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Shrinking dynamics of skyrmions and antiskyrmions

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Abstract

Magnetic skyrmions and antiskyrmions are topologically nontrivial spin textures occurring in various magnetic systems. In chiral magnets, they emerge from the competition between Heisenberg exchange interaction, Dzyaloshinskii-Moriya interaction (DMI), and external magnetic fields. Owing to their nanoscale size, topological stability, and the possibility of their controlled manipulation by electric currents, magnetic fields, or thermal gradients, they are considered promising candidates for future spintronic technologies. Understanding their dynamical stability and collapse mechanisms is therefore of both fundamental and technological relevance. Unstable skyrmions and antiskyrmions shrink in size until their eventual collapse, while sufficiently strong isotropic DMI can stabilize skyrmions. The detailed shrinking dynamics remain insufficiently understood and therefore, this thesis develops an analytical continuum model for the nonequilibrium shrinking and breathing dynamics of both textures.

For skyrmions, a collective-coordinate approach is employed in which the radius and the helicity are treated as time-dependent variables. Using a triangular ansatz for the skyrmion profile together with the Landau-Lifshitz-Gilbert equation, we derive coupled nonlinear differential equations governing the shrinking dynamics. Our analysis reveals a crossover from a Zeeman-dominated exponential decay at large skyrmion sizes to an exchange-driven square-root behavior near the critical collapse time. If the DMI is too small to stabilize the skyrmion, the shrinking is accompanied by rotational breathing dynamics, whereas a sufficiently strong DMI stabilizes damped, oscillatory breathing modes around a finite equilibrium size. The critical collapse time is found to depend logarithmically on the lattice constant and sensitively on the magnetic field, DMI strength, and Gilbert damping.

For antiskyrmions, the continuum framework is extended to include elliptical deformations. These are incorporated to account for the antiskyrmions intrinsic twofold symmetry that gives rise to direction-dependent anisotropic energy contributions in systems with isotropic DMI. The set of collective coordinates is therefore expanded to include the two semi-axes, the helicity, and the in-plane rotation angle of the texture. The resulting theory reveals that elliptic antiskyrmions are energetically favored over circular ones, although they remain unstable. At large sizes they shrink exponentially and in the regime of small antiskyrmion sizes, the exchange interaction drives the antiskyrmion towards circularity and leads again to a square-root-like collapse dynamics. This overall behavior is then superimposed with coupled shrinking, deformation, rotation, and helicity dynamics, that give rise to a characteristic rotational quadrupole breathing mode.

The analytical predictions are systematically validated through micromagnetic simulations on a discrete lattice. Overall, this work establishes an analytical framework for the shrinking, breathing, and deformation dynamics of skyrmions and antiskyrmions, providing a basis for predicting and controlling topological spin textures in future spintronic applications.

Zusammenfassung

Magnetische Skyrmionen und Antiskyrmionen sind topologisch nichttriviale Spintexturen, die in verschiedenen magnetischen Systemen vorkommen. In chiralen Magneten werden sie durch das Zusammenspiel der Heisenberg-Austauschwechselwirkung zusammen mit der Dzyaloshinskii-Moriya-Wechselwirkung (DMI) und externen Magnetfeldern stabilisiert. Aufgrund ihrer Größe im Nanometerbereich, ihrer topologischen Stabilität sowie der Möglichkeit, sie durch elektrische Ströme, Magnetfelder oder thermische Gradienten manipulieren zu können, gelten sie als vielversprechende Kandidaten für zukünftige spintronische Technologien. Daher ist das Verständnis ihrer Stabilität und ihrer Zerfallsmechanismen nicht nur von grundlegender, sondern auch von technologischer Bedeutung. Während instabile Skyrmionen und Antiskyrmionen bis zu ihrem Zerfall schrumpfen, kann eine hinreichend starke isotrope DMI Skyrmionen stabilisieren. Da ihre detaillierte Schrumpfdynamik bislang nicht ausreichend verstanden ist, wird in dieser Dissertation ein analytisches Kontinuumsmodell für die Schrumpfungs- und Atmungs-dynamik beider Texturen im Nichtgleichgewicht entwickelt.

Für die Beschreibung der Skyrmionen wird ein Ansatz kollektiver Koordinaten verwendet, bei dem Radius und Helizität als zeitabhängige Variablen behandelt werden. Mithilfe eines dreieckförmigen Ansatzes für das Skyrmionprofil sowie unter Zuhilfenahme der Landau-Lifshitz-Gilbert-Gleichung leiten wir gekoppelte nichtlineare Differentialgleichungen für die Schrumpfdynamik her. Unsere Analyse zeigt, dass große Skyrmionen zunächst, getrieben durch den Zeeman-Term, exponentiell schrumpfen. Kurz vor der kritischen Zerfallszeit geht dieses Schrumpfen in ein durch den Austauschterm getriebenes Wurzelverhalten über. Reicht die Stärke der DMI nicht aus, um Skyrmionen zu stabilisieren, wird dieses Schrumpfen von einer Atmungs-dynamik begleitet, bei der die Helizität kontinuierlich präzediert. Die kritische Zerfallszeit hängt logarithmisch von der Gitterkonstante ab und ist zudem stark vom Magnetfeld, der DMI-Stärke und der Gilbert-Dämpfung beeinflusst. Bei hinreichend starker, stabilisierender DMI oszillieren sowohl der Skyrmionradius als auch dessen Helizität gedämpft um die Gleichgewichtslage herum.

Anschließend wird der Kontinuumsansatz auf elliptische Deformationen erweitert, um den Zerfall von Antiskyrmionen zu beschreiben. Diese werden eingeführt, da Antiskyrmionen zwei Symmetrieachsen haben, die zu richtungsabhängigen, anisotropen Energiebeiträgen in Systemen mit isotroper DMI führen. Der Satz kollektiver Koordinaten wird daher um die beiden Halbachsen, die Helizität und den Rotationswinkel der Ellipse in der Ebene erweitert. Die daraus resultierende Theorie zeigt, dass elliptische Antiskyrmionen energetisch günstiger sind als kreisförmige. Sie bleiben im betrachteten System jedoch stets instabil und zerfallen bei großen Größen exponentiell. Im Grenzfall kleiner Antiskyrmionen zwingt die Austauschwechselwirkung das Antiskyrmion wieder in eine Kreisform, was erneut zu einem wurzelartigen Zerfall führt. Diese Dynamik wird überlagert von einer gekoppelten Dynamik aus Schrumpfen, Deformation, Rotation und Helizitäts-präzession. Dabei zeigt sich eine charakteristische quadrupolartige Atmungs-mode, bei der die Helizität

kontinuierlich präzediert.

Die analytischen Vorhersagen werden systematisch durch mikromagnetische Simulationen auf einem diskreten Gitter validiert. Diese Arbeit etabliert einen analytischen Rahmen für die Schrumpfungs-, Atmungs- und Deformationsdynamik von Skyrmionen und Antiskyrmionen und liefert damit eine Grundlage für die Vorhersage und Kontrolle topologischer Spintexturen in zukünftigen spintronischen Anwendungen.

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1. Introduction

The rapid growth of digital information has led to an ever-increasing demand for data storage and processing technologies [1]. Conventional information processing technologies are predominantly based on complementary metal-oxide-semiconductor (CMOS) architectures, where information is encoded and manipulated through electronic charge transport [2]. While the continuous miniaturization of the devices has enabled exponential increases in computational power and storage density over the past decades, the CMOS technology is approaching fundamental scaling limits [3]. This has motivated the search for alternative concepts for information storage and processing. In this context, spintronic devices have emerged as promising alternatives. These make use not only of the electron's charge, but also of its spin as an additional degree of freedom. Among the various proposed spintronic devices, the magnetic domain-wall racetrack memory has attracted significant attention as a candidate for non-volatile and high-density information storage [4]. In such systems, information is encoded in the position of magnetic domain walls. They are stable magnetic textures that can be manipulated by electric currents with current densities of the order of $j \sim 10^{11} \text{ A/m}^2$ [5–7] or by spin waves [8].

A further reduction in device dimensions as well as energy consumption is achieved when moving from domain walls to topologically non-trivial spin textures such as magnetic skyrmions. These objects can be manipulated using remarkably low current densities of the order of $j \sim 10^6 \text{ A/m}^2$ [9] and typically have diameters on the nanometer scale [10–12]. These properties combined with their robustness against thermal fluctuations and defects [13] make them particularly attractive candidates for future spintronic devices.

The conceptual foundation of skyrmions dates back to the work of Tony Skyrme in the early 1960s [14], who introduced a nonlinear field theory in particle physics in order to describe baryons as topological solitons. In this framework, the winding number of the field configuration corresponds to a conserved topological charge, which can be identified with baryon number. Although originally formulated in high-energy physics, the skyrmion concept has since found applications across condensed matter systems, including quantum Hall ferromagnets [15] and Bose-Einstein condensates [16]. However, its most prominent realization is found in magnetic systems. Here, skyrmions emerge as stable, localized spin textures.

In magnetic systems, the existence of skyrmions was first theoretically explored by Belavin and Polyakov in 1975 for ferromagnets with purely Heisenberg exchange interaction [17]. In this case, skyrmions appear as scale-invariant, high-energy excitations, with no intrinsic energetic stabilization. Consequently, despite their topological protection in

the continuum limit, the discreteness of real lattice systems renders them unstable. A major breakthrough was then achieved by Bogdanov and Yablonskii in 1989, who demonstrated that the inclusion of the antisymmetric Dzyaloshinskii-Moriya interaction (DMI) together with an external magnetic field can stabilize skyrmions in real materials [18]. The DMI arises in solid-state systems with strong spin-orbit coupling and broken inversion symmetry [19, 20], and plays a central role in stabilizing chiral magnetic textures. A subsequent theoretical study further established the existence of stable magnetic vortices in ferromagnetic models beyond the standard Heisenberg framework [21]. In particular, it was shown that such systems can support stable localized configurations with very small sizes. Furthermore, it was proposed that skyrmion lattices can arise as ground states in certain condensed matter systems, including surfaces, thin films, and bulk compounds [22].

Experimentally, the first observation of a skyrmion lattice was reported in 2009 in the noncentrosymmetric magnet MnSi [23], where neutron scattering revealed the spontaneous formation of a two-dimensional skyrmion lattice phase. This discovery was followed by a series of experiments, including real-space imaging of skyrmion lattices in thin films of $\text{Fe}_{0.5}\text{Co}_{0.5}\text{Si}$ using Lorentz transmission electron microscopy [24], as well as atomic-scale imaging of skyrmion textures using spin-polarized scanning tunneling microscopy [25]. These experiments established skyrmions as experimentally accessible quasiparticles in magnetic materials and opened the way for the exploration of a broader class of topological spin textures. In addition to skyrmions, a variety of related magnetic configurations have since been identified, including their topological counterpart, the antiskyrmion, as well as skyrmionium states, and three-dimensional structures such as hopfions [26–28]. In this thesis, however, the focus will be restricted to isolated magnetic skyrmions and antiskyrmions in ferromagnetic films.

Due to their topological protection and nanoscale size, skyrmions have been proposed as information carriers in future data storage technologies [29], where binary information can be encoded in their presence or absence [29–34]. To understand how such information may be written and erased, considerable research effort has been devoted to the creation, stability, and annihilation of skyrmions [10, 35]. In chiral magnets with uniform DMI, skyrmions typically appear as metastable or stable configurations depending on the conditions given by the environment [13, 31]. Their stability has been studied using numerical techniques such as the geodesic nudged elastic band (GNEB) method, which allows one to determine minimum-energy transition paths between different magnetic states [36–39]. These studies provide insight into the energy barriers governing skyrmion nucleation and annihilation. In addition, the collapse of quantum skyrmions via quantum tunneling in quantum spin systems has also been investigated [40, 41]. Furthermore, their controlled manipulation via electric currents [30, 42], magnetic fields [43], and thermal gradients [44, 45] has been explored. The understanding of the possibilities enabled by such control mechanisms makes skyrmions promising candidates for both fundamental studies of magnetism and technological applications in spintronic devices [29, 32–34, 46, 47].

For technological applications, one major challenge is the skyrmion Hall effect, which causes skyrmions to move at an angle with respect to an applied current [48]. Several strategies have been proposed to mitigate this effect, including the use of fully compensated antiferromagnets, which are predicted to exhibit a vanishing skyrmion Hall angle [49, 50], and ferrimagnets, where a reduction of this angle can be achieved [51, 52]. In ferromagnetic systems, hybrid DMI combined with spin-orbit torques has also been proposed as a route towards eliminating the skyrmion Hall angle [53]. In contrast to the fully rotationally symmetric skyrmions, antiskyrmions exhibit a twofold rotational symmetry arising from their anisotropic internal spin structure [54, 55]. Because of this symmetry, their dynamics can be tuned by adjusting the current direction relative to their internal spin structure, allowing for control of the skyrmion Hall angle down to zero in certain configurations [56]. The stabilization of antiskyrmions requires anisotropic DMI [57–60], which can arise in materials with tetragonal D_{2d} [59, 61] or S_4 [62, 63] crystal symmetries. Their emergence in these materials as well as the possibility of a vanishing skyrmion Hall angle for antiskyrmions broadens the range of materials hosting topological magnetic textures and allows for more possible applications.

Recent studies have further explored current-driven antiskyrmion motion [64, 65], as well as the controlled transition between skyrmion and antiskyrmion states in ferromagnetic thin films via strain-induced modulation of the DMI [66]. Additionally, pairwise creation of skyrmions and antiskyrmions from helical states under increasing magnetic fields [54] or via in-plane currents [35] has been demonstrated. Since the two quasiparticles carry opposite topological charges, this pairwise creation conserves the total topological charge of the system. In an environment with uniform DMI, however, the antiskyrmion is unstable and collapses, such that the topological charge of the skyrmion alone remains. If the DMI strength is not large enough to stabilize the skyrmion, it collapses as well.

This is where this thesis comes in. It aims at describing the shrinking dynamics of skyrmions and antiskyrmions that have been nucleated in the region where they are unstable or where they were brought to by a magnetic field or other means. In the literature, it is well known that unstable skyrmions shrink over time until their size approaches the value of the lattice constant, at which point they collapse on very short time scales that are often treated as instantaneous [67]. The collapse dynamics depend sensitively on the initial size of the skyrmion [68] and external magnetic fields [69]. In particular, large skyrmions in a magnetic field exhibit an approximately exponential reduction in size, whereas smaller skyrmions display collapse times that scale with the fourth power of their initial size, with a logarithmic correction [68, 69]. In this work, we develop an analytical framework to describe the shrinking dynamics of skyrmions and antiskyrmions in systems with isotropic bulk DMI, focusing on the time evolution of their size, their internal structure, as well as their emergent breathing modes, and shape deformations.

The thesis is organized as follows. In Chapter 2, we review the theoretical framework underlying this thesis. We begin by constructing the lattice Hamiltonian describing fer-

romagnetic thin films capable of hosting skyrmions. Next, we introduce the Landau-Lifshitz-Gilbert (LLG) equation governing the magnetization dynamics, in which the system parameters enter through the Hamiltonian. We then take the continuum limit, which enables the development of the analytical framework used throughout our study of skyrmion and antiskyrmion shrinking dynamics. To describe these magnetic textures, we discuss the properties defining and distinguishing them, including their magnetization field parametrization, topology, stability, and internal structure. Finally, we introduce the approximations employed to enable a closed-form analytical treatment and present the collective coordinate approach used throughout the thesis.

In Chapter 3, we study the shrinking dynamics of skyrmions in systems with isotropic DMI. Using the collective coordinate approach, we derive and analyze the coupled equations of motion governing the dynamics of the skyrmion radius and helicity. We show that, in the absence of DMI, the shrinking exhibits a crossover from an exponential decay at large skyrmion sizes to a square-root-like collapse dynamics at small sizes due to the competition between Zeeman and exchange interactions. Furthermore, we demonstrate that the critical collapse time depends logarithmically on the lattice constant, such that smaller lattice constants lead to later collapse times. At finite DMI, we find that additional breathing dynamics emerge, including rotational and oscillatory breathing modes depending on whether the DMI strength is sufficient to stabilize the skyrmion.

In Chapter 4, we extend this framework to antiskyrmions in systems with isotropic bulk DMI. Owing to their anisotropic twofold symmetry, antiskyrmions deviate from the circular symmetry in systems with isotropic DMI, reflecting the direction-dependent energy costs induced by their internal spin structure [54]. Due to this symmetry, the dynamics are described by additional collective coordinates accounting for elliptic deformations and in-plane rotations of the texture. At vanishing DMI, the shrinking exhibits, similarly to the skyrmion case, a crossover from exponential decay at large sizes to a square-root-like collapse dynamics at small sizes. In the presence of DMI, we show that the energy landscape becomes anisotropic and strongly influences the antiskyrmion shrinking dynamics. In particular, the coupling between the semi-axes, helicity, and rotation angle leads to considerably richer dynamics, including a rotational quadrupole breathing mode, orientation-dependent deformations, and in-plane rotations during the shrinking process.

The thesis concludes with a summary and an outlook in Chapter 5.

2. Theoretical background

In this chapter, we establish the theoretical framework underlying the analysis of skyrmion and antiskyrmion dynamics in ferromagnetic films. We begin with a classical description of magnetic moments on a discrete lattice and construct the Hamiltonian, incorporating the Heisenberg exchange interaction, the Dzyaloshinskii-Moriya interaction, and the Zeeman coupling to an external magnetic field. From this model, we derive the effective magnetic field and the Landau-Lifshitz-Gilbert equation, which governs the magnetization dynamics. To enable an analytical treatment of magnetic structures, we subsequently take the continuum limit, transforming the discrete model into a continuum field theory, with a corresponding energy functional and effective field. Within this framework, we introduce the parametrization of the magnetization field used to describe skyrmions and antiskyrmions, and analyze their topological properties, stability, and internal structure in terms of the radial profile, vorticity, and helicity. An approximate triangular ansatz for the radial profile is presented to render the problem analytically tractable. Finally, we outline the collective coordinate approach, which reduces the infinite-dimensional dynamics of the magnetization field to a low-dimensional description in terms of a small set of variables. This approach forms the basis for the analytical investigation of shrinking skyrmions and antiskyrmions in the following chapters.

2.1. Classical spin model on the lattice

The atoms in a magnetic material form a periodic crystal lattice, consisting of repeating unit cells. On each unit cell, a net magnetic moment can be defined, arising from the collective contribution of the atomic spins. Since the characteristic length scale of magnetic textures such as skyrmions is typically in the nanometer range [70, 71], which is much larger than the lattice spacing, the system can be described within the framework of micromagnetics.

Depending on the level of description, the magnetic moments $\mathbf{S}_i$ may be interpreted either as quantum mechanical spin operators or as effective classical vectors. While fully quantum mechanical treatments of lattice spin systems have been considered in the context of quantum skyrmions [72–74], in this work, we restrict ourselves to a classical description, in which the magnetic moments represent the coarse-grained magnetization of each lattice site. The associated magnetic moment is given by [75]

$$\boldsymbol{\mu}_i = -g\mu_B \frac{\mathbf{S}_i}{\hbar}, \quad (2.1)$$

where g is the Landé factor, μ_B is the Bohr magneton, and $\hbar$ is the reduced Planck constant.

It is known that the total length of each local magnetic moment is conserved, due to the strong exchange interactions between the spins in ferromagnets [76, 77]. Hence, in the classical approximation, the magnitude of each local magnetic moment is taken to be constant, and only its orientation is treated as a dynamical variable [78]. We therefore introduce the normalized vectors

$$\mathbf{n}_i = \frac{\boldsymbol{\mu}_i}{|\boldsymbol{\mu}_i|}, \quad |\mathbf{n}_i| = 1, \quad (2.2)$$

so that the magnetic state of the system is fully characterized by the set of unit vectors $\{\mathbf{n}_i\}$.

Each vector $\mathbf{n}_i$ can be parametrized in terms of spherical coordinates by a polar angle $\Theta_i \in [0, \pi]$, measured from the z -axis, and an azimuthal angle $\Phi_i \in [0, 2\pi)$, describing the in-plane orientation. Accordingly, the magnetic moment at lattice site i is represented by a unit vector pointing to a position on the unit sphere as

$$\mathbf{n}_i = \begin{pmatrix} \sin \Theta_i \cos \Phi_i \\ \sin \Theta_i \sin \Phi_i \\ \cos \Theta_i \end{pmatrix}. \quad (2.3)$$

This parametrization is visualized in Figure 2.1 and is particularly useful for describing non-collinear magnetic textures, such as skyrmions and antiskyrmions, where the spatial variation of the angles Θ_i and Φ_i encodes the structure of the magnetization field.

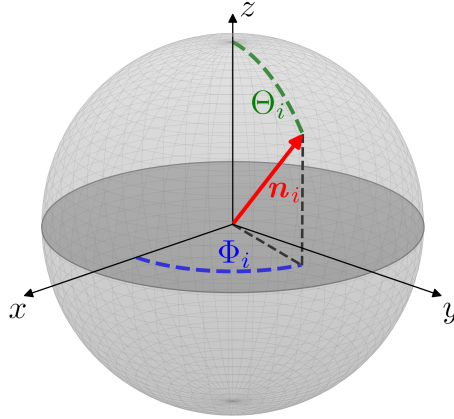


Figure 2.1.: Visualization of the unit vector $\mathbf{n}_i$ at lattice site i pointing to a position on the unit sphere. Its orientation is determined by the polar angle Θ_i and the azimuthal angle Φ_i .

2.2. Hamiltonian, effective magnetic field, and dynamics

The dynamics of localized magnetic moments is determined by their interaction energy, which is described by an effective Hamiltonian defined on a discrete lattice. Starting from the Heisenberg exchange interaction, which constitutes the dominant energy scale in ferromagnetic systems, we introduce the key competing contributions of the Dzyaloshinskii-Moriya interaction (DMI) and the Zeeman coupling to external fields. Together, these interactions enable the stabilization of chiral magnetic textures, such as skyrmions and antiskyrmions, which are in the focus of the present work. We further briefly discuss additional interactions, including magnetic anisotropy, dipolar coupling, and higher-order exchange terms, which may play a role in real materials but are not considered here in detail. The resulting lattice Hamiltonian provides the basis for deriving the effective magnetic field acting on each magnetic moment and, subsequently, the Landau-Lifshitz-Gilbert equation governing their time evolution, thereby forming the foundation for both the numerical simulations and the continuum description developed in Sec. 2.3.

2.2.1. Heisenberg exchange interaction

The Heisenberg exchange interaction is the primary mechanism responsible for ferromagnetism [79]. It is a short-range interaction and is typically restricted to nearest-neighbor sites [80]. Since we consider ferromagnetic systems in this work, the exchange interaction constitutes the leading energy contribution in our model. Microscopically, the exchange interaction originates from the Coulomb repulsion between electrons in combination with the Pauli exclusion principle [81, 82]. According to the Heisenberg model, the interaction energy between neighboring magnetic moments $\mathbf{n}_i$ and $\mathbf{n}_j$ is given by

$$H_{\text{Exch}} = - \sum_{\langle i,j \rangle} J_{ij} \mathbf{n}_i \cdot \mathbf{n}_j. \quad (2.4)$$

Here, J_{ij} denotes the exchange interaction strength between magnetic moments at lattice sites i and j , and the notation $\langle i,j \rangle$ indicates summation over nearest-neighbor pairs. A positive value of J_{ij} favors parallel alignment of the magnetic moments and thus leads to ferromagnetic ordering, while a negative value corresponds to antiferromagnetic coupling. In the present work, we restrict ourselves to the ferromagnetic case, i.e., $J_{ij} > 0$. Assuming a homogeneous ferromagnet on a square lattice with constant exchange strength $J_{ij} = J$, the Hamiltonian simplifies to

$$H_{\text{Exch}} = -J \sum_{\langle i,j \rangle} \mathbf{n}_i \cdot \mathbf{n}_j. \quad (2.5)$$

2.2.2. Dzyaloshinskii-Moriya interaction (DMI)

The main competing interaction to the exchange interaction considered in this study is the Dzyaloshinskii-Moriya interaction (DMI). It was first proposed by Dzyaloshinskii in 1958

based on phenomenological symmetry arguments [20]. He showed that, in the presence of this interaction, a canted alignment of spins can be energetically favorable compared to a collinear one. This provided an explanation for weak ferromagnetism observed in otherwise antiferromagnetic crystals, such as Fe_2O_3 and MnCO_3 . Subsequently, Moriya derived a microscopic theory of the DMI in 1960, identifying it as an antisymmetric exchange interaction arising from relativistic spin-orbit coupling [19, 83]. He further formulated a set of symmetry rules that determine the orientation of the DMI vector depending on the crystal structure [19].

The competition between the exchange interaction, which favors parallel alignment of magnetic moments, and the DMI, which promotes canting, enables the stabilization of a wide range of chiral magnetic textures. These include domain walls [84], spin spirals [85], and magnetic skyrmions [22, 23, 86, 87]. In the following, we discuss two types of DMI that are typically responsible for stabilizing skyrmions.

Interfacial DMI

Interfacial DMI arises in ultrathin magnetic films deposited on heavy-metal substrates with strong spin-orbit coupling and broken inversion symmetry [88], such as Fe/Ir [25], CoFe/Pt, and CoFe/Ta interfaces [89]. This type of DMI stabilizes Néel-type spin spirals and skyrmions, and the corresponding contribution to the Hamiltonian is given by

$$H_{\text{i-DMI}} = -D \sum_{\langle i,j \rangle} (\mathbf{r}_{ij} \times \hat{\mathbf{z}}) \cdot (\mathbf{n}_i \times \mathbf{n}_j), \quad (2.6)$$

where D denotes the DMI strength, $\mathbf{r}_{ij}$ is the unit vector connecting lattice sites i and j , and $\hat{\mathbf{z}}$ is the normal to the interface.

Bulk DMI

Bulk DMI occurs in noncentrosymmetric crystal structures, such as MnSi [23], $\text{Fe}_{1-x}\text{Co}_x\text{Si}$ [24], and FeGe [90], where inversion symmetry is broken in the bulk of the material. This interaction stabilizes Bloch-type spin spirals and skyrmions, and is the type of DMI considered throughout this work. Its contribution to the Hamiltonian is given by

$$H_{\text{b-DMI}} = -D \sum_{\langle i,j \rangle} \mathbf{r}_{ij} \cdot (\mathbf{n}_i \times \mathbf{n}_j). \quad (2.7)$$

While we primarily consider an isotropic DMI as given in Eq. (2.7), which stabilizes circular skyrmions, anisotropic DMI can arise in materials with reduced crystal symmetry, such as tetragonal D_{2d} [59, 61] or S_4 [62, 63] structures. In such cases, the interaction can be described by a second-rank DMI tensor $\mathbf{D}$ with direction-dependent components [58]. This anisotropy can lead to distorted or non-axisymmetric skyrmions. In particular, for vanishing off-diagonal elements and diagonal components satisfying $D_x = -D_y$, the

tensor

$$\mathbf{D} = \begin{pmatrix} D_x & 0 \\ 0 & D_y \end{pmatrix}, \quad (2.8)$$

can stabilize circular antiskyrmions [57–60].

The distinct properties of Néel- and Bloch-type skyrmions, as well as antiskyrmions, will be discussed in more detail in Sec. 2.4.

2.2.3. Zeeman interaction

The last contribution to the Hamiltonian considered in this model is the Zeeman interaction, which describes the coupling of the magnetic moments to an externally applied magnetic field. It tends to align the magnetic moments parallel to the field direction and provides a mechanism to control and stabilize magnetic textures [23]. Its contribution to the Hamiltonian is given by

$$H_{\text{Zee}} = - \sum_i \mathbf{n}_i \cdot \mathbf{B}, \quad (2.9)$$

where constant prefactors, in particular the magnitude of the magnetic moment, have been absorbed into the definition of the field $\mathbf{B}$.

2.2.4. Additional magnetic interactions

In addition to the exchange interaction, DMI, and Zeeman coupling, several other magnetic interactions can play a role in real materials. In this work, we focus on the DMI as the dominant mechanism responsible for stabilizing skyrmions. Nevertheless, alternative contributions such as magnetic anisotropy [91, 92], dipolar interactions [93, 94] and higher-order exchange interactions [95] can also lead to nontrivial magnetic textures in specific systems, but are not considered in the present study.

Magnetic anisotropy describes the dependence of the internal energy on the direction of magnetization. In anisotropic systems, certain directions are energetically preferred, typically referred to as easy axes, while deviations from these directions cost energy. Anisotropy can originate from the crystal structure, the shape of the sample, or surface and interface effects [96]. In particular, perpendicular magnetic anisotropy in thin films is an important ingredient for stabilizing out-of-plane magnetized textures, and can support the formation of skyrmions in combination with exchange interaction and DMI.

Another relevant contribution is the magnetic dipolar interaction [97], which arises from the long-range coupling between magnetic moments. At atomic length scales, it is typically much weaker than the exchange interaction, and therefore does not determine the magnetic ordering in most materials. However, in thin-film geometries, it can become important due to its competition with perpendicular anisotropy. This competition can lead to the formation of stripe and bubble domains, where the magnetization alternates periodically in space. Such magnetic bubble states were historically studied for data storage applications

and, in some cases, can be interpreted as topologically equivalent to skyrmions. Also, it was shown that skyrmion-antiskyrmion pairs are metastable in dipolar magnets [54].

Finally, higher-order exchange interactions can arise from more complex electronic processes beyond the simple Heisenberg model [98], such as higher-order perturbative effects. These include biquadratic and four-spin interactions, involving products of spin scalar products on several lattice sites. Although these contributions are generally much weaker than the leading exchange interaction, they can influence the magnetic ground state and, in certain materials, contribute to the stabilization of nontrivial magnetic textures, such as skyrmion lattices.

2.2.5. Effective magnetic field

The effective magnetic field $\mathbf{B}_i^{\text{eff}}$ models the influence of the surrounding environment on the magnetic moment at site i . It is defined as the negative derivative of the Hamiltonian with respect to the normalized magnetic moment, as

$$\mathbf{B}_i^{\text{eff}} = -\frac{\partial H}{\partial \mathbf{n}_i}, \quad (2.10)$$

where constant prefactors, such as the magnitude of the magnetic moment and the gyromagnetic ratio γ , have been absorbed into the definition of $\mathbf{B}_i^{\text{eff}}$. The complete Hamiltonian used in this study consists of the exchange interaction in Eq. (2.5), the isotropic bulk DMI in Eq. (2.7) and the Zeeman term in Eq. (2.9), and reads

$$H = -J \sum_{\langle i,j \rangle} \mathbf{n}_i \cdot \mathbf{n}_j - D \sum_{\langle i,j \rangle} \mathbf{r}_{ij} \cdot (\mathbf{n}_i \times \mathbf{n}_j) - \sum_i \mathbf{n}_i \cdot \mathbf{B}. \quad (2.11)$$

The effective magnetic field can be obtained from this Hamiltonian by following Eq. (2.10), yielding

$$\mathbf{B}_i^{\text{eff}} = J \sum_{j \in \langle i \rangle} \mathbf{n}_j + D \sum_{j \in \langle i \rangle} (\mathbf{n}_j \times \mathbf{r}_{ij}) + B \hat{\mathbf{z}}. \quad (2.12)$$

Here, $\hat{\mathbf{z}}$ denotes the unit vector in z -direction, since we only apply an external magnetic field along this axis. For the square lattice considered here, the index $j \in \langle i \rangle$ runs over the four nearest neighbors of site i .

2.2.6. Landau-Lifshitz-Gilbert equation

The theoretical description of magnetization dynamics was first formulated by Landau and Lifshitz in 1935 [99], who introduced an equation of motion describing the precessional dynamics of magnetic moments in an effective magnetic field, together with a phenomenological damping term driving the system towards equilibrium. Later, Gilbert proposed an alternative formulation of the damping term based on a Lagrangian approach via the introduction of a Rayleigh dissipation function [100]. This formulation is often preferred in micromagnetic simulations, as it provides a more convenient and stable description of

damping across a wide range of damping parameters [101]. Combining both approaches yields the Landau-Lifshitz-Gilbert (LLG) equation, which is widely used to describe magnetization dynamics and is given by

$$\frac{\partial \mathbf{n}_i}{\partial t} = -\mathbf{n}_i \times \mathbf{B}_i^{\text{eff}} + \alpha \mathbf{n}_i \times \frac{\partial \mathbf{n}_i}{\partial t}, \quad (2.13)$$

where α denotes the Gilbert damping parameter. Throughout the present work, we employ dimensionless units, where time is measured in units of $t_0 = 1/J$ and the gyromagnetic ratio γ is absorbed into the definition of the effective magnetic field $\mathbf{B}_i^{\text{eff}}$.

For numerical implementation, it is convenient to rewrite Eq. (2.13) in explicit form. This is achieved by substituting the time derivative on the right-hand side into itself and collecting terms, yielding

$$\frac{\partial \mathbf{n}_i}{\partial t} = -\frac{1}{1+\alpha^2} \mathbf{n}_i \times \mathbf{B}_i^{\text{eff}} - \frac{\alpha}{1+\alpha^2} \mathbf{n}_i \times (\mathbf{n}_i \times \mathbf{B}_i^{\text{eff}}). \quad (2.14)$$

The dynamical motion of a magnetic moment $\mathbf{n}_i$ induced by the LLG equation is visualized in Figure 2.2. The first term in Eq. (2.14), shown in blue, represents the torque acting on the magnetic moment at site i , which induces a precessional motion around the effective field when $\mathbf{n}_i$ is not aligned with $\mathbf{B}_i^{\text{eff}}$. The second term, shown in orange, accounts for dissipative relaxation towards equilibrium, causing the magnetic moment to gradually align with the effective field until a stationary state is reached. The resulting trajectory on the unit sphere is indicated by the green line.

Magnetization dynamics are commonly studied using established micromagnetic simulation frameworks, such as MuMax3 [102] and OOMMF [103]. These packages numerically

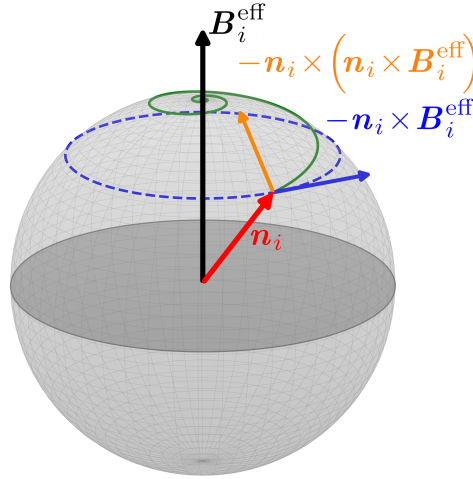


Figure 2.2.: Schematic illustration of the dynamics of a magnetic moment $\mathbf{n}_i$ governed by the LLG equation. The first term (blue) induces a precessional motion around the effective magnetic field $\mathbf{B}_i^{\text{eff}}$, while the second term (orange) describes dissipative relaxation towards alignment with the field. The resulting trajectory (green) approaches a stationary state characterized by $\mathbf{n}_i \parallel \mathbf{B}_i^{\text{eff}}$.

integrate the LLG equation on a discretized magnetic system by computing the effective field from the underlying energy contributions and propagating the magnetic moments in time.

In this work, we instead employ a self-developed GPU-accelerated micromagnetic simulation on a two-dimensional square lattice. A detailed description of the numerical scheme and implementation is provided in Appendix A.

2.3. Continuum limit

While numerical simulations on discrete lattices provide detailed insights into magnetic dynamics, analytical studies of extended magnetic structures require a continuum description [104]. Here, we derive this continuum limit from our discrete lattice model by introducing a continuous magnetization field $\mathbf{n}(\mathbf{r})$ and performing a systematic Taylor expansion of the lattice interactions. Starting from the discrete lattice, we consider the midpoint $\mathbf{r}_k$ between two neighboring lattice sites i and j . The magnetic moments at these sites can then be expressed as $\mathbf{n}_{\mathbf{r}_k \pm \frac{a}{2} \mathbf{e}_s}$, where a denotes the lattice constant and $\mathbf{e}_s$ are unit vectors along the spatial directions $s = \{x, y\}$. Then, we perform a Taylor expansion around $\mathbf{r}_k$, yielding

$$\mathbf{n}_{\mathbf{r}_k \pm \frac{a}{2} \mathbf{e}_s} \approx \mathbf{n}_{\mathbf{r}_k} \pm \frac{a}{2} \partial_s \mathbf{n}_{\mathbf{r}_k}. \quad (2.15)$$

This procedure maps the discrete Hamiltonian onto a continuum energy functional, providing the basis for the derivation of effective field equations governing the dynamics of smooth magnetic textures.

2.3.1. Continuum energy functional

Using the Taylor expansion in Eq. (2.15), the exchange contribution to the Hamiltonian given in Eq. (2.11) can be approximated as

$$-J \sum_{\langle i, j \rangle} \mathbf{n}_i \cdot \mathbf{n}_j \approx \text{const} + \frac{Ja^2}{2} \sum_{k, s} (\partial_s \mathbf{n})^2. \quad (2.16)$$

Here, the indices for all of the $\mathbf{n}_{\mathbf{r}_k}$ have been dropped for readability. Furthermore, the constant term is independent of the magnetization configuration and is neglected in the following. Similarly, the DMI term can be expanded and becomes

$$-D \sum_{\langle i, j \rangle} \mathbf{r}_{ij} \cdot (\mathbf{n}_i \times \mathbf{n}_j) \approx -aD \sum_{k, s} \mathbf{e}_s \cdot (\mathbf{n} \times \partial_s \mathbf{n}). \quad (2.17)$$

The external magnetic field is only applied along the z -direction and therefore, the expansion of the Zeeman contribution is straightforward and yields $-B \sum_i n_i^z$. In the continuum limit, the discrete sum over lattice sites is replaced by an integral according to

$\sum_k \rightarrow \int d^2r/a^2$. Applying this substitution to all energy contributions yields

$$\begin{aligned} \frac{Ja^2}{2} \sum_{k,s} (\partial_s \mathbf{n})^2 &\rightarrow \frac{J}{2} \int d^2r \left[(\partial_x \mathbf{n})^2 + (\partial_y \mathbf{n})^2 \right], \\ -aD \sum_{k,s} \mathbf{e}_s \cdot (\mathbf{n} \times \partial_s \mathbf{n}) &\rightarrow -\frac{D}{a} \int d^2r \mathbf{n} \cdot (\nabla \times \mathbf{n}), \\ -B \sum_i n_i^z &\rightarrow -\frac{B}{a^2} \int d^2r n_z. \end{aligned} \quad (2.18)$$

For convenience, we redefine the parameters $\tilde{D} = D/a$ and $\tilde{B} = B/a^2$, absorbing the lattice constant into the system parameters. The resulting continuum energy functional [39] is then given by

$$E = \int d^2r \left[\frac{J}{2} \left[(\partial_x \mathbf{n})^2 + (\partial_y \mathbf{n})^2 \right] - \tilde{D} \mathbf{n} \cdot (\nabla \times \mathbf{n}) - \tilde{B} n_z \right]. \quad (2.19)$$

The corresponding effective magnetic field $\mathbf{B}_{\text{eff}}$ in the continuum is obtained from the functional derivative of the energy density $\mathcal{E}$ with respect to the magnetization field, yielding

$$\mathbf{B}_{\text{eff}} = -\frac{\delta \mathcal{E}}{\delta \mathbf{n}} = J \Delta \mathbf{n} - 2\tilde{D} (\nabla \times \mathbf{n}) + \tilde{B} \hat{z}, \quad (2.20)$$

where $\mathcal{E}$ denotes the integrand of Eq. (2.19).

2.3.2. Landau-Lifshitz-Gilbert equation in the continuum

In the continuum limit, the time evolution of the magnetization field is governed by the LLG equation, which takes the form

$$\frac{\partial \mathbf{n}}{\partial t} = \frac{a^2}{1 + \alpha^2} [-\mathbf{n} \times \mathbf{B}_{\text{eff}} - \alpha \mathbf{n} \times (\mathbf{n} \times \mathbf{B}_{\text{eff}})]. \quad (2.21)$$

This expression is consistent with derivations based on a Lagrangian formalism including Berry phase contributions, where the lattice constant naturally enters the continuum description [105, 106].

2.4. Properties of skyrmions and antiskyrmions

With the micromagnetic framework in place, we now turn to topologically nontrivial magnetization configurations. In particular, we focus on skyrmions and antiskyrmions, which are localized magnetic textures characterized by a nontrivial topology. This section discusses their static properties, starting with a parametrization of the magnetization field, followed by a detailed analysis of their topology, stability, and internal structure. Special attention is paid to the helicity, which distinguishes different in-plane configurations, and to the radial profile, which characterizes their finite size. These quantities will later serve

as the relevant conjugated variables in the collective coordinate approach introduced in Sec. 2.5.

2.4.1. Magnetization field parametrization

Magnetic skyrmions and antiskyrmions are described by a continuous magnetization field $\mathbf{n}(\mathbf{r})$ defined on a two-dimensional plane $\mathbf{r} = (x, y)$. The magnetization is assumed to be of constant magnitude, $|\mathbf{n}(\mathbf{r})| = 1$, such that it can be represented as a mapping onto the unit sphere. It is therefore convenient to parametrize the vector field in terms of the spherical angles Θ and Φ as

$$\mathbf{n}(\mathbf{r}) = \begin{pmatrix} \sin \Theta(\mathbf{r}) \cos \Phi(\mathbf{r}) \\ \sin \Theta(\mathbf{r}) \sin \Phi(\mathbf{r}) \\ \cos \Theta(\mathbf{r}) \end{pmatrix}. \quad (2.22)$$

Within this representation, each point in the plane is associated with a point on the unit sphere, similar to the visualization in Figure 2.1, i.e., the magnetization field defines a mapping $(x, y) \mapsto (\Theta, \Phi)$.

For isolated skyrmions and antiskyrmions, which exhibit radial symmetry, it is convenient to introduce polar coordinates $\mathbf{r} = (\rho, \varphi)$ centered at the skyrmion core, such that $\rho = 0$ corresponds to the skyrmion center. In this case, the angular variables simplify to $\Theta(\rho)$ and $\Phi(\varphi)$. The azimuthal angle $\Phi(\varphi)$ determines the in-plane rotation of the magnetization and is expressed as

$$\Phi(\varphi) = m\varphi + \varphi_0, \quad (2.23)$$

where $m \in \mathbb{Z}$ denotes the vorticity and φ_0 the helicity [30]. In the present work, the vorticity is restricted to $m = \pm 1$, where $m = 1$ corresponds to a skyrmion and $m = -1$ to an antiskyrmion. This form of Φ distinguishes different magnetic textures, such as Néel- and Bloch-type skyrmions, as depicted in Figure 2.3 (a) and (b), as well as antiskyrmions, as shown in Figure 2.3 (c). Here, the local magnetization direction in the plane is

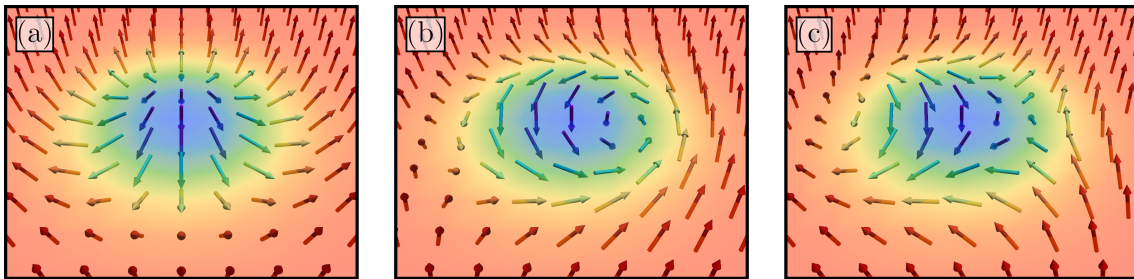


Figure 2.3.: Arrow representation of (anti-)skyrmions. (a) Néel-type skyrmion, (b) Bloch-type skyrmion, and (c) antiskyrmion. The arrows indicate the direction of the magnetization, while the color encodes the out-of-plane component, with blue corresponding to the $-z$ -direction and red to the $+z$ -direction.

represented by arrows, while the out-of-plane component is encoded by the colors, with blue corresponding to the $-z$ -direction and red to the $+z$ -direction. These internal structures are determined by the vorticity and helicity and will be discussed in more detail in Sec. 2.4.3.

The function $\Theta(\rho)$ describes the radial profile of the skyrmion by interpolating between opposite orientations at its center and in the ferromagnetic background [18]. Throughout this work, we assume that the magnetization at the skyrmion center points in the $-z$ -direction, i.e., $\Theta(0) = \pi$, while the ferromagnetic background points in the $+z$ -direction, i.e., $\Theta(\infty) = 0$. The explicit form of $\Theta(\rho)$ is not known in closed analytical form. However, it can be studied by minimizing the energy functional in Eq. (2.19), as shown in Ref. [107]. To this end, the parametrization in Eq. (2.22), together with the ansatz $\Theta = \Theta(\rho)$ and Φ from Eq. (2.23), is inserted into the energy density $\mathcal{E}$. Throughout this work, we consider isotropic bulk DMI and in this case, Bloch-type skyrmions are energetically stabilized, while antiskyrmions are unstable [35]. Consequently, we restrict the following analysis to $m = 1$ and focus on the skyrmion configuration, for which stable energy-minimizing solutions exist.

Using polar coordinates and integrating over the azimuthal coordinate, the two-dimensional energy functional reduces to an effective one-dimensional functional of the form $E = 2\pi \int d\rho \mathcal{L}(\Theta, \Theta', \rho)$ with $\mathcal{L} = \rho \mathcal{E}$. Minimizing this functional with respect to $\Theta(\rho)$ yields the Euler-Lagrange equation

$$\frac{d}{d\rho} \left(\frac{\partial \mathcal{L}}{\partial \Theta'} \right) - \frac{\partial \mathcal{L}}{\partial \Theta} = 0, \quad (2.24)$$

which leads to the nonlinear second-order differential equation

$$J \left[\frac{d^2 \Theta}{d\rho^2} + \frac{1}{\rho} \frac{d\Theta}{d\rho} - \frac{\sin \Theta \cos \Theta}{\rho^2} \right] + 2\tilde{D} \frac{\sin^2 \Theta}{\rho} - \tilde{B} \sin \Theta = 0. \quad (2.25)$$

This equation governs the equilibrium profile of the skyrmion by balancing exchange, DMI, and Zeeman interactions, and thereby determines its size and internal radial structure.

