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Geometry of Effective Field Theory Positivity Cones and Kuranishi Moduli Space of Generalized Complex Structures via Hamilton–Nash–Moser Theory

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Paula Naomi Pilatus

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1. Gutachter: Prof. Dr. Vicente Cortés
2. Gutachter: Prof. Dr. Christophe Grojean
3. Gutachter: Prof. Dr. Roberto Rubio

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Summary

This thesis is divided into two parts, each addressing a separate research problem.

In the first part of the thesis we study so-called positivity bounds – theoretical constraints on the Wilson coefficients of an effective field theory. These bounds arise from the requirement that a given effective field theory must be the low-energy limit of a quantum field theory that satisfies the fundamental principles of unitarity, locality, and causality. The task of deriving these bounds (for Wilson coefficients corresponding to operators that are quartic in momenta and contribute to two-to-two scattering at tree-level) can be reformulated as a convex geometric question, namely the determination of the extremal representation of a closed convex cone. This so-called positivity cone consists of all positive semi-definite tensors in

$$W = \{S \in \text{Sym}^2(\text{Sym}^2 V^*) \oplus \text{Sym}^2(\Lambda^2 V^*) : \tau S = S\} \subset \text{Sym}^2(V^* \otimes V^*),$$

where τ denotes transposition in the second and fourth tensor factor and V is a finite-dimensional real vector space. The dimension of V has the physical interpretation of the number of (particle) flavors under consideration.

In this thesis, we solve this question up to three flavors, i.e. $\dim V \leq 3$, proving a full classification of all extremal elements in these cases. We furthermore study the implications of our findings, deriving the full positivity bounds for amplitudes with and without additional symmetries. In the cases with additional symmetries that we consider, we find that the so-called elastic bounds are sufficient to recover the full positivity bounds. In contrast, if $\dim V \geq 3$ and no additional symmetry conditions are imposed on the amplitude, there will be bounds that cannot be deduced from the elastic positivity bounds. These arise from a new class of extremal elements that appears in the extremal representation of the positivity cone starting from $\dim V = 3$.

In the second part of this thesis, we study deformations of complex and generalized complex structures using Hamilton–Nash–Moser theory. Our main result is an extension of Gualtieri’s theorem on the existence of a locally complete family of deformations of generalized complex structures on exact Courant algebroids to the broader class of $\mathcal{G}$ -flat transitive Courant algebroids. To establish this result, it is necessary to revisit certain analytic aspects of both Gualtieri’s proof and Kuranishi’s classical proof of the existence of a locally complete family of (classical) complex structures.

We therefore provide transparent proofs of Kuranishi’s and Gualtieri’s theorems using Hamilton’s framework of tame Fréchet spaces and the Hamilton–Nash–Moser implicit function theorem. Building on these results, we then extend Gualtieri’s theorem to the setting of $\mathcal{G}$ -flat transitive Courant algebroids. A key ingredient is the construction of an infinite-dimensional Lie group structure on the group of autoequivalences of a $\mathcal{G}$ -flat transitive Courant algebroid, more specifically, a tame Fréchet Lie group structure.

Zusammenfassung

Diese Dissertation gliedert sich in zwei Teile, die jeweils einem eigenständigen Forschungsproblem gewidmet sind.

Im ersten Teil der Dissertation untersuchen wir sogenannte Positivitätsschranken (positivity bounds) – theoretische Einschränkungen für die Wilson-Koeffizienten einer effektiven Feldtheorie. Diese Schranken ergeben sich aus der Forderung, dass eine gegebene effektive

Feldtheorie der Niederenergielimes einer Quantenfeldtheorie sein muss, welche die grundlegenden Prinzipien der Unitarität, Lokalität und Kausalität erfüllt. Die Herleitung dieser Schranken (für Wilson-Koeffizienten, die Operatoren entsprechen, welche quartisch in den Impulsen sind und auf Baumebene zu Zwei-zu-Zwei-Streuamplituden beitragen) lässt sich auf eine Frage der konvexen Geometrie zurückführen, nämlich auf die Bestimmung der Extremaldarstellung eines abgeschlossenen konvexen Kegels. Dieser sogenannte Positivitätskegel besteht aus allen positiv semidefiniten Tensoren in

$$W = \{S \in \text{Sym}^2(\text{Sym}^2 V^*) \oplus \text{Sym}^2(\Lambda^2 V^*) : \tau S = S\} \subset \text{Sym}^2(V^* \otimes V^*),$$

wobei τ die Transposition im zweiten und vierten Tensorfaktor bezeichnet und V ein endlichdimensionaler reeller Vektorraum ist. Die Dimension von V besitzt die physikalische Interpretation der Anzahl der betrachteten (Teilchen-)Flavours.

In dieser Dissertation lösen wir dieses Problem für bis zu drei Flavours, d.h. für $\dim V \leq 3$, und beweisen eine vollständige Klassifikation aller Extremalelemente in diesen Fällen. Darüber hinaus untersuchen wir die Konsequenzen dieser Resultate und leiten die vollständigen Positivitätsschranken für Streuamplituden sowohl mit als auch ohne zusätzliche Symmetrien her. Für die von uns betrachteten Fälle mit zusätzlichen Symmetrien zeigen wir, dass die sogenannten elastischen Schranken (elastic bounds) ausreichen, um sämtliche Positivitätsschranken zu erhalten. Im Gegensatz dazu existieren für $\dim V \geq 3$ ohne zusätzliche Symmetrieannahmen über die Amplitude Schranken, die sich nicht aus den elastischen Positivitätsschranken ableiten lassen. Diese entstehen durch eine neue Klasse von Extremalelementen, die ab $\dim V = 3$ in der Extremaldarstellung des Positivitätskegels auftritt.

Im zweiten Teil der Dissertation untersuchen wir Deformationen komplexer und verallgemeinerter komplexer Strukturen mithilfe der Hamilton–Nash–Moser-Theorie. Unser Hauptergebnis ist eine Verallgemeinerung von Gualtieris Satz über die Existenz einer lokal vollständigen Familie von Deformationen verallgemeinerter komplexer Strukturen auf exakten Courant-Algebroiden auf die umfassendere Klasse der $\mathcal{G}$ -flachen transitiven Courant-Algebroide. Zum Nachweis dieses Resultats ist es notwendig, bestimmte analytische Aspekte sowohl von Gualtieris Beweis als auch von Kuranishis klassischem Beweis der Existenz einer lokal vollständigen Familie von (klassischen) komplexen Strukturen erneut zu untersuchen.

In dieser Dissertation entwickeln wir transparente Beweise der Sätze von Kuranishi und Gualtieri unter Verwendung von Hamiltons Theorie zahmer Fréchet-Räume sowie des Hamilton–Nash–Moser-Implizite-Funktionensatzes. Aufbauend auf diesen Resultaten erweitern wir anschließend Gualtieris Satz auf den Kontext $\mathcal{G}$ -flacher transitiver Courant-Algebroide. Ein wesentlicher Bestandteil des Beweises ist die Konstruktion einer unendlichdimensionalen Liegruppenstruktur auf der Gruppe der Autoäquivalenzen eines $\mathcal{G}$ -flachen transitiven Courant-Algebroids, genauer gesagt einer zahmen Fréchet-Liegruppenstruktur.

Publication list and relevant contributions

Part of this thesis is based on the following coauthored publication and preprint:

1. Q. Bonnefoy, V. Cortés, E. Gendy, K. Ritter von Merkl, P. N. Pilatus. Geometry of Effective Field Theory Positivity Cones. *Annales Henri Poincaré* (2026). 10.1007/s00023-026-01687-y (cited as [Bon+26])
2. V. Cortés, P. N. Pilatus. Deformation theory of complex structures via Hamilton–Nash–Moser theory. arXiv.2606.16834 (cited as [CP26])

Other publications that were coauthored during this time are:

1. M. J. D. Hamilton, D. Kotschick, P. N. Pilatus. Born geometry via Künneth structures and recursion operators. *Communications in Mathematical Physics*. **406**, 123 (2025). 10.1007/s00220-025-05296-4
2. A. Gil-García, P. N. Pilatus. Born Lie algebras. *Journal of Geometry and Physics*. **214**, 105518 (2025). 10.1016/j.geomphys.2025.105518

Chapter 4 of the thesis is based on [Bon+26]. This project grew out of a discussion of V. Cortés, E. Gendy and C. Grojean after a presentation of E. Gendy on earlier work by himself, Q. Bonnefoy and C. Grojean at an internal meeting of the Quantum Universe cluster of excellence. After discussions over several weeks with Q. Bonnefoy, E. Gendy and C. Grojean, V. Cortés wrote a note in which he proved the results of Section 3.1–3.2.1 of the paper (Section 4.2.1–4.2.3 in the thesis). K. Ritter von Merkl obtained Theorem 4.2 of the paper (part of Theorem 4.15 in the thesis) in his master thesis (with a different proof). Building on that, the results of Section 3.2.2–4.4 of the paper (Section 4.2.5–4.3 of the thesis) including the main result were obtained largely independently by the author of this thesis. Several draft versions of the manuscript were critically discussed in complete detail with V. Cortés, and brought into the final shape over these meetings.

The suggestion to look at amplitudes with additional symmetries came from Q. Bonnefoy who conjectured that the main result can be used to show that the elastic bounds are the strongest positivity bounds for pion scattering. Following this suggestion and discussions with Q. Bonnefoy and C. Grojean, the author of the thesis proved the results in Section 5 of the paper (Section 4.4–4.5 of the thesis). Q. Bonnefoy also contributed Figure 1 (Figure 4.1 in the thesis) and significantly to the writing of the appendix as well as large portions of the introduction of the paper. C. Grojean helped with the writing of the paper. This project benefited a lot from regular meetings among the six authors where ideas were suggested and earlier papers on related topics were critically reviewed.

The Chapters 7 and 8 are based on [CP26] and joint work in progress with V. Cortés. This project was actively initiated by Vicente Cortés who during his lecture course on generalized complex geometry noticed some crucial analytic details that were not addressed in the works by Kuranishi [Kur65] and Gualtieri [Gua11]. He suggested the use of Hamilton–Nash–Moser theory in order to revisit the proofs in [Kur65] and [Gua11] and eventually extend Gualtieri’s result to larger families of Courant algebroids. Starting from there, the results of [CP26] and Chapter 8 were obtained largely independently by the author of this thesis with early contributions by J. Heck [Hec25], confirming V. Cortés’ suggestion that Hamilton–Nash–Moser theory was the right framework to work in. The results were then discussed with V. Cortés who pointed out some mistakes in earlier versions of the proofs and helped with the writing of the paper.

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Chapter 1

Introduction

A common phenomenon that we observe in nature is that of separation of scales. It is the reason why humans were able to build houses thousands of years before knowing about quantum mechanics and why they could study fluid dynamics before the discovery of molecules. Effective field theories (EFTs) underlie that same principle. They are physical theories describing the low-energy limit of some more fundamental quantum theory. In many situations this "full" quantum theory at high energies is not known. In that case, EFTs can also serve to extend a theory that is known to be accurate at low energies – such as the Standard Model of particle physics – by adding higher-dimension operators to the Lagrangian, which parametrize the effects of physics at higher energy scales. This approach is often called the *bottom-up approach*.

The physical parameters of an effective field theory are so-called *Wilson coefficients*. The values of the Wilson coefficients that best describe nature need to be experimentally determined. If the "full" quantum theory is unknown, they could a priori take any value. However, it was shown in [Ada+06] that imposing the mild assumption that the unknown theory should be compatible with fundamental principles of physics such as causality, unitarity and locality yields constraints on the Wilson coefficients, the so-called (*two-to-two forward scattering*) *positivity bounds*. They are a powerful tool in order to reduce the landscape of physically meaningful EFTs.

These positivity bounds are associated with a specific kind of scattering, known as two-to-two scattering: two incoming particles interact and yield two outgoing particles. Additionally, the forward limit – namely, identical incoming and outgoing kinematics – is invoked. In the literature, often only *elastic (positivity) bounds* are considered, i.e. positivity bounds that arise from two-to-two scattering processes, where the initial and the final state are identical. However, in [ZZ20] it was pointed out that in general there will be additional constraints, which cannot be deduced from the elastic bounds.

As it was argued in [Li+21], the full set of positivity bounds can be described abstractly as elements in a closed convex cone in a space of four-tensors on a real vector space V . These four-tensors have symmetries prescribed by the symmetries of the (two-to-two forward) scattering amplitude. We will refer to this cone as the *positivity cone*. Many elements of the positivity cone in fact yield redundant positivity bounds. To obtain a complete set of bounds it is sufficient to restrict to *extremal* elements, i.e. elements that cannot be written as a non-trivial positive combination of other elements in the cone. Some of these extremal elements will give rise to the aforementioned elastic bounds. It was already argued in [Li+21] that if the vector space V is two-dimensional in fact all extremal elements are of this kind.

In the first part of this thesis we will classify all extremal elements of the positivity cone in the case that V is three-dimensional. This gives rise to all (two-to-two forward scattering) positivity bounds for scattering processes involving up to three flavors. In particular, we will see that, starting from dimension three, there is a new class of extremal elements that gives rise to positivity bounds that cannot be deduced from elastic bounds. On the other hand, we will show that when imposing additional symmetries on the amplitude, the elastic positivity bounds are in fact the strongest (two-to-two forward scattering) positivity bounds. This in particular holds true for the physical example of pion scattering in quantum chromodynamics.

The second research problem addressed in this thesis arises in the field of *generalized geometry*. Morally speaking, the starting point of generalized geometry is to replace the tangent bundle of a manifold M by the *generalized tangent bundle* $\mathbb{T}M = TM \oplus T^*M$ or, more generally, a *Courant algebroid*. These are vector bundles equipped with the data of a scalar product, a bracket on sections (called *Dorfman bracket*) and a vector bundle morphism to the tangent bundle (called *anchor map*), satisfying compatibility conditions.

A Courant algebroid structure on the generalized tangent bundle is given by the canonical projection onto TM as anchor map, the natural pairing

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(X))$$

as scalar product, and a Dorfman bracket defined in terms of the Lie derivative and the exterior derivative together with a closed three-form.

Generalized geometry has applications to string theory and to supergravity, the EFT of string theory: It can be used to "geometrize" certain supergravity theories by reformulating the equations of motion as a *generalized Einstein equation* [CSW11; CSW14; Gar14]. The well-posedness of the initial value problem for the generalized Einstein equations was recently established in [Sch26] (see [CS25] for a geometric description of the constraint equations). Moreover, the renormalization group flow of certain supergravity/string theories may be identified with *generalized Ricci flow* [Str17]. Generalized geometry also serves as a good framework to understand T-duality in string theory [CD23; CG10; CW26; Šev15].

There are also applications to complex (especially non-Kähler) geometry and (higher) gauge theory (see e.g. [GG25; GJS23; Sil+25]). A key ingredient is that certain geometric flows or systems of geometric partial differential equations may be recast as versions of generalized Ricci-flow or the vanishing of generalized curvature. Last but not least, an important feature of generalized geometry is that it may be used to simultaneously generalize a priori different geometric structures.

Generalized complex structures are a particular instance of this phenomenon, as they encompass both complex and symplectic structures as special cases. In fact, this unification of complex and symplectic geometry was the starting point of Hitchin [Hit03] to initiate the field of generalized geometry. Generalized complex structures appear naturally in the context of supersymmetric compactifications [GHR84; Gua14] and the study of deformations of generalized complex structures is relevant to moduli problems in supergravity theories (see e.g. [But+08; Kap04]).

The deformation theory of generalized complex structures on *exact* Courant algebroids, i.e. those isomorphic to the generalized tangent bundle, has already been studied by Gualtieri in [Gua03; Gua11]. Whereas deformations of (classical) complex structures are studied up to the action of the diffeomorphism group, deformations of *generalized* complex structures

must be studied up to so-called *Courant autoequivalences*: vector bundle equivalences that preserve the Courant algebroid structure. Within this setting, Gualtieri proved the existence of a locally complete family of generalized complex deformations. His approach was inspired by Kuranishi's simplified proof in [Kur65] of the existence of a locally complete family of (classical) complex deformations.

A long-term objective (as part of a project which goes beyond the scope of this thesis) is to eventually extend Gualtieri's theorem to the larger class of *transitive* Courant algebroids, i.e. Courant algebroids with a surjective anchor map. These are isomorphic to Courant algebroids with underlying vector bundle $TM \oplus \mathcal{G} \oplus T^*M$, where $\mathcal{G}$ is a bundle of quadratic Lie algebras. This has applications to supergravity theories, where in the presence of additional gauge fields, it is often not enough to only consider sections to the generalized tangent bundle [Gar14].

We will see that in order to achieve such an extension, some analytic details in the classical proofs by Kuranishi [Kur65] and Gualtieri [Gua03; Gua11] need to be revisited. An appropriate framework for this is Hamilton–Nash–Moser theory. This will set the foundation for an extension of Gualtieri's theorem to generalized complex structures on $\mathcal{G}$ -flat transitive Courant algebroids, a subclass of transitive Courant algebroids for which the Dorfman bracket takes a particularly simple shape. As an important step in the proof of this extension, we will endow the group of autoequivalences of a $\mathcal{G}$ -flat transitive Courant algebroid with an infinite-dimensional Lie group structure, more precisely, a *tame* Lie group structure.

1.1 Overview of Part I

The aforementioned positivity cone takes the explicit form of the cone

$$\mathcal{C}_W := \{S \in W : S \geq 0\},$$

of positive semi-definite elements in

$$W := \{S \in \text{Sym}^2(\text{Sym}^2 V^*) \oplus \text{Sym}^2(\Lambda^2 V^*) : \tau S = S\} \subset V^{*\otimes 4}.$$

Here, V is a finite-dimensional vector space and $(\tau S)(v_1, v_2, v_3, v_4) := S(v_1, v_4, v_3, v_2)$ for all $v_1, \dots, v_4 \in V$. The dimension of the space W grows rapidly with $\dim V =: n$: It is given by

$$\dim W = \frac{1}{8} n(n+1)(n(n+1)+2).$$

In the particular case that $\dim V = 3$ we have $\dim W = 21$.

As explained above, our task is to identify all extremal elements of the cone $\mathcal{C}_W$. In this thesis, we will solve this problem for $\dim V \leq 3$ proving the following theorem.

Theorem (cf. Theorem 4.22). *Suppose $\dim V = 3$. Then $S \in \mathcal{C}_W$ is extremal if and only if it is of one of the following types:*

1. For some $\alpha \in V^* \setminus \{0\}$, $S = \alpha^4$.
2. For linearly independent $\alpha_1, \alpha_2 \in V^*$, $S = (\alpha_1 \otimes \alpha_2)^{\otimes 2} + (\alpha_2 \otimes \alpha_1)^{\otimes 2}$.
3. For a basis $\{\alpha_1, \alpha_2, \alpha_3\}$ of V^* and parameters g, d, h such that $g^2 > 1 - d^2 + dh$ we

have

$$S = S_{\text{tot}} + 2(g^2 + d^2 - 1 - dh) [(\alpha_2 \otimes \alpha_3)^{\otimes 2} + (\alpha_3 \otimes \alpha_2)^{\otimes 2}],$$

where

$$\begin{aligned} S_{\text{tot}} = & \alpha_1^4 + 6\alpha_1^2\alpha_2^2 + 6\alpha_1^2\alpha_3^2 + 12d\alpha_1\alpha_2^2\alpha_3 + 12g\alpha_1\alpha_2\alpha_3^2 + 4h\alpha_1\alpha_3^3 + (1+d^2)\alpha_2^4 \\ & + 4dg\alpha_2^3\alpha_3 + 6(1+dh)\alpha_2^2\alpha_3^2 + 4g(d+h)\alpha_2\alpha_3^3 + (1+g^2+h^2)\alpha_3^4. \end{aligned}$$

The first two cases in the theorem correspond to the aforementioned elastic positivity bounds, while the third class of extremal elements gives rise to new bounds that cannot be deduced from the elastic ones. This gives rise to all (two-to-two forward scattering) positivity bounds for scattering processes involving up to three flavors.

To prove this result, we make use of the fact that $\mathcal{C}_W$ is a *linear spectrahedron*: the intersection of a linear subspace of the symmetric bilinear forms on some vector space with the cone of positive semi-definite ones. The extremal elements of linear spectrahedra have been characterized in [RG95] as those elements which have "maximal kernel" among all non-trivial elements in the cone (in the sense that there is no non-trivial other element of the cone whose kernel properly contains that of the extremal element). Therefore, to identify the extremal elements we will distinguish the elements in the cone by their rank.

Moreover, it will be useful to consider the restriction maps

$$R: W \longrightarrow \text{Sym}^2(\text{Sym}^2 V^*), \quad R': W \longrightarrow \text{Sym}^2(\Lambda^2 V^*).$$

We will see that the map R is in fact an isomorphism so that an element $S \in W$ is completely determined by its projection $R(S) \in \text{Sym}^2(\text{Sym}^2 V^*)$. Moreover, $S \in \mathcal{C}_W$ if and only if $R(S) \geq 0$ and $R'(S) \geq 0$. Rather than just considering the rank of an element $S \in W$, to decide whether S is extremal, we will need to consider both the rank of $R(S)$ and $R'(S)$ separately. Furthermore, we need to consider the dimension of certain characteristic subspaces of the kernel of the element S .

Our strategy can be summarized in the following steps:

1. We show that an element in $\mathcal{C}_W$ is an extremal element of the first or second kind if and only if $\text{rank } R(S) = 1$. This is true in arbitrary dimensions of V and can be seen as an alternative interpretation of the elastic positivity bounds.
2. We find bounds on the rank of extremal elements. For $\dim V = 3$, an extremal element in $\mathcal{C}_W$ must have

$$\dim \ker S > 3, \quad \dim \ker R(S) \geq 1, \quad \dim \ker R'(S) \geq 1.$$

3. We distinguish elements by the dimension of certain characteristic subspaces of their kernels.
4. It turns out that a (non-elastic) element $S \in \mathcal{C}_W$ is extremal if and only if

- (a) $\dim(\ker S \cap \text{Sym}^2 V) = 3$,
- (b) $\dim(\ker S \cap \Lambda^2 V) = 2$

- (c) $\ker S \cap (z \vee V) = 0$, where $z \in V \setminus \{0\}$ is such that $\ker S \cap \Lambda^2 V = z \wedge V$ and $z \vee V = \{z \otimes v + v \otimes z : v \in V\}$.

5. For these elements we can derive the explicit description as stated in the theorem.

We will also study the extremal elements of the cone $(\mathcal{C}_W)^G$ of G -invariant elements of $\mathcal{C}_W$, where $G \in \{O(3), \mathbb{Z}_2^3, SO(2)\}$ and show that every extremal element of $(\mathcal{C}_W)^G$ is a projection of some *elastic* extremal element of $\mathcal{C}_W$. This implies that for amplitudes with such additional symmetry, the elastic positivity bounds are sufficient to describe the complete set of positivity bounds. This is in particular the case for the pion scattering amplitude, which has an $O(3)$ -symmetry.

Finally, we will discuss why, starting from $\dim V \geq 4$, proving an analogous classification result becomes much less feasible and how this is related to the fact that for more than three variables, there exist homogeneous polynomials of degree four which are positive but not sums of squares.

The structure of Part I is as follows:

- In the preliminary Chapter 2 we will give a brief introduction into the physical context in which positivity bounds arise. We will introduce effective field theories and the amplitude. Moreover, we will discuss the fundamental principles of unitarity, causality/analyticity and locality as well as their consequences for the amplitude and how these give rise to the positivity bounds. We will also discuss the example of positivity bounds for two-to-two pion scattering. This chapter does not contain original results.
- In the preliminary Chapter 3 we will introduce the background from convex geometry needed in order to state and prove our main result of Part I. In particular, we will introduce the concept of extremal representations of closed convex cones as well as spectrahedra. We will also discuss the characterization of extremal elements of spectrahedra in terms of their kernels. We will end the chapter with a discussion of Hankel spectrahedra, which are important examples of spectrahedra that arise in the context of sums-of-squares polynomials. These will become relevant in Chapter 4. This chapter does not contain original results.
- The Chapter 4 is the first original chapter of this thesis and contains the results of [Bon+26]. In the chapter, we will prove the main theorem on the classification of all extremal elements of $\mathcal{C}_W$ for $\dim V = 3$. Along the way we will prove several necessary conditions for extremality that are valid for arbitrary dimensions of V as well as an alternative interpretation of the elastic positivity bounds. We will also study amplitudes with additional symmetries showing that in these cases there are no positivity bounds other than those implied by the elastic ones. We end the chapter with a discussion of why it appears that $\dim V = 3$ is the highest dimension in which proving a full classification of all extremal elements of $\mathcal{C}_W$ (with explicit parameterizations) is feasible.

1.2 Overview of Part II

In Part II, we will extend Gualtieri's deformation theorem for generalized complex structures on exact Courant algebroids to a larger class of Courant algebroids, namely $\mathcal{G}$ -flat transitive Courant algebroids. As already mentioned above, to achieve this, we first need to revisit

Gualtieri's proof in [Gua03; Gua11] as well as Kuranishi's proof [Kur65] on the existence of a locally complete family of (classical) complex deformations. Let us briefly recall the setup of Kuranishi's theorem.

Deformations of a (classical) complex structure on a compact manifold M can be described by elements in a neighborhood of 0 in $\mathcal{A}^{0,1}(M, T^{1,0}M)$ (smooth $(0,1)$ -forms with values in $T^{1,0}M$). The almost complex structure corresponding to some deformation $\varphi \in \mathcal{A}^{0,1}(M, T^{1,0}M)$ is integrable if and only if φ satisfies the Maurer–Cartan equation

$$\bar{\partial}\varphi - \frac{1}{2}[\varphi, \varphi] = 0,$$

where $\bar{\partial}$ denotes the Dolbeault differential and the bracket is defined by a combination of the Lie bracket of vector fields and the wedge product. The action of diffeomorphisms on the space of almost complex structures gives rise to a (local) action of diffeomorphisms on the deformations $\varphi \in \mathcal{A}^{0,1}(M, T^{1,0}M)$. Two deformations related in that way are considered equivalent. A smooth family of complex deformations is called *locally complete* if every other smooth family of (sufficiently small) complex deformations can be obtained from it by acting with a family of diffeomorphisms. All of the above will be explained in greater detail in Section 5.1.

Kuranishi proved the following result (we adapted the precise formulation):

Theorem (cf. Theorem 7.9). *There exists an open neighborhood $\mathcal{W} \subset H^{0,1}(M, T^{1,0}M)$ of 0, a family $\tilde{\mathcal{M}} = \{\varphi_t : t \in \mathcal{W}\}$ of almost complex deformations of J , and an analytic obstruction map*

$$\phi : \mathcal{W} \rightarrow H^{0,2}(M, T^{1,0}M)$$

such that the almost complex deformations $\mathcal{M} := \{\varphi_t \in \tilde{\mathcal{M}} : \phi(t) = 0\}$ are precisely the integrable ones. Any sufficiently small complex deformation of J is equivalent to at least one member of the family $\mathcal{M}$. In the case that the obstruction map vanishes, $\mathcal{M}$ is a locally complete family of complex deformations.

The strategy of Kuranishi's proof in [Kur65] can be divided into the following steps:

1. The goal is to write $\mathcal{M}$ as the intersection of a neighborhood of 0 in $\mathcal{A}^{0,1}(M, T^{1,0}M)$ with the set

$$\{\varphi \in \mathcal{A}^{0,1}(M, T^{1,0}M) : \bar{\partial}\varphi - \frac{1}{2}[\varphi, \varphi] = 0, \bar{\partial}^*\varphi = 0\},$$

where $\bar{\partial}^*$ is the formal adjoint of the Dolbeault operator $\bar{\partial}$ with respect to some L^2 -inner product. The equation $\bar{\partial}^*\varphi = 0$ plays the role of a "gauge fixing condition".

2. Kuranishi describes some set $\mathcal{V} \subset \mathcal{A}^{0,1}(M, T^{1,0}M)$ containing $\mathcal{M}$, which can be parameterized by an open neighborhood $\mathcal{W} \subset H^{0,1}(M, T^{1,0}M)$ of 0, and an analytic map $\phi : \mathcal{V} \rightarrow H^{0,2}(M, T^{1,0}M)$ such that $\phi^{-1}(0) = \mathcal{M}$ (to obtain a map $\phi : \mathcal{W} \rightarrow H^{0,2}(M, T^{1,0}M)$ as stated in the theorem, one needs to concatenate with the parametrization of $\mathcal{V}$).
3. It remains to show that for every sufficiently small $\varphi \in \mathcal{A}^0(M, T^{1,0}M)$, we can find some diffeomorphism $f \in \text{Diff}(M)$ such that

$$\bar{\partial}^*(f \cdot \varphi) = 0, \tag{1.1}$$

where $f \cdot \varphi$ denotes the (local) action of diffeomorphisms on deformations, and that the solution f of (1.1) depends smoothly on φ .

To follow through the steps described above in [Kur65], Kuranishi relies on Sobolev spaces and refers to the implicit function theorem on Banach spaces to argue why (1.1) indeed admits a solution. However, crucial analytic details required to argue that this implicit function theorem is indeed applicable are omitted such as the proof of total differentiability of the relevant maps.

In [Gua03; Gua11], Gualtieri followed Kuranishi's proof in [Kur65] to obtain a far reaching extension to a theorem on the existence of a locally complete family of deformations of *generalized* complex structures on exact Courant algebroids, i.e. Courant algebroids isomorphic to the generalized tangent bundle $TM \oplus T^*M$ of a (compact) manifold M . Let us briefly summarize the adaptations compared to the classical complex case described above:

- Every generalized complex structure is associated with a Lie algebroid $L \subset E_{\mathbb{C}}$, i.e. a vector bundle whose space of sections is equipped with a Lie bracket together with a vector bundle morphism $L \rightarrow TM$. In the case of a generalized complex structure, the dual bundle $L^* \rightarrow M$ also has a natural Lie algebroid structure. Deformations of a generalized complex structure are then described by (small) elements in $\Gamma(M, \Lambda^2 L^*)$.
- For every Lie algebroid, we may define a so-called *Lie algebroid differential*

$$d_L: \Gamma(M, \Lambda^k L^*) \rightarrow \Gamma(M, \Lambda^{k+1} L^*).$$

The Lie algebroid differential defines a complex, which is elliptic for generalized complex structures on transitive Courant algebroids, i.e. Courant algebroids with surjective anchor map. In particular, for generalized complex structures on transitive Courant algebroids over compact manifolds, the Lie algebroid cohomology groups $H^k(M, L)$ are finite-dimensional.

- For every Lie algebroid we can moreover define a bracket

$$[\cdot, \cdot]: \Gamma(M, \Lambda^k L^*) \times \Gamma(M, \Lambda^l L^*) \rightarrow \Gamma(M, \Lambda^{k+l-1} L^*),$$

the so-called Schouten bracket. The integrability of a generalized almost complex structure corresponding to a deformation $\varphi \in \Gamma(M, \Lambda^2 L^*)$ is again equivalent to a Maurer–Cartan type equation for φ , involving the Lie algebroid differential and the Schouten bracket.

- While in the classical case (almost) complex structures/ deformations are considered up to diffeomorphisms, in the generalized (almost) complex case, the structures/deformations are identified if they are related by the action of an *autoequivalence*, i.e. a vector bundle equivalence preserving the Courant algebroid structure. Furthermore, to prove local completeness, Gualtieri restricts to the action of the subgroup of *exact* autoequivalences, i.e. autoequivalences whose infinitesimal action is by the Dorfman bracket of the Courant algebroid. This has the advantage that the setup resembles the classical complex case more closely.

In his proof, Gualtieri [Gua03; Gua11] followed through the analogous steps as those of Kuranishi [Kur65], which we summarized above. However, the same crucial analytic details as in [Kur65] are omitted when showing local completeness of the deformation family. Moreover, while Kuranishi was working with local charts for the diffeomorphism group, Gualtieri is working with flows, which raises additional analytic questions.

In Part II of this thesis, we will revisit the classical proofs by Kuranishi [Kur65] and Gualtieri [Gua03; Gua11] in the framework of tame Fréchet spaces using Hamilton's Nash–Moser implicit function theorem. This approach has the advantage of allowing us to remain entirely within the smooth category throughout the proof, avoiding the need to track different Sobolev spaces or invoke additional regularity arguments. Along the way we will furthermore provide the missing analytic details mentioned above. Building on that, we will extend Gualtieri's result to generalized complex structures on the larger class of $\mathcal{G}$ -flat transitive Courant algebroids. More specifically, we will prove the following theorem.

Theorem (cf. Theorem 8.36). *Let $\mathcal{J}$ be a generalized complex structure on a $\mathcal{G}$ -flat transitive Courant algebroid $E \rightarrow M$ with corresponding Lie algebroid L . There exists an open neighborhood $\mathcal{W} \subset H^2(M, L)$ of 0, a family $\tilde{\mathcal{M}} = \{\varphi_t : t \in \mathcal{W}\}$ of generalized almost complex deformations of L , and an analytic obstruction map*

$$\phi: \mathcal{W} \rightarrow H^3(M, L)$$

such that the generalized almost complex deformations $\mathcal{M} := \{\varphi_t \in \tilde{\mathcal{M}} : \phi(t) = 0\}$ are precisely the integrable ones. Any sufficiently small generalized complex deformation of L is equivalent to at least one member of the family $\mathcal{M}$. In the case that the obstruction map vanishes, $\mathcal{M}$ is a locally complete family of generalized complex deformations.

As part of the proof we will endow the group of autoequivalences of a $\mathcal{G}$ -flat transitive Courant algebroid with an infinite-dimensional Lie group structure, more precisely, a tame Fréchet Lie group structure. Moreover, we will show that the exact autoequivalences form a tame Fréchet Lie subgroup. We would like to point out that it was already shown by Rubio and Tipler in [RT20] that the group of autoequivalences of *exact* Courant algebroids has a strong ILH Lie group structure, which is another notion of infinite-dimensional Lie group structure. Moreover, it is shown in [RT20] that the exact autoequivalences of an exact Courant algebroid indeed form a (strong ILH) Lie subgroup and our results build on this fact.

The structure of Part II is as follows:

- In the preliminary Chapter 5, we will introduce the relevant background on complex and generalized (complex) geometry. In particular, we will define Courant algebroids and discuss the standard forms of exact and transitive Courant algebroids. We will also describe the explicit shape the autoequivalences take in these standard forms and define the class of $\mathcal{G}$ -flat transitive Courant algebroids. We will discuss how deformations of complex and generalized complex structures can be described in terms of forms and define the notion of a locally complete family of deformations. Moreover, we will show that the Lie algebroid complex of a generalized complex structure on a transitive Courant algebroid is elliptic. We will also present some formulas that will be relevant for the proofs of the deformation theorems for complex and generalized complex structures in Chapter 7 and 8, respectively.

This chapter does not contain any original results except for the ellipticity of the said Lie algebroid complex (cf. Proposition 5.44) and the identities in Lemma 5.37, which to our knowledge have not appeared in the literature before. The proof of Proposition 5.44 is by a simple extension of an argument by Gualtieri [Gua03; Gua11] for generalized complex structures on exact Courant algebroids.

- In the preliminary Chapter 6, we will introduce the main concepts and results from Hamilton–Nash–Moser theory that we need in Chapter 7 and 8. This includes the definition of tame Fréchet spaces and smooth tame maps, as well as the Nash–Moser theorem and Hamilton’s Nash–Moser implicit function theorem. We will also introduce tame Fréchet manifolds and tame Fréchet Lie groups and give a more detailed discussion of the diffeomorphism group. This chapter does not contain any original results except for a specific result about certain families of operators in Proposition 6.31 which we will apply in Chapter 7 and 8. The result is a simple consequence of a general statement by Hamilton [Ham82] and the Hodge decomposition.
- Chapter 7 is the first original chapter of Part II and contains the results of [CP26]. We revisit Kuranishi’s proof of the existence of a locally complete family of complex deformations on a compact complex manifold using Hamilton–Nash–Moser theory. This includes proving that the action of the diffeomorphisms on the almost complex structures and the corresponding local action on almost complex deformations is smooth tame (cf. Proposition 7.1) and computing a partial derivative of the local expression for the local action (cf. Proposition 7.7). For the proof of the deformation theorem, we will closely follow Kuranishi’s steps outlined above but will employ the techniques of Hamilton–Nash–Moser theory.
- The second original Chapter 8 of Part II is based on joint work in progress with Vicente Cortés. In the chapter, we revisit Gualtieri’s proof of the existence of a locally complete family of generalized complex deformations for generalized complex structures on exact Courant algebroids over a compact manifold using Hamilton–Nash–Moser theory. Moreover, we will extend the result to the larger class of $\mathcal{G}$ -flat transitive Courant algebroids. To establish these results, we will endow the group of autoequivalences of a $\mathcal{G}$ -flat transitive Courant algebroid over a compact manifold with a tame Fréchet Lie group structure. Moreover, we show that the exact autoequivalences form a tame Lie subgroup and prove that the action of the autoequivalences on the generalized almost complex structures is smooth tame.

To prove the main result we will build on the proof of the deformation theorem for classical complex structures constructed in Chapter 7. There are several technical deviations from the classical complex case that are summarized in the beginning of Chapter 8.

Part I

Geometry of Effective Field Theory Positivity Cones

Chapter 2

Positivity in Effective Field Theories

In this chapter, we will describe the physical context in which the research question arises that we will answer in the first part of this thesis. This physical context lies within the field of high-energy physics, more precisely, effective field theories and scattering amplitudes.

First, we will give a brief introduction to quantum field theories and effective field theories. In Section 2.2 we will define the amplitude and discuss its symmetries. Then we will introduce the fundamental principles of *unitarity*, *locality* and *causality/analyticity* and discuss important consequences of these principles for the amplitude. Next, in Section 2.4, we will see how the implications of unitarity, locality and causality yield bounds on the physical parameters of an effective field theory. We will outline how these bounds can be formulated in terms of convex geometry although some of the relevant definitions and concepts will only be introduced in the next chapter. Finally, we will briefly describe how to derive the bounds in the example of pion scattering.

This chapter contains no original results. Most of the content can be found in introductory textbooks of quantum field theory such as [Sch13; Wei95].

2.1 Effective field theories

In this section, we will give a brief introduction to effective field theories. We will first review the foundations of quantum field theories. Then we will introduce effective field theories as a way to approximate a theory of higher energies in terms of its field content at lower energies.

2.1.1 Quantum field theories

Quantum field theories (QFTs) are built on the same formalism as quantum mechanics. *Physical states* are lines in a complex separable Hilbert space. They can be described by sections of some vector bundle over the *spacetime* manifold. For the applications we have in mind, we may assume the spacetime to be the four-dimensional Minkowski space. There is a distinguished physical state called *vacuum state*.

Observables, i.e. physically measurable quantities, can be represented by self-adjoint operators, with the eigenvalues being the possible outcomes of an experiment that is being conducted. The norm-squared of the scalar product (ψ_1, ψ_2) of two normalized vectors ψ_1 and ψ_2 is interpreted as the "probability overlap" of the two physical states represented by ψ_1

and ψ_2 . For instance, in order to determine how likely a certain outcome of an experiment is, we may take the norm-squared of the scalar product of a normalized vector representing a given physical state with a normalized eigenvector corresponding to the measurement value.

Every (elementary) particle of a given theory comes with a *creation* and an *annihilation operator*. An (elementary) one-particle state of the theory is obtained by acting with the corresponding creation operator on the vacuum state. Moreover, acting with any of the annihilation operators on the vacuum state gives 0. The creation and annihilation operators are commonly referred to as *quantum fields*. The collection of all quantum fields in a QFT is referred to as the *field content* of the theory.

We will mainly be interested in *local* theories, i.e. theories that are described by a Lagrangian density¹ that only depends on the values of the quantum fields at a point and finitely many derivatives thereof. Then the Lagrangian density may be written as a sum of terms depending on the different fields and their derivatives, multiplied by a constant² that takes care of the units.

The simplest example of a Lagrangian density is that of a free (real) scalar field ϕ , i.e. $\phi \in C^\infty(\mathbb{R}^4)$, such that ϕ is invariant under Lorentz transformations. The Lagrangian density is given by

$$\mathcal{L}(\phi) = \frac{1}{2} (\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2),$$

where

$$\partial^\mu \phi \partial_\mu \phi = \left(\frac{\partial \phi}{\partial x^0} \right)^2 - \sum_{i=1}^3 \left(\frac{\partial \phi}{\partial x^i} \right)^2.$$

More generally, the Greek indices (e.g. $\partial^\mu \phi$) run from 0 to 3 and refer to four-vectors/momenta. These are paired using the Minkowski inner product of spacetime. The index is lowered using the Minkowski metric $\eta = \text{diag}(1, -1, -1, -1)$, whose components are denoted $\eta_{\mu\nu}$. Explicitly, we write $\partial_\mu \phi = \eta_{\mu\nu} \partial^\nu \phi$. Consistently, the indices are raised using the inverse metric $\eta^{-1} = \text{diag}(1, -1, -1, -1)$, whose components are denoted with raised indices $\eta^{\mu\nu}$. Explicitly, we write $\partial^\mu \phi = \eta^{\mu\nu} \partial_\nu \phi$.

We could also consider n different scalar fields and consider Lagrangian densities with additional terms such as

$$\mathcal{L}(\phi_1, \dots, \phi_n) = \frac{1}{2} \sum_{i=1}^n (\partial^\mu \phi_i \partial_\mu \phi_i - m_i^2 \phi_i^2) + \sum_{i,j,k,l=1}^n \lambda_{ijkl} \partial^\mu \phi_i \partial_\mu \phi_j \partial^\nu \phi_k \partial_\nu \phi_l,$$

where $\lambda_{ijkl} \in \mathbb{R}$. Moreover, we could consider more general quantum fields, which are not scalar such as a connection one-form of a principal G -bundle. This happens for instance in electrodynamics, whose Lagrangian density is given by

$$\mathcal{L}(A) = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

where A is a connection one-form and F is its curvature. Another possibility are (*Dirac*)

¹The Lagrangian density of a theory is a function that when integrated over the spatial directions gives rise to the Lagrangian of the theory. The action functional is obtained by integrating the Lagrangian density over the entire spacetime.

²In practice this constant will "run" with the energy meaning that it depends on the energy scale at which it is probed due to quantum effects and renormalization.

spinors $\psi: \mathbb{R}^4 \rightarrow \mathbb{C}^4$. The (free) Dirac Lagrangian density of a Dirac spinor is given by

$$\mathcal{L}(\psi) = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi,$$

where $\bar{\psi} = \psi^\dagger\gamma^0$, where $\psi^\dagger$ denotes the conjugate transpose, and γ^μ are the gamma-matrices, which may be represented by

$$\begin{aligned}\gamma^0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & \gamma^1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ \gamma^2 &= i \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, & \gamma^3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.\end{aligned}$$

There are many more possibilities for Lagrangian densities. The different terms appearing in the Lagrangian density are also referred to as *operators*, although they are not operators on the Hilbert space of states. The coefficient of an operator $\sim \phi^2$ (containing two fields of the same kind but no derivatives) is related to the mass of the particle corresponding to ϕ . Operators containing more than two fields are relevant for interactions. The coefficients of these operators (e.g. the λ_{ijkl} appearing in the second example above) are also called *coupling constants*. To every operator we may assign a number called *mass dimension* that counts the amount of fields and derivatives in the operator. Together with the value of the coupling constant, the mass dimension of a given operator can serve as an indicator for the energy scale at which the operator contributes.

Furthermore, we will require the Lagrangian density to be Lorentz-invariant so that the corresponding QFT is Lorentz-invariant in the sense that Lorentz transforms of solutions to the equations of motion are again solutions.

2.1.2 Effective Lagrangians

Suppose we are given a quantum field theory that is valid in a certain range of energy and described by some Lagrangian density $\mathcal{L}_0(\psi_1, \dots, \psi_n)$ with some field content $\{\psi_i\}_{i=1}^n$. Suppose furthermore that our given theory is the low energy limit of a (typically unknown) "full theory", which is referred to as the *UV-theory* and described by some Lagrangian density $\mathcal{L}_{\text{full}}(\psi_1, \dots, \psi_n, \varphi_1, \dots, \varphi_k)$, where the fields $\{\varphi_i\}$ are very heavy with masses much greater than the energy scales at which the QFT described by $\mathcal{L}_0$ is valid. Then $\mathcal{L}_0$ is called an *effective Lagrangian (density)* for $\mathcal{L}_{\text{full}}$ and the corresponding QFT is called an *effective field theory (EFT)*.

Conversely, given a QFT described by a Lagrangian density $\mathcal{L}_0$, under appropriate assumptions, the low-energy effects of heavier particles that are not included in the Lagrangian density $\mathcal{L}_0$, may be described by adding operators of higher mass dimension to $\mathcal{L}_0$. Schematically, we write

$$\mathcal{L}_{\text{full,eff}} \approx \mathcal{L}_0 + \sum_{d>4} \sum_i f_i^{(d)} \mathcal{O}_i^{(d)},$$

where $\{\mathcal{O}_i^{(d)}\}$ is a basis of operators of mass dimension d containing the fields $\{\psi_j\}$. The coefficients $f_i^{(d)}$ are called *Wilson coefficients*.

In practice, only finitely many terms up to a specific mass dimension would be added to the Lagrangian density $\mathcal{L}_0$. The number of operators that should be added is usually determined by the precision we want to account for, which is often related to the experimental precision that can be achieved. The resulting theory is an EFT for an unknown UV theory. This is often called the *bottom-up* approach. An important example for an EFT arising in this way is the *Standard Model effective field theory (SMEFT)*.

The *Standard Model* of particle physics is a theory that classifies all experimentally observed elementary particles and describes three out of the four fundamental forces (electromagnetic, weak and strong interaction). Note that there is no possibility of including gravity in the Standard Model. While the Standard Model has been very successful at making experimental predictions it fails to explain certain phenomena observed in nature such as dark matter or the neutrino mass. Therefore, it is believed that the Standard Model is only an EFT of a more fundamental UV theory. In order to study possible extensions of the Standard Model, in the SMEFT additional operators of higher mass dimension are added to the Standard Model Lagrangian (density) that are invariant under the (gauge) symmetries of the Standard Model. The values of the Wilson coefficients of these operators that most closely resemble nature can be determined experimentally. These values then provide information about possible extensions of the Standard Model e.g. by new elementary particles. For this, scattering experiments are performed. These will be reviewed in the following section.

2.2 Two-to-two forward scattering amplitude

In this section, we will introduce the S -matrix and the (two-to-two forward scattering) amplitude. Finally, we will describe some symmetries of the amplitude. These will characterize the mathematical objects that we study in Part I of this thesis.

2.2.1 The S -matrix

Suppose we have some state Ψ_A^+ with particle content A prepared at the "infinite past" that is scattering to some state Ψ_B^- at the "infinite future" with particle content B . We denote by

$$S_{BA} := (\Psi_B^-, \Psi_A^+)$$

the probability amplitude for the transition $A \rightarrow B$. The S -matrix is defined to be the operator S such that

$$S_{BA} = (\Phi_B, S\Phi_A),$$

where Φ_A and Φ_B denote the *free* (i.e. non-interacting) state with particle content A and B , respectively. For instance, given a process where two particles represented by the vectors α and β collide and scatter into two particles represented by γ and δ , we would have $A = \alpha \otimes \beta$ and $B = \gamma \otimes \delta$. The intuition behind this definition is that we assume that at the "infinite past" and "infinite future" the two scattered particles are very far from each other so that they do not interact with each other.

A QFT is called *unitary* if its S -matrix is a unitary operator. Unitarity is related to the preservation of probabilities in scattering processes and thus should be expected to hold in a physically meaningful theory.

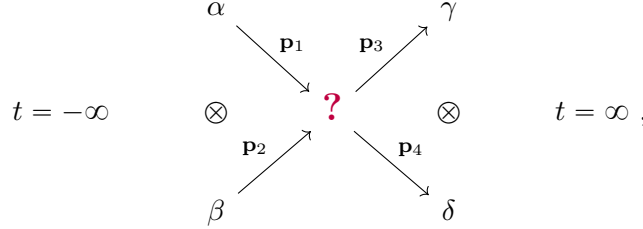
2.2.2 The amplitude and two-to-two scattering

We define the scattering amplitude $\mathcal{A}_{A \rightarrow B}$ of the process $A \rightarrow B$ via

$$i(2\pi)^4 \delta^4(p_B - p_A) \mathcal{A}_{A \rightarrow B} = S_{BA} - \delta(B - A).$$

Here, p_A and p_B denote the total four-momenta of the states A and B , respectively, and we factored out a factor of $\delta^4(p_B - p_A)$, which ensures the conservation of momentum and energy. Moreover, we subtracted $\delta(B - A)$ which would correspond to a trivial scattering. The amplitude describes the part of the S -matrix representing the actual interaction. It can be related to the *total cross-section*, a measurable quantity.

Let us take a closer look at the case of two-to-two particle scattering. In a diagram, we may depict the situation as follows:



where $\mathbf{p}_i$ denote the (three-)momenta of the particles being scattered, which have to satisfy

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4$$

due to momentum conservation. The amplitude $\mathcal{A}_{\alpha \otimes \beta \rightarrow \gamma \otimes \delta}$ depends on the *Mandelstam variables*

$$s := (p_1 + p_2)^2, \quad t := (p_1 - p_3)^2.$$

There is a third Mandelstam variable $u := (p_1 - p_4)^2$, which however is not independent from s and t due to the relation

$$s + t + u = m^2,$$

where m^2 denotes the sum of the squared masses of the four particles.

Once we fix the momentum of a particle, its state is determined by a vector in a finite-dimensional complex vector space $V_{\text{fl},c}$, which we will call *flavor space*. Therefore, when considering two-to-two scattering of particles of the same type, we can think of the amplitude as a map from the Mandelstam variables s and t to four-tensors on the flavor space by setting

$$\mathcal{A}_{ijkl}(s, t) := \mathcal{A}_{\alpha_k \otimes \alpha_l \rightarrow \alpha_i \otimes \alpha_j}(s, t),$$

where the $\{\alpha_i\}$ form a basis of the flavor space and the momenta of the scattered states are encoded in the Mandelstam variables. We will often write $\mathcal{A}(s, t)$ for the corresponding four-tensor such that

$$\mathcal{A}(s, t)(\alpha_i, \alpha_j, \alpha_k, \alpha_l) = \mathcal{A}_{ijkl}(s, t). \quad (2.1)$$

Note that the amplitude $\mathcal{A}(s, t)$ thought of as a map $V_{\text{fl},c}^{\otimes 4} \rightarrow \mathbb{C}$ is complex linear in the first two and complex anti-linear in the second two components. This is due to the sesquilinearity

of the scalar product of the Hilbert space which enters the definition of the amplitude as

$$i(2\pi)^4 \mathcal{A}_{\alpha\otimes\beta\rightarrow\gamma\otimes\delta} = (\alpha\otimes\beta, (S - \text{id})\gamma\otimes\delta).$$

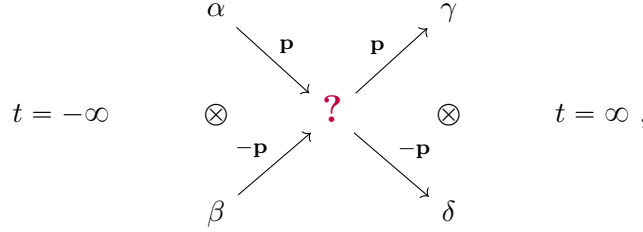
However, the amplitude $\mathcal{A}(s, t)$ will in general have singularities: There will be poles coming from the production of intermediate one-particle states and branch cuts coming from multi-particle intermediate states and loop effects. We will assume that the amplitude has a *mass gap*, i.e. that there is a neighborhood of zero in which the amplitude is analytic. This is the case for processes involving no massless particles.

2.2.3 The forward limit

We will mostly be concerned with the (two-to-two) *forward scattering* amplitude $\mathcal{A}(s)$, which is defined by the limit

$$\mathcal{A}(s) := \lim_{t \rightarrow 0} \mathcal{A}(s, t). \quad (2.2)$$

Intuitively this corresponds to a process in which the kinematics of initial and final state coincide, i.e. in the center-of-mass frame it is of the form



but the forward scattering amplitude is really defined by taking the limit (2.2). Moreover, subtleties in this definition may arise when the masses of the scattered states are different [Rha+18] as well as for particles with spin [Bel17]. We will therefore focus on the scattering of scalar fields that are (at least approximately) of the same mass.

2.2.4 Symmetries of the amplitude

It turns out that not all scattering processes are unrelated and in fact the (two-to-two forward scattering) amplitude has certain symmetries. To state these symmetries, we note that there is a complex conjugation on the Hilbert space of states, defined by taking the anti-particle and suitably reversing momenta and spin. This gives rise to a complex conjugation on the flavor space, which we denote by $\bar{\cdot}$.

The amplitude satisfies the *crossing symmetry*

$$\mathcal{A}(s)(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \mathcal{A}(m^2 - s)(\alpha_1, \bar{\alpha}_4, \alpha_3, \bar{\alpha}_2), \quad (2.3)$$

where m^2 denotes the sum of the four masses squared. Moreover, the amplitude satisfies *CPT invariance*

$$\mathcal{A}(s)(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \mathcal{A}(s)(\bar{\alpha}_3, \bar{\alpha}_4, \bar{\alpha}_1, \bar{\alpha}_2). \quad (2.4)$$

An intuition for why these processes should involve a complex conjugation is that $\mathcal{A}(s)$ is complex linear in the first two entries while it is complex anti-linear in the last two. Therefore "exchanging" in- and out-states must be accompanied by some kind of anti-linear operation.

In addition to crossing symmetry and CPT invariance, we will assume that the amplitude has the symmetry

$$\mathcal{A}(s)(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \mathcal{A}(s)(\alpha_2, \alpha_1, \alpha_4, \alpha_3). \quad (2.5)$$

This symmetry is always satisfied for scalar fields, on which we focus. For particles with spin, it would imply parity conservation, i.e. the symmetry of the amplitude under spatial inversion which would affect the three-momenta as $\mathbf{p} \rightarrow -\mathbf{p}$. If the amplitude is not invariant under parity, we could instead consider the sum of the amplitude and its parity transform in order to obtain a tensor invariant under this additional symmetry [Zha22].

2.3 Unitarity, locality and causality

Previously, we already encountered the fundamental concepts of locality and unitarity. Together with the property of *causality* they give rise to important implications for the amplitude, which we will briefly describe below. These will be the underlying causes for the appearance of *positivity bounds*, which we will derive in the following section.

2.3.1 Causality and analyticity

A QFT is *causal* if its observables and fields commute at space-like distance. Causality is very closely related to *analyticity*, i.e. the possibility to analytically continue the amplitude as a function of s to the whole complex plane³ and we will use both notions interchangeably. In the case of two-to-two scattering of identical scalar particles, building on previous works [BEG64; Leh58; Leh66], it was shown rigorously that the assumptions of causality and unitarity imply that the amplitude $\mathcal{A}(s, t)$ is analytic in s and t (away from the poles of s on the real axis for fixed t) for any $s \in \mathbb{C}$ and $|t| < 4m^2$, where m is the mass of the particles being scattered [Mar66]. In the following, we will always assume that the forward scattering amplitude is analytic in s for all $s \in \mathbb{C} \setminus \mathbb{R}$ (and for real s in a neighborhood of 0 by the assumption of a mass gap).

2.3.2 The optical theorem

Given a scattering amplitude $\mathcal{A}(s, t)$, we define its *discontinuity* at $s \in \mathbb{R}$ as

$$\text{Disc } \mathcal{A}(s, t) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2i} [\mathcal{A}(s + i\varepsilon, t) - \mathcal{A}(s - i\varepsilon, t)].$$

The name is related to the fact that the discontinuity measures the "jump" of the amplitude from above to below its singularities on the real axis.

A direct consequence of unitarity is the (*generalized*) *optical theorem*

$$2 \text{Disc } \mathcal{A}_{A \rightarrow B}(s) = \sum_X \mathcal{A}_{A \rightarrow X}^*(s) \mathcal{A}_{B \rightarrow X}(s), \quad (2.6)$$

³As mentioned before, the amplitude will in general have singularities on the real axis but away from the real axis it will be analytic.

where $\cdot^*$ denotes complex conjugation and $\sum_X$ denotes the sum (or more generally the integral) over all possible intermediate states X of the scattering $A \rightarrow B$.

2.3.3 The Froissart–Martin bound

Now, in addition to unitarity and the analytic properties of the amplitude mentioned above, we assume that the amplitude satisfies the following additional "locality" assumption that for small positive values of the Mandelstam variable t , the two-to-two scattering amplitude $\mathcal{A}(s, t)$ is bounded by some polynomial in s .

Then it was shown that under these assumptions, the forward scattering amplitude cannot grow faster than $s \log s$ for large s . This bound was first proved by Froissart under stronger assumptions [Fro61] and later, by Martin [Mar63] in the setting stated above.

A consequence of the Froissart–Martin bound is that $|\mathcal{A}(s)/s^3| \rightarrow 0$ for $|s| \rightarrow \infty$, which will be useful in the following section, where it allows us to drop the integration of $\mathcal{A}(s)/s^3$ over some contour at infinity.

2.4 Bounds from unitarity, locality and causality

In this section, we will use the fundamental principles discussed in Section 2.3 in order to demonstrate how the positivity bounds arise in EFTs. First, we will derive a condition on the low-energy amplitude of a theory that is unitary, causal and local. These bounds can be translated into bounds on the Wilson coefficients of the EFT, the positivity bounds. We will distinguish *elastic* and *inelastic* positivity bounds. Finally, we will discuss the elastic positivity bounds for the scattering of *pions*, scalar fields that appear in the theory of strong interactions. This section contains no original results but goes beyond classical introductory textbooks on QFT. We will mostly be following the references [Li+21; ZZ20]. Most of the content discussed (in particular Section 2.4.3) can also be found in the appendix of the coauthored publication [Bon+26].

2.4.1 A condition on the amplitude

We will associate a real four-tensor to the two-to-two forward scattering amplitude in the following way. First, we restrict the amplitude to the real subspace $V_{\mathbb{R}}$ of self-conjugate vectors. More explicitly, we fix a basis $\{\alpha_i\}$ of $V_{\mathbb{R},c}$ consisting of self-conjugate vectors $\alpha_i = \overline{\alpha_i}$. Then $V_{\mathbb{R}} = \text{span}_{\mathbb{R}}\{\alpha_i\}$. We consider the forward scattering amplitude

$$\mathcal{A}_{ijkl}(s) = \mathcal{A}(s)(\alpha_i \otimes \alpha_j, \alpha_k \otimes \alpha_l)$$

for the process $\alpha_i \otimes \alpha_j \rightarrow \alpha_k \otimes \alpha_l$.

Then we define

$$M_{ijkl} := \frac{d^2}{ds^2} \mathcal{A}_{ijkl}(s = m^2/2) + \text{c.c.},$$

where m^2 is the sum of the four squared masses of the scattered particles, and c.c. stands for the complex conjugate, so that the M_{ijkl} are real numbers.⁴

⁴Note that here we assumed that $s = m^2/2$ is still within the analytic region of the amplitude. More generally, there might be singularities due to lighter particles, which can be explicitly computed and subtracted [Bel17; Rha+17], so that $\mathcal{A}_{ijkl}$ becomes analytic in $s = m^2/2$ and therefore M_{ijkl} is well-defined.

Since the vectors α_i are self-conjugate, it follows from the crossing symmetry (2.3) and CPT-invariance (2.4) of the amplitude $\mathcal{A}$ that

$$M_{ijkl} = M_{ilkj} = M_{kjil} = M_{klij} .$$

Moreover, the additional symmetry (2.5) we assumed implies $M_{ijkl} = M_{jilk}$.

We may think of the M_{ijkl} as the components of a real four-tensor M on $V_{\mathbb{R}}$. It is worth noticing that, despite the restriction to the real subspace $V_{\mathbb{R}}$, the M_{ijkl} contain information about two-to-two scatterings of arbitrary states (also of those which are represented by vectors that are not self-conjugate).

Now we will assume that $\mathcal{A}$ is the two-to-two forward scattering amplitude of a theory which is the low energy limit of a unitary, local and causal theory. Then, assuming that $s = m^2/2$ is contained in the analytic region of the amplitude, M_{ijkl} may be computed by integrating along a small contour $\mathcal{C}$ encircling $m^2/2$

$$M_{ijkl} = \frac{1}{\pi i} \int_{\mathcal{C}} ds \frac{\mathcal{A}_{ijkl}(s)}{(s - m^2/2)^3} + \text{c.c.} .$$

Using analyticity of the amplitude, one can deform the integration contour to larger values of s as sketched in Figure 2.1. Then the Froissart–Martin bound in Section 2.3.3 allows us

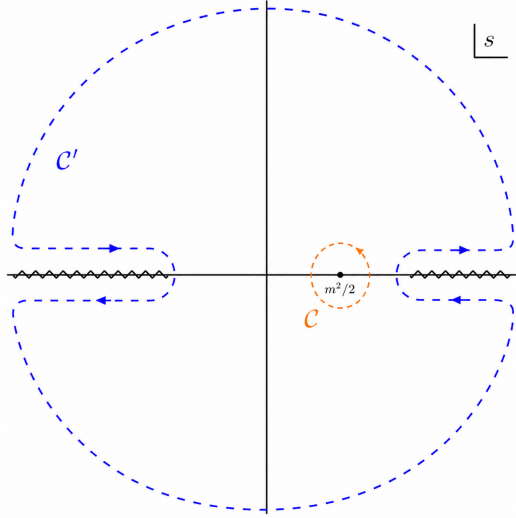


Figure 2.1: Integration contours to derive the positivity bounds. The contour $\mathcal{C}$ may be deformed to the larger contour $\mathcal{C}'$. This figure is a modification of Figure 2 in [RR19].

to drop the integral over the circle at infinity, leaving behind the integration along the real s -axis, which results in an integration over the discontinuities of the amplitude along the axis. Moreover, the discontinuities can be related to other amplitudes $\mathcal{A}_{ij \rightarrow X}$, $\mathcal{A}_{kl \rightarrow X}$ via the generalized optical theorem, while the integral over negative values of s is mapped to the integral over positive values of the amplitude $\mathcal{A}_{ilkj}(s)$ via crossing symmetry. Altogether, one finds

$$M_{ijkl} = \frac{1}{\pi} \int_{\Lambda}^{\infty} ds \frac{\sum_X \mathcal{A}_{ij \rightarrow X} \mathcal{A}_{kl \rightarrow X}^*}{(s - m^2/2)^3} + (j \leftrightarrow l) + \text{c.c.} ,$$

where $\Lambda > m^2/2$, and X refers to arbitrary intermediate states.

Expanding $\mathcal{A}_{ij \rightarrow X} = Q_{X, \mathfrak{R}, ij} + i Q_{X, \mathfrak{S}, ij}$ into real and imaginary parts, one finds that

$$M_{ijkl} = \sum_a Q_{a, ij} Q_{a, kl} + Q_{a, il} Q_{a, kj}, \quad \text{for some } Q_a \in V^{\otimes 2},$$

where a ranges over the combined $(X, \mathfrak{R}/\mathfrak{S})$ labels. Moreover, since we assume the symmetry $M_{ijkl} = M_{jilk}$, the Q tensors can be assumed to be either symmetric or anti-symmetric. To see this, we decompose the tensors Q_a into symmetric and anti-symmetric part $Q_a = Q_a^S + Q_a^A$ and compute

$$\begin{aligned} M_{ijkl} &= \frac{1}{2}(M_{ijkl} + M_{jilk}) \\ &= \frac{1}{2} \sum_a (Q_a^S + Q_a^A)_{ij} (Q_a^S + Q_a^A)_{kl} + (Q_a^S + Q_a^A)_{il} (Q_a^S + Q_a^A)_{kj} \\ &\quad + \frac{1}{2} \sum_a (Q_a^S + Q_a^A)_{ji} (Q_a^S + Q_a^A)_{lk} + (Q_a^S + Q_a^A)_{li} (Q_a^S + Q_a^A)_{jk} \\ &= \sum_a Q_{a, ij}^S Q_{a, kl}^S + Q_{a, ij}^A Q_{a, kl}^A + Q_{a, il}^S Q_{a, kj}^S + Q_{a, il}^A Q_{a, kj}^A, \end{aligned}$$

where we used

$$Q_{a, ij}^S Q_{a, kl}^S = Q_{a, ji}^S Q_{a, lk}^S, \quad Q_{a, ij}^S Q_{a, kl}^A = -Q_{a, ji}^S Q_{a, lk}^A, \quad Q_{a, ij}^A Q_{a, kl}^A = Q_{a, ji}^A Q_{a, lk}^A.$$

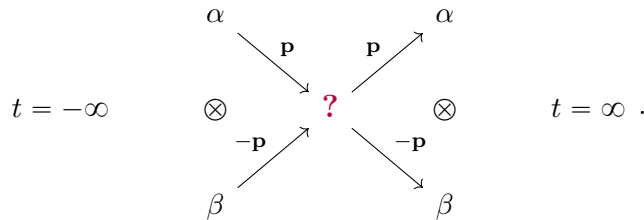
In total, we observe that

$$M \in \mathcal{C} := \left\{ \sum_a Q_a^{\otimes 2} + \tau Q_a^{\otimes 2} : Q_a \in \text{Sym}^2 V \text{ or } Q_a \in \Lambda^2 V \right\}, \quad (2.7)$$

where we defined $(\tau M)(v_1, v_2, v_3, v_4) = M(v_1, v_4, v_3, v_2)$ for some $M \in V^{\otimes 4}$. In the next chapter (in Example 3.8) we will see that $\mathcal{C}$ is a closed convex cone.

2.4.2 Elastic positivity bounds and beyond

One first consequence of (2.7) is the following. Consider the two-to-two forward scattering of an initial state of two particles $\alpha \otimes \beta$ to itself



A scattering process like this, where "nothing happens" is called *elastic* scattering. Then (2.7) implies that

$$\begin{aligned} M(\alpha \otimes \beta, \alpha \otimes \beta) &= \sum_{i,j,k,l} \alpha_i \beta_j \alpha_k \beta_l M_{ijkl} \\ &= 2 \sum_a \sum_{i,j,k,l} \alpha_i \beta_j \alpha_k \beta_l Q_{a,ij} Q_{a,kl} \\ &= 2 \left(\sum_{a,ij} \alpha_i \beta_j Q_{a,ij} \right)^2 \geq 0. \end{aligned}$$

The constraints

$$\sum_{i,j,k,l} \alpha_i \beta_j \alpha_k \beta_l M_{ijkl} \geq 0, \quad \text{for } \alpha, \beta \in V^* \quad (2.8)$$

give rise to a system of linear inequalities (depending on the parameters α and β) that need to be satisfied in order for the amplitude to be compatible with (2.7), which was a necessary condition for the amplitude to be compatible with the assumptions of unitarity, locality and causality of the UV-theory. They are called *elastic (positivity) bounds* and have received the most attention in the literature. However, as it was already pointed out in [Li+21], in general the elastic positivity bounds are not sufficient to constrain the tensor M to be contained in $\mathcal{C}$.

More abstractly, the (linear forms associated to the) inequalities constraining any convex cone $\mathcal{K} \subset U$ inside a vector space U form a closed convex cone by themselves called *dual cone* $\mathcal{K}^*$

$$\mathcal{K}^* = \{S \in U^* : S(M) \geq 0\}.$$

In order to determine all inequalities describing the cone $\mathcal{C}$, we need to understand the shape of the dual cone of $\mathcal{C}$. This cone is given by

$$\mathcal{C}^* = \mathcal{C}_W := \{S \in W : S \geq 0\},$$

where we defined

$$W := \{S \in \text{Sym}^2(\text{Sym}^2 V^*) \oplus \text{Sym}^2(\Lambda^2 V^*) : \tau S = S\} \subset (V^*)^{\otimes 4}$$

where as before $\tau S(v_1, v_2, v_3, v_4) = S(v_1, v_4, v_3, v_2)$ and $S \geq 0$ means that S is positive semidefinite as an element of $\text{Sym}^2(V^* \otimes V^*)$, i.e. $S(Q, Q) \geq 0$ for all $Q \in V \otimes V$. Note that here we used the decomposition of two-tensors into symmetric and anti-symmetric part in order to think of W as a subspace of $\text{Sym}^2(V^* \otimes V^*)$.

The linear inequalities describing $\mathcal{C}$ can be translated to linear inequalities on the Wilson coefficients corresponding to the operators quartic in momenta of the (effective) Lagrangian of the theory underlying the amplitude $\mathcal{A}$ (at tree-level). These inequalities are commonly referred to as *positivity bounds*⁵. In the following, we will also refer to the inequalities constraining the abstract cone $\mathcal{C}$ of four-tensors related to the amplitude as positivity bounds

⁵In general, there will also be positivity bounds on some of the operators at higher mass dimension containing more than four momenta. The positivity bounds we describe here are those constraining Wilson coefficients quartic in momenta that contribute to two-to-two scattering processes at tree-level.

(as we already did for the special case of elastic positivity bounds). Although we focused on EFTs of scalar fields before and later in the application of our main result in Chapter 4, we emphasize that these positivity bounds (and our main result) have a broader range of validity, as it has been demonstrated in [Li+21].

Finally, we point out that many of the positivity bounds obtained by elements in $\mathcal{C}_W$ are in fact redundant. In order to derive a minimal system of inequalities, it is sufficient to only consider those determined by *extreme rays* of $\mathcal{C}_W$. These will be introduced in Chapter 3.

In the following section we will illustrate the procedure of extracting the real tensor M from the amplitude and deriving the elastic positivity bounds using the example of pion scattering.

2.4.3 Elastic positivity bounds for pion scattering

Chiral perturbation theory is an effective field theory for quantum chromodynamics, i.e. the theory of strong interactions between quarks and gluons. It captures the dynamics of quantum chromodynamics below its confinement scale, in terms of the lightest mesons and baryons. In particular, the field theory which describes the three lightest mesons, namely *pions*, is built out of three (self-conjugate) scalar fields $\{\pi^a\}$.

To obtain the four-tensor M associated to two-to-two forward scattering of pions, we first note that the relevant parts of the Lagrangian density (those containing four pion fields and four derivatives) are given by⁶

$$\mathcal{L}_{\text{ChPT},4} \supset \ell_1 \partial_\mu \pi^a \partial^\mu \pi^a \partial_\nu \pi^b \partial^\nu \pi^b + \ell_2 \partial_\mu \pi^a \partial_\nu \pi^a \partial^\mu \pi^b \partial^\nu \pi^b$$

for some parameters ℓ_1 and ℓ_2 .

The two-to-two forward scattering amplitude can then be computed in terms of the parameters ℓ_1 and ℓ_2 at tree-level, using Feynman diagrams (see e.g. [Wan+20]). Taking the second derivative at $2m_\pi^2$, where m_π denotes the pion mass, yields

$$M_{abcd} = \frac{d^2}{ds^2} \mathcal{A}_{abcd}(s = 2m_\pi^2) = 2[(2\ell_1 + \ell_2) \delta_{ab} \delta_{cd} + 2\ell_2 \delta_{ac} \delta_{bd} + (2\ell_1 + \ell_2) \delta_{ad} \delta_{bc}].$$

⁶There will be further operators in the Lagrangian density with more derivatives, some of which are subject to positivity bounds. The operators presented here are those that are quartic in momenta and which contribute to two-to-two pion scattering at tree-level.

Observing that in general for some dual bases $\{e_i\}_{i=1}^3$ and $\{\alpha_i\}_{i=1}^3$ of $\mathbb{R}^3$ and $\mathbb{R}^{3*}$, we have

$$\begin{aligned}\delta_{ab}\delta_{cd} &= \sum_{i,j=1}^3 e_i^{\otimes 2} \otimes e_j^{\otimes 2} (\alpha_a, \alpha_b, \alpha_c, \alpha_d) = \left(\sum_{i=1}^3 e_i^{\otimes 2} \right)^{\otimes 2} (\alpha_a, \alpha_b, \alpha_c, \alpha_d), \\ \delta_{ac}\delta_{bd} &= \sum_{i,j=1}^3 e_i \otimes e_j \otimes e_i \otimes e_j (\alpha_a, \alpha_b, \alpha_c, \alpha_d) \\ &= \left[\sum_{i=1}^3 e_i^{\otimes 4} + 2 \sum_{i<j} (e_i \vee e_j)^{\otimes 2} + 2 \sum_{i<j} (e_i \wedge e_j)^{\otimes 2} \right] (\alpha_a, \alpha_b, \alpha_c, \alpha_d), \\ \delta_{ad}\delta_{bc} &= \sum_{i,j=1}^3 e_i \otimes e_j \otimes e_j \otimes e_i (\alpha_a, \alpha_b, \alpha_c, \alpha_d) \\ &= \left[\sum_{i=1}^3 e_i^{\otimes 4} + 2 \sum_{i<j} (e_i \vee e_j)^{\otimes 2} - 2 \sum_{i<j} (e_i \wedge e_j)^{\otimes 2} \right] (\alpha_a, \alpha_b, \alpha_c, \alpha_d),\end{aligned}$$

we conclude that (after rescaling ℓ_1 and ℓ_2 by a factor of $1/2$)

$$\begin{aligned}M &= (2\ell_1 + 3\ell_2) \{e_1^4 + e_2^4 + e_3^4 + 2(e_1 \vee e_2)^{\otimes 2} + 2(e_1 \vee e_3)^{\otimes 2} + 2(e_2 \vee e_3)^{\otimes 2}\} \\ &\quad + (2\ell_1 + \ell_2)(e_1^2 + e_2^2 + e_3^2)^{\otimes 2} + 2(\ell_2 - 2\ell_1) \{(e_1 \wedge e_2)^{\otimes 2} + (e_1 \wedge e_3)^{\otimes 2} + (e_2 \wedge e_3)^{\otimes 2}\}.\end{aligned}$$

Here, we used the notation/conventions

$$e_i \vee e_j = \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i), \quad e_i \wedge e_j = \frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i).$$

One can show either by exploring directly the inequalities (2.8) or using Proposition 4.45 that the elastic positivity bounds are given by $\ell_2 \geq -\ell_1$, $\ell_2 \geq 0$. These have already been derived in the literature, see e.g. [Wan+20].

Chapter 3

Extremal Representations of Spectrahedra

In the previous chapter, we saw that the fundamental principles of unitarity, locality and causality give rise to bounds on the physical parameters of an EFT and that these bounds can be described by elements in a closed convex cone $\mathcal{C}_W$. We already mentioned that many elements in $\mathcal{C}_W$ in fact lead to redundant bounds and that it is sufficient to only consider bounds corresponding to so-called *extreme rays* of $\mathcal{C}_W$.

In this second preliminary chapter, we will make these statements mathematically precise and introduce the concepts needed for the proof of our main result in Chapter 4. First, we will introduce some notions in convex geometry in Section 3.1, in particular that of an extreme ray and the extremal representation of a closed convex cone. In Section 3.2 we will focus on a particular kind of closed convex cones called *linear spectrahedra*. These are cones inside the space of symmetric bilinear forms of some vector space, and (elements spanning the) extreme rays of linear spectrahedra are characterized by having "maximal kernel" among the non-trivial elements of the cone. We will see that the cone $\mathcal{C}_W$ is a linear spectrahedron so that these characterizations of extreme rays apply. We will end the chapter by discussing an important class of spectrahedra arising in the context of sum-of-squares polynomials. These will also be relevant in Chapter 4.

This chapter does not contain any original results. Although for some statements we included a proof (sometimes because we could not find it in the exact same form in the literature), these statements are known to experts in the field.

3.1 Convex sets

In this section, we will introduce some basic notions and concepts for convex sets and convex cones. In particular, we will define the notion of an *extreme ray* of a closed convex cone in Section 3.1.3.

3.1.1 Basic definitions and notations

In the following, let V be a real finite-dimensional vector space. We will start with a review of some definitions and notations for convex sets and more general subsets of V :

Definition 3.1. Let $\mathcal{S} \subset V$ be a subset. Then the *affine hull* of $\mathcal{S}$ is the smallest affine set $\text{aff}(\mathcal{S})$ containing $\mathcal{S}$.

For convex sets we can use the affine hull in order to define the *dimension* of a convex set.

Definition 3.2. Let $\mathcal{C} \subset V$ be a convex set. Then the *dimension* $\dim \mathcal{C}$ of $\mathcal{C}$ is the dimension of its affine hull $\text{aff}(\mathcal{C})$.

An important notion for the description of unbounded convex sets is that of *directions*:

Definition 3.3. A *direction* in V is an equivalence class of closed half-lines in V under the equivalence relation

$$L_1 \sim L_2 : \Longleftrightarrow L_1 \text{ is a translate of } L_2.$$

The direction of a half-line $L = \{x + \lambda \cdot y : \lambda \geq 0\}$, where $x, y \in V$, $y \neq 0$, is also called the *direction of* y .

Definition 3.4. Let $\mathcal{C}$ be a non-empty convex set in V . Then $\mathcal{C}$ *recedes in the direction* D if every half-line in the direction D starting at a point in $\mathcal{C}$ is contained in $\mathcal{C}$. Furthermore, we say that the empty set recedes in every direction in V .

Next, we will define the *convex hull* of a collection of points and directions.

