectral representation of the Dirac delta function

$$\delta(\zeta) = \int d\nu \rho_\delta(\nu) \Omega_\nu(\zeta), \quad (5.B.2)$$

which is needed both for single-exchange and contact diagrams. Here, we introduced the AdS_{d+1} harmonic function $\Omega_\nu(\zeta)$, which is an eigenfunction of the AdS laplacian, satisfying

$$\square_{\text{AdS}} \Omega_\nu(\zeta) = -\left(\nu^2 + \frac{d^2}{4}\right) \Omega_\nu(\zeta), \quad (5.B.3)$$

as well as the spectral functions

$$\rho_\delta(\nu) = \frac{\Gamma(\frac{d}{2} + i\nu)\Gamma(\frac{d}{2} - i\nu)}{2\pi^d \Gamma(-i\nu)\Gamma(i\nu)}, \quad \rho_\Delta = \frac{1}{\nu^2 + (\Delta - d/2)^2}. \quad (5.B.4)$$

These relations are supplemented by the so-called ‘split representation’ of the Harmonic function

$$\Omega_\nu(\zeta(x_1, y_1, x_2, y_2)) = \int d^d x K_{d/2+i\nu}(x_1, y_1; x) K_{d/2-i\nu}(x_2, y_2; x), \quad (5.B.5)$$

which allows one to trade integrals with bulk point dependence by integrals with only three-boundary points, which can always be performed with the help of the mythical AdS 3-point integrals

$$\int \sqrt{g} d^{d+1} y K_{\Delta_1}(y; x_1) K_{\Delta_2}(y; x_2) K_{\Delta_3}(y; x_3) = \frac{\lambda_{\Delta_1, \Delta_2, \Delta_3}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{13}^{\Delta_1+\Delta_3-\Delta_2} x_{23}^{\Delta_2+\Delta_3-\Delta_1}}, \quad (5.B.6)$$

where here and henceforth we simply use the shorthand notation y_i and x_i to denote a bulk and boundary point, respectively. We have also defined the bulk-to-boundary propagator in Poincaré coordinates

$$K_{\Delta_i}(y; x_i) = \left(\frac{y}{y^2 + (x - x_i)^2} \right)^{\Delta_i}, \quad (5.B.7)$$

as well as the AdS 3-point factor

$$\lambda_{\Delta_1, \Delta_2, \Delta_3} = \frac{\pi^{d/2} \Gamma(\frac{\Delta_1+\Delta_2-\Delta_3}{2}) \Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2}) \Gamma(\frac{\Delta_1+\Delta_3-\Delta_2}{2}) \Gamma(\frac{\Delta_1+\Delta_2+\Delta_3-d}{2})}{2\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)}. \quad (5.B.8)$$

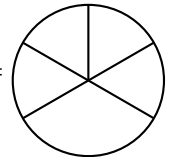
We will also use the more compact notation

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle \equiv \frac{1}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{13}^{\Delta_1+\Delta_3-\Delta_2} x_{23}^{\Delta_2+\Delta_3-\Delta_1}}, \quad (5.B.9)$$

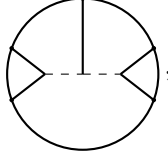
for readability in intermediate steps of the calculations. Let us now see this machinery in action.

Contact diagram The simplest contact diagrams, which have non-derivative couplings of ϕ^5 type, are given by

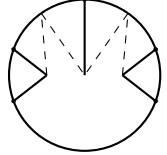
$$G_{\text{ctc}}(x_i) = \int d^{d+1} y \prod_{i=1}^5 K_{\Delta_i}(y; x_i) = \text{Diagram}, \quad (5.B.10)$$



we then introduce two auxiliary integration points which we connect to the original integration point with Dirac delta functions (denoted diagrammatically by a dashed line)

$$\int d^{d+1}y d^{d+1}y' d^{d+1}y'' (K_{\Delta_1}(y', x_1) K_{\Delta_2}(y', x_2) \delta^{d+1}(y - y') K_{\Delta_3}(y, x_3) \delta^{d+1}(y - y'') K_{\Delta_4}(y'', x_4) K_{\Delta_5}(y'', x_5)) =$$

(5.B.11)

in a way that reflects our chosen OPE channel 12-3-45. This also makes it clear that certain diagrams will be harder to express in a given channel as they will require crossing moves. We subsequently use the spectral-split representation for the Dirac deltas, yielding

$$\int d^{d+1}y d^{d+1}y' d^{d+1}y'' d^d w d^d w' d\nu_1 d\nu_2 \rho_\delta(\nu_1) \rho_\delta(\nu_2) (K_{\Delta_1}(y', x_1) K_{\Delta_2}(y', x_2) K_{d/2+i\nu_1}(y', w) K_{d/2-i\nu_1}(y, w) K_{\Delta_3}(y, x_3) K_{d/2+i\nu_2}(y, w') K_{d/2-i\nu_2}(y'', w') K_{\Delta_4}(y'', x_4) K_{\Delta_5}(y'', x_5)) =$$

(5.B.12)

At this stage, all the AdS integrals are standard three-point integrals which can readily be performed yielding

$$G_{\text{ctc}} = \int d\nu_1 d\nu_2 \rho_\delta(\nu_1) \rho_\delta(\nu_2) a_{\Delta_1, \Delta_2, d/2+i\nu_1} a_{d/2-i\nu_1, \Delta_3, d/2+i\nu_2} a_{d/2-i\nu_2, \Delta_4, \Delta_5} \left[\int dw dw' \right.$$
(5.B.13)

$$\left. \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_{d/2+i\nu_1}(w) \rangle \langle \mathcal{O}_{d/2-i\nu_1}(w) \mathcal{O}_3(x_3) \mathcal{O}_{d/2+i\nu_2}(w') \rangle \langle \mathcal{O}_{d/2-i\nu_2}(w') \mathcal{O}_4(x_4) \mathcal{O}_5(x_5) \rangle \right],$$

where in the second line we identify the 5-point conformal partial wave (CPW)

$$\Psi_{\nu_1, \nu_2}^{12,3,45}(x_i) = \int dw dw' \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_{d/2+i\nu_1}(w) \rangle \langle \mathcal{O}_{d/2-i\nu_1}(w) \mathcal{O}_3 \mathcal{O}_{d/2+i\nu_2}(w') \rangle \langle \mathcal{O}_{d/2-i\nu_2}(w') \mathcal{O}_4 \mathcal{O}_5 \rangle,$$
(5.B.14)

and can therefore compactly write the spectral representation of the 5-point contact diagram

$$G_{\text{ctc}}(x_i) = \int d\nu_1 d\nu_2 \rho_{\text{ctc}}(\nu_1, \nu_2) \Psi_{\nu_1, \nu_2}^{12,3,45}(x_i),$$
(5.B.15)

with

$$\rho_{\text{ctc}}(\nu_1, \nu_2) = \rho_\delta(\nu_1) \rho_\delta(\nu_2) a_{\Delta_1, \Delta_2, d/2+i\nu_1} a_{d/2-i\nu_1, \Delta_3, d/2+i\nu_2} a_{d/2-i\nu_2, \Delta_4, \Delta_5}.$$
(5.B.16)

The spectral representation is related to the conformal block decomposition by deformation of the spectral contour on the principal series. Indeed, physical operators

have real scaling dimensions which are captured by the poles of the spectral function $\rho(\nu_1, \nu_2)$. However, the partial wave is not a pure conformal block, it is instead a linear combination of conformal blocks all four possible combination of their shadows ($\nu_i \rightarrow -\nu_i$) weighted by the so-called shadow coefficients. See for example [88, 232] for a detailed discussion.

Once the dust settles, the OPE coefficients are simply obtained as residues of the spectral function

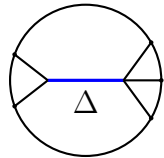
$$a_{\Delta_L, \Delta_R} = 4 \operatorname{Res}_{\nu_2 \rightarrow \Delta_R} \operatorname{Res}_{\nu_1 \rightarrow \Delta_L} K_{d/2-i\nu_1}^{\Delta_3, d/2-i\nu_2} K_{d/2-i\nu_2}^{\Delta_3, d/2+i\nu_1} \rho(\nu_1, \nu_2), \quad (5.B.17)$$

where K are the shadow factors. With this expression in hand, we can analyse the pole structure and detect what families of operators are exchanged in this correlation function. There are three families of exchanges

- $\Delta_L = \Delta_1 + \Delta_2 + 2n_L, \Delta_R = \Delta_4 + \Delta_5 + 2n_R$, ‘double-trace, double-trace’
- $\Delta_L = \Delta_1 + \Delta_2 + 2n_L, \Delta_R = \Delta_1 + \Delta_2 + \Delta_3 + 2n_L + 2m$, ‘double-trace, triple-trace’
- $\Delta_L = \Delta_3 + \Delta_4 + \Delta_5 + 2n_R + 2m, \Delta_R = \Delta_4 + \Delta_5 + 2n_R$, ‘triple-trace, double-trace’

corresponding exactly to the two most interesting families of functionals discussed in the main text. The respective OPE coefficients are obtained precisely by computing the residues in (5.B.17) at these values of scaling dimensions.

Single- and double-exchange diagrams (direct channel) It is very straightforward to adapt the previous derivation to single- and double-exchange diagrams where the channel matches our choice of OPE 12-3-45. The only difference is that either one or two of the Dirac delta functions are replaced by bulk-to-bulk propagators, and therefore at the level of the spectral decomposition we simply need to perform the replacement $\rho_\delta \rightarrow \rho_\Delta$. This simple rule can also be seen in the formalism used in the main text, since the bulk-to-bulk propagator is an eigenfunction of the laplacian operator. Concretely, we have



$$= G_{\text{sing}, \Delta} = \int d\nu_1 d\nu_2 \rho_{\text{sing}, \Delta}(\nu_1, \nu_2) \Psi_{\nu_1, \nu_2}^{12, 3, 45}(x_i), \quad (5.B.18)$$

with the spectral function

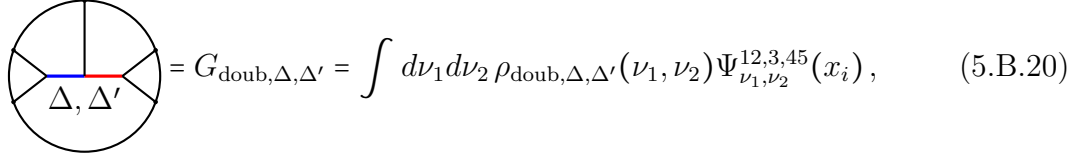
$$\rho_{\text{sing}, \Delta}(\nu_1, \nu_2) = \rho_\Delta(\nu_1) \rho_\delta(\nu_2) a_{\Delta_1, \Delta_2, d/2+i\nu_1} a_{d/2-i\nu_1, \Delta_3, d/2+i\nu_2} a_{d/2-i\nu_2, \Delta_4, \Delta_5}. \quad (5.B.19)$$

while the change to the spectral function is superficially small, it introduces new families of exchanged operators because of the new pole present in ρ_Δ . Apart from the families enumerated for the contact diagram, we also have

- $\Delta_L = \Delta, \Delta_R = \Delta_4 + \Delta_5 + 2n_R$ ‘single-trace, double-trace’
- $\Delta_L = \Delta, \Delta_R = \Delta_3 + \Delta + 2n_R$ ‘single-trace, exchange-double-trace’

which are related to 4-point functional with Δ as an external operator, as discussed in the main text.

Similarly, for the double exchange we have



$$= G_{\text{doub}, \Delta, \Delta'} = \int d\nu_1 d\nu_2 \rho_{\text{doub}, \Delta, \Delta'}(\nu_1, \nu_2) \Psi_{\nu_1, \nu_2}^{12,3,45}(x_i), \quad (5.B.20)$$

with the spectral function

$$\rho_{\text{doub}, \Delta, \Delta'}(\nu_1, \nu_2) = \rho_{\Delta}(\nu_1) \rho_{\Delta'}(\nu_2) a_{\Delta_1, \Delta_2, d/2+i\nu_1} a_{d/2-i\nu_1, \Delta_3, d/2+i\nu_2} a_{d/2-i\nu_2, \Delta_4, \Delta_5}. \quad (5.B.21)$$

which again introduces new families of exchanged operators

- $\Delta_L = \Delta, \Delta_R = \Delta'$ ‘single-trace, single-trace’
- $\Delta_L = \Delta_1 + \Delta_2 + 2n_L, \Delta_R = \Delta'$ ‘double-trace, single-trace’
- $\Delta_L = \Delta_3 + \Delta' + 2n_L, \Delta_R = \Delta'$ ‘exchange-double-trace, single-trace’

We conclude by noting that single- and double- exchange diagrams in other channels can also be obtained in this formalism at the cost of introducing crossing-kernels/6j-symbols in the spectral functions. In the main text, we instead use the Mellin formalism, integrated vertex identities, and the equation of motion/Laplacian to derive recursion relations for the OPE coefficients of these diagrams.

5.C Integrated vertex identities

In this appendix, we discuss a standard tool for the evaluation of Witten diagrams: Integrated vertex identities. Concretely, we first give the scalar integrated vertex identity in arbitrary d (App. 5.C.1) and then showcase its usage for the evaluation of $d = 1$ exchange Witten diagrams, that are relevant for the construction of 5-point Polyakov blocks discussed in this chapter (App. 5.C.2). Finally, we describe a notebook shared in the ancillary files provided on the arxiv page of this publication, which contains further integrated vertex identity results on 1d 5-point Witten diagrams (App. 5.C.3).

5.C.1 Scalar integrated vertex identity

Integrated vertex identities allow us to express AdS_2 vertices involving bulk-to-bulk and bulk-to-boundary propagators integrated over a common point in AdS as a sum of bulk-to-boundary propagators. The concrete example discussed in this appendix is the integrated vertex identity expressing the integral

$$I(y'; x_1, x_2) = \int d^{d+1}y K_{\Delta_1}(y; x_1) K_{\Delta_2}(y; x_2) G_{BB}^\Delta(y, y') \quad (5.C.1)$$

as the sum

$$\begin{aligned} I(y'; x_1, x_2) &= \sum_{i=0}^{\infty} (x_{12})^{2i} T_i(\Delta, \Delta_1, \Delta_2) K_{\Delta_1+1}(y'; x_1) K_{\Delta_2+1}(y'; x_2) \\ &\quad + \sum_{i=0}^{\infty} (x_{12})^{\Delta-\Delta_1-\Delta_2+2i} Q_i(\Delta, \Delta_1, \Delta_2) K_{\frac{\Delta+\Delta_1-\Delta_2}{2}+i}(y', x_1) K_{\frac{\Delta-\Delta_1+\Delta_2}{2}+i}(y', x_2), \end{aligned} \quad (5.C.2)$$

where

$$T_i(\Delta, \Delta_1, \Delta_2) = \frac{(\Delta_1)_i (\Delta_2)_i}{(\Delta - \Delta_{12})(\Delta - \tilde{\Delta}_{12}) \left(\frac{\Delta_{12}-\Delta+2}{2}\right)_i \left(\frac{\Delta_{12}-\tilde{\Delta}+2}{2}\right)_i} \quad (5.C.3)$$

and

$$\begin{aligned} Q_i(\Delta, \Delta_1, \Delta_2) &= \frac{(-1)^i \Gamma\left(\frac{d-2i-2\Delta}{2}\right) \sin\left(\frac{\pi(d-2\Delta)}{2}\right) \Gamma\left(\frac{-d+\Delta+\Delta_1+\Delta_2}{2}\right) \Gamma\left(\frac{-\Delta-\Delta_1+\Delta_2+2}{2}\right)}{4\pi \Gamma(i+1) \Gamma(\Delta_1) \Gamma(\Delta_2)} \\ &\quad \times \frac{\Gamma\left(\frac{\Delta-\Delta_1+\Delta_2}{2}\right) \Gamma\left(\frac{\Delta+\Delta_1-\Delta_2}{2}\right) \Gamma\left(\frac{-\Delta+\Delta_1+\Delta_2}{2}\right) \Gamma\left(\frac{-\Delta+\Delta_1-\Delta_2+2}{2}\right)}{\Gamma\left(\frac{-\Delta+\Delta_1-\Delta_2-2i+2}{2}\right) \Gamma\left(\frac{-\Delta-\Delta_1+\Delta_2-2i+2}{2}\right)}. \end{aligned} \quad (5.C.4)$$

For a derivation of this result and similar identities see [233].

5.C.2 Exchange diagrams

Using Identity (5.C.2), tree level scalar exchange Witten diagrams can be expressed as sums of contact diagrams, which can then directly be decomposed into conformal blocks. Concretely, this leads to a decomposition into two infinite sums of contact

diagrams

$$\begin{aligned}
 & \Delta_4 = \sum_{i=0}^{\infty} (x_{12})^{2i} T_i(\Delta, \Delta_1, \Delta_2) \\
 & + \sum_{i=0}^{\infty} (x_{12})^{\Delta - \Delta_1 - \Delta_2 + 2i} Q_i(\Delta, \Delta_1, \Delta_2)
 \end{aligned}
 \tag{5.C.5}$$

for single exchange diagrams and analogously four double infinite sums of contact diagrams for double exchange diagrams.

5.C.3 Comments on `five_point_iv_data.nb`

The purpose of this appendix is to guide the reader through the ancillary mathematica note book `five_point_iv_data.nb`, which contains expressions for coefficients occuring in the decomposition of 5-point single and double exchange diagrams into conformal blocks that were obtained using the decomposition into contact diagrams discussed in the previous two subsections.

The notebook is structured into the following sections.

1. **Preliminary Definitions.** This section defines useful functions such as the coefficients T_i and Q_i of eq. (5.C.4) and (5.C.3).
2. **Double Exchange: Raw Coefficients.** The coefficient of the leading blocks in the decomposition of exchange diagrams into 12-3-45 conformal blocks can directly be read off from the decomposition of exchange diagrams into contact diagrams as the normalisation of three-point functions obtained by taking OPE limits of the contact diagrams. The second section of the notebook contains these coefficients for the 12-3-45, 12-4-35, 14-2-35 and 14-3-25 double exchange diagrams. All other double exchange diagrams are related to one of these four by relabeling of the external legs.

3. **Double Exchange: Sums.** The data collected in the second section of the notebook is then arranged into series expressions in the third section of the notebook, which can be evaluated by truncating the sum at some finite order.
4. **Double Exchange: Integrals.** The series encountered can be viewed as two variable hypergeometric functions evaluated at 1. The convergence of these series is quite poor which makes simply performing truncated sums unsuitable for a high precision numerical evaluation. Instead, the fourth section of the notebook converts the sums into standard contour integral representations of the relevant hypergeometric functions, which can be evaluated more efficiently.
5. **Double Exchange: Discontinuity.** The integral contours can be deformed to further simplify the integrand to its discontinuity before numerical evaluations. This strategy is implemented in the fifth section of the notebook.
6. **Double Exchange: The Special Case $\Delta_\phi = 1$.** The integrals obtained in the previous step simplify considerably in the case $\Delta_\phi = 1$. To exploit this simplification, this case is implemented separately.
7. **Single Exchange: Raw Coefficients.** Finally, the above strategy is repeated for the single exchange diagrams.
8. **Single Exchange: Simplified Coefficients.** In the case of single exchange diagrams only single variable hypergeometric functions, which are by default already efficiently implemented in Mathematica, need to be evaluated. We can rely on those and do not need to resort to rewriting the sums as integrals in this case.

5.D Recurrence relations for CC Witten diagrams

In this appendix, we develop recurrence relations which make use of Casimir operators to determine the conformal block expansions of Cross-Channel (CC) Witten diagrams in terms of certain (families of) seed blocks. These relations were used as cross-checks of the Mellin and integrated vertex methods used above, and can also be used to speed up the determination of functionals with higher values of the labels n_1, n_2 . A summary of the results (as well as many relations omitted below) is given in the ancillary notebook `casimir5ptrecrel.nb`.

5.D.1 Four-point recap

We start by reviewing the logic in the case of 4-point exchange Witten diagrams. Using the fact that the bulk-to-bulk propagator is a Green's function of the AdS

Laplacian

$$(\square_{\text{AdS}_2} - \Delta(\Delta - 1))G_{BB}(y_1, y_2) = \frac{1}{\sqrt{g}}\delta^{(2)}(y_1 - y_2), \quad (5.D.1)$$

as well as AdS isometry/boundary conformal invariance at the boundary-boundary-bulk vertex, we can rewrite the action of the bulk Laplacian in terms of the boundary quadratic Casimir associated to the pair of points attached to the integration vertex. Hence, when acting with the s -channel Casimir $\mathcal{C}_{12}$ on an s -channel exchange diagram, we have

$$(\mathcal{C}_{12} - \Delta(\Delta - 1))A^{s-\Delta}(x_i) = A^{\text{ctc}}(x_i). \quad (5.D.2)$$

Now, we can use the s -channel OPE and the fact that the s -channel conformal blocks are eigenfunctions of the Casimir of the respective channel, to linearly relate the OPE coefficients of the exchange diagram to those of the contact

$$(C_n^{(s-\Delta)})^2 = \kappa \frac{(C_n^{(\text{ctc})})^2}{\Delta_n(\Delta_n - 1) - \Delta(\Delta - 1)}, \quad (5.D.3)$$

where Δ_n denotes the dimension of double-twist operators and we allowed for a constant κ to soak up normalisations of the propagators. If instead we consider a cross-channel exchange diagram, say in the t -channel, we can use the t -channel Casimir $\mathcal{C}_{13}$ to obtain a contact diagram. However, the s -channel conformal blocks G_Δ are not eigenfunctions of the t -channel Casimir. Instead, when acted by this cross-channel differential operator, they become a linear combination of blocks with integer shifts in the scaling dimension $\Delta - 1, \Delta, \Delta + 1$, given by

$$\mathcal{C}_{13}G_\Delta = \mu_-(\Delta)G_{\Delta-1} + \mu_0(\Delta)G_\Delta + \mu_+(\Delta)G_{\Delta+1}, \quad (5.D.4)$$

with the coefficients

$$\mu_-(\Delta) = -(\Delta - 2\Delta_\phi)^2, \quad \mu_0(\Delta) = \left(\frac{c_\Delta}{2} - 2c_{\Delta_\phi}\right), \quad \mu_+(\Delta) = -\frac{\Delta^2(2\Delta_\phi + \Delta - 1)^2}{4(\Delta^2 - 1)}, \quad (5.D.5)$$

where Δ_ϕ is the external dimension and we introduced the shorthand notation $c_\Delta \equiv \Delta(\Delta - 1)$. Now, matching the conformal block expansions, we can derive a recurrence relation for the OPE coefficients

$$\mu_-(\Delta_n + 1)(C_{\Delta_{n+1}}^{t,\Delta})^2 + \mu_0(\Delta_n)(C_{\Delta_n}^{t,\Delta})^2 + \mu_+(\Delta_n - 1)(C_{\Delta_{n-1}}^{t,\Delta})^2 - c_\Delta(C_{\Delta_n}^{t,\Delta})^2 = \kappa(C_{\Delta_n}^{\text{ctc}})^2, \quad (5.D.6)$$

which can be solved to a desired order (or in this case, exactly), up to a seed coefficient, that of the leading double twist.

5.D.2 Shift relations for five-point blocks

To derive recurrence relations for the 5-point double- and single-exchange diagrams we need to understand the action of the different Casimir operators on the 5-point

blocks in the 12-3-45 OPE channel. By definition, the block is an eigenfunction of the Casimirs $\mathcal{C}_{12}$ and $\mathcal{C}_{45}$, satisfying the eigenvalue equations

$$\mathcal{C}_{12}G_{\Delta_1,\Delta_2}(\chi_1,\chi_2) = c_{\Delta_1}G_{\Delta_1,\Delta_2}(\chi_1,\chi_2), \quad (5.D.7)$$

$$\mathcal{C}_{45}G_{\Delta_1,\Delta_2}(\chi_1,\chi_2) = c_{\Delta_2}G_{\Delta_1,\Delta_2}(\chi_1,\chi_2). \quad (5.D.8)$$

Instead, when acted by the Casimir $\mathcal{C}_{13}$, it acquires shifts by $-1, 0$ and 1 units in the scaling dimension Δ_1 , (which will eventually lead to three term recurrence relations)

$$\mathcal{C}_{13}G_{\Delta_1,\Delta_2} = \nu_{-}(\Delta_1,\Delta_2)G_{\Delta_1-1,\Delta_2} + \nu_0(\Delta_1,\Delta_2)G_{\Delta_1,\Delta_2} + \nu_{+}(\Delta_1,\Delta_2)G_{\Delta_1+1,\Delta_2}, \quad (5.D.9)$$

where the coefficients are given by

$$\nu_{-}(\Delta_1,\Delta_2) = -(\Delta_1 - \Delta_\phi)(\Delta_1 - \Delta_2 - \Delta_\phi), \quad \nu_0(\Delta_1,\Delta_2) = \frac{1}{2}(c_{\Delta_2} - c_{\Delta_1} + 3c_{\Delta_\phi}), \quad (5.D.10)$$

$$\nu_{+}(\Delta_1,\Delta_2) = -\frac{(\Delta_1 + \Delta_2 - \Delta_\phi)(\Delta_1 - \Delta_2 - \Delta_\phi)(\Delta_1 + \Delta_2 + \Delta_\phi - 1)(\Delta_1 + 2\Delta_\phi - 1)}{4(\Delta_1^2 - 1)},$$

with a similar relation for $\mathcal{C}_{23}$, and equivalent ones with $\Delta_1 \leftrightarrow \Delta_2$ for $\mathcal{C}_{34}$ and $\mathcal{C}_{45}$. Finally, we can consider Casimir operators with points attached to both of the end-points of the OPE diagram, i.e. $\mathcal{C}_{14}, \mathcal{C}_{15}, \mathcal{C}_{24}$ and $\mathcal{C}_{25}$. In this case, there are shifts by $-1, 0, 1$ units in Δ_1, Δ_2 in all possible combinations, leading to nine terms, for example

$$\begin{aligned} \mathcal{C}_{15}G_{\Delta_1,\Delta_2} = & \nu_{--}(\Delta_1,\Delta_2)G_{\Delta_1-1,\Delta_2-1} + \nu_{0-}(\Delta_1,\Delta_2)G_{\Delta_1,\Delta_2-1} + \nu_{+-}(\Delta_1,\Delta_2)G_{\Delta_1+1,\Delta_2-1} \\ & + \nu_{-0}(\Delta_1,\Delta_2)G_{\Delta_1-1,\Delta_2} + \nu_{00}(\Delta_1,\Delta_2)G_{\Delta_1,\Delta_2} + \nu_{+0}(\Delta_1,\Delta_2)G_{\Delta_1+1,\Delta_2} \\ & + \nu_{-+}(\Delta_1,\Delta_2)G_{\Delta_1-1,\Delta_2+1} + \nu_{0+}(\Delta_1,\Delta_2+1)G_{\Delta_1,\Delta_2+1} + \nu_{++}(\Delta_1,\Delta_2)G_{\Delta_1+1,\Delta_2+1}, \end{aligned} \quad (5.D.11)$$

where the coefficients are given by

$$\begin{aligned} \nu_{--}(\Delta_1,\Delta_2) &= (\Delta_1 - 2\Delta_\phi)(\Delta_2 - 2\Delta_\phi), \quad \nu_{0-}(\Delta_1,\Delta_2) = \frac{1}{2}(\Delta_1 - 2\Delta_\phi)(\Delta_1 - \Delta_2 - \Delta_\phi), \\ \nu_{+-}(\Delta_1,\Delta_2) &= \frac{(\Delta_1 - 2\Delta_\phi)(\Delta_\phi - \Delta_1 + \Delta_2)(\Delta_\phi - \Delta_1 + \Delta_2 + 1)(2\Delta_\phi + \Delta_2 - 1)}{4(\Delta_2^2 - 1)}, \\ \nu_{-0}(\Delta_1,\Delta_2) &= -\frac{1}{2}(\Delta_2 - 2\Delta_\phi)(\Delta_1 - \Delta_2 + \Delta_\phi), \quad \nu_{00}(\Delta_1,\Delta_2) = \frac{1}{4}(9c_{\Delta_\phi} - c_{\Delta_1} - c_{\Delta_2}), \\ \nu_{+0}(\Delta_1,\Delta_2) &= \frac{(\Delta_1 + \Delta_2 - \Delta_\phi)(\Delta_\phi - \Delta_1 + \Delta_2)(\Delta_\phi + \Delta_1 + \Delta_2 - 1)(2\Delta_\phi + \Delta_2 - 1)}{8(4\Delta_2^2 - 1)}, \\ \nu_{+-}(\Delta_1,\Delta_2) &= \frac{(\Delta_2 - 2\Delta_\phi)(\Delta_\phi + \Delta_1 - \Delta_2)(\Delta_\phi + \Delta_1 - \Delta_2 + 1)(2\Delta_\phi + \Delta_1 - 1)}{4(4\Delta_1^2 - 1)}, \\ \nu_{+0}(\Delta_1,\Delta_2) &= \frac{(\Delta_1 + \Delta_2 - \Delta_\phi)(\Delta_\phi + \Delta_1 - \Delta_2)(\Delta_\phi + \Delta_1 + \Delta_2 - 1)(2\Delta_\phi + \Delta_1 - 1)}{8(4\Delta_1^2 - 1)}, \\ \nu_{++} &= \frac{((\Delta_1 + \Delta_2)^2 - \Delta_\phi^2)((\Delta_1 + \Delta_2)^2 - (\Delta_\phi - 1)^2)(2\Delta_\phi + \Delta_1 - 1)(2\Delta_\phi + \Delta_2 - 1)}{16(4\Delta_1^2 - 1)(4\Delta_2^2 - 1)} \end{aligned} \quad (5.D.12)$$

with similar identities for the remaining Casimirs.

5.D.3 Recurrence relations for five-point Witten diagrams

Having derived the shift relations for the conformal blocks we can now apply them to reduce the number of exchanges in Witten diagrams from two to one and from one to zero (i.e. to a contact diagram).

Double-exchange $\rightarrow$ Single-exchange We begin by analysing double-exchange diagrams in terms of single-exchanges. Clearly, we can derive two independent reduction formulae by acting with the Casimirs in the two-cubic vertices. Treating the single-exchange OPE coefficients as known (we will determine them later), we can then work out how many seed double-exchange coefficients are needed to reconstruct the full diagram. We will discuss the diagrams in increasing order of complexity from the point of view of the recursion relations.

The easiest diagrams are in the channels $12-3-45$, $12-5-34$, $12-4-35$, $13-2-45$, $23-1-45$ since they are completely determined by the single-exchange diagrams and no additional seed coefficients are needed. For example, we have the relation

$$a_{\Delta_n, \Delta_m}^{12-5-34} = \kappa_{\Delta_1} \frac{a_{\Delta_n, \Delta_m}^{125-34, \Delta_2}}{c_{\Delta_n} - c_{\Delta_1}}, \quad (5.D.13)$$

and similarly for the remaining channels. The second hardest diagrams are in the channels $13-5-24$, $13-4-25$, $14-5-23$, $15-4-23$, $15-2-34$, $25-1-34$, $14-2-35$, $24-1-35$, and can be reduced to single-exchange coefficients, up to a single seed coefficient with $n = m = 0$. These diagrams satisfy one three-term recurrence relation and one nine-term recurrence relation. An example of a three-term recurrence relation is

$$\begin{aligned} \nu_{-}(\Delta_n + 1, \Delta_m) a_{\Delta_n+1, \Delta_m}^{13-4-25} + \nu_0(\Delta_n, \Delta_m) a_{\Delta_n, \Delta_m}^{13-4-25} + \nu_{+}(\Delta_n - 1, \Delta_m) a_{\Delta_n-1, \Delta_m}^{13-4-25} \\ = c_{\Delta_1} a_{\Delta_n, \Delta_m}^{13-4-25} + \kappa_{\Delta_1} a_{\Delta_n, \Delta_m}^{134-25, \Delta_2} \end{aligned} \quad (5.D.14)$$

and similarly in the other channels. The nine term recurrence relations are instead similar to the ones given below for the hardest diagrams. These are the diagrams in the channels $15-3-24$, $14-3-25$, which satisfy two nine-term recurrence relations such that two seed blocks are needed, for example those with $n = m = 0$ and $n = 1, m = 0$. An example of these relations is

$$\begin{aligned} \nu_{--}(\Delta_n + 1, \Delta_m + 1) a_{\Delta_n+1, \Delta_m+1}^{15-3-24} + \nu_{-0}(\Delta_n + 1, \Delta_m) a_{\Delta_n+1, \Delta_m}^{15-3-24} \\ + \nu_{0-}(\Delta_n, \Delta_m + 1) a_{\Delta_n, \Delta_m+1}^{15-3-24} + \nu_{00}(\Delta_n, \Delta_m) a_{\Delta_n, \Delta_m}^{15-3-24} + \nu_{0+}(\Delta_n, \Delta_m - 1) a_{\Delta_n, \Delta_m-1}^{15-3-24} \\ + \nu_{+-}(\Delta_n - 1, \Delta_m + 1) a_{\Delta_n-1, \Delta_m+1}^{15-3-24} + \nu_{+0}(\Delta_n - 1, \Delta_m) a_{\Delta_n-1, \Delta_m}^{15-3-24} \\ + \nu_{-+}(\Delta_n + 1, \Delta_m - 1) a_{\Delta_n+1, \Delta_m-1}^{15-3-24} + \nu_{++}(\Delta_n - 1, \Delta_m - 1) a_{\Delta_n-1, \Delta_m-1}^{15-3-24} \\ = c_{\Delta_1} a_{\Delta_n, \Delta_m}^{15-3-24} + \kappa_{\Delta_1} a_{\Delta_n, \Delta_m}^{153-24, \Delta_2}, \end{aligned} \quad (5.D.15)$$

and similarly for the other coefficients and in the other channel.

Single-exchange $\rightarrow$ Contact Reducing the double- to single-exchanges is already a drastic reduction in complexity, as both the Mellin (only single pole) and integrated vertex methods (only single integral) become much more manageable in this case. Nonetheless, nothing stops us from further simplifying the problem by using recurrence relations to reduce OPE coefficients to those of the contact diagram.

The nature of this reduction is now very different, since we have only one instead of two Casimir operators to act with, the number of independent recurrence relations is lower and in some channels, we will be left with an infinite number of seed blocks. Fortunately, even in the worst case scenario we reduce the ‘doubly infinite’ number of coefficients (labeled by n, m) to a ‘single infinity’ labeled by either $n = 0, m \geq 0$ or $n \geq 0, m = 0$ (or both). We again proceed by increasing level of complexity.

The easiest channels are the 12 – 345 and 123 – 45 exchanges which are completely determined in terms of the contact diagram. For example

$$a_{\Delta_n, \Delta_m}^{12-345, \Delta_1} = \kappa_{\Delta_1} \frac{a_{\Delta_n, \Delta_m}^{\text{ctc}}}{c_{\Delta_n} - c_{\Delta_1}}, \quad (5.D.16)$$

and similarly for the other diagram. The channels 125 – 34 and 124 – 35 require as boundary conditions the full $n \geq 0, m = 0$ set of coefficients and satisfy recursion relations of the form

$$\begin{aligned} \nu_{-}(\Delta_n, \Delta_m + 1) a_{\Delta_n, \Delta_m + 1}^{125-34} + \nu_0(\Delta_n, \Delta_m) a_{\Delta_n, \Delta_m}^{125-34} + \nu_{+}(\Delta_n, \Delta_m - 1) a_{\Delta_n, \Delta_m - 1}^{125-34}, \\ = c_{\Delta_1} a_{\Delta_n, \Delta_m}^{125-34} + \kappa_{\Delta_1} a_{\Delta_n, \Delta_m}^{\text{ctc}} \end{aligned} \quad (5.D.17)$$

and similarly the channels 13 – 245 and 23 – 145 require as boundary conditions the full $n = 0, m \geq 0$ set of coefficients and satisfy the same recursion relations with the role of n and m swapped. Finally, the most complicated diagrams are in the channels 14 – 235, 15 – 234, 24 – 135 and 25 – 134, and require as seeds the full set of coefficients satisfying $nm = 0$. They satisfy recursion relations of the form

$$\begin{aligned} \nu_{--}(\Delta_n + 1, \Delta_m + 1) a_{\Delta_n + 1, \Delta_m + 1}^{14-235} + \nu_{-0}(\Delta_n + 1, \Delta_m) a_{\Delta_n + 1, \Delta_m}^{14-235} \\ + \nu_{0-}(\Delta_n, \Delta_m + 1) a_{\Delta_n, \Delta_m + 1}^{14-235} + \nu_{00}(\Delta_n, \Delta_m) a_{\Delta_n, \Delta_m}^{14-235} + \nu_{0+}(\Delta_n, \Delta_m - 1) a_{\Delta_n, \Delta_m - 1}^{14-235} \\ + \nu_{+-}(\Delta_n - 1, \Delta_m + 1) a_{\Delta_n - 1, \Delta_m + 1}^{14-235} + \nu_{+0}(\Delta_n - 1, \Delta_m) a_{\Delta_n - 1, \Delta_m}^{14-235} \\ + \nu_{-+}(\Delta_n + 1, \Delta_m - 1) a_{\Delta_n + 1, \Delta_m - 1}^{14-235} + \nu_{++}(\Delta_n - 1, \Delta_m - 1) a_{\Delta_n - 1, \Delta_m - 1}^{14-235} \\ = c_{\Delta_1} a_{\Delta_n, \Delta_m}^{14-235} + \kappa_{\Delta_1} a_{\Delta_n, \Delta_m}^{\text{ctc}}, \end{aligned} \quad (5.D.18)$$

and similarly for the other channels. A complete list of all the recursion relations that can be used to reduced the computations to the minimal amount of seed coefficients is given in `casimir5ptrecrel.nb`.

Chapter 6

Future perspectives

– Solving the input data problem & other future directions.

The previous three chapters have laid the foundation for a higher-point extension of the numerical conformal bootstrap and the analytic Polyakov–Mellin bootstrap. Chapter 3 established the existence of rigorous six-point bounds, while Chapter 4 made these bounds practically accessible. Chapter 5 initiated the construction of analytic multi-point functionals.

Despite the significant progress reported in this thesis, these results should primarily be viewed as only a proof of concept for new multi-point techniques. Substantial work remains in order to fully realise the potential of these approaches to the bootstrap. In particular, we have so far addressed only the first two of the three challenges formulated in Section 1.2, namely the **computability problem** and the **convergence problem**, but not the **input data problem**.

This short chapter outlines several possible directions for future research in the multi-point bootstrap. Concretely, Section 6.1 discusses future objectives for the numerical six-point bootstrap. In particular, Subsection 6.1.1 proposes a strategy to address the input data problem by combining six-point and four-point crossing equations. Section 6.2 outlines future directions for the analytic multi-point bootstrap.

6.1 Future multi-point numerical bootstrap

Chapter 3 took the first steps towards establishing a rigorous numerical bootstrap programme for the six-point correlator. In it, the crossing equation was formulated in a way amenable to using SDP methods, albeit for infinite-dimensional matrices. The infinite nature of the problem constituted a major technical challenge, effectively restricting us to deriving only a few low-derivative-order bounds. As an important

check, these were compared to free field theory correlators. Most bounds in Chapter 3 were found to be saturated by these simple correlation functions.

Chapter 4 introduced sparse SDP techniques to the numerical six-point bootstrap, resulting in significant performance gains and making the many advantages of the conformal bootstrap's standard SDP solver **SDPB** accessible in multi-point applications. With this technical improvement, new bootstrap targets came within reach. Concretely, Section 4.3 discussed deformations of the generalised free boson and fermion that are accessible via the six-point bootstrap, which are related to ϕ^4 and ϕ^6 deformations of a massive free boson in AdS_2 . While the improvement presented in Chapter 4 constitutes a key step in the technical advancement of the higher-point numerical bootstrap, there remain many important challenges and necessary future steps for the development of this programme.

This section proposes several directions for future research on the six-point bootstrap. The long-term goal behind these proposals is to advance SDP-based multi-point bootstrap techniques from the proof-of-concept stage of this thesis to a versatile and widely applicable tool for the bootstrap community.

6.1.1 The input data problem

The six-point bootstrap proposed in Chapter 3 is based on the decomposition of correlation functions involving six insertions of a scalar field ϕ into sums of products of four-point functions of the form $\langle\phi\phi\phi\mathcal{O}\rangle\langle\mathcal{O}^\dagger\phi\phi\phi\rangle$, see e.g. Figure 4.1.1. In this procedure, the exchange of ϕ itself, i.e. the $\mathcal{O} = \phi$ contribution, is rather special. Let us make three related observations on the ϕ -exchange.

1. **Positivity** of $\langle\phi\phi\phi\phi\rangle$. As opposed to generic four-point functions $\langle\phi\phi\phi\mathcal{O}\rangle$ occurring in the decomposition of the six-point correlator, the four-point function $\langle\phi\phi\phi\phi\rangle$ is manifestly positive. The proper constraint to impose for the ϕ -exchange in the six-point bootstrap is hence *copositivity* not *positive semi-definiteness*. This weaker property of an inner product is defined as:

$$M \text{ copositive} \Leftrightarrow v^T M v \geq 0 \quad \forall v \geq 0. \quad (6.1.1)$$

More concretely, the ϕ -exchange contribution is

$$\sum_{\mathcal{O}_1, \mathcal{O}_3} C_{\phi\phi\mathcal{O}_1}^2 G_{\Delta_1, \Delta_\phi, \Delta_3}(\chi_1, \chi_2, \chi_3) C_{\phi\phi\mathcal{O}_3}^2 \subseteq G(\chi_1, \chi_2, \chi_3), \quad (6.1.2)$$

where G is the comb channel six-point block. Thus, the correct property to impose in order to derive a bound is copositivity¹ of linear combinations of derivatives of $G_{\Delta_1, \Delta_\phi, \Delta_3}$ (or more precisely of the crossing vector $F_{\Delta_1, \Delta_\phi, \Delta_3}$ obtained by subtracting the appropriate crossed-channel block from $G_{\Delta_1, \Delta_\phi, \Delta_3}$).

¹As a matrix (or operator) with indices Δ_1 and Δ_3

2. **Unboundedness** of problems like gap maximisation in $\phi \times \phi \times \phi \sim \phi + \mathcal{O}_* + \dots$. In principle we could try to ignore the nuance about the positivity structure of the ϕ -exchange pointed out above and simply impose positive semi-definiteness of the ϕ -contribution, just as we successfully imposed positive semi-definiteness on the $\mathcal{O} \neq \phi$ contributions in Chapters 3 and 4. While this would most likely not produce optimal bounds, it is conceivable that bootstrap problems can be formulated in which this approach would produce at least some bounds. The author of this thesis has, as of now, not found any bounds that can be produced in this manner, which is why he resorted to specifying $\langle \phi \phi \phi \phi \rangle$ as input in Chapters 3 and 4. This is the origin of the input data problem that has been repeatedly mentioned throughout this thesis.
3. **Inaccessibility** of four-point data. The six-point bootstrap does not resolve four-point functions into three-point functions. This implies two shortcomings of the procedure. First, it cannot use gap assumptions in $\phi \times \phi$ OPE to improve bounds. Second, it cannot directly produce bounds on individual OPE coefficients and the $\phi \times \phi$ spectrum.

The second and third of these observations are limitations of the six-point bootstrap. The first observation, on the other hand, is a resource that might help us to overcome these limitations. To make use of that resource, we need to treat the ϕ -contribution to

$$\phi \times \phi \times \phi \sim \phi + \mathcal{O}_* + \dots \quad (6.1.3)$$

separately from all other exchanges. For the latter, i.e. for contributions of $\Delta \geq \Delta_*$ operators, we should impose positivity of the descendant-space matrix $\alpha_{nm}[\Delta]$ as in Chapter 3 and 4.

For $\Delta = \Delta_\phi$, we can impose a relaxation of copositivity, such as the positivity of $F_{\Delta_1, \Delta_\phi, \Delta_3}$ (as individual numbers) for all values of Δ_i . This particular relaxation is not only easy to implement but also allows us to quite naturally combine the six-point with the four-point bootstrap. Indeed, we can simply add crossing vectors $F_{\Delta}^{(n)}$ of the four-point function $\langle \phi \phi \phi \phi \rangle$ to derivatives of $F_{\Delta_1, \Delta_\phi, \Delta_3}$ and then try to impose

$$\sum_{a,b,c} F_{\Delta_1, \Delta_\phi, \Delta_3}^{(a,b,c)} + \sum_{d,k} (F_{\Delta_1}^{(d)} \Delta_3^k + F_{\Delta_3}^{(d)} \Delta_1^k) \geq 0 \quad \forall \Delta_1, \Delta_3 \in \phi \times \phi. \quad (6.1.4)$$

Any linear combination of four-point and six-point crossing vectors that satisfies this positivity constraint, as well as positive semi-definiteness of six-point descendant-space matrices for $\Delta \geq \Delta_*$ implies a universal bound that does not require theory specific four-point input. In this way, the 4+6-point mixed-correlator bootstrap solves the input data problem.

Furthermore, the procedure allows us to study interesting new questions such as: How do bounds on four-point data (OPE coefficients and four-point gaps) depend on assumptions on the double-OPE $\phi \times \phi \times \phi$ gap? Or vice versa: what bounds do four-point gap assumptions impose on the six-point function? The author of this thesis hopes to report on these new bounds in the near future.

6.1.2 Other future directions

This short subsection comments on three further directions for future multi-point numerical bootstrap research, namely, the extension to higher dimensional CFTs, the extrapolation to the flat space limit, and the extraction of extremal spectra.

Six-point bootstrap in two dimensions. The restriction to 1d poses a significant limitation to the applicability of the numerical methods described in this thesis. It would therefore be highly desirable to extend the present formalism to higher dimensional CFTs. Formulating a numerical six-point bootstrap in 2d seems to be the most natural realistic first step in this direction. Such a generalisation would increase the set of discrete parameters from the descendant level n encountered in 1d to the two independent levels $n, \bar{n}$ and the primary spin J encountered in 2d. Importantly, taking derivatives w.r.t. the holomorphic and antiholomorphic cross-ratios available in 2d preserves the band structure that was crucial to the success of the 1d six-point bootstrap. It therefore seems plausible that, while numerically more demanding, a generalisation of the approach formulated here should be implementable in 2d rather straightforwardly from a conceptual point of view.

Flat space limit. A problem that could be studied already with the 1d methods described in this thesis is the extrapolation to large scaling dimensions and 2d flat-space scattering amplitudes. Reference [60] studied 1d bootstrap problems similar to those we encountered in Section 4.3.3. Through a two-fold extrapolation procedure, first extrapolating to infinite derivative order of the bootstrap, then extrapolating to large conformal weights, the authors were able to compute the flat space limit of the dual 2d S-matrix. In the companion paper [234] the result of this conformal bootstrap analysis was then related to extremal S-matrices determined from the S-matrix bootstrap. It would be very interesting to use the numerical bootstrap as a tool to learn more about multi-particle scattering amplitudes, especially since a multi-particle S-matrix bootstrap is so far missing (see however [235] for some steps towards a numerical S-matrix analysis of n -to- m -particle scattering amplitudes).

Spectrum extraction. The extraction of extremal spectra from the functionals obtained through SDP has been of vital importance since the early days of the numerical conformal bootstrap [236, 237]. It would be highly desirable to likewise

systematically extract extremal spectra from the functionals of the six-point bootstrap by studying their kernels. In the framework presented in this thesis, it should be noted that spectrum extraction is potentially more sensitive to the choice of X matrices (see Section 4.2.2) than the bounds themselves. Indeed, while we heuristically observed that relaxations to SDPs with a reduced number of X matrices can be applied without a strong impact on the bounds, the relaxation can lead to unphysical null-vectors of the extremal functionals, which annihilate only certain fixed descendant level n blocks at a fixed exchange dimension Δ .

6.2 Future multi-point analytic bootstrap

The work presented in Chapter 5 took the first steps to extend the Polyakov bootstrap approach to higher-point correlation functions, focusing on the 5-point function in one dimension. Crossing symmetric Polyakov blocks were constructed, and the associated sum-rules derived as well as extensively tested. Finally, the sum-rules were applied in a truncated bootstrap i.e. in the context of the primal approach to crossing reviewed in Section 2.2.1. This section lists several unexplored extensions and applications of these results in increasing order of complexity.

More on the 5-point 1d bootstrap Our sum-rules are a natural way to study $\mathbb{Z}_2$ breaking models, such as ϕ^3 theory in AdS_2 or certain symmetry breaking line defects. It would be nice to study the sum-rule equations in perturbation theory where many functional actions vanish. It is also straightforward to generalise our analysis to the case of non-identical scalars which is useful to study the lightest correlator in $\mathbb{Z}_2$ preserving theories: $\langle \phi\phi\phi^2\phi\phi \rangle$. Finally, it would be interesting to understand the formal properties of our functionals more systematically, namely their asymptotics and analytic structure.

Generalisation to six-points and fermions From our analysis, it becomes clear that constructing higher-point Polyakov blocks should be straightforward but technically very involved, due to the necessity of subtraction terms corresponding to higher-point vertices Φ^n in AdS_2 which are relevant deformations preserving the UV features of the solution. A simple way to bypass this is to consider the fermionic version of the Polyakov blocks, which will not involve vertices with more than three legs, since these correspond to irrelevant interactions. Of course, additional technical subtleties due to the fermionic nature of the Witten diagrams are to be expected in that case.

Higher-dimensions Observing the success of our functionals in the one-dimensional truncated Bootstrap makes the three-dimensional analysis of [90–92] a particularly

appealing target for the Polyakov Bootstrap. With modest resources, the three-dimensional truncated bootstrap produced reasonable estimates for OPE coefficients involving two stress-tensors in the 3d Ising model. A higher-dimensional generalisation of our approach has clear potential to significantly improve these predictions. The additional technical difficulties of including spin and multiple tensor structures in the OPE are, however, not to be underestimated.

S-matrix Bootstrap Finally, it should be mentioned that our results are intimately tied to S-matrices in two-dimensions, with $2 \rightarrow 3$ scattering amplitudes being the key observable. On the one hand, the flat-space limit of conformal 4-point functions is known to map to S-matrices in one higher-dimension [238]. The greater control over the analytic structure of conformal correlators can be used as an alternative to standard axiomatic approaches to scattering amplitudes in deriving fundamental properties of these objects [239, 240]. The QFT₂/CFT₁ setup [186, 209] is particularly powerful in this regard, and our work opens up an avenue to investigate the analytic properties of five-point amplitudes which, even in two space-time dimensions, are far from well understood. On the other hand, the Polyakov Bootstrap makes crossing symmetry manifest and therefore has a closely related cousin at the level of scattering amplitudes: the so-called crossing-symmetric dispersion relations (CSDR), which have proven to be a very useful tool in recent bootstrap applications [215, 216]. The synergy between these approaches might help to make higher-point amplitudes more amenable to a non-perturbative understanding.

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Part II

Pure NSNS AdS_3 holography

Chapter 7

Tensionless strings in symmetric product orbifolds

– New examples of exact holography.

What is known about the AdS_2 branes in pure NSNS AdS_3 backgrounds? How do their open string spectra simplify in the minimal NSNS flux limit? Can we extend the exact holographic dictionary available in this special limit and identify the precise defect or interface of the dual CFT corresponding to an AdS_2 brane?

This chapter begins with a lightning review of the $\text{SL}(2; \mathbb{R})_k$ model and its maximally symmetric boundary states. In the second section, Section 7.2, we then discuss how the $\text{SL}(2; \mathbb{R})_k$ model and its boundaries arise in the worldsheet description of pure NSNS AdS_3 string backgrounds and their AdS_2 branes, putting special emphasis on the minimal NSNS flux limit. As we review in Section 7.3, the latter is known to have an exact holographic dual involving a grand-canonical ensemble of symmetric product orbifolds. We end the chapter with a few brief comments on what is known about the holography of branes in this context, which provides the necessary background to then tread into the unknown and construct the missing holographic dual of AdS_2 branes in Chapters 8 and 9.

7.1 Two dimensional CFTs

Two dimensional CFTs famously possess Virasoro symmetry, an infinite dimensional extension of the global conformal symmetry discussed in Sec. 2.1. The current Section begins by briefly reviewing elementary properties of local conformal symmetry (Sec. 7.1.1). The established standard textbook on the subject is the monolithic Ref. [241]. For a lightweight introduction to conformal field theory on the plane, that provides an interesting bootstrap flavoured perspective on the subject, see Ref. [242]. The brief introduction to 2d CFT sets the stage for a subsequent discussion of the

$\mathrm{SL}(2; \mathbb{R})_k$ model (Sec. 7.1.2). The Section ends with a review of conformal boundary conditions (Sec. 7.1.3), focussing particularly on the maximally symmetric boundaries of $\mathrm{SL}(2; \mathbb{R})_k$. A systematic treatment of boundary conformal field theory is provided e.g. by the textbook [243].

7.1.1 The Virasoro algebra

In two dimensions, conformal symmetry is particularly powerful since any holomorphic or antiholomorphic map f of a Riemann surface is angle preserving with conformal factor $|f'|^2$. Indeed,

$$f^* dz^2 = df(z)^2 = |f'(z)|^2 dz^2. \quad (7.1.1)$$

However, the conformal group of a surface only consists of a small subset of these maps, the biholomorphic (and antiholomorphic) automorphisms. In the case of the sphere, the finite dimensional Lie group of holomorphic automorphisms is for instance the group of Möbius transformations

$$z \mapsto \frac{az + b}{cz + d} \quad \text{with} \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0. \quad (7.1.2)$$

Its Lie algebra $\mathfrak{sl}(2; \mathbb{C})$ is spanned by three complex vector fields. On the torus, only one of these survives, while the automorphism group of higher genus Riemann surfaces is zero dimensional¹. However, at least in a local QFT, constraints on the symmetry group reflecting the global geometry of spacetime should not directly affect the symmetry algebra acting on the Hilbert space. The latter should instead be based on the Lie algebra that generates infinitesimal symmetries of the quantisation time slices.

The Witt algebra. To determine the symmetry algebra of a 2d CFT in radial quantisation, we should hence not look for globally defined small (anti-)holomorphic deformations of the identity map, but only consider such deformations on a thin annulus containing the unit circle. The maps

$$z \mapsto z - \epsilon z^{n+1} = (1 + \epsilon \ell_n)z \quad \text{with} \quad \ell_n = -z^{n+1} \partial_z, \quad \text{for any} \quad n \in \mathbb{Z} \quad (7.1.3)$$

and their antiholomorphic counter-parts form a basis of such deformations. The generators ℓ_n of this Lie algebra, the Witt algebra, satisfy the commutation relation

$$[\ell_n, \ell_m] = (n - m) \ell_{m+n}. \quad (7.1.4)$$

To describe the symmetries of the quantum theory, we as always need to consider central extensions of this classical symmetry algebra.

¹Famously, Hurwitz's automorphisms theorem allows us to make the even sharper statement that it is a finite group with at most $84(g - 1)$ elements.

Central extension of the Witt algebra. The central extension of the Witt algebra is the Virasoro algebra. In a basis consisting of generators $\{L_n\}_{n \in \mathbb{Z}}$ that are in one-to-one correspondence with the generators $\{\ell_n\}_{n \in \mathbb{Z}}$ of the Witt algebra and a central generator $\hat{c}$ commuting with all L_n , the algebra is given by

$$[L_n, L_m] = (n - m)L_{n+m} + \delta_{n+m}^0 \frac{\hat{c}}{12} (n - 1)n(n + 1). \quad (7.1.5)$$

The Hilbert space of every local 2d CFT carries two chiral representations of this algebra, one implementing (infinitesimal) holomorphic and one implementing anti-holomorphic conformal transformations. Typically, we consider theories in whose representations of the Virasoro algebra the central generator $\hat{c}$ of both chiral copies simply acts by multiplication with a complex number c , the central charge of the theory. Below, we will mildly deviate from this premise by considering direct sums of CFTs with different central charges, which promotes $\hat{c}$ to a genuine operator whose insertion into correlation functions has a non-trivial effect.

The stress tensor. In the introductory paragraph of this subsection, we already gave a qualitative argument for the statement that local two-dimensional QFTs with global conformal symmetry should always enjoy full Virasoro symmetry. We can do better and in fact explicitly use one of the defining features of local theories, the existence of a local stress tensor, to construct representations of the Virasoro algebra on the radial quantisation Hilbert space. Indeed, global conformal symmetry implies that the stress tensor has only two non-trivial components T and $\bar{T}$, which satisfy the OPE

$$T(z)T(w) \sim \frac{\hat{c}/2}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{\partial T(w)}{z - w} + \dots \quad (7.1.6)$$

In particular, the singular part of the OPE is fully fixed up to the weight zero operator $\hat{c}$. In the canonical examples, this operator is a multiple of $\mathbb{1}$. Using eq. (7.1.6), one can directly show that the operators

$$L_n := \frac{1}{2\pi i} \oint dz z^{n+1} T(z) \quad \text{and} \quad \bar{L}_n := \frac{1}{2\pi i} \oint d\bar{z} \bar{z}^{n+1} \bar{T}(\bar{z}) \quad (7.1.7)$$

generate two copies of the Virasoro algebra.

Global and local Ward identities. By definition, the correlation functions of 2d CFTs are invariant under the action of the $n \in \{-1, 0, 1\}$ global conformal generators. We discussed the constraints which these global Ward identities put on CFT correlators already in Section 2.1. The generators of the Virasoro algebra allow us to furthermore restrict our attention to Virasoro primaries i.e. operators annihilated by all positive n Virasoro modes. Indeed, N -point correlators of descendants i.e. operators generated by acting with negative Virasoro modes on a primary, reduce

to correlation functions of primaries with additional insertions of the stress-energy tensor. Using the identity

$$\langle T(z) \prod_{i=1}^N V_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \rangle = \sum_{i=1}^N \left(\frac{h_i}{(z - z_i)^2} + \frac{1}{z - z_i} \partial_z \right) \langle \prod_{i=1}^N V_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \rangle, \quad (7.1.8)$$

the latter however reduce to N -point correlators of primaries. This gives rise to additional local Ward identities, relating correlation functions of Virasoro descendants to those of Virasoro primaries.

Symmetric product orbifolds The class of 2d CFTs that will be the most relevant in the two chapters 8 and 9 is that of symmetric product orbifolds. The symmetric product orbifold of degree N with seed theory $\mathcal{M}$ is the 2d CFT $\text{Sym}^N(\mathcal{M})$ obtained by taking a tensor product of N copies of the CFT $\mathcal{M}$ and gauging the S_N global symmetry that permutes the individual tensor factors. That is,

$$\text{Sym}^N(\mathcal{M}) = \mathcal{M}^{\otimes N} / S_N. \quad (7.1.9)$$

In the two chapters below, we take complementary perspectives on these models. In Chapter 8, we will take an algebraic perspective that focuses on the Hilbert space of the theory, which can be constructed by applying a projection onto S_N invariant states to the Hilbert space of the product theory $\mathcal{M}^{\otimes N}$ and its S_N twisted analogues. This procedure is reviewed in detail, in Section 8.1.1. Chapter 9, on the other hand, places the focus on correlation and partition functions instead of states in the Hilbert space. In particular, Section 9.1.2 provides an alternative definition of symmetric orbifolds in terms of a lift of CFT observables along ramified covering maps. To avoid redundancies, this paragraph, instead of providing a detailed review, simply refers the reader to the two complementary Sections 8.1.1 and 9.1.2 for further comments on symmetric orbifolds.

7.1.2 The $\text{SL}(2; \mathbb{R})_k$ model

In this section, we introduce the $\text{SL}(2; \mathbb{R})_k$ model. We first discuss the $\widehat{\mathfrak{sl}}(2, \mathbb{R})_k$ current algebra and its representations and build the spectrum of the model out of these. Then, we turn to its three-point functions. We introduce a sub-sector of the latter as defined by analytic continuation of the three-point functions of a simpler CFT, the exactly solved H_3^+ model. Finally, we sketch how symmetry and consistency conditions can be used to determine the remaining three-point functions, providing a fully explicit definition of the $\text{SL}(2; \mathbb{R})_k$ model via its CFT data.

Establishing the consistency of this construction by showing that it leads to crossing symmetric four-point functions is still an open problem, even though an explicit proposal for formulas expressing any four-point function of the $\text{SL}(2; \mathbb{R})_k$ model as

integral transforms of (analytic continuations of) crossing symmetric H_3^+ correlators is available [129]. In particular, it remains challenging to, in full generality, factorise the proposed formulas for four-point functions into the known three-point correlators [128], see however [244] for recent progress on this problem. For a detailed and pedagogical review of the current state of the art, we refer the reader to Reference [94].

The current algebra. One of the defining features of the $\mathrm{SL}(2; \mathbb{R})_k$ model is its $\widehat{\mathfrak{sl}}(2; \mathbb{R})_k$ current algebra. This level k affine Lie algebra is a central extension of the loop algebra $\mathfrak{sl}(2, \mathbb{R}) \otimes \mathbb{C}[z, z^{-1}]$. Using the generators $(j^a)_{a \in \{+, -, 3\}}$, normalised s.t.

$$[j^a, j^b] = f^{ab}{}_c j^c \quad \text{with} \quad f^{+-}{}_3 = -2, \quad f^{3+}{}_+ = 1 \quad \text{and} \quad f^{3-}{}_- = -1, \quad (7.1.10)$$

as basis of $\mathfrak{sl}(2, \mathbb{R})$ and using the notation $j_n^a := j^a \otimes z^n$, we can write down the Lie bracket of this algebra as

$$[j_n^a, j_m^b] = f^{ab}{}_c j_{n+m}^c + kn \frac{1}{2} K^{ab} \delta_{n+m, 0}. \quad (7.1.11)$$

Here, K is the Killing form of $\mathfrak{sl}(2; \mathbb{R})$, normalised by the dual Coxeter number, i.e.

$$K^{ab} := \frac{1}{4} \mathrm{Tr}[Ad_{j_0^a} Ad_{j_0^b}] \quad (7.1.12)$$

and its non-zero components are hence

$$K^{33} = -1 \quad \text{and} \quad K^{+-} = K^{-+} = 2. \quad (7.1.13)$$

The $\mathrm{SL}(2; \mathbb{R})_k$ Hilbert space carries two representations ρ and $\bar{\rho}$ of $\widehat{\mathfrak{sl}}(2; \mathbb{R})_k$. Following Reference [96], we precompose the right-moving representation $\bar{\rho}$ with the outer automorphism

$$\omega(j_n^\pm) = j_n^\mp, \quad \omega(j_n^3) = -j_n^3 \quad (7.1.14)$$

of $\mathfrak{sl}(2; \mathbb{R})$ and hence define the left- and right-moving $\widehat{\mathfrak{sl}}(2; \mathbb{R})_k$ generators acting on the Hilbert space as

$$J_n^a := \rho(j_n^a) \quad \text{and} \quad \bar{J}_n^a := \bar{\rho}(\omega(j_n^a)). \quad (7.1.15)$$

These modes generate the left- and right-moving currents

$$J^a(z) := \sum_{n \in \mathbb{Z}} z^{-n-1} J_n^a \quad \text{and} \quad \bar{J}^a(\bar{z}) := \sum_{n \in \mathbb{Z}} \bar{z}^{-n-1} \bar{J}_n^a. \quad (7.1.16)$$

In terms of these currents, the commutation relations (7.1.11) can be efficiently combined into the OPE

$$J^a(z) J^b(w) \sim \frac{K^{ab} k / 2}{(z - w)^2} + \frac{f^{ab}{}_c J^c(w)}{z - w} \quad (7.1.17)$$

and its right-moving analogue. The stress tensor of the $\mathrm{SL}(2; \mathbb{R})_k$ model is built from these currents via the Sugawara² construction.

Sugawara construction. The universal enveloping algebra of any affine Lie algebra contains a copy of the Virasoro algebra [246, 247]. Its generators are the modes of the Sugawara stress energy tensor, which is morally the quadratic Casimir of the affine Lie algebra. Applied to $\mathrm{SL}(2; \mathbb{R})_k$, the Sugawara construction leads to the stress tensor

$$T(z) = \frac{K_{ab} : J^a(z) J^b(z) :}{2(k-2)}. \quad (7.1.18)$$

It is instructive to explicitly verify the construction by deducing the stress tensor OPE from the OPE (7.1.17) of the $\mathrm{SL}(2; \mathbb{R})_k$ currents. In particular, this leads to the conclusion that the central charge of the $\mathrm{SL}(2; \mathbb{R})_k$ model is given by

$$c = \frac{3k}{k-2}, \quad (7.1.19)$$

and that the appearance of the number -2 is due to normal ordering i.e. quantum effects. This result is nicely consistent with the semi classical picture that the large k limit of a WZNW model based on a Lie algebra $\mathfrak{g}$ should describe the motion of point particles on a $d = \dim(\mathfrak{g})$ dimensional manifold (see eq. (7.2.12)), which in the case at hand matches the large k limit $c = 3$ of eq. (7.1.19).

Representations. The basic building blocks of the $\mathrm{SL}(2; \mathbb{R})_k$ Hilbert space are the principal discrete and continuous series representations $\mathcal{D}_j^\pm$ and $\mathcal{C}_j^\alpha$ of $\mathfrak{sl}(2; \mathbb{R})$. Let us briefly recollect the definition of these representations (for a systematic treatment consult e.g. Chapter 5 of Ref. [248]).

- The lowest weight representation $\mathcal{D}_j^+$ is generated by a state $|jj\rangle$, which is both an eigenstate of J_0^3 with eigenvalue j and an element of the kernel of J_0^- , together with an infinite collection of states $|jm\rangle$ for all values of $m \in j + \mathbb{N}$ that can be reached from $|jj\rangle$ by acting with J_0^+ .
- The highest weight representation $\mathcal{D}_j^-$ is constructed completely analogously, based on a state $|jj\rangle$ annihilated by J_0^+ .
- In the continuous series representations $\mathcal{C}_j^\alpha$, the eigenvalues of J_0^3 are independent of the eigenvalue of the quadratic Casimir. Instead they are determined

²In line with the historical remarks on the origin of string theory in hadronic physics made in Section 1.1, it might be of interest to note the curious fact that Sugawara's work preceded the advent of WZNW models, string theory and 2d CFTs. Instead, the original context that Sugawara operated in was that of hadronic physics and field theory in 3+1 dimensions [245].

by the parameter α and take values in $\alpha + \mathbb{Z}$. There is no preferred (e.g. highest or lowest) value among these, and in particular, $\mathcal{C}_j^\alpha = \mathcal{C}_j^{1+\alpha}$. We can hence restrict α to the range $0 \leq \alpha < 1$.

The representations $\mathcal{D}_j^\pm$ are unitary for $j > 0$, whereas $\mathcal{C}_j^\alpha$ is unitary for any $j \in \frac{1}{2} + i\mathbb{R}$. $\mathcal{D}_j^\pm$ and $\mathcal{C}_j^\alpha$ lift to representations $\hat{\mathcal{D}}_j^\pm$ and $\hat{\mathcal{C}}_j^\alpha$ of the full current algebra. This is achieved by declaring elements of $\mathcal{D}_j^\pm$ and $\mathcal{C}_j^\alpha$ as affine primaries, i.e. defining

$$J_n^a |jm\rangle = 0 \quad \text{for all } n > 0 \quad (7.1.20)$$

and then freely acting with negative frequency modes to generate the full affine module, similar to how we defined conformal representations induced by finite-dimensional representations of $\mathfrak{so}(d)$ in Section 2.1.1. The quadratic Casimir determines the conformal weight of the affine primaries in these representation w.r.t. the Sugawara stress tensor (7.1.18). Concretely,

$$\Delta_j = -\frac{j(j-1)}{k-2} \quad (7.1.21)$$

for both discrete and continuous representations.