Insight into the structure of the solution can be gained by analyzing its asymptotic behavior, following the approach of Bogdanov and Hubert in Ref. [107]. Close to the skyrmion center, $\rho \rightarrow 0$, the magnetization must vary smoothly to avoid singularities. Expanding $\Theta(\rho)$ around $\rho = 0$ shows that the solution behaves linearly as

$$\Theta(\rho) \approx \pi - \left. \frac{d\Theta}{d\rho} \right|_{\rho=0} \rho + \mathcal{O}(\rho^2), \quad (2.26)$$

where $d\Theta/d\rho|_{\rho=0}$ is a finite constant, ensuring regularity of the solution and cancellation of the potentially divergent terms in Eq. (2.25). In this regime, the profile is dominated by the exchange interaction, which enforces a smooth spatial variation of the magnetization. In contrast, far away from the skyrmion core, $\rho \rightarrow \infty$, the angle Θ becomes small and the

equation can be linearized. In this limit, the DMI contribution becomes negligible, and the balance between exchange and Zeeman interactions leads to an exponential decay of the form

$$\Theta(\rho) \sim \exp\left(-\rho\sqrt{\tilde{B}/J}\right). \quad (2.27)$$

This demonstrates that the skyrmion asymptotically approaches the ferromagnetic background over a characteristic length scale set by the ratio of exchange and Zeeman energies.

While these limiting behaviors provide qualitative insight, the full profile requires a numerical solution of Eq. (2.25) using the boundary conditions $\Theta(0) = \pi$ and $\Theta(\infty) = 0$. The resulting solution is shown in Figure 2.4. It confirms the asymptotic analysis: In the skyrmion core, the profile exhibits an approximately linear behavior governed primarily by exchange and at large distances, the solution approaches the ferromagnetic background exponentially due to the Zeeman term.

In the intermediate region, where $\Theta \approx \pi/2$ and all three interactions compete, the profile deviates from these simple limits. In particular, the DMI contribution is maximal, since $\sin^2 \Theta$ attains its maximum in the vicinity of $\Theta = \pi/2$, thereby enhancing the effective twisting of the magnetization. This results in a characteristic shoulder-like feature in $\Theta(\rho)$. The interplay between exchange, DMI, and Zeeman interactions ultimately determines the finite size and detailed radial structure of the skyrmion. A more detailed analysis of the parameter dependence of $\Theta(\rho)$ can be found in Ref. [107]. For the purposes of the present work, however, it is sufficient to consider a representative profile capturing the general features of the solution as depicted in Figure 2.4.

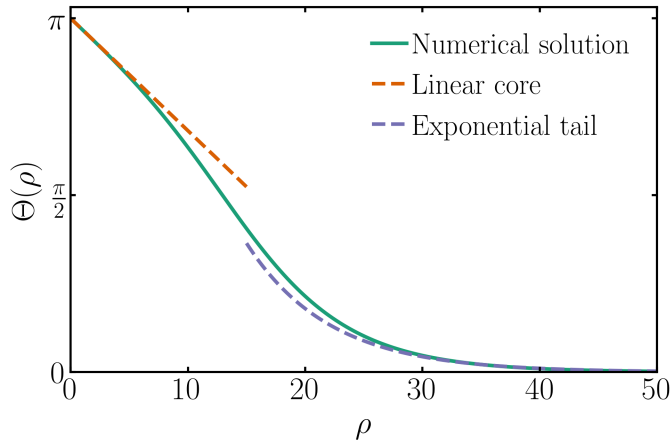


Figure 2.4.: Numerical solution of Eq. (2.25) with boundary conditions $\Theta(0) = \pi$ and $\Theta(\infty) = 0$, showing the radial profile $\Theta(\rho)$ of a Bloch-type skyrmion. The solution exhibits an exchange-dominated linear behavior in the skyrmion core, described by Eq. (2.26) (dashed orange line), a DMI-induced shoulder in the intermediate region, and an exponential decay towards the ferromagnetic background at large distances from the center, described by Eq. (2.27) (dashed purple line). Parameters: $\tilde{D}/J = 0.2$, $\tilde{B}/J = 0.02$.

2.4.2. Topology and stability

Magnetic skyrmions and antiskyrmions are not only characterized by their internal structure, but also by their topological properties and stability. An early version of what is now referred to as a skyrmion was proposed by Belavin and Polyakov in 1975 [17]. They showed that a two-dimensional ferromagnet with only exchange interaction allows for non-trivial excitations in which the magnetization at the center points opposite to the uniform background. These configurations can be understood geometrically as a mapping from the two-dimensional plane onto the sphere, where the continuous magnetization field wraps the sphere [108]. This mapping is illustrated for the example of a Bloch-type skyrmion in Figure 2.5 and defines how many times the sphere is covered by the plane. This integer quantity is referred to as the topological charge and distinguishes skyrmions from topologically trivial states such as the uniformly magnetized ferromagnet [13]. In particular, a skyrmion configuration cannot be continuously deformed into the ferromagnetic state without introducing singularities in the magnetization field. The topological charge is strictly well-defined in the continuum limit, where neighboring magnetic moments are nearly parallel, and is conserved under smooth deformations of the field [109].

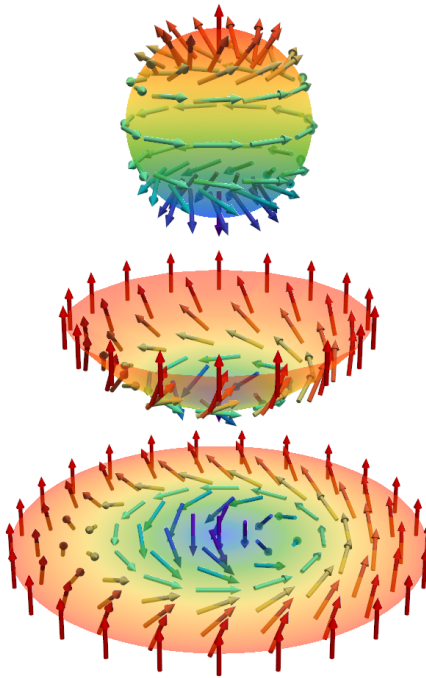


Figure 2.5.: Visualization of the mapping of a Bloch-type skyrmion from the two-dimensional plane onto the sphere. The colors encode the z -component of the magnetization field, with blue corresponding to the $-z$ -direction and red to the $+z$ -direction, while the arrows indicate the local orientation of the field. The continuous magnetization field covers the sphere exactly once, such that each magnetization direction is represented once.

Based on this mapping, the topological charge can be expressed as

$$\mathcal{Q} = \frac{1}{4\pi} \int \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial x} \times \frac{\partial \mathbf{n}}{\partial y} \right) d^2 \mathbf{r}, \quad (2.28)$$

which takes integer values $\mathcal{Q} \in \mathbb{Z}$. The corresponding integrand

$$q(\mathbf{r}) = \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial x} \times \frac{\partial \mathbf{n}}{\partial y} \right), \quad (2.29)$$

is referred to as the topological charge density and can be interpreted as a local measure of how strongly the magnetization field twists in space. For radially symmetric configurations, this expression simplifies considerably and allows for an explicit evaluation of $\mathcal{Q}$. Transforming to polar coordinates and employing the parametrization introduced in Eq. (2.22), the spatial derivatives can be rewritten as

$$\frac{\partial \mathbf{n}}{\partial \rho} = \Theta'(\rho) \frac{\partial \mathbf{n}}{\partial \Theta}, \quad \frac{\partial \mathbf{n}}{\partial \varphi} = m \frac{\partial \mathbf{n}}{\partial \Phi}, \quad (2.30)$$

This form makes explicit that radial variations are governed by the profile $\Theta(\rho)$, while angular variations are entirely encoded in the vorticity m . Inserting these expressions into Eq. (2.28) yields

$$\mathcal{Q} = \frac{m}{4\pi} \int_0^\infty \int_0^{2\pi} \Theta'(\rho) \sin \Theta(\rho) d\varphi d\rho. \quad (2.31)$$

Since the integrand is independent of the azimuthal angle φ , the angular integration simply produces a factor of 2π , reflecting the rotational symmetry of the configuration. Performing this integration and introducing the substitution $u = \Theta(\rho)$, one obtains

$$\mathcal{Q} = \frac{m}{2} \int_{\Theta(0)}^{\Theta(\infty)} \sin u du. \quad (2.32)$$

Here, it is evident that the boundary conditions determine the sign of the topological charge. In the present work, we assume that the skyrmion is embedded in a ferromagnetic background pointing along the $+z$ -direction, with the magnetization in the core of the skyrmion pointing in the opposite direction, i.e., $\Theta(0) = \pi$ and $\Theta(\infty) = 0$. Therefore, $\mathcal{Q}$ evaluates to

$$\mathcal{Q} = -m. \quad (2.33)$$

Thus, for a skyrmion with $m = 1$, the topological charge evaluates to $\mathcal{Q} = -1$, while for an antiskyrmion with $m = -1$, one obtains $\mathcal{Q} = +1$. Higher vorticities m correspond to higher topological charges and skyrmions of topological charges up to $\mathcal{Q} = 10$ have been reported in Ref. [110]. Here, we only discuss skyrmions and antiskyrmions with $m = \pm 1$, which, for the boundary conditions chosen here, implies $\mathcal{Q} = \mp 1$. The integer-valued nature of the topological charge $\mathcal{Q}$, and in particular its nonzero value for skyrmions and antiskyrmions, implies that these configurations are topologically protected. In particular, they cannot

be continuously deformed into the homogeneous ferromagnetic state with $Q = 0$ without introducing singularities in the magnetization field.

The quasiparticle proposed by Belavin and Polyakov arises only in a system with purely Heisenberg exchange interaction and is as such a scale-invariant excitation with no intrinsic size [17]. By the addition of DMI and an external magnetic field, Bogdanov and Yablonskii were then able to propose properties for real materials that would host skyrmions as thermodynamically stable states [18]. The inclusion of these interactions breaks the scale invariance and stabilizes skyrmions with a finite size.

In real materials, the magnetization is not a continuous field, but rather described by a lattice model. Here, topological protection does not imply strict conservation of the topological charge, but instead results in a large energy cost when changing this quantity [109, 111]. Thus, although skyrmions are characterized by a nontrivial topological charge, their stability is not purely topological but of energetic origin. In the systems considered here, stability is governed by the competition between exchange interaction, DMI, and the Zeeman field. This interplay is already reflected in the Euler-Lagrange equation (2.25), which determines the radial profile $\Theta(\rho)$ and thereby fixes both the size and internal radial structure of the skyrmion.

On the level of the energy functional in Eq. (2.19), it is convenient to redefine the Zeeman contribution by subtracting a constant energy density $\tilde{B}$, yielding

$$E = \int d^2r \left[\frac{J}{2} \left[(\partial_x \mathbf{n})^2 + (\partial_y \mathbf{n})^2 \right] - \tilde{D} \mathbf{n} \cdot (\nabla \times \mathbf{n}) - \tilde{B} [n_z - 1] \right]. \quad (2.34)$$

This constant shift in the Zeeman term subtracts the energy density of the ferromagnetic ground state, such that the uniformly magnetized ferromagnetic state, corresponding to $\Theta(\rho) = 0$, carries zero energy and E measures the excess energy of the magnetic texture. This provides a natural reference point, such that all magnetic configurations can be directly compared to the ferromagnetic ground state.

The exchange interaction favors a homogeneous magnetization, while the DMI promotes chiral twists of the magnetization field. Furthermore, the Zeeman term stabilizes the ferromagnetic background and penalizes deviations from it. The competition between these interactions results in a finite equilibrium size of skyrmions. Small skyrmions are energetically unfavorable due to large exchange gradients, whereas large skyrmions have a higher Zeeman energy. The DMI lowers the energy of twisted configurations and thereby enables, for sufficiently strong DMI, the formation of a stable skyrmion with finite radius. The resulting profile $\Theta(\rho)$, discussed in Sec. 2.4.1, reflects this balance through its characteristic shape and length scale.

Notably, the final expression for the topological charge depends only on the boundary conditions and the vorticity m , but is independent of the helicity φ_0 introduced in Eq. (2.23). This reflects the fact that the topological charge characterizes the global mapping of the magnetization field onto the unit sphere and is therefore insensitive to the detailed in-plane rotational structure of the texture. Consequently, skyrmions with differ-

ent helicities belong to the same topological class and share the same value of $\mathcal{Q}$, despite exhibiting distinct internal configurations. The role of the helicity in determining these internal structures will be discussed in the following section.

2.4.3. Helicity

The internal structure of skyrmions and antiskyrmions is parametrized by the polar angle Θ and the azimuthal angle Φ , as discussed in Sec. 2.4.1. As shown in Figure 2.4, the polar angle $\Theta(\rho)$ describes the radial profile of the texture, interpolating from $\Theta(0) = \pi$ at the skyrmion center to $\Theta(\infty) = 0$ in the ferromagnetic background. In contrast, the azimuthal angle $\Phi(\varphi)$ describes the in-plane rotation of the magnetization field and is therefore characterized by the vorticity m and the helicity φ_0 [30].

The pictograms in Figure 2.6 illustrate the different types of skyrmions and anti-skyrmions by showing the in-plane component of the magnetization. They can be interpreted as a top-down view of the magnetic texture, where the skyrmion center is located at the center of each pictogram and the arrows indicate the in-plane magnetization direction. The skyrmions shown in the upper row correspond to $m = 1$ in Eq. (2.23), where the helicity φ_0 determines the rotational sense of the in-plane magnetization. For Néel-type skyrmions ($\varphi_0 = 0$ and $\varphi_0 = \pi$), the magnetization points along the radial direction, whereas for Bloch-type skyrmions ($\varphi_0 = \pi/2$ and $\varphi_0 = 3\pi/2$), it is oriented perpendicular to it. More specifically, for $\varphi_0 = 0$ the magnetization points radially outward from the skyrmion center, while for $\varphi_0 = \pi$ it points inward. In contrast, for $\varphi_0 = \pi/2$ the magnetization rotates counterclockwise around the skyrmion center, whereas for $\varphi_0 = 3\pi/2$ the rotation is clockwise. To distinguish between these structures, we adopt the naming convention that the cases $\varphi_0 = 0$ and $\varphi_0 = \pi$ are referred to as Néel- and anti-Néel-type skyrmions, respectively, while the cases $\varphi_0 = \pi/2$ and $\varphi_0 = 3\pi/2$ are referred to as Bloch-

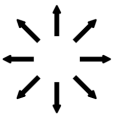
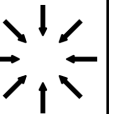
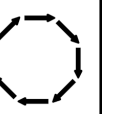
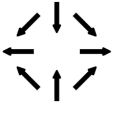
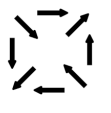
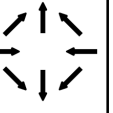
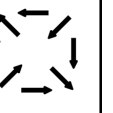
φ_0	0	$\pi/2$	π	$3\pi/2$
Skyrmion $m = 1$				
Antiskyrmion $m = -1$				

Figure 2.6.: Pictograms illustrating the in-plane magnetization of skyrmions and anti-skyrmions for different values of the helicity φ_0 . The arrows represent the in-plane component of the magnetization, while the skyrmion center is located at the center of each pictogram. Upper row: skyrmions ($m = 1$) for varying helicities, showing Néel-type ($\varphi_0 = 0$), Bloch-type ($\varphi_0 = \pi/2$), anti-Néel-type ($\varphi_0 = \pi$), and anti-Bloch-type ($\varphi_0 = 3\pi/2$) windings. Lower row: antiskyrmions ($m = -1$), where changes in helicity correspond to a global rotation of the entire texture rather than a modification of its internal structure.

and anti-Bloch-type skyrmions. Accordingly, the skyrmion displayed in Figure 2.3 (a) corresponds to a Néel-type skyrmion and the one shown in Figure 2.3 (b) to a Bloch-type skyrmion.

For antiskyrmions, corresponding to $m = -1$, the situation is qualitatively different. In this case, varying the helicity does not alter the internal structure of the texture, but instead corresponds to a global rotation of the entire configuration [54]. This is a direct consequence of the fact that an antiskyrmion inherently exhibits regions with Néel-, Bloch-, anti-Néel-, and anti-Bloch-like windings, depending on the azimuthal angle. This structure is illustrated by the pictograms in the lower row of Figure 2.6, which also show the reduced twofold rotational symmetry of antiskyrmions compared to the full rotational symmetry of skyrmions [13, 55].

2.4.4. Radius and triangular approximation of the radial profile

In Sec. 2.4.1, it was shown that the radial profile $\Theta(\rho)$ can be obtained by numerically solving the differential equation given in Eq. (2.25). However, a closed-form analytical expression that solves this equation is not known. Nevertheless, approximate expressions have been proposed that reproduce the numerical profile with high accuracy. These are based on descriptions of magnetic domain walls [70, 71, 112, 113] and a commonly used parametrization is given by [71]

$$\Theta(\rho) = 2 \arctan \left[\frac{\sinh(\rho_0/w)}{\sinh(\rho/w)} \right], \quad (2.35)$$

where ρ_0 denotes the skyrmion radius and w the domain wall width. Here, the radius is defined as the distance from the skyrmion center $\rho = 0$ to the contour where the out-of-plane magnetization vanishes, i.e., $n_z(\rho_0) = 0$. This expression captures both the exponential decay into the ferromagnetic background and, due to the finite domain wall width, the characteristic shoulder of the profile, appearing in Figure 2.4.

Despite its accuracy, the complexity of Eq. (2.35) prevents a fully analytical treatment of the problem pursued in this work. Therefore, we employ a simpler approximation, originally proposed in 1989 by Bogdanov and Yablonskii [18], which we refer to as the triangular ansatz. It is defined using the Heaviside function $\mathcal{H}(x)$ as

$$\Theta(\rho) = (\pi - \xi)\mathcal{H}(\pi - \xi), \quad \text{with} \quad \xi = \frac{\pi\rho}{2\rho_0}. \quad (2.36)$$

Compared to the original definition in Ref. [18], we introduce a factor of 1/2 in ξ such that the profile decreases linearly from $\Theta(0) = \pi$ to $\Theta(2\rho_0) = 0$. This ensures that the condition $\Theta(\rho_0) = \pi/2$, and thus $n_z(\rho_0) = 0$, coincides with the definition of the skyrmion radius used in Eq. (2.35). This is also depicted in Figure 2.7, where the upper panel shows the exact radial profile $\Theta(\rho)$ as the green line with dots and the triangular approximation of this profile is plotted with the orange line. The condition for the definition of the skyrmion

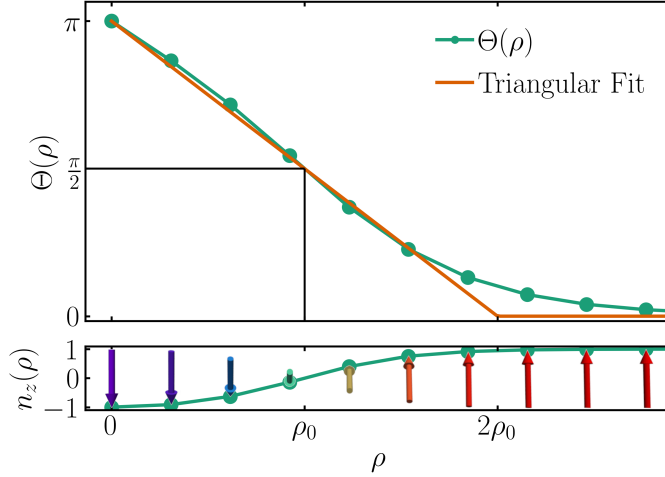


Figure 2.7.: Radial profile of a Bloch-type skyrmion. Upper panel: The green line with dots represents the exact $\Theta(\rho)$ -profile, while the orange line depicts the triangular profile that approximates the real profile. The skyrmion radius is defined as the radial position where the z -component vanishes, i.e., $n_z(\rho_0) = 0$ and $\Theta(\rho_0) = \pi/2$, as indicated by the black rectangle. Lower panel: The green line with dots depicts the z -component as a function of the radial distance ρ . The arrows represent the projected magnetic moments, with the blue arrow in the center of the skyrmion at $\rho = 0$ pointing into the $-z$ -direction and the red arrows for $\rho \rightarrow \infty$ pointing along the $+z$ -direction, indicating the ferromagnetic background. [Figure adapted from Ref. [114]]

radius is then indicated by the black rectangle. Additionally, the lower panel depicts the z -component of the skyrmion in radial direction, obtained via $n_z(\rho) = \cos[\Theta(\rho)]$, with the arrows indicating the projected magnetic moments, that point in the skyrmion center downwards (blue arrows at $\rho = 0$) and transition into the ferromagnetic background far away from the center (red arrows for $\rho \rightarrow \infty$).

The triangular ansatz for the radial profile allows for fully analytical expressions of the skyrmion energy and enables tractable dynamical calculations within the LLG framework. For antiskyrmions, the situation is more intricate compared to the skyrmion case, but their analytical treatment is still based on and enabled by the triangular ansatz. Despite its usefulness, this approximation has clear limitations. In particular, it imposes a fixed relation between the skyrmion radius and the effective wall width, whereas in reality the latter is determined by the competition between exchange, DMI, and, if present, anisotropy contributions [71, 115]. Furthermore, the smooth asymptotic tails of the exact solution are neglected. Consequently, the triangular ansatz provides only a qualitative description of the internal structure of skyrmions and antiskyrmions, as well as of their dynamics studied in Chapters 3 and 4. Quantitative results, which incorporate all internal deformations, are therefore obtained from full numerical simulations, while the analytical approach primarily serves to provide physical insight into the underlying mechanisms governing the dynamics.

2.5. Collective coordinate approach

Having established the LLG equation, which governs the dynamics of magnetic textures, as well as the properties of skyrmions and antiskyrmions in the previous sections, we now discuss how such systems can be treated analytically. In the continuum limit derived in Sec. 2.3, the LLG equation describes the time evolution of a general continuous magnetization field $\mathbf{n}(\mathbf{r}, t)$, which in principle possesses infinitely many degrees of freedom. This renders a fully analytical treatment generally intractable.

A powerful approach to overcome this complexity exploits the fact that the dynamics of topological magnetic textures often exhibit a separation of timescales between the fast relaxation of internal degrees of freedom and the slower evolution of a few macroscopic parameters. When such a separation holds, the complex, infinite-dimensional dynamics of the magnetization field can be effectively reduced to a low-dimensional system describing only the soft modes of the structure, i.e., modes with long relaxation times [116]. This forms the essence of the collective coordinate approach. The central idea is to parametrize the magnetization field in terms of a set of time-dependent variables as

$$\mathbf{n}(\mathbf{r}, t) = \mathbf{n}(\mathbf{r}, \mathbf{q}(t)), \quad \mathbf{q}(t) = \{q_1(t), q_2(t), \dots\}, \quad (2.37)$$

where the coordinates $\mathbf{q}(t)$ describe the dominant, low-energy modes of the system. Restricting the number of these modes allows for a drastic simplification of the problem and enables the derivation of a limited set of coupled differential equations of motion [116].

A systematic derivation of equations of motion for collective coordinates was first proposed by Thiele [117, 118], who showed that for rigid magnetic bubble domains, the LLG equation can be projected onto its translational modes. The resulting Thiele equation, given by

$$\mathbf{G} \times \dot{\mathbf{R}} + \alpha \mathcal{D} \dot{\mathbf{R}} + \mathbf{F} = 0, \quad (2.38)$$

expresses the balance of forces acting on the texture, involving the gyrocoupling vector $\mathbf{G}$, the Gilbert damping α and dissipation tensor $\mathcal{D}$, and an external force $\mathbf{F}$. Within this framework, the collective coordinate is the position $\mathbf{R}$ of the texture, while its internal structure is assumed to remain rigid during motion. This concept has been generalized to describe the dynamics of magnetic configurations in terms of a finite set of collective coordinates associated with the soft modes of the system [119–121].

For example, the dynamics of field- and current-driven domain walls have been successfully described within this framework [122]. In these systems, the collective coordinate approach captures the essential features of the motion, such as the velocity and internal tilting of the wall, in terms of a small number of parameters. The Thiele approach has also been applied to skyrmions, for instance to study their current-driven motion [48, 123], where it naturally accounts for the transverse deflection of skyrmions, commonly referred to as the skyrmion Hall effect [108, 124]. In addition, mechanisms such as spin-transfer torque induced rotations of the magnetic texture have been analyzed within this framework

[121].

A closely related formulation can be obtained within a Hamiltonian framework, where the collective coordinates are treated as canonical variables. In this picture, the dynamics follow from an effective energy depending on the collective coordinates, together with a nontrivial geometric structure originating from the spin Berry phase [125]. Within this formalism, both ferromagnetic and antiferromagnetic domain wall dynamics [126] as well as skyrmion and antiskyrmion breathing modes have been investigated [125].

In the subsequent chapters, we adopt a conceptually similar strategy to describe the dynamics of shrinking skyrmions and antiskyrmions, but without assuming a rigid magnetization profile. Here, the time dependence of the magnetization field enters explicitly through the collective coordinates. Applying the chain rule yields

$$\frac{\partial \mathbf{n}}{\partial t} = \sum_k \frac{\partial \mathbf{n}}{\partial q_k} \dot{q}_k, \quad (2.39)$$

which expresses the dynamics of the full field in terms of the time evolution of the reduced set of variables $\mathbf{q}(t)$.

For skyrmions, we consider spatially stationary configurations, such that the position remains constant, and choose the skyrmion radius and helicity as the relevant time-dependent collective coordinates. The magnetization field is then expressed using the triangular ansatz introduced in Sec. 2.4.4, leading to explicit contributions proportional to $\dot{\rho}_0$ and $\dot{\varphi}_0$ in Eq. (2.39). Substituting the parametrized magnetization field into the LLG equation in Eq. (2.21) yields a second expression for $\partial \mathbf{n} / \partial t$. Equating this expression with Eq. (2.39) and performing a spatial integration with appropriate weighting functions results in a closed set of coupled differential equations governing the shrinking dynamics of skyrmions.

For antiskyrmions, the situation is qualitatively similar but more intricate, as they exhibit pronounced shape deformations in systems with isotropic bulk DMI [54]. To capture this behavior, we introduce an elliptic ansatz characterized by the two semi-axes and a rotation angle describing the orientation of the structure in the plane. Together with the helicity, these parameters form the set of collective coordinates, enabling the description of both internal deformations and rotational dynamics during the shrinking process.

The resulting coupled nonlinear ordinary differential equations form the basis of the subsequent analysis, allowing us to systematically investigate how material parameters, external fields, and the value of the lattice constant influence the shrinking dynamics of these topological textures. The numerical solutions of these coupled equations of motion are obtained by integrating the system using the Runge-Kutta method [127].

3. Dynamics of skyrmion shrinking

This chapter is based on Ref. [114], in which the results described here have been published. An important aspect in the context of spintronic devices based on individual skyrmions in thin-film materials is the understanding of the stability of skyrmions and their possible decay dynamics. In the absence of stabilizing mechanisms, an unstable skyrmion is generally expected to shrink over time, with its radius decreasing until it reaches the order of the lattice constant. At this point, the skyrmion undergoes a rapid collapse, which is often treated as effectively instantaneous [67]. The collapse dynamics have been studied in relation to the initial skyrmion size [68] and the presence or absence of an external magnetic field [69]. For sufficiently large skyrmions in the presence of a magnetic field, the radius is found to decrease exponentially in time, whereas for smaller skyrmions, the collapse time scales with the fourth power of their size, with a logarithmic correction [68, 69]. In this chapter, we investigate the stability and shrinking behavior of Bloch-type skyrmions in systems with either vanishing or finite isotropic bulk DMI.

3.1. The energy of rotationally symmetric skyrmions

Skyrmions are rotationally symmetric in systems with isotropic DMI and therefore, we can reduce the problem such that the polar angle is just a function of the radial coordinate $\Theta(\rho)$ and the azimuthal angle is a function of the azimuthal coordinate $\Phi(\varphi)$ [13, 30]. The latter one is given by Eq. (2.23) with $m = 1$ to obtain a skyrmion configuration. Since the exact functional dependence of $\Theta(\rho)$ cannot be expressed by known functions, we employ the simplistic form of the triangular profile in Eq. (2.36) [18]. This approach enables us to study a circular skyrmion analytically, such that we can determine how its total energy depends on its radius ρ_0 and its helicity φ_0 . Using this triangular approximation, we determine its energetically preferred configuration in an environment exhibiting isotropic bulk DMI. We then compare the results with those obtained using the domain-wall approximation in Eq. (2.35), which is more accurate but not analytically tractable.

The magnetization field can be expressed for a general polar angle $\Theta(\rho)$ together with the azimuthal angle $\Phi(\varphi)$ as

$$\mathbf{n}(\rho, \varphi) = \begin{pmatrix} \sin \Theta(\rho) \cos \Phi(\varphi) \\ \sin \Theta(\rho) \sin \Phi(\varphi) \\ \cos \Theta(\rho) \end{pmatrix}. \quad (3.1)$$

The energy density $\mathcal{E} = \mathcal{E}(\rho, \Theta(\rho), \Phi(\varphi))$ is the integrand of Eq. (2.34) and can be evaluated by use of Eq. (3.1) to represent a skyrmion configuration as

$$\mathcal{E} = \frac{J}{2} \left[\Theta'^2 + \frac{\sin^2 \Theta}{\rho^2} \right] + \tilde{D} \left[\Theta' + \frac{\sin(2\Theta)}{2\rho} \right] \sin \varphi_0 + \tilde{B} [1 - \cos \Theta]. \quad (3.2)$$

Here, the prime indicates the derivative with respect to ρ , i.e., $\Theta' = d\Theta/d\rho$. For the triangular skyrmion profile defined in Eq. (2.36), this derivative reads

$$\Theta'(\rho) = -\frac{\pi}{2\rho_0} [\mathcal{H}(\pi - \xi) + (\pi - \xi)\delta(\pi - \xi)] = \begin{cases} -\frac{\pi}{2\rho_0}, & 0 < \rho < 2\rho_0, \\ 0, & \rho > 2\rho_0, \end{cases} \quad (3.3)$$

where the delta-term vanishes, i.e., $(\pi - \xi)\delta(\pi - \xi) = 0$. In the following, the integration in Eq. (2.34) will be performed only over $0 \leq \rho \leq 2\rho_0$ and therefore, the Heaviside function $\mathcal{H}(\pi - \xi)$ can be omitted. Consequently, we set $\Theta = \pi - \xi$ and $\Theta' = -\pi/(2\rho_0)$. With this, the energy density for the triangular skyrmion becomes

$$\begin{aligned} \mathcal{E}(\xi, \rho_0, \varphi_0) = & \frac{J}{2} \left[\frac{\pi^2}{4\rho_0^2} + \frac{\pi^2 \sin^2 \xi}{4\rho_0^2 \xi^2} \right] \\ & - \tilde{D} \left[\frac{\pi}{2\rho_0} + \frac{\pi \sin(2\xi)}{2\rho_0 2\xi} \right] \sin \varphi_0 + \tilde{B} [\cos \xi + 1]. \end{aligned} \quad (3.4)$$

Here, $\xi(\rho) = \pi\rho/(2\rho_0)$ contains the dependence on the radial coordinate. The total energy is obtained from the spatial integral over the energy density. To simplify the radial integration, we perform a change of variables from ρ to ξ , which maps the interval $\rho \in [0, 2\rho_0]$ onto $\xi \in [0, \pi]$. To this end, we rewrite

$$\xi = \frac{\pi\rho}{2\rho_0} \Leftrightarrow \rho = \frac{2\rho_0}{\pi}\xi, \quad (3.5)$$

such that the corresponding integration factor is given by

$$d\rho = \frac{2\rho_0}{\pi} d\xi. \quad (3.6)$$

Using these relations, the radial integration becomes

$$\int_0^{2\rho_0} \rho d\rho = \frac{4\rho_0^2}{\pi^2} \int_0^\pi \xi d\xi. \quad (3.7)$$

Accordingly, the total energy can be expressed via

$$E = E(\rho_0, \varphi_0) = \frac{4\rho_0^2}{\pi^2} \int_0^{2\pi} \int_0^\pi \xi \mathcal{E}(\xi, \rho_0, \varphi_0) d\xi d\varphi, \quad (3.8)$$

and yields an energy that depends only on the skyrmion radius ρ_0 and helicity φ_0 in the

explicit form [114]

$$E(\rho_0, \varphi_0) = J \left[\frac{\pi^3}{2} + \frac{\pi}{2} (\ln(2\pi) + C - \text{Ci}(2\pi)) \right] - 2\pi^2 \rho_0 \tilde{D} \sin \varphi_0 + \left(\pi - \frac{4}{\pi} \right) 4\rho_0^2 \tilde{B}, \quad (3.9)$$

where C is the Euler-Mascheroni constant and Ci the cosine integral [128]. The term in square brackets evaluates to 19.3322.

The exchange contribution is independent of both the skyrmion radius and helicity, whereas the Zeeman term depends quadratically on the radius. The DMI term, however, scales linearly with ρ_0 and exhibits an explicit dependence on the helicity φ_0 via the sine term. Given the bulk DMI, this term stabilizes Bloch-type skyrmions ($\varphi_0 = \pi/2$), since $\sin(\pi/2) = 1$, thereby lowering the total energy and creating together with the Zeeman term an energy minimum at a finite radius. The total energy is constructed such, that the homogeneous ferromagnetic state is at zero energy, i.e., $E_{\text{ferr}} = 0$. Therefore, a sufficiently large DMI can yield a skyrmion of finite size that is energetically below the ferromagnetic background. For large radii, however, the Zeeman term dominates and drives the skyrmion energy above that of the ferromagnetic state. Néel- and anti-Néel-type skyrmions are degenerate, as $\sin 0 = \sin \pi = 0$, such that the DMI contribution vanishes and for the anti-Bloch-type configuration, the DMI term increases the total energy because $\sin(3\pi/2) = -1$. Hence, the latter three configurations (Néel-, anti-Néel-, and anti-Bloch-type) are always energetically higher than the homogeneous ferromagnetic state. This behavior is illustrated in Figure 3.1, which shows the energy landscape of Eq. (3.9) for $\tilde{D} = 0.2$ and $\tilde{B} = 0.02$ (in units of J). The pictograms next to the plot indicate the skyrmion configurations for the given helicity, where green and red represent Bloch- and anti-Bloch-type skyrmions, respectively, and orange corresponds to Néel- and anti-Néel-type configurations. Throughout this work, the reference lattice constant a_{ref} is used as the fundamental unit of length. The lattice spacing is therefore given by aa_{ref} , where a denotes the dimensionless lattice constant. Consequently, quantities such as the skyrmion radius ρ_0 are expressed in units of a_{ref} .

For the Bloch-type skyrmion configuration, the radius ρ_0^{st} that minimizes the energy of the triangular skyrmion ansatz is given by [114]

$$\rho_0^{\text{st}} = \frac{\pi^3 \tilde{D}}{4(\pi^2 - 4) \tilde{B}}. \quad (3.10)$$

This expression corresponds to the energy-minimizing radius reported in Ref. [129], in our case for vanishing anisotropy. With increasing DMI strength, ρ_0^{st} increases, whereas it decreases with increasing external magnetic field.

Although the triangular ansatz is analytically convenient, it constitutes a rather rough approximation of the actual skyrmion profile, as it neglects both the domain wall-like shoulder and the smooth exponential decay into the ferromagnetic background at large distances from the skyrmion center, as discussed in Sec. 2.4.4. To get a better understand-

ing of the accuracy of the triangular ansatz, we compare its energy dependence on the skyrmion radius with that obtained from the more accurate domain wall approximation shown in Eq. (2.35) [71]. Figure 3.2 shows the energy of the triangular Bloch-type skyrmion

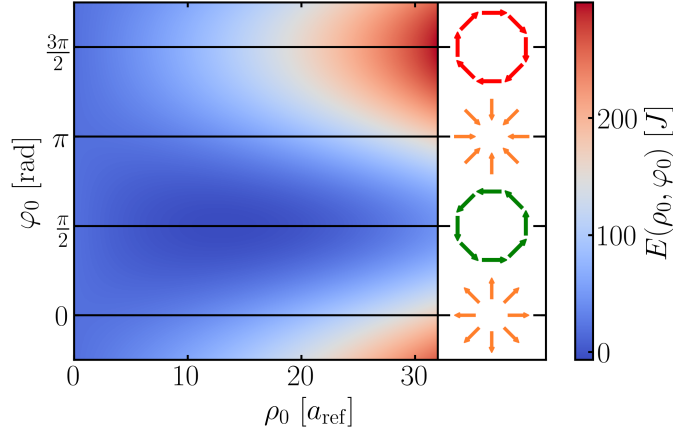


Figure 3.1.: Energy landscape $E(\rho_0, \varphi_0)$ of the triangular skyrmion obtained from Eq. (3.9), with colors indicating the total energy. The pictograms next to the plot illustrate the corresponding skyrmion configurations with green and red representing Bloch- and anti-Bloch-type skyrmions, respectively, while orange corresponds to Néel- ($\varphi_0 = 0$) and anti-Néel-type ($\varphi_0 = \pi$) configurations. The Bloch-type skyrmion exhibits an energy minimum at a finite radius below that of the homogeneous ferromagnetic state ($E_{\text{ferr}} = 0$). In contrast, the Néel-, anti-Néel-, and anti-Bloch-type configurations remain energetically higher, with the anti-Bloch-type configuration being the highest in energy. The two Néel-type configurations are degenerate in energy. Parameters: $\tilde{D} = 0.2$, $\tilde{B} = 0.02$.

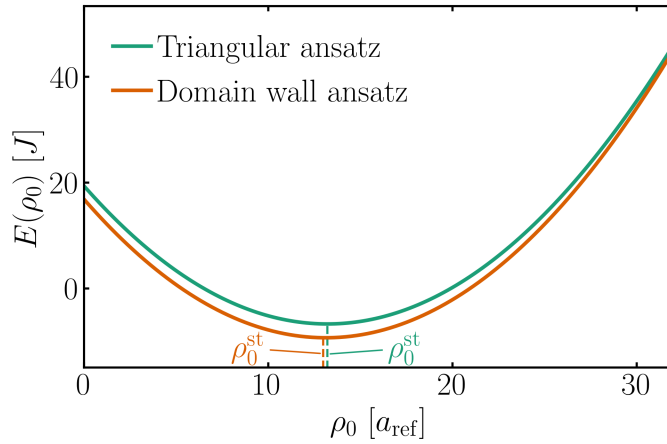


Figure 3.2.: Comparison of the energy of a Bloch-type skyrmion obtained from the triangular ansatz of Eq. (3.9) (green line) and from the domain wall approximation (orange line). The latter one is calculated via numerical integration of Eq. (3.2) with Eq. (2.35). The triangular ansatz slightly overestimates the energy compared to the domain wall approximation, but the overall energy curve shape and the location of the energy minimum are reproduced well. Parameters: $\tilde{D} = 0.2$, $\tilde{B} = 0.02$, $\varphi_0 = \pi/2$. [Figure adapted from Ref. [114]]

(green line) alongside that obtained from numerical integration of Eq. (3.2) using the domain wall approximation (orange line). In general, the triangular ansatz slightly overestimates the energy compared to the domain wall approximation, but the overall shape of the energy curve is well reproduced. In particular, the energetically most favorable skyrmion radii obtained from the minima of both curves are in very good agreement. This is illustrated by the dashed vertical lines, which mark the positions of the respective minima and highlight the close correspondence between the two approximations [114].

Hence, the close agreement between the two models suggests that the triangular ansatz constitutes a reliable approximation, suitable for describing the skyrmion shrinking dynamics analyzed in the following sections.

3.2. Derivation of the coupled equations of motion for the skyrmion shrinking

Since the skyrmion radius and helicity have proven to effectively describe skyrmion breathing modes in ferromagnetic materials with bulk DMI [125], we adopt these quantities as the collective coordinates to describe the shrinking dynamics of circular skyrmions within the framework of the triangular ansatz. Therefore, we derive a set of coupled non-linear differential equations by use of the LLG and the time derivative of the magnetization field.

The magnetization field $\mathbf{n}$ in Eq. (3.1), parametrized by the radius ρ_0 and helicity φ_0 , takes the form

$$\mathbf{n} = \begin{pmatrix} \sin \xi \cos \Phi \\ \sin \xi \sin \Phi \\ -\cos \xi \end{pmatrix}, \quad (3.11)$$

where the polar angle follows the triangular ansatz of Eq. (2.36) and the azimuthal angle is given by Eq. (2.23) with $m = 1$, corresponding to a skyrmion configuration.

To derive the equations of motion for the skyrmion radius and helicity, we calculate the time derivative of this magnetization field [114] denoted in the following as the left-hand side (LHS). The time dependence enters through $\rho_0(t)$ and $\varphi_0(t)$. Applying the chain rule yields

$$\dot{\mathbf{n}}^{\text{LHS}}(\rho_0(t), \varphi_0(t)) = \begin{pmatrix} -\xi \frac{\dot{\rho}_0}{\rho_0} \cos \xi \cos \Phi - \dot{\varphi}_0 \sin \xi \sin \Phi \\ -\xi \frac{\dot{\rho}_0}{\rho_0} \cos \xi \sin \Phi + \dot{\varphi}_0 \sin \xi \cos \Phi \\ -\xi \frac{\dot{\rho}_0}{\rho_0} \sin \xi \end{pmatrix}. \quad (3.12)$$

Furthermore, the LLG equation in Eq. (2.21) needs to be expressed in terms of the triangular ansatz to obtain the corresponding right-hand side (RHS) expressions. To this end, the effective magnetic field from Eq. (2.20) is evaluated accordingly, giving

$$\mathbf{B}_{\text{eff}} = \frac{\pi^2}{4\rho_0^2} \begin{pmatrix} J \cos(\varphi + \varphi_0) \left(\frac{\cos \xi}{\xi} - \sin \xi - \frac{\sin \xi}{\xi^2} \right) - \frac{4\tilde{D}\rho_0 \sin \xi \sin \varphi}{\pi} \\ J \sin(\varphi + \varphi_0) \left(\frac{\cos \xi}{\xi} - \sin \xi - \frac{\sin \xi}{\xi^2} \right) + \frac{4\tilde{D}\rho_0 \sin \xi \cos \varphi}{\pi} \\ J \left(\cos \xi + \frac{\sin \xi}{\xi} \right) - \frac{4\tilde{D}\rho_0}{\pi} \left(\cos \xi + \frac{\sin \xi}{\xi} \right) \sin \varphi_0 \end{pmatrix} + \tilde{B} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (3.13)$$

Plugging this expression into the LLG equation results in lengthy terms depending on ρ , φ , ρ_0 , and φ_0 . These are displayed in Appendix B together with more details on how to derive Eq. (3.13). From these derivations, we obtain a set of coupled non-linear differential equations governing the collective coordinates $\rho_0(t)$ and $\varphi_0(t)$, which are subsequently analyzed and solved numerically for both vanishing and finite DMI. The influence of the lattice constant, DMI strength, external magnetic field, and Gilbert damping on the dynamics is investigated in detail.

3.2.1. Radius equation of motion

To derive the dynamical equation for the skyrmion radius, we isolate $\dot{\rho}_0$ from the components of Eq. (3.12) using appropriate weighting functions and subsequent integration over the azimuthal coordinate φ [114].

The z -component depends already only on $\dot{\rho}_0$, and integrating over φ yields

$$\int_0^{2\pi} d\varphi [\dot{\mathbf{n}}]_z^{\text{LHS}} = -2\pi \frac{\dot{\rho}_0}{\rho_0} \xi \sin \xi. \quad (3.14)$$

As expected for a rotationally symmetric skyrmion, the x - and y -components vanish upon integration, because the radially symmetric azimuthal contributions cancel each other out around the circle. This leaves no net x - or y -component after full angular averaging, i.e.,

$$\begin{aligned} \int_0^{2\pi} d\varphi [\dot{\mathbf{n}}]_x^{\text{LHS}} &= 0, \\ \int_0^{2\pi} d\varphi [\dot{\mathbf{n}}]_y^{\text{LHS}} &= 0. \end{aligned} \quad (3.15)$$

However, weighting the x - and y -components by $\cos \Phi$ and $\sin \Phi$, respectively, and integrating over φ yields nonzero results for the terms containing $\dot{\rho}_0$, while the $\dot{\varphi}_0$ terms vanish. Subsequent summation of the x - and y -component then yields

$$\int_0^{2\pi} d\varphi [\dot{\mathbf{n}}]_x^{\text{LHS}} \cdot \cos \Phi + \int_0^{2\pi} d\varphi [\dot{\mathbf{n}}]_y^{\text{LHS}} \cdot \sin \Phi = -2\pi \frac{\dot{\rho}_0}{\rho_0} \xi \cos \xi. \quad (3.16)$$

To obtain the RHS expressions, we apply the same procedure to the corresponding components of the LLG equation displayed in Eqs. (B.10)-(B.12) in Appendix B. The derivations

yield for the z -component

$$\int_0^{2\pi} d\varphi [\dot{\mathbf{n}}]_z^{\text{RHS}} = \frac{a^2\alpha}{1+\alpha^2} \left(2\pi\tilde{B}\sin^2\xi + \tilde{D}\frac{2\pi^2}{\rho_0} \left[-\sin^2\xi\frac{1}{\alpha}\cos\varphi_0 - \frac{\sin^3\xi}{\xi}\sin\varphi_0 \right] \right. \\ \left. + J\frac{\pi^3}{2\rho_0^2} \left[\frac{\sin\xi}{\xi} - \frac{\sin^2\xi\cos\xi}{\xi^2} \right] \right), \quad (3.17)$$

and for the sum of the x - and y -components

$$\int_0^{2\pi} d\varphi [\dot{\mathbf{n}}]_x^{\text{RHS}} \cdot \cos\Phi + \int_0^{2\pi} d\varphi [\dot{\mathbf{n}}]_y^{\text{RHS}} \cdot \sin\Phi = \frac{a^2\alpha}{1+\alpha^2} \left(2\pi\tilde{B}\sin\xi\cos\xi \right. \\ \left. + \tilde{D}\frac{2\pi^2}{\rho_0} \left[-\sin\xi\cos\xi\frac{1}{\alpha}\cos\varphi_0 - \frac{\sin^2\xi\cos\xi}{\xi}\sin\varphi_0 \right] \right. \\ \left. + J\frac{\pi^3}{2\rho_0^2} \left[\frac{\cos\xi}{\xi} - \frac{\sin\xi\cos^2\xi}{\xi^2} \right] \right). \quad (3.18)$$

Note, that dividing Eqs. (3.14) and (3.17) derived from $[\dot{\mathbf{n}}]_z$ by $\sin\xi$ and Eqs. (3.16) and (3.18) derived from the sum of $[\dot{\mathbf{n}}]_x$ and $[\dot{\mathbf{n}}]_y$ by $\cos\xi$ results in the same expressions. Thus, equating Eqs. (3.14) and (3.17), or equivalently Eqs. (3.16) and (3.18), and subsequently rearranging, yields [114]

$$\dot{\rho}_0\xi = \frac{a^2\alpha}{1+\alpha^2} \left(-\rho_0\tilde{B}\sin\xi + \tilde{D}\pi \left[\sin\xi\frac{1}{\alpha}\cos\varphi_0 + \frac{\sin^2\xi}{\xi}\sin\varphi_0 \right] \right. \\ \left. - J\frac{\pi^2}{4\rho_0} \left[\frac{1}{\xi} - \frac{\sin\xi\cos\xi}{\xi^2} \right] \right). \quad (3.19)$$

To preserve the triangular shape of $\Theta(\rho)$, we project both sides of Eq. (3.19) onto the ξ -dependence of its left-hand side. This projection ensures that no additional ξ -dependence is introduced into the dynamics, effectively enforcing that $\Theta(\rho)$ retains its shape over time. The procedure yields a set of dimensionless coefficients c_i , which quantify the weighted contributions of the ξ -dependent terms through integration over the radial coordinate. Applying the change of variables for this integration done in Eq. (3.7), these coefficients evaluate to

$$c_1 = \frac{\int_0^\pi d\xi \xi^2 \sin\xi}{\int_0^\pi \xi d\xi \xi^2} = \frac{\pi^2 - 4}{\pi^4/4} \approx 0.241029, \\ c_2 = \frac{\pi \int_0^\pi d\xi \xi^2 \sin\xi}{\int_0^\pi \xi d\xi \xi^2} = \pi c_1 \approx 0.757215, \\ c_3 = \frac{\pi \int_0^\pi d\xi \xi \sin^2\xi}{\int_0^\pi \xi d\xi \xi^2} = \frac{\pi^3/4}{\pi^4/4} = \frac{1}{\pi} \approx 0.318310, \\ c_4 = \frac{\frac{\pi^2}{4} \int_0^\pi d\xi (\xi - \sin\xi\cos\xi)}{\int_0^\pi \xi d\xi \xi^2} = \frac{\pi^4/8}{\pi^4/4} = 0.5, \quad (3.20)$$

which finally leads to the non-linear ordinary differential equation governing the time evolution of the skyrmion radius [114]

$$\dot{\rho}_0 = \frac{a^2 \alpha}{1 + \alpha^2} \left(-c_1 \tilde{B} \rho_0 + c_3 \tilde{D} \sin \varphi_0 - c_4 J \frac{1}{\rho_0} \right) + \frac{a^2}{1 + \alpha^2} c_2 \tilde{D} \cos \varphi_0. \quad (3.21)$$

This dynamical equation shows that the skyrmion radius evolves due to the interplay of the Zeeman, exchange, and DMI contributions, with $\varphi_0(t)$ entering explicitly through the DMI term.