Definition 3.5. Let $S_0 \subset V$ be a set of points and S_1 a set of directions in V . Then the *convex hull* of $S = S_0 \cup S_1$ is the smallest convex set $\text{conv}(S)$ containing S_0 and such that $\text{conv}(S)$ recedes in all the directions in S_1 .

Remark 3.6 (Section 17 in [Roc70]). Let $S = S_0 \cup S_1$ as in Definition 3.5. Then every element $x \in \text{conv}(S)$ is of the form

$$x = \lambda_1 x_1 + \cdots + \lambda_k x_k + \lambda_{k+1} x_{k+1} + \cdots + \lambda_m x_m,$$

for some $k \geq 1$, where $x_i \in S_0$ for $i \leq k$, and for $k+1 \leq i \leq m$, $x_i \in V$ such that the direction of x_i is in S_1 , and $\lambda_i \geq 0$ such that

$$\lambda_1 + \cdots + \lambda_k = 1.$$

An expression of this form is called a *convex combination of m points and directions* in S .

A related notion is that of the *conical hull*.

Definition 3.7. The *conical hull* of a subset $S \subset V$ is the smallest convex cone containing S and 0. It is given by

$$\text{cone}(S) = \left\{ \sum_i \lambda_i x_i : \lambda_i \geq 0, x_i \in S \right\}.$$

Note that if $S \subset V$ does not contain 0, its conical hull $\text{cone}(S)$ is the convex hull of 0 and the directions of all elements in S .

Example 3.8. Recall from Section 2.4 the cone

$$\mathcal{C} = \mathcal{C}_W^* = \left\{ \sum_i Q_i^{\otimes 2} + \tau Q_i^{\otimes 2} : Q_i \in \text{Sym}^2 V \text{ or } Q_i \in \Lambda^2 V \right\}.$$

Then $\mathcal{C}$ is the convex cone

$$\mathcal{C} = \text{cone} \{Q^{\otimes 2} + \tau Q^{\otimes 2} : Q \in \text{Sym}^2 V \text{ or } Q \in \Lambda^2 V\}.$$

We claim that $\mathcal{C}$ is closed.

First, we consider the cone

$$\mathcal{C}_1 := \left\{ \sum_i Q_i^{\otimes 2} : Q_i \in \text{Sym}^2 V \text{ or } \Lambda^2 V \right\} \subset \text{Sym}^2(\text{Sym}^2 V) \oplus \text{Sym}^2(\Lambda^2 V).$$

Note that this is just a product of the two cones

$$\mathcal{C}'_1 := \left\{ \sum_i Q_i^{\otimes 2} : Q_i \in \text{Sym}^2 V \right\}, \quad \mathcal{C}''_1 := \left\{ \sum_i Q_i^{\otimes 2} : Q_i \in \Lambda^2 V \right\}.$$

These are in fact examples of *sum-of-squares cones*, which are known to be closed (see Section 3.2.2). Then we note that

$$\mathcal{C} = (1 + \tau)\mathcal{C}_1.$$

The linear map

$$\frac{1}{2}(\text{id} + \tau) : V^{\otimes 4} \longrightarrow V^{\otimes 4}$$

is the projection to the $+1$ -eigenspace of the involution τ . The kernel of this projection is the space of elements that are in the -1 -eigenspace of the involution. This space has trivial intersection with $\mathcal{C}_1$. In particular, this shows that $\mathcal{C}$ is a closed convex cone, since it is the image of the closed convex cone $\mathcal{C}_1$ (which contains no lines) under a linear projection whose kernel has trivial intersection with $\mathcal{C}_1$ [Roc70, Thm 9.1].

The following theorem tells us that we can in fact express every element in $\text{conv}(S)$ by a convex combination of at most $n + 1$ points and directions in S :

Theorem 3.9 (Carathéodory's theorem, Theorem 17.1 in [Roc70]). *Let S be a set of points and directions in a real vector space V , $\dim V = n$, and let $\mathcal{C} = \text{conv}(S)$. Then $x \in \mathcal{C}$ if and only if it can be expressed as a convex combination of at most $n + 1$ points and directions in S .*

3.1.2 Topological notions for convex sets

For a subset $\mathcal{S} \subset V$ of a real finite-dimensional vector space V , we will denote by $\text{cl}(\mathcal{S})$ and $\text{int}(\mathcal{S})$ its closure and interior, respectively. For convex sets we may furthermore define the notion of a *relative interior*. This is often more useful to consider, since the interior of a convex set $\mathcal{C}$ in V with $\dim \mathcal{C} < \dim V$ is always empty.

Definition 3.10. Let $\mathcal{C}$ be a convex set. The *relative interior* $\text{relint}(\mathcal{C})$ of $\mathcal{C}$ is defined as the interior of $\mathcal{C}$ with respect to the induced topology of $\text{aff}(\mathcal{C})$. The *relative boundary* of $\mathcal{C}$ is defined as $\partial_{\text{rel}} \mathcal{C} := \text{cl}(\mathcal{C}) \setminus \text{relint}(\mathcal{C})$.

Remark 3.11 (Section 6 in [Roc70]). While the inclusion $\mathcal{C}_1 \supset \mathcal{C}_2$ implies $\text{cl}(\mathcal{C}_1) \supset \text{cl}(\mathcal{C}_2)$ and $\text{int}(\mathcal{C}_1) \supset \text{int}(\mathcal{C}_2)$, in general it does not imply $\text{relint}(\mathcal{C}_1) \supset \text{relint}(\mathcal{C}_2)$ for convex $\mathcal{C}_1, \mathcal{C}_2$: If $\mathcal{C}_1$ is a simplex in $\mathbb{R}^2$ and $\mathcal{C}_2$ is one of its edges, then $\text{relint}(\mathcal{C}_1)$ and $\text{relint}(\mathcal{C}_2)$ are both non-empty but disjoint.

In the following, we will list some properties of the closure and relative interior of convex sets:

Proposition 3.12 (Theorem 6.3 and Corollary 6.3.1–6.3.3 in [Roc70]). *Let $\mathcal{C}, \mathcal{C}'$ be convex sets. Then the following hold:*

1. $\text{cl}(\text{relint}(\mathcal{C})) = \text{cl}(\mathcal{C})$ and $\text{relint}(\text{cl}(\mathcal{C})) = \text{relint}(\mathcal{C})$,
2. $\text{cl}(\mathcal{C}) = \text{cl}(\mathcal{C}') \iff \text{relint}(\mathcal{C}) = \text{relint}(\mathcal{C}') \iff \text{relint}(\mathcal{C}) \subset \mathcal{C}' \subset \text{cl}(\mathcal{C})$,
3. every open set which meets $\text{cl}(\mathcal{C})$ also meets $\text{relint}(\mathcal{C})$,
4. if $\mathcal{C}'$ is contained in the relative boundary of $\mathcal{C}$, then $\dim \mathcal{C}' < \dim \mathcal{C}$.

3.1.3 Faces and extreme rays

In this section, we will define the notion of extreme rays and extremal elements of a closed convex cone. Before we can introduce these, we need to first understand the notion of a *face* of a convex set.

Definition 3.13. A *face* of a convex set $\mathcal{C}$ is a convex subset $\mathcal{C}'$ of $\mathcal{C}$ such that every (closed) line segment in $\mathcal{C}$ with a relative interior point in $\mathcal{C}'$ has both endpoints in $\mathcal{C}'$.

In particular, we will consider the whole convex set $\mathcal{C}$ a face of itself. The following proposition lists several important properties of the faces of a convex set.

Proposition 3.14 (Theorem 18.1 and Corollary 18.1.1–3 in [Roc70]). *Let $\mathcal{C}$ be a convex set, and F a face of $\mathcal{C}$. Then the following hold:*

1. If $\mathcal{C}'$ is a convex set in $\mathcal{C}$ such that $\text{relint} \mathcal{C}' \cap F \neq \emptyset$, then $\mathcal{C}' \subset F$.
2. F is closed.
3. If F' is another face of $\mathcal{C}$ such that $\text{relint} F \cap \text{relint} F' \neq \emptyset$, then $F = F'$.
4. If $F \neq \mathcal{C}$ then $F \subset \partial_{\text{rel}} \mathcal{C}$.

By Proposition 3.12, this implies that $\dim F < \dim \mathcal{C}$ for every face $F \neq \mathcal{C}$ of a convex set $\mathcal{C}$. Zero-dimensional faces are called *extreme points*.

Theorem 3.15 (Theorem 18.5 in [Roc70]). *Let $\mathcal{C}$ be a non-empty convex set, and let $\mathcal{U}$ be the collection of all relative interiors of non-empty faces of $\mathcal{C}$. Then $\mathcal{U}$ is a partition of $\mathcal{C}$, i.e. the sets in $\mathcal{U}$ are disjoint and their union is $\mathcal{C}$.*

In particular, this means that every point in $\mathcal{C}$ is contained in the relative interior of precisely one face of $\mathcal{C}$.

Definition 3.16. For $x \in \mathcal{C}$ the unique face $F_{\mathcal{C}}(x)$ such that $x \in \text{relint} F_{\mathcal{C}}(x)$ is called the *minimal face* of $x \in \mathcal{C}$.

Remark 3.17. By Proposition 3.12, the minimal face of $x \in \mathcal{C}$ has the smallest dimension among the faces containing x .

We will furthermore introduce the notion of *exposedness* of a face of a convex set. For this, we need to define *supporting hyperplanes* of convex sets:

Definition 3.18. A *supporting half-space* to a convex set $\mathcal{C}$ is a closed half-space which contains $\mathcal{C}$ and has a point of $\mathcal{C}$ in its boundary. A *supporting hyperplane* to $\mathcal{C}$ is a hyperplane which is the boundary of a supporting half-space to $\mathcal{C}$.

Definition 3.19. A face of a convex set $\mathcal{C}$ is called *exposed* if it is $\mathcal{C}$ itself or its intersection $\mathcal{C} \cap H$ with a supporting hyperplane H of $\mathcal{C}$.

Example 3.20 (Section 18 in [Roc70]). Faces of a convex set are not always exposed: An example is given by the convex hull $\mathcal{C}_T := \text{conv}(T^2) \subset \mathbb{R}^3$ of the two-torus embedded in $\mathbb{R}^3$. The two-dimensional faces of $\mathcal{C}_T$ are the two discs on the sides of $\mathcal{C}_T$. The points on the boundaries of these discs are zero-dimensional faces of $\mathcal{C}_T$, which are not exposed, as any supporting hyperplane containing one of these zero-faces contains the whole two-face in whose relative boundary the zero-face is contained.

We will now give a definition of *extreme rays* of a convex cone in terms of faces:

Definition 3.21. An *extreme ray* of a convex cone $\mathcal{C}$ is a face of $\mathcal{C}$ which is of the form $\text{ray}(x)$ for some $x \in \mathcal{C} \setminus \{0\}$, where

$$\text{ray}(x) = \mathbb{R}_{\geq 0} x \cap \mathcal{C}.$$

An element $x \in \mathcal{C} \setminus \{0\}$ generating an extreme ray is called *extremal*.

Example 3.22. The extreme rays of a polyhedral cone are its edges. The extreme rays of an "ice cream cone" (i.e. the cone over the two-disc) are the half-lines in the boundary starting at the origin. They are parametrized by a circle. See Figure 3.1.

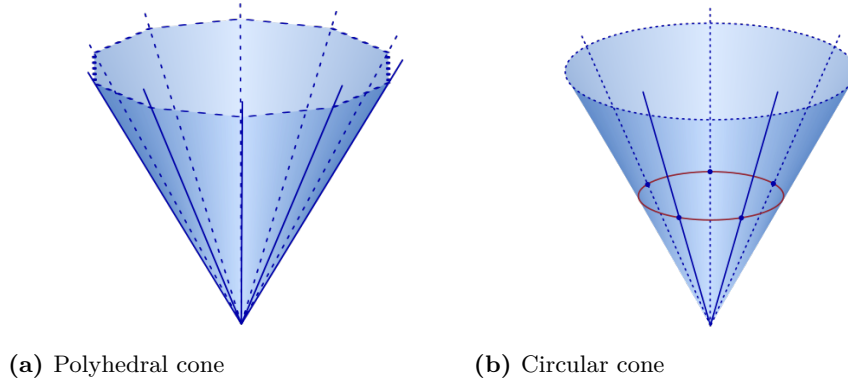


Figure 3.1: Extreme rays of a polyhedral and a circular cone. The extreme rays of a polyhedral cone are its edges. The extreme rays of a circular cone are parametrized by a circle.

Proposition 3.23. Let $\mathcal{C}$ be a convex cone containing more than just the origin but no lines. Then $x \in \mathcal{C} \setminus \{0\}$ is extremal if and only if for every $x_1, x_2 \in \mathcal{C} \setminus \{0\}$ such that $x = x_1 + x_2$ it follows $x_i \in \mathbb{R}_{>0} x$ for $i = 1, 2$.

Proof. If $x \in \mathcal{C} \setminus \{0\}$ generates an extreme ray, every closed line segment

$$\{(1-t)y_1 + ty_2 : t \in [0, 1]\} \subset \mathcal{C}$$

starting at some $x \neq y_1 \in \mathcal{C}$, ending at some $x \neq y_2 \in \mathcal{C}$ and passing through x must be contained in $\text{ray}(x)$, which is only possible if $y_1, y_2 \in \mathbb{R}_{>0} \cdot x$.

Conversely, if x is not extremal, there is a line segment

$$L := \{(1-t)y_1 + ty_2 : t \in [0, 1]\} \subset \mathcal{C},$$

whose interior intersects $\text{ray}(x)$ in precisely one point. Then the set of vectors $\{x, y_1, y_2\} \subset \mathcal{C}$ is linearly dependent. We may furthermore assume that, up to translating and rescaling L , we have $L \cap \text{ray}(x) = \{x\}$ and conclude that $x = (1-s) \cdot y_1 + s \cdot y_2$ for some $s \in (0, 1)$. $\square$

The reason why it is interesting to study the extreme rays of a convex cone is the following:

Theorem 3.24 ([Kle57], see also Corollary 18.5.2 in [Roc70]). *Let $\mathcal{C}$ be a closed convex cone containing more than just the origin but no lines. Let T be any set of vectors in $\mathcal{C}$ such that each extreme ray of $\mathcal{C}$ is generated by some $x \in T$. Then $\mathcal{C}$ is the convex cone generated by T .*

The following is a consequence of Theorem 3.24 and will be useful for us.

Proposition 3.25. *Let V be a real finite-dimensional vector space and $\mathcal{C} \subset V$ a closed convex cone. Consider a linear projection $p: V \rightarrow U$ to a subspace $U \subset V$. Suppose that the projection of the cone $\tilde{\mathcal{C}} := p(\mathcal{C})$ is closed. Then every extreme ray of $\tilde{\mathcal{C}}$ is the projection of some extreme ray in $\mathcal{C}$.*

Proof. The case where $\tilde{\mathcal{C}} = 0$ is trivial. We may therefore assume that $\tilde{\mathcal{C}} \neq 0$. Let $x \in \tilde{\mathcal{C}} \setminus \{0\}$ be an extremal element. Let $y \in \mathcal{C}$ such that $p(y) = x$. Suppose that $y = y_0 + y_1$ for some $y_0, y_1 \in \mathcal{C}$. Then $x = p(y_0) + p(y_1)$. We may furthermore assume that $p(y_0) \neq 0$. Since by assumption x is extremal, it follows from Proposition 3.23 that $p(y_0) = \lambda x$ for some $\lambda > 0$. In particular, $1/\lambda y_0$ projects onto x . Since by Theorem 3.24 every element in $\mathcal{C}$ is a finite positive linear combination of extremal elements, we may iterate the procedure described above (reducing in each step the number of terms in the linear combination) to obtain an extremal element projecting onto $x \in \tilde{\mathcal{C}}$. $\square$

3.2 Spectrahedra

We will now focus on a particular kind of convex sets, namely (linear) spectrahedra. Given some real, finite-dimensional vector space V , we denote by

$$\text{Sym}_+^2 V^* = \{Q \in \text{Sym}^2 V^* : Q(v, v) \geq 0 \ \forall v \in V\}$$

the positive semi-definite bilinear forms on V .

Definition 3.26. Let V be a finite-dimensional real vector space. An intersection of $\text{Sym}_+^2 V^*$ with an affine subspace of $\text{Sym}^2 V^*$ is called a *spectrahedron*. A spectrahedron is called *linear* if it is the intersection of $\text{Sym}_+^2 V^*$ with a linear subspace of $\text{Sym}^2 V^*$.

Remark 3.27 (see e.g. [Vin14], [RG95]). 1. A spectrahedron can always be parametrized as

$$\mathcal{Q} = \left\{ (t_1, \dots, t_k) \in \mathbb{R}^k : Q_0 + \sum_{i=1}^k t_i \cdot Q_i \geq 0 \right\}$$

for some $Q_0, \dots, Q_k \in \text{Sym}^2 V^*$. The spectrahedron is linear if and only if we may choose $Q_0 = 0$.

2. If a spectrahedron is linear, then it is a closed convex cone.

Example 3.28. Recall from Section 2.4 the cone

$$\mathcal{C}_W := \{S \in W \mid S \geq 0\},$$

where

$$W := \{S \in \text{Sym}^2(\text{Sym}^2 V^*) \oplus \text{Sym}^2(\Lambda^2 V^*) : \tau S = S\} \subset \text{Sym}^2(V^* \otimes V^*),$$

for some finite-dimensional real vector space V , and where $\tau S(v_1, v_2, v_3, v_4) = S(v_1, v_4, v_3, v_2)$ for all $v_1, \dots, v_4 \in V$. Then $\mathcal{C}_W = W \cap \text{Sym}_+^2(V^* \otimes V^*)$ and thus is a linear spectrahedron.

3.2.1 The faces and extreme rays of a (linear) spectrahedron

In [RG95], the faces and extreme rays of (linear) spectrahedra were characterized. In order to state the result, given a symmetric bilinear form $b \in \text{Sym}^2 V^*$ on some finite-dimensional vector space V , we will denote by

$$\ker b := \ker(b : V \rightarrow V^*)$$

the *kernel* of b and by

$$\text{rank } b := \dim V - \dim(\ker b)$$

the *rank* of b . Then the faces of a spectrahedron are characterized as follows.

Theorem 3.29 (Theorem 1 and Corollary 1 in [RG95]). *Let V be a finite-dimensional real vector space, $U \subset \text{Sym}^2 V^*$ an affine subspace. Let $\mathcal{C}$ be the spectrahedron $\mathcal{C} = U \cap \text{Sym}_+^2 V^*$. Then the following hold:*

1. *For every $x \in \mathcal{C}$, the minimal face $F_{\mathcal{C}}(x)$ and its affine hull are given by*

$$\begin{aligned} F_{\mathcal{C}}(x) &= \{z \in \mathcal{C} : \ker z \supset \ker x\}, \\ \text{aff}(F_{\mathcal{C}}(x)) &= \{z \in U : \ker z \supset \ker x\}; \end{aligned}$$

2. *The kernel $\ker x$ is constant over the relative interior of any face in $\mathcal{C}$;*

3. *Every face in $\mathcal{C}$ is exposed.*

We also note the following description of the relative boundary of the minimal face of a point $x \in \mathcal{C}$:

Corollary 3.30. *Let $\mathcal{C}$ be a spectrahedron and $x \in \mathcal{C}$. Then*

$$\partial_{\text{rel}} F_{\mathcal{C}}(x) = \{z \in \mathcal{C} : \ker z \supsetneq \ker x\}.$$

Proof. Let $x \in \mathcal{C}$, $y \in \partial_{\text{rel}} F_{\mathcal{C}}(x)$. Since $\mathcal{C}$ is closed, by Proposition 3.14 the face $F_{\mathcal{C}}(x)$ is closed. Therefore, $y \in F_{\mathcal{C}}(x)$ and by Theorem 3.29, it follows that $\ker y \supset \ker x$. If $\ker y = \ker x$, Theorem 3.29 implies that also $x \in F_{\mathcal{C}}(y)$, which is impossible since y was assumed to be in the relative boundary of $F_{\mathcal{C}}(x)$ and therefore $\dim F_{\mathcal{C}}(y) < \dim F_{\mathcal{C}}(x)$. Conversely, if $\ker y \supsetneq \ker x$ for some $y \in \mathcal{C}$, then $y \in F_{\mathcal{C}}(x) \setminus \text{relint } F_{\mathcal{C}}(x) = \partial_{\text{rel}} F_{\mathcal{C}}(x)$. $\square$

In the case of a linear spectrahedron, the first property in Theorem 3.29 yields a criterion to determine whether an element $x \in \mathcal{C}$ is extremal in $\mathcal{C}$:

Corollary 3.31 (Corollary 4 in [RG95]). *Let $\mathcal{C} = U \cap \text{Sym}_+^2 V^*$ be a linear spectrahedron. Then $x \in \mathcal{C} \setminus \{0\}$ is extremal if and only if for every $z \in U$ the inclusion $\ker x \subset \ker z$ implies $z \in \mathbb{R}x$.*

The following is a straightforward consequence of the above characterizations and was essentially already shown in [RG95].

Corollary 3.32. *Let $\mathcal{C} = U \cap \text{Sym}_+^2 V^*$ be a linear spectrahedron. Then the following are equivalent:*

1. $x \in \mathcal{C} \setminus \{0\}$ is extremal;
2. for every $z \in \mathcal{C}$ the proper inclusion $\ker x \subsetneq \ker z$ implies $z = 0$;
3. for every $z \in U$ the proper inclusion $\ker x \subsetneq \ker z$ implies $z = 0$.

Proof. It is clear from Theorem 3.29 that the first assertion implies the third and it is obvious that the third assertion implies the second. It remains to show that the second assertion implies the first.

By Corollary 3.30, the property that for every $z \in \mathcal{C}$ the inclusion $\ker x \subsetneq \ker z$ implies $z = 0$ is equivalent to saying that the relative boundary of the minimal face $F_{\mathcal{C}}(x)$ of x is the point $0 \in \mathcal{C}$. We claim that this is equivalent to the statement that $\dim F_{\mathcal{C}}(x) = 1$. Indeed, if $\dim F_{\mathcal{C}}(x) > 1$, then there would be some $y \in \text{relint } F_{\mathcal{C}}(x)$ linearly independent of x . We consider the line L in $U \subset \text{Sym}_+^2 V^*$ through x and y . Since by assumption x is linearly independent of y , this line does not intersect the origin and hence it does not intersect the boundary of $F_{\mathcal{C}}(x)$. Therefore, $L \subset \text{relint } F_{\mathcal{C}}(x)$, which is impossible, since $\text{Sym}_+^2 V^*$ is line-free and so must be $\mathcal{C} \subset \text{Sym}_+^2 V^*$. $\square$

The facial structure of (linear) spectrahedra $\mathcal{C} \subset \text{Sym}_+^2 V^*$ suggests that in order to identify its extreme points (or extremal elements if it is linear) it is useful to study the dimension of the kernel or equivalently the rank of extreme points (or extremal elements). For extreme points, the rank is bounded by the *Pataki inequality*.

Proposition 3.33 (Pataki inequality [Pat00], see also [Sch22]). *Let V be a real vector space with $\dim V = n < \infty$, $U \subset \text{Sym}_+^2 V^*$ an affine subspace, $\dim U = m$ and $\mathcal{C} = U \cap \text{Sym}_+^2 V^*$. Let $x \in \mathcal{C}$ be an extreme point of $\mathcal{C}$ and $r = \text{rank } x$. Then*

$$r \leq -\frac{1}{2} + \frac{\sqrt{(2n+1)^2 - 8m}}{2}.$$

The Pataki inequality can also be used to determine bounds on the ranks of extremal elements of a linear spectrahedron as follows: Suppose $\mathcal{C} = U \cap \text{Sym}_+^2 V^*$ is a linear spectrahedron. Let $x \in \mathcal{C} \setminus \{0\}$ be an extremal element of $\mathcal{C}$. Then we can pick an affine subspace $\tilde{U} \subset U$ containing x but not the origin and such that $\dim \tilde{U} = \dim U - 1$. Then $\tilde{\mathcal{C}} = \tilde{U} \cap \text{Sym}_+^2 V^*$ is a spectrahedron and $x \in \tilde{\mathcal{C}}$ is an extreme point of $\tilde{\mathcal{C}}$ so that Proposition 3.33 can be applied.

Corollary 3.34. *Let V be a real vector space $\dim V = n < \infty$, $U \subset \text{Sym}^2 V^*$ a linear subspace, $\dim U = m$ and $\mathcal{C} = U \cap \text{Sym}_+^2 V^*$. Let $x \in \mathcal{C}$ be an extremal element of $\mathcal{C}$ and $r = \text{rank } x$. Then*

$$r \leq -\frac{1}{2} + \frac{\sqrt{(2n+1)^2 - 8m + 8}}{2}.$$

3.2.2 Hankel spectrahedra and SOS polynomials

Important examples of spectrahedra arise in the following context. Denote by

$$P_{n,2d} = \{p \in \mathbb{R}[x_1, \dots, x_n]_{2d} : p(x) \geq 0 \ \forall x \in \mathbb{R}^n\}$$

the cone of nonnegative real homogeneous polynomials of degree $2d$ in n variables and by $\Sigma_{n,2d} \subset P_{n,2d}$ the cone of polynomials in n variables that can be written as a sum of squares of homogeneous polynomials of degree d

$$\Sigma_{n,2d} = \left\{ p \in \mathbb{R}[x_1, \dots, x_n]_{2d} : p = \sum_i p_i^2, \ p_i \in \mathbb{R}[x_1, \dots, x_n]_d \right\}.$$

In 1885, during the oral defense of his dissertation, Minkowski proposed that there exist nonnegative polynomials that are not sums of squares. Although Minkowski did not present a full proof, Hilbert, who was one of the examiners, was convinced by Minkowski's arguments. Three years later, in [Hil88] he resolved the question proving the following theorem:

Theorem 3.35 (Hilbert's theorem [Hil88]). *The equality $P_{n,2d} = \Sigma_{n,2d}$ holds if and only if (n, d) are as in one of the following cases:*

- $n = 2$
- $2d = 2$
- or $n = 3, 2d = 4$.

Although Hilbert proved that in all other cases there exist nonnegative (homogeneous) polynomials that are not sums of squares, he did not provide explicit examples of such. It was not until almost 80 years later that the first example of a nonnegative polynomial that is not a sum of squares was found by Motzkin [Mot67]. Hilbert's proof was later revisited and generalized by Reznick in [Rez07] where many more examples were presented.

A way to certify the difference between $P_{n,2d}$ and $\Sigma_{n,2d}$ is by looking at their dual cones $P_{n,2d}^* \subset \mathbb{R}[x_1, \dots, x_n]_{2d}^*$ and $\Sigma_{n,2d}^* \subset \mathbb{R}[x_1, \dots, x_n]_{2d}^*$. Given $v \in \mathbb{R}^n$, we denote by $\ell_v \in \mathbb{R}[x_1, \dots, x_n]_{2d}^*$ the linear functional given by evaluation at v :

$$\ell_v(p) = p(v) \quad \text{for } p \in \mathbb{R}[x_1, \dots, x_n]_{2d}.$$

Linear functionals of the form ℓ_v are called *point evaluations*.

Proposition 3.36 (See e.g. [Ble12a]). *The dual cone of $P_{n,2d}$ is the conical hull of point evaluations ℓ_v with $v \in S^{n-1}$*

$$P_{n,2d}^* = \text{cone}(\{\ell_v : v \in S^{n-1}\}) = \left\{ \sum_i \lambda_i \ell_{v_i} : \lambda_i \in \mathbb{R}_{\geq 0}, v_i \in S^{n-1} \right\}.$$

Moreover the extreme rays of $P_{n,2d}^*$ are spanned by ℓ_v , $v \in S^{n-1}$.

The dual cone of $\Sigma_{n,2d}$ is a spectrahedron. To see this, we consider the space $\mathcal{S}_{n,d}$ of real quadratic forms on $\mathbb{R}[x_1, \dots, x_n]_d$ as well as the cone $\mathcal{S}_{n,d}^+$ of positive semidefinite forms in $\mathcal{S}_{n,d}$. To every linear functional $\ell \in \mathbb{R}[x_1, \dots, x_n]_{2d}^*$ we can associate a quadratic form $Q_\ell \in \mathcal{S}_{n,d}$ via $Q_\ell(p) = \ell(p^2)$. The symmetric matrix representing a quadratic form Q_ℓ associated to $\ell \in \mathbb{R}[x_1, \dots, x_n]_{2d}^*$ with respect to the monomial basis of $\mathbb{R}[x_1, \dots, x_n]_d$ is called a *Hankel matrix*. One can show that $\ell \in \Sigma_{n,2d}^*$ if and only if its associated quadratic form is positive semi-definite $Q_\ell \geq 0$. Therefore, we conclude the following.

Proposition 3.37 (See e.g. [Ble12a]). *The cone $\Sigma_{n,2d}^*$ is given by the intersection*

$$\Sigma_{n,2d}^* = \mathcal{S}_{n,d}^+ \cap \mathbb{R}[x_1, \dots, x_n]_{2d}^*.$$

In particular, it is a linear spectrahedron called Hankel spectrahedron.

Extreme rays of $\Sigma_{n,2d}^*$ give rise to linear functionals that are nonnegative on SOS polynomials. In particular, extreme rays of $\Sigma_{n,2d}^*$ which are not extreme rays for $P_{n,2d}^*$ can help us distinguish polynomials that are nonnegative but not sums of squares.

Note that quadratic forms in $\mathcal{S}_{n,d}$ corresponding to point evaluations have rank one and conversely, rank one (positive semi-definite) quadratic forms in $\mathcal{S}_{n,d}$ correspond to point evaluations (see e.g. [Ble12a]).

Corollary 3.38. *We have $\Sigma_{n,2d} = P_{n,2d}$ if and only if all extremal elements of $\Sigma_{n,2d}^*$ have rank one.*

Moreover, the following is known about the kernel of extremal elements of $\Sigma_{n,2d}^*$.

Proposition 3.39 ([Ble12b], see also [Ble12a]). *Let Q_ℓ be the quadratic form corresponding to an extremal element ℓ of $\Sigma_{n,2d}^*$. Then either $\text{rank } Q_\ell = 1$ or the polynomials in the kernel of Q_ℓ have no common zeros (real or complex).*

In the special cases of $\Sigma_{3,6}^*$ and $\Sigma_{4,4}^*$ (the first cases where $\Sigma_{n,2d} \neq P_{n,2d}$) the extremal elements are completely characterized by the following theorems.

Theorem 3.40 ([Ble12b], see also [Ble12a]). *Suppose $\ell \in \Sigma_{3,6}^*$ is an extremal element of $\Sigma_{3,6}^*$. Then $\text{rank } Q_\ell = 1$ or $\text{rank } Q_\ell = 7$. Conversely, if $Q_\ell \in \mathcal{S}_{3,3}$ is a positive semi-definite quadratic form such that $\text{rank } Q_\ell = 7$, the element $\ell \in \Sigma_{3,6}^*$ is extremal if and only if the polynomials in the kernel of Q_ℓ have no common zeros (real or complex).*

Theorem 3.41 ([Ble12b], see also [Ble12a]). *Suppose $\ell \in \Sigma_{4,4}^*$ is an extremal element of $\Sigma_{4,4}^*$. Then $\text{rank } Q_\ell = 1$ or $\text{rank } Q_\ell = 6$. Conversely, if $Q_\ell \in \mathcal{S}_{4,2}$ is a positive semi-definite quadratic form such that $\text{rank } Q_\ell = 6$, the element $\ell \in \Sigma_{4,4}^*$ is extremal if and only if the polynomials in the kernel of Q_ℓ have no common zeros (real or complex).*

However, there is no known "parametrization" of all extreme rays of $\Sigma_{3,6}^*$ and $\Sigma_{4,4}^*$ as it was possible for $P_{n,2d}^*$. The closest we can get to a description of the set of extremal elements is the following.

Theorem 3.42 (Theorem 5 in [Ble+12]). *The Hankel matrices of rank ≤ 7 representing elements in $\mathcal{S}_{3,3}$ constitute a rational projective variety of dimension 21 and degree 2640. The Hankel matrices of rank ≤ 6 representing elements in $\mathcal{S}_{4,2}$ constitute a rational projective variety of dimension 24 and degree 28 314.*

Chapter 4

Extremal Elements of the Positivity Cone

In this first original chapter of the thesis, we will study the extremal elements of the cone $\mathcal{C}_W$. In particular, we will prove the main result of Part I of the thesis, namely the full classification of all extremal elements of the cone $\mathcal{C}_W \subset V^{*\otimes 4}$ for $\dim V \leq 3$. This will give rise to all (forward scattering) positivity bounds of any scattering process involving up to three flavors. In particular, we will see that, while for $\dim V = 2$ there are no positivity bounds other than those implied by the elastic ones, starting from $\dim V = 3$, there is a new class of extremal elements, which gives rise to additional bounds that cannot be deduced from elastic positivity bounds. We will then consider amplitudes (involving three flavors) that are invariant under additional symmetries and show that in these cases the elastic positivity bounds are sufficient in order to describe the complete set of positivity bounds. In particular, we will see that this is true in the case of pion scattering discussed in Section 2.4.3.

The structure of this chapter is as follows. After fixing some notation and discussing some auxiliary results in Section 4.1, we will describe our general setup in Section 4.2. In particular, we will study some properties of the space W in which $\mathcal{C}_W$ lies and give an alternative interpretation of the extremal elements of $\mathcal{C}_W$ corresponding to elastic positivity bounds. We will also relate the cone $\mathcal{C}_W \cap \text{Sym}^4 V^*$ to the Hankel spectrahedra $\Sigma_{n,4}^*$ introduced in Section 3.2.2 and deduce that for $\dim V \leq 3$, elements in $\mathcal{C}_W \cap \text{Sym}^4 V^*$ are extremal in $\mathcal{C}_W$ if and only if they correspond to elastic positivity bounds. Moreover, we will prove some general necessary conditions for elements in $\mathcal{C}_W$ to be extremal as well as a classification of all extremal elements of $\mathcal{C}_W$ in the case where $\dim V = 2$.

In Section 4.3 we will state and prove the classification of all extremal elements in $\mathcal{C}_W$ in the case where $\dim V = 3$. We will then take a closer look at the corresponding positivity bounds in Section 4.4, noting that starting from $\dim V = 3$, additional positivity bounds arise that are not already captured by the elastic ones. In Section 4.5 we will see however that when imposing certain additional symmetries on the amplitude, there are no positivity bounds that do not follow from the elastic ones. Finally, in Section 4.6, we will briefly discuss what can be said for the case that $\dim V > 3$ and point out why proving an analogous statement to our classification result in the case $\dim V = 3$ appears to be significantly less feasible.

Except for Sections 4.2.4 and 4.6, up to minor changes, all sections of this chapter are taken from the coauthored publication [Bon+26].

4.1 Notation and auxiliary results

In this chapter, we will use the following convention for the symmetric and anti-symmetric product of vectors $u, v \in V$ in some finite-dimensional real vector space V :

$$u \vee v = \frac{1}{2}(u \otimes v + v \otimes u), \quad u \wedge v = \frac{1}{2}(u \otimes v - v \otimes u)$$

so that

$$u \otimes v = u \vee v + u \wedge v.$$

In the second part of the thesis we will return to the more common definition for the wedge product without the factor $1/2$. But in the context of this chapter the above convention will be more natural and improve the readability of our formulas.

Moreover, for some vector $v \in V$ and subspace $U \subset V$, we will denote by $v \vee U$ the space

$$v \vee U := \{v \vee u : u \in U\}$$

and similarly, we write

$$v \wedge U := \{v \wedge u : u \in U\}.$$

We will furthermore denote the totally symmetric product of one-forms β_i , $i = 1, \dots, k$ by

$$\beta_1 \beta_2 \cdots \beta_k := \frac{1}{k!} \sum_{\sigma \in S_k} \beta_{\sigma(1)} \otimes \beta_{\sigma(2)} \otimes \cdots \otimes \beta_{\sigma(k)}.$$

When $\beta_1 = \beta_2 = \cdots = \beta_k =: \beta$, we will also write $\beta^k = \beta \beta \cdots \beta$.

As before, given a symmetric bilinear form b on some vector space V , we will denote by

$$\ker b := \ker(b: V \rightarrow V^*), \quad \text{rank } b := \dim V - \dim(\ker b)$$

the kernel and rank of b , respectively, and we will extend these notions to anti-symmetric bilinear forms.

In the following, we will list some useful general results on symmetric bilinear forms:

Lemma 4.1. *Let V be a real two-dimensional vector space. Then any subspace $U \subset \text{Sym}^2 V$ such that $\dim U \geq 2$ contains an element of the form $u \vee v$ for $u, v \in V$.*

Proof. Let V, U as above. If $\dim U = 3$, the statement is trivial. Suppose $\dim U = 2$ and that U contains no element of rank one (otherwise, the statement is also trivial). Let $\{u_1^{\otimes 2} + \varepsilon_1 v_1^{\otimes 2}, u_2^{\otimes 2} + \varepsilon_2 v_2^{\otimes 2}\}$ be a basis for U , where $u_i, v_i \in V$ are linearly independent, $\varepsilon_i = \pm 1$ for $i \in \{1, 2\}$. If $\varepsilon_i = -1$ for $i = 1$ or $i = 2$, we observe $u^{\otimes 2} - v^{\otimes 2} = (u + v) \vee (u - v)$ and the assertion follows. In the case where $\varepsilon_1 = 1 = \varepsilon_2$, we note that $u_1^{\otimes 2} + v_1^{\otimes 2}$ defines a metric on V^* . Let $A \in \text{O}(V^*)$ with respect to that metric, such that, possibly after rescaling u_2 and v_2 , we have $A^*(u_1) = u_2$. Since $A^*(u_1)^{\otimes 2} + A^*(v_1)^{\otimes 2} = u_1^{\otimes 2} + v_1^{\otimes 2}$, it follows that $v_2^{\otimes 2} - A^*(v_1)^{\otimes 2} \in U$. $\square$

Lemma 4.2. *Let V be a real vector space such that $\dim V = 3$. Then any subspace $U \subset \text{Sym}^2 V$ such that $\dim U \geq 2$ has an element $T \in U$ such that $\text{rank } T < 3$.*

Proof. Consider two linearly independent elements $T_1, T_2 \in U$. Suppose that $\text{rank } T_i = 3$ for $i = 1, 2$. We consider the polynomial

$$\begin{aligned} \mathbb{R} \ni t &\mapsto p(t) := \det(T_1 + tT_2) \\ &= \det(T_2)t^3 + \mu_2(T_1, T_2)t^2 + \mu_1(T_1, T_2)t + \det(T_1), \end{aligned}$$

where the coefficients $\mu_i \in \mathbb{R}$ depend on T_1, T_2 . Since by assumption $\det(T_2) \neq 0$, there is some $t_0 \in \mathbb{R}$ such that $p(t_0) = 0$. In particular, $T := T_1 + t_0 T_2 \in U$ has a non-trivial kernel. $\square$

We also recall the following well-known property of semi-definite bilinear forms:

Lemma 4.3. *Suppose b is a positive semi-definite bilinear form on some vector space V . Then $v \in \ker(b: V \rightarrow V^*)$ if and only if $b(v, v) = 0$.*

Finally, we note the following useful formula for elements $S \in W$ evaluated on a pair $(u^{\otimes 2}, v^{\otimes 2})$ for some $u, v \in V$:

Lemma 4.4. *Let $S \in W$, $u, v \in V$. Then*

$$S(u^{\otimes 2}, v^{\otimes 2}) = S(u \vee v, u \vee v) - S(u \wedge v, u \wedge v).$$

Proof. For $S \in W$, $u, v \in V$, we find

$$\begin{aligned} S(u^{\otimes 2}, v^{\otimes 2}) &= S(u \otimes v, v \otimes u) \\ &= S(u \vee v + u \wedge v, u \vee v - u \wedge v) \\ &= S(u \vee v, u \vee v) - S(u \wedge v, u \wedge v), \end{aligned}$$

where we used the invariance under τ in the first step and the fact that $\Lambda^2 V$ and $\text{Sym}^2 V$ are orthogonal with respect to S for the last equality. $\square$

4.2 General properties of the positivity cone

Let V be a real n -dimensional vector space. Like in Section 2.4, we denote by W the vector space

$$W := \{S \in \text{Sym}^2(\text{Sym}^2 V^*) \oplus \text{Sym}^2(\Lambda^2 V^*) : \tau S = S\} \subset V^{*\otimes 4},$$

where $(\tau S)(v_1, v_2, v_3, v_4) := S(v_1, v_4, v_3, v_2)$ for all $v_1, \dots, v_4 \in V$. In Section 4.2.1, we will construct two natural linear maps, R and R' , associated with W , which are defined via restriction to certain subspaces. Then, we will study the properties of the space W and of the restriction maps R and R' . Thereafter, we will consider the cone

$$\mathcal{C}_W := \{S \in W : S \geq 0\}$$

of positive semi-definite elements of $W \subset \text{Sym}^2(V^* \otimes V^*)$ introduced in Section 2.4. In Section 4.2.3 we will give an alternative characterization of the extremal elements of $\mathcal{C}_W$ which give rise to the elastic positivity bounds. Moreover, in Section 4.2.4 we will take a closer look at the cone $\mathcal{C}_W \cap \text{Sym}^4 V^*$ and show that it can be identified with the Hankel spectrahedron $\Sigma_{n,4}^*$ introduced in Section 3.2.2. We will conclude that an element $S \in \mathcal{C}_W \cap \text{Sym}^4 V^*$ is extremal in $\mathcal{C}_W \cap \text{Sym}^4 V^*$ (and hence in $\mathcal{C}_W$) if and only if $\text{rank } S = 1$. Finally, in

Section 4.2.5 we will establish some properties of the extremal elements of $\mathcal{C}_W$ as well as a complete classification of all extreme rays in the case where $\dim V = 2$.

4.2.1 Relevant spaces and linear maps

We denote by R and R' the restriction maps

$$R: W \longrightarrow \text{Sym}^2(\text{Sym}^2 V^*), \quad R': W \longrightarrow \text{Sym}^2(\Lambda^2 V^*),$$

respectively. We furthermore consider the total symmetrization map

$$\text{tot}: W \rightarrow \text{Sym}^4 V^*$$

on W and denote by $K_W \subset W$ its kernel. Then we have the following decomposition

$$W = \text{Sym}^4 V^* \oplus K_W.$$

In the following, we will study some properties of the vector space W and the restriction maps R and R' . We first note that the restriction map R is in fact an isomorphism.

Proposition 4.5. *The restriction map*

$$R: W \rightarrow \text{Sym}^2(\text{Sym}^2 V^*)$$

is an isomorphism, which restricts to the identity map on $\text{Sym}^4 V^$.*

Proof. It follows from the symmetries of W that $W \cap \text{Sym}^2(\text{Sym}^2 V^*) = \text{Sym}^4 V^*$. Therefore R restricts to the identity on $\text{Sym}^4 V^*$. To prove that R is an isomorphism, we will start by showing that R is injective. For this, we need to show that $W \cap \text{Sym}^2(\Lambda^2 V^*) = \{0\}$. Using the symmetries, we find that for every $S \in W \cap \text{Sym}^2(\Lambda^2 V^*)$, we have

$$\begin{aligned} S(v_1, v_2, v_3, v_4) &= S(v_1, v_4, v_3, v_2) \\ &= -S(v_4, v_1, v_3, v_2) \\ &= -S(v_4, v_2, v_3, v_1) \\ &= S(v_2, v_4, v_3, v_1) \\ &= S(v_2, v_1, v_3, v_4) \\ &= -S(v_1, v_2, v_3, v_4) \end{aligned}$$

and therefore, $S = 0$. In order to prove that R is surjective, we show that it has a right-inverse. We set

$$\begin{aligned} R^{-1}: \text{Sym}^2(\text{Sym}^2 V^*) &\longrightarrow W \\ T &\longmapsto T + h(T) \end{aligned}$$

where

$$\begin{aligned} h: \text{Sym}^2(\text{Sym}^2 V^*) &\longrightarrow \text{Sym}^2(\Lambda^2 V^*) \\ T &\longmapsto h(T) \end{aligned}$$

is defined by

$$h(T)(e_i \wedge e_j, e_k \wedge e_l) = T(e_i \vee e_l, e_j \vee e_k) - T(e_i \vee e_k, e_j \vee e_l).$$

We show that $S := R^{-1}(T)$ is indeed in W as claimed. All the defining symmetries of W but the symmetry under τ follow immediately from the definition. It remains to verify the symmetry under τ , i.e. $S(v_1, v_2, v_3, v_4) = S(v_1, v_4, v_3, v_2)$:

$$\begin{aligned} S(v_1, v_2, v_3, v_4) &= T(v_1 \vee v_2, v_3 \vee v_4) + h(T)(v_1 \wedge v_2, v_3 \wedge v_4) \\ &= T(v_1 \vee v_2, v_3 \vee v_4) + T(v_1 \vee v_4, v_2 \vee v_3) - T(v_1 \vee v_3, v_2 \vee v_4) \\ &= T(v_1 \vee v_4, v_3 \vee v_2) + T(v_1 \vee v_2, v_4 \vee v_3) - T(v_1 \vee v_3, v_4 \vee v_2) \\ &= T(v_1 \vee v_4, v_3 \vee v_2) + h(T)(v_1 \wedge v_4, v_3 \wedge v_2) \\ &= S(v_1, v_4, v_3, v_2), \end{aligned}$$

where we used the definition of S in the last step. This now shows that $T + h(T) \in W$ for all $T \in \text{Sym}^2(\text{Sym}^2 V^*)$ as desired. Since clearly $R(R^{-1}(T)) = T$, surjectivity of R follows. $\square$

Using that $\dim \text{Sym}^2 U = \dim U(\dim U + 1)/2$ for any vector space U , we conclude the following.

Corollary 4.6. *Let $\dim V = n$. Then the dimension of W is given by*

$$\dim W = \frac{n(n+1)(n(n+1)+2)}{8}.$$

Next, we will study the kernel and image of R' .

Proposition 4.7. *The restriction map*

$$R' : W \longrightarrow \text{Sym}^2(\Lambda^2 V^*)$$

is an epimorphism onto the space of algebraic curvature tensors

$$K(V) := \left\{ S \in \text{Sym}^2(\Lambda^2 V^*) : \sum_{\mathfrak{S}} S(v_1, v_2, v_3) = 0 \quad \text{for all } v_1, v_2, v_3 \in V \right\},$$

where $\mathfrak{S}$ indicates the cyclic sum. The kernel of R' is $\text{Sym}^4 V^$. As a consequence,*

$$R'|_{K_W} : K_W \longrightarrow K(V)$$

is an isomorphism.

Proof. It follows from $W \cap \text{Sym}^2(\text{Sym}^2 V^*) = \text{Sym}^4 V^*$ that the kernel of R' is the space $\text{Sym}^4 V^*$. To see that the image $R'(K_W)$ is contained in $K(V)$, notice that for $S \in K_W$ we

have

$$\begin{aligned}
0 &= \text{tot}(S)(v_1, v_2, v_3, v_4) \\
&= \frac{1}{24} \sum_{\sigma \in S_4} (\sigma S)(v_1, v_2, v_3, v_4) \\
&= \frac{1}{3} [S(v_1, v_2, v_3, v_4) + S(v_2, v_3, v_1, v_4) + S(v_3, v_1, v_2, v_4)] \\
&= \frac{1}{3} \sum_{\mathfrak{S}} S(v_1, v_2, v_3)(v_4),
\end{aligned}$$

where we have used the symmetries of S to reorder the arguments for every permutation $\sigma \in S_4$ from the second line to obtain one of the permutations in the third line. By doing so, we find every permutation eight times since this is the order of the symmetry group of elements in W . As a result, we see that the cyclic sum over elements in K_W vanishes. Since the other symmetries are inherited as well, we indeed end up in $K(V)$ after restricting to $\Lambda^2 V$. This can be seen explicitly by performing the cyclic sum over $R'(S)(v_1, v_2, v_3, v_4) = (S(v_1, v_2, v_3, v_4) - S(v_2, v_1, v_3, v_4))/2$, which decomposes into two (independently vanishing) cyclic sums for S .

To show that R' maps W surjectively onto $K(V)$, we will provide an explicit formula for the inverse of $R'|_{K_W}$. For $A \in K(V)$ we define a $(0, 4)$ -tensor σ_A by

$$\sigma_A := \frac{2}{3}(A + \tau A).$$

We claim that $\sigma_A = (R'|_{K_W})^{-1}(A)$. We first show that $\sigma_A \in W$. By construction $\tau \sigma_A = \sigma_A$. To see that $\sigma_A \in \text{Sym}^2(V^* \otimes V^*)$ it suffices to check that $\tau A \in \text{Sym}^2(V^* \otimes V^*)$. We compute

$$(\tau A)(v_3, v_4, v_1, v_2) = A(v_3, v_2, v_1, v_4) = A(v_1, v_4, v_3, v_2) = (\tau A)(v_1, v_2, v_3, v_4),$$

proving that indeed $\tau A \in \text{Sym}^2(V^* \otimes V^*)$. To prove that $\sigma_A \in W$ it remains only to check that $\sigma_A(x \wedge y, z \vee w) = 0$ for all $x, y, z, w \in V$. We compute

$$\begin{aligned}
\sigma_A(x \wedge y, z \vee w) &= \frac{1}{4}(\sigma_A(x, y, z, w) - \sigma_A(y, x, z, w) + \sigma_A(x, y, w, z) - \sigma_A(y, x, w, z)) \\
&= \frac{1}{6}(A(x, y, z, w) - A(y, x, z, w) + A(x, y, w, z) - A(y, x, w, z) \\
&\quad + A(x, w, z, y) - A(y, w, z, x) + A(x, z, w, y) - A(y, z, w, x)) \\
&= 0,
\end{aligned}$$

where we have used only the symmetries of $A \in \text{Sym}^2(\Lambda^2 V^*)$. Using in addition the Bianchi identity (i.e. the vanishing cyclic sum), we can easily check that $R'(\sigma_A) = A$:

$$\begin{aligned}
\sigma_A(x \wedge y, z \wedge w) &= \frac{1}{6}(4A(x, y, z, w) - 2A(w, x, z, y) - 2A(x, z, w, y)) \\
&= A(x, y, z, w).
\end{aligned}$$

Finally, we note that indeed $\sigma_A \in K_W$, i.e. $\text{tot}(\sigma_A) = 0$, since $\text{tot}(A) = \text{tot}(\tau A) = 0$, due to the skew-symmetry of A in at least one pair of arguments. $\square$

4.2.2 The positivity cone

We will now consider the cone

$$\mathcal{C}_W := \{S \in W : S \geq 0\}$$

of positive semi-definite elements of $W \subset \text{Sym}^2(V^* \otimes V^*)$. We furthermore denote by

$$\mathcal{C}_{\text{Sym}} := \{S \in \text{Sym}^2(\text{Sym}^2 V^*) : S \geq 0\}$$

the cone of positive semi-definite symmetric bilinear forms on $\text{Sym}^2 V^*$. The following is clear:

Lemma 4.8. *Let $S \in W$. Then $S \in \mathcal{C}_W$ if and only if both $R(S) \geq 0$ and $R'(S) \geq 0$.*

In the following proposition we show that elements in $\mathcal{C}_W$ must always have some non-vanishing totally symmetric part.

Proposition 4.9. *The subspace K_W has trivial intersection with the cone $\mathcal{C}_W$, i.e. $K_W \cap \mathcal{C}_W = 0$.*

Proof. By the proof of Proposition 4.7, any element of K_W is of the form σ_A for some $A \in K(V)$. The tensor σ_A belongs to $\mathcal{C}_W$ if and only if $R(\sigma_A) \geq 0$ and $R'(\sigma_A) = A \geq 0$. The latter condition implies $0 \leq A(u \wedge v, u \wedge v) = A(u, v, u, v)$ for all $u, v \in V$, while the former implies

$$\begin{aligned} 0 &\leq \sigma_A(u^2 + v^2, u^2 + v^2) = \sigma_A(u, u, v, v) + \sigma_A(v, v, u, u) \\ &= \frac{2}{3}(A(u, v, v, u) + A(v, u, u, v)) = -\frac{4}{3}A(u, v, u, v). \end{aligned}$$

This proves that $A(u, v, u, v) = 0$ for all $u, v \in V$ and hence, $A = 0$ by Lemma 4.3. This proves Proposition 4.9. $\square$

In what follows we will develop criteria for determining whether a given element of $\mathcal{C}_W$ is extremal, based on simple properties of the element. We begin by studying the elements of $\mathcal{C}_W$ which project to extremal elements in $\mathcal{C}_{\text{Sym}}$. Then we will establish several necessary conditions for an element of $\mathcal{C}_W$ to be extremal, formulated in terms of the dimension of its kernel. We will conclude the section with a full classification of all extremal elements of $\mathcal{C}_W$ in the special case where $\dim V = 2$.

4.2.3 Extreme rays that project to extreme rays

We already saw in Lemma 4.8 that $R(\mathcal{C}_W) \subset \mathcal{C}_{\text{Sym}}$. We will now consider elements in $\mathcal{C}_W$ projecting onto extremal elements in $\mathcal{C}_{\text{Sym}}$ under R .

Proposition 4.10. *Let $S \in \mathcal{C}_W$ be such that $R(S)$ is extremal in $\mathcal{C}_{\text{Sym}}$. Then S is extremal in $\mathcal{C}_W$ and there exists $\alpha, \beta \in V^*$ such that*

$$S = (\alpha \vee \beta)^{\otimes 2} + (\alpha \wedge \beta)^{\otimes 2} = \frac{1}{2}(\alpha \otimes \beta \otimes \alpha \otimes \beta + \beta \otimes \alpha \otimes \beta \otimes \alpha). \quad (4.1)$$

In particular, every element of $\mathcal{C}_W$ that is of the form as in (4.1) for some $\alpha, \beta \in V^ \setminus \{0\}$ is extremal.*

Remark 4.11. Note that extremal elements of the form as in (4.1) give rise to the elastic positivity bounds mentioned in Section 2.4. Therefore, we will also refer to them as *elastic extremal elements*.

To prove Proposition 4.10, we will need the following lemmas.

Lemma 4.12. *The map $h := R' \circ R^{-1} : \text{Sym}^2(\text{Sym}^2 V^*) \rightarrow \text{Sym}^2(\Lambda^2 V^*)$ is given by:*

$$h(\gamma_1 \vee \gamma_2)(x, y, z, w) = \frac{1}{2}(\gamma_1(x, w)\gamma_2(z, y) + \gamma_2(x, w)\gamma_1(z, y) - \gamma_1(y, w)\gamma_2(z, x) - \gamma_2(y, w)\gamma_1(z, x)),$$

for all $\gamma_1, \gamma_2 \in \text{Sym}^2 V^*$.

Proof. It follows from the proof of Proposition 4.5 that for every $T \in \text{Sym}^2(\text{Sym}^2 V^*)$, we have $R' \circ R^{-1}(T) = h(T)$, where

$$h(T)(e_i \wedge e_j, e_k \wedge e_l) = T(e_i \vee e_l, e_j \vee e_k) - T(e_i \vee e_k, e_j \vee e_l).$$

Applying this to $T = \gamma_1 \vee \gamma_2$ for $\gamma_1, \gamma_2 \in \text{Sym}^2 V^*$, we arrive at the desired formula. $\square$

Remark 4.13. The map h is related to the well-known Kulkarni–Nomizu product by

$$h(\gamma_1 \vee \gamma_2) = -\frac{1}{2}\gamma_1 \oslash \gamma_2.$$

Lemma 4.14. *Let $\gamma \in \text{Sym}^2 V^*$ and consider $\rho_\gamma := h(\gamma^{\otimes 2})$. If $\rho_\gamma \geq 0$, then $\text{rank } \gamma \leq 2$. Moreover, if $\text{rank } \gamma = 2$, γ has signature of type $(+, -, 0, \dots, 0)$.*

Proof. Let $E \subseteq V$ be a plane with basis (x, y) . Then

$$\begin{aligned} \rho_\gamma(x \wedge y, x \wedge y) &= \rho_\gamma(x, y, x, y) \\ &= \gamma(x, y)^2 - \gamma(x, x)\gamma(y, y) \\ &= -\det \begin{pmatrix} \gamma(x, x) & \gamma(x, y) \\ \gamma(x, y) & \gamma(y, y) \end{pmatrix} \end{aligned}$$

This is non-negative if and only if γ restricted to E is indefinite or degenerate. Since this must hold for all planes E , the claim follows. $\square$

We are now ready to prove Proposition 4.10.

Proof of Proposition 4.10. First, we show that every such element must be of the form as in equation (4.1). To see this, note that the extremal elements in $\mathcal{C}_{\text{Sym}}$ are precisely those of rank one. Suppose $S \in \mathcal{C}_W$ is such that $\text{rank } R(S) = 1$. Then we can write $R(S) = \gamma^{\otimes 2}$ for some $\gamma \in \text{Sym}^2 V^*$. Moreover, since $S \in \mathcal{C}_W$, we must have $R'(S) = \rho_\gamma \geq 0$. By the previous lemma, the latter condition is satisfied precisely if γ can be written as $\gamma = \alpha \vee \beta$ for some $\alpha, \beta \in V^*$. It follows that $S = \gamma^{\otimes 2} + h(\gamma^{\otimes 2})$ is of the form as in (4.1), since $h(\gamma^{\otimes 2}) = (\alpha \wedge \beta)^2$.

Next, we show that every element of this form is extremal in $\mathcal{C}_W$, using Corollary 3.32. Let $S \in \mathcal{C}_W$ be of the form (4.1) and $S' \in \mathcal{C}_W$ such that $\ker S \subsetneq \ker S'$. Then in particular $\ker R(S) \subset \ker R(S')$. Since $\text{rank } R(S) = 1$ this implies that $R(S') = \lambda R(S)$ for some $\lambda \geq 0$. By Proposition 4.5 this means that $S' = \lambda S$ for some $\lambda \geq 0$ and hence $\ker S \subsetneq \ker S'$ readily implies $S' = 0$. $\square$

4.2.4 Extremal elements in $\text{Sym}^4 V^* \cap \mathcal{C}_W$

Recall that by choosing a basis $\{e_i\}$ of V , elements $T \in \text{Sym}^d V$ are identified with homogeneous polynomials of degree d in $n = \dim V$ variables via

$$p_T(x_1, \dots, x_n) := T(x, \dots, x),$$

where $x = \sum_i x_i e_i \in V$. Using this identification for $d = 4$ and $d = 2$ as well as the identification of elements in $\mathbb{R}[x_1, \dots, x_n]_{2d}^*$ with their quadratic forms we observe that

$$\text{Sym}^4 V^* \cap \mathcal{C}_W = \Sigma_{n,4}^*,$$

where $\Sigma_{n,4}^*$ is the Hankel spectrahedron introduced in Section 3.2.2. Note that the property of being an SOS polynomial is invariant under the action of $\text{GL}(V)$, i.e. $p(x)$ is SOS if and only if $p(Ax)$ is SOS for $A \in \text{GL}(V)$. Therefore, the identification of $\text{Sym}^4 V^* \cap \mathcal{C}_W$ with $\Sigma_{n,4}^*$ is independent of the choice of basis.

In particular, we observe that we can use Hilbert's theorem to prove the following:

Theorem 4.15. *Suppose $\dim V \leq 3$. Then all extremal elements in $\text{Sym}^4 V^* \cap \mathcal{C}_W$ are of the form α^4 for some $\alpha \in V^* \setminus \{0\}$. If $\dim V > 3$, there exist extremal elements $S \in \text{Sym}^4 V^* \cap \mathcal{C}_W$ such that $\text{rank } S > 1$.*

Proof. This is a direct consequence of Theorem 3.35 and Corollary 3.38. $\square$

4.2.5 Some necessary conditions for extremality

We will now study some necessary conditions for extremal elements of $\mathcal{C}_W$. Since the case where $\dim V = 1$ is trivial, we will assume that $\dim V \geq 2$ throughout the section. First, we observe that for extremal elements $S \in \mathcal{C}_W$ the kernel of the projection $R(S)$ must be non-trivial.

Lemma 4.16. *Let $S \in \mathcal{C}_W$ be extremal. Then $\ker R(S) \neq \{0\}$.*

Proof. Let $S \in \mathcal{C}_W$. If $R(S)$ is strictly positive, then for every $\alpha \in V^*$, we have

$$\ker \alpha^{\otimes 4} \supsetneq \ker S.$$

Indeed, $\ker \alpha^{\otimes 4} \supset \Lambda^2 V \supset \ker S$, since $R(S)$ has full rank and we have

$$\text{rank } \alpha^{\otimes 4} = 1 < \dim \text{Sym}^2 V \leq \text{rank } S.$$

By Corollary 3.32 it follows that S is not extremal. $\square$

A similar statement can be made concerning the kernel of $R'(S)$:

Lemma 4.17. *Let $S \in \mathcal{C}_W$ such that $R'(S)$ has full rank and such that $R(S)$ is not extremal in $\mathcal{C}_{\text{Sym}}$. Then S is not extremal in $\mathcal{C}_W$. Moreover, if $\dim V > 2$, then for extremal $S \in \mathcal{C}_W$, $R'(S)$ has a non-trivial kernel.*

Proof. Let $S \in \mathcal{C}_W$ as above. Then there is some $T \in \mathcal{C}_{\text{Sym}} \setminus \{0\}$ such that $\ker R(S) \subsetneq \ker T$. Now if $\ker R'(S) = \{0\}$, it follows that $T + h(T) \in W \setminus \{0\}$ has $\ker(T + h(T)) \supsetneq \ker S$ and thus S cannot be extremal by Corollary 3.32.

Now suppose $\dim V > 2$. By Proposition 4.10, elements $S \in \mathcal{C}_W$ such that $R(S)$ is extremal in $\mathcal{C}_{\text{Sym}}$ are of the form

$$S = (\alpha \vee \beta)^{\otimes 2} + (\alpha \wedge \beta)^{\otimes 2}$$

for some $\alpha, \beta \in V^*$. Therefore, for such S we have $\text{rank } R'(S) = 1$. Since for $\dim V > 2$ we have $\dim \Lambda^2 V > 1$, in this case $R'(S)$ has a non-trivial kernel. $\square$

Next, we will establish a lower bound for the kernel of extremal elements.

Proposition 4.18. *Let $\dim V = n \geq 2$. If $S \in \mathcal{C}_W$ is extremal, then $\dim(\ker S) \geq n$. For odd dimensions n , we even have $\dim(\ker S) > n$.*

Proof. By Lemma 4.16, we know that for extremal $S \in \mathcal{C}_W$, we have $\ker S \cap \text{Sym}^2 V \neq \{0\}$. If $S \in \mathcal{C}_W \cap \text{Sym}^4 V^*$, then we furthermore know that $\Lambda^2 V \subset \ker S$. Therefore,

$$\dim \ker S > \dim \Lambda^2 V \geq n - 1 \quad \text{for } n \geq 2.$$

Similarly, if $R(S)$ is extremal in $\mathcal{C}_{\text{Sym}}$, then

$$\dim \ker S \geq \dim \ker R(S) = \frac{n(n+1)}{2} - 1 \geq n \quad \text{for } n \geq 2.$$

Now suppose $S \in \mathcal{C}_W \setminus \text{Sym}^4 V^*$ such that $R(S)$ is not extremal in $\mathcal{C}_{\text{Sym}}$. Let $k = \dim \ker S$ and $\{Q_i\}_{i=1}^k$ a basis for $\ker S$. We may assume that each $Q_i \in \text{Sym}^2 V$ or $Q_i \in \Lambda^2 V$. We pick some $\alpha \in V^* \setminus \{0\}$ and consider the maps

$$Q_i(\alpha, \cdot): V^* \longrightarrow \mathbb{R}.$$

Then for $k < n$, we find

$$\bigcap_{i=1}^k \ker Q_i(\alpha, \cdot) \neq \{0\}.$$

It follows that if $k < n$ there is some $\beta \in V^* \setminus \{0\}$ such that

$$Q_i(\alpha, \beta) = \alpha \otimes \beta(Q_i) = 0 \quad \text{for } i = 1, \dots, k.$$

Since the Q_i are all either symmetric or anti-symmetric, we also have

$$\beta \otimes \alpha(Q_i) = 0 \quad \text{for } i = 1, \dots, k.$$

It follows that

$$(\alpha \otimes \beta)^{\otimes 2} + (\beta \otimes \alpha)^{\otimes 2}|_{\ker S} \equiv 0.$$

Therefore, by Corollary 3.32, for extremal S , we must have $k \geq n$.

For odd dimensions n , we can furthermore use that $\ker S \cap \Lambda^2 V \neq \{0\}$ by Lemma 4.17. Therefore, for some i we have $Q_i \in \Lambda^2 V$ and since V is odd-dimensional, this Q_i is degenerate. Choosing $\alpha \in \ker(Q_i: V^* \rightarrow V)$ the same argument as above implies that $n > k$. $\square$

Note that in the case where $\dim V = 3$, this yields a slightly improved bound as compared to the Pataki inequality: Proposition 4.18 implies $\dim \ker S > 3$ for extremal elements S , while Corollary 3.34 only yields the bound $\dim \ker S \geq 3$.

Corollary 4.19. *Suppose $\dim V = 3$. Then for extremal $S \in \mathcal{C}_W$, we have*

$$\dim(\ker S \cap \operatorname{Sym}^2 V) \geq 2.$$

Proof. By Proposition 4.18 we know that $\dim \ker S \geq 4$. If $\dim(\ker S \cap \Lambda^2 V) < 3$, this implies $\dim(\ker S \cap \operatorname{Sym}^2 V) \geq 2$. If $\dim(\ker S \cap \Lambda^2 V) = 3$, then $S \in \mathcal{C}_W \cap \operatorname{Sym}^4 V^*$ and the assertion follows from Theorem 4.15. $\square$

Corollary 4.20. *Suppose $\dim V = 3$. If $S \in \mathcal{C}_W$ is extremal, then there is some $Q \in \operatorname{Sym}^2 V \cap \ker S$ such that $\operatorname{rank} Q < 3$.*

Proof. This follows from Corollary 4.19 and Lemma 4.2. $\square$

The following theorem already appeared in [Li+21], but with a different proof.

Theorem 4.21. *Suppose $\dim V = 2$. Then $S \in \mathcal{C}_W$ is extremal if and only if $R(S)$ is extremal in $\mathcal{C}_{\operatorname{Sym}}$.*

Proof. Suppose $S \in \mathcal{C}_W$ is extremal. If $R(S)$ is not extremal in $\mathcal{C}_{\operatorname{Sym}}$, then since $\dim \Lambda^2 V = 1$, we know by Lemma 4.17 that $R'(S) \equiv 0$. Therefore, $S \in \operatorname{Sym}^4 V^* \cap \mathcal{C}_W$ and it follows from Theorem 4.15 that it must be extremal in $\mathcal{C}_{\operatorname{Sym}}$, a contradiction. $\square$

4.3 Extremal elements of $\mathcal{C}_W$ for $\dim V = 3$

In this section, we will determine the extremal elements of $\mathcal{C}_W$ for three-dimensional V . Our goal is to prove the following theorem.

Theorem 4.22. *Suppose $\dim V = 3$. Then $S \in \mathcal{C}_W$ is extremal if and only if it is of one of the following types:*

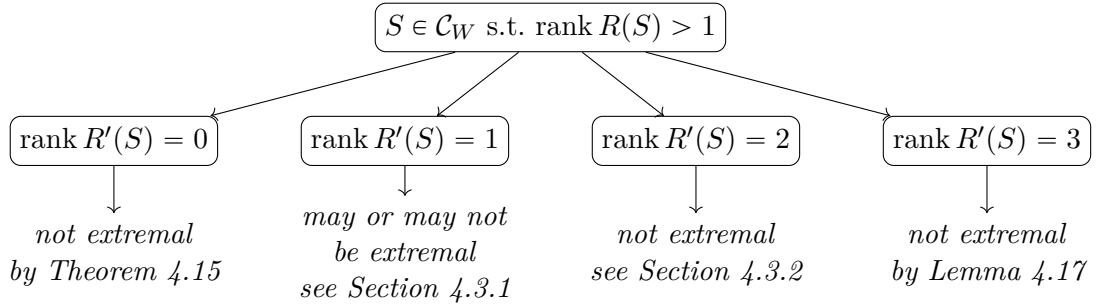
1. *For some $\alpha \in V^* \setminus \{0\}$, $S = \alpha^4$.*
2. *For linearly independent $\alpha_1, \alpha_2 \in V^*$, $S = (\alpha_1 \vee \alpha_2)^{\otimes 2} + (\alpha_1 \wedge \alpha_2)^{\otimes 2}$.*
3. *For a basis $\{\alpha_1, \alpha_2, \alpha_3\}$ of V^* and parameters g, d, h such that $g^2 > 1 - d^2 + dh$ we have*

$$S = S_{\text{tot}} + 2(g^2 + d^2 - 1 - dh) [(\alpha_2 \otimes \alpha_3)^{\otimes 2} + (\alpha_3 \otimes \alpha_2)^{\otimes 2}],$$

where

$$\begin{aligned} S_{\text{tot}} = & \alpha_1^4 + 6\alpha_1^2\alpha_2^2 + 6\alpha_1^2\alpha_3^2 + 12d\alpha_1\alpha_2^2\alpha_3 + 12g\alpha_1\alpha_2\alpha_3^2 + 4h\alpha_1\alpha_3^3 + (1 + d^2)\alpha_2^4 \\ & + 4dg\alpha_2^3\alpha_3 + 6(1 + dh)\alpha_2^2\alpha_3^2 + 4g(d + h)\alpha_2\alpha_3^3 + (1 + g^2 + h^2)\alpha_3^4. \end{aligned}$$

It will be useful to distinguish the cases of different rank of $R'(S)$ for $S \in \mathcal{C}_W$. Since $\dim \Lambda^2 V = 3$, we have $0 \leq \operatorname{rank} R'(S) \leq 3$. By Lemma 4.17, we know that $S \in \mathcal{C}_W$ is not extremal if $\operatorname{rank} R'(S) = 3$. In view of Proposition 4.10, we will furthermore restrict our study to $S \in \mathcal{C}_W$ such that $R(S)$ is not extremal in $\mathcal{C}_{\operatorname{Sym}}$, i.e. we assume $\operatorname{rank} R(S) > 1$. The general structure is outlined in the following diagram:

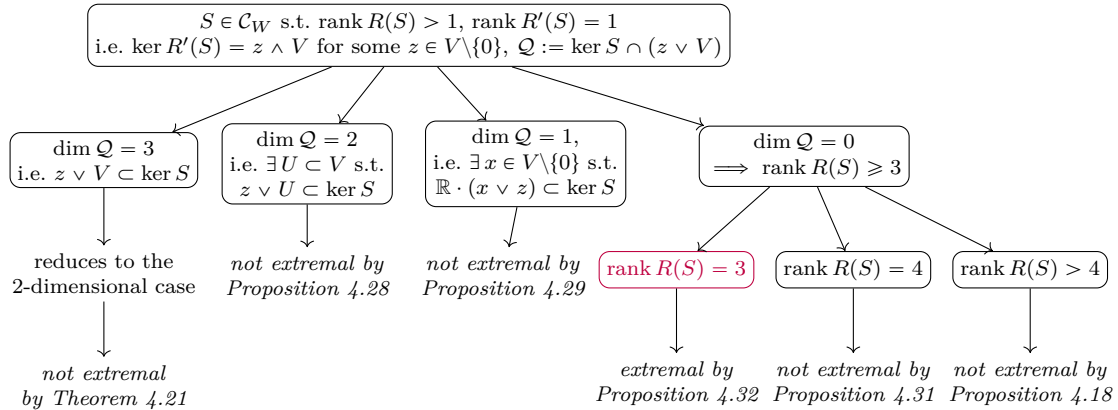


4.3.1 Case $\text{rank } R'(S) = 1$

We will now study the case where $\dim V = 3$ and $S \in \mathcal{C}_W$ is such that $\text{rank } R'(S) = 1$. In this case $R'(S) = (\alpha \wedge \beta)^{\otimes 2}$ for some linearly independent $\alpha, \beta \in V^*$ and $\ker R'(S) = z \wedge U$ for $z \in (\ker \alpha \cap \ker \beta) \setminus \{0\}$. It will be useful to study the cases of different dimensions of the intersection $\ker S \cap (z \vee V)$ separately. The aim of this section is to prove the following:

Theorem 4.23. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ such that $R'(S) = (\alpha \wedge \beta)^{\otimes 2} \neq 0$ for some $\alpha, \beta \in V^*$ and such that $R(S)$ is not extremal in $\mathcal{C}_{\text{Sym}}$. Let $z \in V$ such that $\ker \alpha \cap \ker \beta = \mathbb{R}z$. Then S is extremal in $\mathcal{C}_W$ if and only if S is such that $\ker S \cap (z \vee V) = \{0\}$ and $\text{rank } R(S) = 3$.*

The following diagram shows an outline of the structure of this section:



As a first step, we will exclude the presence of certain elements in $\ker S$ for extremal S with $\text{rank } R'(S) = 1$:

Lemma 4.24. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ such that $R'(S) = (\alpha \wedge \beta)^{\otimes 2} \neq 0$ for some $\alpha, \beta \in V^*$. Let furthermore $z \in V$ such that $\ker \alpha \cap \ker \beta = \mathbb{R} \cdot z$. If $z^{\otimes 2} \in \ker S$, then S is not extremal unless $R(S)$ is extremal in $\mathcal{C}_{\text{Sym}}$.*

Proof. If $z^{\otimes 2} \in \ker S$, then by Lemma 4.4 we find for every $v \in V$

$$0 = S(v^{\otimes 2}, z^{\otimes 2}) = S(v \vee z, v \vee z) - S(v \wedge z, v \wedge z) = S(v \vee z, v \vee z),$$

where we used $z \in \ker \alpha \cap \ker \beta$ in the last step. It follows that $z \vee v \in \ker S$ for every $v \in V$ and hence the problem reduces to the two-dimensional case, which is covered in Theorem 4.21. $\square$

Lemma 4.25. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ be such that $R'(S) = (\alpha \wedge \beta)^{\otimes 2} \neq 0$ for some $\alpha, \beta \in V^*$. Let furthermore $z \in V$ such that $\ker \alpha \cap \ker \beta = \mathbb{R}z$. If there are some $x, y \in V$, linearly independent of z and such that either $x^{\otimes 2} + z^{\otimes 2} \in \ker S$ or $x^{\otimes 2} + y^{\otimes 2} + z^{\otimes 2} \in \ker S$, then also $z^{\otimes 2} \in \ker S$. In particular, S is not extremal unless $R(S)$ is extremal in $\mathcal{C}_{\text{Sym}}$.*

Proof. If $x^{\otimes 2} + z^{\otimes 2} \in \ker S$, then

$$S(z^{\otimes 2}, \cdot) = -S(x^{\otimes 2}, \cdot).$$

Using Lemma 4.4, we deduce

$$0 \leq S(z^{\otimes 2}, z^{\otimes 2}) = -S(x^{\otimes 2}, z^{\otimes 2}) = -S(x \vee z, x \vee z) + S(x \wedge z, x \wedge z) = -S(x \vee z, x \vee z) \leq 0,$$

where we used $z \in \ker \alpha \cap \ker \beta$. Similarly, if $x^{\otimes 2} + y^{\otimes 2} + z^{\otimes 2} \in \ker S$, we find

$$0 \leq S(z^{\otimes 2}, z^{\otimes 2}) = -S(x^{\otimes 2}, z^{\otimes 2}) - S(y^{\otimes 2}, z^{\otimes 2}) = -S(x \vee z, x \vee z) - S(y \vee z, y \vee z) \leq 0.$$

In both cases, it follows that $z^{\otimes 2} \in \ker S$ and the assertion follows from Lemma 4.24. $\square$

As described above, we want to consider the cases of different dimensions of the intersection $\ker S \cap (z \vee V)$ with $S \in \mathcal{C}_W$ and $z \in V$ as above. When this intersection is two-dimensional, by Lemma 4.24 it is enough to consider the situation where $z \vee U \subset \ker S$ for some $U \subset V$ complementary to the line $\mathbb{R}z$. It will therefore be useful to study the intersection $\ker S \cap \text{Sym}^2 U$ for some $U \subset V$ complementary to $\mathbb{R}z$.

Lemma 4.26. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ such that $R'(S) = (\alpha \wedge \beta)^{\otimes 2} \neq 0$ for some $\alpha, \beta \in V^*$, and let $U \subset V$ complementary to $\ker \alpha \cap \ker \beta$. If $\{x, y\} \subset U$ is a basis of U , then*

$$\dim(\ker S \cap (x \vee \text{span}\{x, y\})) \leq 1.$$

In particular, the intersection $\text{Sym}^2 U \cap \ker S$ is at most two-dimensional. Moreover, if the intersection $\ker S \cap \text{Sym}^2 U$ is two-dimensional, it contains an element of rank 1.

Proof. Clearly, $\dim(\ker S \cap (x \vee \text{span}\{x, y\})) \leq 1$ when $\dim(\text{Sym}^2 U \cap \ker S) \leq 1$. Now suppose $\dim(\text{Sym}^2 U \cap \ker S) > 1$. By Lemma 4.1 we know that there is an element of the form $x \vee y \in \text{Sym}^2 U \cap \ker S$, for some $x, y \in U \setminus \{0\}$.