Spectral flow. The $\widehat{\mathfrak{sl}}(2; \mathbb{R})_k$ algebra has a family of automorphisms $(\rho^w)_{w \in \mathbb{Z}}$, which act on the basis elements J_n^a as

$$\rho^w(J_n^3) = J_n^3 - \frac{k}{2}w\delta_{n,0} \quad \text{and} \quad \rho^w(J_n^\pm) = J_{n \pm w}^\pm. \quad (7.1.22)$$

Extending these spectral flow automorphisms to the universal enveloping algebra of the current algebra, we find that their action on the Sugawara Virasoro generators is

$$\rho^w(L_n) = L_n + wJ_n^3 - \frac{k}{4}w^2\delta_{n,0}. \quad (7.1.23)$$

Spectral flow allows us to construct new representations $\hat{\mathcal{D}}_j^{\pm,w}$ and $\hat{\mathcal{C}}_j^{\alpha,w}$, which on the level of vector spaces are canonically isomorphic to $\hat{\mathcal{D}}_j^\pm$ and $\hat{\mathcal{C}}_j^\alpha$ via a map $|jm\rangle \mapsto |jm\rangle^{(w)}$. Concretely, we define

$$J_n^a |\psi\rangle^{(w)} := (\rho^{-w}(J_n^a) |\psi\rangle)^{(w)}, \quad (7.1.24)$$

such that the canonical identification between the flowed and unflowed representations is an intertwiner twisted by the action of ρ^w i.e.

$$(J_n^a |\psi\rangle)^{(w)} = \rho^w(J_n^a) |\psi\rangle^{(w)}. \quad (7.1.25)$$

For generic values of the spin j and the level k , these spectrally flowed representations are almost all genuinely distinct. The only generic isomorphism between spectrally flowed representations is the $\mathrm{SL}(2; \mathbb{R})$ series identification

$$\hat{\mathcal{D}}_j^{\pm, w} \cong \hat{\mathcal{D}}_{\frac{k}{2}-j}^{\mp, w \pm 1}. \quad (7.1.26)$$

However, while they are not directly isomorphic, representations with $w < 0$ are related to $w > 0$ representations via the outer automorphism $J^3 \rightarrow -J^3$, $J^\pm \rightarrow J^\mp$ of $\mathfrak{sl}(2; \mathbb{R})$ of the current algebra. To describe the features of spectrally flowed representations it is hence sufficient to without loss of generality restrict to $w > 0$.

The action of a positive amount of spectral flow on affine primaries $|\psi\rangle = |jm\rangle$ produces states $|jm\rangle^{(w)}$ with the following properties

- The state $|jm\rangle^{(w)}$ is generically not an affine primary, since $J_n^+ |jm\rangle^{(w)} = (J_{n-w} |jm\rangle)^{(w)}$ generically does not vanish for $1 \leq n \leq w$.
- The state $|jm\rangle^{(w)}$, however, is a Virasoro primary with weight

$$\Delta_{j,m}^{(w)} = -\frac{j(j-1)}{k-2} - mw - \frac{k}{4}w^2. \quad (7.1.27)$$

Indeed, $L_n |jm\rangle^{(w)} = (L_n - wJ_n^3)^{(w)} = 0$ for $n > 0$ and

$$L_0 |jm\rangle^{(w)} = ((L_0 - wJ_0^3 - \frac{k}{4}w^2) |jm\rangle)^{(w)} = \Delta_{j,m}^{(w)} |jm\rangle^{(w)}. \quad (7.1.28)$$

- The state $|jm\rangle^{(w)}$ is a lowest-weight state generating a $\mathcal{D}_{m+\frac{k}{2}w}^+$ subrepresentation of the zero-mode algebra. Indeed,

$$J_0^3 |jm\rangle^{(w)} = ((J_0^3 + \frac{k}{2}w) |jm\rangle)^{(w)} = (m + \frac{k}{2}w) |jm\rangle^{(w)} \quad (7.1.29)$$

and furthermore

$$J_0^- |jm\rangle^{(w)} = (J_w^+ |jm\rangle)^{(w)} = 0. \quad (7.1.30)$$

Crucially, the last of the above observations, i.e. that we end up with a discrete $\mathcal{D}_{m+\frac{k}{2}w}^+$ representation of the zero modes inside of the spectrally flowed representations of the current algebra, is true independent of which unflowed representation we started with. As we will discuss in Section 7.2.2, this is the reason why spectrally flowed continuous representations are compatible with an interpretation of the $\mathrm{SL}(2; \mathbb{R})_k$ model as a sector of a consistent worldsheet theory, even though unflowed continuous representations would lead to an unbounded spectrum and are hence incompatible with a consistent, stable string background.

Spectrum. With the chiral representations described in the previous paragraphs, we can now state that the Hilbert space of the $\mathrm{SL}(2; \mathbb{R})_k$ model is [96]

$$\mathcal{H} = \bigoplus_{w \in \mathbb{Z}} [\mathcal{H}_w^{\mathcal{D}} \oplus \mathcal{H}_w^{\mathcal{C}}]. \quad (7.1.31)$$

The subspaces $\mathcal{H}_w^{\mathcal{D}}$ and $\mathcal{H}_w^{\mathcal{C}}$ in this definition are

$$\mathcal{H}_w^{\mathcal{D}} = \int_{\frac{1}{2}}^{\frac{k-1}{2}} dj \hat{\mathcal{D}}_j^{+,w} \otimes \hat{\mathcal{D}}_j^{+,w} \quad \text{and} \quad \mathcal{H}_w^{\mathcal{C}} = \int_{\frac{1}{2}+i\mathbb{R}} dj \int_0^1 d\alpha \hat{C}_j^{\alpha,w} \otimes \hat{C}_j^{\alpha,w}. \quad (7.1.32)$$

What motivates this choice? There are two important requirements that we have on the $\mathrm{SL}(2; \mathbb{R})_k$ model, which justify, and in fact uniquely single out, the definition given in eq. (7.1.31). The first of these is of course that the model ought to be a consistent CFT with $\widehat{\mathfrak{sl}}(2; \mathbb{R})_k$ symmetry. This requirement alone, however, does not force us to consider a spectrum as rich as (7.1.31). Instead, the canonical example for a CFT with $\widehat{\mathfrak{sl}}(2; \mathbb{R})_k$ symmetry is rather the structurally simpler H_3^+ model. As we review in the next paragraph, the latter actually plays a crucial role in the construction of the $\mathrm{SL}(2; \mathbb{R})_k$ model. The second, much stronger, requirement on the $\mathrm{SL}(2; \mathbb{R})_k$ model is that it provides a consistent worldsheet description of (bosonic) string theory on AdS_3 . This aspect of the $\mathrm{SL}(2; \mathbb{R})_k$ model is the topic of Section 7.2.2 below.

Unflowed correlators. The $\mathrm{SL}(2; \mathbb{R})_k$ model is partially defined by analytically continuing the H_3^+ model. The latter is a 2d CFT with the same $\widehat{\mathfrak{sl}}(2; \mathbb{R})_k$ current algebra as the $\mathrm{SL}(2; \mathbb{R})_k$ model, whose spectrum however only consists of unflowed continuous series representations [249]. The H_3^+ model was solved by Jörg Teschner in the late nineties [250, 251] by computing all two- and three-point functions in closed form, using the same logic as in his bootstrap of Liouville theory and its conformal boundaries, which we discussed in Section 2.2.1. Teschner later also solidified the proposed results by proving that they indeed define a fully crossing symmetric CFT [252]. The unflowed discrete series correlation functions of the $\mathrm{SL}(2; \mathbb{R})_k$ model are the analytic continuation of these results to the values $j \in [\frac{1}{2}, \frac{k-2}{2}]$.

Spectral flow preserving correlators. It is useful to factor out a radius $R = \sqrt{k}$ compact free boson from the $\mathrm{SL}(2; \mathbb{R})_k$ model, reducing the non-trivial dynamics to that of the cigar $\mathrm{SL}(2; \mathbb{R})_k/\mathrm{U}(1)_{\sqrt{k}}$ [253] with an asymptotic circle whose radius coincides with that of the $\mathrm{U}(1)$ factor. On the level of vertex operators $V_{j,m}^{(w)}$ corresponding to affine primary states acting upon with w units of spectral flow, this amounts to a factorisation

$$V_{j,m}^{(w)}(z) = \Psi_{jm}(z) e^{(m+\frac{k}{2}w)\sqrt{\frac{2}{k}}\phi(z)}. \quad (7.1.33)$$

The non-trivial part of the task to evaluate correlation functions of the $\text{SL}(2; \mathbb{R})_k$ model is hence to compute correlators of cigar vertex operators $\Psi_{jm\bar{m}}$. Crucially, spectral flow only changes the $\text{U}(1)$ factor of the vertex operators. We can hence use our knowledge of unflowed correlators to immediately determine also some correlation functions of spectrally flowed operators. However, as a consequence of the charge conservation condition imposed in the $\text{U}(1)$ factor, only a sub-sector of all cigar correlation functions can in this way be related to analytic continuations of H_3^+ correlation functions. The upshot is that only spectral flow preserving correlators, i.e. correlation functions of N spectrally flowed vertex operators satisfying

$$\sum_{i=1}^N w_i = 0 \quad (7.1.34)$$

are directly accessible via this approach.

General m -basis correlators. If spectral flow was a conserved quantity, then the method described in the previous paragraph would fully solve the $\text{SL}(2; \mathbb{R})_k$ model. The true spectral flow selection rule is however not eq. (7.1.34) but a weaker condition which, for correlation functions involving only continuous series operators (see Ref. [98] eq. (D.7)), reads

$$\left| \sum_{i=1}^N w_i \right| \leq N - 2. \quad (7.1.35)$$

In particular, this means that, in addition to the spectral flow conserving three-point correlators, we also need to fix the value of three-point functions that violate spectral flow by one unit. Closed form expressions for spectral flow conserving three-point functions in terms of ${}_3F_2$ hypergeometric functions can be obtained from Ref. [254]. The spectral flow violating three-point functions can be obtained from the factorisation of spectral flow conserving four-point functions [98].

General x -basis correlators. In principle the spectrum given above and the m -basis three-point functions fully define the $\text{SL}(2; \mathbb{R})_k$ model. For holographic applications we are, in practice, however not necessarily interested in the correlation functions of the affine primaries and their spectrally flowed cousins. As Section 7.3 argues in more detail, natural worldsheet quantities in that context are the correlation functions of so-called x -basis vertex operators

$$V_{jh\bar{h}}^{(w)}(z, \bar{z}, x, \bar{x}) := e^{xJ_0^+ + \bar{x}\bar{J}_0^+} V_{jm\bar{m}}^{(w)}(z, \bar{z}) e^{-xJ_0^+ - \bar{x}\bar{J}_0^+}, \quad (7.1.36)$$

where $h = m + \frac{k}{2}w$ is the associated spacetime conformal weight. Importantly, the state $V_{jh\bar{h}}^{(w)}(0, 0, x, \bar{x})|0\rangle$ is, for $x, \bar{x} \neq 0$, not contained in the $\hat{\mathcal{D}}_j^{\pm, w}$ (or $\hat{\mathcal{C}}_j^{\alpha, w}$) sub-space of the Hilbert space. Said differently, $V_{jh\bar{h}}^{(w)}(0, 0, x, \bar{x})|0\rangle$ does not have a well defined

spectral flow quantum number w.r.t. the mode expansion of the $\mathrm{SL}(2; \mathbb{R})_k$ currents around $z = 0$. Since their spectral flow label in this sense has an inherently different meaning from that of m -basis operators, there is no reason to expect the selection rule eq. (7.1.35) to hold for these operators. Instead, they satisfy a much weaker selection rule, which in the case $w_i \geq 0 \forall i$ reads (see Ref. [98] eq. (D.8))

$$w_i - \sum_{i \neq j} w_j \leq N - 2 \quad \forall 1 \leq i \leq N. \quad (7.1.37)$$

Furthermore, while the definition (7.1.36) in terms of m -basis operators in principle allows us to use our knowledge of m -basis correlators to compute those of x -basis operators, it is in practice challenging to derive useful expressions for x -basis correlators in this way. Explicit formulas for three-point [128] and four-point [129] functions of x -basis vertex operators were derived only comparably recently, using recursion relations implied by $\mathrm{SL}(2; \mathbb{R})_k$ Ward identities.

7.1.3 Boundary conformal field theory

This section reviews some basic aspects of conformal boundaries in two dimensions. The reader is referred to Ref. [255] for a concise general introduction to the topic and Ref. [243] for a deeper systematic treatment that emphasises the application to D-branes.

Introducing boundaries to a physical system generally enriches the dynamics, adding new local degrees of freedom. In a scale-invariant setting, a generalisation of the state-operator-correspondence reviewed in Section 2.1.2 allows us to identify these local excitations with states in a Boundary Conformal Field Theory (BCFT) Hilbert space. The first two paragraphs of the section describe this boundary operator state correspondence and the symmetry algebra of the BCFT Hilbert space. In the third paragraph, we then review how the bulk operator state correspondence can be used to algebraically classify conformal boundaries.

After these general comments on BCFT, we turn our attention to the specific case of the $\mathrm{SL}(2; \mathbb{R})_k$ model of Section 7.1.2. As a first step, we review the general conditions imposed by demanding that a boundary not only preserves conformal symmetry, but instead the full diagonal subalgebra of the $\widehat{\mathfrak{sl}}(2; \mathbb{R})_k \otimes \widehat{\mathfrak{sl}}(2; \mathbb{R})_k$ current algebra. We then discuss concrete boundary conditions of the H_3^+ and $\mathrm{SL}(2; \mathbb{R})_k$ model that satisfy these constraints. This provides the basis for the discussion of branes in Lorentzian AdS_3 on which we embark in Section 7.2.

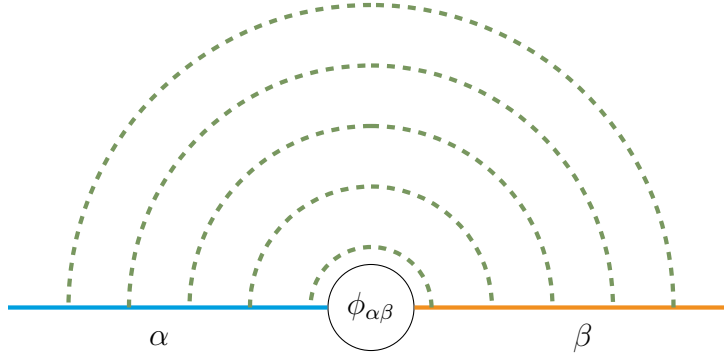


Figure 7.1.1: BCFT radial quantisation. Boundary conditions α and β are imposed on the negative real (cyan) and positive real (orange) line. The Hilbert space $\mathcal{H}_{\alpha\beta}$ is defined in terms of a foliation by semicircles (green dashed lines) centred in the origin. Scale invariance provides a one-to-one map between boundary condition changing operators $\phi_{\alpha\beta}$ inserted at the origin and states in $\mathcal{H}_{\alpha\beta}$.

Boundary operator state correspondence. For a CFT living on the Upper Half Plane (UHP), radial quantisation time slices are semi-circles, centred in the origin, whose endpoints are on the real line. Generically, the boundary conditions imposed at these two ends may differ. Let us impose boundary condition α on the negative, and condition β on the positive half of the real numbers, as illustrated in Fig. 7.1.1. If α and β preserve scale invariance, then the resulting BCFT Hilbert space $\mathcal{H}_{\alpha\beta}$ can directly be identified with the set of boundary condition changing operators that can be inserted between boundary segments of type α and β .

Boundary conformal symmetry. The presence of a boundary breaks conformal symmetry down to (at most) a diagonal subalgebra of $\text{Vir} \oplus \overline{\text{Vir}}$. Indeed, the conformal transformations of the UHP need to map the real line onto itself. On the classical level, the local symmetries of a CFT on the UHP are therefore described by the subalgebra of the sum of left and right moving Witt algebras, consisting of vector fields that are tangential to the real line. A basis of this subalgebra is given by the vector fields $\frac{1}{2}(\ell_n + \bar{\ell}_n)$, which in cartesian coordinates $z = x + iy$ restrict to

$$\frac{1}{2}(\ell_n + \bar{\ell}_n) = -x^{n+1} \partial_x \quad (7.1.38)$$

on the real line. On the quantum level, we hence conclude that $\mathcal{H}_{\alpha\beta}$ only carries a representation of a single chiral Virasoro algebra, which can be identified with the diagonal subalgebra of the bulk algebra $\text{Vir} \oplus \overline{\text{Vir}}$.

Boundary states. Using conformal symmetry, we can map the upper-half-plane to the unit disc, and thus identify a boundary condition α with a state $|\alpha\rangle\rangle$ in the

radial quantisation Hilbert space, its Cardy state. In this configuration, the classical symmetry algebra consists of vector fields that are tangential to the unit circle. The projection down to this subalgebra of the left and right moving Witt algebras is best performed by changing coordinates from $(z, \bar{z})$ to polar coordinates (r, θ) , with $z = re^{i\theta}$. The restriction of the vector fields ℓ_n and $\bar{\ell}_n$ to the unit circle is given by

$$\ell_n = -\frac{1}{2}e^{in\theta}(\partial_r - i\partial_\theta) \quad \text{and} \quad \bar{\ell}_n = -\frac{1}{2}e^{-in\theta}(\partial_r + i\partial_\theta). \quad (7.1.39)$$

Hence, the subalgebra preserving the boundary is the one generated by $\{\ell_n - \bar{\ell}_{-n}\}_{n \in \mathbb{Z}}$. On the quantum level, a conformally invariant boundary state therefore has to satisfy

$$(L_n - \bar{L}_{-n})|\alpha\rangle\rangle = 0 \quad \text{for all } n \in \mathbb{Z}. \quad (7.1.40)$$

Solutions to these conditions are in one-to-one correspondence with spin 0 conformal primary states $|h, \bar{h} = h\rangle$. That is, for each spin 0 primary $|h, h\rangle$ there is a unique (infinite) linear combination $|h\rangle\rangle$ of descendants of $|h, h\rangle$ that satisfies

$$(L_n - \bar{L}_{-n})|h\rangle\rangle = 0 \quad \text{for all } n \in \mathbb{Z}. \quad (7.1.41)$$

Such an $|h\rangle\rangle$ is called an Ishibashi state [256]. Conformal boundary states are characterised by the set of coefficients α_h that define their decomposition

$$|\alpha\rangle\rangle = \sum_h \alpha_h |h\rangle\rangle \quad (7.1.42)$$

into Ishibashi states.

Energy flow across the boundary. Condition (7.1.40) implies that

$$T(z)|\alpha\rangle\rangle = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n |\alpha\rangle\rangle = z^{-4} \bar{T}(1/z) |\alpha\rangle\rangle = \bar{T}(z) |\alpha\rangle\rangle \quad (7.1.43)$$

holds at the boundary i.e. on the unit circle. Mapping back to the UHP, we conclude

$$T(x) = \bar{T}(x) \quad \text{for } z = \bar{z} = x. \quad (7.1.44)$$

The conformal invariance condition (7.1.40) of the boundary state therefore has the interpretation of ensuring that no energy dissipates out of the UHP on the real line.

SL(2; $\mathbb{R}$) gluing conditions. In theories with extended current algebras it is natural to focus on boundary states constructed from extended Ishibashi states that not only preserve the diagonal Virasoro algebra but a chiral half of the full current algebra. More precisely, for any choice of automorphism Ω and primary j of the current algebra, there is a unique Ishibashi state $|j, \Omega\rangle\rangle$ that satisfies the analogue

$$J^a(z)|j, \Omega\rangle\rangle = \Omega^a_b \bar{J}^b(z)|j, \Omega\rangle\rangle \quad \text{at } |z| = 1 \quad (7.1.45)$$

of eq. (7.1.43). The Sugawara construction guarantees that these extended Ishibashi states automatically also satisfies the conformal invariance condition (7.1.40). On the level of modes, condition (7.1.45) reads³

$$(J_n^a + \Omega_b^a \bar{J}_{-n}^b)|j, \Omega\rangle = 0. \quad (7.1.46)$$

In the case of the $\text{SL}(2; \mathbb{R})_k$ model, we are mainly interested in two types of Ishibashi states: Those corresponding to trivially gluing the currents and those corresponding to gluing with a twist via the outer automorphism ω defined in eq. (7.1.14). We refer to the former as spherical, and the latter as AdS_2 Ishibashi states. Since we work in a convention where the right-moving current algebra is already implicitly acted upon by the (involutive) automorphism ω , see eq. (7.1.15), the roles of these two classes on a first glance seems to be interchanged. That is, the AdS_2 Ishibashi states satisfy

$$(J_n^\pm + \bar{J}_{-n}^\pm)|j, \omega\rangle \quad \text{and} \quad (J_n^3 + \bar{J}_{-n}^3)|j, \omega\rangle \quad (7.1.47)$$

whereas the spherical Ishibashi states satisfy

$$(J_n^\pm + \bar{J}_{-n}^\mp)|j, id\rangle \quad \text{and} \quad (J_n^3 - \bar{J}_{-n}^3)|j, id\rangle. \quad (7.1.48)$$

Boundary states of the H_3^+ and $\text{SL}(2; \mathbb{R})_k$ model. Maximally symmetric boundary states of the H_3^+ and $\text{SL}(2; \mathbb{R})_k$ model received a significant amount of attention during the early 2000s. Let us end this section, by briefly summarising some of these developments, and guiding the reader to the relevant literature.

As we review in Section 7.2 below, the $\text{SL}(2; \mathbb{R})_k$ model is intimately connected to string theory on pure NSNS AdS_3 backgrounds. The study of D-branes in that context was initiated in 1999 by Ref. [257], focussing on symmetric branes in $\text{AdS}_3 \times \text{S}_3 \times \mathbb{T}^4$, and Ref. [258] which extended the analysis to $\text{AdS}_3 \times \text{S}_3 \times \text{S}_3 \times \text{S}_1$. Shortly afterwards, Ref. [126] provided a systematic semi-classical analysis of maximally symmetric boundary conditions of the $\text{SL}(2; \mathbb{R})_k$ model and the stability of the branes that they describe. The physically most important branes listed in this classification are the so called AdS_2 and H_2 branes. The AdS_2 branes satisfy the ω -gluing conditions of the AdS_2 Ishibashi states (7.1.47). In the large volume limit $k \rightarrow \infty$ of the $\text{SL}(2; \mathbb{R})_k$ limit, they describe branes whose worldvolume is an AdS_2 subspace of AdS_3 (see Fig. 7.1.2, l.h.s.). Similarly, the H_2 branes satisfy trivial gluing conditions such as those defining the spherical Ishibashi states eq. (7.1.48). As opposed to the AdS_2 branes, the H_2 branes are instantonic. Their large k worldvolume is a euclidean H_2 slice of AdS_3 (see Fig. 7.1.2, r.h.s.).

³Note the different relative sign between the left and rightmoving modes as compared to eq. (7.1.40). In general the radial inversion that identifies z and $\bar{z}$ on the unit circle, introduces a factor of $(-1)^J$ to the gluing condition of a spin J current.

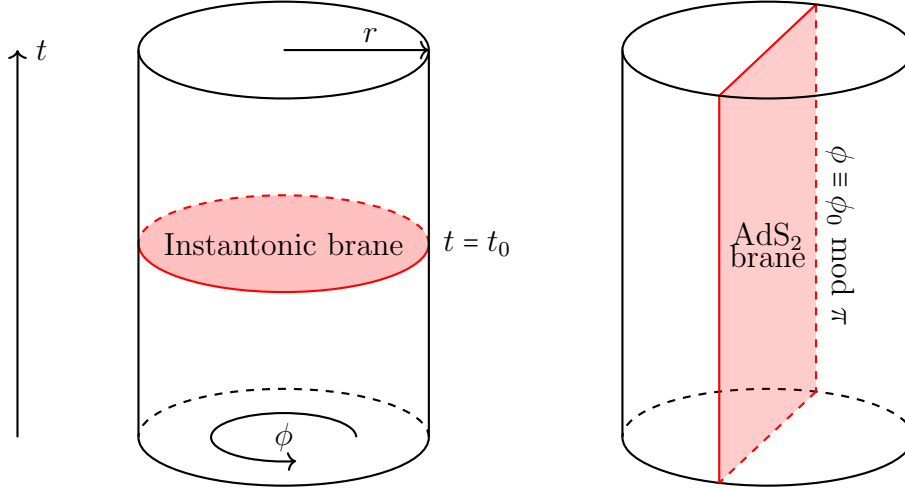


Figure 7.1.2: Left: Instantonic H_2 brane at fixed time $t = t_0$. Right: Flat AdS_2 brane at fixed angle $\phi = \phi_0$. This illustration is an adaptation of a figure from Ref. [124].

Taking a step back from the $SL(2; \mathbb{R})_k$ to the H_3^+ model, a detailed fully quantum analysis of $\widehat{\mathfrak{sl}}(2; \mathbb{R})_k$ symmetric branes was first provided in Reference [259]. The latter in particular contains explicit expressions for consistent boundary states built out of the spherical and AdS_2 Ishibashi states of the H_3^+ model. In the large volume limit, these boundary states describe branes whose worldvolume are euclidean AdS_2 and spherical submanifolds of H_3^+ respectively, which justifies their names. The euclidean AdS_2 branes of the H_3^+ model were independently also described in Ref. [260], which via analytic continuation connected them to AdS_2 branes of the $SL(2; \mathbb{R})_k$ model, leading to a computation of the open string spectrum for AdS_2 branes in AdS_3 . For an inherently Lorentzian discussion see e.g. Reference [261], which constructed maximally symmetric boundary states of the $SL(2; \mathbb{R})_k$ model from discrete series Ishibashi states.

A renewed interest in pure NSNS AdS_3 string theory, and hence the $SL(2; \mathbb{R})_k$ model, arose in the beginning of the 2020s due to the exact holographic duality that we will review in Section 7.3. In that context, explicit boundary states for H_2 and AdS_2 branes in $AdS_3 \times S_3 \times \mathbb{T}^4$ [124] and $AdS_3 \times S_3 \times S_3 \times S_1$ [262] at minimal NSNS flux (i.e. minimal level of the current algebra) were recently constructed. We comment in more detail on these recent developments in Section 7.3.

7.2 AdS_2 branes in pure NSNS AdS_3 string theory

This section is an introduction to string theory on pure NSNS AdS_3 backgrounds. We start in Sec. 7.2.1 by reviewing how a pure NSNS $AdS_3 \times S_3 \times \mathbb{T}^4$ background can be

obtained as a near horizon limit of NS5 branes and F1 strings in type IIB superstring theory. In Sec. 7.2.2, we then slightly shift gears and review how the $SL(2; \mathbb{R})_k$ model arises as part of the worldsheet description of (super-)strings propagating on any $AdS_3 \times X$ background containing an AdS_3 factor threaded by k units of NSNS flux. Finally, Section 7.2.3 reviews how the boundary states of the $SL(2; \mathbb{R})_k$ model that we have discussed in Section 7.1.3 can be applied in this context to describe open strings ending on branes in AdS_3 .

7.2.1 $AdS_3 \times S_3 \times \mathbb{T}^4$ as a near-horizon geometry

The two Chapters 8 and 9, to which the current chapter serves as an introduction, discuss a class of interfaces in symmetric product orbifolds and its broader relevance for holography of various pure NSNS AdS_3 backgrounds. When it comes to concrete, fully explicit applications, we will, however, always focus on one specific example: string theory on $AdS_3 \times S_3 \times \mathbb{T}^4$ with one unit of NSNS flux. In this section, we sketch how this string background can be obtained as a near-horizon limit of a stack of NS5-branes and F1-strings in type II superstring theory.

The brane configuration. We begin with strings propagating on $\mathbb{R}^{1,4} \times S_1 \times \mathbb{T}^4$ in the presence of NS5-branes filling the $S_1 \times \mathbb{T}^4$ factor and fundamental strings that are wrapped around S_1 and smeared over $\mathbb{T}^4$. In this setting, both the NS5 branes and the F1-strings appear point-like from the perspective of the Minkowski factor, where we place them all at the origin, i.e. at $r = 0$ in polar coordinates. The situation is described by the following parameters

- The number of fundamental strings n_1 .
- The number of NS5 branes n_5 .
- The radius of the S_1 factor R .
- The volume of $\mathbb{T}^4$ in units of string length V .
- The string coupling on the celestial sphere g_s .
- The inverse string tension α' .

The metric ds^2 of the pure NSNS super-gravity solution parametrised by the quantities listed above is [263–265]

$$ds^2 = f_1^{-1}(r)(dy^2 - dt^2) + f_5(r)(dr^2 + r^2 d\Omega_3^2) + ds_{\mathbb{T}^4}^2. \quad (7.2.1)$$

Here, the coordinate y of the circle is defined such that $y \sim 2\pi R$. Furthermore,

$$f_i(r) = 1 + \frac{r_i^2}{r^2} \quad \text{with} \quad r_1^2 = \frac{g_s^2 \alpha' n_1}{V} \quad \text{and} \quad r_5^2 = \alpha' n_5. \quad (7.2.2)$$

The dilaton Φ and NSNS 3-form H take the form

$$e^{2\Phi} = g_s^2 \frac{f_5}{f_1} \quad \text{and} \quad H = 2n_5 \omega_3 + 2 \frac{g_s^2 n_1}{V} \frac{f_1}{f_5} \star \omega_3. \quad (7.2.3)$$

In the expression for H at fixed r , we used the volume form ω_3 of the radius r sphere in the transverse Minkowski space and the Hodge star operator $\star$ of $\mathbb{R}^{4,1} \times S_1$. The smearing approximation that we made for the F1-strings is justified as long as the transverse distance is much greater than the spacing between the F1-strings. Assuming the latter to be uniformly distributed and the diameter of the torus to be of order $V^{\frac{1}{4}}$, we should hence restrict ourselves to the region

$$r/\sqrt{\alpha'} \gg (V/n_1)^{\frac{1}{4}}. \quad (7.2.4)$$

The near horizon limit. If we restrict ourselves to the region $r \ll r_1, r_5$, then the metric simplifies to

$$ds^2 = \frac{r^2}{r_1^2} (dy^2 - dt^2) + \frac{r_5^2}{r^2} (dr^2 + r^2 d\Omega_3^2) + ds_{\mathbb{T}^4}^2, \quad (7.2.5)$$

or by rescaling $r \rightarrow r\sqrt{r_1 r_5 / \alpha'}$,

$$ds^2 = n_5 \left[\frac{r^2}{\alpha'} (dy^2 - dt^2) + \frac{\alpha'}{r^2} dr^2 + \alpha' d\Omega_3^2 \right] + ds_{\mathbb{T}^4}^2. \quad (7.2.6)$$

Thus, the near horizon geometry is $AdS_3 \times S_3 \times \mathbb{T}^4$ where the AdS_3 radius and the radius of the three sphere coincide, both given by n_5 in units of the string length.

The NSNS three form becomes self-dual, leading to enhanced SUSY, and the dilaton becomes constant in the near horizon limit. Concretely, they take the form

$$e^{2\Phi} = g_s^2 \frac{r_5^2}{r_1^2} = \frac{n_5}{n_1} V \quad \text{and} \quad H = 2n_5 (\omega_3 + \star \omega_3). \quad (7.2.7)$$

We hence read off that the effective 6d string coupling is

$$g_6^2 = e^{2\Phi} / V = \frac{n_5}{n_1}. \quad (7.2.8)$$

Furthermore, the three-sphere (corresponding to the ω_3 component of H) and the AdS_3 factor (corresponding to the $\star \omega_3$ component) are both supporting the same amount of NSNS flux, which like the size of the 6d geometry is controlled by n_5 .

Computationally controlled regime. In order to ensure the existence of a region where both $r \ll r_1, r_5$ and condition (7.2.4) are satisfied, we need to impose

$$V \ll n_5^2 n_1 \quad \text{and} \quad V \ll g_s^{\frac{4}{3}} n_1. \quad (7.2.9)$$

Setting $g_s^2 \sim 1/n_1$, the second condition is equivalent to $V \ll n_1^{1/3}$.

Beyond these constraints on V imposed by the consistency of the smearing approximation in supergravity, the fact that the 10d string coupling is given by $V n_5/n_1$ also implies a breakdown of string perturbation theory at $V \sim n_1$. Reference [266] provides a more detailed discussion of this breakdown, with a particular focus on $n_5 = 1$ and how, in that case, the fact that the string theory becomes strongly coupled is reflected in the holographic dual.

To have a weakly coupled string theory, we also need to impose $g_6^2 \ll 1$ i.e. $n_5 \ll n_1$. Since n_5 also controls the AdS radius in string units, a weakly-coupled low-energy supergravity description is thus available in the regime

$$1 \ll n_5 \ll n_1. \quad (7.2.10)$$

However, as we will review in the next section, the relation between this string background and WZNW models actually allows us to treat the α' dependence of string scattering amplitudes exactly. This allows us to probe the much larger regime

$$n_5 \ll n_1, \quad (7.2.11)$$

going all the way down to $n_5 = 1$.

7.2.2 Closed Strings and pure NSNS AdS_3

In this section, we review the relation between the $\text{SL}(2; \mathbb{R})_k$ model of Section 7.1.2 and string theory on pure NSNS AdS_3 backgrounds, and discuss some of its implications for the physical string spectrum.

$\text{SL}(2; \mathbb{R})$ and AdS_3 . Recall that the WZNW action is a special case of a non-linear sigma model coupling to the metric G and a closed 3-form H of a target space $\mathcal{M}$ [267]. More concretely, it is the case where $\mathcal{M}$ is a group manifold with G and H canonically induced by the group structure, leading to the action

$$\frac{k}{8\pi\alpha'} \left(\int d^2\sigma \text{Tr}[\xi^\alpha \xi_\alpha] - \frac{i}{4} \int d^3\sigma \epsilon_{\alpha\beta\gamma} \text{Tr}[\tilde{\xi}^\alpha \tilde{\xi}^\beta \tilde{\xi}^\gamma] \right) \quad \text{with} \quad \xi^\alpha := g^{-1} \partial^\alpha g. \quad (7.2.12)$$

In the case of $\text{SL}(2; \mathbb{R})$, supplementing this action with the usual bc ghost system and an “internal” CFT $\mathcal{X}$ with central charge

$$c_{\mathcal{X}} = 26 - \frac{3k}{k-2} \quad (7.2.13)$$

provides a worldsheet description of bosonic strings propagating on $AdS_3 \times X$ with k units of background NSNS-flux. This is because three-dimensional AdS is the universal cover of the group-manifold $SL(2; \mathbb{R})$, a fact that is most manifestly visible in the parametrisation

$$g = \begin{pmatrix} X_{-1} + X_1 & X_0 - X_2 \\ -X_0 - X_2 & X_{-1} - X_1 \end{pmatrix}, \quad (7.2.14)$$

in which the condition $\det(g) = 1$ becomes the quadratic equation

$$X_{-1}^2 + X_0^2 - X_1^2 - X_2^2 = 1, \quad (7.2.15)$$

defining a three-dimensional hyperboloid in $\mathbb{R}^{2,2}$ with compact time direction. Using the parametrisation

$$X_{-1} = \cos t \cosh \rho, \quad X_0 = \sin t \cosh \rho, \quad (7.2.16)$$

$$X_1 = \cos \phi \cosh \rho, \quad X_2 = \sin \phi \sinh \rho \quad (7.2.17)$$

one can straightforwardly verify that the action (7.2.12) hence describes strings coupling to the metric

$$ds^2 = k(-dX_{-1}^2 - dX_0^2 + dX_1^2 + dX_2^2) = k(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2) \quad (7.2.18)$$

of AdS_3 at radius k with a B -field

$$B = k \sinh^2 \rho dt \wedge d\phi. \quad (7.2.19)$$

Of course, the bosonic strings described in this manner could also arise as the bosonic subsector of superstrings propagating on an AdS_3 background. In particular, we can conclude by comparing eqs.(7.2.18) and eq. (7.2.19) to the analogous expressions in Section 7.2.1 that the $SL(2; \mathbb{R})_k$ model is part of the worldsheet description of superstrings probing the near horizon limit of $n_5 \sim k$ NS5 branes and $n_1 \gg n_5$ F1 strings.

The concrete relation between k and n_5 depends on how we choose to treat the spacetime supersymmetries. In the RNS-formalism, the AdS_3 and S_3 factors of the geometry are described by two separate WZNW models, namely the $\mathcal{N} = 1$ worldsheet supersymmetric $SL(2; \mathbb{R})_{n_5}$ and $SU(3)_{n_5}$ models. The fermions of these models can be decoupled, leading to an equivalent description in terms of the bosonic $SL(2; \mathbb{R})_{n_5+2}$ WZNW model, the bosonic $SU(3)_{n_5-2}$ WZNW model and six free fermions. In the hybrid formalism, the AdS_3 and S_3 factors are instead described in a manifestly spacetime supersymmetric way by the $PSU(1,1|2)_k$ WZNW model. In the latter, the bosonic $\widehat{\mathfrak{sl}}(2; \mathbb{R})$ and $\widehat{\mathfrak{su}}(3)$ subalgebras are both at level $k = n_5$.

Virasoro constraints and no-ghost theorem. The representations of $\mathrm{SL}(2; \mathbb{R})$ discussed in Section 7.1.2 include negative norm states. For the specific choice of representations that constitute the $\mathrm{SL}(2; \mathbb{R})_k$ Hilbert space, see eq. (7.1.31), Maldacena and Ooguri [96] have derived a no-ghost theorem: Imposing the Virasoro constraints

$$(L_n + L_n^{\mathcal{X}} - \delta_n^0)|\psi\rangle = 0 \quad \forall n \in \mathbb{N}_0 \quad (7.2.20)$$

leads to a ghost free Hilbert space. However, the Virasoro constraints alone do not directly provide us with a perturbatively stable string theory. The reason for this is that $w = 0$ continuous series representations lead to tachyonic modes that survive the projection. This subtlety arises not only for the bosonic, but also for the superstring theory. To obtain a consistent model, it is hence also in the latter case necessary to by hand remove the $w = 0$ continuous representations. It is a non-trivial fact that the fusion rules allow for this truncation, see Ref. [68], Sec. 5.4 for further comments on the issue. In a similar spirit, Ref. [268] recently constructed, for the special case $k = 3$, a (non-geometric) bosonic string theory based on a modified $\mathrm{SL}(2; \mathbb{R})_3$ model in which the $w = 0$ sector can be consistently removed, providing a rare example of a tachyon-free bosonic string theory.

Structure of the long string spectrum. The mass-shell condition implies that physical long string Virasoro primary states have spacetime weight

$$h = \frac{1}{w} \left[-\frac{j(j-1)}{k-2} + \frac{k}{4}w^2 + h_{\mathcal{X}} - 1 \right]. \quad (7.2.21)$$

Now if we formally introduce a coupling constant b , background charge Q and momentum α defined as

$$b = \frac{1}{\sqrt{k-2}}, \quad Q = b^{-1} - b = \frac{k-3}{\sqrt{k-2}} \quad \text{and} \quad \alpha = \frac{j + \frac{k}{2} - 2}{\sqrt{k-2}}, \quad (7.2.22)$$

then this can be rewritten as

$$h = \frac{6k}{24} \left(w - \frac{1}{w} \right) + \frac{1}{w} (h_{\mathcal{X}} + \alpha(Q - \alpha)). \quad (7.2.23)$$

This looks like an excitation of the w -twisted vacuum in a symmetric product orbifold of a 2d CFT with central charge $c_{\text{seed}} = 6k$. The term $h_{\mathcal{X}} + \alpha(Q - \alpha)$ furthermore suggests, that the underlying seed theory state is a product of a state in $\mathcal{X}$ and a momentum α excitation of a linear dilaton $\mathbb{R}_Q$ with background charge Q . This suggestion is supported by the observation that $\mathcal{X} \times \mathbb{R}_Q$ indeed has the necessary central charge

$$c_{\mathcal{X}} + (1 + 6Q^2) = 26 - \frac{3k}{k-2} + 1 + 6\frac{(k-3)^2}{k-2} = 6k. \quad (7.2.24)$$

It is thus tempting to conjecture that the subsector of the holographic dual, corresponding to the long string subsector on $AdS_3 \times \mathcal{X}$, is closely related to the symmetric orbifold $Sym^N(\mathbb{R}_Q \times \mathcal{X})$ with $N \sim n_1$ and $b = 1/\sqrt{n_5}$. As Section 7.3 describes in more detail, this expectation is very precisely realised in the case $k = 3$ where the linear dilaton decouples. However, for generic values of k , this cannot be the full story: The correlation function of the symmetric orbifold generically cannot reproduce the appropriate string scattering amplitudes. A refined proposal including a marginal deformation of the symmetric product orbifold that perturbatively captures the dual string scattering amplitudes close to $k = 3$ was introduced by Reference [269]. The all-orders perturbative match was established in Ref. [270], see also Ref. [271].

7.2.3 Open strings and AdS_2 branes

This section reviews basic features of AdS_2 branes in pure NSNS AdS_3 backgrounds and their open string spectra, mostly following Refs. [260, 272]. Holographic aspects of these branes are discussed in Section 7.3.3.

Semi-classical geometry of AdS_2 branes. The fact that, on the quantum level, AdS_2 branes impose ω -twisted gluing conditions of the $\widehat{\mathfrak{sl}}(2; \mathbb{R})_k$ current algebra is reflected in the semi-classical geometry of the branes at large k . To make this statement more precise, recall that the outer automorphism ω of $\mathfrak{sl}(2; \mathbb{R})$ given by eq. (7.1.14) lifts to the automorphism

$$g \mapsto \omega_0 g \omega_0 \quad \text{with} \quad \omega_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (7.2.25)$$

of the Lie group $SL(2; \mathbb{R})$. The precise statement is now that the semi-classical world volume of the AdS_2 brane is an ω_0 -twisted orbit of $SL(2; \mathbb{R})$. In other words, it is the solution set of the equation

$$\text{Tr}(\omega_0 g) = 2 \sinh \psi_0, \quad (7.2.26)$$

where the parameter ψ_0 determines by how much the brane bends away from the centre of AdS_3 . In particular, $\psi_0 = 0$ corresponds to a straight brane that goes directly through the centre of AdS_3 . For a $\psi_0 \neq 0$ configuration to be stable, there has to be a competing force acting on the brane that compensates the inherent tension, which on its own would always straighten it out back to $\psi_0 = 0$. In a pure NSNS background, the only origin for this force can be a coupling to the B -field. Hence, the AdS_2 brane has to carry a non-zero fundamental string charge Q to allow for $\psi_0 \neq 0$. The semi-classical DBI-analysis of Ref. [126] shows that the precise relation between Q and ψ_0 is

$$\sinh \psi_0 = g_s Q. \quad (7.2.27)$$

In particular, ψ_0 is quantised.

Quantum AdS₂ branes. In the quantum theory, the geometry of the AdS₂ brane is defined by the scattering amplitudes of probe wave-packets. The geometric parameter ψ_0 is hence encoded on the quantum level in how the coupling of the brane to probe strings depends on their momentum j . In the conventions of [260], this j dependence takes the simple form⁴

$$U_{\psi_0}^{\pm}(j) \sim \Gamma\left(1 - \frac{2j-1}{k-2}\right) \nu^{\frac{1}{2}-j} e^{\pm\psi_0(2j-1)} \quad (7.2.28)$$

The signs $+$ and $-$ labelling $U_{\psi_0}^{\pm}(j)$ play the role of distinguishing between the coupling to strings incoming from either of the two halves in which the AdS₂ brane splits AdS₃. In particular, we can easily convince ourselves that $U_{-\psi_0}^{\pm}(j) = U_{\psi_0}^{\mp}(j)$.

The spectrum of defect excitations supported by a flat, i.e. $\psi_0 = 0$, AdS₂ brane is simply a chiral half of the bulk $\text{SL}(2; \mathbb{R})_k$ spectrum. For $\psi_0 \neq 0$ the situation is slightly more subtle and the open string spectrum of a single brane only contains even winding states. To restore the full spectral flow structure, a system involving a second brane with $\psi'_0 = -\psi_0$ has to be considered. The two brane system contains open strings with an arbitrary amount of spectral flow, with odd w strings connecting the two branes, whereas an even amount of spectral flow leads to open strings whose ends are both on the same brane. See Section 5.3 of Ref. [260] for more details.

Results in Euclidean vs. Lorentzian signature. As we already commented towards the end of Section 7.1.3, AdS₂ branes are well understood in Euclidean signature. Both an explicit algebraic description in terms of boundary states of the H_3^+ model [259, 260], and a Lagrangian description in terms of a boundary action [273] are available. In fact, the boundary version of FZZ duality [274] reduces these branes to the exactly solved conformal boundaries of Liouville theory [275, 276]. Much less is known, however, about AdS₂ branes of the $\text{SL}(2; \mathbb{R})_k$ model. In particular, a disc analogue of the $\text{SL}(2; \mathbb{R})_k$ correlation functions derived in [128, 129] is still missing in the literature. We will make further comments on this in Chapter 10.

7.3 AdS₃/CFT₂ at minimal NSNS flux

As we discussed in Section 7.2.1, the relation between string theory on AdS₃ and the $\text{SL}(2; \mathbb{R})_k$ model allows us to study the near horizon limit of a configuration of n_1 F1 strings and n_5 NS5 branes with the only constraint being $n_5 \ll n_1$, i.e. without

⁴The functions $U^{\pm}$ given by eq. (7.2.28) strictly speaking describe the coupling of unflowed continuous series states to a Euclidean AdS₂ brane of the H_3^+ model. The coupling of Lorentzian AdS₂ branes in the unflowed subsector of the $\text{SL}(2; \mathbb{R})$ model, which as stated in eq. (7.1.31) only contains discrete series states, can be obtained via analytic continuation in j . We make further comments on Lorentzian and Euclidean signature in the next paragraph.

having to take the further limit $1 \ll n_5$. In particular the string theory is under full computational control all the way down to the extreme limit $n_5 = 1$, in which the string length approaches the AdS radius and the putative holographically dual CFT_2 should become a free gauge theory. In this section, we focus on the special features of this tensionless string limit.

More precisely, Section 7.3.1 discusses the emergence of higher spin massless states in the string theory at $n_5 = 1$ and the dramatic simplification of the worldsheet theory that comes with that. The second section, Sec. 7.3.2, sketches the detailed derivation of AdS_3/CFT_2 that is made possible by having both a bulk string theory described via a solvable worldsheet theory, and a holographically dual spacetime CFT that is a free theory. Finally, Section 7.3.3 ends this review chapter by discussing what is known about the holographic dual of probe branes in this context, and outlining the gaps in our knowledge that will be filled in Chapters 8 and 9.

7.3.1 The $n_5 = 1$ limit

In this section, we first discuss the emergence of physical massless higher spin states in (bosonic) string theory on AdS_3 , described by the $SL(2; \mathbb{R})_k$ model, in the limit of $k = 3$. To interpret this result as a statement about the $n_5 = 1$ case of the F1-NS5-brane system, the $SL(2; \mathbb{R})_{n_5+2}$ based RNS description has to, as we discuss in the second paragraph of the section, be replaced by the $PSU(1, 1|2)_{n_5}$ based hybrid description. This technical subtlety however changes nothing about the conclusion of the discussion: At $n_5 = 1$ higher spin currents arise that decouple a large portion of the spectrum, leading to a significantly simplified worldsheet theory.

Massless higher spin states. The AdS_3 mass associated to a state with boundary conformal weights $(h, \bar{h})$ is given by

$$M^2 = (h + \bar{h} - |h - \bar{h}|)(h + \bar{h} + |h - \bar{h}| - 2) = (\Delta - |J|)(\Delta + |J| - 2). \quad (7.3.1)$$

Let us discuss what physical massless higher spin states exist in the range $k > 2$ where the $SL(2; \mathbb{R})_k$ model has a positive central charge. The unitarity bound $\Delta \geq |J|$ of the spacetime CFT implies that the second factor in eq. (7.3.1) cannot vanish for $J > 1$ states. Hence, higher spin massless fields arise at $\Delta = J$ i.e. $h = 0$ or $\bar{h} = 0$. Since the spectrum of the $SL(2; \mathbb{R})_k$ model is invariant under $h \leftrightarrow \bar{h}$, we can without loss of generality focus on the case $\bar{h} = 0$. For $\bar{h} = 0$, the mass-shell condition of the right movers amounts to

$$\frac{k}{4}w^2 + \bar{h}_{\mathcal{X}} + \bar{N} - 1 = \frac{j(j-1)}{k-2}. \quad (7.3.2)$$

Let us first discuss solutions in the short string sector where

$$j \leq \frac{k-1}{2}, \quad \text{and thus} \quad \frac{k}{4}w^2 + \bar{h}_{\mathcal{X}} + \bar{N} - 1 = \frac{j(j-1)}{k-2} \leq \frac{j}{2}, \quad (7.3.3)$$

which for $k > 2$, $\bar{N}, \bar{h}_{\mathcal{X}} \geq 0$ can only be satisfied if $w^2 = 0$ or $w^2 = 1$. It is reasonable to discuss the two cases $w = 0$ and $w = \pm 1$ separately.

- $w = 0$ If $w = 0$ then $\bar{N} = \bar{m} + \bar{N} \geq j$ implies

$$j - 1 \leq \bar{h}_{\mathcal{X}} + \bar{N} - 1 = \frac{j(j-1)}{k-2} \leq \frac{j}{2} \quad (7.3.4)$$

and hence $j \leq 2$. But $j \leq 2$ in turn leads to $h_{\mathcal{X}} + \bar{N} - 1 \leq 1$ and thus $\bar{N} = 0$ or $\bar{N} = 1$. The former option is excluded by $\bar{N} \geq j \geq 1/2$. Thus only $\bar{N} = 1$, which in particular implies $j \leq 1$, remains feasible. Since

$$0 \leq \bar{h}_{\mathcal{X}} = h_{\mathcal{X}} + \bar{N} - 1 = \frac{j(j-1)}{k-2}, \quad (7.3.5)$$

we, however, also know that $j \geq 1$. We conclude that $j = 1$ and therefore $\bar{h}_{\mathcal{X}} = 0$. To summarise, the unflowed sector can only accommodate massless higher spin modes for

$$\bar{N} = 1, \quad j = 1 \quad \text{and} \quad \bar{h}_{\mathcal{X}} = 0. \quad (7.3.6)$$

By the level matching condition

$$w(h - \bar{h}) = N - \bar{N} + h_{\mathcal{X}} - \bar{h}_{\mathcal{X}}, \quad (7.3.7)$$

we furthermore conclude that the two combinations

$$N = 1 \quad \text{and} \quad h_{\mathcal{X}} = 0 \quad \text{or} \quad N = 0 \quad \text{and} \quad h_{\mathcal{X}} = 1 \quad (7.3.8)$$

are the only two allowed values for the left-moving data. In the former case, i.e. $h_{\mathcal{X}} = 0$, these massless states are the supergravity modes, massless fluctuations of the graviton, dilaton and B -field that exist at any value of k .

- $w = \pm 1$ For $w = \pm 1$, the inequality (7.3.3) implies

$$\bar{h}_{\mathcal{X}} + \bar{N} - 1 \leq \frac{1}{2}(j - \frac{k}{2}) \leq -\frac{1}{4} \quad (7.3.9)$$

and hence $\bar{N} = 0$ and $\bar{h}_{\mathcal{X}} \leq 3/4$. But then $\mp \frac{k}{2} = \bar{N} + \bar{m} \geq j$ immediately rules out $w = 1$. Thus only $w = -1$ remains feasible. However, the level matching condition implies

$$h = -N - h_{\mathcal{X}} + \bar{h}_{\mathcal{X}} \leq \frac{3}{4} \quad (7.3.10)$$

We hence conclude that no higher spin massless states can be found for $w^2 = 1$.

To summarise, the only higher-spin massless modes in the short string sector are the supergravity modes, and these persist at any k . Let us now consider the long string sector, where the analogue of eq. (7.3.3) is

$$j = \frac{1}{2} + is \quad \text{and thus} \quad \frac{k}{4}w^2 + \bar{h}_{\mathcal{X}} + \bar{N} - 1 = -\frac{1}{k-2}\left(\frac{1}{4} + s^2\right) \leq 0, \quad (7.3.11)$$

which directly tells us that $\bar{N} = 0$ and $w = \pm 1$ (since we exclude the problematic $w = 0$ continuous series representations, as explained in the second paragraph of Section 7.2.2). We can furthermore deduce that

$$0 \leq \bar{h}_{\mathcal{X}} = 1 - \frac{1}{4}\left(k + \frac{1}{k-2}\right) - \frac{s^2}{k-2}. \quad (7.3.12)$$

However, the minimal value of $\frac{1}{4}\left(k + \frac{1}{k-2}\right)$ in the range $k > 2$ is 1. This minimum is positioned at $k = 3$. We thus conclude that $k = 3$, $s = 0$ and $\bar{h}_{\mathcal{X}} = 0$. For $w = -1$, this implies $h = h_{\mathcal{X}} = N = 0$. We hence find that higher-spin massless long string states occur if and only if

$$k = 3, \quad w = 1, \quad s = 0, \quad \bar{h}_{\mathcal{X}} = 0, \quad \bar{N} = 0 \quad \text{and} \quad h = N + h_{\mathcal{X}}. \quad (7.3.13)$$

Note that $k = 3$ exactly corresponds to the minimal number $n_5 = k - 2 = 1$ of NS5 branes. We thus learn that $n_5 = 1$ is a tensionless string limit in the sense that it gives rise to an infinite tower of massless higher spin modes.

PSU(1, 1|2)₁ and the hybrid formalism. The analysis of the previous paragraph has an important shortcoming when applied to super-strings on $AdS_3 \times S_3 \times T^4$. In the RNS-formalism, the $AdS_3 \times S_3$ factor in the near horizon limit of $n_1 \gg 1$ F1 strings and n_5 NS5 branes is described by the product

$$SL(2; \mathbb{R})_{n_5+2} \times SU(2)_{n_5-2} \quad (7.3.14)$$

of WZNW models. At $n_5 = 1$ the level of the $SU(2)$ factor becomes negative, signalling a breakdown of the RNS formalism. Despite this technical complication, the conclusion of the previous paragraph that $n_5 = 1$ is a tensionless string limit was valid and can be derived more rigorously in the hybrid formalism.

The latter amounts to a manifestly spacetime supersymmetric description of the $AdS_3 \times S_3$ factor, via the $PSU(1, 1|2)_{n_5}$ super WZNW model. In the hybrid formalism, the $n_5 \rightarrow 1$ limit is perfectly well behaved, allowing for a controlled analysis of the emergence of higher spin massless modes and the accompanying significant simplification of the spectrum. The upshot of the analysis is that no short strings, and only long strings at the bottom of the continuum i.e. at $j = 1/2$ survive at $n_5 = 1$.

This result is nicely compatible with the physical picture of the $n_5 = 1$ regime as the limit where the geometry becomes string sized. On the one hand, a string sized

geometry simply has no room to accommodate for bound states (i.e. short strings) localised to some region around the centre of AdS_3 , which suggests that these states should be absent from the spectrum. On the other hand, strings cannot resolve any details of such a small geometry, and it should hence not be possible to construct wave packets of long strings combining modes with different momenta: Only a single value of j should persist in the spectrum. For more details on the $\text{PSU}(1,1|2)_1$ spectrum, we refer the reader to Section 8.2.1.

7.3.2 The holographic match

Towards the end of Section 7.2.2, we already commented on the relation between long strings in the bosonic $\text{SL}(2;\mathbb{R})_k$ model and a marginal deformation of a symmetric product orbifold. In the tensionless limit, this relation is especially clear. As shown by Eberhardt, Gaberdiel and Gopakumar [68, 69, 277, 278], string theory on the pure NSNS $\text{AdS}_3 \times \text{S}_3 \times \mathbb{T}^4$ background at $n_5 = 1$, is exactly dual to a (grand-canonical ensemble of) symmetric product orbifold(s) of the small $\mathcal{N} = (4,4)$ supersymmetric sigma model with target space $\mathbb{T}^4$.

At the heart of this duality is an identification between the string worldsheet and branched coverings of ∂AdS_3 that compute partition functions and scattering amplitudes in the symmetric orbifold. In this section, we first provide a heuristic motivation for this covering map picture based on the Wakimoto representation of the $\text{SL}(2;\mathbb{R})_k$ model. Then, we discuss the concrete realisation of these principles in the holography of tensionless AdS_3 closed string scattering amplitudes.

First order formulation of the $\text{SL}(2;\mathbb{R})_k$ model. We can parametrise an arbitrary $g \in \text{SL}(2;\mathbb{R})$ via real fields ϕ , γ_1 and γ_2 as

$$g = \begin{pmatrix} e^{-\phi} + e^{\phi}\gamma\bar{\gamma} & e^{\phi}\gamma \\ e^{\phi}\bar{\gamma} & e^{\phi} \end{pmatrix} \quad \text{with} \quad \gamma := \gamma_1 + \gamma_2, \quad \text{and} \quad \bar{\gamma} := \gamma_1 - \gamma_2. \quad (7.3.15)$$

Using this parametrisation, the WZNW action (7.2.12) of the $\text{SL}(2;\mathbb{R})_k$ model takes the simple form

$$S = \frac{k}{\pi} \int d^2z \left(\partial\phi\bar{\partial}\phi + e^{2\phi}\partial\bar{\gamma}\bar{\partial}\gamma \right). \quad (7.3.16)$$

This formulation of the model also makes the relation between $\text{SL}(2;\mathbb{R})$ and H_3^+ particularly clear: Since in the $\text{SL}(2;\mathbb{R})$ case we consider a σ -model with Lorentzian target space, the kinetic term for γ_2 has the wrong sign in the action, making the path integral ill defined. A target space Wick-rotation $\gamma_2 \rightarrow i\gamma_2$ fixes this issue. The Wick-rotated action, in which γ and $\bar{\gamma}$ are a conjugate pair of complex fields, is the action of the H_3^+ model.

Rescaling ϕ , γ and $\bar{\gamma}$ by $1/\sqrt{2k}$ and introducing Lagrange multipliers β and $\bar{\beta}$, we can rewrite (7.3.16) as

$$S = \frac{1}{2\pi} \int d^2z \left(\partial\phi\bar{\partial}\phi + \bar{\beta}\partial\bar{\gamma} + \beta\bar{\partial}\gamma - e^{-\sqrt{\frac{2}{k}}\phi}\beta\bar{\beta} \right) + \dots, \quad (7.3.17)$$

where the ellipses denote quantum corrections that vanish as $k \rightarrow \infty$ and can be derived by considering the effect of the Lagrange multipliers on the level of the full path integral (instead of only the equations of motion).

Wakimoto representation. The action (7.3.17) suggests that the large ϕ regime i.e. the physics near the asymptotic boundary, is captured by a free boson and a $\beta\gamma$ -system, with, by the equations of motion, *holomorphic* fields $\beta, \gamma, \partial\phi$ satisfying

$$\partial\phi(z)\partial\phi(0) \sim -\frac{1}{z^2} \quad \text{and} \quad \beta(z)\gamma(0) \sim -\frac{1}{z}. \quad (7.3.18)$$

And indeed, consistent with this picture, the $\widehat{\mathfrak{sl}}(2; \mathbb{R})_k$ current algebra possess a free-field realisation, known as the Wakimoto representation [279], which is given by

$$J^+ = \beta, \quad J^3 = \beta\gamma - \sqrt{\frac{k-2}{2}}\partial\phi, \quad J^- = \beta\gamma^2 - \sqrt{2(k-2)}\gamma\partial\phi - k\partial\gamma. \quad (7.3.19)$$

To produce a stress tensor with the sugawara central charge, a non-trivial background charge for the free boson is necessary. Concretely, the Wakimoto stress tensor

$$T(z) = \beta\partial\gamma - \frac{1}{2}\partial\phi\partial\phi - \frac{Q_\phi}{\sqrt{2}}\partial^2\phi \quad (7.3.20)$$

coincides with the sugawara stress tensor exactly for

$$Q_\phi = \sqrt{\frac{1}{6} \left(\frac{3k}{k-2} - 3 \right)} = \frac{1}{\sqrt{k-2}}. \quad (7.3.21)$$

Demanding consistency with the Wakimoto picture allows us to determine the full quantum version of eq. (7.3.17), namely

$$S = \frac{1}{2\pi} \int d^2z \left(\partial\phi\bar{\partial}\phi + \frac{Q_\phi}{2\sqrt{2}}R\phi + \bar{\beta}\partial\bar{\gamma} + \beta\bar{\partial}\gamma - e^{-\sqrt{\frac{2}{k-2}}\phi}\beta\bar{\beta} \right). \quad (7.3.22)$$

The spectral flow automorphism of the current algebra amounts to the transformations

$$\gamma(z) \rightarrow z^w\gamma(z), \quad \beta(z) \rightarrow z^{-w}\beta(z), \quad \phi(z, \bar{z}) \rightarrow \phi(z, \bar{z}) - w\sqrt{\frac{k-2}{2}}\log|z|^2 \quad (7.3.23)$$

of the Wakimoto fields. According to Refs. [280, 281], the spectrally flowed vertex operators can be expressed as

$$V_{jh}^w(x, z) = e^{(j-1+\frac{k-2}{2}w)\sqrt{\frac{2}{k-2}}\phi(z)} \left(\frac{1}{w!} \partial_z^2(\gamma(z) - x) \right)^{\frac{k}{2}w+j-h-1} \prod_{n=0}^{w-1} \delta(\partial^n(\gamma(z) - x)). \quad (7.3.24)$$

Localisation to covering maps. Crucially, the OPE between $w > 0$ operators and γ is regular:

$$\gamma(z_0)V_{jh}^w(x, z) = xV_{jh}^w(x, z) + z^wV_{j,h-1}^w(x, 0). \quad (7.3.25)$$

This has important implications for the insertion of γ into a (non-vanishing) correlator of $\mathrm{SL}(2; \mathbb{R})_k$ vertex operators: on the one hand, the function

$$\mathcal{G}(z) := \langle \gamma(z) \prod_{i=1}^n V_{j_i h_i}^{w_i}(x_i, z_i) \rangle \quad (7.3.26)$$

is a holomorphic map of $\overline{\mathbb{C}}$ onto itself which at the insertion points z_i behaves as

$$\mathcal{G}(z_i + \delta z) = (x_i + a_i \delta z^{w_i} + O(\delta z^{w_i+1})) \langle \prod_{i=1}^n V_{j_i h_i}^{w_i}(x_i, z_i) \rangle. \quad (7.3.27)$$

On the other hand, we expect that the derivative of $\mathcal{G}$ (or $1/\mathcal{G}$, depending on which of the two coordinate patches of the Riemann sphere a point is mapped to) is non-vanishing around generic points, rendering $\mathcal{G}$ locally invertible. The upshot is that $\mathcal{G}$ is a branched covering with ramification index w_i at the ramification point z_i that is mapped to the branch point x_i (see Section 9.1.1 for short review of the mathematics of such maps). We can also divide out $\langle \prod_{i=1}^n V_{j_i h_i}^{w_i}(x_i, z_i) \rangle$ to write this result as

$$\langle \gamma(z) \prod_{i=1}^n V_{j_i h_i}^{w_i}(x_i, z_i) \rangle = \Gamma(z) \langle \prod_{i=1}^n V_{j_i h_i}^{w_i}(x_i, z_i) \rangle, \quad (7.3.28)$$

where Γ is a branched covering map that implicitly depends on the data of all vertex operators in the correlator. Crucially, if three points $(z_1, z_2, z_3) = (x_1, x_2, x_3) = (0, 1, \infty)$ are fixed, only a finite set of such maps exists. Furthermore, the existence of a branched-covering map places strong constraints on the x_i, z_i and w_i . We are hence led to the proposal that those correlation functions of the $\mathrm{SL}(2; \mathbb{R})_k$ model that are adequately described by the near boundary picture, should localise to discrete sums of the form

$$\langle \prod_{i=1}^n V_{j_i h_i}^{w_i}(x_i, z_i) \rangle = \sum_{\Gamma} \prod_{i=1}^n a_i^{-h_i} \bar{a}_i^{-\bar{h}_i} \prod_{i=4}^n \delta(x_i - \Gamma(z_i)) W_{\Gamma}(z_4, \dots, z_n). \quad (7.3.29)$$

Local Ward identities and localisation condition. On physical grounds, the analysis of the previous paragraph, and in particular the conclusion in eq. (7.3.29), should hold at least in the tensionless limit, where all strings are long AdS_3 sized objects that live at the asymptotic boundary. A more rigorous statement is that Reference [69] established (7.3.29) as a solution of the $\widehat{\mathfrak{sl}}(2; \mathbb{R})_k$ local Ward identities under the assumption that

$$\sum_{i=1}^n j_i = \frac{k}{2}(n-2) - (n-3) \quad (7.3.30)$$

holds. In particular, at $k = 3$ and $j_i = 1/2$ all n -point functions satisfy condition (7.3.30), which is consistent with our expectation that the tensionless limit is fully described by the Wakimoto picture. In addition to a more detailed bottom up analysis of correlation functions based on the near boundary action eq. (7.3.22) provided by Refs. [282], it was also shown from a top down perspective, that the $j = 1/2$ truncated $\text{SL}(2; \mathbb{R})_3$ spectrum with covering map localised correlation functions defines a consistent, crossing symmetric CFT with $\widehat{\mathfrak{sl}}(2; \mathbb{R})_3$ current algebra [268]:

The correlation functions of the symmetric product orbifold of any $c = 17$ CFT $\mathcal{X}$ exactly coincide with the string scattering amplitudes provided by the (non-geometric) bosonic string theory whose worldsheet CFT is given by the product of this localising $\text{SL}(2; \mathbb{R})_3$ model, the CFT $\mathcal{X}$ and the bc -ghost system.

Application to $\text{AdS}_3 \times \text{S}_3 \times \mathbb{T}^4$. Supersymmetry does not significantly affect the mechanism outlined in this section. Worldsheet correlators of x -basis vertex operators describing superstrings on $\text{AdS}_3 \times \text{S}_3 \times \mathbb{T}^4$ at $n_5 = 1$ contain delta-functions that localise the worldsheet insertion-points z_i to preimages of the spacetime positions x_i under holomorphic covering maps, as in eq. (7.3.29). These delta-functions essentially trivialise the integrals over worldsheet moduli that promote worldsheet correlators to string scattering amplitudes. The latter end up being exactly the sums of correlators of $\mathbb{T}^4$ evaluated on covering surfaces of the holographic boundary, that compute correlation functions of the dual symmetric product orbifold $\text{Sym}^N(\mathbb{T}^4)$. By a detailed description of this mechanism, References [68, 69, 277, 278] derived this instance of the $\text{AdS}_3/\text{CFT}_2$ correspondence.