3.2.2. Helicity equation of motion

The dynamical equation for the skyrmion helicity can only originate from the x - and y -components of Eq. (3.12) [114], as these are the only components with explicit Φ -dependence. Weighting the x - and y -components by $\sin \Phi$ and $\cos \Phi$, respectively, and integrating over φ , isolates $\dot{\varphi}_0$ by subsequent subtraction of the components according to

$$\int_0^{2\pi} d\varphi [\dot{\mathbf{n}}]_x^{\text{LHS}} \cdot \sin \Phi - \int_0^{2\pi} d\varphi [\dot{\mathbf{n}}]_y^{\text{LHS}} \cdot \cos \Phi = -2\pi \dot{\varphi}_0 \sin \xi. \quad (3.22)$$

The same weighting yields for the LLG components

$$\begin{aligned} \int_0^{2\pi} d\varphi [\dot{\mathbf{n}}]_x^{\text{RHS}} \cdot \sin \Phi - \int_0^{2\pi} d\varphi [\dot{\mathbf{n}}]_y^{\text{RHS}} \cdot \cos \Phi = & \frac{a^2}{1 + \alpha^2} \left(-2\tilde{B}\pi \sin \xi \right. \\ & + 2\pi^2 \tilde{D} \frac{1}{\rho_0} \left[-\alpha \sin \xi \cos \varphi_0 + \frac{\sin^2 \xi}{\xi} \sin \varphi_0 \right] \\ & \left. + J \frac{\pi^3}{2\rho_0^2} \left[-\frac{1}{\xi} + \frac{\sin \xi \cos \xi}{\xi^2} \right] \right). \end{aligned} \quad (3.23)$$

Equating and rearranging these two equations yields [114]

$$\begin{aligned} \dot{\varphi}_0 \sin \xi = & \frac{a^2}{1 + \alpha^2} \left(\tilde{B} \sin \xi - \pi \tilde{D} \frac{1}{\rho_0} \left[-\alpha \sin \xi \cos \varphi_0 + \frac{\sin^2 \xi}{\xi} \sin \varphi_0 \right] \right. \\ & \left. + J \frac{\pi^2}{4\rho_0^2} \left[\frac{1}{\xi} - \frac{\sin \xi \cos \xi}{\xi^2} \right] \right). \end{aligned} \quad (3.24)$$

Since the helicity $\varphi_0(t)$ is assumed to be isotropic and therefore independent of the radial coordinate, we project both sides of Eq. (3.24) onto the ξ -dependence of its left-hand side. This projection yields another set of dimensionless coefficients c_i , representing the weighted contributions of the ξ -dependent terms, obtained through integration over the radial coordinate

$$\begin{aligned} c_5 &= \frac{\int_0^\pi d\xi \xi \sin^2 \xi}{\int_0^\pi \xi d\xi \sin^2 \xi} = 1, \\ c_6 &= \frac{\pi \int_0^\pi d\xi \xi \sin^2 \xi}{\int_0^\pi \xi d\xi \sin^2 \xi} = \pi, \\ c_7 &= \frac{\pi \int_0^\pi d\xi \sin^3 \xi}{\int_0^\pi \xi d\xi \sin^2 \xi} = \frac{16}{3\pi} \approx 1.697653, \end{aligned} \quad (3.25)$$

$$c_8 = \frac{\frac{\pi^2}{4} \int_0^\pi d\xi \left(\sin \xi - \frac{\sin^2 \xi \cos \xi}{\xi} \right)}{\int_0^\pi \xi d\xi \sin^2 \xi} \approx 1.709585. \quad (3.26)$$

The resulting non-linear ordinary differential equation describing the time evolution of the skyrmion helicity reads [114]

$$\dot{\varphi}_0 = \frac{a^2}{1 + \alpha^2} \left(c_5 \tilde{B} - \tilde{D} \frac{1}{\rho_0} [-\alpha c_6 \cos \varphi_0 + c_7 \sin \varphi_0] + c_8 J \frac{1}{\rho_0^2} \right). \quad (3.27)$$

Here, the dependencies on $\rho_0(t)$ enter through the DMI and exchange terms.

To summarize, the weighting procedure used to isolate both collective coordinates from the x - and y -components of Eq. (3.12) can be expressed in compact matrix form as

$$\int_0^{2\pi} d\varphi \left[\begin{pmatrix} \sin \Phi & -\cos \Phi \\ \cos \Phi & \sin \Phi \end{pmatrix} \begin{pmatrix} [\dot{\mathbf{n}}]_x^{\text{LHS}} \\ [\dot{\mathbf{n}}]_y^{\text{LHS}} \end{pmatrix} \right] = \begin{pmatrix} -2\pi \dot{\varphi}_0 \sin \xi \\ -2\pi \frac{\dot{\rho}_0}{\rho_0} \xi \cos \xi \end{pmatrix}. \quad (3.28)$$

This represents a rotation by $R(\pi/2 - \Phi)$ into the local frame defined by the azimuthal angle $\Phi(\varphi)$ [130]. The same transformation applies to the right-hand sides derived from the LLG equation. Projecting the resulting relations onto the respective ξ -dependencies of their left-hand sides ensures the preservation of both the triangular radial profile and the isotropic helicity throughout the time evolution. The dynamical equation for the radius, obtained from the z -component, requires no weighting and reproduces the same result as the one derived from the x - and y -components, confirming the robustness of this procedure.

3.3. Analysis and solution of the coupled equations of motion

Analyzing the structure of the equations of motion for the skyrmion radius and helicity in Eqs. (3.21) and (3.27), respectively, already provides insights into the general time evolution expected for the two collective coordinates [114].

In Eq. (3.21), it is evident that for large skyrmions the Zeeman term dominates the dynamics, since in this regime $c_1 \tilde{B} \rho_0 \gg c_4 J / \rho_0$. Consequently, the equation predicts an exponentially decreasing skyrmion radius,

$$\rho_0(t) \sim \rho_{0,0} \exp \left[-\frac{c_1 a^2 \alpha}{1 + \alpha^2} \tilde{B} t \right], \quad (3.29)$$

where $\rho_{0,0}$ is the initial skyrmion radius at $t = 0$. Note that $\tilde{B} = B/a^2$ and therefore, the shrinking rate is independent on the lattice constant a and depends on the external magnetic field strength, and the Gilbert damping, with larger values of both parameters leading to faster shrinking.

At small skyrmion sizes, the exchange term in Eq. (3.21) becomes dominant, as $c_4 J / \rho_0 \gg c_1 \tilde{B} \rho_0$ in this regime. The radius then follows a square-root dependence

$$\rho_0(t) \sim \sqrt{\frac{2c_4 a^2 \alpha}{1 + \alpha^2}} J(t_c - t), \quad (3.30)$$

driven by the exchange interaction strength, the lattice constant, and the Gilbert damping. Here, t_c denotes the critical collapse time at which the skyrmion radius vanishes. It is important to note that, within the continuum model, the topological charge is conserved [109]. Therefore, the vanishing of the radius does not imply the destruction of the topological object, after which only the topologically trivial ferromagnetic state with $Q = 0$ would remain. Instead, the skyrmion collapses into a delta-function-like object. Consequently, integration over space still yields $Q = -1$. Thus, the theory cannot capture the annihilation of the skyrmion, but rather describes how the skyrmion size evolves from an exponential decay at large radii to a square-root-like behavior at small radii until t_c is reached.

Overall, this shrinking behavior is modulated, particularly at intermediate skyrmion sizes, by the oscillatory DMI contribution, which depends sinusoidally on the time-dependent helicity $\varphi_0(t)$. The DMI enters Eq. (3.21) through two distinct terms, both proportional to the DMI strength and the lattice constant (recall that $\tilde{D} = D/a$). The term proportional to $\sin \varphi_0$ is dependent on the damping and thus represents a dissipative contribution to the dynamics, while the $\cos \varphi_0$ term remains finite even in the limit of vanishing damping. Together, they determine the helicity dependent modulation of the skyrmion shrinking. At vanishing Gilbert damping, the overall dynamics of the skyrmion radius is solely governed by the DMI term proportional to $\cos \varphi_0$ and thus determined by the time evolution of $\varphi_0(t)$.

Just as for the skyrmion radius, the dynamical equation for the helicity in Eq. (3.27) shows that its time evolution is dominated by the Zeeman term for large skyrmions. Consequently, it is primarily driven by the external magnetic field strength $\tilde{B}$, with only small corrections at finite damping constants α . This leads to a precessional behavior of the form

$$\varphi_0(t) \sim \varphi_{0,0} + \frac{c_5 a^2}{1 + \alpha^2} \tilde{B} t, \quad (3.31)$$

where $\varphi_{0,0}$ denotes the initial skyrmion helicity at $t = 0$.

For small skyrmion sizes, the exchange term becomes dominant. In this regime, combining Eq. (3.27) with the radius evolution in Eq. (3.30) yields a logarithmic time dependence of the helicity

$$\varphi_0(t) \sim -\frac{c_8}{2c_4 \alpha} \ln(t_c - t), \quad (3.32)$$

indicating a diverging precession frequency proportional to $1/\alpha$ as the skyrmion radius collapses at the critical time t_c .

The DMI gives rise to two terms in the dynamical equation for the helicity. These are both proportional to the DMI strength, the lattice constant, and inversely proportional to the skyrmion radius $\rho_0(t)$. Together, they set the helicity dependent modulation of the dynamics at intermediate radii. The term proportional to $\cos \varphi_0$ is multiplied by α and is

therefore dissipative, while the term proportional to the $\sin \varphi_0$ remains finite when $\alpha \rightarrow 0$. In this zero damping limit, the Zeeman and exchange term remain finite in Eq. (3.27) as well. At large skyrmion sizes, the Zeeman term dominates, driving a precession of the helicity linear in time that is superimposed with a sinusoidal modulation originating from the DMI. Consequently, when $\alpha \rightarrow 0$, the skyrmion radius $\rho_0(t)$ exhibits a breathing mode, driven by the DMI contribution in Eq. (3.21) through the precession of the helicity.

It is well established that, in the absence of other stabilization mechanisms, the DMI is essential for stabilizing skyrmions [13, 30], consistent with our energy analysis of the triangular skyrmion in Sec. 3.1. Several studies have demonstrated that the DMI strength can be externally tuned by electric fields or applied voltages [131–133]. In the following, we therefore investigate the skyrmion shrinking dynamics for varying DMI strengths. We first examine the shrinking for vanishing DMI, where the non-linear ordinary differential equation for the radius (Eq. (3.21)) can be solved analytically. We then investigate the finite-DMI regime by numerically integrating the coupled differential equations (3.21) and (3.27). In this regime, we first study the case of vanishing Gilbert damping to understand the breathing behavior of the skyrmion, before addressing its stability and shrinking in the dissipative regime.

3.3.1. Skyrmion shrinking at vanishing DMI

At vanishing DMI, the dynamical equation for the skyrmion radius (Eq. (3.21)) decouples from the helicity equation (Eq. (3.27)). Thus, if the DMI is quenched to zero at time $t = 0$, i.e., $\tilde{D} \rightarrow 0$, the differential equation for the skyrmion radius can be solved analytically, yielding [114]

$$\rho_0(t) = \left[\rho_{0,0}^2 \exp \left(-\frac{2c_1 a^2 \alpha}{1 + \alpha^2} \tilde{B} t \right) + \frac{c_4 J}{c_1 \tilde{B}} \left(\exp \left(-\frac{2c_1 a^2 \alpha}{1 + \alpha^2} \tilde{B} t \right) - 1 \right) \right]^{\frac{1}{2}}. \quad (3.33)$$

The critical time t_c at which the skyrmion radius vanishes follows as

$$t_c = \frac{1 + \alpha^2}{2c_1 a^2 \alpha \tilde{B}} \ln \left[1 + \rho_{0,0}^2 \frac{c_1 \tilde{B}}{c_4 J} \right]. \quad (3.34)$$

Equation (3.33) describes an exponential decay of $\rho_0(t)$ at early times, where the first term in the brackets dominates and the dynamics are governed by the exponential prefactor. As $t \rightarrow t_c$, a Taylor expansion of the expression in brackets around t_c yields a linear dependence on $t_c - t$. This results in the scaling law $\rho_0(t) \propto \sqrt{t_c - t}$, which recovers the square-root behavior of Eq. (3.30). Moreover, Eq. (3.34) reveals, when considering $\tilde{B} = B/a^2$, that the critical time t_c increases logarithmically with decreasing lattice constant a . For small a

$$t_c \sim \ln \left(\frac{c_1 B}{c_4 J} \right) + 2 \ln \left(\frac{\rho_{0,0}}{a} \right), \quad (3.35)$$

with the first term being constant, such that the critical time follows $t_c \sim \ln(\rho_{0,0}/a)$ [114]. This implies that, with decrease lattice constant, the radius vanishes at a later time t_c and in the limit $a \rightarrow 0$ the critical time diverges, i.e., $t_c \rightarrow \infty$.

In the following, we discuss the solutions of the coupled differential equations. The DMI strength $\tilde{D}$ and the external magnetic field $\tilde{B}$ are expressed in units of the exchange constant J . Unless stated otherwise, the initial conditions for the skyrmion radius and helicity are set to $\rho_{0,0} = 100a$ and to a Bloch-type configuration, $\varphi_{0,0} = \pi/2$, respectively.

Figure 3.3 (a) shows the time evolution of the skyrmion radius obtained from the analytical solution of Eq. (3.33). The corresponding evolution of the helicity, shown in Figure 3.3 (b), is obtained from numerical integration of Eq. (3.27) using a Runge-Kutta method of order 5(4) [127], which uses the time-dependent radius from Eq. (3.33) as input. The parameters were set to $\tilde{D} = 0$, $\tilde{B} = 0.02$, $\alpha = 0.1$ for different values of the lattice constant a . The shrinking of the skyrmion radius shows the predicted initial exponential

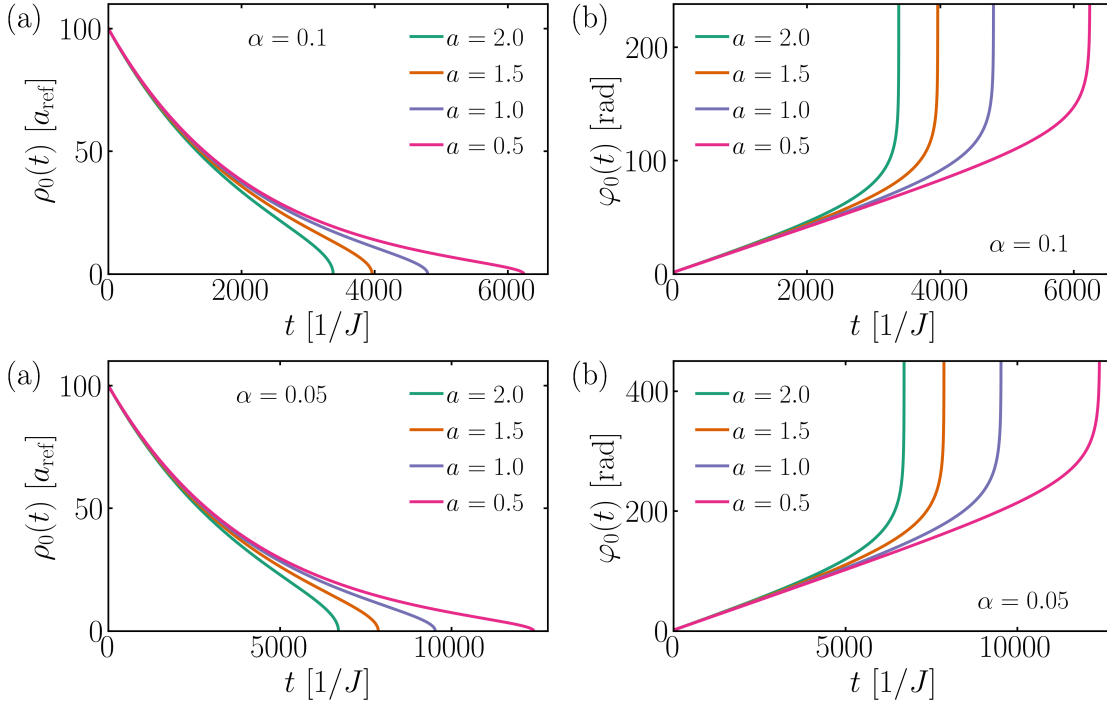


Figure 3.3.: Panel (a) shows the time evolution of the skyrmion radius $\rho_0(t)$ obtained from Eq. (3.33) and panel (b) shows the corresponding time evolution of its helicity $\varphi_0(t)$ obtained from numerical integration of Eq. (3.27) for different values of the lattice constant a in the zero DMI regime at $\alpha = 0.1$. At large skyrmion sizes, the radius follows an initial exponential decay that crosses over to a square-root behavior close to its collapse at the critical time t_c . The helicity exhibits linear precession at large radii and logarithmic divergence as $t \rightarrow t_c$. Smaller lattice constants result in a later collapse of the skyrmion radius, reflecting the logarithmic dependence of t_c on a . Panels (c) and (d) show the same dynamics at smaller Gilbert damping $\alpha = 0.05$, where the overall slower dynamics preserve the same characteristic features as in panels (a) and (b). Parameters: $\tilde{D} = 0$, $\tilde{B} = 0.02$.

decay, which crosses over to the square-root behavior close to the collapse at t_c . The time evolution of the helicity similarly shows the predicted linear precession at large radii, driven by the external magnetic field, and a logarithmic divergence as $t \rightarrow t_c$, in agreement with Eqs. (3.31) and (3.32). The logarithmic dependence of t_c on the lattice constant a , as seen in Eq. (3.35), is also evident, with smaller lattice constants corresponding to a later collapse of the skyrmion radius.

Panels (c) and (d) show the same dynamics at reduced Gilbert damping. Here, the slower energy dissipation leads to a delayed collapse of the radius, in agreement with Eq. (3.34). As expected, the characteristic features of the time evolution of the radius and helicity are preserved.

3.3.2. Skyrmion breathing dynamics at finite DMI and vanishing Gilbert damping

In Ref. [125], the authors studied the breathing dynamics, characterized by periodic expansion and contraction of the skyrmion radius, in systems with the DMI being large enough to stabilize the skyrmion. They describe two distinct types of breathing dynamics with one being an oscillatory breathing mode. This is the breathing mode where the helicity oscillates around its equilibrium configuration, that is typically referred to in the literature [13, 134, 135]. The other mode is a rotational breathing mode, at higher energies, where the helicity precesses, such that the skyrmion continuously switches between Bloch- and Néel-type configurations. For the rotational mode they predict a linear energy dissipation rate and for the oscillatory mode an exponential one.

To test the robustness of our analytical model, we investigate these breathing modes within our framework. Therefore, we study our system at large DMI and vanishing Gilbert damping. At finite DMI, the equation for the skyrmion radius couples to the time evolution of the helicity. In general, the coupled system of Eqs. (3.21) and (3.27) cannot be solved analytically. Therefore, we solve the system of equations numerically in the following.

As discussed above, in the limit of vanishing Gilbert damping only the cosine DMI term contributes to the skyrmion-radius dynamics, so Eq. (3.21) becomes

$$\dot{\rho}_0 = a^2 c_2 \tilde{D} \cos \varphi_0, \quad (3.36)$$

whereas the helicity equation (3.27) reduces to

$$\dot{\varphi}_0 = a^2 \left(c_5 \tilde{B} - c_7 \tilde{D} \frac{1}{\rho_0} \sin \varphi_0 + c_8 J \frac{1}{\rho_0^2} \right). \quad (3.37)$$

The DMI and exchange contributions scale as $1/\rho_0$ and $1/\rho_0^2$, respectively. Because the DMI term enters with opposite sign to the Zeeman and exchange terms, there can be a range of radii for which the DMI counteracts the other torques and the helicity becomes

quasi-stationary, i.e., $\varphi_0 \approx \text{const}$. Setting $\dot{\varphi}_0 = 0$ yields the quadratic form

$$c_5 \tilde{B} \rho_0^2 - c_7 \tilde{D} \rho_0 \sin \varphi_0 + c_8 J = 0, \quad (3.38)$$

with the roots

$$\rho_0^\pm = \frac{c_7 \tilde{D} \sin \varphi_0 \pm \sqrt{(c_7 \tilde{D} \sin \varphi_0)^2 - 4c_5 \tilde{B} c_8 J}}{2c_5 \tilde{B}}. \quad (3.39)$$

Real roots require $(c_7 \tilde{D} \sin \varphi_0)^2 \geq 4c_5 \tilde{B} c_8 J$ and, to find physical results, at least one root must be positive. For Néel- and anti-Néel-type helicities ($\varphi_0 = 0, \pi$), the sine evaluates to $\sin \varphi_0 = 0$, such that the discriminant reduces to $-4c_5 \tilde{B} c_8 J < 0$, yielding no physical solutions. For the anti-Bloch-type configuration ($\varphi_0 = 3\pi/2$) the roots are also always negative and as such they give no physical solution. The absence of physical solutions for the two Néel-type and the anti-Bloch-type configurations implies that a skyrmion initially prepared in one of these configurations will not stay in this configurations for any given initial radius. However, a Bloch-type helicity ($\varphi_0 = \pi/2$) can yield two positive roots $\rho_0^- < \rho_0^+$ under the condition that $(c_7 \tilde{D})^2 \geq 4c_5 \tilde{B} c_8 J$. This implies that it is possible to prepare a Bloch-type skyrmion at $t = 0$ at a specific radius, such that $\varphi_0(t) = \pi/2$ and also $\rho_0(t) = \text{const}$. This is consistent with the bulk DMI underlying the model that stabilizes only Bloch-type skyrmions.

To further analyze the Bloch-type configuration, we differentiate $\dot{\varphi}_0$ with respect to φ_0 and evaluate it at $\varphi_0 = \pi/2$ yielding

$$\left. \frac{\partial \dot{\varphi}_0}{\partial \varphi_0} \right|_{\varphi_0=\pi/2} = -\frac{a^2 c_7 \tilde{D} \cos \pi/2}{\rho_0} = 0. \quad (3.40)$$

This shows that the linear term in φ_0 vanishes, such that a linear stability analysis is inconclusive. The character of the stationary point is determined by the coupled non-linear dynamics of $\rho_0(t)$ and $\varphi_0(t)$. A full linear stability analysis around the fixed points $(\rho_0^\pm, \pi/2)$, where $\dot{\rho}_0 = 0$ and $\dot{\varphi}_0 = 0$, reveals their distinct characters. The Jacobian matrix $\mathcal{J}$, derived from Eqs. (3.36) and (3.37), takes the form

$$\mathcal{J} = a^2 \left(\begin{array}{cc} \frac{\partial \dot{\rho}_0}{\partial \rho_0} & \frac{\partial \dot{\rho}_0}{\partial \varphi_0} \\ \frac{\partial \dot{\varphi}_0}{\partial \rho_0} & \frac{\partial \dot{\varphi}_0}{\partial \varphi_0} \end{array} \right) \bigg|_{\rho_0=\rho_0^\pm, \varphi_0=\pi/2} = a^2 \left(\begin{array}{cc} 0 & -c_2 \tilde{D} \\ \frac{c_7 \tilde{D}}{(\rho_0^\pm)^2} - \frac{2c_8 J}{(\rho_0^\pm)^3} & 0 \end{array} \right), \quad (3.41)$$

with eigenvalues $\varepsilon = \pm a^2 \sqrt{XY}$, where $X = -c_2 \tilde{D}$ and $Y = c_7 \tilde{D}/(\rho_0^\pm)^2 - 2c_8 J/(\rho_0^\pm)^3$. Using the fixed-point condition $c_5 \tilde{B}(\rho_0^\pm)^2 - c_7 \tilde{D} \rho_0^\pm + c_8 J = 0$ to simplify, we find

$$Y = \frac{c_5 \tilde{B}(\rho_0^\pm)^2 - c_8 J}{(\rho_0^\pm)^3}. \quad (3.42)$$

For the smaller root ρ_0^- , the exchange energy dominates, leading to $c_5\tilde{B}(\rho_0^-)^2 - c_8J < 0$. From Eq. (3.42), this implies $Y < 0$. Since X is always negative, the product $XY > 0$. Consequently, the eigenvalues ε are real and of opposite signs, identifying ρ_0^- as a saddle point. Conversely, for the larger root ρ_0^+ , the Zeeman energy dominates, resulting in $c_5\tilde{B}(\rho_0^+)^2 - c_8J > 0$ and thus $Y > 0$. This leads to $XY < 0$, yielding purely imaginary eigenvalues ε and identifying ρ_0^+ as a center.

Figure 3.4 shows the time evolution of the skyrmion radius and helicity in the limit of vanishing damping for different initial radii $\rho_{0,0}$. The dashed black horizontal lines indicate the corresponding initial values. For the chosen parameters, $\tilde{D} = 0.24$, $\tilde{B} = 0.02$, and $\varphi_{0,0} = \pi/2$, Eq. (3.39) yields $\rho_0^- = 5.911$ and $\rho_0^+ = 14.461$. The DMI strength is intentionally chosen such that Eq. (3.39) yields physical solutions.

The numerical solution reveals that the global phase space structure is organized by the saddle point at ρ_0^- , which acts as a separatrix between distinct dynamical regimes, while the center at ρ_0^+ governs the local oscillatory behavior. We observe three clear types of breathing dynamics: For $\rho_{0,0} > \rho_0^+$, the system exhibits a rotational breathing mode, where $\varphi_0(t)$ undergoes a full 2π precession while $\rho_0(t)$ oscillates. For $\rho_{0,0} \in [\rho_0^-, \rho_0^+]$, the system exhibits oscillatory breathing mode, with trajectories orbiting the center at ρ_0^+ and $\varphi_0(t)$ oscillating weakly around $\pi/2$. Finally, for $\rho_{0,0} < \rho_0^-$, the system exhibits pulse-like rotational breathing mode, where the helicity undergoes rapid jumps between equivalent Bloch-type configurations.

The interval $[\rho_0^-, \rho_0^+]$ thus defines the basin of attraction for oscillatory dynamics, with the center at ρ_0^+ acting as the stable focus around which the skyrmion performs small amplitude oscillations of both radius and helicity. This is the oscillatory breathing mode described in Ref. [125].

For large skyrmions ($\rho_{0,0} > \rho_0^+$), the rotational breathing mode can be understood from the energy landscape shown in Figure 3.1. In the absence of damping, no energy is dissipated. However, the external magnetic field drives a continuous precession of the helicity $\varphi_0(t)$. Consequently, the skyrmion leaves its energetically most favorable Bloch-type configuration ($\varphi_0 = \pi/2$) and rotates through higher-energy states, i.e., first the anti-Néel-type ($\varphi_0 = \pi$) and then the least favorable anti-Bloch-type ($\varphi_0 = 3\pi/2$) configuration. Along this trajectory, maintaining a constant radius would imply an energy increase. Therefore, the skyrmion shrinks, reaching its minimum size when $\varphi_0 = 3\pi/2$. As φ_0 continues to rotate through the Néel-type configurations ($\varphi_0 = 2\pi$ and 0), the skyrmion expands again and recovers its initial size at the Bloch-type configuration.

When the initial radius lies within the oscillatory basin $[\rho_0^-, \rho_0^+]$, the radius oscillates around ρ_0^+ and the helicity oscillations are phase-shifted by $\pi/2$ relative to the radius oscillations, such that the radius is maximal or minimal when $\varphi_0 = \pi/2$, and intermediate when the helicity is at its extrema.

For small skyrmions ($\rho_{0,0} < \rho_0^-$), below the oscillatory basin, the dynamics are driven primarily by the exchange term. The radius $\rho_0(t)$ remains nearly constant most of the

time but exhibits rapid, steep dips when the helicity $\varphi_0(t)$ jumps between preferred Bloch-type configurations ($\varphi_0 = \pi/2 + 2\pi\lambda$, $\lambda \in \mathbb{Z}$). At finite damping, such small initial radii would lead to the radius vanishing, even though the DMI stabilizes the skyrmion. Thus, this behavior reflects energy conservation in the zero-damping limit: the radius cannot vanish completely, so the system accommodates the dynamics through rapid helicity jumps coupled to brief radius contractions.

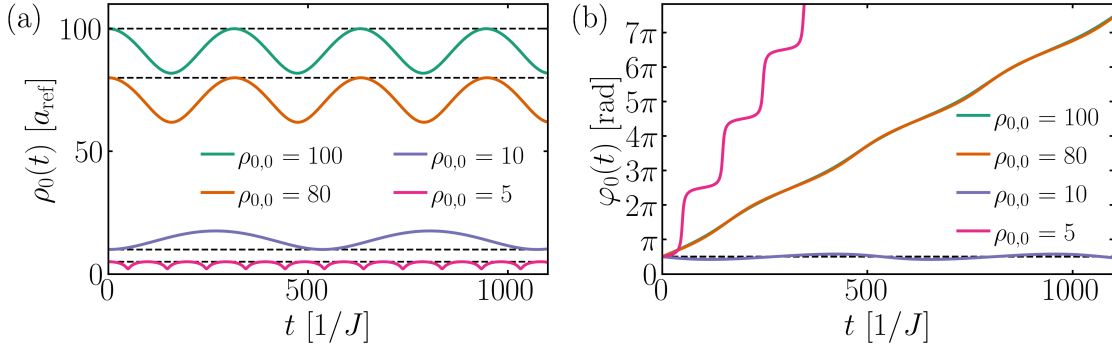


Figure 3.4.: Panel (a) shows the time evolution of the skyrmion radius $\rho_0(t)$ and panel (b) the one of the helicity $\varphi_0(t)$ for different initial radii $\rho_{0,0}$ in the limit of vanishing damping ($\alpha = 0$). The dashed black horizontal lines indicate the corresponding initial values. The DMI strength and magnetic field are chosen such that the fixed points $\rho_0^- = 5.911$ and $\rho_0^+ = 14.461$ define a basin of attraction for oscillatory dynamics, visualizing three distinct breathing regimes: Large skyrmions ($\rho_{0,0} > \rho_0^+$) exhibit a rotational breathing mode where $\rho_0(t)$ oscillates while $\varphi_0(t)$ rotates through Bloch-, anti-Néel-, and anti-Bloch-type configurations. The radius is minimal when φ_0 reaches $3\pi/2$ and recovers its initial value at $\varphi_0 = 2\pi$. Skyrmions with intermediate initial radii within this basin experience an oscillatory breathing mode, with $\rho_0(t)$ and $\varphi_0(t)$ oscillating out of phase by $\pi/2$ around their energy-minimizing values. Small skyrmions ($\rho_{0,0} < \rho_0^-$) show pulse-like rotational breathing, characterized by a nearly constant radius with rapid dips accompanying abrupt helicity jumps. Parameters: $\tilde{D} = 0.24$, $\tilde{B} = 0.02$, $a = 1$, $\alpha = 0$.

3.3.3. Skyrmion shrinking at finite DMI and finite Gilbert damping

At finite Gilbert damping ($\alpha > 0$), the system can dissipate energy, leading to relaxation dynamics absent in the conservative ($\alpha = 0$) limit. A central question is whether the skyrmion relaxes to a stable stationary state or collapses, which depends critically on the DMI strength. We therefore analyze the existence and stability of stationary solutions to identify the parameter regime for skyrmion stability.

The stationary points are given by the roots of Eq. (3.27) for finite-damping, which generalizes Eq. (3.39) to

$$\rho_0^\pm = \frac{\tilde{D}(c_7 \sin \varphi_0 - \alpha c_6 \cos \varphi_0) \pm \sqrt{\tilde{D}^2 (\alpha c_6 \cos \varphi_0 - c_7 \sin \varphi_0)^2 - 4c_5 \tilde{B} c_8 J}}{2c_5 \tilde{B}}. \quad (3.43)$$

For Bloch-type ($\varphi_0 = \pi/2$) and anti-Bloch-type ($\varphi_0 = 3\pi/2$) configurations, the $\cos \varphi_0$ term vanishes, reducing Eq. (3.43) to the zero-damping result in Eq. (3.39). The Bloch-type configuration yields the range $\rho_0 \in [\rho_0^-, \rho_0^+]$, while the anti-Bloch-type configuration again produces unphysical, negative roots.

For Néel- and anti-Néel-type configurations ($\varphi_0 = 0$ and π , respectively), the $\sin \varphi_0$ term vanishes, leading to

$$\rho_0^\pm = -\frac{\alpha c_6 \tilde{D} \cos \varphi_0 \pm \sqrt{(\alpha c_6 \tilde{D} \cos \varphi_0)^2 - 4c_5 \tilde{B} c_8 J}}{2c_5 \tilde{B}}. \quad (3.44)$$

The discriminant may become positive for sufficiently small external magnetic fields $\tilde{B}$. However, since both α and $\tilde{D}$ are typically small, this requires weak fields and even if the discriminant is positive, its value is always smaller than the magnitude of the leading term. Thus, for the Néel-type configuration ($\cos 0 = 1$), Eq. (3.44) yields negative, unphysical radii, while for the anti-Néel-type ($\cos \pi = -1$), it results in positive roots.

To determine which of these stationary points are stable and thus physically observable, we perform a linear stability analysis. Differentiating $\dot{\varphi}_0$ with respect to φ_0 gives

$$\frac{\partial \dot{\varphi}_0}{\partial \varphi_0} = -\frac{a^2 \tilde{D}}{(1 + \alpha^2) \rho_0} (c_7 \cos \varphi_0 + \alpha c_6 \sin \varphi_0). \quad (3.45)$$

Evaluating this at $\varphi_0 = \pi$ (anti-Néel-type) yields

$$\left. \frac{\partial \dot{\varphi}_0}{\partial \varphi_0} \right|_{\varphi_0=\pi} = \frac{a^2 c_7 \tilde{D}}{(1 + \alpha^2) \rho_0} > 0, \quad (3.46)$$

indicating an unstable stationary point. In contrast, for the Bloch-type configuration at $\varphi_0 = \pi/2$, the derivative is

$$\left. \frac{\partial \dot{\varphi}_0}{\partial \varphi_0} \right|_{\varphi_0=\pi/2} = -\frac{a^2 \alpha c_6 \tilde{D}}{(1 + \alpha^2) \rho_0} < 0, \quad (3.47)$$

confirming a stable stationary point. Therefore, only the Bloch-type skyrmion provides a stable and physical stationary solution, consistent with the underlying bulk DMI.

To complete the stability analysis, we also examine the radial direction. For the Bloch-type configuration ($\varphi_0 = \pi/2$), the derivative of Eq. (3.21) with respect to ρ_0 is

$$\left. \frac{\partial \dot{\rho}_0}{\partial \rho_0} \right|_{\rho_0=\rho_0^\pm, \varphi_0=\pi/2} = \frac{a^2 \alpha}{1 + \alpha^2} \left(-c_1 \tilde{B} + \frac{c_4 J}{(\rho_0^\pm)^2} \right). \quad (3.48)$$

At the fixed points $\rho_0^\pm$, this yields a finite value whose sign depends on the parameter values. If $c_4 J / (\rho_0^\pm)^2 < c_1 \tilde{B}$, the ρ_0 direction is stable, while if $c_4 J / (\rho_0^\pm)^2 > c_1 \tilde{B}$, it is unstable. Since $\rho_0^- < \rho_0^+$, we have $c_4 J / (\rho_0^-)^2 > c_4 J / (\rho_0^+)^2$, making ρ_0^- more likely to be unstable and ρ_0^+ more likely to be stable in the radial direction. For typical parameters

where the exchange term dominates at small radii and the Zeeman term dominates at large radii, and from the zero damping analysis, we expect ρ_0^- to be a saddle point, that is unstable in both directions and ρ_0^+ to be a stationary point, that is stable in both directions.

The condition for real roots in the Bloch-type case can be derived from Eq. (3.43) as

$$(c_7 \tilde{D})^2 \geq 4c_5 \tilde{B} c_8 J, \quad (3.49)$$

and defines a critical DMI strength $\tilde{D}_{\text{crit}}$ necessary for the existence of a stationary skyrmion. This separates two dynamical regimes: For $\tilde{D} \geq \tilde{D}_{\text{crit}}$, a stable fixed point exists. Finite damping causes trajectories to relax towards it, in contrast to the persistent oscillations at $\alpha = 0$. And for $\tilde{D} < \tilde{D}_{\text{crit}}$, no finite-radius stationary solution exists. The skyrmion is unstable, and finite damping enables its radius to vanish at a critical time t_c .

In the following, we investigate this collapse dynamics for sub-critical DMI, where the interplay of damping and the lack of a stationary state leads to the skyrmion shrinking until its radius vanishes. Figure 3.5 shows the numerical solution of the coupled system of Eqs. (3.21) and (3.27) at finite DMI and finite Gilbert damping for different values of the lattice constant a . The general time evolution and its dependence on a are similar to the zero DMI limit. In panel (a), the exponential decay of the radius crosses over to a square-root-like behavior. This time evolution is superimposed with a breathing mode, consistent with our earlier analysis. Since the DMI is chosen to be too small to stabilize

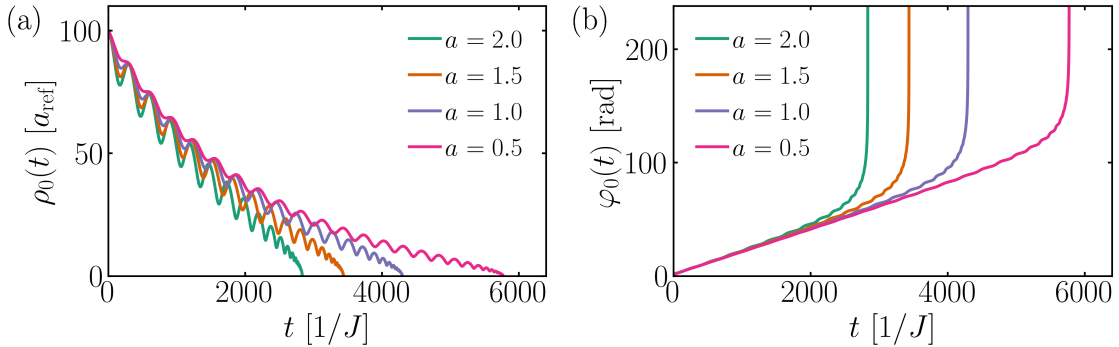


Figure 3.5.: Time evolution of (a) skyrmion radius $\rho_0(t)$ and (b) helicity $\varphi_0(t)$ for finite DMI with varying lattice constant a . The radius exhibits exponential decay crossing over to square-root-like behavior, superimposed with rotational breathing oscillations. The amplitude of the rotational breathing mode decreases linearly with decreasing lattice constant. The helicity shows a precessional motion with linear evolution at large radii (Zeeman-dominated) crossing over to a logarithmic divergence near the collapse of the radius (exchange-dominated). In the latter regime, the oscillation periods of $\rho_0(t)$ shorten, driven by the accelerated precession of the helicity. Furthermore, the helicity oscillations diverge logarithmically in time as the radius decreases, consistent with exchange-driven dynamics at small skyrmion sizes. Parameters: $\tilde{D} = 0.1$, $\tilde{B} = 0.02$, $\alpha = 0.1$.

the skyrmion, only the rotational breathing mode persists, as revealed by the helicity evolution in panel (b). The breathing amplitude decreases with a , matching Eq. (3.21) where the DMI driven oscillatory terms scale linearly with a . The breathing frequency however is independent of a .

At large radii, the Zeeman term dominates, causing the helicity to evolve approximately linearly in time, in agreement with Eq. (3.31) for this regime. As the radius decreases, the exchange term becomes increasingly dominant and drives $\varphi_0(t)$ towards a logarithmic divergence, following Eq. (3.32). This is reflected in the radius dynamics, where the rotational breathing mode oscillates with progressively shorter periods.

The shrinking dynamics of the skyrmion for different values of the external magnetic field $\tilde{B}$ are displayed in Figure 3.6. As the magnetic field increases, the slope of $\varphi_0(t)$ becomes steeper, consistent with Eq. (3.31), since φ_0 evolves proportionally to the field strength. Here, $c_5 = 1$ and $\tilde{B} = B/a^2$, implying a direct proportionality with only a minor correction from Gilbert damping. Consequently, the frequency of the rotational breathing mode of $\rho_0(t)$ increases. This results in shorter decay times t_c , as predicted by Eq. (3.35). Furthermore, the breathing amplitude decreases at higher fields.

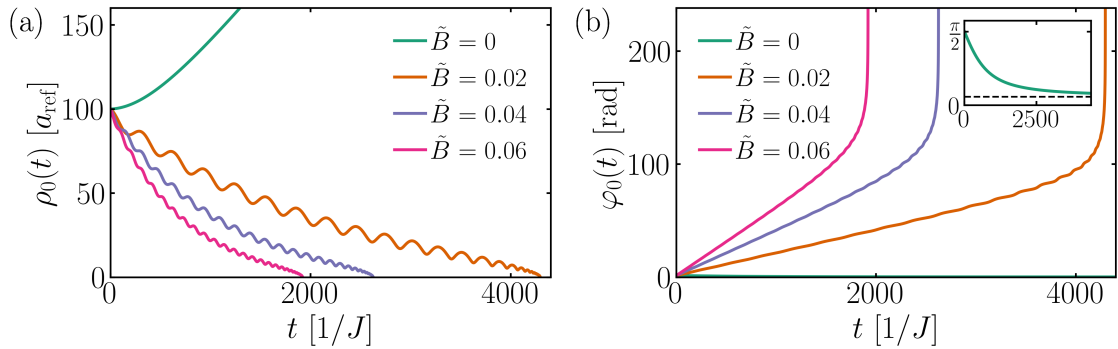


Figure 3.6.: Time evolution of (a) the skyrmion radius $\rho_0(t)$ and (b) the helicity $\varphi_0(t)$ for varying external magnetic fields $\tilde{B}$ at finite DMI. The slope of the helicity time evolution at large radii is directly proportional to the field strength. Increasing $\tilde{B}$ reduces the rotational breathing amplitude, increases the frequency, and shortens the collapse time t_c of $\rho_0(t)$. At $\tilde{B} = 0$, the radius grows indefinitely, analogous to the Belavin-Polyakov skyrmion, while the helicity asymptotically approaches the stable stationary value $\varphi_{0,\text{st}} = 0.183$, that is derived from the DMI dominated dynamics at large skyrmion sizes. This is depicted in the inset in (b), where the dashed black line corresponds to the value of $\varphi_{0,\text{st}}$. Parameters: $\tilde{D} = 0.1$, $a = 1$, $\alpha = 0.1$.

In the absence of an external magnetic field, the skyrmion size increases continuously. This behavior is reminiscent of the Belavin-Polyakov case [17], where scale invariance prevents the existence of an energetically preferred size. Without terms that break this invariance, such as a magnetic field, the skyrmion lacks a characteristic length scale and can expand indefinitely, consistent with the behavior observed here.

Setting the external field strength to zero in Eq. (3.27) shows that with increasing

skyrmion size, the DMI term becomes dominant compared to the exchange term. For $\rho_0 \rightarrow \infty$, we can identify the long-term stationary point for $\varphi_0(t \rightarrow \infty)$ by setting the bracket in the DMI term equal to zero, i.e.,

$$-\alpha c_6 \cos \varphi_0 + c_7 \sin \varphi_0 = 0. \quad (3.50)$$

Solving for φ_0 yields

$$\varphi_{0,\text{st}} = \arctan\left(\frac{\alpha c_6}{c_7}\right) + \pi\lambda, \quad \lambda \in \mathbb{Z}. \quad (3.51)$$

For the Gilbert damping $\alpha = 0.1$ and $\lambda = 0$, this evaluates to $\varphi_{0,\text{st}} = 0.183$. The stability is confirmed by

$$\begin{aligned} \left. \frac{\partial \dot{\varphi}_0}{\partial \varphi_0} \right|_{\varphi_0 = \varphi_{0,\text{st}}} &= - \frac{a^2 \tilde{D}}{(1 + \alpha^2) \rho_0} (-\alpha c_6 \sin \varphi_{0,\text{st}} + c_7 \cos \varphi_{0,\text{st}}) \\ &= - \frac{a^2 \tilde{D}}{(1 + \alpha^2) \rho_0} \cdot 1.612 < 0, \end{aligned} \quad (3.52)$$

This long-term behavior of $\varphi_0(t)$ at zero magnetic field strength is also depicted in the inset in Figure 3.6 (b), where the dashed black line represents $\varphi_{0,\text{st}} = 0.183$. It is evident that, while the skyrmion expands indefinitely, φ_0 approaches this value asymptotically.