We first consider the case where x and y are linearly dependent, i.e. $x^{\otimes 2} \in \ker S$. By Lemma 4.4, we find that for every $w \in U \setminus \{0\}$ linearly independent of x ,

$$0 = S(x^{\otimes 2}, w^{\otimes 2}) = S(x \vee w, x \vee w) - S(x \wedge w, x \wedge w). \quad (4.2)$$

By the assumption $R'(S) = (\alpha \wedge \beta)^{\otimes 2} \neq 0$, we know that $\ker R'(S) = z \wedge U$. In particular, we have $S(x \wedge w, x \wedge w) > 0$ and it follows from equation (4.2) that $x \vee w \notin \ker S$ and therefore $\dim(\ker S \cap (x \vee \text{span}\{x, w\})) \leq 1$ for every $w \in U$.

Similarly, if $x \vee y \in \ker S$ for linearly independent $x, y \in U$, then Lemma 4.4 implies

$$S(x^{\otimes 2}, y^{\otimes 2}) = S(x \vee y, x \vee y) - S(x \wedge y, x \wedge y) = -S(x \wedge y, x \wedge y) < 0.$$

Therefore, neither $x^{\otimes 2}$ nor $y^{\otimes 2}$ can be in the kernel of S and we conclude that

$$\text{Sym}^2 U \cap \ker S = \text{span}\{x \vee y, x^{\otimes 2} + \lambda y^{\otimes 2}\}$$

for some $\lambda \neq 0$. We claim that $\lambda > 0$:

Indeed, since $x^{\otimes 2} + \lambda y^{\otimes 2} \in \ker S$, we have

$$0 = S(x^{\otimes 2} + \lambda y^{\otimes 2}, x^{\otimes 2}) = S(x^{\otimes 2}, x^{\otimes 2}) + \lambda S(x^{\otimes 2}, y^{\otimes 2}).$$

Using again Lemma 4.4 and that $x \vee y \in \ker S$, we obtain

$$0 < S(x^{\otimes 2}, x^{\otimes 2}) = -\lambda S(x^{\otimes 2}, y^{\otimes 2}) = \lambda S(x \wedge y, x \wedge y).$$

Since $S(x \wedge y, x \wedge y) > 0$, we must have $\lambda > 0$.

We may therefore write

$$\text{Sym}^2 U \cap \ker S = \text{span}\{x \vee y, x^{\otimes 2} + \lambda y^{\otimes 2}\} = \text{span}\{x \vee y, (x + \sqrt{\lambda}y)^{\otimes 2}\}.$$

This shows that $\text{Sym}^2 U \cap \ker S$ contains an element of rank one. $\square$

Lemma 4.27. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ such that $R'(S) = (\alpha \wedge \beta)^{\otimes 2} \neq 0$ for some $\alpha, \beta \in V^*$, and let $U \subset V$ complementary to $\ker \alpha \cap \ker \beta$. Then there exist $\tilde{\alpha}, \tilde{\beta} \in \text{span}\{\alpha, \beta\}$ such that $\tilde{\alpha} \vee \tilde{\beta}$ vanishes on $\ker S \cap \text{Sym}^2 U$.*

Proof. If $\text{Sym}^2 U \cap \ker S$ is one-dimensional, then

$$\dim(\text{Ann}(\text{Sym}^2 U \cap \ker S) \cap \text{Sym}^2(\text{span}\{\alpha, \beta\})) \geq 2$$

and by Lemma 4.1 there are some $\tilde{\alpha}, \tilde{\beta} \in \text{span}\{\alpha, \beta\}$ such that $\tilde{\alpha} \vee \tilde{\beta} \in \text{Ann}(\text{Sym}^2 U \cap \ker S)$.

Now suppose $\dim(\text{Sym}^2 U \cap \ker S) > 1$. By Lemma 4.26 we know that in this case $\dim(\text{Sym}^2 U \cap \ker S) = 2$ and there is an element of the form $u^{\otimes 2} \in \text{Sym}^2 U \cap \ker S \setminus \{0\}$. Since $\text{Sym}^2 U \cap \ker S$ is two-dimensional, we have

$$\dim(\text{Ann}(\text{Sym}^2 U \cap \ker S) \cap \text{Sym}^2(\text{span}\{\alpha, \beta\})) \geq 1,$$

i.e. $\text{Ann}(\text{Sym}^2 U \cap \ker S) \cap \text{Sym}^2(\text{span}\{\alpha, \beta\})$ contains a non-trivial element γ . By Sylvester's theorem, it will be either of the form $\gamma = \tilde{\alpha}^{\otimes 2}$ for some $\tilde{\alpha} \in \text{span}\{\alpha, \beta\}$ or of the form $\gamma = \bar{\alpha}^{\otimes 2} \pm \bar{\beta}^{\otimes 2}$ for linearly independent $\bar{\alpha}, \bar{\beta} \in \text{span}\{\alpha, \beta\}$. Furthermore, since there is a rank one element $u^{\otimes 2} \in \text{Sym}^2 U \cap \ker S$ and $u \notin \ker \alpha \cap \ker \beta$, we may exclude the case $\gamma = \bar{\alpha}^{\otimes 2} + \bar{\beta}^{\otimes 2}$ for linearly independent $\bar{\alpha}, \bar{\beta}$. Recalling that

$$\bar{\alpha}^{\otimes 2} - \bar{\beta}^{\otimes 2} = (\bar{\alpha} + \bar{\beta}) \vee (\bar{\alpha} - \bar{\beta}),$$

it follows that there are $\tilde{\alpha}, \tilde{\beta} \in \text{span}\{\alpha, \beta\}$ such that $\tilde{\alpha} \vee \tilde{\beta} \in \text{Ann}(\text{Sym}^2 U \cap \ker S)$. $\square$

We will now show that S as above with $\text{rank } R(S) > 1$ cannot be extremal if for some $U \subset V$ complementary to the line $\mathbb{R}z$ we have $z \vee U \subset \ker S$, where $z \in V$ is as above:

Proposition 4.28. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ such that $R'(S) = (\alpha \wedge \beta)^{\otimes 2} \neq 0$ for some $\alpha, \beta \in V^*$. Let furthermore $z \in \ker \alpha \cap \ker \beta \setminus \{0\}$. If for some $U \subset V$ complementary to $\mathbb{R}z$ we have $z \vee U \subset \ker S$, then S is not extremal unless $R(S)$ is extremal in $\mathcal{C}_{\text{Sym}}$.*

Proof. If $z \vee v \in \ker S$ for every $v \in U$, observing that for all $v, w \in U$

$$S(v \vee w, z^{\otimes 2}) = S(v \vee z, w \vee z) = 0,$$

we find that $z^{\otimes 2} \perp_S \text{Sym}^2 U$. By Lemma 4.24, we may furthermore assume that $z^{\otimes 2}$ is not in the kernel of S . It follows that there is a basis $\{Q_i\}$ of $\ker R(S)$ such that $Q_1, Q_2 \in z \vee U$ and $Q_i \in \text{Sym}^2 U$ for $i > 2$.

By Lemma 4.27, there exist $\tilde{\alpha}, \tilde{\beta} \in \text{span}\{\alpha, \beta\}$ such that $\tilde{\alpha} \vee \tilde{\beta} \in \text{Ann}(\text{Sym}^2 U \cap \ker S)$. Since $z \in \ker \alpha \cap \ker \beta$, it follows that

$$\tilde{\alpha} \vee \tilde{\beta}(Q_1) = 0 = \tilde{\alpha} \vee \tilde{\beta}(Q_2)$$

and hence $\tilde{\alpha} \vee \tilde{\beta}|_{\ker S} \equiv 0$. Since

$$h((\tilde{\alpha} \vee \tilde{\beta})^{\otimes 2}) = (\tilde{\alpha} \wedge \tilde{\beta})^{\otimes 2} \in \mathbb{R}(\alpha \wedge \beta)^{\otimes 2},$$

which vanishes on $\ker S$, it follows that

$$((\tilde{\alpha} \otimes \tilde{\beta})^{\otimes 2} + (\tilde{\beta} \otimes \tilde{\alpha})^{\otimes 2})|_{\ker S} \equiv 0.$$

The statement follows now from Corollary 3.32. $\square$

We will now improve the statement of Proposition 4.28 by showing that S as above with $\text{rank } R(S) > 1$ cannot be extremal if $\ker S \cap (z \vee V) \neq \{0\}$:

Proposition 4.29. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ such that $R'(S) = (\alpha \wedge \beta)^{\otimes 2} \neq 0$ for some $\alpha, \beta \in V^*$. Let furthermore $z \in \ker \alpha \cap \ker \beta \setminus \{0\}$. If for some $x \in V \setminus \{0\}$ we have $x \vee z \in \ker S$, then S is not extremal unless $R(S)$ is extremal in $\mathcal{C}_{\text{Sym}}$.*

Proof. By Lemma 4.24 we may assume that x is linearly independent of z . Using Proposition 4.28 we may furthermore assume that

$$\ker S \cap (z \vee V) = \mathbb{R}x \vee z.$$

Since by assumption also $x \wedge z \in \ker S$, it follows that

$$x \otimes z = x \vee z + x \wedge z \in \ker S.$$

Thus, we observe

$$S(x \vee v, z^{\otimes 2}) = S(x \otimes v, z^{\otimes 2}) = S(x \otimes z, z \otimes v) = 0,$$

where we used that $\Lambda^2 V$ and $\text{Sym}^2 V$ are perpendicular with respect to S for the first equality and the invariance under τ for the second equality.

Similarly, we find

$$S(x^{\otimes 2}, v \vee z) = S(x^{\otimes 2}, v \otimes z) = S(x \otimes z, v \otimes x) = 0$$

for every $v \in V$.

Furthermore, we observe for every $v, w \in V$ that

$$S(x \vee v, w \vee z) = S(x \otimes v, w \otimes z) = S(x \otimes z, w \otimes v) = 0, \quad (4.3)$$

where we used that $w \wedge z \in \ker S$ for every $w \in V$.

With these relations in mind, we complete $\{x, z\} \subset V$ to a basis $\{x, y, z\}$ of V such that $S(z, z, z, y) = 0$. To see that this is possible, we observe that the linear map

$$S(z, z, z, \cdot): V \rightarrow \mathbb{R}$$

has a two-dimensional kernel, since we know $z \notin \ker(S(z, z, z, \cdot))$, as we assumed that $z^{\otimes 2}$ is not in the kernel and therefore by Lemma 4.3, it follows that $S(z^{\otimes 2}, z^{\otimes 2}) \neq 0$. We furthermore know that $x \in \ker(S(z, z, z, \cdot))$, since by assumption $x \vee z \in \ker S$. Choosing a vector $y \in \ker(S(z, z, z, \cdot))$ linearly independent of x , we obtain a basis $\{x, y, z\}$ of V such that $S(z, z, z, y) = 0$.

Then, by equation (4.3), we have

$$(z \vee V) \perp_S (x \vee \text{span}\{x, y\}).$$

Furthermore, there is a line $L \subset \text{Sym}^2 V$, which is complementary and orthogonal to $(z \vee V) \oplus (x \vee \text{span}\{x, y\})$ in $\text{Sym}^2 V$. Let $T \in \text{Sym}^2 V$ such that $L = \mathbb{R}T$. We obtained a decomposition

$$\text{Sym}^2 V = (z \vee V) \oplus (x \vee \text{span}\{x, y\}) \oplus \mathbb{R}T$$

of $\text{Sym}^2 V$ into pairwise orthogonal subspaces.

We would now like to study the kernel of S in terms of this decomposition. For the first factor we know by assumption that $\ker S \cap (z \vee V) = \mathbb{R}x \vee z$. For the second factor we recall that, by Lemma 4.26, the intersection $\ker S \cap x \vee \text{span}\{x, y\}$ can at most be one-dimensional.

If $T \notin \ker S$, it follows that $\ker S \cap \text{Sym}^2 V = \text{span}\{x \vee z, x \vee u\}$ for some $u \in \text{span}\{x, y\}$. Therefore, choosing some $\gamma \in V^* \setminus \{0\}$ such that $\gamma(x) = 0$, we find that $\gamma^{\otimes 4}|_{\ker S} \equiv 0$ and hence S is not extremal by Corollary 3.32.

Now suppose $T \in \ker S$, i.e. $\ker S \cap \text{Sym}^2 V = \text{span}\{x \vee z, x \vee u, T\}$ for some $u \in \text{span}\{x, y\}$. We will study separately the cases, where T is of rank one, two or three.

Firstly, we observe that under the assumptions we made, we must have $\text{rank } T > 1$: If $T = w^{\otimes 2}$ for some $w \in V \setminus \{0\}$, then we observe that, since T is complementary to $(z \vee V) \oplus (x \vee \text{span}\{x, y\})$, w must be linearly independent of x and z . Using Lemma 4.4 and the fact that $w \wedge z \in \ker S$, we find

$$0 = S(w^{\otimes 2}, z^{\otimes 2}) = S(w \vee z, w \vee z) - S(w \wedge z, w \wedge z) = S(w \vee z, w \vee z),$$

in contrast to the assumption that $\ker S \cap (z \vee V) = \mathbb{R}x \vee z$.

Suppose now that $\text{rank } T = 2$. In this case T may either be of the form $v \vee w$ or $v^{\otimes 2} + w^{\otimes 2}$ for some $v, w \in V \setminus \{0\}$. If $T = v \vee w$, then we may choose $\gamma \in V^* \setminus \{0\}$ such that $\gamma(x) = 0 = \gamma(v)$ and conclude that $\gamma^{\otimes 4}|_{\ker S} \equiv 0$. If $T = v^{\otimes 2} + w^{\otimes 2}$ for linearly independent $v, w \in V \setminus \{0\}$, we show that there exist $\tilde{\alpha}, \tilde{\beta} \in \text{span}\{\alpha, \beta\}$ such that $\tilde{\alpha} \vee \tilde{\beta}|_{\ker S} \equiv 0$. For this, we observe first that

$$\text{Sym}^2(\text{span}\{\alpha, \beta\}) \subset \text{Ann}(x \vee z),$$

and hence,

$$\text{Ann}(\ker S \cap \text{Sym}^2 V) \cap \text{Sym}^2(\text{span}\{\alpha, \beta\}) = \text{Ann}(\text{span}\{x \vee u, T\}) \cap \text{Sym}^2(\text{span}\{\alpha, \beta\}).$$

Therefore,

$$\dim(\text{Ann}(\ker S \cap \text{Sym}^2 V) \cap \text{Sym}^2(\text{span}\{\alpha, \beta\})) \geq 1.$$

Since by assumption $T = v^{\otimes 2} + w^{\otimes 2}$ for linearly independent $v, w \in V \setminus \{0\}$, it cannot be annihilated by an element of the form $\tilde{\alpha}^{\otimes 2} + \tilde{\beta}^{\otimes 2}$ for linearly independent $\tilde{\alpha}, \tilde{\beta} \in \text{span}\{\alpha, \beta\}$. Therefore, there must be some

$$\tilde{\alpha} \vee \tilde{\beta} \in \text{Ann}(\ker S \cap \text{Sym}^2 V) \cap \text{Sym}^2(\text{span}\{\alpha, \beta\}).$$

Arguing as in the proof of Proposition 4.28, it follows that

$$((\tilde{\alpha} \otimes \tilde{\beta})^{\otimes 2} + (\tilde{\beta} \otimes \tilde{\alpha})^{\otimes 2})|_{\ker S} \equiv 0.$$

By Corollary 3.32 it follows that S cannot be extremal in $\mathcal{C}_W$.

Now, suppose that $\text{rank } T = 3$. By Lemma 4.25 we know that T must be indefinite. Then T determines a Lorentzian metric g_T on V , which, up to changing the sign of T , we may assume to be of mostly positive signature. The vector $x \in V$ can satisfy $g_T(x, x) > 0$, $g_T(x, x) = 0$ or $g_T(x, x) < 0$.

Case 1: If $g_T(x, x) > 0$, then after rescaling T , we may assume that $T = x^{\otimes 2} + v^{\otimes 2} - w^{\otimes 2}$ for some $v, w \in V$, such that $\{x, v, w\} \subset V \setminus \{0\}$ are linearly independent. Choosing $\gamma \in V^* \setminus \{0\}$ such that $\gamma(x) = 0$, $\gamma(v) = \gamma(w)$ we find $\gamma^{\otimes 4}|_{\ker S} \equiv 0$ and hence S is not extremal in $\mathcal{C}_W$ by Corollary 3.32.

Case 2: If $g_T(x, x) = 0$, there exist $v, w \in V \setminus \{0\}$ such that $\{x, v, w\} \subset V$ are linearly independent and such that $T = x \vee v + w^{\otimes 2}$. Choosing $\gamma \in V^* \setminus \{0\}$ such that $\gamma(x) = \gamma(w) = 0$ we find $\gamma^{\otimes 4}|_{\ker S} \equiv 0$ and conclude that S is not extremal.

Case 3: If $g_T(x, x) < 0$, then there exist $v, w \in V \setminus \{0\}$ such that $\{x, v, w\} \subset V$ are linearly independent and such that $T = v^{\otimes 2} + w^{\otimes 2} - x^{\otimes 2}$. We claim that this is not possible under the assumptions of Proposition 4.29. Indeed, if $v^{\otimes 2} + w^{\otimes 2} - x^{\otimes 2} \in \ker S$, then using Lemma 4.4 and the fact that $z \wedge V \subset \ker S$, yields

$$0 = S(x^{\otimes 2}, z^{\otimes 2}) = S(v^{\otimes 2} + w^{\otimes 2}, z^{\otimes 2}) = S(v \vee z, v \vee z) + S(w \vee z, w \vee z) > 0,$$

as we assumed that $\ker S \cap (z \vee V) = \mathbb{R}x \vee z$. $\square$

Next, we will focus on $S \in \mathcal{C}_W$ as above such that $\ker S \cap (z \vee V) = \{0\}$. We will first study some properties of elements in the kernel of such S :

Lemma 4.30. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ such that $R'(S) = (\alpha \wedge \beta)^{\otimes 2} \neq 0$ for some $\alpha, \beta \in V^*$. Let furthermore $z \in \ker \alpha \cap \ker \beta \setminus \{0\}$ and suppose that $\ker S \cap (z \vee V) = \{0\}$. Then the following hold:*

1. *For every $U \subset V$ complementary to $\mathbb{R}z$ the intersection $\ker S \cap \text{Sym}^2 U$ is at most one-dimensional.*
2. *Every non-trivial element in $\ker R(S)$ is indefinite. In particular, there are no elements of rank one in $\ker R(S)$.*

Proof. Let $S \in \mathcal{C}_W$ as in Lemma 4.30. We will start excluding the presence of certain elements in $\ker S$.

We note that if $x^{\otimes 2} \in \ker S$ for some $x \in V$, then using Lemma 4.4 and the fact that $z \wedge V \subset \ker S$, we obtain

$$0 = S(x^{\otimes 2}, z^{\otimes 2}) = S(x \vee z, x \vee z),$$

which is impossible by the assumption that $\ker S \cap (z \vee V) = \{0\}$. Similarly, if $x^{\otimes 2} + y^{\otimes 2} \in \ker S$ for some $x, y \in V$ linearly independent of z , we find

$$0 < S(x \vee z, x \vee z) = S(x^{\otimes 2}, z^{\otimes 2}) = -S(y^{\otimes 2}, z^{\otimes 2}) = -S(y \vee z, y \vee z) < 0,$$

a contradiction.

Thus, we observe that if there is some $T \in \ker S \cap \text{Sym}^2 U \setminus \{0\}$ for some $U \subset V$ complementary to $\ker \alpha \cap \ker \beta$, then it must be indefinite. In fact, any element $T \in \ker R(S)$ must be indefinite:

If $\text{rank } T = 3$, then T must be indefinite by Lemma 4.25. If $T = x^{\otimes 2} + y^{\otimes 2}$ for some $x, y \in V \setminus \{0\}$, then we must have $z \in \text{span}\{x, y\}$, since otherwise $T \in \text{Sym}^2 U$ for $U := \text{span}\{x, y\}$ complementary to $\mathbb{R} \cdot z$, which is in contrast to our above observation. Therefore, we may assume after applying a rotation that $y \in \mathbb{R} \cdot z$. Again, this cannot happen by Lemma 4.25. Similarly, if $\text{rank } T = 1$, then by the above argument we must have $T = \lambda z^{\otimes 2}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$, which is excluded by our assumptions.

Therefore, $\ker R(S)$ consists of indefinite elements. In particular, there are no rank one elements in the kernel of $R(S)$.

Next, we claim that for every $U \subset V$ complementary to $\ker \alpha \cap \ker \beta$ the intersection $\ker S \cap \text{Sym}^2 U$ can at most be one-dimensional.

Indeed, suppose that $\text{span}\{T_1, T_2\} \subset \ker S \cap \text{Sym}^2 U$ for linearly independent $T_1, T_2 \in \text{Sym}^2 U$. Since T_1 is indefinite, there exist linearly independent $u_1, w_1 \in U$ such that $T_1 = u_1 \vee w_1$. Similarly, we know that also T_2 defines a Lorentzian metric g_{T_2} on U . If u_1 is null for g_{T_2} , then there exists some w linearly independent of u_1 such that $T_2 = u_1 \vee w$. If u_1 is not null for g_{T_2} then we know that, up to rescaling by a non-zero factor, $T_2 = u_1^{\otimes 2} - w^{\otimes 2}$ for some $w \in U$ linearly independent of u_1 :

Case 1: If $T_1 = u_1 \vee w_1$ and $T_2 = u_1 \vee w$, where $u_1, w \in U$ are linearly independent, we may write $w = \lambda u_1 + \mu w_1$ for some $\lambda, \mu \in \mathbb{R} \setminus \{0\}$. Then, we have

$$u_1 \vee w = \lambda u_1^{\otimes 2} + \mu u_1 \vee w_1 \in \ker S$$

which implies $u_1^{\otimes 2} \in \ker S$ in contrast to our above observations.

Case 2: If $T_1 = u_1 \vee w_1$, $T_2 = u_1^{\otimes 2} - w^{\otimes 2}$, where $u_1, w \in U$ are linearly independent, we again write $w = \lambda u_1 + \mu w_1$ for some $\lambda, \mu \in \mathbb{R}$, $\mu \neq 0$. Then,

$$u_1^{\otimes 2} - w^{\otimes 2} = (1 - \lambda^2) u_1^{\otimes 2} - \mu^2 w_1^{\otimes 2} - 2\lambda\mu u_1 \vee w_1 \in \ker S,$$

and hence, $(1 - \lambda^2) u_1^{\otimes 2} - \mu^2 w_1^{\otimes 2} \in \ker S$. By our observation that elements in $\ker S \cap \text{Sym}^2 U$ must be indefinite, we must have $1 - \lambda^2 > 0$. Therefore, after rescaling w_1 , we conclude that $u_1^{\otimes 2} - w_1^{\otimes 2} \in \ker S$. Then, we apply Lemma 4.4 and use $u_1 \vee w_1 \in \ker S$ to obtain

$$0 < S(u_1^{\otimes 2}, u_1^{\otimes 2}) = S(u_1^{\otimes 2}, w_1^{\otimes 2}) = -S(u_1 \wedge w_1, u_1 \wedge w_1) < 0,$$

a contradiction. □

If $\ker S \cap (z \vee V) = \{0\}$ for S and z as above, then $\text{rank } R(S) \geq 3$. In the following proposition we show that if $\text{rank } R(S) = 4$, then S is not extremal.

Proposition 4.31. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ such that $R'(S) = (\alpha \wedge \beta)^{\otimes 2} \neq 0$ for some $\alpha, \beta \in V^*$ and such that $\text{rank } R(S) = 4$. Let furthermore $z \in \ker \alpha \cap \ker \beta \setminus \{0\}$ and suppose that $\ker S \cap (z \vee V) = \{0\}$. Then S is not extremal in $\mathcal{C}_W$.*

Proof. By assumption, we have $\dim(\ker R(S)) = 2$. By Lemma 4.30, we know that all the elements in $\ker S \cap \text{Sym}^2 V$ are indefinite.

We write $\ker S \cap \text{Sym}^2 V = \text{span}\{T_1, T_2\}$. By Lemma 4.2 we may assume that $\text{rank } T_1 < 3$ and since T_1 must be indefinite, this means that $T_1 = u_1 \vee v_1$ for some linearly independent $u_1, v_1 \in V \setminus \{0\}$. If also $\text{rank } T_2 < 3$, i.e. $T_2 = u_2 \vee v_2$ for some linearly independent $u_2, v_2 \in V \setminus \{0\}$, then choosing $\gamma \in V^* \setminus \{0\}$ such that $\gamma(u_1) = 0 = \gamma(u_2)$, we find $\gamma^{\otimes 4}|_{\ker S} \equiv 0$ and therefore, S cannot be extremal by Corollary 3.32.

Now suppose $\text{rank } T_2 = 3$. Then T_2 defines a Lorentzian metric g_{T_2} on V . Up to changing the sign of T_2 , we may furthermore assume that g_{T_2} is of mostly positive signature.

Case 1: If $g_{T_2}(u_1, u_1) > 0$, then there exist some $v, w \in V \setminus \{0\}$ such that $\{u_1, v, w\} \subset V$ are linearly independent and such that

$$T_2 = u_1^{\otimes 2} + v^{\otimes 2} - w^{\otimes 2}.$$

Choosing $\gamma \in V^* \setminus \{0\}$ such that

$$\gamma(u_1) = 0, \quad \gamma(v) = \gamma(w)$$

we have $\gamma^{\otimes 4}|_{\ker S} \equiv 0$, hence S is not extremal in $\mathcal{C}_W$ by Corollary 3.32.

Case 2: If $g_{T_2}(u_1, u_1) = 0$, then there exist some $v, w \in V \setminus \{0\}$ such that $\{u_1, v, w\} \subset V$ are linearly independent and such that

$$T_2 = u_1 \vee v + w^{\otimes 2}.$$

Then for $\gamma \in V^* \setminus \{0\}$ satisfying $\gamma(u_1) = 0 = \gamma(w)$, we find $\gamma^{\otimes 4}|_{\ker S} \equiv 0$.

Case 3: Now suppose $g_{T_2}(u_1, u_1) < 0$. There exist some $v, w \in V \setminus \{0\}$ such that $\{u_1, v, w\} \subset V$ are linearly independent and such that

$$T_2 = v^{\otimes 2} + w^{\otimes 2} - u_1^{\otimes 2}.$$

If $g_{T_2}(v_1, v_1) \geq 0$, we can exchange the roles of u_1 and v_1 so that we are in case one or two. Therefore, suppose without loss of generality that also $g_{T_2}(v_1, v_1) < 0$. After applying a rotation to v and w , we may then assume that some non-trivial element in the line $\mathbb{R}v_1$ is related to u_1 by a boost in the direction of v , i.e. for some $\lambda \in \mathbb{R} \setminus \{0\}$, $\kappa \in \mathbb{R} \setminus \{0\}$, $v' \in V$ we have

$$\begin{pmatrix} \lambda v_1 \\ v' \end{pmatrix} = \begin{pmatrix} \cosh \kappa & \sinh \kappa \\ \sinh \kappa & \cosh \kappa \end{pmatrix} \begin{pmatrix} u_1 \\ v \end{pmatrix}$$

Therefore, for some $\lambda \in \mathbb{R} \setminus \{0\}$, $\kappa \in \mathbb{R} \setminus \{0\}$

$$\lambda v_1 = \cosh \kappa (u_1 + \tanh \kappa \cdot v).$$

We set $t := \tanh \kappa$. In particular, $0 < t^2 < 1$. Then, after rescaling v_1 , we find

$$v_1 = u_1 + t v.$$

Solving for v , this yields

$$v = \frac{1}{t} (v_1 - u_1)$$

and hence

$$T_2 = \frac{1}{t^2} v_1^{\otimes 2} + w^{\otimes 2} + \left(\frac{1}{t^2} - 1 \right) u_1^{\otimes 2} - \frac{2}{t} u_1 \vee v_1.$$

We define

$$\tilde{T}_2 := T_2 + \frac{2}{t} T_1 = \frac{1}{t^2} v_1^{\otimes 2} + w^{\otimes 2} + \frac{1-t^2}{t^2} u_1^{\otimes 2} \in \ker S.$$

Since $t^2 < 1$, $\tilde{T}_2$ is positive definite. But this is in contrast to our observation in Lemma 4.30. Therefore, we cannot have both $g_{T_2}(u_1, u_1) < 0$ and $g_{T_2}(v_1, v_1) < 0$. $\square$

Finally, we will show that if for S and z as above $\ker S \cap (z \vee V) = \{0\}$ and if $\text{rank } R(S) = 3$, then S is extremal.

Proposition 4.32. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ such that $R'(S) = (\alpha \wedge \beta)^{\otimes 2} \neq 0$ for some $\alpha, \beta \in V^*$ and such that $\text{rank } R(S) = 3$. Let furthermore $z \in \ker \alpha \cap \ker \beta \setminus \{0\}$ and suppose that $\ker S \cap (z \vee V) = \{0\}$. Then S is extremal in $\mathcal{C}_W$.*

Proof. Let S as in Proposition 4.32. We need to show that every element $S' \in \mathcal{C}_W$ such that $\ker S' \supsetneq \ker S$ has to vanish.

For this, we note that every element $S' \in \mathcal{C}_W$ such that $\ker S' \supsetneq \ker S$ must either have $R'(S') = 0$, i.e. S' is fully symmetric, or its kernel contains an element $u \vee z \neq 0$: Indeed, the proper inclusion $\ker S' \supsetneq \ker S$ implies that either $\ker R'(S') \supsetneq \ker R'(S)$ or $\ker R(S') \supsetneq \ker R(S)$. If $\ker R'(S') \supsetneq \ker R'(S)$, then we must have $R'(S') = 0$, since $\text{rank } R'(S) = 1$ by assumption. If $\ker R(S') \supsetneq \ker R(S)$ by the assumption $\text{rank } R(S) = 3$ we must have $\text{rank } R(S') \leq 2$, i.e. $\dim(\ker R(S')) \geq 4$ and hence $\ker R(S') \cap (z \vee V) \neq \{0\}$.

We have seen in Theorem 4.15 and Proposition 4.29 that in both such cases S' is not extremal unless it projects to an extremal element in $\mathcal{C}_{\text{Sym}}$. It is therefore enough to consider the case where $\text{rank } R(S') = 1$, i.e. S' is of the form $S' = \gamma^{\otimes 4}$ for some $\gamma \in V^*$ or of the form $S' = (\tilde{\alpha} \vee \tilde{\beta})^{\otimes 2} + (\tilde{\alpha} \wedge \tilde{\beta})^{\otimes 2}$ for linearly independent $\tilde{\alpha}, \tilde{\beta} \in V^*$ such that $\tilde{\alpha} \wedge \tilde{\beta}|_{\ker R(S)} \equiv 0$, i.e. $\tilde{\alpha}, \tilde{\beta} \in \text{span}\{\alpha, \beta\}$.

In Proposition 4.42, we show that the kernel of $R(S)$ for an element $S \in \mathcal{C}_W$ as in Proposition 4.32 is of one of the following three forms:

1. for parameters $d, g, h \in \mathbb{R}$ such that $d, g \neq 0$ and $g^2 > 1 - d^2 + dh$ and some $x, y \in V$ such that $\{x, y, z\}$ is a basis of V

$$\begin{aligned} \ker R(S) = \text{span}\{ & -x^{\otimes 2} + z^{\otimes 2} + d y \vee z, \\ & h x^{\otimes 2} - d y^{\otimes 2} + (d - h) z^{\otimes 2} + d g x \vee z, \\ & - (g^2 - d h) x^{\otimes 2} - d^2 y^{\otimes 2} + (d^2 + g^2 - d h) z^{\otimes 2} + d g x \vee y \}; \end{aligned}$$

2. for parameters $g, h \in \mathbb{R}$ such that $g^2 > 1$ and some $x, y \in V$ such that $\{x, y, z\}$ is a basis of V

$$\ker R(S) = \text{span}\{y \vee (x - g z), -x^{\otimes 2} + z^{\otimes 2}, g x^{\otimes 2} - g y^{\otimes 2} + h x \vee y + g^2 x \vee z\};$$

3. for parameters $d, h \in \mathbb{R}$ such that $d(d - h) > 1$ and some $x, y \in V$ such that $\{x, y, z\}$ is a basis of V

$$\begin{aligned} \ker R(S) = \text{span}\{ & -x^{\otimes 2} + y^{\otimes 2} + (d - h) \cdot y \vee z, \ x \vee (y - d z), \\ & h x^{\otimes 2} - d y^{\otimes 2} + (d - h) z^{\otimes 2}\}. \end{aligned}$$

We will first consider the case where $S' = \gamma^{\otimes 4}$ for some $\gamma \in V^*$. Finding $\gamma \in V^* \setminus \{0\}$ such that $\gamma^{\otimes 2}|_{\ker S} \equiv 0$ would in each case amount to finding a non-trivial real solution $(\gamma_x, \gamma_y, \gamma_z) \in \mathbb{R}^3$ to the respective system of equations:

1. for parameters $d, g, h \in \mathbb{R}$ such that $d, g \neq 0$ and $g^2 > 1 - d^2 + dh$,

$$\begin{aligned} -\gamma_x^2 + \gamma_z^2 + d\gamma_y\gamma_z &= 0, \\ h\gamma_x^2 - d\gamma_y^2 + (d-h)\gamma_z^2 + dg\gamma_x\gamma_z &= 0, \\ -(g^2 - dh)\gamma_x^2 - d^2\gamma_y^2 + (d^2 + g^2 - dh)\gamma_z^2 + dg\gamma_x\gamma_y &= 0; \end{aligned}$$

2. for parameters $g, h \in \mathbb{R}$ such that $g^2 > 1$,

$$\begin{aligned} \gamma_y(\gamma_x - g\gamma_z) &= 0, \\ -\gamma_x^2 + \gamma_z^2 &= 0, \\ g\gamma_x^2 - g\gamma_y^2 + h\gamma_x\gamma_y + g^2\gamma_x\gamma_z &= 0; \end{aligned}$$

3. for parameters $d, h \in \mathbb{R}$ such that $d(d-h) > 1$,

$$\begin{aligned} -\gamma_x^2 + \gamma_y^2 + (d-h)\gamma_y\gamma_z &= 0, \\ \gamma_x(\gamma_y - d\gamma_z) &= 0, \\ h\gamma_x^2 - d\gamma_y^2 + (d-h)\gamma_z^2 &= 0. \end{aligned}$$

However, these systems of equations do not have a non-trivial solution.

Now we consider the case where $S' = (\tilde{\alpha} \vee \tilde{\beta})^{\otimes 2} + (\tilde{\alpha} \wedge \tilde{\beta})^{\otimes 2}$ for some $\tilde{\alpha}, \tilde{\beta} \in \text{span}\{\alpha, \beta\}$. We set

$$\tilde{\alpha}(x) =: \tilde{\alpha}_x, \quad \tilde{\alpha}(y) =: \tilde{\alpha}_y, \quad \tilde{\beta}(x) =: \tilde{\beta}_x, \quad \tilde{\beta}(y) =: \tilde{\beta}_y.$$

Then in each case, $(\tilde{\alpha} \vee \tilde{\beta})|_{\ker R(S)} \equiv 0$ implies that

$$\begin{aligned} \tilde{\alpha}_x\tilde{\beta}_y + \tilde{\beta}_x\tilde{\alpha}_y &= 0 \\ \tilde{\alpha}_x\tilde{\beta}_x - \tilde{\alpha}_y\tilde{\beta}_y &= 0. \end{aligned}$$

This implies that either $\tilde{\alpha} = 0$ or $\tilde{\beta} = 0$.

We conclude that every $S' \in \mathcal{C}_W$ such that $\ker S' \supsetneq \ker S$ must vanish. Hence, S is extremal by Corollary 3.32. $\square$

We are now prepared to prove Theorem 4.23:

Proof of Theorem 4.23. Let $S \in \mathcal{C}_W$ such that $R'(S) = (\alpha \wedge \beta)^{\otimes 2}$ for some $\alpha, \beta \in V^*$ and such that $R(S)$ is not extremal in $\mathcal{C}_{\text{Sym}}$. Let furthermore $z \in (\ker \alpha \cap \ker \beta) \setminus \{0\}$.

If $\dim((z \vee V) \cap \ker S) > 0$, then S is not extremal by Proposition 4.29.

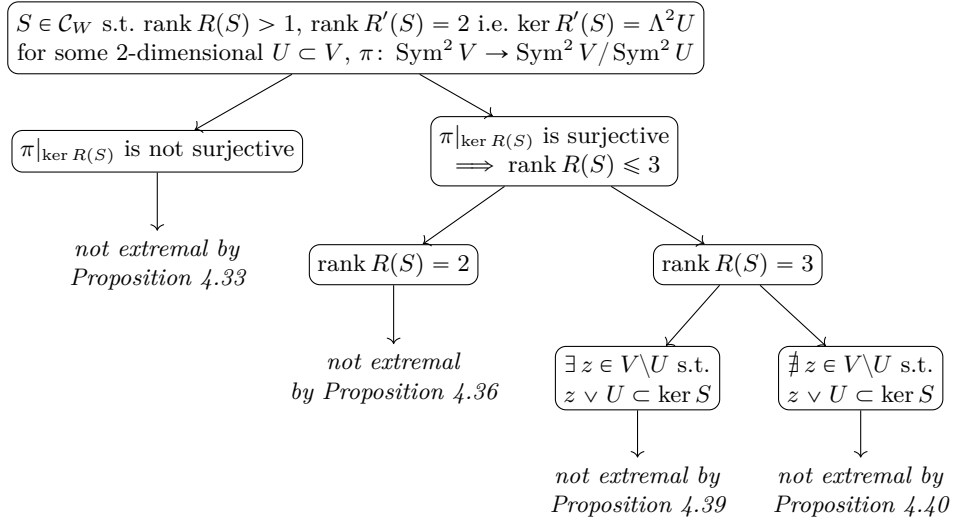
Now suppose $\dim((z \vee V) \cap \ker S) = 0$. Then in particular we must have $\dim \ker R(S) \leq 3$. Moreover, by Proposition 4.18, we must have $\dim \ker S > 3$. Since $\dim \ker R'(S) = 2$, this implies that $\dim \ker R(S) > 1$ and hence, $\text{rank } R(S) \in \{3, 4\}$. If $\text{rank } R(S) = 4$, then S is not extremal by Proposition 4.31, while if $\text{rank } R(S) = 3$, then S is extremal by Proposition 4.32. This proves the assertion. $\square$

4.3.2 Case $\text{rank } R'(S) = 2$

We will now consider $S \in \mathcal{C}_W$ such that $\text{rank } R'(S) = 2$. In this case, we know by Proposition 4.10 that $R(S)$ cannot be extremal in $\mathcal{C}_{\text{Sym}}$ and therefore, $\text{rank } R(S) \geq 2$. Moreover, since V is three-dimensional, $\dim(\ker R'(S)) = 1$ and hence $\ker R'(S) = \mathbb{R} \cdot x \wedge y$ for some linearly independent $x, y \in V$ or, equivalently, $\ker R'(S) = \Lambda^2 U$ for some two-dimensional subspace $U \subset V$ of V . Our strategy will be to distinguish whether the restriction of the projection map

$$\pi: \text{Sym}^2 V \longrightarrow \text{Sym}^2 V / \text{Sym}^2 U$$

to the kernel of $R(S)$ is surjective or not. We will see that S as above cannot be extremal if the restriction is not surjective. This will reduce the list of possible candidates for extremal elements $S \in \mathcal{C}_W$ with $\text{rank } R'(S) = 2$ to a few cases, in which we are able to give an explicit description of the kernel. Eventually, it will turn out that none of these $S \in \mathcal{C}_W$ is extremal. The structure of the section is as follows:



We will start proving that $S \in \mathcal{C}_W$ with $\text{rank } R'(S) = 2$ cannot be extremal if the restriction of the projection onto $\text{Sym}^2 V / \text{Sym}^2 U$ to $\ker R(S)$ is not surjective.

Proposition 4.33. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ such that $\text{rank } R'(S) = 2$. Let $U \subset V$ such that $\ker R'(S) = \Lambda^2 U$ and consider the projection*

$$\pi: \text{Sym}^2 V \longrightarrow \text{Sym}^2 V / \text{Sym}^2 U.$$

We denote the image of $\ker R(S)$ under the projection π by $\overline{\ker R(S)} := \pi(\ker R(S))$. If

$$\dim \left(\overline{\ker R(S)} \right) \leq 2,$$

i.e. the restriction of π to $\ker R(S)$ is not surjective, then S is not extremal.

Proof. Let $\gamma \in \text{Ann}(U) \setminus \{0\}$. Then $\gamma \vee V^*$ annihilates $\text{Sym}^2 U$. For $S \in \mathcal{C}_W$ as above, let $\{[T_1], [T_2]\}$ be a generating set of $\ker R(S)$, where $[T_i]$ is represented by some $T_i \in \text{Sym}^2 V$

for $i = 1, 2$. Then

$$(\gamma \vee V^*) \cap \text{Ann}(T_1) \cap \text{Ann}(T_2) \neq \{0\}.$$

Therefore, there is some $\eta \in V^* \setminus \{0\}$ such that $\gamma \vee \eta(T_i) = 0$ for $i = 1, 2$. Since $\gamma \vee V^*$ annihilates $\text{Sym}^2 U$, $\gamma \vee \eta(\tilde{T}_i) = 0$ holds for any $\tilde{T}_i$ representing $[T_i] \in \overline{\ker R(S)}$.

It follows that $\gamma \vee \eta|_{\ker S} \equiv 0$. Since $\gamma \in \text{Ann}(U)$, also $\gamma \wedge \eta|_{\ker S} \equiv 0$. We conclude that

$$((\gamma \otimes \eta)^{\otimes 2} + (\eta \otimes \gamma)^{\otimes 2})|_{\ker S} \equiv 0.$$

By Corollary 3.32 this shows that S cannot be extremal. $\square$

Next, we will consider $S \in \mathcal{C}_W$ as above such that $\pi|_{\ker R(S)}$ is surjective. Then we must have $\text{rank } R(S) \leq 3$. If $\text{rank } R(S) = 2$, the intersection $\ker S \cap \text{Sym}^2 U$ is one-dimensional. The following lemma shows that in this situation, we have $\ker S \cap \text{Sym}^2 U = \mathbb{R} \cdot (u \vee v)$ for linearly independent $u, v \in U$:

Lemma 4.34. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ such that $\text{rank } R'(S) = 2$. Let $U \subset V$ such that $\ker R'(S) = \Lambda^2 U$. If $\dim(\ker S \cap \text{Sym}^2 U) = 1$, then $\ker S \cap \text{Sym}^2 U = \mathbb{R} \cdot (u \vee v)$ for linearly independent $u, v \in U$.*

Proof. For S as above, we know that for $u, v \in U$, we have $u \wedge v \in \ker S$ and, hence, by Lemma 4.4,

$$S(u^{\otimes 2}, v^{\otimes 2}) = S(u \vee v, u \vee v).$$

Therefore, if $u^{\otimes 2} \in \ker S$ for some $u \in U$, it follows that $u \vee v \in \ker S$ for every $v \in U$.

Similarly, if $u^{\otimes 2} + v^{\otimes 2} \in \ker S$ for linearly independent $u, v \in U$, then

$$0 \leq S(u^{\otimes 2}, u^{\otimes 2}) = -S(u^{\otimes 2}, v^{\otimes 2}) = -S(u \vee v, u \vee v) \leq 0,$$

and hence, $\text{Sym}^2 U \subset \ker S$.

Thus, the intersection $\ker S \cap \text{Sym}^2 U$ can only be one-dimensional if it is of the form $\ker S \cap \text{Sym}^2 U = \mathbb{R} \cdot (u \vee v)$ for linearly independent $u, v \in U$. $\square$

With the help of Lemma 4.34, we can deduce further information about $\ker R(S)$, when $S \in \mathcal{C}_W$ is as above with $\pi|_{\ker R(S)}$ surjective and $\text{rank } R(S) = 2$:

Lemma 4.35. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ such that $\text{rank } R'(S) = 2$ and $\text{rank } R(S) = 2$. Suppose that $\dim(\ker S \cap \text{Sym}^2 U) = 1$, where $U \subset V$ such that $\ker R'(S) = \Lambda^2 U$. Then there is a $z \in V \setminus U$ such that $z \vee U \subset \ker S$.*

Proof. By Lemma 4.34 we know that $\ker S \cap \text{Sym}^2 U = \mathbb{R} \cdot u \vee v$ for some linearly independent $u, v \in U$. Since also $u \wedge v \in \ker S$ this implies $u \otimes v \in \ker S$ and $v \otimes u \in \ker S$. It follows that

$$S(u^{\otimes 2}, v^{\otimes 2}) = S(u \otimes v, v \otimes u) = 0. \quad (4.4)$$

Furthermore, we observe that for every $w \in V$:

$$S(u^{\otimes 2}, v \vee w) = S(u^{\otimes 2}, v \otimes w) = S(u \otimes w, v \otimes u) = 0 \quad (4.5)$$

and, similarly,

$$S(v^{\otimes 2}, u \vee w) = 0. \quad (4.6)$$

By our assumption that $\dim(\ker R(S)) = 4$ and $\dim(\ker S \cap \text{Sym}^2 U) = 1$, we know that the projection

$$\pi: \ker S \longrightarrow \text{Sym}^2 V / \text{Sym}^2 U$$

is surjective. Hence, for $\bar{z} \in V \setminus U$, there must be $Q_i \in \text{Sym}^2 U$, $i = 1, 2, 3$ such that

$$\bar{z} \vee u + Q_1, \bar{z} \vee v + Q_2 \in \ker S$$

Since $u \vee v \in \ker S$, we may furthermore assume that

$$Q_i = \lambda_i u^{\otimes 2} + \mu_i v^{\otimes 2}$$

for some $\lambda_i, \mu_i \in \mathbb{R}$.

We claim that $\mu_1 = 0$: Indeed, by assumption we have

$$\bar{z} \vee u + \lambda_1 u^{\otimes 2} + \mu_1 v^{\otimes 2} = (\bar{z} + \lambda_1 u) \vee u + \mu_1 v^{\otimes 2} \in \ker S.$$

Applying equation (4.6) for $w = \bar{z} + \lambda_1 u$, we observe that $v^{\otimes 2}$ and $(\bar{z} + \lambda_1 u) \vee u$ are orthogonal with respect to S . Since S is semi-definite, this implies that both $\mu_1 \cdot v^{\otimes 2} \in \ker S$ and $(\bar{z} + \lambda_1 u) \vee u \in \ker S$. Since by Lemma 4.34, we know that $v^{\otimes 2} \notin \ker S$, it follows that $\mu_1 = 0$.

An analogous argument shows that $\lambda_2 = 0$. Therefore, we obtain

$$\text{span}\{(\bar{z} + \lambda_1 u) \vee u, (\bar{z} + \mu_2 v) \vee v\} \subset \ker S.$$

Defining

$$z := \bar{z} + \lambda_1 u + \mu_2 v \in V \setminus \{0\},$$

we find

$$\begin{aligned} z \vee u - \mu_2 u \vee v \in \ker S &\implies z \vee u \in \ker S \\ z \vee v - \lambda_1 u \vee v \in \ker S &\implies z \vee v \in \ker S, \end{aligned}$$

where we used that $u \vee v \in \ker S$. It follows that $z \vee U \subset \ker S$. $\square$

Combining Lemma 4.34 and 4.35, we know enough about $\ker R(S)$ for $S \in \mathcal{C}_W$ as above with $\pi|_{\ker R(S)}$ surjective and $\text{rank } R(S) = 2$ to conclude that such S cannot be extremal:

Proposition 4.36. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ such that $\text{rank } R'(S) = 2$ and $\text{rank } R(S) = 2$. Suppose that $\dim(\ker S \cap \text{Sym}^2 U) = 1$, where $U \subset V$ such that $\ker R'(S) = \Lambda^2 U$. Then S is not extremal in $\mathcal{C}_W$.*

Proof. Let $S \in \mathcal{C}_W$ as in Proposition 4.36. Then we know by Lemma 4.34 that for some linearly independent $u, v \in U \setminus \{0\}$, we have

$$\ker S \cap \text{Sym}^2 U = \mathbb{R} u \vee v.$$

Furthermore, by Lemma 4.35, there is some $z \in V \setminus U$ such that $z \vee U \subset \ker S$.

Therefore, for some $\lambda, \mu \in \mathbb{R}$,

$$\ker R(S) = \text{span}\{u \vee v, z \vee u, z \vee v, z^{\otimes 2} + \lambda u^{\otimes 2} + \mu v^{\otimes 2}\}.$$

We may assume that

$$S(u^{\otimes 2}, u^{\otimes 2}) = 1 = S(v^{\otimes 2}, v^{\otimes 2}).$$

Using Lemma 4.4, we find

$$\begin{aligned} S(u^{\otimes 2}, v^{\otimes 2}) &= S(u \vee v, u \vee v) - S(u \wedge v, u \wedge v) = 0, \\ S(z \wedge u, z \wedge u) &= -S(z^{\otimes 2}, u^{\otimes 2}) = \lambda S(u^{\otimes 2}, u^{\otimes 2}) + \mu S(v^{\otimes 2}, u^{\otimes 2}) = \lambda, \\ S(z \wedge v, z \wedge v) &= -S(z^{\otimes 2}, v^{\otimes 2}) = \lambda S(u^{\otimes 2}, v^{\otimes 2}) + \mu S(v^{\otimes 2}, v^{\otimes 2}) = \mu. \end{aligned}$$

In particular, $\lambda, \mu > 0$.

We set

$$\alpha := \alpha_1 + \sqrt{\lambda}\alpha_3, \quad \beta := \alpha_1 - \sqrt{\lambda}\alpha_3,$$

where $\{\alpha_1, \alpha_2, \alpha_3\}$ is the basis of V^* dual to $\{u, v, z\}$. Then $v \in \ker \alpha \cap \ker \beta$ and therefore

$$(\alpha \vee \beta)(u \vee v) = 0 = (\alpha \vee \beta)(z \vee v), \quad (\alpha \wedge \beta)(u \wedge v) = 0.$$

Moreover, we compute

$$\begin{aligned} (\alpha \vee \beta)(z \vee u) &= 0, \\ (\alpha \vee \beta)(z^{\otimes 2} + \lambda u^{\otimes 2} + \mu v^{\otimes 2}) &= -\lambda + \lambda = 0. \end{aligned}$$

Therefore,

$$((\alpha \vee \beta)^{\otimes 2} + (\alpha \wedge \beta)^{\otimes 2})|_{\ker S} \equiv 0.$$

By Corollary 3.32 it follows that S is not extremal. $\square$

Next, we will focus on the situation where $S \in \mathcal{C}_W$ is as above with $\pi|_{\ker R(S)}$ surjective and $\text{rank } R(S) = 3$. In this case it is in general no longer true that we can find some $z \in V \setminus U$ such that $z \vee U \subset \ker S$. However, in any case we get a relatively nice description of the kernel of $R(S)$.

Lemma 4.37. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ such that $\text{rank } R'(S) = 2$ and $\text{rank } R(S) = 3$. Suppose that $\ker S \cap \text{Sym}^2 U = \{0\}$, where $U \subset V$ such that $\ker R'(S) = \Lambda^2 U$. Then either there is some $z \in V \setminus U$ such that $z \vee U \subset \ker S$ or there are $z \in V \setminus U$ and linearly independent $u, v \in U \setminus \{0\}$ such that $\text{span}\{z \vee u, z \vee v + v^{\otimes 2} - u^{\otimes 2}\} \subset \ker S$.*

Proof. Let $S \in \mathcal{C}_W$ as in Lemma 4.37. Let $\bar{z} \in V \setminus U$. By the assumption $\ker S \cap \text{Sym}^2 U = \{0\}$ we know that there is a two-dimensional subspace of $\ker S \cap \text{Sym}^2 V$ which projects surjectively onto $\pi(\bar{z} \vee U) \subset \text{Sym}^2 V / \text{Sym}^2 U$, where π is the projection of $\text{Sym}^2 V$ to the quotient space $\text{Sym}^2 V / \text{Sym}^2 U$. In particular by Lemma 4.2 this subspace contains a non-trivial element T_1 of rank < 3 , which must be of the form $T_1 = (\bar{z} + v) \vee u$ for some $u, v \in U$, $u \neq 0$. We set $z := \bar{z} + v \in V \setminus U$. Then $T_1 = z \vee u \in \ker S$.

Next, we observe that for every $v, w \in U$,

$$S(z \vee v, w \vee u) = S(z \otimes v, w \otimes u) = S(z \otimes u, w \otimes v) = S(z \vee u, w \vee v) = 0, \quad (4.7)$$

where we used that $\Lambda^2 U \subset \ker S$, the symmetry of S under τ and $z \vee u \in \ker S$. Using again that $z \vee u \in \ker S$, this shows that for every $v \in U$

$$z \vee v \perp_S (u \vee V). \quad (4.8)$$

Moreover, since $\ker R(S)$ projects surjectively onto $\text{Sym}^2 V / \text{Sym}^2 U$, we know that for every $v \in U \setminus \{0\}$ linearly independent of u there should be some $Q \in \text{Sym}^2 U$ such that $T_2 := z \vee v + Q \in \ker S$.

Case 1: If $\text{rank } Q < 2$, then $Q = w^{\otimes 2}$ for some $w \in U$. Then

$$0 = S(z \vee v + w^{\otimes 2}, u^{\otimes 2}) = S(w^{\otimes 2}, u^{\otimes 2}) = S(u \vee w, u \vee w), \quad (4.9)$$

where we used (4.8) and $u \wedge w \in \ker S$. Since by assumption $\ker S \cap \text{Sym}^2 U = \{0\}$, it follows that $w = 0$ and hence $Q = 0$. It follows that $T_2 = z \vee v \in \ker S$. Since also $T_1 = z \vee u \in \ker S$ and because we assumed that u and v are linearly independent, we must have $z \vee U \subset \ker S$.

Case 2: Now suppose $\text{rank } Q = 2$. By an analogous calculation as in (4.9), it follows that Q cannot be definite as an element of $\text{Sym}^2 U$. Moreover, it cannot be of the form $Q = u \vee w$ for some $w \in U \setminus \{0\}$: Otherwise, using (4.8), we would find

$$0 = S(z \vee v + u \vee w, u \vee w) = S(u \vee w, u \vee w),$$

in contrast to the assumption that $\ker S \cap \text{Sym}^2 U = \{0\}$. Therefore, for some $w \in U$, we have $\pm Q = w^{\otimes 2} - \lambda u^{\otimes 2}$, for some $\lambda > 0$. After rescaling we may assume that $\lambda = 1$ and after replacing T_2 by a linear combination of T_1 and T_2 , we may further assume $v = w$. Then, possibly after changing $T_2 \rightarrow -T_2$ and $w \rightarrow -w$, we obtain

$$\text{span}\{z \vee u, z \vee w + w^{\otimes 2} - u^{\otimes 2}\} \subset \ker S.$$

□

Remark 4.38. The two cases described in Lemma 4.37 cannot occur simultaneously: Indeed, if for some $z \in V \setminus U$ we have $z \vee U \subset \ker S$, then any $\bar{z} \in V \setminus U$ such that $\bar{z} \vee v \in \ker S$ for some $v \in U$ must satisfy $\bar{z} \in \mathbb{R}z$, since otherwise, there would be a contradiction with the assumption that $\ker S \cap \text{Sym}^2 U = \{0\}$. Similarly, if $z \vee U \subset \ker S$, there cannot be any linearly independent $u, w \in U$ such that $z \vee w + w^{\otimes 2} - u^{\otimes 2} \in \ker S$.

We can now apply Lemma 4.37 to explicitly describe the kernel of $R(S)$ for $S \in \mathcal{C}_W$ as above with $\pi|_{\ker R(S)}$ surjective and $\text{rank } R(S) = 3$. We will first consider the case where $z \vee U \subset \ker S$ for some $z \in V \setminus U$, $U \subset V$ as above. We will then conclude that such S is not extremal:

Proposition 4.39. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ such that $\text{rank } R'(S) = 2$ and $\text{rank } R(S) = 3$. Suppose that $\ker S \cap \text{Sym}^2 U = \{0\}$, where $U \subset V$ such that $\ker R'(S) = \Lambda^2 U$. Suppose furthermore that there is some $z \in V \setminus U$ such that $z \vee U \subset \ker S$. Then S is not extremal in $\mathcal{C}_W$.*

Proof. Let $S \in \mathcal{C}_W$, $U \subset V$ and $z \in V \setminus U$ as in Proposition 4.39. Then for some basis $\{u, v\}$ of U and $\lambda, \mu, \delta \in \mathbb{R}$ we have

$$\ker S = \text{span}\{u \wedge v, z \vee u, z \vee v, z^{\otimes 2} + \lambda u^{\otimes 2} + \mu v^{\otimes 2} + \delta u \vee v\}.$$

We may furthermore assume that $\delta = 0$, possibly after changing to a different basis of U .

Case 1: If $\lambda \geq 0$, we may proceed as in the proof of Proposition 4.36 to find that

$$\left[\left((\alpha_1 + \sqrt{\lambda} \alpha_3) \vee (\alpha_1 - \sqrt{\lambda} \alpha_3) \right)^{\otimes 2} + \left((\alpha_1 + \sqrt{\lambda} \alpha_3) \wedge (\alpha_1 - \sqrt{\lambda} \alpha_3) \right)^{\otimes 2} \right] \Big|_{\ker S} \equiv 0,$$

where $\{\alpha_1, \alpha_2, \alpha_3\}$ is the basis of V^* dual to $\{u, v, z\}$. Hence, S is not extremal by Corollary 3.32.

Case 2: Now suppose that both λ and μ are negative. We will show that this case cannot occur. For this, we assume without loss of generality that $\lambda = -1 = \mu$. Then

$$S(u^{\otimes 2}, \cdot) = S(z^{\otimes 2}, \cdot) - S(v^{\otimes 2}, \cdot)$$

and, since we assumed $\ker S \cap \text{Sym}^2 U = \{0\}$, it follows that

$$\begin{aligned} 0 &< S(u^{\otimes 2}, u^{\otimes 2}) \\ &= S(z^{\otimes 2}, u^{\otimes 2}) - S(v^{\otimes 2}, u^{\otimes 2}) \\ &= S(z \vee u, z \vee u) - S(z \wedge u, z \wedge u) - S(u \vee v, u \vee v) + S(u \wedge v, u \wedge v) \\ &= -S(z \wedge u, z \wedge u) - S(u \vee v, u \vee v) < 0, \end{aligned}$$

a contradiction. Here, we used Lemma 4.4 for the second equality and $z \vee u, u \wedge v \in \ker S$, $z \wedge u, u \vee v \notin \ker S$ and $S \geq 0$ in the last step.

We conclude that S as in Proposition 4.39 is not extremal in $\mathcal{C}_W$. $\square$

Finally, we apply Lemma 4.37 to $S \in \mathcal{C}_W$ as above with $\pi|_{\ker R(S)}$ surjective, $\text{rank } R(S) = 3$ and such that there is no $z \in V \setminus U$ such that $z \vee U \subset \ker S$, where $\pi: \text{Sym}^2 V \rightarrow \text{Sym}^2 V / \text{Sym}^2 U$ and $U \subset V$ are as above, to show that such S is not extremal:

Proposition 4.40. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W$ such that for some basis $\{u, v, z\}$ of V we have*

$$\ker S = \text{span}\{u \wedge v, z \vee u, z \vee v + v^{\otimes 2} - u^{\otimes 2}, z^{\otimes 2} + \lambda u^{\otimes 2} + \mu v^{\otimes 2} + \delta u \vee v\}$$

for some $\lambda, \mu, \delta \in \mathbb{R}$. Then S is not extremal in $\mathcal{C}_W$.

Proof. Let $S \in \mathcal{C}_W$ as in Proposition 4.40. Note that the same arguments used in Lemma 4.37 to arrive at (4.8) apply in the setting of Proposition 4.40. Thus, we have

$$S(z \vee w, u \vee x) = 0$$

for every $w \in U, x \in V$. Using that $z \vee v + v^{\otimes 2} - u^{\otimes 2} \in \ker S$, we therefore find

$$\begin{aligned} 0 &= S(z \vee v, u^{\otimes 2}) = S(u^{\otimes 2}, u^{\otimes 2}) - S(v^{\otimes 2}, u^{\otimes 2}), \\ 0 &= S(z \vee v, u \vee v) = S(u^{\otimes 2}, u \vee v) - S(v^{\otimes 2}, u \vee v). \end{aligned}$$

We may furthermore assume that

$$1 = S(u^{\otimes 2}, u^{\otimes 2}) = S(u^{\otimes 2}, v^{\otimes 2}) = S(u \vee v, u \vee v).$$

We set

$$q_1 := S(v^{\otimes 2}, v^{\otimes 2}), \quad q_2 := S(u^{\otimes 2}, u \vee v) = S(v^{\otimes 2}, u \vee v).$$

Since $R(S)$ is positive semi-definite we find

$$q_1 \geq 1, \quad -1 \leq q_2 \leq 1. \tag{4.10}$$

Since $u \wedge z \notin \ker S$, we must furthermore have

$$0 < S(u \wedge z, u \wedge z) = -S(u^{\otimes 2}, z^{\otimes 2}). \quad (4.11)$$

Using that $z^{\otimes 2} + \lambda u^{\otimes 2} + \mu v^{\otimes 2} + \delta u \vee v \in \ker S$, we compute

$$S(u^{\otimes 2}, z^{\otimes 2}) = S(-\lambda u^{\otimes 2} - \mu v^{\otimes 2} - \delta u \vee v, u^{\otimes 2}) = -\lambda - \mu - \delta q_2.$$

Hence, equation (4.11) becomes

$$0 < \lambda + \mu + \delta q_2. \quad (4.12)$$

We show that there exist $\alpha, \beta \in V^* \setminus \{0\}$ such that

$$((\alpha \wedge \beta)^{\otimes 2} + (\alpha \vee \beta)^{\otimes 2})|_{\ker S} \equiv 0. \quad (4.13)$$

For this, we make the ansatz

$$\alpha(u) = \beta(u) = 1, \quad \alpha(v) = \beta(v) = \varepsilon, \quad \alpha(z) = -\beta(z)$$

with

$$\varepsilon = \begin{cases} 1 & \text{for } \delta \geq 0 \\ -1 & \text{for } \delta < 0 \end{cases}.$$

Then we find

$$\begin{aligned} (\alpha \wedge \beta)(u \wedge v) &= 0 = (\alpha \vee \beta)(z \vee u) = (\alpha \vee \beta)(z \vee v + v^{\otimes 2} - u^{\otimes 2}), \\ (\alpha \vee \beta)(z^{\otimes 2} + \lambda u^{\otimes 2} + \mu v^{\otimes 2} + \delta u \vee v) &= -\alpha(z)^2 + \lambda + \mu + |\delta|. \end{aligned}$$

Therefore, α and β satisfy equation (4.13) if and only if $\alpha(z)$ solves

$$-\alpha(z)^2 + \lambda + \mu + |\delta| = 0.$$

Since by (4.10) and (4.12) we have

$$\lambda + \mu + |\delta| \geq \lambda + \mu + \delta q_2 > 0,$$

we can always find a real solution for $\alpha(z)$. Using Corollary 3.32, we conclude that S is not extremal. $\square$

We summarize our results in the following theorem:

Theorem 4.41. *Suppose $\dim V = 3$. Let $S \in \mathcal{C}_W \setminus \{0\}$ such that $R(S)$ is not extremal in $\mathcal{C}_{\text{Sym}}$. Then S is extremal in $\mathcal{C}_W$ if and only if it is of the type as in Proposition 4.32, i.e. $\text{rank } R'(S) = 1$, $\text{rank } R(S) = 3$ and $\ker S \cap (z \vee V) = \{0\}$, where $z \in V$ such that $\ker R'(S) = z \wedge V$.*

4.3.3 Explicit description of the extremal elements

We will now derive an explicit form for the extremal elements of the type as in Theorem 4.41.

Proposition 4.42. *Let $\dim V = 3$, $S \in \mathcal{C}_W$ such that $\text{rank } R(S) = 3$, $\ker R'(S) = z \wedge V$ for some $z \in V \setminus \{0\}$ and such that $\ker S \cap (z \vee V) = \{0\}$. Then for some parameters $d, g, h \in \mathbb{R}$ such that $g^2 > 1 - d^2 + dh$, and for some $e_2, e_3 \in V$ such that $\{z =: e_1, e_2, e_3\}$ is a basis of V , in terms of the basis $\{e_1^{\otimes 2}, e_2^{\otimes 2}, e_3^{\otimes 2}, e_1 \vee e_2, e_1 \vee e_3, e_2 \vee e_3\}$ of $\text{Sym}^2 V$, we have the representation*

$$R(S) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 + d^2 & 1 + dh & dg & 0 & d \\ 1 & 1 + dh & 1 + g^2 + h^2 & g(d + h) & g & h \\ 0 & dg & g(d + h) & d^2 + g^2 & d & g \\ 0 & 0 & g & d & 1 & 0 \\ 0 & d & h & g & 0 & 1 \end{pmatrix}.$$

The kernel of $R(S)$ is given by the following:

If $d, g \neq 0$:

$$\begin{aligned} \ker R(S) = \text{span}\{ & e_1^{\otimes 2} - e_2^{\otimes 2} + d e_1 \vee e_3, \\ & (d - h) e_1^{\otimes 2} + h e_2^{\otimes 2} - d e_3^{\otimes 2} + dg e_1 \vee e_2, \\ & (d^2 + g^2 - dh) e_1^{\otimes 2} - (g^2 - dh) e_2^{\otimes 2} - d^2 e_3^{\otimes 2} + dg e_2 \vee e_3\}. \end{aligned}$$

If $d = 0$:

$$\ker R(S) = \text{span}\{(g e_1 - e_2) \vee e_3, e_1^{\otimes 2} - e_2^{\otimes 2}, g e_2^{\otimes 2} - g e_3^{\otimes 2} + h e_2 \vee e_3 + g^2 e_1 \vee e_2\}.$$

If $g = 0$:

$$\begin{aligned} \ker R(S) = \text{span}\{ & -e_2^{\otimes 2} + e_3^{\otimes 2} + (d - h) e_1 \vee e_3, (d e_1 - e_3) \vee e_2, \\ & (d - h) e_1^{\otimes 2} + h e_2^{\otimes 2} - d e_3^{\otimes 2}\}. \end{aligned}$$

Proof. Since $\ker S \cap (z \vee V) = \{0\}$ and $z \wedge V \subset \ker S$, we deduce that $S(z^{\otimes 2}, \cdot)$ defines a positive definite symmetric bilinear form on V . Indeed, for any $v \in V$, using Lemma 4.4, we observe

$$S(z^{\otimes 2}, v^{\otimes 2}) = S(z \vee v, z \vee v) - S(z \wedge v, z \wedge v) = S(z \vee v, z \vee v) > 0.$$

We complete $\{z\}$ to an orthonormal base $\{z =: e_1, e_2, e_3\} \subset V$ of V with respect to $S(z^{\otimes 2}, \cdot)$. In addition, we require that

$$S(e_2^{\otimes 2}, e_2 \vee z) = 0.$$

To see that this is possible, let $\{\bar{x}, \bar{y}\}$ be a basis of the orthogonal complement of $\mathbb{R} \cdot z$ with respect to $S(z^{\otimes 2}, \cdot)$. We may assume that

$$S(\bar{y}^{\otimes 2}, \bar{y} \vee z) \neq 0$$

Then

$$p_{\bar{x}, \bar{y}}(t) := S(((\bar{x} + t\bar{y})^{\otimes 2}, (\bar{x} + t\bar{y}) \vee z))$$

is a cubic polynomial in t and therefore, for some $t_0 \in \mathbb{R}$, we have $p_{\bar{x}, \bar{y}}(t_0) = 0$. Then $e_2 := \bar{x} + t_0 \bar{y}$ has the desired property.

We set

$$\begin{aligned}
q &:= S(e_2 \wedge e_3, e_2 \wedge e_3) > 0, \\
a &:= S(e_2^{\otimes 2}, e_2^{\otimes 2}), \\
b &:= S(e_2^{\otimes 2}, e_3^{\otimes 2}) = S(e_2 \vee e_3, e_2 \vee e_3) - S(e_2 \wedge e_3, e_2 \wedge e_3) = S(e_2 \vee e_3, e_2 \vee e_3) - q, \\
c &:= S(e_2^{\otimes 2}, e_2 \vee e_3), \\
d &:= S(e_2^{\otimes 2}, e_1 \vee e_3) = S(e_2 \vee e_3, e_1 \vee e_2), \\
e &:= S(e_3^{\otimes 2}, e_3^{\otimes 2}), \\
f &:= S(e_3^{\otimes 2}, e_2 \vee e_3), \\
g &:= S(e_3^{\otimes 2}, e_1 \vee e_2) = S(e_2 \vee e_3, e_1 \vee e_3), \\
h &:= S(e_3^{\otimes 2}, e_1 \vee e_3).
\end{aligned}$$

With these choices of parameters, we write the quadratic form on $\text{Sym}^2 V$ defined by $R(S)$ in terms of the basis $\{e_1^{\otimes 2}, e_2^{\otimes 2}, e_3^{\otimes 2}, e_1 \vee e_2, e_1 \vee e_3, e_2 \vee e_3\}$ as

$$R(S) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & a & b & 0 & d & c \\ 1 & b & e & g & h & f \\ 0 & 0 & g & 1 & 0 & d \\ 0 & d & h & 0 & 1 & g \\ 0 & c & f & d & g & b+q \end{pmatrix}.$$

Requiring the fourth principal minors of $R(S)$ to vanish, the first, second and third principal minors to be non-negative and the parameter q to be positive, yields

$$\begin{aligned}
a &= 1 + d^2, & b &= 1 + dh, & c &= dg, & e &= 1 + g^2 + h^2, & f &= g(d + h) \\
b + q &= d^2 + g^2, & g^2 &> 1 - d^2 + dh.
\end{aligned}$$

Furthermore, with these choices of parameters, we obtain the following basis of the kernel of $R(S)$:

If $d, g \neq 0$:

$$\begin{aligned}
\ker R(S) &= \text{span}\{e_1^{\otimes 2} - e_2^{\otimes 2} + d e_1 \vee e_3, \\
&\quad (d - h) e_1^{\otimes 2} + h e_2^{\otimes 2} - d e_3^{\otimes 2} + dg e_1 \vee e_2, \\
&\quad (d^2 + g^2 - dh) e_1^{\otimes 2} - (g^2 - dh) e_2^{\otimes 2} - d^2 e_3^{\otimes 2} + dg e_2 \vee e_3\}.
\end{aligned}$$

If $d = 0$ (this implies $g^2 > 1$):

$$\ker R(S) = \text{span}\{(g e_1 - e_2) \vee e_3, e_1^{\otimes 2} - e_2^{\otimes 2}, g e_2^{\otimes 2} - g e_3^{\otimes 2} + h e_2 \vee e_3 + g^2 e_1 \vee e_2\}.$$

If $g = 0$ (this implies $d \neq 0, d \neq h$):

$$\begin{aligned}
\ker R(S) &= \text{span}\{-e_2^{\otimes 2} + e_3^{\otimes 2} + (d - h) \cdot e_1 \vee e_3, (d e_1 - e_3) \vee e_2, \\
&\quad (d - h) e_1^{\otimes 2} + h e_2^{\otimes 2} - d e_3^{\otimes 2}\}.
\end{aligned}$$

In particular, the kernel of $R(S)$ is three-dimensional and hence $\text{rank } R(S) = 3$ as re-

quired. $\square$

Proposition 4.43. *Let $\dim V = 3$, $S \in \mathcal{C}_W$ such that $\text{rank } R(S) > 1$. Then S is extremal if and only if it is of the following form: For some basis $\{\alpha_1, \alpha_2, \alpha_3\}$ of V^* and some parameters $g, h, d \in \mathbb{R}$ such that $g^2 > 1 - d^2 + dh$ we have*

$$S = S_{\text{tot}} + 2(g^2 + d^2 - 1 - dh) [(\alpha_2 \otimes \alpha_3)^{\otimes 2} + (\alpha_3 \otimes \alpha_2)^{\otimes 2}],$$

where $S_{\text{tot}} \in \text{Sym}^4 V^*$ is given by

$$\begin{aligned} S_{\text{tot}} = & \alpha_1^4 + 6\alpha_1^2\alpha_2^2 + 6\alpha_1^2\alpha_3^2 + 12d\alpha_1\alpha_2^2\alpha_3 + 12g\alpha_1\alpha_2\alpha_3^2 + 4h\alpha_1\alpha_3^3 + (1 + d^2)\alpha_2^4 \\ & + 4dg\alpha_2^3\alpha_3 + 6(1 + dh)\alpha_2^2\alpha_3^2 + 4g(d + h)\alpha_2\alpha_3^3 + (1 + g^2 + h^2)\alpha_3^4. \end{aligned}$$

Proof. Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be the basis of V^* dual to the basis $\{e_1, e_2, e_3\}$ in Proposition 4.42. Note that, with our conventions for the symmetric and the anti-symmetric product, we have

$$2(\alpha_i \vee \alpha_j)(e_i \vee e_j) = 1, \quad 2(\alpha_i \wedge \alpha_j)(e_i \wedge e_j) = 1 \quad \text{for } i \neq j.$$

Using this, it follows from Proposition 4.42 that

$$\begin{aligned} S = & \alpha_1^{\otimes 4} + 2\alpha_1^2 \vee \alpha_2^2 + 2\alpha_1^2 \vee \alpha_3^2 + (1 + d^2)\alpha_2^{\otimes 4} + 2(1 + dh)\alpha_2^2 \vee \alpha_3^2 \\ & + 2d\alpha_2^2 \vee (2\alpha_1 \vee \alpha_3) + 2dg\alpha_2^2 \vee (2\alpha_2 \vee \alpha_3) + (1 + g^2 + h^2)\alpha_3^{\otimes 4} \\ & + 2g\alpha_3^2 \vee (2\alpha_1 \vee \alpha_2) + 2h\alpha_3^2 \vee (2\alpha_1 \vee \alpha_3) + 2g(d + h)\alpha_3^2 \vee (2\alpha_2 \vee \alpha_3) \\ & + (2\alpha_1 \vee \alpha_2)^{\otimes 2} + 2d(2\alpha_1 \vee \alpha_2) \vee (2\alpha_2 \vee \alpha_3) + (2\alpha_1 \vee \alpha_3)^{\otimes 2} \\ & + 2g(2\alpha_1 \vee \alpha_3) \vee (2\alpha_2 \vee \alpha_3) + (d^2 + g^2)(2\alpha_2 \vee \alpha_3)^{\otimes 2} \\ & + (d^2 + g^2 - 1 - dh)(2\alpha_2 \wedge \alpha_3)^{\otimes 2}. \end{aligned}$$

Observe that for any $\beta, \gamma, \delta \in V^*$, we have

$$\begin{aligned} 4\beta^2 \vee (\beta \vee \gamma) &= 4\beta^3\gamma \\ 4(\beta \vee \gamma)^{\otimes 2} + 2\beta^2 \vee \gamma^2 &= 6\beta^2\gamma^2 \\ 4\beta^2 \vee (\gamma \vee \delta) + 8(\beta \vee \gamma) \vee (\beta \vee \delta) &= 12\beta^2\gamma\delta. \end{aligned}$$

After applying these identities and rearranging the terms, we find

$$\begin{aligned} S &= S_{\text{tot}} + 4(g^2 + d^2 - 1 - dh) [(\alpha_2 \vee \alpha_3)^{\otimes 2} + (\alpha_2 \wedge \alpha_3)^{\otimes 2}] \\ &= S_{\text{tot}} + 2(g^2 + d^2 - 1 - dh) [(\alpha_2 \otimes \alpha_3)^{\otimes 2} + (\alpha_3 \otimes \alpha_2)^{\otimes 2}], \end{aligned}$$

where

$$\begin{aligned} S_{\text{tot}} = & \alpha_1^4 + 6\alpha_1^2\alpha_2^2 + 6\alpha_1^2\alpha_3^2 + 12d\alpha_1\alpha_2^2\alpha_3 + 12g\alpha_1\alpha_2\alpha_3^2 + 4h\alpha_1\alpha_3^3 + (1 + d^2)\alpha_2^4 \\ & + 4dg\alpha_2^3\alpha_3 + 6(1 + dh)\alpha_2^2\alpha_3^2 + 4g(d + h)\alpha_2\alpha_3^3 + (1 + g^2 + h^2)\alpha_3^4. \end{aligned}$$

$\square$

In conclusion, we obtain the classification of extremal elements of $\mathcal{C}_W$ for three-dimensional V as stated in Theorem 4.22.