7.3.3 Branes in tensionless $\text{AdS}_3/\text{CFT}_2$

The previous section ended on the conclusion that string theory on $\text{AdS}_3 \times \text{S}_3 \times \mathbb{T}^4$ at $n_5 = 1$ is known to be precisely dual to a grand-canonical ensemble of symmetric orbifolds of $\mathbb{T}^4$ on the level of local observables i.e. the spectrum of local operators and their correlation functions. In this short section, we briefly summarise what is known about the nature of branes, boundaries and interfaces i.e. non-local observables in this duality and in which directions the next two chapters extend the picture.

Holography for instantonic branes. In Reference [124], Gaberdiel, Knighton and Vošmera explicitly constructed maximally symmetric boundary states of the $\text{PSU}(1, 1|2)_1$ model. From the worldsheet perspective, their analysis provides a fully explicit description of both instantonic branes, corresponding to boundary states of the $\text{SL}(2; \mathbb{R})_k$ model satisfying the trivial gluing conditions (7.1.48), and AdS_2 branes, corresponding to the twisted gluing conditions (7.1.47).

In the case of the instantonic branes, Ref. [124] furthermore makes a concrete proposal for a holographic dual in terms of boundary states of the dual symmetric orbifold. The construction of these boundary states is reviewed in the paragraph “Reflecting part of the interfaces” of Section 8.1.2. As evidence for this proposal, Ref. [124] shows that the overlaps of these boundary states precisely match the cylinder amplitudes of the dual instantonic branes. At this stage it however remained unclear what a suitable candidate for a holographic dual of AdS_2 branes could be.

The missing dual of the AdS_2 brane. In Ref. [125], Martinec qualitatively described an open string extension of the perturbative duality between long strings on AdS_3 and a marginally deformed symmetric product orbifold that we have discussed towards the end of Section 7.2.2. He proposed that the dual description of AdS_2 branes in that context should involve an ensemble of interfaces with varying degree of transmissivity in the symmetric product orbifold, that should be deformed by a marginal interface changing operator across which the degree of transmissivity jumps by one. Ref. [125] does not explicitly construct these interfaces, which however, in particular, is necessary in order to provide quantitative evidence for the proposal by holographically matching string theory computations.

In the next Chapter, Ch. 8, we sharpen the statement of the duality by explicitly constructing a family of interfaces between symmetric product orbifolds that has the desired properties. This, in particular, allows us to exactly reproduce the open string spectrum from the dual CFT in the undeformed symmetric orbifold case. We then, in Chapter 9, develop covering map techniques to systematically evaluate the correlation functions of these interfaces. As an application we at this stage precisely fix the chemical potentials of the interface ensemble in terms of the dual string coupling. This provides a basis on which the open string generalisation can be formulated and derived at the same level of detail with which the closed string case was treated in the literature. Further comments on this and other future directions of research can be found in Chapter 10.

Chapter 8

Holographic interfaces in symmetric product orbifolds

– Boundary states and annulus amplitudes.

Whereas the focus of the introductory Chapter 7 was on pure NSNS AdS_3 string theory and the $SL(2;\mathbb{R})_k$ model, we now move to the other side of the holographic duality and focus on the symmetric product orbifold, aiming for a construction of the holographic dual for the AdS_2 branes that we have introduced in Section 7.2.

Symmetric product orbifolds of two-dimensional conformal field theories have been studied extensively for more than thirty years starting with [283]. They provide families of two-dimensional CFTs whose central charge grows linearly with the number N of copies of the underlying seed theory $\mathcal{M}$. Correlation functions of local operators in these theories share many features with those of higher-dimensional gauge theories. In particular, they possess a diagrammatic large N expansion that is very similar to the 't Hooft expansion of non-Abelian gauge theories [284]. Just as it is the case for their higher-dimensional cousins, these features of symmetric product orbifolds are suggestive of their holographic correspondence with string theory in AdS_3 backgrounds, as was first suggested in [285–287]. As we have reviewed in the previous chapter, the work of Eberhardt, Gaberdiel and Gopakumar [68] established that the symmetric product orbifold of a supersymmetric four-torus is dual to string theory on AdS_3 with one unit of pure NSNS flux. This has become one of the key examples for a holographic duality and could help us to uncover the inner workings of the AdS/CFT correspondence.

The study of symmetric product orbifolds has largely focused on the spectrum and correlation functions of local fields. There are only relatively few papers that deal with boundary conditions, defects or interfaces, see [124, 125, 288–291]. This is in some contrast with higher-dimensional gauge theories, and in particular in $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory, where the study of e.g. Karch-Randall

interfaces and Maldacena-Wilson lines has received very considerable attention, both for their roles in gauge theory and for the insights they provide on non-perturbative aspects of string theory in anti-de Sitter (AdS) space. In this chapter, we initiate a study of interfaces in symmetric product orbifolds that can be considered the lower-dimensional cousins of both Karch-Randall interfaces and of Wilson lines at the same time.

Before we go into the features of the interfaces between symmetric product orbifolds, it is instructive to briefly recall basic facts about the higher-dimensional analogues¹. The Karch-Randall interface connects two maximally SYM theories with gauge groups $SU(N_+)$ and $SU(N_-)$ on either side of the interface. It arises from a D3-D5 brane system [293], where D3 branes end on D5 branes.² Wilson lines in $\mathcal{N} = 4$ SYM theory possess different holographic realisations depending on a choice of representation of the gauge group. The most standard case is that of the fundamental representation in which case the brane realisation is in terms of a single fundamental string in AdS_5 . For representations that arise through the tensor product of k fundamental representations with k of order N , which is more relevant in our context, the brane realisation uses D-branes instead, either D3 or D5 branes depending on whether the representation is symmetric or antisymmetric.³ In both cases, the charge of k fundamental strings is carried by the electric flux of the D-brane, see e.g. [294] and references therein, in particular [295–298]. Such branes with non-vanishing electric flux approach the boundary of AdS at an angle that depends on the amount of flux. This modulus of the branes is the analogue of the parameter ψ_0 that we have introduced in eq. (7.2.26) to describe the semi-classical geometry of AdS_2 branes.

Our description of AdS_2 branes in Section 7.2 was mostly focussed on the $SL(2; \mathbb{R})_k$ model. It is instructive to take a step back from this microscopic perspective and discuss their realisation in super gravity as we have done for the higher dimensional analogue in the previous paragraph. This requires us to replace the $\mathcal{N} = 4$ SYM stack of D3 branes by the F1-NS5 brane system discussed in Section 7.2.1, or the S -dual D1-D5 system. We can create a defect in this setting by inserting a probe D1 brane into the near-horizon AdS_3 background. The case relevant for AdS_2 branes is of course that where the probe brane is localised along an AdS_2 inside AdS_3 . Beyond considering a D1 probe brane, we can also generalise the setup to $(n, 1)$ -strings, i.e. bound states of a D1 brane and n fundamental strings, see e.g. [299].

¹We would like to refer the reader to [292] for a recent overview of various top-down and bottom-up examples of holographic defects and boundaries.

²More precisely, one considers configurations in which q of the N_+ D3 branes that realise the $SU(N_+)$ SYM theory end on the D5 brane while $N_+ - q = N = N_-$ run through it to realise the $SU(N_-)$ SYM theory on the other side.

³Other representations of the gauge group whose Young diagram have more than one row or column require multiple probe branes.

As in the case of Wilson lines reviewed in the previous paragraph, one may think of these $(n, 1)$ strings as a D1 brane with a worldvolume electric field turned on with a strength that is determined by n . When we allow n fundamental strings to run along the worldvolume of the probe D1 brane, the number of fundamental strings that appear in the brane realisation of the symmetric product orbifold can jump from one side of the D1 brane junction to the other, just as in the case of Karch-Randall interfaces in gauge theory. This motivates the expectation that the holographic dual of AdS_2 branes should involve interfaces at which the degree of the symmetric product orbifold can jump from N_- to N_+ .

The precise realisation of this idea constructed in this chapter, is a family of interfaces between any pair of symmetric product orbifolds of rank N_- and N_+ that are parametrised by some integer $p = 1, \dots, \min(N_-, N_+)$. The interfaces $\mathcal{I}_{[a_{\pm}]}^{(p)}$ that we define are transmissive in p components of the product theory and reflective in the remaining ones with a reflective behaviour that depends on the choice of a boundary conditions $a_{\pm}$ of the underlying seed conformal field theory $\mathcal{M}$. A precise definition is given in section 8.1.2, see eq. (8.1.38) and the explanation of notation below that formula. Our description uses the folding trick and hence it describes the interface in terms of a boundary state of the folded theory, as shown in figure 8.0.1. As usual, the boundary state encodes the entire set of one-point couplings between the interface and bulk fields of the CFT. From this description of the interfaces, we then determine the annulus amplitude and thereby the exact spectrum of interface changing operators between any pair of interfaces in section 8.1.4. This key result is stated in eqs. (8.1.54 - 8.1.59). As is familiar in the context symmetric product orbifolds since the seminal paper of Dijkgraaf-Moore-Verlinde-Verlinde [300], it is easiest to state the result in terms of a grand canonical partition function. In the setup under consideration, this grand canonical partition function involves four chemical potentials $\mu_{\pm}$ and ρ_A with $A \in \{L, R\}$. The first two, i.e. μ_- and μ_+ , correspond to the order $N_{\pm}$ of the symmetric product orbifold on the two sides of the interfaces. The other two chemical potentials, ρ_L and ρ_R , are associated with the numbers p_L and p_R to the left and the right of the interface changing operator, see figure 8.0.2.

While for generic values of the level k there exists a continuous family of AdS_2 branes, the tensionless limit of IIB superstring theory has a unique AdS_2 brane in the AdS_3 factor of the target space.⁴ With our holographic proposal, we extend the match of bulk partition functions [68] to this unique AdS_2 brane. Our matching of spectra is based on formulas for the partition function of open strings on AdS_2 branes in (global and) thermal AdS_3 backgrounds that we derive in section 8.2, see in particular eq. (8.2.34). For a concise summary of the holographic relation we

⁴In [124] the authors constructed two boundary states. Here we only admit one special linear combination thereof in order to restore the rotational symmetry with respect to half-rotations of AdS_3 . This additional feature makes the boundary state unique.

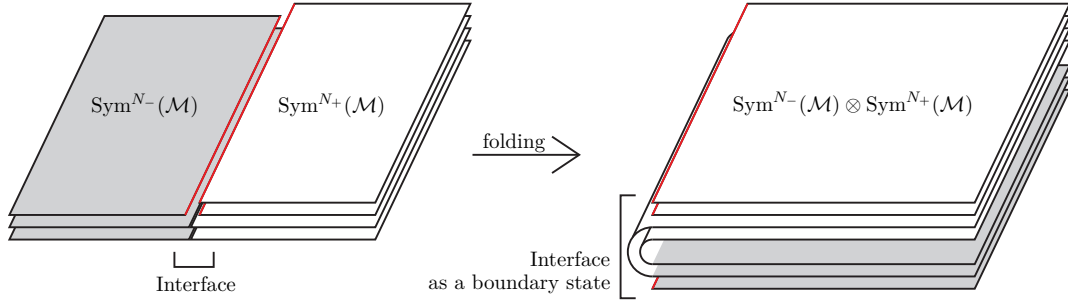


Figure 8.0.1: Interface between $\text{Sym}^{N_-}\mathcal{M}$ and $\text{Sym}^{N_+}\mathcal{M}$ as a boundary state of the folded theory. The red lines indicate reflective boundaries. This figure is for $N_- = 3$, $N_+ = 4$, $p = 2$.

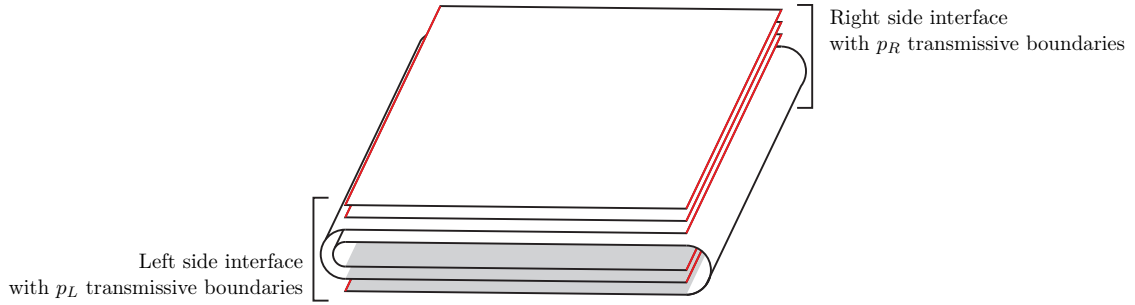


Figure 8.0.2: Contribution to the torus partition function in the presence of a pair of interfaces with parameters $N_- = 3$, $N_+ = 4$, $p_L = 2$, $p_R = 1$. The red lines indicate reflective boundaries. As discussed in Section 8.1.4, the full partition function is a symmetrised sum over terms of the type illustrated in this figure.

propose, the reader might also find the box on page 269 useful.

Let us now briefly outline the plan of this chapter. In many respects, section 8.1 contains the main new results. After a short review 8.1.1 of symmetric product orbifolds that we use to set up notations, section 8.1.2 constructs the interfaces $\mathcal{I}_{|a_{\pm}}^{(p)}$. As a small interlude, we compute the transmissivity and reflectivity of these interfaces in terms of the parameter p in section 8.1.3. Then, in section 8.1.4, we calculate the spectrum of interface changing operators. While our construction of the interface may seem somewhat artificial, it is guided by the intention to construct a holographic dual of AdS_2 branes in AdS_3 . To show that this objective is indeed achieved, at least in the tensionless limit of type IIB superstrings on $AdS_3 \times S^3 \times \mathbb{T}^4$, we discuss the string theory of AdS_2 branes in section 8.2.1. The boundary states of the world sheet theory corresponding to AdS_2 branes have been constructed in [124]. We briefly review

this construction and then, in section 8.2.2, use it to determine the spectrum of worldsheet boundary excitations associated to open string states on the AdS_2 branes in global AdS_3 . The result serves as an input to section 8.2.3, where we perform an orbifold construction to pass to thermal AdS_3 . After integration over the modular parameter of the toroidal worldsheet, we finally obtain the partition function of open superstrings in thermal AdS_3 . The two strands of the discussion are merged in section 8.3 where we compare the string partition function with the connected contribution to the torus partition function of our interface in the symmetric product orbifold of the supersymmetric four-torus to find complete agreement between these two quantities. The section also contains some qualitative discussion of correlation functions in the symmetric product orbifold and the corresponding scattering amplitudes of open and closed strings in the dual string theory. The chapter contains three appendices in which we provide some explicit calculations to prove formulas that are stated in the main text.

8.1 Interfaces of symmetric product orbifolds

The arguably simplest type of interfaces are those that are purely reflecting or purely transmitting. Purely reflecting interfaces can be obtained by choosing boundary conditions for the two CFTs which are separated by the interface, resulting simply in a product of two essentially decoupled boundary theories. In the specific context of symmetric product orbifolds, a detailed analysis of the spectrum of boundary states has been provided by [288]. The most prominent example of a purely transmitting interface is the trivial interface between two identical CFTs. For symmetric product orbifolds, a broader class of purely transmitting defects was studied in [289].

In this section, we introduce a more general family of interfaces between two different symmetric product orbifolds $\text{Sym}^{N_{\pm}}(\mathcal{M})$ of the same seed theory $\mathcal{M}$ that interpolates between the purely transmitting and purely reflecting extremes. For this purpose, section 8.1.1 establishes the general notation which we use to describe symmetric product orbifolds. Section 8.1.2 gives a precise definition of our interfaces. Section 8.1.3 serves the purpose of making our definitions more accessible by demonstrating how to compute the reflectivity and transmissivity coefficients of our interface. It does not contain any information that is strictly necessary for the rest of the discussion and may be skipped. Finally, section 8.1.4 determines partition functions of interface changing operators that act within the class of interfaces that we have proposed.

8.1.1 Notation and conventions

This section sets up the necessary notation to work with symmetric product orbifolds $\text{Sym}^N(\mathcal{M})$ of a CFT $\mathcal{M}$. The Hilbert space of the seed theory $\mathcal{M}$ is denoted by

$\mathcal{H}_{\mathcal{M}}$. It comes equipped with two commuting actions of the Virasoro algebra along with possibly further modes of some extended chiral algebra $\mathcal{W}$. Under the action of this chiral algebra, the state space decomposes into sectors which we label by letters $i, j, \dots$ of the Roman alphabet. In a unitary CFT, these are lowest weight representations. The partition function of the seed theory $\mathcal{M}$ is given by

$$Z(t, \bar{t}) = \text{Tr}_{\mathcal{H}_{\mathcal{M}}} [x^{L_0 - \frac{c_{\mathcal{M}}}{24}} \bar{x}^{\bar{L}_0 - \frac{c_{\mathcal{M}}}{24}}], \quad x = e^{2\pi i t}, \quad \bar{x} = e^{-2\pi i \bar{t}}. \quad (8.1.1)$$

Here, $c_{\mathcal{M}}$ denotes the central charge of the Virasoro algebra, as usual. With these basic ingredients of the seed theory set up, we move on to the symmetric product orbifold.

States of the symmetric product orbifold $\text{Sym}^N(\mathcal{M})$ are obtained by the standard rules of the orbifold construction. In a first step, we construct twisted sectors $\mathcal{H}^g$ for the N^{th} power $\mathcal{M}^{\otimes N}$ of the seed theory. These are labelled by group elements $g \in S_N$ of the symmetric group. For a given element g , we define the centraliser subgroup

$$\mathcal{C}_g = \mathcal{C}_g^N = \{ \sigma \in S_N \mid \sigma g \sigma^{-1} = g \} \subseteq S_N. \quad (8.1.2)$$

The twisted sector $\mathcal{H}^g$ carries a representation of $\mathcal{C}_g$ and, as part of the orbifold procedure, we are instructed to project to its $\mathcal{C}_g$ invariant subspace. The projection can be performed by averaging over the orbits of $\mathcal{C}_g$. Hence, the torus partition function of the symmetric product orbifold is given by⁵

$$Z_N(t, \bar{t}) := \frac{1}{N!} \sum_{g \in S_N} \sum_{h \in \mathcal{C}_g} \text{Tr}_{\mathcal{H}^g} [h x^{L_0 - \frac{N c_{\mathcal{M}}}{24}} \bar{x}^{\bar{L}_0 - \frac{N c_{\mathcal{M}}}{24}}]. \quad (8.1.3)$$

One can straightforwardly express the partition function of the orbifold theory in terms of the seed theory partition function by directly evaluating the trace. The most elegant formula for the partition function [300] passes through the grand canonical ensemble

$$\mathcal{Z}(\kappa; t, \bar{t}) := \sum_{N=0}^{\infty} \kappa^N Z_N(t, \bar{t}) = \exp \left[\sum_{k=1}^{\infty} \kappa^k T_k Z(t, \bar{t}) \right]. \quad (8.1.4)$$

The terms in the exponent on the right hand side are obtained from the partition function $Z(t, \bar{t})$ of the seed theory through application of the following Hecke operators

$$T_k Z(t, \bar{t}) := \frac{1}{k} \sum_{w|k} \sum_{j=0}^{w-1} Z \left(\frac{kt}{w^2} + \frac{j}{w}, \frac{k\bar{t}}{w^2} + \frac{j}{w} \right). \quad (8.1.5)$$

By expanding the exponent on the right hand side in powers of κ , it is straightforward to obtain the partition functions Z_N of symmetric product orbifolds for any finite integer N .

⁵The normalisation factor is $1/N!$ because we divide both by the size of the centraliser $|\mathcal{C}_g|$ (due to the averaging over $\mathcal{C}_g$ orbits) as well as by the size of the conjugacy class of g .

It is important to discuss formula (8.1.3) in more detail. We have written the formula as a sum over all elements g of the symmetric group S_N . But many of the twisted sectors $\mathcal{H}^g$ we are summing over give rise to equivalent representations of the chiral algebra. By construction, two twisted sectors $\mathcal{H}^{g_1}$ and $\mathcal{H}^{g_2}$ give rise to equivalent representations if g_1 and g_2 belong to the same conjugacy class. We denote the conjugacy class of $g \in S_N$ by

$$[g] := \{hgh^{-1} : h \in S_N\} \subseteq S_N \quad (8.1.6)$$

and more generally

$$[g_1, g_2, \dots, g_n] := \{(hg_1h^{-1}, hg_2h^{-1}, \dots, hg_nh^{-1}) : h \in S_N\}. \quad (8.1.7)$$

Famously, the conjugacy classes $[g]$ of the symmetric group S_N are in one to one correspondence with integer partitions $(w_1, \dots, w_\ell)$ of $N = \sum_{r=1}^\ell w_r$ or, equivalently, Young diagrams Y with N boxes. The correspondence is easy to spell out more explicitly. Any element $g \in S_N$ possesses a unique (up to reordering of the factors) decomposition

$$g = \omega_1 \omega_2 \cdots \omega_\ell \quad (8.1.8)$$

into cyclic permutations ω_r of length w_r such that the cycles constitute a partition $\{\omega_r\}_{r=1}^\ell$ of $\{1, \dots, N\}$ into ordered sets. The integer partition $\{w_r\}_{r=1}^\ell$ of N is graphically represented by the Young diagram Y_g associated to $g \in S_N$. Conversely, any two group elements that possess a factorisation of the form (8.1.8) with ℓ cycles of order w_r are conjugate to each other. Given a Young diagram Y with N boxes we can always pick some representative g_Y of the corresponding conjugacy class of S_N .

With the help of the factorisation (8.1.8) of the group element g into a product of cyclic permutations, it is easy to give a more explicit description of the centraliser group $\mathcal{C}_g^N$. Concretely, $\mathcal{C}_g^N$ is given by

$$\mathcal{C}_g^N := \left\{ \sigma = \pi \prod_{r=1}^\ell \omega_r^{\nu_r} \mid \nu_r \in \mathbb{Z} \text{ and } \pi \in S_\ell \text{ such that } w_{\pi r} = w_r \right\} \simeq \prod_{w=1}^N S_{m_w} \ltimes \mathbb{Z}_w^{m_w}, \quad (8.1.9)$$

where the sequence $(m_w)_{w \in \mathbb{N}}$ is the weight of the diagram Y_g i.e. m_w is the number of rows of length w . By the orbit-stabiliser theorem, the order $|S_N| = N!$ of the symmetric group factorises as

$$|S_N| = |\mathcal{C}_g^N| \cdot |Y_g|, \quad (8.1.10)$$

where $|Y_g|$ denotes the number of elements in the conjugacy class Y_g of g . After this preparation, it is natural to rewrite our formula (8.1.3) for the partition function of the symmetric product orbifold as

$$Z_N(t, \bar{t}) = \sum_{|Y|=N} \text{Tr}_{\mathcal{H}^{g_Y}} \left[\Pi_0^{g_Y} x^{L_0 - \frac{Nc_M}{24}} \bar{x}^{\bar{L}_0 - \frac{Nc_M}{24}} \right] \quad \text{where} \quad \Pi_0^{g_Y} := \frac{1}{|\mathcal{C}_{g_Y}^N|} \sum_{\sigma \in \mathcal{C}_{g_Y}^N} \sigma. \quad (8.1.11)$$

In performing the sum over Young diagrams Y with N boxes, we can pick a single representative $g = g_Y \in S_N$ and then construct the twisted sector $\mathcal{H}^g$, the centraliser $\mathcal{C}_g$ and the projection Π_0^g for that representative. The result is obviously independent of the representative we choose. Thereby, we have rewritten the partition function of the symmetric product orbifold as a sum over partitions of N rather than group elements $g \in S_N$.

In order to illustrate the difference between the formulas (8.1.3) and (8.1.11), let us briefly discuss the example of $N = 3$. In this case, the symmetric group S_3 has six elements and the original sum in eq. (8.1.3) runs over all these six elements. But the six group elements fall into only three conjugacy classes which are associated with the three possible partitions of $N = 3$. The conjugacy class for the partition $Y = [1, 1, 1]$ consists of the unit element $g = e$ and its centraliser group $\mathcal{C}_e^3 \simeq S_3 \ltimes \mathbb{Z}_1^3$ is the entire symmetric group S_3 . Next, let us consider the partition $Y = [1, 2]$. The associated conjugacy class contains three elements, namely $g_1 = (1)(23)$, $g_2 = (2)(13)$ and $g_3 = (3)(12)$. Their centraliser subgroups $\mathcal{C}_{g_i}^3 \cong \mathbb{Z}_2$ contain two elements each. Finally, for the trivial partition $Y = [3]$, the conjugacy class consists of the elements $\omega_1 = (123)$ and $\omega_2 = (132)$. Their centraliser group, on the other hand, is now given by $\mathcal{C}_{\omega_i}^3 \cong \mathbb{Z}_3$.

In the following, we shall continue to think of the twisted sectors $\mathcal{H}^g$ as being associated with elements $g \in S_N$ rather than with the Young diagrams Y_g . This is correct as long as we always remember not to sum but to average over the elements of a conjugacy class. Now given a group element g with the factorisation (8.1.8) of length ℓ , states of the associated twisted sector $\mathcal{H}^g$ take the form

$$|(\psi_1, \dots, \psi_\ell)\rangle_g \quad \text{where} \quad \psi_r \in \mathcal{H}_{\mathcal{M}} \quad (8.1.12)$$

are ℓ states in the seed theory. By abuse of notation, we simply write

$$|\psi_r\rangle_g \quad \text{instead of} \quad |(\psi_r)_{r=1}^\ell\rangle_g. \quad (8.1.13)$$

Let us assume that these ℓ states ψ_r , $r = 1, \dots, \ell$ of the seed theory have been chosen to be eigenstates of $(L_0, \bar{L}_0)$ with eigenvalues $(h_r, \bar{h}_r)$. Then the twisted sector state (8.1.12) is an eigenstate of the symmetric product orbifold Virasoro generators $(L_0, \bar{L}_0)$ with eigenvalues

$$h^g = \sum_{r=1}^{\ell} \left(\frac{h_r}{w_r} + \frac{c_{\mathcal{M}}}{24} \left(w_r - \frac{1}{w_r} \right) \right) \quad \text{and} \quad \bar{h}^g = \sum_{r=1}^{\ell} \left(\frac{\bar{h}_r}{w_r} + \frac{c_{\mathcal{M}}}{24} \left(w_r - \frac{1}{w_r} \right) \right). \quad (8.1.14)$$

In particular, its spin is given by

$$J^g = \bar{h}^g - h^g = \sum_{r=1}^{\ell} \left(\frac{\bar{h}_r - h_r}{w_r} \right). \quad (8.1.15)$$

Only states (8.1.12) for which the spin is an integer correspond to local operators of the symmetric product orbifold. In order to find the subset of physical states within the twisted sector $\mathcal{H}^g$, one needs to impose the projection to $\mathcal{C}_g$ invariants, see the construction in Π_0 in eq. (8.1.11). The action of $\sigma \in \mathcal{C}_g$ on twisted sector states (8.1.12) is given by

$$\sigma|\psi_r\rangle_g = \prod_{r=1}^{\ell} e^{2\pi i \frac{\nu_r}{w_r} (\bar{h}_r - h_r)} |\psi_{\pi r}\rangle_g = |e^{2\pi i \nu_{\pi r} (\bar{L}_0 - L_0)} \psi_{\pi r}\rangle_g. \quad (8.1.16)$$

Here, we have represented $\sigma \in \mathcal{C}_g$ as a product $\sigma = \pi \omega_1^{\nu_1} \dots \omega_{\ell}^{\nu_{\ell}}$ as explained in eq. (8.1.9). Furthermore, the Virasoro elements on the right hand side are understood to be those of the symmetric product orbifold i.e. operators with a spectrum as given in (8.1.14). The exponents ν_r of the single cycles ω_r only enter the phase factor, while π permutes the r labels. In order to help the reader to get used to the notation for symmetric product orbifolds introduced in this section, we use it in appendix 8.A to compute the torus partition function.

8.1.2 Definition of the interfaces

This section constructs a class of interfaces between $\text{Sym}^{N_-}(\mathcal{M})$ and $\text{Sym}^{N_+}(\mathcal{M})$. We think of these interfaces as separating the upper and lower half of the complex plane with $\text{Sym}^{N_-}(\mathcal{M})$ living in the lower half plane and $\text{Sym}^{N_+}(\mathcal{M})$ in the upper. The definition of our interfaces involves selecting a pair of boundary states $|a_{\pm}\rangle$ of the seed theory $\mathcal{M}$. These boundary states of the seed theory give rise to a purely reflecting interface, in which all the $N_-[N_+]$ copies of the seed theory in the lower[upper] half plane are simply reflected⁶ back. This is obviously a bit too trivial to be interesting. Instead, we propose to consider interfaces in which $p \leq \min(N_-, N_+)$ components can pass through the interface while the remaining ones are reflected.

Qualitative overview of the construction. Before we discuss the precise construction of the interfaces, let us first give a qualitative description of all steps that constitute it. To spell out a concrete formula for the interfaces $\mathcal{I} = \mathcal{I}_{|a_{\pm}\rangle}^{(p)}$, we utilize the folding trick and describe the interfaces as boundary states of

$$\text{Sym}^{N_-}(\mathcal{M}) \otimes \text{Sym}^{N_+}(\mathcal{M}) = \mathcal{M}^{N_-+N_+} / S_{N_-} \times S_{N_+}. \quad (8.1.17)$$

In our construction, we first choose a gauge in which p components in each of the theories that we want to be transmitting are singled out. This procedure manifestly breaks the $S_{N_-} \times S_{N_+}$ symmetry of the folded theory down to $S_{N_- - p} \times S_p \times S_p \times S_{N_+ - p}$.

⁶See section 8.1.3 below for a more precise definition of reflectivity and transmissivity in this context.

After this first gauge choice, we impose the reflecting boundary condition $|a_- \rangle$ along $N_- - p$ components from the first factor and likewise $|a_+ \rangle$ along $N_+ - p$ components from the second factor. The remaining copies are glued together with transmitting boundary conditions. Such a boundary condition is stabilised by the subgroup

$$S_{N_- - p} \times S_{p, \text{diag}} \times S_{N_+ - p} \subseteq S_{N_- - p} \times S_p \times S_p \times S_{N_+ - p} \subseteq S_{N_-} \times S_{N_+} \quad (8.1.18)$$

and thus the selection of a particular way to transmit the selected p copies of the seed theory in the lower half plane to p copies in the upper half plane constitutes a second gauge choice that breaks $S_p \times S_p$ to the diagonal subgroup $S_{p, \text{diag}} := \{(g, g) | g \in S_p\} \subseteq S_p \times S_p$. To restore the full $S_{N_-} \times S_{N_+}$ symmetry, we finally need to average over the gauge orbits.

In the construction of boundary states of the folded theories, we can select characters for each of the three factors of the stabiliser subgroup within the orbifold group $S_{N_-} \times S_{N_+}$. We denote these characters by (χ_-, χ_p, χ_+) . The boundary states we are about to construct are denoted by

$$|p, a_{\pm}; \chi_{\pm}, \chi_p \rangle = |p, a_{\pm}; \chi_{\pm}, \chi_p \rangle_{\mathcal{M}}^{N_{\pm}}. \quad (8.1.19)$$

We shall often drop the sub- and superscripts that refer to the bulk data. In addition, throughout most of our discussion we will set the characters $\chi_{\pm}, \chi_p$ to be trivial, drop them from the arguments and write

$$|p, a_{\pm}; N_{\pm} \rangle = |p, a_{\pm}; \chi_{\pm} = \mathbf{1}, \chi_p = \mathbf{1} \rangle_{\mathcal{M}}^{N_{\pm}}. \quad (8.1.20)$$

The interfaces obtained from the trivial representation of the stabiliser subgroup turn out to be the ones that are relevant for holography. This is not merely an a posteriori observation, but rather to be expected from first principles: From the string perspective, the representations associated to the characters $\chi_{\pm}$ and χ_p are the representations of the symmetric group that govern the statistics of multi-string states. If we would like to describe multi-string ensembles with Bose statistics, we should hence make use only of the trivial characters in the dual symmetric orbifold, see also [291]. Moreover, the restriction to fully symmetric representations of the ‘gauge group’ S_N is also consistent with the situation for Maldacena-Wilson lines in four-dimensional $N = 4$ SYM theory. In the higher dimensional context, Wilson lines in other than symmetric traceless tensor representations require to consider multiple $D3$ branes in the bulk [301] (or $D5$ branes).

Reflective part of the interfaces. After this first qualitative description of our interfaces, let us now proceed with their systematic construction. The boundary states $|a_{\pm} \rangle$ of the seed theory constitute the most non-trivial data that enters the construction. Let us recall that the sectors of the seed theory, i.e. the irreducible

representations of its chiral algebra $\mathcal{W}$, are labelled by some index j . Since the gluing condition for chiral fields relates the label of the right- and left-moving representations, the index j also labels the Ishibashi states $|j\rangle\rangle$ that can contribute to the boundary state.⁷ Thus, a boundary state $|a\rangle$ of the seed theory is described by a set of coefficients a_j in the expansion

$$|a\rangle = \sum_j a_j |j\rangle\rangle. \quad (8.1.21)$$

We can uplift this boundary state of the seed theory to a boundary state of the symmetric product orbifold $\text{Sym}^M(\mathcal{M})$ for any $M \in \mathbb{N}$. Specifically we are interested in the cases $M = N_- - p$ or $M = N_+ - p$. Once again, the boundary state of the symmetric product orbifold is a sum over Ishibashi states. For a fixed gauge $\rho \in S_M$, the relevant Ishibashi state is given by

$$|j_r\rangle\rangle_\rho^M := |\{ |j_r\rangle\rangle \}_\rho^M. \quad (8.1.22)$$

Here, ρ is assumed to possess a factorisation of the form (8.1.8) and the twisted sector states that appear on the right hand side are obtained from ℓ Ishibashi states $|j_r\rangle\rangle$, $r = 1, \dots, \ell$ of the seed theory in the spirit of eq. (8.1.12). The only difference is that Ishibashi states live in some appropriate completion of $\mathcal{H}^\rho$ rather than $\mathcal{H}^\rho$ itself.⁸ The action of elements $\sigma \in \mathcal{C}_\rho$ in the centraliser subgroup on the Ishibashi states takes the form

$$\sigma |j_r\rangle\rangle_\rho = |j_{\pi r}\rangle\rangle_\rho, \quad (8.1.23)$$

where $\pi = \pi_\sigma$ is a permutation that exchanges two cycles of the same length. Note that the action of the factors $\omega_r^{\nu_r}$ in the factorisation formula for σ is trivial since contributions from holomorphic and anti-holomorphic components cancel each other. The overlap of any two of these Ishibashi states is given by

$${}_{\rho'} \langle\langle j'_r | \sigma \hat{x}^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{12})} | j_r \rangle\rangle_\rho = \delta_{j'_r, j_{\pi r}}^{(\ell)} \delta_{\rho', \rho} \prod_{r=1}^{\ell} \chi_{j_r}(\frac{\hat{t}}{w_r}). \quad (8.1.24)$$

Here, $\sigma \in \mathcal{C}_\rho^M$ is in the centraliser subgroup of ρ and $\pi = \pi_\sigma$, as before. Furthermore⁹,

$$\chi_j(t) := \text{Tr}_{\mathcal{H}_{j,j}} x^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{12})}. \quad (8.1.25)$$

⁷To be more precise, the sectors of the seed theory are labelled by pairs $(j, \bar{j})$ of representations for the (anti-)holomorphic chiral fields. The specification of a boundary state involves picking a gluing automorphism Ω for the chiral algebra. This automorphism induces a map on representation labels $j \mapsto j_\Omega$. Given the choice of Ω , Ishibashi states only exist for those sectors of the theory for which $\bar{j} = j_\Omega$. In this sense, we only need to specify a single representation label in order to specify the Ishibashi state, see [243] for details.

⁸For instance, they may be viewed as discontinuous linear functionals on $\mathcal{H}$.

⁹In accordance with footnote 7, we could also have written j_Ω instead of $\bar{j}$ in eq. (8.1.25).

Let us stress that the Ishibashi states $|j_r\rangle\rangle_\rho$ we have introduced are not yet projected to the subspace of $\mathcal{C}_\rho$ invariant states. Given our Ishibashi states $|j_r\rangle\rangle_\rho$, we construct the linear combination

$$|a\rangle_\rho = \sum_{\{j_r\}} \frac{a_{j_1} \cdots a_{j_\ell}}{\sqrt{|\mathcal{C}_\rho^M|}} |j_r\rangle\rangle_\rho^M \quad (8.1.26)$$

with coefficients formed from products of the coefficients that appeared in the boundary state $|a\rangle$ of the seed theory. The overlap of any two of these states can be easily computed from the overlap (8.1.24). Note that the states $|a\rangle_\rho$ are actually within the subspace of $\mathcal{C}_\rho$ invariant elements, i.e. the projector Π_0^ρ acts trivially on these states. The normalisation we chose is the one that is appropriate for Cardy consistent boundary states, see e.g. [288] or Section 4.3 of [124]. The states (8.1.26) constitute the first ingredient we shall use in constructing our interfaces.

Transmissive part of the interfaces. As a second ingredient of our construction, we need to discuss conformal interfaces for $\text{Sym}^p(\mathcal{M})$. After the folding trick, such an interface is described as a boundary state in which the holomorphic fields of the theory on the upper half plane are glued to the anti-holomorphic fields of the second theory that was folded up from the lower half plane and vice versa. Ishibashi states for the “permutation boundary states” [302] in $\text{Sym}^p(\mathcal{M}) \times \text{Sym}^p(\mathcal{M})$ will be denoted by

$$|j_r, i_s\rangle\rangle_{\tau_-, \tau_+}^{p,p} \quad (8.1.27)$$

with $\tau_\pm \in S_p$ that are conjugate to each other, i.e. $[\tau_-] = [\tau_+]$. More concretely, if we formally¹⁰ write

$$|j_r\rangle\rangle_\tau = |j_r\rangle_\tau \overline{|j_r\rangle_\tau}, \quad (8.1.28)$$

then

$$|j_r, i_s\rangle\rangle_{\tau_-, \tau_+}^{p,p} = |j_r\rangle_{\tau_-} \overline{|i_s\rangle_{\tau_-}} |i_s\rangle_{\tau_+} \overline{|j_r\rangle_{\tau_+}}. \quad (8.1.29)$$

The action of the centraliser of τ_+ and τ_- on these Ishibashi states is given by

$$\sigma_+ \sigma_- |j_r, i_s\rangle\rangle_{\tau_-, \tau_+}^{p,p} = |\alpha^{\nu_{\pi^-}^-} j_{\pi^- r}\rangle_{\tau_-} \overline{|\alpha^{\nu_{\pi^-}^-} i_{\pi^- s}\rangle_{\tau_-}} |\alpha^{\nu_{\pi^+}^+} i_{\pi^+ s}\rangle_{\tau_+} \overline{|\alpha^{\nu_{\pi^+}^+} j_{\pi^+ r}\rangle_{\tau_+}}, \quad (8.1.30)$$

where

$$\alpha = e^{-2\pi i L_0}. \quad (8.1.31)$$

¹⁰Of course $|j_r\rangle\rangle_\tau$ is not a pure tensor product of left and right movers, but a sum of such products. Our notation captures the structure of the individual pure tensor summands.

The overlaps of these Ishibashi states take the form

$$\tau'_\pm \langle\langle j'_r, i'_s | \sigma_- \sigma_+ \hat{x}^{L_0 + \bar{L}_0 - \frac{c}{12}} | j_r, i_s \rangle\rangle_{\tau_\pm} = \delta_{i'_k, i_{\pi_\pm k}}^{(2\ell)} \delta_{j'_k, j_{\pi_\pm k}}^{(2\ell)} \delta_{\tau'_\pm, \tau_\pm}^{(2)} \chi_{i, \sigma_+, \sigma_-}(t) \chi_{j, \sigma_-, \sigma_+}(t). \quad (8.1.32)$$

In particular, they are only non-vanishing if $j_{\pi^- r} = j_{\pi^+ r}$. To spell out $\chi_{j, \sigma_-, \sigma_+}(t)$, recall that $\pi^\pm$ is a product of permutations that shuffle twisted sectors associated to cyclic permutations of length w . Hence, if we write $\pi^\pm$ as a product $\pi^\pm = \pi_1^\pm \pi_2^\pm \dots$ of cyclic permutations, then each permutation $\pi_a^\pm$ has two integers associated to it, namely the winding $w_{\pi_a^\pm}$ of the sectors it permutes and the length $\ell_{\pi_a^\pm} = |\pi_a^\pm|$.

The same is true for $\pi := \pi_+ \pi_-^{-1} = \pi_1 \dots \pi_m$, i.e. for each π_a we have pair (w_a, ℓ_a) of integers. The condition $j_{\pi^- r} = j_{\pi^+ r}$ is equivalent to the statement that $j_r = j_{\pi r}$ i.e. $j_r = j_{\pi_a r}$ for all a . In particular, if we denote by $\mathbf{a}$ the set of labels appearing in the cycle π_a , then we can introduce $j_{\mathbf{a}}$ as being equal to j_r for some not further specified $r \in \mathbf{a}$ and this is a well defined prescription. In terms of

$$\chi_j(t, \bar{t}) := \text{Tr}_{\mathcal{H}_{j, \bar{j}}} x^{L_0 - \frac{c}{24}} \bar{x}^{\bar{L}_0 - \frac{c}{24}}, \quad (8.1.33)$$

$\chi_{j, \sigma_-, \sigma_+}(t)$ can then be expressed as

$$\chi_{j, \sigma_-, \sigma_+}(t) = \prod_{a=1}^m \chi_{j_{\mathbf{a}}} \left(\frac{\ell_a t + \sum_{k \in \mathbf{a}} \nu_k^-}{w_a}, \frac{\ell_a t + \sum_{k \in \mathbf{a}} \nu_k^+}{w_a} \right). \quad (8.1.34)$$

Once again, we introduce a linear combination of these twisted Ishibashi states by summing over the labels j_r, i_s . But in this case, the sum over the labels is not sufficient in order to ensure $\mathcal{C}_{\tau_\pm}$ invariance. Hence, we also have to perform the relevant projection

$$|\mathbb{I}\rangle_{\tau_-, \tau_+}^{p, p} = \Pi_0^{\tau_+} \Pi_0^{\tau_-} \sum_{j_r, j_s} |j_r, i_s\rangle_{\tau_-, \tau_+}^{p, p} \quad \text{where} \quad \Pi_0^{\tau_\pm} = \frac{1}{|\mathcal{C}_{\tau_\pm}^p|} \sum_{\sigma_\pm \in \mathcal{C}_{\tau_\pm}^p} \sigma_\pm. \quad (8.1.35)$$

The overlap between any two such states can be computed with the help of formula (8.1.32). This completes the discussion of the second ingredient for our construction of interfaces.

Averaging over gauge orbits. As the final step in the construction of the boundary states (8.1.19), we now perform the averaging over gauge orbits. This gives the gauge invariant state

$$|p, a_\pm\rangle_{\mathcal{O}_\pm}^{N_\pm} = \frac{1}{\sqrt{|\mathcal{O}_-| |\mathcal{O}_+|}} \sum_{(\rho_\pm, \tau_\pm) \in \mathcal{O}_\pm} |a_-\rangle_{\rho_-} \cdot |\mathbb{I}\rangle_{\tau_-, \tau_+}^{p, p} \cdot |a_+\rangle_{\rho_+}, \quad (8.1.36)$$

where

$$\mathcal{O}_\pm := [\tau, \rho_\pm]_p^{N_\pm} = \{(h\tau h^{-1}, h\rho_\pm h^{-1}) : h \in S_{N_\pm}\} \quad (8.1.37)$$

are $S_{N_{\pm}}$ gauge orbits associated to a choice of $\rho_{\pm} \in S_{N_{\pm}-p}$ and $\tau \in S_p$. We added the subscript p to remind ourselves, that τ and $\rho_{\pm}$ are elements of $S_{N_{\pm}-p}$ and S_p and that the associated subgroup is an extra piece of data that we explicitly keep track of. That is, we distinguish for instance between $id_{S_p} \in S_p$ and $h \cdot id_{S_p} \cdot h^{-1} = id_{hS_p} \in hS_p$.

Let us briefly comment on the choice of normalisation that we made in the definition of $|p, a_{\pm}\rangle_{\mathcal{O}_{\pm}}^{N_{\pm}}$. Ultimately, we do not have an a priori principle that tells us what the “correct” normalisation should be. We can only justify our choice by showing that its overlaps lead to the physical partition functions that we are interested in, which indeed will turn out to be the case. However, we can at least argue, already before any computation, that our guess is very natural one: Since we normalised appropriately the mutually orthogonal states $|a_{\pm}\rangle_{\rho_{\pm}}$ and $|\mathbb{I}\rangle_{\tau_{-}, \tau_{+}}$ to represent reflective/transmissive boundaries, we should simply divide out the square root of the size of the orbits that we sum over such that the sum does not alter the already correct norm.

Now that we have constructed the boundary states $|p, a_{\pm}\rangle_{\mathcal{O}_{\pm}}^{N_{\pm}}$ associated to some choice of $S_{N_{\pm}}$ orbits, we finally perform a weighted sum over all choices of $\mathcal{O}_{\pm}$ to obtain the boundary state

$$|p, a_{\pm}; \chi_{-}, \chi_p, \chi_{+}\rangle = \sum_{\tau \in S_p} \sum_{\rho_{\pm} \in S_{N_{\pm}-p}} \frac{\chi_{-}(\rho_{-}) \chi_p(\tau) \chi_{+}(\rho_{+})}{|[\tau]| |[\rho_{+}]| |[\rho_{-}]|} |p, a_{\pm}\rangle_{\mathcal{O}_{\pm}}^{N_{\pm}}. \quad (8.1.38)$$

As advertised with eq. (8.1.19) at the end of our first qualitative description of the construction, formulating eq. (8.1.38) is the purpose of this section.

Summary of the construction. Let us finally summarise all ingredients of eq. (8.1.38) in a concise manner. $\chi_{\pm}, \chi_p$ are characters of the $S_{N_{\pm}-p}$ and S_p , respectively. The normalising prefactors involving the numbers $|[\rho_{\pm}]|$ and $|[\tau]|$ of elements in the conjugacy classes of $\rho_{\pm} \in S_{N_{\pm}-p}$ and $\tau \in S_p$, respectively, appear here because we are summing over group elements rather than conjugacy classes. The states we sum over are defined in eq. (8.1.36) through a weighted average over elements $g_{\pm}$ of $S_{N_{\pm}}$. Their definition involves a product of three states. Two of them are the states $|a_{\pm}\rangle_{\rho'_{\pm}}$, which are purely reflective with reflection coefficients that are specified by the coefficients of boundary states $|a_{\pm}\rangle$ in the seed theory, see eq. (8.1.26). The third factor in the states we average over is purely transmitting, see eq. (8.1.35).

8.1.3 Reflectivity and transmissivity of the interfaces

Now that we have defined the interfaces in eq. (8.1.38) let us pause for a moment and compute a first physical quantity that may help to better understand them. Concretely, we determine the (stress-energy) reflectivity and transmissivity as proposed in [303]. It measures e.g. how much energy gets transmitted through the interface

or, after passing to the folded setup, how much energy gets passes from e.g. from $+$ -components of the folded CFT to the $-$ -component and vice versa. The interpretation of the reflectivity is similar. More concretely, after folding the theory to the upper half plane we need to compute correlations of the product $T^\eta(z)\bar{T}^\eta(\bar{z})$ with $\eta \in \{+, -\}$ in the presence of the interface boundary state (8.1.38). Replacing the Virasoro fields by the corresponding states, the quantity in question is

$$R_{\eta\eta'} = \frac{\langle 0|L_2^\eta \bar{L}_2^{\eta'}|p, a_\pm; \chi_-, \chi_p, \chi_+\rangle}{\langle 0|p, a_\pm; \chi_-, \chi_p, \chi_+\rangle}. \quad (8.1.39)$$

The overlaps that appear in this expressions are not that difficult to compute. The simplest is clearly the overlap in the denominator which is obviously given by

$$\langle 0|p, a_\pm; \chi_-, \chi_p, \chi_+\rangle = \chi_{r_-}(id)\chi_p(id)\chi_{r_+}(id)\sqrt{\binom{N_-}{p}\binom{N_+}{p}}\frac{(a_-)_0^{N_- - p}(a_+)_0^{N_+ - p}}{\sqrt{(N_- - p)!(N_+ - p)!}}. \quad (8.1.40)$$

The overlap in the numerator is not that much harder to compute. Note that the Virasoro fields $T^\pm$ and $\bar{T}^\pm$ of the two symmetric product orbifold CFTs are in the untwisted sector and hence the overlap in the numerator sees essentially the same coefficients as that in the denominator. In formulas this means only the term with $\tau = \rho_\pm = id$ in the expansion of the state (8.1.38) can contribute to give

$$\langle 0|L_2^\eta \bar{L}_2^{\eta'}|p, a_\pm; \chi_-, \chi_p, \chi_+\rangle = \chi_{r_-}(id)\chi_p(id)\chi_{r_+}(id)\langle 0|L_2^i \bar{L}_2^j|p, a_\pm\rangle_{[id, id]_p}^{N_\pm}. \quad (8.1.41)$$

The state on the right hand side is a special case of eq. (8.1.36) which consists of a single term only and has trivial coefficient,

$$|p, a_\pm\rangle_{[id, id]_p}^{N_\pm} = |a_- \rangle \cdot |\mathbb{I}\rangle^{p, p} \cdot |a_+ \rangle. \quad (8.1.42)$$

Let us now first address the off-diagonal elements of the reflection matrix R , i.e. the matrix elements $R_{+-} = R_{-+}$. Each of the $N_- - p$ components within the state $|a_\pm\rangle$ contributes a factor of $(a_\pm)_0$. Moreover, every transmitting component gives rise to a term $c_{\mathcal{M}}2(2^2 - 1)/12 = c_{\mathcal{M}}/2$ from the commutations relations of the Virasoro generators in the seed theory. Since we need to sum over all the transmitting components of which there are p , we find the simple relation

$$\langle 0|L_2^\pm \bar{L}_2^\mp|p, a_\pm; \chi\rangle = p\frac{c_{\mathcal{M}}}{2}\langle 0|p, a_\pm; \chi\rangle. \quad (8.1.43)$$

between the overlaps in the numerator and denominator. Here, we used the shorthand $\chi = (\chi_-, \chi_p, \chi_+)$ to denote the triple of characters. For the diagonal elements of R , the transmissive boundaries do not contribute at all. The contribution of all the reflective factors, on the other hand, sum to

$$\langle 0|L_2^\pm \bar{L}_2^\pm|p, a_\pm; \chi\rangle = (N_\pm - p)\frac{c_{\mathcal{M}}}{2}\langle 0|p, a_\pm; \chi\rangle. \quad (8.1.44)$$

Plugging the previous two relations back into the definition (8.1.39) of the reflection matrix R we conclude that

$$R = \frac{c_{\mathcal{M}}}{2} \begin{pmatrix} N_- - p & p \\ p & N_+ - p \end{pmatrix}. \quad (8.1.45)$$

From the matrix elements, we read off the reflectivity of our interface, which is given by

$$\mathcal{R} := \frac{2}{c_{\mathcal{M}}(N_- + N_+)} (R_{--} + R_{++}) = 1 - \frac{2p}{N_- + N_+}, \quad (8.1.46)$$

and the transmissivity

$$\mathcal{T} = \frac{2p}{N_- + N_+}. \quad (8.1.47)$$

Note that the “unitarity condition” $\mathcal{R} + \mathcal{T} = 1$ is indeed satisfied. We also observe that $p = 0$ corresponds to a purely reflective interface and $N_- = N_+ = p$ is purely transmissive, as expected. Let us end this short interlude with three minor comments on the result.

First, note that the formula (8.1.45) for R may be brought into its standard form

$$R = \frac{c_{\mathcal{M}}}{2} \frac{N_- N_+}{N_- + N_+} \left[\begin{pmatrix} \frac{N_-}{N_+} & 1 \\ 1 & \frac{N_+}{N_-} \end{pmatrix} + \omega_b \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right] \quad (8.1.48)$$

from which we read off

$$\omega_b = 1 - \frac{p}{N_-} - \frac{p}{N_+}. \quad (8.1.49)$$

Second, note that the reflectivity and transmissivity coefficients are insensitive to most of the data that we used to construct the interfaces in the previous subsection. In fact, only the number p of transmissive components and the numbers $N_{\pm} - p$ of reflective components enter. The choice of the boundary conditions $a_{\pm}$ or the characters $\chi_{\pm}$ and χ_p on the other hand has no effect. The reason for this is rather clear from the derivation: The stress-energy reflectivity and transmissivity only receive contributions from the untwisted sector of the symmetric product orbifold. In this sense, $\mathcal{R}$ and $\mathcal{T}$ are very coarse grained characterisations of the interfaces $\mathcal{I}_{|a_{\pm}}^{(p)}$.

Let us finally make a somewhat related observation. We decided to determine the matrix R above from the concrete formula (8.1.38). However, the authors of [303] noted already that for a certain class of interfaces, the matrix R is rather easy to compute by more general arguments. They considered a situation in which the chiral algebras $\mathcal{A}_-$ and $\mathcal{A}_+$ of the two CFTs the interface interpolates between possess a common subalgebra $\mathcal{C}$. If the interface $\mathcal{I}$ in question is completely transmissive for

the common subalgebra $\mathcal{C}$ and reflective for all other degrees of freedom, then the chiral algebra of the folded theory is broken down to a product

$$\mathcal{A}_- \otimes \mathcal{A}_+ \rightarrow \mathcal{A}_-/\mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{A}_+/\mathcal{C}. \quad (8.1.50)$$

by the presence of the interface $\mathcal{I}$. Denoting the corresponding boundary state by $|\mathcal{I}\rangle$, we can easily compute

$$\begin{aligned} \langle 0|L_2^\pm \bar{L}_2^\mp|\mathcal{I}\rangle &= \langle 0|(L_2^{(\mathcal{A}/\mathcal{C})^\pm} + L_2^{\mathcal{C}_\pm})(\bar{L}_2^{(\mathcal{A}/\mathcal{C})^\mp} + \bar{L}_2^{\mathcal{C}_\mp})|\mathcal{I}\rangle \\ &= \langle 0|(L_2^{(\mathcal{A}/\mathcal{C})^\pm} + L_2^{\mathcal{C}_\pm})(L_{-2}^{(\mathcal{A}/\mathcal{C})^\mp} + L_{-2}^{\mathcal{C}_\mp})|\mathcal{I}\rangle = \frac{c}{2} \end{aligned} \quad (8.1.51)$$

and

$$\begin{aligned} \langle 0|L_2^\pm \bar{L}_2^\pm|\mathcal{I}\rangle &= \langle 0|(L_2^{(\mathcal{A}/\mathcal{C})^\pm} + L_2^{\mathcal{C}_\pm})(\bar{L}_2^{(\mathcal{A}/\mathcal{C})^\pm} + \bar{L}_2^{\mathcal{C}_\pm})|\mathcal{I}\rangle \\ &= \langle 0|(L_2^{(\mathcal{A}/\mathcal{C})^\pm} + L_2^{\mathcal{C}_\pm})(L_{-2}^{(\mathcal{A}/\mathcal{C})^\pm} + L_{-2}^{\mathcal{C}_\mp})|\mathcal{I}\rangle = \frac{c_\pm - c}{2}, \end{aligned} \quad (8.1.52)$$

where $c_\pm$ are the central charges of $\mathcal{A}_\pm$ and c the central charge of the chiral subalgebra $\mathcal{C}$. In the two short calculations we have split the Virasoro elements $L^\pm$ and $\bar{L}^\pm$ of the chiral algebras $\mathcal{A}_\pm$ according the factorisation (8.1.50). Then, we used the Ishibashi conditions of the interface boundary state $|\mathcal{I}\rangle$ to replace anti-holomorphic components by holomorphic ones in passing from the first to the second line. In the last step we used commutation relations of the Virasoro elements to evaluate the overlap in terms of the central charges. In our case, the chiral algebras $\mathcal{A}_\pm$ are those of the symmetric product orbifolds $\text{Sym}^{N_\pm}(\mathcal{M})$ while the common subalgebra $\mathcal{C}$ that is preserved by the interface is the chiral algebra of $\text{Sym}^p(\mathcal{M})$. Plugging in the associated values of the central charges $c_\pm = c_{\mathcal{M}}N_\pm$ and $c = c_{\mathcal{M}}p$ reproduces eq. (8.1.45).

8.1.4 Partition functions of interface changing operators

A very interesting object related to the interfaces that we defined in section 8.1.2 is the partition function that enumerates interface changing operators. To be concrete, let us choose one interface $|p^R, a_\pm^R; \chi^R\rangle$ to the right of the origin and another $|p^L, a_\pm^L; \chi^L\rangle$ to the left. The associated partition function for interface changing operators is obtained from the associated overlap as follows

$$\mathcal{Z}_{(p^L, a_\pm^L; \chi^L), (p^R, a_\pm^R; \chi^R)}^{N_\pm}(t) = \text{Tr}[x^{L_0 - \frac{c}{24}}] = \langle p^L, a_\pm^L; \chi^L | \hat{x}^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{12})} | p^R, a_\pm^R; \chi^R \rangle. \quad (8.1.53)$$

Here, $c = (N_+ + N_-)c_{\mathcal{M}}$ is the total central charge of the folded theory and the parameter $\hat{x}$ on the right is related to $x = \exp(2\pi it)$ by modular transformation $\hat{x} = \exp(2\pi i \hat{t})$ with $\hat{t} = -1/t$. We continue to use the shorthand χ for the entire triple (χ_-, χ_p, χ_+) of characters. These overlaps could certainly be computed in full

generality, but since we have no need for such a general result, we restrict to the case in which all the characters $\chi^{L/R}$ are trivial. This turns out to be the only one that is relevant for the holographic relation with AdS_2 branes.

As usual, the partition functions of the type defined in eq. (8.1.53) are somewhat inconvenient to describe individually, but as whole organise into a rather simple grand canonical partition function. Concretely, we propose that the grand canonical partition function

$$\mathcal{Z}_{a_{\pm}^{L/R}}[\mu_{\pm}, \rho_{L/R}; t] := \sum_{N_{\pm}=0}^{\infty} \sum_{p^{L/R}=0}^{\min(N_{\pm})} \mu_{+}^{N_{+}} \mu_{-}^{N_{-}} \rho_L^{N_{+}+N_{-}-2p^L} \rho_R^{N_{+}+N_{-}-2p^R} \mathcal{Z}_{(p^L, a_{\pm}^L; \mathbf{1}), (p^R, a_{\pm}^R; \mathbf{1})}^{N_{\pm}}(t) \quad (8.1.54)$$

is given by the exponential

$$\mathcal{Z}_{a_{\pm}^{L/R}}[\mu_{\pm}, \rho_{L/R}; t] = \exp(\mathcal{Z}_C[\mu_{\pm}, \rho_{L/R}; t] + \mathcal{Z}_O[\mu_{\pm}, \rho_{L/R}; t]) \quad (8.1.55)$$

of the sum of a “connected¹¹ closed string” part

$$\mathcal{Z}_C[\mu_{\pm}, \rho_{L/R}; t] = \sum_{k=1}^{\infty} \mu_{-}^k \mu_{+}^k T_k Z_c(t) \quad (8.1.56)$$

and a “connected open string” part

$$\mathcal{Z}_O[\mu_{\pm}, \rho_{L/R}; t] = \sum_{A, B \in \{L, R\}} \sum_{k=1}^{\infty} \sum_{w|k} \rho_A^w \rho_B^w \mu_{-}^{k-w} \delta_A^R \delta_B^L \mu_{+}^{k-w} \delta_A^L \delta_B^R \frac{1}{w} \hat{Z}_o^{AB} \left(\frac{(2k-w+w\delta_A^B)t}{w^2} \right). \quad (8.1.57)$$

$\mathcal{Z}_C$ and $\mathcal{Z}_O$ involve two types of partition functions for the bulk theory. On the one hand, there is the “closed string” partition function Z_c defined as

$$Z_c(t) = Z_{\mathcal{M}}(t, \bar{t})_{t=\bar{t}} \quad (8.1.58)$$

which is the modular invariant bulk partition function of the seed theory $\mathcal{M}$ restricted to the diagonal $t = \bar{t}$. On the other hand, there are the open string partition functions Z_o^{AB} defined as

$$Z_o^{LL} = Z_{a_{-}^L, (a_{+}^L)^*}, \quad Z_o^{LR} = Z_{a_{-}^L, a_{+}^R}, \quad Z_o^{RL} = Z_{a_{+}^L, a_{+}^R} \quad \text{and} \quad Z_o^{RR} = Z_{(a_{+}^R)^*, a_{-}^R} \quad (8.1.59)$$

where $Z_{a,b}(t) = Z_{b^*, a^*}(t)$ is the partition function of the annulus with boundary conditions a and b imposed along the two components of the boundary. It is famously related to the overlap of the boundary states in the seed theory through

$$Z_{a,b}(t) = \langle a | \hat{x}^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c\mathcal{M}}{12})} | b \rangle = \hat{Z}_{a,b}(\hat{t}). \quad (8.1.60)$$

¹¹The term *connected* is motivated in this context by the covering map approach to symmetric product orbifolds, in which the exponent of the grand canonical partition function is generated by covering maps with a connected covering space.

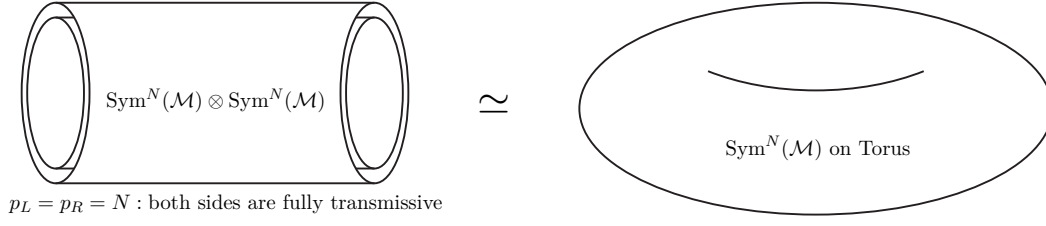


Figure 8.1.1: Coincidence of partition function of fully transmissive boundaries and torus partition function.

with $\hat{t} = -1/t$ as usual. Strictly speaking, formula (8.1.55) is a conjecture. A full proof of this formula is given in Chapter 9, using covering map techniques. In appendix 8.B, we also carefully derive it for the special case $\rho_L = 0$ using the more algebraic methods of the present chapter. Considering this special case is necessary in order to verify that we made the correct normalisation choices in section 8.1.2. It is also sufficient in the sense that the case of $\rho_L = 0$ is fully sensitive to all normalisations that we chose in the construction of the interface (8.1.38). Assuming that, with the correct choice of normalisation, the grand canonical partition function exponentiates, eq. (8.1.55) then follows by computing the “connected world sheet” sums $\mathcal{Z}_C$ and $\mathcal{Z}_O$ in the exponent. The calculation of these “connected world sheet” sums is the most illuminating part of the computation. This is why we dedicate the remainder of the current section to this task.

Let us start by discussing the limit $\rho_L = \rho_R = 0$ in which only transmissive boundaries are present. When both ρ parameters vanish, so does $\mathcal{Z}_O$. This means that only the first term in the exponent contributes in the limit. Hence, our result (8.1.55) simplifies to

$$\sum_{N=0}^{\infty} \kappa^N \langle N | x^{\frac{1}{2}(L_0^{(1)} + \bar{L}_0^{(2)} - \frac{c}{12})} \bar{x}^{\frac{1}{2}(\bar{L}_0^{(1)} + L_0^{(2)} - \frac{c}{12})} | N \rangle = \exp \left[\sum_{k=1}^{\infty} \mu_-^k \mu_+^k T_k Z(t, \bar{t}) \right], \quad (8.1.61)$$

where $Z = Z_c$ is the torus partition function of the seed theory, as before, T_k is the usual Hecke operator (see eq. (8.1.5)), and $|N\rangle$ is the fully transmissive boundary state

$$|N\rangle = |p = N, a_{\pm}; N_+ = N = N_-\rangle. \quad (8.1.62)$$

Obviously, this state does not depend on the choice of $|a_{\pm}\rangle$ since we do not allow for any reflection. The state $|N\rangle$ is associated with the trivial defect line of the symmetric product orbifold. Hence, we expect the partition function to coincide with the torus partition function (8.1.4), as shown in figure 8.1.1. This is indeed what we observe.

To verify eq. (8.1.61), we start by plugging in the definition (8.1.38) of $|N\rangle$ on the left hand side of the equation and simplifying, which gives

$$\langle N | x^{\frac{1}{2}(L_0^{(1)} + \bar{L}_0^{(2)})} \bar{x}^{\frac{1}{2}(\bar{L}_0^{(1)} + L_0^{(2)})} | N \rangle = \frac{1}{N!} \sum_{\tau \in S_N} \sum_{\sigma \in \mathcal{C}_\tau^N} \sum_{j_r, i_s} \langle j_r, i_s | x^{L_0} \bar{x}^{\bar{L}_0} \sigma | j_r, i_s \rangle_{\tau, \tau}^{N, N}. \quad (8.1.63)$$

This formula for the overlap manifestly is a sum of contributions each associated to a permutation $\tau \in S_N$ and an element σ of the centraliser of τ . As explained in section 8.1.1, σ includes a factor π_σ that permutes j and i labels associated to cycles of the same length within the permutation τ . Only those terms in the sum for which $j_r = j_{\pi_\sigma r}$ and $i_s = i_{\pi_\sigma s}$ are non-vanishing. The full answer factorises into individual contributions associated to the cycles of τ and π_σ . This suggests¹² that the full grand canonical partition function is an exponential of sums of connected contributions $Z_{(w, \ell)}$ associated to a π_σ that is a single cycle of length ℓ and a τ that is a product of ℓ cycles of length w such that $N = w\ell$. To compute the connected contributions $Z_{(w, \ell)}$, let us define

$$\tau_{(w, \ell)} = \prod_{k=0}^{\ell-1} (kw + 1, kw + 2, \dots, kw + w). \quad (8.1.64)$$

From eq. (8.1.63), we can then directly conclude that

$$Z_{(w, \ell)} = \frac{|\tau_{(w, \ell)}|}{N!} \sum_{\sigma \in \mathcal{C}_{\tau_{(w, \ell)}}} \sum_{j_r, i_s} \langle j_r, i_s | x^{L_0 - \frac{c}{24}} \bar{x}^{\bar{L}_0 - \frac{c}{24}} \sigma | j_r, i_s \rangle_{\tau_{(w, \ell)}, \tau_{(w, \ell)}}^{N, N}. \quad (8.1.65)$$

The action of σ on the states was spelled out in eq. (8.1.30) which leads to

$$Z_{(w, \ell)} = \frac{1}{w^\ell \ell!} (\ell - 1)! \sum_{i_1=0}^{w-1} \dots \sum_{i_{\ell-1}=0}^{w-1} Z_{\mathcal{M}} \left(\frac{\ell t + \sum_{k=1}^{\ell} i_k}{w}, \frac{\ell \bar{t} + \sum_{k=1}^{\ell} i_k}{w} \right). \quad (8.1.66)$$

Using that the partition function $Z_{\mathcal{M}}$ of the bulk theory is invariant under modular T transformations, i.e. under shifts of the argument by arbitrary integers, the previous formula simplifies to

$$Z_{(w, \ell)} = \frac{1}{w^\ell} \sum_{i=0}^{w-1} Z_{\mathcal{M}} \left(\frac{\ell t + i}{w}, \frac{\ell \bar{t} + i}{w} \right) = \frac{1}{N} \sum_{i=0}^{w-1} Z_{\mathcal{M}} \left(\frac{Nt}{w^2} + \frac{i}{w}, \frac{N\bar{t}}{w^2} + \frac{i}{w} \right). \quad (8.1.67)$$

This is what we wanted to show.