The solution of the coupled differential equations for different values of the DMI strength is displayed in Figure 3.7. Increasing the DMI strength shortens the critical time t_c for radius collapse [114]. Furthermore, the amplitude of the rotational breathing modes increases with increasing DMI strength. If $\tilde{D} < \tilde{D}_{\text{crit}}$, the breathing mode is always rotational as depicted in panel (b) and the radius vanishes at t_c as shown in panel (a). If $\tilde{D} \geq \tilde{D}_{\text{crit}}$, the skyrmion exhibits rotational breathing at large radii. As it shrinks, it

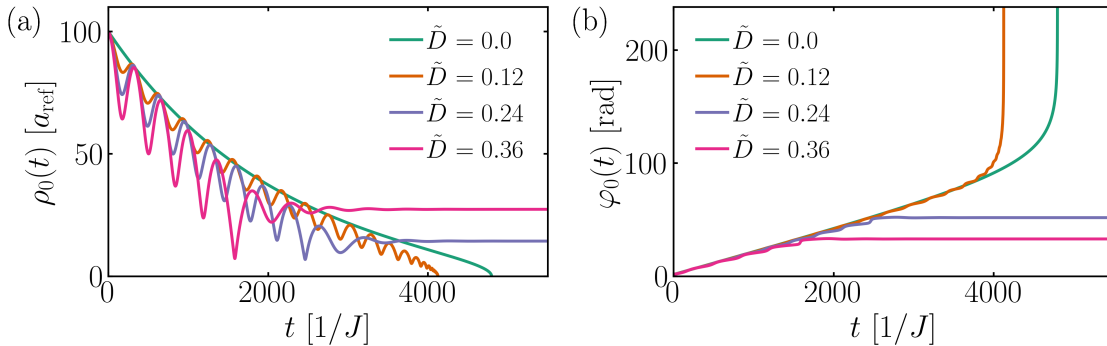


Figure 3.7.: Time evolution of (a) the skyrmion radius $\rho_0(t)$ and (b) the helicity $\varphi_0(t)$ for varying DMI strength $\tilde{D}$. The amplitude of the oscillations in the radius increases with $\tilde{D}$. For $\tilde{D} < \tilde{D}_{\text{crit}}$, the radius collapses at t_c with rotational breathing. For $\tilde{D} \geq \tilde{D}_{\text{crit}}$, the dynamics transitions from a rotational breathing mode at large radii to an oscillatory breathing mode around the stable Bloch-type configuration. The oscillatory breathing mode is slower in frequency and is damped such that the skyrmion relaxes to a finite size at $t \rightarrow \infty$. Parameters: $\tilde{B} = 0.02$, $a = 1$, $\alpha = 0.1$. [Figure adapted from Ref. [114]]

enters the regime of oscillatory breathing, where $\varphi_0(t)$ stops rotating and instead oscillates around a value equal to the Bloch-type skyrmion configuration ($\varphi_0 = \pi/2 + 2\pi\lambda$, $\lambda \in \mathbb{Z}$). In this regime, the skyrmion radius $\rho_0(t)$ shows damped oscillations around its finite minimal-energy size to which it relaxes for $t \rightarrow \infty$.

It is evident that in the rotational breathing regime, the breathing frequency is independent of the DMI strength. If the DMI is large enough to stabilize the skyrmion, there is a transition from rotational to oscillatory breathing, where the breathing frequency slows down, as oscillatory breathing is generally slower than rotational breathing, all in agreement with the earlier analysis on breathing characterization. This is also shown in Figure 3.8 where the data from Figure 3.7 (b) are wrapped within the interval $[0, 2\pi]$. In the beginning of the time evolution of the helicity, the rotation is independent of $\tilde{D}$, as its slope is determined by the external field strength as given by Eq. (3.31). As the rotational mode evolves into the oscillatory mode at stabilizing DMI, the rotation in φ_0 slows down [114]. On the right-hand side of the plot, the corresponding skyrmion configurations are displayed as pictograms.

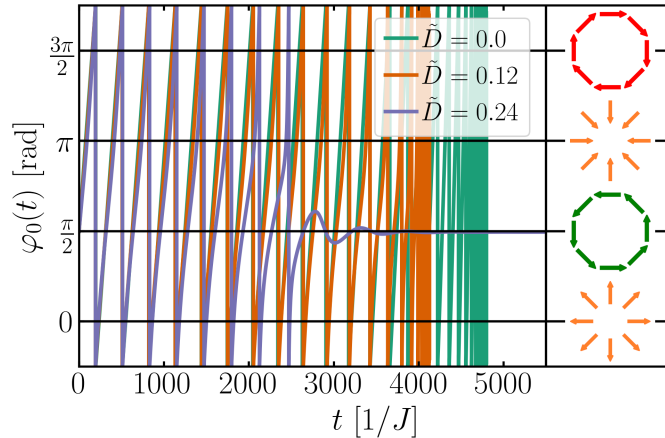


Figure 3.8.: The data of $\varphi_0(t)$ from Figure 3.7 (b) wrapped within the interval $[0, 2\pi]$ are displayed. The pictograms on the right-hand side of the plot correspond to the skyrmion configuration at the given helicities. The rotational breathing mode is independent of $\tilde{D}$. If the DMI is large enough to stabilize the skyrmion, the radius enters the oscillatory breathing regime. Here, the breathing frequency slows down as it transitions to oscillations around $\pi/2$. For sub-critical DMI, continuous rotation persists until the skyrmion radius collapses. Parameters: $\tilde{B} = 0.02$, $a = 1$, $\alpha = 0.1$. [Figure adapted from Ref. [114]]

3.4. Comparison with numerical simulations

In this section, we compare the analytical findings from our continuum theory, obtained using the triangular ansatz for the skyrmion profile with radius and helicity as collective coordinates, with data from direct numerical solutions of the LLG equation in Eq. (2.14) on a square grid, as described in Sec. 2.2.6. For more details on the simulation software, see Appendix A. Throughout all simulations, the exchange constant $J = 1$ serves as the

energy unit and the reference lattice constant $a_{\text{ref}} = 1$ as the length unit, while the DMI strength D , external magnetic field B , Gilbert damping α , and lattice constant a are varied. The dimensionless lattice constant a is decreased by increasing the number of grid points for a fixed system size. The system is quadratic ($L_x = L_y = L$), where L denotes the number of lattice sites along one spatial direction. Accordingly, L is adjusted when varying a to maintain a fixed system size. The parameters are then rescaled with the lattice constant to J/a^2 and D/a . This ensures that the skyrmion maintains the same spatial extent and energy across different grid resolutions.

To prepare a Bloch-type skyrmion as the initial state for all simulations, we initialize the skyrmion on the grid using Eq. (2.3) together with Eqs. (2.35) and (2.23), which describe the skyrmion radial profile and the azimuthal angle at lattice site i , respectively. In the latter one, we set $m = 1$, to obtain a skyrmion. For initialization, we use $\rho_0 = 15a$ and $w = \rho_0/2$. The artificially created skyrmion is then relaxed at a bulk DMI of $D = 0.2$ and an external field of $B = 0.02$ until it reaches its minimum energy as a stationary, isotropic skyrmion in the two-dimensional plane.

If this skyrmion is then subject to an instantaneous parameter change at $t = 0$, it responds to the sudden quench, and its time evolution, specifically of its radius and helicity, can be tracked. In the following, we first describe how the skyrmion radius and helicity are extracted from the numerical data. Since we study skyrmion decay, we then explain how the decay time t_c is determined using the evolution of the topological charge $\mathcal{Q}(t)$. Finally, we discuss skyrmion decay at vanishing DMI and subsequently at finite DMI under the variation of the lattice constant, the Gilbert damping, and the external magnetic field and also include the case where the DMI is large enough to stabilize the skyrmion.

3.4.1. Determination of the skyrmion radius and helicity

To track the evolution of the skyrmion radius over the course of the simulation, we fit the numerical data to the approximated function in Eq. (2.35) [114], which is designed to capture the domain wall-like shoulder appearing in typical skyrmion profiles well [71]. The skyrmion is isotropic in the plane and therefore, the fit is performed starting from the skyrmion center along an arbitrary radial direction with the fitting parameters being the skyrmion radius ρ_0 and the width of the domain wall-like shoulder w . The skyrmion center is defined as the position where $n_z = -1$. If this position lies between grid points, it is determined via interpolation. This fitting is performed at each time point during the simulation, such that we can extract and plot the radius time evolution $\rho_0(t)$.

To extract the time evolution of the skyrmion helicity from the simulation data, we compute the averaged helicity at each time step over all $\varphi_{0,i}$ angles on the grid as

$$\langle \varphi_0 \rangle = \frac{\sum_i \mathcal{W}_i \cdot \varphi_{0,i}}{\sum_i \mathcal{W}_i}, \quad (3.53)$$

where $\varphi_{0,i}$ is weighted using the function $\mathcal{W}_i = 1 - |n_i^z|$. This way, magnetic moments

pointing almost parallel or anti-parallel to the z -axes contribute less to the mean since their azimuthal angle is less sharply defined [114].

Employing these procedures, we can follow the time evolutions of the radius and helicity from the numerical simulations and can compare those to the findings from the theory, where we used these two quantities as the collective coordinates.

3.4.2. Determination of the collapse time

To determine the collapse time t_c , we analyze the time evolution of the system's topological charge. The topological charge of the skyrmion is evaluated on the simulation grid using a discretized form of Eq. (2.28), where the spatial derivatives $\partial_x \mathbf{n}$ and $\partial_y \mathbf{n}$ are approximated by central finite differences. For each grid point (i, j) , the local topological charge density is computed as

$$q_{ij} = \frac{1}{4\pi} \mathbf{n}_{ij} \cdot \left[\frac{\mathbf{n}_{i+1,j} - \mathbf{n}_{i-1,j}}{2} \times \frac{\mathbf{n}_{i,j-1} - \mathbf{n}_{i,j+1}}{2} \right]. \quad (3.54)$$

The total topological charge Q is then obtained by summing q_{ij} over all lattice sites

$$Q \approx \sum_{i,j} q_{ij}. \quad (3.55)$$

This procedure is carried out at each simulation time step and yields, e.g., the results shown in Figure 3.9, where the time evolution of the total topological charge is displayed for different lattice constants and Gilbert dampings. In the simulations shown, the skyrmion is at $t = 0$ exposed to an environment with vanishing DMI ($D = 0$), rendering it unstable.

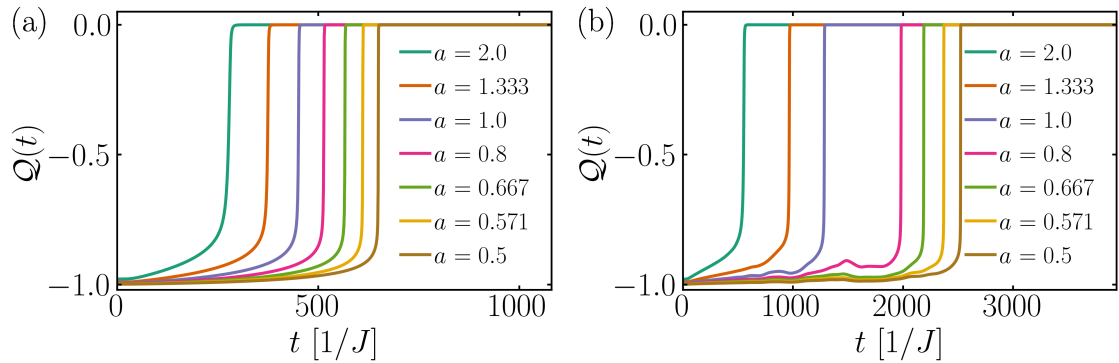


Figure 3.9.: Time evolution of the total topological charge $Q(t)$ for a Gilbert damping of (a) $\alpha = 0.1$ and (b) $\alpha = 0.01$ for different lattice constants. In all simulations, the skyrmion is initialized at $t = 0$ in an environment with vanishing DMI, which renders it unstable. The topological charge starts from $Q \approx -1$, corresponding to a single skyrmion, and abruptly changes to $Q \approx 0$ as the system relaxes into the topologically trivial ferromagnetic state. Smaller lattice constants lead to later occurrences of this transition, indicating an increase of the skyrmion decay time t_c with decreasing lattice spacing. Also, a larger Gilbert damping results in shorter decay times for a fixed lattice constant. Parameters: $D = 0$, $B = 0.02$. [Figure adapted from Ref. [114]]

Initially, the skyrmion possesses a nontrivial topological charge of $\mathcal{Q} \approx -1$. As time progresses, $\mathcal{Q}$ abruptly changes to $\mathcal{Q} \approx 0$, corresponding to the topologically trivial ferromagnetic state. This shows that, in contrast to the continuum model, the simulation on the grid does not conserve the topological charge and as such allows for a destruction of the skyrmion with a residual topological charge of $\mathcal{Q} \approx 0$. We find that smaller lattice constants lead to later occurrences of this abrupt change in the topological charge, indicating an increase of the skyrmion collapse time t_c with decreasing lattice constant [114]. Comparison of the different dampings in (a) and (b) further shows that the larger damping in (a) results in shorter collapse times for fixed a lattice constant. These observations are both in agreement with Eq. (3.34) and thus, already confirm predictions from our analytical model, which we will discuss in more detail in the following sections.

Figure 3.10 (a) shows a zoomed-in view of the time evolution of the total topological charge from Figure 3.9 in the vicinity of the time of decay of the skyrmion. The green curve represents $\mathcal{Q}(t)$ near the moment of skyrmion collapse, while the orange line shows its numerical time derivative, $\mathcal{Q}'(t)$, which is used to identify the inflection point in the time evolution. The black vertical line indicates the time at which $\mathcal{Q}'(t)$ reaches its maximum. Figure 3.10 (b) displays the radial profile of the z -component of the magnetization field through the skyrmion center at various times around this inflection point. The blue line corresponds to the profile at the inflection point itself, while the green and orange lines show earlier times and the magenta, yellow, and gray lines show later times. It is evident

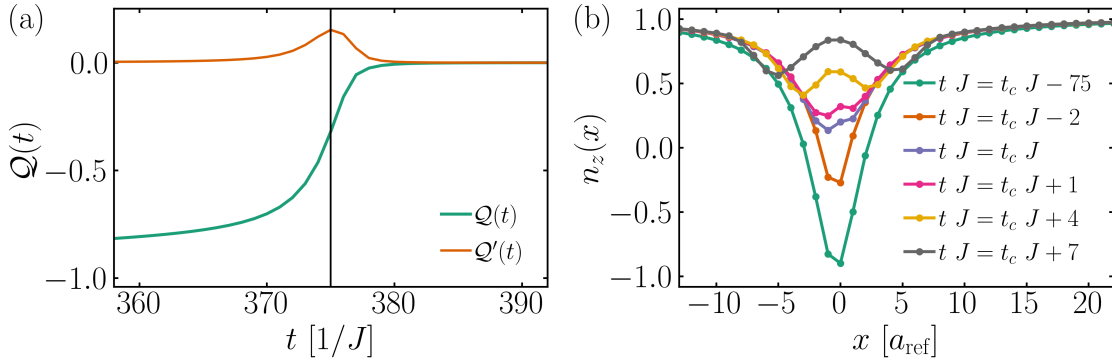


Figure 3.10.: Panel (a) shows a zoomed-in view of the time evolution of the total topological charge $\mathcal{Q}(t)$ around the skyrmion decay time. The green curve shows $\mathcal{Q}(t)$, while the orange line represents its numerical time derivative $\mathcal{Q}'(t)$. The black vertical line marks the time at which $\mathcal{Q}'(t)$ reaches its maximum, identifying the inflection point of $\mathcal{Q}(t)$. Panel (b) displays the radial profiles of the z -component of the magnetization through the skyrmion center at different times around the inflection point. The blue curve corresponds to the profile at the inflection point, green and orange to earlier times, and magenta, yellow, and gray to later times. The inflection point marks the last moment before the profile becomes non-monotonic at the origin, defining the skyrmion decay time t_c . Parameters: $D = 0$, $B = 0.02$, $a = 1.333$, $\alpha = 0.1$. [Figure adapted from Ref. [114]]

that the identified time point marks the last instant before the $n_z(x)$ profile becomes non-monotonic at the origin, defining the skyrmion decay time t_c [114]. This behavior is consistent across all simulations, demonstrating that the procedure provides a robust determination of t_c .

In Figure 3.11, we study how the skyrmion collapse depends on the lattice constant. To this end, we shift the data from Figure 3.9 (a) so that each collapse time t_c coincides with $t = 0$. This highlights that the drop in $Q(t)$ becomes steeper for smaller lattice constants, resembling the fact that smaller lattice constants are closer to the continuum. Since the collapse time is identified from the maximum of the derivative $Q'(t)$, the steepness of the jump in $Q(t)$ is quantified by extracting the full width at half maximum (FWHM) from a Lorentzian fit to $Q'(t)$. The resulting dependence of the FWHM on the lattice constant is shown for $\alpha = 0.1$ and $\alpha = 0.01$ in the inset of Figure 3.11. It is evident that the FWHM decreases with decreasing lattice constant, confirming the above observation. In contrast, the FWHM values for different damping strengths do not differ significantly, indicating that the decay process itself is largely independent of the Gilbert damping α [114]. Thus, α primarily affects the collapse time t_c , whereas the lattice constant influences both t_c and the rate at which the collapse occurs.

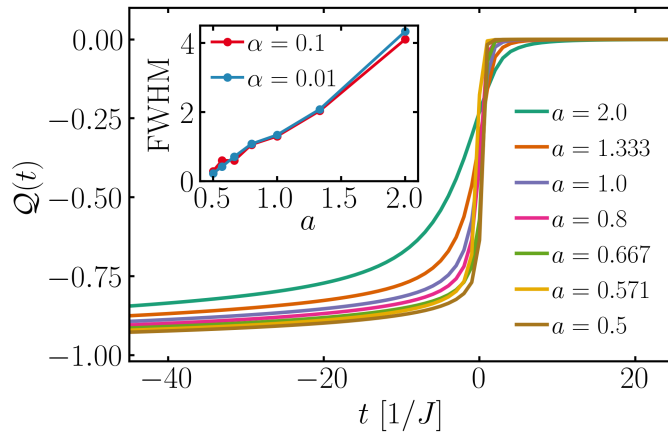


Figure 3.11.: Time evolution of the total topological charge $Q(t)$ for different lattice constants, aligned such that the skyrmion collapse time t_c coincides with $t = 0$. The steeper jump in $Q(t)$ for smaller lattice constants indicates a faster collapse. The collapse time t_c is determined from the maximum of the time derivative $Q'(t)$, and the steepness of the jump is quantified by the full width at half maximum (FWHM) of a Lorentzian fit to $Q'(t)$. The inset shows the FWHM as a function of the lattice constant for $\alpha = 0.1$ and $\alpha = 0.01$, based on the data from Figure 3.9 (a) and (b), indicating that the collapse becomes faster for smaller lattice constants, while the damping α does not significantly affect the collapse rate. Parameters: $D = 0$, $B = 0.02$, $\alpha = 0.1$. [Figure adapted from Ref. [114]]

3.4.3. Skyrmion decay at vanishing DMI

In this section, we present the simulation results obtained in the limit of vanishing DMI ($D = 0$). We first examine the time evolution of the skyrmion radius and helicity after the DMI is abruptly turned off, highlighting the distinct dynamical regimes and their correspondence with the continuum predictions. To further assess the agreement between the simulations and our model, we then extend the analysis to different lattice constants and damping parameters to explore their influence on the skyrmion decay.

The time evolution of the skyrmion radius and helicity, extracted from the simulation data, is shown for an example simulation in Figure 3.12 (a) and (b), respectively. The initial skyrmion is in a Bloch-type configuration, and at $t = 0$, the DMI is abruptly turned off. As is typical for quenches, the system reacts with a delay to the sudden change of parameters. Consequently, the dynamics initially follows a quadratic time dependence, which can be fitted by a parabola at early times. After the system adjusts to the new conditions, the subsequent evolution agrees with the predictions of our continuum model.

At large skyrmion sizes, the dynamics are dominated by the external magnetic field, as the magnetic moments collectively rotate in time. As a result, the skyrmion averaged helicity $\langle\varphi_0\rangle$ is modulated while the system transitions through the different skyrmion configurations: Bloch-, anti-Néel-, anti-Bloch-, and Néel-type [114]. As predicted in Eq. (3.31), this rotation is linear in time with a slope equal to the magnetic field strength, $B = 0.02$. Simultaneously, the skyrmion size ρ_0 decreases exponentially until, at small radii where the exchange term dominates, it crosses over to a square-root-like decay. In this small-

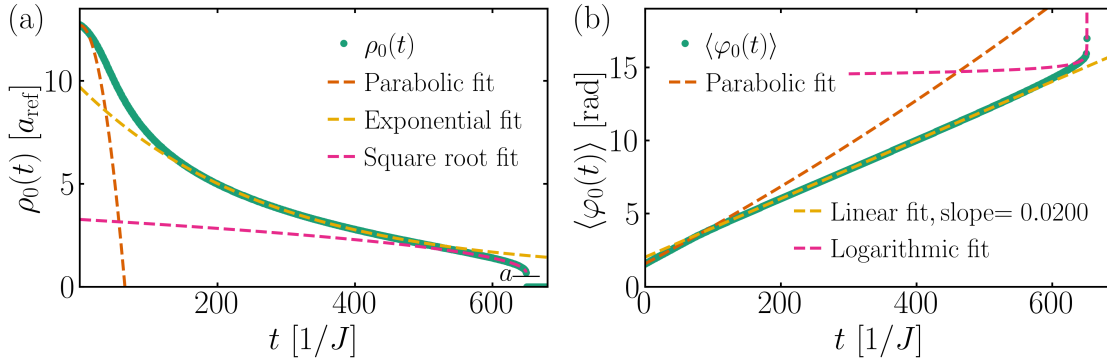


Figure 3.12.: Time evolution of the skyrmion radius (a) and helicity (b) extracted from simulation data after the DMI is abruptly switched off at $t = 0$. The initial skyrmion is Bloch-type and, after a short transitional quadratic regime, follows the time evolution predicted by the continuum model. At large radii, the Zeeman term dominates the dynamics, and the helicity rotates linearly in time with slope $B = 0.02$, while the radius decreases exponentially. At small skyrmion sizes, the exchange term dominates. Consequently, the helicity diverges logarithmically in time, while the radius follows a square-root-like decay. The horizontal black line in (a) marks the lattice constant a . When the skyrmion radius reaches a , it collapses abruptly. Parameters: $D = 0$, $B = 0.02$, $a = 0.5$, $\alpha = 0.1$. [Figure (a) adapted from Ref. [114]]

radius regime, the helicity diverges logarithmically in time. All these observations are in quantitative agreement with our model predictions.

Notably, the last time point where the skyrmion still exists corresponds to the final simulated time step for which its radius remains larger than the lattice constant [114]. The horizontal black line in Figure 3.12 (a) marks the lattice constant used in the simulation. Hence, the decay time t_c is reached when the skyrmion radius would become smaller than the lattice constant.

Figure 3.13 displays the time evolution of the skyrmion radius and helicity for different lattice constants and damping parameters at vanishing DMI. The simulation data correspond to the topological charge evolutions shown in Figure 3.9. Panels (a) and (c) show $\rho_0(t)$, while panels (b) and (d) display $\langle\varphi_0(t)\rangle$ for $\alpha = 0.1$ and $\alpha = 0.01$, respectively. The horizontal black lines indicate the lattice constant used in each simulation.

The analysis and fitting procedure shown in Figure 3.12 can be reproduced for all simulations performed at the larger Gilbert damping $\alpha = 0.1$ in Figure 3.13 (a). In contrast,

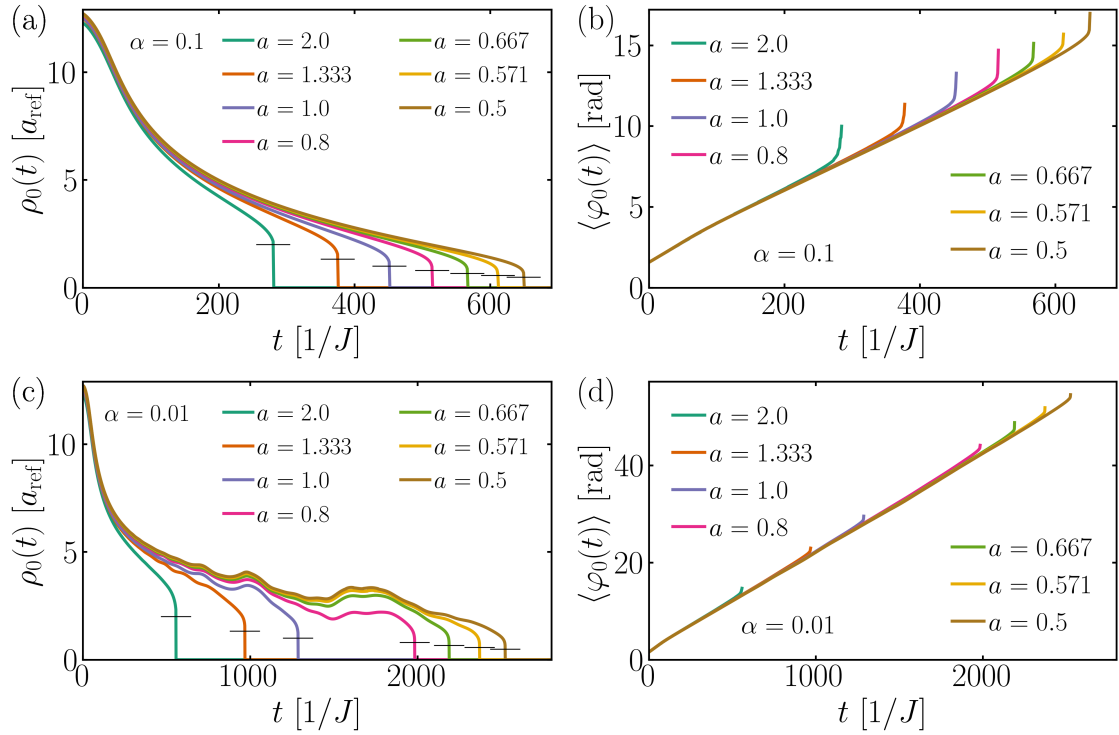


Figure 3.13.: Time evolution of the skyrmion radius $\rho_0(t)$ and helicity $\langle\varphi_0(t)\rangle$ for different lattice constants and Gilbert damping values at vanishing DMI. Panels (a) and (b) correspond to $\alpha = 0.1$ and (c) and (d) to $\alpha = 0.01$. For large damping, the skyrmion shrinks monotonically, following the same quantitative time evolution for ρ_0 and $\langle\varphi_0\rangle$ as in Figure 3.12, whereas for small damping, nonperiodic, breathing-like oscillations appear due to the insufficient suppression of internal vibrational modes. Decreasing lattice constants and Gilbert damping parameters, delay the decay times t_c . Horizontal black lines in (a) indicate the lattice constant value used in each simulation. Parameters: $D = 0$, $B = 0.02$. [Figures (a) and (c) adapted from Ref. [114]]

for smaller damping (Figure 3.13 (c)), the skyrmion radius does not decrease monotonically but exhibits alternating non-periodical shrinking and expansion before vanishing. In this case, the damping is insufficient to suppress internal collective vibrational modes of the magnetic moments, leading to breathing-like oscillations. Stronger damping, as in (a), suppresses these modes, resulting in a monotonically shrinking skyrmion [114].

To investigate the dependence of t_c on the Gilbert damping, additional simulations were performed for several values of α . Figure 3.14 shows the decay time t_c as a function of the lattice constant on a logarithmic scale. For small damping values ($\alpha \leq 0.04$), breathing modes appear, while they are suppressed for larger damping ($\alpha \geq 0.06$). The linear behavior observed in this large-damping regime suggests an overall logarithmic dependence of t_c on the lattice constant a [114], in accordance with the predictions from our analytical model found in Eq. (3.34). At lower damping, the presence of breathing modes delays the skyrmion's collapse time, leading to deviations from the logarithmic trend. In our analytical model, the azimuthal angle is forced to remain isotropic around the skyrmion center at all times, as given by Eq. (2.23). This Ansatz does not allow for higher internal modes and therefore cannot capture the internal vibrational modes responsible for the non-periodic breathing observed in the low-damping regime in the simulations. The simulation results also show that the decay time t_c decreases with increasing α at a fixed lattice constant, which is consistent with our analytical results obtained within the continuum model.

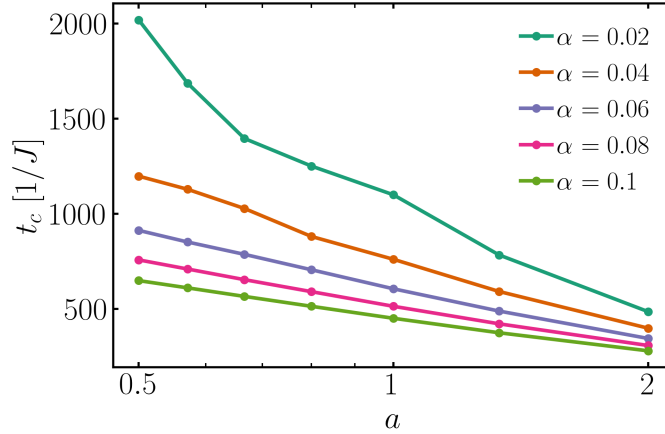


Figure 3.14.: Logarithmic plot of the decay time t_c as a function of the lattice constant for different Gilbert damping parameters α . For small damping ($\alpha \leq 0.04$), non-periodic breathing modes lead to deviations from the logarithmic behavior predicted by Eq. (3.34), while for larger damping ($\alpha \geq 0.06$) the critical times t_c exhibit a logarithmic dependence on the lattice constant. Increasing α reduces t_c at a fixed lattice constant. Parameters: $D = 0$, $B = 0.02$. [Figure adapted from Ref. [114]]

3.4.4. Skyrmion decay at finite DMI

Next, we present the simulation results for finite DMI, focusing on the dynamical behavior of the skyrmion radius and helicity under different conditions. We first analyze the case of a DMI strength that is insufficient to stabilize the skyrmion at a finite size, revealing a rotational breathing mode characterized by periodic expansion and contraction of the radius accompanied by a continuous helicity rotation. Subsequently, we study the influence of the lattice constant, damping, and external magnetic field on the decay dynamics and breathing frequency, as well as the sequence of internal transitions between distinct skyrmion configurations during the breathing cycle. Finally, we explore how the skyrmion stability and breathing characteristics evolve with increasing DMI strength, identifying the transition from rotational to oscillatory breathing modes once the DMI becomes strong enough to stabilize the skyrmion at a finite equilibrium radius.

The time evolution of the skyrmion radius and helicity for different lattice constants and damping parameters is displayed in Figure 3.15. The DMI strength is set to a value not large enough to stabilize the skyrmion at a finite size. Panels (a) and (c) display $\rho_0(t)$, while panels (b) and (d) show $\langle\varphi_0(t)\rangle$ for $\alpha = 0.1$ and $\alpha = 0.01$, respectively. The breathing behavior, that superimposes the general exponential to square-root-like decay, is visible in $\rho_0(t)$, where the skyrmion radius alternately contracts and expands [114]. The breathing is accompanied by a continuous rotation of $\langle\varphi_0(t)\rangle$ with step-like jumps, synchronized with the breathing of the radius. We observe these step-like jumps in the time evolution of $\varphi_0(t)$ from our analytical model as well, see, e.g., Figure 3.5. These results confirm our model predictions that the skyrmion exhibits rotational breathing modes while shrinking in size, until its decay. However, although the helicity diverges logarithmically in time as the skyrmion approaches collapse, the breathing frequency does not increase. This is because the skyrmion radius reaches the magnitude of the lattice constant and decays abruptly before the breathing mode frequency can rise further.

While the decay time t_c generally increases with decreasing lattice constant, its dependence on a cannot be described as logarithmic in the presence of finite DMI. Instead, the decay time is primarily determined by the amplitude of the skyrmion's breathing motion and the value for the lattice constant [114]. When the oscillation amplitude becomes large enough for the skyrmion radius to reach the lattice constant of the respective simulation, the skyrmion collapses. Consequently, for larger lattice constants, this critical radius is reached earlier during the simulation, resulting in shorter decay times.

The time evolution of the skyrmion radius at finite DMI and a fixed lattice constant for different values of the external magnetic field, with $\alpha = 0.1$ (a) and $\alpha = 0.01$ (b), is depicted in Figure 3.16. Increasing the magnetic field reduces the amplitude of the rotational breathing mode while increasing its frequency [114], in agreement with the findings from our analytical model in Eq. (3.31). Our model predicts further, that a larger magnetic field leads to a faster exponential decay at large radii as seen in Eq. (3.29). This is also observed in these simulation results, where a larger external field yields in general

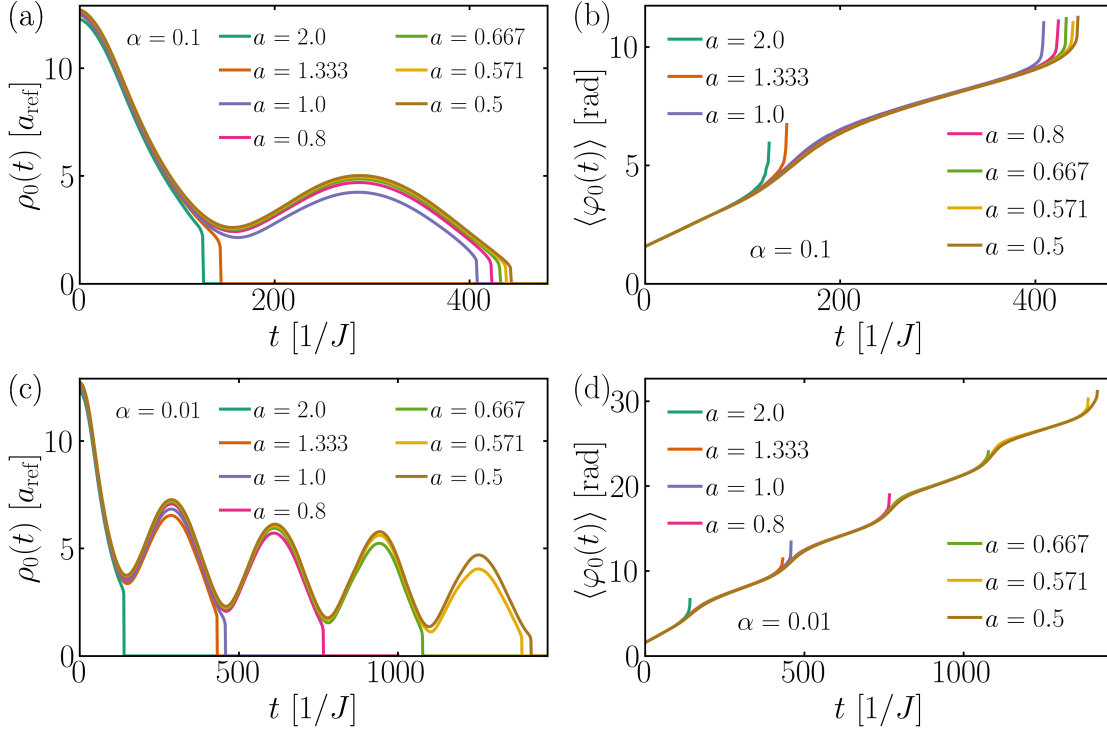


Figure 3.15.: Time evolution of the skyrmion radius $\rho_0(t)$ and helicity $\langle\varphi_0(t)\rangle$ for different lattice constants and Gilbert damping parameters at finite DMI, which is not large enough to stabilize the skyrmion at a finite size. Panels (a) and (b) correspond to $\alpha = 0.1$, while (c) and (d) show the results for $\alpha = 0.01$. The skyrmion exhibits a rotational breathing mode characterized by periodic contractions and expansions of $\rho_0(t)$ synchronized with a rotation of $\langle\varphi_0(t)\rangle$ superimposed with step-like jumps, that are synchronized with the radius breathing. As the skyrmion radius approaches the lattice constant, it collapses abruptly. While the decay time t_c generally increases with decreasing lattice constant, its dependence on a is not logarithmic in the presence of DMI. Instead, t_c is determined by the amplitude of the breathing motion: when the oscillation amplitude becomes large enough for the radius to reach the lattice constant of the corresponding simulation, the skyrmion decays. Consequently, for larger lattice constants, this critical radius is reached earlier, leading to shorter decay times. Parameters: $D = 0.04$, $B = 0.02$. [Figures (a) and (c) adapted from Ref. [114]]

shorter decay times t_c , even though the breathing amplitude of $\rho_0(t)$ decreases.

For $B = 0$, the skyrmion size ρ_0 increases continuously, reproducing the behavior of a Belavin-Polyakov skyrmion [17]. In this case, our model predicts no rotation of the helicity. Instead, $\varphi_0(t)$ relaxes asymptotically to a value given by Eq. (3.51). Because the simulations use a finite system with closed boundary conditions (fixed magnetic moments with $n_z = 1$ at the edges), the skyrmion cannot expand indefinitely. The time evolution of the averaged helicity $\langle\varphi_0(t)\rangle$ for the zero-field simulations from (a) and (b) within a time frame before edge effects influence the skyrmion are shown in Figure 3.16 (c). The dashed lines indicate the theoretical long-term values $\varphi_{0,\text{st}}$ from Eq. (3.51) for the corresponding

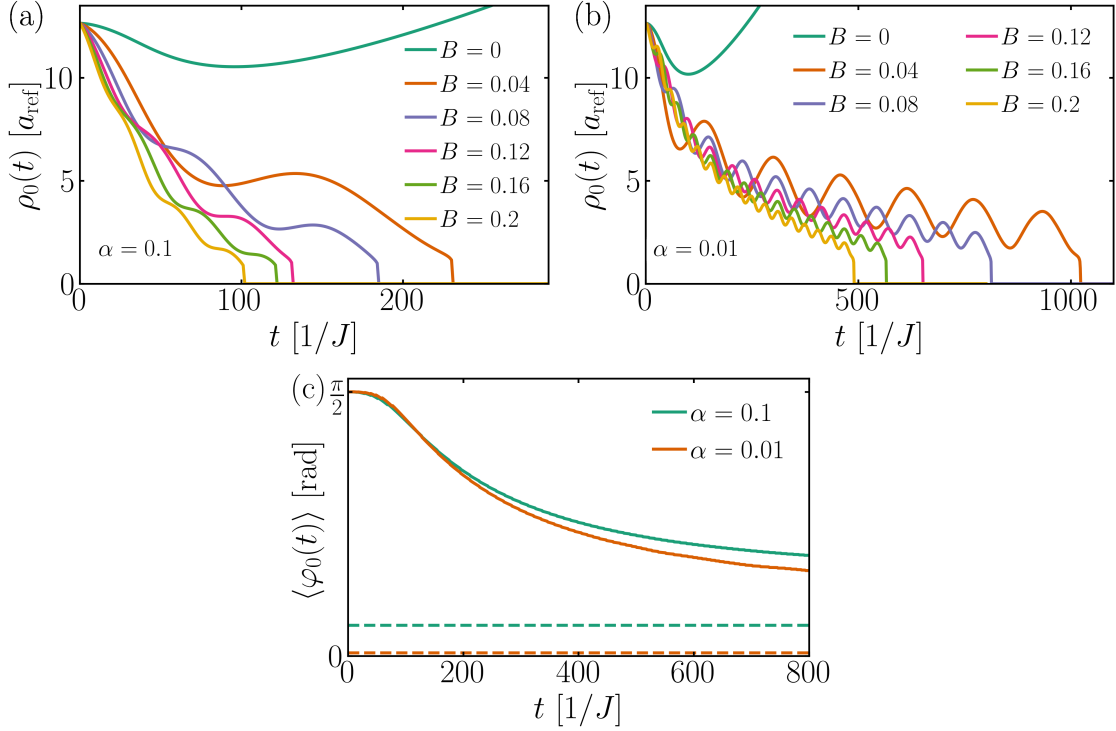


Figure 3.16.: Time evolution of the skyrmion radius $\rho_0(t)$ at finite DMI and a fixed lattice constant for different external magnetic fields. Panels (a) and (b) correspond to Gilbert damping $\alpha = 0.1$ and $\alpha = 0.01$, respectively. Increasing the magnetic field reduces the amplitude of the rotational breathing mode while increasing its frequency, leading to generally shorter decay times t_c . For $B = 0$, the skyrmion radius grows continuously, reproducing the Belavin-Polyakov behavior, and the helicity $\langle \varphi_0(t) \rangle$ in panel (c) relaxes asymptotically towards a value $\langle \varphi_{0,\text{st}} \rangle$. The theoretically predicted values for $\varphi_{0,\text{st}}$ obtained from Eq. (3.51) are indicated as the dashed lines for the corresponding damping strengths. Parameters: $D = 0.04$, $a = 0.8$. [Figures (a) and (b) adapted from Ref. [114]]

damping strengths. While the model underestimates the simulation results, both the model and the simulation agree that the helicity relaxes to an asymptotic value between the Bloch- and Néel-type configurations, and confirm that a larger Gilbert damping yields a larger $\langle \varphi_{0,\text{st}} \rangle$.

To study the dependence of the breathing frequency of ρ_0 on the external magnetic field strength B , we extracted the average frequency from the simulation results for $\alpha = 0.01$ displayed in Figure 3.16 (b). The resulting dependence is shown in Figure 3.17 and reveals a linear dependence of the breathing frequency on B [114]. This dependence is in agreement with the predictions from our analytical model, as the radius breathing is predicted to be synchronized with the time evolution of the helicity, that is rotating through the different skyrmion configurations. Eq. (3.31) shows that $\varphi_0(t)$ is linearly proportional to the magnetic field strength and this is reflected in the linear dependence of the breathing frequency of the radius on the field strength, as displayed here.

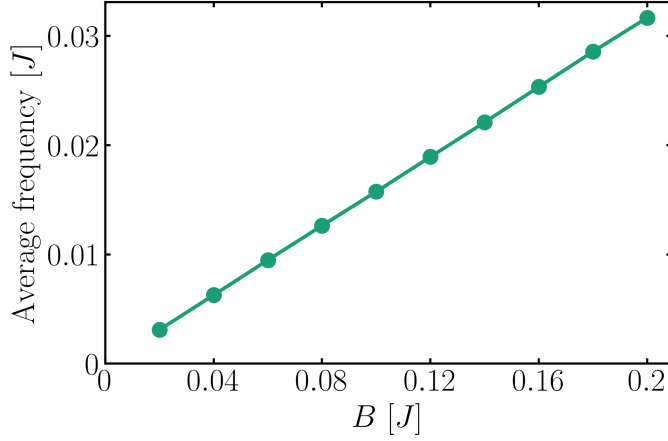


Figure 3.17.: Frequency of the rotational breathing mode of the skyrmion as a function of the external magnetic field strength B for $\alpha = 0.01$. The results reveal a linear increase of the oscillation frequency with the field, in agreement with the predictions of the analytical model for large skyrmion radii. Parameters: $D = 0.04$, $a = 0.8$, $\alpha = 0.01$. [Figure adapted from Ref. [114]]

To further illustrate the internal transitions between different skyrmion types during the rotational breathing mode, the temporal evolution of the spin configurations is shown in Figure 3.18. The pictograms indicate the corresponding skyrmion configurations, with green and red representing Bloch- and anti-Bloch-type skyrmions, respectively, while orange corresponds to Néel- and anti-Néel-type configurations.

Initially, at $t = 0$, the skyrmion is in its energetically favorable Bloch-type configuration with $\varphi_0 = \pi/2$, as shown in subpanel (1). Driven by the external Zeeman field, the helicity $\varphi_0(t)$ increases and cycles through the different skyrmion configurations, leading to the breathing of the radius. The radius reaches periodically a local maximum (2), corresponding again to the Bloch-type configuration. From there, the skyrmion departs from its energetically most favorable state and passes through higher-energy configurations, causing a reduction of its size. It first transitions through the anti-Néel-type configuration ($\varphi_0 = \pi$) at the inflection point (3), and then reaches the least favorable anti-Bloch-type configuration ($\varphi_0 = 3\pi/2$) at its local minimum radius (4). Subsequently, it continues through the Néel-type configuration at the next inflection point (5) and expands again until it recovers the Bloch-type configuration and its corresponding local maximum in size (6).

This sequence of transitions is fully consistent with the energy landscape shown in Figure 3.1 and with the characterization of the breathing dynamics discussed in Sec. 3.3.2 based on the continuum model. On longer time scales, the total energy decreases exponentially due to Gilbert damping, leading to an overall exponential decay of $\rho_0(t)$. All these observations are in agreement with the predictions of the coupled differential equations derived from the analytical model.

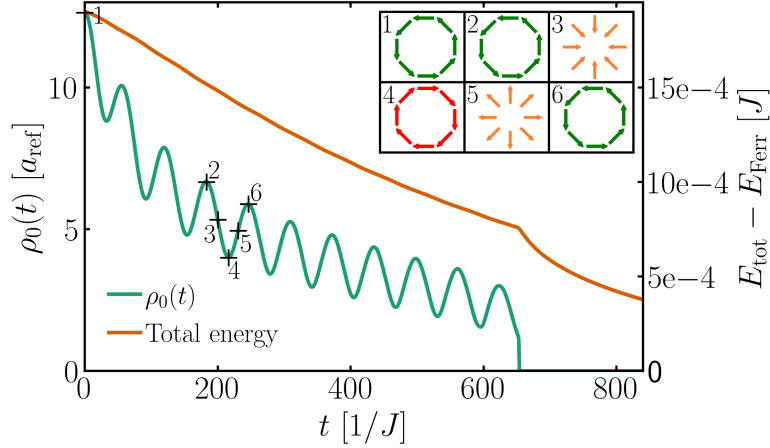


Figure 3.18.: Time evolution of the spin configurations during the rotational breathing mode of the skyrmion. Initially (1), the skyrmion is in its favorable Bloch-type configuration. The time points (2)-(6) illustrate the sequence of helicity-driven transitions between different skyrmion types. Green and red pictograms represent Bloch- and anti-Bloch-type configurations, respectively, while orange indicates Néel- and anti-Néel-type configurations. Starting from the energetically favorable Bloch-type state (2), the helicity $\varphi_0(t)$ increases continuously, driving the skyrmion through successive configurations and a periodic size oscillation until it returns to the Bloch-type state (6). The observed sequence agrees with the rotational breathing dynamics characterization based on the continuum-model analysis in Sec. 3.3.2 together with the energy landscape in Figure 3.1. The energy (orange line) dissipates on longer time scales leading to an overall exponential decay of the skyrmion radius. Parameters: $D = 0.08$, $B = 0.1$, $a = 0.8$, $\alpha = 0.01$. [Figure adapted from Ref. [114]]

Finally, we investigate the dependence of the skyrmion decay on the DMI strength. The time evolution of the skyrmion radius for varying DMI values is shown in Figure 3.19 for damping parameters $\alpha = 0.1$ (a) and $\alpha = 0.01$ (b). For DMI values that are not large enough to stabilize the skyrmion, the breathing amplitude increases proportionally with the DMI strength [114]. In Eq. (3.21), this behavior originates from the sinusoidal terms, which are proportional to the DMI and thus determine the amplitude of the oscillations. Consequently, the radius is more likely to reach the size of the lattice constant at an earlier oscillation cycle for larger DMI values, leading to an earlier skyrmion collapse.