4.4 Positivity bounds

In this section, we will use the extreme rays of the cone $\mathcal{C}_W$ in order to determine a complete set of inequalities describing the dual cone

$$\mathcal{C}_W^* = \{M \in W^* : M(S) \geq 0 \quad \forall S \in \mathcal{C}_W\}$$

of $\mathcal{C}_W$, where $M(S)$ denotes the natural pairing. Note that in order to describe the dual cone $\mathcal{C}_W^*$ it is enough to require $M(S)$ to be non-negative for all extremal elements S of $\mathcal{C}_W$. In this way, every extremal element in the classification in Theorem 4.22 gives rise to a linear inequality constraining $\mathcal{C}_W^*$. Since $\mathcal{C}_W^*$ is precisely the cone that was denoted $\mathcal{C}$ in Section 2.4, these inequalities have the physical interpretation of constraining the two-to-two forward scattering amplitudes of an EFT that is the low-energy limit of a QFT compatible with the physical principles of unitarity, locality and causality. In particular, Theorem 4.22 gives rise to the complete set of (two-to-two forward scattering) positivity bounds for scattering processes with three flavors of particles, i.e. $\dim V = 3$.

Recall that the bounds coming from the extreme rays of the first and the second type in Theorem 4.22 are referred to as elastic (positivity) bounds. Positivity bounds that are not elastic are called *inelastic*. In the case where $\dim V = 3$ these are the bounds corresponding to the extreme ray of the third kind in Theorem 4.22. In the following (unless stated otherwise), we will assume that $\dim V = 3$. First, we will give an alternative description of all elastic bounds and show that it depends on fewer parameters. We will also discuss the effect of the inelastic bounds.

4.4.1 The elastic bounds

Recall that the elastic bounds are the inequalities corresponding to the extreme rays of the first and second kind. Using the symmetries of W^* , they can be summarized as

$$M(\alpha \otimes \beta, \alpha \otimes \beta) \geq 0, \quad \forall \alpha, \beta \in V^* \quad (4.14)$$

for every $M \in (\mathcal{C}_W)^*$. Note that the constraint (4.14) does not change under (non-zero) rescaling of α and β . Thus, the family of elastic bounds is parametrized by $\mathbb{RP}^2 \times \mathbb{RP}^2$ and is therefore four-dimensional.

In the following, we will see that the number of parameters needed to describe all bounds can be reduced further to two. For this, we fix a basis $\{\alpha_1, \alpha_2, \alpha_3\}$ and write $M_{ijkl} = M(\alpha_i, \alpha_j, \alpha_k, \alpha_l)$. Moreover, we use the isomorphism

$$\Phi: \mathbb{R}^9 \longrightarrow V^* \otimes V^* \quad (4.15)$$

defined by the basis $\{\alpha_i \otimes \alpha_j\}_{i,j=1}^3$ in alphabetic ordering. The following result is well-known:

Lemma 4.44. *A vector $u \in \mathbb{R}^9 \cong V^* \otimes V^*$ represents a decomposable tensor $\gamma \in V^* \otimes V^*$, i.e. $\gamma = \alpha \otimes \beta$ for some $\alpha, \beta \in V^*$, if and only if it satisfies the following relations:*

$$\begin{aligned} u_1 u_5 - u_2 u_4 &= 0, & u_1 u_6 - u_3 u_4 &= 0, & u_1 u_8 - u_2 u_7 &= 0, \\ u_1 u_9 - u_3 u_7 &= 0, & u_2 u_6 - u_3 u_5 &= 0, & u_2 u_9 - u_3 u_8 &= 0, \\ u_4 u_8 - u_5 u_7 &= 0, & u_4 u_9 - u_6 u_7 &= 0, & u_5 u_9 - u_6 u_8 &= 0. \end{aligned}$$

Proof. Given an element $\gamma \in V^* \otimes V^*$ we can think of it as a linear map $\gamma: V \rightarrow V^*$. Using the basis $\{\alpha_i\}_{i=1}^3$ we may then represent γ by a matrix. The condition that $\gamma = \alpha \otimes \beta$ for some $\alpha, \beta \in V^*$ is then equivalent to the statement that the determinant of all submatrices of order two must vanish. This yields precisely the above conditions on the components u_i of the vector $u \in \mathbb{R}^9 \cong V^* \otimes V^*$ representing γ . $\square$

Proposition 4.45. *Let $M \in W^*$. Then M satisfies the elastic bounds if and only if the following hold:*

$$\begin{aligned} M_{1111} &\geq 0, & M_{1212} &\geq 0, & M_{1313} &\geq 0, & M_{2222} &\geq 0, & M_{2323} &\geq 0, & M_{3333} &\geq 0, \\ q_1 q_4 - q_2^2 &\geq 0, & q_1 q_6 - q_3^2 &\geq 0, & q_4 q_6 - q_5^2 &\geq 0, \\ q_1 q_4 q_6 + 2 q_2 q_3 q_5 - q_1 q_5^2 - q_2^2 q_6 - q_3^2 q_4 &\geq 0, \end{aligned}$$

where the q_i are real quadratic polynomials in two variables given by

$$\begin{aligned} q_1(t_1, t_2) &:= M_{1111} + 2t_1 M_{1112} + 2t_2 M_{1113} + t_1^2 M_{1212} + 2t_1 t_2 M_{1213} + t_2^2 M_{1313}, \\ q_2(t_1, t_2) &:= M_{1112} + 2t_1 M_{1122} + 2t_2 M_{1123} + t_1^2 M_{1222} + 2t_1 t_2 M_{1223} + t_2^2 M_{1323}, \\ q_3(t_1, t_2) &:= M_{1113} + 2t_1 M_{1123} + 2t_2 M_{1133} + t_1^2 M_{1232} + 2t_1 t_2 M_{1233} + t_2^2 M_{1333}, \\ q_4(t_1, t_2) &:= M_{1212} + 2t_1 M_{1222} + 2t_2 M_{1232} + t_1^2 M_{2222} + 2t_1 t_2 M_{2223} + t_2^2 M_{2323}, \\ q_5(t_1, t_2) &:= M_{1213} + 2t_1 M_{1223} + 2t_2 M_{1233} + t_1^2 M_{2223} + 2t_1 t_2 M_{2233} + t_2^2 M_{2333}, \\ q_6(t_1, t_2) &:= M_{1313} + 2t_1 M_{1323} + 2t_2 M_{1333} + t_1^2 M_{2323} + 2t_1 t_2 M_{2333} + t_2^2 M_{3333}. \end{aligned}$$

Before proving Proposition 4.45, we will introduce the following notation for better readability:

$$\begin{aligned} m_1 &= M_{1111}; & m_2 &= M_{1112} = M_{1121} = M_{1211} = M_{2111}; \\ m_3 &= M_{1113} = M_{1131} = M_{1311} = M_{3111}; & m_4 &= M_{1122} = M_{1221} = M_{2112} = M_{2211}; \\ m_5 &= M_{1212} = M_{2121}; \\ m_6 &= M_{1123} = M_{1132} = M_{2311} = M_{3211} = M_{1321} = M_{3112} = M_{1231} = M_{2113}; \\ m_7 &= M_{1213} = M_{1312} = M_{2131} = M_{3121}; & m_8 &= M_{1133} = M_{1331} = M_{3113} = M_{3311}; \\ m_9 &= M_{1313} = M_{3131}; & m_{10} &= M_{1222} = M_{2122} = M_{2212} = M_{2221}; \\ m_{11} &= M_{1223} = M_{2132} = M_{2213} = M_{1322} = M_{2231} = M_{3122} = M_{3221} = M_{2312}; \\ m_{12} &= M_{1232} = M_{3212} = M_{2123} = M_{2321}; \\ m_{13} &= M_{1233} = M_{1332} = M_{2133} = M_{3123} = M_{3312} = M_{3213} = M_{3321} = M_{2331}; \\ m_{14} &= M_{1323} = M_{2313} = M_{3132} = M_{3231}; & m_{15} &= M_{1333} = M_{3133} = M_{3313} = M_{3331}; \\ m_{16} &= M_{2222}; & m_{17} &= M_{2223} = M_{2232} = M_{2322} = M_{3222}; \\ m_{18} &= M_{2233} = M_{2332} = M_{3223} = M_{3322}; & m_{19} &= M_{2323} = M_{3232}; \\ m_{20} &= M_{2333} = M_{3233} = M_{3323} = M_{3332}; & m_{21} &= M_{3333}. \end{aligned}$$

Proof of Proposition 4.45. By Lemma 4.44, $M \in W^*$ satisfies the elastic bounds if and only

if for every $u \in \mathbb{R}^9$ satisfying the constraints in Lemma 4.44 we have

$$\begin{aligned}
0 &\leq M(\Phi(u), \Phi(u)) \\
&= u_1^2 m_1 + 2u_1(u_2 + u_4) m_2 + 2u_1(u_3 + u_7) m_3 + 2(u_2 u_4 + u_1 u_5) m_4 + (u_2^2 + u_4^2) m_5 \\
&\quad + 2(u_3 u_4 + u_1 u_6 + u_2 u_7 + u_1 u_8) m_6 + 2(u_2 u_3 + u_4 u_7) m_7 + 2(u_3 u_7 + u_1 u_9) m_8 \\
&\quad + (u_3^2 + u_7^2) m_9 + 2(u_2 + u_4) u_5 m_{10} + 2(u_3 u_5 + u_2 u_6 + u_5 u_7 + u_4 u_8) m_{11} \\
&\quad + 2(u_4 u_6 + u_2 u_8) m_{12} + 2(u_6 u_7 + u_3 u_8 + u_2 u_9 + u_4 u_9) m_{13} + 2(u_3 u_6 + u_7 u_8) m_{14} \\
&\quad + 2(u_3 + u_7) u_9 m_{15} + u_5^2 m_{16} + 2u_5(u_6 + u_8) m_{17} + 2(u_6 u_8 + u_5 u_9) m_{18} \\
&\quad + (u_6^2 + u_8^2) m_{19} + 2(u_6 + u_8) u_9 m_{20} + u_9^2 m_{21},
\end{aligned}$$

where Φ is as in (4.15). Henceforth, we will use the shorter notation $M(u, u) = M(\Phi(u), \Phi(u))$. If all u_i but one vanish, this implies

$$m_1 \geq 0, \quad m_5 \geq 0, \quad m_9 \geq 0, \quad m_{16} \geq 0, \quad m_{19} \geq 0, \quad m_{21} \geq 0. \quad (4.16)$$

Now suppose that $u_i \neq 0$ for precisely two indices i . Note that not all choices of pairs of indices are allowed here, since e.g. the relation $u_1 u_5 - u_2 u_4 = 0$ prevents us from choosing $u_1, u_5 \neq 0$ when $u_2 = 0$. But, for allowed choices such as $u_1, u_2 \neq 0$, all other $u_i = 0$, the relations in Lemma 4.44 are trivially satisfied for all (non-zero) values of u_1 and u_2 . Therefore, in this case the condition $M(u, u) \geq 0$ is the same as requiring the restriction of M to the two-dimensional subspace $\text{span}\{e_1, e_2\} \subset \mathbb{R}^9$, where e_i are the standard orthonormal basis vectors of $\mathbb{R}^9$, to be positive semi-definite. Proceeding like this for all allowed choices of u with only two non-vanishing components we obtain the additional constraints

$$\begin{aligned}
m_1 m_5 - m_2^2 &\geq 0, \quad m_1 m_9 - m_3^2 \geq 0, \quad m_5 m_9 - m_7^2 \geq 0, \\
m_5 m_{16} - m_{10}^2 &\geq 0, \quad m_5 m_{19} - m_{12}^2 \geq 0, \quad m_9 m_{19} - m_{14}^2 \geq 0, \\
m_9 m_{21} - m_{15}^2 &\geq 0, \quad m_{19} m_{16} - m_{17}^2 \geq 0, \quad m_{19} m_{21} - m_{20}^2 \geq 0.
\end{aligned} \quad (4.17)$$

For allowed u with only three u_i non-vanishing the same reasoning yields the additional constraints

$$\begin{aligned}
2m_2 m_3 m_7 + m_1 m_5 m_9 - m_3^2 m_5 - m_1 m_7^2 - m_2^2 m_9 &\geq 0, \\
2m_{10} m_{12} m_{17} + m_{16} m_{19} m_5 - m_{12}^2 m_{16} - m_{10}^2 m_{19} - m_{17}^2 m_5 &\geq 0, \\
2m_{14} m_{15} m_{20} + m_{19} m_{21} m_9 - m_{15}^2 m_{19} - m_{14}^2 m_{21} - m_{20}^2 m_9 &\geq 0.
\end{aligned}$$

When four components of u are non-vanishing there will always remain a non-trivial constraint. For instance, if all components of u vanish but u_1, u_2, u_4 and u_5 , we set

$$a := u_1 u_5 = u_2 u_4 \quad \implies \quad u_5 = a/u_1, \quad u_4 = a/u_2.$$

The elastic constraints become

$$\begin{aligned}
0 &\leq a^2 (u_1^2 m_5 + 2u_1 u_2 m_{10} + u_2^2 m_{16}) + 2a (u_1^3 u_2 m_2 + 2u_1^2 u_2^2 m_4 + u_1 u_2^3 m_{10}) \\
&\quad + u_1^4 u_2^2 m_1 + 2u_1^3 u_2^3 m_2 + u_1^2 u_2^4 m_5
\end{aligned} \quad (4.18)$$

for all $a, u_1, u_2 \neq 0$. Note that by the constraint (4.17), the inequality also holds for $a = 0$,

as well as

$$u_1^2 m_5 + 2u_1 u_2 m_{10} + u_2^2 m_{16} \geq 0.$$

Thinking of the right hand-side of (4.18) as a quadratic form evaluated on $(1, a)$ we find that we get the additional constraint that for all $u_1, u_2 \in \mathbb{R} \setminus \{0\}$, we have

$$0 \leq (u_1^2 m_1 + 2u_1 u_2 m_2 + u_2^2 m_5) (u_1^2 m_5 + 2u_1 u_2 m_{10} + u_2^2 m_{16}) - (u_1^2 m_2 + 2u_1 u_2 m_4 + u_2^2 m_{10})^2.$$

Using again the previous constraints (or alternatively by continuity), we see that this still holds true for $u_1 = 0$ or $u_2 = 0$. We assume without loss of generality that $u_1 = 1$ and set $u_2 := t$. Then, after introducing the notation

$$P_{i,j,k}(t) := m_i + 2t m_j + t^2 m_k,$$

the constraint becomes

$$P_{1,2,5}(t) P_{5,10,16}(t) - P_{2,4,10}(t)^2 \geq 0.$$

Proceeding similarly for other u with precisely four non-vanishing components, we find moreover

$$\begin{aligned} P_{1,3,9}(t) P_{5,12,19}(t) - P_{2,6,14}(t)^2 &\geq 0, & P_{1,2,5}(t) P_{9,14,19}(t) - P_{3,6,12}(t)^2 &\geq 0, \\ P_{1,3,9}(t) P_{9,15,21}(t) - P_{3,8,15}(t)^2 &\geq 0, & P_{5,7,9}(t) P_{16,17,19}(t) - P_{10,11,14}(t)^2 &\geq 0, \\ P_{5,7,9}(t) P_{19,20,21}(t) - P_{12,13,15}(t)^2 &\geq 0, & P_{5,10,16}(t) P_{9,14,19}(t) - P_{7,11,16}(t)^2 &\geq 0, \\ P_{5,12,19}(t) P_{9,15,21}(t) - P_{7,13,20}(t)^2 &\geq 0, & P_{16,17,19}(t) P_{19,20,21}(t) - P_{17,18,20}(t)^2 &\geq 0. \end{aligned}$$

Note that these already imply the quadratic constraints (4.17) above.

There are no u satisfying the relations of Lemma 4.44 with precisely 5, 7 or 8 non-vanishing components. If there are precisely six non-vanishing components, e.g. $u_i \neq 0$ for $i \leq 6$, $u_i = 0$ else, we set

$$\begin{aligned} a &:= u_1 u_5 = u_2 u_4, & u_1 u_6 - u_3 u_4 &= 0, & u_2 u_6 - u_3 u_5 &= 0 \\ \implies & u_4 = a/u_2, & u_5 &= a/u_1, & u_6 &= \frac{u_3}{u_1 u_2} a. \end{aligned}$$

Following the same strategy as above, we find that for all $t_1, t_2 \in \mathbb{R}$

$$P_{1,2,3,5,7,9}(t_1, t_2) P_{5,10,12,16,17,19}(t_1, t_2) - P_{2,4,6,10,11,14}(t_1, t_2)^2 \geq 0,$$

where

$$P_{i,j,k,l,p,q}(t_1, t_2) := m_i + 2t_1 m_j + 2t_2 m_k + t_1^2 m_l + 2t_1 t_2 m_p + t_2^2 m_q.$$

Similarly, we find that for all $t_1, t_2 \in \mathbb{R}$ it holds that

$$\begin{aligned} P_{1,2,3,5,7,9}(t_1, t_2) P_{9,14,15,19,20,21}(t_1, t_2) - P_{3,6,8,12,13,15}(t_1, t_2)^2 &\geq 0, \\ P_{5,10,12,16,17,19}(t_1, t_2) P_{9,14,15,19,20,21}(t_1, t_2) - P_{7,11,13,17,18,20}(t_1, t_2)^2 &\geq 0. \end{aligned}$$

Note that when setting either $t_1 = 0$ or $t_2 = 0$, these imply the quadratic constraints above.

In the case where none of the components of u vanish we set

$$a := u_1 u_5 = u_2 u_4, \quad b := u_1 u_8 = u_2 u_7.$$

Using all constraints from Lemma 4.44 we find that

$$u_4 = \frac{a}{u_2}, \quad u_5 = \frac{a}{u_1}, \quad u_6 = \frac{u_3}{u_1 u_2} a, \quad u_7 = \frac{b}{u_2}, \quad u_8 = \frac{b}{u_1}, \quad u_9 = \frac{u_3}{u_1 u_2} b.$$

We assume without loss of generality that $u_1 = 1$ and set $u_2 := t_1$, $u_3 := t_2$. Then the elastic constraints become

$$\begin{aligned} 0 \leq & P_{1,2,3,5,7,9}(t_1, t_2) + 2a \frac{1}{t_1} P_{2,4,6,10,11,14}(t_1, t_2) + 2b \frac{1}{t_1} P_{3,6,8,12,13,15}(t_1, t_2) \\ & + a^2 \frac{1}{t_1^2} P_{5,10,12,16,17,19}(t_1, t_2) + 2ab \frac{1}{t_1^2} P_{7,11,13,17,18,20}(t_1, t_2) + b^2 \frac{1}{t_1^2} P_{9,14,15,19,20,21}(t_1, t_2) \end{aligned}$$

for all $t_1, t_2 \neq 0$. Using the previous constraints, this is equivalent to

$$\begin{aligned} 0 \leq & P_{1,2,3,5,7,9}(t_1, t_2) P_{5,10,12,16,17,19}(t_1, t_2) P_{9,14,15,19,20,21}(t_1, t_2) \\ & + 2 P_{2,4,6,10,11,14}(t_1, t_2) P_{3,6,8,12,13,15}(t_1, t_2) P_{7,11,13,17,18,20}(t_1, t_2) \\ & - P_{1,2,3,5,7,9}(t_1, t_2) P_{7,11,13,17,18,20}(t_1, t_2)^2 - P_{2,4,6,10,11,14}(t_1, t_2)^2 P_{9,14,15,19,20,21}(t_1, t_2) \\ & - P_{3,6,8,12,13,15}(t_1, t_2)^2 P_{5,10,12,16,17,19}(t_1, t_2) \end{aligned}$$

for all $t_1, t_2 \in \mathbb{R} \setminus \{0\}$. Moreover, by the previous constraints, the inequality remains true also for $t_1 = 0$ or $t_2 = 0$. Furthermore, we note that this constraint implies the cubic constraints above when setting $t_1, t_2 = 0$, or setting $t_1 = 0$, $t_2 \rightarrow \infty$ or $t_2 = 0$, $t_1 \rightarrow \infty$. Observing that the q_i in the statement agree with the P 's appearing in the constraints, the assertion follows. $\square$

4.4.2 The inelastic bounds

After fixing some basis $\{\alpha_1, \alpha_2, \alpha_3\}$ of V^* we can parameterize the extreme rays of the third kind by the parameters d, g, h as above plus some $A \in \text{SL}(V) \cong \text{SL}(3, \mathbb{R})$ via

$$S(A, d, g, h) = A^* S(\text{id}, d, g, h),$$

where

$$\begin{aligned} S(\text{id}, d, g, h) = & \alpha_1^4 + 6\alpha_1^2\alpha_2^2 + 6\alpha_1^2\alpha_3^2 + 12d\alpha_1\alpha_2^2\alpha_3 + 12g\alpha_1\alpha_2\alpha_3^2 + 4h\alpha_1\alpha_3^3 + (1+d^2)\alpha_2^4 \\ & + 4dg\alpha_2^3\alpha_3 + 6(1+dh)\alpha_2^2\alpha_3^2 + 4g(d+h)\alpha_2\alpha_3^3 + (1+g^2+h^2)\alpha_3^4 \\ & + 2(g^2+d^2-1-dh)[(\alpha_2 \otimes \alpha_3)^{\otimes 2} + (\alpha_3 \otimes \alpha_2)^{\otimes 2}]. \end{aligned}$$

Then the inelastic bounds become

$$0 \leq S(A, d, g, h)(M) = A^* S(\text{id}, d, g, h)(M) = S(\text{id}, d, g, h)(A \cdot M),$$

for all $A \in \text{SL}(V)$ and $d, g, h \in \mathbb{R}$ such that $g^2 + d^2 - 1 - dh > 0$. Here, $A \cdot M$ denotes the induced action of A on M .

4.4.3 Elastic vs. inelastic bounds

The extremal elements of the third kind in Theorem 4.22 give rise to new constraints. A way to see this is as follows: We denote by $\mathcal{C}_{\text{el}}$ the closed convex cone generated by the directions of the elastic extreme rays of $\mathcal{C}_W$. Then $\mathcal{C}_{\text{el}} \subsetneq \mathcal{C}_W$, since the extreme rays of $\mathcal{C}_W$ of the third kind in Theorem 4.22 are not contained in $\mathcal{C}_{\text{el}}$. Suppose now that the inelastic extreme rays in $\mathcal{C}_W$ do not give rise to any new constraint on the dual cone $\mathcal{C}_W^*$, i.e. $\mathcal{C}_W^* = \mathcal{C}_{\text{el}}^*$. Then we have $\mathcal{C}_W^{**} = \mathcal{C}_{\text{el}}^{**}$. Since both $\mathcal{C}_W$ and $\mathcal{C}_{\text{el}}$ are closed convex cones, it follows that

$$\mathcal{C}_W = \mathcal{C}_W^{**} = \mathcal{C}_{\text{el}}^{**} = \mathcal{C}_{\text{el}},$$

a contradiction.

Figure 4.1 illustrates the difference between $\mathcal{C}_W^*$ and $\mathcal{C}_{\text{el}}^*$.

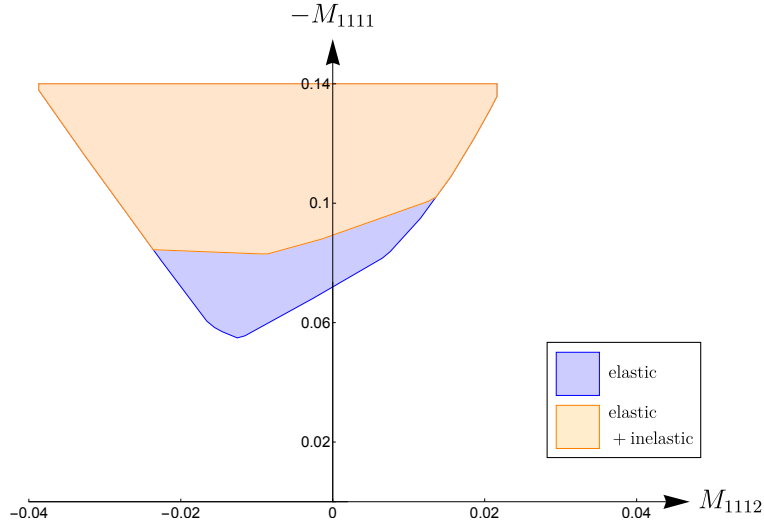


Figure 4.1: Regions of parameters consistent with positivity bounds. The union of the blue and orange regions corresponds to points which abide by the elastic bounds and the orange region corresponds to imposing all (elastic and inelastic) bounds. The 21-dimensional parameter space has been restricted to M_{1111} and M_{1112} , more precisely to the perturbations in their direction around the following reference point: $M_{1111} = 0.332, M_{1112} = 0.022, M_{1113} = 0.12, M_{1122} = 0.056, M_{1123} = 0.086, M_{1133} = -0.2, M_{1212} = 0.428, M_{1213} = -0.252, M_{1222} = 0.12, M_{1223} = -0.184, M_{1232} = -0.172, M_{1233} = -0.04, M_{1313} = 0.356, M_{1323} = -0.05, M_{1333} = -0.124, M_{2222} = 0.476, M_{2223} = 0.12, M_{2233} = -0.14, M_{2323} = 0.626, M_{2333} = -0.028, M_{3333} = 0.332$. The exact regions have been approximated by numerically sampling extreme rays, with a focus on the neighborhood of the extreme rays which vanish when evaluated on the reference point. The choice of reference point, as well as that of the directions of the perturbations around it, were heavily inspired by [Li+21]. Consequently, this figure is qualitatively similar to Figure 3 in the supplemental material of [Li+21].

4.5 Bounds on amplitudes with symmetries

We will now study the spaces of amplitudes that are invariant under the action of some symmetry group. More precisely, for some group G acting on the space W^* , we would like

to determine the intersection of the space

$$(\mathcal{C}_W^*)^G = \{M \in \mathcal{C}_W^* : A \cdot M = M \ \forall A \in G\} = \mathcal{C}_W^* \cap (W^*)^G$$

of G -invariant elements in W^* with the cone $\mathcal{C}_W^*$. Note that $(\mathcal{C}_W^*)^G = (W^G \cap \mathcal{C}_W^*)^*$, where W^G denotes the space of elements in W that are invariant under the action of G induced by the action on W^* .

As in Section 4.4 we will assume that $\dim V = 3$ and study the cases where $G = \mathrm{O}(3)$, $G = \mathbb{Z}_2^3$ and $G = \mathrm{SO}(2)$ and such that the action on W^* is induced by some action of G on V . In each of these cases we will see that the elastic bounds are sufficient in order to describe the cone $(\mathcal{C}_W^*)^G$.

4.5.1 $\mathrm{O}(3)$ -invariant amplitudes and chiral perturbation theory

We will now consider amplitudes that are invariant under some $\mathrm{O}(3)$ -symmetry. Fixing a basis $\{e_i\}_{i=1}^3$ of V we will identify $V \cong \mathbb{R}^3$ and consider the group $\mathrm{O}(3) = \mathrm{O}(V)$. The action of $\mathrm{O}(3)$ on V induces an action on W^* . In the following, we will consider the cone $(\mathcal{C}_W^*)^{\mathrm{O}(3)}$ of invariant elements in $\mathcal{C}_W^*$.

Theorem 4.46. *Let $M \in (W^*)^{\mathrm{O}(3)}$. Then $M \in \mathcal{C}_W^*$ if and only if it satisfies the elastic positivity bounds.*

Proof. Every four-tensor $M \in V^{*\otimes 4}$ that is invariant under the action of $\mathrm{O}(3)$ is of the form [Jef62]

$$\begin{aligned} M &= a_1 (e_1^2 + e_2^2 + e_3^2)^{\otimes 2} + a_2 (e_1^4 + e_2^4 + e_3^4 + 2(e_1 \vee e_2)^{\otimes 2} + 2(e_1 \vee e_3)^{\otimes 2} + 2(e_2 \vee e_3)^{\otimes 2}) \\ &\quad + a_3 ((e_1 \wedge e_2)^{\otimes 2} + (e_1 \wedge e_3)^{\otimes 2} + (e_2 \wedge e_3)^{\otimes 2}) \\ &= (a_1 + a_2) (e_1^4 + e_2^4 + e_3^4) \\ &\quad + 2a_1 (e_1^2 \vee e_2^2 + 2(e_1 \vee e_2)^{\otimes 2} + e_1^2 \vee e_3^2 + 2(e_1 \vee e_3)^{\otimes 2} + e_2^2 \vee e_3^2 + 2(e_2 \vee e_3)^{\otimes 2}) \\ &\quad + (2a_2 - 4a_1) ((e_1 \vee e_2)^{\otimes 2} + (e_1 \vee e_3)^{\otimes 2} + (e_2 \vee e_3)^{\otimes 2}) \\ &\quad + a_3 ((e_1 \wedge e_2)^{\otimes 2} + (e_1 \wedge e_3)^{\otimes 2} + (e_2 \wedge e_3)^{\otimes 2}). \end{aligned}$$

Note that the terms with the prefactors $(a_1 + a_2)$ and $2a_1$ in the last equality are fully symmetric tensors. This shows that M is invariant under τ (and therefore contained in W^*) if and only if $a_3 = 2a_2 - 4a_1$.

From now on, we will assume that $a_3 = 2a_2 - 4a_1$. Then, using Proposition 4.45, it follows that the elastic bounds for M are equivalent to $a_2 \geq |a_1|$.

Next, we would like to derive the inelastic bounds. Note that since M is invariant under $\mathrm{O}(3)$, using the polar decomposition it is enough to consider $A \in \mathrm{Sym}^2(V)$, $A > 0$ in the parameterization given in Section 4.4.2. We may furthermore assume without loss of generality that the orthonormal basis vectors $\{e_i\}_{i=1}^3$ are eigenvectors of A with eigenvalues $\lambda_i > 0$. Then we find

$$\begin{aligned} A \cdot M &= (a_1 + a_2) \sum_i \lambda_i^4 e_i^4 + 2a_1 \sum_{i < j} \lambda_i^2 \lambda_j^2 e_i^2 \vee e_j^2 + 2a_2 \sum_{i < j} \lambda_i^2 \lambda_j^2 (e_i \vee e_j)^{\otimes 2} \\ &\quad + (2a_2 - 4a_1) \sum_{i < j} \lambda_i^2 \lambda_j^2 (e_i \wedge e_j)^{\otimes 2}. \end{aligned}$$

Hence, the inelastic bounds become

$$\begin{aligned} 0 &\leq S(\text{id}, d, g, h)(A \cdot M) \\ &= (a_1 + a_2) \{ \lambda_1^4 + 2\lambda_1^2\lambda_2^2 + 2\lambda_1^2\lambda_3^2 + (1 + d^2) \lambda_2^4 + 2(1 + dh) \lambda_2^2\lambda_3^2 + (1 + g^2 + h^2) \lambda_3^4 \} \\ &\quad + 4(a_2 - a_1)(g^2 + d^2 - 1 - dh) \lambda_2^2\lambda_3^2 \end{aligned}$$

for all $g, d, h \in \mathbb{R}$ such that $g^2 + d^2 - 1 - dh > 0$ and $\lambda_i > 0$. However, these inequalities already follow from the elastic bounds: If $a_2 \geq |a_1|$, we find that

$$\begin{aligned} S(\text{id}, d, g, h)(A \cdot M) &\geq (a_1 + a_2) \{ d^2 \lambda_2^4 + 2dh \lambda_2^2\lambda_3^2 + h^2 \lambda_3^4 \} \\ &\geq (a_1 + a_2) \{ d^2 \lambda_2^4 - 2|dh| \lambda_2^2\lambda_3^2 + h^2 \lambda_3^4 \} \\ &= (a_1 + a_2)(|d| \lambda_2^2 - |h| \lambda_3^2)^2 \geq 0. \end{aligned}$$

□

Application 4.47 (Chiral perturbation theory). Recall from Section 2.4 that the four-tensor M associated to the amplitude for pion two-to-two scattering at low energies is given by

$$\begin{aligned} M &= (2\ell_1 + 3\ell_2) \{ e_1^4 + e_2^4 + e_3^4 + 2(e_1 \vee e_2)^{\otimes 2} + 2(e_1 \vee e_3)^{\otimes 2} + 2(e_2 \vee e_3)^{\otimes 2} \} \\ &\quad + (2\ell_1 + \ell_2)(e_1^2 + e_2^2 + e_3^2)^{\otimes 2} + 2(\ell_2 - 2\ell_1) \{ (e_1 \wedge e_2)^{\otimes 2} + (e_1 \wedge e_3)^{\otimes 2} + (e_2 \wedge e_3)^{\otimes 2} \}, \end{aligned}$$

where ℓ_1 and ℓ_2 are free parameters. Note that the four-tensor M is $O(3)$ -invariant (as is the pion scattering amplitude). By Theorem 4.46 it follows that the elastic positivity bounds $\ell_2 \geq -\ell_1$, $\ell_2 \geq 0$ are the strongest positivity bounds that can be derived from two-to-two forward scattering of pions.

4.5.2 $\mathbb{Z}_2^3$ -invariant amplitudes

We will now study the case where the amplitude is invariant under the action of $\mathbb{Z}_2^3$ on V generated by $Q_i \in \text{End}(V)$, $i \in \{1, 2, 3\}$, which are represented by the matrices

$$Q_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with respect to some fixed basis $\{e_i\}_{i=1}^3$ of V . Introducing the notation

$$U_0 := \text{span}\{e_1^2, e_2^2, e_3^2\}, \quad U_{ij} := \text{span}\{e_i \otimes e_j, e_j \otimes e_i\}, \quad i < j,$$

we have

$$(\text{Sym}^2(V^{\otimes 2}))^{\mathbb{Z}_2^3} = \text{Sym}^2 U_0 \oplus \text{Sym}^2 U_{12} \oplus \text{Sym}^2 U_{13} \oplus \text{Sym}^2 U_{23},$$

and $M \in (W^*)^{\mathbb{Z}_2^3}$ if and only if $M \in (\text{Sym}^2(V^{\otimes 2}))^{\mathbb{Z}_2^3}$ and for all i, j we have

$$M(\alpha_i^{\otimes 2}, \alpha_j^{\otimes 2}) = M(\alpha_i \otimes \alpha_j, \alpha_j \otimes \alpha_i), \quad M(\alpha_i \otimes \alpha_j, \alpha_i \otimes \alpha_j) = M(\alpha_j \otimes \alpha_i, \alpha_j \otimes \alpha_i),$$

where $\{\alpha_i\}_{i=1}^3$ denotes the dual basis of $\{e_i\}_{i=1}^3$.

As in the case of the $O(3)$ -symmetric amplitudes, it will turn out that the elastic bounds are sufficient to describe the whole cone $(\mathcal{C}_W^*)^{\mathbb{Z}_2^3}$. Since $\mathbb{Z}_2^3$ is a subgroup of $O(3)$ this is a strict relaxation of Theorem 4.46. For the proof, we will however follow a different approach. First, we prove the following lemma.

Lemma 4.48. *Let $p: W \rightarrow W^{\mathbb{Z}_2^3}$ be the orthogonal projection. Then for every $S \in \mathcal{C}_W$, we have $\text{rank } p(S) \geq \text{rank } S$, where $\text{rank } p(S)$ denotes the rank of $p(S)$ as an element in $\text{Sym}^2(V^* \otimes V^*)$. Similarly, we have $\text{rank } R(p(S)) \geq \text{rank } R(S)$ and $\text{rank } R'(p(S)) \geq \text{rank } R'(S)$, where R and R' are as defined in Section 4.2.*

Proof. Let $S \in \mathcal{C}_W$. Observe that the projection $p(S)$ is given by removing the entries of S away from the blocks $\text{Sym}^2 U_0$ and $\text{Sym}^2 U_{ij}$ in the above notation. Thus, $p(S)$ is block-diagonal of the form

$$p(S) = S|_{U_0} + S|_{U_{12}} + S|_{U_{13}} + S|_{U_{23}}.$$

Suppose that $\text{rank } S = k$. Since by assumption S is positive semi-definite this means that there must be at least one principal submatrix A of order k which is positive definite (see e.g. [HJ13]). The corresponding principal submatrix of $p(S)$ will be block-diagonal and its determinant will be a product of principal minors of A . Hence, if it was zero, this would imply that at least one of the principal minors of A must vanish, in contrast to the assumption that A is positive definite. Therefore, $p(S)$ has a non-vanishing principal minor of order k and hence $\text{rank } p(S) \geq k = \text{rank } S$. The statement about $\text{rank } R(p(S))$ and $\text{rank } R'(p(S))$ follows analogously, after decomposing the spaces U_{ij} into the symmetric and antisymmetric part, which are orthogonal with respect to $p(S)$ since $S \in W$. $\square$

Theorem 4.49. *Let $M \in (W^*)^{\mathbb{Z}_2^3}$. Then $M \in \mathcal{C}_W^*$ if and only if it satisfies the elastic bounds.*

Proof. First, we observe that $(\mathcal{C}_W^*)^{\mathbb{Z}_2^3} = (\mathcal{C}_W^{\mathbb{Z}_2^3})^*$. We prove that the projection of an extremal element of the third kind in Theorem 4.22 is never extremal. Since by Proposition 3.25 every extremal element in $\mathcal{C}_W^{\mathbb{Z}_2^3}$ is the projection of some extremal element in $\mathcal{C}_W$, this will show that every extremal element in $\mathcal{C}_W^{\mathbb{Z}_2^3}$, and therefore every necessary inequality describing $(\mathcal{C}_W^*)^{\mathbb{Z}_2^3}$, comes from an elastic extremal element in $\mathcal{C}_W$.

Let $S \in \mathcal{C}_W$ be some extremal element in $\mathcal{C}_W$ of the third kind in Theorem 4.22. Then $\text{rank } S = 4$, $\text{rank } R(S) = 3$, $\text{rank } R'(S) = 1$ (cf. Theorem 4.41), and it follows from Lemma 4.48 that $\text{rank } p(S) \geq \text{rank } S = 4$, $\text{rank } R(p(S)) \geq 3$ and $\text{rank } R'(p(S)) \geq 1$. We will consider separately the cases of different rank of the restrictions $p(S)|_{U_0}$ and $p(S)|_{U_{ij}}$.

Suppose first that $p(S)|_{U_0} > 0$ or $p(S)|_{U_{ij}} > 0$ for some indices i, j with $i < j$. Then, there are some $i, j \in \{1, 2, 3\}$ such that

$$\ker [(\alpha_i \otimes \alpha_j)^{\otimes 2} + (\alpha_j \otimes \alpha_i)^{\otimes 2}] \supsetneq \ker p(S),$$

where $\{\alpha_i\}$ is the basis of V^* dual to $\{e_i\}$. By Corollary 3.32 this shows that $p(S)$ is not extremal in $p(\mathcal{C}_W)$. We will from now on assume that $\text{rank } p(S)|_{U_0} \leq 2$ and $\text{rank } p(S)|_{U_{ij}} \leq 1$ for all $i < j$.

The case where $\text{rank } p(S)|_{U_0} = 2$ and $\text{rank } p(S)|_{U_{ij}} \leq 1$ for all $i < j$ will be treated in Proposition 4.50 below, where we show that such $p(S)$ is not extremal.

Finally, suppose that $\text{rank } p(S)|_{U_0} = 1$ and $\text{rank } p(S)|_{U_{ij}} \leq 1$. Since $\text{rank } p(S) \geq 4$, this implies $\text{rank } p(S)|_{U_{ij}} = 1$, for all i, j . We show that this is impossible for S as above. To see this, we observe that for $p(S) \in \mathcal{C}_W^{\mathbb{Z}_2^3}$ satisfying $\text{rank } p(S)|_{U_0} = 1 = \text{rank } p(S)|_{U_{ij}}$ for all $i < j$, there must be $\lambda_i \neq 0$, $i = 1, 2, 3$ such that

$$p(S) = \left(\sum_{i=1}^3 \lambda_i \alpha_i^{\otimes 2} \right)^{\otimes 2} + \sum_{i < j} |\lambda_i \lambda_j| (\alpha_i \otimes \alpha_j + \text{sgn}(\lambda_i \lambda_j) \alpha_j \otimes \alpha_i)^{\otimes 2}.$$

If $\lambda_i \lambda_j > 0$ for all $i, j \in \{1, 2, 3\}$, then we observe that $R'(p(S)) = 0$, which is impossible as we must have $\text{rank } R'(p(S)) \geq 1$. If there is some pair $i < j$ such that $\lambda_i \lambda_j < 0$ there must be precisely two such pairs and we find $\text{rank } R(p(S)) = 2$ (as $\text{rank } p(S) = 4$), in contrast to $\text{rank } R(p(S)) \geq 3$. This completes the proof. $\square$

Proposition 4.50. *Let $S \in \mathcal{C}_W$ be an extremal element of the third kind in Theorem 4.22. Suppose furthermore that $\text{rank } p(S)|_{U_0} = 2$ and $\text{rank } p(S)|_{U_{ij}} \leq 1$ for all $i < j$. Then $p(S)$ is not extremal in $p(\mathcal{C}_W)$.*

Proof. We will first show that we must have $\text{rank } p(S)|_{U_{ij}} = 1$ for every pair $i < j$. Indeed, recall that by Theorem 4.41 every inelastic extremal element S in $\mathcal{C}_W$ has $\text{rank } R(S) = 3$, $\ker R'(S) = z \wedge V$ for some $z \in V \setminus \{0\}$ and $z \vee V \cap \ker S = \{0\}$. If there was some pair $i < j$ such that $p(S)|_{U_{ij}} = 0$ (there can be at most one such pair since $\text{rank } p(S) \geq 4$), then we would have $\{e_i \vee e_j, e_i \wedge e_j\} \subset \ker p(S)$ and hence also $\{e_i \vee e_j, e_i \wedge e_j\} \subset \ker S$. Moreover, by Lemma 4.48 and the properties of S we know that $\text{rank } R(p(S)) \geq 3$ and $\text{rank } R'(p(S)) \geq 1$, hence $\text{rank } R(p(S)) = 3$ and $\text{rank } R'(p(S)) = 1$, as we assumed that $\text{rank } p(S) = 4$. In particular, for the unique $k \in \{1, 2, 3\}$ such that $i \neq k \neq j$, we must have $e_i \wedge e_k \in \ker p(S)$ or $e_j \wedge e_k \in \ker p(S)$ (here we have used that the kernel of $p(S)$ is the sum of the kernels of its restrictions to U_0, U_{12}, U_{13} and U_{23}). Hence, either $\{e_i \wedge V, e_i \vee e_j\} \subset \ker S$ or $\{e_j \wedge V, e_i \vee e_j\} \subset \ker S$, a contradiction.

Next, we observe that in the case where $\text{rank } p(S)|_{U_0} = 2$, $\text{rank } p(S)|_{U_{ij}} = 1$ for all i, j , there are some linearly independent vectors $(\lambda_i), (\mu_i) \in \mathbb{R}^3 \setminus \{0\}$ such that

$$p(S) = \left(\sum_{i=1}^3 \lambda_i \alpha_i^{\otimes 2} \right)^{\otimes 2} + \left(\sum_{i=1}^3 \mu_i \alpha_i^{\otimes 2} \right)^{\otimes 2} + \sum_{i < j} |\lambda_i \lambda_j + \mu_i \mu_j| (\alpha_i \otimes \alpha_j + \text{sgn}(\lambda_i \lambda_j + \mu_i \mu_j) \alpha_j \otimes \alpha_i)^{\otimes 2}.$$

Note that since $\text{rank } p(S)|_{U_{ij}} = 1$ for all pairs $i < j$, we must have $\lambda_i \lambda_j + \mu_i \mu_j \neq 0$.

Case 1: If $\text{sgn}(\lambda_i \lambda_j) = \text{sgn}(\lambda_i \lambda_j + \mu_i \mu_j)$ for all i, j , we note that

$$T := \left(\sum_{i=1}^3 \lambda_i \alpha_i^{\otimes 2} \right)^{\otimes 2} + \sum_{i < j} |\lambda_i \lambda_j| (\alpha_i \otimes \alpha_j + \text{sgn}(\lambda_i \lambda_j) \alpha_j \otimes \alpha_i)^{\otimes 2}$$

satisfies $\ker T \supsetneq \ker p(S)$. Since $T \in p(\mathcal{C}_W)$ this shows that $p(S)$ is not extremal in $p(\mathcal{C}_W)$. In the case where $\text{sgn}(\mu_i \mu_j) = \text{sgn}(\lambda_i \lambda_j + \mu_i \mu_j)$ for all i, j it follows similarly that $p(S)$ is not extremal.

Case 2: Now suppose that $\lambda_i \neq 0$ and $\mu_i \neq 0$ for all i and that there are $i \neq j, k \neq l \in \{1, 2, 3\}$ such that $\text{sgn}(\lambda_i \lambda_j + \mu_i \mu_j) \neq \text{sgn}(\lambda_i \lambda_j)$ and $\text{sgn}(\mu_k \mu_l) \neq \text{sgn}(\lambda_k \lambda_l + \mu_k \mu_l)$. Up to

a permutation we will assume without loss of generality that these two pairs are $(1, 2)$ and $(1, 3)$. Since we assumed that $\lambda_i, \mu_i \neq 0$ for all i , we must have $\text{sgn}(\lambda_1\lambda_2 + \mu_1\mu_2) = \text{sgn}(\mu_1\mu_2)$ and $\text{sgn}(\lambda_1\lambda_3) = \text{sgn}(\lambda_1\lambda_3 + \mu_1\mu_3)$. Then there is some $t > 1$, such that for some pair $k < l$ we have $\lambda_k\lambda_l + t\mu_k\mu_l = 0$ and, for the other pairs i, j we have $\lambda_i\lambda_j + t\mu_i\mu_j = 0$ or $\text{sgn}(\lambda_i\lambda_j + t\mu_i\mu_j) = \text{sgn}(\lambda_i\lambda_j + \mu_i\mu_j)$ (note that the pair (k, l) can be only $(1, 3)$ or $(2, 3)$). In particular, we find that

$$T_t := \left(\sum_{i=1}^3 \lambda_i \alpha_i^{\otimes 2} \right)^{\otimes 2} + \left(\sum_{i=1}^3 \sqrt{t} \mu_i \alpha_i^{\otimes 2} \right)^{\otimes 2} + \sum_{i < j} |\lambda_i \lambda_j + t \mu_i \mu_j| (\alpha_i \otimes \alpha_j + \text{sgn}(\lambda_i \lambda_j + t \mu_i \mu_j) \alpha_j \otimes \alpha_i)^{\otimes 2}$$

satisfies $\ker T_t \supsetneq \ker p(S)$ and hence $p(S)$ cannot be extremal in $p(\mathcal{C}_W)$.

Case 3: It remains to consider the case where there is some i such that $\lambda_i = 0$ or $\mu_i = 0$. If only one of the λ_i is non-vanishing, we may assume after relabeling that $\lambda_1 \neq 0, \lambda_2 = 0 = \lambda_3$. But then it follows that $\ker \alpha_1^{\otimes 4} \supsetneq \ker p(S)$ and thus $p(S)$ is not extremal. If only one of the λ_i is vanishing, we assume after relabeling that it is λ_3 . Then we know that $\text{sgn}(\lambda_1\lambda_3 + \mu_1\mu_3) = \text{sgn}(\mu_1\mu_3)$ and $\text{sgn}(\lambda_2\lambda_3 + \mu_2\mu_3) = \text{sgn}(\mu_2\mu_3)$. If also $\text{sgn}(\mu_1\mu_2) = \text{sgn}(\lambda_1\lambda_2 + \mu_1\mu_2)$, we are in case 1 above and we already know that $p(S)$ is not extremal. If $\text{sgn}(\mu_1\mu_2) \neq \text{sgn}(\lambda_1\lambda_2 + \mu_1\mu_2)$ then since $\lambda_1\lambda_2 + \mu_1\mu_2 \neq 0$, we must have $\text{sgn}(\lambda_1\lambda_2) = \text{sgn}(\lambda_1\lambda_2 + \mu_1\mu_2)$. But then it follows that

$$T := \left(\sum_{i=1}^3 \lambda_i \alpha_i^{\otimes 2} \right)^{\otimes 2} + \sum_{i < j} |\lambda_i \lambda_j| (\alpha_i \otimes \alpha_j + \text{sgn}(\lambda_i \lambda_j) \alpha_j \otimes \alpha_i)^{\otimes 2}$$

satisfies $\ker T \supsetneq \ker p(S)$ and hence S is not extremal. The case where $\mu_i = 0$ for some $i \in \{1, 2, 3\}$ works analogously. $\square$

Remark 4.51. In the appendix of [Li+21], a classification of all extreme rays of the cone of $\mathbb{Z}_2^3$ -invariant amplitudes is given. Theorem 4.49 can then alternatively be proved providing explicit elastic extremal elements in $\mathcal{C}_W$ projecting to the extremal elements in the classification result of [Li+21]. More precisely, in [Li+21], it is shown that if $T \in \mathcal{C}_W^{\mathbb{Z}_2^3}$ is extremal in $\mathcal{C}_W^{\mathbb{Z}_2^3}$ then either there is some $\mu \in \mathbb{R}, i \neq j$ such that

$$T = \mu [(\alpha_i \otimes \alpha_j)^{\otimes 2} + (\alpha_j \otimes \alpha_i)^{\otimes 2}]$$

or for some $(\lambda_i) \in \mathbb{R}^3 \setminus \{0\}$ we have

$$T = \left(\sum_{i=1}^3 \lambda_i \alpha_i^{\otimes 2} \right)^{\otimes 2} + \sum_{i < j} |\lambda_i \lambda_j| (\alpha_i \otimes \alpha_j + \text{sgn}(\lambda_i \lambda_j) \alpha_j \otimes \alpha_i)^{\otimes 2}.$$

In the first case T is obviously the projection of an elastic extremal element in $\mathcal{C}_W$. Moreover, if in the second case only one of the λ_i is non-vanishing, then clearly $T = \mu \alpha_i^{\otimes 4}$ for some $i \in \{1, 2, 3\}$, which again means that it is the projection of an elastic extremal element in $\mathcal{C}_W$.

Now suppose that there is precisely one $i \in \{1, 2, 3\}$ such that $\lambda_i = 0$. We may assume

without loss of generality that $\lambda_3 = 0$, $\lambda_1 \lambda_2 \neq 0$. If $\lambda_1 \lambda_2 > 0$ then we observe that

$$T = p \left((\sqrt{|\lambda_1|} \alpha_1 + \sqrt{|\lambda_2|} \alpha_2)^{\otimes 4} \right).$$

If $\lambda_1 \lambda_2 < 0$, then we set

$$\alpha := \sqrt{|\lambda_1|} \alpha_1 + \sqrt{|\lambda_2|} \alpha_2, \quad \beta := \sqrt{|\lambda_1|} \alpha_1 - \sqrt{|\lambda_2|} \alpha_2$$

and observe that

$$T = p \left((\alpha \vee \beta)^{\otimes 2} + (\alpha \wedge \beta)^{\otimes 2} \right).$$

Finally we suppose that $\lambda_i \neq 0$ for all $i \in \{1, 2, 3\}$. Then either $\lambda_i \lambda_j > 0$ for all $i, j \in \{1, 2, 3\}$ and in this case we find

$$T = p \left((\sqrt{|\lambda_1|} \alpha_1 + \sqrt{|\lambda_2|} \alpha_2 + \sqrt{|\lambda_3|} \alpha_3)^{\otimes 4} \right)$$

or $\lambda_i \lambda_j < 0$ for precisely two pairs $i \neq j$. In the latter case we may assume that $\lambda_1, \lambda_2 > 0$ and $\lambda_3 < 0$. Then for

$$\alpha := \sqrt{\lambda_1} \alpha_1 + \sqrt{\lambda_2} \alpha_2 + \sqrt{-\lambda_3} \alpha_3, \quad \beta := \sqrt{\lambda_1} \alpha_1 + \sqrt{\lambda_2} \alpha_2 - \sqrt{-\lambda_3} \alpha_3$$

we find

$$p(S) = p \left((\alpha \vee \beta)^{\otimes 2} + (\alpha \wedge \beta)^{\otimes 2} \right).$$

4.5.3 SO(2)-invariant amplitudes

Given some basis $\{e_1, e_2, e_3\}$ of V , we will consider the subgroup $\text{SO}(2) \subset \text{O}(3)$ given by rotations with axis e_3 and its action on W^* . In the following, we will show that the cone $(\mathcal{C}_W^*)^{\text{SO}(2)}$ is completely described by the elastic bounds.

Theorem 4.52. *Let $M \in (W^*)^{\text{SO}(2)}$. Then $M \in \mathcal{C}_W^*$ if and only if it satisfies the elastic bounds.*

Proof. We show that every element in $(W^*)^{\text{SO}(2)}$ is contained in $(W^*)^{\mathbb{Z}_2^3}$. First, we observe that since every four-tensor is invariant under the action of $-1 \in \text{O}(3)$ and since every $M \in (W^*)^{\text{SO}(2)}$ is in particular invariant under the action of $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we find that every $M \in (W^*)^{\text{SO}(2)}$ is invariant under the action of Q_3 in Section 4.5.2.

In order to see that $M \in (W^*)^{\text{SO}(2)}$ is also invariant under the action of Q_1 and Q_2 , we observe that the condition that M is invariant under the infinitesimal rotation $\epsilon_{ij3} \in \mathfrak{so}(2) \subset \mathfrak{so}(3)$ around e_3 translates to the condition

$$0 = \sum_{p=1}^3 M_{pjkl} \epsilon_{ip3} + M_{ipkl} \epsilon_{jp3} + M_{ijpl} \epsilon_{kp3} + M_{ijkp} \epsilon_{lp3} \quad (4.19)$$

for every $i, j, k, l \in \{1, 2, 3\}$. Writing out equation (4.19) for $i = j = k = l = 1$, we find

$$0 = M_{2111} + M_{1211} + M_{1121} + M_{1112} = 4 M_{1112},$$

where we used the symmetries of W^* in the last step. Similarly, applying (4.19) for $i = j = k = l = 2$, we find that $M_{1222} = 0$. Moreover, for $i = j = 1, k = l = 3$, equation (4.19) yields

$$0 = M_{2133} + M_{1233} = 2 M_{1233}$$

and, similarly, choosing $i = k = 1$ and $j = l = 3$ in (4.19) yields $M_{1323} = 0$. Since we already know that M is invariant under Q_3 , we conclude that M_{ijkl} must vanish, if the index 1 or 2 appears an odd number of times. Thus, M is also invariant under the action of Q_1 and Q_2 and therefore $M \in (W^*)^{\mathbb{Z}_2^3}$. The statement follows now from Theorem 4.49. $\square$

4.6 Beyond $\dim V = 3$

Starting from $\dim V = 4$ understanding the extremal elements of $\mathcal{C}_W$ becomes significantly harder. One reason for this is that the dimension of W increases rapidly with the dimension of V (cf. Corollary 4.6). Another reason is that our proof of Theorem 4.22 relies very heavily on the fact that when $\dim V = 3$, every non-trivial element of $\Lambda^2 V$ has rank one and can therefore be written as $x \wedge y$ for some $x, y \in V \setminus \{0\}$.

A more fundamental issue that arises for $\dim V > 3$ is that there will be extremal elements of $\mathcal{C}_W \cap \text{Sym}^4 V^*$ of rank larger than one. In the case $\dim V = 4$, Theorem 3.42 tells us that the extremal elements in $\mathcal{C}_W \cap \text{Sym}^4 V^*$ of larger rank are contained in a rational projective variety. However, being 24-dimensional and of degree 28314, this projective variety is already very complicated. Therefore, proving a full classification result like in Theorem 4.22 that gives explicit parametrizations of all possible kinds of extremal elements of the full cone $\mathcal{C}_W$ does not appear to be a feasible task.

On the other hand, the extremal elements described in Theorem 4.22 thought of as four-tensors on a three-dimensional subspace of a higher-dimensional vector space V , still give rise to extremal elements of $\mathcal{C}_W$. In particular, the extremal elements of the third kind in Theorem 4.22 give rise to an explicitly parametrized set of inelastic positivity bounds on scattering processes involving four or more flavors.

Other possible future directions are a more systematic exploration of the difference between the elastic versus the full positivity bounds, or studying positivity bounds for more general scattering processes such as n -to- n for $n > 2$ (see e.g. [CRS18]).

Part II

Kuranishi Moduli Space of Generalized Complex Structures via Hamilton–Nash–Moser Theory

Chapter 5

Generalized Complex Geometry

In this preliminary chapter, we will review some background on (generalized) complex geometry needed in Chapter 7 and 8. First, in Section 5.1 we will fix some notation and briefly recall relevant formulas and results from complex geometry needed for Chapter 7. In Section 5.2, we will introduce Courant algebroids. We will also recall the standard forms for exact and transitive Courant algebroids as well as the shape the group of autoequivalences takes in this standard form. Finally, in Section 5.3, we will define generalized complex structures and recall how to describe deformations of generalized complex structures. Moreover, we will show that the Lie algebroid complex of a generalized complex structure on a transitive Courant algebroid is elliptic as well as some formulas needed in Chapter 8.

This chapter does not contain any original results except for the ellipticity of the said Lie algebroid complex (cf. Proposition 5.44) and the identities in Lemma 5.37, which to our knowledge have not appeared in the literature before. The proof of Proposition 5.44 is by a simple extension of an argument by Gualtieri [Gua03; Gua11].

5.1 Complex structures

Let M be a smooth manifold. Recall that an *almost complex structure* is an endomorphism $J \in \Gamma(M, \text{End } TM)$ such that $J^2 = -\text{id}$. Given an almost complex structure J on M , the complexified tangent bundle decomposes into the i - and $-i$ -eigenbundles of J , which we denote by $T^{1,0}(M, J)$ and $T^{0,1}(M, J)$, respectively. This gives rise to a bigrading on forms. We will denote the (p, q) -bundles with respect to J by $\Lambda^{p,q}(M, J)$ and, for some open subset $U \subset M$, we denote the smooth (p, q) -forms on U with respect to J by $\mathcal{A}_{M,J}^{p,q}(U)$.

Recall that an almost complex structure is called *integrable* or a *complex structure* if its $\pm i$ -eigenbundles are integrable in the sense that they are closed under the Lie bracket of vector fields. Recall furthermore that this is equivalent to any of the following

- $T^{1,0}(M, J)$ is integrable;
- $d = \partial + \bar{\partial}$, where $\partial := \text{pr}_{p+1,q} \circ d : \mathcal{A}_{M,J}^{p,q} \rightarrow \mathcal{A}_{M,J}^{p+1,q}$, $\bar{\partial} := \text{pr}_{p,q+1} \circ d : \mathcal{A}_{M,J}^{p,q} \rightarrow \mathcal{A}_{M,J}^{p,q+1}$;
- $\bar{\partial}^2 = 0$;
- $N_J = 0$, where N_J denotes the Nijenhuis tensor of J .

By the Newlander–Nirenberg theorem an almost complex structure J on a smooth manifold M is integrable if and only if it comes from a holomorphic atlas on M . In this case we will call (M, J) a *complex manifold*.

Finally, recall that for a complex structure J , the bundle $T^{1,0}(M, J)$ is a holomorphic vector bundle, i.e. a complex vector bundle with biholomorphic transition functions, called the *holomorphic tangent bundle*. Given any holomorphic vector bundle $E \rightarrow M$ on a complex manifold, we denote the space of (p, q) -forms with values in E by $\mathcal{A}_{M,J}^{p,q}(U, E)$.

In the following and later on in the text, we will often fix a manifold M and consider the gradings with respect to a fixed complex structure J . We will then often omit writing M and J and instead denote e.g. $T^{1,0}(M, J) = T^{1,0}$, $\Lambda^{p,q}(M, J) = \Lambda^{p,q}$, $\mathcal{A}_{M,J}^{p,q} = \mathcal{A}^{p,q}$ etc.

This section is structured as follows: In Section 5.1.1, we will define a bracket on forms with values in the holomorphic tangent bundle and discuss some formulas for this bracket and the Dolbeault differential. In Section 5.1.2, we will see how deformations of complex structures can be described via $(0, 1)$ -forms with values in the holomorphic tangent bundle and we will introduce the notion of a *locally complete family of complex deformations*. Finally, in Section 5.1.3, we will recall some results from Hodge theory on compact Hermitian manifolds.

5.1.1 Forms with values in the holomorphic tangent bundle

In the following, we fix a complex manifold (M, J) and consider (p, q) -forms with values in the holomorphic tangent bundle. Recall that the *Dolbeault differential* $\bar{\partial}_E$ of a holomorphic vector bundle E is the unique $\mathbb{C}$ -linear operator $\bar{\partial}_E: \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q+1}(E)$ such that $\bar{\partial}_E^2 = 0$ and which satisfies the Leibniz rule $\bar{\partial}_E(fs) = (\bar{\partial}f)s + f\bar{\partial}_E s$. We will denote the Dolbeault differential of $T^{1,0}$ by $\bar{\partial}$.

Furthermore, we define a bracket

$$[\cdot, \cdot]: \mathcal{A}^{0,p}(T^{1,0}) \times \mathcal{A}^{0,q}(T^{1,0}) \longrightarrow \mathcal{A}^{0,p+q}(T^{1,0})$$

by applying the wedge product on the form parts and the complexified Lie bracket on the $T^{1,0}$ -parts. More precisely, in local coordinates, we may write $\alpha = d\bar{z}^I \otimes \alpha_I^j \partial_j$, $\beta = d\bar{z}^K \otimes \beta_K^l \partial_l$, where $|I| = p$, $|K| = q$, and obtain

$$[\alpha, \beta] := d\bar{z}^I \wedge d\bar{z}^K \otimes [\alpha_I^j \partial_j, \beta_K^l \partial_l],$$

where we used multi-index notation and the Einstein convention. The bracket can be defined more generally on (p, q) -forms. The following lemma lists some properties of the bracket and Dolbeault differential:

Lemma 5.1 ([KNS58]). *Let $U \subset M$, $\varphi \in \mathcal{A}^{0,p}(U, T^{1,0})$, $\psi \in \mathcal{A}^{0,q}(U, T^{1,0})$, $\eta \in \mathcal{A}^{0,r}(U, T^{1,0})$. Then*

1. $[\varphi, \psi] = (-1)^{pq+1}[\psi, \varphi];$
2. $\bar{\partial}[\varphi, \psi] = [\bar{\partial}\varphi, \psi] + (-1)^p[\varphi, \bar{\partial}\psi];$
3. $(-1)^{pr}[\varphi, [\psi, \eta]] + (-1)^{pq}[\psi, [\eta, \varphi]] + (-1)^{qr}[\eta, [\varphi, \psi]] = 0.$

We will furthermore need the following explicit formulas for the brackets and Dolbeault derivatives of 0- and $(0, 1)$ -forms with values in $T^{1,0}$, which can be directly computed using the local expressions.

Lemma 5.2. *Let $\varphi \in \mathcal{A}^{0,1}(M, T^{1,0})$, $\xi \in \mathcal{A}^0(M, T^{1,0})$, $X, Y \in \Gamma(M, T^{0,1})$. Then*

1. $\bar{\partial}\xi(X) = \text{pr}_{1,0}[X, \xi];$
2. $\bar{\partial}\varphi(X, Y) = \text{pr}_{1,0}[X, \varphi(Y)] - \text{pr}_{1,0}[Y, \varphi(X)] - \varphi([X, Y]);$
3. $[\varphi, \xi](X) = [\varphi(X), \xi] - \varphi(\text{pr}_{0,1}[X, \xi]);$
4. $[\varphi, \varphi](X, Y) = 2[\varphi(X), \varphi(Y)] - 2\varphi(\text{pr}_{0,1}[\varphi(X), Y]) - 2\varphi(\text{pr}_{0,1}[X, \varphi(Y)]).$

5.1.2 Deformations of complex structures

Let now (M, J) be a compact complex manifold.

Definition 5.3. An almost complex structure J' on M has *finite distance* to J if there exists some $\varphi_{J'} \in \mathcal{A}^{0,1}(M, T^{1,0})$ such that for every $p \in M$

$$T^{0,1}(J')_p = \{-\varphi_{J'}(X_p) + X_p : X_p \in T_p^{0,1}\}.$$

Conversely, given $\varphi \in \mathcal{A}^{0,1}(M, T^{1,0})$, we can define an almost complex structure J_φ on M with finite distance to J if the subbundle

$$L_\varphi = \{-\varphi(X) + X : X \in T^{0,1}\} \subset T_{\mathbb{C}}$$

satisfies the condition

$$(L_\varphi)_p \cap \overline{(L_\varphi)_p} = \{0\} \quad \forall p \in M. \quad (5.1)$$

Then φ defines an almost complex structure on M via $T^{0,1}(J_\varphi) := L_\varphi$. For small enough $\varphi \in \mathcal{A}^{0,1}(M, T^{1,0})$ the condition (5.1) is always satisfied.

Definition 5.4. We will call a section $\varphi \in \mathcal{A}^{0,1}(M, T^{1,0})$ satisfying (5.1) an *almost complex deformation* of the complex structure J .

It was shown in [KNS58] that the condition of the deformed structure to be integrable translates into a partial differential equation:

Theorem 5.5 (Equation (3') in [KNS58]). *The almost complex structure J_φ obtained from a complex structure J and an almost complex deformation $\varphi \in \mathcal{A}^{0,1}(M, T^{1,0})$ is integrable if and only if*

$$\bar{\partial}\varphi - \frac{1}{2}[\varphi, \varphi] = 0. \quad (5.2)$$

Definition 5.6. An almost complex deformation φ is called a *complex deformation* if it satisfies equation (5.2).

Remark 5.7. Note that above we adapted Kuranishi's convention of defining L_φ to be the graph of $-\varphi$ (thought of as a homomorphism from $T^{0,1}$ to $T^{1,0}$). Alternatively, we could have chosen a different convention, where L_φ is defined to be the graph of φ . In this case, the sign in front of the second term of (5.2) changes.

Definition 5.8. A (smooth) family of almost complex deformations of a complex structure J is a smooth map

$$\varphi: S \longrightarrow \mathcal{A}^{0,1}(M, T^{1,0})$$

for some open neighborhood S of 0 in some finite-dimensional vector space such that $\varphi(0) = 0$ and such that for every $s \in S$, the form $\varphi(s)$ satisfies condition (5.1). Smoothness is defined here as smoothness of φ considered as a map $M \times S \rightarrow \Lambda^{0,1} T^* \otimes T^{1,0}$. The family $\{\varphi(s) : s \in S\}$ is a (*smooth*) *family of complex deformations* if in addition for every $s \in S$, the form $\varphi(s)$ satisfies equation (5.2).

The group of diffeomorphisms acts on the space of almost complex structures. We will see in Section 7.1 that for a diffeomorphism f with small enough one-jet, its action on an almost complex structure J_φ with finite distance from J has again finite distance from J and we will denote the corresponding almost complex deformation by $f \cdot \varphi$. We will consider two (almost) complex deformations related in this way by a diffeomorphism as equivalent.

Definition 5.9. We say that a family of complex deformations $\varphi' : S' \rightarrow \mathcal{A}^{0,1}(M, T^{1,0})$ is *obtainable from* the family of complex deformations $\varphi : S \rightarrow \mathcal{A}^{0,1}(M, T^{1,0})$ if there is a smooth map $\tau : S' \rightarrow S$ such that $\tau(0) = 0$ and a family of diffeomorphisms $\{f_{s'}\}_{s' \in S'}$ of M depending smoothly¹ on s' , with $f_0 = \text{id}_M$ such that for every $s' \in S'$

$$\varphi(\tau(s')) = f_{s'} \cdot \varphi'(s').$$

Definition 5.10. A family of complex deformations $\{\varphi(s) : s \in S\}$ is called *locally complete* (or *versal* [AGV12]) if for every family of complex deformations the restriction to some neighborhood of 0 is obtainable from $\{\varphi(s) : s \in S\}$.

Remark 5.11. In [Kur65], Kuranishi instead defined the notion of *complex analytic families of deformations*, where S is allowed to be an analytic set in some open neighborhood in a finite-dimensional complex vector space and the family $\varphi(s)$ should depend holomorphically on $s \in S$ in an appropriate sense.

5.1.3 Hodge theory for compact Hermitian manifolds

Let now E be a holomorphic vector bundle over a compact complex manifold (M, J) . We fix a Hermitian metric on M and a Hermitian bundle metric on E . These induce Hermitian bundle metrics $\langle \cdot, \cdot \rangle$ on $\Lambda^{p,q} \otimes E$. We define a Hermitian product on the spaces $\mathcal{A}^{p,q}(M, E)$, via

$$(\alpha, \beta) := \int_M \langle \alpha, \beta \rangle \text{vol},$$

where vol denotes the metric volume form.

Let $\bar{\partial}^*$ be the formal adjoint of the Dolbeault differential $\bar{\partial}$ of E with respect to $(\cdot, \cdot)$. We denote by

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

the Laplace operator and by

$$\begin{aligned} \mathcal{H}^k(M, E) &:= \{\alpha \in \mathcal{A}^k(M, E) : \bar{\partial}\alpha = 0 = \bar{\partial}^*\alpha\} = \{\alpha \in \mathcal{A}^k(M, E) : \Delta_{\bar{\partial}}\alpha = 0\} \\ \mathcal{H}^{p,q}(M, E) &:= \{\alpha \in \mathcal{A}^{p,q}(M, E) : \bar{\partial}\alpha = 0 = \bar{\partial}^*\alpha\} = \{\alpha \in \mathcal{A}^{p,q}(M, E) : \Delta_{\bar{\partial}}\alpha = 0\}. \end{aligned}$$

¹Here we endowed the diffeomorphisms with the Whitney C^∞ -topology or, equivalently, the topology defined by the local charts introduced in Section 6.3.1.

the spaces of harmonic forms.

Theorem 5.12 (Hodge decomposition, see e.g. [Wel07]). *There exists a natural orthogonal decomposition*

$$\mathcal{A}^{p,q}(M, E) = \bar{\partial}\mathcal{A}^{p,q-1}(M, E) \oplus \mathcal{H}^{p,q}(M, E) \oplus \bar{\partial}^*\mathcal{A}^{p,q+1}(M, E).$$

Moreover, the canonical projection $\mathcal{H}^{p,q}(M, E) \rightarrow H^{p,q}(M, E)$ onto the Dolbeault cohomology is an isomorphism.

We denote by $\text{pr}_{\mathcal{H}}: \mathcal{A}^\bullet \rightarrow \mathcal{H}^\bullet$ the orthogonal projection and by $G_{\bar{\partial}}: \mathcal{A}^\bullet \rightarrow \mathcal{A}^\bullet$ the Green's operator satisfying

$$\text{im } G_{\bar{\partial}} = \ker \text{pr}_{\mathcal{H}}, \quad \ker G_{\bar{\partial}} = \text{im } \text{pr}_{\mathcal{H}}, \quad \text{pr}_{\mathcal{H}} + G_{\bar{\partial}} \Delta_{\bar{\partial}} = \text{id}.$$

Furthermore, we define the operator $Q_{\bar{\partial}} := \bar{\partial}^* G_{\bar{\partial}}$. We will frequently use the following standard identities (see e.g. [Wel07]).

Proposition 5.13. *The operators $\text{pr}_{\mathcal{H}}$, $G_{\bar{\partial}}$ and $Q_{\bar{\partial}}$ satisfy*

1. $G_{\bar{\partial}} \bar{\partial} = \bar{\partial} G_{\bar{\partial}}, G_{\bar{\partial}} \bar{\partial}^* = \bar{\partial}^* G_{\bar{\partial}};$
2. $\text{pr}_{\mathcal{H}} + \bar{\partial} Q_{\bar{\partial}} + Q_{\bar{\partial}} \bar{\partial} = \text{id};$
3. $Q_{\bar{\partial}}^2 = \bar{\partial}^* Q_{\bar{\partial}} = Q_{\bar{\partial}} \bar{\partial}^* = \text{pr}_{\mathcal{H}} Q_{\bar{\partial}} = Q_{\bar{\partial}} \text{pr}_{\mathcal{H}} = 0.$

5.2 Courant algebroids

Generalized complex structures are defined as special endomorphisms on *Courant algebroids*, vector bundles equipped with extra data. In Section 5.2.1, we will state the precise definition of a Courant algebroid, discuss some basic properties and define the notion of an *autoequivalence* of a Courant algebroid. In Section 5.2.2 and 5.2.3, we will focus on the special classes of *exact* and *transitive* Courant algebroids, respectively. We will see that these can always be brought to a standard form via Courant algebroid isomorphisms. We will also discuss the shape that the autoequivalences take in the standard form.

5.2.1 Definition and basic properties of Courant algebroids

We will now introduce the notion of a *Courant algebroid*.

Definition 5.14. A *Courant algebroid* on a manifold M is a vector bundle $E \rightarrow M$ equipped with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle \in \Gamma(\text{Sym}^2(E^*))$ (called the *scalar product*), a bilinear operation $[\cdot, \cdot]$ (called the *Dorfman bracket*) on the space of smooth sections $\Gamma(E)$ of E and a homomorphism of vector bundles $\pi: E \rightarrow TM$ (called the *anchor*), such that the following conditions are satisfied: for all sections $u, v, w \in \Gamma(E)$,

1. $[u, [v, w]] = [[u, v], w] + [v, [u, w]],$
2. $\pi(u)\langle v, w \rangle = \langle [u, v], w \rangle + \langle v, [u, w] \rangle,$
3. $\langle [u, v] + [v, u], w \rangle = \pi(w)\langle u, v \rangle.$

The following consequences are well-known.

Proposition 5.15. *The bracket and anchor of a Courant algebroid $E \rightarrow M$ satisfy the following:*

1. $[u, fv] = \pi(u)(f)v + f[u, v],$
2. $\pi[u, v] = \mathcal{L}_{\pi u}(\pi v),$

for all $u, v \in \Gamma(E)$, $f \in C^\infty(M)$.

Definition 5.16. A Courant algebroid E is called

1. *transitive* if the anchor map $\pi: E \rightarrow TM$ is surjective;
2. *exact* if the following sequence is exact

$$0 \rightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \rightarrow 0.$$

Definition 5.17. An *equivalence of Courant algebroids* between Courant algebroids $(E_1, \pi_1, [\cdot, \cdot]_1, \langle \cdot, \cdot \rangle_1)$ and $(E_2, \pi_2, [\cdot, \cdot]_2, \langle \cdot, \cdot \rangle_2)$ over the same base manifold M is an equivalence of vector bundles $F: E_1 \rightarrow E_2$ covering a diffeomorphism f such that for all $u, v \in \Gamma(M, E_1)$

$$\begin{aligned} df \circ \pi_1 &= \pi_2 \circ F \\ \langle u, v \rangle_1 \circ f &= \langle F(u), F(v) \rangle_2 \\ F([u, v]_1) &= [F(u), F(v)]_2. \end{aligned}$$

An *autoequivalence of Courant algebroids* is an equivalence from a Courant algebroid to itself. An *automorphism of Courant algebroids* is an autoequivalence covering the identity.

For Courant autoequivalences, the first property in Definition 5.17 follows from the second and third.

Definition 5.18. Let E be a Courant algebroid. A *derivation* of E covering $X \in \mathfrak{X}(M)$ is an endomorphism $\mathcal{D} \in \text{End}(\Gamma(M, E))$ such that

$$\begin{aligned} \mathcal{D}(fu) &= f\mathcal{D}u + (\mathcal{L}_X f)u, \\ X\langle \cdot, \cdot \rangle &= \langle \mathcal{D}\cdot, \cdot \rangle + \langle \cdot, \mathcal{D}\cdot \rangle \\ \mathcal{D}[\cdot, \cdot] &= [\mathcal{D}\cdot, \cdot] + [\cdot, \mathcal{D}\cdot]. \end{aligned}$$

Particular examples of derivations are the maps $\text{ad}(u) := [u, \cdot]$ covering $\pi(u)$ for some $u \in \Gamma(M, E)$. The derivations of a Courant algebroid form a Lie subalgebra of $\text{End}(\Gamma(E))$, which we denote by $\text{Der}(E)$. If $\mathcal{D}_1$ is a derivation covering X_1 and $\mathcal{D}_2$ is a derivation covering X_2 , then $[\mathcal{D}_1, \mathcal{D}_2]$ covers $\mathcal{L}_{X_1}X_2$. The subalgebra $\text{ad } \Gamma(E) \subset \text{Der}(E)$ is called the Lie algebra of *inner derivations*.

5.2.2 Splittings and autoequivalences of exact Courant algebroids

Recall that a Courant algebroid $(E, \pi, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ over a manifold M is called exact if the sequence

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0 \quad (5.3)$$

is exact.

Exactness of (5.3) forces the scalar product of E to be of split signature. By choosing an isotropic splitting of the sequence (5.3), every exact Courant algebroid can be brought to the following standard form:

Proposition 5.19. *Every exact Courant algebroid is isomorphic to the generalized tangent bundle $E = TM \oplus T^*M$ with anchor map the canonical projection to TM , scalar product defined by*

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(X))$$

and with Dorfman bracket given by

$$[X + \alpha, Y + \beta] = \mathcal{L}_X Y + \mathcal{L}_X \beta - \iota_Y d\alpha + H(X, Y, \cdot),$$

where $H \in \Omega^3(M)$ is a fixed closed three-form, $X, Y \in \Gamma(TM)$ and $\alpha, \beta \in \Gamma(T^*M)$.