Let us now look at the more generic case, where only ρ_L is sent to 0 in our general formula (8.1.55). In this case, we obtain

$$\mathcal{Z}_{aL/R}[\mu_{\pm}, 0, \rho_R; \hat{t}] = \exp \left(\sum_{k=1}^{\infty} \mu_-^k \mu_+^k T_k Z_c(\hat{t}) + \sum_{k=1}^{\infty} \sum_{w|k} \mu_-^k \mu_+^k \rho_R^{2w} \frac{1}{w} \hat{Z}_o^{RR} \left(2 \frac{k\hat{t}}{w^2} \right) \right). \quad (8.1.68)$$

¹²As mentioned before, the computation in appendix 8.B proves the claim rigorously.

We now sketch how this result is derived. Again, just by plugging in the definition (8.1.38) and performing a few elementary simplification steps, one can deduce that

$$\langle N|\hat{x}^{L_0}|p, a\rangle = \frac{1}{p!(N-p)!} \sum_{\tau \in S_p} \sum_{\rho \in S_{N-p}} \sum_{\sigma \in \mathcal{C}_{\tau\rho}^N} \sum_{j_r, i_s} {}^{N,N}_{\rho\tau, \tau\rho} \langle j_r, i_s | \hat{x}^{L_0} \sigma | a \rangle_\rho |\mathbb{I}\rangle_{\tau, \tau} | a \rangle_\rho, \quad (8.1.69)$$

where σ acts only on the states of the symmetric orbifold living in the lower half plane. At this point, we can also unfold using the notation introduced in eq. (8.1.29) and obtain

$$\langle N|\hat{x}^{L_0}|p, a\rangle = \frac{1}{p!(N-p)!} \sum_{\tau \in S_p} \sum_{\rho \in S_{N-p}} \sum_{\sigma \in \mathcal{C}_{\tau\rho}^N} \sum_{j_r, i_s} \rho \langle a | (i_s |_\tau^p \langle j_r |_\tau^p \hat{x}^{L_0} \sigma | i_s \rangle_\tau^p | j_r \rangle_\tau^p | a \rangle_\rho. \quad (8.1.70)$$

The overlap can be interpreted as a sum of products of “open string” contributions (which involve reflecting boundaries) and “closed string” contributions (which do not involve reflecting boundaries) that arise from the interplay of different τ , ρ and σ . In the special case $p = 0$, there are no closed string contributions.

Furthermore, for $p = 0$, the only connected contributions come from the conjugacy class $[\rho] = [(1 \dots N)]$ of the maximal cyclic permutation. The choice of σ does not matter: every element of the commutator acts trivially. Accordingly, these contributions to the partition are simply

$$\frac{|[\rho]| |\mathcal{C}_\rho|}{N!} \rho \langle a | \hat{x}^{2L_0 - \frac{c}{12}} | a \rangle_\rho = \rho \langle a | \hat{x}^{2L_0 - \frac{c}{12}} | a \rangle_\rho = \frac{1}{N} \hat{Z}_o \left(\frac{2t}{N} \right), \quad (8.1.71)$$

where Z_o is defined in eq. (8.1.59). The factor $1/N$ on the right hand side of the equation originates from our choice to normalise $|a\rangle_\rho$ by $1/\sqrt{\mathcal{C}_\rho}$, see eq. (8.1.26).

More generally, we can find connected contributions at $p = w(\ell - 1)$ for $N = w\ell$. They arise for

$$|\pi_\sigma| = \ell, \quad [\tau] = [\tau_{(w, \ell-1)}], \quad [\rho] = [((\ell - 1)w + 1 \dots \ell w)]. \quad (8.1.72)$$

Their contribution to the partition function is

$$\frac{|[\tau]| |[\rho]| |\{\sigma \in \mathcal{C}_{\tau\rho}^N : |\pi_\sigma| = \ell\}|}{p!(N-p)! |\mathcal{C}_\rho^{N-p}|} \hat{Z}_o \left(\frac{2\ell t}{w} \right) = \frac{|\{\sigma \in \mathcal{C}_{\tau\rho}^N : |\pi_\sigma| = \ell\}|}{|\mathcal{C}_\tau^p| |\mathcal{C}_\rho^{N-p}|^2} \hat{Z}_o \left(\frac{2\ell t}{w} \right) = \frac{1}{w} \hat{Z}_o \left(\frac{2Nt}{w^2} \right). \quad (8.1.73)$$

Assuming that the grand canonical partition function is obtained by the exponentiation of these connected contributions, we obtain eq. (8.1.68). Appendix 8.B provides a detailed computation that establishes this result more rigorously.

Before we close this section, let us add a few comments concerning the formula (8.1.57). First of all, we have introduced $\mu_\pm$ and $\rho_{L/R}$ in order to unify the discussion

of the various special cases obtained by sending some of the chemical potentials to zero in this section. However, we will ultimately be especially interested in the case

$$\mathcal{Z}_O[\mu, \hat{t}] := \mathcal{Z}_O[\mu, \mu, 1, 1, \hat{t}] = \sum_{A, B \in \{L, R\}} \sum_{k=1}^{\infty} \sum_{w|k} \frac{1}{w} \mu^{2k+w\delta_A^B-w} \hat{Z}_o^{AB} \left(\frac{(2k-w+w\delta_A^B)\hat{t}}{w^2} \right), \quad (8.1.74)$$

which is more conveniently described by summing over

$$\ell = 2\frac{k}{w} + \delta_A^B - 1 \quad (8.1.75)$$

instead of k . Indeed,

$$\mathcal{Z}_O[\mu, \hat{t}] = \sum_{A, B \in \{L, R\}} \sum_{\ell, w=1}^{\infty} \frac{1}{2w} (1 - e^{i\pi(\ell+\delta_A^B)}) \mu^{w\ell} \hat{Z}_o^{AB} \left(\frac{\ell\hat{t}}{w} \right). \quad (8.1.76)$$

In this equation, as in all other partition functions above, we have used the “closed string” modular parameter $\hat{t}$ as appropriate for the overlap of boundary states. Given the interpretation as a counting function for boundary/interface changing operators, it may be more natural to perform a modular S transformation and rewrite the grand canonical partition function in terms of the dual modular parameter $t = -1/\hat{t}$. Using $Z_o(t) = \hat{Z}_o(\hat{t})$, we obtain

$$\mathcal{Z}_O[t] = \sum_{A, B \in \{L, R\}} \sum_{\ell, w=1}^{\infty} \frac{1}{2\ell} (1 - e^{i\pi(w+\delta_A^B)}) Z_o^{AB} \left(\frac{\ell t}{w} \right). \quad (8.1.77)$$

In rewriting the expression we also exchanged w and ℓ .

In the remainder of this chapter, our focus is on the holographic relation with AdS_2 branes for tensionless superstring theory in $AdS_3 \times S^3 \times \mathbb{T}^4$. In this theory, the spacetime symmetry is enhanced from the usual Virasoro algebra to the small $\mathcal{N} = 4$ superconformal algebra and hence states of the CFT can be distinguished by their R-symmetry charge that is given by the eigenvalue of the Cartan generators K_0 and $\bar{K}_0$ of the R-symmetry group. It is then natural to include these charges in the counting functions for states. At the level of the seed theory, this means that the bulk and boundary partition functions (8.1.58) and (8.1.59) become functions of another variable ζ in addition to t , i.e.

$$Z_c(t) \rightarrow Z_c(t, \zeta), \quad Z_o^{AB}(t) \rightarrow Z_o^{AB}(t, \zeta). \quad (8.1.78)$$

The corresponding change to formula (8.1.57) is given by the substitution

$$\hat{Z}_o^{AB} \left(\frac{\ell\hat{t}}{w} \right) \rightarrow \hat{Z}_o^{AB} \left(\frac{\ell\hat{t}}{w}, \ell\hat{\zeta} \right). \quad (8.1.79)$$

The function $\hat{Z}$ is obtained from the partition function Z through the prescription

$$\hat{Z}(\hat{t}, \hat{\zeta}) = Z(-1/\hat{t}, \hat{\zeta}/\hat{t}) = Z(t, \zeta). \quad (8.1.80)$$

We complement these qualitative remarks with more explicit formulas in section 8.3.1 after establishing some more background on the supersymmetric four-torus in section 8.2.2.

8.2 AdS_2 branes in tensionless AdS_3 backgrounds

In this section, we turn to the dual side and discuss the string theory of AdS_2 branes in AdS_3 , or more concretely, in type IIB superstring theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ with one unit of NS-NS flux. In the hybrid formulation or Berkovits-Vafa-Witten [304], the latter involves a $\mathfrak{psu}(1,1|2)_k$ WZNW model at level $k = 1$ along with a topologically twisted four-torus and some ghosts, see below.

In the first subsection, section 8.2.1, we review the relevant boundary states for AdS_2 branes in the $\mathfrak{psu}(1,1|2)_1$ WZNW model following [124]. After adding the remaining factors of the worldsheet model, i.e. the topologically twisted $\mathbb{T}^4$ and the ghost factors, we, in section 8.2.2, compute the overlaps of the full boundary states and interpret the resulting quantity as a counting function for boundary operators. The worldsheet partition function we end up with counts physical open string vertex operators in the tensionless $AdS_3 \times S^3 \times \mathbb{T}^4$ background. In order to compute the partition function of the associated spacetime string theory, we must integrate over the modulus τ of the worldsheet torus. We do this first for global AdS_3 and then, anticipating our comparison with the dual CFT we studied in section 8.1, for thermal AdS_3 in section 8.2.3. Note that the boundary of global AdS_3 is a sphere whereas the boundary of thermal AdS_3 is a torus. The latter is the geometry that is relevant for comparison with the CFT partition functions we computed above.

8.2.1 Boundary states for AdS_2 branes in tensionless AdS_3

According to the work of Berkovits, Vafa and Witten [304], the description of superstrings in $AdS_3 \times S^3$ in the hybrid formulation involves a $\mathfrak{psu}(1,1|2)_1$ WZNW model. The latter has been studied in the past, see in particular [305, 306] and references therein to earlier work in string theory and quantum Hall plateau transitions, as well as [307] for some later extensions. Here, we restrict to the case of $k = 1$. Our exposition is purposefully kept minimalistic and we refer the reader to sections 3 and 4 of [124] for more detail.

The even part of the $\mathfrak{psu}(1,1|2)_1$ current algebra is generated by the affine currents $J^a, a = 1, 2, 3$, of an $\mathfrak{sl}(2, \mathbb{R})_1$ WZNW model along with the currents $K^a, a = 1, 2, 3$, of an $\mathfrak{su}(2)_1$ WZNW model. These six bosonic currents are associated with the six directions in the bosonic base of the supergroup $PSU(1,1|2)$. In addition, the supergroup also has eight fermionic directions which give rise to the fermionic currents

$S^{\alpha\beta\gamma}$ in the WZNW model. Here α, β, γ are spinor indices that take the values $\alpha = \pm 1$, respectively. These currents satisfy the relations of an affine $\mathfrak{psu}(1, 1|2)_1$ Kac-Moody algebra at level $k = 1$. Since we do not need these relations, we do not spell them out here, see e.g. section 3 of [68] for a complete list.

The state space of the bulk theory is a direct sum of representations of this worldsheet current algebra. Following standard conventions, we shall denote by $\mathcal{F}_\lambda$ the representations of the $\mathfrak{psu}(1, 1|2)_1$ algebra whose spectrum of conformal weights is bounded from below. Here, the index $\lambda \in [0, 1[$ determines the quantization of the zero mode J_0^3 of the current J^3 of the non-compact current algebra $\mathfrak{sl}(2, \mathbb{R})_1$. For generic level k , the representations $\mathcal{F}_\lambda$ would carry other labels that keep track of angular momenta but these are all removed by the null-vectors at $k = 1$. In addition to the representations $\mathcal{F}_\lambda$, the worldsheet model also includes representations $\mathcal{F}_\lambda^w$ that are obtained by application of the spectral flow automorphism σ^w , that is $\mathcal{F}_\lambda^w = \sigma^w(\mathcal{F}_\lambda)$. Recall that the space of ground states from which $\mathcal{F}_\lambda$ is obtained by the action of the negative $\mathfrak{psu}(1, 1|2)_1$ modes is the short multiplet

$$(\mathcal{C}_{\lambda+\frac{1}{2}}^1, \mathbf{1}) \quad (\mathcal{C}_{\lambda}^{\frac{1}{2}}, \mathbf{2}) \quad (\mathcal{C}_{\lambda+\frac{1}{2}}^0, \mathbf{1}). \quad (8.2.1)$$

Here, the symbol $(\mathcal{C}_\lambda^j, \mathbf{m})$ denotes an irreducible representation of the subalgebra $\mathfrak{sl}(2, \mathbb{R})_1 \oplus \mathfrak{su}(2)_1$ which is the maximal bosonic subalgebra of $\mathfrak{psu}(1, 1|2)_1$. These representations are constructed from the $\mathbf{m}$ -dimensional highest weight representation $\mathbf{m}$ of the compact algebra $\mathfrak{su}(2)_1$, along with the continuous series representations $\mathcal{C}_\lambda^j$ of the non-compact $\mathfrak{sl}(2, \mathbb{R})_1$. The labels j and λ of $\mathcal{C}_\lambda^j$ capture the eigenvalue $-j(j-1)$ of the quadratic Casimir element and the fractional part of the J_0^3 eigenvalues, respectively. As we stressed before, in the case $k = 1$, the label j is entirely fixed and hence we dropped it from the symbol $\mathcal{F}_\lambda$. In terms of representations of the $\mathfrak{psu}(1, 1|2)_1$ current algebra, the Hilbert space of the $\mathfrak{psu}(1, 1|2)_1$ WZNW is¹³ the direct sum

$$\mathcal{H}^{\text{WZNW}} = \bigoplus_{w \in \mathbb{Z}} \int_0^1 d\lambda \, \sigma^w(\mathcal{F}_\lambda) \otimes \overline{\sigma^w(\mathcal{F}_\lambda)}. \quad (8.2.2)$$

Let us now turn to the boundary theory. The gluing conditions for the $\mathfrak{sl}(2, \mathbb{R})$ currents $J^a(z)$, the $\mathfrak{su}(2)$ currents $K^a(z)$ and the Fermionic currents $S^{\alpha\beta\gamma}(z)$ associated to the AdS_2 branes take the form

$$J^a(z) = \bar{J}^a(\bar{z}), \quad K^a(z) = \bar{K}^a(\bar{z}), \quad S^{\alpha\beta\gamma}(z) = \varepsilon \bar{S}^{\alpha\beta\gamma}(\bar{z}) \quad (8.2.3)$$

¹³Strictly speaking the Hilbert space of the WZNW model does not directly include $\mathcal{F}_\lambda$ for $\lambda = \frac{1}{2}$. The representation (8.2.2) is however equivalent to the true Hilbert space in the Grothendieck ring. Since we only compute partition functions, the special nature of the $\lambda = \frac{1}{2}$ representation is not relevant for our analysis. See [68] for more details on the atypical $\lambda = \frac{1}{2}$ representation.

at $z = \bar{z}$. The parameter ε that enters the gluing conditions we impose on the fermionic currents is a sign i.e. $\varepsilon \in \{+, -\}$. The corresponding Ishibashi states $|w, \lambda, \varepsilon\rangle$ in the sectors of the bulk decomposition (8.2.2) are characterised by

$$(J_n^a + \bar{J}_{-n}^a)|w, \lambda, \varepsilon\rangle = 0, \quad (K_n^a + \bar{K}_{-n}^a)|w, \lambda, \varepsilon\rangle = 0, \quad (S_n^{\alpha\beta\gamma} + \varepsilon \bar{S}_{-n}^{\alpha\beta\gamma})|w, \lambda, \varepsilon\rangle = 0. \quad (8.2.4)$$

These conditions imply $w = 0$ and $\lambda \in \{0, 1/2\}$. In [124], Gaberdiel et al. suggested to consider the following two linear combinations of Ishibashi states

$$\|\varepsilon\rangle_A = \frac{1}{\sqrt{2}} \sum_{\lambda=0,1/2} e^{2\pi i(\lambda-1/2)\delta_A^L} |0, \lambda, \varepsilon\rangle, \quad (8.2.5)$$

where the label¹⁴ A is chosen from $A \in \{L, R\}$. The two boundary states impose Dirichlet conditions for the angular coordinate of AdS_3 forcing open strings to end “left” or “right” of the centre of the AdS_2 brane respectively. In this sense, they only describe half branes and the worldsheet boundary state corresponding to the full AdS_2 brane is

$$\|\varepsilon\rangle = \|\varepsilon\rangle_L + \|\varepsilon\rangle_R. \quad (8.2.6)$$

For generic $k > 1$, the splitting of $\|\varepsilon\rangle$ into $\|\varepsilon\rangle_L$ and $\|\varepsilon\rangle_R$ is unnatural since the ends of open strings can cross the centre of AdS_3 . It is a special feature of the tensionless limit, that it is useful to keep track of $\|\varepsilon\rangle_L$ and $\|\varepsilon\rangle_R$ separately. Indeed, since long open strings in the tensionless limit cannot probe the centre of AdS_3 , an open string end on the left/right asymptotic end of the AdS_2 brane will always stay on the left/right. We illustrate the situation in figures 8.2.1 and 8.2.2.

As we stressed above, the full worldsheet description of superstrings in the hybrid formulation also involves a topologically twisted four-torus and ghosts. Hence, the boundary states of the $\mathfrak{psu}(1, 1|2)_1$ WZNW model we have described here must still be multiplied with further contributions from these additional sectors to obtain the following boundary states for AdS_2 branes in $AdS_3 \times S^3 \times \mathbb{T}^4$,

$$\|u, \varepsilon\rangle = \|\varepsilon\rangle \|u, R, \varepsilon\rangle_{\mathbb{T}^4} \|\text{ghost}, \varepsilon\rangle. \quad (8.2.7)$$

Corresponding to the splitting (8.2.6), we also define

$$\|u, \varepsilon\rangle_A = \|\varepsilon\rangle_A \|u, R, \varepsilon\rangle_{\mathbb{T}^4} \|\text{ghost}, \varepsilon\rangle. \quad (8.2.8)$$

The additional factors are discussed in detail in appendix B.1 of [124]. Here, it suffices to say that the letter u refers to the freedom we have in selecting a boundary condition on the four-torus $\mathbb{T}^4$, such as the brane’s dimension, orientation and position. It is not necessary for us to specify this choice any further.

¹⁴In [124], this label is called Θ and chosen from $\{0, 1\}$.

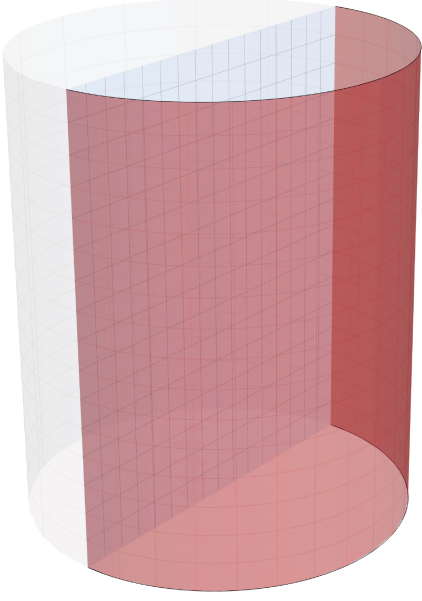


Figure 8.2.1: Open string winding half way around AdS_3 at $k = 1$. Since the worldsheet (red) is pinned to ∂AdS_3 , the end points of the string remain on either of the two asymptotic boundaries of the AdS_2 brane (blue).

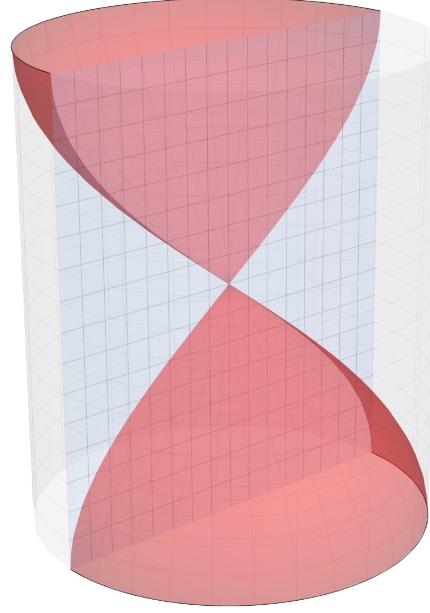


Figure 8.2.2: Open string winding half way around AdS_3 for large k . The string can fall into the interior of AdS_3 and re-emerge with its end-point on opposite asymptotic boundaries of the AdS_2 brane.

Regarding the R label, recall that in the hybrid formalism, the fermions of the original sigma model on $\mathbb{T}^4$ with small $\mathcal{N} = (4, 4)$ supersymmetry mix with the currents of the $\mathfrak{sl}(2, \mathbb{R})_1$ and $\mathfrak{su}(2)_1$ WZNW models to generate the supercurrents of the $\mathfrak{psu}(1, 1|2)_1$ model. The subsequent decoupling of torus degrees of freedom from the $\mathfrak{psu}(1, 1|2)_1$ WZNW model leaves us with a topologically twisted $\mathbb{T}^4$. While the bosonic degrees of freedom of the topologically twisted sigma model coincide with that of the untwisted theory, part of the fermionic degrees of freedom are removed and we are essentially only left with the RR sector. Consequentially, the boundary condition that we pick for the topologically twisted sector is a boundary condition for RR fermions. The purpose of the label R is to remind us of this fact.

8.2.2 Partition function for AdS_2 branes in global AdS_3

Our goal in this subsection is to compute the overlap of any two of the boundary states (8.2.7) for AdS_2 branes and to apply a modular transformation in order to interpret this overlap in terms of the boundary spectrum. More precisely, the quantity

in question is

$$\hat{Z}_{u|v}^{\text{WS}}(\hat{t}, \hat{\zeta}; \hat{\tau}) := \langle\langle u, \mp \| \hat{q}^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{12})} \hat{x}^{\frac{1}{2}(J_0^3 - \bar{J}_0^3)} \hat{y}^{\frac{1}{2}(K_0^3 - \bar{K}_0^3)} \| v, \pm \rangle\rangle = \sum_{A, B \in \{L, R\}} \hat{Z}_{A, u|B, v}^{\text{WS}}(\hat{t}, \hat{\zeta}; \hat{\tau}), \quad (8.2.9)$$

where on the right hand side, we split the brane into a left and right half according to (8.2.6), that is

$$\hat{Z}_{A, u|B, v}^{\text{WS}}(\hat{t}, \hat{\zeta}; \hat{\tau}) := {}_A \langle\langle u, \mp \| \hat{q}^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{12})} \hat{x}^{\frac{1}{2}(J_0^3 - \bar{J}_0^3)} \hat{y}^{\frac{1}{2}(K_0^3 - \bar{K}_0^3)} \| v, \pm \rangle\rangle_B \quad (8.2.10)$$

counts the open strings that stretch between the A -half of the u -brane and the B -half of the v -brane. The overlap factorises into three contributions from the $\mathfrak{psu}(1, 1|2)_1$ WZNW model, the four-torus $\mathbb{T}^4$ and the ghosts respectively. In order to spell these out explicitly, we need a bit of notation. It is convenient to use the conventions of [38] for Jacobi theta-functions. That is, we define

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\zeta | \tau) := \sum_{n \in \mathbb{Z}} \exp \left(i\pi(n + \alpha)^2 \tau + 2\pi i(n + \alpha)(\zeta + \beta) \right). \quad (8.2.11)$$

Together with the standard Dedekind η function, one can use these theta-functions to compute the supercharacter [124]

$$\widetilde{\text{ch}}[\mathcal{F}_\lambda](t, z; \tau) := \text{tr}_{\mathcal{F}_\lambda} \left[(-1)^F q^{L_0 - \frac{c}{24}} x^{J_0^3} y^{K_0^3} \right] = \sum_{r \in \mathbb{Z} + \lambda} \frac{x^r}{\eta(\tau)^4} \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \left(\frac{t+\zeta}{2} \middle| \tau \right) \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \left(\frac{t-\zeta}{2} \middle| \tau \right), \quad (8.2.12)$$

which by performing the sum over r can also be written as

$$\widetilde{\text{ch}}[\mathcal{F}_\lambda](t, z; \tau) = x^\lambda \sum_{m \in \mathbb{Z}} \frac{\delta(t - m)}{\eta(\tau)^4} \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \left(\frac{t+\zeta}{2} \middle| \tau \right) \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \left(\frac{t-\zeta}{2} \middle| \tau \right). \quad (8.2.13)$$

By the identity

$$\vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \left(\frac{m+\zeta}{2} \middle| \tau \right) = e^{i\pi \lceil \frac{m}{2} \rceil} \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1+e^{i\pi m}}{4} \end{bmatrix} \left(\pm \frac{\zeta}{2} \middle| \tau \right), \quad (8.2.14)$$

this simplifies further to

$$\widetilde{\text{ch}}[\mathcal{F}_\lambda](t, z; \tau) = \sum_{m \in \mathbb{Z}} e^{2\pi i \lambda m} \frac{\delta(t - m)}{\eta(\tau)^4} \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1+e^{i\pi m}}{4} \end{bmatrix} \left(\frac{\zeta}{2} \middle| \tau \right) \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1+e^{i\pi m}}{4} \end{bmatrix} \left(-\frac{\zeta}{2} \middle| \tau \right). \quad (8.2.15)$$

The supercharacter can be used to determine the overlap of the boundary states for AdS_2 branes in the $\mathfrak{psu}(1, 1|2)_1$ WZNW model,

$$\langle\langle 0, \lambda', \mp | \hat{q}^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{12})} \hat{x}^{\frac{1}{2}(J_0^3 - \bar{J}_0^3)} \hat{y}^{\frac{1}{2}(K_0^3 - \bar{K}_0^3)} | 0, \lambda, \pm \rangle\rangle = \delta_{\lambda, \lambda'} \widetilde{\text{ch}}[\mathcal{F}_\lambda](\hat{t}, \hat{\zeta}; \hat{\tau}). \quad (8.2.16)$$

Combining the explicit expression (8.2.15) for the super character with the definition (8.2.5) of the boundary states representing the AdS_2 brane, we obtain

$$\begin{aligned}\hat{Z}_{A|B}^{\text{psu}}(\hat{t}, \hat{\zeta}; \hat{\tau}) &= \frac{1}{2} \sum_{\lambda=0,1/2} e^{2\pi i(1/2-\lambda)(\delta_A^L - \delta_B^L)} \widetilde{\text{ch}}[\mathcal{F}_\lambda](\hat{t}, \hat{\zeta}; \hat{\tau}) \\ &= \sum_{m=1}^{\infty} \frac{e^{i\pi m} - e^{i\pi \delta_A^B}}{2\eta(\hat{\tau})^4} \delta(\hat{t} - m) \vartheta \left[\frac{\frac{1}{2}}{\delta_{A,B}} \right] \left(-\frac{\hat{\zeta}}{2} \middle| \hat{\tau} \right) \vartheta \left[\frac{\frac{1}{2}}{\delta_{A,B}} \right] \left(+\frac{\hat{\zeta}}{2} \middle| \hat{\tau} \right).\end{aligned}\quad (8.2.17)$$

The two remaining contributions from the overlaps of the boundary states in the four-torus and the ghost factor are the same as in the case of spherical branes that were fully analysed in [124]. We use the same conventions and only slightly different notation. For instance, we write the four-torus factor in the overlap using the notation for Jacobi theta-functions introduced in eq. (8.2.11) as

$$\hat{Z}_{u|v}^{\mathbb{T}^4} \left[\begin{matrix} \alpha \\ \beta \end{matrix} \right] (\hat{\zeta} | \hat{t}) := \frac{\hat{\Theta}_{u|v}^{\mathbb{T}^4}(\hat{t})}{\eta(\hat{t})^6} \vartheta \left[\begin{matrix} \alpha \\ \beta \end{matrix} \right] \left(+\frac{\hat{\zeta}}{2} \middle| \hat{t} \right) \vartheta \left[\begin{matrix} \alpha \\ \beta \end{matrix} \right] \left(-\frac{\hat{\zeta}}{2} \middle| \hat{t} \right). \quad (8.2.18)$$

Here, the quantity $\hat{\Theta}_{u|v}^{\mathbb{T}^4}$ in the numerator of the first factor stems from the bosonic directions on the four-torus and it depends on the choice of boundary states on $\mathbb{T}^4$, see appendix B.1 of [124] for details. Instead of the bracket notation it is also common to write, as Gaberdiel et al. do in [124],

$$\hat{Z}_{u|v}^{\mathbb{T}^4} \left[\begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} \right] = \hat{Z}_{u|v, \tilde{R}}^{\mathbb{T}^4}, \quad \hat{Z}_{u|v}^{\mathbb{T}^4} \left[\begin{matrix} \frac{1}{2} \\ 0 \end{matrix} \right] = \hat{Z}_{u|v, R}^{\mathbb{T}^4}, \quad \hat{Z}_{u|v}^{\mathbb{T}^4} \left[\begin{matrix} 0 \\ 0 \end{matrix} \right] = \hat{Z}_{u|v, \text{NS}}^{\mathbb{T}^4}, \quad \hat{Z}_{u|v}^{\mathbb{T}^4} \left[\begin{matrix} 0 \\ \frac{1}{2} \end{matrix} \right] = \hat{Z}_{u|v, \tilde{\text{NS}}}^{\mathbb{T}^4}, \quad (8.2.19)$$

where R and NS refer to the Ramond and Neveu-Schwarz sector of the fermions, respectively, and the tilde on top indicates whether we count fermions with additional signs or not, i.e. whether we insert a factor $(-1)^F$ in the trace. Since

$$\|u, R, \mp\rangle = (-1)^F \|u, R, \pm\rangle, \quad (8.2.20)$$

the insertion of $(-1)^F$ is effectively achieved by choosing opposite ε for the two states whose overlap we compute. With the notations in place, we can now state that the relevant overlap of the boundary states on the four-torus is

$$\langle\langle u, R, \mp | \hat{q}^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{12})} | v, R, \pm \rangle\rangle = \frac{\hat{\Theta}_{u|v}^{\mathbb{T}^4}(\hat{\tau})}{\eta(\hat{\tau})^6} \vartheta \left[\begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} \right] (0 | \hat{\tau}) \vartheta \left[\begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} \right] (0 | \hat{\tau}). \quad (8.2.21)$$

The last contribution we need is that from the overlap of boundary states in the ghost sector which we also take from [124]

$$\hat{Z}_{\text{ghost}}(\hat{\tau}) = \frac{\eta(\hat{\tau})^4}{\vartheta \left[\begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} \right] (0 | \hat{\tau}) \vartheta \left[\begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} \right] (0 | \hat{\tau})}. \quad (8.2.22)$$

Multiplying eqs. (8.2.22), (8.2.21) and (8.2.17), we conclude that the overlap (8.2.10) is given by

$$\hat{Z}_{A,u|B,v}^{\text{WS}}(\hat{t}, \hat{\zeta}; \hat{\tau}) = \sum_{m=1}^{\infty} \frac{1}{2} (e^{i\pi m} - e^{i\pi \delta_A^B}) \delta(\hat{t} - m) \hat{Z}_{u|v}^{\mathbb{T}^4} \left[\begin{array}{c} \frac{1}{2} \\ \frac{\delta_{A,B}}{2} \end{array} \right] (\hat{\zeta} | \hat{\tau}). \quad (8.2.23)$$

We placed a superscript WS on this quantity to remind us that this is a partition of the worldsheet theory that describes strings in $AdS_3 \times S^3 \times \mathbb{T}^4$. In particular, the modular parameter $\hat{\tau} = -1/\tau$ is that of the annulus worldsheet.

The overlap we have just computed is still written in terms of closed string parameters $\hat{t}, \hat{\zeta}$ and $\hat{\tau}$. In order to reinterpret the overlap in terms of open string vertex operators, we need to perform a modular S transformation. This transformation acts as

$$\hat{\tau} = -\frac{1}{\tau}, \quad \hat{t} = \frac{t}{\tau}, \quad \hat{\zeta} = \frac{\zeta}{\tau} \quad (8.2.24)$$

on the chemical potentials. We can rewrite our result (8.2.23) as a function of the dual variables with the help of the following transformation formula for the overlap of boundaries on the four-torus

$$\hat{Z}_{u|v}^{\mathbb{T}^4} \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] (\hat{\zeta} | \hat{\tau}) = e^{\frac{\pi i \zeta^2}{2\tau}} Z_{u|v}^{\mathbb{T}^4} \left[\begin{array}{c} \beta \\ \alpha \end{array} \right] (\zeta | \tau) \quad (8.2.25)$$

with $\alpha, \beta \in \{0, \frac{1}{2}\}$. Here, following [124], we absorb the sign from the modular transformation of the theta functions into the definition

$$Z_{u|v}^{\mathbb{T}^4} \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] (\zeta | t) := e^{4\pi i \alpha \beta} \frac{\Theta_{u|v}^{\mathbb{T}^4}(t)}{\eta(t)^6} \vartheta \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] \left(+\frac{\zeta}{2} \middle| t \right) \vartheta \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] \left(-\frac{\zeta}{2} \middle| t \right). \quad (8.2.26)$$

This allows us to rewrite eq. (8.2.23) as

$$\hat{Z}_{A,u|B,v}^{\text{WS}}(\hat{t}, \hat{\zeta}; \hat{\tau}) = \sum_{m=1}^{\infty} \frac{1}{2} (e^{i\pi m} - e^{i\pi \delta_A^B}) \delta\left(\frac{t}{\tau} - m\right) e^{\frac{\pi i \zeta^2}{2\tau}} Z_{u|v}^{\mathbb{T}^4} \left[\begin{array}{c} \frac{\delta_{A,B}}{2} \\ \frac{1}{2} \end{array} \right] (\zeta | \tau). \quad (8.2.27)$$

As usual, we interpret the right hand side as the partition function that counts operators that can be inserted along the boundary of the worldsheet or, equivalently, the state space of the theory in an infinite strip, i.e.

$$\begin{aligned} Z_{A,u|B,v}^{\text{WS}}(t, \zeta; \tau) &:= \text{Tr}_{\mathcal{H}_{A,u|B,v}^{\text{WZNW}}} \left((-1)^F q^{L_0 - \frac{c}{24}} e^{2\pi i t J_0^3} e^{2\pi i \zeta K_0^3} \right) \\ &= \frac{\tau^2}{t} \sum_{w=1}^{\infty} \frac{1}{2} (e^{i\pi w} - e^{i\pi \delta_A^B}) \delta\left(\frac{t}{w} - \tau\right) e^{\frac{\pi i \zeta^2}{2\tau}} Z_{u|v}^{\mathbb{T}^4} \left[\begin{array}{c} \frac{\delta_{A,B}}{2} \\ \frac{1}{2} \end{array} \right] (\zeta | \tau). \end{aligned} \quad (8.2.28)$$

Relative to the previous result in eq. (8.2.27), there is a prefactor $\frac{\tau^2}{t}$ that results from rewriting the argument of the δ function.

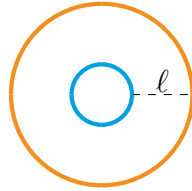
Integrating over the worldsheet modulus τ using the integral measure $d\hat{\tau} = d\tau/\tau^2$ yields

$$\int_0^{i\infty} \frac{d\tau}{\tau^2} Z_{A,u|B,v}^{\text{WS}}(t, \zeta; \tau) = \frac{1}{t} \sum_{w=1}^{\infty} \frac{1}{2} (e^{i\pi w} - e^{i\pi \delta_A^B}) e^{\frac{\pi i w \zeta^2}{2t}} Z_{u|v}^{\mathbb{T}^4} \left[\begin{matrix} \frac{\delta_{A,B}}{2} \\ \frac{1}{2} \end{matrix} \right] (\zeta | \frac{t}{w}). \quad (8.2.29)$$

Formulas (8.2.28) and its integrated cousin (8.2.29) are the main result of this subsection. We discuss these further in section 8.3 when we compare with the expressions for the partition function of the spacetime CFT.

8.2.3 Partition function for AdS_2 brane in thermal AdS_3

We finally compute the boundary spectrum of AdS_2 branes in thermal AdS_3 along the lines of [278]. As is well known, thermal AdS is obtained from the global one through an orbifold construction with the orbifold group $\mathbb{Z}$. The $\mathbb{Z}$ action enforces a periodic identification along the Euclidean time direction. Our starting point for the construction is the boundary partition function we computed in eq. (8.2.28). The associated orbifold partition function is obtained as a sum over twisted sector contributions



$$= \mathcal{N} Z_{A,u|B,v}^{\text{WS}}(\ell t, \ell \zeta; \tau). \quad (8.2.30)$$

Here, the orange and cyan solid lines are the two boundaries of the world sheet annulus on which the states labelled by (A, u) and (B, v) are inserted. Contributions with different ℓ correspond (after integration over τ) to a trace of a segment of global Euclidean AdS_3 of length ℓt over the physical open string Hilbert space¹⁵. In the equation, we introduced a normalisation constant $\mathcal{N}$ that we shall determine later. Summing over ℓ , we obtain the worldsheet partition function

$$Z_{A,u|B,v}^{\text{WS } AdS_3^T}(t, \zeta; \tau) = \mathcal{N} \frac{\tau^2}{\ell t} \sum_{w, \ell=1}^{\infty} \frac{1}{2} (e^{i\pi w} - e^{i\pi \delta_A^B}) \delta(\frac{\ell}{w} t - \tau) e^{\frac{\pi i \ell^2 \zeta^2}{2\tau}} Z_{u|v}^{\mathbb{T}^4} \left[\begin{matrix} \frac{\delta_{A,B}}{2} \\ \frac{1}{2} \end{matrix} \right] (\ell \zeta | \tau). \quad (8.2.31)$$

Upon integration over the modulus τ of the worldsheet, we obtain

$$Z_{A,u|B,v}^{\text{ST } AdS_3^T}(t, \zeta) = \frac{\mathcal{N}}{t} \sum_{w, \ell=1}^{\infty} \frac{1}{2\ell} (e^{i\pi w} - e^{i\pi \delta_A^B}) e^{\frac{\pi i w \ell \zeta^2}{2t}} Z_{u|v}^{\mathbb{T}^4} \left[\begin{matrix} \frac{\delta_{A,B}}{2} \\ \frac{1}{2} \end{matrix} \right] (\ell \zeta | \frac{\ell}{w} t). \quad (8.2.32)$$

¹⁵Note that this can be rephrased as the statement that we compute the annulus partition function describing the boundary spectrum of a brane in the $\mathbb{Z}$ -orbifolded world sheet theory that has additional couplings to $\mathbb{Z}$ -twisted states i.e. thermal winding modes.

In terms of $N = w\ell$, we can also rewrite this as

$$Z_{A,u|B,v}^{\text{ST } AdS_3^T}(t, \zeta) = \frac{\mathcal{N}}{t} \sum_{N=1}^{\infty} \sum_{w|N} \frac{w}{N} \frac{1}{2} (e^{i\pi w} - e^{i\pi \delta_A^B}) e^{\frac{\pi i N \zeta^2}{2t}} Z_{u|v}^{\mathbb{T}^4} \left[\begin{array}{c} \frac{\delta_{A,B}}{2} \\ \frac{1}{2} \end{array} \right] \left(\frac{N}{w} \zeta \middle| \frac{N}{w^2} t \right). \quad (8.2.33)$$

Finally, we conclude that the full open string partition function, counting strings that can start and end on both halves of the AdS_2 brane, is

$$Z_{u|v}^{\text{ST } AdS_3^T}(t, \zeta) = \frac{\mathcal{N}}{t} \sum_{A,B \in \{L,R\}} \sum_{w,\ell=1}^{\infty} \frac{1}{2\ell} (e^{i\pi w} - e^{i\pi \delta_A^B}) e^{\frac{\pi i w \ell \zeta^2}{2t}} Z_{u|v}^{\mathbb{T}^4} \left[\begin{array}{c} \frac{\delta_{A,B}}{2} \\ \frac{1}{2} \end{array} \right] (\ell \zeta \middle| \frac{\ell}{w} t). \quad (8.2.34)$$

Here, t is the modular parameter of the spacetime torus and ζ is a chemical potential associated with the R-charge. The only non-trivial ingredient in the summands are the functions $Z_{u|v}^{\mathbb{T}^4}$ that we defined in eq. (8.2.26).

Before carrying on and matching the result of this string computation to the symmetric orbifold grand canonical partition function in section 8.3, let us pause and give a geometric interpretation of the formulas we obtained. Looking back at our formula (8.2.31), we infer from the argument of the δ function that the worldsheet modulus τ localises to $\frac{\ell}{w}t$ with two integers ℓ and w . We can relate this observation to the geometric interpretation that was given to a similar closed string computation in [278]. Eberhardt explained that the one-to-one correspondence between solutions to the equation

$$t = \frac{a\tau + b}{b\tau + d}, \quad \tau = \frac{dt - b}{-ct + a} \quad (8.2.35)$$

with integer coefficients $a, b, c, d \in \mathbb{Z}$ and holomorphic coverings of the t torus by the τ torus can be understood by realising that such a covering must lift to an affine map $\tilde{\Gamma} : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, $z \mapsto \alpha z + \beta$ such that $\tilde{\Gamma}(\Lambda_\tau) \subseteq \Lambda_t$ holds. The latter condition in particular implies $\alpha, \beta \in \Lambda$ and $\alpha\tau + \beta \in \Lambda_t$ and hence enforces that τ is a fraction of elements of Λ_t . Consequentially, τ can be written as

$$\tau = \frac{dt - b}{-ct + a}. \quad (8.2.36)$$

For the situation at hand, we describe coverings of a cylinder by a cylinder obtained by cutting the tori along the imaginary axis (because the connected components of the boundary of the AdS_2 brane are parallel thermal cycles in the boundary torus of thermal AdS_3). Thus, we want that $\tilde{\Gamma}(i\mathbb{R}) \subseteq i\mathbb{R}$, which implies $\alpha \in \mathbb{R}$ and $\beta \in i\mathbb{R}$. However, in combination with the fact that t is purely imaginary, $\alpha \in \mathbb{R}$ implies that c is zero. But if c is zero and t, τ are purely imaginary then b must also be zero. Hence, we are restricted to

$$\tau = \frac{d}{a}t. \quad (8.2.37)$$

Furthermore, the prefactor of $\frac{1}{\ell}$ in the partition function is to be expected. w counts the wrapping of the world sheet around the spatial cycle, while ℓ counts the wrapping around the thermal cycle. The translation symmetry along the spatial cycle is broken by the presence of the AdS_2 brane. But we still have full translation symmetry along the thermal cycle. Hence, there is a $\mathbb{Z}_\ell$ symmetry that the partition function needs to reflect by a factor of $\frac{1}{\ell}$.

8.3 Holographic matching of interfaces with AdS_2 branes

The goal of this section is to collect evidence supporting the claim that the interfaces we have constructed in section 8.1 indeed provide a holographic description of AdS_2 branes in a theory of tensionless type IIB superstrings in $AdS_3 \times S^3 \times \mathbb{T}^4$. In order to do so, we first, in section 8.3.1, compare our results on open string spectra, most notably formula (8.2.34) for a thermal partition function of open strings ending on an AdS_2 brane, with the counting function (8.1.57) for interface changing operators. To fully establish the correspondence, it is necessary to furthermore match the interface symmetric orbifold correlators with string scattering amplitudes in the presence of the brane. In section 8.3.2, we review the holographic match in the absence of branes. We then, in section 8.3.3 take first steps towards establishing the holographic duality between amplitudes involving open and closed strings in the presence of the AdS_2 brane and corresponding correlation functions of the symmetric orbifold that involve certain interface changing operators. Concretely, we show that the large N expansion of correlation functions takes the form of a string theoretic genus expansion, for a certain class of our interface changing operators.

8.3.1 Partition functions and string amplitudes

As promised at the end of section 8.1.4, we now use the notation for partition functions of the supersymmetric four-torus established in section 8.2.2, to spell out eq. (8.1.76) more concretely for the special case in which the seed theory $\mathcal{M}$ is given by the supersymmetric four-torus $\mathbb{T}^4$. This entails making a particular choice for the boundary conditions $a_\pm^{L/R}$. Our proposal is

$$|a_-^L\rangle = \|u, R, \varepsilon\|_{\mathbb{T}^4} \quad \text{and} \quad |a_+^L\rangle = \|v^*, R, \varepsilon\|_{\mathbb{T}^4} \quad (8.3.1)$$

as well as

$$|a_-^R\rangle = -\|v, R, -\varepsilon\|_{\mathbb{T}^4} \quad \text{and} \quad |a_+^R\rangle = -\|u^*, R, -\varepsilon\|_{\mathbb{T}^4}, \quad (8.3.2)$$

which leads to

$$\hat{Z}_o^{LL}(\hat{t}, \hat{\zeta}) = \hat{Z}_o^{RR}(\hat{t}, \hat{\zeta}) = \hat{Z}_{u|v}^{\mathbb{T}^4} \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] (\hat{\zeta}|\hat{t}) \quad \text{and} \quad \hat{Z}_o^{LR}(\hat{t}, \hat{\zeta}) = \hat{Z}_o^{RL}(\hat{t}, \hat{\zeta}) = -\hat{Z}_{u|v}^{\mathbb{T}^4} \left[\begin{array}{c} \frac{1}{2} \\ 0 \end{array} \right] (\hat{\zeta}|\hat{t}). \quad (8.3.3)$$

We thus conclude that

$$\mathcal{Z}_O[\hat{t}, \hat{\zeta}] = \sum_{A,B \in \{L,R\}} \sum_{\ell,w=1}^{\infty} \frac{1}{2w} (e^{i\pi\ell} - e^{i\pi\delta_A^B}) \hat{Z}_{u|v}^{\mathbb{T}^4} \left[\begin{array}{c} \frac{1}{2} \\ \frac{\delta_{A,B}}{2} \end{array} \right] (\ell\hat{\zeta}|\frac{\ell}{w}\hat{t}). \quad (8.3.4)$$

Applying eq. (8.2.25), as well as swapping w and ℓ , gives

$$\mathcal{Z}_O[t, \zeta] = \sum_{A,B \in \{L,R\}} \sum_{\ell,w=1}^{\infty} \frac{1}{2\ell} (e^{i\pi\ell} - e^{i\pi\delta_A^B}) e^{\frac{\pi i w \ell \zeta^2}{2t}} Z_{u|v}^{\mathbb{T}^4} \left[\begin{array}{c} \frac{\delta_{A,B}}{2} \\ \frac{1}{2} \end{array} \right] (\ell\zeta|\frac{\ell}{w}t). \quad (8.3.5)$$

This partition function is manifestly identical to eq. (8.2.34) if we choose the normalisation $\mathcal{N} = t$. Furthermore, eq. (8.1.56) directly tells us that $\mathcal{Z}_C$ is identical to the $t = \bar{t}$ restriction of the space time CFT torus partition function. Making the appropriate choice of spin structure, the latter has been matched with the thermal AdS_3 closed string partition function in [278]. The rest of this section is devoted to a qualitative description of the holographic duality underlying this equality.

The equality of partition functions is equivalent to the statement that the spectrum (i.e. the conformal weights and the eigenvalues of K_0 as well as the dimensions of the eigenspaces) of open strings stretching between two AdS_2 branes coincides with the spectrum of interface changing operators of our symmetric orbifold interfaces. This observation leads to a natural proposal for the holographic description of the tensionless string dynamics in the presence of an AdS_2 brane. Let us first sketch this proposal in a rough qualitative manner and then explore its precise quantitative implications.

1. We recall from [277] that in the tensionless limit, long *closed superstrings* of type IIB theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ possess a holographic description in terms of twisted sector bulk operators in the symmetric product orbifolds $\text{Sym}^N(\mathbb{T}^4)$.
2. Based on the equality between the partition functions (8.2.34) and (8.3.5), we conjecture that the AdS_2 brane is dual to a weighted sum of the interfaces of $\text{Sym}^{N_{\pm}}(\mathbb{T}^4)$ constructed in section 8.1.2.
3. We furthermore propose that the *open strings* that end on the AdS_2 brane are dual to interface changing operators between those interfaces.

In order to simplify the navigation through the rest of this section, we divided it into short paragraphs that provide the interested reader with further details on various aspects of our proposal. The paragraphs can be read somewhat independently.

Thermal versus global AdS_3 . In the three claims above, we do not specify whether we consider string theory on thermal or global AdS_3 . Let us start our discussion of the claims by filling in some more detail on this. Recall that we constructed the open string partition function on thermal AdS_3 from the partition function of global AdS_3 , which we computed in eq. (8.2.29), by including additional contributions originating from world sheets that wrap the thermal cycle $\ell = N/w$ times. Vice versa, we can pass from the thermal AdS_3 open string partition function (8.2.34) to its analogue for global AdS_3 simply by dropping all the $\ell > 1$ terms¹⁶. Because of this, the above holographic matching of the thermal AdS_3 partition function directly implies that we can also obtain the global AdS_3 partition function from our interface construction. What this requires is to adjust the boundary states (8.1.38) that describe our interfaces or, more precisely, their building blocks defined in eq. (8.1.36). These states involve the boundary states (8.1.26) which contain a sum in the ρ -twisted sectors. In order to recover the partition function of global AdS_2 , we need to restrict the sum over $\rho_{\pm}$ to the trivial term $\rho_{\pm} = id$. This manifestly removes the $\ell > 1$ terms in the partition function of defect changing operators.

Absence of disk contributions. Note that, in sharp contrast to the bulk discussion [308] as well as that of spherical branes [124], our holographic matching of the open string partition function did not rely on the extrapolation of large k results for (disconnected) genus 0 contributions to $k = 1$. Instead, the one-loop (annulus) string computation directly reproduces the exact symmetric orbifold partition function of interface changing operators (up to an overall normalisation factor)¹⁷. In fact, this remarkable absence of the somewhat problematic disk contributions is to be expected on simple geometric grounds. As opposed to the spherical branes, the AdS_2 branes are not contractible inside thermal AdS_3 . Thus, the existence of contractible world-sheets whose boundary winds one of the boundaries of the AdS_2 brane is topologically obstructed.

The set of string scattering states. Having commented on the absence of disk contributions and clarified the roles of thermal and global AdS_3 , let us now expand more generally on part three of our proposal. In order to formulate an actual prescription for holographic computations of open string scattering amplitudes in the tensionless limit of superstring theory on $AdS_3 \times S^3 \times \mathbb{T}^4$, the next paragraphs describe how to label and construct the interface changing operators dual to string scattering states. Recall from eqs. (8.1.55 - 8.1.59) that these operators fall into two classes,

¹⁶As also explained at the end of section 8.2.3, the situation for the open strings is in this sense simpler than the closed string analogue discussed in [278]. Indeed, for the closed strings it is not true that the global AdS_3 partition function is obtained from the thermal AdS_3 one by simply dropping the $\ell > 1$ terms.

¹⁷We thank Andrea Dei for encouraging us to emphasise this point more explicitly.

corresponding to the two classes of string states. On the one hand, the string theory contains long closed strings. Their quantum numbers include the winding number w , which counts how many times the string winds the spatial cycle of the boundary torus. In addition, the string theory also contains long open strings. These possess two new labels $A, B \in \{L, R\}$ as well as the familiar winding number w . The labels $A, B \in \{L, R\}$ refer to the two components of the boundary of the AdS_2 brane to which two endpoints of the open strings are attached. While the interface changing operators that are dual to long closed strings are simply obtained by restricting familiar bulk operators to the one-dimensional locus of the interface, those that are associated with long open strings are new. In the following few paragraphs we will therefore mostly focus on the latter.

Interface changing operators dual to open strings. When we discussed bulk fields of symmetric product orbifolds in section 8.1, we pointed out that it is useful to work with a larger set of twist fields that are labelled by group elements of the orbifold group rather than the physical fields which are labelled by conjugacy classes. The physical operators are obtained from the former by averaging over the orbits of the $S_{N_{\pm}}$ action. We follow a similar strategy for the interface changing operators and describe the physical operators

$$\sigma_w^{AB} = \mathcal{N} \sum_{\gamma} \sigma_{\gamma}^{AB} \quad (8.3.6)$$

as a sum over integer sequences $\gamma = (\gamma_i)_{i=1}^w$ of length w . More precisely, γ is a sequence of entries that we choose alternating between the two sets $\underline{N}_{\pm} = \{1, \dots, N_{\pm}\}$ with the ‘boundary’ conditions

$$\gamma_1 \in \underline{N}_{-} \text{ if } A = L \quad \text{and} \quad \gamma_1 \in \underline{N}_{+} \text{ if } A = R \quad (8.3.7)$$

$$\gamma_w \in \underline{N}_{-} \text{ if } B = R \quad \text{and} \quad \gamma_w \in \underline{N}_{+} \text{ if } B = L. \quad (8.3.8)$$

Repeated entries are not allowed, i.e. $\gamma_i \neq \gamma_j$ for $|i - j|$ even. Let us also note that the boundary condition on γ_w can only be satisfied for sequences of even length w if $A = B$ and for sequences of odd length w if $A \neq B$. The labelling of open string states through sequences of integers was described in [125] already. Martinec used red and blue colours to distinguish between elements of $\underline{N}_{\pm}$.

Having described the label γ , it is rather straightforward to state how the actual operator σ_{γ}^{AB} acts. To do so it makes sense to first briefly reconsider the construction of the interfaces on which σ_w^{AB} is supposed to be inserted. Recall that in section 8.1 the interfaces are described as gauge invariant sums (8.1.36) of states that couple to one specific pair $g_{\pm} = \rho_{\pm} \tau_{\pm} \in S_{N_{\pm}}$ of twisted sectors in the bulk of the product theory. To each cycle in g_{-} , these gauge fixed states either associate a reflective boundary condition or a cycle in g_{+} to which it is transmitted. The individual gauge fixed

contributions to the interface hence contain two pieces of data: The distribution of elements of $\underline{N}_\pm$ into cycles of permutations in $S_{N_\pm}$ and the attribution of the property “reflective” or “transmissive”. As we shall argue in the next paragraph, the detailed distribution into cycles is actually not important to decide whether a given summand σ_γ^{AB} from the sum (8.3.6) can contribute to a correlation function. What is important are the subsets of elements $T_+ \subseteq \underline{N}_+$ and $T_- \subseteq \underline{N}_-$ of colours that are transmitted along the interface as well as the precise pairing of colours. We encode the latter in bijections $\pi_\pm : T_\pm \rightarrow T_\mp$ with π_+ and π_- inverse to each other, i.e. $\pi_- \circ \pi_+ = id$. In the following we shall often drop the lower index on π that tells us on which of the two subsets $T_\pm$ it acts. We construct one triple (T_+, T_-, π) for each summand in the gauge invariant sums (8.1.36). Even though these triples forget the information about the distribution of colours in cycles, i.e. about the elements $g_\pm$, we now argue that $T_\pm \subseteq \underline{N}_\pm$ and π contain enough information to decide whether σ_γ^{AB} can be inserted at the transition point between two defects.

From a path integral perspective, correlation functions of the interface symmetric product orbifold are computed by (locally) integrating over $N_\pm$ copies of the fields of the seed theory on the upper and lower half plane labelled by the elements of $\underline{N}_\pm$. On fields whose $\underline{N}_\pm$ labels are elements of reflected cycles, i.e. contained in $\underline{N}_\pm \setminus T_\pm$, reflecting boundary conditions are imposed locally at the interface. On fields whose $\underline{N}_\pm$ labels are elements of transmissive cycles, i.e. contained in $T_\pm \subset \underline{N}_\pm$, we locally impose transmissive boundary conditions. How the transmitted and reflected elements of $\underline{N}_\pm$ are distributed among different cycles has no direct impact on the local boundary conditions imposed at the interface, but affects the global structure of how the different copies of the seed theory fields are glued together¹⁸. Local interface changing operators describe how the boundary conditions imposed at the interface change locally at their insertion point. Hence, we do not need to reference the global structure encoded in the distribution of $\underline{N}_\pm$ labels among different $S_{N_\pm}$ cycles to define them.

Let us now choose two interfaces $\mathcal{I}^{p^L}$ and $\mathcal{I}^{p^R}$ to the left (i.e. on the L interface) and the right (i.e. on the R interface) of some interface changing operator σ_w^{AB} . As we argued above, we can associate a triple (T_+, T_-, π) to each term in the sum (8.1.36). Since we have an interfaces on both sides of the insertion, we have two triples (T_+^L, T_-^L, π^L) and (T_+^R, T_-^R, π^R) . A summand σ_γ^{AB} can contribute to the corresponding correlation functions provided the following conditions are satisfied

- **Reflective end points.** The first label γ_1 must be in the set $\underline{N}_\pm \setminus T_\pm^A$ of

¹⁸From a global perspective the path integral is performed over sections of $S_{N_\pm}$ fibre bundles that only in local trivialisations can be identified with $N_\pm$ copies of the fields of the seed theory. The distribution of the $\underline{N}_\pm$ labels among different $S_{N_\pm}$ cycles described the monodromy of these fibre bundles along the interface.

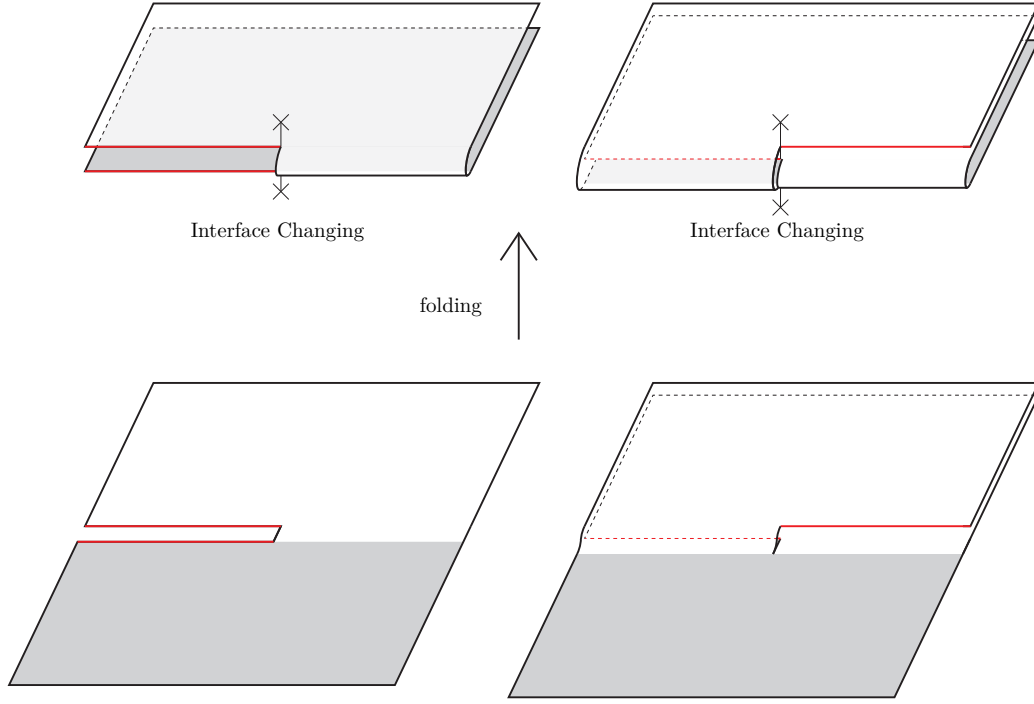


Figure 8.3.1: Visualisation of interface changing operators. Red lines indicate reflective boundaries. Gray sheets are representing the lower half-plane on which $\text{Sym}^{N_-}(\mathcal{M})$ lives. White sheets are upper half-planes that carry $\text{Sym}^{N_+}(\mathcal{M})$. On the left hand side, we portray the operators σ_2^{LL} . The right hand side represents σ_3^{RL} .

colours that are reflected on the A interface. Similarly, the last entry γ_w must be taken from the set $\underline{N}_\pm \setminus T_\pm^B$ that are reflected on the B interface.

- **Transmissive interior.** If $A = L$ then the entries γ_i and γ_{i+1} for odd integer i are glued together at the R interface, i.e. $\gamma_i \in T_-^R$ and $\gamma_{i+1} = \pi^R(\gamma_i) \in T_+^R$. For even i they are glued at the L interface, i.e. $\gamma_i \in T_+^L$ and $\gamma_{i+1} = \pi^L(\gamma_i) \in T_-^L$. For $A = R$ the prescription is reversed.
- **Unaffected complement.** The boundary condition imposed on colours that do not appear in γ are unchanged at the insertion point of σ_w^{AB} . If we denote the set of colours from $\underline{N}_\pm$ that appear in γ by $S_\gamma^\pm$, the formal condition reads

$$T_\pm^L \setminus S_\gamma^\pm = T_\pm^R \setminus S_\gamma^\pm. \quad (8.3.9)$$

In addition the restriction of π^L and π^R must agree on the two intersections.

Interface changing operators dual to closed strings. Now that we have completed our description of the interface changing operators we propose to be dual to

open strings, let us briefly comment on the operators σ_w dual to closed strings. These are more familiar.

Again, they possess a representation akin to eq. (8.3.6) in terms of a sum of gauge fixed operators σ_γ . The only difference between σ_γ and σ_γ^{AB} is that σ_γ describes no reflective i.e. only transmissive boundary conditions. More concretely, it is obtained by dropping the “reflective end points” condition formulated in the previous paragraph and instead use the “transmissive interior” condition with the periodic identification $\gamma_{w+1} = \gamma_1$. Upon performing such a periodic identification any choice of the labels A and B describes the same operator and hence these labels become obsolete. Notice that since we had chosen neighbouring entries from the upper and lower half plane in an alternating fashion, w must be even in order to admit the periodic identification. The fields described in this manner are essentially bulk twist fields with twist $w/2$ inserted on the interface.

Taking stock. Let us pause a moment to take stock of what we have formulated in the previous paragraphs. We have constructed two types of interface changing operators σ_w^{AB} and σ_w that carry the same labels as open strings ending on AdS_2 branes and closed strings winding the boundary of AdS_3 respectively. All interface changing operators between any two of the interfaces of section 8.1 can be obtained by taking products of the elementary operators we described.

We have seen that the string partition function computed by summing over connected world sheets describing open strings ending the AdS_2 brane and propagating around the non-contractible cycle of thermal AdS_3 matches the connected partition function $\mathcal{Z}_O$ of interface changing operators in the symmetric product orbifold. By exponentiating both sides, we can deduce that the second quantised string partition function coincides with the trace over the Hilbert space of interface changing operators in a grand canonical ensemble of symmetric product orbifolds. This Hilbert space manifestly has the form of a Fock space generated from a single particle Hilbert space that is spanned by the single cycle twist fields discussed above. Thereby we have provided strong evidence for the conjecture that the two families of operators we have introduced in the previous paragraphs are dual to the single particle open and closed string states of the bulk theory.

General string scattering amplitudes. Of course, the proposed duality should eventually be supported through a comparison of arbitrary scattering amplitudes of open and closed strings with correlation functions of the interface changing operators σ_w^{AB} and σ_w we introduced above. At this point, we can at least state somewhat more explicitly what this relation is expected to look like.

On the string side, we encounter scattering amplitudes that can be written as inte-

grals over the moduli space $\mathcal{M}_{b,g,n_c,n_o}$ of Riemann surfaces Σ_{b,g,n_c,n_o} of genus g with b boundary components as well as n_c punctures in the interior and n_o punctures on the boundary. The associated integrand is a correlation function of n_c closed string vertex operators and n_o open string vertex operators of the worldsheet theory. Thus, such a string scattering amplitude takes the form

$$\mathcal{A} = \int_{\mathcal{M}_{b,g,n_c,n_o}} \left\langle \prod_{i=1}^{n_c} V_{w_i, h_i, \bar{h}_i}(z_i, \bar{z}_i, x_i, \bar{x}_i) \prod_{j=1}^{n_o} V_{w_j, \Delta_j}^{AB}(y_j, t_j) \right\rangle_{\Sigma_{b,g,n_c,n_o}}. \quad (8.3.10)$$

Here, w_i label spectral flow of the closed string vertex operators, $(h_i, \bar{h}_i)$ are the holomorphic and anti-holomorphic space time conformal weights, $(z_i, \bar{z}_i)$ labels a point in the interior of the Riemann surface, $(x_i, \bar{x}_i)$ labels a point on the holographic boundary of the target space and t_i labels a point on the boundary of the AdS_2 brane. The role of t_i is completely analogous to that of $(x_i, \bar{x}_i)$ in the x -basis vertex operators of [69], i.e. they are introduced by an appropriate deformation of the ordinary vertex operators. In the closed string scenario discussed in [69], the deformation is generated by the zero modes J_0^+ and $\bar{J}_0^+$ of one holomorphic and one anti-holomorphic current of the $SL(2, \mathbb{R})$ WZNW model. The AdS_2 boundary condition glues these currents together along the boundary so that only one zero modes survives. This zero mode generates the deformation by the parameter t . Moreover, the open string vertex operators are inserted at the points y_j on the boundary of the worldsheet and they carry only a single conformal weight Δ_j .

On the symmetric orbifold side, these string scattering amplitudes should be reproduced by correlation functions

$$\left\langle \prod_{i=1}^{n_c} \sigma_{w_i, h_i, \bar{h}_i}(x_i, \bar{x}_i) \prod_{j=1}^{n_o} \sigma_{w_j, \Delta_j}^{AB}(t_j) \right\rangle_{g,b} \quad (8.3.11)$$

of interface changing operators. Here, we have not only included the labels which specify the ground state of the twisted sectors in which the operators live, but also added labels $h_i, \bar{h}_i$ and Δ_j that keep track of the excitations. The label g, b instructs us to restrict to the contribution to the correlator computed through covering maps whose covering space is an element of $\mathcal{M}_{b,g,n_c,n_o}$.

We will formulate the conjecture in more detail and collect evidence for it in Chapter 9 below. Here, we shall content ourselves with a few basic statements. Concretely, we show in section 8.3.3, for a special subset of boundary operators, that the N scaling of the large N -expansion of the boundary symmetric orbifold matches the scaling of the dual string scattering amplitudes in the string coupling.

