On the structure of concentrated atmospheric vortices in a gradient wind regime and its motion on synoptic scales

Dissertation

Zur Erlangung des Doktorgrades der Naturwissenschaften
im Department Geowissenschaften der Universität Hamburg

vorgelegt von
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aus
Frankfurt (Oder)

Hamburg, 2007

Als Dissertation angenommen vom Department Geowissenschaften der Universität Hamburg

Auf Grund der Gutachten von Prof. Dr. M. Claußen
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Hamburg, den 06.02.2007

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Zusammenfassung

Dreidimensionale, konzentrierte, atmosphärische Wirbel die sich vertikal durch die gesamte Troposphäre erstrecken und dessen mittleren horizontalen Abmessungen unterhalb der synoptischen Skala im Bereich des Gradientwind-Regimes liegen, sind Gegenstand der Untersuchungen in der vorliegenden Arbeit. Ein typisches Beispiel für derartige atmosphärische Strömungsphänomene sind tropische Zyklonen mit zentrumsnahen Winden die Hurrikanstärke erreicht haben. In den letzten Jahren haben Forschungsergebnisse unabhängiger wissenschaftlicher Studien gezeigt, dass die Bewegung, Struktur und Entwicklung derartiger konzentrierter Wirbel sehr stark von atmosphärischen Prozessen beeinflußt werden, die auf unterschiedlichen Raum- und Zeitskalen auftreten und miteinander wechselwirken. Es ist allgemein bekannt, dass diese wechselwirkenden Prozesse beispielsweise durch den Einfluss der Erdrotation, einer Umgebungsströmung und kleinskaliger konvektiver Systeme hervorgerufen werden können. Die Inhalte der vorliegenden Dissertation sollen zu einer weiteren Vertiefung dieser Erkenntnisse beitragen. Insbesondere wurden dafür reduzierte Modellgleichungen hergeleitet, die herangezogen werden können, um den Einfluss kleinskaliger Prozesse die die mesoskalige Struktur des Wirbels bestimmen, auf die Bewegung des Wirbels über synoptisch skalige Distanzen und umgekehrt zu beschreiben. Dabei wird der Einfluss einer vertikal gescherten Hintergrundströmung und der Einfluss diabatischer Effekte aufgrund von Feuchteumwandlungsprozessen mitberücksichtigt.

Es besteht ein grosses Interesse darin, jene Mechanismen besser zu verstehen, die die Bewegung und Struktur konzentrierter atmosphärischer Wirbel bestimmen. Für den operationellen Betrieb ist beispielsweise eine korrekte Vorhersage der Wirbeltrajektorie von enormer Wichtigkeit, um mögliche Katastrophen im Falle sich der Küste nähernder Hurrikane rechtzeitig abzuwenden. Die in dieser Arbeit hergeleiteten Modellgleichungen könnten eine Grundlage für die Entwicklung neuartiger Vorhersagemodelle darstellen, die einen Beitrag zur Verbesserung der Vorhersagen für die Wirbeltrajektorie leisten könnten.

Abstract

Three-dimensional concentrated atmospheric vortices with vertical extensions throughout the whole troposphere and diameters corresponding to the sub-synoptic gradient wind regime are studied in this work. Hurricane-like vortices are representative examples for this type of atmospheric flow phenomena. Research in recent years have shown that the complex interplay between atmospheric processes acting on different time and length scales strongly affect the motion, structure and development of hurricane-like vortices. It is well known that these interacting processes arise among others from the earth rotation, the environmental flow and small scale convective systems. It is against this background that this dissertation aims to derive reduced model equations that elucidate how scale interactions influence the motion and structure of concentrated atmospheric vortices. In particular reduced model equations are derived that describe how the mesoscale structure of the vortex itself affects the synoptic scale vortex motion and vice versa, while taking the influence of a vertically sheared environmental flow and diabatic effects due to moisture conversion processes into account. For the derivation of such reduced model equations multiple scales asymptotic analysis based on matched asymptotic expansions are used.

For various reasons a better understanding of the mechanisms determining the motion and structure of atmospheric vortices is of great interest. In operational use, for instance, an accurate forecast of the vortex trajectory is needed to avoid potential disasters caused by a landfalling storm systems. The reduced model equations derived in this work can be used to design hurricane track models that might contribute to improvements of hurricane track forecasts.