Different choices of isotropic splittings are related by the action of some two-form $B \in \Omega^2(M)$ acting on $TM \oplus T^*M$ via

$$F_B := \begin{pmatrix} \text{id} & B \\ 0 & \text{id} \end{pmatrix}.$$

Such an action of a two-form B is called *B-field transformation*. The three-form $\hat{H}$ defining the Dorfman bracket in the new splitting is given by $\hat{H} = H + dB$. In particular, the isomorphism class of a Courant algebroid is completely determined by the *Ševera class* $[H] \in H_{\text{dR}}^3(M)$ and closed B -field transformations give rise to automorphisms of an exact Courant algebroid. In fact every automorphism of an exact Courant algebroid is a closed B -field transformations.

In order to describe autoequivalences covering a diffeomorphism $f: M \rightarrow M$ we consider the lift of f to $TM \oplus T^*M$ given by

$$F_f := \begin{pmatrix} df & 0 \\ 0 & df^{-1*} \end{pmatrix}.$$

Then $F_{f,B} := F_f F_B$ is an autoequivalence if and only if $H - f^*H = dB$. Every autoequivalence of an exact Courant algebroid can be expressed in this form.

Proposition 5.20 (Proposition 2.2 in [Gua03; Gua11]). *Let F be an orthogonal equivalence of the generalized tangent bundle preserving the H -twisted Courant bracket and covering a diffeomorphism $f: M \rightarrow M$. Then $F = F_f F_B$ for a unique 2-form $B \in \Omega^2(M)$ satisfying $H - f^*H = dB$.*

We conclude the following:

Theorem 5.21. *Let E be the generalized tangent bundle with H -twisted Dorfman bracket. Then its group of autoequivalences is given by*

$$\text{Auteq}(E) = \{F_{f,B} : H - f^*H = dB\}$$

with multiplication given by

$$F_{f,B} F_{f',B'} = F_{f'',B''},$$

where $f'' = f \circ f'$ and $B'' = B' + f'^* B$

We will also denote the group of autoequivalences of the generalized tangent bundle with H -twisted Dorfman bracket by $\text{Auteq}(\mathbb{T}M, H)$. This gives rise to the group of autoequivalences of an arbitrary exact Courant algebroid E with Ševera class $[H]$: It is obtained by conjugating the group $\text{Auteq}(\mathbb{T}M, H)$ with the Courant algebroid isomorphism from E to $(\mathbb{T}M, H)$.

5.2.3 Bisections of transitive Courant algebroids

We will see that similarly as in the case of exact Courant algebroids, every transitive Courant algebroid is isomorphic to a *standard transitive Courant algebroid*. Before we can state this, we need to introduce some notation.

Let $(\mathcal{G} \rightarrow M, \langle \cdot, \cdot \rangle_{\mathcal{G}}, [\cdot, \cdot]_{\mathcal{G}})$ be a bundle of quadratic Lie algebras, i.e. a vector bundle $\mathcal{G} \rightarrow M$ with a scalar product $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ and a Lie bracket $[\cdot, \cdot]_{\mathcal{G}}$ on each fiber such that for every $p \in M$ and $X, Y, Z \in \mathcal{G}_p$ we have

$$\langle [X, Y]_{\mathcal{G}}, Z \rangle_{\mathcal{G}} + \langle Y, [X, Z]_{\mathcal{G}} \rangle_{\mathcal{G}} = 0.$$

We denote by $\text{O}(\mathcal{G})$ the group of orthogonal equivalences of $\mathcal{G}$, i.e. vector bundle morphisms covering a diffeomorphism which preserve the scalar product of $\mathcal{G}$. Moreover, we denote by $\text{Auteq}(\mathcal{G})$ the group of *autoequivalences* of $\mathcal{G}$, i.e. vector bundle morphisms covering a diffeomorphism, which preserve the Lie bracket and scalar product of $\mathcal{G}$. We denote by $\text{Aut}(\mathcal{G})$ the group of *automorphisms*, i.e. autoequivalences covering the identity.

Moreover, we denote by $\Omega^p(M, \mathcal{G})$ the p -forms with values in $\mathcal{G}$. We define the map

$$\langle \cdot \wedge \cdot \rangle_{\mathcal{G}}: \Omega^p(M, \mathcal{G}) \times \Omega^q(M, \mathcal{G}) \longrightarrow \Omega^{p+q}(M)$$

via

$$\langle \mathbf{A} \wedge \mathbf{B} \rangle_{\mathcal{G}}(X_1, \dots, X_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} (-1)^{|\sigma|} \langle \mathbf{A}(X_{\sigma(1)}, \dots, X_{\sigma(p)}), \mathbf{B}(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}) \rangle_{\mathcal{G}}$$

and the map

$$[\cdot, \cdot]_{\mathcal{G}}: \Omega^p(M, \mathcal{G}) \times \Omega^q(M, \mathcal{G}) \longrightarrow \Omega^{p+q}(M, \mathcal{G})$$

via

$$[\mathbf{A}, \mathbf{B}]_{\mathcal{G}}(X_1, \dots, X_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} (-1)^{|\sigma|} [\mathbf{A}(X_{\sigma(1)}, \dots, X_{\sigma(p)}), \mathbf{B}(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)})]_{\mathcal{G}}.$$

For $\mathbf{A} \in \Omega^p(M, \mathcal{G})$ we furthermore define $\text{ad}_{\mathbf{A}} \in \Omega^p(M, \text{End } \mathcal{G})$ via

$$\text{ad}_{\mathbf{A}}(X_1, \dots, X_p)(\mathbf{r}) = [\mathbf{A}(X_1, \dots, X_p), \mathbf{r}]_{\mathcal{G}}.$$

An autoequivalence $\nu \in \text{Auteq}(\mathcal{G})$ covering $f \in \text{Diff}(M)$ acts on the sections $\Gamma(M, \mathcal{G})$ via $\nu \cdot \mathbf{r} = \nu \circ \mathbf{r} \circ f^{-1}$ and on forms $\mathbf{A} \in \Omega^k(M, \mathcal{G})$ via

$$(\nu \cdot \mathbf{A})(X_1, \dots, X_k) = \nu \left(\mathbf{A}(df^{-1}X_1, \dots, df^{-1}X_k) \right).$$

Furthermore, given a connection ∇ on $\mathcal{G}$, the autoequivalence ν acts as

$$(\nu \cdot \nabla)_X r = \nu \cdot \nabla_{f_*^{-1}X}(\nu^{-1} \cdot \mathbf{r}),$$

where $f_*X = df \circ X \circ f^{-1}$ denotes the push-forward.

Definition 5.22. A *standard transitive Courant algebroid* $S[\nabla, \mathbf{R}, H]$ over a manifold M is a transitive Courant algebroid whose underlying vector bundle is $TM \oplus \mathcal{G} \oplus T^*M$ for some bundle of quadratic Lie algebras $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \langle \cdot, \cdot \rangle_{\mathcal{G}})$ with anchor map the projection to TM , inner product

$$\langle X + \mathbf{r} + \alpha, Y + \mathbf{s} + \beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(X)) + \langle \mathbf{r}, \mathbf{s} \rangle_{\mathcal{G}}$$

and Dorfman bracket depending on $H \in \Omega^3(M)$, $\mathbf{R} \in \Omega^2(M, \mathcal{G})$ and some connection ∇ on $\mathcal{G}$ satisfying the conditions

$$\begin{aligned} X\langle \mathbf{r}, \mathbf{s} \rangle_{\mathcal{G}} &= \langle \nabla_X \mathbf{r}, \mathbf{s} \rangle_{\mathcal{G}} + \langle \mathbf{r}, \nabla_X \mathbf{s} \rangle_{\mathcal{G}} \\ \nabla_X [\mathbf{r}, \mathbf{s}]_{\mathcal{G}} &= [\nabla_X \mathbf{r}, \mathbf{s}]_{\mathcal{G}} + [\mathbf{r}, \nabla_X \mathbf{s}]_{\mathcal{G}} \\ d_{\nabla} \mathbf{R} &= 0 \\ d_{\nabla}^2 &= \text{ad}_{\mathbf{R}} \\ dH &= \frac{1}{2} \langle \mathbf{R} \wedge \mathbf{R} \rangle_{\mathcal{G}}. \end{aligned}$$

The various components of the Dorfman bracket are given by

$$\begin{aligned} [X, Y] &= i_Y i_X H + \mathbf{R}(X, Y) + \mathcal{L}_X Y \\ [X, \mathbf{r}] &= -[\mathbf{r}, X] = -2\langle \mathbf{r}, \iota_X \mathbf{R} \rangle_{\mathcal{G}} + \nabla_X \mathbf{r} \\ [X, \alpha] &= \mathcal{L}_X \alpha \\ [\alpha, X] &= -i_X d\alpha \\ [\mathbf{r}, \mathbf{s}] &= 2\langle \mathbf{s}, \nabla \mathbf{r} \rangle_{\mathcal{G}} + [\mathbf{r}, \mathbf{s}]_{\mathcal{G}} \\ [\alpha, \mathbf{r}] &= [\mathbf{r}, \alpha] = [\alpha, \beta] = 0 \end{aligned}$$

for $\mathbf{r}, \mathbf{s} \in \Gamma(\mathcal{G})$, $X, Y \in \Gamma(TM)$ and $\alpha, \beta \in \Gamma(T^*M)$.

Definition 5.23. A *bisection* of a Courant algebroid E is a Courant isomorphism from E to a standard Courant algebroid.

Theorem 5.24 ([CSX13]). *Every transitive Courant algebroid admits a bisection.*

Remark 5.25. Explicitly, the isomorphism is given as follows: Let $\mathcal{G} := \ker \pi / (\ker \pi)^{\perp}$, where the orthogonal complement is taken with respect to the scalar product of the Courant algebroid E . Then it turns out that $\mathcal{G}$ together with the induced scalar product and bracket is a bundle of quadratic Lie algebras. It was shown in [CSX13] that there exist

1. a vector bundle morphism $\lambda: TM \longrightarrow E$ such that $\pi \circ \lambda = \text{id}$ and such that $\text{im } \lambda$ is isotropic with respect to $\langle \cdot, \cdot \rangle$,
2. and a vector bundle morphism $\sigma: \mathcal{G} \longrightarrow \ker \pi$ such that $\text{pr}_{\mathcal{G}} \circ \sigma = \text{id}$ such that $\text{im } \sigma$ is orthogonal to $\text{im } \lambda$ with respect to $\langle \cdot, \cdot \rangle$.

The isomorphism is then given by

$$\begin{aligned} TM \oplus \mathcal{G} \oplus T^*M &\longrightarrow E \\ X + \mathbf{r} + \alpha &\longmapsto \lambda(X) + \sigma(\mathbf{r}) + \frac{1}{2}\pi^*\alpha. \end{aligned}$$

Theorem 5.26 ([CSX13]). *Let E be a transitive Courant algebroid, $S[\nabla, \mathbf{R}, H]$ and $S[\hat{\nabla}, \hat{\mathbf{R}}, \hat{H}]$ different choices of bisections of E . Then there is some $\nu \in \mathcal{O}(\mathcal{G})$ covering the identity, $\mathbf{A} \in \Omega^1(M, \mathcal{G})$ and $B \in \Omega^2(M)$, such that the isomorphism $\delta: S[\nabla, \mathbf{R}, H] \longrightarrow S[\hat{\nabla}, \hat{\mathbf{R}}, \hat{H}]$ describing the change of bisection is given by*

$$\delta = \begin{pmatrix} \text{id} & 0 & 0 \\ \nu \circ \mathbf{A} & \nu & 0 \\ B - \langle \mathbf{A}, \mathbf{A} \rangle_{\mathcal{G}} & -2\langle \mathbf{A}, \cdot \rangle_{\mathcal{G}} & \text{id} \end{pmatrix}.$$

Moreover, the data defining the different standard transitive Courant algebroids is related by

$$\begin{aligned} \hat{\nabla} &= \nu \cdot (\nabla - \text{ad}_{\mathbf{A}}) \\ \hat{\mathbf{R}} &= \nu \cdot (\mathbf{R} - \text{d}^{\nabla} \mathbf{A} + \frac{1}{2}[\mathbf{A}, \mathbf{A}]_{\mathcal{G}}) \\ \hat{H} &= H - \text{d}B - 2\langle \mathbf{A} \wedge \mathbf{R} \rangle_{\mathcal{G}} + \langle \mathbf{A} \wedge \text{d}^{\nabla} \mathbf{A} \rangle_{\mathcal{G}} - \frac{1}{3}\langle \mathbf{A} \wedge [\mathbf{A}, \mathbf{A}]_{\mathcal{G}} \rangle_{\mathcal{G}}. \end{aligned}$$

An orthogonal equivalence $\nu \in \mathcal{O}(\mathcal{G})$ of $\mathcal{G}$ covering a diffeomorphism $f \in \text{Diff}(M)$ defines a vector bundle morphism F_{ν} from E to itself via

$$F_{\nu} := \begin{pmatrix} df & & \\ & \nu & \\ & & (df^{-1})^* \end{pmatrix}$$

in the splitting $E = TM \oplus \mathcal{G} \oplus T^*M$.

Given a two-form $B \in \Omega^2(M)$ and a one-form $\mathbf{A} \in \Omega^1(M, \mathcal{G})$ with values in $\mathcal{G}$ we can furthermore define vector bundle morphisms F_B and $F_{\mathbf{A}}$ from E to itself via

$$F_B := \begin{pmatrix} \text{id} & 0 & 0 \\ 0 & \text{id} & 0 \\ B & 0 & \text{id} \end{pmatrix}, \quad F_{\mathbf{A}} := \begin{pmatrix} \text{id} & 0 & 0 \\ \mathbf{A} & \text{id} & 0 \\ -\langle \mathbf{A}, \mathbf{A} \rangle_{\mathcal{G}} & -2\langle \mathbf{A}, \cdot \rangle_{\mathcal{G}} & \text{id} \end{pmatrix}.$$

Note that F_{ν} , F_B and $F_{\mathbf{A}}$ all preserve the scalar product $\langle \cdot, \cdot \rangle$ on E as well as the anchor map. We will denote the morphism obtained by composing F_{ν} , $F_{\mathbf{A}}$ and F_B by

$$F_{\nu, \mathbf{A}, B} := F_{\nu} \circ F_{\mathbf{A}} \circ F_B.$$

The group $\mathcal{O}(E, \pi, \langle \cdot, \cdot \rangle)$ of equivalences of E preserving the anchor map and the scalar product is given by [Cou17]

$$\mathcal{O}(E, \pi, \langle \cdot, \cdot \rangle) = \{F_{\nu, \mathbf{A}, B} : \nu \in \mathcal{O}(\mathcal{G}), \mathbf{A} \in \Omega^1(M, \mathcal{G}), B \in \Omega^2(M)\}.$$

The group of autoequivalences of E is the subgroup of elements in $\mathcal{O}(E, \pi, \langle \cdot, \cdot \rangle)$ that are compatible with the Dorfman bracket of E . An explicit description is given in the following theorem.

Theorem 5.27 (Theorem 2.37 in [Cou17]). *Let E be a standard transitive Courant algebroid. Then the group of autoequivalences of E is given by*

$$\begin{aligned} \text{Auteq}(E) &= \{F_{\nu, \mathbf{A}, B} : \nu \in \text{Auteq}(\mathcal{G}) \text{ covers } f \in \text{Diff}(M), \mathbf{A} \in \Omega^1(M, \mathcal{G}), B \in \Omega^2(M), \\ &\quad \nabla = \nu \cdot (\nabla - \text{ad}_{\mathbf{A}}), \\ &\quad \mathbf{R} = \nu \cdot (\mathbf{R} - d_{\nabla} \mathbf{A} + \tfrac{1}{2}[\mathbf{A}, \mathbf{A}]_{\mathcal{G}}), \\ &\quad H - f^*H = dB + 2\langle \mathbf{A} \wedge \mathbf{R} \rangle_{\mathcal{G}} - \langle \mathbf{A} \wedge d_{\nabla} \mathbf{A} \rangle_{\mathcal{G}} - \tfrac{1}{3}\langle \mathbf{A} \wedge [\mathbf{A}, \mathbf{A}]_{\mathcal{G}} \rangle_{\mathcal{G}}\}, \end{aligned}$$

with the product

$$F_{\nu, \mathbf{A}, B} \circ F_{\nu', \mathbf{A}', B'} = F_{\nu'', \mathbf{A}'', B''},$$

where $\nu'', \mathbf{A}'', B''$ are given by

$$\begin{aligned} \nu'' &= \nu\nu', \\ \mathbf{A}'' &= \nu'^{-1} \cdot \mathbf{A} + \mathbf{A}', \\ B'' &= f'^*B + B' + \langle \nu'^{-1} \cdot \mathbf{A} \wedge \mathbf{A}' \rangle_{\mathcal{G}} \end{aligned}$$

and the identity element is $((\text{id}_M, 0), \text{id}, 0)$. The inversion map is given by

$$(\nu, \mathbf{A}, B) \longmapsto (\nu^{-1}, -\nu \cdot \mathbf{A}, -(f^{-1})^*B)$$

Remark 5.28. Note that in [Cou17, Thm 2.37] there are some additional factors of $1/2$ appearing. This is due to a different convention for the scalar product of a standard transitive Courant algebroid. Moreover, we remark that the theorems stated above are special cases of more general results in [Cou17; CSX13] for the class of regular Courant algebroids (i.e. Courant algebroids whose anchor map has constant rank).

Like in the case of exact Courant algebroids, the group of autoequivalences of a standard transitive Courant algebroid gives rise to a description of the group of autoequivalences of an arbitrary transitive Courant algebroid E by conjugating with the isomorphism to E .

Later, we will consider the following subclass of transitive Courant algebroids for which the bisections take a particularly simple form.

Definition 5.29. A $\mathcal{G}$ -flat standard transitive Courant algebroid is a standard transitive Courant algebroid such that $\mathcal{G} \cong M \times \mathbb{R}^K$ is a trivial vector bundle with trivial Lie bracket on the fibers and such that ∇ is flat with trivial holonomy and $\mathbf{R} = 0$. A transitive Courant algebroid is called $\mathcal{G}$ -flat if it is isomorphic to a $\mathcal{G}$ -flat standard transitive Courant algebroid.

Proposition 5.30. *If $S(\nabla, \mathbf{R}, H)$ is a standard $\mathcal{G}$ -flat transitive Courant algebroid, then H is closed.*

Proof. This follows from the condition that $dH = \frac{1}{2}\langle \mathbf{R} \wedge \mathbf{R} \rangle_{\mathcal{G}} = 0$. □

5.3 Generalized complex structures

In this section we will introduce generalized complex structures. First, in Section 5.3.1, we will state their definition and show that it encompasses both complex and symplectic

structures as special cases. We will see that every generalized complex structure is associated with a Lie algebroid L . In Section 5.3.2, we will see how deformations of a generalized complex structure can be encoded into forms $\varphi \in \Gamma(M, \Lambda^2 L^*)$.

We will moreover introduce the *Schouten bracket* and Lie algebroid differential of a Lie algebroid, which are relevant to study the integrability of the deformed structure. We will also discuss some formulas for the Schouten bracket and Lie algebroid differential of a generalized complex structure. In Section 5.3.3 we will introduce the notion of a *locally complete family of generalized complex deformations*.

Finally, in Section 5.3.4 we will take a closer look at the complex defined by the Lie algebroid differential of a generalized complex structure. We will see that for generalized complex structures on transitive Courant algebroids this complex is elliptic. For generalized complex structures on transitive Courant algebroids over compact manifolds, we will recall some results and formulas from Hodge theory.

5.3.1 Definition and examples of generalized complex structures

We will now state the definition of a *generalized complex structure*.

Definition 5.31. A *generalized almost complex structure* on a Courant algebroid $E \rightarrow M$ is an endomorphism field $\mathcal{J} \in \Gamma(\text{End } E)$ such that $\mathcal{J}^2 = -\text{id}$ and such that $\mathcal{J}$ is skew-symmetric with respect to the scalar product of E . It is called *integrable* or a *generalized complex structure* if the subbundles $L(\mathcal{J}), \bar{L}(\mathcal{J})$ of $E_{\mathbb{C}}$ given by the $+i$ - and $-i$ -eigenspaces of $\mathcal{J}$ are involutive with respect to the Dorfman bracket.

An equivalent definition of a generalized complex structure on a Courant algebroid can be formulated directly in terms of the subbundle $L \subset E_{\mathbb{C}}$:

Definition 5.32. A *generalized almost complex structure* on E is a maximally isotropic subbundle $L \subset E_{\mathbb{C}}$ of the complexification $E_{\mathbb{C}}$ of E such that $E_{\mathbb{C}} = L \oplus \bar{L}$. A generalized almost complex structure is called *integrable* or a *generalized complex structure* if it is closed under the Dorfman bracket.

Given a generalized complex structure L , note that it follows from the third axiom in Definition 5.14 and the fact that L and $\bar{L}$ are isotropic that the restriction of the Dorfman bracket to L and $\bar{L}$, respectively, is in fact anti-symmetric. As a consequence, the bundles L and $\bar{L}$ together with the restrictions of π and the Dorfman bracket of E form Lie algebroids.

As the name suggests, generalized complex structures are generalizations of complex structures, as the following example shows.

Example 5.33. Let (M, J) be an almost complex manifold. Then the endomorphism

$$\mathcal{J}_J := \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix} \in \Gamma(M, \text{End } \mathbb{T}M)$$

defines a generalized almost complex structure on the generalized tangent bundle of M . The generalized almost complex structure $\mathcal{J}_J$ is integrable with respect to $[\cdot, \cdot]_H$ for $H \in \Omega_{\text{cl}}^3(M)$ if and only if J is integrable and H is of type $(1, 2) + (2, 1)$.

Further examples of generalized complex structures are symplectic structures.

Example 5.34. Let ω be a non-degenerate two-form on M . We think of ω as a map $\omega: TM \rightarrow T^*M$, $X \mapsto \omega(X, \cdot)$ and denote by $\omega^{-1}: T^*M \rightarrow TM$ its inverse. Then the endomorphism

$$\mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \in \Gamma(M, \text{End } \mathbb{T}M)$$

defines a generalized almost complex structure on the generalized tangent bundle of M . The generalized almost complex structure $\mathcal{J}_\omega$ is integrable with respect to $[\cdot, \cdot]_H$ for $H \in \Omega_{\text{cl}}^3(M)$ if ω is closed and $H = 0$.

5.3.2 Deformations of generalized complex structures

Let L be a generalized complex structure on some Courant algebroid E . Every maximal isotropic having trivial intersection with $\bar{L}$ is described as the graph of some homomorphism $\varphi: L \rightarrow \bar{L}$ satisfying $\langle \varphi u, v \rangle + \langle u, \varphi v \rangle = 0$ for all $u, v \in \Gamma(L)$. After identifying $\bar{L} \cong L^*$ via the scalar product $\langle \cdot, \cdot \rangle$ of E , we may also think of such φ as a form $\varphi \in \Gamma(M, \Lambda^2 L^*)$ and the new isotropic is given by

$$L_\varphi := (1 + \varphi)L = \{u + \iota_u \varphi : u \in L\}.$$

The isotropic L_φ will be a generalized almost complex structure if and only if $L_\varphi \oplus \overline{L_\varphi} = E_{\mathbb{C}}$. Since this is an open condition, it will be satisfied for every $\varphi \in \Gamma(M, \Lambda^2 L^*)$ in a small enough (C^0) -neighborhood of zero. We will call such φ a *generalized almost complex deformation*.

Remark 5.35. Note the different sign in front of the form φ as compared to our convention for (classical) complex deformations in Section 5.1.2. The reason why we chose the different sign conventions is that, while in Section 5.1.2 we adopted Kuranishi's convention, here we follow Gualtieri's convention. Gualtieri's convention has the advantage that we can interpret L_φ as the graph of φ .

Next, we will study under which conditions the deformed generalized almost complex structure L_φ is integrable. Before stating the condition, we need to introduce a differential and a bracket associated to the Lie algebroids L and $\bar{L}$.

For every Lie algebroid L , we may define the *Lie algebroid differential*

$$d_L: \Gamma(\Lambda^k L^*) \longrightarrow \Gamma(\Lambda^{k+1} L^*)$$

via

$$\begin{aligned} (d_L \varphi)(u_0, \dots, u_k) &:= \sum_i (-1)^i \pi(u_i) \varphi(u_0, \dots, \hat{u}_i, \dots, u_k) \\ &\quad + \sum_{i < j} (-1)^{i+j} \varphi([u_i, u_j], u_0, \dots, \hat{u}_i, \dots, \hat{u}_j, \dots, u_k) \end{aligned}$$

for $\varphi \in \Gamma(\Lambda^k L^*)$, and $u_0, \dots, u_k \in \Gamma(L)$.

Lemma 5.36. *Let L be a Lie algebroid. Then $(\Gamma(\Lambda^\bullet L^*), d_L)$ forms a differential graded algebra, i.e. d_L is a first order linear differential operator of degree one that satisfies*

$$1. (d_L)^2 = 0$$

2. $d_L(\varphi \wedge \psi) = (d_L\varphi) \wedge \psi + (-1)^p \varphi \wedge d_L\psi$ for all $\varphi \in \Gamma(\Lambda^p L^*)$, $\psi \in \Gamma(\Lambda^q L^*)$.

Furthermore, for every Lie algebroid L the *Schouten bracket*

$$[\cdot, \cdot]_L: \Gamma(\Lambda^p L) \times \Gamma(\Lambda^q L) \longrightarrow \Gamma(\Lambda^{p+q-1} L)$$

is defined as follows: For $u_1, \dots, u_p, v_1, \dots, v_q \in \Gamma(L)$, we set

$$\begin{aligned} & [u_1 \wedge \dots \wedge u_p, v_1 \wedge \dots \wedge v_q]_L \\ &:= \sum_{i,j} (-1)^{i+j} [u_i, v_j] \wedge u_1 \wedge \dots \wedge \hat{u}_i \wedge \dots \wedge u_p \wedge v_1 \wedge \dots \wedge \hat{v}_j \wedge \dots \wedge v_q. \end{aligned}$$

For $u \in \Gamma(L)$, $f \in C^\infty(M)$, we set

$$[u, f]_L = -[f, u]_L := \pi(u)(f).$$

Finally, we set

$$\begin{aligned} & [u_1 \wedge \dots \wedge u_p, f]_L = (-1)^p [f, u_1 \wedge \dots \wedge u_p]_L \\ &:= (-1)^p ([f, u_1]u_2 \wedge \dots \wedge u_p - u_1 \wedge [f, u_2]u_3 \wedge \dots \wedge u_p \\ &\quad + \dots + (-1)^{p-1} u_1 \wedge \dots \wedge [f, u_p]). \end{aligned}$$

Lemma 5.37. *Let L be a generalized complex structure. We identify $\bar{L} \cong L^*$. Then for $U \subset M$, $\varphi \in \Gamma(U, \Lambda^p L^*)$, $\psi \in \Gamma(U, \Lambda^q L^*)$, $\eta \in \Gamma(U, \Lambda^r L^*)$ the following hold*

1. $[\varphi, \psi]_{L^*} = -(-1)^{(p-1)(q-1)} [\psi, \varphi]_{L^*}$
2. $[\varphi, [\psi, \eta]_{L^*}]_{L^*} = [[\varphi, \psi]_{L^*}, \eta]_{L^*} + (-1)^{(p-1)(q-1)} [\psi, [\varphi, \eta]_{L^*}]_{L^*}$
3. $d_L[\varphi, \psi]_{L^*} = [d_L\varphi, \psi]_{L^*} + (-1)^{p-1} [\varphi, d_L\psi]_{L^*}$.

Proof. Properties 1 and 2 are general properties of the Schouten bracket. Property 3 follows from the fact that L and $\bar{L} \cong L^*$ form a Lie bialgebroid (see Theorem 2.6 in [LWX97]). $\square$

We will now state the condition for integrability:

Theorem 5.38 (Theorem 6.1 in [LWX97]). *The generalized almost complex structure L_φ corresponding to a generalized almost complex deformation $\varphi \in \Gamma(M, \Lambda^2 L^*)$ is integrable if and only if φ satisfies the Maurer–Cartan equation*

$$d_L\varphi + \frac{1}{2}[\varphi, \varphi]_{L^*} = 0. \quad (5.4)$$

In this case φ is called a generalized complex deformation.

We will from now on drop the subscript L and L^* of the Schouten bracket. In the following we will collect some identities for generalized complex structures that will become very useful in Chapter 8.

Lemma 5.39. *Let L be a generalized complex structure. We identify $\bar{L} \cong L^*$. Then for $U \subset M$, $\varphi, \varphi' \in \Gamma(U, \Lambda^2 L^*)$, $\xi \in \Gamma(U, L^*)$, $u, v, w \in \Gamma(U, L)$ the following hold*

1. $d_L\xi(u, v) = -\langle [\xi, u], v \rangle$

$$2. \quad d_L \varphi(u, v, w) = \langle [u, \iota_v \varphi] + [\iota_u \varphi, v] - \iota_{[u, v]} \varphi, w \rangle$$

$$3. \quad [\varphi, \xi](u, v) = \langle [\iota_u \varphi, \xi] + \iota_{\text{pr}_L[\xi, u]} \varphi, v \rangle$$

4.

$$\begin{aligned} [\varphi, \varphi'](u, v, w) &= \langle [\iota_u \varphi, \iota_v \varphi'] - [\iota_v \varphi, \iota_u \varphi'], w \rangle \\ &\quad + \frac{1}{2} \langle \iota_{\text{pr}_L}([v, \iota_u \varphi] - [\iota_u \varphi, v] - [u, \iota_v \varphi] + [\iota_v \varphi, u]) \varphi', w \rangle \\ &\quad + \frac{1}{2} \langle \iota_{\text{pr}_L}([v, \iota_u \varphi'] - [\iota_u \varphi', v] - [u, \iota_v \varphi'] + [\iota_v \varphi', u]) \varphi, w \rangle \end{aligned}$$

in particular,

$$[\varphi, \varphi](u, v, w) = 2 \langle [\iota_u \varphi, \iota_v \varphi], w \rangle + \langle \iota_{\text{pr}_L}([v, \iota_u \varphi] - [\iota_u \varphi, v] - [u, \iota_v \varphi] + [\iota_v \varphi, u]) \varphi, w \rangle.$$

Proof. For the first equation, we plug in the definition of d_L and compute

$$\begin{aligned} d_L \xi(u, v) &= \pi(u) \xi(v) - \pi(v) \xi(u) - \xi([u, v]) \\ &= \pi(u) \langle \xi, v \rangle - \pi(v) \langle \xi, u \rangle - \langle \xi, [u, v] \rangle \\ &= \langle [u, \xi], v \rangle - \pi(v) \langle \xi, u \rangle \\ &= -\langle [\xi, u], v \rangle, \end{aligned}$$

where we used the second and the third axiom in Definition 5.14.

Similarly, for the second identity, we compute

$$\begin{aligned} d_L \varphi(u, v, w) &= \pi(u) (\varphi(v, w)) - \pi(v) (\varphi(u, w)) + \pi(w) (\varphi(u, v)) \\ &\quad - \varphi([u, v], w) + \varphi([u, w], v) - \varphi([v, w], u) \\ &= \pi(u) \langle \iota_v \varphi, w \rangle - \pi(v) \langle \iota_u \varphi, w \rangle + \pi(w) \langle \iota_u \varphi, v \rangle \\ &\quad - \langle \iota_{[u, v]} \varphi, w \rangle - \langle \iota_v \varphi, [u, w] \rangle + \langle \iota_u \varphi, [v, w] \rangle \\ &= \langle [u, \iota_v \varphi] + [\iota_u \varphi, v] - \iota_{[u, v]} \varphi, w \rangle. \end{aligned}$$

Note that to show the third equation, we may assume that $\varphi = \eta \wedge \psi$ for some $\eta, \psi \in \Gamma(U, L^*)$. Using the definition of the Schouten bracket we compute

$$\begin{aligned} [\eta \wedge \psi, \xi](u, v) &= ([\eta, \xi] \wedge \psi - [\psi, \xi] \wedge \eta)(u, v) \\ &= \langle [\eta, \xi], u \rangle \langle \psi, v \rangle - \langle [\eta, \xi], v \rangle \langle \psi, u \rangle \\ &\quad - \langle [\psi, \xi], u \rangle \langle \eta, v \rangle + \langle [\psi, \xi], v \rangle \langle \eta, u \rangle \\ &= (-\pi(\xi) \langle \eta, u \rangle + \langle \eta, [\xi, u] \rangle) \langle \psi, v \rangle + (\pi(\xi) \langle \eta, v \rangle - \langle \eta, [\xi, v] \rangle) \langle \psi, u \rangle \\ &\quad + (\pi(\xi) \langle \psi, u \rangle - \langle \psi, [\xi, u] \rangle) \langle \eta, v \rangle + (-\pi(\xi) \langle \psi, v \rangle + \langle \psi, [\xi, v] \rangle) \langle \eta, u \rangle \\ &= \pi(\xi) (-\langle \eta, u \rangle \langle \psi, v \rangle + \langle \eta, v \rangle \langle \psi, u \rangle) + \eta \wedge \psi(\text{pr}_L[\xi, u], v) - \eta \wedge \psi(\text{pr}_L[\xi, v], u). \end{aligned}$$

Observing that

$$\begin{aligned} \langle [\iota_u(\eta \wedge \psi), \xi], v \rangle &= \langle [\langle \eta, u \rangle \psi - \langle \psi, u \rangle \eta, \xi], v \rangle \\ &= -\pi(\xi) \langle \langle \eta, u \rangle \psi - \langle \psi, u \rangle \eta, v \rangle + \langle \langle \eta, u \rangle \psi - \langle \psi, u \rangle \eta, [\xi, v] \rangle \\ &= \pi(\xi) (-\langle \eta, u \rangle \langle \psi, v \rangle + \langle \eta, v \rangle \langle \psi, u \rangle) - \eta \wedge \psi(\text{pr}_L[\xi, v], u) \end{aligned}$$

we obtain the third identity.

For the last identity we may again assume that $\varphi = \eta \wedge \psi$, $\varphi' = \eta' \wedge \psi'$ for $\eta, \eta', \psi, \psi' \in \Gamma(M, L^*)$. We start computing

$$\begin{aligned}
& \langle [\iota_u(\eta \wedge \psi), \iota_v(\eta' \wedge \psi')] , w \rangle \\
&= \langle [\langle \eta, u \rangle \psi - \langle \psi, u \rangle \eta, \langle \eta', v \rangle \psi' - \langle \psi', v \rangle \eta'] , w \rangle \\
&= \pi(\psi) (\langle \eta', v \rangle) \langle \psi', w \rangle \langle \eta, u \rangle - \pi(\psi) (\langle \psi', v \rangle) \langle \eta, u \rangle \langle \eta', w \rangle \\
&\quad - \pi(\eta) (\langle \eta', v \rangle) \langle \psi', w \rangle \langle \psi, u \rangle + \pi(\eta) (\langle \psi', v \rangle) \langle \eta', w \rangle \langle \psi, u \rangle \\
&\quad - \pi(\psi') (\langle \eta, u \rangle) \langle \psi, w \rangle \langle \eta', v \rangle + \pi(\psi') (\langle \psi, u \rangle) \langle \eta, w \rangle \langle \eta', v \rangle \\
&\quad + \pi(\eta') (\langle \eta, u \rangle) \langle \psi, w \rangle \langle \psi', v \rangle - \pi(\eta') (\langle \psi, u \rangle) \langle \eta, w \rangle \langle \psi', v \rangle \\
&\quad + \langle \eta, u \rangle \langle \eta', v \rangle \langle [\psi, \psi'], w \rangle - \langle \eta, u \rangle \langle \psi', v \rangle \langle [\psi, \eta'], w \rangle \\
&\quad - \langle \psi, u \rangle \langle \eta', v \rangle \langle [\eta, \psi'], w \rangle + \langle \psi, u \rangle \langle \psi', v \rangle \langle [\eta, \eta'], w \rangle,
\end{aligned}$$

where we used the Leibniz rule for the bracket (Proposition 5.15) in combination with the skew-symmetry of the bracket when restricted to sections of the isotropic bundle $\bar{L}$. Using the second axiom in Definition 5.14, we find

$$\begin{aligned}
& \langle [\iota_u(\eta \wedge \psi), \iota_v(\eta' \wedge \psi')] , w \rangle \\
&= -\langle [\psi, \psi'], v \rangle \langle \eta, u \rangle \langle \eta', w \rangle + \langle [\psi', \psi], u \rangle \langle \eta, w \rangle \langle \eta', v \rangle + \langle [\psi, \psi'], w \rangle \langle \eta, u \rangle \langle \eta', v \rangle \\
&\quad + \langle [\psi, \eta'], v \rangle \langle \psi', w \rangle \langle \eta, u \rangle - \langle [\eta', \psi], u \rangle \langle \eta, w \rangle \langle \psi', v \rangle - \langle [\psi, \eta'], w \rangle \langle \eta, u \rangle \langle \psi', v \rangle \\
&\quad + \langle [\eta, \psi'], v \rangle \langle \eta', w \rangle \langle \psi, u \rangle - \langle [\psi', \eta], u \rangle \langle \psi, w \rangle \langle \eta', v \rangle - \langle [\eta, \psi'], w \rangle \langle \psi, u \rangle \langle \eta', v \rangle \\
&\quad - \langle [\eta, \eta'], v \rangle \langle \psi', w \rangle \langle \psi, u \rangle + \langle [\eta', \eta], u \rangle \langle \psi, w \rangle \langle \psi', v \rangle + \langle [\eta, \eta'], w \rangle \langle \psi, u \rangle \langle \psi', v \rangle \\
&\quad + \langle \eta', [\psi, v] \rangle \langle \psi', w \rangle \langle \eta, u \rangle - \langle \psi', [\psi, v] \rangle \langle \eta, u \rangle \langle \eta', w \rangle \\
&\quad - \langle \eta', [\eta, v] \rangle \langle \psi', w \rangle \langle \psi, u \rangle + \langle \psi', [\eta, v] \rangle \langle \eta', w \rangle \langle \psi, u \rangle \\
&\quad - \langle \eta, [\psi', u] \rangle \langle \psi, w \rangle \langle \eta', v \rangle + \langle \psi, [\psi', u] \rangle \langle \eta, w \rangle \langle \eta', v \rangle \\
&\quad + \langle \eta, [\eta', u] \rangle \langle \psi, w \rangle \langle \psi', v \rangle - \langle \psi, [\eta', u] \rangle \langle \eta, w \rangle \langle \psi', v \rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \langle [\iota_u(\eta \wedge \psi), \iota_v(\eta' \wedge \psi')] , w \rangle - \langle [\iota_v(\eta \wedge \psi), \iota_u(\eta' \wedge \psi')] , w \rangle \\
&= ([\psi, \psi'] \wedge \eta \wedge \eta' - [\psi, \eta'] \wedge \eta \wedge \psi' - [\eta, \psi'] \wedge \psi \wedge \eta' + [\eta, \eta'] \wedge \psi \wedge \psi') (u, v, w) \\
&\quad + \langle \psi', w \rangle (\langle \eta', [\psi, v] \rangle \langle \eta, u \rangle - \langle \eta', [\psi, u] \rangle \langle \eta, v \rangle - \langle \eta', [\eta, v] \rangle \langle \psi, u \rangle + \langle \eta', [\eta, u] \rangle \langle \psi, v \rangle) \\
&\quad - \langle \eta', w \rangle (\langle \psi', [\psi, v] \rangle \langle \eta, u \rangle - \langle \psi', [\psi, u] \rangle \langle \eta, v \rangle - \langle \psi', [\eta, v] \rangle \langle \psi, u \rangle + \langle \psi', [\eta, u] \rangle \langle \psi, v \rangle) \\
&\quad + \langle \psi, w \rangle (\langle \eta, [\psi', v] \rangle \langle \eta', u \rangle - \langle \eta, [\psi', u] \rangle \langle \eta', v \rangle - \langle \eta, [\eta', v] \rangle \langle \psi', u \rangle + \langle \eta, [\eta', u] \rangle \langle \psi', v \rangle) \\
&\quad - \langle \eta, w \rangle (\langle \psi, [\psi', v] \rangle \langle \eta', u \rangle - \langle \psi, [\psi', u] \rangle \langle \eta', v \rangle - \langle \psi, [\eta', v] \rangle \langle \psi', u \rangle + \langle \psi, [\eta', u] \rangle \langle \psi', v \rangle).
\end{aligned}$$

On the other hand, for any $\epsilon \in \Gamma(M, L^*)$, we have

$$\begin{aligned}
& \iota_{\text{pr}_L}([\iota_u(\eta \wedge \psi), v] - [v, \iota_u(\eta \wedge \psi)]) \epsilon \\
&= \langle \epsilon, [\langle \eta, u \rangle \psi, v] - [\langle \psi, u \rangle \eta, v] \rangle - \langle \epsilon, [v, \langle \eta, u \rangle \psi] - [v, \langle \psi, u \rangle \eta] \rangle \\
&= \pi(\epsilon) (\langle \eta, u \rangle \langle \psi, v \rangle - \langle \psi, u \rangle \langle \eta, v \rangle) - 2 \langle \epsilon, [v, \langle \eta, u \rangle \psi] - [v, \langle \psi, u \rangle \eta] \rangle \\
&= \pi(\epsilon) (\langle \eta, u \rangle \langle \psi, v \rangle - \langle \psi, u \rangle \langle \eta, v \rangle) - 2 \langle \eta, u \rangle \langle \epsilon, [v, \psi] \rangle + 2 \langle \psi, u \rangle \langle \epsilon, [v, \eta] \rangle \\
&= \pi(\epsilon) (\langle \eta, u \rangle \langle \psi, v \rangle - \langle \psi, u \rangle \langle \eta, v \rangle) - 2 \langle \eta, u \rangle \pi(\epsilon) (\langle v, \psi \rangle) + 2 \langle \psi, u \rangle \pi(\epsilon) (\langle v, \eta \rangle) \\
&\quad + 2 \langle \eta, u \rangle \langle \epsilon, [\psi, v] \rangle - 2 \langle \psi, u \rangle \langle \epsilon, [\eta, v] \rangle.
\end{aligned}$$

It follows that

$$\begin{aligned} & \iota_{\text{pr}_L}([\iota_u(\eta \wedge \psi), v] - [\iota_v(\eta \wedge \psi), u] + [\iota_u(\eta \wedge \psi), u]) \epsilon \\ &= 2\langle \eta, u \rangle \langle \epsilon, [\psi, v] \rangle - 2\langle \psi, u \rangle \langle \epsilon, [\eta, v] \rangle - 2\langle \eta, v \rangle \langle \epsilon, [\psi, u] \rangle + 2\langle \psi, v \rangle \langle \epsilon, [\eta, u] \rangle \end{aligned}$$

and hence

$$\begin{aligned} & \frac{1}{2} \langle \iota_{\text{pr}_L}([\iota_u(\eta \wedge \psi), v] - [\iota_v(\eta \wedge \psi), u] + [\iota_u(\eta \wedge \psi), u]) (\eta' \wedge \psi'), w \rangle \\ &= (\langle \eta', [\psi, v] \rangle \langle \eta, u \rangle - \langle \eta', [\psi, u] \rangle \langle \eta, v \rangle - \langle \eta', [\eta, v] \rangle \langle \psi, u \rangle + \langle \eta', [\eta, u] \rangle \langle \psi, v \rangle) \langle \psi', w \rangle \\ & \quad - (\langle \psi', [\psi, v] \rangle \langle \eta, u \rangle - \langle \psi', [\psi, u] \rangle \langle \eta, v \rangle - \langle \psi', [\eta, v] \rangle \langle \psi, u \rangle + \langle \psi', [\eta, u] \rangle \langle \psi, v \rangle) \langle \eta', w \rangle. \end{aligned}$$

Putting everything together, we obtain

$$\begin{aligned} & \langle [\iota_u \varphi, \iota_v \varphi'] - [\iota_v \varphi, \iota_u \varphi'], w \rangle \\ &+ \frac{1}{2} \langle \iota_{\text{pr}_L}([v, \iota_u \varphi] - [\iota_u \varphi, v] - [u, \iota_v \varphi] + [\iota_v \varphi, u]) \varphi', w \rangle \\ &+ \frac{1}{2} \langle \iota_{\text{pr}_L}([v, \iota_u \varphi'] - [\iota_u \varphi', v] - [u, \iota_v \varphi'] + [\iota_v \varphi', u]) \varphi, w \rangle \\ &= ([\psi, \psi'] \wedge \eta \wedge \eta' - [\psi, \eta'] \wedge \eta \wedge \psi' - [\eta, \psi'] \wedge \psi \wedge \eta' + [\eta, \eta'] \wedge \psi \wedge \psi') (u, v, w) \\ &= [\varphi, \varphi'](u, v, w). \end{aligned}$$

□

5.3.3 Families of deformations

Next, we will introduce *smooth families of generalized almost complex deformations*. For this, let S be an open neighborhood of 0 in some finite-dimensional vector space.

Definition 5.40. A (smooth) family of generalized almost complex deformations of a generalized complex structure L is a smooth map

$$\varphi: S \longrightarrow \Gamma(M, \Lambda^2 L^*)$$

such that $\varphi(0) = 0$ and such that for every $s \in S$, the form $\varphi(s)$ is a generalized almost complex deformation. Like in the complex case, smoothness is defined as smoothness of φ considered as a map $M \times S \rightarrow \Lambda^2 L^*$. The family $\{\varphi(s) : s \in S\}$ is a (smooth) family of generalized complex deformations if in addition for every $s \in S$, the form $\varphi(s)$ satisfies equation (5.4).

The group of autoequivalences $\text{Auteq}(E)$ of a Courant algebroid E acts on the space of generalized almost complex structures $\text{GAC}(E)$ on E via

$$\begin{aligned} \mu_{\text{GAC}}: \text{Auteq}(E) \times \text{GAC}(E) &\longrightarrow \text{GAC}(E) \\ (F, \mathcal{J}) &\longmapsto F \cdot \mathcal{J} := F \circ \mathcal{J} \circ F^{-1}. \end{aligned}$$

If the (1-jet of the) autoequivalence F is close enough to the identity in an appropriate sense, then $F \cdot \mathcal{J}$ can be described by a generalized (almost) complex deformation $\Theta(F) \in \Gamma(M, \Lambda^2 L^*)$ of $\mathcal{J}$ (see Section 8.1.3 for more details). Similarly, given a generalized almost complex deformation $\varphi \in \Gamma(M, \Lambda^2 L^*)$ of $\mathcal{J}$, for F close enough to the identity, the generalized almost complex structure $F \cdot \mathcal{J}_\varphi$ is again described by a generalized almost

complex deformation $\Theta(F, \varphi) \in \Gamma(M, \Lambda^2 L^*)$. We will also denote $F \cdot \varphi := \Theta(F, \varphi)$. We will consider two generalized complex deformations equivalent if they are related by the action of some autoequivalence.

Definition 5.41. A family $\varphi': S' \rightarrow \Gamma(M, \Lambda^2 L^*)$ of generalized complex deformations is *obtainable from* $\varphi: S \rightarrow \Gamma(M, \Lambda^2 L^*)$ if there is a smooth map $\tau: S' \rightarrow S$ such that $\tau(0) = 0$ and a family of autoequivalences $\{F_{s'}\}_{s' \in S'}$ depending smoothly² on s' , with $F_0 = \text{id}$ such that for every $s' \in S'$

$$\varphi(\tau(s')) = F_{s'} \cdot \varphi'(s').$$

Definition 5.42. A family of deformations $\{\varphi(s) : s \in S\}$ is called *locally complete* if for every family of deformations the restriction to some neighborhood of 0 is obtainable from $\{\varphi(s) : s \in S\}$.

5.3.4 The Lie algebroid complex of a generalized complex structure

Next, we will take a closer look at the Lie algebroid complex

$$d_L: \Gamma(\Lambda^k L^*) \rightarrow \Gamma(\Lambda^{k+1} L^*). \quad (5.5)$$

We will see that if L is a generalized complex structure on a transitive Courant algebroid, then this complex is elliptic. First, we will show the following Lemma:

Lemma 5.43. *We denote by $\Re: TM_{\mathbb{C}} \rightarrow TM$ the projection to the real part of $TM_{\mathbb{C}}$. Suppose E is a transitive Courant algebroid with a generalized complex structure $L \subset E_{\mathbb{C}}$. Then the map*

$$\Re \circ \pi: L \rightarrow TM$$

is surjective.

Proof. Recall that $L \subset E_{\mathbb{C}}$ is the i -eigen-bundle of the generalized almost complex structure $\mathcal{J}$ on E , which is given by

$$L = \{u - i\mathcal{J}u : u \in E\}.$$

By the assumption that E is transitive, we obtain

$$\Re \circ \pi(L) = \pi(E) = TM.$$

□

Proposition 5.44. *If E is transitive, then the Lie algebroid complex $(\Gamma(\Lambda^\bullet L^*), d_L)$ is elliptic.*

Proof. For $\alpha \in T^*M$, the symbol is given by

$$s_\alpha(d_L) = \text{pr}_{L^*}(\pi^* \alpha) \wedge \cdot.$$

²Here we endow the autoequivalences with the Whitney C^∞ -topology. For the group of autoequivalences of $\mathcal{G}$ -flat transitive Courant algebroids this is equivalent to the topology of the tame manifold structure constructed in Section 8.1.2.

We choose $v \in L$ and compute

$$\mathrm{pr}_{L*}(\pi^*\alpha)(v) = \langle \pi^*\alpha, v \rangle = \alpha \circ \pi(v).$$

Suppose $\alpha \circ \pi(v) = 0$ for all $v \in L$. By decomposing the equation into real and imaginary part and using Lemma 5.43, this implies that $\alpha = 0$. In particular, this shows that the symbol sequence is exact and therefore the complex $(\Gamma(\Lambda^\bullet L^*), d_L)$ is elliptic. $\square$

Corollary 5.45. *Let L be a generalized complex structure on a transitive Courant algebroid $E \rightarrow M$ over a compact manifold M . Then the cohomology $H^\bullet(M, L)$ of the complex (5.5), called Lie algebroid cohomology, is a finite-dimensional graded ring.*

For the remainder of this section, we will assume that L is a generalized complex structure on a transitive Courant algebroid $E \rightarrow M$ over a compact orientable manifold M . We fix an auxiliary Riemannian metric on M and a Hermitian bundle metric h_L on L . These induce a scalar product $(\cdot, \cdot)$ on $\Gamma(M, \Lambda^\bullet L^*)$ via

$$(\alpha, \beta) := \int_M h_L(\alpha, \beta) * 1.$$

We denote by d_L^* the formal adjoint of d_L with respect to $(\cdot, \cdot)$ and define the Laplace operator

$$\Delta_L := d_L d_L^* + d_L^* d_L.$$

Moreover, we denote by

$$\mathcal{H}_L^k(M) := \ker \Delta_L|_{\Gamma(M, \Lambda^k L^*)} = \ker d_L|_{\Gamma(M, \Lambda^k L^*)} \cap \ker d_L^*|_{\Gamma(M, \Lambda^k L^*)}$$

the Δ_L -harmonic forms.

Theorem 5.46 (Hodge decomposition, see e.g. [Wel07]). *There exists a natural orthogonal decomposition*

$$\Gamma(M, \Lambda^k L^*) = d_L \Gamma(M, \Lambda^{k-1} L^*) \oplus \mathcal{H}_L^k(M) \oplus d_L^* \Gamma(M, \Lambda^{k+1} L^*).$$

Moreover, the canonical projection $\mathcal{H}_L^k(M) \rightarrow H^k(M, L)$ is an isomorphism.

Definition 5.47. We define the orthogonal projection $\mathrm{pr}_{\mathcal{H}_L}: \Gamma(M, \Lambda^\bullet L^*) \rightarrow \mathcal{H}_L^\bullet(M)$, the Green's operator $G_L: \Gamma(M, \Lambda^\bullet L^*) \rightarrow \Gamma(M, \Lambda^\bullet L^*)$, satisfying

$$\mathrm{im} G_L = \ker \mathrm{pr}_{\mathcal{H}_L}, \quad \ker G_L = \mathrm{im} \mathrm{pr}_{\mathcal{H}_L}, \quad \mathrm{pr}_{\mathcal{H}_L} + G_L \Delta_L = \mathrm{id}$$

and the operator $Q_L := d_L^* G_L$.

Again, the operators satisfy the standard identities as in Proposition 5.13 for the case of holomorphic vector bundles.

Proposition 5.48 (See e.g. [Wel07]). *The operators $\mathrm{pr}_{\mathcal{H}_L}$, G_L and Q_L satisfy*

1. $G_L d_L = d_L G_L$, $G_L d_L^* = d_L^* G_L$;
2. $\mathrm{pr}_{\mathcal{H}_L} + d_L Q_L + Q_L d_L = \mathrm{id}$;
3. $Q_L^2 = d_L^* Q_L = Q_L d_L^* = \mathrm{pr}_{\mathcal{H}_L} Q_L = Q_L \mathrm{pr}_{\mathcal{H}_L} = 0$.

Chapter 6

Hamilton–Nash–Moser Theory

A typical analytic challenge that arises in moduli space problems in differential geometry is that the linear spaces involved are usually spaces of smooth sections or functions, which are not Banach spaces but only Fréchet spaces and the Lie groups involved such as the diffeomorphism group of a manifold are not Banach Lie groups but Fréchet Lie groups. As a consequence, the inverse function theorem and implicit function theorem for Banach spaces are not applicable.

One way around this is the passage through Sobolev spaces. Another approach is to work with Hamilton’s framework of tame Fréchet spaces and smooth tame maps. These form the so-called Nash–Moser category, which enjoy the Nash–Moser inverse function theorem as well as an implicit function theorem. For vector bundles over compact manifolds, the space of smooth sections is a tame Fréchet space and the diffeomorphism group of a compact manifold is a tame Lie group. Working in this framework thus allows us to remain entirely within the smooth category throughout the proof, without the need to track different Sobolev spaces or apply additional regularity arguments.

In this chapter we review the main concepts and results from Hamilton–Nash–Moser theory that are required for Chapters 7 and 8. In Section 6.1, we will introduce tame Fréchet spaces and smooth tame maps. In Section 6.2, we will state the Nash–Moser theorem and Hamilton’s Nash–Moser implicit function theorem. We will also review some results on families of linear inverses, which are needed to apply the Nash–Moser theorem. In particular, we employ the general theory to establish a specific result about families of operators in Proposition 6.31 which is applied to the deformation family in Section 8.2.1. Finally, in Section 6.3, we will introduce tame Fréchet manifolds and Lie groups and discuss the important example of the diffeomorphism group of a compact manifold.

This chapter does not contain any original results except for Proposition 6.31, which is a consequence of a general statement by Hamilton [Ham82] and the Hodge decomposition. A more specialized version of this Proposition has already appeared in the coauthored preprint [CP26]. We also prove that the action of the diffeomorphism group on endomorphisms of the tangent bundle and the pull-back of differential forms are smooth tame. These statements are known to the experts in the field but we could not find these proofs in the literature, which is why we include them here.

6.1 Tame Fréchet spaces and the Nash–Moser category

In this section, we will introduce *tame Fréchet spaces* and *smooth tame maps* which together form the so-called *Nash–Moser category*. Tame Fréchet spaces are special kinds of *graded Fréchet spaces*, which will be defined in Section 6.1.1. In Section 6.1.2, we will recall the notion of *smoothness* for maps between Fréchet spaces and fix some notation. Finally, in the Sections 6.1.3 and 6.1.4 we will introduce the notion of *tameness* for maps between graded Fréchet spaces and define tame Fréchet spaces.

6.1.1 Graded Fréchet spaces

In this section, we will introduce *graded Fréchet spaces*. We will start recalling the definition of a *locally convex* topological vector space.

Definition 6.1. A topological vector space $\mathcal{F}$ is called *locally convex* if its topology is induced by a collection $\{\|\cdot\|_i\}_{i \in \mathcal{I}}$ of semi-norms, in the sense that it is the coarsest translational invariant topology for which the functions $\|\cdot\|_i$ are continuous.

It follows that a sequence $\{f_j\} \subset \mathcal{F}$ converges to $f \in \mathcal{F}$ if and only if $\|f_j - f\|_i \rightarrow 0$ for all $i \in \mathcal{I}$. It is easy to see that a locally convex topological vector space is Hausdorff if and only if $\|f\|_i = 0$ for every $i \in \mathcal{I}$ implies $f = 0$.

Definition 6.2. A Hausdorff locally convex topological vector space is called *metrizable* if the collection of semi-norms inducing its topology can be chosen a countable collection.

Remark 6.3. Given a metrizable Hausdorff locally convex vector space $(\mathcal{F}, \{\|\cdot\|_i\}_{i \in \mathbb{N}})$ we can define a metric on $\mathcal{F}$ via

$$d(f, g) := \sum_{i \in \mathbb{N}} \frac{1}{2^i} \frac{\|f - g\|_i}{1 + \|f - g\|_i}$$

for $f, g \in \mathcal{F}$. The metric d is called *Fréchet metric* and the topology induced by d is precisely the topology of $\mathcal{F}$.

Definition 6.4. A *Cauchy sequence* in a locally convex topological vector space $\mathcal{F}$ is a sequence $\{f_n\} \subset \mathcal{F}$ such that $\|f_j - f_k\|_i \rightarrow 0$ as $j, k \rightarrow \infty$ for all $i \in \mathcal{I}$. The space $\mathcal{F}$ is called *complete* if every Cauchy sequence converges.

We are now ready to give the definition of a Fréchet space:

Definition 6.5. A *Fréchet space* is a complete Hausdorff metrizable locally convex topological vector space.

Remark 6.6 (Section I.1.2 in [Ham82]). 1. Closed subspaces of Fréchet spaces and quotients by closed subspaces are again Fréchet spaces.

2. Countable products of Fréchet spaces are Fréchet spaces.

3. The (topological) dual of a Fréchet space that is not a Banach space is *never* a Fréchet space.

Next, we will introduce *gradings* and define *graded Fréchet spaces*.

Definition 6.7. A *grading* on a Fréchet space $\mathcal{F}$ is a collection $\{\|\cdot\|_n\}_{n \in \mathbb{N}_0}$ of semi-norms on $\mathcal{F}$ inducing the topology such that for every $f \in \mathcal{F}$

$$\|f\|_0 \leq \|f\|_1 \leq \|f\|_2 \leq \dots$$

Two gradings $\{\|\cdot\|_n\}$ and $\{\|\cdot\|'_n\}$ on a Fréchet space $\mathcal{F}$ are *tamely equivalent* of degree $r \in \mathbb{Z}$ and base $b \in \mathbb{N}_0$ if for all $f \in \mathcal{F}$ and $n \geq b$ we have

$$\|f\|_n \leq C(n)\|f\|'_{n+r} \quad \text{and} \quad \|f\|'_n \leq C'(n)\|f\|_{n+r}$$

for some constants $C(n), C'(n) > 0$. A *graded Fréchet space* is a Fréchet space with a choice of grading.

The prototypical example [Ham82] of a graded Fréchet space is the space $(\Sigma(\mathcal{B}), \{\|\cdot\|_n\}_{n \in \mathbb{N}_0})$ of rapidly decreasing sequences $\{f_k\}$ in a Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, that is sequences such that

$$\|\{f_k\}\|_n := \sum_{k=0}^{\infty} e^{nk} \|f_k\|_{\mathcal{B}} < \infty$$

for every n .

The most important example of a (graded) Fréchet space for us is the space $\Gamma(M, V)$ of smooth global sections of a vector bundle V over a compact oriented manifold M . Fixing bundle metrics and connections on V and TM and denoting by $D^j s$ the j -th covariant derivative of $s \in \Gamma(V)$ the $W^{k,2}$ -Sobolev norms defined by

$$\|f\|_{k,2} := \left(\sum_{i=0}^k \int_M \langle D^i f, D^i f \rangle \text{vol} \right)^{1/2}$$

constitute a grading on $\Gamma(M, V)$, where $\langle \cdot, \cdot \rangle$ denotes the induced bundle metric on $(T^*M)^{\otimes k} \otimes V$ and vol denotes the metric volume form. An equivalent grading is given by the C^k -norms

$$\|s\|_{k,\infty} := \sum_{j=0}^k \sup_{x \in M} \|D^j s(x)\|,$$

as follows from the Sobolev embedding theorem. Indeed, by the Sobolev embedding theorem for $r > \dim M/2$, we have

$$\|f\|_{n,\infty} \leq C\|f\|_{n+r,2}.$$

Since also $\|f\|_{n,2} \leq C'\|f\|_{n,\infty} \leq C''\|f\|_{n+r,\infty}$, the two gradings are equivalent.

6.1.2 Directional derivatives and families of linear maps

We recall the following definition of *continuous differentiability* on Fréchet spaces.

Definition 6.8. Let $\mathcal{F}$ and $\mathcal{G}$ be Fréchet spaces, $\mathcal{U} \subset \mathcal{F}$ open and $P: \mathcal{U} \rightarrow \mathcal{G}$ a continuous map. The *derivative* of P at $f \in \mathcal{U}$ in the direction $h \in \mathcal{F}$ is defined by

$$DP(f)h = \lim_{t \rightarrow 0} \frac{P(f + th) - P(f)}{t}.$$

We say P is *continuously differentiable* or C^1 on $\mathcal{U}$ if the limit exists for all $f \in \mathcal{U}$ and all $h \in \mathcal{F}$ and if

$$DP: \mathcal{U} \times \mathcal{F} \longrightarrow \mathcal{G}$$

is continuous.

Here, we use the notation $L(f, g) = L(f)g$ to indicate that the map L is linear in the second variable. We also refer to such maps as *families of linear maps* and denote families of multi-linear maps as $L(f, g_1, \dots, g_n) = L(f)\{g_1, \dots, g_n\}$. It is worth emphasizing that families of linear maps are not considered as maps to $\text{Hom}(\mathcal{G}, \mathcal{H})$, which in general is not a Fréchet space.

The directional derivative on Fréchet spaces satisfies the usual chain rule [Ham82], i.e. given Fréchet spaces $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$, open subsets $\mathcal{U} \subset \mathcal{F}$, $\mathcal{V} \subset \mathcal{G}$ and C^1 -maps $P: \mathcal{U} \rightarrow \mathcal{V}$, $Q: \mathcal{V} \rightarrow \mathcal{H}$, the composition $Q \circ P$ is a C^1 -map and

$$D[Q \circ P](f)h = DQ(P(f)) DP(f)h.$$

For maps that depend on several variables we can define partial differentiability in the usual way. Maps of class C^n are defined iteratively and the n -th derivative of a C^n -map is a family of n -linear maps. A map of Fréchet spaces is called *smooth* if it is C^n for every $n \in \mathbb{N}$.

For a C^1 -map, the following property holds.

Proposition 6.9 (Theorem I.3.2.2 in [Ham82]). *Let $P: (\mathcal{U} \subset \mathcal{F}) \rightarrow \mathcal{G}$ be a C^1 -map between Fréchet spaces, $f, h \in \mathcal{F}$ such that $f + th \in \mathcal{U}$ for all $t \in [0, 1]$. Then*

$$P(f + h) - P(f) = \int_0^1 DP(f + th)h \, dt.$$

Here, $\int_a^b$ denotes the Riemann integral of a continuous map $f: [a, b] \rightarrow \mathcal{F}$ to a Fréchet space $\mathcal{F}$, which is defined as follows.

Definition 6.10 (Theorem I.2.2.1 in [Ham82]). Let $\mathcal{F}$ be a Fréchet space,

$$f: [a, b] \longrightarrow \mathcal{F}$$

a continuous map. Then we define the *Riemann integral* $\int_a^b f(t)dt \in \mathcal{F}$ to be the unique element in $\mathcal{F}$ such that for every linear functional $\ell: \mathcal{F} \rightarrow \mathbb{R}$, it holds

$$\ell\left(\int_a^b f(t)dt\right) = \int_a^b \ell(f(t)) \, dt.$$

We also note the following properties of the Riemann integral.

Theorem 6.11 (Theorem I.2.2.1 in [Ham82]). *The Riemann integral has the following properties for continuous $f, g: [a, b] \rightarrow \mathcal{F}$, $c \in \mathbb{R}$:*

1. *for every continuous semi-norm on $\mathcal{F}$, we have*

$$\left\| \int_a^b f(t)dt \right\| \leq \int_a^b \|f(t)\|dt;$$

$$\begin{aligned}
2. \quad & \int_a^c f(t)dt + \int_c^b f(t)dt = \int_a^b f(t)dt; \\
3. \quad & \int_a^b [f(t) + c \cdot g(t)]dt = \int_a^b f(t)dt + c \int_a^b g(t)dt.
\end{aligned}$$

Later on we will use the following simplified version of Taylor's formula:

Theorem 6.12 (Theorem I.3.5.6 in [Ham82]). *Let $\mathcal{F}, \mathcal{G}$ be Fréchet spaces, $\mathcal{U} \subset \mathcal{F}$ open and $P: \mathcal{U} \rightarrow \mathcal{G}$ a C^2 -map, $f, h \in \mathcal{F}$ such that $f + th \in \mathcal{U}$ for all $t \in [0, 1]$. Then*

$$P(f + h) = P(f) + DP(f)h + R(h),$$

where R is a C^2 -map such that $R(sh) = s^2 \tilde{R}(h, s)$ for small enough $s \in \mathbb{R}$, with $\tilde{R}(h, s)$ depending continuously on s . If P is smooth, $\tilde{R}(h, s)$ depends smoothly on s .

6.1.3 Tame linear maps and tame Fréchet spaces

Choosing a grading on a Fréchet space allows us to define *tame estimates* for linear maps.

Definition 6.13. Let $b \in \mathbb{N}_0$, $r \in \mathbb{Z}$. A linear map $L: \mathcal{F} \rightarrow \mathcal{G}$ of graded Fréchet spaces satisfies a *tame estimate* of degree r and base b if for every $f \in \mathcal{F}$ and $n \geq b$ we have

$$\|Lf\|_n \leq C(n)\|f\|_{n+r}$$

for some constant $C(n) > 0$. We say L is *tame linear* if it satisfies a tame estimate for some r and b . We will often leave the dependence of $C(n)$ on n implicit and simply write C .

We include the following examples to be used later.

Example 6.14 (Example I.1.2.2(3) in [Ham82]). A linear partial differential operator L of order r (on a vector bundle V) over a compact manifold is tame linear: $\|Ls\|_{n,2} \leq C\|s\|_{n+r,2}$ and similarly $\|Ls\|_{n,\infty} \leq C\|s\|_{n+r,\infty}$, for all $s \in \Gamma(M, V)$.

Example 6.15. Let M be a compact complex manifold. Then the bracket $[\cdot, \cdot]: \mathcal{A}^{0,p}(M, T^{1,0}) \times \mathcal{A}^{0,q}(M, T^{1,0}) \rightarrow \mathcal{A}^{p+q}(M, T^{1,0})$ as defined in Section 5.1.1 is tame linear in both arguments separately. More precisely, we have $\|[\alpha, \beta]\|_{n,2} \leq C\|\alpha\|_{n+1,2}\|\beta\|_{n+1,2}$ for $n \geq \dim M/2 + 1$ and similarly $\|[\alpha, \beta]\|_{n,\infty} \leq C\|\alpha\|_{n+1,\infty}\|\beta\|_{n+1,\infty}$ for all $n \in \mathbb{N}_0$. The analogous inequalities hold for the Schouten bracket of a Lie algebroid over a compact manifold.

Example 6.16. The Green's operator G of an elliptic linear partial differential operator of order r on a vector bundle V over a compact manifold satisfies $\|Gv\|_{n,2} \leq \|v\|_{n-r,2}$ for all $n \in \mathbb{N}_0$ and is therefore tame linear [Kod86, Thm 7.4 in appendix]. In particular, the operator $Q_{\bar{\partial}} = \bar{\partial}^* G_{\bar{\partial}}$ given in Section 5.1.3 is tame linear as it satisfies $\|Q_{\bar{\partial}}\xi\|_{n,2} \leq C\|\xi\|_{n-1,2}$ for every $\xi \in \mathcal{A}^\bullet(M, E)$ and for all $n \in \mathbb{N}_0$. The analogous inequality holds for Q_L as defined in Section 5.3.4 and for $Q_{dR} = d^* G_{dR}$, where G_{dR} is the Green's operator for $\Delta_{dR} = dd^* + d^*d$ and d^* denotes the formal adjoint of the de Rham differential d .

In order to define tame Fréchet spaces, we need the notion of a *tame direct summand*.

Definition 6.17. Let $\mathcal{F}$ and $\mathcal{G}$ be graded spaces. Then $\mathcal{F}$ is a *tame direct summand* of $\mathcal{G}$ if there are tame linear maps $L: \mathcal{F} \rightarrow \mathcal{G}$ and $M: \mathcal{G} \rightarrow \mathcal{F}$ such that the composition

$$\mathcal{F} \xrightarrow{L} \mathcal{G} \xrightarrow{M} \mathcal{F}$$

is the identity.

Definition 6.18. A *tame* Fréchet space is a graded Fréchet space that is a tame direct summand of the Fréchet space $\Sigma(\mathcal{B})$ of rapidly decreasing sequences for some Banach space $\mathcal{B}$. A *tame (Fréchet) subspace* $\mathcal{F}$ of a tame Fréchet space $\mathcal{G}$ is a graded Fréchet space $\mathcal{F}$ together with a tame linear inclusion $\mathcal{F} \hookrightarrow \mathcal{G}$ of base 0.

The motivation behind this definition is as follows: The graded Fréchet space $\Sigma(\mathcal{B})$ for some Banach space $\mathcal{B}$ admits so-called *smoothing operators*, a property which is inherited to tame direct summands. This property is essential for the proof of the Nash–Moser theorem, which is why it makes sense to restrict to graded Fréchet spaces that are *tame*.

Example 6.19. The Fréchet space $\Gamma(M, V)$ with the (equivalent) gradings discussed in the end of Section 6.1.1 is a tame Fréchet space [Ham82, Cor II.1.3.9].

Note that tame direct summands and products of tame Fréchet spaces (with the induced grading) are tame Fréchet spaces [Ham82, Lem II.1.3.3-4]. As a consequence, we also note the following important examples of tame subspaces of spaces of sections.

Example 6.20. For some elliptic complex $d_i: \Gamma(M, E_i) \rightarrow \Gamma(M, E_{i+1})$ on a compact manifold M , such that $\Gamma(M, E_i)$ are equipped with L^2 -inner products, it follows from the Hodge decomposition that for every i , the closed subspaces $d_i\Gamma(M, E_i)$ and $d_i^*\Gamma(M, E_{i+1})$ of $\Gamma(M, E_i)$ are tame direct summands and hence tame Fréchet spaces.

6.1.4 Smooth tame maps and the Nash–Moser category

We will now define *tameness* for arbitrary continuous maps between graded Fréchet spaces.

Definition 6.21. Let $\mathcal{F}$ and $\mathcal{G}$ be graded Fréchet spaces, $P: (\mathcal{U} \subset \mathcal{F}) \rightarrow \mathcal{G}$ a map and $b \in \mathbb{N}_0$, $r \in \mathbb{Z}$. We say that P satisfies a *tame estimate* of degree r and base b if for all $f \in \mathcal{U}$ and $n \geq b$ it holds

$$\|P(f)\|_n \leq C(n) (1 + \|f\|_{n+r})$$

for some constant $C(n) > 0$. We say that P is *tame* if it is defined on an open set, is continuous and satisfies a tame estimate in a neighborhood of each point.

This is a generalization of the notion of tame linear maps, since a map between graded Fréchet spaces is a tame linear map if and only if it is linear and tame [Ham82, Thm II.2.1.5]. For tame maps of several variables, we allow different degrees in the different variables. More specifically, we say that a map $P(f, g)$ of graded Fréchet spaces satisfies a tame estimate of degree r in f and s in g and base b if

$$\|P(f, g)\|_n \leq C(n)(1 + \|f\|_{n+r} + \|g\|_{n+s})$$

for all $n \geq b$ for some constants $C(n)$ independent of f and g .

For tame families of linear maps, we have the following.

Lemma 6.22 (Lemma II.2.1.7 in [Ham82]). *Let $L(f)h$ be a continuous family of linear maps that satisfies a tame estimate of degree r in f and s in h in a $\|\cdot\|_{b+r}$ -neighborhood of f_0 and*

for all h in a $\|\cdot\|_{b+s}$ -neighborhood of 0. Then for all $n \geq b$, f in a $\|\cdot\|_{b+s}$ -neighborhood of f_0 and all h it satisfies the estimate

$$\|L(f)h\|_n \leq C(n)(\|f\|_{n+r}\|h\|_{b+s} + \|h\|_{n+s})$$

for constants $C(n)$ independent of f and h .

Definition 6.23. Let $\mathcal{F}$ and $\mathcal{G}$ be graded Fréchet spaces. A map $P: (\mathcal{U} \subset \mathcal{F}) \rightarrow \mathcal{G}$ is *smooth tame* if it is smooth and all derivatives of P are tame.

The set of tame Fréchet spaces together with the smooth tame maps form a category, the *Nash–Moser category*. An important class of smooth tame maps are *non-linear partial differential operators*:

Definition 6.24. Let M be a compact manifold, V and W vector bundles over M . A *non-linear partial differential operator* of order r is a map

$$P: (\mathcal{U} \subset \Gamma(M, V)) \longrightarrow \Gamma(M, W)$$

which factorizes through a smooth fiber-preserving map $U \subset J^r V \rightarrow W$, where U is an open subset of the vector bundle $J^r V$ of r -jets of sections of V . Non-linear partial differential operators of order zero we will also refer to as *non-linear vector bundle operators*.

Proposition 6.25 (Theorem II.2.2.7 in [Ham82]). *A non-linear partial differential operator is smooth tame.*

An application of Proposition 6.25 is the following proposition.

Proposition 6.26. *Let M be a compact manifold, V , V_1 , V_2 and V_3 vector bundles over M . Then the maps*

$$\begin{aligned} \Gamma(M, \text{Hom}(V_2, V_3)) \times \Gamma(M, \text{Hom}(V_1, V_2)) &\longrightarrow \Gamma(M, \text{Hom}(V_1, V_3)) \\ (A, B) &\longmapsto A \circ B \end{aligned}$$

and

$$\begin{aligned} \Gamma(M, \text{Aut } V) &\subset \Gamma(M, \text{Hom}(V, V)) \longrightarrow \Gamma(M, \text{Aut } V) \\ A &\longmapsto A^{-1} \end{aligned}$$

are smooth tame.

Proof. This follows by observing that the concatenation and inversion of vector bundle homomorphisms are non-linear vector bundle operators. $\square$

6.2 The Nash–Moser theorem and families of inverses

We will now state the Nash–Moser theorem, which is a generalization of the inverse function theorem on Banach spaces to the Nash–Moser category.

Theorem 6.27 (Nash–Moser theorem, Theorem III.1.1.1 in [Ham82]). *Let $\mathcal{F}$ and $\mathcal{G}$ be tame Fréchet spaces and $P: (\mathcal{U} \subset \mathcal{F}) \rightarrow \mathcal{G}$ a smooth tame map. Suppose that the equation for the*

derivative $DP(f)h = k$ has a unique solution $h = VP(f)k$ for all $f \in \mathcal{U}$ and $k \in \mathcal{G}$ and that the family of inverses $VP: \mathcal{U} \times \mathcal{G} \rightarrow \mathcal{F}$ is a smooth tame map. Then P is locally invertible and each local inverse P^{-1} is a smooth tame map.

Note that in contrast to the inverse function theorem of Banach spaces, where the derivative only needs to be invertible in a point, the Nash–Moser theorem requires invertibility in an open neighborhood, and that this family of inverses is a smooth tame family of linear maps. To check that it is smooth tame it is in fact enough to show that it is continuous and tame, as the following proposition tells us.

Proposition 6.28 (Theorem II.3.1.1 in [Ham82]). *Let $L: (\mathcal{U} \subset \mathcal{F}) \times \mathcal{H} \rightarrow \mathcal{K}$ be a smooth tame family of linear maps. Suppose that $L(f)$ is invertible for every $f \in \mathcal{U}$ and the family of inverses*

$$V: (\mathcal{U} \subset \mathcal{F}) \times \mathcal{K} \rightarrow \mathcal{H}$$

is continuous and tame as a map. Then V is a smooth tame family of linear maps.

The following generalization of the Nash–Moser theorem is due to Hamilton [Ham82].

Theorem 6.29 (Implicit function theorem, cf. Theorem III.3.3.1 in [Ham82]). *Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be tame Fréchet spaces and let*

$$A: (\mathcal{U} \subset \mathcal{F} \times \mathcal{G}) \longrightarrow \mathcal{H}$$

be a smooth tame map. Suppose that the partial derivative $D_f A(f, g)$ is surjective, and there is a smooth tame family of linear maps

$$V: (\mathcal{U} \subset \mathcal{F} \times \mathcal{G}) \times \mathcal{H} \longrightarrow \mathcal{F}$$

such that for every $(f, g) \in \mathcal{U}$ and $h \in \mathcal{H}$, we have

$$D_f A(f, g)V(f, g)h = h.$$

Then if $A(f_0, g_0) = 0$ for some $(f_0, g_0) \in \mathcal{U}$ we can find neighborhoods $\mathcal{V}_1 \subset \mathcal{F}$ and $\mathcal{V}_2 \subset \mathcal{G}$ of f_0 and g_0 , respectively, such that for all $g \in \mathcal{V}_2$ we can find some $f \in \mathcal{V}_1$ with $A(f, g) = 0$. Moreover the solution $f = B(g)$ is defined by a smooth tame map B .