In contrast, when the DMI is sufficiently strong to stabilize the skyrmion ($D \geq 0.12$) at a finite asymptotic radius $\rho_0(\infty)$, the breathing amplitude depends on the deviation of the initial radius from its energy-minimizing value. In this regime, the skyrmion exhibits damped breathing oscillations and gradually relaxes towards the equilibrium radius [114].

The time evolution of the averaged helicity $\langle \varphi_0(t) \rangle$, wrapped within the interval $[0, 2\pi]$, for simulations from Figure 3.19 (b) are presented in Figure 3.20. The corresponding skyrmion configurations are indicated by black horizontal lines, with their schematic illustrations shown to the right of the graph.

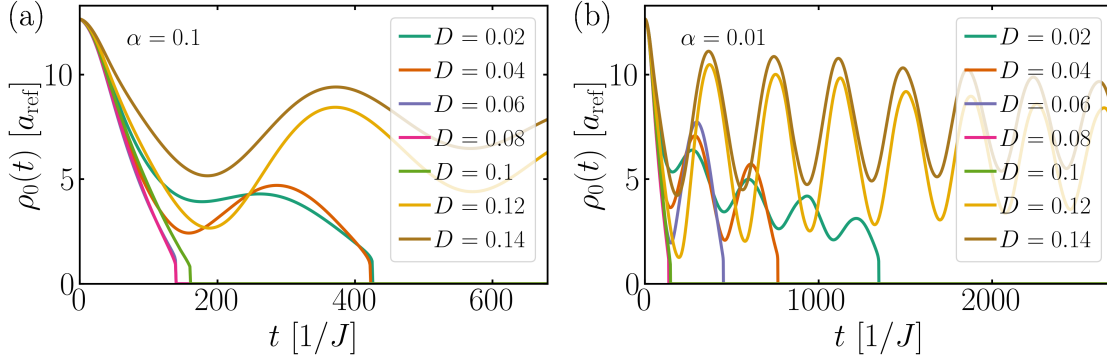


Figure 3.19.: Time evolution of the skyrmion radius for different DMI strengths at (a) $\alpha = 0.1$ and (b) $\alpha = 0.01$. For weak DMI, the breathing amplitude increases with D , leading to an earlier collapse. For strong DMI ($D \geq 0.12$), the skyrmion stabilizes at a finite asymptotic radius and exhibits damped breathing oscillations towards its energy minimizing equilibrium state. Parameters: $B = 0.02$, $a = 0.8$. [Figure adapted from Ref. [114]]

This highlights the distinct breathing regimes emerging during the dynamics: for DMI values too small to stabilize the skyrmion, a rotational breathing mode occurs, in which radius oscillations are accompanied by a continuous rotation of the helicity. For sufficiently large, stabilizing DMI, an oscillatory breathing mode emerges, characterized by slower radius oscillations synchronized with helicity oscillations around the energy-minimizing Bloch-type configuration [114]. These findings are consistent with the breathing mode characterization derived from the continuum theory and align with the observations reported in Ref. [125].

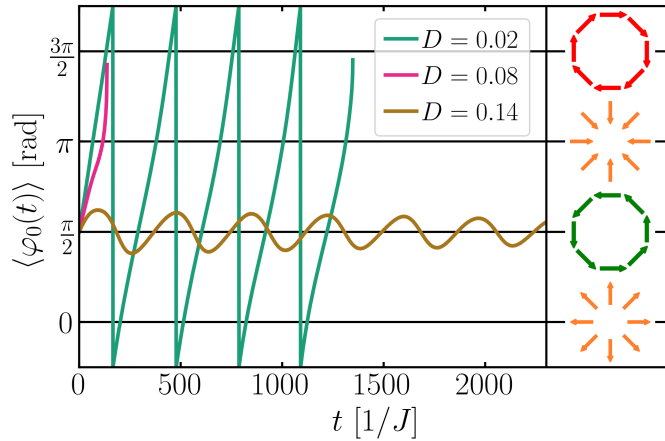


Figure 3.20.: Time evolution of the averaged helicity $\langle \varphi_0(t) \rangle$ for different DMI strengths, corresponding to the simulation results in Figure 3.19 (b). The black horizontal lines and the pictograms next to the graph indicate the respective skyrmion configurations. For weak DMI, a rotational breathing mode is observed, while for strong DMI, an oscillatory breathing mode with helicity oscillations around the Bloch-type configuration emerges. Parameters: $B = 0.02$, $a = 0.8$, $\alpha = 0.01$. [Figure adapted from Ref. [114]]

3.5. Conclusion

In this chapter, we have systematically investigated the stability and dynamical shrinking behavior of magnetic skyrmions in systems with isotropic bulk DMI. By employing a collective coordinate approach with the skyrmion radius $\rho_0(t)$ and helicity $\varphi_0(t)$ as the dynamical variables, and utilizing a triangular ansatz for the skyrmion profile, we derived a set of coupled non-linear differential equations governing the skyrmion's time evolution.

Our analysis, complemented by numerical solutions of the coupled equations of motion and micromagnetic simulations on a grid, has revealed rich dynamical behaviors. The energy landscape, as a function of radius and helicity, confirms that the bulk DMI stabilizes Bloch-type skyrmions at a finite equilibrium radius, while Néel-type and anti-Bloch-type configurations are energetically unfavorable.

The interplay of exchange, Zeeman, and DMI terms leads to distinct shrinking behaviors. At large radii, the Zeeman term dominates, leading to an exponential decay of the radius and a linear precession of the helicity. At small radii, the exchange interaction governs the dynamics, resulting in a square-root-like collapse of the radius and a logarithmic divergence of the helicity precession. The DMI introduces modulations in these trends, giving rise to breathing dynamics, especially at intermediate radii. In the conservative limit ($\alpha = 0$), we identified two fundamental breathing modes which can be transferred to the damped dynamics: a rotational breathing mode, where the helicity undergoes full 2π precession, and an oscillatory breathing mode, where both the radius and helicity perform oscillations around their equilibrium values. The existence of a stable skyrmion state is determined by a critical DMI strength. For a small DMI strength, that is not large enough to stabilize a skyrmion, and at finite Gilbert damping, the skyrmion is unstable and its radius collapses, with its shrinking superimposed by a rotational breathing mode. For large, stabilizing DMI, the skyrmion relaxes to a stable finite size, exhibiting a damped oscillatory breathing mode. The dynamics are further impacted by the system parameters. We identify a critical time t_c at which the skyrmion radius vanishes, which depends logarithmically on the lattice constant a , with smaller lattice constants yielding later collapse times. This critical time depends further on the strength of the external magnetic field, the DMI and the damping.

In conclusion, the collective coordinate model provides a powerful and qualitatively accurate framework for describing the complex shrinking and breathing dynamics of skyrmions, successfully capturing the transition from unstable decay to stable breathing oscillations and the intricate coupling between the skyrmion's size and its internal helicity degree of freedom. These findings offer profound insights into the nonequilibrium dynamics of magnetic skyrmions and establish a reliable theoretical foundation for predicting their behavior in experimental settings.

4. Dynamics of antiskyrmion shrinking

This chapter is based on Ref. [136], in which the results described here have been published. Antiskyrmions carry a topological charge opposite to that of skyrmions and possess an azimuthal twofold symmetry [13, 54, 55], which fundamentally distinguishes them from their rotationally symmetric skyrmion counterparts. This symmetry arises from the magnetization field exhibiting Bloch- and anti-Bloch-type configurations along two orthogonal in-plane axes, while along axes rotated by 45° Néel- and anti-Néel-type configurations appear. In systems with isotropic bulk DMI, as studied here, these distinct windings lead to direction-dependent energy contributions. Consequently, the antiskyrmion tends to deform from its circular shape in order to reduce regions associated with energetically unfavorable windings [54]. This deformation implies that the polar profile of the antiskyrmion is no longer purely radial, but acquires an explicit dependence on the azimuthal coordinate φ , i.e., $\Theta(\rho, \varphi)$. In this chapter, we investigate the shrinking dynamics of antiskyrmions in such environments with isotropic bulk DMI, where they are intrinsically unstable.

4.1. Elliptic triangular approximation of the polar angle

To account for the elliptical deformation of the antiskyrmion, the triangular approximation of the radial profile introduced in Eq. (2.36) must be generalized to include the explicit azimuthal dependence, yielding

$$\Theta(\rho, \varphi) = (\pi - \xi) \mathcal{H}(\pi - \xi), \quad \text{with } \xi = \frac{\pi \rho}{2\rho_0(\varphi)}. \quad (4.1)$$

Here, $\rho_0(\varphi)$ represents the azimuthally varying radius of the antiskyrmion, which captures its directional anisotropy. To model this dependence, we employ an elliptical ansatz for $\rho_0(\varphi)$. We begin by recalling the equation of an ellipse centered at the origin with semi-axes a_0 and b_0

$$\frac{x^2}{a_0^2} + \frac{y^2}{b_0^2} = 1. \quad (4.2)$$

Then, we express the coordinates of the antiskyrmion boundary in polar form, where $x = \rho_0 \cos \varphi$ and $y = \rho_0 \sin \varphi$, and obtain

$$\rho_0^2 \left(\frac{\cos^2 \varphi}{a_0^2} + \frac{\sin^2 \varphi}{b_0^2} \right) = 1. \quad (4.3)$$

Solving the ellipse equation for ρ_0 yields the radial distance from the origin as a function of the azimuthal angle

$$\rho_0(\varphi) = \pm \frac{a_0 b_0}{\sqrt{b_0^2 \cos^2(\varphi) + a_0^2 \sin^2(\varphi)}}. \quad (4.4)$$

Only the positive branch is physically meaningful. To allow the ellipse to align its major axis with the preferred in-plane direction of the spin winding, set by the helicity φ_0 , we introduce an additional in-plane rotation by an angle ω . This rotation accounts for the possible misalignment between the semi-axes of the ellipse and the coordinate axes. The resulting azimuthally dependent radius is [136]

$$\rho_0(\varphi) = \frac{a_0 b_0}{\sqrt{b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega)}}. \quad (4.5)$$

Figure 4.1 illustrates the z -component of the magnetization field $\mathbf{n}$, parametrized via the elliptical ansatz for $\rho_0(\varphi)$, given by

$$\mathbf{n}(\rho, \varphi) = \begin{pmatrix} \sin \Theta(\rho, \varphi) \cos \Phi(\varphi) \\ \sin \Theta(\rho, \varphi) \sin \Phi(\varphi) \\ \cos \Theta(\rho, \varphi) \end{pmatrix}. \quad (4.6)$$

The plot visualizes the semi-axes a_0 and b_0 , as well as the rotation angle ω . Together with the helicity φ_0 , which enters via $\Phi(\varphi)$ in Eq. (2.23) with $m = -1$, these parameters fully define the elliptic triangular antiskyrmion. It is known from recent studies of skyrmions in curved magnetic films [137, 138] that an elliptic deformation of the out-of-plane mag-

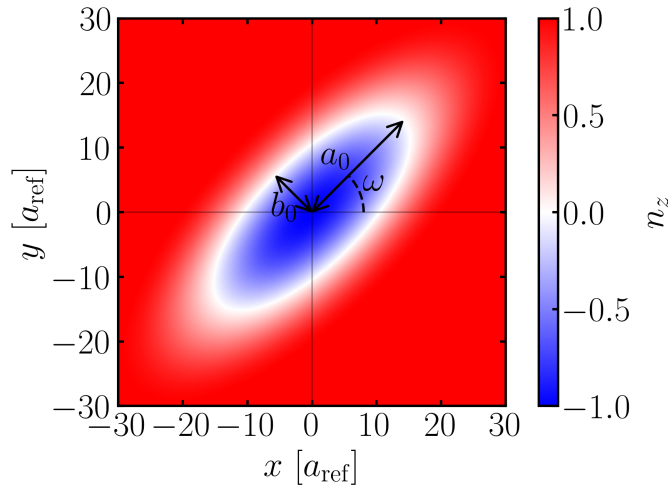


Figure 4.1.: Color map of the z -component of the magnetization field in Eq. (4.6) parametrized via the elliptic triangular approach. Depicted are the major and minor semi-axes a_0 and b_0 , respectively, as well as the rotation angle ω . Parameters: $a_0 = 20$, $b_0 = 8$, $\omega = \pi/4$. [Figure adapted from Ref. [136]]

netization, as introduced in Eq. (4.5), also induces an elliptic deformation of the in-plane component. However, our ansatz for $\Phi(\varphi)$ in Eq. (2.23) does not allow for such deformations of the azimuthal angle. As discussed in Sec. 2.4.4, the triangular profile of the polar angle already imposes a strongly simplified description of the antiskyrmion, neglecting the smooth internal structure of the domain wall-like shoulder. Therefore, incorporating a highly detailed modulation of Φ would exceed the overall accuracy of the model. At the same time, the simplified ansatz for Φ enables an analytical treatment of the dynamical equations presented here. Thus, this framework captures the qualitative trends of the antiskyrmion's shrinking behavior in a material with isotropic bulk DMI within a simplified collective-coordinate approach.

4.2. The energy of an elliptic antiskyrmion

Having established the geometric form of the elliptic triangular ansatz, we now derive its total energy based on the energy functional introduced in Eq. (2.34). So, the general set-up is identical to that in Chapter 3, with the system parameters being the exchange interaction J , the isotropic bulk DMI $\tilde{D}$, and the external Zeeman field $\tilde{B}$. Throughout this work, all energies are expressed in units of J , such that $J = 1$ and the reference lattice constant a_{ref} is used as the fundamental unit of length.

We assume that the azimuthal angle $\Phi(\varphi)$ depends only on the azimuthal coordinate φ , such that it is given by Eq. (2.23). The antiskyrmion is then formed via $m = -1$. The energy density, $\mathcal{E} = \mathcal{E}[\rho, \Theta(\rho, \varphi), \Phi(\varphi)]$, is the integrand of the total energy in Eq. (2.34) and, for an arbitrary radial profile $\Theta(\rho, \varphi)$, reads

$$\mathcal{E} = \frac{J}{2} \left[(\partial_\rho \Theta)^2 + \frac{(\partial_\varphi \Theta)^2}{\rho^2} + \frac{\sin^2 \Theta}{\rho^2} \right] + \tilde{D} \left[\left(\frac{\sin(2\Theta)}{2\rho} - (\partial_\rho \Theta) \right) \sin(2\varphi - \varphi_0) - \frac{(\partial_\varphi \Theta)}{\rho} \cos(2\varphi - \varphi_0) \right] + \tilde{B} - \tilde{B} \cos \Theta. \quad (4.7)$$

Here, $\partial_\rho \Theta$ and $\partial_\varphi \Theta$ denote the partial derivatives of the radial profile with respect to ρ and φ , respectively. By substituting the triangular approximation in Eq. (4.1) together with the elliptical ansatz in Eq. (4.5), these derivatives become

$$\partial_\rho \Theta = -\frac{\pi}{2\rho_0}, \quad (4.8)$$

$$\partial_\varphi \Theta = \frac{(b_0^2 - a_0^2) \pi \rho \rho_0}{4 a_0^2 b_0^2} \sin(2(\varphi - \omega)). \quad (4.9)$$

The total energy is then obtained by integrating the energy density over polar coordinates

$$E(a_0, e_0, \varphi_0, \omega) = \int_0^{2\pi} d\varphi \int_0^{2\rho_0} d\rho \rho \mathcal{E}(a_0, e_0, \varphi_0, \omega, \rho, \varphi). \quad (4.10)$$

The upper bound of the integration over the radial coordinate is $2\rho_0$ and therefore depends explicitly on the azimuthal coordinate φ . Because of this, it is crucial to integrate first over ρ . Performing this integration yields

$$\begin{aligned}
E(a_0, b_0, \varphi_0, \omega) = \int_0^{2\pi} d\varphi \left\{ J \left[\frac{1}{4} (\pi^2 + \ln(2\pi) + C - \text{Ci}(2\pi)) \right. \right. \\
+ \frac{(a_0^2 - b_0^2)^2 \pi^2 \sin^2(2\varphi - 2\omega)}{16 (b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega))^2} \Big] \\
+ \tilde{D} \frac{a_0 b_0 \pi [(a_0^2 + b_0^2) \sin(2\varphi - \varphi_0) + (a_0^2 - b_0^2) \sin(\varphi_0 - 2\omega)]}{2 (b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega))^{3/2}} \\
\left. + \tilde{B} \frac{2a_0^2 b_0^2 (\pi^2 - 4)}{\pi^2 (b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega))} \right\}. \quad (4.11)
\end{aligned}$$

The remaining integration over φ generally has no closed-form expression. However, for the simplified case $a_0 = b_0$, corresponding to a circular antiskyrmion, the integral can be evaluated analytically, yielding

$$E(a_0) = J \left[\frac{\pi^3}{2} + \frac{\pi}{2} (\ln(2\pi) + C - \text{Ci}(2\pi)) \right] + \left(\pi - \frac{4}{\pi} \right) 4a_0^2 \tilde{B}. \quad (4.12)$$

This energy contains the same contributions from the exchange interaction and the external magnetic field as in the isotropic skyrmion case, Eq. (3.9). The dependence on the rotation angle ω vanishes, reflecting the rotational symmetry of the system. In contrast to the skyrmion, the circular antiskyrmion's energy is independent of the DMI and the helicity [136]. This can be understood by considering the twofold symmetry of the antiskyrmion and the fact that it inherently includes regions of Bloch-, anti-Bloch-, Néel-, and anti-Néel-type windings, independent of the value of φ_0 . This is illustrated in Figure 4.2 (b). Figure 4.2 shows color maps of the energy density of the antiskyrmion in the circular limit. Panel (a) displays the total energy density, while panels (b)-(d) show the isolated DMI, exchange, and Zeeman contributions, respectively. By construction, the homogeneous ferromagnet corresponds to zero energy. The helicity is fixed at $\varphi_0 = \pi/2$, and in (b), the corresponding antiskyrmion configuration is indicated by the overlaid pictogram, emphasizing the different windings. Along the x -direction, the antiskyrmion exhibits Bloch-like winding, and the DMI contribution to the energy density $\mathcal{E}_{\text{DMI}}$ is lower than the ferromagnetic background. Orthogonally, along the y -axis, the antiskyrmion has an anti-Bloch-like configuration, where the energy is increased. In between, the (anti-)Néel-like configurations are energetically degenerate with the ferromagnetic background.

Thus, in the circular limit and in an environment with isotropic bulk DMI, the corresponding contributions of the different windings to the total energy are evenly distributed and average to zero. Therefore, Eq. (4.12) consists only of the constant energy shift induced by the exchange interaction strength J and a parabolic energy increase with increasing radius a_0 induced by the external field $\tilde{B}$. Accordingly, panels (c) and (d) in

Figure 4.2 show that the exchange and Zeeman contributions are radially isotropic. Thus, an antiskyrmion with a finite size will always have a higher energy than the homogeneous ferromagnetic state.

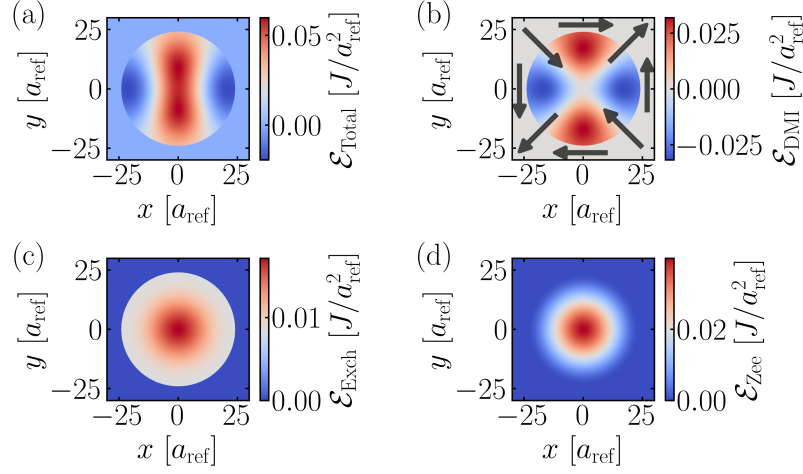


Figure 4.2.: Energy density distribution of a circular antiskyrmion. Panel (a) shows the total energy density, while panels (b)-(d) display the DMI, exchange, and Zeeman contributions, respectively. In (b), the overlaid pictogram indicates the underlying antiskyrmion configuration, showing Bloch-like winding along the x -axis and anti-Bloch-like winding along the y -axis. The (anti-)Néel-like regions in between are energetically degenerate with the ferromagnetic background, which is defined as zero energy. The exchange (c) and Zeeman (d) contributions are radially isotropic. Parameters: $\tilde{D} = 0.2$, $\tilde{B} = 0.02$, $a_0 = 12$, $b_0 = 12$, $\varphi_0 = \pi/2$.

In an elliptic antiskyrmion, however, the areas of different windings are not evenly distributed. Consequently, different values of the antiskyrmion rotation angle ω at a fixed φ_0 lead to different total energies [136]. To investigate this further, we numerically integrate Eq. (4.11). Figure 4.3 shows color maps of $E(\varphi_0, \omega)$ for different semi-axes values and DMI strengths. While changing these parameters affects the absolute energy values in panels (a) and (b), the overall structure of the color maps remains the same. The paths of minimum and maximum energy are indicated by blue and red colors, respectively. The energy minima are located at

$$\varphi_0(\omega) = 2\omega - (4\ell + 1)\frac{\pi}{2}, \quad \ell \in \mathbb{Z}. \quad (4.13)$$

In general, the dependence of the helicity on the rotation angle is linear, i.e., $\varphi_0(\omega) \propto 2\omega$. Intuitively, this follows from the π -symmetry of the ellipse: while the helicity recovers the same configuration after a full 2π rotation, the ellipse requires only a half rotation of π . Thus, the energy-minimizing rotation angle varies with half the slope of the helicity. These grooves can be attributed to the DMI term in Eq. (4.11), which contains the argument

$$\sin(\varphi_0 - 2\omega) = -\sin\left((4\ell + 1)\frac{\pi}{2}\right) = -1, \quad \ell \in \mathbb{Z}, \quad (4.14)$$

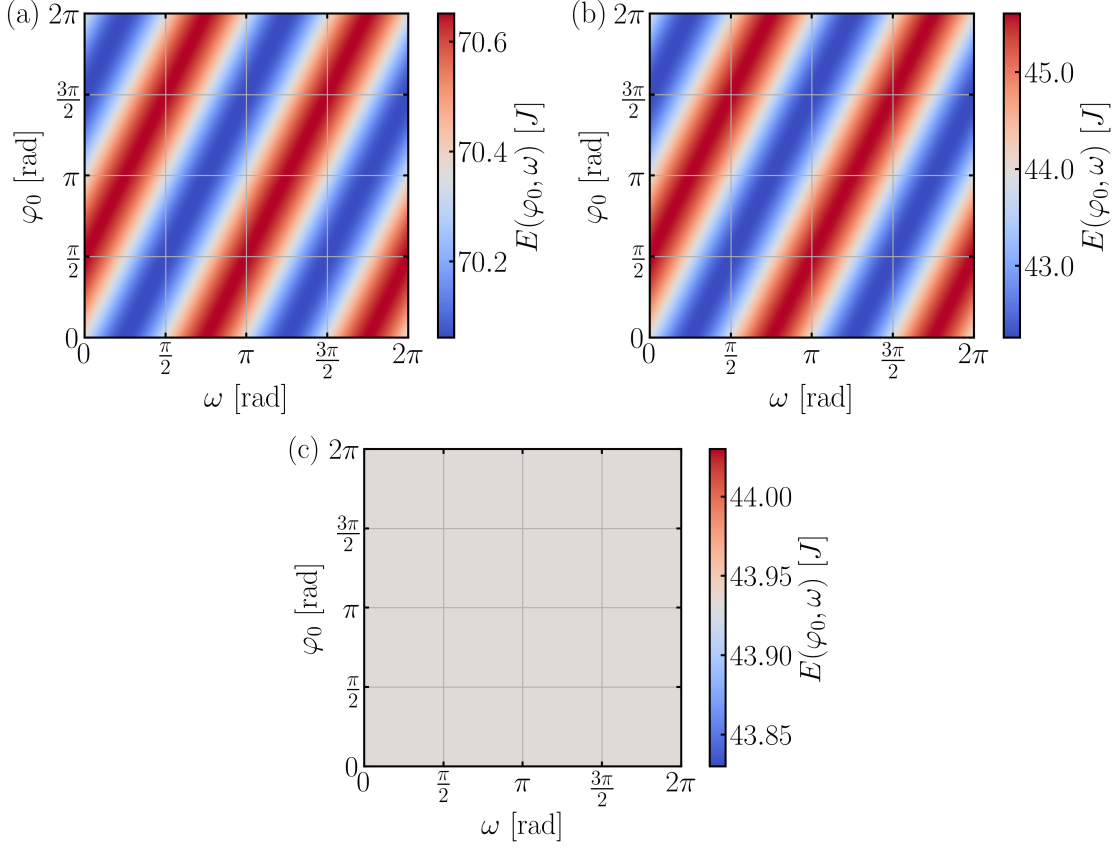


Figure 4.3.: Color maps of the total energy $E(\varphi_0, \omega)$ of an elliptic antiskyrmion obtained from numerical integration of Eq. (4.11). The plots illustrate how the energy depends on the helicity φ_0 and the rotation angle ω for different parameter sets. Panel (a) corresponds to $\tilde{D} = 0.02$, $\tilde{B} = 0.02$, $a_0 = 20$, and $b_0 = 17$, while panel (b) shows the case $\tilde{D} = 0.04$, $\tilde{B} = 0.02$, $a_0 = 15$, and $b_0 = 4$. In both cases, the overall structure of the energy landscape remains the same and reflects the interplay between the 2π -periodicity of the helicity and the π -periodicity of the ellipse. Blue and red regions indicate the minima and maxima paths given by Eqs. (4.13) and (4.15), respectively. Panel (c) shows the color map for the same parameter values as in (b), but for vanishing DMI ($\tilde{D} = 0$). In the absence of DMI, the energy landscape flattens and remains constant throughout the (φ_0, ω) parameter space, reflecting the energetic degeneracy of all orientations. [Figures (a) and (b) adapted from Ref. [136]]

which highlights its significant role in the energy functional. Likewise, the maximum energy paths in Figure 4.3 are given by

$$\varphi_0(\omega) = 2\omega - (4\ell + 3)\frac{\pi}{2}, \quad \ell \in \mathbb{Z}, \quad (4.15)$$

for which the sine term in the DMI contribution becomes maximal,

$$\sin(\varphi_0 - 2\omega) = -\sin\left((4\ell + 3)\frac{\pi}{2}\right) = 1, \quad \ell \in \mathbb{Z}. \quad (4.16)$$

Additionally, panel (c) shows the energy landscape in the absence of DMI, illustrating

that the energy becomes completely flat, i.e., $E(\varphi_0, \omega) = \text{const.}$ Consequently, the distinct minima and maxima observed in panels (a) and (b) can be attributed solely to the DMI contribution. Thus, the DMI introduces a directional energy preference in the (φ_0, ω) parameter space.

The dependence of the total energy on the semi-axes for fixed values of the helicity and rotation angle is shown in Figure 4.4 (a). Here, numerical integration of Eq. (4.11) yields a color map of $E(a_0, b_0)$, with slices at constant values of a_0 indicated by colored dashed lines, which are displayed in the $E(b_0)$ plot in panel (b).

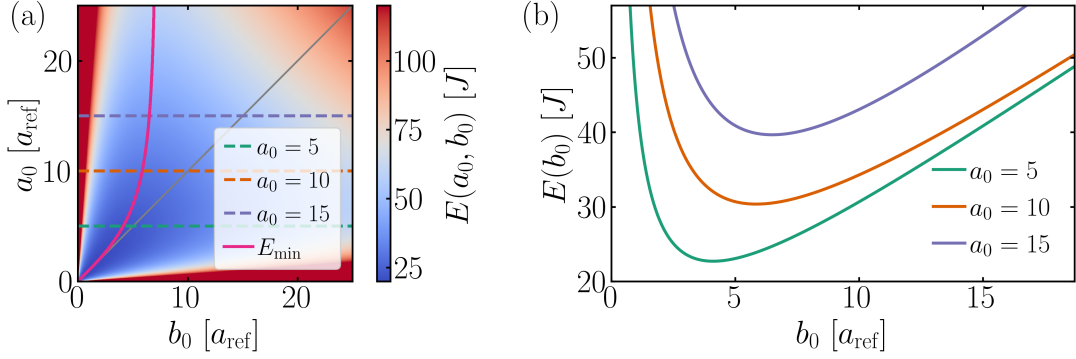


Figure 4.4.: Color map of the total energy $E(a_0, b_0)$ of an elliptic antiskyrmion obtained from numerical integration of Eq. (4.11) for fixed values of the helicity and the rotation angle. Panel (a) shows the full energy surface as a function of the semi-axes a_0 and b_0 , with the gray line indicating the circular limit $a_0 = b_0$ and the magenta line marking the minimum energy path. The colored dashed lines are slices at constant a_0 , shown in panel (b), illustrating the energy as a function of b_0 . Energy minima occur at finite $b_0 < a_0$, indicating that elliptic configurations are energetically favored over circular ones. Parameters: $\tilde{D} = 0.02$, $\tilde{B} = 0.02$, $\varphi_0 = 0$, $\omega = \pi/2$. [Figure adapted from Ref. [136]]

While the antiskyrmion energy is always above that of the ferromagnetic ground state, panel (b) reveals an energy minimum at $b_0 < a_0$ for a finite $b_0 > 0$ at a fixed a_0 . Thus, although an elliptic antiskyrmion remains energetically higher than the homogeneous ferromagnet, it lowers its energy compared to a circular configuration [136]. The gray line in panel (a) corresponds to $a_0 = b_0$, while the magenta line indicates the minimum energy path through the energy surface. This path can be mirrored across the $a_0 = b_0$ line. The plot also shows that the larger a_0 is, the more pronounced the ellipticity of its minimum energy configuration is. Consequently, the minimum energy configuration of a smaller antiskyrmion is closer to the circular limit.

Figure 4.5 illustrates the dependence of the energy density of the elliptic triangular antiskyrmion on its rotation angle. Here, the energy density for an antiskyrmion is shown for rotation angles of $\omega = 0$ and $\omega = \pi/2$. The helicity is in both cases fixed to $\varphi_0 = \pi/2$ and the corresponding antiskyrmion configuration is indicated by the overlaid pictogram in panel (a). Panels (a) and (b) display the total energy density including all contributions, demonstrating that the energy density is not rotationally invariant, in agreement with

Figure 4.3. Panels (c) and (d) show the DMI contribution, panels (e) and (f) the exchange contribution, and panels (g) and (h) the Zeeman contribution. The exchange and Zeeman contributions are rotationally invariant, while the DMI contribution varies with ω [136]. In particular, the $\omega = \pi/2$ rotation reduces the area of unfavorable anti-Bloch- and (anti-) Néel-type regions compared to the $\omega = 0$ case, leading to a lower overall energy density, also in agreement with Figure 4.3.

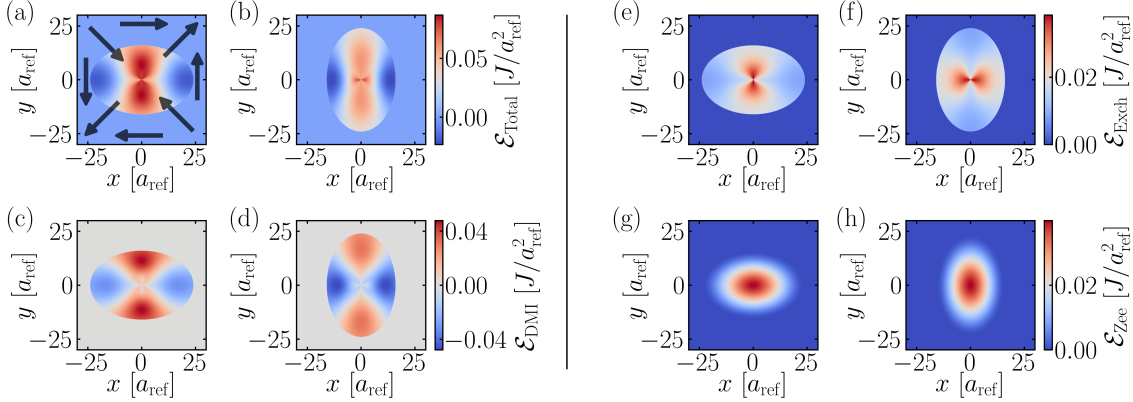


Figure 4.5.: Energy density of an elliptic triangular antiskyrmion, obtained from numerical evaluation of Eq. (4.11), for rotation angles $\omega = 0$ (panels (a), (c), (e), (g)) and $\omega = \pi/2$ (panels (b), (d), (f), (h)). The helicity is fixed at $\varphi_0 = \pi/2$, and the corresponding antiskyrmion configuration is overlaid in panel (a). Panels (a) and (b) show the total energy density, while panels (c) and (d) display the DMI contribution, panels (e) and (f) the exchange contribution, and panels (g) and (h) the Zeeman contribution. The exchange and Zeeman contributions are rotationally invariant, whereas the DMI term varies with ω . The $\omega = \pi/2$ rotation reduces the area of unfavorable anti-Bloch- and (anti-)Néel-type regions compared to $\omega = 0$, leading to a lower overall energy density. Parameters: $\tilde{D} = 0.2$, $\tilde{B} = 0.02$, $a_0 = 12$, $b_0 = 8$, $\varphi_0 = \pi/2$. [Figure adapted from Ref. [136]]

4.3. The dynamical equations for the antiskyrmion shrinking

In the last section, we discussed how the twofold symmetry of the antiskyrmion leads to lower energies for elliptic shapes compared to circular ones. Consequently, we now extend the circular ansatz to an elliptic one, which captures this anisotropic deformation. To this end, the antiskyrmion configuration is described by four collective quantities: the helicity φ_0 , the semi-axes a_0 and b_0 , and the rotation angle ω [136]. The helicity enters the model through the azimuthal angle, given by Eq. (2.23), where for antiskyrmions we set $m = -1$ such that $\Phi = -\varphi + \varphi_0$. The remaining parameters enter through the elliptic ansatz for the antiskyrmion radius in Eq. (4.5), which is inserted into the polar profile in Eq. (4.1).

In this section, we derive a set of coupled non-linear ordinary differential equations that govern the time evolution of these collective coordinates. This procedure follows closely the approach used in Sec. 3.2 for the shrinking dynamics of circular skyrmions, but is now

generalized to account for the elliptic deformation.

The first step is to compute the time derivative of the magnetization field in Eq. (4.6) using the chain rule and we denote this derivative as the left-hand side (LHS)

$$\dot{\mathbf{n}}^{\text{LHS}}(a_0(t), b_0(t), \varphi_0(t), \omega(t)) = \begin{pmatrix} \dot{\Theta} \cos \Theta \cos \Phi - \dot{\Phi} \sin \Theta \sin \Phi \\ \dot{\Theta} \cos \Theta \sin \Phi + \dot{\Phi} \sin \Theta \cos \Phi \\ -\dot{\Theta} \sin \Theta \end{pmatrix}. \quad (4.17)$$

The time derivative of the azimuthal angle is straightforward, $\dot{\Phi}(\varphi_0(t)) = \dot{\varphi}_0$, while the time derivative of the polar angle governs the dynamics of the semi-axes and the rotation angle, and thus reads

$$\dot{\Theta}(a_0(t), b_0(t), \omega(t)) = \frac{\partial \Theta}{\partial a_0} \dot{a}_0 + \frac{\partial \Theta}{\partial b_0} \dot{b}_0 + \frac{\partial \Theta}{\partial \omega} \dot{\omega}. \quad (4.18)$$

The full expression of the components of $[\dot{\mathbf{n}}]_{x,y,z}^{\text{LHS}}$ that are used throughout the subsequent derivations are displayed in Appendix C.1.

To extract the equations of motion for the collective coordinates, we follow a similar weighting procedure introduced for the circular skyrmion case in Sec. 3.2. Equation (3.28) already suggests an appropriate starting point for isolating the time-dependent quantities. However, due to the elliptic deformation, we introduce an additional geometric weighting factor

$$\eta = \pi (b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega))^2, \quad (4.19)$$

which modifies the angular contributions to the projection integrals. This factor ensures that each direction is properly weighted according to the elliptic geometry of the ansatz and makes integrals solvable. Since the upper bound of the ρ -integration, $2\rho_0(\varphi)$, depends explicitly on φ , the integration over ρ must be performed first.

Taking this into account and following Eq. (3.28), the helicity can be isolated from the x - and y -components of Eq. (4.17), that are the components that gather terms proportional to $\dot{\varphi}_0$. The isolation procedure yields

$$\int_0^{2\pi} d\varphi \int_0^{2\rho_0} d\rho \rho \left[[\dot{\mathbf{n}}]_x^{\text{LHS}} \eta \sin \Phi - [\dot{\mathbf{n}}]_y^{\text{LHS}} \eta \cos \Phi \right] = -4\pi a_0^2 b_0^2 (a_0^2 + b_0^2) \dot{\varphi}_0. \quad (4.20)$$

To obtain the evolution of the semi-axes a_0 and b_0 , we follow Eq. (3.28) and combine projections of the x - and y -components of $\dot{\mathbf{n}}^{\text{LHS}}$ using the same weighting factor η . Integration yields

$$\int_0^{2\pi} d\varphi \int_0^{2\rho_0} d\rho \rho \left[\frac{[\dot{\mathbf{n}}]_x^{\text{LHS}}}{\cos \xi} \eta \cos \Phi + \frac{[\dot{\mathbf{n}}]_y^{\text{LHS}}}{\cos \xi} \eta \sin \Phi \right] = -\frac{4}{3} \pi^3 a_0 b_0 (b_0^3 \dot{a}_0 + a_0^3 \dot{b}_0). \quad (4.21)$$

Since the z -component of Eq. (4.17) also gathers terms proportional to $\dot{a}_0$ and $\dot{b}_0$, an equivalent expression can be derived from that component. The division by $\cos \xi$ in Eq. (4.21)

ensures consistency with the expression obtained from the z -component, when dividing it with $\sin \xi$ as

$$\int_0^{2\pi} d\varphi \int_0^{2\rho_0} d\rho \rho \frac{[\dot{\mathbf{n}}]_z^{\text{LHS}}}{\sin \xi} \eta = -\frac{4}{3}\pi^3 a_0 b_0 \left(b_0^3 \dot{a}_0 + a_0^3 \dot{b}_0 \right). \quad (4.22)$$

This weighting by $\cos \xi$ and $\sin \xi$ mirrors the derivation of the differential equation for the skyrmion radius and the equivalence of Eqs. (4.21) and (4.22) confirms the internal consistency of the projection method. However, in this case this method isolates the semi-axes from the helicity and the rotation angle, but the semi-axes themselves are not isolated from each other, yet. Therefore, a second combination of the same components but this time additionally weighted by $\cos(2\Phi)/\sin(2(\varphi_0 - \omega))$ and $\sin(2\Phi)/\cos(2(\varphi_0 - \omega))$ yields another expression for the semi-axes

$$\begin{aligned} \int_0^{2\pi} d\varphi \int_0^{2\rho_0} d\rho \rho \left\{ \left[\frac{[\dot{\mathbf{n}}]_x^{\text{LHS}}}{\cos \xi} \cos \Phi + \frac{[\dot{\mathbf{n}}]_y^{\text{LHS}}}{\cos \xi} \sin \Phi \right] \frac{\eta \cos(2\Phi)}{\sin(2(\varphi_0 - \omega))} \right. \\ \left. + \left[\frac{[\dot{\mathbf{n}}]_x^{\text{LHS}}}{\cos \xi} \cos \Phi + \frac{[\dot{\mathbf{n}}]_y^{\text{LHS}}}{\cos \xi} \sin \Phi \right] \frac{\eta \sin(2\Phi)}{\cos(2(\varphi_0 - \omega))} \right\} \\ = \frac{4}{3}\pi^3 a_0 b_0 \csc(4(\varphi_0 - \omega)) \left(a_0^3 \dot{b}_0 - b_0^3 \dot{a}_0 \right). \end{aligned} \quad (4.23)$$

As before, an identical expression is obtained from the z -component, again demonstrating the robustness of the method

$$\begin{aligned} \int_0^{2\pi} d\varphi \int_0^{2\rho_0} d\rho \rho \left[\frac{[\dot{\mathbf{n}}]_z^{\text{LHS}}}{\sin \xi} \frac{\eta \cos(2\Phi)}{\sin(2(\varphi_0 - \omega))} + \frac{[\dot{\mathbf{n}}]_z^{\text{LHS}}}{\sin \xi} \frac{\eta \sin(2\Phi)}{\cos(2(\varphi_0 - \omega))} \right] \\ = \frac{4}{3}\pi^3 a_0 b_0 \csc(4(\varphi_0 - \omega)) \left(a_0^3 \dot{b}_0 - b_0^3 \dot{a}_0 \right). \end{aligned} \quad (4.24)$$

From Eqs. (4.21) and (4.23) or, equivalently, Eqs. (4.22) and (4.24), $\dot{a}_0$ and $\dot{b}_0$ can be isolated explicitly as

$$\dot{a}_0 = -\frac{3}{8\pi^3 a_0 b_0^4} \left(\text{Eq. (4.21)} + \frac{\text{Eq. (4.23)}}{\csc(4(\varphi_0 - \omega))} \right), \quad (4.25)$$

$$\dot{b}_0 = -\frac{3}{8\pi^3 a_0^4 b_0} \left(\text{Eq. (4.21)} - \frac{\text{Eq. (4.23)}}{\csc(4(\varphi_0 - \omega))} \right). \quad (4.26)$$

Finally, the time derivative of the rotation angle ω is obtained by similarly constructing a linear combination of integrals weighted by $\sin(2\Phi)/\sin(2(\varphi_0 - \omega))$ and $\cos(2\Phi)/\cos(2(\varphi_0 - \omega))$. This procedure isolates the terms proportional to $\dot{\omega}$ from the x - and y -components

while cancelling those associated with $\dot{a}_0$ and $\dot{b}_0$, yielding

$$\begin{aligned} \int_0^{2\pi} d\varphi \int_0^{2\rho_0} d\rho \rho \left\{ \left[\frac{[\dot{\mathbf{n}}]_x^{\text{LHS}}}{\cos \xi} \cos \Phi + \frac{[\dot{\mathbf{n}}]_y^{\text{LHS}}}{\cos \xi} \sin \Phi \right] \frac{\eta \sin(2\Phi)}{\sin(2(\varphi_0 - \omega))} \right. \\ \left. - \left[\frac{[\dot{\mathbf{n}}]_x^{\text{LHS}}}{\cos \xi} \cos \Phi + \frac{[\dot{\mathbf{n}}]_y^{\text{LHS}}}{\cos \xi} \sin \Phi \right] \frac{\eta \cos(2\Phi)}{\cos(2(\varphi_0 - \omega))} \right\} \\ = \frac{4}{3} \pi^3 a_0^2 b_0^2 (a_0^2 - b_0^2) \csc(4(\varphi_0 - \omega)) \dot{\omega}. \end{aligned} \quad (4.27)$$

The z -component also gathers terms proportional to $\dot{\omega}$ and thus the same relation can again be verified from the z -component as

$$\begin{aligned} \int_0^{2\pi} d\varphi \int_0^{2\rho_0} d\rho \rho \left[\frac{[\dot{\mathbf{n}}]_z^{\text{LHS}}}{\sin \xi} \frac{\eta \sin(2\Phi)}{\sin(2(\varphi_0 - \omega))} - \frac{[\dot{\mathbf{n}}]_z^{\text{LHS}}}{\sin \xi} \frac{\eta \cos(2\Phi)}{\cos(2(\varphi_0 - \omega))} \right] \\ = \frac{4}{3} \pi^3 a_0^2 b_0^2 (a_0^2 - b_0^2) \csc(4(\varphi_0 - \omega)) \dot{\omega}. \end{aligned} \quad (4.28)$$

Together, these projections yield a complete pathway to derive a set of four coupled non-linear ordinary differential equations for the quantities $a_0(t)$, $b_0(t)$, $\varphi_0(t)$, and $\omega(t)$, which describe the collective dynamics of the elliptic antiskyrmion [136].