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Chapter 1

Introduction

Tropical cyclones as well as extratropical cyclones are representative examples for three-dimensional, concentrated atmospheric vortices considered in this work. Tropical cyclones form over the warm tropical oceans within a global band between 10 and 30 degrees, called Intertropical Convergence Zone (ITCZ). They can be viewed as giant vertical heat engines whose primary energy source is the release of the heat of condensation from water vapor condensing at high altitudes. Tropical cyclones are refered to by different names depending on its intensity measured in terms of its sustained surface winds and location. For example, cyclones with maximum winds of 17 m/s are called tropical depression, whereas cyclones with typical winds between 17 m/s $\leq u_{\max} < 33$ m/s are called tropical storm. Above 33 m/s they are refered to hurricane in the North Atlantic Ocean, the Northeast Pacific Ocean east of the dateline and the South Pacific Ocean east of 160E, or typhoon in the Northwest Pacific Ocean west of the dateline. Extratropical cyclones are storm systems that form in the westwind zone of the northern and southern Hemisphere. Here the cyclones derive their energy from horizontal temperature differences. In the southern Hemisphere, for instance, it is not uncommon that extratropical cyclones reach hurricane like intensities with $u_{\max} \geq 33$ m/s. That is why seafaring men called the southern higher latitudes the roaring 40's, furious 50's or screaming 60's. A famous example for a northern hemispheric severe extratropical storm system is the winter storm 'Lothar'[1] in which maximum winds up to 55 m/s were observed. A typical life-span of both tropical and extratropical cyclones ranges between 1-30 days.

In the last decades much effort has been made to study concentrated atmospheric vortices described above. Notably the motion, structure and de-

[1] In 1999 (24-26 December) the storm devastated many regions in northern France, southern Germany and northern Switzerland.

velopment of these vortices has been a favourite subject among atmospheric scientists. It is well known that a variety of parameters, such as the environmental flow conditions, the Coriolis force, diabatic effects and the atmospheric stratification strongly affect these vortex properties. For instance, it is observed that an unfavourable condition for a hurricane to develop or survive is given by a strong vertical shear in the environmental flow. One common explanation for this is that the dispersion of heat as a consequence of disruption of organized pattern of convection by strong winds is responsible for a weakening or limiting of the development of mature storms. Observational evidence for such interactions between hurricane-like vortices and its environmental flows can be found during a major El Niño[2] event, which is characterized by strong winds aloft over the tropical Atlantic. During this time one observes usually fewer Atlantic hurricanes than normal (Ahrens, 1999).

Studies on the motion and structure of atmospheric vortices is the central theme of this work. The overall goal is to derive a reduced set of model equations that can be used to gain deeper insights into the mechanisms that determine the motion and three-dimensional structure of atmospheric vortices. In particular, a vortex embedded in an environmental flow with vertical shear is considered in an attempt to answear the following two research questions:

(1) How do scale interactions between the flow on vortex scales and a large scale vertically sheared environmental flow determine the vortex motion and its structural features? and

(2) In this context, what is the role of diabatic effects due to moisture conversion processes?

Research in recent years have shown that multi-scale processes play a nontrivial role in tropical cyclone (TC) development, motion and structure. In a review about the current status of TC structure and intensity changes, Wang & Wu (2004) summarize the main results of current research focusing on multi-scale interactions as follows: "While the motion is mostly controlled by the steering flow associated with the large-scale environment, as well as the beta-gyres[3] and the upper-tropospheric negative potential vorticity anomalies, ..., the structure and intensity changes are affected at any time by large and complex arrays of physical processes that govern the inner core structure and the interaction between the storm and both the underlying ocean and its atmospheric environ-

[2] An extensive ocean warming that begins along the coast of Peru and Ecuador. Major El Niño events occur once every 2 to 7 years as a current of nutrient-poor tropical water moves southward along the west coast of South America (from Ahrens, 1999).

[3] Beta-gyres characterize a secondary dipole circulation in the vicinity of the cyclone center. A dipole is a pair of counter-rotating vortices that mutually advect each other. Therewith they provide a mechanism of self-propagation (beta-drift).

ment". With an increasing awareness that complex scale interactions play an important role in determining the intensity and structure of TCs, Wang & Wu (2004) proposed that future work should focus on improving the understanding of complex interactions between different scales. It is against this background that this work focuses on the above research questions.