It is worth pointing out that in [Ham82, Thm III.3.3.1], Hamilton proves a more general theorem that allows for a "quadratic error" when inverting the partial derivative away from $(f, g) \in \mathcal{U}$ such that $A(f, g) = 0$. Again we stress that in contrast to the implicit function theorem of Banach spaces it is not sufficient to show surjectivity of the derivative in a point. This can be challenging in general but is much less problematic when dealing for instance with families of elliptic linear partial differential operators. These will be discussed in the following section.

6.2.1 Smooth tame families of elliptic linear partial differential operators

The following theorem appears in a very similar form in [Ham82] as Theorem II.3.3.3.

Theorem 6.30 (Theorem II.3.3.3 in [Ham82]). *Let $V \rightarrow M$, $W \rightarrow M$ be vector bundles over some compact manifold M . Let moreover $\mathcal{F}$ be a tame Fréchet space, $\mathcal{U} \subset \mathcal{F}$ an open*

subset and let V_1, V_2 be finite-dimensional vector spaces together with continuous linear maps

$$i: V_1 \rightarrow \Gamma(M, W), \quad j: \Gamma(M, V) \rightarrow V_2.$$

Consider a smooth tame family of linear maps

$$\ell: \mathcal{U} \times \Gamma(M, V) \longrightarrow \Gamma(M, W)$$

such that for every $f \in \mathcal{U}$, $\ell(f)$ is an elliptic linear partial differential operator of degree r . Suppose that for some norm on V_2 we have $\|j(h)\| \leq C\|h\|_{r-1}$ for every $h \in \Gamma(M, V)$. We define the family of linear maps

$$\begin{aligned} \tilde{\ell}: \mathcal{U} \times \Gamma(M, V) \times V_1 &\longrightarrow \Gamma(M, W) \times V_2 \\ (f, h, x) &\longmapsto (\ell(f)h + i(x), j(h)). \end{aligned}$$

If for some $f_0 \in \mathcal{U}$ the linear map $\tilde{\ell}(f_0)$ is invertible, there is an open neighborhood $\tilde{\mathcal{U}} \subset \mathcal{U}$ of f_0 such that $\tilde{\ell}(f)$ is an isomorphism for every $f \in \tilde{\mathcal{U}}$ and the corresponding family of inverses is a smooth tame family of linear maps.

The following application of Theorem 6.30 will be used later.

Proposition 6.31. *Let $\mathcal{U} \subset \mathcal{F}$ be an open neighborhood of 0 in a tame Fréchet space $\mathcal{F}$. Let M be a compact manifold, $d_i: \Gamma(M, E_i) \rightarrow \Gamma(M, E_{i+1})$ an elliptic complex such that d_i is of order one for every i and for a fixed $j \in \mathbb{Z}$ let*

$$A: \mathcal{U} \times \Gamma(M, E_j) \longrightarrow \Gamma(M, E_j)$$

be a smooth tame family of linear maps such that

$$A(\varphi)\alpha = \alpha + \delta(\varphi)\alpha,$$

where $\delta(\varphi)$ is such that $\alpha \mapsto \Delta_j \delta(\varphi)\alpha$ is a linear partial differential operator of order ≤ 2 , $\delta(\varphi)\alpha \in \text{im } d_j^$ for every $(\varphi, \alpha) \in \mathcal{U} \times \Gamma(M, E_j)$ and such that $\delta(0) = 0^1$. Then there is an open neighborhood $\mathcal{U}' \subset \mathcal{U}$ of 0 such that the restriction*

$$A(\varphi): (\text{im } d_j^* \cap \Gamma(M, E_j)) \longrightarrow (\text{im } d_j^* \cap \Gamma(M, E_j))$$

is invertible for every $\varphi \in \mathcal{U}'$ and the family of inverses is smooth tame. It follows that $A(\varphi)$ is invertible for every $\varphi \in \mathcal{U}'$ and the family of inverses is smooth tame.

Proof. Since the restriction

$$\Delta_j: \text{im } d_j^* \longrightarrow \text{im } d_j^*$$

is invertible, solving $A(\varphi)\alpha = \beta$ for $\alpha \in \text{im } d_j^*$, where $\beta \in \text{im } d_j^*$ is equivalent to solving

$$\Delta_j A(\varphi)\alpha = \Delta_j \alpha + \Delta_j \delta(\varphi)\alpha = \Delta_j \beta =: \tilde{\beta}$$

¹An example for a map δ with such properties that we will encounter in Chapter 7 is $\delta(\varphi)\alpha := G_{\bar{\partial}} \bar{\partial}^*[\varphi, \alpha]$, where $\varphi, \alpha \in \mathcal{A}^{0,1}(M, T^{1,0})$.

for $\alpha \in \text{im } d_j^*$. We define the smooth tame family of linear maps

$$\begin{aligned} \ell: \mathcal{U} \times \Gamma(M, E_j) &\longrightarrow \Gamma(M, E_j), \\ (\varphi, \alpha) &\longmapsto \Delta_j \alpha + \Delta_j \delta(\varphi) \alpha. \end{aligned}$$

Since $\delta(0) = 0$, we observe that $\ell(0) = \Delta_j$ and therefore, there is an open neighborhood $\mathcal{U}' \subset \mathcal{U}$ of 0 such that $\ell: \mathcal{U}' \times \Gamma(M, E_j)$ is a smooth tame family of elliptic operators of degree 2. We define

$$\begin{aligned} \tilde{\ell}: \mathcal{U}' \times \Gamma(M, E_j) \times \mathcal{H}_j(M) &\longrightarrow \Gamma(M, E_j) \times \mathcal{H}_j(M) \\ (\varphi, \alpha, \beta) &\longmapsto (\ell(\varphi) \alpha + i(\beta), j(\alpha)), \end{aligned}$$

where $\mathcal{H}_j$ denotes the harmonic forms of degree j and

$$i: \mathcal{H}_j(M) \hookrightarrow \Gamma(M, E_j), \quad \text{pr}_{\mathcal{H}}: \Gamma(M, E_j) \rightarrow \mathcal{H}_j(M),$$

denote the inclusion and orthogonal projection map, respectively. Let $\|\cdot\|$ be the norm on $\mathcal{H}_j(M)$ induced by $\|\cdot\|_{0,2}$ on $\Gamma(M, E_j)$. Then by Hodge theory

$$\|\text{pr}_{\mathcal{H}}(\alpha)\| \leq \|\alpha\|_{0,2} \leq \|\alpha\|_{1,2}.$$

Since $\ell(0) = \Delta_j$ and $\ker \Delta_j = \mathcal{H}_j(M) = \text{coker } \Delta_j$, we see from the definition of $\tilde{\ell}$ that $\tilde{\ell}(0)$ is invertible. Therefore, by Theorem 6.30 it follows that there is a neighborhood $\mathcal{U}'' \subset \mathcal{U}'$ such that $\tilde{\ell}(\varphi)$ is invertible for every $\varphi \in \mathcal{U}''$ and the family $\tilde{p}(\varphi)$ of inverses is smooth tame.

Next we show that as a consequence the restriction $\ell(\varphi): \text{im } d_j^* \rightarrow \text{im } d_j^*$ is invertible. Since

$$\ell(\varphi)|_{\text{im } d_j^*} = \tilde{\ell}(\varphi)|_{\text{im } d_j^* \oplus 0}: \text{im } d_j^* \oplus 0 \rightarrow \text{im } d_j^* \oplus 0,$$

$\ell(\varphi)(\text{im } d_{j-1}) \subset \text{im } d_{j-1} \oplus \text{im } d_j^*$ and $\ell(\varphi)(\text{im } d_{j-1}) \cap \text{im } d_j^* = 0$, it follows that $\ell(\varphi): \text{im } d_j^* \rightarrow \text{im } d_j^*$ is invertible and its family of inverses is a smooth tame family of linear maps given by the restriction

$$p(\varphi) := \tilde{p}(\varphi)|_{\text{im } d_j^* \oplus 0}: \text{im } d_j^* \cap \Gamma(M, E_j) \longrightarrow \text{im } d_j^* \cap \Gamma(M, E_j).$$

As explained above this implies the invertibility of $A(\varphi)|_{\text{im } d_j^*}: \text{im } d_j^* \rightarrow \text{im } d_j^*$ with a smooth tame family of inverses.

For the last assertion, given $\alpha \in \Gamma(M, E_j)$, $\beta \in \Gamma(M, E_j)$, we write

$$\alpha = \alpha_0 + d_{j-1} \alpha_1 + d_j^* \alpha_2, \quad \beta = \beta_0 + d_{j-1} \beta_1 + d_j^* \beta_2,$$

where $\Delta_j \alpha_0 = 0 = \Delta_j \beta_0$. Then the equation $A(\varphi) \alpha = \beta$ is equivalent to

$$\begin{aligned} \alpha_0 = \beta_0, \quad d_{j-1} \alpha_1 = d_{j-1} \beta_1, \quad d_j^* \alpha_2 + \delta(\varphi) d_j^* \alpha_2 &= d_j^* \beta_2 - \delta(\varphi) \{\alpha_0 + d_{j-1} \alpha_1\} \\ \implies A(\varphi) d_j^* \alpha_2 &= d_j^* \beta_2 - \delta(\varphi) \{\beta_0 + d_{j-1} \beta_1\}, \end{aligned}$$

which has a smooth tame family of solutions by the previous statement. $\square$

6.3 Tame Fréchet manifolds and tame Lie groups

We will not only be dealing with linear spaces but also with *tame manifolds*, topological spaces that are locally modeled on tame Fréchet spaces.

Definition 6.32. A *tame (Fréchet) manifold* is a Hausdorff topological space with an atlas of coordinate charts taking values in tame Fréchet spaces such that the transition maps are smooth tame maps.

We will also need a notion of *tame submanifolds*.

Definition 6.33. A *tame (Fréchet) submanifold* $\mathcal{N}$ of a tame Fréchet manifold $\mathcal{M}$ is a subset that can be covered by (smooth tame) charts $\{(\mathcal{U}_i, \Phi_i: \mathcal{U}_i \longrightarrow E_i \oplus F_i)\}$ of $\mathcal{M}$, where $E_i \oplus F_i$ is a direct sum of graded Fréchet spaces such that $\Phi_i(\mathcal{N} \cap \mathcal{U}_i) = \Phi_i(\mathcal{U}_i) \cap E_i \times \{0\}$.

Smooth tame maps between tame Fréchet manifolds are defined in the usual way using local charts.

Definition 6.34. A map $P: \mathcal{M} \rightarrow \mathcal{N}$ between tame Fréchet manifolds is called a *smooth tame* map of Fréchet manifolds if there exist charts around every point in $\mathcal{M}$ and its image in $\mathcal{N}$ such that the local representative of P in these charts is a smooth tame map of graded Fréchet spaces.

Example 6.35 (Corollary II.2.3.2 and Theorem II.2.3.3 in [Ham82]). Let M and N be finite-dimensional manifolds such that M is compact. Then the space $C^\infty(M, N)$ of smooth maps of M into N is a tame Fréchet manifold. If N' is another finite-dimensional manifold and N is also compact, then the composition map

$$\begin{aligned} C^\infty(N, N') \times C^\infty(M, N) &\longrightarrow C^\infty(M, N') \\ (f, g) &\longmapsto f \circ g \end{aligned}$$

is a smooth tame map of tame Fréchet manifolds.

Finally, we introduce *tame (Fréchet) Lie groups*.

Definition 6.36. A *tame (Fréchet) Lie group* is a tame Fréchet manifold $\mathcal{G}$ with a group structure such that the multiplication and the inversion map are smooth tame maps. A *tame (Fréchet) Lie subgroup* of a tame Lie group $\mathcal{G}$ is a tame Fréchet submanifold, which is a subgroup of $\mathcal{G}$.

The diffeomorphism group of a compact manifold is a tame Fréchet Lie group [Ham82]. We will explicitly describe a tame atlas in the following section. Another example of a tame Lie group is the automorphism bundle of a vector bundle over a compact manifold (cf. Proposition 6.26).

6.3.1 The diffeomorphism group of a compact manifold

Let M be a compact manifold. Fix an auxiliary Riemannian metric h on M . Then we denote by

$$\exp^h: TM \longrightarrow M$$

the exponential map with respect to h and by $\pi: TM \rightarrow M$ the canonical projection. There is a tubular neighborhood $U \subset TM$ of $M \subset TM$ on which the extended map

$$\begin{aligned} \text{Exp}^h: TM &\longrightarrow M \times M \\ v &\longmapsto \left(\pi(v), \exp_{\pi(v)}^h(v) \right) \end{aligned}$$

restricts to a diffeomorphism. This gives rise to a homeomorphism from a neighborhood $\mathcal{U}_{\text{id}} \subset \mathfrak{X}(M)$ of 0 to a neighborhood $\mathcal{V}_{\text{id}} \subset \text{Diff}(M)$ of $\text{id} \in \text{Diff}(M)$, which we will also denote by

$$\exp^h: \mathcal{U}_{\text{id}} \longrightarrow \mathcal{V}_{\text{id}}.$$

Similarly, for $f \in \text{Diff}(M)$ and some neighborhood $\mathcal{U}_f \subset \Gamma(M, f^*TM)$ of 0 in $\Gamma(M, f^*TM)$, we obtain a homeomorphism

$$\exp^h: \mathcal{U}_f \longrightarrow \mathcal{V}_f$$

to a neighborhood $\mathcal{V}_f \subset \text{Diff}(M)$ of f .

Proposition 6.37 (Theorem III.2.3.1 and Corollary III.2.3.2 in [Ham82]). *The local charts $(\mathcal{V}_f, (\exp^h)^{-1})$ form a smooth tame atlas of $\text{Diff}(M)$. In particular, we may identify $T_f \text{Diff}(M) \cong \Gamma(M, f^*TM)$.*

For $f \in \text{Diff}(M)$ we denote by L_f and R_f the left- and right-multiplication by f , respectively, and by I the inversion map.

Proposition 6.38 (Example I.4.4.5-6 in [Ham82]). *For $f, g \in \text{Diff}(M)$, $X \in \Gamma(M, g^*TM)$ we have*

$$DL_f(g)X = df \circ X, \quad DR_f(g)X = X \circ f, \quad DI(g)X = -dg^{-1} \circ X \circ g^{-1}.$$

The Lie algebra of the diffeomorphism group is given by $(\mathfrak{X}(M), -[\cdot, \cdot])$, where $[\cdot, \cdot]$ is the usual Lie bracket of vector fields. The adjoint action of $\text{Diff}(M)$ on its Lie algebra is given by

$$\begin{aligned} \text{Ad}: \text{Diff}(M) \times \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ (f, X) &\longmapsto \text{Ad}(f)X := df \circ X \circ f^{-1}. \end{aligned}$$

In particular this shows that the push-forward of a vector field by a diffeomorphism is a smooth tame map.

We will also consider the action of diffeomorphisms on endomorphisms of the tangent bundle. In the following we will see that this action is smooth tame.

Lemma 6.39. *The map*

$$\begin{aligned} \text{Diff}(M) \times \Gamma(M, \text{End } TM) &\longrightarrow \Gamma(M, \text{End } TM) \\ (f, \psi) &\longmapsto df \circ \psi \circ df^{-1} \end{aligned}$$

is smooth tame.

Proof. We define the auxiliary map

$$\begin{aligned} A: \operatorname{Diff}(M) \times \Gamma(M, \operatorname{End} TM) \times \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ (f, \psi, X) &\longmapsto df \circ \psi \circ df^{-1}(X). \end{aligned}$$

We observe that

$$\begin{aligned} A(f, \psi, X) &= df \circ \psi \circ df^{-1} \circ X \circ f \circ f^{-1} \\ &= df \circ (\psi \circ (df^{-1} \circ X \circ f)) \circ f^{-1} \\ &= \operatorname{Ad}(f)\{\psi \circ \operatorname{Ad}(f^{-1})X\}. \end{aligned}$$

It follows that A is a composition of smooth tame maps and hence smooth tame. In a local coordinate neighborhood, we can recover the original map by setting

$$(f, \psi) \longmapsto \sum_i A(f, \psi, \partial_i) \otimes dx^i.$$

These patch together to give a well-defined global section to $\operatorname{End} TM$. The patching can be made explicit using a partition of unity. In particular, this yields a smooth tame map. $\square$

Moreover, we will encounter the pull-back of differential forms by diffeomorphisms. In the following we show that this is a smooth tame map.

Proposition 6.40. *The pull-back of a k -form by a diffeomorphism seen as a map*

$$\begin{aligned} \operatorname{Diff}(M) \times \Omega^k(M) &\longrightarrow \Omega^k(M) \\ (f, \varphi) &\longmapsto f^*\varphi. \end{aligned}$$

is smooth tame.

Proof. First, we observe that for $k = 0$, the pull-back of $g \in \Omega^0(M) = C^\infty(M)$ simply becomes the composition map $f^*g = g \circ f$ for $f \in \operatorname{Diff}(M)$, which is a smooth tame map (see [Ham82, Thm II.2.3.3]).

Next, given an exact one-form $\alpha \in \Omega^1(M)$, we can write it as $\alpha = dg$ for some $g \in C^\infty(M)$. The function g is uniquely determined up to a constant and there is a tame linear map $\alpha \mapsto g$ such that $dg = \alpha$, given by $Q_{\operatorname{dR}} = d^* G_{\operatorname{dR}}$, where d^* is the formal adjoint of the de Rham differential with respect to the L^2 -inner product induced by the auxiliary Riemannian metric on M and G_{dR} is the Greens operator of the Laplacian $\Delta = d^*d + dd^*$. It follows from Example 6.16 that Q_{dR} is tame linear. The pull-back of α by $f \in \operatorname{Diff}(M)$ is then given by

$$f^*\alpha = f^*dg = d(g \circ f)$$

and hence smooth tame as a composition of smooth tame maps.

Moreover, for some $g \in C^\infty(M)$ and exact one-forms $\alpha_1, \dots, \alpha_k$, the pull-back of $\varphi := g \alpha_1 \wedge \dots \wedge \alpha_k$ by $f \in \operatorname{Diff}(M)$ is given by

$$f^*\varphi = (g \circ f) f^*\alpha_1 \wedge \dots \wedge f^*\alpha_k.$$

Since the wedge product is a tame multi-linear map, this shows that

$$\begin{aligned} \text{Diff}(M) \times C^\infty(M) \times (\text{d}\Omega^0(M))^k &\longrightarrow \Omega^k(M) \\ (f, g, \alpha_1, \dots, \alpha_k) &\longmapsto f^*(g \alpha_1 \wedge \dots \wedge \alpha_k) \end{aligned} \quad (6.1)$$

is a smooth tame map.

Locally, every k -form can be written as a finite sum of elements of the form $g \alpha_1 \wedge \dots \wedge \alpha_k$, where $g \in C^\infty(M)$ and $\alpha_i \in \text{d}\Omega^0(M)$. These local descriptions can be patched together using a partition of unity and since M is compact, in this way we can write every k -form as a finite sum of elements of the form $g \alpha_1 \wedge \dots \wedge \alpha_k$ as above. In particular, this shows that the pull-back of any k -form by some diffeomorphism is a finite sum of terms of the form as in (6.1) which are smooth tame. It follows that the pull-back is smooth and that all its derivatives satisfy tame estimates. This finishes the proof. $\square$

In the following chapter, we will be interested in the situation where the compact manifold is equipped with a complex structure J . We will often use the isomorphism of (real) vector bundles

$$\begin{aligned} \Psi: T^{1,0}M &\longrightarrow TM \\ \xi &\longmapsto \xi + \bar{\xi}, \end{aligned} \quad (6.2)$$

in order to identify $T_f \text{Diff}(M) \cong \Gamma(M, f^*T^{1,0}M)$.

Lemma 6.41. *For $\xi \in \Psi^{-1}(\mathcal{U}_{\text{id}})$ we denote $F_\xi := \exp^h \circ \Psi(\xi) \in \text{Diff}(M)$. Then*

$$\text{Ad}(F_\xi)X = X - [\Psi\xi, X] + R_{\text{Ad}}(\xi)X,$$

where $R_{\text{Ad}}(s\xi) = s^2 \tilde{R}_{\text{Ad}}(\xi, s)$, for a real parameter s , where $\tilde{R}_{\text{Ad}}(\xi, s)$ depends smoothly on s for small s .

Proof. The statement follows directly from Theorem 6.12 using the facts that $D_0 \exp^h = \text{id}$ and

$$D_f \text{Ad}(f = \text{id})\{X, Y\} = -[Y, X]$$

for $X, Y \in \mathfrak{X}(M)$. $\square$

Chapter 7

Deformations of Complex Structures

We have seen in Section 5.1.2 how deformations of a complex structure can be encoded into forms $\varphi \in \mathcal{A}^{0,1}(M, T^{1,0})$. We have also seen that integrability of the deformed structure is equivalent to a Maurer–Cartan equation for φ . However, as we already mentioned in Section 5.1.2, not every complex deformation φ will give rise to a genuinely new complex structure: some of them might simply arise by acting on the "original" complex structure with a small diffeomorphism. More generally, two deformations might be related to one another by the action of a diffeomorphism. This led to the notion of a *locally complete family of deformations*, a family of complex deformations from which every other family of (sufficiently small) deformations can be obtained by the action of a family of diffeomorphisms.

Building on previous work by Kodaira and Spencer (with contributions by Nirenberg) in [KNS58; KS58a; KS58b; KS60], Kuranishi [Kur62] proved a general theorem on the existence of locally complete families of deformations on compact complex manifolds. Subsequently, in [Kur65], he gave a simplified proof of his theorem. Kuranishi's proof in [Kur65] relies on Sobolev spaces and refers to both the inverse and implicit function theorem on Banach spaces. However, as we already mentioned in the introduction, the presentation in [Kur65] omits proofs of the differentiability of the relevant maps, which is required to argue that the implicit function theorem is indeed applicable.

In this first original chapter of Part II, we will revisit Kuranishi's proof in [Kur65] in the framework of tame Fréchet spaces using the Nash–Moser inverse function theorem as well as Hamilton's Nash–Moser implicit function theorem introduced in Chapter 6. This will set the foundations for our extension of the deformation theory of generalized complex structures to a larger class of Courant algebroids in Chapter 8.

The chapter is structured as follows. In Section 7.1, we will prove that the action of diffeomorphisms on the almost complex structures is smooth tame. We will then consider the corresponding local action on deformations and its local expression in a chart around the identity of the diffeomorphism group, which are again smooth tame maps. Moreover, we will compute a partial derivative of this local expression, which will be relevant to our proof of local completeness. In Section 7.2, we will then prove the "deformation theorem" on the existence of a locally complete family of complex deformations. The proof is split into two parts: First, we construct the family in Section 7.2.1, then we will prove its local completeness in Section 7.2.2. We will furthermore see that if the group of complex automorphisms of the complex manifold is discrete, this gives rise to a miniversal family, i.e. a locally complete

family with a minimal set of parameters.

Up to minor changes to the content and adaptations of the notation, all sections in this chapter are taken from the coauthored preprint [CP26].

7.1 Diffeomorphisms acting on deformations

Let M be a compact manifold. The set of almost complex structures of M can be seen as the space of sections $\text{AC}(M)$ of a fiber subbundle of $\text{End } TM$ and hence is a tame Fréchet submanifold of $\Gamma(M, \text{End } TM)$ [Ham82, Thm II.2.3.1]. The diffeomorphism group $\text{Diff}(M)$ of M acts on the tame Fréchet manifold of almost complex structures on M via

$$(f, J) \longmapsto J_f := df \circ J \circ df^{-1}.$$

This is a restriction of the action of diffeomorphisms on sections of the endomorphism bundle of TM , which is smooth tame by Lemma 6.39.

Proposition 7.1. *The action of the diffeomorphisms on the tame Fréchet manifold $\text{AC}(M)$ of almost complex structures is smooth tame.*

Proof. This follows from Lemma 6.39, since $\text{AC}(M) \subset \Gamma(M, \text{End } TM)$ is a tame Fréchet submanifold. $\square$

It was already mentioned in Section 5.1 that when acting with a diffeomorphism f contained in a small enough C^1 -neighborhood $\mathcal{V}$ of the identity on a complex structure J , the resulting complex structure J_f will again have finite distance from J . We will denote by $\Theta(f) \in \mathcal{A}^{0,1}(M, T^{1,0})$ the corresponding complex deformation. Since $\Theta(f)$ can be thought of as the action of $f \in \text{Diff}(M)$ in a local chart of the tame manifold of almost complex structures, the map $\Theta: \mathcal{V} \rightarrow \mathcal{A}^{0,1}(M, T^{1,0})$ is smooth tame.

Similarly, when acting on an almost complex structure J_φ at finite distance from J , there is a C^1 -neighborhood of the identity $\mathcal{V} \subset \text{Diff}(M)$ such that for $f \in \mathcal{V}$, the almost complex structure $J_{\varphi,f}$ has again finite distance from J and we will denote the corresponding almost complex deformation by $\Theta(f, \varphi) \in \mathcal{A}^{0,1}(M, T^{1,0})$. Again Θ can be thought of as a local description of the action of the diffeomorphisms on $\text{AC}(M)$ and hence, it is smooth tame. In the following we will see an explicit formula for $\Theta(f, \varphi)$. For this, we introduce the following notation for the different components of the $\mathbb{C}$ -linear extension of the differential df of a diffeomorphism f with respect to $T = T^{1,0} \oplus T^{0,1}$ by

$$df = \begin{pmatrix} \partial f & \bar{\partial} f \\ \partial \bar{f} & \bar{\partial} \bar{f} \end{pmatrix}.$$

Proposition 7.2. *There is a C^1 -neighborhood $\mathcal{V}$ of the identity in $\text{Diff}(M)$ and a C^0 -neighborhood $\mathcal{U} \subset \mathcal{A}^{0,1}(M, T^{1,0})$ of 0 such that for every $f \in \mathcal{V}$, $\varphi \in \mathcal{U}$, the almost complex structure $J_{\varphi,f}$ has finite distance from J . The corresponding almost complex deformation $\Theta(f, \varphi) \in \mathcal{A}^{0,1}(M, T^{1,0})$ depends smooth tamely on φ and f and is given by the formula*

$$\Theta(f, \varphi) = -(\bar{\partial} f - \partial \bar{f} \circ \varphi)(\bar{\partial} \bar{f} - \partial \bar{f} \circ \varphi)^{-1}. \quad (7.1)$$

Proof. We only need to prove the explicit formula for $\Theta(\varphi, f)$. For this, we observe that

$$T^{0,1}(J_{\varphi,f})_p = \left\{ df(Y - \varphi(Y)) : Y \in T_{f^{-1}(p)}^{0,1} \right\}.$$

Then

$$\begin{aligned} \text{pr}_{0,1} T^{0,1}(J_{\varphi,f})_p &= \left\{ \bar{\partial}f(Y) - \partial\bar{f}(\varphi(Y)) : Y \in T_{f^{-1}(p)}^{0,1} \right\}, \\ \text{pr}_{1,0} T^{0,1}(J_{\varphi,f})_p &= \left\{ \bar{\partial}f(Y) - \partial\bar{f}(\varphi(Y)) : Y \in T_{f^{-1}(p)}^{0,1} \right\}. \end{aligned}$$

Note that there is a C^0 -neighborhood $\mathcal{U} \subset \mathcal{A}^{0,1}(M, T^{1,0})$ and a C^1 -neighborhood $\mathcal{V} \subset \text{Diff}(M)$ of the identity such that $(\bar{\partial}f - \partial\bar{f} \circ \varphi) : T^{0,1} \rightarrow T^{0,1}$ is invertible for every $(f, \varphi) \in \mathcal{V} \times \mathcal{U}$. Then for every $Y \in T_{f^{-1}(p)}^{0,1}$ we can find a unique $X \in T_p^{0,1}$ such that

$$(\bar{\partial}f - \partial\bar{f} \circ \varphi) Y = X.$$

It follows that

$$T^{0,1}(J_{\varphi,f})_p = \left\{ X + (\bar{\partial}f - \partial\bar{f} \circ \varphi) (\bar{\partial}f - \partial\bar{f} \circ \varphi)^{-1} X : X \in T_p^{0,1} \right\}$$

and hence

$$\Theta(f, \varphi) = -(\bar{\partial}f - \partial\bar{f} \circ \varphi) (\bar{\partial}f - \partial\bar{f} \circ \varphi)^{-1}.$$

□

We will also write $\Theta(f, \varphi) =: f \cdot \varphi$.

Lemma 7.3. *Let $f, g \in \text{Diff}(M)$ and $\varphi \in \mathcal{A}^{0,1}(M, T^{1,0})$ such that $\Theta(g \circ f, \varphi)$ is defined. Then $\Theta(g \circ f, \varphi) = g \cdot \Theta(f, \varphi)$.*

Proof. This follows from

$$J_{g \circ f} = d(g \circ f) \circ J \circ d(g \circ f)^{-1} = dg \circ J_f \circ dg^{-1} = (J_f)_g.$$

□

We set

$$\begin{aligned} \mathcal{U}_\varepsilon^{(k)} &:= \{\xi \in \mathcal{A}^0(M, T^{1,0}) : \|\xi\|_{k,\infty} < \varepsilon\}, \\ \mathcal{W}_\varepsilon^{(k)} &:= \{\varphi \in \mathcal{A}^{0,1}(M, T^{1,0}) : \|\varphi\|_{k,\infty} < \varepsilon\}. \end{aligned}$$

Let h be an auxiliary Riemannian metric on M and Ψ as defined in (6.2). Then for $\varepsilon > 0$ small enough, we may concatenate Θ with the local parametrization $\exp^h \circ \Psi$ of $\text{Diff}(M)$ (cf. Section 6.3.1) to obtain a smooth tame map

$$\begin{aligned} \mathcal{E} : \mathcal{U}_\varepsilon^{(1)} \times \mathcal{W}_\varepsilon^{(0)} &\longrightarrow \mathcal{A}^{0,1}(M, T^{1,0}) \\ (\xi, \varphi) &\longmapsto \mathcal{E}(\xi, \varphi) := F_\xi \cdot \varphi. \end{aligned}$$

Remark 7.4. In [Ham77], Hamilton already stated that a map closely related to $\mathcal{E}$ is smooth tame by arguing that it is a nonlinear partial differential operator. However, this is not clear to us as when substituting $f = \exp^h(\Psi\xi)$ in the formula (7.1) for $\Theta(f, \varphi)$, the section

$\xi \in \mathcal{A}^{0,1}(M, T^{1,0})$ is evaluated in different base points. This means that the map $\mathcal{E}(\xi, \varphi) = \Theta(\exp^h(\Psi\xi), \varphi)$ does not factorize through the jet bundle. Therefore, we chose to give the different proof above.

In the following, we will compute the partial derivative $D_\xi \mathcal{E}(\xi, \varphi)$, which is needed for the proof of local completeness of the deformation family in Section 7.2.2. First, we prove the following proposition.

Proposition 7.5. *Let $(\xi, \varphi) \in \mathcal{U}_\varepsilon^{(1)} \times \mathcal{W}_\varepsilon^{(0)}$. Then for $\varepsilon > 0$ small enough*

$$\mathcal{E}(\xi, \varphi) = \varphi - \bar{\partial}(a_\varphi \xi) + [\varphi, a_\varphi \xi] - \iota_{\bar{\xi}}(\bar{\partial}\varphi - \tfrac{1}{2}[\varphi, \varphi]) + R(\xi, \varphi),$$

where $a_\varphi \xi = \xi + i_{\bar{\xi}}\varphi$ and $R(t\xi, \varphi) = t^2 \tilde{R}(\xi, \varphi, t)$ for a real parameter t with $\tilde{R}(\xi, \varphi, t)$ depending smoothly on t for small t .

Proof. We write

$$\begin{aligned} T_p^{0,1}(J_{\mathcal{E}(\xi, \varphi)}) &= \left\{ \text{Ad}(F_\xi)(X - \varphi(X))_p : X \in \Gamma(M, T^{0,1}) \right\} \\ &= \left\{ X_p - \varphi(X_p) - [\Psi\xi, X - \varphi(X)]_p + R_A(\xi)(X - \varphi(X))_p : X \in \Gamma(M, T^{0,1}) \right\}, \end{aligned}$$

where we used Lemma 6.41. Then

$$\text{pr}_{0,1}(T_p^{0,1}(J_{\mathcal{E}(\xi, \varphi)})) = \{X_p - \text{pr}_{0,1}[\Psi\xi, X - \varphi(X)]_p + \text{pr}_{0,1} R_A(\xi)(X - \varphi(X))_p : X \in \Gamma(M, T^{0,1})\}.$$

We define $B(\xi, \varphi) : \Gamma(M, T^{0,1}) \rightarrow \Gamma(M, T^{0,1})$ by

$$\begin{aligned} B(\xi, \varphi)X &= \text{pr}_{0,1} \text{Ad}(F_\xi)(X - \varphi(X)) \\ &= X - \text{pr}_{0,1}[\Psi\xi, X - \varphi(X)] + \text{pr}_{0,1} R_A(\xi)(X - \varphi(X)). \end{aligned} \tag{7.2}$$

For $\varepsilon > 0$ small enough, the map $B(\xi, \varphi)$ is invertible for every $(\xi, \varphi) \in \mathcal{U}_\varepsilon^{(1)} \times \mathcal{W}_\varepsilon^{(0)}$ and the family of inverses $B(\xi, \varphi)^{-1}Z$ depends smooth tamely on φ and ξ . To see this note that the inverse of $\text{pr}_{0,1} : T^{0,1}(J_{\mathcal{E}(\xi, \varphi)}) \rightarrow T^{0,1}$ is given by $\text{id} - \mathcal{E}(\xi, \varphi) : T^{0,1} \rightarrow T^{0,1}(J_{\mathcal{E}(\xi, \varphi)}) \subset T$, which we already know depends smooth tamely on ξ and φ , and the inverse of $X \mapsto \text{Ad}(F_\xi)(X - \varphi(X))$ is given by $Z \mapsto \text{pr}_{0,1} \text{Ad}(F_\xi^{-1})Y$. Using that $B(\varphi, \xi)^{-1}Z$ depends smoothly on ξ and equation (7.2), we note that

$$B(\xi, \varphi)^{-1}Z = Z + \text{pr}_{0,1}[\Psi\xi, Z - \varphi(Z)] + \text{higher orders of } \xi.$$

Then for $Z \in \Gamma(M, T^{0,1})$ we find that

$$\begin{aligned} T_p^{0,1}(J_{\mathcal{E}(\xi, \varphi)}) &= \{Z_p - \varphi(B(\xi, \varphi)^{-1}(Z)_p) - \text{pr}_{1,0}[\Psi\xi, B(\xi, \varphi)^{-1}(Z) - \varphi(B(\xi, \varphi)^{-1}(Z))]\}_p \\ &\quad + \text{higher orders of } \xi : Z \in \Gamma(M, T^{0,1})\} \\ &= \{Z_p - \varphi(Z_p) - \varphi(\text{pr}_{0,1}[\Psi\xi, Z - \varphi(Z)]_p) - \text{pr}_{1,0}[\Psi\xi, Z - \varphi(Z)]_p \\ &\quad + \text{higher orders of } \xi : Z \in \Gamma(M, T^{0,1})\}. \end{aligned}$$

It follows that

$$\mathcal{E}(\xi, \varphi)(Z_p) = \varphi(Z_p) + \text{pr}_{1,0}[\Psi\xi, Z - \varphi(Z)]_p + \varphi(\text{pr}_{0,1}[\Psi\xi, Z - \varphi(Z)]_p) + R(\xi, \varphi)(Z_p), \tag{7.3}$$

where $R(t\xi, \varphi) = t^2 R(\xi, \varphi, t)$ for a real parameter t with $R(\xi, \varphi, t)$ depending smoothly on t for small t . We compute

$$\begin{aligned} & \text{pr}_{1,0}[\Psi\xi, Z - \varphi(Z)]_p + \varphi(\text{pr}_{0,1}[\Psi\xi, Z - \varphi(Z)]_p) \\ &= \text{pr}_{1,0}[\xi, Z] - [\xi, \varphi(Z)] - \text{pr}_{1,0}[\bar{\xi}, \varphi(Z)] + \varphi([\bar{\xi}, Z]) + \varphi(\text{pr}_{0,1}[\xi, Z]) - \varphi(\text{pr}_{0,1}[\bar{\xi}, \varphi(Z)]) \\ &= -\bar{\partial}\xi(Z) + [\varphi, \xi](Z) - \bar{\partial}\varphi(\bar{\xi}, Z) + \frac{1}{2}[\varphi, \varphi](\bar{\xi}, Z) - \bar{\partial}(\varphi(\bar{\xi}))(Z) + [\varphi, \varphi(\bar{\xi})](Z) \\ &= -\bar{\partial}(a_\varphi\xi)(Z) + [\varphi, a_\varphi\xi](Z) - \iota_{\bar{\xi}}(\bar{\partial}\varphi - \frac{1}{2}[\varphi, \varphi])(Z), \end{aligned}$$

where we used Lemma 5.2 and wrote $a_\varphi\xi = \xi + \iota_{\bar{\xi}}\varphi$. Plugging this back into equation (7.3), we obtain the expression for $\mathcal{E}(\xi, \varphi)$ as stated above. $\square$

Lemma 7.6. *For some $\varepsilon > 0$, $\varphi \in \mathcal{W}_\varepsilon^{(0)}$, let*

$$\begin{aligned} a_\varphi: \mathcal{A}^0(M, T^{1,0}) &\longrightarrow \mathcal{A}^0(M, T^{1,0}) \\ \xi &\longmapsto \xi + \iota_{\bar{\xi}}\varphi. \end{aligned}$$

Then the family of (real-)linear maps

$$\begin{aligned} a: \mathcal{W}_\varepsilon^{(0)} \times \mathcal{A}^{0,1}(M, T^{1,0}) &\longrightarrow \mathcal{A}^0(M, T^{1,0}) \\ (\varphi, \xi) &\longmapsto a_\varphi\xi \end{aligned}$$

is smooth tame. Moreover, for $\varepsilon > 0$ small enough, a_φ is invertible for every $\varphi \in \mathcal{W}_\varepsilon^{(0)}$ and the family of inverses a_φ^{-1} is smooth tame.

Proof. That a is smooth tame of tameness degree 0 follows from Proposition 6.25 and the fact that it is a non-linear partial differential operator of degree 0. Moreover, a_φ is invertible for $\|\varphi\|_{0,\infty}$ small enough. Define $\tilde{\varphi}: T^{1,0} \rightarrow T^{1,0}$ by $\tilde{\varphi}(\xi) := \varphi(\bar{\xi})$. Then $a_\varphi = \text{id} + \tilde{\varphi}$ and the inverse a_φ^{-1} is given by the sum

$$a_\varphi^{-1} = \sum_{k=0}^{\infty} (-\tilde{\varphi})^k,$$

which converges fiberwise for small enough $\|\varphi\|_{0,\infty}$. In particular, the whole family of linear maps $a_\varphi^{-1}\xi$ is a non-linear partial differential operator of degree 0 and therefore smooth tame of degree 0 by Proposition 6.25. $\square$

We will now use Proposition 7.5 to compute $D_\xi\mathcal{E}(\xi, \varphi)$ for almost complex deformations φ :

Proposition 7.7. *Let $(\xi, \varphi) \in \mathcal{U}_\varepsilon^{(1)} \times \mathcal{W}_\varepsilon^{(0)}$. Then for $\varepsilon > 0$ small enough*

$$D_\xi\mathcal{E}(\xi, \varphi)\eta = -\bar{\partial}(b_{(\varphi,\xi)}\eta) + [\mathcal{E}(\xi, \varphi), b_{(\xi,\varphi)}\eta] - \iota_{\overline{c_\xi\eta}}(\bar{\partial}\mathcal{E}(\xi, \varphi) - \frac{1}{2}[\mathcal{E}(\xi, \varphi), \mathcal{E}(\xi, \varphi)]),$$

where $b_{(\xi,\varphi)}\eta = a_{\mathcal{E}(\xi,\varphi)}c_\xi\eta$, with a as defined in Proposition 7.5 and

$$c_\xi\eta = DC_\xi(0)\eta, \quad C_\xi: \mathcal{U} \ni \tilde{\eta} \mapsto \left(\exp^h \circ \Psi\right)^{-1} \left(F_{\xi+\tilde{\eta}}F_\xi^{-1}\right) \in \mathcal{A}^0(M, T^{1,0}),$$

for a sufficiently small C^1 -neighborhood $\mathcal{U} \subset \mathcal{A}^0(M, T^{1,0})$ of zero.

Proof. Let $\xi \in \mathcal{U}_\varepsilon^{(1)}$. Moreover, let $\mathcal{U} \subset \mathcal{U}_\varepsilon^{(1)}$ be an open neighborhood of 0 such that $\xi + \tilde{\eta} \in \mathcal{U}_\varepsilon^{(1)}$ for every $\tilde{\eta} \in \mathcal{U}$. We have:

$$\mathcal{E}(\xi + \tilde{\eta}, \varphi) = F_{\xi + \tilde{\eta}} \cdot \varphi.$$

Note that, possibly after shrinking $\mathcal{U}$, we have

$$F_{\xi + \tilde{\eta}} \circ F_\xi^{-1} \in \exp^h \circ \Psi(\mathcal{U}_\varepsilon^{(1)})$$

for every $\tilde{\eta} \in \mathcal{U}$. We set

$$C_\xi(\tilde{\eta}) := \left(\exp^h \circ \Psi \right)^{-1} \left(F_{\xi + \tilde{\eta}} F_\xi^{-1} \right).$$

Then by Lemma 7.3, we obtain

$$\mathcal{E}(\xi + \tilde{\eta}, \varphi) = \left(F_{C_\xi(\tilde{\eta})} \circ F_\xi \right) \cdot \varphi = F_{C_\xi(\tilde{\eta})} \cdot \mathcal{E}(\xi, \varphi) = \mathcal{E}(C_\xi(\tilde{\eta}), \mathcal{E}(\xi, \varphi)).$$

By Proposition 7.5 and since $C_\xi(0) = 0$, it follows that

$$\begin{aligned} D_\xi \mathcal{E}(\xi, \varphi) \eta &= \lim_{t \rightarrow 0} \frac{1}{t} \cdot [\mathcal{E}(\xi + t\eta, \varphi) - \mathcal{E}(\xi, \varphi)] \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \cdot [\mathcal{E}(C_\xi(t\eta), \mathcal{E}(\xi, \varphi)) - \mathcal{E}(\xi, \varphi)] \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \cdot \left[-\bar{\partial}(a_{\mathcal{E}(\xi, \varphi)} C_\xi(t\eta)) + [\mathcal{E}(\xi, \varphi), a_{\mathcal{E}(\xi, \varphi)} C_\xi(t\eta)] \right. \\ &\quad \left. - \iota_{\overline{C_\xi(t\eta)}} (\bar{\partial} \mathcal{E}(\xi, \varphi) - \tfrac{1}{2} [\mathcal{E}(\xi, \varphi), \mathcal{E}(\xi, \varphi)]) + R(C_\xi(t\eta), \mathcal{E}(\xi, \varphi)) \right] \\ &= -\bar{\partial}(a_{\mathcal{E}(\xi, \varphi)} c_\xi \eta) + [\mathcal{E}(\xi, \varphi), a_{\mathcal{E}(\xi, \varphi)} c_\xi \eta] - \iota_{\overline{c_\xi \eta}} (\bar{\partial} \mathcal{E}(\xi, \varphi) - \tfrac{1}{2} [\mathcal{E}(\xi, \varphi), \mathcal{E}(\xi, \varphi)]) , \end{aligned}$$

where $c_\xi(\eta) = DC_\xi(0)\eta$. After setting $b_{(\xi, \varphi)}\eta = a_{\mathcal{E}(\xi, \varphi)}c_\xi\eta$ we arrive at the expression as stated in Proposition 7.7. $\square$

Lemma 7.8. *For some $\varepsilon > 0$, let $\mathcal{W}_\varepsilon^{(0)}$, $\mathcal{U}_\varepsilon^{(1)}$ as above. We assume that $\varepsilon > 0$ is small enough so that c_ξ and $b_{(\xi, \varphi)}$ as in Proposition 7.7 are defined for $\xi \in \mathcal{U}_\varepsilon^{(1)}$ and $\varphi \in \mathcal{W}_\varepsilon^{(0)}$. Then the families of linear maps*

$$\begin{aligned} c: \mathcal{U}_\varepsilon^{(1)} \times \mathcal{A}^0(M, T^{1,0}) &\longrightarrow \mathcal{A}^0(M, T^{1,0}) \\ (\xi, \eta) &\longmapsto c_\xi \eta \end{aligned}$$

and

$$\begin{aligned} b: \mathcal{U}_\varepsilon^{(1)} \times \mathcal{W}_\varepsilon^{(0)} \times \mathcal{A}^0(M, T^{1,0}) &\longrightarrow \mathcal{A}^0(M, T^{1,0}) \\ (\xi, \varphi, \eta) &\longmapsto b_{(\xi, \varphi)} \eta \end{aligned}$$

are smooth tame. Moreover, for $\varepsilon > 0$ small enough, c_ξ is invertible for every $\xi \in \mathcal{U}_\varepsilon^{(1)}$, and $b_{(\xi, \varphi)}$ is invertible for every $\xi \in \mathcal{U}_\varepsilon^{(2)}$, $\varphi \in \mathcal{W}_\varepsilon^{(1)}$. The families of inverses c_ξ^{-1} and $b_{(\xi, \varphi)}^{-1}$ are smooth tame.

Proof. Recall that

$$c_\xi \eta = DC_\xi(0)\eta, \quad C_\xi: \mathcal{U} \ni \tilde{\eta} \mapsto \left(\exp^h \circ \Psi \right)^{-1} \left(F_{\xi+\tilde{\eta}} F_\xi^{-1} \right) \in \mathcal{A}^0(M, T^{1,0}),$$

where $\mathcal{U} \subset \mathcal{U}_\varepsilon^{(1)}$ is an open neighborhood of 0 such that $\xi + \tilde{\eta} \in \mathcal{U}_\varepsilon^{(1)}$ for every $\tilde{\eta} \in \mathcal{U}$ and $F_{\xi+\tilde{\eta}} \circ F_\xi^{-1} \in \exp^h \circ \Psi(\mathcal{U}_\varepsilon^{(1)})$. Note that C_ξ can be extended to a C^0 -neighborhood $\tilde{\mathcal{U}}$ of 0 such that $\xi + \tilde{\eta} \in (\exp^h \circ \Psi)^{-1}(\mathcal{V}_{\text{id}})$ and $F_{\xi+\tilde{\eta}} F_\xi^{-1} \in \mathcal{V}_{\text{id}}$ for every $\tilde{\eta} \in \tilde{\mathcal{U}}$, where $\mathcal{V}_{\text{id}}$ denotes the local chart neighborhood of $\text{id} \in \text{Diff}(M)$. We may furthermore assume, possibly after shrinking $\varepsilon > 0$, that $\tilde{\mathcal{U}}$ can be chosen independently of ξ , such that we obtain a well-defined map

$$\begin{aligned} C: \mathcal{U}_\varepsilon^{(1)} \times \tilde{\mathcal{U}} &\longrightarrow \mathcal{A}^0(M, T^{1,0}) \\ (\xi, \tilde{\eta}) &\longmapsto C_\xi(\tilde{\eta}). \end{aligned}$$

Then $C_\xi(\tilde{\eta})$ is smooth tame in ξ and $\tilde{\eta}$ as a composition of smooth tame maps and invertible for every small enough ξ . To compute the inverse, we note that

$$C_\xi(\tilde{\eta}) = (\exp^h \circ \Psi)^{-1} \left(R_{F_\xi}^{-1} \circ \exp^h \circ \Psi(\xi + \tilde{\eta}) \right).$$

Since $R_{F_\xi}^{-1} = R_{F_\xi}$, we find that

$$\begin{aligned} C_\xi^{-1}(\chi) &= (\exp^h \circ \Psi)^{-1} \circ R_{F_\xi} \circ \exp^h \circ \Psi(\chi) - \xi \\ &= (\exp^h \circ \Psi)^{-1} (F_\chi F_\xi) - \xi, \end{aligned}$$

which is smooth tame. It follows that the family c of linear maps is smooth tame and c_ξ is invertible for small enough ξ and the family of inverses c_ξ^{-1} is again smooth tame.

For the statements about b we observe that when shrinking ε , for $(\xi, \varphi) \in \mathcal{U}_\varepsilon^{(2)} \times \mathcal{W}_\varepsilon^{(1)}$ we can make $\|\mathcal{E}(\xi, \varphi)\|_{0,\infty}$ arbitrarily small. Indeed, using Proposition 6.9 and the properties of the Fréchet integral in Theorem 6.11, we find

$$\begin{aligned} \|\mathcal{E}(\varphi, \xi)\|_{0,\infty} &= \left\| \varphi + \int_0^1 D_\xi \mathcal{E}(\varphi, t\xi) \xi \, dt \right\|_{0,\infty} \\ &\leq \|\varphi\|_{0,\infty} + \int_0^1 \|D_\xi \mathcal{E}(\varphi, t\xi) \xi\|_{0,\infty} \, dt \\ &\leq \|\varphi\|_{0,\infty} + \int_0^1 C((\|\varphi\|_{1,\infty} + t\|\xi\|_{2,\infty}) \|\xi\|_{1,\infty} + \|\xi\|_{1,\infty}) \, dt \\ &\leq \varepsilon \left(1 + C + \frac{3}{2}\varepsilon \right), \end{aligned}$$

for $(\xi, \varphi) \in \mathcal{U}_\varepsilon^{(2)} \times \mathcal{W}_\varepsilon^{(1)}$, where we used Lemma 6.22 in the third step.

It follows that for small enough $\varepsilon > 0$, we can use Lemma 7.6 to conclude that $a_{\mathcal{E}(\xi, \varphi)} \eta$ is a smooth tame family of linear maps which is invertible for every $\varphi \in \mathcal{W}_\varepsilon^{(1)}$, $\xi \in \mathcal{U}_\varepsilon^{(2)}$ and the family of inverses is again smooth tame. Since $b_{(\xi, \varphi)} \eta = a_{\mathcal{E}(\xi, \varphi)} c_\xi \eta$, this proves the assertion. $\square$

7.2 The deformation theorem

Let now (M, J) be a compact complex manifold. When not stated otherwise, the bi-gradings appearing in this section are with respect to the complex structure J . Our aim is to show the following:

Theorem 7.9 (Deformation theorem). *There exists an open neighborhood $\mathcal{W} \subset \mathcal{H}^{0,1}(M, T^{1,0})$ of 0, a family $\tilde{\mathcal{M}} = \{\varphi_t : t \in \mathcal{W}\}$ of almost complex deformations of J , and an analytic obstruction map*

$$\phi: \mathcal{W} \rightarrow \mathcal{H}^{0,2}(M, T^{1,0})$$

such that the almost complex deformations $\mathcal{M} := \{\varphi_t \in \tilde{\mathcal{M}} : \phi(t) = 0\}$ are precisely the integrable ones. Any sufficiently small complex deformation of J is equivalent to at least one member of the family $\mathcal{M}$. In the case that the obstruction map vanishes, $\mathcal{M}$ is a locally complete family of complex deformations.

Definition 7.10. We will call the family $\mathcal{M}$ in Theorem 7.9 the *Kuranishi family* and the set $\phi^{-1}(0) \subset \mathcal{W}$ the *Kuranishi moduli space*.

The proof of Theorem 7.9 is split into two parts. First, we will construct the family and the obstruction map in Section 7.2.1. Then we will prove that the family is locally complete in Section 7.2.2. We will closely follow Kuranishi's line of thought (in particular in Section 7.2.1) complemented by the techniques of Hamilton–Nash–Moser theory.

7.2.1 Description of the family

Let

$$\tilde{\mathcal{V}}_1 := \{\varphi \in \mathcal{A}^{0,1}(M, T^{1,0}) : \bar{\partial}\varphi - \frac{1}{2}[\varphi, \varphi] = 0 = \bar{\partial}^*\varphi\} \subset \mathcal{A}^{0,1}(M, T^{1,0}).$$

We will see that as a set, the family of complex deformations $\mathcal{M}$ is contained in $\tilde{\mathcal{V}}_1$. Our strategy will be the following: First we show that $\tilde{\mathcal{V}}_1$ is contained in another set $\tilde{\mathcal{V}}_2$ and that for small enough $\varepsilon > 0$ and large enough $k \in \mathbb{N}_0$, the intersection $\mathcal{W}_\varepsilon^{(k)} \cap \tilde{\mathcal{V}}_2$ is parametrized by a neighborhood of 0 in $\mathcal{H}^{0,1}(M, T^{1,0})$. Then we will find an analytic map

$$\phi: \mathcal{W}_\varepsilon^{(k)} \cap \tilde{\mathcal{V}}_2 \longrightarrow \mathcal{H}^{0,2}(M, T^{1,0})$$

such that $\phi(\varphi) = 0$ if and only if φ is integrable and hence contained in $\mathcal{M}$.

In the following lemma, we define the set $\tilde{\mathcal{V}}_2$ and prove that it contains the set $\tilde{\mathcal{V}}_1$ as defined above.

Lemma 7.11. $\tilde{\mathcal{V}}_1 \subset \tilde{\mathcal{V}}_2 := \{\varphi \in \mathcal{A}^{0,1}(M, T^{1,0}) : \varphi - \frac{1}{2} \mathcal{Q}_{\bar{\partial}}[\varphi, \varphi] \in \mathcal{H}^{0,1}(M, T^{1,0})\}$.

Proof. Let $\varphi \in \tilde{\mathcal{V}}_1$. Then

$$\Delta_{\bar{\partial}}\varphi = (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\varphi = \bar{\partial}^*\bar{\partial}\varphi = \frac{1}{2}\bar{\partial}^*[\varphi, \varphi],$$

where we used $\bar{\partial}^*\varphi = 0$ for the second equality and $\bar{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]$ for the last one. Applying the Green's operator $G_{\bar{\partial}}$ to both sides and using $\text{pr}_{\mathcal{H}} + G_{\bar{\partial}}\Delta_{\bar{\partial}} = \text{id}$ yields

$$0 = G_{\bar{\partial}}\Delta_{\bar{\partial}}\varphi - \frac{1}{2}G_{\bar{\partial}}\bar{\partial}^*[\varphi, \varphi] = \varphi - \text{pr}_{\mathcal{H}}\varphi - \frac{1}{2}\mathcal{Q}_{\bar{\partial}}[\varphi, \varphi].$$

Hence,

$$\varphi - \frac{1}{2} Q_{\bar{\partial}}[\varphi, \varphi] = \text{pr}_{\mathcal{H}} \varphi \in \mathcal{H}^{0,1}(M, T^{1,0})$$

and thus, $\varphi \in \tilde{\mathcal{V}}_2$. $\square$

Next, we want to show that for some $\varepsilon > 0$ and $k \in \mathbb{N}_0$, the intersection $\mathcal{W}_{\varepsilon}^{(k)} \cap \tilde{\mathcal{V}}_2$ is parametrized by a neighborhood of 0 in $\mathcal{H}^{0,1}(M, T^{1,0})$. To this end, we define the map

$$\begin{aligned} \sigma: \mathcal{A}^{0,1}(M, T^{1,0}) &\longrightarrow \mathcal{A}^{0,1}(M, T^{1,0}) \\ \varphi &\longmapsto \varphi - \frac{1}{2} Q_{\bar{\partial}}[\varphi, \varphi]. \end{aligned}$$

Lemma 7.12. *The map σ is smooth tame.*

Proof. This follows from Example 6.15 and 6.16. $\square$

Proposition 7.13. *For $\varepsilon > 0$ small enough and large enough $k \in \mathbb{N}_0$, the equation $D\sigma(\varphi)\alpha = \beta$ has a unique solution α for every $\varphi \in \mathcal{W}_{\varepsilon}^{(k)}$, $\beta \in \mathcal{A}^{0,1}(M, T^{1,0})$. Moreover, the family $V(\varphi)\beta = \alpha$ of solutions is a smooth tame family of linear maps.*

Proof. We compute

$$D\sigma(\varphi)\alpha = \alpha - Q_{\bar{\partial}}[\alpha, \varphi] = \alpha - \bar{\partial}^* G_{\bar{\partial}}[\alpha, \varphi].$$

The statement follows then from Proposition 6.31. $\square$

Corollary 7.14. *For $\varepsilon > 0$ small enough and $k \in \mathbb{N}_0$ large enough, σ maps $\mathcal{W}_{\varepsilon}^{(k)}$ bijectively onto a neighborhood $\tilde{\mathcal{W}}_{\varepsilon} \subset \mathcal{A}^{0,1}(M, T^{1,0})$ of 0 such that the inverse map*

$$\sigma^{-1}: \tilde{\mathcal{W}}_{\varepsilon} \longrightarrow \mathcal{W}_{\varepsilon}^{(k)}$$

is smooth tame.

Proof. This follows from Theorem 6.27 and Proposition 7.13. $\square$

In particular, setting $\mathcal{W}_{\varepsilon} := \tilde{\mathcal{W}}_{\varepsilon} \cap \mathcal{H}^{0,1}(M, T^{1,0})$ and $\mathcal{V}_{\varepsilon} := \tilde{\mathcal{V}}_2 \cap \mathcal{W}_{\varepsilon}^{(k)}$, we obtain a smooth tame map

$$\varphi := \sigma^{-1}|_{\mathcal{W}_{\varepsilon}}: \mathcal{W}_{\varepsilon} \longrightarrow \mathcal{V}_{\varepsilon}.$$

Corollary 7.15. *The family $\tilde{\mathcal{M}} := \{\varphi(t) : t \in \mathcal{W}_{\varepsilon}\}$ is a smooth family of almost complex deformations.*

Next, we will identify the obstruction map, whose vanishing set are precisely the integrable deformations in $\mathcal{V}_{\varepsilon}$.

Lemma 7.16. *For $\varepsilon > 0$ small enough, $\varphi \in \mathcal{V}_{\varepsilon}$ is in $\mathcal{S}_{\varepsilon} := \tilde{\mathcal{V}}_1 \cap \mathcal{W}_{\varepsilon}^{(k)}$ if and only if $\phi(\varphi) := \text{pr}_{\mathcal{H}}[\varphi, \varphi] = 0$.*

Proof. We first note that for any $\varphi \in \mathcal{V}_{\varepsilon}$, we have

$$\begin{aligned} \bar{\partial}\varphi - \frac{1}{2}[\varphi, \varphi] &= \frac{1}{2}\bar{\partial}Q_{\bar{\partial}}[\varphi, \varphi] - \frac{1}{2}[\varphi, \varphi] \\ &= -\frac{1}{2}Q_{\bar{\partial}}\bar{\partial}[\varphi, \varphi] - \frac{1}{2}\text{pr}_{\mathcal{H}}[\varphi, \varphi], \end{aligned}$$

where we used $\varphi \in \mathcal{V}_\varepsilon$ for the first equality and $\text{pr}_{\mathcal{H}} + \bar{\partial} Q_{\bar{\partial}} + Q_{\bar{\partial}} \bar{\partial} = \text{id}$ for the second equality. Since the images of $Q_{\bar{\partial}}$ and $\text{pr}_{\mathcal{H}}$ are orthogonal to each other, it follows that

$$\bar{\partial}\varphi - \frac{1}{2}[\varphi, \varphi] = 0 \iff Q_{\bar{\partial}} \bar{\partial}[\varphi, \varphi] = 0 \text{ and } \text{pr}_{\mathcal{H}}[\varphi, \varphi] = 0. \quad (7.4)$$

We claim that for $\varepsilon > 0$ small enough, $\text{pr}_{\mathcal{H}}[\varphi, \varphi] = 0$ already implies $Q_{\bar{\partial}} \bar{\partial}[\varphi, \varphi] = 0$. To see this, suppose $\varphi \in \mathcal{V}_\varepsilon$ such that $\text{pr}_{\mathcal{H}}[\varphi, \varphi] = 0$. Then,

$$Q_{\bar{\partial}} \bar{\partial}[\varphi, \varphi] = -\bar{\partial} Q_{\bar{\partial}}[\varphi, \varphi] + [\varphi, \varphi].$$

Using the compatibility of the bracket $[\cdot, \cdot]$ and $\bar{\partial}$ as stated in Lemma 5.1 and $\varphi \in \tilde{\mathcal{V}}_2$, we compute

$$Q_{\bar{\partial}} \bar{\partial}[\varphi, \varphi] = 2 Q_{\bar{\partial}}[\bar{\partial}\varphi, \varphi] = Q_{\bar{\partial}}[\bar{\partial} Q_{\bar{\partial}}[\varphi, \varphi], \varphi] = Q_{\bar{\partial}}[-Q_{\bar{\partial}} \bar{\partial}[\varphi, \varphi], \varphi] + Q_{\bar{\partial}}[[\varphi, \varphi], \varphi].$$

By the Jacobi identity (see Lemma 5.1), we must have $[[\varphi, \varphi], \varphi] = 0$. Therefore, setting $\eta := Q_{\bar{\partial}} \bar{\partial}[\varphi, \varphi]$, we find

$$\eta = -Q_{\bar{\partial}}[\eta, \varphi].$$

It follows from Example 6.15 and 6.16 that for $k \geq \dim M/2 + 1$

$$\|\eta\|_{k,2} = \|Q_{\bar{\partial}}[\eta, \varphi]\|_{k,2} \leq C\|\eta\|_{k,2}\|\varphi\|_{k,2} \leq C\varepsilon\|\eta\|_{k,2}.$$

In particular, for $\varepsilon < 1/C$, this implies that $\eta = 0$ and hence $Q_{\bar{\partial}} \bar{\partial}[\varphi, \varphi] = 0$. By equation (7.4) this shows that $\varphi \in \mathcal{S}_\varepsilon$. $\square$

7.2.2 Local completeness of the family

In the last step of the proof of Theorem 7.9, we need to show that every sufficiently small complex deformation φ can be obtained from $\mathcal{M}$ via a diffeomorphism, i.e. there is some small $\xi \in \mathcal{A}^0(M, T^{1,0})$ such that $\bar{\partial}^* \mathcal{E}(\xi, \varphi) = 0$. Moreover, given a smooth family of small (almost) complex deformations φ_t , we need to show that there is a family $\xi_t \in \mathcal{A}^0(M, T^{1,0})$ depending smoothly on t such that $\bar{\partial}^* \mathcal{E}(\xi_t, \varphi_t) = 0$.

In the following, we will denote by $\mathcal{H}^0(M, T^{1,0})^\perp$, the orthogonal complement of $\mathcal{H}^0(M, T^{1,0})$ in $\mathcal{A}^0(M, T^{1,0})$. The elements of $\mathcal{H}^0(M, T^{1,0})^\perp$ are the $\bar{\partial}^*$ -exact forms in $\mathcal{A}^0(M, T^{1,0})$.

Proposition 7.17. *There is a neighborhood $\mathcal{W}$ of the origin in $\mathcal{A}^{0,1}(M, T^{1,0})$ and a neighborhood $\mathcal{U}$ of the origin in $\mathcal{H}^0(M, T^{1,0})^\perp$ such that for any $\varphi \in \mathcal{W}$ there is a unique solution $\xi = \xi(\varphi) \in \mathcal{U}$ to the equation*

$$\bar{\partial}^* \mathcal{E}(\xi, \varphi) = 0.$$

The solution map $\xi(\varphi)$ is smooth tame.

Remark 7.18. Note that if $\mathcal{H}^0(M, T^{1,0}) = 0$, Proposition 7.17 implies that every almost complex deformation $\varphi \in \mathcal{W}$ is related to an almost complex deformation φ' such that $\bar{\partial}^* \varphi' = 0$ via a unique diffeomorphism F_ξ , $\xi \in \mathcal{U}$. Thus, the representative φ' is locally unique. If the obstruction map vanishes, Theorem 7.9 yields a so-called *miniversal* family of complex deformations and the condition $\bar{\partial}^* \varphi = 0$ can be thought of as taking a local slice for the action of the diffeomorphisms on the space of (almost) complex structures.

Since $H^0(M, T^{1,0}) \cong \mathcal{H}^0(M, T^{1,0})$ is the Lie algebra of the Lie group of complex automorphisms of M , the condition $H^0(M, T^{1,0}) = 0$ is satisfied if and only if the group of complex automorphisms of M is discrete.

For the proof of Proposition 7.17, we wish to apply Theorem 6.29. For this, note that $\bar{\partial}^* \mathcal{E}(\xi, \varphi) = 0$ if and only if $Q_{\bar{\partial}} \mathcal{E}(\xi, \varphi) = 0$. For some $\varepsilon > 0$, $k \in \mathbb{N}$, let $\mathcal{W}_\varepsilon^{(k)}$ be as before and

$$\tilde{\mathcal{U}}_\varepsilon^{(k)} := \left\{ \xi \in \mathcal{H}^0(M, T^{1,0})^\perp : \|\xi\|_{k,\infty} < \varepsilon \right\}.$$

Then, for ε small enough and large enough k the map

$$\begin{aligned} A: \tilde{\mathcal{U}}_\varepsilon^{(k)} \times \mathcal{W}_\varepsilon^{(k)} &\longrightarrow \mathcal{H}^0(M, T^{1,0})^\perp \\ (\xi, \varphi) &\longmapsto Q_{\bar{\partial}} \mathcal{E}(\xi, \varphi) \end{aligned}$$

is well-defined and smooth tame as a composition of smooth tame maps. Using Proposition 7.7, for $(\xi, \varphi) \in \tilde{\mathcal{U}}_\varepsilon \times \mathcal{W}_\varepsilon$, $\alpha \in \mathcal{H}^0(M, T^{1,0})^\perp$ we compute

$$\begin{aligned} D_\xi A(\xi, \varphi) \alpha &= Q_{\bar{\partial}} D_\xi \mathcal{E}(\xi, \varphi) \alpha \\ &= -Q_{\bar{\partial}} \bar{\partial}(b_{(\xi, \varphi)} \alpha) + Q_{\bar{\partial}} [\mathcal{E}(\varphi, \xi), b_{(\xi, \varphi)} \alpha] \\ &\quad - Q_{\bar{\partial}} \iota_{\overline{c_\xi \alpha}} (\bar{\partial} \mathcal{E}(\xi, \varphi) - \tfrac{1}{2} [\mathcal{E}(\xi, \varphi), \mathcal{E}(\xi, \varphi)]) \\ &= -(\text{id} - \text{pr}_{\mathcal{H}})(b_{(\xi, \varphi)} \alpha) + Q_{\bar{\partial}} [\mathcal{E}(\xi, \varphi), b_{(\xi, \varphi)} \alpha] \\ &\quad - Q_{\bar{\partial}} \iota_{\overline{c_\xi \alpha}} (\bar{\partial} \mathcal{E}(\xi, \varphi) - \tfrac{1}{2} [\mathcal{E}(\xi, \varphi), \mathcal{E}(\xi, \varphi)]), \end{aligned}$$

where we used Proposition 5.13 in the last step as well as the fact that $Q_{\bar{\partial}}$ maps $\mathcal{A}^0(M, T^{1,0})$ to 0.

Lemma 7.19. *For $\varepsilon > 0$ small enough, the map*

$$(\text{id} - \text{pr}_{\mathcal{H}}) \circ b_{(\xi, \varphi)}: \mathcal{H}^0(M, T^{1,0})^\perp \longrightarrow \mathcal{H}^0(M, T^{1,0})^\perp$$

is invertible for every $(\xi, \varphi) \in \tilde{\mathcal{U}}_\varepsilon \times \mathcal{W}_\varepsilon$. The family of inverses is a smooth tame family of linear maps.

Proof. We follow similar ideas as in the proof of [Omo16, Lem III.2.6]. Recall that for $\eta \in \mathcal{A}^0(M, T^{1,0})$, the map $b_{(\xi, \varphi)}$ was defined as

$$b_{(\xi, \varphi)} \eta = c_\xi \eta + \iota_{\overline{c_\xi \eta}} \mathcal{E}(\xi, \varphi),$$

where

$$c_\xi \eta = DC_\xi(0) \eta, \quad C_\xi: \mathcal{U} \ni \tilde{\eta} \mapsto \left(\exp^h \circ \Psi \right)^{-1} \left(F_{\xi + \tilde{\eta}} F_\xi^{-1} \right) \in \mathcal{A}^0(M, T^{1,0}),$$

for a sufficiently small C^0 -neighborhood $\mathcal{U} \subset \mathcal{A}^0(M, T^{1,0})$ of zero. Recall furthermore that by Lemma 7.8, for $\varepsilon > 0$ small enough $b_{(\xi, \varphi)}$ is invertible for every $(\xi, \varphi) \in \tilde{\mathcal{U}}_\varepsilon \times \mathcal{W}_\varepsilon$ and the family of inverses $b_{(\xi, \varphi)}^{-1}$ is smooth tame. This together with the observation that

$$\mathcal{H}^0(M, T^{1,0}) \cap b_{(\xi, \varphi)} \left(\mathcal{H}^0(M, T^{1,0})^\perp \right) = \{0\}$$

for all $(\xi, \varphi) \in \tilde{\mathcal{U}}_\varepsilon \times \mathcal{W}_\varepsilon$ and $\varepsilon > 0$ small enough shows that

$$(\text{id} - \text{pr}_{\mathcal{H}}) \circ b_{(\xi, \varphi)}: \mathcal{H}^0(M, T^{1,0})^\perp \longrightarrow \mathcal{H}^0(M, T^{1,0})^\perp$$

is injective. To show that it is surjective, we introduce several auxiliary families of linear maps. First, we observe that, again for $\varepsilon > 0$ small enough and $(\xi, \varphi) \in \tilde{\mathcal{U}}_\varepsilon \times \mathcal{W}_\varepsilon$,

$$b_{(\xi, \varphi)}^{-1} (\mathcal{H}^0(M, T^{1,0})) \cap \mathcal{H}^0(M, T^{1,0})^\perp = \{0\}.$$

Since $\mathcal{H}^0(M, T^{1,0})$ is finite-dimensional, this implies that

$$\text{pr}_{\mathcal{H}} b_{(\xi, \varphi)}^{-1}: \mathcal{H}^0(M, T^{1,0}) \longrightarrow \mathcal{H}^0(M, T^{1,0})$$

is invertible and we denote the inverse map by $q_{(\xi, \varphi)}$. Being a smooth map to a finite-dimensional vector space, $q_{(\xi, \varphi)}$ is a smooth tame family of linear maps. We furthermore define

$$\begin{aligned} \psi_1(\xi, \varphi): \mathcal{A}^0(M, T^{1,0}) &\longrightarrow \mathcal{H}^0(M, T^{1,0})^\perp, & \eta &\longmapsto \eta - b_{(\xi, \varphi)}^{-1} q_{(\xi, \varphi)} \text{pr}_{\mathcal{H}} \eta, \\ \psi_2(\xi, \varphi): \mathcal{A}^0(M, T^{1,0}) &\longrightarrow b_{(\xi, \varphi)}^{-1} (\mathcal{H}^0(M, T^{1,0})), & \eta &\longmapsto b_{(\xi, \varphi)}^{-1} q_{(\xi, \varphi)} \text{pr}_{\mathcal{H}} \eta, \\ \omega(\xi, \varphi): \mathcal{A}^0(M, T^{1,0}) &\longrightarrow \mathcal{H}^0(M, T^{1,0}), & \eta &\longmapsto \text{pr}_{\mathcal{H}} b_{(\xi, \varphi)} \psi_1(\xi, \varphi) b_{(\xi, \varphi)}^{-1} \eta. \end{aligned}$$

We claim that

$$b_{(\xi, \varphi)} (\mathcal{H}^0(M, T^{1,0})^\perp) = \{\eta + \omega(\xi, \varphi)\eta : \eta \in \mathcal{H}^0(M, T^{1,0})^\perp\}.$$

To see this, given $\eta \in \mathcal{H}^0(M, T^{1,0})^\perp$, we write

$$b_{(\xi, \varphi)}^{-1} \eta = \psi_1(\xi, \varphi) b_{(\xi, \varphi)}^{-1} \eta + \psi_2(\xi, \varphi) b_{(\xi, \varphi)}^{-1} \eta.$$

This implies

$$\begin{aligned} \eta &= (\text{id} - \text{pr}_{\mathcal{H}}) b_{(\xi, \varphi)} \psi_1(\xi, \varphi) b_{(\xi, \varphi)}^{-1} \eta + \text{pr}_{\mathcal{H}} b_{(\xi, \varphi)} \psi_1(\xi, \varphi) b_{(\xi, \varphi)}^{-1} \eta \\ &\quad + b_{(\xi, \varphi)} \psi_2(\xi, \varphi) b_{(\xi, \varphi)}^{-1} \eta. \end{aligned}$$

Since the last two terms on the right-hand side are in $\mathcal{H}^0(M, T^{1,0})$ and $\eta \in \mathcal{H}^0(M, T^{1,0})^\perp$ it follows that

$$\eta + \text{pr}_{\mathcal{H}} b_{(\xi, \varphi)} \psi_1(\xi, \varphi) b_{(\xi, \varphi)}^{-1} \eta = b_{(\xi, \varphi)} \psi_1(\xi, \varphi) b_{(\xi, \varphi)}^{-1} \eta$$

and hence $\eta + \omega(\xi, \varphi)\eta \in b_{(\xi, \varphi)} (\mathcal{H}^0(M, T^{1,0})^\perp)$. Therefore, we have

$$\{\eta + \omega(\xi, \varphi)\eta : \eta \in \mathcal{H}^0(M, T^{1,0})^\perp\} \subset b_{(\xi, \varphi)} (\mathcal{H}^0(M, T^{1,0})^\perp).$$

Observing that the subspaces on both sides have the same (finite) codimension, we conclude that they are in fact equal.

In particular, this shows that $(\text{id} - \text{pr}_{\mathcal{H}}) b_{(\xi, \varphi)}: \mathcal{H}^0(M, T^{1,0})^\perp \rightarrow \mathcal{H}^0(M, T^{1,0})^\perp$ is surjec-

tive and therefore invertible. The family of inverses is given by

$$\begin{aligned} p_{(\xi, \varphi)} : \mathcal{H}^0(M, T^{1,0})^\perp &\longrightarrow \mathcal{H}^0(M, T^{1,0})^\perp \\ \eta &\longmapsto b_{(\xi, \varphi)}^{-1}(\eta + \omega(\xi, \varphi)\eta) \end{aligned}$$

which is smooth tame as a composition of smooth tame maps. $\square$

Lemma 7.20. *For $\varepsilon > 0$ small enough and large enough k , the equation $D_\xi A(\xi, \varphi)\alpha = \beta$ has a unique solution α for every $(\xi, \varphi) \in \tilde{\mathcal{U}}_\varepsilon^{(k)} \times \mathcal{W}_\varepsilon^{(k)}$, $\beta \in \mathcal{H}^0(M, T^{1,0})^\perp$. Moreover, the family of inverse maps is a smooth tame family of linear maps.*

Proof. Setting $\tilde{\alpha} := (\text{id} - \text{pr}_{\mathcal{H}})b_{(\xi, \varphi)}\alpha \in \mathcal{H}^0(M, T^{1,0})^\perp$, we observe that finding a solution to the equation $D_\xi A(\xi, \varphi)\alpha = \beta$ is equivalent to finding a solution of to the equation

$$\tilde{\alpha} - Q_{\bar{\partial}}[\mathcal{E}(\xi, \varphi), b_{(\xi, \varphi)}p_{(\xi, \varphi)}\tilde{\alpha}] + Q_{\bar{\partial}}\iota_{c_{\xi}p_{(\xi, \varphi)}\tilde{\alpha}}(\bar{\partial}\mathcal{E}(\xi, \varphi) - \tfrac{1}{2}[\mathcal{E}(\xi, \varphi), \mathcal{E}(\xi, \varphi)]) = \beta.$$

The statement follows then from Proposition 6.31 $\square$

Proof of Proposition 7.17. By Lemma 7.20, the assumptions of Theorem 6.29 hold. Since $A(0, 0) = 0$, we conclude that there is a neighborhood $\mathcal{W}$ of 0 in $\mathcal{A}^{0,1}(M, T^{1,0})$ and a neighborhood $\mathcal{U}$ of 0 in $\mathcal{H}^0(M, T^{1,0})^\perp$ such that for every $\varphi \in \mathcal{W}$ we can find some $\xi = \xi(\varphi) \in \mathcal{U}$ such that $\bar{\partial}^*\mathcal{E}(\xi, \varphi) = 0$. Since the derivative $D_\xi A(\xi, \varphi)$ is even invertible, it follows furthermore from Theorem 6.27 that the solution $\xi(\varphi)$ is (locally) unique (see also [DL25, Thm A.12]). Moreover, the solution $\xi(\varphi)$ is defined by a smooth tame map. This finishes the proof of Proposition 7.17. $\square$

Proof of Theorem 7.9. By Corollary 7.15, the family $\tilde{\mathcal{M}} = \{\varphi(t) : t \in \mathcal{W}_\varepsilon\}$ is a smooth family of almost complex deformations. The integrable deformations are precisely those for which the analytic map $\phi(\varphi) := \text{pr}_{\mathcal{H}}[\varphi, \varphi]$ vanishes. They are contained in a neighborhood of 0 in $\tilde{\mathcal{V}}_1$. By Proposition 7.17, we know that any sufficiently small complex deformation is equivalent to a complex deformation in a neighborhood of 0 of $\tilde{\mathcal{V}}_1$.