Applying these projections to the components of the LLG in Eq. (2.21), denoted in the following as the right-hand side (RHS), yields then the full expressions. Therefore, the effective magnetic field in Eq. (2.20) needs to be evaluated in terms of the elliptic triangular ansatz to express the LLG within the given framework. The full expressions of the effective magnetic field components are displayed in Appendix C.2 together with the resulting full expressions of the LLG components $[\dot{\mathbf{n}}]_{x,y,z}^{\text{RHS}}$. With these, we derive the right-hand sides to the corresponding isolated collective coordinates. The terms governed by the DMI term lead to expressions during the weighting procedure, that are not integrable with respect to φ . Therefore, we derive and discuss to begin with only the contributions coming from the exchange and Zeeman terms. The finite DMI regime will be treated in the subsequent section. Thus by applying the same weighting as in Eq. (4.20) to the components gathered from the LLG, while setting $\tilde{D} = 0$, we obtain

$$\begin{aligned} \int_0^{2\pi} d\varphi \int_0^{2\rho_0} d\rho \rho \left[[\dot{\mathbf{n}}]_x^{\text{RHS}} \eta \sin \Phi - [\dot{\mathbf{n}}]_y^{\text{RHS}} \eta \cos \Phi \right] = \frac{a^2}{1 + \alpha^2} \left\{ \tilde{B} \pi [-4a_0^4 b_0^2 - 4a_0^2 b_0^4] \right. \\ \left. + J \pi^2 \left[-2a_0^2 b_0^2 \pi + \frac{1}{4} a_0^2 b_0^2 \text{Si}(2\pi) + \frac{3}{8} a_0^4 \text{Si}(2\pi) + \frac{3}{8} b_0^4 \text{Si}(2\pi) \right] \right\}. \end{aligned} \quad (4.29)$$

Equating this expression with Eq. (4.20) and subsequent rearranging yields the equation for the time evolution of the antiskyrmion helicity [136]

$$\dot{\varphi}_0 = \frac{a^2}{1 + \alpha^2} \left\{ \tilde{B} + \frac{J\pi}{a_0^2 + b_0^2} \left[\frac{\pi}{2} - \frac{\text{Si}(2\pi)}{16} - \frac{3a_0^2 \text{Si}(2\pi)}{32b_0^2} - \frac{3b_0^2 \text{Si}(2\pi)}{32a_0^2} \right] \right\}. \quad (4.30)$$

The right-hand side equations for the semi-axes are then derived from the LLG components

following the weighting procedure shown in Eqs. (4.21)-(4.24). $\dot{a}_0$ and $\dot{b}_0$ are then isolated by applying Eqs. (4.25) and (4.26). So, firstly we derive from the x - and y -components

$$\int_0^{2\pi} d\varphi \int_0^{2\rho_0} d\rho \rho \left[\frac{[\dot{\mathbf{n}}]_x^{\text{RHS}}}{\cos \xi} \eta \cos \Phi + \frac{[\dot{\mathbf{n}}]_y^{\text{RHS}}}{\cos \xi} \eta \sin \Phi \right] = \frac{a^2 \alpha}{1 + \alpha^2} \left\{ 4\tilde{B}\pi [a_0^4 b_0^2 + a_0^2 b_0^4] \right. \\ \left. + J\pi^2 \left[2\pi a_0^2 b_0^2 - \frac{1}{4} a_0^2 b_0^2 \text{Si}(2\pi) - \frac{3}{8} a_0^4 \text{Si}(2\pi) - \frac{3}{8} b_0^4 \text{Si}(2\pi) \right] \right\}, \quad (4.31)$$

while correspondingly the z -component yields the same expression

$$\int_0^{2\pi} d\varphi \int_0^{2\rho_0} d\rho \rho \frac{[\dot{\mathbf{n}}]_z^{\text{RHS}}}{\sin \xi} \eta = \frac{a^2 \alpha}{1 + \alpha^2} \left\{ 4\tilde{B}\pi [a_0^4 b_0^2 + a_0^2 b_0^4] \right. \\ \left. + J\pi^2 \left[2\pi a_0^2 b_0^2 - \frac{1}{4} a_0^2 b_0^2 \text{Si}(2\pi) - \frac{3}{8} a_0^4 \text{Si}(2\pi) - \frac{3}{8} b_0^4 \text{Si}(2\pi) \right] \right\}. \quad (4.32)$$

Secondly, by means of the weighting procedure in Eqs. (4.23) and (4.24), we derive their corresponding right-hand sides, both from the x - and y -component as

$$\int_0^{2\pi} d\varphi \int_0^{2\rho_0} d\rho \rho \left\{ \left[\frac{[\dot{\mathbf{n}}]_x^{\text{RHS}}}{\cos \xi} \cos \Phi + \frac{[\dot{\mathbf{n}}]_y^{\text{RHS}}}{\cos \xi} \sin \Phi \right] \frac{\eta \cos(2\Phi)}{\sin(2(\varphi_0 - \omega))} \right. \\ \left. + \left[\frac{[\dot{\mathbf{n}}]_x^{\text{RHS}}}{\cos \xi} \cos \Phi + \frac{[\dot{\mathbf{n}}]_y^{\text{RHS}}}{\cos \xi} \sin \Phi \right] \frac{\eta \sin(2\Phi)}{\cos(2(\varphi_0 - \omega))} \right\} \\ = \frac{a^2 \alpha}{1 + \alpha^2} \csc(4(\varphi_0 - \omega)) \left\{ 4\tilde{B}\pi [a_0^2 b_0^4 - a_0^4 b_0^2] + \frac{J}{2} \pi^2 \text{Si}(2\pi) [a_0^4 - b_0^4] \right\}, \quad (4.33)$$

and accordingly from the z -component as

$$\int_0^{2\pi} d\varphi \int_0^{2\rho_0} d\rho \rho \left[\frac{[\dot{\mathbf{n}}]_z^{\text{RHS}}}{\sin \xi} \frac{\eta \cos(2\Phi)}{\sin(2(\varphi_0 - \omega))} + \frac{[\dot{\mathbf{n}}]_z^{\text{RHS}}}{\sin \xi} \frac{\eta \sin(2\Phi)}{\cos(2(\varphi_0 - \omega))} \right] \\ = \frac{a^2 \alpha}{1 + \alpha^2} \csc(4(\varphi_0 - \omega)) \left\{ 4\tilde{B}\pi [a_0^2 b_0^4 - a_0^4 b_0^2] + \frac{J}{2} \pi^2 \text{Si}(2\pi) [a_0^4 - b_0^4] \right\}. \quad (4.34)$$

Following Eqs. (4.25) and (4.26), the isolated equations of motion for the semi-axes $\dot{a}_0$ and $\dot{b}_0$ follow now by use of the corresponding right-hand side Eqs. (4.31) and (4.33) or, equivalently, Eqs. (4.32) and (4.34) as [136]

$$\dot{a}_0 = -\frac{3}{8\pi^3 a_0 b_0^4} \left(\text{Eq. (4.31)} + \frac{\text{Eq. (4.33)}}{\csc(4(\varphi_0 - \omega))} \right) \\ = \frac{a^2 \alpha}{1 + \alpha^2} \left\{ -3\tilde{B} \frac{a_0}{\pi^2} + J \left[-\frac{3a_0}{4b_0^2} + \frac{21}{64a_0\pi} \text{Si}(2\pi) \right. \right. \\ \left. \left. - \frac{3a_0^3}{64b_0^4\pi} \text{Si}(2\pi) + \frac{3a_0}{32b_0^2\pi} \text{Si}(2\pi) \right] \right\}, \quad (4.35)$$

$$\begin{aligned}
\dot{b}_0 &= -\frac{3}{8\pi^3 a_0^4 b_0} \left(\text{Eq. (4.31)} - \frac{\text{Eq. (4.33)}}{\csc(4(\varphi_0 - \omega))} \right) \\
&= \frac{a^2 \alpha}{1 + \alpha^2} \left\{ -3\tilde{B} \frac{b_0}{\pi^2} + J \left[-\frac{3b_0}{4a_0^2} + \frac{21}{64b_0\pi} \text{Si}(2\pi) \right. \right. \\
&\quad \left. \left. - \frac{3b_0^3}{64a_0^4\pi} \text{Si}(2\pi) + \frac{3b_0}{32a_0^2\pi} \text{Si}(2\pi) \right] \right\}. \tag{4.36}
\end{aligned}$$

Finally, we can find the expression for the rotation angle by following the weighting procedure of Eqs. (4.27) and (4.28) and applying it to the components of the LLG. This yields from the x - and y -components

$$\begin{aligned}
\int_0^{2\pi} d\varphi \int_0^{2\rho_0} d\rho \rho \left\{ \left[\frac{[\dot{\mathbf{n}}]_x^{\text{RHS}}}{\cos \xi} \cos \Phi + \frac{[\dot{\mathbf{n}}]_y^{\text{RHS}}}{\cos \xi} \sin \Phi \right] \frac{\eta \sin(2\Phi)}{\sin(2(\varphi_0 - \omega))} \right. \\
\left. - \left[\frac{[\dot{\mathbf{n}}]_x^{\text{RHS}}}{\cos \xi} \cos \Phi + \frac{[\dot{\mathbf{n}}]_y^{\text{RHS}}}{\cos \xi} \sin \Phi \right] \frac{\eta \cos(2\Phi)}{\cos(2(\varphi_0 - \omega))} \right\} = 0, \tag{4.37}
\end{aligned}$$

with the same equation resulting from the z -component as

$$\int_0^{2\pi} d\varphi \int_0^{2\rho_0} d\rho \rho \left[\frac{[\dot{\mathbf{n}}]_z^{\text{RHS}}}{\sin \xi} \frac{\eta \sin(2\Phi)}{\sin(2(\varphi_0 - \omega))} - \frac{[\dot{\mathbf{n}}]_z^{\text{RHS}}}{\sin \xi} \frac{\eta \cos(2\Phi)}{\cos(2(\varphi_0 - \omega))} \right] = 0. \tag{4.38}$$

This yields trivially together with Eqs. (4.27) and (4.28) that at vanishing DMI, the dynamical equation for the rotation angle is [136]

$$\dot{\omega} = 0, \tag{4.39}$$

implying that $\omega(t) = \text{const.}$ Also, the final expressions for the ordinary linear differential equations of the helicity in Eq. (4.30) and the semi-axes in Eqs. (4.35) and (4.36) show no dependence on the instantaneous value of the antiskyrmion's rotation angle ω . In general, the structure of the vanishing DMI case is reminiscent of the case of the isotropic skyrmion shrinking. While the equations for the semi-axes are coupled to each other, they are decoupled from the helicity, just as the differential equation for the skyrmion shrinking decouples from the equation of the helicity in Eq. (3.21) at zero DMI strength. Furthermore, the antiskyrmion's helicity shows a linear dependence on the Zeeman field, and couples to the semi-axes via the exchange term, just like the skyrmion's helicity does with regards to its radius in Eq. (3.27). In the following section, we will analyze the shrinking dynamics for initially circular and elliptic antiskyrmions in detail.

4.4. Shrinking dynamics at vanishing DMI

To emphasise the structural dependence of the dynamical equations on the semi-axes, we rewrite the Eqs. (4.30), (4.35), (4.36) and (4.39) in terms of the dimensionless ratio $\chi = a_0/b_0$. Throughout the investigation, we take a_0 as the major axis and b_0 as the minor axis, i.e., $a_0 > b_0$. The ratio χ is therefore similar to the generally known eccentricity,

which is a measure of how much the ellipse deviates from the circular shape. However, in contrast to the eccentricity, the circular case is represented by $\chi = 1$, while a larger χ leads to a more elliptic shape. By applying this ratio, the system of coupled differential equations reads [136]

$$\dot{a}_0 = \frac{a^2 \alpha}{1 + \alpha^2} \left\{ -\frac{3}{\pi^2} \tilde{B} a_0 + \frac{3}{4} \frac{\text{Si}(2\pi)}{8\pi} \frac{J}{a_0} \left[\frac{7}{2} + \left(1 - \frac{8\pi}{\text{Si}(2\pi)} \right) \chi^2 - \frac{1}{2} \chi^4 \right] \right\}, \quad (4.40)$$

$$\dot{b}_0 = \frac{a^2 \alpha}{1 + \alpha^2} \left\{ -\frac{3}{\pi^2} \tilde{B} b_0 + \frac{3}{4} \frac{\text{Si}(2\pi)}{8\pi} \frac{J}{b_0} \left[\frac{7}{2} + \left(1 - \frac{8\pi}{\text{Si}(2\pi)} \right) \frac{1}{\chi^2} - \frac{1}{2\chi^4} \right] \right\}, \quad (4.41)$$

$$\dot{\varphi}_0 = \frac{a^2}{1 + \alpha^2} \left\{ \tilde{B} - \frac{\text{Si}(2\pi)}{16\pi} \frac{\pi^2 J}{a_0^2 + b_0^2} \left[1 - \frac{8\pi}{\text{Si}(2\pi)} + \frac{3}{2} \left(\chi^2 + \frac{1}{\chi^2} \right) \right] \right\}, \quad (4.42)$$

$$\dot{\omega} = 0. \quad (4.43)$$

Here, Si is the sine integral [128]. As stated above, the dynamics of the rotation angle suggest $\omega(t) = \text{const}$ and do not enter the other equations and as such Eq. (4.43) is only given for completeness. As expected, Eqs. (4.40) and (4.41) exhibit symmetry under the exchange $a_0 \leftrightarrow b_0$ combined with $\chi \leftrightarrow 1/\chi$, reflecting the invariance of the dynamics under relabelling of the principal axes.

4.4.1. Solution of the coupled differential equations

It is known from the energy considerations in Sec. 4.2 that, in the absence of DMI, the energy of a circular antiskyrmion takes the same form as the energy of a circular skyrmion. Consequently, it is expected that the shrinking dynamics of a circular antiskyrmion are comparable to those of the skyrmion investigated in Sec. 3.4.3. The coupled differential Eqs. (4.40)-(4.43) simplify for an initially circular antiskyrmion [136], where $a_0 = b_0$, such that the two semi-axis equations reduce to a single equation, i.e., $\dot{a}_0 = \dot{b}_0$, and become

$$\dot{a}_0 = \frac{a^2 \alpha}{1 + \alpha^2} \left[-\frac{3}{\pi^2} \tilde{B} a_0 - \varsigma_1 J \frac{1}{a_0} \right], \quad (4.44)$$

with $\varsigma_1 = \frac{3}{4} - \frac{3}{8\pi} \text{Si}(2\pi) \approx 0.58072$. The differential equation for the helicity also simplifies and obeys

$$\dot{\varphi}_0 = \frac{a^2}{1 + \alpha^2} \left[\tilde{B} + \varsigma_2 \frac{J}{a_0^2} \right], \quad (4.45)$$

with $\varsigma_2 = \frac{\pi^2}{4} - \frac{\pi}{8} \text{Si}(2\pi) \approx 1.9105$. Both equations mirror the structure of the corresponding skyrmion shrinking equations in Eqs. (3.21) and (3.27) for the zero DMI case, where a_0 now represents the antiskyrmion radius. Hence, the resulting antiskyrmion dynamics exhibit the same qualitative behavior as the skyrmion dynamics discussed in Sec. 3.4.3.

The numerical solution of the simplified system of Eqs. (4.44) and (4.45), is displayed in

Figure 4.6. The lattice constant is set to $a = 1$ for all plots in this section. In addition to the solution for the simplified equations of $\dot{a}_0$ and $\dot{\varphi}_0$, the time evolution of $b_0(t)$ is shown as the orange dashed line in panel (a), obtained from the full system of coupled differential Eqs. (4.40)-(4.42) with circular initial conditions. Both $a_0(t)$ and $b_0(t)$ evolve identically, confirming that the antiskyrmion remains circular throughout the shrinking process.

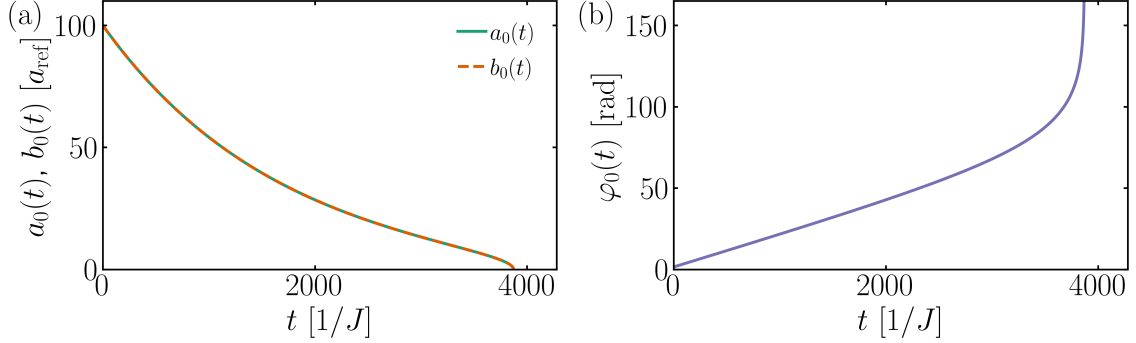


Figure 4.6.: Time evolution of the radius $a_0(t)$ and helicity $\varphi_0(t)$ of a circular antiskyrmion at vanishing DMI. Panel (a) shows the exponential decay of $a_0(t)$ at early times driven by the external magnetic field, followed by the square-root collapse close to the critical time t_c . The orange dashed line represents the evolution of $b_0(t)$ from the full coupled system, confirming that the antiskyrmion preserves its circular shape, i.e., $a_0(t) = b_0(t)$, throughout the shrinking. Panel (b) displays the linear-in-time precession of the helicity at large antiskyrmion sizes and its logarithmic divergence as the dynamics approach t_c . Parameters: $\tilde{B} = 0.02$, $\alpha = 0.1$, $a_{0,0} = 100$, $b_{0,0} = 100$, $\varphi_{0,0} = \pi/2$. [Figure adapted from Ref. [136]]

At large sizes, the external magnetic field drives an exponential decay of the radius $a_0(t)$, as shown in panel (a) and leads to a linear-in-time precession of the helicity $\varphi_0(t)$, as shown in panel (b). As t approaches the critical time t_c , the exchange term dominates, resulting in a square-root-like collapse of the radius and a logarithmic divergence of the helicity. Overall the dynamics of $a_0(t)$ can be expressed analytically similar to Eq. (3.33) as

$$a_0(t) = \left[a_{0,0}^2 \exp\left(-\frac{6a^2\alpha}{\pi^2(1+\alpha^2)}\tilde{B}t\right) + \frac{\pi^2\varsigma_1 J}{3\tilde{B}} \left(\exp\left(-\frac{6a^2\alpha}{\pi^2(1+\alpha^2)}\tilde{B}t\right) - 1 \right) \right]^{\frac{1}{2}}. \quad (4.46)$$

The corresponding critical time t_c follows analogously to the skyrmion case in Eq. (3.34) as

$$t_c = \frac{\pi^2(1+\alpha^2)}{6a^2\alpha\tilde{B}} \ln \left[1 + a_{0,0}^2 \frac{3\tilde{B}}{\pi^2\varsigma_1 J} \right], \quad (4.47)$$

and, as such, increases logarithmically with decreasing lattice constant according to $t_c \sim \ln(a_{0,0}/a)$. Note that, as discussed for the skyrmion case in Sec. 3.3, the topological charge is conserved in the continuum limit. As a result, the antiskyrmion collapses into a delta-function-like object at time t_c , rather than being annihilated into the topologically trivial ferromagnetic state. Consequently, the topological charge remains $\mathcal{Q} = 1$ and the actual

destruction of the antiskyrmion is not captured within this model.

We have shown that the shrinking of an initially circular antiskyrmion is comparable to that of a circular skyrmion in the absence of DMI. However, the system of equations in Eqs. (4.40)-(4.42) also allows us to study the shrinking dynamics of an initially elliptic antiskyrmion [136]. This corresponds to the case $a_{0,0} > b_{0,0}$, where the Zeeman term dominates at large sizes in the equations governing the time evolution of the semi-axes in Eqs. (4.40) and (4.41). In this regime both semi-axes decay exponentially with the same rate Γ yielding

$$a_0(t) \sim a_{0,0} \exp(-\Gamma t), \quad (4.48)$$

$$b_0(t) \sim b_{0,0} \exp(-\Gamma t), \quad (4.49)$$

where the rate Γ , identical to that of the circular case in Eq. (4.46), is given by

$$\Gamma = \frac{a^2 \alpha}{1 + \alpha^2} \frac{3}{\pi^2} \tilde{B}. \quad (4.50)$$

In the same large size regime, the helicity dynamics are also dominantly governed by the Zeeman term, leading to a linear-in-time rotation

$$\varphi_0(t) \sim \varphi_{0,0} + \frac{a^2}{1 + \alpha^2} \tilde{B} t. \quad (4.51)$$

The exchange interaction, on the other hand, becomes relevant at small sizes. In Eqs. (4.40) and (4.41), the exchange terms scale inversely with the semi-axes, i.e., as $1/a_0$ for the major axis and $1/b_0$ for the minor axis. Similarly, in Eq. (4.42), the exchange-driven helicity term scales as $1/(a_0^2 + b_0^2)$. These terms are further rescaled by even powers of the instantaneous ratio $\chi = a_0/b_0$. In particular, the exchange contributions to $\dot{a}_0$ and $\dot{b}_0$ are rescaled asymmetrically, by factors of χ and $1/\chi$, respectively. Therefore, in this exchange-dominated small-size regime, the dynamics are governed by the evolution of the ratio χ itself. Applying the chain rule to compute $\dot{\chi} = \partial_t(a_0(t)/b_0(t))$, we obtain

$$\dot{\chi} = (\dot{a}_0 b_0 - a_0 \dot{b}_0)/b_0^2. \quad (4.52)$$

Substituting $\dot{a}_0$ and $\dot{b}_0$ from Eqs. (4.40) and (4.41) into $\dot{\chi}$ yields

$$\dot{\chi} = -\frac{3a^2 \alpha}{64\pi(1 + \alpha^2)} \frac{J}{b_0^2} \frac{(\chi^2 - 1)}{\chi^3} [\text{Si}(2\pi)(1 + \chi^4) + 2\chi^2(8\pi + 3\text{Si}(2\pi))]. \quad (4.53)$$

This expression only exhibits contributions from the exchange terms of $\dot{a}_0$ and $\dot{b}_0$ as their Zeeman contributions cancel. For $\chi = 1$ the prefactor $(\chi^2 - 1)$ vanishes and hence $\dot{\chi}|_{\chi=1} = 0$, revealing a stationary point. To determine its stability, we evaluate the derivative of Eq. (4.53) at $\chi = 1$, which yields

$$\left. \frac{\partial \dot{\chi}}{\partial \chi} \right|_{\chi=1} = -\frac{3a^2\alpha(2\pi + \text{Si}(2\pi))}{4\pi(1+\alpha^2)} \frac{J}{b_0^2} < 0. \quad (4.54)$$

Since all parameters in this expression are positive, the negative sign confirms that the stationary point is stable [136]. Consequently, in the exchange-dominated regime, the shrinking is enhanced for the major axis a_0 compared to the minor axis b_0 , and the elliptic antiskyrmion is driven towards a circular shape ($\chi \rightarrow 1$). Once $a_0 = b_0$, the dynamics reduce to those of the circular antiskyrmion, as described by Eqs. (4.44) and (4.45). In this regime, the semi-axes obey the square-root collapse

$$a_0(t) = b_0(t) \sim \sqrt{\frac{2\zeta_1 a^2 \alpha}{1 + \alpha^2} J (t_c - t)}, \quad (4.55)$$

while the helicity exhibits a logarithmic divergence

$$\varphi_0(t) \sim -\frac{\zeta_2}{2\zeta_1} \ln(t_c - t). \quad (4.56)$$

The complete dynamics of an initially elliptic antiskyrmion in an environment with vanishing DMI are displayed in Figure 4.7, where the numerical solution of the coupled differential Eqs. (4.40)-(4.42) are shown. Panels (a) and (b) show the time evolution of the semi-axes and helicity, respectively. As described above, the semi-axes initially decay exponentially, and the helicity rotates linearly, both due to the Zeeman field dominating the dynamics at large sizes. Then, at small semi-axes, the exchange term that causes dynamical instability

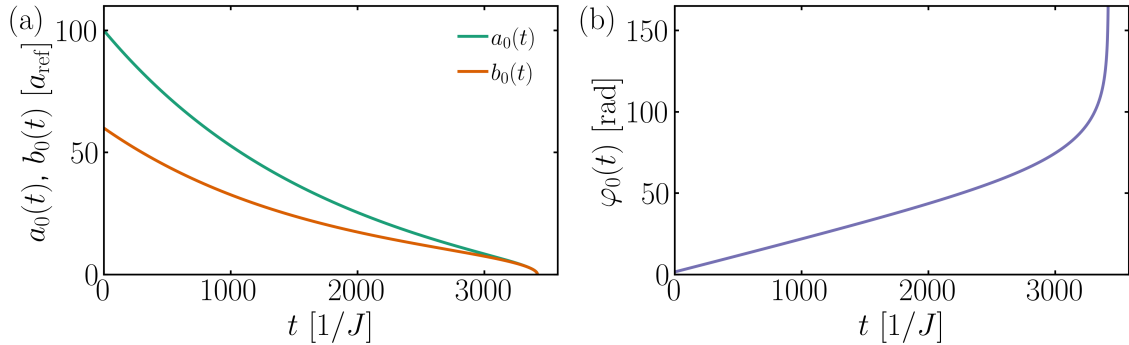


Figure 4.7.: Time evolution of the semi-axes $a_0(t)$ and $b_0(t)$ in panel (a) and the helicity $\varphi_0(t)$ in panel (b) of an initially elliptic antiskyrmion at zero DMI. The semi-axes decay exponentially at large sizes, driven by the Zeeman term. As the system enters the exchange-dominated regime at smaller sizes, the major axis shrinks faster than the minor axis, and the shape approaches a circular configuration ($\chi \rightarrow 1$). Once the antiskyrmion is circular, both semi-axes vanish following a square-root-like behavior. The corresponding helicity dynamics show an initial linear-in-time precession due to Zeeman field, followed by a rapid, exchange-driven logarithmic divergence as the system approaches the collapse time t_c . Parameters: $\bar{B} = 0.02$, $\alpha = 0.1$, $a_{0,0} = 100$, $b_{0,0} = 60$, $\varphi_{0,0} = \pi/2$. [Figure adapted from Ref. [136]]

of the ellipticity dominates and drives the antiskyrmion towards a circular shape as the ratio χ approaches 1. In this small-size regime, the exponential decay transitions to a square-root-like behavior of the semi-axes, and the helicity diverges logarithmically.

4.5. Shrinking dynamics at finite DMI

Above, we derived and solved the coupled dynamical equations for the collective coordinates a_0 , b_0 , φ_0 , and ω in the absence of the DMI. We found that the antiskyrmion's rotation angle ω remains constant in time, and the time evolution of the remaining three collective coordinates is independent of it. This behavior is consistent with the energy landscape shown in Figure 4.3 (c), where the energy $E(\varphi_0, \omega)$ exhibits complete degeneracy over the entire (φ_0, ω) parameter space when the DMI vanishes. Consequently, the differential equations for the semi-axes decouple also from the helicity dynamics. This is expected, since the exchange and Zeeman interactions are isotropic [139] and therefore invariant under rotation of the antiskyrmion, implying that all orientations and helicities are energetically equivalent. In contrast, the DMI breaks this rotational symmetry [54, 55] and introduces directional energy preferences, as shown in Figure 4.3 (a) and (b). Therefore, at finite DMI strength, the dynamics become more intricate, with the DMI contribution inducing dependencies on both the helicity and the rotation angle in the evolution of the semi-axes. In this section, we derive the DMI contribution to the coupled differential equations governing the time evolution of the collective coordinates.

4.5.1. Exact integral representation of the DMI contribution

To derive the DMI contributions to the coupled differential equations given by Eqs. (4.40)-(4.43), we follow the same projection procedure as outlined in Eqs. (4.29)-(4.39). To this end, we take the expressions for $[\dot{\mathbf{r}}]_{x,y,z}^{\text{RHS}}$ displayed in Appendix C.2, set $J = 0$ and $\tilde{B} = 0$, and obtain the DMI-induced dynamical equations of the isolated collective coordinates as [136]

$$\begin{aligned} \dot{a}_0^{\text{DMI}} = & \frac{3\tilde{D}a^2}{8\sqrt{2}b_0^3\pi^2(1+\alpha^2)} \int_0^{2\pi} d\varphi \sqrt{a_0^2 + b_0^2 + (b_0^2 - a_0^2) \cos(2(\varphi - \omega))} \\ & \times [2(a_0^2 + b_0^2) \cos(2\varphi - \varphi_0) - 2(a_0^2 - b_0^2) \cos(\varphi_0 - 2\omega) \\ & + \alpha(a_0^2 + b_0^2 + (b_0^2 - a_0^2) \cos(2(\varphi - \omega))) \sin(2\varphi - \varphi_0)] (2 \cos(2(\varphi - \omega)) + 1), \end{aligned} \quad (4.57)$$

$$\begin{aligned} \dot{b}_0^{\text{DMI}} = & -\frac{3\tilde{D}a^2}{8\sqrt{2}a_0^3\pi^2(1+\alpha^2)} \int_0^{2\pi} d\varphi \sqrt{a_0^2 + b_0^2 + (b_0^2 - a_0^2) \cos(2(\varphi - \omega))} \\ & \times [2(a_0^2 + b_0^2) \cos(2\varphi - \varphi_0) - 2(a_0^2 - b_0^2) \cos(\varphi_0 - 2\omega) \\ & + \alpha(a_0^2 + b_0^2 + (b_0^2 - a_0^2) \cos(2(\varphi - \omega))) \sin(2\varphi - \varphi_0)] (2 \cos(2(\varphi - \omega)) - 1), \end{aligned} \quad (4.58)$$

$$\begin{aligned} \dot{\varphi}_0^{\text{DMI}} = & \frac{\tilde{D}a^2}{4\sqrt{2}a_0b_0(a_0^2+b_0^2)(1+\alpha^2)} \int_0^{2\pi} d\varphi \sqrt{a_0^2+b_0^2+(b_0^2-a_0^2)\cos(2(\varphi-\omega))} \\ & \times [2(a_0^2+b_0^2)\alpha\cos(2\varphi-\varphi_0) - 2(a_0^2-b_0^2)\alpha\cos(\varphi_0-2\omega) \\ & + (-a_0^2-b_0^2+(a_0^2-b_0^2)\cos(2(\varphi-\omega)))\sin(2\varphi-\varphi_0)], \end{aligned} \quad (4.59)$$

$$\begin{aligned} \dot{\omega}^{\text{DMI}} = & -\frac{3\tilde{D}a^2}{2\sqrt{2}a_0b_0(b_0^2-a_0^2)\pi^2(1+\alpha^2)} \int_0^{2\pi} d\varphi \sqrt{a_0^2+b_0^2+(b_0^2-a_0^2)\cos(2(\varphi-\omega))} \\ & \times [2(a_0^2+b_0^2)\cos(2\varphi-\varphi_0) - 2(a_0^2-b_0^2)\cos(\varphi_0-2\omega) \\ & + \alpha(a_0^2+b_0^2+(b_0^2-a_0^2)\cos(2(\varphi-\omega)))\sin(2\varphi-\varphi_0)] \sin(2(\varphi-\omega)). \end{aligned} \quad (4.60)$$

Here, the square-root terms within the integrals prevent an explicit evaluation of the angular integrals over φ . Consequently, the complete system of coupled equations, combining the exchange and Zeeman terms in Eqs. (4.40)-(4.43) with the DMI contributions in Eqs. (4.57)-(4.60), is solved in the following sections by direct numerical integration of the DMI terms at each time step. As anticipated from the energy analysis in Sec. 4.2, the terms involving $\cos(\varphi_0 - 2\omega)$ in Eqs. (4.57) and (4.58) clearly reveal that the DMI contributions couple the dynamics of the semi-axes to both the helicity and the rotation angle. Furthermore, Eq. (4.60) confirms that the rotation angle ω itself acquires a finite DMI-induced contribution to the dynamics.

It is important to note that the prefactor in the expression for $\dot{\omega}^{\text{DMI}}$ contains the term $(b_0^2 - a_0^2)$ in the denominator. As a result, this expression diverges in the circular limit, $a_0 = b_0$. To avoid this divergence in numerical simulations of initially circular antiskyrmions, we introduce a small asymmetry by setting $b_0 = a_0 - \epsilon$, with ϵ being a small parameter.

4.5.2. Approximated analytical DMI contribution

To allow investigation of the shrinking dynamics of an initially circular antiskyrmion analytically, it is reasonable to expand the square-root term under the assumption that $a_0 \approx b_0$ for short simulation times. This approximation allows us to focus on the early-stage dynamics of the shrinking and to address the question of whether the antiskyrmion tends to deform into an elliptic shape at a finite DMI strength and, if so, along which direction its major axis aligns.

We define $\delta = a_0^2 - b_0^2$ as a small quantity and expand the square root to first order, neglecting terms of order $\mathcal{O}(\delta^2)$, leading to

$$\sqrt{a_0^2 + b_0^2 - \delta \cos(2(\varphi - \omega))} \approx \sqrt{a_0^2 + b_0^2} - \frac{\delta \cos(2(\varphi - \omega))}{2\sqrt{a_0^2 + b_0^2}} + \mathcal{O}(\delta^2). \quad (4.61)$$

Substituting this expansion allows analytical integration over φ of Eqs. (4.57)-(4.60). Subsequently reinserting $\delta = a_0^2 - b_0^2$ yields the approximated analytical DMI contributions,

reading [136]

$$\begin{aligned} \dot{a}_{0,\text{approx}}^{\text{DMI}} &= \frac{3\tilde{D}a^2}{32\sqrt{2}\pi b_0^3 \sqrt{a_0^2 + b_0^2} (1 + \alpha^2)} \\ &\times \left[4(a_0^4 + 4a_0^2 b_0^2 + 11b_0^4) \cos(\varphi_0 - 2\omega) - (5a_0^4 + 10a_0^2 b_0^2 + 17b_0^4) \alpha \sin(\varphi_0 - 2\omega) \right], \end{aligned} \quad (4.62)$$

$$\begin{aligned} \dot{b}_{0,\text{approx}}^{\text{DMI}} &= \frac{3\tilde{D}a^2}{32\sqrt{2}\pi a_0^3 \sqrt{a_0^2 + b_0^2} (1 + \alpha^2)} \\ &\times \left[-4(11a_0^4 + 4a_0^2 b_0^2 + b_0^4) \cos(\varphi_0 - 2\omega) + (17a_0^4 + 10a_0^2 b_0^2 + 5b_0^4) \alpha \sin(\varphi_0 - 2\omega) \right], \end{aligned} \quad (4.63)$$

$$\dot{\varphi}_{0,\text{approx}}^{\text{DMI}} = - \frac{\tilde{D}\pi a^2 (a_0^2 - b_0^2)}{8\sqrt{2}a_0 b_0 \sqrt{a_0^2 + b_0^2} (1 + \alpha^2)} [10\alpha \cos(\varphi_0 - 2\omega) + 3\sin(\varphi_0 - 2\omega)], \quad (4.64)$$

$$\begin{aligned} \dot{\omega}_{\text{approx}}^{\text{DMI}} &= \frac{3\tilde{D}a^2}{16\sqrt{2}\pi a_0 b_0 (a_0^2 - b_0^2) \sqrt{a_0^2 + b_0^2} (1 + \alpha^2)} \\ &\times \left[(9a_0^4 + 14a_0^2 b_0^2 + 9b_0^4) \alpha \cos(\varphi_0 - 2\omega) + 16(a_0^2 + b_0^2)^2 \sin(\varphi_0 - 2\omega) \right]. \end{aligned} \quad (4.65)$$

4.5.3. Shrinking of an initially circular antiskyrmion at finite DMI

To investigate whether the system tends towards ellipticity or not, we again introduce the ratio $\chi = a_0/b_0$ to the approximated DMI contributions in Eqs. (4.62)-(4.65). The expressions then read

$$\begin{aligned} \dot{a}_{0,\text{approx}}^{\text{DMI}} &= \frac{3\tilde{D}a^2}{32\sqrt{2}\pi \sqrt{1 + \chi^2} (1 + \alpha^2)} \\ &\times \left[4(\chi^4 + 4\chi^2 + 11) \cos(\varphi_0 - 2\omega) - (5\chi^4 + 10\chi^2 + 17) \alpha \sin(\varphi_0 - 2\omega) \right], \end{aligned} \quad (4.66)$$

$$\begin{aligned} \dot{b}_{0,\text{approx}}^{\text{DMI}} &= \frac{3\tilde{D}a^2}{32\sqrt{2}\pi \chi^3 \sqrt{1 + \chi^2} (1 + \alpha^2)} \\ &\times \left[-4(11\chi^4 + 4\chi^2 + 1) \cos(\varphi_0 - 2\omega) + (17\chi^4 + 10\chi^2 + 5) \alpha \sin(\varphi_0 - 2\omega) \right], \end{aligned} \quad (4.67)$$

$$\dot{\varphi}_{0,\text{approx}}^{\text{DMI}} = - \frac{\tilde{D}\pi a^2 (\chi^2 - 1)}{8\sqrt{2}b_0 \chi \sqrt{1 + \chi^2} (1 + \alpha^2)} [10\alpha \cos(\varphi_0 - 2\omega) + 3\sin(\varphi_0 - 2\omega)], \quad (4.68)$$

$$\begin{aligned} \dot{\omega}_{\text{approx}}^{\text{DMI}} &= \frac{3\tilde{D}a^2}{16\sqrt{2}\pi b_0 \chi (\chi^2 - 1) \sqrt{1 + \chi^2} (1 + \alpha^2)} \\ &\times \left[\alpha (9\chi^4 + 14\chi^2 + 9) \cos(\varphi_0 - 2\omega) + 16(1 + \chi^2)^2 \sin(\varphi_0 - 2\omega) \right]. \end{aligned} \quad (4.69)$$

In the absence of DMI, the equation for $\dot{\chi}$ in Eq. (4.53) exhibits a stable stationary point for the circular configuration at $\chi = 1$. By inserting Eqs. (4.66) and (4.67) into the expression in Eq. (4.52), to derive the additional term that $\dot{\chi}$ acquires in the presence of DMI, we obtain [136]

$$\begin{aligned} \dot{\chi}^{\text{DMI}} = & \frac{3a^2}{32\sqrt{2}\pi(1+\alpha^2)} \frac{\tilde{D}}{b_0} \frac{\sqrt{1+\chi^2}}{\chi^2} \\ & \times \left[4(1+14\chi^2+\chi^4) \cos(\varphi_0 - 2\omega) - (5+22\chi^2+5\chi^4) \alpha \sin(\varphi_0 - 2\omega) \right]. \end{aligned} \quad (4.70)$$

In contrast to the zero DMI case, $\dot{\chi}$ evaluated at $\chi = 1$ reveals that there is no stationary point at $a_0 = b_0$, since

$$\dot{\chi}|_{\chi=1} = -\frac{3a^2}{\pi(1+\alpha^2)} \frac{\tilde{D}}{b_0} [2\cos(\varphi_0 - 2\omega) - \alpha \sin(\varphi_0 - 2\omega)] \neq 0. \quad (4.71)$$

Hence, the DMI drives the system away from the circular configuration $a_0 = b_0$ by providing a finite driving force away from $\chi = 1$. The sign and magnitude of this force depend on the DMI strength, the helicity, the rotation angle and the Gilbert damping. To examine how an initially circular antiskyrmion prepared with a given helicity aligns its major axis when it becomes elliptic, we study the stationary condition ω_{st} for $\dot{\omega}_{\text{approx}}^{\text{DMI}}$ in Eq. (4.65). Stationarity is reached when the bracket vanishes, yielding

$$\omega_{\text{st}} = \frac{1}{2} \left[\varphi_0 + \arctan\left(\frac{\alpha(9+14\chi^2+9\chi^4)}{16(1+\chi^2)^2}\right) - \lambda\pi \right], \quad \lambda \in \mathbb{Z}. \quad (4.72)$$

To investigate the stability of this stationary point, we take the derivative of $\dot{\omega}_{\text{approx}}^{\text{DMI}}$ in Eq. (4.69) with respect to ω and obtain

$$\begin{aligned} \frac{\partial \dot{\omega}_{\text{approx}}^{\text{DMI}}}{\partial \omega} = & \frac{3\tilde{D}a^2}{8\sqrt{2}\pi b_0 \chi (\chi^2 - 1) \sqrt{1+\chi^2} (1+\alpha^2)} \\ & \times \left[\alpha(9\chi^4 + 14\chi^2 + 9) \sin(\varphi_0 - 2\omega) - 16(1+\chi^2)^2 \cos(\varphi_0 - 2\omega) \right]. \end{aligned} \quad (4.73)$$

Inserting ω_{st} into this derivative yields

$$\begin{aligned} \left. \frac{\partial \dot{\omega}_{\text{approx}}^{\text{DMI}}}{\partial \omega} \right|_{\omega=\omega_{\text{st}}} = & \frac{3\tilde{D}a^2}{8\sqrt{2}\pi b_0 \chi (\chi^2 - 1) \sqrt{1+\chi^2} (1+\alpha^2)} \\ & \times \left[\alpha(9\chi^4 + 14\chi^2 + 9) \sin\left(\lambda\pi - \arctan\left(\frac{\alpha(9+14\chi^2+9\chi^4)}{16(1+\chi^2)^2}\right)\right) \right. \\ & \left. - 16(1+\chi^2)^2 \cos\left(\lambda\pi - \arctan\left(\frac{\alpha(9+14\chi^2+9\chi^4)}{16(1+\chi^2)^2}\right)\right) \right]. \end{aligned} \quad (4.74)$$

This expression can be rewritten by defining $A = \arctan(u)$ together with its argument $u = \alpha (9 + 14\chi^2 + 9\chi^4) / 16 (1 + \chi^2)^2$. Using the trigonometric identities

$$\sin(\pi\lambda - A) = \sin(\pi\lambda) \cos(A) - \cos(\pi\lambda) \sin(A) = -(-1)^\lambda \sin(A), \quad (4.75)$$

$$\cos(\pi\lambda - A) = \cos(\pi\lambda) \cos(A) + \sin(\pi\lambda) \sin(A) = (-1)^\lambda \cos(A), \quad (4.76)$$

$$\sin(\arctan(u)) = \frac{u}{\sqrt{u^2 + 1}}, \quad (4.77)$$

$$\cos(\arctan(u)) = \frac{1}{\sqrt{u^2 + 1}}, \quad (4.78)$$

we obtain the simplified form

$$\left. \frac{\partial \dot{\omega}_{\text{approx}}^{\text{DMI}}}{\partial \omega} \right|_{\omega=\omega_{\text{st}}} = -(-1)^\lambda \frac{3\tilde{D}a^2 \sqrt{256(1+\chi^2)^4 + \alpha^2(9\chi^4 + 14\chi^2 + 9)^2}}{8\sqrt{2}\pi b_0 \chi (\chi^2 - 1) \sqrt{1 + \chi^2} (1 + \alpha^2)}. \quad (4.79)$$

Throughout our studies, we assume $\tilde{D} > 0$. Since all other quantities are positive and considering that $\chi \geq 1$ by definition, the overall sign of the expression is determined solely by the integer λ . Consequently, for even λ , Eq. (4.72) yields a stable stationary point, where Eq. (4.79) becomes negative. Accordingly, for odd λ , Eq. (4.79) becomes positive, revealing an unstable stationary point.

The time evolution of the coupled equations of motion for an initially circular antiskyrmion is shown in Figure 4.8 for different values of the DMI strength [136]. These results are obtained by numerically integrating the coupled nonlinear ordinary differential Eqs. (4.40)-(4.43), together with their DMI contributions in Eqs. (4.57)-(4.60). The latter terms are evaluated numerically by integrating over φ at every time step. Recall that $\chi = 1$ leads to a divergence in $\dot{\omega}$. Therefore, a slight asymmetry had to be introduced when analyzing the circular antiskyrmion, which we implemented by setting $b_0 = a_0 - \epsilon$, with $\epsilon = 10^{-3}$. In the following, when we refer to an initially circular antiskyrmion, it is meant to refer to this slightly asymmetric configuration. Panels (a) and (c) show the time evolution of the semi-axes, while panels (b) and (d) display the corresponding time evolution of the helicity and rotation angle. The initial antiskyrmion configuration is always prepared with $\varphi_0 = \pi/2$ and $\omega = \pi/2$ at $t = 0$.

In addition to the dynamics governed by the exact integral expressions of the DMI contributions, the dashed lines in panels (a) and (b) depict the solutions obtained using the approximate representations of Eqs. (4.66)-(4.69). The dynamics generated by the exact and approximate DMI expressions agree with high accuracy, demonstrating that the small-eccentricity approximation yields precise results for the initially circular antiskyrmion at small DMI strength. The case of larger DMI, with larger eccentricities, will be discussed later.

The inset in panel (b) shows $\varphi_0(t)$ and $\omega(t)$ at short simulation times and reveals that the rotation angle rotates away from its initially prepared value of $\omega = \pi/2$ and rapidly relaxes to a value slightly above $\pi/4$. In this time frame, the helicity remains approximately

constant at $\varphi_0 \approx \pi/2$ and thus, ω evolves towards ω_{st} predicted by Eq. (4.72) for the initial conditions $\varphi_0 \approx \pi/2$, $\chi \approx 1$, $\alpha = 0.1$ and $\lambda = 0$, yielding $\omega_{\text{st}} \approx \pi/4 + 0.05$.

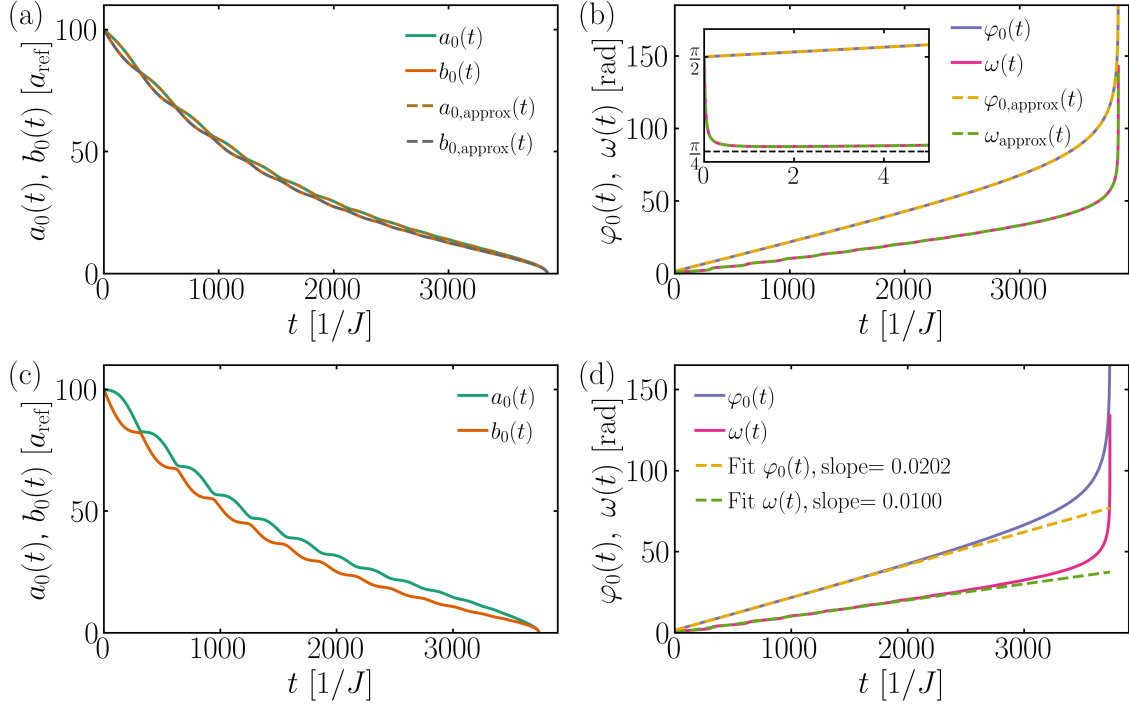


Figure 4.8.: Time evolution of the coupled equations of motion for an initially circular antiskyrmion at a finite DMI of $\tilde{D} = 0.02$ in (a) and (b) and $\tilde{D} = 0.06$ in (c) and (d). Panels (a) and (c) show the evolution of the semi-axes $a_0(t)$ and $b_0(t)$, while panels (b) and (d) display the corresponding helicity $\varphi_0(t)$ and rotation angle $\omega(t)$. The solid lines represent the dynamics obtained using the exact integral expressions of the DMI contributions, whereas the dashed lines in panels (a) and (b) denote the approximate small-eccentricity solutions. The inset in panel (b) highlights the initial rapid relaxation of $\omega(t)$ from its prepared value of $\pi/2$ towards ω_{st} , within a time frame, where φ_0 remains nearly constant. The semi-axes shrink according to an exponential-to-square-root crossover behavior, superimposed with quadrupole-like oscillations. These oscillations increase in amplitude with increasing DMI strength. At large semi-axes, $\varphi_0(t)$ exhibits linear growth governed by the Zeeman field, confirmed by the yellow dashed linear fit in panel (d), while $\omega(t)$ follows with half the slope (green dashed line), reflecting the π rotational symmetry of the elliptic shape. Parameters: $\tilde{B} = 0.02$, $\alpha = 0.1$, $a_{0,0} = 100$, $b_{0,0} = a_{0,0} - \epsilon$, $\epsilon = 10^{-3}$, $\varphi_{0,0} = \pi/2$, $\omega_0 = \pi/2$. [Figure adapted from Ref. [136]]

In general, the time evolution of the semi-axes exhibits an exponential decay at large antiskyrmion sizes. Similar to the skyrmion shrinking behavior discussed in Sec. 3.3.3, this overall trend is superimposed with oscillations. But, in contrast to the isotropic breathing observed in the skyrmion case, the antiskyrmion's oscillations are quadrupole-like. Thus, at $t = 0$, the antiskyrmion is driven away from circularity and becomes elliptic, but subsequently oscillates between nearly circular and elliptic shapes during its shrinking process. These oscillations arise due to the coupling of the semi-axes to both the helicity

and the rotation angle in the presence of DMI. Consequently, their amplitude increases with increasing DMI strength. The origin of the oscillations will be discussed in detail in the following.