8.3.2 Bulk correlation functions and amplitudes - a review

As mentioned towards the end of the last section, now that we have successfully matched the string theory and space time CFT partition functions, the natural next

step is to show that the duality also extends to correlators. Here, we shall content ourselves with the very first step of such an extension. Namely, relating the large N expansion of correlation functions involving bulk and interface changing operators in the presence of our interfaces to a string theoretic genus expansion. This is reminiscent of ‘t Hooft’s analysis of the large N limit for gauge theories. A detailed comparison of the CFT correlators with the scattering amplitudes of string theory in AdS_3 will be performed in a forthcoming publication¹⁹. For bulk correlations of symmetric product orbifolds, the large N behaviour was studied by Pakman, Rastelli and Razamat [284]. To set up notations and illustrate the main ideas, we shall review their arguments first before including interface changing operators in the next subsection.

The covering space method of²⁰ Lunin and Mathur [310] computes correlation functions

$$\langle \sigma_{g_1}(z_1, \bar{z}_1) \dots \sigma_{g_{n_c}}(z_{n_c}, \bar{z}_{n_c}) \rangle \quad (8.3.12)$$

of gauge fixed twist fields in (bulk) symmetric product orbifolds on some two-dimensional base space M from the data of branched coverings

$$\gamma : \Sigma \rightarrow M \quad (8.3.13)$$

of the base space M by a covering space Σ . Here, we shall assume that the elements $g_\nu \in S_N$ are cyclic permutations of length w_ν . The general case can be treated similarly and in fact follows directly from that of cyclic g_ν by taking suitable OPEs. We place a subscript c on the number n_c of bulk field insertions to remind us that these should be in one-to-one correspondence with insertions of vertex operators of closed strings in the dual theory.

Let us explain how to determine the covering space topology from the choice of cyclic permutations g_i . To this end, it is useful to introduce the concept of “active colours” [284]. Given any $g \in S_N$, the subset A_g of active colours consists of all elements in $\{1, \dots, N\}$ on which g acts non-trivially i.e. $A_g := \{i | gi \neq i\}$. Furthermore, we define the active colours A_S of any subset $S \subseteq S_N$ of the permutation group to be the union of the active colours of all its elements. This definition is now applied to the set $S = \{g_1, \dots, g_{n_c}\}$ of cyclic permutations that appear in the correlation function (8.3.12). We call the associated set A_S the *set of active colours of the correlation functions*.

The permutations g_ν that appear in the correlation function we want to evaluate generate a subgroup $H_S \subseteq S_N$ of the permutation group. By construction, H_S acts

¹⁹This detailed comparison will include a discussion of the interface grand-canonical ensemble. In the current chapter, we restrict ourselves to a fixed N analysis of the interface correlators, which suffers from the same issues of the analogous fixed N bulk correlators, see e.g. [266].

²⁰See also [309] for an earlier application of covering space methods to orbifold CFTs.

faithfully on the corresponding set A_S of active colours. Under this action, the set A_S decomposes into orbits which we denote by $\mathcal{O}_1, \dots, \mathcal{O}_m$. Once we have determined the orbits, it is rather easy to infer key features of the covering surface Σ . More specifically, it turns out that

1. The orbits $\mathcal{O}$ are in one-to-one correspondence with the connected components $\Sigma_{\mathcal{O}}$ of the covering surface Σ . In particular, the number of connected components of Σ is equal to the number $m = m(S)$ of orbits.
2. The genus $g_{\mathcal{O}}$ of the connected component $\Sigma_{\mathcal{O}}$ that is associated to the orbit $\mathcal{O}$ can be computed in terms of the number of elements $|\mathcal{O}|$ of the set $\mathcal{O}$ as

$$g_{\mathcal{O}} = 1 - |\mathcal{O}| + \frac{1}{2} \sum_{A_{g_{\nu}} \subseteq \mathcal{O}} (w_{\nu} - 1). \quad (8.3.14)$$

The sum runs over all those indices $\nu \in \{1, \dots, n_c\}$ for which the set of active colours $A_{g_{\nu}}$ of the group element g_{ν} is a subset of the orbit $\mathcal{O}$.

Hence, the covering surface Σ is connected if and only if there is exactly one orbit. Put differently, in order for Σ to be simply connected the subgroup $H_S \subseteq S_N$ that is generated by the elements of $S = \{g_1, \dots, g_{n_c}\}$ must act transitively on the active colours A_S of the correlator. Denoting the number of active colours of the correlator by $|A_S| = n$, the formula for the genus then becomes

$$g = 1 - n + \frac{1}{2} \sum_{\nu=1}^{n_c} (w_{\nu} - 1). \quad (8.3.15)$$

The physical correlation functions to consider in a symmetric product orbifold are of course not the gauge fixed correlators that we have discussed so far, but rather the correlation functions of gauge invariant twist fields

$$\sigma_w = \frac{1}{\sqrt{|[(1\dots w)]|}} \sum_{g \in [(1\dots w)]} \sigma_g = \sqrt{\frac{(N-w)!w}{N!}} \sum_{g \in [(1\dots w)]} \sigma_g. \quad (8.3.16)$$

Clearly, the expansion of the gauge invariant twist fields into gauge fixed twist fields allows us to immediately express correlators of the former in terms of correlators of the latter,

$$\langle \prod_{\nu=1}^{n_c} \sigma_{w_{\nu}}(z_{\nu}, \bar{z}_{\nu}) \rangle = \left(\prod_{\nu=1}^{n_c} \sqrt{\frac{(N-w_{\nu})!w_{\nu}}{N!}} \right) \sum_{g_{\nu} \in [(1\dots w_{\nu})]} \langle \sigma_{g_1}(z_1, \bar{z}_1) \dots \sigma_{g_{n_c}}(z_{n_c}, \bar{z}_{n_c}) \rangle. \quad (8.3.17)$$

The connected part of such a correlator is defined as the restriction of the above sum to those terms that are associated to connected covering spaces Σ . According to the comments that we made earlier in this section, we can state this definition as

$$\langle \prod_{\nu=1}^{n_c} \sigma_{w_{\nu}}(z_{\nu}, \bar{z}_{\nu}) \rangle_c := \left(\prod_{\nu=1}^{n_c} \sqrt{\frac{(N-w_{\nu})!w_{\nu}}{N!}} \right) \sum_{\substack{g_{\nu} \in [(1\dots w_{\nu})] \\ m(\{g_1, \dots, g_{n_c}\})=1}} \langle \sigma_{g_1}(z_1, \bar{z}_1) \dots \sigma_{g_{n_c}}(z_{n_c}, \bar{z}_{n_c}) \rangle. \quad (8.3.18)$$

The restricted sum now runs over all sets $S = \{g_1, \dots, g_{n_c}\}$ of cyclic permutations g_ν for which the active colours A_S of the correlations function form a single orbit under the action of H_S . Remarkably, the large N expansion of the connected part of the correlator takes the form of a string genus expansion. In order to understand the precise scaling with N , we point out that the restricted sum on the right hand side of eq. (8.3.18) contains combinatorial enhancements from the number of ways in which one can pick n active colours from the N colours that are available. Thus, the connected correlator scales as

$$\prod_{\nu=1}^{n_c} \sqrt{\frac{(N-w_\nu)! w_\nu}{N!}} \frac{N!}{n!(N-n)!} \sim N^{n-\frac{1}{2} \sum_{\nu}^{n_c} w_\nu} \sim g_s^{-2n+\sum_{\nu}^{n_c} w_\nu} = g_s^{-2+2g+n_c} . \quad (8.3.19)$$

In the first step, we used Stirling's formula to determine the leading asymptotics of the prefactor as N becomes large. Then, we used the standard relation $g_s^2 \sim 1/N$ between the string coupling g_s and the total number N of colours. Finally, we inserted eq. (8.3.15) in order to rewrite the exponent in terms of the genus g of the covering surface Σ . The right hand side of eq. (8.3.19) can be recognised as the usual dependence of closed string scattering amplitudes on the string coupling.

8.3.3 Boundary genus expansion for AdS_2 branes

In this subsection, we extend the previous analysis to correlation functions involving both bulk and interface changing fields and show that the large N expansion of correlation functions in the presence of our interfaces organises itself in the fashion of a string theory genus expansion. Just as in section 8.3.2, (the connected part of) a gauge invariant correlation function is a sum of gauge dependent correlators each of which can be associated to a covering space with a certain genus g and with n_c punctures associated to the (bulk) operators inserted inside the correlator. In contrast to the covering spaces described in section 8.3.2, the covering spaces that we encounter in this subsection additionally have a possibly non-vanishing number b of boundaries, as well as a number n_o of punctures on the boundary.

In this chapter, we do not provide a full analysis of all interface changing correlation functions, but rather restrict to one specific class of contributions to one specific type of boundary operators. The generalisation to arbitrary covering maps contributing to interface correlation functions of interface changing operators is the subject of Chapter 9 below. Concretely, we restrict to fields $\sigma^\pm$ that interpolate between $\mathcal{I}_{|a_\pm\rangle}^{(p)}$ and $\mathcal{I}_{|a_\pm\rangle}^{(p\pm 1)}$ at $N_- = N_+ = N$ and possess an expansion of the form

$$\sigma^\pm = N^{-\frac{1}{2}} \sum_{i^-=1}^N \sum_{i^+=1}^N \sigma_{i^-i^+}^\pm . \quad (8.3.20)$$

Here, the operator σ_{i^-,i^+}^- cuts the i^- -th copy of the seed theory in the lower hemisphere from the i^+ -th copy in the upper hemisphere and turns on an $|a_\pm\rangle$ boundary for both.

Conversely, the operator $\sigma_{i^+i^-}^+$ removes two copies of the boundary $|a_\pm\rangle$ and glues the i^- -th copy of the seed theory in the lower hemisphere to the i^+ -th copy in the upper hemisphere. Consequently

$$\sigma^+ = \sigma_2^{RR} \quad \text{and} \quad \sigma^- = \sigma_2^{LL} \quad (8.3.21)$$

in the notation used in section 8.3.1. Note that, in eq. (8.3.20), the order of the indices i^+ and i^- is different in σ^+ and σ^- . This is a choice of notation that is convenient in describing the computation of correlation functions

$$\left\langle \prod_{\nu=1}^{n_o} \sigma^{s_\nu}(t_\nu) \prod_{\mu=1}^{n_c} \sigma_{w_\mu}(x_\mu, \bar{x}_\mu) \right\rangle, \quad (8.3.22)$$

where $s_\nu \in \{-, +\}$. We assume the ordering $\nu < \mu \Rightarrow x_\nu < x_\mu$. In general (8.3.22) is a sum over many gauge fixed contributions or said differently, of contributions arising from many different covering maps. Here, we restrict to those contributions for which the boundary condition imposed at infinity is fully transmissive. For gauge fixed contribution to the correlator which fall into this class, any boundaries that should be glued together by a σ^+ first had to be cut open by the insertion of a σ^- and thus, such a correlator with n_o gauge fixed interface changing operators $\sigma^\pm$ has to satisfy

$$\sum_{i=1}^{n_o} s_i = 0 \quad \text{and} \quad \sum_{i=1}^k s_i \leq 0 \quad (8.3.23)$$

for every $k < n_o$. As for the correlation functions of bulk twist fields, we can express the correlator (8.3.22) as a sum over gauge fixed terms. Concretely, inserting the definition (8.3.20) of $\sigma^\pm$, we obtain

$$\left\langle \prod_{\nu=1}^{n_o} \sigma^{s_\nu}(t_\nu) \prod_{\mu=1}^{n_c} \sigma_{w_\mu}(x_\mu, \bar{x}_\mu) \right\rangle = N^{-\frac{n_o}{2}} \left\langle \prod_{\nu=1}^{n_o} \sum_{i_\nu^\pm} \sigma_{i_\nu^{s_\nu}, i_\nu^{s_\nu}}(t_\nu) \prod_{\mu=1}^{n_c} \sigma_{w_\mu}(x_\mu, \bar{x}_\mu) \right\rangle \quad (8.3.24)$$

For the individual terms on the right hand side, all indices $i_\nu^\pm$ must be contracted according to the following rule:

Wick contractions for $\sigma^\pm$ colour indices: The left index i_ν^+ of every boundary creation operator $\sigma_{i_\nu^+, i_\nu^-}^-$ needs to be contracted with the right index i_μ^+ of a boundary annihilation operator $\sigma_{i_\mu^-, i_\mu^+}^+$ such that $x_\nu < x_\mu$. In complete analogy, the left index i_μ^- of a boundary annihilation operator inserted at x_μ needs to be contracted with the right index i_ν^- of a boundary creation operator inserted at $x_\nu < x_\mu$.

Following this contraction procedure, starting from an arbitrary boundary creation operator $\sigma_{c_1^+ c_1^-}^-$, we zigzag back and forth along the boundary line, collecting a string of boundary creation and annihilation operators $\sigma_{c_1^+ c_1^-}^- \sigma_{c_1^- c_2^+}^+ \sigma_{c_2^+ c_2^-}^- \dots \sigma_{c_{w_{n_c}}^- c_1^+}^+$ that ultimately has to close into some cycle bringing us back to c_1^+ . Keeping only the c_i^- ,

we obtain a cyclic permutation $g_{n_c+1} = (c_1^- c_2^- \dots c_{w_{n_c+1}}^-) \in S_{N_-}$ whose length we denote by w_{n_c+1} . This notation reflects the fact that upon fusing all of the operators that formed the cycle, we obtain twist fields in the g_{n_c+1} twisted sector. The latter are analogous to the bulk operators that are indexed by n_c cycles g_ν of length w_ν .

Effectively, the Wick contractions among interface changing operators we just performed thus gave rise to an additional cycle. If the latter did not include all the interface changing operators, we pick one of the remaining ones and form a second cycle g_{n_c+2} of length w_{n_c+2} . We continue this process until we reach the last cycle g_{n_c+b} of length w_{n_c+b} , i.e. until no uncontracted interface changing operators remain.

Once we have completed this contraction process, we are now left with $n_c + b$ cycles g_ν of length w_ν . These data are identical to those that would appear if we had to calculate a $(n_c + b)$ -point correlation function of bulk operators. Consequently, we can now follow essentially the same steps we outlined in the previous subsection. As we did there, we first form the set $S = \{g_1, g_2, \dots, g_{n_c+b}\}$ and the group H_S that is generated by the cycles g_μ . We then let H_S act on the active colours. We shall say that a term in the sum on the right hand side of eq. (8.3.24) contributes to the connected part of the correlation functions if the group H_N acts transitively on the active colours. Furthermore, the terms that are contributing to the connected part of our correlation functions are associated with a connected covering space whose genus g is given by²¹

$$g = 1 - n + \frac{1}{2} \sum_{\mu=1}^{n_c+b} (w_\mu - 1) \quad (8.3.25)$$

where n denotes the number of active colours as before. By construction, the cycles that are associated with the interface changing operators are in one-to-one correspondence with the connected components of the boundary of the covering surface. In particular, the number of connected components of the boundary is b .

We are finally able to compute the large N scaling behaviour of the connected part of the correlation function. There are three factors that contribute. As in the case of bulk correlation functions, we need to select the n active colours from the N colours that are available. In addition, we need to collect the normalisations of bulk and interface changing fields. The large N behaviour of the first two factors we evaluated in eq. (8.3.19) already. Since the interface changing fields we consider each contribute a factor of $N^{-1/2}$, the large N asymptotics of the prefactor reads

$$N^{n-\frac{1}{2} \sum_{\nu=1}^{n_c} w_\nu - \frac{n_o}{2}} = N^{n-\frac{1}{2} \sum_{\nu=1}^{n_c} w_\nu - \frac{1}{2} \sum_{\nu=1}^b w_{n_c+\nu} - \frac{n_o}{4}}. \quad (8.3.26)$$

²¹This follows directly from the fact that the interface changing operators that are associated to a cycle g_{n_c+i} fuse into g_{n_c+i} twist fields upon performing the OPE, which we have already alluded to above.

In going to the right hand side of this equations, we have used the fact that

$$\sum_{\nu=1}^b w_{n_c+\nu} = \frac{n_o}{2}, \quad (8.3.27)$$

which follows from the construction of the w_μ with $\mu > n_c$ since they count the number of interface changing operators that are combined into a cycle and non-vanishing correlators require that all the fields are part of some cycle. Now we can proceed by inserting equation (8.3.25) to express the sums over the cycle lengths w_μ through the genus g of the branching surface. The result is

$$N^{n-\frac{1}{2}\sum_{\nu}^c w_\nu - \frac{1}{2}\sum_{\nu=1}^b w_{n_c+\nu} - \frac{n_o}{4}} = N^{1-g-\frac{b}{2}-\frac{n_c}{2}-\frac{n_o}{4}} \sim g_s^{-2+2g+b+n_c+\frac{n_o}{2}}. \quad (8.3.28)$$

Once again, we have used that $g_s^2 \sim 1/N$ to express the final result for the large N asymptotics of the correlation function in terms of the string coupling g_s rather than $N \sim 1/g_s^2$. The final result indeed has the g_s dependence of a string amplitude for a surface of genus g with b boundary components, n_c bulk punctures and n_o boundary punctures. This is what we wanted to show.

Appendix

8.A Torus partition function of the symmetric product orbifold

In this appendix, we derive eq. (8.1.4). To do so, we need to acknowledge that fixed (g, h) contributions to the partition function (8.1.3) can be identified with ways of wrapping collections of disconnected tori around the spacetime torus. In this picture, we should view $g \in [w_1, \dots, w_\ell]$ as providing the data that, if we were to cut all tori open along a spatial slice, then the corresponding covering of cylinders would be one where the covering space consists of ℓ disconnected cylinders that are wrapped around the spacetime cylinder w_i times respectively. h on the other hand tells us how these cylinders are glued together at the spatial cycles where we performed the cut. The exponential in the grand canonical partition function comes from the fact that the full partition function is obtained from the connected part – i.e. the part corresponding to coverings with a connected covering space – by exponentiation. To be precise, the contribution

$$\frac{1}{N} \sum_{j=0}^{w-1} Z\left(\frac{Nt}{w^2} + \frac{j}{w}\right) \quad (8.A.1)$$

comes from covering the space time torus with a torus that winds w times around the spatial cycle and $\ell = N/w$ times the thermal cycle (where w necessarily has to divide N). That is, it comes from terms in the trace where $g \in [g_{(w,\ell)}] \subseteq S_N$ with

$$g_{(w,\ell)} = \prod_{k=0}^{\ell-1} (wk + 1 \ wk + 2 \ \dots \ wk + w) \quad (8.A.2)$$

and where furthermore $h \in S_\ell \ltimes (\mathbb{Z}_w)^\ell$ has a S_ℓ part π_h that is just a single cycle of length ℓ . There are $(\ell - 1)!$ such cycles π_h and hence we get that the (w, ℓ) winding covers contribute

$$Z_{(w,\ell)} = \frac{1}{N!} \sum_{g \in [g_{(w,\ell)}]} \sum_{\substack{h \in \mathcal{C}_g \\ |\pi_h| = \ell}} \text{Tr}_{\mathcal{H}^g} [h q^{L_0 - \frac{N_c \mathcal{M}}{24}}] = \frac{[g_{(w,\ell)}]}{N!} (\ell - 1)! \sum_{j_1, \dots, j_\ell = 0}^{w-1} Z\left(\frac{\ell t + \sum_{k=1}^{\ell} j_k}{w}\right). \quad (8.A.3)$$

But $||[g_{(w,\ell)}]|| = \frac{N!}{\ell!w^\ell}$ and, since the spectrum only contains integer spin states, the seed partition function is $\mathbb{Z}$ periodic. Hence,

$$Z_{(w,\ell)} = \frac{1}{\ell w} \sum_{j=0}^{w-1} Z\left(\frac{\ell t + j}{w}\right) \quad (8.A.4)$$

and summing over ℓ and w as well as exponentiating, we get the grand canonical partition function

$$\sum_{N=0}^{\infty} \kappa^N Z_N = \exp\left(\sum_{\ell,w=1}^{\infty} Z_{(w,\ell)} \kappa^{w\ell}\right), \quad (8.A.5)$$

which upon replacing ℓ by N/w is the same as eq. (8.1.4).

8.B Interface changing operator partition function: detailed computation

In this appendix, we provide a more detailed derivation of eq. (8.1.68). The starting point is eq. (8.1.69), which written as a sum over conjugacy classes instead of individual group elements, takes the form

$$\langle N | \hat{x}^{L_0} | p, a \rangle = \sum_{[\tau] \in [S_p]} \sum_{[\rho] \in [S_{N-p}]} \sum_{\sigma \in \mathcal{C}_{\tau\rho}^N} \sum_{j_b, i_d} \frac{1}{|\mathcal{C}_{\rho}^{N-p}| |\mathcal{C}_{\tau}^p|} \frac{N, N}{\rho\tau, \tau\rho} \langle j_b, i_d | \hat{x}^{L_0} \sigma | a \rangle_{\rho} |\mathbb{I}\rangle_{\tau, \tau} | a \rangle_{\rho}. \quad (8.B.1)$$

We can describe equivalence classes $[\tau]$ and $[\rho]$ with sequences²² $(t_w)_{w \in \mathbb{N}}$ and $(r_w)_{w \in \mathbb{N}}$ such that t_w gives the number of cycles of length w in $[\tau]$. In this context, let

$$|t| := \sum_{w=1}^{\infty} t_w w. \quad (8.B.2)$$

Using this notation,

$$\mathcal{Z}_a[\mu_{\pm}, 0, \rho_R; \hat{t}] = \sum_{t, r} \mu^{|t+r|} \rho_R^{2|r|} \sum_{\sigma \in \mathcal{C}_{r+t}} \frac{1}{|\mathcal{C}_t| |\mathcal{C}_r|} \sum_{j_b, i_d} \frac{N, N}{\rho\tau, \tau\rho} \langle j_b, i_d | \hat{x}^{L_0} \sigma | a \rangle_{\rho} |\mathbb{I}\rangle_{\tau, \tau} | a \rangle_{\rho}, \quad (8.B.3)$$

where $\mu := \mu_- \mu_+$ and

$$\mathcal{C}_u := \prod_{w=1}^{\infty} (S_{u_w} \ltimes \mathbb{Z}_{u_w}^{u_w}), \quad (8.B.4)$$

²²Since the modular parameter relevant for this section will always be $\hat{t}$ and not the S transformed modular parameter t , we hope that the reader will not be offended by our choice to instead use t in this appendix to refer to the sequence associated to τ .

while ρ and τ are arbitrary representatives of the conjugacy class determined by r and t . Now the contribution of fixed t and r factorises into a product of components that correspond to different cycle lengths according to

$$\mathcal{Z}_a = \sum_{t,r} \prod_{w=1}^{\infty} \left(\mu^{w(t_w+r_w)} \rho_R^{2wr_w} \sum_{\sigma_w \in S_{r_w+t_w} \ltimes \mathbb{Z}_w^{r_w+t_w}} \frac{\prod_{s \in S} Z_c \left(\frac{\ell_s^c \hat{t} + \sum_{i=1}^{\ell_s^c} z_i}{w} \right) \prod_{j=1}^{r_w} \hat{Z}_o \left(\frac{2\ell_j^o \hat{t}}{w} \right)}{t_w! w^{t_w} r_w!^2 w^{2r_w}} \right). \quad (8.B.5)$$

Here, for $\sigma_w = (\pi, (z_1, \dots, z_{r_w+t_w}))$, define ℓ_j^o with $j = 1, \dots, r_w$ as the smallest integer $k > 0$ such that $\pi^k j \leq r_w$. Furthermore, let $\pi = \pi_1 \dots \pi_L$ be the decomposition of π into cycles. We denote the elements of the cycle π_s by $(\pi_s^1 \dots \pi_s^{\ell_s^c})$ and let $S \subseteq \{1, \dots, L\}$ be the set of those indices s for which $\{\pi_s^1, \dots, \pi_s^{\ell_s^c}\}$ has an empty overlap with $\{1, \dots, r_w\}$. Note that

$$\sum_{s \in S} \ell_s^c + \sum_{j=1}^{r_w} \ell_j^o = r_w + t_w. \quad (8.B.6)$$

To simplify further, we can now perform the summation over $\mathbb{Z}_w^{r_w+t_w}$. This gives

$$\mathcal{Z}_a = \sum_{t,r} \prod_{w=1}^{\infty} \left(\mu^{w(t_w+r_w)} \rho_R^{2wr_w} \sum_{\sigma_w \in S_{r_w+t_w}} \frac{w^{t_w-|S|} \prod_{s \in S} Z_c \left(\frac{\ell_s^c \hat{t} + \sum_{i=0}^{w-1} i}{w} \right) w^{r_w} \prod_{j=1}^{r_w} \left(\hat{Z}_o \left(\frac{2\ell_j^o \hat{t}}{w} \right) \right)}{t_w! w^{t_w} r_w!^2 w^{2r_w}} \right). \quad (8.B.7)$$

By the use of eq. (8.B.6), this can be simplified further to

$$\mathcal{Z}_a = \sum_{t,r} \prod_{w=1}^{\infty} \left(\sum_{\sigma_w \in S_{r_w+t_w}} \frac{\prod_{s \in S} \frac{\mu^{w\ell_s^c}}{w} Z_c \left(\frac{\ell_s^c \hat{t} + \sum_{i=0}^{w-1} i}{w} \right) \prod_{j=1}^{r_w} \frac{\mu^{w\ell_j^o} \rho_R^{2w}}{w} \hat{Z}_o \left(\frac{2\ell_j^o \hat{t}}{w} \right)}{t_w! r_w!^2} \right). \quad (8.B.8)$$

Now we perform the sum over $S_{r_w+t_w}$. As a first step towards this goal, we should replace the labels ℓ_j^o and ℓ_s^c of individual cycle lengths by labels L^o and L^c that just count how many cycles of certain length are present. That is

$$L_\ell^o := |\{j \in \{1, \dots, r_w\} : \ell_j^o = \ell\}|, \quad L_\ell^c := |\{s \in S : \ell_s^c = \ell\}|. \quad (8.B.9)$$

Just as we notationally suppressed the dependence of ℓ^o, ℓ^c on σ , we will suppress the dependence of L^o, L^c on σ . In terms of the new labels, the sum becomes

$$\mathcal{Z}_a = \sum_{t,r} \prod_{w=1}^{\infty} \left(\sum_{\sigma_w \in S_{r_w+t_w}} \frac{\prod_{\ell=1}^{\infty} \left(\frac{\mu^{w\ell}}{w} Z_c \left(\frac{\ell \hat{t} + \sum_{i=0}^{w-1} i}{w} \right) \right)^{L_\ell^c} \prod_{\ell=1}^{\infty} \left(\frac{\mu^{w\ell} \rho_R^{2w}}{w} \hat{Z}_o \left(\frac{2\ell \hat{t}}{w} \right) \right)^{L_\ell^o}}{t_w! r_w!^2} \right). \quad (8.B.10)$$

Now, since the terms which we sum only depend on the L^o and L^c labels, we can introduce an equivalence relation $\sim$ on $S_{r_w+t_w}$ that identifies permutations with equal L^o, L^c labels, trading the sum over σ_w with a sum over equivalence classes $[L^o L^c]$. To correctly compute the sum, we have to determine the size of $[L^o L^c]$. Let us first take a step back and try to count permutations with fixed ℓ^o . They need to satisfy $\sigma_w^k(1) > r_w$ for all $0 < k < \ell_1^o$ and $\sigma_w^{\ell_1^o}(1) \leq r_w$. This leads to $\frac{t_w!}{(t_w - \ell_1^o + 1)!} r_w$ choices for $\{\sigma_w^k(1)\}_{k=1}^{\ell_1^o}$. Continuing a counting like this for all $\{\sigma_w^k(i)\}_{k=1}^{\ell_i^o}$ with $1 \leq i \leq r$ gives a factor of

$$r_w! \frac{t_w!}{\left(t_w - \sum_{i=1}^{r_w} (\ell_i^o - 1)\right)!} = r_w! \frac{t_w!}{\left(t_w - \sum_{\ell=0}^{\infty} L_\ell^o (\ell - 1)\right)!} \quad (8.B.11)$$

contributing to the size of $[L^o L^c]$. Additionally, we need to consider how many ways there are to realise the label L^o with labels ℓ^o . This leads to an extra factor of $\frac{r_w!}{\prod_{\ell} L_\ell^o!}$.

Hence, the choice of L^o contributes a factor of

$$\frac{r_w!^2}{\prod_{\ell} L_\ell^o!} \frac{t_w!}{\left(t_w - \sum_{\ell=0}^{\infty} L_\ell^o (\ell - 1)\right)!} \quad (8.B.12)$$

to the size of $[L^o L^c]$. Finally, we need to multiply by the size of the conjugacy class labelled by L^c to obtain

$$|[L^o L^c]| = \frac{r_w!^2}{\prod_{\ell=1}^{\infty} L_\ell^o!} \frac{t_w!}{\left(t_w - \sum_{\ell=1}^{\infty} L_\ell^o (\ell - 1)\right)!} \frac{\left(\sum_{\ell=1}^{\infty} \ell L_\ell^c\right)!}{\prod_{\ell=1}^{\infty} L_\ell^c! \ell^{L_\ell^c}}. \quad (8.B.13)$$

Now

$$\sum_{\ell=1}^{\infty} \ell (L_\ell^c + L_\ell^o) = r_w + t_w \quad \text{and} \quad \sum_{\ell=1}^{\infty} L_\ell^o = r_w \quad (8.B.14)$$

implies

$$t_w - \sum_{\ell=1}^{\infty} L_\ell^o (\ell - 1) = \sum_{\ell=1}^{\infty} \ell L_\ell^c \quad (8.B.15)$$

and therefore

$$|[L^o L^c]| = \frac{r_w! 2t_w!}{\prod_{\ell=1}^{\infty} L_{\ell}^o! L_{\ell}^c! \ell^{L_{\ell}^c}}. \quad (8.B.16)$$

This implies

$$\mathcal{Z}_a = \sum_{t,r} \prod_{w=1}^{\infty} \left(\sum_{[L^o L^c]} \prod_{\ell=1}^{\infty} \frac{\left(\frac{\mu^{w\ell}}{w\ell} Z_c \left(\frac{\ell\hat{t} + \sum_{i=0}^{w-1} i}{w} \right) \right)^{L_{\ell}^c}}{L_{\ell}^c!} \prod_{\ell=1}^{\infty} \frac{\left(\frac{\mu^{w\ell} \rho_R^{2w}}{w} \hat{Z}_o \left(\frac{2\ell\hat{t}}{w} \right) \right)^{L_{\ell}^o}}{L_{\ell}^o!} \right), \quad (8.B.17)$$

where the $[L^o L^c]$ sum runs over the set of equivalence classes $S_{r_w+t_w}/\sim$. Finally, we trade the unconstrained sum over the sequences t and r , which in turn constrains L^o and L^c to correspond to equivalence classes in $S_{r_w+t_w}$ for an unconstrained sum over L^o and L^c without t, r sum. This is achieved by reinterpreting the constraints (8.B.14) as fixing t_w and r_w in terms of L^o and L^c , leading to

$$\mathcal{Z}_a = \prod_{w,\ell=1}^{\infty} \left(\sum_{L_{\ell}^c=0}^{\infty} \left(\frac{\left(\frac{\mu^{w\ell}}{w\ell} Z_c \left(\frac{\ell\hat{t} + \sum_{i=0}^{w-1} i}{w} \right) \right)^{L_{\ell}^c}}{L_{\ell}^c!} \right) \sum_{L_{\ell}^o=0}^{\infty} \left(\frac{\left(\frac{\mu^{w\ell} \rho_R^{2w}}{w} \hat{Z}_o \left(\frac{2\ell\hat{t}}{w} \right) \right)^{L_{\ell}^o}}{L_{\ell}^o!} \right) \right). \quad (8.B.18)$$

The sums now simply give exponential functions

$$\mathcal{Z}_a[\mu_{\pm}, 0, \rho_R; \hat{t}] = \exp \left(\sum_{w,\ell=1}^{\infty} \left(\frac{\mu^{w\ell}}{w\ell} Z_c \left(\frac{\ell\hat{t} + \sum_{i=0}^{w-1} i}{w} \right) \right) + \sum_{w,\ell=1}^{\infty} \left(\frac{\mu^{w\ell} \rho_R^{2w}}{w} \hat{Z}_o \left(\frac{2\ell\hat{t}}{w} \right) \right) \right). \quad (8.B.19)$$

Replacing the summation variable ℓ by $k = w\ell$ finally gives us the result (8.1.68) that we wanted to prove.

Chapter 9

Interface correlators in symmetric product orbifolds

– Covering maps and general correlators.

In this chapter, we continue our analysis of the interfaces introduced in Chapter 8 by providing the necessary tools to efficiently compute their local correlation functions, especially with the aim to establish universal holographic features.

As convincing as the agreement of partition functions that has been established in Section 8.3 may seem, a complete derivation of the holographic correspondence does require to also match CFT correlation functions of bulk and interface operator with scattering amplitudes of tensionless closed and open strings on AdS_2 -branes in AdS_3 . In order to approach such a comparison, we need to be able to compute the relevant CFT correlators. This is the problem we address in the current chapter. In order to achieve this goal, we extend the Lunin-Mathur covering space techniques to interface CFT. This involves two steps. In a first step, the correlation functions in the interface symmetric product orbifold CFT are represented as a sum over an appropriate set of interface branched coverings Γ , see Subsection 9.1.3 and eq. (9.1.33). The summands involve correlation functions in the seed CFT $\mathcal{M}$ on surfaces with boundaries. These are weighted with some coefficients $\alpha[\Gamma]$ that do not depend on the seed CFT. Following the ideas of Lunin-Mathur [310], the correlation functions in the seed CFT can then be calculated with the help of (boundary) Liouville field theory. For pure twist field insertions, the dependence on the seed CFT is restricted to the bulk and boundary entropies, i.e. the central charge c and the g -factors $g_{\pm}$ of the two boundary conditions $a_{\pm}$, see for example eq. (9.3.23).

In order to construct the sum over coverings Γ and compute the weight factors $\alpha[\Gamma]$, we develop a diagrammatic approach that extends the Feynman-like diagrams that were introduced for bulk symmetric product orbifolds by Pakman, Rastelli and Razamat in [284]. In our case, the diagrams are built from five different types of

propagator-lines. Three of them are represented by double lines. We shall refer to them as *aligned* and *misaligned*_±. In addition there are two single lines of opposite orientation. Bulk twist fields of order w are represented by vertices with w misaligned double lines, see Figure 9.2.2a. Interface twist field of order ω , on the other hand, are represented by vertices with two single lines and $\omega - 1$ aligned double lines, see Subfigure a, b, c and d of Figure 9.2.6. In addition to these vertices that represent field insertions there are also two classes of trivalent vertices that are not associated with any field insertions. These additional trivalent vertices allow to split a misaligned double line into either a pair of aligned double lines, see Figure 9.2.2b, or into a pair of single lines, see Subfigure e and f of Figure 9.2.6. In order to describe a valid interface covering map Γ , a diagram built out of these basic ingredients needs to satisfy some simple rules, see Section 9.2.2. Given a valid diagram it is then easy to read off the degrees $N_{\pm}$ of the associated covering map, the degree of transmissivity P of the interface and the symmetry factor $\alpha[\Gamma]$.

If the diagram has no single lines, it actually describes an ordinary bulk covering map. In the corresponding diagrams, it is possible to eliminate all the misaligned double lines by splitting them into pairs of aligned ones. Hence, these diagrams can be drawn with a single type of double line which agrees with the pictorial descriptions proposed by Pakman-Rastelli-Razamat (PRR). After we split the misaligned lines, bulk twist field of order w are represented by a vertex with $2w$ aligned double lines rather than w misaligned ones, see Figure 9.2.3b. Thereby, we recover the graphical rules for PRR diagrams. Let us note that our representation of the aligned double lines looks a bit different from the one in the work of Pakman et al. who used double lines with one dashed and one solid edge.

Another central result that we obtain from our new understanding of interface covering maps concerns the structure of the large $N_{\pm}$ and large P expansion that is shown to match the string genus expansion for scattering amplitudes of closed and open strings to all orders in the string coupling. In the bulk case it has been well understood that the comparison of the two expansions requires passing to a grand-canonical ensemble in which the parameter N is controlled by a dual fugacity μ . The latter scales with the string coupling g_s as $\mu \sim g_s^{2G-2}$ where G denotes the genus of the holographic boundary X . In the grand-canonical ensemble, N is not a number but one can check that the expectation value of N scales as $N \sim g_s^{-2}$. We extend these insights to the interface setup. In particular, we introduce a new grand-canonical ensemble in which $N_{\pm}$ and the degree of transmissivity P are controlled by dual fugacities, see eq. (9.4.21). With some appropriate scaling behaviour of these three fugacities, see eq. (9.4.22), the expectation values of $N_{\pm}$ and P scale as g_s^{-2} while the expectation value of $N_{\pm} - P$ turns out to scale as g_s^{-1} .

The plan for this chapter is as follows. The next section is devoted to covering maps

and their role in computing correlation functions in symmetric product orbifolds. We begin by reviewing the well known notion of branched coverings in Section 9.1.1 before we recall, in Section 9.1.2, how correlation functions on symmetric product orbifolds on a closed surface X can be expressed as a sum over branched coverings of X . The individual summands involve correlation functions of the seed theory on the covering surface Σ . We then extend this discussion to the case of interfaces. In Subsection 9.1.3, we introduce the relevant notion of generalised covering maps before we spell out how to compute correlation functions involving interfaces as a sum over these covering maps, see Subsection 9.1.4 and in particular eq. (9.1.33). We evaluate the right hand side of this equation for some special setup in which we just insert two parallel interfaces along some cycle of the torus, without any further local insertions of either bulk or interface fields. The resulting formula coincides with the interface partition functions we calculated in [4] from an algebraic construction that involves the folding trick and explicit boundary states. This justifies a posteriori the rather natural notion of generalised covering maps we propose in Subsection 9.1.3.

Section 9.2 is devoted to the diagrammatic representation of interface covering maps that we summarised above. After a short review of the PRR diagrams for correlation functions in symmetric product orbifolds in Section 9.2.1, we describe the corresponding interface diagrams in Section 9.2.2. This is followed by a brief discussion of the symmetries of interface PRR diagrams in Sections 9.2.3 and 9.2.4. These symmetries allow to read off the group of deck transformations of the associated branched covering. Finally, we determine the large N behaviour of the expansion in Section 9.2.5.

With all this control over the interface PRR expansion of correlation functions in symmetric product orbifolds with holographic interface insertions, it remains to evaluate its individual diagrams, i.e. the relevant correlation functions of the seed theory on the covering surfaces Σ . This is addressed in Section 9.3 where we extend the ideas proposed by Lunin and Mathur for symmetric product orbifolds on a closed surface to the interface setup. Once again, we start with a short review of the bulk analysis before we discuss its extension to interfaces and several concrete examples. Section 9.4 is about grand-canonical partition functions. After a brief review of the bulk case in Section 9.4.2, we introduce grand-canonical interface partition functions in Section 9.4.3 and show that these indeed have the same structure as partition functions of closed and open strings with the scalings we described above. Some more technical explanations and additional material are collected in five appendices.

9.1 Interfaces in symmetric product orbifolds

In this section, we discuss a family of universal interfaces of symmetric product orbifolds that were introduced in Reference [4]. Using the folding trick, Section 2 of that work gave a fully explicit construction of these interfaces in terms of boundary states, from which in particular a closed formula for the grand-canonical torus partition function with two interface insertions was computed. Some very special cases of the interfaces had been considered previously, for instance in References [288, 289].

However, all these previous discussions, including Reference [4], took a fairly algebraic perspective on symmetric product orbifolds. In this section, we would like to complement the existing literature on the topic by an analytic perspective that focuses on an interpretation of objects in symmetric product orbifolds in terms of branched covering maps, i.e. we represent interface correlation functions in symmetric product orbifolds through a sum over appropriately defined “interface covering maps”.

In order to set the stage, we first review the corresponding description for correlation functions of local fields in symmetric product orbifolds, see Section 9.1.2. To make this chapter self-contained, the relevant mathematical notions concerning branched coverings of Riemann surfaces are collected in Section 9.1.1. Analogously, our discussion of interface correlators starts by introducing a class of generalised covering maps in Section 9.1.3 and proving that they satisfy a modified Riemann-Hurwitz formula. Equipped with this mathematical background, the final Section 9.1.4 provides a new description of the interfaces whose correlation functions are explored in the rest of this chapter. As a first application of the definition in terms of a sum over generalised covering maps, we re-calculate the interface partition functions that were first computed in [4]. Crucially, the new calculation turns out to be considerably easier than the original one, which allows us to study much more general partition functions and correlators in this chapter, than those accessible with the methods of the previous one.

9.1.1 Branched coverings

To prepare for our discussion of correlations of local fields in symmetric product orbifolds, we start with a lightning review of branched coverings. The definition of unramified and branched coverings is stated in the next two paragraphs. This is followed by three paragraphs in which we introduce the notation of monodromy and deck transformations of branched coverings. The final paragraph reviews a classical result of the theory of covering maps: The Riemann-Hurwitz formula.

Unramified coverings. Throughout this section, let Σ and X be Riemann surfaces. Unless stated otherwise, neither Σ nor X are assumed to be closed. Furthermore, let $\Gamma : \Sigma \rightarrow X$ be a holomorphic surjection. If Γ is locally invertible and if every point $x \in X$ has $|\Gamma^{-1}(x)| = N$ preimages, then Γ is called a degree N *unramified covering* of X by Σ .

Branched coverings. Let B be a finite subset of X and $R := \Gamma^{-1}(B)$. If R is a finite set and the restriction of Γ to $\Sigma \setminus R$ is a holomorphic covering of $X \setminus B$ then Γ is a *branched covering* of X . In this case, we refer to elements of B and R as *branch points* and *ramification points* respectively. At each ramification point $z \in \Sigma$, the covering map Γ locally takes the form

$$\Gamma(z + \delta z) = \Gamma(z) + O(\delta z^w). \quad (9.1.1)$$

The integer w is the *ramification index* of Γ in z . For $z \notin R$, the index is one.

Monodromy. For a connected surface X , consider a closed curve $\gamma : [0, 1] \rightarrow X \setminus B$ based in the point $x \in X$, i.e. a curve satisfying $\gamma(0) = \gamma(1) = x$ that does not pass through any of the branch points of the covering map Γ . For each element z of the set $\Gamma^{-1}(x)$ there is a unique lift $\Gamma^*\gamma_z$ of γ to Σ such that

$$\Gamma \circ \Gamma^*\gamma_z = \gamma \quad \text{and} \quad \Gamma^*\gamma_z(0) = z. \quad (9.1.2)$$

The map $z \mapsto \Gamma^*\gamma_z(1)$ is a bijection of the set $\Gamma^{-1}(x)$ onto itself i.e. an element of $\text{Aut}[\Gamma^{-1}(x)]$. It only depends on the homotopy class $[\gamma]$ of the curve γ and therefore defines a group homomorphism

$$M_\Gamma : \pi_1(X, x) \rightarrow \text{Aut}[\Gamma^{-1}(x)] \quad (9.1.3)$$

which sends γ to an automorphism α_γ of the set $\Gamma^{-1}(x) \subseteq \Sigma$. The map M_Γ is the *monodromy action* of the covering map Γ . Its image is the *monodromy group* of Γ .

Deck transformations. A *Deck transformation* of the covering Γ is an automorphism D of Σ with the property $\Gamma \circ D = \Gamma$. If Σ is connected, then the image $D(z)$ of an arbitrary unramified point $z \in \Sigma$ already fixes D completely. In this sense, the group $\text{Deck}[\Gamma]$ of deck transformations can be identified with a subgroup of $\text{Aut}[\Gamma^{-1}(x)]$ for a generic $x \in X$ in this case. Since deck transformations are precisely the maps that preserve Γ , this subgroup is the stabiliser of the monodromy group of Γ under the conjugation of $\text{Aut}[\Gamma^{-1}(x)]$.

Local gauge choice. Given a point $x \in X$ and a set C of degree N coverings of X , for which x is not a branch point, we refer to a choice of labelling the N points in the preimage $\Gamma^{-1}(x)$ of x w.r.t. every covering map $\Gamma \in C$ by the numbers $1, \dots, N$ as

a *local gauge choice*¹ for C at $x \in X$. In particular, a local gauge choice for $C = \{\Gamma\}$ is a bijection

$$\psi : \underline{N} := \{1, 2, \dots, N\} \rightarrow \Gamma^{-1}(x), \quad (9.1.4)$$

that allows to identify elements α of $\text{Aut}[\Gamma^{-1}(x)]$ with elements $\alpha^\psi := \psi^{-1} \circ \alpha \circ \psi$ of the symmetric group S_N . Thus, local gauge choices lead to an identification of the monodromy group and deck transformations with subgroups of S_N . As an application, we can for example use a local gauge choice to define a map

$$M_\Gamma^R : \pi_1(X, x) \rightarrow \mathbb{C}, \quad [\gamma] \mapsto \chi_R(M_\Gamma[\gamma]) := \chi_R(M_\Gamma^\psi[\gamma]), \quad (9.1.5)$$

given a representation R of S_N with character χ_R . Since characters are invariant under the inner automorphisms of S_N , the value of $\chi_R(M_\Gamma[\gamma])$ is independent of the choice of ψ .

Riemann-Hurwitz formula. In this paragraph, we assume that Σ and X are closed. If Γ is unramified, sufficiently fine triangulations of X lift along Γ to triangulations of Σ with $N = \deg[\Gamma]$ times as many vertices, edges and faces. Thus, the Euler characteristics of Σ and X are related by $\chi(\Sigma) = N\chi(X)$. More generally, consider a triangulation of X whose vertices include all branch points of Γ . Then the lifted triangulation still has the expected number of edges and faces. However, $w - 1$ vertices on Σ are lost for each ramification point with index w . Therefore, the Euler characteristic $\chi(\Sigma)$ of Σ is given by

$$\chi(\Sigma) = N\chi(X) - \sum_{i=1}^n (w_i - 1), \quad (9.1.6)$$

where $w_1, \dots, w_n$ are the ramification indices of Γ . This is the *Riemann-Hurwitz formula*.

9.1.2 Symmetric product orbifolds

This section introduces symmetric product orbifolds as two-dimensional CFTs defined by lifting correlation functions of a seed theory $\mathcal{M}$ on a closed Riemann surface X to various covering spaces. It is divided into several paragraphs, each devoted to one aspect of the lift. The first of these provides a definition. The definition is then applied to the computation of vacuum partition functions. These allow us to determine the spectrum of the symmetric orbifold and its local correlation functions.

¹This terminology is justified in Appendix 9.A, where symmetric product orbifolds are discussed in the language of gauge theory, and the local gauge choices we define here are shown to correspond to local trivialisations of the gauge bundle.

Covering space lift. Let us denote by $\mathcal{A}_X[\mathcal{T}]$ the set of observables $\mathcal{O}$ of a CFT $\mathcal{T}$ on the surface X , i.e. the set of all objects that can be inserted in its correlation functions. The most basic elements of $\mathcal{A}_X[\mathcal{T}]$ are products $\mathcal{O}_S = \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)$ of local operators $\mathcal{O}_i$ that are inserted at some finite set of points $x_i \in S \subseteq X$. Beyond those, $\mathcal{A}_X$ also includes non local observables such as line operators.

We define the symmetric product orbifold $\text{Sym}^N(\mathcal{M})$ of the seed theory $\mathcal{M}$ by a prescription that expresses the correlation functions of its observables in terms of correlators of $\mathcal{M}$. This is achieved through the following two steps. First, we associate a set $C_N[\mathcal{O}]$ of N -sheeted branched coverings² of X with each observable $\mathcal{O} \in \mathcal{A}_X[\text{Sym}^N(\mathcal{M})]$.

Second, we associate an observable $\mathcal{O}^\Gamma$ of the seed theory to any pair $(\mathcal{O}, \Gamma)$ of an observable $\mathcal{O}$ of the symmetric product orbifold and a covering map $\Gamma \in C_N[\mathcal{O}]$. For the multipoint observables $\mathcal{O}_S$ we described above, for example, the associated observable in the seed theory is essentially given by some $\mathcal{O}_{\Gamma^{-1}(S)}$. Given these two ingredients, i.e. the space $C_N[\mathcal{O}]$ of covering maps associated with $\mathcal{O}$ and the map

$$\mathcal{O} \mapsto \mathcal{O}^\Gamma \in \mathcal{A}_\Sigma[\mathcal{M}], \quad (9.1.7)$$

that sends $\mathcal{O} \in \mathcal{A}_X[\text{Sym}^N(\mathcal{M})]$ to the observable $\mathcal{O}^\Gamma \in \mathcal{A}_\Sigma[\mathcal{M}]$, we lift expectation values in the symmetric product orbifolds to expectation values in the seed theory as,

$$\langle \mathcal{O} \rangle_{\text{Sym}^N(\mathcal{M})}^{\text{unnormalised}} = \frac{1}{N!} \sum_{\Gamma \in C_N[\mathcal{O}]} \langle \mathcal{O}^\Gamma \rangle_{\mathcal{M}}, \quad (9.1.8)$$

where $\langle \mathcal{O}^\Gamma \rangle_{\mathcal{M}}$ is a correlator of $\mathcal{M}$ on Σ with a metric that is obtained by pulling back the metric of X along Γ . We have placed a subscript ‘unnormalised’ on this correlator since it does not necessarily yield 1 for the one-point functions of the identity operator $\mathbb{1}$. We define its normalised analogue as the quotient

$$\langle \mathcal{O} \rangle_{\text{Sym}^N(\mathcal{M})} = \frac{\langle \mathcal{O} \rangle_{\text{Sym}^N(\mathcal{M})}^{\text{unnormalised}}}{\langle \mathbb{1} \rangle_{\text{Sym}^N(\mathcal{M})}^{\text{unnormalised}}}. \quad (9.1.9)$$

The vacuum. As a first example, let us discuss the concrete realisation of the general equation (9.1.8) for the case $\mathcal{O} = \mathbb{1}$ of the identity, or vacuum operator. The set $C_N[\mathbb{1}]$ is

$$C_N[\mathbb{1}] := \{N \text{ sheeted unramified coverings of } X\}. \quad (9.1.10)$$

²We consider coverings that are related by an isomorphism that identifies their covering spaces as identical. A more precise statement would be that $C_N[\mathcal{O}]$ is a set of equivalence classes of covering maps.

Given some $\Gamma \in C_N[\mathbb{1}]$, the observable $\mathbb{1}^\Gamma$ of the seed theory on the covering surface Σ is the identity operator, up to some normalising prefactor $\alpha[\Gamma]$, i.e.

$$\mathbb{1}^\Gamma = \alpha[\Gamma] \cdot \mathbb{1} \quad \text{where} \quad \alpha[\Gamma] := \frac{\deg[\Gamma]!}{|\text{Deck}[\Gamma]|}. \quad (9.1.11)$$

By definition, the vacuum partition function Z_N^X of $\text{Sym}^N(\mathcal{M})$ on the surface X is

$$Z_N^X := \langle \mathbb{1} \rangle_{\text{Sym}^N(\mathcal{M})}^{\text{unnormalised}}. \quad (9.1.12)$$

Let us compute Z_N^X in the simplest case $X = \mathbb{S}^2$, where the set $C_N[\mathbb{1}]$ has only a single element Γ , which covers the sphere by N disconnected spheres. The group $\text{Deck}[\Gamma]$ consists of the $N!$ permutations of these spheres. Therefore, we obtain $Z_N^{\mathbb{S}^2} = 1/N!$ in a normalisation where the sphere partition function of the seed theory is set to one. A much greater amount of information is encoded in the partition function of higher genus surfaces X . In particular, the torus partition function contains the entire spectral data of local operators and the genus two partition function yields all structure constants.

Torus partition function. Let us compute the torus partition function from def. (9.1.10). Using the Riemann-Hurwitz formula, one can easily conclude that the covering space of an unramified covering of $\mathbb{T}^2$ must be a disjoint union of tori. More precisely, all connected unramified coverings $\Gamma_{w,\ell}^k$ of the torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ arise from linear maps

$$\Gamma : \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto (m\tau + w)z \quad \text{with} \quad m, w \in \mathbb{Z} \quad (9.1.13)$$

by quotienting out the lattices $\mathbb{Z} + t\mathbb{Z}$ and $\mathbb{Z} + \tau\mathbb{Z}$ of the domain and image of Γ respectively. The modular parameter t of the covering space takes the form $t = \frac{\ell\tau + k}{m\tau + w}$ with $\ell, k \in \mathbb{Z}$ and modular transformations allow us to fix $m = 0$, $w, \ell > 0$ and $1 \leq k \leq w$. As anticipated, the coverings are thus labelled by three integers. Since $\text{Deg}[\Gamma_{w,\ell}^k] = w\ell$, we find

$$Z_N^{\mathbb{T}^2} = \frac{1}{N!} \sum_{\Gamma \in C_N[\mathbb{1}]} \alpha[\Gamma] \langle \mathbb{1} \rangle_{\mathcal{M}} = \sum_{\sum w_i \ell_i n_i = N} \sum_{k_i=1}^{w_i} \prod_i \frac{1}{n_i!} \left(\frac{Z(\frac{\ell_i \tau + k_i}{w_i})}{|\text{Deck}[\Gamma_{w_i, \ell_i}^{k_i}]|} \right)^{n_i}. \quad (9.1.14)$$

$\text{Deck}[\Gamma_{w,\ell}^k]$ is the group of translations $z \mapsto z + \frac{a\tau + b}{w}$ with integer a and b , which has size $|\text{Deck}[\Gamma_{w,\ell}^k]| = w\ell$. In the grand-canonical ensemble, the constraint $\sum w_i \ell_i n_i = N$ is lifted and we obtain the simple formula

$$\sum_{N=0}^{\infty} Z_N^{\mathbb{T}^2} \mu^N = \exp \left(\sum_{w,\ell=0}^{\infty} \sum_{k=1}^w \frac{\mu^{w\ell}}{w\ell} Z\left(\frac{\ell\tau + k}{w}\right) \right), \quad (9.1.15)$$

from which the spectrum of the orbifold can be read off. This completes our short computation of the torus partition function in symmetric product orbifolds.

Correlation functions of twist fields. A universal (i.e. seed theory independent) feature of all symmetric product orbifolds is the existence of twist fields – local operators $\sigma_{\mathbf{w}}$ labelled by integer partitions $\mathbf{w} = [w_1, \dots, w_\ell]$ whose conformal weights are

$$h_{\mathbf{w}} = \bar{h}_{\mathbf{w}} = \frac{c_{\mathcal{M}}}{24} \sum_{i=1}^{\ell} \left(w_i - \frac{1}{w_i} \right). \quad (9.1.16)$$

The state corresponding to $\sigma_{\mathbf{w}}$ contributes to the torus partition function (9.1.14) through a particular set of covering maps. These are the maps that, upon cutting the torus along the real cycle, yield coverings of a cylinder X by ℓ disconnected cylinders Σ_i , such that Σ_i winds X exactly w_i times. Therefore,

$$C \left[\prod_{i=1}^n \sigma_{\mathbf{w}_i}(x_i) \right] = \left\{ \begin{array}{l} N \text{ sheeted coverings of } X \text{ with branchpts. at } x_i \text{ lifting to } \\ \ell_i \text{ ramification pts. with ramification indices } w_1, \dots, w_{\ell_i}. \end{array} \right\}. \quad (9.1.17)$$

Observables made up of pure twist fields $\sigma_{\mathbf{w}}$ in the symmetric product orbifolds are mapped to identity fields in the seed theory. More precisely, the prescription is

$$\left[\prod_{i=1}^n \sigma_{\mathbf{w}_i}(z_i) \right]^{\Gamma} = \alpha[\Gamma] \mathbb{1} \quad \text{where} \quad \alpha[\Gamma] = \frac{\deg[\Gamma]!}{|\text{Deck}[\Gamma]|}, \quad (9.1.18)$$

as for the identity field. Partitions $\mathbf{w} = [w, 1, 1, \dots, 1]$ with only a single $w > 1$ play a special role. They are referred to as single cycle twist fields and we denote them simply as σ_w , labelled by the single integer w .

Connected sphere correlators. As an important example, let us discuss correlators of twist fields on the sphere $X = \mathbb{S}^2$ in more detail. Concretely, let us focus on contributions to these correlators that stem from covering maps

$$\Gamma : \Sigma \sqcup [\mathbb{S}^2]^{\sqcup(N-n_c)} \rightarrow X, \quad (9.1.19)$$

where Σ is *connected*, $\Gamma|_{\Sigma}$ has degree n_c and Γ is the identity map when restricted to the sphere factors. We denote this subset of covering maps by $C_{n_c}^{\text{conn}}[\mathcal{O}]$. The resulting partial sum of expectation values in the seed theory is referred to as the *connected part* of the correlator. For general observables $\mathcal{O}$, the connected expectation value is defined as

$$\langle \mathcal{O} \rangle_{\text{Sym}^N(\mathcal{M})}^{\text{conn, unnormalised}} = \frac{1}{N!} \sum_{n_c=1}^N \sum_{\Gamma \in C_{n_c}^{\text{conn}}[\mathcal{O}]} \langle \mathcal{O}^{\Gamma} \rangle_{\mathcal{M}}. \quad (9.1.20)$$

In order to compute a connected correlator

$$\langle \sigma_{w_1}(x_1) \dots \sigma_{w_n}(x_n) \rangle_{\text{Sym}^N(\mathcal{M})}^{\text{conn, unnormalised}} \quad (9.1.21)$$

of a product of single cycle twist fields, one has to determine the factor $\alpha[\Gamma]$ of connected covering maps. In order to do so, pick some unramified point $z \in \Sigma$ and choose n closed curves γ_i based in z such that γ_i winds around the ramification point z_i associated with x_i exactly once and has vanishing winding number with respect to all the other ramification points. Furthermore, let $x := \Gamma(z)$. The monodromy action M_Γ then associates a cyclic permutation $g_i := M_\Gamma[\gamma_i] \in \text{Aut}[\Gamma^{-1}(x)]$ of order $|g_i| = w_i$ to each of the twist fields.

From the collection $g_1, \dots, g_n$ of permutations, we can calculate the symmetry factor $\alpha[\Gamma]$ that appears in the prescription (9.1.18) as

$$\alpha[\Gamma] = |[g_1, \dots, g_n]| \quad \text{with} \quad [g_1, \dots, g_n] = \{(gg_1g^{-1}, \dots, gg_ng^{-1}) | g \in \text{Aut}[\Gamma^{-1}(x)]\} \quad (9.1.22)$$

because $\text{Deck}[\Gamma]$ is precisely the stabiliser of $(g_1, \dots, g_n)$ under conjugation. Let us stress that this formula for $\alpha[\Gamma]$ only holds for connected covering maps Γ .

Wilson loops. Let us now introduce a first family of line operators, namely the *Wilson loop* $\mathcal{W}_R[\gamma] \in \mathcal{A}_X[\text{Sym}^N(\mathcal{M})]$. Here, γ is a closed curve on X that starts and ends at some point $x \in X$ and R labels a representation of the symmetric group S_N . Inserting a Wilson loop $\mathcal{W}_R[\gamma]$ into a correlator does not alter the set $C_N[\mathcal{O}]$ of coverings we sum over, i.e.

$$C_N[\mathcal{O}\mathcal{W}_R[\gamma]] = C_N[\mathcal{O}]. \quad (9.1.23)$$

What changes is only the sum on the right hand side of our defining relation (9.1.8). After insertion of the Wilson line $\mathcal{W}_R[\gamma]$ the formulas reads

$$\langle \mathcal{O}\mathcal{W}_R[\gamma] \rangle_{\text{Sym}^N(\mathcal{M})}^{\text{unnormalised}} = \frac{1}{N!} \sum_{\Gamma \in C_N[\mathcal{O}]} M_\Gamma^R[\gamma] \langle \mathcal{O}^\Gamma \rangle_{\mathcal{M}}. \quad (9.1.24)$$

Here, we have used the map M_Γ^R defined in eq. (9.1.5) as the evaluation of the character of R on the monodromy action of Γ . The only effect of a the Wilson loop insertion into the expectation value of some observable $\mathcal{O}$ of the N -fold symmetric product orbifold is therefore to multiply the individual summands in (9.1.8) by some complex numbers. For a discussion of open Wilson loops i.e. Wilson lines as well as dual 't Hooft lines, which change the structure of covering maps contributing to correlation functions, see App. 9.B.

9.1.3 Interface covering maps

In this section, we generalise the notion of (un)ramified coverings to a broader class of maps, which we call *interface coverings*. As the name suggests, these maps compute correlators in the presence of the interfaces, which we introduce in Sec. 9.1.4. An important result of this section is the generalised Riemann-Hurwitz formula (9.1.28) satisfied by interface coverings.

Unramified interface coverings. Let us consider a connected Riemann surface X without boundary and a based loop $\gamma : [0, 1] \mapsto X$ with $\gamma(0) = x = \gamma(1)$ therein. We denote the image $\gamma([0, 1])$ by s_γ and assume that the complement $X \setminus s_\gamma$ of s_γ in X splits into two connected components $X_\pm^o$. Let us denote their closure in X by $X_\pm = X_\pm^o \cup s_\gamma$. By construction the two sets X_+ and X_- possess the same boundary $\partial X_\pm = s_\gamma$.

An *interface covering map* (for an interface localised along the curve γ) is a surjective holomorphic map $\Gamma : \Sigma \rightarrow X$ whose restrictions $\Gamma_\pm$ to the subspaces $\Sigma_\pm^o := \Gamma^{-1}(X_\pm^o)$ are covering maps in the sense of Sec. 9.1.1. The degrees $N_\pm$ of these covering maps may differ, i.e. the size N_+ of the set $\Gamma^{-1}(x)$ for an element x of X_+^o may not be the same as the number N_- of points in $\Gamma^{-1}(x)$ for $x \in X_-^o$.³ We define the *degree of transmissivity* P of Γ on γ as

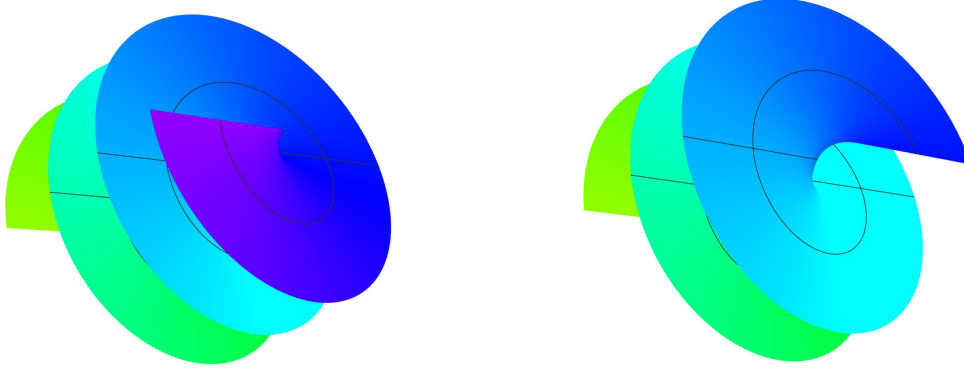
$$P := |\Gamma^{-1}(\gamma(t)) \cap \Sigma^o| \quad (9.1.25)$$

for any $t \in [0, 1]$. Let us stress that the boundary $\partial\Sigma$ of the covering space is allowed to be non-empty. Since $\Gamma(\partial\Sigma) \subseteq s_\gamma$, our definition of the degree of transmissivity implies that $\partial\Sigma$ has at most $N_+ + N_- - 2P$ connected components. Let us also note that $P \leq \min(N_-, N_+)$. While we, for concreteness and ease of notation, restricted to the specific case of a single curve γ splitting X into two connected components here, the definition given above trivially generalises to an arbitrary number of curves splitting X into arbitrarily many components.

Ramification of interface coverings. Let B be a finite subset of X and $R := \Gamma^{-1}(B)$. If R is a finite set and the restriction of $\Gamma : \Sigma \rightarrow X$ to $\Sigma \setminus R$ is an unramified interface covering of $X \setminus B$ then Γ is a *branched interface covering* of X . We refer to elements of B and R as branch points and ramification points respectively. Branch points that lie on the curve γ are called *interface branch points* below. The other branch points are referred to as *bulk branch points*. The ramification points associated with bulk branch points are simply ramification points of the restrictions $\Gamma_\pm$ of Γ to $X_\pm^o$ in the sense of Section 9.1.1.

Ramification points of an interface covering map fall into two different classes: They can either be in the interior of Σ , in which case we speak of *interior ramification points*, or on the boundary $\partial\Sigma$, in which case we speak of *boundary ramification points*. The local behaviour of the covering map near an interior ramification point is captured by eq. (9.1.1). All ramification points associated with bulk branch points are interior ramification points. However, some interior ramification points may map to interface branch points. We will occasionally find it convenient to associate elements of this subset of interior ramification points with an *interface ramification*

³Here it is important that $X \setminus s_\gamma$ is not connected. If it was, the degree of Γ could not jump across γ .



(a) Type 6^{LL} ramification point of a $N_{\pm} = 3$ covering. The degree of transmissivity jumps $N_{+} = 3$ covering. The degree of transmissivity from 2 to 3 at the ramification point. (b) Type 5^{RL} ramification point of a $N_{-} = 2$ covering. The degree of transmissivity is constantly 2.

Figure 9.1.1: Visualisation of coverings in a neighbourhood of interface ramification points. The covering map is the vertical projection down the symmetry axis of the spiral. The preimage of the interface is marked by horizontal black lines on the covering space.

index $\omega := 2w$, where w is the index defined through eq. (9.1.1). If all ramification points of an interface branch point are interior ramification points, then the degree of transmissivity P of the interface covering map is the same on both sides of that branch point and bounds the value of the index as $\omega \leq 2P$.

Let us now turn to boundary ramification points, of which we distinguish four different types. To define these types, holomorphically identify a small neighbourhood of the ramification point z with the upper half complex plane, such that z is identified with 0. Let c be the curve in Σ that, through the identification, corresponds to the curve in the upper half plane which maps $\varphi \in [0, \pi]$ to $\varepsilon e^{i\varphi}$. If Γ is unramified in z , then c lies completely within Σ_{-} or Σ_{+} and we say that z has ramification index 1^{LR} or 1^{RL} respectively. If however Γ is ramified in z , there are $\omega - 1 \geq 1$ transitions from $\Sigma_{\pm}$ to $\Sigma_{\mp}$ along c . In this case, we say Γ has *interface ramification index* ω in z . We furthermore classify z as

$$\text{type } \omega^{LL} \text{ if } c(0) \in \Sigma_{-}, c(\pi) \in \Sigma_{+}, \quad \text{type } \omega^{RR} \text{ if } c(0) \in \Sigma_{+}, c(\pi) \in \Sigma_{-}, \quad (9.1.26)$$

$$\text{type } \omega^{LR} \text{ if } c(0), c(\pi) \in \Sigma_{-}, \quad \text{type } \omega^{RL} \text{ if } c(0), c(\pi) \in \Sigma_{+}. \quad (9.1.27)$$

See Figure 9.1.1 for examples. Note that type ω^{LL} and ω^{RR} ramification points force the degree of transmissivity to jump by one unit.