Furthermore, if $\phi \equiv 0$,

$$\mathcal{M} = \{\varphi(t) \in \tilde{\mathcal{M}} : \phi(\varphi(t)) = 0\} = \tilde{\mathcal{M}}$$

is a smooth family of complex deformations. Given any other smooth family of complex deformations $\{\varphi'(t') : t' \in \mathcal{W}'\}$, it follows again from Proposition 7.17 that (possibly after shrinking $\mathcal{W}'$) there is a family of diffeomorphisms $\{f_{t'}\}_{t' \in \mathcal{W}'}$, depending smoothly on t' , and a smooth map $\tau : \mathcal{W}' \rightarrow \mathcal{W}_\varepsilon$ such that for every $t' \in \mathcal{W}'$, we have

$$f_{t'} \cdot \varphi'(t') = \varphi(\tau(t')).$$

Hence, the family is locally complete. $\square$

Chapter 8

Deformations of Generalized Complex Structures on $\mathcal{G}$ -flat Transitive Courant Algebroids

We have seen in Chapter 5 how deformations of a generalized complex structure L can be described by elements in $\Gamma(M, \Lambda^2 L^*)$. In close analogy to the complex case, the integrability of the deformed generalized almost complex structure is equivalent to a Maurer–Cartan type equation for the corresponding $\varphi \in \Gamma(M, \Lambda^2 L^*)$. Recall that while in the complex case we considered two deformations equivalent if they are related by the action of a diffeomorphism, we will consider two *generalized* complex structures equivalent if they are related by the action of an autoequivalence. This led to the definition of a locally complete family of generalized complex deformations.

In [Gua03; Gua11], Gualtieri followed Kuranishi’s proof in [Kur65] to establish a far-reaching extension of the theorem on the existence of a locally complete family of complex deformations to an analogous result for deformations of generalized complex structures on exact Courant algebroids. However, like in [Kur65], certain analytic details were not addressed in Gualtieri’s proof in [Gua03; Gua11].

In this second original chapter of Part II, like we did in Chapter 7 for Kuranishi’s proof in [Kur65], we will revisit Gualtieri’s proof in [Gua03; Gua11] using Hamilton–Nash–Moser theory. Moreover, we will extend the result to generalized complex structures on the larger class of $\mathcal{G}$ -flat transitive Courant algebroids. There will be some important deviations from the classical complex case as discussed in Chapter 7:

1. While it has been established for a long time that the diffeomorphism group is a (tame) Fréchet Lie group (cf. Section 6.3.1), we need to first endow the group of autoequivalences of $\mathcal{G}$ -flat transitive Courant algebroids with a tame Lie group structure. (For *exact* Courant algebroids the autoequivalences have already been endowed with some notion of infinite-dimensional Lie group structure in [RT20].)
2. When proving the local completeness of the deformation family in Section 8.2.2, we will only consider the action by *exact* autoequivalences. These are autoequivalences whose infinitesimal action on the sections of the Courant algebroid E is by inner derivations $-[u, \cdot] \in \text{End } \Gamma(M, E)$, where $u \in \Gamma(M, E)$. We will see that these form a tame Lie subgroup of the group of autoequivalences. Restricting to the action of this subgroup has the advantage that this setup resembles the classical complex case more closely,

where the infinitesimal action of a diffeomorphism on vector fields is by $-\mathcal{L}_X$ for some $X \in \mathfrak{X}(M)$. However, while the Lie algebra of the diffeomorphism group is the entire space of sections $\mathfrak{X}(M)$ (or, for complex manifolds, the entire space $\Gamma(M, T^{1,0})$), the map $\Gamma(M, E) \ni u \mapsto [u, \cdot] \in \text{End} \Gamma(M, E)$ in general has a kernel so that the Lie algebra of autoequivalences can only be identified with *a subspace of* $\Gamma(M, E)$ or, in the presence of a generalized complex structure L , *a subspace of* $\Gamma(M, L^*)$.

3. Like for the proof of local completeness of the Kuranishi family of classical complex deformations in Section 7.2.2, we will apply Hamilton's Nash–Moser implicit function theorem. To do so, we will need to prove the existence of a smooth tame family of linear right inverses to some smooth tame family of linear maps. Like in Section 7.2.2, to find such family of right inverses, we will relate the given family of linear equations to one that can be solved using elliptic operator theory. However, this will only yield a sensible solution if we can argue that the coexact one-forms $d_L^* \Gamma(M, \Lambda^2 L^*)$ are contained in the subspace of $\Gamma(M, L^*)$ identified with the exact autoequivalences. This can be achieved by a careful choice of bundle metrics defining the L^2 -inner products on $\Gamma(M, E)$ and $\Gamma(M, L^*)$ (and a bisection of E). Note that this subtlety is not present in the classical complex case where the entire space $\mathcal{A}^0(M, T^{1,0}) = \Gamma(M, T^{1,0})$ is identified with the Lie algebra of $\text{Diff}(M)$.
4. Recall that in the complex case, we proved a slightly stronger statement than just local completeness of the deformation family: In Proposition 7.17 we showed that every sufficiently small deformation is related to one in the deformation family by a *unique* diffeomorphism represented by a $\bar{\partial}^*$ -exact form in a neighborhood $\mathcal{U} \subset \mathcal{A}^0(M, T^{1,0})$ of 0. From this we concluded that in the case where the obstruction map vanishes and if $H^0(M, T^{1,0}) = 0$, the Kuranishi family is in fact a miniversal family of deformations. Since in the generalized complex case, to prove local completeness of the deformation family we restrict to the action of exact autoequivalences, we cannot draw an analogous conclusion. Therefore, we will not attempt to prove any sort of uniqueness statement and simply show existence of some exact autoequivalence relating a sufficiently small generalized almost complex deformation to one in our given family (and that this autoequivalence depends smoothly on the given generalized almost complex deformation).

This chapter is structured as follows: In Section 8.1 we endow the autoequivalences of a $\mathcal{G}$ -flat Courant algebroid with a tame Lie group structure and we show that the action on generalized almost complex structures is smooth tame. We will conclude that the local action of exact autoequivalences on generalized almost complex deformations is smooth tame and compute its partial derivative with respect to the component of the exact autoequivalence. In Section 8.2, we will then prove the deformation theorem.

This chapter is based on joint work in progress with Vicente Cortés.

8.1 Groups of autoequivalences as tame Lie groups

In this section, we will endow the group of autoequivalences of a $\mathcal{G}$ -flat transitive Courant algebroid with a tame Fréchet Lie group structure. We will do so in two steps: First, we will consider the group of autoequivalences of an exact Courant algebroid in Section 8.1.1. We will see that these admit a tame Fréchet Lie group structure and there is a tame Fréchet

Lie subgroup of so-called *exact* autoequivalences, which will be relevant to the deformation theory of generalized complex structures in Section 8.2. Building on these results, we will show in Section 8.1.2 that the group of autoequivalences of every $\mathcal{G}$ -flat transitive Courant algebroid is a tame Lie group and there is again a tame subgroup of exact autoequivalences.

In Section 8.1.3 we will then study the action of (exact) autoequivalences on generalized almost complex structures and show that this action is smooth tame. As a consequence, the local action of "small" autoequivalences on generalized almost complex deformations is smooth tame. We will furthermore consider the local expression $\mathcal{E}(\xi, \varphi)$ of the local action of an exact autoequivalence represented by $\xi \in \Gamma(M, L^*)$ on a generalized complex deformation $\varphi \in \Gamma(M, \Lambda^2 L^*)$. Finally, we compute the partial derivative $D_\xi \mathcal{E}(\xi, \varphi)$. This will be important for the proof of the local completeness of the deformation family in Section 8.2.2.

8.1.1 Autoequivalences of exact Courant algebroids

In this section, we will endow the autoequivalences of an exact Courant algebroid E with a tame Fréchet Lie group structure. It follows already from results of Rubio and Tipler [RT20] that they admit a Fréchet Lie group structure. We will give an equivalent description of this Fréchet Lie group structure in Section 8.1.1.1 and argue that it is in fact tame. There is a Lie subalgebra of the Lie algebra of the autoequivalences that is identified with the inner derivations of E . It was already proved in [RT20] that this Lie subalgebra integrates to a Lie subgroup, the group of *exact* autoequivalences. We will see in Section 8.1.1.2 that as a consequence of the result in [RT20], this subgroup is in fact a *tame* Lie subgroup. We will furthermore see that its Lie algebra (i.e. the space of inner derivations) is tamely isomorphic to a tame subspace of $\Gamma(M, E)$.

8.1.1.1 The group of autoequivalences as a tame Lie group

In the following, we will endow the group of autoequivalences of an exact Courant algebroid over a compact manifold with a tame Lie group structure. Note that it was already proved by Rubio and Tipler in [RT20] that the group of autoequivalences of an exact Courant algebroid over a compact manifold is a strong inverse limit Hilbert (ILH) Lie group.

Since the definition of strong ILH Lie groups is even more involved than that of tame Fréchet Lie groups, we will refer to [Omo16] for an introduction to strong ILH Lie groups and simply note that these are special kinds of Fréchet Lie groups that are modeled on an ILH chain, a particular type of (graded) Fréchet space, such that in a local chart around the identity, the group multiplication and the inversion map satisfy certain special properties. Below, we will follow a different approach to describe a (tame) Fréchet Lie group structure for the group of autoequivalences, which has already been laid out in the master thesis of Heck [Hec25]. We will then see that this Fréchet Lie group structure is equivalent to the one described in [RT20].

For the remainder of this section, let $E \rightarrow M$ be the generalized tangent bundle with H -twisted Dorfman bracket over a compact manifold M . Furthermore, we will fix an auxiliary Riemannian metric h on M . We will denote by

$$\mathrm{Diff}(M)_{[H]} := \{f \in \mathrm{Diff}(M) : [f^*H - H] = 0\}$$

the group of diffeomorphisms preserving $[H] \in H^3_{\mathrm{dR}}(M)$. Since for $f \in \mathrm{Diff}(M)$ the class $[H - f^*H] \in H^3_{\mathrm{dR}}(M)$ only depends on the connected component of $f \in \mathrm{Diff}(M)$ we conclude

that it is a tame Fréchet subgroup of $\text{Diff}(M)$ (containing the connected component of the identity in $\text{Diff}(M)$). In particular, it is a tame Fréchet Lie group with atlas defined using the Riemannian exponential map $\exp^h$ of the auxiliary metric h as described in Section 6.3.1.

Note that via

$$\text{Diff}(M)_{[H]} \times \Omega_{\text{cl}}^2(M) \ni (f, b) \longmapsto F_{f, B(f)+b} \in \text{Auteq}(E),$$

where $B(f) := Q_{\text{dR}}(H - f^*H)$, with Q_{dR} as in Example 6.16, the autoequivalences $\text{Auteq}(E)$ are in bijection with $\text{Diff}(M)_{[H]} \times \Omega_{\text{cl}}^2(M)$. We have the following:

Theorem 8.1 ([Hec25]). *Let $E \rightarrow M$ be the generalized tangent bundle of a compact manifold M with H -twisted Dorfman bracket. Then its group of autoequivalences is a tame Fréchet Lie group. The underlying manifold structure is given by $\text{Diff}(M)_{[H]} \times \Omega_{\text{cl}}^2(M)$ and the multiplication is given by*

$$((f_1, b_1), (f_2, b_2)) \longmapsto (f_3, b_3),$$

where $f_3 = f_1 \circ f_2$ and

$$b_3 = f_2^*(b_1 + B(f_1)) + b_2 + B(f_2) + f_2^* Q_{\text{dR}}(H - f_1^*H) - Q_{\text{dR}} f_2^*(H - f_1^*H).$$

The inversion map is given by

$$(f, b) \longmapsto (f^{-1}, -(f^{-1})^*b + I(f)),$$

where

$$\begin{aligned} I(f) &= -(f^{-1})^*(Q_{\text{dR}}(H - f^*H)) - Q_{\text{dR}}(H - (f^{-1})^*H) \\ &= [Q_{\text{dR}}, (f^{-1})^*](H - f^*H). \end{aligned}$$

The Lie algebra $\mathfrak{auteq}(E)$ of the group of autoequivalences is the space $\mathfrak{X}(M) \oplus \Omega_{\text{cl}}^2(M)$ with bracket given by

$$[(X, b), (Y, c)] = (\mathcal{L}_Y X, \mathcal{L}_Y b - \mathcal{L}_X c + (\mathcal{L}_X Q_{\text{dR}} \mathcal{L}_Y - \mathcal{L}_Y Q_{\text{dR}} \mathcal{L}_X)H - Q_{\text{dR}}(\mathcal{L}_{[X, Y]}H)).$$

We will omit the proof of Theorem 8.1 since most of the computations and arguments necessary for the proof will be explained in the proof of the more general Theorem 8.11.

Remark 8.2. To argue that the Fréchet Lie group structure obtained in Theorem 8.1 is indeed equivalent to that described in [RT20], let us briefly review the construction in [RT20].

First Rubio and Tipler consider the group $\text{Diff}(M) \ltimes \Omega^2(M)$ with neutral element $(\text{id}, 0)$, multiplication given by

$$(f_1, B_1)(f_2, B_2) = (f_1 \circ f_2, f_2^* B_1 + B_2)$$

and inversion given by $(f, B)^{-1} = (f^{-1}, -B)$. This is a strong ILH Lie group and in particular a Fréchet Lie group. Moreover, elements of this group are precisely equivalences of the generalized tangent bundle preserving both the anchor map and the scalar product. Then Rubio and Tipler argue that the group of autoequivalences of the generalized tangent bundle E with H -twisted Dorfman bracket is a strong ILH Lie subgroup of $\text{Diff}(M) \ltimes \Omega^2(M)$ by

defining the map¹

$$\begin{aligned}\tilde{\rho}: \Omega^3(M) \times (\text{Diff}(M) \ltimes \Omega^2(M)) &\longrightarrow \Omega^3(M) \\ (h, (f, B)) &\longmapsto f^*h + dB,\end{aligned}$$

observing that $\text{Auteq}(E) = \tilde{\rho}(H, \cdot)^{-1}(H)$ and using an implicit function theorem to show that this indeed has a strong ILH submanifold structure. Again, since a strong ILH Lie group structure induces a Fréchet Lie group structure, this proves that the autoequivalences of the generalized tangent bundle form a Fréchet Lie group.

To see how their construction relates to ours, we consider the (tame) Fréchet Lie group $\text{Diff}(M) \ltimes_H \Omega^2(M)$, whose underlying (tame) Fréchet manifold is $\text{Diff}(M) \times \Omega^2(M)$ with neutral element $(\text{id}, 0)$, multiplication map

$$(f_1, b_1) \circ (f_2, b_2) = (f_1 \circ f_2, b_3),$$

where

$$b_3 = f_2^*(b_1 + B(f_1)) + b_2 + B(f_2) + f_2^* Q_{\text{dR}}(H - f_1^*H) - Q_{\text{dR}} f_2^*(H - f_1^*H)$$

and with inversion map

$$(f, b)^{-1} = (f^{-1}, -(f^{-1})^*b - (f^{-1})^*(Q_{\text{dR}}(H - f^*H)) - Q_{\text{dR}}(H - (f^{-1})^*H)).$$

There is a smooth (tame) isomorphism of Lie groups

$$\begin{aligned}\Phi_H: \text{Diff}(M) \ltimes \Omega^2(M) &\longrightarrow \text{Diff}(M) \ltimes_H \Omega^2(M) \\ (f, B) &\longrightarrow (f, B - Q_{\text{dR}}(H - f^*H)).\end{aligned}$$

We observe that

$$\Phi_H(\tilde{\rho}(H, \cdot)^{-1}(H)) = (\tilde{\rho}(H, \Phi_H^{-1} \cdot))^{-1}(H) = \text{Diff}(M)_{[H]} \times \Omega_{\text{cl}}^2(M).$$

Moreover, the map $\Phi_H^{-1}(f, b) = (f, b + Q_{\text{dR}}(f^*H - H))$ is precisely the bijection that we used in order to identify $\text{Auteq}(E)$ with $\text{Diff}(M)_{[H]} \times \Omega_{\text{cl}}^2(M)$. This shows that the (tame) Fréchet Lie group structure described in Theorem 8.1 is related to the one in [RT20] by a conjugation with Φ_H . Since this is a smooth (tame) map, we conclude that both Fréchet Lie group structures are equivalent.

Finally, note that by choosing a splitting, we can use Theorem 8.1 in order to find that the group of autoequivalences of any exact Courant algebroid is a tame Fréchet Lie group. To see that this is independent of the choice of splitting, recall that different choices of splittings are related by conjugation with a B -field transformation. This gives rise to a tame Lie group isomorphism between the groups of autoequivalences corresponding to different choices of splittings.

¹In [RT20], the map $\tilde{\rho}$ is defined with a minus sign in front of dB , due to a different sign convention for H .

8.1.1.2 Exact autoequivalences of an exact Courant algebroid

In this section, we will introduce a tame Lie subgroup of the Courant autoequivalences, which will be important in the proof of local completeness of the deformation family in Section 8.2. First, we will describe the corresponding Lie subalgebra. To this end, we consider the map

$$\begin{aligned} q: \mathfrak{X}(M) \oplus \Omega_{\text{cl}}^2(M) &\longrightarrow \mathcal{H}^2(M) \\ (X, b) &\longmapsto \text{pr}_{\mathcal{H}}(b - Q_{\text{dR}} \mathcal{L}_X H + \iota_X H). \end{aligned}$$

We denote $\mathfrak{h}_{\text{ex}}(E) := \ker q \subset \mathfrak{auteq}(E)$. Note that since $d Q_{\text{dR}} \mathcal{L}_X H = d Q_{\text{dR}} d \iota_X H = d \iota_X H$, for $b \in \Omega_{\text{cl}}^2(M)$, we have

$$d(b - Q_{\text{dR}} \mathcal{L}_X H + \iota_X H) = 0.$$

Hence, $(X, b) \in \ker q$ if and only if there is some $\alpha \in \Omega^1(M)$ such that

$$b - Q_{\text{dR}} \mathcal{L}_X H + \iota_X H = d\alpha.$$

Proposition 8.3. *The space $\mathfrak{h}_{\text{ex}}(E) := \ker q \subset \mathfrak{auteq}(E)$ is a tame Lie subalgebra of finite codimension.*

Proof. Since $\mathcal{H}^2(M)$ is finite-dimensional and $\mathfrak{auteq}(E) = \mathfrak{h}_{\text{ex}}(E) \oplus \mathcal{H}^2(M)$, it follows that $\mathfrak{h}_{\text{ex}}(E)$ is a tame subspace of finite codimension. To see that it is a Lie subalgebra, we consider $(X, b), (Y, c) \in \mathfrak{h}_{\text{ex}}(E)$. Then there are some $\alpha, \beta \in \Omega^1(M)$ such that

$$b = d\alpha + Q_{\text{dR}} \mathcal{L}_X H - \iota_X H, \quad c = d\beta + Q_{\text{dR}} \mathcal{L}_Y H - \iota_Y H.$$

We compute

$$\begin{aligned} &[(X, b), (Y, c)] \\ &= (\mathcal{L}_Y X, \mathcal{L}_Y(d\alpha + Q_{\text{dR}} \mathcal{L}_X H - \iota_X H) - \mathcal{L}_X(d\beta + Q_{\text{dR}} \mathcal{L}_Y H - \iota_Y H) + \\ &\quad + (\mathcal{L}_X Q_{\text{dR}} \mathcal{L}_Y - \mathcal{L}_Y Q_{\text{dR}} \mathcal{L}_X)H - Q_{\text{dR}}(\mathcal{L}_{\mathcal{L}_X Y} H)) \\ &= (\mathcal{L}_Y X, \mathcal{L}_Y d\alpha - \mathcal{L}_X d\beta - \mathcal{L}_X \iota_Y H + \mathcal{L}_Y \iota_X H - Q_{\text{dR}} \mathcal{L}_{\mathcal{L}_X Y} H) \\ &= (\mathcal{L}_Y X, d(\iota_Y d\alpha - \iota_X d\beta + \iota_X \iota_Y H) + \iota_X \mathcal{L}_Y H - \mathcal{L}_Y \iota_X H - Q_{\text{dR}} \mathcal{L}_{\mathcal{L}_X Y} H) \\ &= (\mathcal{L}_Y X, d(\iota_X d\beta - \iota_Y d\alpha - \iota_X \iota_Y H) + Q_{\text{dR}} \mathcal{L}_{\mathcal{L}_Y X} H - \iota_{\mathcal{L}_Y X} H), \end{aligned}$$

where we used the identities $\mathcal{L}_X = d\iota_X + \iota_X d$ and $[\mathcal{L}_Y, \iota_X] = \iota_{[Y, X]}$, as well as the fact that H is closed. This shows that $[(X, b), (Y, c)] \in \mathfrak{h}_{\text{ex}}(E)$ for all $(X, b), (Y, c) \in \mathfrak{h}_{\text{ex}}(E)$ and hence $\mathfrak{h}_{\text{ex}}(E)$ is a subalgebra. $\square$

The Lie subalgebra $\mathfrak{h}_{\text{ex}}(E)$ of $\mathfrak{auteq}(E)$ in fact integrates to a tame Lie subgroup of $\text{Auteq}(E)$.

Theorem 8.4. *There is a tame Lie subgroup of $\text{Auteq}(E)$, the group of exact Courant autoequivalences $\text{Auteq}^{\text{ex}}(E)$ such that $\text{Lie}(\text{Auteq}^{\text{ex}}(E)) = \mathfrak{h}_{\text{ex}}(E)$.*

Proof. In the setup of [RT20], the Lie algebra $\mathfrak{h}_{\text{ex}}(E)$ corresponds to the Lie (sub-)algebra

$$\{(X, b) \in \mathfrak{X}(M) \oplus \Omega^2(M) : \iota_X H - b = d\alpha \text{ for some } \alpha \in \Omega^1(M)\}$$

with Lie bracket

$$[(X_1, b_1), (X_2, b_2)] = ([X_2, X_1], \mathcal{L}_{X_2} b_1 - \mathcal{L}_{X_1} b_2).$$

It was proved in [RT20] that this Lie algebra integrates to a strong ILH Lie subgroup of the autoequivalences. In particular, this ILH Lie subgroup is a Fréchet submanifold, which under conjugation with Φ_H (cf. Remark 8.2) is mapped to a Fréchet submanifold $\text{Auteq}^{\text{ex}}(E)$ of the tame Fréchet manifold $\text{Auteq}(E)$. Since $\text{Auteq}^{\text{ex}}(E)$ is of finite codimension it follows from [Fre15, Lem 3.5] that it is a tame Fréchet manifold and hence a tame Lie group. $\square$

In the following, it will be useful for us to identify $\mathfrak{h}_{\text{ex}}(E)$ with a tame subspace of $\Gamma(M, E)$.

Proposition 8.5. *There is a tame isomorphism*

$$\mathfrak{h}_{\text{ex}} \cong \{X + \alpha \in \Gamma(M, E) : \alpha \in d^*\Omega^2(M)\} \subset \Gamma(M, E),$$

where d^* is the formal adjoint of d with respect to the L^2 -inner product induced by the auxiliary metric h .

Proof. We may think of $\mathfrak{h}_{\text{ex}}(E)$ as the image of the map

$$\begin{aligned} \tilde{q} : \Gamma(M, E) &\longrightarrow \mathfrak{auteq}(E), \\ X + \alpha &\longmapsto X + d\alpha + Q_{\text{dR}} \mathcal{L}_X H - \iota_X H. \end{aligned}$$

We observe that $\ker \tilde{q} = \Omega_{\text{cl}}^1(M) \subset \Gamma(M, E)$. Therefore, we have

$$\text{im } \tilde{q} \cong (\ker \tilde{q})^\perp = \mathfrak{X}(M) \oplus d^*\Omega^2(M),$$

where the orthogonal complement is taken with respect to the L^2 -inner product defined by the bundle metric $h \oplus h^*$ on $E = TM \oplus T^*M$. To see that the inverse map is tame, note that it is given by

$$\text{im } \tilde{q} \ni (X, b) \longmapsto (X, Q_{\text{dR}}(b - Q_{\text{dR}} \mathcal{L}_X H + \iota_X H)) \in \mathfrak{X}(M) \oplus d^*\Omega^2(M).$$

$\square$

In a neighborhood $\mathcal{V}$ of $\text{id} \in \text{Auteq}^{\text{ex}}(E)$, we consider a chart Φ taking values in some neighborhood $\mathcal{U} \subset \mathfrak{h}_{\text{ex}}(E)$ around 0. We may assume that the differential of this chart at the identity is the identity map on $\mathfrak{h}_{\text{ex}}(E)$, $D\Phi(\text{id}) = \text{id}$. We denote by $F_v := \Phi^{-1}(v)$ the exact autoequivalence in $\mathcal{V}$ corresponding to $v \in \mathcal{U}$ with respect to this chart.

In the following we will see that the Lie algebra of the exact autoequivalences can in fact be identified with the Lie algebra of inner derivations. First, we consider the natural action of autoequivalences on sections $u \in \Gamma(M, E)$, given by

$$\mu_E : (F, u) \longmapsto F \circ u \circ f^{-1},$$

where f is the diffeomorphism covered by F .

Proposition 8.6. *The map $\mu_E : \text{Auteq}(E) \times \Gamma(M, E) \rightarrow \Gamma(M, E)$ is smooth tame.*

Proof. Let $F = (f, b) \in \text{Diff}_{[H]}(M) \times \Omega_{\text{cl}}^2(M)$, $u = X + \alpha \in \Gamma(M, E)$. We compute

$$\begin{aligned} \mu_E(F, u) &= F_f \circ (X + \alpha + \iota_X(b + Q_{\text{dR}}(H - f^*H))) \circ f^{-1} \\ &= df \circ X \circ f^{-1} + (df^{-1})^*(\alpha + \iota_X(b + Q_{\text{dR}}(H - f^*H))) \circ f^{-1} \\ &= f_*X + f^{-1*}(\alpha + \iota_X(b + Q_{\text{dR}}(H - f^*H))). \end{aligned} \tag{8.1}$$

This shows that μ_E is a composition of smooth tame maps and therefore smooth tame. $\square$

For exact autoequivalences we find the following:

Proposition 8.7. *Let $v \in \mathfrak{h}_{\text{ex}}(E)$, $u \in \Gamma(M, E)$. Then we have*

$$D_F \mu_E(F = \text{id}, u)v = -[v, u].$$

In particular, for some small exact autoequivalence Φ_v , we have

$$\mu_E(\Phi_v, u) = u - [v, u] + R_{\mu_E}(v)u,$$

where $R_{\mu_E}(tv) = t^2 \tilde{R}_{\mu_E}(v, t)$, for a real parameter t , where $\tilde{R}_{\mu_E}(v, t)$ depends smoothly on t for small t .

Proof. First, using equation (8.1), we observe that for $(Y, b) \in \mathbf{auteq}(E)$, $u = X + \alpha \in \Gamma(M, E)$, we have

$$D_F \mu_E(F = \text{id}, u)(Y, b) = -\mathcal{L}_Y(X + \alpha) + \iota_X b - \iota_X Q_{\text{dR}} \mathcal{L}_Y H.$$

Now, if $b = d\beta - Q_{\text{dR}} \mathcal{L}_Y H + \iota_Y H$, for some $\beta \in \text{im}(Q_{\text{dR}})$, i.e. in the case that we are taking the derivative in the direction $v = (Y, \beta) \in \mathfrak{h}_{\text{ex}}$, we find

$$\begin{aligned} D_F \mu_E(F = \text{id}, u)v &= -\mathcal{L}_Y(X + \alpha) + \iota_X(d\beta + Q_{\text{dR}} \mathcal{L}_Y H - \iota_Y H) - \iota_X Q_{\text{dR}} \mathcal{L}_Y H \\ &= -\mathcal{L}_Y(X + \alpha) + \iota_X d\beta - \iota_X \iota_Y H \\ &= -[v, u]. \end{aligned}$$

The second statement follow from Theorem 6.12. $\square$

The derivative at the identity of the action μ_E of autoequivalences on sections of E yields an identification of elements in the Lie algebra $\mathbf{auteq}(E)$ of the autoequivalences with the derivations of E .

Corollary 8.8. *Under the identification of $\mathbf{auteq}(E)$ with the derivations of E , we have $\mathfrak{h}_{\text{ex}}(E) = -\text{im}(\text{ad} : \Gamma(M, E) \rightarrow \text{Der}(E))$, where $\text{ad}(u) := [u, \cdot]$.*

Finally, we observe that the map $\tilde{q}$ considered in the proof of Proposition 8.5 is in fact the map $-\text{ad} : \Gamma(E) \rightarrow \text{Der}(E) \cong \mathbf{auteq}(E)$ and hence what we constructed in Proposition 8.5 is a tame isomorphism $\mathfrak{h}_{\text{ex}}(E) \cong (\ker \text{ad})^\perp$, where the orthogonal complement is taken with respect to the L^2 -inner product $(\cdot, \cdot)_E$ defined by the bundle metric $h \oplus h^*$.

8.1.2 Autoequivalences of $\mathcal{G}$ -flat transitive Courant algebroids

We will now focus on $\mathcal{G}$ -flat transitive Courant algebroids in the sense of Definition 5.29. After fixing some conventions and notations in Section 8.1.2.1, we will show in Section 8.1.2.2 that the group of autoequivalences of a $\mathcal{G}$ -flat standard transitive Courant algebroid is a tame Fréchet Lie group and that there exists a tame Lie subgroup of *exact* autoequivalences, whose Lie algebra is identified with the inner derivations of E . Since every $\mathcal{G}$ -flat transitive Courant algebroid is isomorphic to such a standard form, this yields a tame Fréchet Lie group structure for any $\mathcal{G}$ -flat transitive Courant algebroid.

We will also see an explicit description of the group of autoequivalences for a general bisection, which is not necessarily $\mathcal{G}$ -flat and show that for a suitable choice of L^2 -inner products on $\Gamma(M, E)$, the Lie algebra of the exact autoequivalences (i.e. the inner derivations) can be identified with a tame subspace of $\Gamma(M, E)$ via a tame isomorphism. Given a generalized complex structure L on E , we may furthermore identify $\Gamma(M, E) \cong \Gamma(M, L^*)$ via a vector bundle isomorphism. Thus, the Lie algebra of the exact autoequivalences is identified with a tame Fréchet subspace of $\Gamma(M, L^*)$.

On the other hand there is the tame Fréchet subspace $d_L^* \Gamma(M, \Lambda^2 L^*)$ of coexact forms. We will see in Section 8.1.2.4 that for some choice of bisection on E and suitable choices of bundle metrics on E and L^* , the subspace $d_L^* \Gamma(M, \Lambda^2 L^*)$ is in fact contained in the subspace identified with the exact autoequivalences. This will be crucial for our arguments in Section 8.2.2 to prove local completeness of the deformation family.

8.1.2.1 L^2 -inner products, conventions and notation

We fix the following L^2 -inner products. We choose a positive definite bundle metric $\langle \cdot, \cdot \rangle_{\text{Eucl}}$ on $\mathcal{G}$ and denote by $(\cdot, \cdot)_{\mathcal{G}}$ the L^2 -inner products on $\Gamma(M, \mathcal{G})$ and $\Omega^k(M, \mathcal{G})$ defined by $\langle \cdot, \cdot \rangle_{\text{Eucl}}$ and the auxiliary Riemannian metric. Using $\langle \cdot, \cdot \rangle_{\text{Eucl}}$ and the auxiliary Riemannian metric h , we fix a (positive definite) bundle metric on $\Gamma(E) = TM \oplus \mathcal{G} \oplus T^*M$ via $h \oplus \langle \cdot, \cdot \rangle_{\text{Eucl}} \oplus h^*$ and denote by $(\cdot, \cdot)_E$ the L^2 -inner product on $\Gamma(M, E)$ defined by this bundle metric and h .

Note that when the bundle $\mathcal{G}$ is trivial $\mathcal{G} \cong M \times \mathbb{R}^K$, after choosing a global orthonormal frame $\{\mathbf{e}_i\}_{i=1}^K$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{G}}$, we may think of sections $\mathbf{r} \in \Gamma(M, \mathcal{G})$ as vectors of smooth functions $\mathbf{r} = (r_i)_{i=1}^K$, $r_i \in C^\infty(M)$ and of p -forms with values in $\mathcal{G}$ as vectors of p -forms on M . Given $\mathbf{A} = (A_i)_{i=1}^K \in \Omega^p(M, \mathcal{G})$ and $\mathbf{B} = (B_i)_{i=1}^K \in \Omega^q(M, \mathcal{G})$, we note that

$$\langle \mathbf{A} \wedge \mathbf{B} \rangle_{\mathcal{G}} = \sum_{i=1}^K \epsilon_i A_i \wedge B_i,$$

where $\epsilon_i = \langle \mathbf{e}_i, \mathbf{e}_i \rangle_{\mathcal{G}} \in \{\pm 1\}$. Moreover, we may assume that $\{\mathbf{e}_i\}$ is a global parallel orthonormal frame for the connection ∇ on $\mathcal{G}$, since ∇ has trivial holonomy and $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ is parallel by the compatibility conditions. Then we have $\nabla \mathbf{r} = (dr_i)_{i=1}^K =: d\mathbf{r}$, and $d_{\nabla} \mathbf{A} = (dA_i)_{i=1}^K =: d\mathbf{A}$ for $\mathbf{A} \in \Omega^p(M, \mathcal{G})$.

The differential operator $d: \Omega^\bullet(M, \mathcal{G}) \longrightarrow \Omega^{\bullet+1}(M, \mathcal{G})$ defines an elliptic complex. We denote the formal adjoint of d with respect to $(\cdot, \cdot)_{\mathcal{G}}$ by d_{∇}^* , its Laplace operator by $\Delta_{\nabla} = dd_{\nabla}^* + d_{\nabla}^*d$, the Greens operator by G_{∇} and by $Q_{\nabla} := d_{\nabla}^* G_{\nabla}$. Note that by Example 6.16, the map Q_{∇} is tame linear.

Remark 8.9. Given a fixed parallel orthonormal frame $\{\mathbf{e}_i\}_{i=1}^K$ for $\mathcal{G}$, we may define a (constant) positive definite bundle metric $\langle \cdot, \cdot \rangle'_{\text{Eucl}}$ on $\mathcal{G}$ by $\langle \mathbf{e}_i, \mathbf{e}_j \rangle'_{\text{Eucl}} = \delta_{ij}$, i.e. $\{\mathbf{e}_i\}$ is an orthonormal basis for $\langle \cdot, \cdot \rangle'_{\text{Eucl}}$. Together with the auxiliary Riemannian metric h , this bundle metric defines an L^2 -inner product on $\Gamma(M, \mathcal{G})$ and $\Omega^\bullet(M, \mathcal{G})$, which we denote by $(\cdot, \cdot)'_{\mathcal{G}}$. The formal adjoint of $d: \Omega^k(M, \mathcal{G}) \longrightarrow \Omega^{k+1}(M, \mathcal{G})$ with respect to $(\cdot, \cdot)'_{\mathcal{G}}$ is simply given by $d_{\nabla}^* \mathbf{A} = \sum_{i=1}^K (d^* A_i) \mathbf{e}_i$ for $\mathbf{A} = \sum_{i=1}^K A_i \mathbf{e}_i$, where d^* is the formal adjoint of d with respect to the L^2 -inner product induced by h . In particular, the space $\{\mathbf{A} = \sum_{i=1}^K A_i \mathbf{e}_i \in \Omega^k(M, \mathcal{G}) : A_i \in d^* \Omega^{k+1}(M)\}$ is a topological complement of $\Omega_{\text{cl}}^k(M, \mathcal{G}) = \{\mathbf{A} \in \Omega^k(M, \mathcal{G}) : d\mathbf{A} = 0\}$.

Finally, we note that if $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ is of signature $(q, K - q)$, the autoequivalences of $\mathcal{G}$ are

given by $\text{Diff}(M) \times C^\infty(M, \text{O}(q, K - q))$ acting on $(p, v) \in \mathcal{G} = M \times \mathbb{R}^K$ as

$$(f, O)(p, v) = (f(p), O(p)v).$$

These act on sections $\mathbf{r} \in \Gamma(M, \mathcal{G})$ via

$$(f, O) \cdot \mathbf{r} = (O\mathbf{r}) \circ f^{-1},$$

where $(Or)_i(p) = \sum_j O_{ij}(p)r_j(p)$.

For $f \in \text{Diff}(M)$ and $\mathbf{A} \in \Gamma(M, \Lambda^k L^*)$, we will denote by $f^*\mathbf{A} := (f^*A_i)_{i=1}^K$. We observe that

$$df^*\mathbf{A} = \sum_{i=1}^K df^*A_i \mathbf{e}_i = \sum_{i=1}^K f^*dA_i \mathbf{e}_i = f^*d\mathbf{A}. \quad (8.2)$$

8.1.2.2 Autoequivalences of $\mathcal{G}$ -flat standard transitive Courant algebroids

In this section, we will see that the autoequivalences of a $\mathcal{G}$ -flat transitive Courant algebroid form a tame Fréchet Lie group. First, we show the following.

Proposition 8.10. *Let $E = TM \oplus \mathcal{G} \oplus T^*M$ be a $\mathcal{G}$ -flat standard transitive Courant algebroid with $\mathcal{G} \cong M \times \mathbb{R}^K$ and $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ of signature $(q, K - q)$. Then the autoequivalences $\text{Auteq}(E)$ have a tame Fréchet manifold structure as a product*

$$\text{Auteq}(E) = \text{Auteq}(\mathbb{T}M, H) \times \text{O}(q, K - q) \times \Omega_{\text{cl}}^1(M, \mathcal{G}),$$

where $\text{Auteq}(\mathbb{T}M, H)$ denotes the autoequivalences of the H -twisted generalized tangent bundle.

Proof. Using Theorem 5.27, the fact that $[\cdot, \cdot]_{\mathcal{G}} \equiv 0$, $\mathbf{R} = 0$ and that ∇ is flat, we know that

$$\begin{aligned} \text{Auteq}(E) &= \{\Phi_{\nu, \mathbf{A}, B} : \nu \in \text{Auteq}(\mathcal{G}) \text{ covers } f \in \text{Diff}(M), \mathbf{A} \in \Omega^1(M, \mathcal{G}), B \in \Omega^2(M), \\ &\quad \nabla = \nu \cdot \nabla, 0 = \nu \cdot d\mathbf{A}, H - f^*H = dB + \langle \mathbf{A} \wedge d\mathbf{A} \rangle_{\mathcal{G}}\}. \end{aligned}$$

Note that the second equation implies that $d\mathbf{A} = 0$. Therefore, the last equation becomes $H - f^*H = dB$ and hence $F_{f,B} \in \text{Auteq}(\mathbb{T}M, H)$. Finally, let $O \in C^\infty(M, \text{O}(q, K - q))$ such that $\nu = (f, O)$, $X \in \mathfrak{X}(M)$ and $\mathbf{r} \in \Gamma(M, \mathcal{G})$. We compute

$$\begin{aligned} (\nu \cdot \nabla)_X \mathbf{r} &= \nu \cdot \left(\nabla_{f_*^{-1}X} (\nu^{-1} \cdot \mathbf{r}) \right) \\ &= \nu \cdot d(O^{-1}\mathbf{r} \circ f)(f_*^{-1}X) \\ &= \nu \cdot (d(O^{-1}\mathbf{r})X) \circ f \\ &= O(d(O^{-1}\mathbf{r})X) \\ &= \iota_X [(OdO^{-1})\mathbf{r} + d\mathbf{r}]. \end{aligned}$$

Hence, the condition $\nu \cdot \nabla = \nabla$ becomes $OdO^{-1} = 0$, which implies that $dO = 0$ and hence O must be constant. This implies that as a manifold

$$\text{Auteq}(E) = \text{Auteq}(\mathbb{T}M, H) \times \text{O}(q, K - q) \times \Omega_{\text{cl}}^1(M, \mathcal{G}).$$

This finishes the proof. $\square$

Recalling the bijection $\text{Auteq}(\mathbb{T}M, H) \cong \text{Diff}_{[H]}(M) \times \Omega_{\text{cl}}^2(M)$ via

$$\text{Diff}(M)_{[H]} \times \Omega_{\text{cl}}^2(M) \ni (f, b) \longmapsto F_{f, b+B(f)},$$

where $B(f) = Q_{\text{dR}}(H - f^*H)$, which we used in order to define the manifold structure on $\text{Auteq}(\mathbb{T}M, H)$, we find the following:

Theorem 8.11. *Let $E = TM \oplus \mathcal{G} \oplus T^*M$ be a $\mathcal{G}$ -flat standard transitive Courant algebroid with $\mathcal{G} \cong M \times \mathbb{R}^K$ and three-form $H \in \Omega_{\text{cl}}^3(M)$. Then $\text{Auteq}(E)$ is a tame Fréchet Lie group with underlying manifold structure*

$$\text{Auteq}(E) = \text{Diff}(M)_{[H]} \times \Omega_{\text{cl}}^2(M) \times \text{O}(q, K - q) \times \Omega_{\text{cl}}^1(M, \mathcal{G}).$$

The multiplication map is given by

$$(f_1, b_1, O_1, \mathbf{A}_1) \times (f_2, b_2, O_2, \mathbf{A}_2) \longmapsto (f_1 \circ f_2, b_3, O_1 O_2, \mathbf{A}_3),$$

where

$$\begin{aligned} b_3 &= f_2^* b_1 + b_2 + \langle O_2^{-1} f_2^* \mathbf{A}_1 \wedge \mathbf{A}_2 \rangle_{\mathcal{G}} + [f_2^*, Q_{\text{dR}}](H - f_1^* H), \\ \mathbf{A}_3 &= O_2^{-1} f_2^* \mathbf{A}_1 + \mathbf{A}_2, \end{aligned}$$

and the inversion map is given by

$$(f, b, O, \mathbf{A}) \longmapsto (f^{-1}, -(f^{-1})^* b + [Q_{\text{dR}}, (f^{-1})^*](H - f^* H), O^{-1}, -O(f^{-1})^* \mathbf{A}).$$

Proof. It follows from the corresponding expressions in Theorem 5.27 that multiplication on $\text{Auteq}(E) = \text{Auteq}(\mathbb{T}M, H) \times \text{O}(q, K - q) \times \Omega_{\text{cl}}^1(M)$ is given by

$$(F_{f_1, B_1}, O_1, \mathbf{A}_1) \circ (F_{f_2, B_2}, O_2, \mathbf{A}_2) = (F_{f_3, B_3}, O_3, \mathbf{A}_3),$$

where $f_3 = f_1 \circ f_2$, $O_3 = O_1 O_2$ and

$$\begin{aligned} B_3 &= f_2^* B_1 + B_2 + \langle O_2^{-1} (f_2^* \mathbf{A}_1) \wedge \mathbf{A}_2 \rangle_{\mathcal{G}}, \\ \mathbf{A}_3 &= O_2^{-1} f_2^* \mathbf{A}_1 + \mathbf{A}_2 \end{aligned}$$

and the identity element is $(\text{id}, \text{id}, 0)$. The inversion map is given by

$$(F_{f, B}, O, \mathbf{A}) \longmapsto (F_{f, B}^{-1}, O^{-1}, -O(f^{-1})^* \mathbf{A}).$$

We only need to check the formulas for the multiplication and inversion map under the bijection $\text{Auteq}(\mathbb{T}M, H) \cong \text{Diff}(M)_{[H]} \times \Omega_{\text{cl}}^2(M)$. In each case, the only non-trivial part is

the component in $\Omega_{\text{cl}}^2(M)$. For the multiplication, we find

$$\begin{aligned} b_3 &= f_2^* (b_1 + B(f_1)) + b_2 + B(f_2) + \langle O_2^{-1} f_2^* \mathbf{A}_1 \wedge \mathbf{A}_2 \rangle_{\mathcal{G}} - B(f_1 \circ f_2) \\ &= f_2^* b_1 + b_2 + \langle O_2^{-1} f_2^* \mathbf{A}_1 \wedge \mathbf{A}_2 \rangle_{\mathcal{G}} + f_2^* Q_{\text{dR}}(H - f_1^* H) \\ &\quad + Q_{\text{dR}}(H - f_2^* H) - Q_{\text{dR}}(H - f_2^* f_1^* H) \\ &= f_2^* b_1 + b_2 + \langle O_2^{-1} f_2^* \mathbf{A}_1 \wedge \mathbf{A}_2 \rangle_{\mathcal{G}} - [Q_{\text{dR}}, f_2^*](H - f_1^* H). \end{aligned}$$

Similarly, we find that the $\Omega_{\text{cl}}^2(M)$ -component of the inverse of $(f, b, O, \mathbf{A})$ is given by

$$\begin{aligned} & - (f^{-1})^* (b + B(f)) - B(f^{-1}) \\ &= - (f^{-1})^* b - (f^{-1})^* Q_{\text{dR}}(H - f^* H) - Q_{\text{dR}}(H - (f^{-1})^* H) \\ &= - (f^{-1})^* b + [Q_{\text{dR}}, (f^{-1})^*](H - f^* H). \end{aligned}$$

Both the multiplication and the inversion map are compositions of smooth tame maps and hence smooth tame. $\square$

Remark 8.12. Note that for (not necessarily $\mathcal{G}$ -flat) transitive Courant algebroids with rank $\mathcal{G} = 1$ a strong ILH Lie group structure of $\text{Auteq}(E)$ has been constructed in [RT20].

Using the atlas of $\text{Diff}(M)_{[H]}$ in terms of the Riemannian exponential map of the auxiliary metric h , we find that the autoequivalences of a $\mathcal{G}$ -flat standard transitive Courant algebroid are locally modelled on the tame Fréchet space $\mathfrak{X}(M) \oplus \Omega_{\text{cl}}^2(M) \oplus \mathfrak{so}(q, K - q) \oplus \Omega_{\text{cl}}^1(M, \mathcal{G})$.

Proposition 8.13. *The Lie algebra of $\text{Auteq}(E)$, where E is the $\mathcal{G}$ -flat standard transitive Courant algebroid with $\mathcal{G} \cong M \times \mathbb{R}^K$ and three-form H , has underlying vector space*

$$\mathfrak{auteq}(E) = \mathfrak{X}(M) \oplus \Omega_{\text{cl}}^2(M) \oplus \mathfrak{so}(q, K - q) \oplus \Omega_{\text{cl}}^1(M, \mathcal{G})$$

with Lie bracket

$$\begin{aligned} & [X_1 + b_1 + T_1 + \mathbf{A}_1, X_2 + b_2 + T_2 + \mathbf{A}_2] \\ &= \mathcal{L}_{X_2} X_1 + \mathcal{L}_{X_2} b_1 - \mathcal{L}_{X_1} b_2 + 2 \langle \mathbf{A}_1 \wedge \mathbf{A}_2 \rangle_{\mathcal{G}} + (\mathcal{L}_{X_1} Q_{\text{dR}} \mathcal{L}_{X_2} - \mathcal{L}_{X_2} Q_{\text{dR}} \mathcal{L}_{X_1}) H \\ &\quad - Q_{\text{dR}} \mathcal{L}_{\mathcal{L}_{X_1} X_2} H + [T_1, T_2] - T_2 \mathbf{A}_1 + \mathcal{L}_{X_2} \mathbf{A}_1 + T_1 \mathbf{A}_2 - \mathcal{L}_{X_1} \mathbf{A}_2. \end{aligned}$$

Proof. First, given $(f_1, \tilde{b}_1, O_1, \tilde{\mathbf{A}}_1), (f_2, \tilde{b}_2, O_2, \tilde{\mathbf{A}}_2) \in \text{Auteq}(E)$, we determine $b_3 \in \Omega_{\text{cl}}^2(M)$ and $\mathbf{A}_3 \in \Omega_{\text{cl}}^1(M, \mathcal{G})$ such that

$$(f_1, \tilde{b}_1, O_1, \tilde{\mathbf{A}}_1)(f_2, \tilde{b}_2, O_2, \tilde{\mathbf{A}}_2)(f_1, \tilde{b}_1, O_1, \tilde{\mathbf{A}}_1)^{-1} = (f_1 f_2 f_1^{-1}, b_3, O_1 O_2 O_1^{-1}, \mathbf{A}_3).$$

For this, we first compute

$$\begin{aligned} & (f_2, \tilde{b}_2, O_2, \tilde{\mathbf{A}}_2)(f_1, \tilde{b}_1, O_1, \tilde{\mathbf{A}}_1)^{-1} \\ &= (f_2, \tilde{b}_2, O_2, \tilde{\mathbf{A}}_2)(f_1^{-1}, -(f_1^{-1})^* \tilde{b}_1 + [Q_{\text{dR}}, (f_1^{-1})^*](H - f_1^* H), O_1^{-1}, -O_1(f_1^{-1})^* \tilde{\mathbf{A}}_1) \\ &= (f_2 f_1^{-1}, (f_1^{-1})^* \tilde{b}_2 - (f_1^{-1})^* \tilde{b}_1 + [Q_{\text{dR}}, (f_1^{-1})^*](H - f_1^* H) \\ &\quad - \langle O_1(f_1^{-1})^* \tilde{\mathbf{A}}_2 \wedge O_1(f_1^{-1})^* \tilde{\mathbf{A}}_1 \rangle_{\mathcal{G}} - [Q_{\text{dR}}, (f_1^{-1})^*](H - f_2^* H), O_2 O_1^{-1}, O_1(f_1^{-1})^*(\tilde{\mathbf{A}}_2 - \tilde{\mathbf{A}}_1)). \end{aligned}$$

Multiplying from the left by $(f_1, \tilde{b}_1, O_1, \tilde{\mathbf{A}}_1)$ yields

$$\begin{aligned} b_3 &= (f_2 f_1^{-1})^* \tilde{b}_1 + (f_1^{-1})^* (\tilde{b}_2 - \tilde{b}_1) - [\mathbf{Q}_{\text{dR}}, (f_1^{-1})^*] (f_1^* H - f_2^* H) \\ &\quad - [\mathbf{Q}_{\text{dR}}, (f_2 f_1^{-1})^*] (H - f_1^* H) + \langle O_1 O_2^{-1} (f_2 f_1^{-1})^* \tilde{\mathbf{A}}_1 \wedge O_1 (f_1^{-1})^* (\tilde{\mathbf{A}}_2 - \tilde{\mathbf{A}}_1) \rangle_{\mathcal{G}} \\ \mathbf{A}_3 &= O_1 O_2^{-1} (f_2 f_1^{-1})^* \tilde{\mathbf{A}}_1 + O_1 (f_1^{-1})^* (\tilde{\mathbf{A}}_2 - \tilde{\mathbf{A}}_1) \end{aligned}$$

Here, we used $\langle f^* \mathbf{A} \wedge f^* \mathbf{A}' \rangle_{\mathcal{G}} = f^* \langle \mathbf{A} \wedge \mathbf{A}' \rangle_{\mathcal{G}}$ and $\langle O \mathbf{A} \wedge O \mathbf{A}' \rangle_{\mathcal{G}} = \langle \mathbf{A} \wedge \mathbf{A}' \rangle_{\mathcal{G}}$. Setting $f_1(t) = \exp^h(tX_1)$, $\tilde{b}_1(t) = tb_1$, $\tilde{\mathbf{A}}_1(t) = t\mathbf{A}_1$ and $O_1(t) = \exp(tT_1)$ and similarly $f_2(s) = \exp^h(sX_2)$, $\tilde{b}_2(s) = sb_2$, $\tilde{\mathbf{A}}_2(s) = s\mathbf{A}_2$ and $O_2(s) = \exp(sT_2)$ and taking the derivative $\frac{d}{ds}|_{s=0} \frac{d}{dt}|_{t=0}$ yields the expression for the Lie bracket. $\square$

Next, we identify a tame Lie subgroup of $\text{Auteq}(E)$, which generalizes the group of exact autoequivalences of an exact Courant algebroid.

Theorem 8.14. *The tame submanifold $\text{Auteq}^{\text{ex}}(\mathbb{T}M, H) \times \Omega_{\text{ex}}^1(M, \mathcal{G}) \subset \text{Auteq}(E)$ inherits the structure of a tame Lie subgroup of $\text{Auteq}(E)$, the tame Lie subgroup $\text{Auteq}^{\text{ex}}(E)$ of exact autoequivalences of E .*

Proof. Let $(F_1, \mathbf{A}_1), (F_2, \mathbf{A}_2) \in \text{Auteq}^{\text{ex}}(\mathbb{T}M, H) \times \Omega_{\text{ex}}^1(M, \mathcal{G})$. Then applying the restriction of the multiplication map in $\text{Auteq}(E)$ described in Theorem 8.11 yields

$$(F_1, \mathbf{A}_1) \times (F_2, \mathbf{A}_2) \longmapsto \left(F_{\langle f_2^* \mathbf{A}_1 \wedge \mathbf{A}_2 \rangle_{\mathcal{G}}} F_1 F_2, f_2^* \mathbf{A}_1 + \mathbf{A}_2 \right) \in \text{Auteq}(\mathbb{T}M, H) \times \Omega_{\text{cl}}^1(M, \mathcal{G}),$$

where f_2 is the diffeomorphism covered by F_2 . Clearly, $f_2^* \mathbf{A}_1 + \mathbf{A}_2 \in \Omega_{\text{ex}}^1(M, \mathcal{G})$. Moreover, $\langle f_2^* \mathbf{A}_1 \wedge \mathbf{A}_2 \rangle_{\mathcal{G}} \in \Omega_{\text{ex}}^2(M)$ and hence $F_{\langle f_2^* \mathbf{A}_1 \wedge \mathbf{A}_2 \rangle_{\mathcal{G}}}$ is an exact B -field transformation and in particular an exact autoequivalence of $\mathbb{T}M$. Since $\text{Auteq}^{\text{ex}}(\mathbb{T}M, H)$ is closed under multiplication, we conclude that $\text{Auteq}^{\text{ex}}(E)$ is closed under multiplication. Since it is closed under inversion too and clearly contains the identity, it follows that it is a subgroup of $\text{Auteq}(E)$. As it is also a tame submanifold it follows that it is a tame Lie subgroup. $\square$

In the following, we describe the Lie subalgebra of the exact autoequivalences, which again can be identified with a tame subspace of $\Gamma(M, E)$.

Proposition 8.15. *The tame Lie subalgebra corresponding to $\text{Auteq}^{\text{ex}}(E)$ is given by*

$$\mathfrak{h}_{\text{ex}}(E) = \{X + d\alpha + \mathbf{Q}_{\text{dR}} \mathcal{L}_X H - \iota_X H + \mathbf{A} : X \in \mathfrak{X}(M), \alpha \in \Omega^1(M), \mathbf{A} \in \Omega_{\text{ex}}^1(M, \mathcal{G})\}.$$

It is isomorphic to the tame Fréchet space

$$\mathfrak{h}_{\text{ex}}(E) \cong \mathfrak{X}(M) \oplus d^* \Omega^2(M) \oplus d_{\nabla}^* \Omega^1(M, \mathcal{G}) \subset \Gamma(M, E).$$

Proof. Recall that the tangent space at the identity of $\text{Auteq}^{\text{ex}}(\mathbb{T}M, H)$ is given by

$$T_{\text{id}} \text{Auteq}^{\text{ex}}(\mathbb{T}M, H) = \{X + d\alpha + \mathbf{Q}_{\text{dR}} \mathcal{L}_X H - \iota_X H : X \in \mathfrak{X}(M), \alpha \in \Omega^1(M)\}.$$

It follows that

$$T_{\text{id}} \text{Auteq}^{\text{ex}}(E) = \{X + d\alpha + \mathbf{Q}_{\text{dR}} \mathcal{L}_X H - \iota_X H + d\mathbf{r} : X \in \mathfrak{X}(M), \alpha \in \Omega^1(M), \mathbf{r} \in \Omega^0(M, \mathcal{G})\}.$$

We may therefore think of $\mathfrak{h}_{\text{ex}}(E) = T_{\text{id}} \text{Auteq}^{\text{ex}}(E)$ as the image of the tame linear map

$$\begin{aligned} p: \Gamma(M, E) &\longrightarrow \mathfrak{auteq}(E) \\ X + \mathbf{r} + \alpha &\longmapsto X + d\alpha + Q_{\text{dR}} \mathcal{L}_X H - \iota_X H + d\mathbf{r}. \end{aligned}$$

We observe that $\ker p = \Omega_{\text{cl}}^1(M) \oplus \Omega_{\text{cl}}^0(M, \mathcal{G})$. It follows from the Hodge decomposition that with respect to the L^2 -inner product $(\cdot, \cdot)_E$, the orthogonal complement of the kernel of p is given by $(\ker p)^\perp = \mathfrak{X}(M) \oplus d^* \Omega^2(M) \oplus d_{\nabla}^* \Omega^1(M, \mathcal{G})$ and hence

$$\mathfrak{h}_{\text{ex}}(E) = \text{im } p \cong (\ker p)^\perp = \mathfrak{X}(M) \oplus d^* \Omega^2(M) \oplus d_{\nabla}^* \Omega^1(M, \mathcal{G}).$$

To see that this is a tame isomorphism, note that its inverse map is given by

$$\mathfrak{h}_{\text{ex}}(E) \ni X + b + \mathbf{A} \longmapsto X + Q_{\text{dR}}(b - (Q_{\text{dR}} \mathcal{L}_X H - \iota_X H)) + Q_{\nabla} \mathbf{A}.$$

□

Like in the case of exact Courant algebroids, we consider a local chart neighborhood $\mathcal{V}$ of $\text{id} \in \text{Auteq}^{\text{ex}}(E)$, with a chart Φ taking values in some neighborhood $\mathcal{U} \subset \mathfrak{h}_{\text{ex}}(E)$ around 0 and assume that $D\Phi(\text{id}) = \text{id}$. We will again denote by $F_v := \Phi^{-1}(v)$ the exact autoequivalence in $\mathcal{V}$ corresponding to $v \in \mathcal{U}$.

Like in the exact case, the Lie algebra of exact autoequivalences can be identified with the Lie algebra of inner derivations. First, we consider the natural action of autoequivalences on sections $u \in \Gamma(M, E)$, which we denote again by μ_E .

Proposition 8.16. *The map $\mu_E: \text{Auteq}(E) \times \Gamma(M, E) \rightarrow \Gamma(M, E)$ is smooth tame.*

Proof. Given $(f, b, O, \mathbf{A}) \in \text{Auteq}(E)$ and $u = X + \mathbf{r} + \alpha \in \Gamma(M, E)$, we compute

$$\begin{aligned} (f, b, O, \mathbf{A}) \circ u \circ f^{-1} &= F_{f,O} \circ F_{\mathbf{A}} \circ F_{b+B(f)}(X + \mathbf{r} + \alpha) \circ f^{-1} \\ &= F_{f,O} \circ F_{\mathbf{A}}(X + \mathbf{r} + \alpha + \iota_X b + \iota_X Q_{\text{dR}}(H - f^* H)) \circ f^{-1} \\ &= F_{f,O} \circ (X + \mathbf{A}(X) + \mathbf{r} + \alpha + \iota_X b + \iota_X Q_{\text{dR}}(H - f^* H) - \iota_X \langle \mathbf{A}, \mathbf{A} \rangle_{\mathcal{G}} - 2\langle \mathbf{A}, \mathbf{r} \rangle_{\mathcal{G}}) \circ f^{-1} \\ &= f_* X + O(\mathbf{A}(X) + \mathbf{r}) \circ f^{-1} + (f^{-1})^*(\alpha + \iota_X b + \iota_X Q_{\text{dR}}(H - f^* H) - \iota_X \langle \mathbf{A}, \mathbf{A} \rangle_{\mathcal{G}} - 2\langle \mathbf{A}, \mathbf{r} \rangle_{\mathcal{G}}), \end{aligned}$$

which is smooth tame as a composition of smooth tame maps. □

Proposition 8.17. *Let $v \in \mathfrak{h}_{\text{ex}}(E)$, $u \in \Gamma(M, E)$. Then we have*

$$D_F \mu_E(F = \text{id}, u)v = -[v, u].$$

In particular, for any small enough exact autoequivalence Φ_v , we have

$$\mu_E(\Phi_v, u) = u - [v, u] + R_{\mu_E}(v)u,$$

where $R_{\mu_E}(tv) = t^2 \tilde{R}_{\mu_E}(v, t)$, for a real parameter t , where $\tilde{R}_{\mu_E}(v, t)$ depends smoothly on t for small t .

Proof. First, using the formula from the proof of Proposition 8.16, for $(Y + b + T + \mathbf{A}) \in$

$\text{auteq}(E)$ and $u = X + \mathbf{r} + \alpha$, we find

$$\begin{aligned} D_F \mu_E(F = \text{id}, u)\{Y + b + T + \mathbf{A}\} \\ = -\mathcal{L}_Y X + \mathbf{A}(X) + T\mathbf{r} - \mathcal{L}_Y \mathbf{r} - \mathcal{L}_Y \alpha + \iota_X b - \iota_X Q_{\text{dR}} \mathcal{L}_Y H - 2\langle \mathbf{A}, \mathbf{r} \rangle_{\mathcal{G}}. \end{aligned}$$

Setting $T = 0$, $\mathbf{A} = d\mathbf{s}$ for some $\mathbf{s} \in \Omega^0(M, \mathcal{G})$, $b = d\beta + Q_{\text{dR}} \mathcal{L}_Y H - \iota_Y H$, and using $\mathcal{L}_Y \mathbf{r} = d\mathbf{r}(Y)$, we find

$$\begin{aligned} D_F \mu_E(F = \text{id}, u)\{Y + d\beta + Q_{\text{dR}} \mathcal{L}_Y H - \iota_Y H + d\mathbf{s}\} \\ = -\mathcal{L}_Y X + d\mathbf{s}(X) - d\mathbf{r}(Y) - \mathcal{L}_Y \alpha + \iota_X d\beta - \iota_X \iota_Y H - 2\langle d\mathbf{s}, \mathbf{r} \rangle_{\mathcal{G}} \\ = -[Y + \mathbf{s} + \beta, X + \mathbf{r} + \alpha]. \end{aligned}$$

□

Recall again that the derivative at the identity of μ_E gives rise to an identification of $\text{auteq}(E)$ with the derivations of E .

Corollary 8.18. *Under the identification $\text{Der}(E) \cong \text{auteq}(E)$, we have $\mathfrak{h}_{\text{ex}}(E) = -\text{im}(\text{ad})$.*

Remark 8.19. Analogous to the exact case we observe that Proposition 8.15 in fact states that $\mathfrak{h}_{\text{ex}}(E)$ is tamely isomorphic to $(\text{im ad})^\perp$, where the orthogonal complement is taken with respect to the L^2 -inner product on $\Gamma(M, E)$ defined by the bundle metric $h \oplus \langle \cdot, \cdot \rangle_{\text{Eucl}} \oplus h^*$.

The tame Fréchet Lie group structure of the standard $\mathcal{G}$ -flat transitive Courant algebroid gives rise to a tame Fréchet Lie group structure on a general $\mathcal{G}$ -flat transitive Courant algebroid. Given a $\mathcal{G}$ -flat transitive Courant algebroid, there is no unique standard $\mathcal{G}$ -flat transitive Courant algebroid it is isomorphic to: A B -field transformation only changes $H \rightarrow H + dB$ and hence maps standard $\mathcal{G}$ -flat transitive Courant algebroids to standard $\mathcal{G}$ -flat transitive Courant algebroids. Similarly, elements in $\text{Aut}(\mathcal{G}) = C^\infty(M, \text{O}(q, K - q))$ are simply changes of trivializations and again yield a standard $\mathcal{G}$ -flat transitive Courant algebroid.

In contrast, when acting with $\mathbf{A} \in \Omega^1(M, \mathcal{G})$, in general $\mathbf{R}$ no longer vanishes but rather $\mathbf{R} = -d\mathbf{A}$. Later, we will require more flexibility in the choice of bisection and need to also consider those bisections in which $\mathbf{R}$ does not vanish. Therefore, we will be interested in an explicit description of the group of autoequivalences of a standard transitive Courant algebroid obtained from a standard $\mathcal{G}$ -flat Courant algebroid by an $\mathbf{A}$ -field transformation.

8.1.2.3 Autoequivalences in a general choice of bisection

Suppose we change the bisection from a flat standard transitive Courant algebroid by acting with $-\mathbf{S} \in \Omega^1(M, \mathcal{G})$. Since the action by a closed $\mathbf{A}$ -field transform on a standard $\mathcal{G}$ -flat transitive Courant algebroid will again yield a standard $\mathcal{G}$ -flat transitive Courant algebroid, we will assume without loss of generality that $\mathbf{S} = \sum_{i=1}^K S_i \mathbf{e}_i$, with $S_i \in d^* \Omega^2(M)$ (recall that by Remark 8.9 the $\mathbf{S}$ of this kind form a topological complement to $\Omega_{\text{cl}}^1(M)$).

Then the data defining the bracket is as follows: the connection on $\mathcal{G}$ is still flat with trivial holonomy, $\mathbf{R} = d\mathbf{S}$ and the three-form H satisfies $H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}} \in \Omega_{\text{cl}}^3(M)$. The group of autoequivalences of this Courant algebroid is the subgroup of the group of orthogonal equivalences conjugate to the autoequivalences of the standard $\mathcal{G}$ -flat transitive Courant algebroid by the action of $F_{-\mathbf{S}}$. In particular it still has a tame Fréchet Lie group structure, which we explicitly describe in the following theorem.

Theorem 8.20. *Let $E \cong TM \oplus \mathcal{G} \oplus T^*M$ be a standard transitive Courant algebroid with $\mathcal{G} = M \times \mathbb{R}^K$, ∇ a flat connection on $\mathcal{G}$ with trivial holonomy, $\mathbf{R} = d\mathbf{S}$ for some $\mathbf{S} = \sum_{i=1}^K S_i \mathbf{e}_i$, $S_i \in d^*\Omega^2(M)$ and $H \in \Omega^3(M)$ such that $H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}} \in \Omega_{\text{cl}}^3(M)$. Then*

$$\begin{aligned} \text{Auteq}(E) &= \{F_{\nu, \mathbf{A}, B} : \nu = (f, O) \text{ s.t. } f \in \text{Diff}_{[H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}}]}(M), O \in \text{O}(q, K - q), \\ &\quad \mathbf{a} := \mathbf{A} - \mathbf{S} + O^{-1}f^*\mathbf{S} \in \Omega_{\text{cl}}^1(M, \mathcal{G}), \\ &\quad b := B - Q_{\text{dR}}(H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}} - f^*(H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}})) \\ &\quad - \langle \mathbf{a} \wedge (\mathbf{S} + O^{-1}f^*\mathbf{S}) \rangle_{\mathcal{G}} + \langle O^{-1}f^*\mathbf{S} \wedge \mathbf{S} \rangle_{\mathcal{G}} \in \Omega_{\text{cl}}^2(M)\} \\ &\cong \text{Diff}(M)_{[H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}}]} \times \Omega_{\text{cl}}^2(M) \times \text{O}(q, K - q) \times \Omega_{\text{cl}}^1(M, \mathcal{G}). \end{aligned}$$

The multiplication map is given by

$$(f_1, b_1, O_1, \mathbf{a}_1) \circ (f_2, b_2, O_2, \mathbf{a}_2) \longmapsto (f_1 f_2, b_3, O_1 O_2, \mathbf{a}_3),$$

where

$$\begin{aligned} \mathbf{a}_3 &= O_2^{-1} f_2^* \mathbf{a}_1 + \mathbf{a}_2 \\ b_3 &= f_2^* b_1 + b_2 + \langle O_2^{-1} f_2^* \mathbf{a}_1 \wedge \mathbf{a}_2 \rangle_{\mathcal{G}} + [f_2^*, Q_{\text{dR}}](H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}} - f_1^*(H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}})), \end{aligned}$$

and the inversion map is given by

$$(f, b, O, \mathbf{a}) \mapsto (f^{-1}, b', O^{-1}, -O(f^{-1})^* \mathbf{a}),$$

where

$$b' = -(f^{-1})^* b + [Q_{\text{dR}}, (f^{-1})^*](H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}} - f^*(H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}})).$$

Proof. Like in the proofs of Proposition 8.10 and Theorem 8.11, we will first take a closer look at the constraints in Theorem 5.27 on $\nu = (f, O) \in \text{Auteq}(\mathcal{G})$, $\mathbf{A} \in \Omega^1(M, \mathcal{G})$ and $B \in \Omega^2(M)$ to define an autoequivalence $F_{\nu, \mathbf{A}, B}$ of E .

By the same argument as in Proposition 8.10, the first condition implies that O must be constant, i.e. $O \in \text{O}(q, K - q)$. The second condition becomes

$$O^{-1}f^*d\mathbf{S} = d\mathbf{S} - d\mathbf{A},$$

which, using equation (8.2) and that O is constant, this is equivalent to

$$d(\mathbf{A} + O^{-1}f^*\mathbf{S} - \mathbf{S}) = 0.$$

Henceforth, we will write

$$\mathbf{A} = \mathbf{a} + \mathbf{S} - O^{-1}f^*\mathbf{S} \text{ for some } \mathbf{a} \in \Omega_{\text{cl}}^1(M, \mathcal{G}).$$

The last condition in Theorem 5.27 becomes

$$\begin{aligned} H - f^*H &= dB + 2\langle \mathbf{A} \wedge d\mathbf{S} \rangle_{\mathcal{G}} - \langle \mathbf{A} \wedge d\mathbf{A} \rangle_{\mathcal{G}} \\ &= dB + 2\langle (\mathbf{a} + \mathbf{S} - O^{-1}f^*\mathbf{S}) \wedge d\mathbf{S} \rangle_{\mathcal{G}} - \langle (\mathbf{a} + \mathbf{S} - O^{-1}f^*\mathbf{S}) \wedge (d\mathbf{S} - O^{-1}f^*d\mathbf{S}) \rangle_{\mathcal{G}}. \end{aligned}$$

Observing that for general $\mathbf{A}' \in \Omega^k(M, \mathcal{G})$ and $\mathbf{B}' \in \Omega^l(M, \mathcal{G})$, we have

$$\begin{aligned}\langle O^{-1}\mathbf{A}' \wedge O^{-1}\mathbf{B}' \rangle_{\mathcal{G}} &= \langle \mathbf{A}' \wedge \mathbf{B}' \rangle_{\mathcal{G}}, \\ \langle f^*\mathbf{A}' \wedge f^*\mathbf{B}' \rangle_{\mathcal{G}} &= f^* \langle \mathbf{A}' \wedge \mathbf{B}' \rangle_{\mathcal{G}}, \\ d\langle \mathbf{A}' \wedge \mathbf{B}' \rangle_{\mathcal{G}} &= \langle d\mathbf{A}' \wedge \mathbf{B}' \rangle_{\mathcal{G}} + (-1)^k \langle \mathbf{A}' \wedge d\mathbf{B}' \rangle_{\mathcal{G}},\end{aligned}$$

we find

$$H - f^*H = dB - d\langle \mathbf{a} \wedge (\mathbf{S} + O^{-1}f^*\mathbf{S}) \rangle_{\mathcal{G}} - d\langle \mathbf{S} \wedge O^{-1}f^*\mathbf{S} \rangle_{\mathcal{G}} + \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}} - f^*\langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}}.$$

In particular, this shows that $f \in \text{Diff}(M)_{[H-\langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}}]}$ and that

$$B = b + B(f, \mathbf{a}) \text{ for some } b \in \Omega_{\text{cl}}^2(M),$$

where we defined

$$B(f, \mathbf{a}) = Q_{\text{dR}}(H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}}) + \langle \mathbf{a} \wedge (\mathbf{S} + O^{-1}f^*\mathbf{S}) \rangle_{\mathcal{G}} + \langle \mathbf{S} \wedge O^{-1}f^*\mathbf{S} \rangle_{\mathcal{G}}.$$

In total, we obtain a bijection

$$\begin{aligned}\text{Diff}(M)_{[H-\langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}}]} \times \Omega_{\text{cl}}^2(M) \times O(q, K - q) \times \Omega_{\text{cl}}^1(M, \mathcal{G}) &\xrightarrow{\cong} \text{Auteq}(E) \\ (f, b, O, \mathbf{a}) &\longmapsto F_{(f, O), \mathbf{a} + \mathbf{S} - O^{-1}f^*\mathbf{S}, b + B(f, \mathbf{a})}.\end{aligned}$$

To express the multiplication in $\text{Auteq}(E)$ in terms of this bijection, we write

$$(f_1, b_1, O_1, \mathbf{a}_1)(f_2, b_2, O_2, \mathbf{a}_2) = (f_1f_2, b_3, O_1O_2, \mathbf{a}_3).$$

Then, using the formulas for the multiplication in Theorem 5.27, we find that $\mathbf{a}_3$ is given by

$$\begin{aligned}\mathbf{a}_3 &= O_2^{-1}f_2^*(\mathbf{a}_1 + \mathbf{S} - O_1^{-1}f_1^*\mathbf{S}) + \mathbf{a}_2 + \mathbf{S} - O_2^{-1}f_2^*\mathbf{S} - (\mathbf{S} - (O_1O_2)^{-1}(f_1f_2)^*\mathbf{S}) \\ &= O_2^{-1}f_2^*\mathbf{a}_1 + O_2^{-1}f_2^*\mathbf{S} - (O_1O_2)^{-1}(f_1f_2)^*\mathbf{S} + \mathbf{a}_2 + \mathbf{S} - O_2^{-1}f_2^*\mathbf{S} \\ &\quad - (\mathbf{S} - (O_1O_2)^{-1}(f_1f_2)^*\mathbf{S}) \\ &= O_2^{-1}f_2^*\mathbf{a}_1 + \mathbf{a}_2.\end{aligned}$$

Similarly,

$$\begin{aligned}b_3 &= f_2^*(b_1 + B(f_1, \mathbf{a}_1)) + (b_2 + B(f_2, \mathbf{a}_2)) \\ &\quad + \langle O_2^{-1}f_2^*(\mathbf{a}_1 + \mathbf{S} - O_1^{-1}f_1^*\mathbf{S}) \wedge (\mathbf{a}_2 + \mathbf{S} - O_2^{-1}f_2^*\mathbf{S}) \rangle_{\mathcal{G}} - B(f_1f_2, O_2^{-1}f_2^*\mathbf{a}_1 + \mathbf{a}_2).\end{aligned}$$

To simplify this expression, we first compute

$$\begin{aligned}&f_2^*(b_1 + B(f_1, \mathbf{a}_1)) + (b_2 + B(f_2, \mathbf{a}_2)) - B(f_1f_2, O_2^{-1}f_2^*\mathbf{a}_1 + \mathbf{a}_2) \\ &= [f_2^*, Q_{\text{dR}}](H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}} - f_1^*(H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}})) + \langle f_2^*\mathbf{a}_1 \wedge f_2^*\mathbf{S} \rangle_{\mathcal{G}} \\ &\quad - \langle O_1^{-1}(f_1f_2)^*\mathbf{S} \wedge f_2^*\mathbf{S} \rangle_{\mathcal{G}} + \langle \mathbf{a}_2 \wedge O_2^{-1}f_2^*\mathbf{S} \rangle_{\mathcal{G}} - \langle O_2^{-1}f_2^*\mathbf{S} \wedge \mathbf{S} \rangle_{\mathcal{G}} \\ &\quad - \langle O_2^{-1}f_2^*\mathbf{a}_1 \wedge \mathbf{S} \rangle_{\mathcal{G}} - \langle \mathbf{a}_2 \wedge (O_1O_2)^{-1}(f_1f_2)^*\mathbf{S} \rangle_{\mathcal{G}} + \langle (O_1O_2)^{-1}(f_1f_2)^*\mathbf{S} \wedge \mathbf{S} \rangle_{\mathcal{G}}.\end{aligned}$$

Expanding and adding the remaining terms

$$f_2^* b_1 + b_2 + \langle O_2^{-1} f_2^* (\mathbf{a}_1 + \mathbf{S} - O_1^{-1} f_1^* \mathbf{S}) \wedge (\mathbf{a}_2 + \mathbf{S} - O_2^{-1} f_2^* \mathbf{S}) \rangle_{\mathcal{G}}$$

we find that

$$b_3 = f_2^* b_1 + b_2 + \langle O_2^{-1} f_2^* \mathbf{a}_1 \wedge \mathbf{a}_2 \rangle_{\mathcal{G}} + [f_2^*, Q_{\text{dR}}] (H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}} - f_1^* (H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}})).$$

Finally, using these formulas for the multiplication, we confirm that the inversion map is given by the expression stated in the theorem. $\square$

Since $\text{Auteq}(E)$ is conjugate to the group of autoequivalences of a flat standard transitive Courant algebroid by some $\mathbf{A}$ -field transformation, and since the action of $\mathbf{A}$ -field transformations on $\Gamma(M, E)$ is smooth tame, we conclude from Proposition 8.16 that the action of $\text{Auteq}(E)$ on $\Gamma(M, E)$ is smooth tame.

Moreover, $\text{Auteq}(E)$ has a tame Fréchet Lie subgroup, the group $\text{Auteq}^{\text{ex}}(E)$ of exact autoequivalences, which is conjugate to the group of exact autoequivalences of some standard $\mathcal{G}$ -flat transitive Courant algebroid. Its Lie algebra is again the image of $-\text{ad}: \Gamma(M, E) \rightarrow \mathfrak{auteq}(E)$. Using this fact, we will give an explicit description of the underlying tame Fréchet space below.