The helicity evolves linearly in time at large semi-axes. Similar to the skyrmion shrinking case, for DMI strengths that are insufficient to stabilize the skyrmion at a finite size, t_c decreases with increasing DMI. The yellow dashed line in panel (d) shows a linear fit to $\varphi_0(t)$ at large semi-axes, confirming that the helicity is driven by the Zeeman field in this regime, as its slope equals the value of $\tilde{B}$. The rotation angle's evolution $\omega(t)$ exhibits a similar overall trend but with additional step-like jumps that are in phase with the quadrupole oscillations of the semi-axes. The green dashed line shows a linear fit to $\omega(t)$ at large semi-axes and indicates that the rotation angle follows the helicity with half its slope, i.e., $\omega(t) \approx \varphi_0(t)/2$. This reflects the π rotational symmetry of the elliptic shape, which was already evident in the energy landscape shown in Figure 4.3.

The quadrupole oscillations of the semi-axes originate from their coupling to the helicity and the rotation angle. To understand their nature in more detail, we first investigate them in the undamped limit, i.e., at $\alpha = 0$, where the oscillations can be attributed directly to energy conservation [136]. Figure 4.9 shows the undamped time evolution of the system obtained from the coupled ordinary differential equations for a large antiskyrmion, where the Zeeman term dominates.

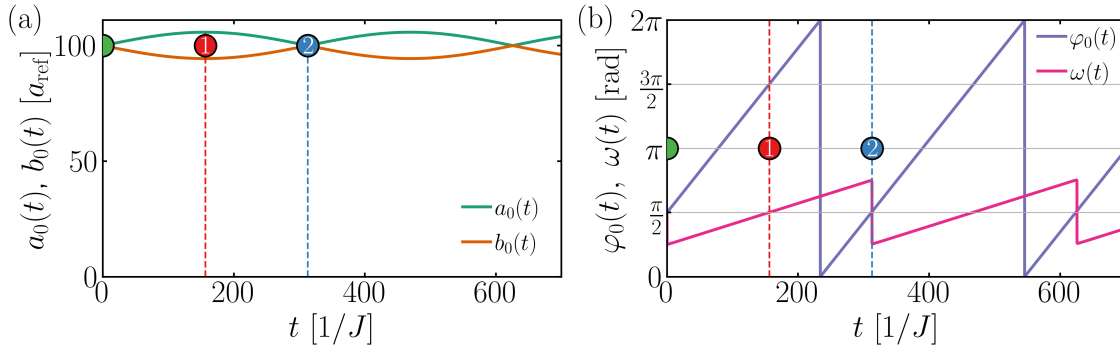


Figure 4.9.: Undamped quadrupole oscillations of the antiskyrmion obtained from the coupled collective-coordinate equations. Panel (a) depicts the time evolution of the semi-axes showing quadrupole oscillations that conserve the averaged radius. In panel (b), the Zeeman term drives a linear precession of the helicity, while the rotation angle follows with one quarter of its slope. The markers ●, ①, and ② indicate the start of the simulation, the point of highest ellipticity, and the return to circularity, respectively. The jumps in $\omega(t)$, realigning to the value prescribed by Eq. (4.72), occur whenever the antiskyrmion passes through the circular configuration. Parameters: $\tilde{D} = 0.06$, $\tilde{B} = 0.02$, $\alpha = 0$, $a_{0,0} = 100$, $b_{0,0} = a_{0,0} - \epsilon$, $\epsilon = 10^{-3}$, $\varphi_{0,0} = \pi/2$, $\omega_0 = \pi/2$. [Figure adapted from Ref. [136]]

The semi-axes in panel (a) exhibit quadrupole oscillations while conserving, due to the absence of damping, the value of the averaged radius, and thus the area of the antiskyrmion remains constant. The Zeeman term in Eq. (4.42) drives a linear precession of the helicity

as depicted in panel (b), where the data are wrapped into the interval $[0, 2\pi]$. It is revealed that the rotation angle follows the helicity with one quarter of its slope. The green dot ● marks the beginning of the simulation, where the antiskyrmion is driven away from circularity due to Eq. (4.71). The antiskyrmion then reaches its highest ellipticity at ❶, and as it returns to the circular state at ❷, the rotation angle exhibits a jump and realigns with the value consistent with Eq. (4.72). Thus, the jumps in $\omega(t)$ are synchronized with the quadrupole oscillations of the semi-axes.

To further understand this behavior, the trajectory of $\varphi_0(t)$ and $\omega(t)$ from Figure 4.9 (b) is projected onto the energy landscape $E(\varphi_0, \omega)$, shown in Figure 4.3 (a) and (b), yielding Figure 4.10 [136]. The absolute energy values of $E(\varphi_0, \omega)$ depend on the instantaneous values of the semi-axes and therefore vary in time. However, the general structure of the landscape remains unchanged as long as the antiskyrmion is not perfectly circular. In the limit $a_0 = b_0$, Eq. (4.12) applies and implies that $E(\varphi_0, \omega) = \text{const}$, as illustrated in Figure 4.3 (c). To avoid the divergence in $\dot{\omega}$, we use an initial antiskyrmion with a small asymmetry, such that this perfectly circular limit is never exactly reached. Nevertheless, the energy landscape becomes increasingly flat as the ellipticity decreases. Therefore, the energy shown in Figure 4.10 is given in arbitrary units.

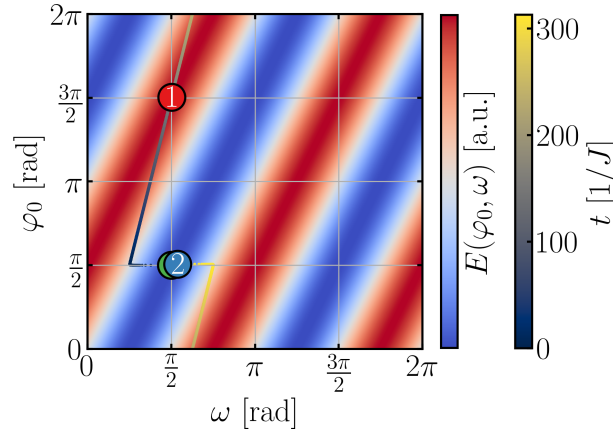


Figure 4.10.: Projection of the trajectory of $\varphi_0(t)$ and $\omega(t)$ from Figure 4.9 (b) onto the energy landscape $E(\varphi_0, \omega)$ for an initially circular antiskyrmion. The blue-to-yellow line represents the collective time evolution of $\varphi_0(t)$ and $\omega(t)$ in the undamped limit ($\alpha = 0$), using the same time markers as in Figure 4.9. The system follows a closed path of constant energy, converting shape deformation into rotational motion. Starting from ●, the antiskyrmion becomes elliptic and aligns its major axis towards $\omega \approx \pi/4$, consistent with Eq. (4.72). It reaches maximal ellipticity at ❶, corresponding to a maximum in the energy landscape, and then returns to a circular state at ❷, where the landscape becomes flat and ω freely realigns. Since the absolute energy depends on the time-varying semi-axes, it is given in arbitrary units. Parameters: $\tilde{D} = 0.06$, $\tilde{B} = 0.02$, $\alpha = 0$, $a_{0,0} = 100$, $b_{0,0} = a_{0,0} - \epsilon$, $\epsilon = 10^{-3}$, $\varphi_{0,0} = \pi/2$, $\omega_0 = \pi/2$. [Figure adapted from Ref. [136]]

The blue-to-yellow line indicates the trajectory of $\varphi_0(t)$ and $\omega(t)$, using the same time markers as in Figure 4.9. The plot reveals that the system evolves along a closed path of constant energy. The antiskyrmion is initially prepared with $\varphi_{0,0} = \pi/2$ and $\omega_0 = \pi/2$, and the trajectory therefore starts at the green dot $\bullet$. As it becomes elliptic, its major axis aligns around $\omega \approx \pi/4$, as predicted by Eq. (4.72). As discussed in Sec. 4.2, the elliptic antiskyrmion is energetically favorable over the circular one. Since no energy dissipation occurs at $\alpha = 0$, the excess energy is converted into rotational motion in the energy landscape $E(\varphi_0, \omega)$, shifting the state towards regions of higher energy.

Driven by the external magnetic field, φ_0 rotates linearly in time while the antiskyrmion becomes more elliptic, and consequently ω adapts to maintain constant energy. At $\textcircled{1}$, the antiskyrmion reaches maximal ellipticity and the trajectory reaches a maximum in the energy landscape. Subsequently, both φ_0 and ω continue to evolve, moving away from the energy maximum, and the energy is converted back into a circular shape at $\textcircled{2}$. In this nearly circular state, the energy landscape becomes flat, allowing ω to realign freely. The alignment along $\omega \approx 3\pi/4$, corresponding to $\lambda = -1$ in Eq. (4.72) resembles an unstable stationary point according to Eq.(4.79) and because of that the rotation angle realigns along $\omega \approx \pi/4$ and the process repeats periodically.

In the regime of finite damping, the overall oscillation mechanism remains qualitatively similar. However, energy dissipation enables the antiskyrmion to shrink, and it no longer returns to its initial circular state after reaching maximal ellipticity. This is illustrated in Figure 4.11, which shows the time evolution of the semi-axes from Figure 4.8 (c), together with the averaged radius $\varrho_0(t)$, defined as

$$\varrho_0(t) = \frac{1}{2} (a_0(t) + b_0(t)). \quad (4.80)$$

The averaged radius, and thus the antiskyrmion area, decays monotonically following an exponential-to-square-root trend and exhibits no oscillations. The inset highlights the first oscillation cycle and demonstrates that the antiskyrmion does not recover its initial circular shape after its first maximal ellipticity. The time point of minimal ellipticity is marked by the vertical dashed blue line, for which the ratio $\chi = a_0/b_0$ equals $\chi = 1.012$, with $a_0 = 83.00$ and $b_0 = 81.99$. This reflects a significantly stronger asymmetry compared to the initial condition at $t = 0$, where $\epsilon = 10^{-3}$. As the shrinking proceeds, this asymmetry at local minima of ellipticity increases further. This behavior alters in particular the time evolution of the antiskyrmion's rotation angle ω .

The time evolution of the collective coordinates from Figure 4.8 (c) and (d), with a focus on the first oscillations during the antiskyrmion shrinking process, is shown in Figure 4.12. The data in panel (b) are wrapped to the interval $[0, 2\pi]$. The green, red, and blue dots mark the beginning of the time evolution, and the time points corresponding to maximal and minimal ellipticity, respectively. At large antiskyrmion sizes, the energy dissipation is dominated by the Zeeman term in Eqs. (4.40)-(4.42), leading to an exponential decay of the averaged radius. In the evolution of the rotation angle, we observe that $\omega(t)$ at $\textcircled{2}$ and

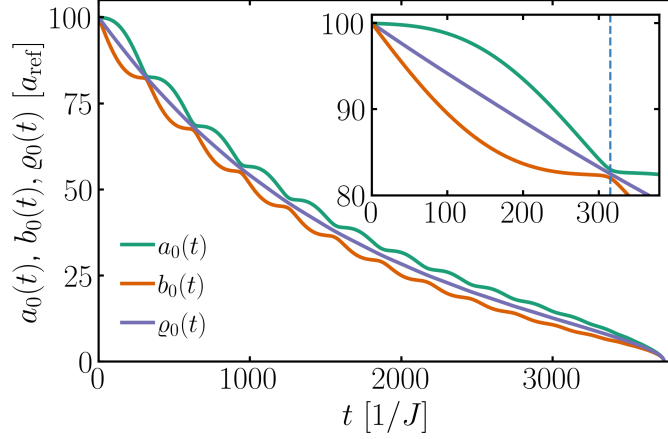


Figure 4.11.: Time evolution of the semi-axes $a_0(t)$ and $b_0(t)$ from Figure 4.8 (c) together with the averaged radius $\varrho_0(t)$ in the finite-damping regime. The averaged radius decays monotonically following an exponential-to-square-root trend without exhibiting oscillations, reflecting the continuous shrinking of the antiskyrmion. The inset highlights the first oscillation cycle and the vertical dashed blue line marks the moment of minimal ellipticity, where the antiskyrmion does not recover a circular shape. Parameters: $\tilde{D} = 0.06$, $\tilde{B} = 0.02$, $\alpha = 0.1$, $a_{0,0} = 100$, $b_{0,0} = a_{0,0} - \epsilon$, $\epsilon = 10^{-3}$, $\varphi_{0,0} = \pi/2$, $\omega_0 = \pi/2$.

④ does not jump back to values close to $\omega \approx \pi/4$, as was evident in the undamped case.

This behavior arises because the antiskyrmion does not recover a circular shape at configurations of minimal ellipticity. This can be further understood from Figure 4.13 (a), where the collective trajectory of $\varphi_0(t)$ and $\omega(t)$ from Figure 4.12 (b) is drawn into the energy landscape $E(\varphi_0, \omega)$ using the same time point markers [136]. Similar to the undamped regime, the energy gain from elliptic deformation is compensated by a rotation of ω into regions of higher energy. Since $\varphi_0(t)$ precesses under the influence of the external magnetic field, $\omega(t)$ rotates as well to compensate for the helicity change. However, after maximal ellipticity is reached at ①, the antiskyrmion no longer relaxes back to a circular state at ②, as shown in Figure 4.11. Consequently, $E(\varphi_0, \omega)$ does not flatten, unlike in the undamped regime. Therefore, the system cannot freely reorient towards $\omega \approx \pi/4$, as passing through the corresponding energy minimum would require dynamically increasing the averaged radius. Instead, the antiskyrmion rotates after ① towards $\omega \approx 5\pi/4$, consistent with Eq. (4.72) for $\lambda = -2$. To traverse this energy barrier while the mean radius continues to decay exponentially and the helicity rotates linearly, the antiskyrmion accelerates its rotation, explaining the step-like jumps observed in $\omega(t)$. Analogously, the antiskyrmion rotates after ③ towards $\omega \approx \pi/4$, as it cannot reorient through the energy minimum towards $5\pi/4$. At longer times, the oscillations of the semi-axes decrease in amplitude, and the antiskyrmion remains elliptic, as shown in Figure 4.11.

Figure 4.13 (b) shows the collective trajectory of $\varphi_0(t)$ and $\omega(t)$ within the $E(\varphi_0, \omega)$ energy landscape for the entire shrinking process depicted in Figure 4.8 (c) and (d). The green dot marks the beginning of the evolution, while the purple dot indicates the time t_c

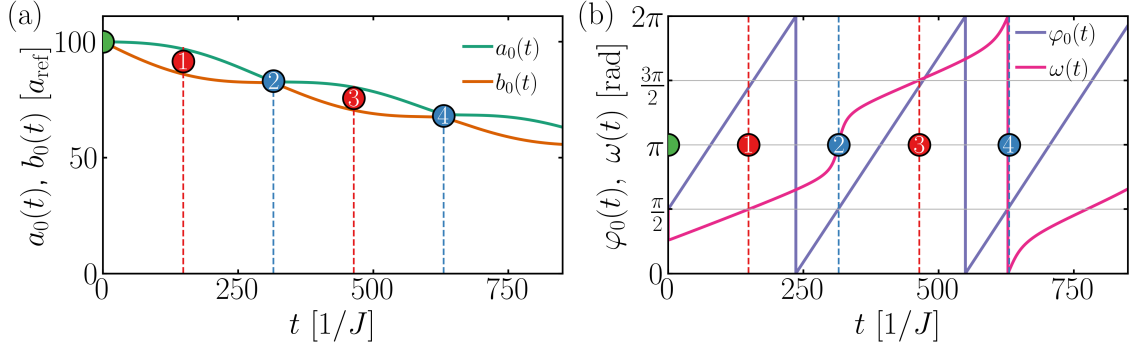


Figure 4.12.: Time evolution of the coupled equations of motion during the initial oscillations of the antiskyrmion shrinking process. Panel (a) shows the quadrupole-like oscillations of the semi-axes, while panel (b) displays the wrapped trajectories of $\varphi_0(t)$ and $\omega(t)$ in the interval $[0, 2\pi]$. The green, red, and blue markers indicate the start of the evolution, and the times of maximal and minimal ellipticity, respectively. In contrast to the undamped case, the rotation angle $\omega(t)$ at ② and ④ does not return to values near $\omega \approx \pi/4$. Parameters: $\tilde{D} = 0.06$, $\tilde{B} = 0.02$, $\alpha = 0.1$, $a_{0,0} = 100$, $b_{0,0} = a_{0,0} - \epsilon$, $\epsilon = 10^{-3}$, $\varphi_{0,0} = \pi/2$, $\omega_0 = \pi/2$. [Figure adapted from Ref. [136]]

at which the antiskyrmion's semi-axes vanish. The trajectory reveals that for long times it evolves near the energy maximum, in accordance with Eq. (4.15) [136].

To summarize, the system dissipates energy primarily through the shrinking of the antiskyrmion size, while the configuration is simultaneously driven away from circularity, in accordance with Eq. (4.71). The helicity exhibits continuous precession and, due to the twofold symmetry of the elliptic antiskyrmion, the energy in an environment with isotropic bulk DMI depends explicitly on the helicity for a fixed rotation angle ω . Thus, the quadrupole oscillations of the semi-axes together with the rotation of ω act to compensate the helicity precession [136]. The breathing behavior of the shrinking antiskyrmion at finite DMI can therefore be characterized as rotational quadrupole-like breathing. Rotational refers to the coupled rotation of both the helicity and the rotation angle during the shrinking process, analogous to the helicity rotation observed in the skyrmion case for non-stabilizing DMI strengths. Quadrupole-like refers to the out-of-phase oscillation of the semi-axes, such that the averaged radius $\varrho_0(t)$ decreases smoothly without exhibiting oscillations.

The differential equation for the ratio of the semi-axes χ scales with $1/b_0^2$ in the exchange term in Eq. (4.53) and with $1/b_0$ in the DMI term in Eq. (4.70). Therefore, at small antiskyrmion sizes, the exchange contribution governs both the evolution of χ and the overall system dynamics. In this regime, the antiskyrmion tends towards a circular configuration in accordance to the stable stationary point for circularity of χ in Eq. (4.53), and $a_0(t)$ and $b_0(t)$ transition from exponential to square-root decay, as given by Eq. (4.55). Simultaneously, $\varphi_0(t)$ exhibits a logarithmic divergence according to Eq. (4.56). Consequently, the rotation angle $\omega(t)$ follows the helicities logarithmic divergence.

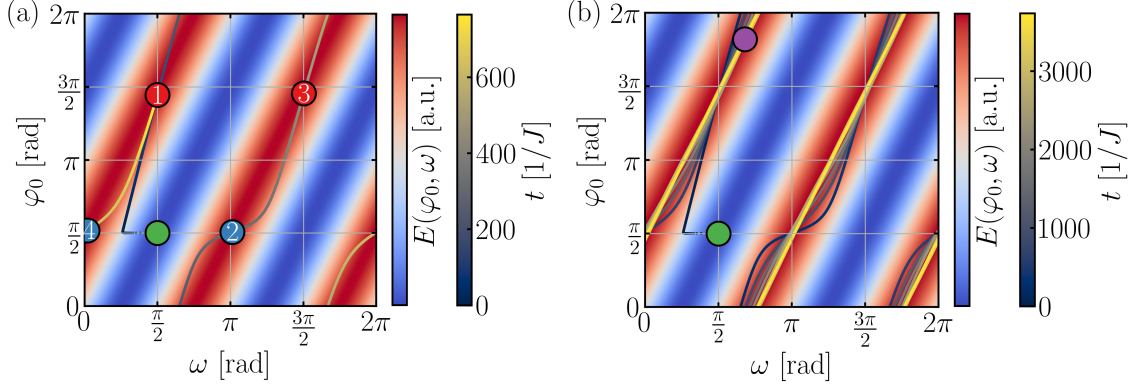


Figure 4.13.: Energy landscape $E(\varphi_0, \omega)$ illustrating the collective trajectory of $\varphi_0(t)$ and $\omega(t)$ during the antiskyrmion shrinking process. Panel (a) shows the early-stage trajectory from Figure 4.12 (b), marked with identical time markers. The energy gain from deformation towards ellipticity is compensated by a rotation into areas of higher energies. Because the system does not relax to a circular state at ② and ④, the energy landscape does not flatten and therefore, the antiskyrmion rotates towards $\omega \approx 5\pi/4$ and $\omega \approx \pi/4$, consistent with Eq. (4.72) for $\lambda = -2$ and $\lambda = 0$, respectively. This results in the step-like jumps observed in $\omega(t)$. Panel (b) displays the full shrinking trajectory from Figure 4.8 (d). The green and purple dots indicate the beginning of the evolution and the collapse time t_c , respectively. For long times, where the antiskyrmion keeps ellipticity, the trajectory remains near the energy maximum in given by Eq. (4.15). Parameters: $\tilde{D} = 0.06$, $\tilde{B} = 0.02$, $\alpha = 0.1$, $a_{0,0} = 100$, $b_{0,0} = a_{0,0} - \epsilon$, $\epsilon = 10^{-3}$, $\varphi_{0,0} = \pi/2$, $\omega_0 = \pi/2$. [Figure adapted from Ref. [136]]

4.5.4. Shrinking of an initially elliptic antiskyrmion at finite DMI

The shrinking dynamics of an initially elliptic antiskyrmion in the presence of DMI are shown in Figure 4.14, for a small initial ellipticity in panels (a) and (b), and for a larger one in panels (c) and (d). To assess the accuracy of the approximated DMI contributions in Eqs. (4.66)-(4.69), the dashed lines in panels (c) and (d) depict the solutions obtained by employing these approximations in the coupled differential equations. Once again, a high level of agreement is observed, demonstrating that the approximation remains valid even for large ellipticities [136].

As in the zero DMI case, the semi-axes exhibit exponential-to-square-root time evolution, while the helicity increases linearly at large antiskyrmion sizes and diverges logarithmically as it approaches t_c . The time evolution of the semi-axes is superimposed with quadrupole-like oscillations because of their coupling to the helicity and the rotation angle via the DMI contributions. This behavior is consistent with the finite DMI shrinking of an initially circular antiskyrmion. The rotation angle, however, evolves differently: it remains pinned during the initial phase. As shown in the inset of panel (b), $\omega(t)$ exhibits small oscillations around its initial value, whose amplitude gradually increases until $\omega(t)$ eventually unlocks and begins to rotate freely.

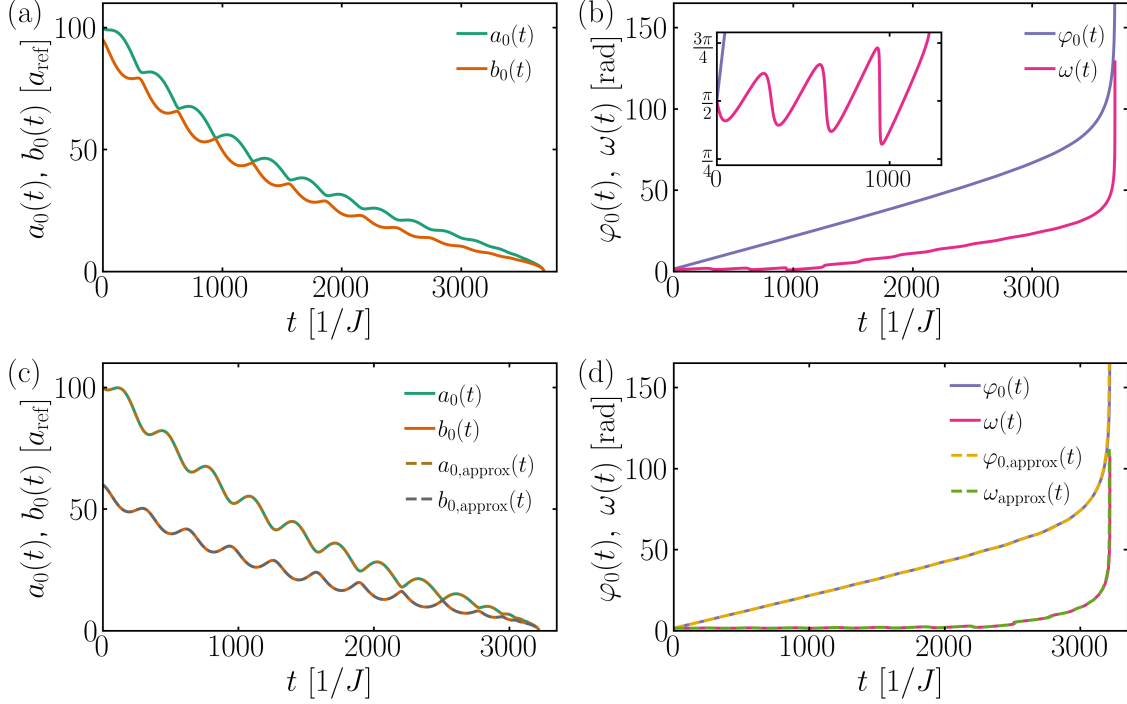


Figure 4.14.: Shrinking dynamics of an initially elliptic antiskyrmion at finite DMI. The initial value of the major axis is $a_{0,0} = 100$ and panels (a) and (b) show the evolution of the semi-axes, helicity, and rotation angle for a small initial ellipticity with $b_{0,0} = 95$, while panels (c) and (d) illustrate the behavior for a larger ellipticity with $b_{0,0} = 60$. The dashed lines in (c) and (d) represent solutions obtained using the approximated DMI contributions from Eqs. (4.66)-(4.69), confirming their accuracy even for strong elliptic deformation. The semi-axes follow the characteristic exponential-to-square-root evolution, superimposed with quadrupole-like oscillations, whereas the helicity grows linearly for large antiskyrmion sizes and diverges logarithmically near t_c . In contrast, the rotation angle $\omega(t)$ remains pinned at early times, exhibiting small oscillations around its initial value, as shown in the inset of panel (b), before unlocking and entering a free rotational regime. Parameters: $\bar{D} = 0.06$, $\bar{B} = 0.02$, $\alpha = 0.1$, $\varphi_{0,0} = \pi/2$ and $\omega_0 = \pi/2$. [Figure adapted from Ref. [136]]

This pinning behavior can be understood by considering the effective potential $V(\omega)$, which follows from Eq. (4.69) via

$$\dot{\omega} = -\frac{\partial V(\omega)}{\partial \omega}. \quad (4.81)$$

The resulting potential reads [136]

$$V(\omega) = \frac{3a^2}{32\sqrt{2}\pi(1+\alpha^2)} \frac{\tilde{D}}{b_0} \frac{1}{\chi(\chi^2-1)\sqrt{1+\chi^2}} \times \left[(9+14\chi^2+9\chi^4) \alpha \sin(\varphi_0-2\omega) - 16(1+\chi^2)^2 \cos(\varphi_0-2\omega) \right], \quad (4.82)$$

which reveals how the torque acting on ω evolves as the antiskyrmion becomes increasingly elliptic, i.e., as χ increases. Figure 4.15 shows $V(\omega)$ for increasing values of ϵ , corresponding to increasing elliptic deformation as $b_0 = a_0 - \epsilon$. The overall amplitude decreases significantly even for small increases in ϵ . Consequently, the torque aligning the rotation angle vanishes rapidly once the shape deviates from circularity, and thus the small amplitude of $V(\omega)$ effectively pins ω during the early stages of the dynamics shown in Figure 4.14.

The gradually increasing oscillations in the inset of Figure 4.14 (b) follow the quadrupole-like breathing period of the semi-axes. With each oscillation cycle, the difference between the semi-axes decreases periodically, i.e., χ is modulated with the same period, enhancing the amplitude of $V(\omega)$. As a result, the torque acting on ω increases progressively, until it becomes sufficiently large to unlock $\omega(t)$, allowing it to rotate freely. In the subsequent dynamics, $\omega(t)$ enters a dynamical regime in which it follows $\varphi_0(t)$ with half its slope, analogous to the behavior observed for an initially circular antiskyrmion.

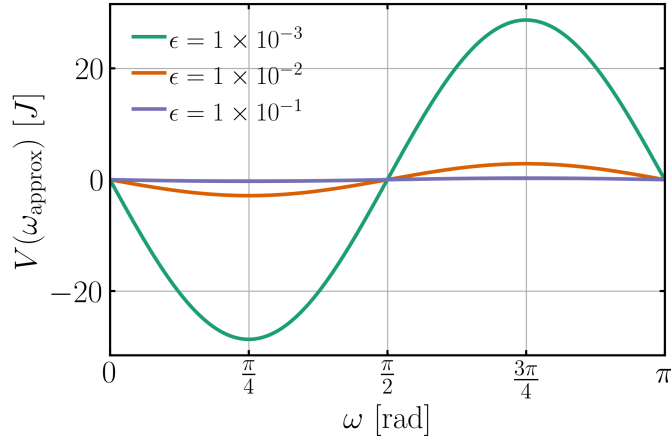


Figure 4.15.: The effective potential $V(\omega)$ governing the torque acting on ω . The curves show $V(\omega)$ for increasing values of ϵ . As ϵ increases, the overall amplitude of the potential decreases rapidly, indicating a weakening of the torque that aligns ω . This reduction in torque effectively pins the rotation angle during the early stages of the dynamics of an initially elliptic antiskyrmion, as observed in Figure 4.14. Parameters: $\tilde{D} = 0.06$, $\tilde{B} = 0.02$, $a_0 = 100$, $b_0 = a_0 - \epsilon$, and $\varphi_0 = \pi/2$. [Figure adapted from Ref. [136]]

The collective trajectory of $\varphi_0(t)$ and $\omega(t)$ from Figure 4.14 (b) is projected onto the energy landscape $E(\varphi_0, \omega)$ in Figure 4.16. The green dot indicates the starting point of the time evolution, while the purple dot marks the critical time t_c , at which the semi-axes vanish. It is evident that, at early times, the torque arising from $V(\omega)$ is not yet strong enough to align ω in accordance with Eq. (4.72). As the semi-axes progressively decrease their ellipticity, the torque increases and, simultaneously, the energy landscape $E(\varphi_0, \omega)$ periodically flattens. Eventually, this enables ω to unlock, and the trajectory subsequently evolves in a manner similar to the initially circular case shown in Figure 4.13 (b), where $\omega(t)$ follows $\varphi_0(t)$ with half its slope.

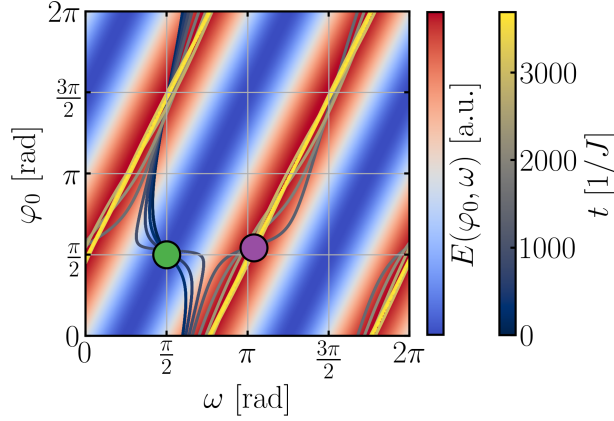


Figure 4.16.: Projection of the collective trajectory of $\varphi_0(t)$ and $\omega(t)$ for the initially elliptical antiskyrmion onto the energy landscape $E(\varphi_0, \omega)$. The green and purple marker denote the starting point of the evolution and the critical time t_c , respectively. At early stages, the torque generated by $V(\omega)$ is insufficient to align ω according to Eq. (4.72). As the ellipticity decreases over time periodically, the torque increases and the energy landscape gradually flattens periodically as well, allowing ω to unlock and the trajectory evolves similar to the circular case. Parameters: $\tilde{D} = 0.06$, $\tilde{B} = 0.02$, $\alpha = 0.1$, $a_{0,0} = 100$, $b_{0,0} = 95$, $\varphi_{0,0} = \pi/2$, $\omega_0 = \pi/2$.

4.6. Comparison with numerical simulations

In this section, we present simulation results obtained from the direct numerical solution of the LLG equation in Eq. (2.14), serving as a comparison to the results from our analytical model. The simulations are performed in complete analogy to Sec. 3.4 on a square grid, with the exchange constant $J = 1$ defining the energy unit and $a_{\text{ref}} = 1$ the unit of length. For more details on the simulation software, see Appendix A. The antiskyrmion is initialized at the center of a quadratic system of size $L_x = L_y = L$, where L serves as the fixed system size and acts as the length scale.

To construct the initially circular antiskyrmion configuration, we assign to each lattice site i the magnetization vector according to Eq. (2.3). At each grid point, the radial profile is specified by employing the domain-wall parametrization from Eq. (2.35) and the azimuthal angle is defined as and Eq. (2.23). For the latter one, we use $m = -1$ to ensure an antiskyrmion configuration. Since simulations start either with an initially circular or an elliptic antiskyrmion, we additionally apply the elliptic deformation described by Eq. (4.5), thereby introducing the semi-axes and rotation angle. The major semi-axis is set to $a_0 = 100$ and for an initially circular antiskyrmion, we choose $b_0 = a_0$, whereas for the elliptic case we set $b_0 = 60$ and $\omega = \pi/2$. The wall width is fixed to $w = a_0/1.8$, which is a representative value obtained by fitting the domain-wall profile to numerically relaxed initial skyrmions in Sec. 3.4. Due to the isotropic nature of the bulk DMI used here, the antiskyrmion is always energetically unfavorable compared to the ferromagnetic ground state. As a consequence, relaxation of the initialized antiskyrmion, as performed for the

skyrmion case, is not feasible. However, in the presence of anisotropic DMI ($D_x = -D_y$), circular antiskyrmions can be stabilized and relaxed, as discussed in Sec. 2.2.2. Notably, this relaxation yields the same wall width value, $w = a_0/1.8$, supporting the choice used here. To enable a direct comparison between initially circular and elliptic antiskyrmions with the same a_0 , both configurations are initialized artificially on the grid.

At $t = 0$, the artificially initialized antiskyrmion is subjected to an environment characterized by the chosen system parameters. It subsequently responds to the sudden parameter quench, and its time evolution can be tracked numerically. In the analytical framework, the semi-axes, rotation angle, and helicity are selected as the collective coordinates. Accordingly, these are also the quantities extracted from the simulation data for direct comparison. The helicity is tracked in the same way as described in Sec. 3.4.1. In the following, we describe how the semi-axes and the rotation angle are obtained from the numerical simulations, before presenting and discussing the results for the antiskyrmion decay at vanishing and finite DMI for both initially circular and elliptic configurations.

4.6.1. Determination of the antiskyrmion semi-axes and rotation angle

To extract the time-dependent ellipse parameters $a_0(t)$, $b_0(t)$, and $\omega(t)$ from the simulation data, we first determine the collapse time t_c of the antiskyrmion as described in Sec. 3.4.2. This is achieved by computing the total topological charge $Q(t)$ at each time step using its discretized form in Eq. (3.55), and identifying the collapse point as the minimum of its temporal derivative $Q'(t)$, obtained via numerical differentiation. Note, that for the antiskyrmion the total topological charge is $Q = 1$ and thus, the topological charge starts at $t = 0$ at 1 and then changes around t_c abruptly to the trivial ferromagnetic topological charge of $Q = 0$.

For all time steps prior to the collapse point, the center of the antiskyrmion is located by identifying the minimum of the out-of-plane magnetization component n_z using interpolation. Owing to this interpolation, the center can be determined with sub-lattice accuracy. The antiskyrmion boundary is then extracted by detecting the longest closed contour corresponding to the level set $n_z = 0$. These boundary coordinates are transformed into polar coordinates with respect to the center. Subsequently, fitting the elliptic ansatz for ρ_0 given in Eq. (4.5) yields the semi-major axis $a_0(t)$, the semi-minor axis $b_0(t)$, and the orientation angle $\omega(t)$. An example snapshot of an elliptic antiskyrmion during its decay is shown in Figure 4.17, where the colormap displays the z -component and the fitted ellipse is indicated by the green line.

To ensure temporal consistency and to suppress artificial discontinuities due to branch switching, the angular parameter $\omega(t)$ is post-processed by selecting the value that minimizes its deviation from the previous time step. After the collapse time t_c , the antiskyrmion no longer exists, and all ellipse parameters are set to zero.

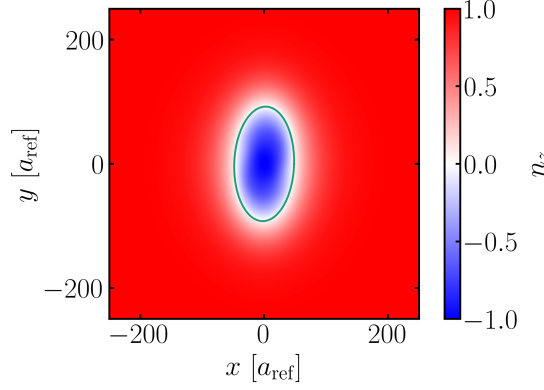


Figure 4.17.: Snapshot of an elliptic antiskyrmion during its decay to illustrate the fitting procedure used at each time step prior to the collapse time t_c . The colormap shows the out-of-plane magnetization component n_z , with blue and red representing negative and positive values, respectively and the green line denotes the fitted ellipse.

4.6.2. Antiskyrmion decay at vanishing DMI

In the absence of DMI, the analytical model predicts that a circular antiskyrmion shrinks while preserving its circular shape. Its temporal evolution is therefore analogous to the isotropic skyrmion case: the radius exhibits an exponential decay at large sizes, which crosses over into a square-root-like collapse in the small-size regime. For an initially elliptic antiskyrmion, Eqs. (4.48) and (4.49) likewise predict an exponential decay of the semi-axes $a_0(t)$ and $b_0(t)$ at large sizes. At small semi-axes, Eq. (4.53) shows that the ratio $\chi = a_0/b_0$ is driven towards $\chi = 1$ in the exchange-dominated regime. Consequently, the antiskyrmion becomes circular and then follows the same square-root-like collapse as in the circular case.

In both scenarios, the helicity evolves linearly in time at large sizes due to the Zeeman term and diverges logarithmically in the exchange-dominated regime at small sizes. For the elliptic case, Eq. (4.39) predicts that the rotation angle will remain constant at its initial value.

These predictions from our analytical model are confirmed qualitatively by the corresponding simulation results [136] shown in Figure 4.18. Panels (a) and (b) display the time evolution of a circular antiskyrmion and panels (c) and (d) the time evolution of an elliptic one at vanishing DMI. As in the skyrmion case, the antiskyrmion exhibits a delayed response to the external environment. After an initial adjustment period, the semi-axes decay exponentially at large sizes in both the circular case in panel (a) and the elliptical case in panel (c), with the latter becoming circular upon entering the small-size regime. In this regime, both simulations show the transition into the square-root-like collapse.

The helicity shown in panels (b) and (d) displays the expected linear time dependence at large antiskyrmion sizes. The deviation from this linear regime near t_c is not observable. In this regime, the antiskyrmion radius approaches the lattice constant and collapses abruptly

before the helicity can exhibit the predicted logarithmic divergence. After collapse, the collective phase φ_0 is lost, and the helicity can no longer be tracked.

Finally, the time evolution of the rotation angle $\omega(t)$ in panel (d) confirms that an elliptic antiskymion does not undergo in-plane rotation in the absence of DMI. The noisy oscillations close to the collapse originate from numerical uncertainty in determining the semi-axes directions when the configuration becomes nearly circular.

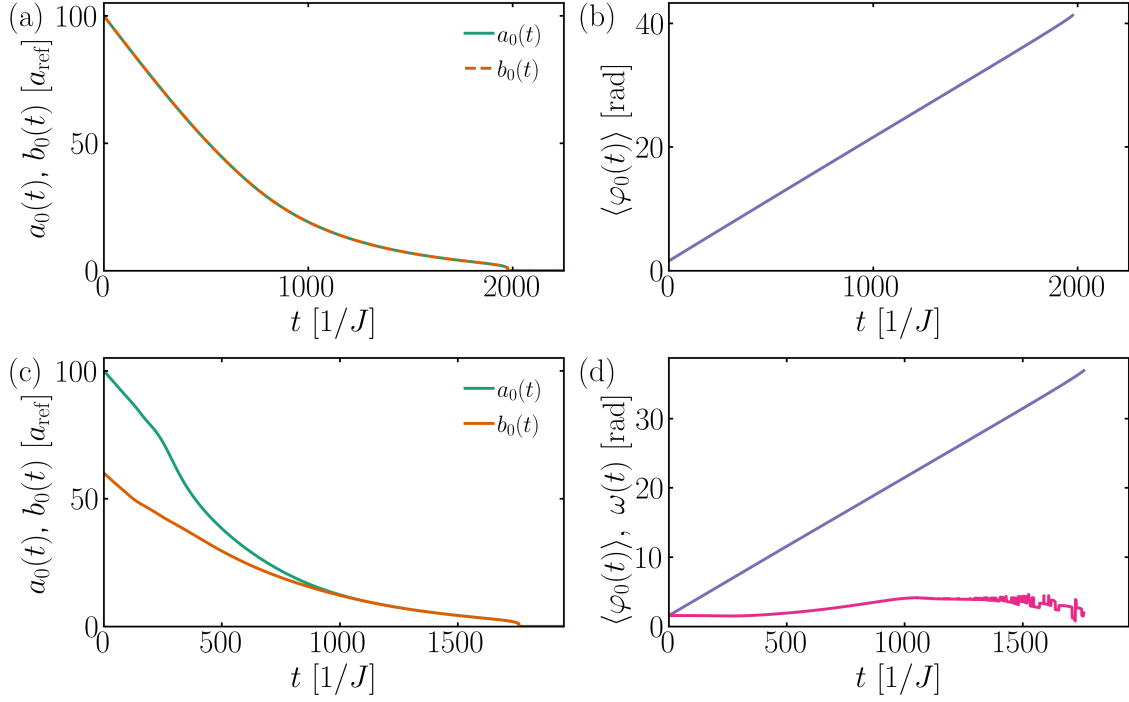


Figure 4.18.: Simulation results for the antiskymion decay in the absence of DMI. Panels (a) and (b) show the decay of the radius and helicity for an initially circular antiskymion, while panels (c) and (d) display the corresponding quantities for an initially elliptic antiskymion. In both cases, the semi-axes decay exponentially at large sizes and transition into a square-root-like collapse in the small-size regime. The elliptic antiskymion becomes circular during the collapse. The helicity evolves linearly at large sizes, but deviations from this behavior at small sizes as the collapse is approached are not observable. The time evolution of the rotation angle in (d) shows no in-plane rotation and the wiggles near t_c are due to numerical difficulties to track the direction of the semi-axes, when the antiskymion is close to circularity. Parameters: $D = 0$, $B = 0.02$, $\alpha = 0.1$, $a_{0,0} = 100$, $b_{0,0} = 100$, $\varphi_{0,0} = \pi/2$, $\omega_0 = \pi/2$. [Figure adapted from Ref. [136]]

4.6.3. Antiskymion decay at finite DMI

In the presence of DMI, our analytical model predicts that the semi-axes follow the same overall time evolution as in the zero DMI case, but with superimposed quadrupole oscillations. These oscillations arise from the coupling between the semi-axes, the helicity, and the rotation angle due to the finite DMI strength. The rotation angle itself is ex-

pected to evolve with half the slope of the helicity and to exhibit step-like jumps that are synchronized with the quadrupole oscillations.

The numerical simulation results at finite DMI are shown in Figure 4.19, with panels (a) and (b) corresponding to an initially circular antiskyrmion and panels (c) and (d) to an initially elliptic one. The simulations display the exponential-to-square-root time evolution of the semi-axes, modulated by quadrupole oscillations [136]. This rotational quadrupole breathing mode is fully consistent with findings from the analytical model: in the initially circular case, the rotation angle ω tracks the helicity with half its slope, as confirmed by the linear fits in panel (b), and exhibits jumps synchronized with the semi-axes oscillations. Moreover, the inset shows, that ω relaxes on a short time scale, while $\varphi_0 \approx \pi/2$, to a value slightly above $\pi/4$, in agreement with Eq. (4.72). At longer times, the antiskyrmion remains elliptic and approaches a circular shape only shortly before its collapse, again consistent with the model.