Interface Riemann-Hurwitz formula. In this final paragraph, we prove that interface coverings of closed Riemann surfaces Σ satisfy the modified Riemann-Hurwitz

formula

$$\chi(\Sigma) = N_- \chi(X_-) + N_+ \chi(X_+) - \sum_{i=1}^{n_c} (w_i - 1) - \frac{1}{2} \sum_{i=1}^{n_o} (\omega_i - 1), \quad (9.1.28)$$

where $w_1, \dots, w_{n_c}$ are the indices of the n_c interior ramification points and $\omega_1, \dots, \omega_{n_o}$ are the indices of the n_o boundary ramification points. To simplify the discussion, we w.l.o.g. assume that all interface branch points correspond to boundary ramification points⁴. To derive eq. (9.1.28), consider a triangulation of X_- that contains all bulk and boundary branch points as vertices and is fine enough that all edges and faces lift to exactly N_- copies on Σ_- . As discussed in Section 9.1.1, while generic vertices too lift to N_- copies on Σ_- , we loose $w_i^- - 1$ vertices on Σ_- for each of the n_c^- interior ramification points. The loss of vertices at boundary ramification points of type LL , RR , LR and RL can be determined completely analogously, leading to the formula

$$\chi(\Sigma_-) = N_- \chi(X_-) - \sum_{i=1}^{n_c^-} (w_i^- - 1) - \sum_{i=1}^{n_o^{LR}} \left(\frac{\omega_i^{LR} + 1}{2} - 1 \right) - \sum_{i=1}^{n_o^{RL}} \left(\frac{\omega_i^{RL} - 1}{2} - 1 \right) - \sum_A \sum_{i=1}^{n_o^{AA}} \left(\frac{\omega_i^{AA}}{2} - 1 \right). \quad (9.1.29)$$

Combining this result with the analogous formula for $\chi(\Sigma_+)$, we obtain

$$\chi(\Sigma_-) + \chi(\Sigma_+) = N_- \chi(X_-) + N_+ \chi(X_+) - \sum_{i=1}^{n_c} (w_i - 1) - \sum_{i=1}^{n_o} (\omega_i - 2). \quad (9.1.30)$$

Here $n_c = n_c^+ + n_c^-$ and similarly n_o counts the total number of boundary ramification points. We can view $\Sigma_- \cap \Sigma_+$ as a graph whose vertices are the boundary ramification points. The valency of each vertex is $\omega - 1$, where ω is the corresponding ramification index. Therefore,

$$\chi(\Sigma_- \cap \Sigma_+) = n_o - \frac{1}{2} \sum_{i=1}^{n_o} (\omega_i - 1). \quad (9.1.31)$$

The result (9.1.28) follows directly from the standard identity

$$\chi(\Sigma) = \chi(\Sigma_-) + \chi(\Sigma_+) - \chi(\Sigma_- \cap \Sigma_+). \quad (9.1.32)$$

This concludes our proof of the interface Riemann-Hurwitz formula (9.1.28).

The central role that this result will play in the rest of this chapter is analogous to the significance that the standard Riemann-Hurwitz formula has for symmetric product orbifolds in the absence of interfaces. It will most prominently be applied to derive topological expansions of interface correlators in Sections 9.2.5 and 9.4.3, but is more generally also a useful tool to decide which topologies are allowed to contribute to a given correlation function.

⁴Covering maps with interior ramification points that are mapped to interface branch points can be obtained as limits of coverings without such points. Hence eq. (9.1.28) also directly follows for these maps from the arguments given in this paragraph.

9.1.4 Holographic interfaces

We now use the interface covering maps introduced in Section 9.1.3 to define the interfaces of symmetric product orbifolds that are studied in this chapter. As an application of the new description presented here, we use it to compute a grand-canonical torus partition function with two interface insertions. In the new formalism, the full derivation is rather simple and can conveniently be described in less than a page. Compared to this, the derivation in [4] (which moreover only partially established the result) relied on rather lengthy and complicated combinatorial arguments. This new computational power is one of the key improvements of this chapter and ultimately, in Section 9.4, allows us to compute much more general partition functions that were out of reach with the approach of [4] .

Normalisation of interface correlators. In the setup described in Section 9.1.3, consider $\text{Sym}^{N_{\pm}}(\mathcal{M})$ on $X_{\pm}$. To obtain a well defined theory on X , we need to provide CFT glueing data along each of the n connected components $\gamma_1, \dots, \gamma_n$ of $\partial X_{\pm}$.⁵ Here, we only study interfaces that preserve the lifting property that defines symmetric orbifolds. In the presence of such an interface $\mathcal{I}$, the set $C[\mathcal{IO}]$ of covering maps is no longer restricted to consist of usual branched covers, but generically also includes interface coverings of the type introduced in the previous subsection. With this in mind, the formula for unnormalised interface correlators has exactly the same form (9.1.9) as in the case of local operator insertions,

$$\langle \mathcal{IO} \rangle_{\text{Sym}^{N_-|N_+}(\mathcal{M})}^{\text{unnormalised}} = \frac{1}{N_-!N_+!} \sum_{\Gamma \in C[\mathcal{IO}]} \langle \mathcal{IO}^{\Gamma} \rangle_{\mathcal{M}} \quad (9.1.33)$$

Note, however, that we use the convention that the factor $N!$ appearing in eq. (9.1.9) is replaced by $N_-!N_+!$. As before, we also define normalised interface correlators

$$\langle \mathcal{IO} \rangle_{\text{Sym}^{N_-|N_+}(\mathcal{M})} := \frac{\langle \mathcal{IO} \rangle_{\text{Sym}^{N_-|N_+}(\mathcal{M})}^{\text{unnormalised}}}{\langle \mathcal{I} \rangle_{\text{Sym}^{N_-|N_+}(\mathcal{M})}^{\text{unnormalised}}}. \quad (9.1.34)$$

In contrast to (9.1.9), the interface prescription (9.1.34) is to normalise by an interface partition function that contains the insertion of $\mathcal{I}$ instead of a vacuum bulk partition function.

Definition of the interfaces. Let us define a specific class of interface operators $\mathcal{I}$ that can be inserted along the connected components of $\partial X_{\pm}$ in the setup of the previous paragraph. The definition we give here will later be shown to coincide

⁵We allow for a straightforward generalisation of the discussion in the previous subsection in which the boundary $\partial X_+ = \partial X_-$ can possess several disconnected components γ_i so that the interface $\mathcal{I}$ is a product $\mathcal{I} = \mathcal{I}_1 \cdots \mathcal{I}_n$.

with the interfaces $\mathcal{I}_{|a_{\pm}\rangle}^{(P)}$ of symmetric product orbifolds that were introduced in [4]. The interfaces are parametrised by a non-negative integer $P \leq \min(N_-, N_+)$, a pair of boundary states $|a_-\rangle$ and $|a_+\rangle$ of the seed theory $\mathcal{M}$, as well as characters $\chi_{\pm}$ of $S_{N_{\pm}-P}$ and χ_P of S_P .⁶ To simplify notation, we shall denote the collection of these data by $\varpi = (P, |a_-\rangle, |a_+\rangle; \chi_-, \chi_+, \chi_P)$. Correlation functions with $\mathcal{I}_{\varpi}[\gamma]$ inserted along some curve γ are computed by summing over all interface coverings Γ whose degrees on $X_{\pm}$ are $N_{\pm}$ and whose degree of transmissivity on γ is P . The relevant correlation functions in the seed theory $\mathcal{M}$ on the covering surfaces Σ depend on the choice of boundary states $|a_{\pm}\rangle$, which describe the boundary condition imposed on the potentially non-trivial boundary $\partial\Sigma$. This boundary splits into disjoint subsets

$$R_{\pm}^{\gamma} := \partial\Sigma_{\pm} \cap \Gamma^{-1}(\gamma) \cap \partial\Sigma . \quad (9.1.35)$$

The intersection with $\Gamma^{-1}(\gamma)$ is redundant in case where $\partial X_{\pm}$ consists of a single connected component γ . The intersection with $\partial\Sigma$ removes the part of $\Gamma^{-1}(\gamma)$ that runs through the interior of Σ , i.e. the subsets

$$T^{\gamma} := \Gamma^{-1}(\gamma) \cap \Sigma^{\circ} . \quad (9.1.36)$$

By construction, every element of $\Gamma^{-1}(\gamma)$ is either contained in $R_{\pm}^{\gamma}$ or in T^{γ} , i.e.

$$\Gamma^{-1}(\gamma) = R_-^{\gamma} \sqcup T^{\gamma} \sqcup R_+^{\gamma} . \quad (9.1.37)$$

The freedom to choose three characters and two boundary states in the data ϖ that specifies the interface stems from this decomposition of $\Gamma^{-1}(\gamma)$ into three, and its overlap with $\partial\Sigma$ into two, disjoint sets. Concretely, the boundary states $|a_{\pm}\rangle$ describe the boundary conditions of the seed theory that are imposed along the components $R_{\pm}^{\gamma}$ of $\partial\Sigma$. As for the characters, note that the monodromy of the covering map Γ along the curve γ preserves the decomposition (9.1.37), i.e.

$$M_{\Gamma}[\gamma] \in \text{Aut}[S \cap R_-^{\gamma}] \times \text{Aut}[S \cap T^{\gamma}] \times \text{Aut}[S \cap R_+^{\gamma}] \subseteq \text{Aut}[S] \quad (9.1.38)$$

where $S := \Gamma^{-1}(\gamma(0))$. In other words, the monodromy factorises into a product

$$M_{\Gamma}[\gamma] = M_{\Gamma}^{-}[\gamma] M_{\Gamma}^P[\gamma] M_{\Gamma}^{+}[\gamma] . \quad (9.1.39)$$

The characters $\chi_{\pm}$ and χ_P enter correlations of $\mathcal{I}_{\varpi}[\gamma]$ with $M_{\Gamma}^{\pm}[\gamma]$ and $M_{\Gamma}^P[\gamma]$ as their arguments⁷. More explicitly, correlation functions

$$\langle \prod_{i=1}^n \mathcal{I}_{\varpi_i}[\gamma_i] \rangle_{\text{Sym}^{N_-|N_+}(\mathcal{M})}^{\text{unnormalised}} \quad \text{with} \quad \varpi_i := (P^i, |a_-^i\rangle, |a_+^i\rangle; \chi_-^i, \chi_+^i, \chi_P^i) \quad (9.1.40)$$

⁶The dependence on the characters was suppressed in the notation of [4] since we were mostly considering the case in which all three characters are trivial.

⁷As explained in the paragraph “local gauge choice” of Subsection 9.1.1, the three factors may be identified with elements of the permutation groups $S_{N_{\pm}-P}$ and S_P , respectively, to evaluate the characters without ambiguity i.e. independent of the choice of identification. See in particular eq. (9.1.5).

of products of interfaces are defined through eq. (9.1.33) with

$$C \left[\prod_{i=1}^n \mathcal{I}_{\varpi_i}[\gamma_i] \right] := \left\{ \begin{array}{l} N_{\pm} \text{ sheeted interface coverings of } X = X_+ \cup X_- \\ \text{with degree of transmissivity } P^i \text{ along } \gamma_i. \end{array} \right\} \quad (9.1.41)$$

and the characters $\chi_{\pm}^i$ and χ_P^i enter through the lift

$$\left[\prod_{i=1}^n \mathcal{I}_{\varpi_i}[\gamma_i] \right]^{\Gamma} := \alpha[\Gamma] \left(\prod_{i=1}^n \chi_{-}^i(M_{\Gamma}^{-}[\gamma_i]) \chi_P^i(M_{\Gamma}^P[\gamma_i]) \chi_{+}^i(M_{\Gamma}^{+}[\gamma_i]) \right) \mathbb{1}, \quad (9.1.42)$$

where $\mathbb{1}$ is the identity operator of $\mathcal{M}$ on Σ and the coefficient $\alpha[\Gamma]$ is given by

$$\alpha[\Gamma] = \frac{\deg[\Gamma_-]! \deg[\Gamma_+]!}{|\text{Deck}[\Gamma]|}. \quad (9.1.43)$$

The only data that do not manifestly appear on the r.h.s. of eq. (9.1.42) and (9.1.41) are the boundary states $|a_{\pm}^i\rangle$. These enter in the evaluation of the seed theory expectation value of the r.h.s. of eq. (9.1.42). As mentioned below eq. (9.1.37), these boundary states describe the boundary conditions we impose along the component $R_{\pm}^{\gamma_i}$ of the boundary $\partial\Sigma$. The prescription of how to evaluate the vacuum expectation value (9.1.40) of an arbitrary product of interfaces on an arbitrary Riemann surface X fully defines the interface $\mathcal{I}_{\varpi}$, just as eqs. (9.1.10) and (9.1.11) for arbitrary vacuum partition functions already fully define the bulk theory. Details on the relation of this definition to the one given in Reference [4] are discussed in Appendix 9.C.

Vacuum expectation value. As a first example of an interface correlation function let us consider the simplest setup, namely, a single interface $\mathcal{I}_{\varpi}$ with $\varpi = (P, |a_{\pm}\rangle; \chi_{\pm}, \chi_P)$ inserted along the equator γ of the sphere. The interface splits $X = \mathbb{S}^2$ into two disks $X_{\pm} = \mathbb{D}_{\pm}$. The set $C[\mathcal{I}_{\varpi}]$ consists of a single map Γ , which covers $\mathbb{S}^2$ by a disjoint union of P spheres and $N_{\pm} - P$ disks, that are mapped to the upper/lower hemisphere. Every deck transformations is a product of one of the $P!$ permutations of spheres and two permutations of disks chosen from the $(N_{\pm} - P)!$ possible options. Thus we conclude that

$$Z_{N_{\pm}, \varpi}^{\mathbb{D}_{-}, \mathbb{D}_{+}} := \langle \mathcal{I}_{\varpi} \rangle_{\text{Sym}^{N_{-}-|N_{+}|}(\mathcal{M})}^{\text{unnormalised}} = \frac{\chi_{-}(id) \chi_P(id) \chi_{+}(id) g_{-}^{N_{-}-P} g_{+}^{N_{+}-P}}{(N_{-} - P)!(N_{+} - P)!P!}, \quad (9.1.44)$$

where $g_{\pm} := \langle 0|a_{\pm}\rangle$ are the g -factors of the two boundary conditions $a_{\pm}$ of the seed theory that are imposed along $R_{\pm}^{\gamma}$. Note that the latter consists of $N_{\pm} - P$ disconnected components. The expectation value of the seed theory is evaluated in a normalisation in which the sphere partition function of the seed theory $\mathcal{M}$ is set to 1. The same result can alternatively be obtained from the construction in [4], see eq. (9.C.4).

Torus partition function – statement of the result. Let us now discuss a much richer example of an interface correlator, namely a pair of parallel interfaces inserted on a torus $X = \mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. The interfaces are located along two curves $\gamma_L : s \mapsto \tau s$ and $\gamma_R : s \mapsto \tau s + 1/2$. These cut the torus into two annuli $X_{\pm}$. The interface data ϖ^L and ϖ^R can involve two different indices of transmissivity P_L and P_R and two different pairs of boundary conditions $a_{\pm}^L$ and $a_{\pm}^R$. To avoid notational clutter, we shall assume the characters $\chi^{L/R}$ to be trivial, though it would not be difficult to consider the more general case. According to the general definition, the interface correlation function takes the form

$$Z_{N_{\pm}, P_{L/R}}^{\mathbb{T}^2} = \langle \mathcal{I}^{(P_L)}[\gamma_L] \mathcal{I}^{(P_R)}[\gamma_R] \rangle_{\text{Sym}^{N_-|N_+}(\mathcal{M})}^{\text{unnormalised}} = \sum_{\Gamma \in C[\mathcal{I}^L \mathcal{I}^R]} \frac{1}{|\text{Deck}[\Gamma]|} Z_{\mathcal{M}}^{\Sigma}, \quad (9.1.45)$$

where $Z_{\mathcal{M}}^{\Sigma}$ is the vacuum partition function of $\mathcal{M}$ on Σ with boundary conditions $|a_{\pm}^{L/R}\rangle$ imposed on $\partial\Sigma$. It was found in [4] that these interface partition functions take a particularly useful shape if we pass to the following grand-canonical partition function

$$Z_{\mu_{L/R}, \nu_{L/R}, \pm}^{\mathbb{T}^2} := \sum_{N_{\pm}=0}^{\infty} \sum_{P_{L/R}=0}^{\min(N_-, N_+)} Z_{N_{\pm}, P_{L/R}}^{\mathbb{T}^2} \prod_{A \in \{L, R\}} \mu_A^{P_A} \nu_{A,-}^{N_- - P_A} \nu_{A,+}^{N_+ - P_A} \quad (9.1.46)$$

where the dependence of the partition function on the integers $N_{\pm}$ and $P_{L/R}$ is traded for a dependence on the fugacities $\mu_{L/R}$ and $\nu_{L/R, \pm}$. The formula put forward in [4] reads⁸

$$Z_{\mu_{L/R}, \nu_{L/R}, \pm}^{\mathbb{T}^2} = \exp \left(\sum_{w, \ell=1}^{\infty} \sum_{k=1}^w \frac{\mu^{w\ell}}{w\ell} Z\left(\frac{\ell\tau+k}{w}\right) + \sum_{\substack{\omega, \ell \\ A, B}} \frac{1}{\ell} \left(\frac{\mu^{\omega} \nu_{AB}}{\mu_A \mu_B} \right)^{\ell} Z_o^{AB}\left(\frac{\ell\tau}{\omega}\right) \right) \quad (9.1.47)$$

where the sum in the second terms runs over $A, B \in \{L, R\}$, $\omega \in 2\mathbb{Z}_{>0} - 1 + \delta_A^B$ and $\ell \geq 1$. The variables μ and ν_{AB} are given by $\mu := \mu_L \mu_R$ and

$$\nu_{LL} := \nu_{L,-} \nu_{L,+}, \quad \nu_{LR} := \nu_{L,-} \nu_{R,-}, \quad \nu_{RL} := \nu_{R,+} \nu_{L,+} \quad \text{and} \quad \nu_{RR} := \nu_{R,-} \nu_{R,+}. \quad (9.1.48)$$

The dependence of the grand-canonical partition function on the seed theory and the choice of boundary conditions $a_{\pm}^{L/R}$ enters through the overlaps

$$Z_{a,b}(t) = \langle a | \hat{x}^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c\mathcal{M}}{12})} | b \rangle = \hat{Z}_{a,b}(\hat{t}) \quad \hat{t} := -\frac{1}{t} \quad \hat{x} := e^{2\pi i \hat{t}}, \quad (9.1.49)$$

between a pair of boundary states $|a\rangle$ and $|b\rangle$ of the seed theory. Through modular transformation, these overlaps are related to annulus partition functions $\hat{Z}$, as usual. There are four different annulus partition functions that appear in eq. (9.1.47) which we denoted by

$$Z_o^{LL} = Z_{a_-^L, (a_+^L)^*}, \quad Z_o^{LR} = Z_{a_-^L, a_+^R}, \quad Z_o^{RL} = Z_{a_+^L, a_+^R} \quad \text{and} \quad Z_o^{RR} = Z_{(a_+^R)^*, a_-^R}. \quad (9.1.50)$$

⁸We chose a slightly modified but equivalent way of writing down the result of [4] here.

This concludes our description of the grand-canonical partition function (9.1.47) for two interfaces $\mathcal{I}[\gamma_L]$ and $\mathcal{I}[\gamma_R]$ on the torus $\mathbb{T}^2$, restricted to the case of trivial characters.

Torus partition function – derivation. Let us now derive the formula (9.1.47) from the general prescription in eq. (9.1.45). The first ingredient there is the sum over interface covering maps $\Gamma \in C[\mathcal{I}^L \mathcal{I}^R]$. The relevant covering surfaces Σ can be decomposed as

$$\Sigma = \Sigma_c \sqcup \Sigma_o^{LL} \sqcup \Sigma_o^{LR} \sqcup \Sigma_o^{RL} \sqcup \Sigma_o^{RR} \quad (9.1.51)$$

where the restriction of the covering map Γ to Σ_c is one of the coverings of the torus by tori discussed in Sec. 9.1.2 and each connected component of the surfaces Σ_o^{AB} is a cylinder

$$\Sigma_{\omega,\ell} := [0, \frac{\omega}{2}] \times [0, \ell|\tau|]/(x, 0) \sim (x, \ell|\tau|). \quad (9.1.52)$$

The restriction of $\Gamma_o^{AB} : \Sigma_o^{AB} \rightarrow \mathbb{T}^2$ to the connected component $\Sigma_{\omega,\ell}$ is given by

$$\Gamma_{\omega,\ell}^{AB} : \Sigma_{\omega,\ell} \rightarrow \mathbb{T}^2, \quad (x, y) \mapsto \left[\left(x + \frac{1}{2}\delta_A^R, y \frac{\tau}{|\tau|} \right) \right], \quad (9.1.53)$$

where the value of ω is restricted to $\omega \in 2\mathbb{Z} + 1 - \delta_A^B$. Furthermore, $\text{Deck}[\Gamma_{\omega,\ell}^{AB}] \cong \mathbb{Z}_\ell$ consists of translations $(x, y) \mapsto (x, y + k|\tau|)$. Let us denote by $C_{w,\ell}^k$ the number of copies of the surface $\Sigma_{\omega,\ell}^k$ that appear within Σ_c and similarly denote by $O_{\omega,\ell}^{AB}$ the number of components of type $\Sigma_{\omega,\ell}$ that appear in Σ_o^{AB} . Then the number of Deck transformations is given by

$$|\text{Deck}[\Gamma]| = \prod_{w,\ell,k} C_{w,\ell}^k! (w\ell)^{C_{w,\ell}^k} \prod_{\substack{\omega,\ell \\ A,B}} O_{\omega,\ell}^{AB}! \ell^{O_{\omega,\ell}^{AB}}. \quad (9.1.54)$$

Inserting this formula into the general definition (9.1.45) of the interface partition functions on the torus we obtain

$$Z_{N_\pm, P_{L/R}}^{\mathbb{T}^2} = \sum_{\text{allowed } C, O^{AB}} \left(\prod_{w,\ell,k} \frac{\left(\frac{1}{w\ell} Z\left(\frac{\ell\tau+k}{w}\right) \right)^{C_{w,\ell}^k}}{C_{w,\ell}^k!} \prod_{\substack{\omega,\ell \\ A,B}} \frac{\left(\frac{1}{\ell} Z_o^{AB}\left(\frac{\ell\tau}{w}\right) \right)^{O_{\omega,\ell}^{A,B}}}{O_{\omega,\ell}^{AB}!} \right) \quad (9.1.55)$$

where the allowed coefficients C and $O^{A,B}$ are exactly those satisfying the constraints

$$N_- - P_L = \sum_{\omega,\ell,B} O_{\omega,\ell}^{LB} \ell, \quad N_+ - P_L = \sum_{\omega,\ell,A} O_{\omega,\ell}^{AL} \ell, \quad N_- - P_R = \sum_{\omega,\ell,A} O_{\omega,\ell}^{AR} \ell, \quad (9.1.56)$$

$$N_+ - P_R = \sum_{\omega,\ell,B} O_{\omega,\ell}^{RB} \ell, \quad \text{and} \quad P_A = \sum_{w,\ell} \sum_k w\ell C_{w,\ell}^k + \sum_{\omega,\ell} \sum_{A,B} (\omega - \delta_I^A - \delta_J^A) \ell O_{\omega,\ell}^{I,J}. \quad (9.1.57)$$

For the grand-canonical partition function (9.1.46), these constraints are lifted and by performing the unconstrained sum over $C_{w,\ell}^k$ and $O_{\omega,\ell}^{AB}$ one immediately obtains eq. (9.1.47).

Spectrum of interface changing operators. In this paragraph, we rewrite eq. (9.1.47) in a form that allows us to directly read off the spectrum of interface changing operators. To do so, we start by spelling out the seed partition functions in terms of the bulk and boundary spectra I and I_o^{AB} as well as the multiplicities $d_{h,\bar{h}}$ and d_Δ of the corresponding representations in the Hilbert space, i.e.

$$Z(\tau, \bar{\tau}) = \sum_{(h,\bar{h}) \in I} d_{h,\bar{h}} q^{h - \frac{c\mathcal{M}}{24}} \bar{q}^{\bar{h} - \frac{c\mathcal{M}}{24}} \quad \text{and} \quad Z_o^{AB}(\tau) = \sum_{\Delta \in I_o^{AB}} d_\Delta^{AB} q^{\Delta - \frac{c\mathcal{M}}{24}}. \quad (9.1.58)$$

Setting $\bar{\tau} = \tau$ for the bulk partition function and using the identities

$$\exp\left(\alpha \sum_{\ell=1}^{\infty} \frac{x^\ell}{\ell}\right) = (1-x)^{-\alpha} \quad \text{and} \quad \frac{1}{w} \sum_{k=1}^w e^{2\pi i \frac{k}{w}(x-\bar{x})} = \begin{cases} 1 & \text{if } x - \bar{x} \equiv 0 \pmod{w} \\ 0 & \text{else} \end{cases} \quad (9.1.59)$$

then allows us to rewrite eq. (9.1.47) as

$$Z_{\mu_{L/R}, \nu_{L/R}, \pm}^{\mathbb{T}^2} = \prod_{w>0} \prod_{\substack{(h,\bar{h}) \in I \\ h-\bar{h} \equiv 0 \pmod{w}}} \left(1 - \mu^w q^{\frac{1}{w}(h+\bar{h} - \frac{c\mathcal{M}}{12})}\right)^{-d_{h,\bar{h}}} \prod_{A,B} \prod_{\omega} \prod_{\Delta \in I_o^{AB}} \left(1 - \frac{\mu^\omega \nu_{AB}}{\mu_A \mu_B} q^{\frac{1}{\omega}(\Delta - \frac{c\mathcal{M}}{24})}\right)^{-d_\Delta^{AB}}, \quad (9.1.60)$$

where ω runs over $2\mathbb{Z}_{>0} - 1 + \delta_A^B$. The scaling dimensions of interface changing operators can be read off from eq. (9.1.60) by extracting the fixed $N_\pm$ partition functions and adding $\frac{c\mathcal{M}}{24}(N_- + N_+)$ to the exponents of q . From the grand-canonical perspective, this amounts to shifting the exponent of q for the factors of the product over w by $\frac{c\mathcal{M}}{24}2w$ and likewise, shifting the factors of the ω product by $\frac{c\mathcal{M}}{24}\omega$. We conclude that the spectrum of local operators that can be inserted between our interfaces consists of products of single particle fields falling into the five families $\mathcal{O}_w$ and $\mathcal{O}_\omega^{AB}$ with $A, B \in \{L, R\}$. The former can simply be interpreted as bulk operators inserted on the transmissive part of the interfaces, while the latter are intrinsically attached to the interface. Quantitatively, the arguments given above tell us that for each $w > 0$ and each seed theory bulk operator $\mathcal{O}$ of dimension $\bar{h} + h$, and spin $\bar{h} - h$ divisible by w , we obtain a single particle interface operator with dimension

$$\Delta_w = \frac{\bar{h} + h}{w} + \frac{c\mathcal{M}}{12} \left(w - \frac{1}{w}\right), \quad (9.1.61)$$

while each weight $\Delta \in I_o^{AB}$ leads to a family of fields with dimension

$$\Delta_\omega^{AB} = \frac{\Delta}{\omega} + \frac{c\mathcal{M}}{24} \left(\omega + \frac{1}{\omega}\right). \quad (9.1.62)$$

In the special case where the boundary states that enter the construction of the interfaces are chosen to coincide, i.e. $a_+ = a_-$, the seed theory boundary spectra that are captured by the partition functions (9.1.50) contain an identity operator,

i.e. $0 \in I_o^{AB}$, and thus the spectrum of the symmetric product orbifold contains *interface twist fields* σ_ω^{AB} whose conformal weight is given by eq. (9.1.62) with $\Delta = 0$. Studying correlation functions of these interface twist fields is the main focus of our work and hence we shall mostly consider interfaces with $|a_+\rangle = |a_-\rangle =: |a\rangle$ (and $\chi_\pm = \chi_P = 1$). The associated interface operators will be denoted by $\mathcal{I}_{|a\rangle}^{(P)}$.

Interface changing correlators. We are finally ready to address the main topic of this chapter, namely correlation functions including twist field insertions along the interface (and in the bulk). To keep things simple we return to the case in which an interface operator $\mathcal{I}[\gamma]$ is inserted along a single closed curve γ so that we are strictly in the setup we considered previously in Section 9.1.3. The extension to multiple interface insertions is straightforward. For simplicity we also restrict to the case of trivial characters χ and set $|a_+\rangle = |a_-\rangle =: |a\rangle$, see our discussion at the end of the previous paragraph. This leaves the degree of transmissivity P as the main parameter of the interface, besides the choice of the boundary state $|a\rangle$.

As we explained above, some interface twist fields can only be inserted at points at which the degree of transmissivity changes. To be more precise, the interface twist fields σ_ω^{LL} and σ_ω^{RR} can only be inserted at points along the curve γ at which the degree of transmissivity P jumps by one unit to $P \pm 1$, see also Figure 9.1.1. Put differently, if we want to insert these fields we have to break up the curve γ into smaller segments and insert interface operators with different degree of transmissivity along these segments. Since this is notationally a bit cumbersome we shall keep using the interface operator $\mathcal{I}_{|a\rangle}^{(P)}[\gamma]$ where P denotes the degree of transmissivity of $\mathcal{I}[\gamma]$ at the base point $\gamma(0) = \gamma(1)$ of the curve. We shall sometimes refer to this base point as the *reference point* and denote it by t^* in the rest of this chapter. It is then understood that this degree jumps whenever γ passes through the insertion point of an interface changing twist field $\sigma_\omega^{LL}(t)$ or $\sigma_\omega^{RR}(t)$.

With this comment on notation in mind let us now spell out the family of observables $\mathcal{O}$ whose expectation values we want to compute. These are given by

$$\mathcal{O} = \mathcal{I}_{|a\rangle}^{(P)}[\gamma] \prod_{i=1}^{n_c} \sigma_{w_i}(x_i) \prod_{i=1}^{n_o} \sigma_{\omega_i}^{A_i B_i}(t_i) \quad (9.1.63)$$

where we assume that the reference point $t^* = \gamma(0)$ is not the insertion point of a local operator.

As in our previous discussions, the expectation values of the observables $\mathcal{O}$ we consider now can be evaluated by summing certain expectation values on the seed theory over some set of covering maps $\Gamma \in C[\mathcal{O}]$. The insertion of the interface changing operators $\sigma_\omega^{AB}(t_i)$ into the expectation value enforces that covering maps that contribute to the correlator have a type ω^{AB} interface ramification point that maps to

the branch point $t_i \in X$. Thus,

$$C[\mathcal{O}] = \left\{ \begin{array}{l} N_{\pm} \text{ sheeted interface coverings of } X \text{ with deg. of transmissivity } P \\ \text{at } \gamma(0) \text{ and bulk branchpts. } x_i \text{ lifting to ramification pts. with} \\ \text{ramification index } w_i \text{ as well as interface branchpts. } t_i \text{ lifting to} \\ \text{interface ramification pts. with ramification index } \omega_i^{A_i B_i}. \end{array} \right\}. \quad (9.1.64)$$

Like the bulk vacuum twist fields, the interface vacuum twist fields lift to the identity operator along the covering map, up to some coefficient as in eq. (9.1.42). Since we assumed that all three characters $\chi_{\pm}$ and χ_P are trivial, the factors that depend on the monodromies of the covering map is also trivial and hence we are left with $\alpha[\Gamma]$. In Sec. 9.2, we determine $C[\mathcal{O}]$ for any correlator with $X = \mathbb{S}^2$ explicitly. The explicit evaluation of the correlators from these data is then addressed in Sec. 9.3.

9.2 Generalised Pakman-Rastelli-Razamat diagrammatics

In this section, we introduce diagrammatic rules that provide a systematic classification of the covering maps that contribute to any specific interface correlator. In the first subsection, Sec. 9.2.1, we review the analogous diagrams for bulk correlators which were introduced by Pakman, Rastelli and Razamat in [311]. The main subsection, 9.2.2, introduces the interface generalisation. Sec. 9.2.3 describes how to determine the symmetry group of an interface covering map, i.e. the group $\text{Deck}[\Gamma]$ of its deck transformations, directly from the corresponding diagram. Subsection 9.2.4 provides an alternative perspective on the same problem and determines the symmetry factors by counting different ways to label faces of diagrams, an approach that more directly connects to the bulk discussion of Reference [311]. The final subsection, 9.2.5, is devoted to the large- N limit of interface correlators.

9.2.1 Pakman-Rastelli-Razamat diagrams

This section reviews the Pakman-Rastelli-Razamat (PRR) diagrammatic rules. We first discuss how to construct the associated PRR diagram given a covering map. To do so, we take a slight detour of first constructing auxiliary diagrams with redundant extra vertices and then reducing these to their PRR form. While this makes the discussion of the case without interfaces slightly less efficient than the original description in Reference [311], the approach will pay off in Section 9.2.2 since it is well aligned with the interface generalisation.

After we have explained how to extract diagrams from covering maps, the second paragraph of the section formulates constructive rules that allow us to generate the set of PRR diagrams without knowledge of the covering maps. Finally, we showcase

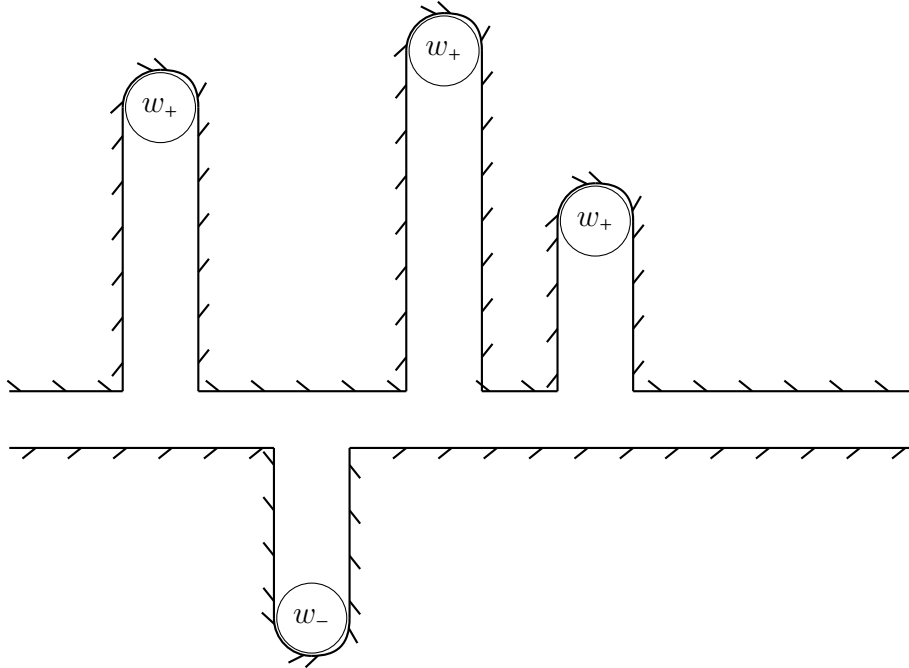


Figure 9.2.1: Lines on $\mathbb{S}^2$ whose preimage w.r.t. a covering map Γ produce $\mathcal{D}_\varepsilon[\Gamma]$. Branch points in the upper and lower hemisphere are denoted by w_+ and w_- . Orientations are represented by half-arrows extending into the disk on whose boundary they are drawn.

how to apply the rules in simple examples. Appendix 9.E sketches a proof of the validity of the rules.

Diagrams from covering maps. To construct the PRR diagram $\mathcal{D}[\Gamma]$ of a covering map $\Gamma : \Sigma \rightarrow \mathbb{S}^2$, we perform the following three steps.

1. Draw lines with the same orientation, at distance ε above and below the equator, deformed as in Fig. 9.2.1 such that all branch points of Γ are in the strip between them.
2. The preimage $\mathcal{D}_\varepsilon[\Gamma]$ of this configuration with respect to Γ is a network of oriented lines on Σ . Take the limit $\mathcal{D}[\Gamma] := \lim_{\varepsilon \rightarrow 0} \mathcal{D}_\varepsilon[\Gamma]$.
3. Each face of $\mathcal{D}[\Gamma]$ is either the preimage of the upper or the lower hemisphere with respect to Γ . Label the faces with “+” and “−” respectively.

If Γ has degree N then $\mathcal{D}[\Gamma]$ has $2N$ faces, all of them with a consistently oriented boundary and exactly half labelled by +. The diagram has four classes of vertices, two of which are shown in Fig. 9.2.2. The other two may be obtained simply by flipping all signs $\pm$ in the faces. In the $\varepsilon \rightarrow 0$ limit, the preimages of the lines on

$\mathbb{S}^2$, reaching out to branch points from the equator of the sphere, degenerate into double lines which end on the branch points. We shall refer to these double lines as misaligned since the arrows along them point in opposite directions. Each misaligned double line lifts to w copies on Σ , leading to the vertex shown in Fig. 9.2.2a.

Note that the faces that touch at a misaligned double line possess the same sign. Furthermore, each region where such a fold leaves the equator becomes a trivalent vertex of the type shown in Fig. 9.2.2b. We refer to the latter type of vertices as redundant vertices for reasons that will become clear in a moment. Let us first note, however, that these trivalent vertices involve a single misaligned double line and two aligned double lines for which the arrows on both sides of the double line point in the same direction. Faces joined by aligned double lines carry opposite signs.

As illustrated in Fig. 9.2.3a, all edges emanating from a vertex that corresponds to a ramification point end on redundant vertices. We can therefore simplify the diagrams without losing any information by moving the trivalent vertices like a zip toward the ramification vertex until we have entirely eliminated the misaligned double line. For each misaligned double line that was entering the ramification vertex before, this reduction produces two aligned double lines that enter the vertex, with the sign of the faces flipping each time we cross an aligned double line. This fusion of ramification and redundant vertices, leads to diagrams that only have one type of vertices, namely those shown in Fig. 9.2.3b. The reduced graphs are the standard PRR diagrams.

Let us close this subsection with further comments on this relation to the diagrams of Pakman, Rastelli and Razamat. The original PRR diagrams use a single type of double lines composed of one dashed and one solid line with no displayed orientation. But in checking consistency of PRR diagrams it becomes important that one goes through the dashed loops in clockwise order and through solid loops in anti-clockwise order. In this sense, the lines of PRR diagrams inherit an orientation from the orientation of Σ . We have decided here to make this orientation manifest in our graphics. The PRR faces that are enclosed by solid loops correspond to the faces we label with $+$, while those surrounded by dashed loops are the ones we label with $-$. The main difference in our approach is that, initially, we have three types of double lines. Our aligned double lines correspond to the PRR double lines. But in addition we have two more double lines, namely the misaligned double lines that run between two faces with label $+$, and the misaligned double lines that run between two faces that carry a $-$ label. Following PRR graphics, one can also picture these as double lines with two solid and two dashed lines, respectively. Our redundant vertex would then be represented as a vertex in which the double line with e.g. two solid lines splits into two double lines of PRR type, i.e. with one dashed and one solid line. As long as there is no interface located at the equator, the misaligned double lines can be removed by the reduction procedure described in the previous paragraph. This

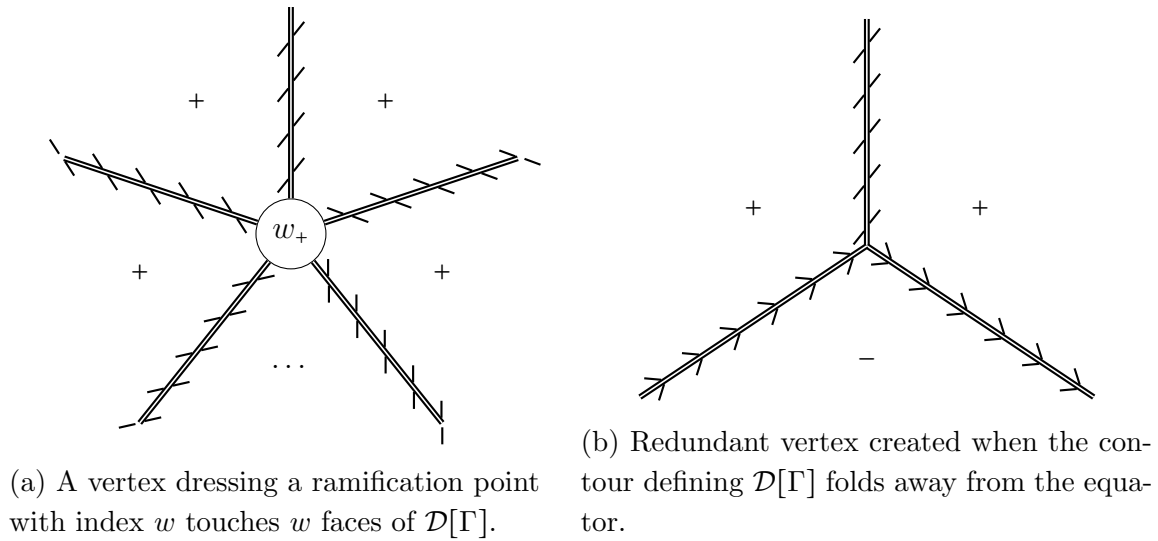


Figure 9.2.2: This figure shows half of the vertices of the PRR diagrams. Two more vertices can be generated by turning around all orientations and exchanging $+\leftrightarrow -$.

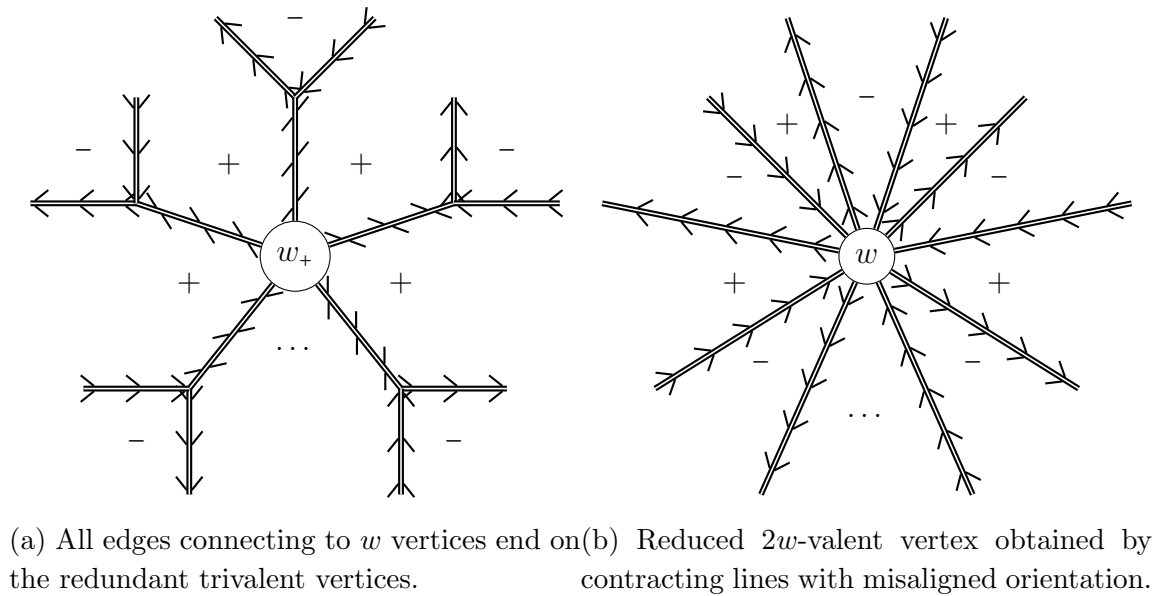


Figure 9.2.3: Reduction to standard PRR diagrams.

leaves us with diagrams that only contain the aligned double lines used by Pakman, Rastelli and Razamat.

Covering maps from diagrams. The PRR diagrams for covering maps Γ of degree N can be constructed by using the vertices in Fig. 9.2.3b, as well as those with flipped signs of the faces, choosing an order of the branch points and imposing

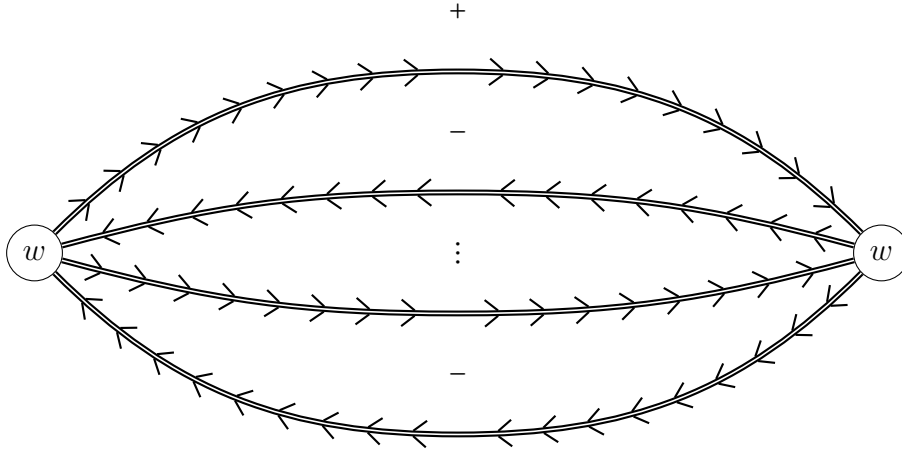


Figure 9.2.4: Diagram corresponding to the covering map that computes the two-point function of twist w fields.

the following three rules.

1. The boundary of a face contains each ramification point at most once.
2. Traversing the boundary of a face along its orientation, one encounters the ramification points in the order of the corresponding branch points.
3. Each face is simply connected and the number of faces equals the degree N of the covering map.

Clearly, the diagrams that classify equivalence classes of covering maps introduced in the previous paragraph satisfy the three rules above. In Appendix 9.E, we show that vice versa every diagram constructed in this way corresponds to an equivalence class of covering maps.

Example: Two-point correlator. As a simple example, Figure 9.2.4 shows the covering map Γ which computes the two-point function $\langle \sigma_w \sigma_w \rangle$ of two twist w operators. This map is a covering of the sphere by the sphere with two ramification points, both of them having ramification index w . Thus, the diagram is a graph embedded in the sphere with two vertices that are connected by double lines of alternating orientation. The graph has $2w$ faces, half of which with a “−” and half with a “+” label.

9.2.2 Interface PRR diagrams

Following the same strategy as in Section 9.2.1, we first construct diagrams from interface covering maps. We then present a set of rules that allow us to systematically

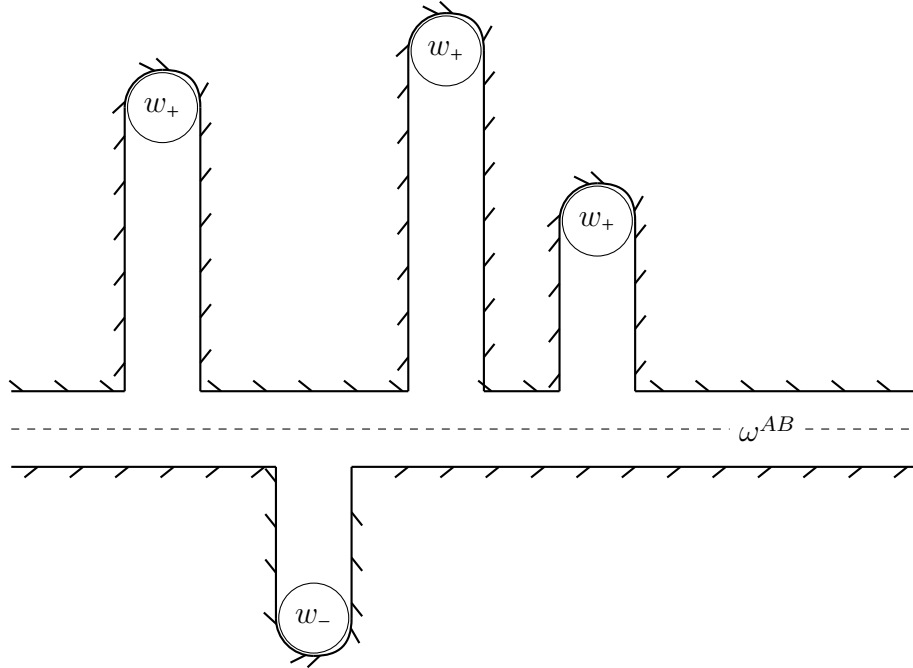


Figure 9.2.5: When the interface is inserted along the equator of the sphere, the contours defining $\mathcal{D}_\varepsilon[\Gamma]$ are identical to those shown in Fig. 9.2.1.

construct all possible diagrams given a set of prescribed ramification points. Finally, we demonstrate the rules in several examples. A proof for the validity of the rules is sketched in App. 9.E.

Diagrams from interface covering maps. The following three steps construct a diagram⁹ $\mathcal{D}[\Gamma]$ on a Riemann surface Σ given an interface covering map $\Gamma : \Sigma \rightarrow \mathbb{S}^2$.

1. Draw lines with the same orientation, at distance ε above and below the interface, deformed as in Fig. 9.2.5 such that all branch points of Γ are in the strip between them.
2. The preimage $\mathcal{D}_\varepsilon[\Gamma]$ of this configuration with respect to Γ is a network of oriented lines on Σ . Take the limit $\mathcal{D}[\Gamma] := \lim_{\varepsilon \rightarrow 0} \mathcal{D}_\varepsilon[\Gamma]$.
3. Each face of $\mathcal{D}[\Gamma]$ is either the preimage of the upper or the lower hemisphere with respect to Γ . Label the faces with “+” and “−” respectively.

In the interface case, the vertices of Fig. 9.2.2 are augmented by the vertices of Fig. 9.2.6. The first four vertices originate from boundary ramification points of the interface covering map. They involve two oriented (single) lines and $\omega - 1$ aligned

⁹In mathematical terms the object that we construct is a combinatorial map i.e. a graph with additional data describing the cyclic configuration of edges at each vertex.

double lines. Since all the double lines are aligned, the label on the faces alternates. There are four different assignments of signs in the two faces that involve the single line. These assignments distinguish the four vertices corresponding to the four classes of boundary ramification points. Additionally, there are two vertices in which a misaligned double line splits into a pair of single lines. The two variants of these vertices are distinguished by the sign of the two faces touching the misaligned line, which can be either both $+$ or both $-$.

The trivalent vertices that we encountered in the absence of the interface were redundant since there was a face with the opposite sign that we could use to split the misaligned double line into a pair of aligned ones. Now, however, the interface can cause the surface to rip open. When this happens, there is no face with opposite sign on the other side. Therefore, these vertices cannot be removed.

As in the case without interface, the two halves of each double line originating in bulk ramification points have opposite orientation, i.e. these double lines are misaligned. Again, we can remove all ‘redundant’ trivalent vertices where misaligned double lines split into a pair of aligned double lines by moving them on top of a bulk ramification point. In this process we can remove some of the misaligned double lines. But if there is a non-trivial interface, it is no longer possible to remove all the misaligned double lines through this process. Hence, the interface diagrams do involve three different types of double lines in general, the aligned ones that also appear in the bulk PRR diagrams and the two types of misaligned ones that run between faces labelled by $+$ and by $-$.

The bulk vertices involve both aligned and misaligned double lines. Their associated ramification index w fixes the overall number of these double lines but not how they are distributed among the two types. More precisely, $2w$ is the number of aligned double lines plus twice the number of misaligned ones.

Covering maps from diagrams. The interface PRR diagrams of interface covering maps Γ of degrees $N_{\pm}$ equipped with a consistent choice of order for the branch points are the diagrams built out of the vertices in Figures 9.2.2 and 9.2.6, satisfying the three rules given below. In this context, “consistent choice of order” means that the restriction of the ordering to interface branch points coincides with the order induced by the orientation of the interface.

1. The boundary of a face contains each ramification point at most once.
2. Traversing the boundary of a face along its orientation, one encounters the ramification points in the order of the corresponding branch points.
3. Each face is simply connected and the number of “ $\pm$ ”-faces is $N_{\pm}$.

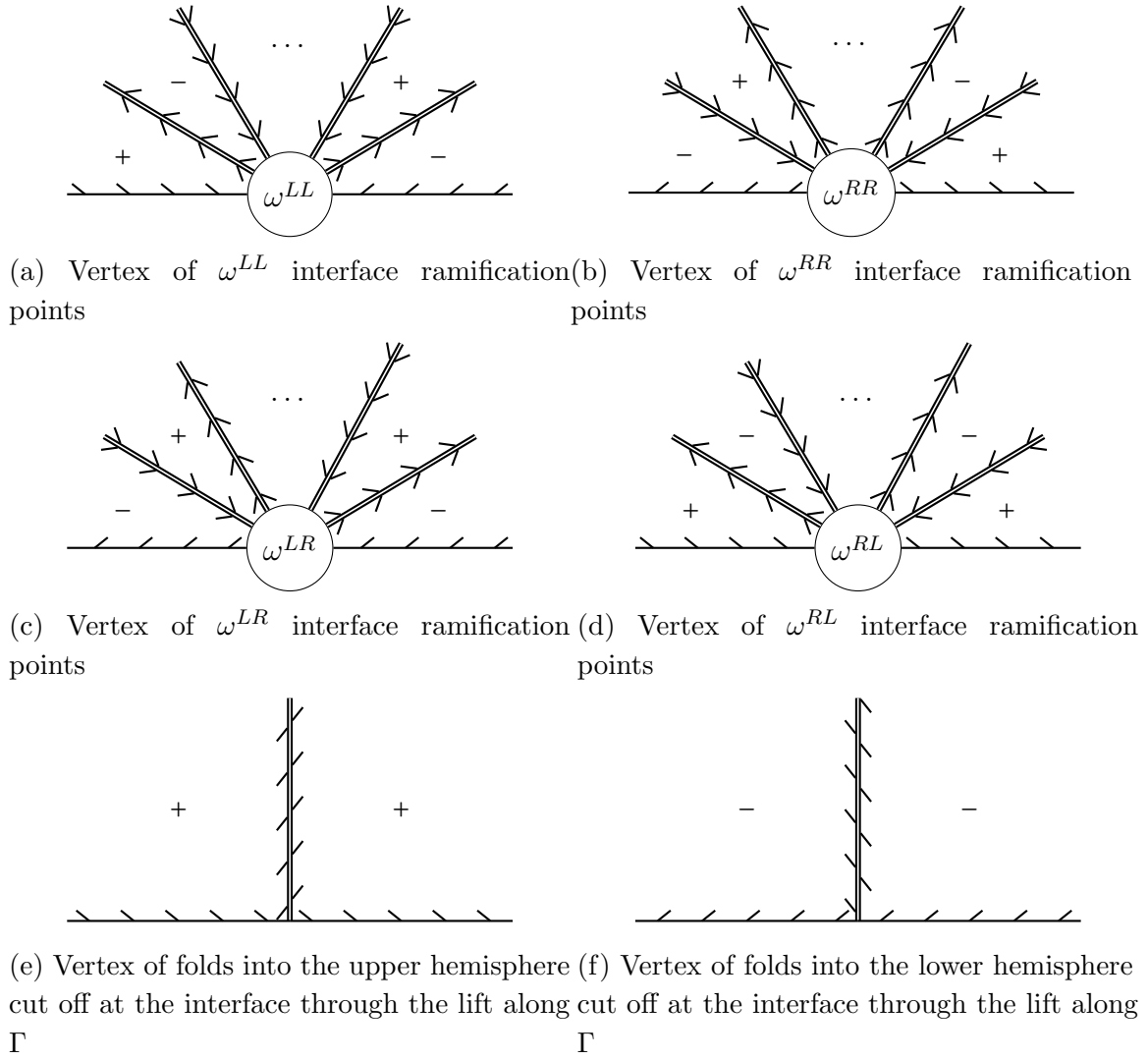


Figure 9.2.6: Additional vertices for interface covering maps.

Clearly, the diagrams that classify equivalence classes of covering maps introduced in the previous paragraph satisfy the three rules above. In Appendix 9.E we show that, conversely, all interface PRR diagrams actually correspond to interface covering maps.

Example 1. As a first example, let us consider the diagrammatics for the interface covering maps Γ that can contribute to the correlation function

$$\langle \sigma_2^{RR} \sigma_2(z) \sigma_2^{LL} \rangle. \quad (9.2.1)$$

The only connected covering space contributing to this correlator is a disk. If z is in the upper hemisphere, the corresponding interface covering map has $N_- = 1$ and $N_+ = 2$. The assignment is reversed if z is in the lower hemisphere, i.e. $N_- = 2$ and

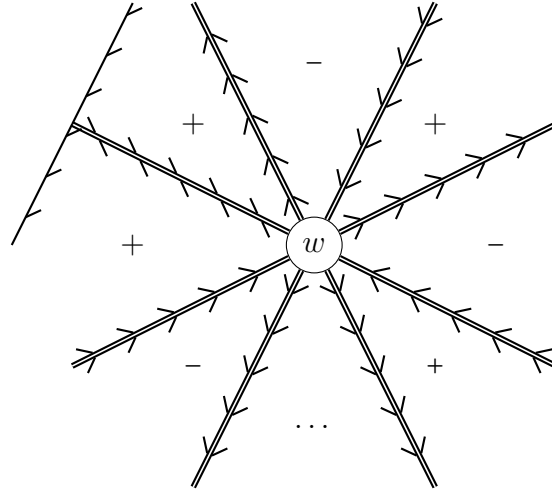


Figure 9.2.7: Bulk vertex in a reduced diagram obtained by removing redundant vertices. While this picture shows only a single misaligned line emanating from the bulk vertex to the boundary of Σ , there could in general be up to w such lines attached to the same vertex.

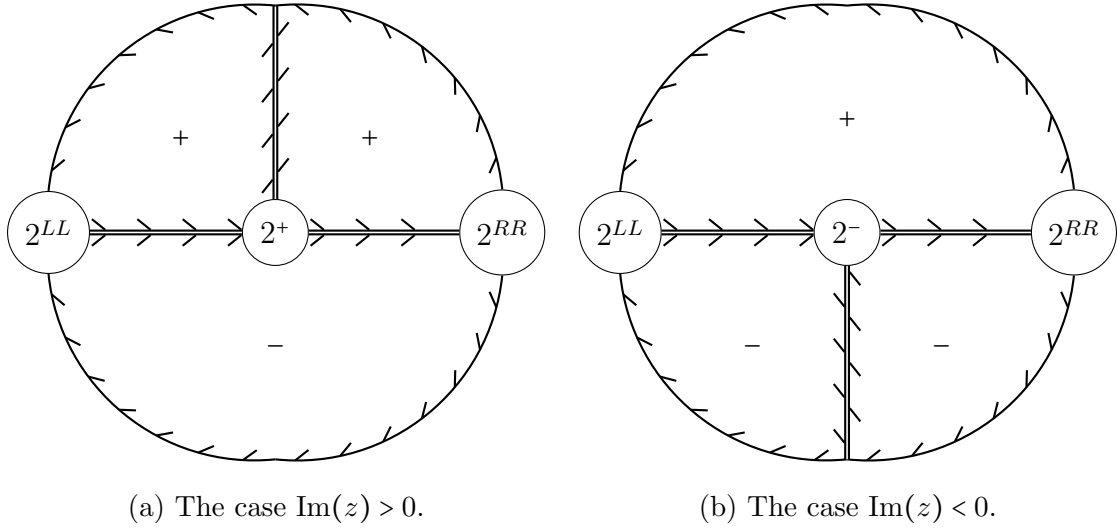


Figure 9.2.8: Diagram of $\langle \sigma_2^{RR} \sigma_2(z) \sigma_2^{LL} \rangle$ with order $2^{LL} < 2 < 2^{RR}$.

$N_+ = 1$ in that case. The two corresponding diagrams are shown in Figure 9.2.8. The correlator is a good example to highlight the relevance of the order that is assigned to the bulk branch points through the contour in Figure 9.2.5. The diagrams shown in Figure 9.2.8 correspond to the choice of ordering $2^{LL} < 2 < 2^{RR}$. The other option, i.e. $2^{LL} < 2^{RR} < 2$, is shown in Figure 9.2.9. Let us stress that the diagrams arising from different choices of ordering do not correspond to different types of covering maps that contribute to the correlator (9.2.1). Instead they are different graphical representations of the same covering map. Finally, let us also draw the attention of

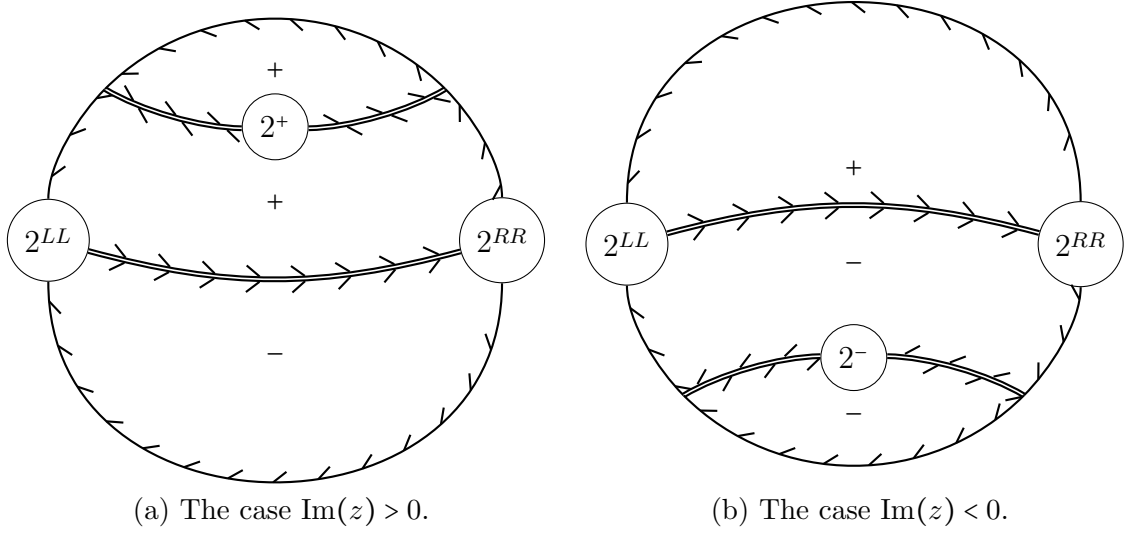


Figure 9.2.9: Diagram of $\langle \sigma_2^{RR} \sigma_2(z) \sigma_2^{LL} \rangle$ with order $2^{LL} < 2^{RR} < 2$.

the reader to the subtle feature that performing a reflection of the diagrams along the vertical axis that goes through the vertex corresponding to the bulk field does *not* result in valid diagrams. The vertices of valid diagrams are precisely those listed in Figure 9.2.6. In particular, a curve moving around a valid ω^{LL} vertex anticlockwise in a small semi-circle has to start on the boundary of a “-”-face and end on the boundary of a “+”-face. The analogous statement with “+” and “-” exchanged holds for the ω^{RR} vertices.

Example 2. Let us now discuss a less trivial example, where multiple diagrams contribute, namely the four-point correlation function

$$\langle \sigma_5^{RL} \sigma_5^{RL} \sigma_2^{RR} \sigma_2^{LL} \rangle \quad (9.2.2)$$

of interface operators. Whereas the ordering of bulk operators along the interface is an arbitrary (but necessary) choice that does not have any direct physical meaning, the ordering of interface operators is unambiguous (i.e. different orderings correspond to physically different correlation functions). With the ordering fixed in eq. (9.2.2), there is a unique disk diagram corresponding to an interface covering map that contributes to the correlator. This diagram is illustrated in Figure 9.2.10. Furthermore, there are two distinct cylinder diagrams that contribute to the connected part of the correlator. These are shown in Figure 9.2.11.

Example 3. As the complexity of correlators increases, the number of diagrams that contributes for each covering surface also grows. Figure 9.2.12 shows only one

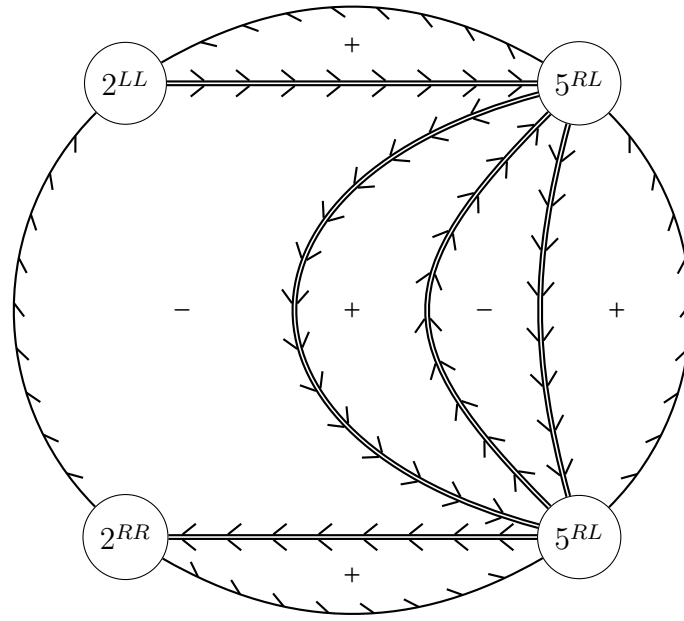
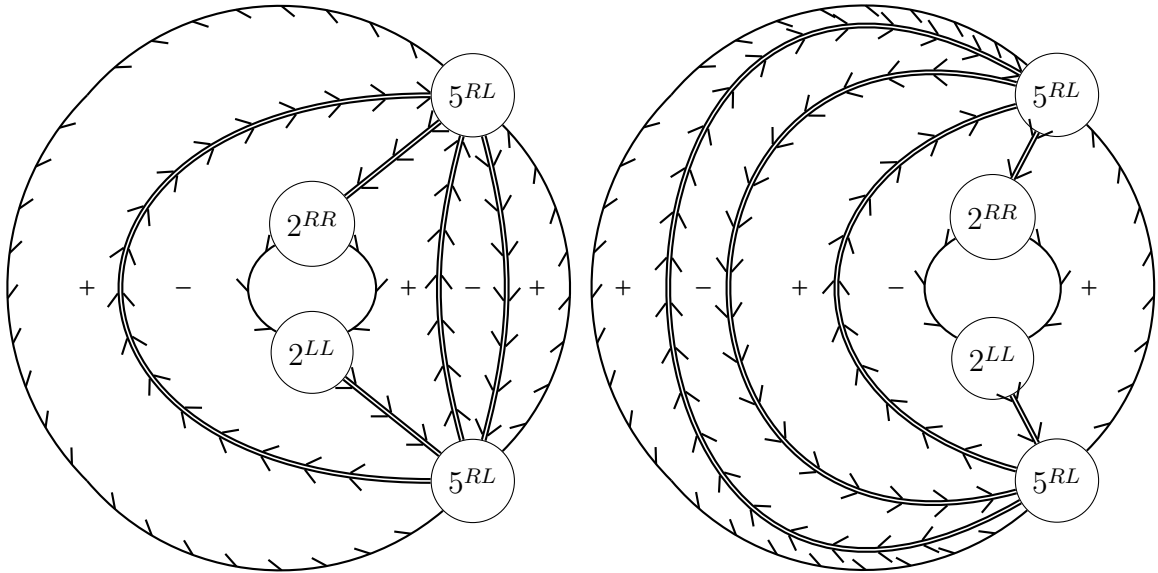


Figure 9.2.10: Disk contribution to the correlator (9.2.2) at $N_- = 2$ and $N_+ = 4$.