There have been some attempts to derive reduced model equations for the motion of atmospheric vortices. Considering a one-layer, inviscid and homogenous atmosphere on an f-plane, Morikawa (1960) applied an ordering approximation procedure to the shallow water equations in order to derive approximate (reduced) systems of equations that are more manageable than the parent equations. As a key step to solve the lowest-order approximate system Morikawa (1960) applied the concept of geostrophic point vortices. In so doing he was able to derive equations for the motion of a single geostrophic vortex embedded in a continuous-flow field that represents a background flow. The equations state that the vortex motion is determined by the continous flow field evaluated at the vortex point, i.e. the vortex is steered by the background flow. The concept of geostrophic point vortices has also been used by Reznik (1992) in order to derive equations for the vortex motion. However, in order to account for the meridional variations of the Coriolis parameter (i.e. $\partial_y f = \beta \neq 0$), the governing equations of Reznik's studies are the shallow water equations on a β-plane. Although Reznik's approach to derive equations for the vortex motion differs in some points from the approach used by Morikawa (1960), Reznik (1992) managed to extend Morikawa's theory. In particular, Reznik found that the above mentioned continous-flow field determining the vortex motion may have some contributions that are due to the β-effect generated by the vortex flow itself. According to Morikawa (1960), however, "care and ingenuity must be used" in applying the results derived on the basis of the concept of geostrophic point vortices to actual flows. That is why the approximate representation of a circularly rotating vortex by a geostrophic vortex breaks down in the immediate vicinity of the point vortex, since the winds satisfying the geostrophic balance[4] only blow along straight paths parallel to the isobars.

The concept of geostrophic point vortices to derive equations for the vortex motion has also been used by Ling & Ting (1988). However, their approach differs from those of Morikawa and Reznik in that they apply matched asymptotic techniques for their derivations, where the concept of geostrophic point vortices is only used in order to derive the so called outer flow solutions. The equations for the vortex motion are derived by matching the outer solution with the so called inner solution describing the flow on smaller spatial scales within

[4]The geostrophic wind balance is a balance between the pressure gradient force and the Coriolis force.

the vicinity of the vortex core. One advantage of such an ansatz is that the singularities in the flow field induced by the point vortex are removed. Another advantage of Ting & Ling's matching ansatz becomes obvious if one looks at their matching results obtained from two-time scale inner solutions. Here they used a faster and slower time scale to describe the temporal evolution of the inner core and the outer flow solutions including the vortex motion. In so doing they found solutions describing a geostrophic vortex which induces an oscillatory motion in addition to moving with the background flow. The important point here is that the period, amplitude and the deviation from the mean trajectory depend on the smaller scale core structure itself and the initial conditions. Thus, the method of matched asymptotic expansions turned out to be a useful tool to derive equations for the vortex motion that takes into account the interaction between the smaller scale flow within the vortex core region and an environmental (outer) flow.

In this doctoral thesis, the approach of Ling & Ting (1988) is extended from a 2D vortex case described with the aid of the shallow water equations to a 3D case based on the three-dimensional Euler equation on the rotating earth that also include diabatic source terms. This is done within the framework of an unified approach to meteorological modelling recently developed by Klein (2004). Such an extension allows the use of the method of matched asymptotic expansions in order to derive equations for the motion and structure of three-dimensional atmospheric vortices under the influence of a vertically sheared environmental flow and diabatic effects, the latter representing the consequences of moisture conversion processes occuring in convective cloud systems. The results obtained by such an ansatz may contribute to a deeper understanding of vortex motion and structure mechanisms.