Proposition 8.21. *Let E be as in Theorem 8.20. Then the Lie algebra of $\text{Auteq}^{\text{ex}}(E)$ is given by*

$$\begin{aligned} \mathfrak{h}_{\text{ex}}(E) = \{X + b + \mathbf{a} \in \mathfrak{auteq}(E) : \exists \alpha \in \Omega^1(M), \exists \mathbf{r} \in \Gamma(M, \mathcal{G}) \text{ s.t. } \mathbf{a} = d\mathbf{r} + d\iota_X \mathbf{S}, \\ b = d(\alpha - \langle 2\mathbf{r} + \iota_X \mathbf{S}, \mathbf{S} \rangle_{\mathcal{G}}) + Q_{\text{dR}} \mathcal{L}_X (H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}}) - \iota_X (H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}})\}, \end{aligned}$$

where as above $\mathbf{S} = \sum_{i=1}^K S_i \mathbf{e}_i$ is such that $\mathbf{R} = d\mathbf{S}$, $S_i \in d^* \Omega^2(M)$.

Proof. This can be proved by first computing $D_F \mu_E (F = \text{id}, Y + \mathbf{r} + \beta) \{X + b + T + \mathbf{a}\}$ for general $X + b + T + \mathbf{a} \in \mathfrak{auteq}(E)$ and then comparing to the expression of $-[X + \mathbf{r} + \alpha, Y + \mathbf{s} + \beta]$. For both expressions to agree with each other, we must have $T = 0$, $\mathbf{a} = d\mathbf{r} + d\iota_X \mathbf{S}$ and

$$b = d(\alpha - \langle 2\mathbf{r} + \iota_X \mathbf{S}, \mathbf{S} \rangle_{\mathcal{G}}) + Q_{\text{dR}} \mathcal{L}_X (H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}}) - \iota_X (H - \langle \mathbf{S} \wedge d\mathbf{S} \rangle_{\mathcal{G}}).$$

$\square$

As before, we will identify $\mathfrak{h}_{\text{ex}}(E)$ with a tame Fréchet subspace of $\Gamma(M, E)$.

Proposition 8.22. *Let $\{\mathbf{e}_i\}_{i=1}^K$ be some parallel orthonormal frame of $\mathcal{G}$ and $S_i \in d^* \Omega^2(M)$ such that $\mathbf{S} = \sum_i S_i \mathbf{e}_i$, where $\mathbf{S}$ is such that $\mathbf{R} = d\mathbf{S}$. We denote by $(G_{ij}) \in \mathbb{R}^{K^2}$ the (positive definite symmetric) matrix defined by $G_{ij} = (\mathbf{e}_i, \mathbf{e}_j)_{\mathcal{G}}$, where $(\cdot, \cdot)_{\mathcal{G}}$ denotes the L^2 -inner product on $\mathcal{G}$ defined in the beginning of Section 8.1.2. There is a tame isomorphism*

$$\begin{aligned} \mathfrak{h}_{\text{ex}}(E) &\cong (\ker \text{ad})^{\perp} \\ &= \left\{ X + \mathbf{r} - 2 \sum_{i,j} G_{ji}^{-1} (\epsilon_i S_i, \alpha) \mathbf{e}_j + \alpha : X \in \mathfrak{X}(M), \mathbf{r} \in d_{\nabla}^* \Omega^1(M, \mathcal{G}), \alpha \in d^* \Omega^2(M) \right\}, \end{aligned}$$

where the orthogonal complement of $\ker \text{ad}$ is taken with respect to the L^2 -inner product $(\cdot, \cdot)_E$ on $\Gamma(M, E)$ (defined in the beginning of Section 8.1.2) and $(\cdot, \cdot)$ denotes the L^2 -inner product induced by the auxiliary Riemannian metric h .

Proof. First, we observe by inspecting the expression of the Dorfman bracket that

$$Y + \mathbf{s} + \beta \in \ker(-\text{ad}) = \ker(\text{ad}) \iff Y = 0, \text{ds} = 0, \beta - 2\langle \mathbf{s}, \mathbf{S} \rangle_{\mathcal{G}} \in \Omega_{\text{cl}}^1(M).$$

We claim that the orthogonal complement of $\ker(\text{ad})$ with respect to $(\cdot, \cdot)_E$ exists and that it is given by

$$(\ker \text{ad})^\perp = \left\{ X + \mathbf{r} - 2 \sum_{i,j} G_{ji}^{-1}(\epsilon_i S_i, \alpha) \mathbf{e}_j + \alpha : X \in \mathfrak{X}(M), \mathbf{r} \in \text{d}_{\nabla}^* \Omega^1(M, \mathcal{G}), \alpha \in \text{d}^* \Omega^2(M) \right\}.$$

To see that it is orthogonal, we observe that for $\mathbf{s} + \beta \in \ker(\text{ad})$ and $X + \mathbf{r} - 2 \sum_{i,j} G_{ji}^{-1}(\epsilon_i S_i, \alpha) \mathbf{e}_j + \alpha$ with $\mathbf{r} \in \text{d}_{\nabla}^* \Omega^1(M, \mathcal{G}), \alpha \in \text{d}^* \Omega^2(M)$ we have

$$\left(\mathbf{s} + \beta, X + \mathbf{r} - 2 \sum_{i,j} G_{ji}^{-1}(\epsilon_i S_i, \alpha) \mathbf{e}_j + \alpha \right) = -2 \sum_i s_i (\epsilon_i S_i, \alpha) + 2(\alpha, \langle \mathbf{s}, \mathbf{S} \rangle_{\mathcal{G}}) = 0,$$

where we used that s_i is constant for every i since $\text{ds} = 0$ and $\text{pr}_{\text{imd}^*} \beta = 2 \text{pr}_{\text{imd}^*} \langle \mathbf{s}, \mathbf{S} \rangle_{\mathcal{G}}$.

To see that it is an algebraic complement, we need to show that for every $\mathbf{t} + \gamma \in \Gamma(M, \mathcal{G}) \oplus \Omega^1(M)$ there exist $\mathbf{s} \in \Omega_{\text{cl}}^0(M, \mathcal{G}), \mathbf{r} \in \text{d}_{\nabla}^* \Omega^1(M, \mathcal{G}), \tilde{\beta} \in \Omega_{\text{cl}}^1(M)$ and $\alpha \in \text{d}^* \Omega^2(M)$ such that

$$\mathbf{t} + \gamma = \mathbf{s} + \tilde{\beta} + 2\langle \mathbf{s}, \mathbf{S} \rangle_{\mathcal{G}} + \mathbf{r} + \alpha - 2 \sum_{i,j} G_{ji}^{-1}(\epsilon_i S_i, \alpha) \mathbf{e}_j.$$

Writing $\mathbf{t} = \mathbf{t}_0 + \mathbf{t}_1$ with $\mathbf{t}_0 \in \Omega_{\text{cl}}^0(M, \mathcal{G}), \mathbf{t}_1 \in \text{d}_{\nabla}^* \Omega^1(M, \mathcal{G})$ and $\gamma = \gamma_0 + \gamma_1$ with $\gamma_0 \in \Omega_{\text{cl}}^1(M), \gamma_1 \in \text{d}^* \Omega^2(M)$, this is equivalent to the system

$$\begin{aligned} \gamma_0 &= \tilde{\beta}, \quad \gamma_1 = \alpha + 2\langle \mathbf{s}, \mathbf{S} \rangle_{\mathcal{G}}, \quad \mathbf{t}_1 = \mathbf{r}, \\ \mathbf{t}_0 &= \mathbf{s} - 2 \sum_{i,j} G_{ji}^{-1}(\epsilon_i S_i, \alpha) \mathbf{e}_j = \mathbf{s} - 2 \sum_{i,j=1}^K G_{ji}^{-1}(\epsilon_i S_i, \gamma_1 - 2\langle \mathbf{s}, \mathbf{S} \rangle_{\mathcal{G}}) \mathbf{e}_j, \end{aligned} \tag{8.3}$$

where we substituted the equation for γ_1 in the second line and used that $\text{pr}_{\text{imd}^*} \langle \mathbf{s}, \mathbf{S} \rangle_{\mathcal{G}} = \langle \mathbf{s}, \mathbf{S} \rangle_{\mathcal{G}}$, since $S_i \in \text{d}^* \Omega^2(M)$ and s_i are constant. Then (8.3) has a (unique) solution if and only if

$$\Omega_{\text{cl}}^0(M, \mathcal{G}) \ni \mathbf{s} \longmapsto \mathbf{s} + 4 \sum_{i,j=1}^K G_{ji}^{-1}(\epsilon_i S_i, \langle \mathbf{s}, \mathbf{S} \rangle_{\mathcal{G}}) \mathbf{e}_j \in \Omega_{\text{cl}}^0(M, \mathcal{G}) \tag{8.4}$$

is invertible. This is equivalent to the invertibility of the matrix $A \in \mathbb{R}^{K^2}$ with

$$A_{ik} = \delta_{ik} + 4 \sum_{i,j} G_{ij}^{-1}(\epsilon_j S_j, \epsilon_k S_k).$$

Note that the matrix defined by $(\epsilon_i S_i, \epsilon_j S_j)$ is positive semi-definite. Indeed, given $(\lambda_i) \in \mathbb{R}^K$, we have

$$\sum_{i,j} \lambda_i (\epsilon_i S_i, \epsilon_j S_j) \lambda_j = \left(\sum_i \lambda_i \epsilon_i S_i, \sum_j \lambda_j \epsilon_j S_j \right) \geq 0.$$

Since G is positive-definite, it follows that GA is invertible as a sum of a positive-definite and positive semi-definite matrix. Hence, A is invertible.

Note that both $\ker \text{ad}$ and $(\ker \text{ad})^\perp$ are closed subspaces of $\Gamma(M, E)$ and therefore graded Fréchet spaces. Moreover, the projections of $\Gamma(M, E)$ to the respective subspaces are tame linear maps, which can be seen from the fact that the solution $(\mathbf{s}, \alpha, \mathbf{r}, \tilde{\beta})$ of (8.3) depends tame linearly on $\mathbf{t} + \gamma$ (note that the inverse map of (8.4) is automatically tame linear, being a linear map between finite-dimensional vector spaces). This shows that $\ker \text{ad}$ and $(\ker \text{ad})^\perp$ are tame direct summands of $\Gamma(M, E)$ and thus tame Fréchet spaces.

We conclude that

$$\begin{aligned} \mathfrak{h}_{\text{ex}}(E) &\cong (\ker \text{ad})^\perp \\ &= \left\{ X + \mathbf{r} - 2 \sum_{i,j} G_{ji}^{-1}(\epsilon_i S_i, \alpha) \mathbf{e}_j + \alpha : X \in \mathfrak{X}(M), \mathbf{r} \in \text{d}^* \Omega^1(M, \mathcal{G}), \alpha \in \text{d}^* \Omega^2(M) \right\}, \end{aligned}$$

where the map $(\ker \text{ad})^\perp \longrightarrow \mathfrak{h}_{\text{ex}}(E)$ is given by the restriction of $\text{ad}: \Gamma(M, E) \longrightarrow \mathfrak{h}_{\text{ex}}(E)$. To see that this is a tame isomorphism, using Proposition 8.21 we observe that its inverse is given by

$$\mathfrak{h}_{\text{ex}}(E) \ni X + b + \mathbf{a} \longmapsto X + \mathbf{r}(X, \mathbf{a}) - 2 \sum_{i,j} G_{ji}^{-1}(\epsilon_i S_i, \alpha(X, b, \mathbf{a})) \mathbf{e}_j + \alpha(X, b, \mathbf{a}),$$

where

$$\begin{aligned} \mathbf{r}(X, \mathbf{a}) &= Q_\nabla(\mathbf{a} - \text{d}\iota_X \mathbf{S}), \\ \alpha(X, b, \mathbf{a}) &= Q_{\text{dR}}(b + \text{d}\langle 2\mathbf{r}(X, \mathbf{a}) + \iota_X \mathbf{S}, \mathbf{S} \rangle_{\mathcal{G}} - Q_{\text{dR}} \mathcal{L}_X(H - \langle \mathbf{S} \wedge \text{d}\mathbf{S} \rangle_{\mathcal{G}}) + \iota_X(H - \langle \mathbf{S} \wedge \text{d}\mathbf{S} \rangle_{\mathcal{G}})). \end{aligned}$$

Here we use that for $\mathbf{A} \in \text{d}\Omega^0(M, \mathcal{G})$, we have $\text{d}Q_\nabla \mathbf{A} = \mathbf{A}$ and similarly, for $B \in \text{d}\Omega^1(M)$, we have $\text{d}Q_{\text{dR}} B = B$. $\square$

In view of Proposition 8.22 (and Remark 8.19) we will from now on always identify

$$\mathfrak{h}_{\text{ex}}(E) \cong (\ker \text{ad})^\perp \subset \Gamma(M, E). \quad (8.5)$$

Last, we observe that given some generalized complex structure L , we may use the isomorphism of real vector bundles

$$\begin{aligned} \Psi: \bar{L} &\longrightarrow E \\ \xi &\longmapsto \xi + \bar{\xi} \end{aligned} \quad (8.6)$$

to describe exact autoequivalences by elements in $\Psi^{-1}(\mathfrak{h}_{\text{ex}}(E)) \subset \Gamma(M, L^*)$. For $\xi \in \Psi^{-1}(\mathfrak{h}_{\text{ex}}(E))$, we will then also denote $F_\xi := F_{\Psi\xi}$.

8.1.2.4 Choices of bisection and bundle metrics

Next, we consider $\mathcal{G}$ -flat transitive Courant algebroids with a generalized complex structure. Note that the existence of a generalized complex structure forces the scalar product $\langle \cdot, \cdot \rangle$ of the Courant algebroid to be of signature $(2k, 2\ell)$, where $2k = \dim M + q$ and $2\ell = \dim M + K - q$. In other words, it forces K and $\dim M + q$ to be even.

Recall that a *generalized (Riemannian) metric* on a Courant algebroid E is a positive definite scalar product $\mathfrak{H}$ on E such that the endomorphism $\mathfrak{H}^{\text{end}}$ defined by $\mathfrak{H} = \langle \mathfrak{H}^{\text{end}} \cdot, \cdot \rangle$

is an involution. We denote by $E_{\pm} \subset E$ the ± 1 -eigenbundles of $\mathfrak{H}^{\text{end}}$.

Lemma 8.23. *Given a generalized complex structure $\mathcal{J}$ on a Courant algebroid E , there exists a generalized Hermitian metric with respect to $\mathcal{J}$, i.e. a generalized metric $\mathfrak{H}$ such that $\mathfrak{H}(\mathcal{J}\cdot, \mathcal{J}\cdot) = \mathfrak{H}(\cdot, \cdot)$.*

Proof. This follows from the fact that every $U(k, \ell)$ -structure admits a reduction to the maximal compact subgroup $U(k) \times U(\ell) \subset U(k, \ell)$. $\square$

Remark 8.24. Explicitly, a generalized Hermitian metric can be obtained by constructing $\mathcal{J}$ -invariant subspaces $E_{\pm}$ in the following way: Take an orthonormal vector $u_1 \in E$, say $\langle u_1, u_1 \rangle = \epsilon \in \{\pm 1\}$. Then

$$\langle \mathcal{J}u_1, \mathcal{J}u_1 \rangle = \langle u_1, u_1 \rangle = \epsilon.$$

Hence, $\text{span}\{u_1, \mathcal{J}u_1\}$ is either a positive or a negative subbundle of E .

Next, we pick some vector $u_2 \in (\text{span}\{u_1, \mathcal{J}u_1\})^{\perp}$ and normalize such that $|\langle u_2, u_2 \rangle| = 1$. Proceeding as before and iterating the procedure yields two subbundles $E_{\pm}$ that are $\mathcal{J}$ -invariant and define a generalized metric that is in particular a generalized Hermitian metric.

A statement similar to the following lemma already appeared in [JV16] but with a different proof.

Lemma 8.25. *Let $\mathfrak{H}$ be a generalized metric on a transitive Courant algebroid E . Then there is a bisection of E in which $\mathfrak{H} = \frac{1}{2}g \oplus \langle \cdot, \cdot \rangle_{\text{Eucl}} \oplus \frac{1}{2}g^*$, where g is a Riemannian metric on M and $\langle \cdot, \cdot \rangle_{\text{Eucl}}$ is a positive definite scalar product on $\mathcal{G}$.*

Proof. We apply similar ideas as in [Gar14]. Since $\mathfrak{H}$ is positive definite, we must have

$$E_+ \cap T^*M = \{0\} = E_- \cap T^*M$$

and it follows that the restrictions

$$\text{pr}_{\mathcal{G}}|_{\ker \pi \cap E_{\pm}} : \ker \pi \cap E_{\pm} \longrightarrow \mathcal{G}$$

are injective. We denote $\mathcal{G}_{\pm} = \text{pr}_{\mathcal{G}} E_{\pm} \cap \ker \pi$. Note that $\langle \cdot, \cdot \rangle_{\mathcal{G}}|_{\mathcal{G}_+} > 0$ and $\langle \cdot, \cdot \rangle_{\mathcal{G}}|_{\mathcal{G}_-} < 0$. It follows that

$$\mathcal{G}_+ \cap \mathcal{G}_- = \{0\}, \quad \text{rank } \mathcal{G}_+ \leq q, \quad \text{rank } \mathcal{G}_- \leq K - q.$$

Moreover, we denote by $(E_+ \cap \ker \pi)^{\perp} \subset E_+$ the orthogonal complement of $E_+ \cap \ker \pi$. Then the anchor map defines an injective vector bundle morphism

$$\pi_{\perp} = \pi|_{(E_+ \cap \ker \pi)^{\perp}} : (E_+ \cap \ker \pi)^{\perp} \longrightarrow TM.$$

In particular, $\text{rank}(E_+ \cap \ker \pi)^{\perp} \leq \dim M$ and we find

$$\dim M \geq \text{rank}(E_+ \cap \ker \pi)^{\perp} = \text{rank } E_+ - \text{rank } \mathcal{G}_+ \geq \dim M + q - q = \dim M$$

and hence

$$\text{rank } \mathcal{G}_+ = q, \quad \text{rank}(E_+ \cap \ker \pi)^{\perp} = \dim M.$$

A similar argument shows that $\text{rank } \mathcal{G}_- = K - q$. Hence $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$ and $\pi_{\perp}$ is in fact an isomorphism. Moreover, we obtain isomorphisms

$$\text{pr}_{\mathcal{G}_{\pm}} = \text{pr}_{\mathcal{G}}|_{\ker \pi \cap E_{\pm}} : \ker \pi \cap E_{\pm} \longrightarrow \mathcal{G}_{\pm}.$$

The isomorphism $\pi_\perp$ determines a Riemannian metric on TM via

$$g(\cdot, \cdot) := \mathfrak{H}(\pi_\perp^{-1}\cdot, \pi_\perp^{-1}\cdot) = \langle \pi_\perp^{-1}\cdot, \pi_\perp^{-1}\cdot \rangle.$$

We define the map

$$\begin{aligned} \lambda: TM &\longrightarrow E \\ X &\longmapsto \pi_\perp^{-1}(X) - \frac{1}{2}\pi^*g(X). \end{aligned}$$

Then clearly $\pi \circ \lambda = \text{id}$ and for $X, Y \in TM$ we find

$$\begin{aligned} \langle \lambda(X), \lambda(Y) \rangle &= \langle \pi_\perp^{-1}(X) - \frac{1}{2}\pi^*g(X), \pi_\perp^{-1}(Y) - \frac{1}{2}\pi^*g(Y) \rangle \\ &= g(X, Y) - \frac{1}{2}g(X)(Y) - \frac{1}{2}g(Y)(X) \\ &= g(X, Y) - g(X, Y) \\ &= 0. \end{aligned}$$

Moreover, for every $\mathbf{r} \in \mathcal{G}$ we write $\mathbf{r} = \mathbf{r}_+ + \mathbf{r}_-$ and define

$$\begin{aligned} \sigma: \mathcal{G} &\longrightarrow \ker \pi \\ \mathbf{r} &\longmapsto \text{pr}_{\mathcal{G}_+}^{-1}(\mathbf{r}_+) + \text{pr}_{\mathcal{G}_-}^{-1}(\mathbf{r}_-). \end{aligned}$$

Then for every $X \in TM$ and $\mathbf{r} \in \mathcal{G}$, we find

$$\langle \sigma(\mathbf{r}), \lambda(X) \rangle = \langle \text{pr}_{\mathcal{G}_+}^{-1}(\mathbf{r}_+), \pi_\perp^{-1}(X) \rangle = 0.$$

Therefore, (λ, σ) define a bisection of E . Since

$$\begin{aligned} E_+ &= \{ \pi_\perp^{-1}(X) + \text{pr}_{\mathcal{G}_+}^{-1}(\mathbf{r}_+) : X \in TM, \mathbf{r}_+ \in \mathcal{G}_+ \} \\ &= \{ \lambda(X) + \frac{1}{2}\pi^*g(X) + \text{pr}_{\mathcal{G}_+}^{-1}(\mathbf{r}_+) : X \in TM, \mathbf{r}_+ \in \mathcal{G}_+ \}, \end{aligned}$$

we observe that in the bisection defined by (λ, σ) we have

$$\begin{aligned} E_+ &= \{ X + g(X) + \mathbf{r}_+ : X \in TM, \mathbf{r}_+ \in \mathcal{G}_+ \}, \\ E_- &= \{ X - g(X) + \mathbf{r}_- : X \in TM, \mathbf{r}_- \in \mathcal{G}_- \}. \end{aligned}$$

It follows that in this splitting $\mathfrak{H}$ takes the form $\mathfrak{H} = \frac{1}{2}g \oplus \langle \cdot, \cdot \rangle_{\text{Eucl}} \oplus \frac{1}{2}g^*$, where

$$\langle \cdot, \cdot \rangle_{\text{Eucl}} = \langle \cdot, \cdot \rangle_{\mathcal{G}}|_{\mathcal{G}_+} - \langle \cdot, \cdot \rangle_{\mathcal{G}}|_{\mathcal{G}_-}.$$

□

Note that in the case where $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ is positive definite we must have $\langle \cdot, \cdot \rangle_{\text{Eucl}} = \langle \cdot, \cdot \rangle_{\mathcal{G}}$. Combining Lemma 8.23 and Lemma 8.25, we obtain the following.

Corollary 8.26. *Suppose that E is a $\mathcal{G}$ -flat transitive Courant algebroid with a generalized complex structure $\mathcal{J}$. Then there exists a generalized Hermitian metric $\mathfrak{H}$ and a choice of bisection such that $\mathfrak{H} = h \oplus \langle \cdot, \cdot \rangle_{\text{Eucl}} \oplus h^*$.*

Proof. This is a direct consequence of Lemma 8.23 and Lemma 8.25 by defining $h := \frac{1}{2}g$. $\square$

Given a $\mathcal{G}$ -flat transitive Courant algebroid with a generalized complex structure, we will fix a bisection and bundle metrics on TM , L and $\bar{L}$ in the following way: First, we choose a generalized Hermitian metric $\mathfrak{H}$ on E . Then we choose a bisection $E \cong TM \oplus \mathcal{G} \oplus T^*M$ and a Riemannian metric h on M such that the generalized Hermitian metric is given by $\mathfrak{H} = h \oplus \langle \cdot, \cdot \rangle_{\text{Eucl}} \oplus h^*$. Restricting the complex sesqui-linear extension of $\mathfrak{H}$ to the subbundles L and $\bar{L}$ yields Hermitian bundle metrics on these subbundles.

In the following, we will always assume that the L^2 -inner products on the spaces $\Gamma(M, \Lambda^\bullet L^*)$, as well as on the spaces $\Omega^\bullet(M)$ and $\mathfrak{X}(M)$ are defined in terms of these choices of bundle metrics and with respect to the Riemannian metric h . Moreover, the Riemannian metric defining the charts for $\text{Diff}(M)_{[H]}$ will be chosen to be h .

The advantage of these choices of bundle metrics is the following.

Lemma 8.27. *Let E be a $\mathcal{G}$ -flat transitive Courant algebroid with a generalized complex structure L . With the above choices of bundle metrics defining the L^2 -inner products and the identification $\mathfrak{h}_{\text{ex}}(E) \cong (\ker \text{ad})^\perp$ as in (8.5), it holds that*

$$\Psi^{-1} \mathfrak{h}_{\text{ex}}(E) \supset d_L^* (\Gamma(M, \Lambda^2 L^*)).$$

Moreover, $d_L^* (\Gamma(M, \Lambda^2 L^*))$ has finite codimension in $\Psi^{-1}(\mathfrak{h}_{\text{ex}}(E))$ and if $H^1(M, L) = 0$, then

$$\Psi^{-1}(\mathfrak{h}_{\text{ex}}(E)) = d_L^* (\Gamma(M, \Lambda^2 L^*)).$$

Proof. Under the identification $\mathfrak{h}_{\text{ex}}(E) \cong (\ker \text{ad})^\perp$, we have an orthogonal decomposition

$$\Gamma(M, E) = \mathfrak{h}_{\text{ex}}(E) \oplus \ker(\text{ad}).$$

Since $\Psi^{-1} = \text{pr}_{\bar{L}} : \Gamma(M, E) \longrightarrow \Gamma(M, \bar{L})$, with our choices of bundle metrics Ψ^{-1} is in fact a homothety and we obtain an orthogonal decomposition

$$\Gamma(M, \bar{L}) = \Psi^{-1} \mathfrak{h}_{\text{ex}}(E) \oplus \Psi^{-1} \ker(\text{ad}).$$

On the other hand recall that by the Hodge decomposition

$$(d_L^* (\Gamma(M, \Lambda^2 L^*)))^\perp = \ker (d_L : \Gamma(M, L^*) \longrightarrow \Gamma(M, \Lambda^2 L^*)).$$

We consider $\xi \in \Gamma(M, L^*) \cong \Gamma(M, \bar{L})$ such that $\Psi\xi = \xi + \bar{\xi} \in \ker \text{ad}$, and $u, v \in \Gamma(M, L)$. Then by Lemma 5.39 we have

$$d_L \xi(u, v) = -\langle [\xi, u], v \rangle = -\langle [\xi + \bar{\xi}, u], v \rangle = 0.$$

It follows that $\xi \in \ker d_L$ and therefore $\ker (d_L : \Gamma(M, L^*) \longrightarrow \Gamma(M, \Lambda^2 L^*)) \supset \Psi^{-1} \ker(\text{ad})$. This implies

$$d_L^* (\Gamma(M, \Lambda^2 L^*)) = (\ker (d_L : \Gamma(M, L^*) \longrightarrow \Gamma(M, \Lambda^2 L^*)))^\perp \subset (\Psi^{-1} \ker \text{ad})^\perp = \Psi^{-1} \mathfrak{h}_{\text{ex}}(E).$$

We claim that $d_L C^\infty(M) \subset \Psi^{-1} \ker(\text{ad})$. Given $f \in C^\infty(M)$ and $u \in \Gamma(M, L)$, we compute

$$d_L f(u) = \pi(u)[f] = \text{pr}_{\bar{L}} \pi^* df(u).$$

Since

$$\mathrm{pr}_L \pi^* df = \Psi^{-1}(\pi^* df) \in \Psi^{-1} \ker(\mathrm{ad}),$$

we conclude that

$$\mathrm{d}_L C^\infty(M) \subset \Psi^{-1} \ker(\mathrm{ad}) \implies \Psi^{-1} \mathfrak{h}_{\mathrm{ex}}(E) \cap \mathrm{d}_L C^\infty(M) = \{0\}.$$

Hence

$$\Psi^{-1} \mathfrak{h}_{\mathrm{ex}}(E) \cap \ker(\mathrm{d}_L : \Gamma(M, L^*) \longrightarrow \Gamma(M, \Lambda^2 L^*)) \subset \mathcal{H}_L^1(M).$$

□

8.1.3 Action on generalized complex deformations

In the following, let E be a $\mathcal{G}$ -flat transitive Courant algebroid over a compact manifold. Generalized almost complex structures on E can be seen as sections of a fiber subbundle of $\mathrm{End} E$, whose fibers are isomorphic to $\mathrm{O}(2k, 2\ell)/\mathrm{U}(k, \ell)$. In particular the space of generalized almost complex structures on E is the space of all sections $\mathrm{GAC}(E)$ of this fiber bundle and thus a tame Fréchet manifold [Ham82, Thm II.2.3.1].

Recall that the group of autoequivalences acts on the tame Fréchet manifold of generalized almost complex structures

$$\mathrm{Auteq}(E) \times \mathrm{GAC}(E) \ni (F, \mathcal{J}) \longmapsto F \cdot \mathcal{J} := F \circ \mathcal{J} \circ F^{-1}.$$

For exact E , it was already shown in [Hec25] that this action is smooth tame. In the following, we will see this more generally in the case of $\mathcal{G}$ -flat transitive Courant algebroids. For this, we need the following lemma.

Lemma 8.28. *The map*

$$\begin{aligned} \mu_{\mathrm{End}} : \mathrm{Auteq}(E) \times \Gamma(M, \mathrm{End} E) &\longrightarrow \Gamma(M, \mathrm{End} E) \\ (F, A) &\longmapsto F \circ A \circ F^{-1} \end{aligned}$$

is smooth tame.

Proof. We consider the auxiliary map

$$\begin{aligned} \mathrm{Auteq}(E) \times \Gamma(M, \mathrm{End} E) \times \Gamma(M, E) &\longrightarrow \Gamma(M, E) \\ (F, A, u) &\longmapsto F \circ A \circ F^{-1} u. \end{aligned}$$

Then we observe that this map is smooth tame since

$$\begin{aligned} F \circ A \circ F^{-1} u &= F \circ A \circ F^{-1} \circ u \circ f \circ f^{-1} \\ &= \mu_E(F, A \mu_E(F, u)), \end{aligned}$$

where f is the diffeomorphism covered by F and we use that μ_E is smooth tame (cf. Proposition 8.16). In a locally trivial neighborhood $U \subset M$ for E , the map μ_{End} can be recovered from the auxiliary map using a local frame $\{e_i\}$ for E and its dual local frame $\{\alpha_i\}$ for E^*

via

$$\mu_{\text{End}}(F, A)|_U = \sum_i \mu_E(F, A\mu_E(F, e_i)) \otimes \alpha_i.$$

These patch together to the global section $\mu_{\text{End}}(F, A)$ of $\text{End } E$. The patching can be made explicit using a partition of unity. In total, this yields a smooth tame map. $\square$

Proposition 8.29. *The action of $\text{Auteq}(E)$ on $\text{GAC}(E)$ is smooth tame.*

Proof. This follows from Lemma 8.28 since $\text{GAC}(E)$ is a tame Fréchet submanifold of $\Gamma(M, \text{End } E)$. $\square$

Consider now a generalized almost complex structure $\mathcal{J}$ on E and an autoequivalence F . Then we have

$$\begin{aligned} L_p(F \cdot \mathcal{J}) &= \{Fu_{f^{-1}(p)} : u_{f^{-1}(p)} \in L_{f^{-1}(p)}\} \\ &= \{F \circ u \circ f^{-1}(p) : u \in \Gamma(M, L)\} \\ &= \{[\mu_E(F, u)]_p : u \in \Gamma(M, L)\}. \end{aligned}$$

For F in a small enough C^1 -neighborhood $\mathcal{V}^{(1)}$ of $\text{id} \in \text{Auteq}(E)$, we find that

$$L(F \cdot \mathcal{J}) \cap \bar{L} = \{0\}$$

and hence there is some $\Theta(F) \in \Gamma(M, \Lambda^2 L^*)$ such that

$$L(F \cdot \mathcal{J}) = \text{graph}(\Theta(F)).$$

In particular, $\Theta(F)$ is the generalized (almost) complex deformation corresponding to $F \cdot \mathcal{J}$. Similarly, possibly after shrinking $\mathcal{V}^{(1)}$ further, and for some small enough C^0 -neighborhood $\mathcal{W}^{(0)}$ of 0 in $\Gamma(M, \Lambda^2 L^*)$, for every $F \in \mathcal{V}^{(1)}$, $\varphi \in \mathcal{W}^{(0)}$, the generalized almost complex structure $F \cdot \mathcal{J}_\varphi$ corresponds to a generalized almost complex deformation $\Theta(F, \varphi)$. Since we can think of Θ as the local expression for the action of $\text{Auteq}(E)$ on $\text{GAC}(E)$ in a local chart around $\mathcal{J}$, it is a smooth tame map. We will state this in the following proposition.

Proposition 8.30. *The map*

$$\begin{aligned} \Theta: \mathcal{V}^{(1)} \times \mathcal{W}^{(0)} &\longrightarrow \Gamma(M, \Lambda^2 L^*) \\ (F, \varphi) &\longmapsto \Theta(F, \varphi) \end{aligned}$$

is smooth tame. It is given by the formula

$$\Theta(F, \varphi) = (F_{\bar{L}L} + F_{\bar{L}\bar{L}} \circ \varphi)(F_{LL} + F_{L\bar{L}} \circ \varphi)^{-1},$$

where we wrote

$$F = \begin{pmatrix} F_{LL} & F_{L\bar{L}} \\ F_{\bar{L}L} & F_{\bar{L}\bar{L}} \end{pmatrix} \quad \text{w.r.t.} \quad E = L \oplus \bar{L}.$$

Proof. We only need to prove the formula for $\Theta(F, \varphi)$. First, we write

$$\begin{aligned} L_p(F \cdot \mathcal{J}_\varphi) &= \{F(u + i_u \varphi)_{f^{-1}(p)} : u \in \Gamma(M, L)\} \\ &= \{(F_{LL} + F_{L\bar{L}} \circ \varphi)u_{f^{-1}(p)} + (F_{\bar{L}L} + F_{\bar{L}\bar{L}} \circ \varphi)u_{f^{-1}(p)} : u \in \Gamma(M, L)\}. \end{aligned}$$

Note that for F in a small C^1 -neighborhood $\mathcal{V}^{(1)}$ of the identity and φ in a small C^0 -neighborhood $\mathcal{W}^{(0)}$ of 0, the map

$$F_{LL} + F_{L\bar{L}} \circ \varphi : L \longrightarrow L$$

is invertible. Therefore, for every $u \in \Gamma(M, L)$ there is some $v \in \Gamma(M, L)$ such that for every $p \in M$ we have

$$v_p = (F_{LL} + F_{L\bar{L}} \circ \varphi)u_{f^{-1}(p)}.$$

It follows that

$$L_p(F \cdot \mathcal{J}_\varphi) = \{v_p + (F_{\bar{L}L} + F_{\bar{L}\bar{L}} \circ \varphi)(F_{LL} + F_{L\bar{L}} \circ \varphi)^{-1}v_p : v \in \Gamma(M, L)\}$$

and thus we find

$$\Theta(F, \varphi) = (F_{\bar{L}L} + F_{\bar{L}\bar{L}} \circ \varphi)(F_{LL} + F_{L\bar{L}} \circ \varphi)^{-1}.$$

□

From now on, we will also denote $\Theta(F, \varphi) := F \cdot \varphi$.

Lemma 8.31. $\Theta(F_1 \circ F_2, \varphi) = F_1 \cdot \Theta(F_2, \varphi)$.

Proof. This follows from

$$\begin{aligned} (F_1 \circ F_2) \cdot \mathcal{J} &= (F_1 \circ F_2) \circ \mathcal{J} \circ (F_1 \circ F_2)^{-1} \\ &= F_1 \circ (F_2 \circ \mathcal{J} \circ F_2^{-1}) \circ F_1^{-1} \\ &= F_1 \cdot (F_2 \cdot \mathcal{J}). \end{aligned}$$

□

We denote by

$$\mathcal{W}_\varepsilon^{(k)} := \{\varphi \in \Gamma(M, \Lambda^2 L^*) : \|\varphi\|_{k, \infty} < \varepsilon\}, \quad \mathcal{U}_\varepsilon^{(k)} := \{\xi \in \Psi^{-1} \mathfrak{h}_{\text{ex}}(E) \subset \Gamma(M, L^*) : \|\xi\|_{k, \infty} < \varepsilon\}.$$

Then for $\varepsilon > 0$ small enough, restricting Θ to exact autoequivalences and concatenating with the local chart Φ of $\text{Auteq}^{\text{ex}}(E)$ yields a smooth tame map

$$\mathcal{E} : \mathcal{U}_\varepsilon^{(1)} \times \mathcal{W}_\varepsilon^{(0)} \longrightarrow \Gamma(M, \Lambda^2 L^*).$$

Like in the complex case, for our proof of the local completeness of the deformation family in Section 8.2.2, we need to compute the partial derivative $D_\xi \mathcal{E}(\xi, \varphi)$. Proceeding similarly as in Section 7.1, we first prove the following proposition.

Proposition 8.32. *Let $\varphi \in \mathcal{W}_\varepsilon^{(0)}$ be a generalized almost complex deformation, F_ξ an exact autoequivalence, where $\xi \in \mathcal{U}_\varepsilon^{(1)}$. Then for $\varepsilon > 0$ small enough, $F_\xi \cdot \varphi$ is well-defined and we have*

$$\mathcal{E}(\xi, \varphi) := F_\xi \cdot \varphi = \varphi + d_L(a_\varphi \xi) + [\varphi, a_\varphi \xi] - \iota_{\bar{\xi}}(d_L \varphi + \tfrac{1}{2}[\varphi, \varphi]) + R(\varphi, \xi),$$

where $a_\varphi \xi := \xi - \iota_{\bar{\xi}} \varphi$ and $R(\varphi, t\xi) = t^2 \tilde{R}(\varphi, \xi, t)$ for a real parameter t with $\tilde{R}(\varphi, \xi, t)$ depending smoothly on t for small t .

Proof. We know that

$$\begin{aligned} L_{F_\xi \cdot \varphi, p} &= \{\mu_E(F_\xi, u + \iota_u \varphi)_p : u \in \Gamma(M, L)\} \\ &= \{u_p + \varphi(u_p) - [\Psi \xi, u + \varphi(u)]_p + R_{\mu_E}(\Psi \xi)(u + \iota_u \varphi)_p : u \in \Gamma(M, L)\}, \end{aligned}$$

where we used Proposition 8.17 where also $\tilde{R}$ is defined and $\Psi: \bar{L} \rightarrow \Gamma(M, E)$ is the vector bundle isomorphism defined in (8.6). It follows that

$$\begin{aligned} \text{pr}_L(L_{F_\xi \cdot \varphi, p}) &= \{u_p - \text{pr}_L[\bar{\xi} + \xi, u + \iota_u \varphi]_p + \text{pr}_L R_{\mu_E}(\Psi \xi)(u + \iota_u \varphi)_p : u \in \Gamma(M, L)\} \\ &= \{u_p - \text{pr}_L[\bar{\xi} + \xi, u]_p - \text{pr}_L[\bar{\xi}, \iota_u \varphi]_p + \text{pr}_L R_{\mu_E}(\Psi \xi)(u + \iota_u \varphi)_p : u \in \Gamma(M, L)\}. \end{aligned}$$

We define by $P(\xi, \varphi): \Gamma(M, L) \longrightarrow \Gamma(M, L)$ by

$$\begin{aligned} P(\xi, \varphi)u &:= \text{pr}_L(\mu_E(F_\xi, u + \iota_u \varphi)) \\ &= u - \text{pr}_L[\bar{\xi} + \xi, u] - \text{pr}_L[\bar{\xi}, \iota_u \varphi] + \text{pr}_L R_{\mu_E}(\Psi \xi)(u + \iota_u \varphi). \end{aligned}$$

Then $P(\xi, \varphi)$ is invertible for $(\xi, \varphi) \in \mathcal{U}_\varepsilon^{(1)} \times \mathcal{W}_\varepsilon^{(0)}$ with inverse

$$P(\xi, \varphi)^{-1}u = u + [\bar{\xi}, u] + \text{pr}_L[\xi, u] + \text{pr}_L[\bar{\xi}, \iota_u \varphi] + \text{higher orders of } \xi,$$

where we used that the family of inverse maps $P(\xi, \varphi)^{-1}u$ depends smoothly on ξ . To see that $P(\xi, \varphi)^{-1}u$ is indeed smooth in ξ , analogous to the complex case we note that the inverse of

$$\text{pr}_L: L_{F_\xi \cdot \varphi} \rightarrow L$$

is given by

$$\text{id} + \mathcal{E}(\xi, \varphi): L \rightarrow L_{F_\xi \cdot \varphi} \subset E,$$

which we already know depends smoothly on ξ , and the inverse of

$$u \mapsto \mu_E(F_\xi, u + \iota_u \varphi)$$

is given by

$$v \mapsto \text{pr}_L \mu_E(F_\xi^{-1}, v).$$

Putting everything together, it follows that

$$\begin{aligned} L_{F_\xi \cdot \varphi, p} &= \left\{ u_p + \varphi \left(P(\xi, \varphi)^{-1} u \right)_p - \text{pr}_{\bar{L}}[\Psi \xi, P(\xi, \varphi)^{-1} u + \varphi \left(P(\xi, \varphi)^{-1} u \right)]_p \right. \\ &\quad \left. + \text{higher orders of } \xi : u \in \Gamma(M, L) \right\} \\ &= \left\{ u_p + \varphi \left(u_p + [\bar{\xi}, u]_p + \text{pr}_L[\xi, u]_p + \text{pr}_L[\bar{\xi}, \iota_u \varphi]_p \right) - \text{pr}_{\bar{L}}[\xi, u]_p - \text{pr}_{\bar{L}}[\xi + \bar{\xi}, \iota_u \varphi]_p \right. \\ &\quad \left. + \text{higher orders of } \xi : u \in \Gamma(M, L) \right\}. \end{aligned}$$

This shows that

$$\mathcal{E}(\xi, \varphi)(u) = \iota_{u+[\bar{\xi}, u] + \text{pr}_L[\xi, u] + \text{pr}_L[\bar{\xi}, \iota_u \varphi]} \varphi - \text{pr}_{\bar{L}}[\xi, u] - \text{pr}_{\bar{L}}[\xi + \bar{\xi}, \iota_u \varphi] + \text{higher orders of } \xi.$$

On the other hand, using Lemma 5.39, we find that

$$\begin{aligned} &\iota_u \left(\varphi + d_L(\xi - \iota_{\bar{\xi}} \varphi) + [\varphi, \xi - \iota_{\bar{\xi}} \varphi] - \iota_{\bar{\xi}}(d_L \varphi + \frac{1}{2}[\varphi, \varphi]) \right) \\ &= \iota_u \varphi - \text{pr}_{\bar{L}}[\xi, u] + \text{pr}_{\bar{L}}[\iota_{\bar{\xi}} \varphi, u] + [\iota_u \varphi, \xi - \iota_{\bar{\xi}} \varphi] + \iota_{\text{pr}_L[\xi - \iota_{\bar{\xi}} \varphi, u]} \varphi \\ &\quad - \text{pr}_{\bar{L}}[\iota_{\bar{\xi}} \varphi, u] - \text{pr}_{\bar{L}}[\bar{\xi}, \iota_u \varphi] + \iota_{[\bar{\xi}, u]} \varphi - [\iota_{\bar{\xi}} \varphi, \iota_u \varphi] \\ &\quad + \frac{1}{2} \iota_{\text{pr}_L([\bar{\xi}, \iota_u \varphi] - [\iota_u \varphi, \bar{\xi}] - [u, \iota_{\bar{\xi}} \varphi] + [\iota_{\bar{\xi}} \varphi, u])} \varphi \\ &= \iota_{u+[\bar{\xi}, u] + \text{pr}_L[\xi, u]} \varphi - \text{pr}_{\bar{L}}[\xi, u] - \text{pr}_{\bar{L}}[\xi + \bar{\xi}, \iota_u \varphi] \\ &\quad + \frac{1}{2} \iota_{\text{pr}_L([\bar{\xi}, \iota_u \varphi] - [\iota_u \varphi, \bar{\xi}] - [u, \iota_{\bar{\xi}} \varphi] - [\iota_{\bar{\xi}} \varphi, u])} \varphi. \end{aligned}$$

We need to show that

$$-\text{pr}_L([\iota_u \varphi, \bar{\xi}] + [u, \iota_{\bar{\xi}} \varphi] + [\iota_{\bar{\xi}} \varphi, u]) = \text{pr}_L[\bar{\xi}, \iota_u \varphi].$$

To this end, for any $\psi \in \bar{L}$, we compute

$$\begin{aligned} &\langle [\bar{\xi}, \iota_u \varphi] + [\iota_u \varphi, \bar{\xi}], \psi \rangle + \langle [u, \iota_{\bar{\xi}} \varphi] + [\iota_{\bar{\xi}} \varphi, u], \psi \rangle \\ &= \pi(\psi) \langle \bar{\xi}, \iota_u \varphi \rangle + \pi(\psi) \langle u, \iota_{\bar{\xi}} \varphi \rangle \\ &= \pi(\psi) (\varphi(u, \bar{\xi}) + \varphi(\bar{\xi}, u)) = 0. \end{aligned}$$

This concludes the proof. $\square$

Lemma 8.33. *For some $\varepsilon > 0$, $\varphi \in \mathcal{W}_\varepsilon^{(0)}$, let*

$$\begin{aligned} a_\varphi : \Gamma(M, L^*) &\longrightarrow \Gamma(M, L^*) \\ \xi &\longmapsto \xi - \iota_{\bar{\xi}} \varphi. \end{aligned}$$

Then the family of linear maps

$$\begin{aligned} a : \mathcal{W}_\varepsilon^{(0)} \times \Gamma(M, L^*) &\longrightarrow \Gamma(M, L^*) \\ (\varphi, \xi) &\longmapsto a_\varphi \xi \end{aligned}$$

is smooth tame. Moreover, for $\varepsilon > 0$ small enough, a_φ is invertible for every $\varphi \in \mathcal{W}_\varepsilon^{(0)}$ and the family of inverses a_φ^{-1} is smooth tame.

Proof. The proof is analogous to that of Lemma 7.6. First, we note that a is a non-linear vector bundle operator and therefore smooth tame by Proposition 6.25. Moreover, a_φ is invertible for $\|\varphi\|_{0,\infty}$ small enough. We set $\tilde{\varphi}: \bar{L} \rightarrow \bar{L}$ via $\tilde{\varphi}(\xi) := \varphi(\bar{\xi})$. Then,

$$a_\varphi^{-1} = \sum_{k=0}^{\infty} \tilde{\varphi}^k$$

where we used that the sum converges fiberwise for small enough $\|\varphi\|_{0,\infty}$. This shows that, when it is defined, the family $a_\varphi^{-1}\xi$ is a non-linear vector bundle operator and hence smooth tame. $\square$

Using Proposition 8.32, we will now compute $D_\xi \mathcal{E}(\xi, \varphi)$ for $(\xi, \varphi) \in \mathcal{U}_\varepsilon^{(1)} \times \mathcal{W}_\varepsilon^{(0)}$.

Proposition 8.34. *Let $\varphi \in \mathcal{W}_\varepsilon^{(0)}$ be a generalized almost complex deformation. Then*

$$D_\xi \mathcal{E}(\xi, \varphi)\eta = d_L(b_{(\xi, \varphi)}\eta) + [\mathcal{E}(\xi, \varphi), b_{(\xi, \varphi)}\eta] - \iota_{\overline{c_\xi \eta}}(d_L \mathcal{E}(\xi, \varphi) + \frac{1}{2}[\mathcal{E}(\xi, \varphi), \mathcal{E}(\xi, \varphi)]),$$

where $b_{(\xi, \varphi)}\eta = a_{\mathcal{E}(\xi, \varphi)}c_\xi\eta$, with a as defined in Proposition 8.32 and

$$c_\xi\eta = DC_\xi(0)\eta, \quad C_\xi: \mathcal{U} \ni \tilde{\eta} \mapsto \Psi^{-1}\Phi(F_{\xi+\tilde{\eta}}F_\xi^{-1}) \in \Psi^{-1}\mathfrak{h}_{\text{ex}}(E),$$

for a sufficiently small C^1 -neighborhood $\mathcal{U} \subset \Psi^{-1}\mathfrak{h}_{\text{ex}}(E)$ of zero.

Proof. Let $\xi \in \mathcal{U}_\varepsilon^{(1)}$, $\varphi \in \mathcal{W}_\varepsilon^{(0)}$. Let $\mathcal{U} \subset \mathcal{U}_\varepsilon^{(1)}$ an open C^1 -neighborhood of 0 in $\Gamma(M, L^*)$ such that $\xi + \tilde{\eta} \in \mathcal{U}_\varepsilon^{(1)}$ for every $\tilde{\eta} \in \mathcal{U}$ and such that $F_{\xi+\tilde{\eta}} \circ F_\xi^{-1} \in \Phi^{-1} \circ \Psi(\mathcal{U}_\varepsilon^{(1)})$ for every $\tilde{\eta} \in \mathcal{U}$. Then we define

$$\begin{aligned} C_\xi: \mathcal{U} &\longrightarrow \mathcal{U}_\varepsilon^{(1)} \\ \tilde{\eta} &\longmapsto \Psi^{-1} \circ \Phi(F_{\xi+\tilde{\eta}} \circ F_\xi^{-1}). \end{aligned}$$

Then

$$\begin{aligned} \mathcal{E}(\xi + \tilde{\eta}, \varphi) &= F_{\xi+\tilde{\eta}} \cdot \varphi \\ &= (F_{C_\xi(\tilde{\eta})} \circ F_\xi) \cdot \varphi \\ &= F_{C_\xi(\tilde{\eta})} \cdot \mathcal{E}(\xi, \varphi). \end{aligned}$$

By Proposition 8.32 and since $C_\xi(0) = 0$, we find

$$\begin{aligned} D_\xi \mathcal{E}(\xi, \varphi) &= \lim_{t \rightarrow 0} \frac{1}{t} \{ \mathcal{E}(\xi + t\eta, \varphi) - \mathcal{E}(\xi, \varphi) \} \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \cdot [\mathcal{E}(C_\xi(t\eta), \mathcal{E}(\xi, \varphi)) - \mathcal{E}(\xi, \varphi)] \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \{ d_L(a_{\mathcal{E}(\xi, \varphi)}C_\xi(t\eta)) + [\mathcal{E}(\xi, \varphi), a_{\mathcal{E}(\xi, \varphi)}C_\xi(t\eta)] \\ &\quad - \iota_{\overline{C_\xi(t\eta)}}(d_L \mathcal{E}(\xi, \varphi) + \frac{1}{2}[\mathcal{E}(\xi, \varphi), \mathcal{E}(\xi, \varphi)]) \} \\ &= d_L(a_{\mathcal{E}(\xi, \varphi)}c_\xi\eta) + [\mathcal{E}(\xi, \varphi), a_{\mathcal{E}(\xi, \varphi)}c_\xi\eta] - \iota_{\overline{c_\xi\eta}}(d_L \mathcal{E}(\xi, \varphi) + \frac{1}{2}[\mathcal{E}(\xi, \varphi), \mathcal{E}(\xi, \varphi)]), \end{aligned}$$

where $c_\xi\eta = DC_\xi(0)\eta$. Setting $b_{(\xi, \varphi)}\eta := a_{\mathcal{E}(\xi, \varphi)}c_\xi\eta$ we arrive at the expression as stated above. $\square$

Lemma 8.35. *For some $\varepsilon > 0$, let $\mathcal{W}_\varepsilon^{(0)}$, $\mathcal{U}_\varepsilon^{(1)}$ as above. We assume that $\varepsilon > 0$ is small enough so that c_ξ and $b_{(\xi, \varphi)}$ as in Proposition 8.34 are defined for $\xi \in \mathcal{U}_\varepsilon^{(1)}$ and $\varphi \in \mathcal{W}_\varepsilon^{(0)}$. Then the families of linear maps*

$$\begin{aligned} c: \mathcal{U}_\varepsilon^{(1)} \times \Psi^{-1} \mathfrak{h}_{\text{ex}}(E) &\longrightarrow \Psi^{-1} \mathfrak{h}_{\text{ex}}(E) \\ (\xi, \eta) &\longmapsto c_\xi \eta \end{aligned}$$

and

$$\begin{aligned} b: \mathcal{U}_\varepsilon^{(1)} \times \mathcal{W}_\varepsilon^{(0)} \times \Psi^{-1} \mathfrak{h}_{\text{ex}}(E) &\longrightarrow \Gamma(M, L^*) \\ (\xi, \varphi, \eta) &\longmapsto b_{(\xi, \varphi)} \eta \end{aligned}$$

are smooth tame. Moreover, for $\varepsilon > 0$ small enough, c_ξ is invertible for every $\xi \in \mathcal{U}_\varepsilon^{(1)}$. The family of inverses c_ξ^{-1} is smooth tame.

Proof. The proof is analogous to that of Lemma 7.8. Recall that

$$c_\xi \eta = DC_\xi(0)\eta, \quad C_\xi: \mathcal{U} \ni \tilde{\eta} \mapsto (\Phi^{-1} \circ \Psi)^{-1} (F_{\xi+\tilde{\eta}} F_\xi^{-1}) \in \Psi^{-1} \mathfrak{h}_{\text{ex}}(E),$$

where $\mathcal{U} \subset \mathcal{U}_\varepsilon^{(1)}$ is an open neighborhood of 0 such that $\xi + \tilde{\eta} \in \mathcal{U}_\varepsilon^{(1)}$ for every $\tilde{\eta} \in \mathcal{U}$ and $F_{\xi+\tilde{\eta}} \circ F_\xi^{-1} \in \Phi(\mathcal{U}_\varepsilon^{(1)})$. Note that C_ξ can be extended to a C^0 -neighborhood $\tilde{\mathcal{U}}$ of 0 such that $\xi + \tilde{\eta} \in \Psi^{-1} \circ \Phi(\mathcal{V}_{\text{id}})$, and $F_{\xi+\tilde{\eta}} F_\xi^{-1} \in \mathcal{V}_{\text{id}}$ for every $\tilde{\eta} \in \tilde{\mathcal{U}}$, where $\mathcal{V}_{\text{id}}$ denotes the local chart neighborhood of $\text{id} \in \text{Auteq}^{\text{ex}}(E)$. We may furthermore assume, possibly after shrinking $\varepsilon > 0$, that $\tilde{\mathcal{U}}$ can be chosen independently of ξ , such that we obtain a well-defined map

$$\begin{aligned} C: \mathcal{U}_\varepsilon^{(1)} \times \tilde{\mathcal{U}} &\longrightarrow \Psi^{-1} \mathfrak{h}_{\text{ex}}(E) \\ (\xi, \tilde{\eta}) &\longmapsto C_\xi(\tilde{\eta}). \end{aligned}$$

Then $C_\xi(\tilde{\eta})$ is smooth tame in ξ and $\tilde{\eta}$ as a composition of smooth tame maps and invertible for every small enough ξ . To compute the inverse, we note that

$$C_\xi(\tilde{\eta}) = (\Phi^{-1} \circ \Psi)^{-1} \left(R_{F_\xi^{-1}} \circ \Phi^{-1} \circ \Psi(\xi + \tilde{\eta}) \right).$$

Since $R_{F_\xi^{-1}} = R_{F_\xi}$, we find that

$$\begin{aligned} C_\xi^{-1}(\chi) &= (\Phi^{-1} \circ \Psi)^{-1} \circ R_{F_\xi} \circ \Phi^{-1} \circ \Psi(\chi) - \xi \\ &= (\Phi^{-1} \circ \Psi)^{-1} (F_\chi F_\xi) - \xi, \end{aligned}$$

which is smooth tame. It follows that the family c of linear maps is smooth tame and c_ξ is invertible for small enough ξ and the family of inverses c_ξ^{-1} is again smooth tame.

For the statements about b we observe that by Proposition 8.30 and since c is smooth tame, b is a composition of smooth tame maps and hence a smooth tame family of linear maps. $\square$

Note that in contrast to Lemma 7.8 we cannot expect $b_{(\xi, \varphi)}$ to be invertible. The reason for this is that in general $\Psi^{-1} \mathfrak{h}_{\text{ex}}(E)$ is a proper subspace of $\Gamma(M, L^*)$, while in Lemma 7.8 we may identify the Lie algebra of the diffeomorphism group with the entire space $\Gamma(M, T^{1,0})$.

8.2 The deformation theorem

We now state the main theorem of this chapter.

Theorem 8.36. *Let L be a generalized complex structure on a $\mathcal{G}$ -flat transitive Courant algebroid $E \rightarrow M$. There exists an open neighborhood $\mathcal{W} \subset H^2(M, L)$ of 0, a family $\tilde{\mathcal{M}} = \{\varphi_t : t \in \mathcal{W}\}$ of generalized almost complex deformations of L , and an analytic obstruction map*

$$\phi : \mathcal{W} \rightarrow H^3(M, L)$$

such that the generalized almost complex deformations $\mathcal{M} := \{\varphi_t \in \tilde{\mathcal{M}} : \phi(t) = 0\}$ are precisely the integrable ones. Any sufficiently small generalized complex deformation of L is equivalent to at least one member of the family $\mathcal{M}$. In the case that the obstruction map vanishes, $\mathcal{M}$ is a locally complete family of generalized complex deformations.

In analogy to the complex case we call the family $\mathcal{M}$ in Theorem 8.36 the *Kuranishi family* and the set $\phi^{-1}(0) \subset \mathcal{W}$ the *Kuranishi moduli space*.

The proof of Theorem 8.36 is split into two parts. First, we will construct the family and the obstruction map in Section 8.2.1. Then we will prove that the family is locally complete in Section 8.2.2. In Section 8.2.1, we will closely follow Gualtieri's proof in [Gua11]. Since Gualtieri followed Kuranishi's proof in [Kur65], there are similarities with this work, too, as well as with Section 7.2, which set the foundation for the proofs in Section 8.2.2.

8.2.1 Description of the family

Let

$$\tilde{\mathcal{V}}_1 := \{\varphi \in \Gamma(M, \Lambda^2 L^*) : d_L \varphi + \frac{1}{2}[\varphi, \varphi] = 0 = d_L^* \varphi\} \subset \Gamma(M, \Lambda^2 L^*)$$

We will see that as a set, the generalized complex deformations $\mathcal{M}$ in the family $\tilde{\mathcal{M}}$ are contained in $\tilde{\mathcal{V}}_1$. Our strategy will be the same as in [Gua11] (and as in Section 7.2): First we show that $\tilde{\mathcal{V}}_1$ is contained in another set $\tilde{\mathcal{V}}_2$ and that for small enough $\varepsilon > 0$ and large enough $k \in \mathbb{N}_0$, the intersection $\mathcal{W}_\varepsilon^{(k)} \cap \tilde{\mathcal{V}}_2$ is parametrized by a neighborhood of 0 in $\mathcal{H}_L^2(M)$. Then we will find an analytic map

$$\phi : \mathcal{W}_\varepsilon^{(k)} \cap \tilde{\mathcal{V}}_2 \longrightarrow \mathcal{H}_L^3(M)$$

such that $\phi(\varphi) = 0$ if and only if φ is integrable and hence contained in $\mathcal{M}$.

In the following lemma, we define the set $\tilde{\mathcal{V}}_2$ and prove that it contains the set $\tilde{\mathcal{V}}_1$ as defined above.

Lemma 8.37 ([Gua03; Gua11]). $\tilde{\mathcal{V}}_1 \subset \tilde{\mathcal{V}}_2 := \{\varphi \in \Gamma(M, \Lambda^2 L^*) : \varphi + \frac{1}{2} Q_L[\varphi, \varphi] \in \mathcal{H}_L^2(M)\}$.

Proof. This exact statement has already been proved in [Gua03; Gua11], but for the sake of completeness we will repeat the argument here. Let $\varphi \in \tilde{\mathcal{V}}_1$. Then

$$\Delta_L \varphi = (d_L d_L^* + d_L^* d_L) \varphi = d_L^* d_L \varphi = -\frac{1}{2} d_L^* [\varphi, \varphi],$$

where we used $d_L^* \varphi = 0$ for the second equality and $d_L \varphi + \frac{1}{2}[\varphi, \varphi] = 0$ for the last equality. Hence,

$$0 = G(\Delta_L \varphi + \frac{1}{2} d_L^* [\varphi, \varphi]) = \varphi - \text{pr}_{\mathcal{H}_L} \varphi + \frac{1}{2} Q_L[\varphi, \varphi],$$

where we used that $\text{pr}_{\mathcal{H}_L} + G_L \Delta_L = \text{id}$. It follows that

$$\varphi + \frac{1}{2} Q_L[\varphi, \varphi] = \text{pr}_{\mathcal{H}_L} \varphi \in \mathcal{H}_L^2(M)$$

and thus $\varphi \in \tilde{\mathcal{V}}_2$. □

We consider the smooth tame map

$$\begin{aligned} \sigma: \Gamma(M, \Lambda^2 L^*) &\longrightarrow \Gamma(M, \Lambda^2 L^*), \\ \varphi &\longmapsto \varphi + \frac{1}{2} Q_L[\varphi, \varphi]. \end{aligned}$$

Proposition 8.38. *For $\varepsilon > 0$ small enough and large enough $k \in \mathbb{N}_0$, σ maps $\mathcal{W}_\varepsilon^{(k)}$ bijectively onto some neighborhood $\tilde{\mathcal{W}}_\varepsilon \subset \Gamma(M, \Lambda^2 L^*)$ such that*

$$\sigma^{-1}: \tilde{\mathcal{W}}_\varepsilon \longrightarrow \mathcal{W}_\varepsilon^{(k)}$$

is smooth tame.

To prove Proposition 8.38, we need the following Lemma:

Lemma 8.39. *For $\varepsilon > 0$ small enough and large enough $k \in \mathbb{N}_0$, the equation $D\sigma(\varphi)\alpha = \beta$ has a unique solution α for every $\varphi \in \mathcal{W}_\varepsilon^{(k)}$, $\beta \in \Gamma(M, \Lambda^2 L^*)$. Moreover, the family of solutions is a smooth tame family of linear maps.*

Proof. Let $\varphi \in \mathcal{W}_\varepsilon^{(k)}$, $\beta \in \Gamma(M, \Lambda^2 L^*)$. We need to find some $\alpha \in \Gamma(M, \Lambda^2 L^*)$ such that

$$\beta = \alpha - Q_L[\alpha, \varphi] = \alpha - d_L^* G_L[\alpha, \varphi]. \quad (8.7)$$

The statement follows now from Proposition 6.31, since $\alpha \mapsto \Delta_L d_L^* G_L[\alpha, \varphi] = d_L^*[\alpha, \varphi]$ is a linear partial differential operator of degree two. □

Proof of Proposition 8.38. This follows from Theorem 6.27 and Lemma 8.39. □

Set

$$\mathcal{W}_\varepsilon := \tilde{\mathcal{W}}_\varepsilon \cap \mathcal{H}_L^2(M), \quad \mathcal{V}_\varepsilon := \tilde{\mathcal{V}}_2 \cap \mathcal{W}_\varepsilon^{(k)}.$$

Then σ yields a smooth tame map

$$\varphi := \sigma^{-1}|_{\mathcal{W}_\varepsilon}: \mathcal{W}_\varepsilon \longrightarrow \mathcal{V}_\varepsilon.$$

Corollary 8.40. *The family $\tilde{\mathcal{M}} := \{\varphi(t) : t \in \mathcal{W}_\varepsilon\}$ is a smooth family of generalized almost complex deformations.*

To identify which elements in the family are integrable, i.e. contained in $\tilde{\mathcal{V}}_2$, we note the following:

Lemma 8.41 ([Gua03; Gua11]). *For $\varepsilon > 0$ small enough, $\varphi \in \mathcal{V}_\varepsilon$ is in $\mathcal{S}_\varepsilon := \tilde{\mathcal{V}}_1 \cap \mathcal{W}_\varepsilon^{(k)}$ if and only if*

$$\phi(\varphi) := \text{pr}_{\mathcal{H}_L}[\varphi, \varphi] = 0.$$

Proof. This exact statement has already been proved in [Gua03; Gua11], but for the sake of completeness we will repeat the argument here. We first note that for any $\varphi \in \mathcal{V}_\varepsilon$, we have

$$\begin{aligned} d_L \varphi + \frac{1}{2}[\varphi, \varphi] &= -\frac{1}{2}d_L Q_L[\varphi, \varphi] + \frac{1}{2}[\varphi, \varphi] \\ &= \frac{1}{2}Q_L d_L[\varphi, \varphi] + \frac{1}{2}\text{pr}_{\mathcal{H}_L}[\varphi, \varphi], \end{aligned}$$

where we used $\varphi \in \mathcal{V}_\varepsilon$ for the first equality and $\text{pr}_{\mathcal{H}_L} + d_L Q_L + Q_L d_L = \text{id}$ for the second equality. Since the images of Q_L and $\text{pr}_{\mathcal{H}_L}$ are orthogonal to each other, it follows that

$$d_L \varphi + \frac{1}{2}[\varphi, \varphi] = 0 \iff Q_L d_L[\varphi, \varphi] = 0 \text{ and } \text{pr}_{\mathcal{H}_L}[\varphi, \varphi] = 0. \quad (8.8)$$

We claim that for $\varepsilon > 0$ small enough, $\text{pr}_{\mathcal{H}_L}[\varphi, \varphi] = 0$ already implies $Q_L d_L[\varphi, \varphi] = 0$. To see this, suppose $\varphi \in \mathcal{V}_\varepsilon$ such that $\text{pr}_{\mathcal{H}_L}[\varphi, \varphi] = 0$. Then,

$$d_L Q_L[\varphi, \varphi] = -Q_L d_L[\varphi, \varphi] + [\varphi, \varphi].$$

Using the compatibility of the bracket $[\cdot, \cdot]$ and d_L as stated in Lemma 5.37 and $\varphi \in \tilde{\mathcal{V}}_2$, we compute

$$Q_L d_L[\varphi, \varphi] = 2Q_L[d_L \varphi, \varphi] = -Q_L[d_L Q_L[\varphi, \varphi], \varphi] = Q_L[Q_L d_L[\varphi, \varphi], \varphi] - Q_L[[\varphi, \varphi], \varphi].$$

By the Jacobi identity (see Lemma 5.37), we must have $[[\varphi, \varphi], \varphi] = 0$. Therefore, setting $\eta := Q_L d_L[\varphi, \varphi]$, we find

$$\eta = Q_L[\eta, \varphi].$$

It follows from Example 6.15 and 6.16 that for $k \geq \dim M/2 + 1$ we have

$$\|\eta\|_{k,2} = \|Q_L[\eta, \varphi]\|_{k,2} \leq C\|\eta\|_{k,2}\|\varphi\|_{k,2} \leq C\varepsilon\|\eta\|_{k,2}.$$

In particular, for $\varepsilon < 1/C$, this implies that $\eta = 0$ and hence $Q_L d_L[\varphi, \varphi] = 0$. By equation (8.8) this shows that $\varphi \in \mathcal{S}_\varepsilon$. $\square$

8.2.2 Local completeness of the family

To finish the proof of Theorem 8.36, we need to show that every sufficiently small generalized complex deformation φ can be obtained from $\mathcal{M}$ via an (exact) autoequivalence, i.e. there is some small $\xi \in \Gamma(M, L^*)$ such that $d_L^* \mathcal{E}(\xi, \varphi) = 0$. Moreover, given a smooth family of small generalized (almost) complex deformations φ_t , we need to show that there is a family $\xi_t \subset \Gamma(M, L^*)$ depending smoothly on t such that $d_L^* \mathcal{E}(\xi_t, \varphi_t) = 0$.

Recall that by Lemma 8.27 we have

$$\Psi^{-1} \mathfrak{h}_{\text{ex}}(E) \supset d_L^* (\Gamma(M, \Lambda^2 L^*)).$$

Proposition 8.42. *There are neighborhoods $\mathcal{W} \subset \Gamma(M, \Lambda^2 L^*)$ of $0 \in \Gamma(M, \Lambda^2 L^*)$, $\mathcal{U} \subset \Gamma(M, L^*)$ of $0 \in \Gamma(M, L^*)$ such that for every $\varphi \in \mathcal{W}$ there is a smooth tame family of solutions $\xi = \xi(\varphi) \in \mathcal{U}$ of the equation*

$$d_L^* \mathcal{E}(\xi, \varphi) = 0.$$

Like in the complex case, for the proof of Proposition 8.42 we wish to apply Theorem 6.29. For some $\varepsilon > 0$, $k \in \mathbb{N}$, let $\mathcal{W}_\varepsilon^{(k)}$ and $\mathcal{U}_\varepsilon^{(k)}$ be as above. For ε small enough and large enough k the map

$$\begin{aligned} A: \mathcal{U}_\varepsilon^{(k)} \times \mathcal{W}_\varepsilon^{(k)} &\longrightarrow d_L^* (\Gamma(M, \Lambda^2 L^*)) \\ (\xi, \varphi) &\longmapsto Q_L \mathcal{E}(\xi, \varphi) \end{aligned}$$

is well-defined and smooth tame as a composition of smooth tame maps. In analogy with the complex case we observe that $d_L^* \mathcal{E}(\xi, \varphi) = 0$ if and only if $A(\xi, \varphi) = 0$. Therefore, like in the complex case, our strategy will be to apply Theorem 6.29 to solve the equation $A(\xi, \varphi) = 0$. In the following lemma, we will show that the assumptions of Theorem 6.29 are indeed satisfied.

Lemma 8.43. *For $\varepsilon > 0$ small enough and large enough k , the equation $D_\xi A(\xi, \varphi)\alpha = \beta$ has a solution α for every $(\xi, \varphi) \in \mathcal{U}_\varepsilon^{(k)} \times \mathcal{W}_\varepsilon^{(k)}$, $\beta \in d_L^* (\Gamma(M, \Lambda^2 L^*))$. Moreover, there is a smooth tame family of linear maps $V(\xi, \varphi): d_L^* (\Gamma(M, \Lambda^2 L^*)) \rightarrow \Psi^{-1} \mathfrak{h}_{\text{ex}}(E)$ such that $D_\xi A(\xi, \varphi)V(\xi, \varphi)\beta = \beta$ for every $\beta \in d_L^* (\Gamma(M, \Lambda^2 L^*))$.*

Proof. Using Proposition 8.34, for $(\xi, \varphi) \in \mathcal{U}_\varepsilon^{(k)} \times \mathcal{W}_\varepsilon^{(k)}$, $\alpha \in \Psi^{-1} \mathfrak{h}_{\text{ex}}$ we compute

$$\begin{aligned} D_\xi A(\xi, \varphi)\alpha &= Q_L D_\xi \mathcal{E}(\xi, \varphi)\alpha \\ &= Q_L d_L(b_{(\xi, \varphi)}\alpha) + Q_L[\mathcal{E}(\xi, \varphi), b_{(\xi, \varphi)}\alpha] - Q_L i_{\overline{c_\xi \alpha}}(d_L \mathcal{E}(\xi, \varphi) + \tfrac{1}{2}[\mathcal{E}(\xi, \varphi), \mathcal{E}(\xi, \varphi)]), \end{aligned}$$

where as before $b_{(\xi, \varphi)}\alpha = a_{\mathcal{E}(\xi, \varphi)}c_\xi \alpha = c_\xi \alpha - \iota_{\overline{c_\xi \alpha}} \mathcal{E}(\xi, \varphi)$.

Lemma 8.44. *For $\varepsilon > 0$ small enough, the restriction*

$$Q_L d_L a_{\mathcal{E}(\xi, \varphi)}: d_L^* (\Gamma(M, \Lambda^2 L^*)) \longrightarrow d_L^* (\Gamma(M, \Lambda^2 L^*))$$

is invertible for every $(\xi, \varphi) \in \mathcal{U}_\varepsilon^{(k)} \times \mathcal{W}_\varepsilon^{(k)}$. The family of inverse maps $p_{(\xi, \varphi)}$ is smooth tame.

Proof. Let $\eta \in d_L^* (\Gamma(M, \Lambda^2 L^*))$. Then

$$Q_L d_L a_{\mathcal{E}(\xi, \varphi)}\eta = Q_L d_L \eta - Q_L d_L \iota_{\bar{\eta}} \mathcal{E}(\xi, \varphi) = \eta - Q_L d_L \iota_{\bar{\eta}} \mathcal{E}(\xi, \varphi).$$

Since

$$\Delta_L Q_L d_L \iota_{\bar{\eta}} \mathcal{E}(\xi, \varphi) = d_L^* d_L \iota_{\bar{\eta}} \mathcal{E}(\xi, \varphi),$$

for $\eta, \tilde{\beta} \in d_L^* (\Gamma(M, \Lambda^2 L^*))$, the equation has a unique solution by Proposition 6.31 and the solution map is a smooth tame family of linear maps. $\square$

Now suppose that $c_\xi \alpha \in d_L^* (\Gamma(M, \Lambda^2 L^*))$ and let $\eta \in d_L^* (\Gamma(M, \Lambda^2 L^*))$ be such that

$$\eta = Q_L d_L a_{\mathcal{E}(\xi, \varphi)} c_\xi \alpha,$$

i.e. $\alpha = c_\xi^{-1} p_{(\xi, \varphi)} \eta$. If we can find $\eta \in d_L^* (\Gamma(M, \Lambda^2 L^*))$ such that

$$\eta + Q_L[\mathcal{E}(\xi, \varphi), b_{(\xi, \varphi)} c_\xi^{-1} p_{(\xi, \varphi)} \eta] - Q_L \iota_{\overline{p_{(\xi, \varphi)} \eta}}(d_L \mathcal{E}(\xi, \varphi) + \tfrac{1}{2}[\mathcal{E}(\xi, \varphi), \mathcal{E}(\xi, \varphi)]) = \beta$$

this will provide us with a solution to $D_\xi A(\xi, \varphi)\alpha = \beta$. We observe by Proposition 6.31, that this equation has a unique solution $\eta \in d_L^*(\Gamma(M, \Lambda^2 L^*))$ for every $\beta \in d_L^*(\Gamma(M, \Lambda^2 L^*))$, provided that $(\xi, \varphi) \in \mathcal{U}_\varepsilon^{(k)} \times \mathcal{W}_\varepsilon^{(k)}$ for small enough $\varepsilon > 0$ and large enough k , and the solution map $s_{(\xi, \varphi)}: d_L^*(\Gamma(M, \Lambda^2 L^*)) \rightarrow d_L^*(\Gamma(M, \Lambda^2 L^*))$ is smooth tame. This gives rise to a smooth tame family of linear maps

$$V(\xi, \varphi) := c_\xi^{-1} p_{(\xi, \varphi)} s_{(\xi, \varphi)}: d_L^*(\Gamma(M, \Lambda^2 L^*)) \longrightarrow \Psi^{-1} \mathfrak{h}_{\text{ex}}(E),$$

such that $D_\xi A(\xi, \varphi)V(\xi, \varphi)\beta = \beta$ for every $\beta \in d_L^*(\Gamma(M, \Lambda^2 L^*))$. $\square$

Proof of Proposition 8.42. By Lemma 8.43, the assumptions of Theorem 6.29 hold. Since $A(0, 0) = 0$, we conclude that there is a neighborhood $\mathcal{W}$ of 0 in $\Gamma(M, \Lambda^2 L^*)$ and a neighborhood $\mathcal{U}$ of 0 in $\Psi^{-1} \mathfrak{h}_{\text{ex}}(E)$ such that for every $\varphi \in \mathcal{W}$ we can find some $\xi = \xi(\varphi) \in \mathcal{U}$ such that $d_L^* \mathcal{E}(\xi, \varphi) = 0$. Moreover, the solution $\xi(\varphi)$ is defined by a smooth tame map. This finishes the proof of Proposition 8.42. $\square$

Proof of Theorem 8.36. By Corollary 8.40, the family $\tilde{\mathcal{M}} = \{\varphi(t) : t \in \mathcal{W}_\varepsilon\}$ is a smooth family of generalized almost complex deformations. The integrable deformations are precisely those for which the analytic map $\phi(\varphi) := \text{pr}_{\mathcal{H}}[\varphi, \varphi]$ vanishes. They are contained in a neighborhood of 0 in $\tilde{\mathcal{V}}_1$. Moreover, by Proposition 8.42, we know that any sufficiently small generalized complex deformation is equivalent to a generalized complex deformation in a neighborhood of 0 of $\tilde{\mathcal{V}}_1$.

Furthermore, if $\phi \equiv 0$, then

$$\mathcal{M} = \{\varphi(t) \in \tilde{\mathcal{M}} : \phi(\varphi(t)) = 0\} = \tilde{\mathcal{M}}$$

is a smooth family of generalized complex deformations. Given any other smooth family of generalized complex deformations $\{\varphi'(t') : t' \in \mathcal{W}'\}$, it follows again from Proposition 7.17 that (possibly after shrinking $\mathcal{W}'$) there is a family of exact autoequivalences $\{F_{t'}\}_{t' \in \mathcal{W}'}$, depending smoothly on t' , and a smooth map $\tau: \mathcal{W}' \rightarrow \mathcal{W}_\varepsilon$ such that for every $t' \in \mathcal{W}'$, we have

$$F_{t'} \cdot \varphi'(t') = \varphi(\tau(t')).$$

Hence, the family is locally complete. $\square$

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Eidesstattliche Erklärung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Paula Naomi Pilatus
Hamburg, 30.06.2026