(a) Diagram with separated boundaries.

(b) Diagram with adjacent boundaries.

Figure 9.2.11: Cylinder contributions to the correlator (9.2.2) at $N_- = 2$ and $N_+ = 3$.

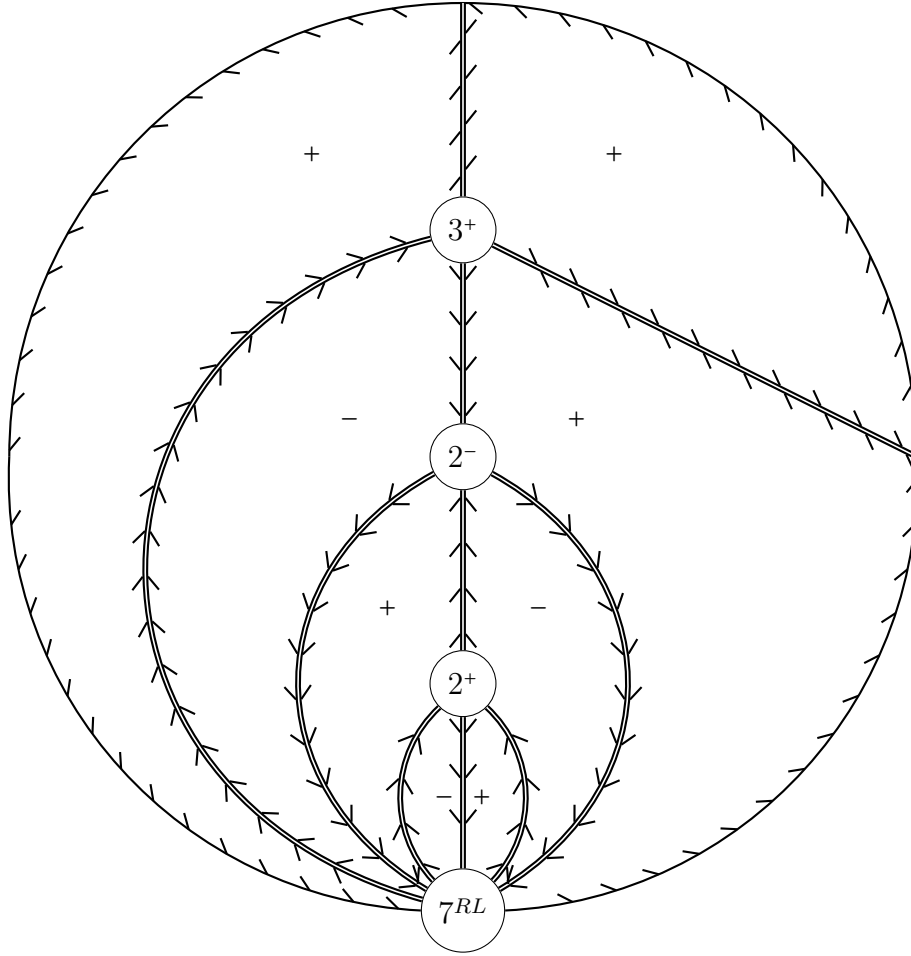


Figure 9.2.12: Diagram for one of the disk covering maps contributing to (9.2.3).

of many diagrams that contribute at disk level to the four-point function

$$\langle \sigma_7^{RL} \sigma_2(z_2^+) \sigma_3(z_3^+) \sigma_2(z_2^-) \rangle. \quad (9.2.3)$$

We hope that these three example suffice to illustrate the diagrammatic rules that we have introduced in order to describe interface covering maps.

Diagrams with reference point. By construction, we can read off $N_{\pm}$ from a diagram by counting the faces labelled by $+$ and $-$. We now explain how one can read off the degree of transmissivity P . Recall that, when moving along the interface, P changes by one unit whenever we cross an interface branch point of type LL or RR , see Section 9.1. Thus, the integer P is not a global property of the covering map, but a local property associated to a reference point t^* on the interface, as also discussed below eq. (9.1.63). Given a choice of t^* , we decorate our diagrams with the elements of the preimage $\Gamma^{-1}(t^*)$ under the covering map. The degree of transmissivity P at t^* is then given by the number of these preimages occurring in the interior of the

covering space, i.e. on double lines. The relevant data encoded in the choice of t^* is its position in the ordering of interface and bulk branch points. Using this data, the reference points have to be distributed on the diagrams according to the following three rules

1. The boundary of each face includes exactly one reference point.
2. Moving along the boundary of a face along its orientation, one encounters not only the ramification points but also the reference point in the correct order.
3. Reference points cannot occur on misaligned double lines.

As an example, Fig. 9.2.13 shows a t^* dressed version of the diagram shown in Fig. 9.2.10. The decoration shown here amounts to the ordering $t^* < 5^{RL} < 5^{RL} < 2^{RR} < 2^{LL}$ for which we read off that $P = 2$ since two of the black dots are placed on double lines. The reader is invited to draw the diagram that is associated with the order $5^{RL} < 5^{RL} < 2^{RR} < t^* < 2^{LL}$ and to verify that this returns $P = 1$.

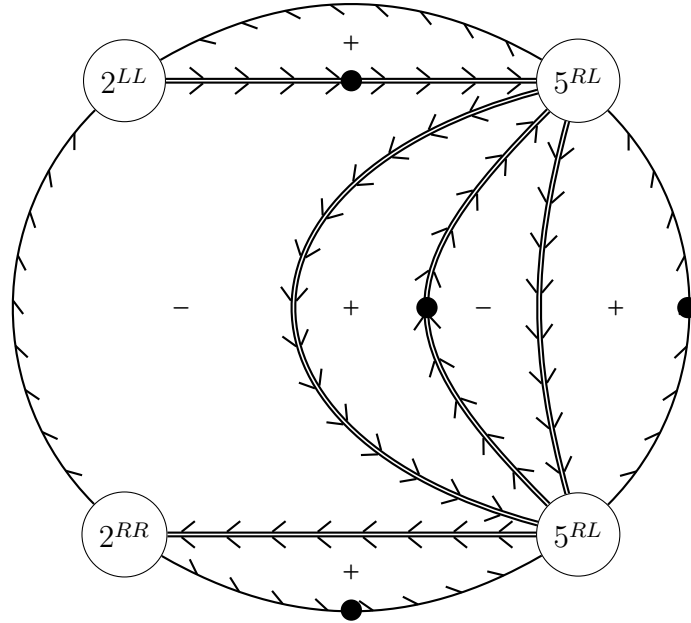


Figure 9.2.13: Dressed version of Fig. 9.2.10. Preimages $\Gamma^{-1}(t^*)$ of the reference point t^* under the covering map are depicted by black dots.

9.2.3 Symmetry group of interface PRR diagrams

In this short subsection, we discuss how to determine the symmetry group of an (interface) covering map from the corresponding diagram. To do so, we first define the automorphism group of (interface) PRR diagrams. From the definition, it will

directly be clear that the automorphism group of a diagram is isomorphic to the group of deck transformations of the covering map. We end the section by applying this isomorphism in two examples.

Definition of the symmetry group. As already mentioned above, an interface PRR diagram is not a just graph but comes with some additional structure. To begin with it is a fat graph – the edges and faces that meet at any vertex come with a cyclic order. In addition, the vertices of the diagram are labelled by the associated branch points of the covering map. For covering maps that arise from single-cycle twist fields, this labelling is injective, i.e. no two vertices possess the same label. However, in case of multi-cycle twist fields, vertices that carry the same label can arise.

Now that we understand the relevant structure of the interface PRR diagram, we can define its symmetry group: It is the group of all permutations of (vertices), edges and faces of the diagram that respect the labelling of vertices, the cyclic ordering at each vertex and of course the structure of the underlying graph. In terms of an explicit embedding representing a PRR diagram in Σ , the symmetries of the diagram are exactly the permutations of (vertices), edges and faces that descend from automorphisms of Σ which preserve the embedded labelled graph. Let us also note that the symmetry group of an interface PRR diagram is much smaller than the symmetry group of an associated graph which one obtains from the diagram by dropping the additional structure.

Identification with deck transformations. We now argue that the symmetry group of an interface PRR diagram is isomorphic to the group of deck transformations of the corresponding covering maps. Given a diagram, we pick a representative Γ of the equivalence class of covering maps that the diagram corresponds to. Clearly, each deck transformation gives a symmetry of the associated interface PRR diagram. Vice versa, if ϕ is an automorphism of Σ that descends to an symmetry of the diagram, then Γ and $\Gamma \circ \phi$ are two holomorphic maps that by construction coincide on the vertices and edges of the embedded interface PRR diagram. But since both Γ and $\Gamma \circ \phi$ are holomorphic (and the faces of diagrams are simply connected), this means that they also need to coincide on the faces, which implies that ϕ is a deck transformation of Γ .

Examples. The only diagram with a non trivial symmetry group discussed so far in this section is the one shown in Fig. 9.2.4. The symmetry group of this diagram is the $\mathbb{Z}_w$ sub-group of the rotations of the sphere that fix the two ramification points and cyclically permute the edges of the graph. On higher genus covering spaces the symmetry groups can be more general transformations than rotations. For instance,

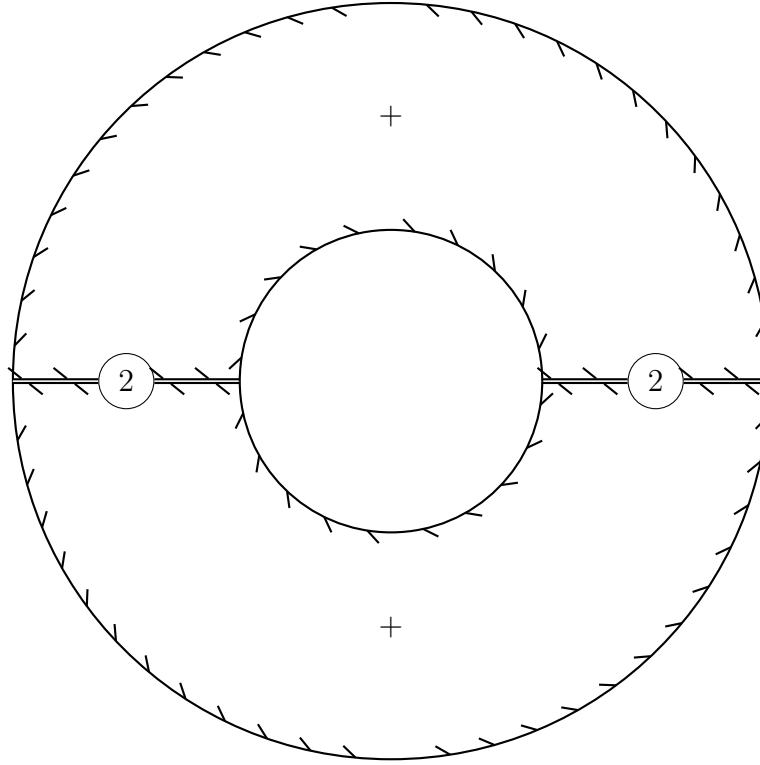


Figure 9.2.14: Cylinder contribution to $\langle \sigma_2 \sigma_2 \rangle$ at $N_+ = 2$, $N_- = 0$.

the cylinder diagram contributing to the two-point function $\langle \sigma_2 \sigma_2 \rangle$ of bulk twist 2 operators shown in Fig. 9.2.14 has a $\mathbb{Z}_2$ symmetry group that is generated by a modular S -transformation of the cylinder which flips the two boundaries but leaves the ramification points fixed.

9.2.4 Gauge fixing of interface PRR diagrams

In equation (9.1.22), we identified the coefficient $\alpha[\Gamma]$ that weighs the contribution of the covering map Γ to a connected sphere correlator, see eq. (9.1.18), with the size of a particular orbit under an S_N action. This subsection provides an analogous interpretation for the interface coefficient $\alpha[\Gamma]$, see eqs. (9.1.42) and (9.1.43), as the size of an orbit of some $S_{N_-} \times S_{N_+}$ action. It is straight forward to obtain such an interpretation from a generalisation of the monodromy based arguments that led to eq. (9.1.22). However, now that we have developed the PRR diagrammatics of interface covering maps we can do better and provide a more transparent description in terms of graphs.

Gauge fixed PRR diagrams. Consider an interface covering map Γ with $\deg[\Gamma_{\pm}] = N_{\pm}$. As in the case of bulk covering maps, we refer to a bijective assignment of integers in $\underline{N}_{\pm} := \{1, \dots, N_{\pm}\}$ to sheets of the covering as a gauge choice. In terms of

interface PRR diagrams, such a gauge choice assigns integers to the $\pm$ -faces. We call a PRR diagram whose faces are labelled in such a way *gauge fixed*.

Gauge fixed ramification indices. Gauge fixed PRR diagrams associate ramification points of Γ with cyclic permutations of $\underline{N}_- \sqcup \underline{N}_+$. These permutations are determined by circumnavigating the associated vertices of the diagram anti-clockwise and reading off the encountered $\underline{N}_\pm$ labels of the traversed faces. We refer to these permutations as *gauge fixed ramification indices*. To give a more detailed description it is useful to introduce different variables j and a to denote elements of $\underline{N}_-$ and $\underline{N}_+$, respectively.

With this convention, every ramification point of Γ with ramification index w whose branch point is in the upper half plane, is associated with a permutation $\mathbf{g} = (a_1 j_1 a_2 a_3 \dots a_w)$ of length $|\mathbf{g}|$. The length is constrained by $w \leq |\mathbf{g}| \leq 2w$. There must be exactly w a -labels. In addition one may have up to w j -labels which, however, must always be separated by a -labels in the permutation. The labels of interior ramification points whose branch point is in the lower half plane are defined analogously with a and j labels interchanged. The labels corresponding to branch points on the interface are also defined in the same manner. In this case, however, the associated permutation necessarily alternates between a - and j -labels and has length $|\mathbf{g}| = 2w$.

Finally, every type ω^{AB} boundary ramification point is associated with a sequence γ^{AB} of length ω , whose entries alternate between a - and j -labels. We choose sequences instead of permutations to describe gauge fixed boundary ramification since the boundary case involves more data than just the cyclic ordering: Boundary ramification points also distinguish a particular first and last element of the associated sequence of $\underline{N}_\pm$ labelled faces. At these end-point the sequence stops since the boundary of the covering space is reached.

Examples of various types of gauge fixed ramification indices are illustrated in Fig. 9.2.15.

Gauge fixed twist fields. Gauge fixed ramification indices allow us to define gauge-fixed twist fields labelled by such indices. These are non-local operators whose correlators are computed by a restriction of the sum over covering maps of local twist fields to gauge fixed PRR diagrams compatible with the gauge fixed ramification indices. Clearly, the constraint can be lifted by taking a linear combination of all possible ways to fix the gauge. Thus, local twist fields can be expressed as linear combinations of their gauge fixed counterpart. In fact, this is how we initially defined the interface changing operators in [4].

Gauge fixed interfaces. The gauge fixed PRR diagram also allows us to associate a permutation to the interface itself. For a covering map with degree of transmissivity

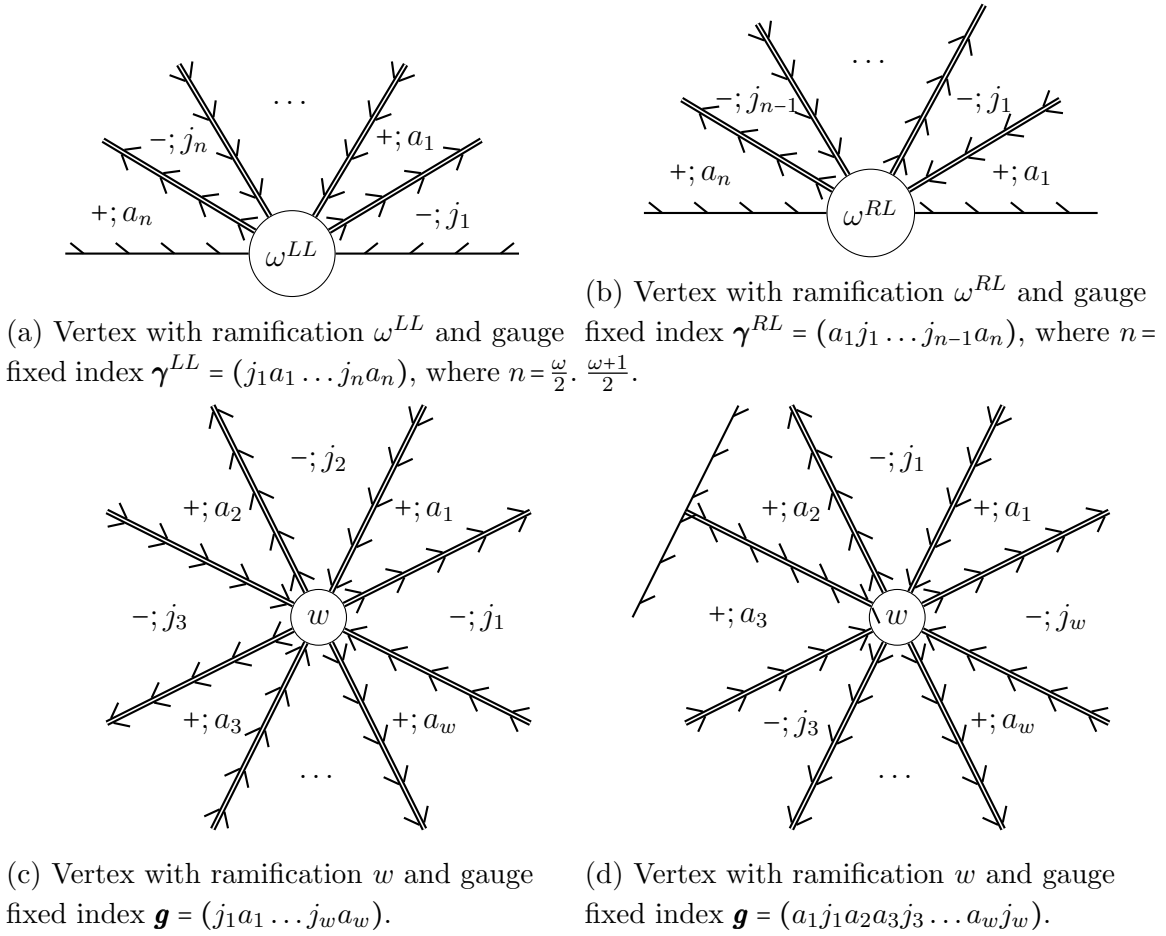


Figure 9.2.15: Examples of vertices and corresponding sequences.

P , the permutation ι that describes the gauge fixed glueing condition of the interface at the reference point t^* is a product $\iota = (j_1 a_1)(j_2 a_2) \dots (j_P a_P)$ of P commuting transpositions $(j_i a_i)$. Recall from the discussion at the end of Section 9.2.2 that the preimage $\Gamma^{-1}(t^*)$ can be represented by placing black dots on the edges of the PRR diagram according to some rules we spelled out before. For an interface covering with the degree of transmissivity P at t^* , there will be exactly P black dots on aligned double lines. Hence we can associate P pairs $(j_i a_i)$ to the choice of t^* . Note that the permutations that would describe the analogous data for any other choices of the reference point is fixed by the gauge fixed ramification indices along with the ι of one specific choice t^* of the reference point.

Symmetry factors from gauge fixed diagrams. Now we are actually ready to determine the weight factor $\alpha(\Gamma)$ we are after. Consider an arbitrary gauge fixed PRR diagram for Γ and read off the corresponding gauge fixed interior ramification indices $\mathbf{g}_1, \dots, \mathbf{g}_{n_c}$ and gauge fixed boundary ramification indices $\gamma_1^{A_1 B_1}, \dots, \gamma_{n_o}^{A_{n_o} B_{n_o}}$ as well as the permutation ι describing the glueing data at the reference point t^* .

Given all this data, the weight is given by

$$\alpha[\Gamma] = \frac{\deg[\Gamma_-]! \deg[\Gamma_+]!}{|\text{Deck}[\Gamma]|} = \left| [g_1, \dots, g_{n_c}, \iota, \gamma_1^{A_1 B_1}, \dots, \gamma_{n_o}^{A_{n_o} B_{n_o}}] \right|. \quad (9.2.4)$$

The number on the r.h.s. is the size of the orbit

$$[g_1, \dots, \iota, \gamma_1^{A_1 B_1}, \dots] := \{(g g_1 g^{-1}, \dots, g \iota g^{-1}, g \cdot \gamma_1^{A_1 B_1}, \dots) | g \in S_{N_-} \times S_{N_+}\}. \quad (9.2.5)$$

that is obtained from the gauged fixed ramification indices and ι by simultaneous action with elements g in the product $S_{N_-} \times S_{N_+}$ of the permutation groups. On the interior ramification indices g and on ι elements g act through the adjoint action. Recall that g_i and ι are themselves elements of $S_{N_-} \times S_{N_+}$ so that the adjoint action is well defined. The action of g on the gauged fixed boundary ramification indices γ_i is through the defining action of $S_{N_-} \times S_{N_+}$ on the entries of the $\gamma_i^{A_i B_i}$ sequences.

Diagrammatically, this group action just corresponds to the relabelling of faces in the gauge fixed PRR diagram. The action of a deck transformation on the covering space amounts to a permutation of faces of the diagram. Therefore, a gauge choice for the diagram allows us to identify $\text{Deck}[\Gamma]$ with a subgroup of $S_{N_-} \times S_{N_+}$. Since deck transformations leave the structure of the covering map invariant, this subgroup of $S_{N_-} \times S_{N_+}$ is exactly the stabiliser of the orbit (9.2.5) under the permutation action. Hence, equation (9.2.4) directly follows from the orbit-stabiliser theorem.

9.2.5 The large N expansion

In this section, we discuss a particular limit in which the computation of our interface correlation functions simplifies and non-planar diagrams are suppressed. Concretely, we study connected correlators on the sphere $X = \mathbb{S}^2$ with an interface $\mathcal{I}_{|a\rangle}^{(P)}$ inserted along the equator and the parameters $N_{\pm}$ and P scaling as

$$N_{\pm} \sim N, \quad P \sim N \quad \text{and} \quad N_{\pm} - P \sim N^{\frac{1}{2}} \quad (9.2.6)$$

with an expansion parameter N that we will take to be large. Our main finding is that contributions of genus g covering spaces with b boundaries to connected correlators of n_c bulk and n_o interface twist fields scale, to leading order in N , as

$$N^{1-g-\frac{b}{2}-\frac{n_c}{2}-\frac{n_o}{4}} \quad (9.2.7)$$

if the two point-functions of the operators are normalised to 1. Equation (9.2.7) is suggestive of an identification $N \sim g_s^{-2}$ of the parameter N with the string coupling of a potential holographic dual. As in the bulk case, the true relationship between the parameters of the symmetric orbifold and the string coupling is however slightly more subtle. In Section 9.4.3, we will show that the resemblance of eq. (9.2.7) to a string genus expansion can be traced back to the fact that eq. (9.2.6) with $N \sim g_s^{-2}$ describes the expectation values of $N_{\pm}$ and P in the truly holographic grand-canonical ensemble of interfaces to leading order in small g_s .

Connected sphere correlators. Towards the end of Section 9.1.2, we defined connected sphere correlators for bulk twist fields in symmetric product orbifolds. Connected correlation functions in the presence of the interface $\mathcal{I}_{|a}^{(P)}$ are defined analogously, i.e.

$$\langle \mathcal{I}_{|a}^{(P)} \mathcal{O} \rangle_{\text{Sym}^{N_-|N_+}(\mathcal{M})}^{\text{conn, unnormalised}} := \frac{1}{N_-!N_+!} \sum_{\substack{n_{\pm}=0 \\ n_-+n_+ \neq 0}}^{N_{\pm}} \sum_{p=0}^{\min(n_{\pm}, P)} \sum_{\Gamma \in C_{n_{\pm}, p}^{\text{conn}}[\mathcal{I}_{|a}^{(P)} \mathcal{O}]} \langle \mathcal{O}^{\Gamma} \rangle_{\mathcal{M}}, \quad (9.2.8)$$

where the third sum runs over the subset of connected interface coverings maps Γ . By definition, these are interface covering maps

$$\Gamma : \overline{\mathbb{C}}^{\sqcup(P-p)} \sqcup \mathbb{D}_-^{\sqcup(N_- - P - b_-)} \sqcup \mathbb{D}_+^{\sqcup(N_+ - P - b_+)} \sqcup \Sigma \rightarrow \overline{\mathbb{C}}, \quad (9.2.9)$$

with $b_{\pm} = n_{\pm} - p$ that satisfy three conditions: First of all, they restrict to the identity map on the $P - p$ spheres $\overline{\mathbb{C}}$. Furthermore they are also required to restrict to the identity embedding of the upper and lower hemispheres on the $N_{\pm} - P - b_{\pm}$ disconnected discs $\mathbb{D}_{\pm}$. Finally, we impose that Σ should be connected. The parameters $n_{\pm}$ and p correspond to the degrees and transmissivity of the restriction $\Gamma|_{\Sigma}$. For $\Gamma \in C_{n_{\pm}, p}^{\text{conn}}[\mathcal{I}_{|a}^{(P)} \mathcal{O}]$, we have

$$|\text{Deck}[\Gamma]| = (N_- - P - b_-)!(N_+ - P - b_+)(P - p)! |\text{Deck}[\Gamma|_{\Sigma}]| \quad (9.2.10)$$

which we can use to partially evaluate the correlator $\langle \mathcal{O}^{\Gamma} \rangle_{\mathcal{M}}$ on the r.h.s. of eq. (9.2.8) as

$$\langle \mathcal{O}^{\Gamma} \rangle_{\mathcal{M}} = \frac{N_-!N_+! g^{N_-+N_+-2P-(b_-+b_+)}}{(N_- - P - b_-)!(N_+ - P - b_+)(P - p)!} \frac{\langle \mathcal{O}^{\Gamma|_{\Sigma}} \rangle_{\mathcal{M}}}{n_-!n_+!}. \quad (9.2.11)$$

The factors in front of the remaining expectation value $\langle \mathcal{O}^{\Gamma|_{\Sigma}} \rangle_{\mathcal{M}}$ account for the contribution of the trivial connected components of the covering space. When inserted into eq. (9.2.8), the formula (9.2.11) leads to

$$\langle \mathcal{I}_{|a}^{(P)} \mathcal{O} \rangle_{\text{Sym}^{N_-|N_+}(\mathcal{M})}^{\text{conn}} = \sum_{\substack{n_{\pm}=0 \\ n_-+n_+ \neq 0}}^{N_{\pm}} \sum_{p=0}^{\min(n_{\pm}, P)} \sum_{\Gamma \in C_{n_{\pm}, p}^{\text{conn}}[\mathcal{I}_{|a}^{(P)} \mathcal{O}]} C_{\Gamma}(N_{\pm}, P) \frac{g^{-(b_-+b_+)}}{n_-!n_+!} \langle \mathcal{O}^{\Gamma|_{\Sigma}} \rangle_{\mathcal{M}} \quad (9.2.12)$$

for normalised connected correlators, where

$$C_{\Gamma}(N_{\pm}, P) := \frac{(N_- - P)!(N_+ - P)!P!}{(N_- - P - b_-)!(N_+ - P - b_+)(P - p)!} \quad (9.2.13)$$

Normalisation of twist fields. Eq. (9.2.12) makes the $N_{\pm}$ and P dependence and hence, with the scaling (9.2.6), the N dependence of correlators fully manifest, since $\langle \mathcal{O}^{\Gamma|_{\Sigma}} \rangle_{\mathcal{M}}$ is independent of the parameter N . To have a well behaved large

N expansion, we would like to replace bulk and interface twist fields by normalised versions

$$\hat{\sigma}_\omega^{AB} = \mathcal{N}_\omega^{AB} \sigma_\omega^{AB} \quad \text{and} \quad \hat{\sigma}_w(x) = \mathcal{N}_w^s \sigma_w(x) \quad (9.2.14)$$

whose two-point functions are of order 1 in the large N limit. The superscript s we placed on the normalization constants $\mathcal{N}_w^s$ is the sign of the imaginary part of the insertion point x . It can take one of the three values $-$, 0 and $+$, depending on whether x is in the lower half-plane, on the real line or in the upper half-plane. The correlators defining the normalizations $\mathcal{N}_\omega^{AB}$ of the interface operators are

$$\langle \sigma_\omega^{RR}(0) \sigma_\omega^{LL}(1) \rangle, \quad \langle \sigma_\omega^{LR}(0) \sigma_\omega^{LR}(1) \rangle \quad \text{and} \quad \langle \sigma_\omega^{RL}(0) \sigma_\omega^{RL}(1) \rangle, \quad (9.2.15)$$

each of which is computed by a single covering map. Let us refer to the three covering maps as $\Gamma_\omega^{RR} = \Gamma_\omega^{LL}$, Γ_ω^{LR} and Γ_ω^{RL} . With this notation, we have

$$\mathcal{N}_\omega^{AB} := C_{\Gamma_\omega^{AB}}(N_\pm, P)^{-\frac{1}{2}} \quad (9.2.16)$$

with C as defined in eq. (9.2.13). For $A \neq B$, the index ω is odd and our formula for C_Γ evaluates to

$$\mathcal{N}_\omega^{LR} = \sqrt{\frac{(P - \frac{\omega-1}{2})!}{(N_- - P)P!}} \quad \text{and} \quad \mathcal{N}_\omega^{RL} = \sqrt{\frac{(P - \frac{\omega-1}{2})!}{(N_+ - P)P!}}, \quad (9.2.17)$$

independently of the choice of reference point defining P and p . In the first case with $AB = LR$, for example, the degrees $n_\pm$ of Γ^{AB} are given by $n_- = \omega/2 + 1/2$ and $n_+ = \omega/2 - 1/2$ while the transmissivity is $p = \omega/2 - 1/2$. Hence, the parameters $b_\pm$ take the values $b_- = 1$ and $b_+ = 0$. Insertion into eq. (9.2.13) gives the first formula in eq. (9.2.17). The second formula is obtained in the same way with the role of $\pm$ exchanged. If $A = B$, on the other hand, the label ω must be even and the resulting normalizations

$$\mathcal{N}_\omega^{AA} = \begin{cases} \sqrt{\frac{(P - \frac{\omega-2}{2})!}{(N_- - P)(N_+ - P)P!}} & \text{if } t^* \in]0, 1[\\ \sqrt{\frac{(P - \frac{\omega}{2})!}{P!}} & \text{else} \end{cases} \quad (9.2.18)$$

explicitly depend on the choice of the reference point t^* . The derivation is similar to the previous one but now the degrees $n_\pm$ coincide, i.e. $n_- = n_+ = \omega/2$. The two possible decorations of the corresponding interface PRR diagrams with black dots lead to a degree of transmissivity $p = n_\pm + 1$ and $p = n_\pm$, respectively. This gives the two possible values of the normalization constant.

For bulk fields $\hat{\sigma}_w$ the normalization is determined along the same lines. The resulting normalization constants are given by

$$\mathcal{N}_w^\pm = \sqrt{\frac{w(N_\pm - w)!}{N_\pm!}} \quad \text{and} \quad \mathcal{N}_w^0 = \sqrt{\frac{w(P - w)!}{P!}}. \quad (9.2.19)$$

Large N expansion. If we let $N_{\pm}$ and P scale with a large parameter N as in eq. (9.2.6), then, to leading order in N ,

$$\mathcal{N}_{\omega}^{AB} = N^{-\frac{\omega}{4}}, \quad \mathcal{N}_w^s = N^{-\frac{w}{2}} \quad \text{and} \quad C_{\Gamma}(N_{\pm}, P) \sim N^{\frac{n_- + n_+}{2}}, \quad (9.2.20)$$

independently of the choice of the reference point, as one can directly read off from eqs. (9.2.13), (9.2.17), (9.2.18) and (9.2.19). Thus, we find that the contribution of elements $\Gamma \in C_{n_{\pm}, P}^{\text{conn}}$ to the correlator

$$\langle \mathcal{I}_{|a}^{(P)} \prod_{i=1}^{n_o} \hat{\sigma}_{\omega_i}^{A_i B_i} \prod_{i=1}^{n_c} \hat{\sigma}_{w_i} \rangle_{\text{Sym}^{N_- | N_+}(\mathcal{M})}^{\text{conn}} \quad (9.2.21)$$

scales, to leading order in N , as

$$N^{-\sum_{i=1}^{n_o} \frac{\omega_i}{4} - \sum_{i=1}^{n_c} \frac{w_i}{2} + \frac{n_- + n_+}{2}} = N^{1-g-\frac{b}{2}-\frac{n_c}{2}-\frac{n_o}{4}}, \quad (9.2.22)$$

where g and b are the genus and number of boundaries of the non-trivial connected component Σ of the covering space associated with Γ . The equality between the l.h.s. and r.h.s. of eq. (9.2.22) follows directly from the interface Riemann-Hurwitz formula (9.1.28). As advertised in the beginning of this section, higher genus and multi-boundary covering spaces are hence systematically suppressed in the large N limit.

9.3 Generalised Lunin-Mathur construction

In this short section, we present an interface version of the Lunin-Mathur construction to compute correlation functions of the seed theory on the covering surface Σ or rather partition functions which are relevant for pure twist field insertions in the symmetric product orbifold. Its first half, i.e. Subsection 9.3.1, is devoted to a review of the bulk method. In the second half, i.e. Subsection 9.3.2, we introduce the interface generalisation and then illustrate its usage in several examples. Appendix 9.D contains detailed computations supporting the main text. Throughout this section, whenever we write c without any further specifications, we mean the central charge $c_{\mathcal{M}}$ of the seed theory.

9.3.1 Review of the bulk method

As reviewed in Section 9.1, correlation functions in symmetric orbifolds are defined in terms of the data of associated covering maps. More specifically, correlators of (single cycle) twist fields on a Riemann surface X can, through the definitions (9.1.17) and (9.1.18), be expressed as a sum

$$\langle \sigma_{w_1}(x_1) \dots \sigma_{w_n}(x_n) \rangle_{\text{Sym}^N(\mathcal{M})}^{\text{unnormalised}} = \sum_{\Gamma} Z[\Gamma] \quad \text{with} \quad Z[\Gamma] := \frac{Z_1^{\Sigma}}{\text{Deck}[\Gamma]} \quad (9.3.1)$$

where Z_1^Σ denotes the partition function Z_1^Σ of the seed theory on the covering space Σ . As before, the sum runs over coverings $\Gamma : \Sigma \rightarrow X$ of X whose branch points are exactly $x_1, \dots, x_n$ with indices $w_1, \dots, w_n$. In this subsection, we review the Lunin-Mathur method [310] for the evaluation of Z_1^Σ .

General setup. We assume that for each conformal equivalence class of Riemannian structures ds_Σ^2 on the covering surface Σ , there is a representative ds_Σ^2 for which the value of the vacuum partition function of the seed theory is known. The problem whose solution we describe in the paragraphs below is how to express Z_1^Σ in terms of $Z_1^{\hat{\Sigma}}$. Two steps need to be performed to achieve this goal. First, we need to compute the conformal factor ϕ in

$$ds_\Sigma^2 = e^\phi ds_{\hat{\Sigma}}^2, \quad (9.3.2)$$

relating the metric $ds_\Sigma^2 = \Gamma^* ds_X^2$ to the reference metric $ds_{\hat{\Sigma}}^2$. Once ϕ has been computed, we need to calculate the value of the Liouville action on ϕ in a second step,

$$S_L[\phi] = \frac{c}{96\pi} \int_\Sigma d^2t \sqrt{g_\Sigma} \left(\partial_\mu \phi \partial_\nu \phi g_\Sigma^{\mu\nu} + 2R^{\hat{\Sigma}} \phi \right). \quad (9.3.3)$$

The partition function Z_1^Σ that appears on the right hand side of eq. (9.3.1) is then given by

$$Z_1^\Sigma = e^{S_L[\phi]} Z_1^{\hat{\Sigma}}. \quad (9.3.4)$$

It is thus reduced to the reference partition function $Z_1^{\hat{\Sigma}}$ which we assumed to be known.

The sphere. For concreteness, we focus on the case where both X and Σ are spheres. Following [310], we introduce a family of metrics ds_δ^2 on the Riemann sphere $\overline{\mathbb{C}}$ which are parametrised by some $\delta > 0$. On $\mathbb{C}$, these metrics are given by

$$ds_\delta^2 = \begin{cases} dzd\bar{z} & \text{if } |\delta z| < 1 \\ \frac{dzd\bar{z}}{|z\delta|^4} & \text{else} \end{cases}. \quad (9.3.5)$$

They extend continuously to the Riemann sphere. Given a seed conformal field theory with central charge c , its partition function on $\overline{\mathbb{C}}$ equipped with ds_δ^2 is

$$Z_1^{\overline{\mathbb{C}}_\delta} = Q \delta^{-\frac{c}{3}}, \quad (9.3.6)$$

where Q is an arbitrary normalisation constant [310]. We take ds_δ^2 as the reference metric on $\hat{\Sigma} = \overline{\mathbb{C}}_\delta$. When dealing with the sphere X that is covered by Σ we shall use the same metric, but denote the parameter by η instead of δ , i.e.

$$X = \overline{\mathbb{C}}_\eta \quad \text{for some small} \quad \eta > 0. \quad (9.3.7)$$

We fix the normalization constant to $Q = \eta^{\frac{c}{3}}$ such that the sphere partition function of the seed theory is trivial, $Z_1^X = 1$. Without loss of generality, we assume that all twist fields of the correlator that we would like to compute are inserted in the region $|\eta z| < 1$. Since we are only interested in covering maps up to the action of automorphisms of Σ , we can fix

$$\Gamma(\infty) = \infty. \quad (9.3.8)$$

As explained above, we compute Z_1^Σ by relating it to the partition function $Z_1^{\hat{\Sigma}}$ with $\hat{\Sigma} = \overline{\mathbb{C}}_\delta$ for an appropriately chosen δ .

Regulating the Liouville action. Consider the subset $B_{\delta^{-1}} \subseteq \mathbb{C}$ that consists of all points $z \in X$ with $|z\delta| < 1$. We denote the preimage of this ball with respect to Γ by $\Sigma_0 := \Gamma^{-1}(B_{\delta^{-1}})$. On the subset $\Sigma_0 \subseteq \Sigma$, the metric is given by

$$ds_\Sigma^2|_{\Sigma_0} = \Gamma^* ds_X^2|_{|z\delta|<1} = d\Gamma(t)d\bar{\Gamma}(t) = \left| \frac{d\Gamma}{dt} \right|^2 dt d\bar{t} \quad (9.3.9)$$

and therefore

$$\phi = \log \left(\frac{d\Gamma}{dt} \right) + \text{h.c.}, \quad (9.3.10)$$

which diverges if the derivative Γ' vanishes and the metric of Σ degenerates. Thus, at the ramification points $R[\Gamma]$ of Γ , it is necessary to regulate the Liouville action. We explain the details of this regularisation in App. 9.D.1 following [310]. It involves cutting out small neighbourhoods of each ramification point from Σ . The result of the detailed computation performed in that appendix is

$$Z_1^\Sigma = \eta^{-\frac{c}{3}(\text{Deg}[\Gamma]-1)} \prod_{r \in R[\Gamma]} w^{-\frac{c}{12}(w-1)} |a_r|^{-\frac{c}{12}(1-\frac{1}{w})} \prod_{p \in P[\Gamma]} |b_p|^{-\frac{c}{6}}, \quad (9.3.11)$$

where η parametrises the metric of X (see eq. (9.3.7)), $R[\Gamma]$ is the set of ramification points of Γ , $P[\Gamma]$ is the set of poles of Γ in $\Sigma \setminus \{\infty\}$ and the complex numbers a_r and b_p are defined via the local behaviour

$$\Gamma(r+t) = \Gamma(r) + a_r t^w + O(t^{w+1}) \quad \text{and} \quad \Gamma(p+t) = \frac{b_p}{t} + O(1) \quad (9.3.12)$$

of the covering map Γ near the ramification points and poles. While formula (9.3.11) was derived in a few examples in the original work [310] an explicit, fully detailed derivation was only recently published in [282]. While the latter uses somewhat different methods, the derivation we spell out in Appendix 9.D.1 follows exactly the route of [310]. It will also turn out to directly extend to the interface case. Before we go there, let us illustrate the Lunin-Mathur construction through an example.

Example: Two-point functions. As an application of eq. (9.3.11), let us compute the two-point functions of two bulk fields σ_w with identical index $w_1 = w = w_2$ in the symmetric product orbifold. Without loss of generality, we can insert the first field at $z_1 = 0$ and then denote the insertion point of the second twist field by $z_2 = z$. In this setup, only a single class of coverings contributes. We choose the representative

$$\Gamma(t) = z \frac{t^w}{t^w - (t-1)^w}. \quad (9.3.13)$$

This covering map has two ramification points $R[\Gamma] = \{0, 1\}$, around which it expands as

$$\Gamma(t) = 0 + z(-1)^{w+1}t^w + O(t^{w+1}) \quad \text{and} \quad \Gamma(1+t) = z + zt^w + O(t^{w+1}). \quad (9.3.14)$$

Moreover, Γ has $w-1$ finite poles which are located at the zeroes of the denominator,

$$t_k := \left(1 - e^{2\pi i \frac{k}{w}}\right)^{-1} \quad (9.3.15)$$

where k runs from $k=1$ to $k=w-1$, i.e. $P[\Gamma] = \{t_1, \dots, t_{w-1}\}$. The residues at these poles are easy to determine,

$$b_k = \frac{z}{4w \sin^2\left(\frac{k\pi}{w}\right)}. \quad (9.3.16)$$

Now we have assembled all the data we need in order to evaluate the formula (9.3.11). The result is given by

$$Z_1^\Sigma = |z|^{-2\Delta^{(w)}} w^{\frac{c}{3}} \eta^{-\frac{c}{3}(w-1)} \quad (9.3.17)$$

with

$$\Delta^{(w)} = \frac{c}{12} \left(w - \frac{1}{w}\right). \quad (9.3.18)$$

The last ingredient we need in order to evaluate the unnormalised correlator (9.3.1) is the number of Deck transformations, which is given by

$$|\text{Deck}[\Gamma]| = w(N-w)! . \quad (9.3.19)$$

Hence, the the unnormalised two-point function is given by

$$\langle \sigma_w(0) \sigma_w(z) \rangle_{\text{Sym}^N(\mathcal{M})}^{\text{unnormalised}} = \frac{w^{\frac{c}{3}} \eta^{-\frac{c}{3}(w-1)}}{w(N-w)!} |z|^{-2\Delta^{(w)}}. \quad (9.3.20)$$

We use the term “unnormalised” in the sense of definition (9.1.9). In particular, the normalised two-point function is

$$\langle \sigma_w(0) \sigma_w(z) \rangle_{\text{Sym}^N(\mathcal{M})} = N! \langle \sigma_w(0) \sigma_w(z) \rangle_{\text{Sym}^N(\mathcal{M})}^{\text{unnormalised}} \quad (9.3.21)$$

and not just $|z|^{-2\Delta^{(w)}}$.

9.3.2 The interface generalisation

Let us now turn to the interface generalisation of the Lunin-Mathur construction. The interface generalisation introduces two modifications to the evaluation of correlators which we detail in the next two paragraphs. The first modification arises from the fact that we now need to evaluate the Liouville action on surfaces with boundaries which requires in particular to add an additional boundary term, see eq. (9.3.22). The second modification is that bulk covering maps are replaced by interface covering maps. This can be dealt with by applying the method of images. After generally discussing disk correlators in the third paragraph of the section, we conclude by computing several example correlation functions.

Boundary Liouville action. As in Section 9.3.1, we assume that there is a reference geometry $\hat{\Sigma}$ such that the partition function $Z_1^{\hat{\Sigma}}$ on $\hat{\Sigma}$ is known. As before, the partition function Z_1^{Σ} on Σ with the induced metric (9.3.2) is then given by eq. (9.3.4). The difference to the bulk case is that the Liouville action now also has a boundary contribution. Concretely,

$$S_L[\phi] = \frac{c}{96\pi} \int_{\Sigma} d^2t \sqrt{g_{\hat{\Sigma}}} \left(\partial_{\mu} \phi \partial_{\nu} \phi g_{\hat{\Sigma}}^{\mu\nu} + 2R^{\hat{\Sigma}} \phi \right) + \frac{c}{24\pi} \int_{\partial\Sigma} \sqrt{\gamma_{\hat{\Sigma}}} K^{\hat{\Sigma}} \phi, \quad (9.3.22)$$

where γ is the induced metric on the boundary $\partial\Sigma$ of the covering surface and $K^{\hat{\Sigma}}$ is the extrinsic curvature of the boundary.

The method of images. In the case where X is a disk, we can use the method of images in order to reduce the discussion of interface covering maps $\Gamma : \Sigma \rightarrow X$ to that of coverings Γ' of the sphere by the Schottky¹⁰ double $\Sigma \# \bar{\Sigma}$. The restriction of the doubled map to Σ coincides with Γ . On $\bar{\Sigma}$, the covering Γ' is given by $\Gamma'(\bar{z}) := \overline{\Gamma(z)}$. The doubled map Γ' has two copies of each interior ramification point of Γ . Additionally, each boundary ramification point with index ω leads to a single ramification point with index $w = \omega$ for Γ' . Below, we use the method of images for two purposes. First, it allows us to construct interface covering maps as restrictions of bulk covering maps. Second, it allows us to evaluate the bulk part of the Liouville action (9.3.22) by applying eq. (9.3.11) to Γ' .

The hemisphere. In the case of coverings of the upper hemisphere in $\bar{\mathbb{C}}_{\eta}$ by another hemisphere, where we take the reference metric to again be of the type ds_{δ}^2 , the extrinsic curvature vanishes along the entire boundary of the covering space. Thus, only the bulk part contributes to the Liouville action. In Appendix 9.D.2,

¹⁰The Schottky double is a closed Riemann surface obtained by taking two copies Σ and $\bar{\Sigma}$ of the Riemann surface Σ and glueing them along the boundary.

we evaluate this bulk part using the method of images. This leads to the interface generalisation

$$Z_1^\Sigma = \eta^{-\frac{\varepsilon}{6}(N_- + N_+ - 1)} \left(\prod_{r \in R_B[\Gamma]} w^{w-1} a_r^{1-\frac{1}{w}} \prod_{p \in P_B[\Gamma]} b_p^2 \right)^{-\frac{c}{12}} \left(\prod_{r \in R_I[\Gamma]} \omega^{w-1} a_r^{1-\frac{1}{w}} \prod_{p \in P_I[\Gamma]} b_p^2 \right)^{-\frac{c}{24}} \quad (9.3.23)$$

of eq. (9.3.11) for interface coverings of the upper half plane by itself. Here, N_+ and N_- are the degrees of the covering map Γ on the upper and lower half plane. The sets $R_B[\Gamma]$ and $R_I[\Gamma]$ contain the bulk and interface ramification points of Γ . Likewise, the sets $P_B[\Gamma]$ and $P_I[\Gamma]$ contain all complex and real poles, except for ∞ (which we however continue to w.l.o.g. assume to be a pole). The coefficients a_r and b_p for interface ramification points and poles are defined as we defined the analogous bulk coefficients in eq. (9.3.12).

Example: Bulk one-point functions. Let us consider the one-point function

$$\langle \mathcal{I}_{|a_\pm}^{(P)} \sigma_w(z) \rangle_{\text{Sym}^{N_-|N_+}(\mathcal{M})}^{\text{unnormalised}} \quad (9.3.24)$$

of a bulk twist field $\sigma_w(z)$ inserted on the upper half plane, with the interface $\mathcal{I}_{|a_\pm}^{(P)}$ inserted along the real line. The correlation function can be non-trivial only if $N_+ - P \geq w$. In that case, it evaluates to

$$\langle \mathcal{I}_{|a_\pm}^{(P)} \sigma_w(z) \rangle_{\text{Sym}^{N_-|N_+}(\mathcal{M})}^{\text{unnormalised}} = \frac{g_-^{N_- - P} g_+^{N_+ - P - w} Z_{|a_+}^{(w)}}{P!(N_- - P)!(N_+ - P - w)!w}, \quad (9.3.25)$$

where $g_\pm$ are the g -functions of $|a_\pm\rangle$ and $Z_{|a_+}^{(w)}$ captures the contribution of a w -fold cover of the upper half plane by itself with a ramification point at z . To compute $Z_{|a_\pm}^{(w)}$, we choose the explicit covering map

$$\Gamma(t) := z \frac{(it - 1)^w}{(it - 1)^w - (it + 1)^w} + \bar{z} \frac{(-it - 1)^w}{(-it - 1)^w - (-it + 1)^w}. \quad (9.3.26)$$

The way Γ is written above makes manifest two important properties, namely

$$\Gamma(\bar{t}) = \overline{\Gamma(t)} \quad \text{and} \quad \Gamma(i) = z. \quad (9.3.27)$$

A more economical way of writing Γ is

$$\Gamma(t) = \frac{z(it - 1)^w - \bar{z}(it + 1)^w}{(it - 1)^w - (it + 1)^w}. \quad (9.3.28)$$

The coefficient associated with the ramification point i is

$$a_i = \frac{z - \bar{z}}{(2i)^w}. \quad (9.3.29)$$

The poles and residues are

$$P[\Gamma] = \left\{ \cot\left(\frac{k\pi}{w}\right) \right\}_{k=1}^{w-1} \quad \text{and} \quad b_k = \frac{i(z - \bar{z})}{2w \sin^2\left(\frac{k\pi}{w}\right)}. \quad (9.3.30)$$

Plugging this into eq. (9.3.23) gives

$$Z_{|a_+}^{(w)} = g_+ \eta^{-\frac{c}{6}(w-1)} |z - \bar{z}|^{-\Delta^{(w)}} w^{\frac{c}{6}} \quad (9.3.31)$$

and therefore,

$$\langle \mathcal{I}_{|a_\pm}^{(P)} \sigma_w(z) \rangle_{\text{Sym}^{N_-|N_+}(\mathcal{M})}^{\text{unnormalised}} = \frac{g_-^{N_- - P} g_+^{N_+ - P - (w-1)} \eta^{-\frac{c}{6}(w-1)} w^{\frac{c}{6}}}{P!(N_- - P)!(N_+ - P - w)! w} |z - \bar{z}|^{-\Delta^{(w)}}. \quad (9.3.32)$$

Example: Interface two-point functions. The evaluation of the two-point functions

$$\langle \mathcal{I}_{|a}^{(P)} \sigma_\omega^{RR}(0) \sigma_\omega^{LL}(x) \rangle_{\text{Sym}^{N_-|N_+}(\mathcal{M})}^{\text{unnormalised}} \quad \text{and} \quad \langle \mathcal{I}_{|a}^{(P)} \sigma_\omega^{RL}(0) \sigma_\omega^{RL}(x) \rangle_{\text{Sym}^{N_-|N_+}(\mathcal{M})}^{\text{unnormalised}} \quad (9.3.33)$$

directly reduces to that of the bulk two-point function (9.3.20) for even and odd $w = \omega$ respectively via the method of images. We thus deduce that the two correlators above are given by¹¹

$$\langle \mathcal{I}_{|a}^{(P)} \sigma_\omega^{RR}(0) \sigma_\omega^{LL}(x) \rangle_{\text{Sym}^{N_-|N_+}(\mathcal{M})}^{\text{unnormalised}} = \frac{g^{N_- + N_+ - 2P + 1} \omega^{\frac{c}{6}} \eta^{-\frac{c}{6}(\omega-1)}}{(N_- - P)!(N_+ - P)!(P - \frac{\omega}{2})!} |x|^{-2\Delta_\omega^{AB}}, \quad (9.3.34)$$

where the reference point for the interface is -1 and, with the same reference point,

$$\langle \mathcal{I}_{|a}^{(P)} \sigma_\omega^{RL}(0) \sigma_\omega^{RL}(x) \rangle_{\text{Sym}^{N_-|N_+}(\mathcal{M})}^{\text{unnormalised}} = \frac{g^{N_- + N_+ - 2P} \omega^{\frac{c}{6}} \eta^{-\frac{c}{6}(\omega-1)}}{(N_- - P)!(N_+ - P - 1)!(P - \frac{\omega-1}{2})!} |x|^{-2\Delta_\omega^{AB}}. \quad (9.3.35)$$

These results follow from applying (9.3.23) to the covering map (9.3.13) with real $z = x$. Here,

$$\Delta_\omega^{AB} = \frac{c}{24} \left(\omega - \frac{1}{\omega} \right), \quad (9.3.36)$$

which agrees with the special case $\Delta = 0$ of eq. (9.1.62). For $\omega = 2$, the interface two-point function was essentially already computed in [125], see Section 5.3 of that reference.

¹¹Assuming of course a sufficiently large number of sheets N_- and N_+ on the upper/lower hemisphere as well as a sufficient number of reflecting and transmitting boundary conditions to ensure that the correlator is non vanishing.

Example: Interface four-point functions. The disk contribution to general interface four-point functions

$$\langle \mathcal{I}_{|a}^{(P)} \sigma_{\omega_1}^{A_1, B_1}(x_1) \sigma_{\omega_2}^{A_2, B_2}(x_2) \sigma_{\omega_3}^{A_3, B_3}(x_3) \sigma_{\omega_4}^{A_4, B_4}(x_4) \rangle_{\text{Sym}^{N-|N_+}(\mathcal{M})}^{\text{unnormalised}} \quad (9.3.37)$$

can be evaluated using section 2 of the notebook `CoveringMaps.nb` provided in the ancillary files of this publication. The corresponding covering maps, via the method of images, reduce to covering maps of the sphere in the absence of an interface, which satisfy the extra constraint $\Gamma(s) = \overline{\Gamma(\bar{s})}$. The latter can be computed by first using the approach described in Appendix 9.D.3 to compute generic covering maps and then restricting to the subset that satisfies the additional reality constraint. By applying $SL(2; \mathbb{R})$ transformations of the upper hemisphere, we can w.l.o.g. ensure that $x_1 = 0$, $x_3 = 1$ and $x_4 = -1$ with corresponding ramification points $s_i = x_i$ for $i \neq 2$. The value of s_2 on the other hand is a non trivial function of the parameter x_2 . Sections 2.2 and 2.3 of the notebook automatically generate plots from the covering maps, which resemble the diagrams discussed in Section 9.2. Using eq. (9.3.23) it is straight forward to evaluate correlation functions by simply determining the poles as well as the local expansions around ramification points of the explicitly computed covering maps. For eq. (9.3.23) to be applicable, one has to, however, ensure that $\Gamma(\infty) = \infty$ i.e. pick an arbitrary pole and conformally map it to ∞ . This operation is performed in Section 2.4. of the notebook.

Example: Interface-Interface-Bulk three-point functions. To study interface-interface-bulk three-point functions, we fix the $SL(2, \mathbb{R})$ symmetry of the disk by mapping the two interface operators to 0 and 1 and the real part of the bulk operator to $\frac{1}{2}$. The remaining kinematical freedom consists then of a single positive real number χ , the imaginary part of the position of the bulk twist field. Hence, the correlators take the form

$$\langle \mathcal{I}_{|a}^{(P)} \sigma_{\omega_1}^{A_1, B_1}(0) \sigma_w(\frac{1}{2} + i\chi) \sigma_{\omega_2}^{A_2, B_2}(1) \rangle_{\text{Sym}^{N-|N_+}(\mathcal{M})}^{\text{unnormalised}} \quad (9.3.38)$$

and are computable by determining the bulk covering maps of the sphere with ramification points 0 and 1 of index ω_1 and ω_2 that are mapped to 0 and 1, as well as ramification points at $\frac{1}{2} \pm is$ for some real number s , which map to $\frac{1}{2} \pm i\chi$ with ramification index w . The maps that contribute non-trivially to the correlator are again those that satisfy $\Gamma(z) = \overline{\Gamma(\bar{z})}$. They are computed and visualised in Section 3 of the `CoveringMaps.nb` notebook.

Example: Bulk two-point functions. The last type of correlators whose evaluation reduces to the computation of sphere coverings with four ramification points are bulk two-point functions. Again, the kinematical freedom reduces to a single real number, which we choose to be the imaginary part of the insertion points of the two

bulk fields after using the $\mathrm{SL}(2, \mathbb{R})$ symmetry to fix the real parts to -1 and 1 and sending the intersection of the line through the two insertion points with $\mathrm{Im}(z) = 0$ to ∞ . The resulting correlator

$$\langle \mathcal{I}_{[a]}^{(P)} \sigma_{w_1}(-1 + i\chi) \sigma_{w_2}(1 + i\chi) \rangle_{\mathrm{Sym}^{N_-|N_+}(\mathcal{M})}^{\mathrm{unnormalised}} \quad (9.3.39)$$

can be computed from covering maps with ramification points at $-1 \pm is$ and $1 \pm is$ for some $s > 0$, which are mapped to $-1 \pm i\chi$ and $1 \pm i\chi$ with ramification index w_1 and w_2 . Section 4 of the `CoveringMaps.nb` notebook computes and visualises the relevant $\Gamma(z) = \overline{\Gamma(\bar{z})}$ maps.

9.4 String theory and the interface GCE

So far, we have described correlation functions at fixed $N_{\pm}$ and P . The purpose of this section is to discuss a Grand-Canonical Ensemble (GCE) in which the parameters $N_{\pm}$ and P are averaged with a fixed chemical potential. As reviewed in Subsection 9.4.1, studying such an ensemble is strongly motivated by holography. In fact, super string theory on $\mathrm{AdS}_3 \times S^3 \times \mathbb{T}^4$ at minimal pure NS-NS flux $k = 1$ is known to be precisely dual to a grand-canonical ensemble of symmetric product orbifolds whose construction we review in Subsection 9.4.2. The key result of this section is a novel grand-canonical ensemble of interfaces introduced in Subsection 9.4.3. Using this new construction, we show that the grand-canonical ensemble of correlation functions studied here structurally matches the genus expansion of string scattering amplitudes. We discuss the expectation values of $N_{\pm}$ and P at the end of Subsection 9.4.3, and show the leading-order agreement to the canonical ensemble result shown in Sec. 9.2.5.

9.4.1 Holographic motivation

Pure NSNS $\mathrm{AdS}_3 \times S^3 \times \mathbb{T}^4$ can be obtained from $\mathbb{R}^6 \times \mathbb{T}^4$ as the near horizon limit of Q_1 fundamental strings and Q_5 NS5 branes, with four directions of the latter wrapping $\mathbb{T}^4$. String perturbation theory around this pure NSNS background is described in the hybrid formalism by a world sheet theory based on the $\mathrm{PSU}(1,1|2)_k$ WZNW model. The level k of the model can be directly identified with the number Q_5 of the brane construction. However, the naive expectation that the string coupling g_s is fixed in terms of Q_1 is not realised. Instead of being determined by the expectation values of fields in the target space, the string coupling is a continuous parameter of the world sheet theory¹². This parameter can be interpreted as a chemical potential controlling the averaging over different values of Q_1 in a grand-canonical ensemble.

¹²For more details consult e.g. [312–315] on pure NSNS AdS_3 strings in the RNS formalism for $Q_5 \geq 2$ and [103, 278] on the special case $Q_5 = 1$ in the hybrid formalism.

Since string perturbation theory therefore does not compute scattering amplitudes in a fixed Q_1 background, but in the grand-canonical ensemble, one should expect the space time CFT description of these amplitudes to analogously be given in terms of correlation functions in a grand-canonical ensemble. For minimal NSNS flux $k = 1$, this expectation has been confirmed in great detail. Concretely, it was established [69, 270, 277, 278, 308] that string theory on $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$ at $k = 1$ is dual to a grand-canonical ensemble of free symmetric orbifold theories $\text{Sym}^{Q_1}(\mathbb{T}^4)$. Section 3 of Reference [266] provides a detailed description of this grand-canonical ensemble, which we partially review in the next subsection, as well as a broader discussion of the string theoretic context that we have sketched above.

Reference [4] provided evidence strongly suggesting that AdS_2 branes in the $k = 1$ background are similarly not dual to a single interface, but rather an ensemble of the interfaces $\mathcal{I}^{(P)}$. We conjecture this to be true not only in the sense that $N_{\pm}$ need to be averaged over since the bulk theory is not a fixed N symmetric product orbifold, but also in the sense that P is averaged over all its possible values for each fixed $N_{\pm}$.

To qualitatively motivate this expectation, consider that, in string theory, the AdS_2 branes are realised by the insertion of a probe string on which the fundamental strings which create the $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$ background can end [299]. The fixed P interface $\mathcal{I}^{(P)}$ corresponds to a setup where P parallel fundamental strings go through the probe string and another $Q_1^- - P$ and $Q_1^+ - P$ end on it extending into either of the two directions parallel to the other fundamental strings and orthogonal to the probe. As recalled above, the issue that the $\text{PSU}(1,1|2)_1$ world sheet theory at first glance cannot account for the parameter Q_1 , or for that matter $Q_1^{\pm}$, is resolved by understanding that it describes a grand-canonical ensemble. Since the AdS_2 boundary states of the world sheet theory do not have a parameter describing the value of P , it is natural to also expect this parameter to be controlled by a chemical potential, which is set by to the only available world sheet parameter – the string coupling. While the result for the torus partition function obtained in Reference [4] is consistent with this expectation, it is not possible to directly infer the precise relation between the chemical potential controlling P and the string coupling from it. The purpose of this section is to go beyond these prior results by fixing the interface chemical potentials in terms of the string coupling, and showing that the ensemble defined in this way produces correlators that structurally reproduce the string genus expansion.

9.4.2 The bulk grand-canonical ensemble

Among the first attempts to connect symmetric product orbifolds to potentially dual string theories it was common to consider correlators of twist fields $\tilde{\sigma}_w$ with the

normalisation

$$\tilde{\sigma}_w = \sqrt{\frac{(N-w)!w}{N!}} \sigma_w \quad (9.4.1)$$

at fixed N . Here, σ_w is the twist field with the canonical normalisation used in Sec. 9.1.2. At leading order in $N \gg 1$, the connected part of these correlation functions indeed shares some features of scattering amplitudes in string perturbation theory. As observed in [284, 310], for instance, the contribution of genus g covering spaces to a connected correlation function of bulk twist fields scales as

$$\left\langle \prod_{i=1}^{n_c} \tilde{\sigma}_{w_i} \right\rangle_{\text{conn}}^{N,g} \sim N^{1-g-\frac{n_c}{2}} \quad (9.4.2)$$

with N . Upon the identification $g_s^2 \sim N^{-1}$, this matches the scaling of genus g contributions to string scattering amplitudes with g_s . However, this proposal breaks down as soon as subleading N corrections or disconnected contributions are considered.

These issues can be fixed by working in the grand-canonical ensemble, as argued in Sec. 3.3 of [266]. To arrive at this conclusion, consider the partition function

$$Z_N^X[J] := \left\langle \exp \left(\sum_{w=1}^{\infty} \int d^2z g_s^w J_w(z) \sigma_w(z) \right) \right\rangle_{\text{Sym}^N(\mathcal{M})}^{\text{unnormalised}} \quad (9.4.3)$$

of $\text{Sym}^N(\mathcal{M})$ with sources $J_1, \dots, J_w$ on a Riemann surface X of genus G defined using the unnormalised symmetric orbifold correlators of eq. (9.1.9). Here, g_s is the parameter that turns out to control the genus expansion of symmetric orbifold correlators. It does not only multiply the sources in the definition (9.4.3) but, more importantly, also parametrises the chemical potential μ which controls N in the grand-canonical ensemble. More concretely, the grand-canonical partition function on X is defined as

$$Z_{g_s}^X[J] := \sum_{N=0}^{\infty} \mu^N Z_N^X[J] \quad \text{where} \quad \mu := g_s^{2G-2}. \quad (9.4.4)$$

The key feature of this ensemble is that grand-canonical correlation functions

$$\left\langle \prod_{i=1}^{n_c} \sigma_{w_i}(z_i) \right\rangle_{g_s}^X := \left[\prod_{i=1}^{n_c} \frac{\delta}{\delta J_{w_i}(z_i)} \right] \frac{Z_{g_s}^X[J]}{Z_{g_s}^X[0]} \Big|_{J=0} \quad (9.4.5)$$

have a genus expansion that, to all orders in g_s and for both connected and disconnected contributions, structurally matches the genus expansion of string perturbation theory with string coupling g_s . This can be shown in three simple steps.

First, we expand the exponential in eq. (9.4.3) and express the resulting correlation functions in terms of covering maps, which leads to

$$Z_N^X[J] = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{w=1}^{\infty} \int d^2z J_w(z) \right)^n \sum_{\Gamma} Z_{g_s}[\Gamma]. \quad (9.4.6)$$

The sum runs over all coverings Γ of X with the appropriate¹³ ramification points and

$$Z_{g_s}[\Gamma] := \frac{g_s^{w[\Gamma]}}{\text{Deck}[\Gamma]} \langle \mathbb{1} \rangle_{\mathcal{M}}^{\Sigma}, \quad (9.4.7)$$

where $w[\Gamma]$ is the total ramification index of Γ i.e. the sum of the indices of all ramification points. Each covering space that is involved has

$$c_{\Gamma} = \sum_{i=1}^k n_i \quad (9.4.8)$$

connected components falling into k isomorphism classes of sizes $n_1, \dots, n_k$. The restriction Γ_i of Γ to any element of the i^{th} isomorphism class has some degree N_i and is associated with the contribution $Z_{g_s}[\Gamma_i]$ to the correlation function. In terms of this decomposition,

$$\text{Deck}[\Gamma] = \prod_{i=1}^k S_{n_i} \times \text{Deck}[\Gamma_i]^{n_i} \quad (9.4.9)$$

and thus $Z_{g_s}[\Gamma]$ can be expressed as

$$Z_{g_s}[\Gamma] = \prod_{i=1}^k \frac{1}{n_i!} (Z_{g_s}[\Gamma_i])^{n_i}. \quad (9.4.10)$$

Therefore,

$$Z_N^X[J] = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{w=1}^{\infty} \int d^2 z J_w(z) \right)^n \sum_{\Gamma} \prod_{i=1}^k \frac{1}{n_i!} (Z_{g_s}[\Gamma_i])^{n_i}. \quad (9.4.11)$$

Note that, since the overall degree of Γ is N , n_i and N_i are constrained by

$$N = \sum_{i=1}^k n_i N_i. \quad (9.4.12)$$

Next, observe that since N is no longer fixed in the grand-canonical ensemble, the sum over n_i and N_i becomes unconstrained for $Z_{g_s}^X$. Upon inserting eq. (9.4.11) into the definition of Z_{g_s} , one therefore finds that the grand-canonical partition function is an exponential

$$Z_{g_s}^X[J] = \exp \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{w=1}^{\infty} \int d^2 z J_w(z) \right)^n \sum_{\Gamma \text{ conn.}} \mu^N Z_{g_s}[\Gamma] \right) \quad (9.4.13)$$

¹³There is some slight abuse of notation in eq. (9.4.6). In a more literal formulation, we should have expanded the n^{th} power of sums over source terms, into a sum of product of sources, corresponding to twist fields whose order and insertion location determines the ramification points.

of a sum over connected coverings with covering maps Γ of arbitrary degree N .