For various reasons a better understanding of the mechanisms determining the motion and structure of atmospheric vortices embedded in an environmental flow with vertical shear is of great interest. In operational use, for instance, an accurate forecast of the vortex trajectory is needed to avoid potential disasters caused by a landfalling storm system. Here, an accurate forecast not only includes a correct prediction of the long-term track but also information about the specific paths of the vortices. Long-range observations based on modern satellite techniques show that the actual path of a hurricane may vary considerably. In particular, tropical cyclones tend to meander about a mean path, where these meanders cover a wide range of scales and take on several forms (Holland & Lander, 1992). Some hurricanes, for example, take erratic paths and make odd turns that occasionally catch weather forecasters by surprise (see **Figure 1.1**; Ahrens, 1999). Holland & Lander (1992) suggest that many meanders occur from interactions with mesoscale vortices and small scale convective systems within the cyclone circulation. In this thesis attempts are made to understand such kind of interactions using the method of matched asymptotics for the derivation of reduced model equations for the vortex motion. In addition to these small scale impacts, processes acting on larger scales may also have an influence on the mesoscale vortex structure. As noted earlier, it is well known that tropical cyclones have little chance of surviving differential advection caused by the vertical shear of an environmental flow in which the vortex is embedded in. In particular, large vertical shear (i.e. above a treshold of approximately 12.5-15 m/s in the 850-200 mb layer) frequently inhibits their formation or results in a loss of the vortex coherence leading to a weakening or limiting of the development of mature vortices (Zehr, 1992; Frank & Ritchie, 2001). Under environmental flow conditions with weak to medium vertical shear, however, vortices seem to have a greater chance to withstand the differential advection (Reasor & Montgomery, 2001 and 2004 ; Frank & Ritchie, 1999). These observations naturally lead to the need to understand what mechanisms are responsible for such vortex behaviours. There is a hope that the reduced model equations for the vortex motion derived in the present work can be used to find out whether there are favourable structural features (and the mechanisms that cause them) that help the vortex to maintain its coherence in an environmental flow with vertical shear. This in turn might be helpful for weather forecasters to differentiate between situations where tropical storms have the potential to maintain and/or increase its intensity.

In addition to purposes described so far, reduced model equations for the vortex motion might be of interest to the scientific area of climate modelling. It has long been recognized that an accurate description of a meridional flux of mass, momentum and energy caused by travelling storm systems (eddies) in the midlatitudes is an essential prerequisite for a realistic generation of climate scenarios. In Earth System Model of Intermediate Complexity (EMIC), for example, the CLIMBER model developed at the Potsdam Institute for Climate Impact Research (PIK), the net-effects of large scale eddy transports on time scales relevant for the climate have to be parameterized because of the low temporal and spatial resolution of the model. For that reason the quality of such models hinges critically on the choice of a parameterization scheme for eddy transports of heat and momentum (Egger, 1992). Even though different parameterization schemes are available (e.g. the diffusive parameterization ansatz proposed by Green (1970)), an opportunity for the development of novel parameterization schemes is given by employing reduced model equations for the large scale vortex motion as derived in the dissertation. In particular, such a reduced model may be used to generate datasets describing realistic large scale

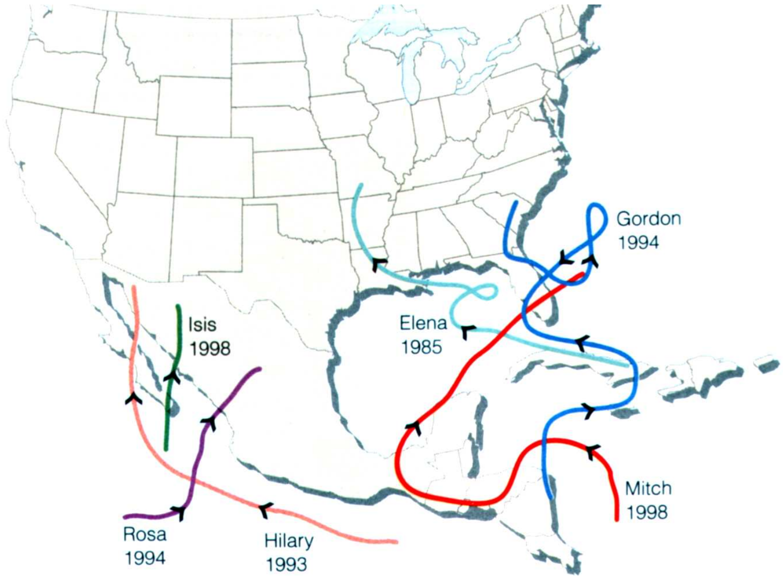

Figure 1.1: Some erratic paths taken by Hurricanes. As an example, Hurricane Elena, with peak winds of 90 knots, moved nothwestward into the Gulf of Mexico on August 29, 1985. It then veered eastward toward the west coast of Florida. After stalling offshore, it headed northwest. After weakening, it then moved onshore near Biloxi, Mississippi, on the morning of September 2. (Ahrens 1999, Fig. 16.9)

fluctuations which in turn strongly depend on a realistic description of travelling large scale eddies. Then, statistical-empirical techniques can be applied to such datasets in order to construct parameterizations schemes appropriate for a realistic description of the eddy transport of mass and momentum.