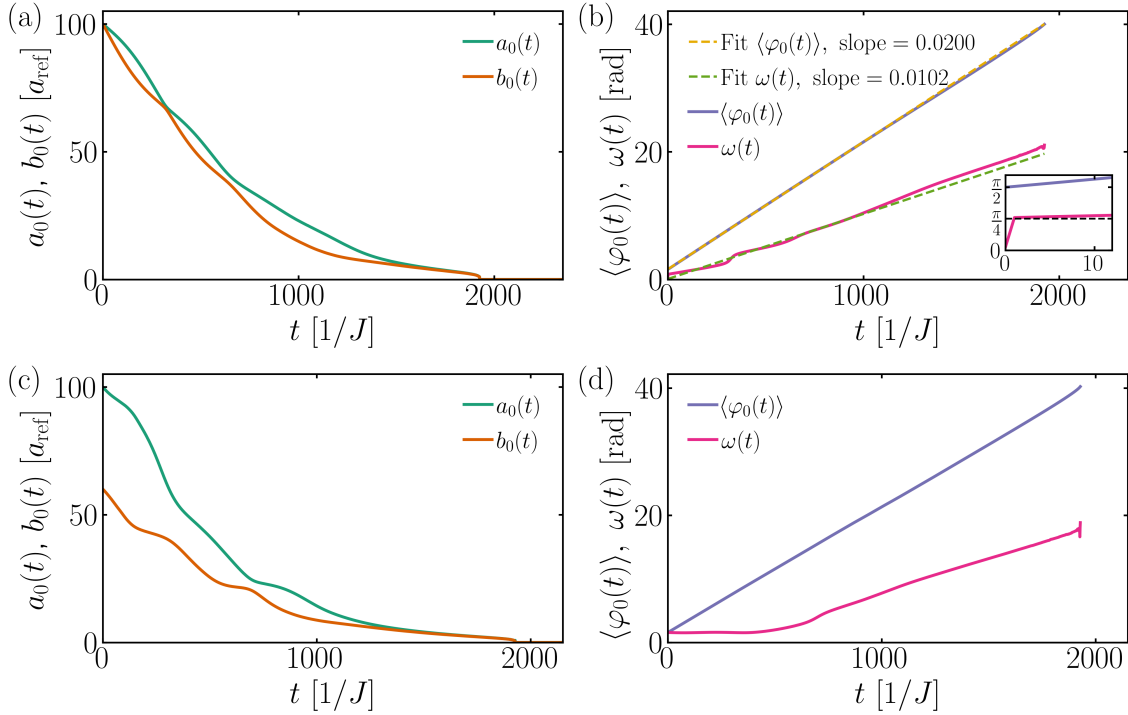


Figure 4.19.: Time evolution of the semi-axes, helicity, and rotation angle obtained from numerical simulations at finite DMI. Panels (a) and (b) show the initially circular antiskyrmion ($b_{0,0} = 100$), while panels (c) and (d) correspond to the initially elliptic case ($b_{0,0} = 60$). The semi-axes exhibit the characteristic exponential-to-square-root decay, superimposed with quadrupole oscillations. In the initially circular case, the rotation angle ω displays synchronized jumps and follows the helicity with half its slope. The inset in panel (b) shows, that at early simulation times ω relaxes rapidly to a value slightly above $\pi/4$. In the initially elliptic case $\omega(t)$ is pinned until the semi-axes come sufficiently close to each other that ω unlocks and begins to rotate. Parameters: $D = 0.02$, $B = 0.02$, $\alpha = 0.1$, $a_{0,0} = 100$, $\varphi_{0,0} = \pi/2$, $\omega_0 = \pi/2$. [Figure adapted from Ref. [136]]

In the initially elliptic case, the system also follows the model predictions: the rotation angle remains approximately pinned to its initial value until the quadrupole oscillations bring the semi-axes sufficiently close together. At that moment, $\omega(t)$ unlocks and resumes following the helicity with half its slope.

In Sec. 4.1, we discussed that a detailed modulation of the azimuthal angle Φ , accounting for elliptic deformations of the in-plane component of the magnetization field, is not included in our analytical model. Instead, we use the simple form $\Phi(\varphi)$ given by Eq. (2.23) to enable an analytical treatment of the coupled ordinary differential equations, while maintaining a level of detail consistent with the triangular profile assumed for Θ , as discussed in Sec. 2.4.4.

To assess the extent to which neglecting the elliptic deformation of the in-plane component affects the dynamics compared to the full simulation results, we analyze simulation data during the shrinking process. For this purpose, we extract $\Phi(\varphi)$ at different values of ρ from the simulation results shown in Figure 4.19 (a) and (b). The results [136] are presented in Figure 4.20 (a), (c), and (e) for a weakly elliptic antiskyrmion at $tJ = 250$, and in panels (b), (d), and (f) for a strongly elliptic antiskyrmion at $tJ = 1100$.

It is evident that the dominant contribution to the azimuthal angle remains the linear dependence, as assumed in Eq. (2.23). Panels (a) and (b) show $\Phi(\varphi)$, with dashed black lines indicating the linear approximation obtained using the averaged helicity $\langle\varphi_0\rangle$ at the respective time points. This dominant linear behavior, however, is superimposed by a weak sinusoidal modulation, reflecting the elliptic deformation of the in-plane component induced by the deformation of the out-of-plane component. To highlight this modulation around the averaged helicity, panels (c) and (d) show $\Phi(\varphi) + \varphi$. Panels (e) and (f) display the z -component of the magnetization, with dashed lines indicating the values of ρ used in the analysis.

The observed modulations demonstrate that the simple, linear ansatz for $\Phi(\varphi)$ used in our analytical model neglects details of the internal structure of the antiskyrmion. However, these modulations remain weak throughout the shrinking process in all simulations considered here. Moreover, as discussed in Sec. 4.1, this approximation is consistent with the level of detail provided by the triangular profile assumed for Θ and is in favor for analytical tractability within the continuum model.

The full dynamical simulation results shown in Figure 4.20 (d) further reveal that the curves corresponding to different values of ρ are slightly shifted with respect to one another [136]. This indicates a radial dependence of Φ and shows that, although this effect remains weak, the antiskyrmion deviates from a perfectly elliptic shape during its shrinking. Our analytical model enforces a triangular radial profile and a linear azimuthal angle without any radial dependence at all times, thereby inherently suppressing such effects. Consequently, the model represents a simplified collective-coordinate approach that captures the main dynamical behavior of the antiskyrmion shrinking without resolving all internal deformation modes quantitatively.

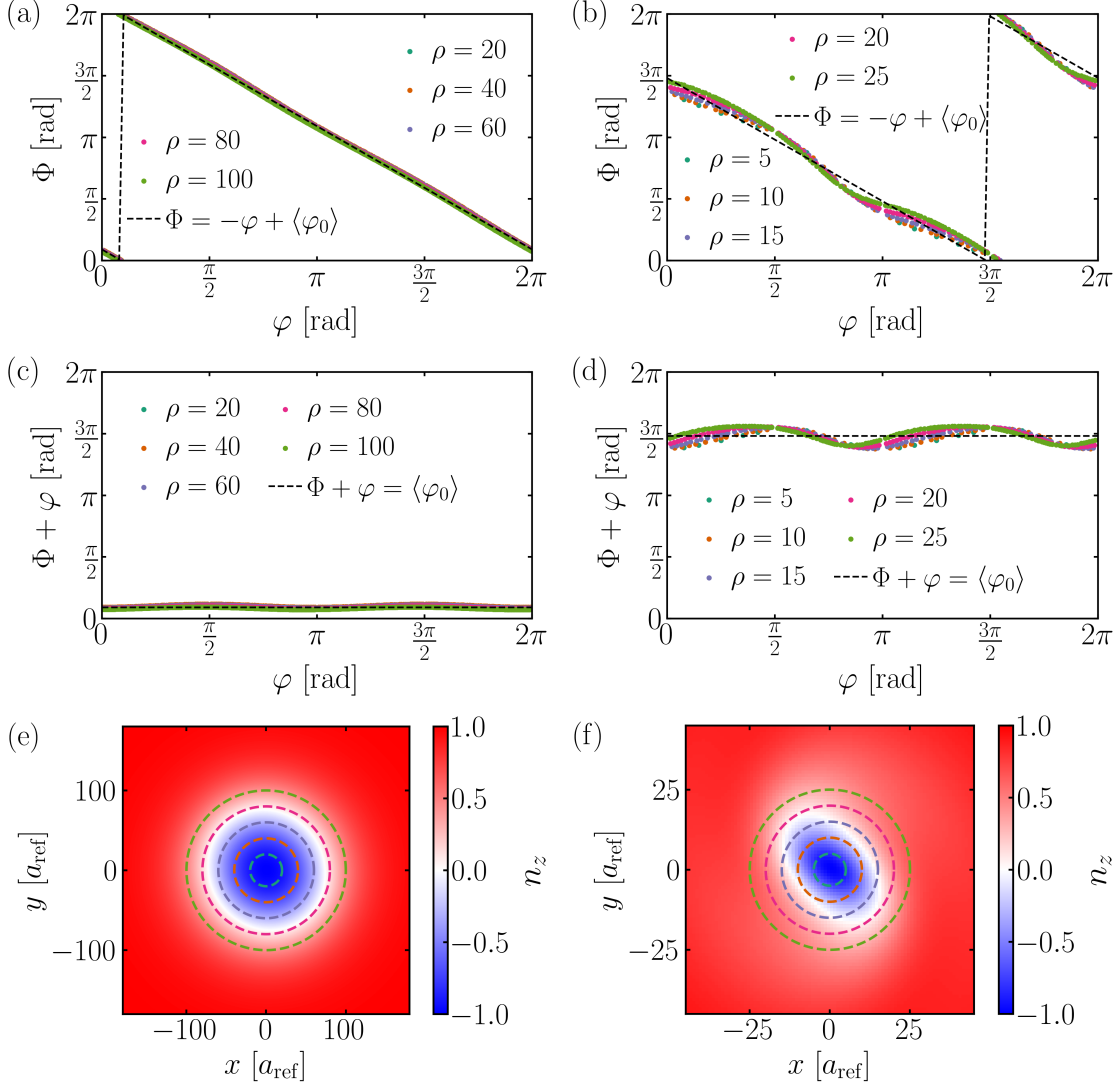


Figure 4.20.: The azimuthal angle $\Phi(\varphi)$ at different radial positions ρ extracted from simulation data during the antiskyrmion shrinking shown in Figure 4.19 (a) and (b). Panels (a), (c), and (e) correspond to a weakly elliptic antiskyrmion at $tJ = 250$, while panels (b), (d), and (f) show a strongly elliptic antiskyrmion at $tJ = 1100$. Panels (a) and (b) display $\Phi(\varphi)$, with dashed black lines indicating the linear approximation based on the averaged helicity $\langle\varphi_0\rangle$. Panels (c) and (d) highlight the sinusoidal modulation around the averaged helicity by showing $\Phi(\varphi) + \varphi$. Panels (e) and (f) show the corresponding z -component of the magnetization, with dashed lines marking the radial positions ρ used for the analysis. Parameters: $D = 0.02$, $B = 0.02$, $\alpha = 0.1$, $a_{0,0} = 100$, $b_{0,0} = 100$, $\varphi_{0,0} = \pi/2$, $\omega_0 = \pi/2$. [Figure adapted from Ref. [136]]

4.7. Conclusion

In this chapter, we have presented our findings on the shrinking dynamics of magnetic antiskyrmions in systems with isotropic bulk DMI. Owing to the antiskyrmion's twofold symmetry, its dynamics can be described using four collective coordinates within the tri-

angular ansatz of its polar profile: the semi-axes $a_0(t)$ and $b_0(t)$, the rotation angle $\omega(t)$, and the helicity $\varphi_0(t)$. This choice of dynamical variables naturally allows for deformation from a circular to an elliptic shape. Within this continuum approach, we examined the energy landscape with respect to these variables and subsequently derived and analyzed a set of coupled nonlinear differential equations governing the antiskyrmion's time evolution. Our analytic predictions for the shrinking dynamics are supported by micromagnetic simulations on a discrete grid.

The energy landscape as a function of the semi-axes shows that an antiskyrmion nucleated in an isotropic bulk DMI material is always energetically higher than the homogeneous ferromagnetic state and is therefore unstable. However, it also reveals that a deformation from a circular to an elliptic shape lowers the antiskyrmion's energy, rendering the elliptic configuration energetically favorable over the circular one, though it remains still unstable overall. Additional analysis of the energy with respect to the rotation angle and helicity shows that different in-plane orientations of an elliptic antiskyrmion at a fixed helicity yield different energies and vice versa. This agrees with the literature [13, 54, 55] as it is known that the antiskyrmion's twofold symmetry causes direction-dependent winding structures that lead to different direction dependent energetic costs. As a result, the energy landscape exhibits a π -periodic structure that is independent of the strength of the non-zero DMI, the Zeeman field, and the lengths of the semi-axes. In the absence of DMI, the energy landscape flattens and all orientations become degenerate in energy.

These energetic dependencies are reflected in the dynamical shrinking of an initially circular antiskyrmion. At vanishing DMI, the shrinking closely resembles the skyrmion case with a Zeeman-field-induced exponential decay of the semi-axes at large sizes, which crosses over into an exchange-dominated square-root-like collapse at small sizes. At finite DMI, however, the dynamics become considerably more intricate. From the dynamical equation for the rotation angle, we extract the initial orientation along which the antiskyrmion aligns its major axis once it becomes elliptical for a given helicity. The DMI breaks rotational symmetry, couples all four dynamical equations, and gives rise to a rotational quadrupole breathing mode: the semi-axes oscillate out of phase while the helicity undergoes continuous precession, and the rotation angle follows with half the helicity's slope to compensate for the energy changes induced by the elliptic deformation. The rotation angle also exhibits step-like jumps synchronized with the semi-axis oscillations. At small antiskyrmion sizes, the exchange energy dominates and restores circularity, reducing the dynamics to the circular case, where the semi-axes vanish at the critical time following a square-root law, while helicity and rotation angle diverge logarithmically.

An initially elliptic antiskyrmion in the absence of DMI exhibits no in-plane rotation and shows exponential decay of both semi-axes at large sizes. At small sizes, the antisymmetric dependence of the exchange contribution on the ratio of the semi-axes drives the system towards circularity, again producing a square-root collapse of the semi-axes. The helicity follows the characteristic linear-to-logarithmic divergence. At finite DMI,

quadrupole oscillations of the semi-axes persist, but the rotation angle becomes pinned. As the semi-axes periodically reduce their difference, the effective potential governing the orientation increases until the torque becomes large enough to overcome the pinning, at which point the in-plane rotation resumes.

In conclusion, we have developed a comprehensive analytical framework for the shrinking dynamics of antiskyrmions in an environment with isotropic bulk DMI and validated it through micromagnetic simulations. By establishing how the coupled evolution of the semi-axes, rotation angle, and helicity governs both the deformation and collapse behavior, we have clarified the characteristic dynamical modes, most notably the rotational quadrupole breathing mode, and their dependence on the underlying energy landscape. These findings deepen the understanding of antiskyrmion deformation and collapse pathways and provide a foundation for predicting and controlling their behavior in realistic materials and device settings.

5. Summary

In this thesis, we have developed a theoretical and computational framework to investigate the nonequilibrium dynamics of topological magnetic textures, focusing on the shrinking behavior of magnetic skyrmions and antiskyrmions in systems with isotropic bulk Dzyaloshinskii-Moriya interaction (DMI). By employing a collective coordinate approach in combination with a triangular ansatz for the radial profile, we derived reduced sets of coupled nonlinear differential equations that capture the essential dynamical behaviors of these quasiparticles. The analytical predictions obtained from this approach were systematically validated through micromagnetic simulations, demonstrating its robustness and reliability.

For skyrmions, the dynamics are governed by the coupled time evolution of the radius and the helicity. The interplay of exchange, Zeeman, and Dzyaloshinskii-Moriya interactions gives rise to a rich phenomenology, including an exponential shrinking of the skyrmion at large radii and a square-root collapse of its radius at small sizes. In the presence of DMI this time evolution is superimposed with a breathing mode. The breathing mode in the regime where skyrmions are stabilized, i.e., for DMI strengths above a critical value, is inherently different compared to the breathing mode in the unstable shrinking regime at lower DMI. In the stabilizing regime, a damped oscillatory breathing mode emerges, where the helicity and radius perform damped oscillations around the energy minimizing skyrmion configuration. Below the critical DMI, the skyrmion radius shrinks and vanishes at a critical time, accompanied by rotational breathing dynamics, where the helicity exhibits full 2π rotations through the different skyrmion configurations. These findings highlight the central role of helicity as an internal degree of freedom that is strongly coupled to the skyrmion size and governs its dynamical response.

In contrast, the dynamics of antiskyrmions is fundamentally enriched by their intrinsic twofold symmetry, which makes a description in terms of two semi-axes, a rotation angle and the helicity necessary. This extended set of collective coordinates naturally captures shape deformations and orientation-dependent effects. Our analysis shows that, unlike skyrmions, antiskyrmions in isotropic bulk DMI systems are always energetically unstable, although elliptic deformations lower their energy relative to the circular configuration. The broken rotational symmetry introduced by the DMI leads to a complex, anisotropic energy landscape, which in turn gives rise to qualitatively new dynamical phenomena. Most notably, we identify a rotational quadrupole-like breathing mode, characterized by out-of-phase oscillations of the semi-axes, continuous helicity precession, and a coupled evolution of the rotation angle. The interplay between deformation, orientation, and in-

ternal structure results in intricate dynamics, including step-like rotational behavior of the in-plane rotation angle and pinning of the rotation angle for initially elliptic antiskyrmions, which have no direct analogue in the skyrmion case.

A central outcome of this work is the identification of universal features shared by both topological textures. In particular, the shrinking dynamics shows for both quasiparticles a crossover from Zeeman-dominated exponential decay at large radii to exchange-driven square-root collapse at small sizes. This universal behavior reflects the underlying micro-magnetic energy contributions and is largely independent of the specific topology. Additionally, for both textures, the critical collapse time increases logarithmically with a decreasing lattice constant. At the same time, the differences between skyrmions and antiskyrmions, which are most prominently the stability of skyrmions versus the inherent instability of antiskyrmions in the studied environment, as well as the isotropic versus anisotropic nature of their dynamics, underscore the crucial role of symmetry and topology in determining their behavior.

Overall, the collective coordinate framework developed in this thesis provides a powerful tool for describing complex magnetization dynamics in a closed analytical form. It bridges the gap between full micromagnetic simulations and analytical understanding, enabling the identification of key mechanisms governing stability, instability and deformation. The insights gained here deepen the fundamental understanding of topological magnetic textures and also have direct implications for their controlled manipulation in future spintronic applications, where stability, dynamics, and geometry-dependent effects are of central importance.

Outlook

A natural extension of the present work would be the generalization of the collective-coordinate framework to systems with anisotropic bulk and interfacial DMI. Such a description could be introduced through the DMI tensor in Eq. (2.8), together with the bulk and interfacial DMI contributions in Eqs. (2.7) and (2.6), respectively. In this context, the elliptic ansatz introduced in Eq. (4.6) could be employed for both skyrmions and antiskyrmions, allowing the study of deformation effects and anisotropic shrinking dynamics for both magnetic textures within a unified framework. The resulting behaviors would then depend on the choice of the DMI parameters D_x and D_y .

For bulk DMI with $D_x = D_y$, the analytical approach would reproduce the equations of motion derived here, while additionally allowing the investigation of initially elliptic skyrmions in isotropic bulk DMI systems. Although circular skyrmions in isotropic bulk DMI are well understood from the present work, the relaxation and breathing dynamics of initially deformed skyrmions have not been treated. For interfacial DMI with $D_x = D_y$, the equations of motion are expected to mirror those obtained for bulk DMI, but with Néel-type skyrmions as the energetically stabilized configurations. Particularly interesting are then the anisotropic systems with $D_x = -D_y$, where antiskyrmions may become en-

ergetically stabilized [57–60]. While both this stabilization and the existence of rotational and oscillatory breathing modes have already been established in the literature [125], an analytical description of the associated shrinking dynamics is currently still missing. In such systems, one may expect isotropic shrinking for antiskyrmions analogous to those found here for skyrmions. Skyrmions, on the other hand, would remain always unstable and show elliptic deformations during their shrinking due to the direction-dependent DMI energy contributions.

In a more general setup, with an anisotropic DMI obeying $|D_x| \neq |D_y|$, nonsymmetric skyrmions and antiskyrmions may be stabilized within the same material simultaneously [58]. Although the present collective-coordinate approach does not describe interactions between multiple quasiparticles directly, it could still identify parameter regimes in which both topological textures are individually stable. An open question concerns the stability criteria of skyrmions and antiskyrmions under variations of the anisotropic DMI strengths. By simultaneously or independently reducing D_x and D_y , one may investigate whether both textures lose stability at identical critical DMI values or whether distinct collapse thresholds emerge due to the anisotropy. Such a study could reveal symmetry relations, or asymmetries, between the stability properties of skyrmions and antiskyrmions in anisotropic DMI systems. Micromagnetic simulations could then be employed to further investigate their coexistence and dynamics within the same ferromagnetic film.

Another promising direction concerns hybrid DMI systems containing both bulk and interfacial contributions [53]. Such systems stabilize so-called intermediate skyrmions, whose helicity continuously interpolates between Bloch- and Néel-type configurations. In Ref. [53], the dynamics of intermediate skyrmions was treated by mapping them onto Néel-type skyrmions through a global spin rotation, thereby allowing their motion to be described within an effective interfacial DMI framework. However, this description was largely restricted to approximately circular and rigid skyrmion textures. Extending the present collective-coordinate framework to hybrid DMI systems could therefore allow the study of elliptic intermediate skyrmions and antiskyrmions with a continuously varying size and helicity.

Beyond modifications of the DMI itself, the framework may also be generalized to include translational degrees of freedom, enabling the study of coupled shrinking and drift dynamics under external currents [9, 140, 141], spin waves [142] or magnetic field gradients [43, 143]. While current-driven skyrmion motion has been extensively studied, the interplay between translational dynamics and the internal breathing and shrinking dynamics identified here has received comparatively little attention. The specific question of whether spin-transfer or spin-orbit torques stabilize or destabilize these modes remains open.

Together, these directions would extend the predictive power of reduced models and support the development of skyrmion- and antiskyrmion-based spintronic devices.

A. Numerical simulation of the LLG dynamics

The analytical studies on skyrmion and antiskyrmion shrinking presented in Chapter 3 and Chapter 4, respectively, are validated using numerical simulations obtained from a micromagnetic simulation program. These simulations are performed with a self-written C++ code employing GPU acceleration via *CUDA* [144], enabling an efficient parallel evaluation of the dynamics of the magnetic moments. The simulated system is defined on a two-dimensional square lattice with a magnetic moment at each lattice site. The time evolution is governed by the LLG Eq. (2.14) in its explicit form. As input, an initial magnetization configuration is specified as an array of size $N_x \times N_y \times 3$, where N_x and N_y denote the number of lattice sites in the x - and y -directions, respectively. At each site, the magnetic moment is represented by three real-valued components. For computational efficiency, this array is flattened into a one-dimensional data structure with $3N_xN_y$ entries. Additional input parameters are the exchange interaction strength J , the DMI strength D , the external magnetic field strength B , the Gilbert damping strength α , and the value of the lattice constant a . All quantities are expressed in dimensionless units consistent with the main text, where energies are scaled by J , time by $t_0 = 1/J$ and lengths by the reference lattice constant a_{ref} . The time evolution at each lattice site is governed by the influence of the effective magnetic field, which in the simulations takes the form

$$\mathbf{B}_i^{\text{eff}} = \frac{J}{a^2} (\mathbf{n}_{i+\hat{x}} + \mathbf{n}_{i-\hat{x}} + \mathbf{n}_{i+\hat{y}} + \mathbf{n}_{i-\hat{y}}) + \mathbf{B}_i^{\text{DMI}} + \mathbf{B}. \quad (\text{A.1})$$

It includes contributions from exchange and DMI, both involving nearest-neighbor coupling, as well as the external magnetic field. In our studies, the external magnetic field is applied along the z -direction, i.e., $\mathbf{B} = (0, 0, B)^T$. We consider systems with bulk DMI, for which the contribution to the effective magnetic field is given by

$$\mathbf{B}_i^{\text{DMI}} = \frac{D}{a} \begin{pmatrix} (n_{i-\hat{y}}^z - n_{i+\hat{y}}^z) \\ (n_{i+\hat{x}}^z - n_{i-\hat{x}}^z) \\ (n_{i-\hat{x}}^y - n_{i+\hat{x}}^y) + (n_{i+\hat{y}}^x - n_{i-\hat{y}}^x) \end{pmatrix}. \quad (\text{A.2})$$

Consequently, the dynamics at each lattice site depend not only on the local magnetic moment but also on its nearest neighbors via finite-difference approximations of spatial derivatives.

Time integration is performed using a fourth-order Runge-Kutta (RK4) scheme [145], which advances the system from time t to $t + \Delta t$ by evaluating the LLG equation at several

intermediate stages within each time step. As an explicit integration method, RK4 provides a numerical approximation to the solution of the differential equations while achieving high accuracy due to its fourth-order convergence in the time step size. After each integration step, the magnetic moments are renormalized to unit length to compensate for numerical deviations. The flattened representation of the magnetization field enables an efficient data-parallel implementation in which all magnetic moments are updated simultaneously using the *thrust* library [146]. In combination with the `odeint` module of the *boost* library [147, 148], the RK4 scheme is applied to all degrees of freedom in parallel.

To incorporate nearest-neighbor interactions within this parallel framework, the lattice is duplicated and shifted by one lattice spacing in the $+x$, $-x$, $+y$, and $-y$ directions. These shifted copies provide direct access to neighboring magnetic moments without requiring explicit indexing during the update step. While this approach simplifies the evaluation of finite-difference expressions and improves computational efficiency, it increases memory usage, as each additional neighbor direction requires storing a full copy of the lattice. Throughout all simulations, closed boundary conditions are employed by fixing the magnetic moments at the system edges, i.e., they are not updated during the time integration, ensuring that no dynamics occur at the boundaries.

B. Dynamics of skyrmion shrinking

In Sec. 3.2.1, we derive the coupled nonlinear ordinary differential equations for the skyrmion radius and helicity, given in Eqs. (3.21) and (3.27), respectively. These are obtained by isolating $\dot{\rho}_0$ and $\dot{\varphi}_0$ from the components of the with respect to time-derived vector field in Eq. (3.12) on the one hand, and by combining them with the components of the LLG equation in Eq. (2.21) on the other hand. The terms originating from the latter, denoted as $[\dot{\mathbf{n}}]_{x,y,z}^{\text{RHS}}$, are not displayed in the main text. In the following, we show them explicitly.

First, we derive the effective magnetic field in Eq. (2.20) in terms of the triangular skyrmion. Each component of $\mathbf{B}_{\text{eff}}$ is generally given by

$$B_{\text{eff},x} = J \Delta n_x - 2\tilde{D} \partial_y n_z, \quad (\text{B.1})$$

$$B_{\text{eff},y} = J \Delta n_y + 2\tilde{D} \partial_x n_z, \quad (\text{B.2})$$

$$B_{\text{eff},z} = J \Delta n_z - 2\tilde{D} (\partial_x n_y - \partial_y n_x) + \tilde{B}. \quad (\text{B.3})$$

Since the skyrmion is radially symmetric, it is convenient to work in polar coordinates. The Laplacian acting on fields $f(\rho, \varphi)$ in polar coordinates reads

$$\Delta f = \partial_\rho^2 f + \frac{1}{\rho} \partial_\rho f + \frac{1}{\rho^2} \partial_\varphi^2 f, \quad (\text{B.4})$$

and the spatial derivatives are given by

$$\partial_x f = \cos \varphi \partial_\rho f - \frac{1}{\rho} \sin \varphi \partial_\varphi f, \quad (\text{B.5})$$

$$\partial_y f = \sin \varphi \partial_\rho f + \frac{1}{\rho} \cos \varphi \partial_\varphi f. \quad (\text{B.6})$$

Note that we consider a two-dimensional system, such that all derivatives with respect to z vanish, i.e., $\partial_z f = 0$. The resulting effective magnetic field expressed in terms of the triangular skyrmion in Eq. (3.11) is displayed in the main text in Eq. (3.13).

In general, the components of the continuum LLG Eq. (2.21) can be evaluated as

$$\begin{aligned} [\dot{\mathbf{n}}]_x^{\text{RHS}} = & -\frac{a^2}{1+\alpha^2} \left[n_y B_{\text{eff},z} - n_z B_{\text{eff},y} \right] \\ & -\frac{a^2 \alpha}{1+\alpha^2} \left[n_y (n_x B_{\text{eff},y} - n_y B_{\text{eff},x}) - n_z (n_z B_{\text{eff},x} - n_x B_{\text{eff},z}) \right], \end{aligned} \quad (\text{B.7})$$

$$[\dot{\mathbf{n}}]_y^{\text{RHS}} = -\frac{a^2}{1+\alpha^2} \left[n_z B_{\text{eff},x} - n_x B_{\text{eff},z} \right] - \frac{a^2 \alpha}{1+\alpha^2} \left[n_z (n_y B_{\text{eff},z} - n_z B_{\text{eff},y}) - n_x (n_x B_{\text{eff},y} - n_y B_{\text{eff},x}) \right], \quad (\text{B.8})$$

$$[\dot{\mathbf{n}}]_z^{\text{RHS}} = -\frac{a^2}{1+\alpha^2} \left[n_x B_{\text{eff},y} - n_y B_{\text{eff},x} \right] - \frac{a^2 \alpha}{1+\alpha^2} \left[n_x (n_z B_{\text{eff},x} - n_x B_{\text{eff},z}) - n_y (n_y B_{\text{eff},z} - n_z B_{\text{eff},y}) \right]. \quad (\text{B.9})$$

Inserting the components of the magnetization field and the effective magnetic field, given by Eqs. (3.11) and (3.13), respectively, into Eqs. (B.7)-(B.9) yields the components

$$\begin{aligned} [\dot{\mathbf{n}}]_x^{\text{RHS}} = & J \frac{\pi^2 a^2}{4(1+\alpha^2) \rho_0^2} \left[\frac{1}{\xi} - \frac{\cos \xi \sin \xi}{\xi^2} \right] \left(\alpha \cos \xi \cos(\varphi + \varphi_0) - \sin(\varphi + \varphi_0) \right) \\ & - \tilde{D} \frac{\pi a^2}{(1+\alpha^2) \rho_0} \left[\alpha \xi \sin \xi \cos^2 \xi \cos \varphi_0 \sin(\varphi + \varphi_0) \right. \\ & + \frac{\sin^2 \xi}{\xi} \left(\alpha \xi \sin \xi \cos \varphi_0 - \sin \varphi_0 \right) \sin(\varphi + \varphi_0) \\ & + \left. \frac{\sin \xi \cos \xi}{\xi} \left(\xi \cos \varphi + \sin \varphi_0 (\alpha \cos(\varphi + \varphi_0) \sin \xi - \xi \sin(\varphi + \varphi_0)) \right) \right] \\ & + \tilde{B} \frac{a^2}{1+\alpha^2} \left[\alpha \sin \xi \cos \xi \cos(\varphi + \varphi_0) - \sin \xi \sin(\varphi + \varphi_0) \right], \end{aligned} \quad (\text{B.10})$$

$$\begin{aligned} [\dot{\mathbf{n}}]_y^{\text{RHS}} = & J \frac{\pi^2 a^2}{4(1+\alpha^2) \rho_0^2} \left[\frac{1}{\xi} - \frac{\cos \xi \sin \xi}{\xi^2} \right] \left(\alpha \cos \xi \sin(\varphi + \varphi_0) + \cos(\varphi + \varphi_0) \right) \\ & + \tilde{D} \frac{\pi a^2}{(1+\alpha^2) \rho_0} \left[\alpha \xi \sin \xi \cos^2 \xi \cos \varphi_0 \cos(\varphi + \varphi_0) \right. \\ & + \frac{\sin^2 \xi}{\xi} \left(\alpha \xi \sin \xi \cos \varphi_0 - \sin \varphi_0 \right) \cos(\varphi + \varphi_0) \\ & - \left. \frac{\sin \xi \cos \xi}{\xi} \left(\xi \sin \varphi + \sin \varphi_0 (\alpha \sin \xi \sin(\varphi + \varphi_0) + \xi \cos(\varphi + \varphi_0)) \right) \right] \\ & + \tilde{B} \frac{a^2}{1+\alpha^2} \left[\alpha \sin \xi \cos \xi \sin(\varphi + \varphi_0) + \sin \xi \cos(\varphi + \varphi_0) \right], \end{aligned} \quad (\text{B.11})$$

$$\begin{aligned} [\dot{\mathbf{n}}]_z^{\text{RHS}} = & J \frac{\pi^2 a^2 \alpha}{4(1+\alpha^2) \rho_0^2} \left[\frac{\sin \xi}{\xi} - \frac{\cos \xi \sin^2 \xi}{\xi^2} \right] \\ & - \tilde{D} \frac{\pi a^2 \alpha}{(1+\alpha^2) \rho_0} \left[\sin^2 \xi \frac{1}{\alpha} \cos \varphi_0 + \frac{\sin^3 \xi}{\xi} \sin \varphi_0 \right] \\ & + \tilde{B} \frac{a^2 \alpha}{1+\alpha^2} \sin^2 \xi. \end{aligned} \quad (\text{B.12})$$

C. Dynamics of antiskyrmion shrinking

In Sec. 4.3, we derive the coupled nonlinear ordinary differential equations for the semi-axes, rotation angle, and helicity of the antiskyrmion in the absence of DMI, given in Eqs. (4.40)-(4.43). The additional contributions arising from a finite DMI are derived in Sec. 4.5 and are given in Eqs. (4.57)-(4.60).

All dynamical equations are obtained by isolating $\dot{a}_0$, $\dot{b}_0$, $\dot{\omega}$, and $\dot{\varphi}_0$ from the components of the time-derived vector field in Eq. (4.17) on the one hand, and combining them with the components of the LLG equation in Eq. (2.21) on the other hand. The contributions originating from the former are denoted by $[\dot{\mathbf{n}}]_{x,y,z}^{\text{LHS}}$, while those originating from the latter are denoted by $[\dot{\mathbf{n}}]_{x,y,z}^{\text{RHS}}$ and are not shown explicitly in the main text. In the following, we present them in full detail.

C.1. Time derivative of the vector field

The time derivative of the magnetization field in Eq. (4.17) governs the dynamics of the antiskyrmion's helicity via the time derivative of the azimuthal angle, while the dynamics of the semi-axes and the rotation angle are governed by the time derivative of the polar angle, as given in Eq. (4.18). The resulting time derivatives of the x -, y -, and z -components of the magnetization field are given by

$$\begin{aligned}
 [\dot{\mathbf{n}}]_x^{\text{LHS}} = & \sin \xi \sin(\varphi - \varphi_0) \dot{\varphi}_0 - \frac{\pi \rho \cos \xi \cos(\varphi - \varphi_0)}{4a_0^2 b_0^2 \sqrt{b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega)}} \\
 & \times \left[2a_0^3 \sin^2(\varphi - \omega) \dot{b}_0 + a_0^3 b_0 \sin(2(\varphi - \omega)) \dot{\omega} \right. \\
 & \left. + b_0^3 (2 \cos^2(\varphi - \omega) \dot{a}_0 - a_0 \sin(2(\varphi - \omega)) \dot{\omega}) \right], \tag{C.1}
 \end{aligned}$$

$$\begin{aligned}
 [\dot{\mathbf{n}}]_y^{\text{LHS}} = & \sin \xi \cos(\varphi - \varphi_0) \dot{\varphi}_0 + \frac{\pi \rho \cos \xi \sin(\varphi - \varphi_0)}{4a_0^2 b_0^2 \sqrt{b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega)}} \\
 & \times \left[2a_0^3 \sin^2(\varphi - \omega) \dot{b}_0 + a_0^3 b_0 \sin(2(\varphi - \omega)) \dot{\omega} \right. \\
 & \left. + b_0^3 (2 \cos^2(\varphi - \omega) \dot{a}_0 - a_0 \sin(2(\varphi - \omega)) \dot{\omega}) \right], \tag{C.2}
 \end{aligned}$$

$$\begin{aligned}
[\dot{\mathbf{n}}]_z^{\text{LHS}} = & - \frac{\pi \rho \sin \xi}{4a_0^2 b_0^2 \sqrt{b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega)}} \\
& \times \left[2a_0^3 \sin^2(\varphi - \omega) \dot{b}_0 + a_0^3 b_0 \sin(2(\varphi - \omega)) \dot{\omega} \right. \\
& \left. + b_0^3 (2 \cos^2(\varphi - \omega) \dot{a}_0 - a_0 \sin(2(\varphi - \omega)) \dot{\omega}) \right].
\end{aligned} \tag{C.3}$$

C.2. Derivation of the LLG components

To derive the effective magnetic field $\mathbf{B}_{\text{eff}}$ in Eq. (2.20) for the elliptic, triangular anti-skyrmion, we follow the general expressions given in Eqs. (B.1)-(B.3). This yields the following components

$$\begin{aligned}
B_{\text{eff},x} = & \frac{1}{\rho^2} \left\{ 2\tilde{D}\rho \sin \Theta \left[\cos \varphi \partial_\varphi \Theta + \rho \sin \varphi \partial_\rho \Theta \right] \right. \\
& - J \left[2 \sin(\varphi - \varphi_0) \cos \Theta \partial_\varphi \Theta + \cos(\varphi - \varphi_0) \sin \Theta (\partial_\varphi \Theta)^2 \right. \\
& + \cos(\varphi - \varphi_0) \left(\sin \Theta - \cos \Theta \partial_\varphi^2 \Theta - \rho \cos \Theta \partial_\rho \Theta \right. \\
& \left. \left. + \rho^2 \sin \Theta (\partial_\rho \Theta)^2 - \rho^2 \cos \Theta \partial_\rho^2 \Theta \right) \right] \left. \right\},
\end{aligned} \tag{C.4}$$

$$\begin{aligned}
B_{\text{eff},y} = & \frac{1}{\rho^2} \left\{ -2\tilde{D}\rho \sin \Theta \left[-\sin \varphi \partial_\varphi \Theta + \rho \cos \varphi \partial_\rho \Theta \right] \right. \\
& + J \left[-2 \cos(\varphi - \varphi_0) \cos \Theta \partial_\varphi \Theta + \sin(\varphi - \varphi_0) \sin \Theta (\partial_\varphi \Theta)^2 \right. \\
& + \sin(\varphi - \varphi_0) \left(\sin \Theta - \cos \Theta \partial_\varphi^2 \Theta - \rho \cos \Theta \partial_\rho \Theta \right. \\
& \left. \left. + \rho^2 \sin \Theta (\partial_\rho \Theta)^2 - \rho^2 \cos \Theta \partial_\rho^2 \Theta \right) \right] \left. \right\},
\end{aligned} \tag{C.5}$$

$$\begin{aligned}
B_{\text{eff},z} = & \frac{1}{\rho^2} \left\{ 2\tilde{D}\rho \left[\cos(2\varphi - \varphi_0) \cos \Theta \partial_\varphi \Theta + \sin(2\varphi - \varphi_0) \left(\rho \cos \Theta \partial_\rho \Theta - \sin \Theta \right) \right] \right. \\
& - J \left[\cos \Theta (\partial_\varphi \Theta)^2 + \sin \Theta \partial_\varphi^2 \Theta \right. \\
& \left. \left. + \rho \left(\sin \Theta \partial_\rho \Theta + \rho \cos \Theta (\partial_\rho \Theta)^2 + \rho \sin \Theta \partial_\rho^2 \Theta \right) \right] \right\} + \tilde{B}.
\end{aligned} \tag{C.6}$$

Here, the derivatives of the polar angle Θ with respect to the coordinates ρ and φ are given by

$$\partial_\rho \Theta = -\frac{\pi \sqrt{b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega)}}{2a_0 b_0}, \quad (\text{C.7})$$

$$\partial_\rho^2 \Theta = 0, \quad (\text{C.8})$$

$$\partial_\varphi \Theta = \frac{(b_0^2 - a_0^2) \pi \rho \sin(2(\varphi - \omega))}{4 a_0 b_0 \sqrt{b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega)}}, \quad (\text{C.9})$$

$$\partial_\varphi^2 \Theta = \frac{(b_0^2 - a_0^2) \pi \rho (b_0^2 \cos^4(\varphi - \omega) + a_0^2 \sin^4(\varphi - \omega))}{2 a_0 b_0 (b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega))^{3/2}}. \quad (\text{C.10})$$

The components of the LLG equation can then be evaluated using Eqs. (B.7)-(B.9). Due to the length of the resulting expressions, we decompose each component into contributions from exchange, DMI, and Zeeman interactions

$$[\dot{\mathbf{n}}]_x^{\text{RHS}} = [\dot{\mathbf{n}}]_x^{\text{Exch}} + [\dot{\mathbf{n}}]_x^{\text{DMI}} + [\dot{\mathbf{n}}]_x^{\text{Zee}}, \quad (\text{C.11})$$

$$[\dot{\mathbf{n}}]_y^{\text{RHS}} = [\dot{\mathbf{n}}]_y^{\text{Exch}} + [\dot{\mathbf{n}}]_y^{\text{DMI}} + [\dot{\mathbf{n}}]_y^{\text{Zee}}, \quad (\text{C.12})$$

$$[\dot{\mathbf{n}}]_z^{\text{RHS}} = [\dot{\mathbf{n}}]_z^{\text{Exch}} + [\dot{\mathbf{n}}]_z^{\text{DMI}} + [\dot{\mathbf{n}}]_z^{\text{Zee}}. \quad (\text{C.13})$$

We thus obtain the following contributions to the x -component

$$\begin{aligned} [\dot{\mathbf{n}}]_x^{\text{Exch}} = & J \frac{\pi a^2}{1 + \alpha^2} \left[\frac{a_0 b_0 (\alpha \cos \xi \cos(\varphi - \varphi_0) + \sin(\varphi - \varphi_0))}{2 \rho (b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega))^{3/2}} \right. \\ & + \frac{(a_0^2 - b_0^2) \cos \xi \sin(2(\varphi - \omega)) (\cos \xi \cos(\varphi - \varphi_0) - \alpha \sin(\varphi - \varphi_0))}{2 \rho a_0 b_0 \sqrt{b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega)}} \\ & \left. - \frac{\cos \xi \sin \xi}{\pi \rho^2} (\alpha \cos \xi \cos(\varphi - \varphi_0) + \sin(\varphi - \varphi_0)) \right], \end{aligned} \quad (\text{C.14})$$

$$\begin{aligned}
[\dot{\mathbf{n}}]_x^{\text{DMI}} = & \tilde{D} \frac{a^2}{1 + \alpha^2} \left[-\frac{2}{\rho} \sin^2 \xi \sin(2\varphi - \varphi_0) \left(\alpha \cos \xi \cos(\varphi - \varphi_0) + \sin(\varphi - \varphi_0) \right) \right. \\
& - \frac{\pi \sin \xi}{4a_0 b_0} \left(\cos \varphi + \cos(3\varphi - 2\varphi_0) + 2\alpha (\sin \varphi - \sin(3\varphi - 2\varphi_0)) \right) \\
& \times \sqrt{b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega)} \\
& + \sin \xi \sin(2\varphi - \varphi_0) \sin(2(\varphi - \omega)) \\
& \left. \times \left(\frac{\pi(a_0^2 - b_0^2)(\cos \xi \cos(\varphi - \varphi_0) - \alpha \sin(\varphi - \varphi_0))}{2a_0 b_0 \sqrt{b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega)}} \right) \right], \tag{C.15}
\end{aligned}$$

$$[\dot{\mathbf{n}}]_x^{\text{Zee}} = \tilde{B} \frac{a^2}{1 + \alpha^2} \sin \xi \left[\sin(\varphi - \varphi_0) + \alpha \cos \xi \cos(\varphi - \varphi_0) \right]. \tag{C.16}$$

For the y -component, we obtain

$$\begin{aligned}
[\dot{\mathbf{n}}]_y^{\text{Exch}} = & J \frac{\pi a^2}{1 + \alpha^2} \left[\frac{a_0 b_0 (\cos(\varphi - \varphi_0) - \alpha \cos \xi \sin(\varphi - \varphi_0))}{2\rho (b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega))^{3/2}} \right. \\
& - \frac{(a_0^2 - b_0^2) \cos \xi \sin(2(\varphi - \omega)) (\cos \xi \sin(\varphi - \varphi_0) + \alpha \cos(\varphi - \varphi_0))}{2\rho a_0 b_0 \sqrt{b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega)}} \\
& \left. + \frac{\cos \xi \sin \xi}{\pi \rho^2} (\alpha \cos \xi \sin(\varphi - \varphi_0) - \cos(\varphi - \varphi_0)) \right], \tag{C.17}
\end{aligned}$$

$$\begin{aligned}
[\dot{\mathbf{n}}]_y^{\text{DMI}} = & \tilde{D} \frac{a^2}{1 + \alpha^2} \left[\frac{2}{\rho} \sin^2 \xi \sin(2\varphi - \varphi_0) \left(\alpha \cos \xi \sin(\varphi - \varphi_0) - \cos(\varphi - \varphi_0) \right) \right. \\
& + \frac{\pi \sin \xi}{4a_0 b_0} \left(-\sin \varphi + \sin(3\varphi - 2\varphi_0) + 2\alpha (\cos \varphi + \cos(3\varphi - 2\varphi_0)) \right) \\
& \times \sqrt{b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega)} \\
& - \sin \xi \sin(2\varphi - \varphi_0) \sin(2(\varphi - \omega)) \\
& \left. \times \left(\frac{\pi(a_0^2 - b_0^2)(\cos \xi \sin(\varphi - \varphi_0) + \alpha \cos(\varphi - \varphi_0))}{2a_0 b_0 \sqrt{b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega)}} \right) \right], \tag{C.18}
\end{aligned}$$

$$[\dot{\mathbf{n}}]_y^{\text{Zee}} = \tilde{B} \frac{a^2}{1 + \alpha^2} \sin \xi \left[\cos(\varphi - \varphi_0) - \alpha \cos \xi \sin(\varphi - \varphi_0) \right]. \tag{C.19}$$

Finally, for the z -component, we find

$$\begin{aligned}
[\dot{\mathbf{n}}]_z^{\text{Exch}} = & J \frac{\pi a^2}{1 + \alpha^2} \left[\frac{\sin \xi}{4 \rho a_0 b_0 (b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega))^{3/2}} \left[2a_0^2 b_0^2 \alpha \cos^4(\varphi - \omega) \right. \right. \\
& + 4b_0^2(a_0^2 - b_0^2) \cos \xi \cos^3(\varphi - \omega) \sin(\varphi - \omega) \\
& + a_0^2 b_0^2 \alpha (3 + \cos(2(\varphi - \omega))) \sin^2(\varphi - \omega) \\
& \left. \left. + 4a_0^2(a_0^2 - b_0^2) \cos \xi \cos(\varphi - \omega) \sin^3(\varphi - \omega) \right] - \alpha \frac{\sin^2 \xi}{2\pi \rho^2} \right], \tag{C.20}
\end{aligned}$$

$$\begin{aligned}
[\dot{\mathbf{n}}]_z^{\text{DMI}} = & \tilde{D} \frac{a^2}{1 + \alpha^2} \left[\frac{\pi \sin^2 \xi ((a_0^2 + b_0^2) \cos(2\varphi - \varphi_0) - (a_0^2 - b_0^2) \cos(\varphi_0 - 2\omega))}{2a_0 b_0 \sqrt{b_0^2 \cos^2(\varphi - \omega) + a_0^2 \sin^2(\varphi - \omega)}} \right. \\
& \left. - \frac{2\alpha \sin^3 \xi \sin(2\varphi - \varphi_0)}{\rho} \right], \tag{C.21}
\end{aligned}$$

$$[\dot{\mathbf{n}}]_z^{\text{Zee}} = \tilde{B} \frac{a^2 \alpha}{1 + \alpha^2} \sin^2 \xi. \tag{C.22}$$

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