Finally, we infer from Definition (9.4.7) that

$$Z_{g_s}[\Gamma] = g_s^{w[\Gamma]} Z[\Gamma] \quad \text{with} \quad w[\Gamma] = \sum_{i=1}^{n_c} w_i \quad \text{and} \quad Z[\Gamma] := Z_1[\Gamma]. \quad (9.4.14)$$

Thus, the power of g_s weighing the contribution of Γ to Z_{g_s} can be expressed using the Riemann-Hurwitz formula (9.1.6) as

$$N(2G - 2) + \sum_{i=1}^{n_c} w_i = -2 + 2g + n_c, \quad (9.4.15)$$

where G is the genus of X . Putting all this together we obtain the following formula for the grand-canonical partition function

$$Z_{g_s}^X[J] = \exp \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{w=1}^{\infty} \int d^2 z J_w(z) \right)^n \sum_{\Gamma \text{ conn.}} g_s^{-2+2g+n_c} Z[\Gamma] \right). \quad (9.4.16)$$

We hence conclude that the logarithm of the grand-canonical partition function is the sum over all connected coverings, or equivalently, connected PRR diagrams. The individual contributions to this sum are weighted with the parameter g_s taken to the power of (minus) the Euler characteristic of the surface Σ on which the PRR diagram is drawn. In the equation, we have expressed the Euler characteristic in terms of number n_c of bulk vertices of the diagram or punctures of the surface Σ and the genus g of Σ . Correlation functions in the grand-canonical ensemble therefore possess a topological expansion which structurally resembles the 't Hooft large N expansion and, crucially, string perturbation theory. This the result that we aimed to establish.

9.4.3 The grand-canonical ensemble for interfaces

As we have shown in Section 9.2.5, it is possible to fix the normalisation of the interface changing operators of our interfaces in such a way that the corresponding connected correlation functions, to leading order in a $1/N$ expansion, match the structure of string perturbation theory. However, this approach leads to the same issues as the bulk fixed N genus expansion, i.e. the match is restricted to the leading large N contribution. In the three paragraphs of this section, we first introduce the interface grand-canonical ensemble, then show that our definition remedies this deficiency, leading to a fully consistent topological expansion, and finally reinterpret the results of Section 9.2.5 as a saddle-point approximation of the full ensemble.

Definition of the interface grand-canonical partition function. This paragraph constructs an interface analogue of the grand-canonical ensemble reviewed

in Sec. 9.4.2. In Sec. 9.1.4, we defined the interface $\mathcal{I}_{|a)}^{(P)}$ as interpolating between $\text{Sym}^{N_-}(\mathcal{M})$ and $\text{Sym}^{N_+}(\mathcal{M})$ on Riemann surfaces $X_\pm$ of genus $G_\pm$ that are glued along their boundaries to form a closed Riemann surface X of genus G . In order to study correlators with a single interface insertion, let us restrict to $X_\pm$ with boundaries

$$\partial X_- \cong \partial X_+ \cong \mathbb{S}^1. \quad (9.4.17)$$

In this setup, we use the notation introduced in Section 9.1.4 to define

$$Z_{N_\pm, P}^{X_-, X_+}[J] := \left\langle \mathcal{I}_{|a)}^{(P)} \exp \left(\oint g_s^w J_w \sigma_w + \oint \sqrt{g_s^\omega} J_\omega^{AB} \sigma_\omega^{AB} \right) \right\rangle_{\text{Sym}^{N_-|N_+}(\mathcal{M})}^{\text{unnormalised}} \quad (9.4.18)$$

with

$$\oint g_s^w J_w \sigma_w := \sum_{w=1}^{\infty} \int_X d^2 z g_s^w J_w(z) \sigma_w(z) \quad (9.4.19)$$

and

$$\oint \sqrt{g_s^\omega} J_\omega^{AB} \sigma_\omega^{AB} := \sum_{A, B \in \{L, R\}} \sum_{\omega} \int_{\partial X_\pm} dx \sqrt{g_s^\omega} J_\omega^{AB}(x) \sigma_\omega^{AB}(x) \quad (9.4.20)$$

where ω runs over $2\mathbb{Z}_{>0} - 1 + \delta_A^B$. Given this definition (9.4.18) of the interface partition functions we now introduce its grand-canonical counterpart which we define as

$$Z_{g_s}^{X_-, X_+}[J] := \sum_{N_\pm=0}^{\infty} \sum_{P=0}^{\min(N_-, N_+)} \mu^P \nu_-^{N_- - P} \nu_+^{N_+ - P} Z_{N_\pm, P}^{X_-, X_+}[J]. \quad (9.4.21)$$

It involves three fugacities $\mu, \nu_\pm$ which are associated to the integers $N_\pm$ that determine the symmetric product orbifolds on both sides of the interface and on the degree of transmissivity P that characterizes the interface itself. We suggest to control these three fugacities through a single loop counting parameter g_s as

$$\mu := g_s^{2G-2}, \quad \nu_\pm := g_s^{2G_\pm-1} \quad (9.4.22)$$

where G and $G_\pm$ denote the genus of the surface $X = X_+ \cup_{\mathbb{S}^1} X_-$ and of the two components $X_\pm$ that X is glued from.

The string genus expansion. We now show that the g_s expansion of correlation functions in the grand-canonical ensemble (9.4.21) has the structure of a string genus expansion describing the scattering of open and closed strings in the presence of a D -brane.

As in the bulk case, our first step is to expand the exponential in the fixed $N_\pm$ partition function (9.4.18) and to express the result in terms of covering maps. This leads to

$$Z_{N_\pm, P}^{X_-, X_+}[J] = \sum_{n_o, n_c=0}^{\infty} \frac{(\oint J_w)^{n_c} (\oint J_\omega^{AB})^{n_o}}{n_o! n_c!} \sum_{\Gamma} Z_{g_s}[\Gamma]. \quad (9.4.23)$$

Here, the sum runs over all coverings Γ of X with the appropriate ramification points and

$$Z_{g_s}[\Gamma] := \frac{g_s^{w[\Gamma] + \omega[\Gamma]}}{\text{Deck}[\Gamma]} \langle 1 \rangle_{\mathcal{M}}^{\Sigma}, \quad (9.4.24)$$

where $w[\Gamma]$ is the sum of bulk ramification indices and $\omega[\Gamma]$ is half the sum of boundary ramification indices of Γ .

As in the bulk case, the covering maps generically have several connected components, which fall into k isomorphism classes $n_1, \dots, n_k$. Let us denote by $N_i^{\pm}$ the order of the covering maps Γ_i in the i^{th} isomorphism class restricted to the preimage of $X^{\pm}$. Analogously, let p_i be the degree of transmissivity of Γ_i at a reference point t_* . Then,

$$\mu^P \nu_-^{N_- - P} \nu_+^{N_+ - P} Z_{g_s}[\Gamma] = \sum_{i=1}^k \frac{1}{n_i!} \left(\mu^{p_i} \nu_-^{N_i^- - p_i} \nu_+^{N_i^+ - p_i} Z_{g_s}[\Gamma_i] \right)^{n_i}. \quad (9.4.25)$$

From the powers of μ and $\nu_{\pm}$ we can read off the constraints

$$\sum_{i=1}^k n_i (N_i^{\pm} - p_i) = N_{\pm} - P \quad \text{and} \quad \sum_{i=1}^k n_i p_i = P \quad (9.4.26)$$

on n_i , $N_i^{\pm}$ and p_i . The sum over P and $N_{\pm}$ in the grand-canonical ensemble lifts these constraints such that the grand-canonical partition function becomes the exponential

$$Z_{g_s}^{X_-, X_+}[J] = \exp \left(\sum_{n_o, n_c=0}^{\infty} \frac{(\oint J_w)^{n_c} (\oint J_w^{AB})^{n_o}}{n_o! n_c!} \sum_{\Gamma \text{ conn.}} \mu^P \nu_-^{N_- - P} \nu_+^{N_+ - P} Z_{g_s}[\Gamma] \right) \quad (9.4.27)$$

of a sum of connected covering maps with orders $N_{\pm}$ on the preimage of $X_{\pm}$ and the degree of transmissivity P at t^* .

Applied to this setup, the interface Riemann-Hurwitz formula (9.1.28) gives

$$N_-(2G_- - 1) + N_+(2G_+ - 1) + \sum_{i=1}^{n_c} w_i + \frac{1}{2} \sum_{i=1}^{n_o} \omega_i = -2 + 2g + b + n_c + \frac{1}{2} n_o, \quad (9.4.28)$$

where g is the genus of the covering space b and is the number of its boundaries. With

$$Z_{g_s}[\Gamma] = g_s^{w[\Gamma] + \omega[\Gamma]} Z[\Gamma], \quad \text{where} \quad w[\Gamma] = \sum_{i=1}^{n_c} w_i, \quad \omega[\Gamma] = \frac{1}{2} \sum_{i=1}^{n_o} \omega_i \quad (9.4.29)$$

and $Z[\Gamma] := Z_1[\Gamma]$ carries no more dependence on the coupling g_s . Using the definitions (9.4.22) of the fugacities in terms of g_s we conclude that

$$Z_{g_s}^{X_-, X_+}[J] = \exp \left(\sum_{n_o, n_c=0}^{\infty} \frac{(\oint J_w)^{n_c} (\oint J_w^{AB})^{n_o}}{n_o! n_c!} \sum_{\Gamma \text{ conn.}} g_s^{-2+2g+b+n_c+\frac{n_o}{2}} Z[\Gamma] \right), \quad (9.4.30)$$

which is the result that we wanted to derive. Once again it contains a sum over connected covering maps, or interface PRR diagrams, in the exponent. Each contribution is weighted by a specific power of the parameter g_s that can be read off from the diagram. It involves the number n_c and n_o of bulk and interface operator insertions, respectively, as well as the genus g and number b of boundary components for the surface Σ the interface PRR diagram is drawn on. As we have anticipated, the exponent mimics precisely that of a string genus expansion for scattering of open strings in the presence of a D -brane.

Expectation value and saddle point approximation. Before we conclude this section we would like to make contact with our analysis in Section 9.2.5. To this end we compute the expectation values of $N_{\pm}$ and P in the grand-canonical ensemble through saddle-point approximation of the interface partition function at small g_s . We will see that the leading contribution to this saddle point approximation agrees with the results of Section 9.2.5.

For the rest of this section, we focus on the sphere case $X = \mathbb{S}^2$, $X_{\pm} = \mathbb{D}_{\pm}$. Let us first consider the vacuum grand-canonical partition function

$$Z_{g_s}^{\mathbb{D}_{\pm}}[0] = \exp[\mu z + \nu_- g_- + \nu_+ g_+], \quad (9.4.31)$$

where z is the sphere partition function of the seed theory, and $g_{\pm} = \langle 0|a_{\pm} \rangle$ are the boundary g -functions of two boundary states $|a_{\pm} \rangle$ therein. The expectation value of P (the number of seed spheres) and $N_{\pm} - P$ (the number of disks $\mathbb{D}_{\pm}$) can directly be evaluated as

$$\overline{P} = \mu \partial_{\mu} \log(Z_{g_s}^{\mathbb{D}_{\pm}}[0]) = \mu z, \quad \overline{N_{\pm} - P} = \nu_{\pm} \partial_{\nu_{\pm}} \log(Z_{g_s}^{\mathbb{D}_{\pm}}[0]) = \nu_{\pm} g_{\pm}. \quad (9.4.32)$$

After setting $\mu = \nu_{\pm}^2 = g_s^{-2}$ and $z = 1$, we find that $\overline{N_{\pm} - P}$ is proportional to $(\overline{P})^{1/2}$ with a constant of proportionality controlled by the boundary conditions $a_{\pm}$. Hence, the scaling behaviour for the averages agrees with the one we used for the symbols $N_{\pm}, P$ and $N_{\pm} - P$ in eq. (9.2.6) with $N \sim g_s^{-2}$. Let us also note that the ratio

$$\frac{\overline{N_+ - P}}{\overline{N_- - P}} = \frac{\langle 0|a_+ \rangle}{\langle 0|a_- \rangle}. \quad (9.4.33)$$

is controlled by the ratio of boundary entropies in the seed theory. To leading order in g_s , the normalised correlation functions

$$\left\langle \prod_{i=1}^{n_o} \sigma_{\omega_i}^{A_i B_i}(t_i) \prod_{i=1}^{n_c} \sigma_{w_i}(z_i) \right\rangle_{g_s}^{\mathbb{D}_{\pm}} = \left[\prod_{i=1}^{n_o} \frac{\delta}{\delta J_{\omega_i}^{A_i B_i}(t_i)} \prod_{i=1}^{n_c} \frac{\delta}{\delta J_{w_i}(z_i)} \right] \frac{Z_{g_s}^{\mathbb{D}_{\pm}}[J]}{Z_{g_s}^{\mathbb{D}_{\pm}}[0]} \Big|_{J=0} \quad (9.4.34)$$

of the grand-canonical theory can be approximated by fixing $N_{\pm}$ and P to their expectation values. If we saddle-point approximate¹⁴ the numerator $\delta_J Z_{g_s}^{\mathbb{D}_{\pm}}[J]|_{J=0}$ and the denominator $Z_{g_s}^{\mathbb{D}_{\pm}}[0]$, correlators of twist fields in the normalisation of (9.4.19) and (9.4.20) reproduce the results of Sec. 9.2.5 to leading order in $g_s \sim 1/\sqrt{N}$.

¹⁴Approximating the sum

$$Z_{g_s}^{\mathbb{D}_{\pm}}[0] = \sum_{P=0}^{\infty} \sum_{N_{\pm}-P=0}^{\infty} \frac{(\mu z)^P (\nu_- g_-)^{N_- - P} (\nu_+ g_+)^{N_+ - P}}{P! (N_- - P)! (N_+ - P)!} \quad (9.4.35)$$

by the largest summand, which occurs at $P \sim \mu z$, $N_{\pm} - P \sim \nu_{\pm} g_{\pm}$, produces

$$Z_{g_s}^{\mathbb{D}_{\pm}}[0] \sim \prod_{\Lambda \in \{\mu z, \nu_- g_-, \nu_+ g_+\}} \frac{e^{\Lambda}}{\sqrt{2\pi\Lambda}}, \quad (9.4.36)$$

which underestimates the full sum by the Gaussian fluctuation prefactor $\sqrt{2\pi\Lambda}$. However, this does not affect our leading-order estimation since the numerator and denominator have the same width and the prefactor cancels in the normalised ratio.

Appendix

9.A Symmetric product orbifolds as discrete gauge theories

Symmetric product orbifolds are discrete gauge theories and therefore we often use gauge theory language in the main text to describe objects in these theories. At first glance, the formulation in terms of covering maps provided in Section 9.1.2 however might seem very different from the usual construction of a gauge theory. The purpose of this appendix is to bridge this gap. We first formulate symmetric product orbifolds in the usual language of gauge theory i.e. using sections of associated bundles of principal bundles. We then discuss how to recover the covering map description for vacuum partition functions. Finally, we extend the match to also include twist fields. The overall dictionary that we end up with is summarised in the box below.

N sheeted covering maps $\Gamma \leftrightarrow S_N$ principal bundles P
Group of Deck transformations $\text{Deck}[\Gamma] \leftrightarrow$ Gauge group $\mathcal{G}(P)$
Measure monodromy of coverings $\leftrightarrow$ Insert Wilson lines
Change covering via cutting and glueing $\leftrightarrow$ Insert 't Hooft lines
S_N twist fields $\leftrightarrow S_N$ monopoles

Symmetric product orbifolds and S_N principal bundles. For simplicity, let us assume that the seed theory can be formulated in terms of some real scalar field ϕ with Lagrangian density $\mathcal{L}[\phi]$ (consider for instance $\mathcal{M}$ to be the free boson). For every S_N principal bundle $\pi : P \rightarrow X$, we can define an associated fibre bundle

$$\Gamma_P : \Sigma_P \rightarrow X \quad \text{with} \quad \Sigma_P := (P \times \underline{N})/S_N \quad (9.A.1)$$

whose fibres are sets of N discrete points,

$$\underline{N} = \{1, 2, \dots, N\}. \quad (9.A.2)$$

In the construction, we let S_N act on $\underline{N}$ in the canonical way by permutations. By considering the elements of $\underline{N}$ as basis vectors of an $\mathbb{R}$ vector-space, we can also

define an $\mathbb{R}^N$ vector-bundle $\mathbb{R}_P^N$ over X . In local trivialisations, sections of $\mathbb{R}_P^N$ are maps

$$\Phi : U \subseteq X \rightarrow U \times \mathbb{R}^N, \quad z \mapsto (z, \Phi^{(1)}(z), \Phi^{(2)}(z), \dots, \Phi^{(N)}(z)). \quad (9.A.3)$$

The action of $\text{Sym}^N(\mathcal{M})$ is a functional on the space of such sections, which we define in local coordinates by using the Lagrangian $\mathcal{L}$ of $\mathcal{M}$:

$$S_X[\Phi] = \int_X d^2z \sum_{i=1}^N \mathcal{L}[\Phi^{(i)}]. \quad (9.A.4)$$

Note that the action is well defined, and in particular not dependent on the choice of trivialisation, since the coordinate changes by construction act as permutations on the components of ϕ and the sum $\sum_{i=1}^N \mathcal{L}[\phi^{(i)}]$ is permutation invariant.

Vacuum partition functions. As usual in gauge theory, the vacuum partition function is a sum over all gauge bundles P weighted by the volumes of their gauge groups $\mathcal{G}(P)$. For the symmetric orbifold on a space time X , it is defined as the path integral

$$Z_N^X := \sum_P \frac{1}{|\mathcal{G}(P)|} \int_{\text{Sections of } \mathbb{R}_P^N} D\Phi e^{-S_X[\Phi]}. \quad (9.A.5)$$

To translate this into the formulation in terms of covering maps, note that there is a natural identification of sections Φ of $\mathbb{R}_P^N$ with functions $\phi : \Sigma_P \rightarrow \mathbb{R}$, namely

$$\Phi(z) = (z, \Phi^{(1)}(z), \dots, \Phi^{(n)}(z)) \quad \leftrightarrow \quad \phi(z, i) = \Phi^{(i)}(z). \quad (9.A.6)$$

Defining this map does not require a choice of trivialisation. Instead we are using that, by construction, there is a natural basis of the fibre $(\mathbb{R}_P^N)_z$ of $\mathbb{R}_P^N$ over z given by the fibre $(\Sigma_P)_z$ of Σ_P over z . Through this identification, we can rewrite eq. (9.A.5) as

$$Z_X^N = \sum_P \frac{1}{|\mathcal{G}(P)|} \int_{\phi: \Sigma_P \rightarrow \mathbb{R}} D\phi e^{-S_{\Sigma_P}[\phi]}, \quad (9.A.7)$$

where S_{Σ_P} is just the action of the seed theory $\mathcal{M}$ on Σ_P . This already almost looks like the sum over covering spaces in e.g. equation (9.1.14). To complete the match, we need to replace the sum over S_N principal bundles by a sum over unramified covering maps. For later convenience, let us directly write down an extended version of this correspondence that also holds for branched covering maps.

Singular S_N principal bundles and branched covering maps. There is a natural one-to-one correspondence between the N sheeted branched coverings of X whose set of branch points is contained in a finite subset B of X and singular S_N principal bundles $\pi : P \rightarrow X$ over X whose singular locus¹⁵ is a subset of B .

To show this, let us first construct a covering map from the principal bundle P . By construction, the projection map Γ_P of the associated fibre bundle $(P \times \underline{N})/S_N$ to the base space $X \setminus B$ is a N sheeted covering map. We can induce a complex structure of P by pulling back the complex structure on X along Γ_P . Then, Γ_P is trivially holomorphic. There is a unique analytic continuation of Γ_P to a ramified covering map of X , whose branch points are a subset of B .

Clearly, this construction is reversible. Starting with a branched covering map Γ , we can remove the ramification points from its domain to obtain a $\underline{N}$ bundle over $X \setminus B$. The structure group of this fibre bundle is S_N and the associated S_N principal bundle is the bundle P_Γ we are after.

Since furthermore $\text{Deck}[\Gamma_P]$ is identical to the group of bundle automorphisms of Σ_P which is isomorphic to $\mathcal{G}(P)$, we can use the above bijection to rewrite eq. (9.A.6) as a sum

$$Z_X^N = \sum_P \frac{1}{|\mathcal{G}(P)|} \int_{\phi: \Sigma_P \rightarrow \mathbb{R}} D\phi e^{-S_{\Sigma_P}[\phi]} = \sum_\Gamma \frac{\langle \mathbb{1} \rangle_{\mathcal{M}}^\Sigma}{|\text{Deck}[\Gamma]|}, \quad (9.A.8)$$

over N sheeted covering maps Γ and thus reproduce the prescription of Section 9.1.2.

Gauge transformations. In gauge theory, the name “gauge transformation” is used for several types of operations, which are distinguished in the language of principal bundles as

1. **Small gauge transformations** i.e. the action of isomorphisms of the gauge bundle that are homotopic to the identity map.
2. **Large gauge transformation** i.e. the action of topologically non trivial isomorphisms of the gauge bundle.
3. **Coordinate changes** of a trivialisation of the bundles, which are sometimes also considered as “small gauge transformations”.
4. **Changes of the global structure of the bundles**, which are sometimes also considered “large gauge transformations”.

¹⁵For our purposes, a singular principal bundle over X is a principle bundle over $X \setminus S$, where S is a finite subset of X , which we call the singular locus of the bundle.

Since the gauge group S_N is discrete, the only small gauge transformation is the identity. Relatedly, discreteness of the fibre implies that every automorphism of the gauge bundle is already fixed over each connected component of X by the action on a single fibre. Thus, if X has n connected components, then the group of large gauge transformations $\mathcal{G}(P)$ is isomorphic to a subgroup of S_N^n .

If $(U_i, s_i)_{i \in I}$ is a trivialisation of P (i.e. a collection of local sections) then the coordinate changes are described by transition functions

$$t_{ij} : U_i \cap U_j \rightarrow S_N, \quad (9.A.9)$$

such that $s_j(z) = s_i(z)t_{ij}(z)$. The transition functions satisfy the cocycle condition

$$t_{ik}(z) = t_{ij}(z)t_{jk}(z) \quad \text{for all} \quad z \in U_i \cap U_j \cap U_k \quad (9.A.10)$$

and fully encode the structure of the bundle P .

We can change the global structure of the gauge bundle by altering the trivialisation through a cut and glue procedure that allows us to act with oriented closed curves on X labelled by permutations $g \in S_N$ on the set of S_N principal bundles. Physically, this corresponds to the insertion of a 't Hooft line of the product theory. The procedure consists of the following three steps.

1. **Setup.** Given an oriented non-self-intersecting closed curve γ on X , a permutation $g \in S_N$ and an S_N principal bundle $\pi : P \rightarrow X$, we define a new bundle P_g^γ .
2. **Trivialisation of P .** Since $\gamma([0, 1]) \cong \mathbb{S}^1$, we can pick a trivialisation of P such that only two charts U and V have a non trivial intersection with $\gamma([0, 1])$. Also, we can ensure that U, V are simply connected, the intersection of U and V with $\gamma([0, 1])$ is connected and that $U \cap V$ has exactly two connected components UV and $\widetilde{UV}$. The transition function t_{UV} is specified by two permutations: Its values on UV and $\widetilde{UV}$ respectively. W.l.o.g. let the value on $\widetilde{UV}$ be 1 and denote the value on UV by h .
3. **Trivialisation of P_g^γ .** The curve γ splits both U and V in two connected components $U'_\pm$ and $V'_\pm$ such that $U'_\pm \cap V'_\mp = \emptyset$. Thicken $U'_\pm$ and $V'_\pm$ slightly to obtain open sets $U_\pm$ and $V_\pm$ that cover γ but whose intersection $U_\pm \cap V_\mp$ is still disjoint from all charts of P except for U and V . If g commutes with h , then replacing (U, t_U) and (V, t_V) by the four local sections

$$(U_-, s_U|_{U_-}) \quad (V_-, s_V|_{V_-}) \quad (U_+, (s_U|_{U_+}) \cdot g) \quad \text{and} \quad (V_+, (s_V|_{V_+}) \cdot g), \quad (9.A.11)$$

leads to

$$t_{U_+V_+} = g^{-1} \cdot (t_{UV}|_{U_+ \cap V_+}) \cdot g = t_{UV}|_{U_+ \cap V_+} \quad \text{and} \quad t_{U_-V_-} = t_{UV}|_{U_- \cap V_-}. \quad (9.A.12)$$

Therefore, we can use the transition functions of P to consistently construct local sections over the entire connected component of X containing γ from the four new local sections defined in eq. (9.A.11). Complementing this with the local sections of P over the rest of X yields the bundle P_g^γ .

Note that, with the choice of trivialisation of P made above, the permutation h is exactly the monodromy of P along γ . The condition that h and g need to commute can therefore be phrased as the statement that we can only act with g along γ on P if the S_N monodromy of P along γ commutes with g .

Monopoles. The field Φ whose action we defined in eq. (9.A.4) is the lift of ϕ to the untwisted sector of the symmetric orbifold. In addition, the spectrum of local excitations also includes monopoles i.e. operators whose insertion in the path integral restricts the sum over gauge bundles by fixing a certain monodromy around their insertion point. Under the identification between S_N gauge bundles and degree N covering maps, this translates to the statements that the insertion of monopoles creates ramification points with a fixed index. Thus, the S_N monopoles are exactly the twist fields as introduced in Sec. 9.1.2.

9.B Wilson and 't Hooft lines in symmetric orbifolds

In this appendix, we first use Wilson loops to obtain an alternative definition of gauge fixed twist fields. We then discuss the 't Hooft lines that implement an S_N action on the global structure of covering maps. These provide us with a different perspective on gauge choices and gauge invariance in symmetric orbifolds and also allow us to quite conveniently describe other permutation orbifolds.

Wilson lines. The data defining a Wilson line $\mathcal{W}[\gamma]$ is a curve $\gamma : [0, 1] \rightarrow X$. The insertion of such a line does not affect the set of covering maps associated with an operator $\mathcal{O} \in \mathcal{A}_X[\text{Sym}^N(\mathcal{M})]$, that is

$$C[\mathcal{O}\mathcal{W}[\gamma]] = C[\mathcal{O}] \quad \text{for all } \mathcal{O} \in \mathcal{A}_X[\text{Sym}^N(\mathcal{M})]. \quad (9.B.1)$$

It only acts through its lift, which is a bijection from $\Gamma^{-1}(\gamma(0))$ to $\Gamma^{-1}(\gamma(1))$, i.e.

$$\mathcal{W}[\gamma]^\Gamma \in \text{Bij}[\Gamma^{-1}(\gamma(0)), \Gamma^{-1}(\gamma(1))]. \quad (9.B.2)$$

Concretely, the value of $\mathcal{W}[\gamma]^\Gamma$ is the map defined by parallel transport along $\Gamma^*\gamma$. Through local gauge choices of $C[\mathcal{O}]$ at $\gamma(0)$ and $\gamma(1)$, we can identify $\mathcal{W}[\gamma]^\Gamma$ with an element $\mathcal{W}[\gamma]_{\text{gf}}^\Gamma$ of S_N . If we change the local gauge at $\gamma(0)$ by acting with a permutation g_0 and that at $\gamma(1)$ by acting with a permutation g_1 , then $\mathcal{W}[\gamma]_{\text{gf}}^\Gamma$ transforms as

$$\mathcal{W}[\gamma]_{\text{gf}}^\Gamma \rightarrow g_1 \mathcal{W}[\gamma]_{\text{gf}}^\Gamma g_0^{-1}. \quad (9.B.3)$$

Wilson loops. For closed Wilson lines i.e. Wilson loops, $\gamma(0) = \gamma(1)$ and we only make a single local gauge choice. Thus, Wilson loops transform by conjugation under a change of the gauge. We can hence define a $\mathbb{C}$ valued operator

$$\mathcal{W}_R[\gamma] := \chi_R(\mathcal{W}[\gamma]) \quad (9.B.4)$$

for every closed loop γ and every representation R of S_N with character χ_R by making an arbitrary gauge choice to evaluate $\chi_R(\mathcal{W}[\gamma]^\Gamma)$. As usual in gauge theory, these topological loops are not the only gauge invariant extended objects that can be built from $\mathcal{W}[\gamma]_{\text{gf}}^\Gamma$. Another option is to consider Wilson lines whose end-points are dressed with operators charged under S_N .

Gauge fixed twist fields. Using Wilson lines, we can define gauge fixed versions σ_g of the twist fields $\sigma_{[g]}$:

$$\sigma_g(z) := \lim_{\epsilon \rightarrow 0} \delta_{\mathcal{W}[\gamma_\epsilon]_{\text{gf}}, g} \sigma_{[g]}(z), \quad (9.B.5)$$

where γ_ϵ is a small loop winding once around z clockwise at distance ϵ . The gauge dependence of σ_g comes from the choice of how to identify $\mathcal{W}[\gamma_\epsilon]$ with an element of S_N . In particular, the transformation behaviour (9.B.3) of the Wilson loop implies that under a change h of the local gauge choice, σ_g transforms as

$$\sigma_g \mapsto \lim_{\epsilon \rightarrow 0} \delta_{h\mathcal{W}[\gamma_\epsilon]_{\text{gf}}h^{-1}, g} \sigma_{[g]} = \lim_{\epsilon \rightarrow 0} \delta_{\mathcal{W}[\gamma_\epsilon]_{\text{gf}}, h^{-1}gh} \sigma_{[g]} = \sigma_{h^{-1}gh}. \quad (9.B.6)$$

Note that the linear combination

$$\hat{\sigma}_{[g]} = \sum_{\tilde{g} \in [g]} \sigma_{\tilde{g}} \quad (9.B.7)$$

is gauge invariant. But of course, $\hat{\sigma}_{[g]}$ is simply $\sigma_{[g]}$

$$\hat{\sigma}_{[g]} = \sum_{\tilde{g} \in [g]} \lim_{\epsilon \rightarrow 0} \delta_{\mathcal{W}[\gamma_\epsilon]_{\text{gf}}, \tilde{g}} \sigma_{[\tilde{g}]} = \sigma_{[g]}. \quad (9.B.8)$$

't Hooft lines. The insertion of Wilson lines does not affect the set of covering maps associated with a correlation function and only acts through the lift of the lines along the covering maps. Let us now construct a dual set of lines that only act by changing the structure of covering maps but lift to the trivial defect in the seed theory. These oriented topological lines $\mathcal{L}_g[\gamma]$ are labeled by a permutation $g \in S_N$ and a non-self intersecting curve γ . They depend on a local gauge choice at $\gamma(0)$ only.

The set $C[\mathcal{OL}_g[\gamma]]$ of covering maps computing correlators in the presence of $\mathcal{L}_g[\gamma]$ is given by applying a cutting and glueing procedure that maps each $\Gamma \in C[\mathcal{O}]$ to a new covering which we shall denote by $\mathcal{L}_g[\gamma] \cdot \Gamma$. To construct $\mathcal{L}_g[\gamma] \cdot \Gamma$, first cut out

the pre-image $\Gamma^{-1}(\gamma([0, 1]))$ of the image of γ under Γ from the covering space Σ to obtain

$$\tilde{\Sigma} := \Sigma \setminus \Gamma^{-1}(\gamma([0, 1])). \quad (9.B.9)$$

By parallel transport of the local gauge choice at $\gamma(0)$ along γ , every point on $\Gamma^{-1}(\gamma([0, 1]))$ has a label $i \in \{1, \dots, N\}$. We now glue $\tilde{\Sigma}$ back together along its cut by identifying the i^{th} sheet with the gi^{th} sheet upon travelling along the cut from below (w.r.t. the orientation of γ). In particular, this identifies several of the N points in $\Gamma^{-1}(\gamma(0))$ (and $\Gamma^{-1}(\gamma(1))$) among each other, namely all those that are elements of the same g orbit. Let the Riemann surface constructed in this manner be denoted by $\mathcal{L}_g[\gamma] \cdot \Sigma$. We define $\mathcal{L}_g[\gamma] \cdot \Gamma$ as the analytic extension of Γ from $\tilde{\Sigma}$ to $\mathcal{L}_g[\gamma] \cdot \Sigma$.

Changing the local gauge at $\gamma(0)$ by acting with a permutation h transforms $\mathcal{L}_g[\gamma]$ as

$$\mathcal{L}_g[\gamma] \rightarrow \mathcal{L}_{hgh^{-1}}[\gamma] \quad (9.B.10)$$

since glueing the i^{th} sheet to that labelled by gi is the same as first relabelling $i \rightarrow i' = hi$ and then glueing the sheet with the new label i' to that carrying the new label $h(gi) = hgh^{-1}i'$.

Thus, a simple way to construct gauge invariant lines from $\mathcal{L}_g[\gamma]$ is to average over characters χ_R of S_N . This leads to the lines

$$\mathcal{L}_R[\gamma] := \sum_{g \in S_N} \chi_R(g) \mathcal{L}_g[\gamma]. \quad (9.B.11)$$

Note that if we average over a particular conjugacy class,

$$\mathcal{L}_{[g]}[\gamma] := \sum_{\tilde{g} \in [g]} \mathcal{L}_{\tilde{g}}[\gamma], \quad (9.B.12)$$

the resulting non-local operators are in effect pairs of $[g]$ -twist fields inserted at $\gamma(0)$ and $\gamma(1)$ that are coupled to each other in a non-local fashion.

There is an obstruction to inserting $\mathcal{L}_g$ along a closed curve γ . The operator $\mathcal{L}_g[\gamma]$ is only well defined for closed γ if the monodromy along γ as measured by $\mathcal{W}[\gamma]$ commutes with g . Thus, we define $\mathcal{L}_g[\gamma] \cdot \Gamma$ for closed curves γ as

$$\mathcal{L}_g[\gamma] \cdot \Gamma = \begin{cases} \text{as in the } \gamma(0) \neq \gamma(1) \text{ case} & \text{if } \mathcal{W}[\gamma]^\Gamma \in \mathcal{C}_g \\ \Gamma & \text{else.} \end{cases} \quad (9.B.13)$$

Global gauge choices and gauge invariance. As made precise in Appendix 9.A, the sums over covering maps in symmetric product orbifolds are sums over

gauge bundles of S_N gauge theories. The closed permutation lines introduced at the end of the previous paragraphs can in this sense be thought of as implementing large gauge transformations. Gauge invariance of the symmetric product orbifold is the statement that for every closed curve γ , every $g \in S_N$ and every $\mathcal{O} \in \mathcal{A}_X[\text{Sym}^N(\mathcal{M})]$,

$$C[\mathcal{O}] = C[\mathcal{O}\mathcal{L}_g]. \quad (9.B.14)$$

As an example, let us verify that the vacuum of the symmetric product orbifold as defined in eq. (9.1.10) is gauge invariant. Given any holomorphic covering map $\Gamma : \Sigma \rightarrow X$ and a closed curve γ on X , there are two cases to consider. The first case is that the monodromy of Γ along γ does not commute with g . In this case, we have defined the permutation line as acting trivially. In the second case $\mathcal{L}_g[\gamma]$ maps Γ into another holomorphic covering map without changing the monodromy along γ . Thus, $\mathcal{L}_g[\gamma]$ simply permutes the different holomorphic covering maps among each other, leaving the set $C[\mathbb{1}]$ invariant.

Other permutation orbifolds. As a side comment, let us briefly note that it is also possible to consider a variation of the above procedure where one only imposes invariance under a certain subset of the operators $\mathcal{L}_g$. This, together with an appropriate analogue of eq. (9.1.10), leads to more general permutation orbifolds.

For example, we could choose to impose no invariance at all. Together with

$$C_{\mathcal{M}^{\otimes N}}[\mathbb{1}] := \{\text{trivial covering of } X \text{ by } X^N\}, \quad (9.B.15)$$

this leads to the product theory. While the lines $\mathcal{L}_R[\gamma]$ defined in eq. (9.B.11) are trivial in the symmetric product orbifold if γ is closed, they do have interesting correlation functions in the product theory.

9.C Comments on the normalisation of the interfaces

The purpose of this appendix is to compare the normalisation of the interfaces as defined in eq. (9.1.33) to the normalisation fixed through the explicit expressions for boundary states of the folded theory given in Reference [4].

In [4], the normalisation of the boundary states $|p^{L/R}\rangle'$ was fixed by demanding that

$$Z_{N_{\pm}, p^{L/R}}^{\text{torus}} := \langle p^L |' \exp(\pi i \hat{t} (L_0 + \bar{L}_0 - \frac{c}{12})) | p^R \rangle' \quad (9.C.1)$$

leads to formula (9.1.47) for the grand-canonical torus partition function (9.1.45). Since we derived the very same formula using the normalisation of interfaces fixed by eq. (9.1.33), the approach presented in this chapter clearly agrees with the approach of [4] for the two interface torus partition function.

However, to obtain a fully consistent picture, the boundary states should be rescaled

$$|p\rangle := \sqrt{N_-!} \sqrt{N_+!} |p\rangle' \quad (9.C.2)$$

and the overlap should be divided by $N_-!N_+!$, so that the torus partition function

$$Z_{N_\pm, p_{L/R}}^{\text{torus}} := \frac{1}{N_-!N_+!} \langle p^L | \exp(2\pi i \hat{t}(L_0 + \bar{L}_0 - \frac{c}{12}) | p^R \rangle \quad \text{with} \quad c = (N_- + N_+)c_{\mathcal{M}} \quad (9.C.3)$$

remains unchanged. As long as we are only interested in $Z_{N_\pm, p_{L/R}}^{\text{torus}}$, the question of whether we work with $|p\rangle'$ or $|p\rangle$ is of course irrelevant and purely a matter of taste. But once we move on to other partition functions, we should work with $|p\rangle$. Let us for instance consider the sphere partition function with a single interface insertion, splitting the sphere into the two disks $\mathbb{D}_\pm$. The fixed $N_\pm, p$ partition function in terms of $|p\rangle$ is

$$Z_{N_\pm, p}^{\mathbb{D}_-, \mathbb{D}_+} = \frac{\langle p|0\rangle}{N_-!N_+!} = \frac{g_-^{N_- - p} g_+^{N_+ - p}}{(N_- - p)!(N_+ - p)!p!}, \quad (9.C.4)$$

where we obtained the r.h.s. from an expression for $\langle 0|p\rangle'$ given in eq. (2.40) of [4]. This result agrees with the expression derived from the interface covering map perspective in the main text, see eq. (9.1.44). As another indication towards the fact that (9.C.4) is the correct prescription, observe that the grand-canonical partition function

$$Z_{g_s}^{\mathbb{D}_-, \mathbb{D}_+} := \sum_{N_\pm=0}^{\infty} \sum_{p=0}^{\min(N_-, N_+)} \mu^p \nu^{N_- + N_+ - 2p} Z_{N_\pm, p}^{\mathbb{D}_-, \mathbb{D}_+} \quad (9.C.5)$$

with $\mu := g_s^{-2}$ and $\nu := g_s^{-1}$ exponentiates as¹⁶

$$Z_{g_s}^{\mathbb{D}_-, \mathbb{D}_+} = \exp(\mu + \nu g_- + \nu g_+) \quad (9.C.6)$$

only if we work with $|p\rangle$ and not $|p'\rangle$.

9.D Generalised LM – computational details

This appendix contains detailed derivations and computations underlying the results discussed in Section 9.3.

¹⁶As a consistency check, note that setting $\nu = 0$ reproduces $Z_{g_s}^{\text{sphere}}[0]$ in a normalisation where the sphere partition function of the seed theory is set to 1.

9.D.1 Details on the regularisation procedure

In this appendix, we derive eq. (9.3.11). In the first short paragraph, we briefly describe the setup of the derivation to make the appendix self-contained. We then begin with the derivation by dividing the covering space Σ into three regions. One of the three regions contains a pathological subregion, which we cut out in order to define the regularised Liouville action (9.D.10). The next three paragraphs evaluate the Liouville action on each individual region. After this, we add the three contributions to obtain the regularised vacuum partition function (9.D.21). The regularised partition function still depends on an unphysical regulator, which we remove in the final paragraph of the section by rescaling the twist fields. This final step directly produces the result (9.3.11) that we want to show.

The setup. In Section 9.3.1, we introduced the family

$$ds_\delta^2 = \begin{cases} dzd\bar{z} & \text{if } |\delta z| < 1 \\ \frac{dzd\bar{z}}{|z\delta|^4} & \text{else} \end{cases} \quad (9.D.1)$$

of continuous metrics on the Riemann sphere parametrised by $\delta > 0$. While ds_δ^2 is continuous, it is not smooth. In particular, its scalar curvature is the distribution

$$R_\delta = 4\delta \left(|z| - \frac{1}{\delta} \right) \quad (9.D.2)$$

whose support is the circle $|z\delta| = 1$. We noted in the main text that the partition function of a CFT with central charge c on $\overline{\mathbb{C}}$ equipped with ds_δ^2 is

$$Z_1^{\overline{\mathbb{C}}_\delta} = Q\delta^{-\frac{c}{3}}, \quad (9.D.3)$$

where Q is an arbitrary constant, which we fixed to $Q = \eta^{\frac{c}{3}}$ so that $X = \overline{\mathbb{C}}_\eta$ has $Z_1^X = 1$. In this appendix, we show in detail how to evaluate Z_1^Σ for a sphere Σ with metric induced via a covering map $\Gamma : \Sigma \rightarrow X$, which is unramified at $\Gamma^{-1}(\infty)$ and satisfies $\Gamma(\infty) = \infty$. As explained in the main text, we can do so (for an appropriate $\delta > 0$ specified in the next paragraph) via the relation

$$Z_1^\Sigma = e^{S_L[\phi]} Z_1^{\overline{\mathbb{C}}_\delta} \quad \text{with} \quad S_L[\phi] = \frac{c}{96\pi} \int_\Sigma d^2t \sqrt{g_\delta} \left(\partial_\mu \phi \partial_\nu \phi g_\delta^{\mu\nu} + 2R_\delta \phi \right), \quad (9.D.4)$$

where ϕ is the Weyl factor relating Σ to $\overline{\mathbb{C}}_\delta$.

Divide and cover. To organise the computation, we divide Σ into three disjoint regions and evaluate the Liouville action separately on each of these. The first of the regions is

$$\mathcal{B} = \{t \in \mathbb{C} : |\eta\Gamma(t)| < 1\} \subseteq \Sigma. \quad (9.D.5)$$

We pick δ sufficiently small to ensure that

$$\mathcal{B} \subseteq \Sigma^0 := \{t \in \mathbb{C} : |t\delta| < 1\}. \quad (9.D.6)$$

Furthermore, we denote the complement of Σ^0 by Σ^∞ . Our decomposition of Σ is

$$\Sigma = \mathcal{B} \sqcup (\Sigma^0 \setminus \mathcal{B}) \sqcup \Sigma^\infty. \quad (9.D.7)$$

We use this decomposition to evaluate the Liouville action

$$S_L[\phi] = S_L^{\mathcal{B}}[\phi] + S_L^{\Sigma^0 \setminus \mathcal{B}}[\phi] + S_L^{\Sigma^\infty}[\phi] \quad (9.D.8)$$

for the Weyl factor ϕ relating metric ds^2 , induced on Σ via Γ , to the metric ds_δ^2 .

Regularisation of $S_L[\phi]$. The region $\mathcal{B}$ contains all ramification points of Γ . At the ramification points, the induced metric degenerates, the Weyl factor diverges and the Liouville action needs to be regularised. To do so, we introduce a regulator $\varepsilon > 0$ that is sufficiently small to ensure that

$$R_\varepsilon := \bigcup_{r \in R[\Gamma]} \{t : |\Gamma(t) - \Gamma(r)| < \varepsilon\} \subseteq \mathcal{B}. \quad (9.D.9)$$

We regularise the Liouville action by removing R_ε from $\mathcal{B}$. That is, we define $\mathcal{B}_\varepsilon := \mathcal{B} \setminus R_\varepsilon$ and use $S_L^{\mathcal{B}_\varepsilon}[\phi]$ as a regularised version of $S_L^{\mathcal{B}}[\phi]$, leading to

$$S_L^\varepsilon[\phi] := S_L^{\mathcal{B}_\varepsilon}[\phi] + S_L^{\Sigma^0 \setminus \mathcal{B}}[\phi] + S_L^{\Sigma^\infty}[\phi] \quad (9.D.10)$$

as a regularised version of (9.D.8).

Evaluation of $S_L^{\mathcal{B}_\varepsilon}[\phi]$ On $\mathcal{B}_\varepsilon$ the metric $ds_\delta^2 = dt d\bar{t}$ is flat. Thus, only the kinetic term of the Liouville action contributes. The latter evaluates to

$$S_L^{\mathcal{B}_\varepsilon}[\phi] = \frac{c}{96\pi} \int_{\mathcal{B}_\varepsilon} d^2t 4\partial\phi\bar{\partial}\phi = \frac{c}{96\pi} \int_{\mathcal{B}_\varepsilon} d^2t 4\partial(\phi\bar{\partial}\phi) = \frac{c}{96\pi} \int_{\partial\mathcal{B}_\varepsilon} ds\phi\partial_n\phi. \quad (9.D.11)$$

via Stokes theorem, where n is the normal vector of $\partial\mathcal{B}_\varepsilon$. In the second step, we used that

$$\phi = \log(\partial\Gamma) + \log(\bar{\partial}\bar{\Gamma}) \quad \text{and hence} \quad \partial\bar{\partial}\phi = 0. \quad (9.D.12)$$

To further evaluate the integral, note that each connected component of $\partial\mathcal{B}$ either belongs to a pole $p \in P[\Gamma]$ of Γ in the complex plane, the point ∞ , or to a ramification point $r \in R[\Gamma]$. Around these three different types of point, Γ behaves as

$$-\Gamma(p+t) = \frac{b_p}{t} + O(1), \quad \Gamma(t) = b_\infty t + O(1) \quad \text{and} \quad \Gamma(r+t) = \Gamma(r) + a_r t^w + O(t^{w+1}). \quad (9.D.13)$$

Thus, a pole p in the complex plane contributes

$$-\int_0^{2\pi} ds b_p \eta (-2 \log(b_p \eta^2)) (-4 b_p^{-1} \eta^{-1}) = -16\pi \log(b_p \eta^2) \quad (9.D.14)$$

to (9.D.11), the point ∞ contributes nothing and a ramification point r contributes

$$-\int_0^{2\pi} ds \varepsilon_r (-2 \log(w a_r \varepsilon_r^{w-1})) (-2(w-1) \varepsilon_r^{-1}) = -8\pi(w-1) \log(w a_r \varepsilon_r^{w-1}) \quad (9.D.15)$$

to leading order in ε and η . Here, $\varepsilon_r := \left(\frac{\varepsilon}{a_r}\right)^{\frac{1}{w}}$. In total, we conclude

$$S_L^{\mathcal{B}_\varepsilon}[\phi] = -\frac{c}{12} \sum_{r \in R[\Gamma]} (w-1) \log\left(w \varepsilon \left(\frac{a_r}{\varepsilon}\right)^{\frac{1}{w}}\right) - \frac{c}{6} \sum_{p \in P[\Gamma]} \log(b_p \eta^2) \quad (9.D.16)$$

Evaluation of $S_L^{\Sigma^0 \setminus \mathcal{B}}[\phi]$. On $\Sigma^0 \setminus \mathcal{B}$ the metric $ds_\delta^2 = \frac{dt d\bar{t}}{|\delta t|^4}$ is flat. Thus, only the kinetic term is non-vanishing. Two regions contribute to the corresponding integral. The first consists of disks around the poles of Γ . Their contribution is suppressed by δ^2 . To leading order, $S_L^{\Sigma^0 \setminus \mathcal{B}}[\phi]$ is thus fully determined by the integral over the second region: The annulus around ∞ . There, $\phi = -\log(b_\infty^2 |\delta t|^4)$ and we get

$$S_L^{\Sigma^0 \setminus \mathcal{B}}[\phi] = \frac{c}{96\pi} \int_{(\Sigma^0 \setminus \mathcal{B})_\infty} d^2 t 4\partial\phi\bar{\partial}\phi = \frac{c}{24\pi} \int_{\frac{1}{b_\infty\eta}}^{\frac{1}{\delta}} dr \int_0^{2\pi} d\theta r b_\infty \frac{4}{r^2} = -\frac{c}{3} \log\left(\frac{\delta}{b_\infty\eta}\right) \quad (9.D.17)$$

Evaluation of $S_L^{\Sigma^\infty}[\phi]$. On Σ^∞ the Weyl factor $\phi = \log\left(\frac{\delta^4}{b_\infty^2 \eta^4}\right)$ is constant. Thus, the kinetic term vanishes and $S_L^{\Sigma^\infty}[\phi]$ is fully determined by the curvature term, which yields

$$S_L^{\Sigma^\infty}[\phi] = \frac{c}{96\pi} 16\pi\phi = \frac{c}{3} \log \delta^2 - \frac{c}{3} \log(b_\infty \eta^2) \quad (9.D.18)$$

Putting everything together. Combining eqs. (9.D.16), (9.D.17) and (9.D.18), we conclude

$$S_L^\varepsilon[\phi] = -\frac{c}{12} \sum_{r \in R[\Gamma]} (w-1) \log\left(w \varepsilon \left(\frac{a_r}{\varepsilon}\right)^{\frac{1}{w}}\right) - \frac{c}{6} \sum_{p \in P[\Gamma]} \log(b_p \eta^2). \quad (9.D.19)$$

Therefore, the regularised version of Z_1^Σ is

$$Z_{1;\varepsilon}^\Sigma := e^{S_L^\varepsilon[\phi]} Z_1^{\bar{\mathcal{C}}_\delta} = \left(\prod_{r \in R[\Gamma]} w^{w-1} \varepsilon^{\frac{(1-w)^2}{w}} a_r^{1-\frac{1}{w}} \prod_{p \in P[\Gamma]} b_p^2 \eta^4 \right)^{-\frac{c}{12}}. \quad (9.D.20)$$

Since ∞ is not a branch point of Γ , we have $|P[\Gamma]| = \text{Deg}[\Gamma] - 1$. Thus,

$$Z_{1;\varepsilon}^\Sigma = \eta^{-\frac{c}{3}(\text{Deg}[\Gamma]-1)} \left(\prod_{r \in R[\Gamma]} w^{w-1} \varepsilon^{\frac{(1-w)^2}{w}} a_r^{1-\frac{1}{w}} \prod_{p \in P[\Gamma]} b_p^2 \right)^{-\frac{c}{12}}. \quad (9.D.21)$$

Rescaling the twist fields. The regularisation scheme introduced an explicit dependence of $Z_{1;\varepsilon}^\Sigma$ on the unphysical parameter ε . To remove the regulator dependence, we need to rescale the twist fields i.e. replace σ_w by $\mathcal{N}_w \sigma_w$. We make the minimal choice

$$\mathcal{N}_w = \varepsilon^{\frac{c}{12} \frac{(1-w)^2}{w}} \quad (9.D.22)$$

so that, after rescaling, $Z_{1;\varepsilon}^\Sigma$ becomes the physical quantity

$$Z_1^\Sigma = \eta^{-\frac{c}{3}(\text{Deg}[\Gamma]-1)} \prod_{r \in R[\Gamma]} w^{-\frac{c}{12}(w-1)} a_r^{-\frac{c}{12}(1-\frac{1}{w})} \prod_{p \in P[\Gamma]} b_p^{-\frac{c}{6}} \quad (9.D.23)$$

which is independent of ε .

9.D.2 Regularisation for the upper half plane

In this appendix, we regulate the Liouville action for interface coverings of the upper half complex plane by the upper half complex plane.

The setup. The setup treated in this appendix is almost identical to that described in the beginning of Appendix 9.D.1. In particular, the reference metric that we consider is the restriction of the metric ds_δ^2 defined in eq. (9.D.1) to the upper half plane. Generically, we would have to evaluate the extrinsic curvature contribution in addition to the bulk Liouville action in the interface case. For the upper half plane equipped with ds_δ^2 , the derivative of the normal vector w.r.t. the tangent vector of the boundary is however always proportional to the normal vector and hence the extrinsic curvature vanishes. Accordingly, only the bulk Liouville action is relevant. As we argue in the next paragraph, the latter can be evaluated using the method of images.

Method of images. Given an interface covering map Γ that covers the upper half complex plane by itself, we can readily construct a covering map

$$\Gamma' : \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto \begin{cases} \Gamma(z) & \text{if } \text{Im}(z) > 0 \\ \overline{\Gamma(\bar{z})} & \text{else.} \end{cases} \quad (9.D.24)$$

For each bulk ramification point of Γ , the doubled map Γ' has two ramification points with the same index. For each interface ramification point of index ω , it has a bulk ramification point with index $w = \omega$ on the real line. The bulk part of the regulated Liouville action for the conformal factor ϕ relating the upper half plane with metric induced by Γ' to the upper half plane with metric ds_δ^2 is given by

$$S_L^\varepsilon[\phi]_{\text{Bulk}} = \frac{1}{2} S_L^\varepsilon[\phi'] \quad (9.D.25)$$

in terms of the Liouville action for the conformal factor ϕ' of the doubled covering map. We can therefore directly generalise eq. (9.D.21) to

$$Z_{1;\varepsilon}^\Sigma = \Omega(\eta) \left(\prod_{r \in R_B[\Gamma]} w^{w-1} \varepsilon^{\frac{(1-w)^2}{w}} a_r^{1-\frac{1}{w}} \prod_{p \in P_B[\Gamma]} b_p^2 \right)^{-\frac{c}{12}} \left(\prod_{r \in R_I[\Gamma]} \omega^{\omega-1} \varepsilon^{\frac{(1-\omega)^2}{\omega}} a_r^{1-\frac{1}{\omega}} \prod_{p \in P_I[\Gamma]} b_p^2 \right)^{-\frac{c}{24}}, \quad (9.D.26)$$

where $\Omega(\eta) := \eta^{-\frac{c}{6}(N_- + N_+ - 1)}$ and $N_\pm = \text{Deg}_\pm[\Gamma]$. Furthermore, $R_B[\Gamma]$ and $R_I[\Gamma]$ are the sets of bulk and interface ramification points of Γ respectively. Likewise, $P_B[\Gamma]$ and $P_I[\Gamma]$ are the sets of poles in the bulk and on the interface, not including ∞ .

Rescaling the twist fields. The result (9.D.26) is consistent with the bulk twist field normalisation found in eq. (9.D.22). To fully remove the regulator dependence, we need to also redefine the normalisation of the interface twist fields. We work with the minimal bulk and interface normalisations

$$\mathcal{N}_w^B = \varepsilon^{\frac{c}{12} \frac{(1-w)^2}{w}} \quad \text{and} \quad \mathcal{N}_\omega^I = \varepsilon^{\frac{c}{24} \frac{(1-\omega)^2}{\omega}}. \quad (9.D.27)$$

In this normalisation, we obtain the physical partition function

$$Z_1^\Sigma = \eta^{-\frac{c}{6}(N_- + N_+ - 1)} \left(\prod_{r \in R_B[\Gamma]} w^{w-1} a_r^{1-\frac{1}{w}} \prod_{p \in P_B[\Gamma]} b_p^2 \right)^{-\frac{c}{12}} \left(\prod_{r \in R_I[\Gamma]} \omega^{\omega-1} a_r^{1-\frac{1}{\omega}} \prod_{p \in P_I[\Gamma]} b_p^2 \right)^{-\frac{c}{24}}. \quad (9.D.28)$$

9.D.3 Computing covering maps

This appendix discusses how covering maps are computed in the first section of the `mathematica` notebook `CoveringMaps.nb` uploaded in the ancillary files associated with the arXiv page of this publication. We focus on coverings $\Gamma : \mathbb{C} \rightarrow \mathbb{C}$ that have branch points

$$x_1 = 0, \quad x_2, \quad x_3 = 1, \quad \text{and} \quad x_4 = \infty \quad (9.D.29)$$

which correspond to ramification points

$$s_1 = 0, \quad s_2, \quad s_3 = 1, \quad \text{and} \quad s_4 = \infty \quad (9.D.30)$$

with ramification indices $w_1, \dots, w_4$. The covering map Γ satisfies

$$\Gamma(s_i + s) = x_i + a_i s^{w_i}. \quad (9.D.31)$$

The choices $s_1 = x_1 = 0$ and $x_4 = s_4 = \infty$ ensure that we can directly satisfy two of these conditions by making the Ansatz

$$\Gamma(s) = \frac{q(s)}{p(s)} \quad (9.D.32)$$

with polynomials q and p that satisfy

$$q(s) = a_1 s^{w_1} + O(s^{w_1+1}), \quad p(s) = 1 + O(s) \quad \text{and} \quad \deg(p) - \deg(q) = w_4. \quad (9.D.33)$$

Since the degree of the covering map coincides with the degree of the polynomial p , we can fix it through the Riemann-Hurwitz formula. This leads to

$$\deg(p) = \frac{1}{2}(w_1 + w_2 + w_3 + w_4) - 1 \quad \text{and} \quad \deg(q) = \frac{1}{2}(w_1 + w_2 + w_3 - w_4) - 1. \quad (9.D.34)$$

Fixing the polynomials p and q hence amounts to computing $w_2 + w_3 - 1$ coefficients and the additional unknown s_2 . This is achieved in the notebook by solving the $w_2 + w_3$ equations

$$\partial_s^n (\Gamma(s) - x_i)|_{s=s_i} = 0 \quad \text{for} \quad i \in \{2, 3\} \quad \text{and} \quad n = 0, 1, \dots, w_i - 1. \quad (9.D.35)$$

9.E Proving the diagrammatic rules

In this appendix, we sketch a proof for the validity of interface PRR diagrammatics. In the first paragraph, we introduce sharpened terminology that allows us to distinguish between embeddings of the diagrams and the intrinsic data of the diagrams more clearly than in the main text. With this preparation, we then show that the map from equivalence classes of covering maps to abstract PRR diagrams is injective i.e. that two covering maps producing the same diagram are equivalent. Next, we show that the map is also surjective i.e. that every abstract PRR diagram indeed arises from a covering map. Together, these two steps imply that there is indeed a one-to-one correspondence between PRR diagrams and equivalence classes of covering maps.

Embedded and abstract PRR diagrams.

Definition. *Embedded (interface) PRR diagrams* are the graphs embedded in Riemann surfaces that can be constructed from the vertices in Sec. 9.2.1 and Sec. 9.2.2 with the constraints outlined in those sections. Consider two embedded diagrams $\mathcal{D}$ and $\mathcal{D}'$ on Riemann surfaces Σ and Σ' whose vertices are labelled by the same branch points. We say that the diagrams are isomorphic if there is an isomorphism between Σ and Σ' that restrict to an isomorphism of graphs for the diagrams $\mathcal{D}$ and $\mathcal{D}'$ preserving the labels of the vertices.

Definition. *Abstract (interface) PRR diagrams* are equivalence classes of embedded diagrams w.r.t. the notion of isomorphism introduced in the previous paragraph.

Injectivity of PRR diagrammatics.

Claim. *If taking the preimage of the contour 9.2.5 for two covering maps Γ and Γ' with the same branch points produces the same abstract PRR diagram, then Γ and Γ' are equivalent i.e. identical up to an isomorphism of the covering space.*

Proof. If the two embedded PRR diagrams $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic, then there is an isomorphism $\phi : \Sigma \rightarrow \Sigma'$ such that $\phi(\mathcal{D}) = \mathcal{D}'$. We now show that ϕ is also an isomorphism of the associated covering maps i.e. $\Gamma = \tilde{\Gamma} := \Gamma' \circ \phi$. By definition, ϕ identifies ramification points that are labelled by the same branch-point. Thus, $\Gamma(r) = \tilde{\Gamma}(r)$ for every ramification point $r \in \Sigma$ of Γ . Furthermore, $\Gamma(x) = \tilde{\Gamma}(x)$ for every point x on an edge of $\mathcal{D}$, since the value at the vertices and the fact that Γ and $\tilde{\Gamma}$ have to be both length and orientation preserving already uniquely characterises these functions on the edges. Finally, the restrictions of Γ and $\tilde{\Gamma}$ to a face of $\mathcal{D}$ are restrictions of two holomorphic maps between disks that coincide on the boundary of the disk. Therefore, Γ and $\tilde{\Gamma}$ also coincide on the faces of $\mathcal{D}$.

Surjectivity of PRR diagrammatics.

Claim. *Every abstract PRR diagram that can be constructed using the rules of Section 9.2 has a representative that is the preimage of the contour 9.2.5 w.r.t. an interface covering map.*

Proof. Take an arbitrary realisation of the PRR diagram on a surface Σ . We now define a map Γ by first setting its value at each ramification-point to the branch-point by which it is labelled and then picking an arbitrary continuous, bijective, orientation preserving identification between each edge of the diagram and the interval of the contour on the sphere that connects the branch-points corresponding to the ramification points at the ends of the edge. The PRR rules enforce that the restriction of the map Γ constructed in this way to any face of the diagram is a homeomorphism that identifies that face with the boundary of a disk on the sphere. For each face, extend Γ to the interior such that it becomes a homeomorphism between the face and a disk on the sphere. Away from the ramification points, Γ is a local homeomorphism and therefore, we can use it to induce a metric and complex structure on Σ , which makes the restriction of Γ to the complement of the ramification points a holomorphic local isometry by construction. Clearly, the PRR diagram corresponding to Γ is the diagram that we used to construct Γ .

Chapter 10

Future perspectives beyond symmetric product orbifolds

– Open string amplitudes & other future directions.

The previous two chapters constructed a new family of interfaces between symmetric product orbifolds (Chapter 8) and developed the necessary covering map machinery to efficiently analyse interface correlation functions of symmetric product orbifolds on arbitrary Riemann surfaces (Chapter 9). These methods open up the possibility of addressing several interesting questions about the nature of interfaces in symmetric product orbifolds and the holography of AdS_2 branes in pure NSNS AdS_3 string backgrounds. This brief concluding chapter sketches some of the potential directions for future research.

Deriving $\text{AdS}_3/\text{CFT}_2$ for open strings. Section 9.4 established a structural match between the genus expansion of interface correlators in the symmetric product orbifold and string perturbation theory. To complete the derivation of the duality between the interfaces and AdS_2 branes in minimal pure NSNS AdS_3 , it is further necessary to show that open string scattering amplitudes quantitatively match the proposed dual correlators at each order in the string genus expansion. The analogous derivation of the duality for closed string scattering amplitudes provided in the seminal paper [69] was based on a localisation of the string worldsheet to ramified coverings of the holographic boundary. It was already conjectured in [124] that this localisation property should extend to the open string case. Now that the class of interface covering maps to which the worldsheet is expected to localise in that case has been identified, this conjecture can be fully established.

A derivation of the bulk correspondence that directly matches CFT correlation functions with string scattering amplitudes, without the need to compute either explicitly, was proposed in [270]. It passes through a deformation of the string background that involves AdS_3 backgrounds with NSNS flux $k > 1$. The dual CFT is a non-rational

symmetric product orbifold that includes a Liouville-like direction [269]. In [270], the H_3^+ -Liouville correspondence of [110, 111] was run in reverse to reconstruct string theory on AdS_3 from the non-rational symmetric product orbifold; see also [271] for related work based on path integral representations. In the limit $k \rightarrow 1$, an $\mathcal{N} = 4$ supersymmetric Liouville field theory decouples from the CFT_2 , leaving a rational symmetric product orbifold for the supersymmetric 4-torus and hence recovering the holographic relation of [69]. It would be interesting to extend this approach to interfaces and open strings using previous work on a boundary version of the H_3^+ -Liouville relation [273, 274].

While it is conceptually appealing that the above approach does not require explicit evaluation of the scattering amplitudes, there is also intrinsic value in computing string scattering amplitudes in the presence of AdS_2 branes for processes involving a small number of closed and/or open strings. Building on the closed string techniques reviewed in Section 7.1.2, it should be possible to determine new explicit formulas for boundary correlation functions of the $SL(2; \mathbb{R})_k$ model with arbitrary spectral flow at generic k .

Finally, it might be instructive to analyse boundary conditions of the localising AdS_3 sigma model of Reference [268]. Just as the relatively simple structure of bulk correlation functions, compared to the standard $SL(2; \mathbb{R})$ model, allowed for a detailed proof of crossing symmetry, similar simplifications of boundary correlators and partition functions should permit a detailed discussion of the full Cardy-Lewellen constraints [316] of the BCFT.

Worldsheet Ryu–Takayanagi formula.¹ In Section 9.1, we defined symmetric product orbifolds via lifts of seed theory data along branched covering maps. This discussion revolved around the covering space lift of partition functions, correlators, boundaries, and interfaces of the seed theory. When extended to density matrices, the covering space lift relates von Neumann entropies in symmetric orbifolds to properties of seed theory ensembles. The details of this relation can be directly inferred by inserting cuts in the known covering maps that compute partition functions and correlators of the orbifold theory.

As a concrete example, the thermal density matrix can be obtained by cutting open the torus partition function along a spatial cycle. Thus, the relation between the torus partition function of the symmetric orbifold and products of seed theory partition functions on covering tori (see e.g. the discussion around page 296) implies that the thermal density matrix of a symmetric product orbifold can be expressed as a sum of products of seed theory thermal density matrices at various temperatures,

¹I would like to thank Matthias Gaberdiel, Christian Northe, and Volker Schomerus for comments and discussions on this point.

corresponding to the admissible modular parameters of the covering space tori.

While, at the level of symmetric product orbifolds, these are somewhat trivial observations, the existence of a covering-space lift of seed theory density matrices also suggests a similar lift to the string worldsheet via the identification between the latter and the covering space in tensionless $\text{AdS}_3/\text{CFT}_2$ holography. It would be interesting to, guided by the covering-space lift, work out the details of this putative worldsheet analogue. Such a result could be interpreted as a microscopic version of the coarse-grained Ryu–Takayanagi formula [122].

Thermal correlators and superficial holography. Reference [317] reports universal behaviour of thermal correlation functions in symmetric product orbifolds, compatible with a BTZ black hole as a semi-classical gravity dual. The result implies that conditions on the analysed thermal correlators of local operators cannot be sufficient criteria to determine whether a given CFT has a geometric bulk dual. It would be interesting to investigate how the inclusion of extended operators such as $\mathcal{I}^{(P)}|_{a\pm}$ fits into this picture—presumably leading to the construction of a superficial semi-classical gravity dual of the interfaces.

SUSY interface covering maps. Recently, a supersymmetric extension of the covering map approach to symmetric product orbifolds was formulated [318]. The construction of interface covering maps provided in Chapter 9 should naturally extend to coverings of super Riemann surfaces, using the formalism developed in [318] for the bulk case.

Interface deformations of symmetric product orbifolds. Reference [319] provides a detailed description of a broad class of generalised symmetries in symmetric product orbifolds, especially emphasising constraints on deformations by operators from twisted sectors of the theory. It would be interesting to generalise these results to the interface set-up and systematically analyse deformations by interface twist fields. Another direction now accessible is to study the effect of marginal bulk deformations on the spectrum of interface operators. Of particular relevance in this context is the deformation that drives the system toward the geometric regime in which the interacting CFT_2 possesses a dual supergravity description. The effect of such deformations on the bulk spectrum has been addressed, e.g., in [320–322]. The interfaces studied in this thesis appear to be ideal new probes for this deformation.

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