This thesis is organised as follows. A general introduction of the method used to derive reduced model equations is given in the second chapter. A particular application of this method to derive reduced model equations for the motion and structure of concentrated atmospheric vortices is presented in chapter 3. In chapter 4 and 5, some aspects of adiabatic and diabatic vortices, respectively, in vertically sheared environmental flow are given. For ease of reading, both chapter 4 and 5 are divided into two parts. The first part gives a literature review on the current understanding of adiabatic and diabatic vortices in vertically sheared environmental flows. The second part presents the derivation of reduced equations governing their motion and structure and a discussion of these results. Finally, a summary of the key results of this doctoral thesis is presented in chapter 6.

Chapter 2

Unified approach to meteorological modelling

The subject of the present work is to derive simplified model equations to explore the physical mechanism that determine the motion and structure of concentrated atmospheric vortices. In doing so an unified approach to meteorological modelling is employed, which has been recently introduced by Klein (2004). The fundamentals of this approach are based on perturbation methods which are frequently used in applied mathematics to solve problems arising from physical problems. The difficulty one wants to overcome in using perturbation methods is that in most instances the governing equations of physical problems are nonlinear, inhomogeneous and multidimensional such that the derivation of closed-form solutions proves to be difficult. Nonetheless, in order to handle such problems the underlying idea of perturbation methods is to exploit the situation that most physical problems involve a small parameter ε, which may appear either in the governing differential equations of the problem or in its boundary conditions. Then, based on the assumption that the solutions of the problem have an asymptotic expansion in terms of that small parameter, an asymptotic analysis formalism is used to construct reasonable accurate approximations to the solution of the problem. There are various reasons why perturbation methods can be regarded as an analysis tool that is at least as useful as numerical methods in order to obtain solutions of a problem. For instance, the most important advantage is that analytical approximate solutions of a problem are more suitable to get a better understanding for the physics of the problem, than to try an interpretation of a model output obtained by numerical simulations of the same problem. The unified approach to meteorological modelling by Klein (2004) uses perturbation methods and is a helpful framework to atmospheric scientists who are interested in deriving simplified model equations in a systematic way. Moreover, one outstanding feature of the approach is that with the use of multiple scale perturbation methods the opportunity is given to study scale interactions of different atmospheric flow phenomena acting on different length and time scales. The following sections shall provide an overview about the key steps of the method.

2.1 Governing equations

The unified approach to meteorological modelling builds up on the conservation equations for mass, momentum and energy. On the rotating earth they have the following form

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \vec{\nabla}_h \cdot (\rho\ \vec{v}_h) + \frac{\partial(\rho w)}{\partial z} \qquad\qquad\qquad\qquad\qquad\quad &= 0 \\
\frac{\partial(\rho \vec{v}_h)}{\partial t} + \vec{\nabla}_h \cdot (\rho \vec{v}_h \circ \vec{v}_h) + \frac{\partial(\rho \vec{v}_h w)}{\partial z} + \vec{\nabla}_h p + (\vec{\Omega} \times \rho \vec{v})_h &= D_{\rho \vec{v}_h} \\
\frac{\partial(\rho w)}{\partial t} + \vec{\nabla}_h \cdot (\rho \vec{v}_h w) + \frac{\partial(\rho w^2)}{\partial z} + \frac{\partial p}{\partial z} + (\vec{\Omega} \times \rho \vec{v})_\perp + \rho &= D_{\rho w} \\
\frac{\partial(\rho e)}{\partial t} + \vec{\nabla}_h \cdot ([\rho e + p]\ \vec{v}_h) + \frac{\partial([\rho e + p]\ w)}{\partial z} \qquad\qquad\quad &= D_{\rho e} + \rho Q
\end{aligned}
\tag{2.1}
$$

Here, the variables $\rho, \vec{v}_h, w, p, e$ are functions of (x, y, z, t) space and denote respectively the density, the horizontal and vertical velocity, the pressure and the total energy. $\vec{\Omega}$ is the vector of earth rotation, and $D_{\rho \vec{v}_h}, D_{\rho w}, D_{\rho e}$ represent effects of microscopical transport of momentum and energy. The diabatic source term ρQ summarizes heating effects due to chemical reactions, radiation and moisture related processes which include among others latent heat release due to condensation. In cartesian coordinates the horizontal velocity vector $\vec{v}_h$ is given by $\vec{v}_h = u\,\vec{i} + v\,\vec{j}$, whereas u and v are horizontal wind components in $\vec{i}$ and $\vec{j}$ direction, respectively. Hence, u and v have to be considered as parallel to a tangential or beta plane approximating the surface of the globe. The horizontal Nabla operator $\vec{\nabla}_h$ has the form $\vec{\nabla}_h = \left(\vec{i}\ \partial/\partial x + \vec{j}\ \partial/\partial y\right)$. Note that the total energy ρe is defined as a sum of internal energy, the kinetic energy and the potential energy, i.e. $\rho e = c_v T + (1/2)\rho \vec{v}^2 + \rho g z$, with the specific heat c_v, the temperature T and the acceleration of gravity g. Using the state equation

$$
p = \rho R T \quad , \tag{2.2}
$$

the gas constant $R = c_p - c_v$ for dry air, and the isentropic exponent $\gamma = c_p/c_v$, the equation for the total energy takes the form

$$\rho e = \frac{p}{\gamma - 1} + \frac{1}{2}\rho\vec{v}^2 + \rho g z \tag{2.3}$$

Hence, equation (2.3) closes the equations set (2.1), if appropriate parameterizations for $D_{\rho\vec{v}_h}, D_{\rho w}, D_{\rho e}$ and Q are given. Note that in this work the effects described by $D_{\rho\vec{v}_h}, D_{\rho w}, D_{\rho e}$ are neglected. Moreover, (2.3) is used in order to rewrite the energy equation $(2.1)_4$ into an evolution equation for the atmospheric pressure p. Then, the equation set (2.1) takes the following form

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \vec{\nabla}_h \cdot (\rho\, \vec{v}_h) + \frac{\partial(\rho w)}{\partial z} &= 0 \\
\frac{\partial(\rho\vec{v}_h)}{\partial t} + \vec{\nabla}_h \cdot (\rho\vec{v}_h \circ \vec{v}_h) + \frac{\partial(\rho\vec{v}_h w)}{\partial z} + \vec{\nabla}_h p + (\vec{\Omega} \times \rho\vec{v})_h &= 0 \\
\frac{\partial(\rho w)}{\partial t} + \vec{\nabla}_h \cdot (\rho\vec{v}_h w) + \frac{\partial(\rho w^2)}{\partial z} + \frac{\partial p}{\partial z} + (\vec{\Omega} \times \rho\vec{v})_\perp + \rho &= 0 \\
\frac{\partial p}{\partial t} + \vec{v}_h \cdot \vec{\nabla}_h p + w\frac{\partial p}{\partial z} + \gamma p\, (\vec{\nabla}_h \cdot \vec{v}_h + \frac{\partial w}{\partial z}) &= \tilde{Q}
\end{aligned} \tag{2.4}$$

Here, the diabatic source term is given by $\tilde{Q} = (\gamma - 1)\rho Q$.

2.2 Nondimensionalization

To nondimensionalize equations means traditionally to remove the units from a mathematical equation. This can be done by a suitable substitution of the independent and dependent variables, i.e. by use of

$$a' = \frac{a}{a_{\text{ref}}} \tag{2.5}$$

Here a denotes the quantity to nondimensionalize, a_{ref} a reference quantity and a' the corresponding dimensionless quantity. The technique of nondimensionalization is closely related to dimensional analysis. The former on uses units such as SI units for nondimensionalization. For a dimensional analysis, however, units that refer to quantities charactersitic for the system are used. In that case a' denotes a nondimensional quantity scaled relative to a_{ref}.

The unified approach to meteorological modelling uses either SI units nor reference quantities characteristic for the system in the sense, that they are related to characteristic length and time scales of a particular atmospheric flow phenomena under consideration. However, it can be said that the unified approach uses reference quantities for the nondimensionalization of the governing equations (2.4) that are

a) intrinisc to the rotating earth, i.e. the earth's rotation frequency Ω_{ref}, the acceleration of gravity g and the radius of earth a with its characteristic values

$$\Omega_{\text{ref}} = 10^{-4}\ 1/\text{s}, \quad g = 10\ \text{m/s}^2, \quad a = 6 \cdot 10^6\ \text{m} \tag{2.6}$$

b) and intrinsic to a wide range of atmospheric flow conditions, given by the thermodynamic pressure p_{ref}, the air density ρ_{ref} and the air flow velocity u_{ref} with its characteristic values

$$p_{\text{ref}} = 10^5\ \text{kg/(m s}^2), \quad \rho_{\text{ref}} = 1\ \text{kg / m}^3, \quad u_{\text{ref}} = 10\ \text{m/s} \tag{2.7}$$

A particular combination of these reference quantities allows the definition of a reference length h_{sc} denoting the pressure scale height (vertical distance with significant pressure drop) and reference time t_{ref} given by

$$h_{\text{sc}} = \frac{p_{\text{ref}}}{g\ \rho_{\text{ref}}} = 10^4\ m \qquad \text{and} \qquad t_{\text{ref}} = \frac{h_{\text{sc}}}{u_{\text{ref}}} = 10^3\ s \tag{2.8}$$

Note that employing these universally valid reference quantities independent on the length and time scales of any particular atmospheric flow phenomena made it possible for Klein, to construct a generalized formal approach to atmosphere modelling. Based on the general substitution (2.5) and the given reference quantities (2.6) - (2.7) nondimensionalization of the equations (2.4) yields

$$\begin{array}{rcl}
\dfrac{\partial \rho'}{\partial t'} + \vec{\nabla}'_h \cdot (\rho'\ \vec{v}'_h) + \dfrac{\partial(\rho' w')}{\partial z'} & = & 0 \\
\dfrac{\partial(\rho'\vec{v}'_h)}{\partial t'} + \vec{\nabla}'_h \cdot (\rho'\vec{v}'_h \circ \vec{v}'_h) + \dfrac{\partial(\rho'\vec{v}'_h w')}{\partial z'} + \dfrac{\vec{\nabla}'_h p'}{\text{M}^2} + \dfrac{(\vec{\Omega}' \times \rho'\vec{v}')_h}{\text{Ro}_{h_{\text{sc}}}} & = & 0 \\
\dfrac{\partial(\rho' w')}{\partial t'} + \vec{\nabla}'_h \cdot (\rho'\vec{v}'_h w') + \dfrac{\partial(\rho' w'^2)}{\partial z'} + \dfrac{1}{\text{M}^2}\dfrac{\partial p'}{\partial z'} + \dfrac{(\vec{\Omega}' \times \rho'\vec{v}')_\perp}{\text{Ro}_{h_{\text{sc}}}} + \dfrac{\rho'}{\text{Fr}^2} & = & 0 \\
\dfrac{\partial p'}{\partial t'} + \vec{v}'_h \cdot \vec{\nabla}'_h p' + w'\dfrac{\partial p'}{\partial z'} + \gamma p'\ (\vec{\nabla}'_h \cdot \vec{v}'_h + \dfrac{\partial w'}{\partial z'}) & = & \tilde{Q}'
\end{array} \tag{2.9}$$

with $\tilde{Q}' = (\gamma - 1)\rho' Q'$. Note that $\tilde{Q}$ denotes a heating rate with the units K/s. Hence, the relation $T_{\text{ref}} = p_{\text{ref}}/(R\rho_{\text{ref}})$ (state equation: $p = \rho RT$) can be used to make $\tilde{Q}$ dimensionless. In terms of the flow numbers Mach (M), Rossby ($\text{Ro}_{h_{\text{sc}}}$) and Froude (Fr), a number of more or less small parameters appear in (2.9). These dimensionless parameters are defined by

$$
\mathrm{M} = \frac{v_{\mathrm{ref}}}{\sqrt{p_{\mathrm{ref}}/\rho_{\mathrm{ref}}}} \sim \frac{1}{30} \ , \qquad \mathrm{Fr} = \frac{v_{\mathrm{ref}}}{\sqrt{g\ h_{\mathrm{sc}}}} \sim \frac{1}{30}
$$

$$
\mathrm{Ro}_{h_{\mathrm{sc}}} = \frac{v_{\mathrm{ref}}}{2\ \Omega_{\mathrm{ref}}\ h_{\mathrm{sc}}} \sim 5 \tag{2.10}
$$

The magnitudes of the reference values M, $\mathrm{Ro}_{h_{\mathrm{sc}}}$ and Fr are determined based on (2.6) - (2.7).

Using perturbation methods, differential equations with more than one small parameters have to be treated more carefully than equations with only one small parameter. But before going on to explain how the unified approach to meteorological modelling handles this, a discussion on the fundamental differences between scale analysis and asymptotic methods is given first. Since the application of the former is common in theoretical meteorology to derive simplified model equations it seems instructive to point out briefly where the advantage of the latter lies. This will give a better understanding why asymptotic techniques serve as the technical basis of the unified approach to meteorological modelling.

2.2.1 Scale analysis vs. asymptotics

The general case is considered where the nondimensional governing equations of a system include only one small parameter given by ε with $\varepsilon << 1$. Proponents of scale analysis techniques argue that terms multiplied with ε can be neglected with respect to the others. Then, a favourable condition would be if neglecting such terms results in simplified equations in the sense that the derivation of solutions is more tractable compared to the original problem. However, such a solution may be considered only as a first approximation and it is unclear how to determine a correction to the approximate solution (Holmes, 1995). In other words there is no way to find out to what extent the inclusion of the omitted terms would change the approximation that has been made in the absence of them. The asymptotic methods overcome these difficulties by assuming that the solution f of the problem can be expanded in terms of a small parameter ε, e.g.

$$
f \sim f^{(0)} + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \dots \tag{2.11}
$$

Then a formal analysis procedure is to substitute such an asymptotic expansion ansatz into the governing equations of the problem under consideration and evaluate terms of corresponding powers of ε . In general this procedure allows step by step a derivation of solutions for $f^{(0)}$ and $f^{(i)}$ (with $i = 1, 2, ..., n$), whereas the leading order term $f^{(0)}$ is similar to the first approximation one obtains using scale analysis arguments. Eventually, the derivation of higher order corrections $f^{(i)}$ makes it possible to estimate how well the leading order term $f^{(0)}$ approximates the real solution.

Eqn. (2.11) just denotes one example for an asymptotic expansion. A general definition for an asymptotic expansion is given through

$$f = \sum_{k=1}^{m} f^{(k)} \phi_k(\varepsilon) + o(\phi_m) \qquad \text{for} \qquad m = 1, ..., n \quad \text{as} \quad \varepsilon \to \varepsilon_0 \tag{2.12}$$

The $\phi_k(\varepsilon)$ are called basis functions. They form an asymptotic sequence $\phi_1, \phi_2, \phi_3, ...$ as $\varepsilon \to \varepsilon_0$ if and only if $\phi_n = o(\phi_m)$ for all m and n that satisfy $m < n$ (Holmes, 1995), where o denotes the Landau[1] symbol 'little o'. Note that in the given example (2.11), the asymptotic sequence reads $\phi_1 = 1,\ \phi_2 = \varepsilon,\ \phi_3 = \varepsilon^2, ...$ It is worth to point out, however, that depending on the problem to be studied other forms of the basis functions may arise.

2.3 Small parameter ε and distinguished limit

The identification of a dimensionless small parameter ε is a key step in Klein's development of an unified approach to meteorological modelling. Dimensionless numbers (M, $\mathrm{Ro}_{h_{sc}}$, Fr) characterizing atmospheric flow conditions are discussed in **Section 2.2**. However, a further dimensionless parameter can be derived by a combination of the reference quantities (2.6) related to the rotating earth. In doing so one obtains a dimensionless number denoting the ratio of the centripetal acceleration on earth's surface to the acceleration of gravity at earth's surface, i.e.

$$\kappa = \frac{a\Omega^2}{g} \sim \frac{1}{512} \cdots \frac{1}{216} \tag{2.13}$$

Hence, the dimensionless numbers (2.10) together with (2.13) characterize general atmospheric flow conditions on a rotating earth.

If physical problems with more than one small parameter are considered, Klein (2004) points out that the asymptotic equations depend strongly on the path on which the parameters are to approach their respective limiting values. For instance, that means that an asymptotic expansion using M as expansion parameter for a formal asymptotic analysis would lead to different results than an expansion with κ or even both M and κ as expansion parameters. Eventually, this problem leads to the idea of a distinguished limit, such that the parameters (2.10) and (2.13) are related to each other in the following way

$$\varepsilon \sim \kappa^{\frac{1}{3}} \sim \frac{1}{\mathrm{Ro}_{h_{sc}}} \sim \sqrt{\mathrm{M}} \sim \sqrt{\mathrm{Fr}} \tag{2.14}$$

[1] Landau symbols are commonly used asymptotic notation for comparing functions. Mathematically the Landau symbol o is defined by $f(\varepsilon) = o(g(\varepsilon)) : \quad \lim_{\varepsilon \to \varepsilon_0} f(\varepsilon)/g(\varepsilon) = 0.$

where $\varepsilon \sim 1/8...1/6$. Other distinguished limits are conceivable. However, the successful employment of (2.14) that allows to rederive a number of well-known classical models in theoretical theory (see **Section 2.4.1**), gives evidence that Klein (2004) was guided by the right intuition by choosing the distinguished limit (2.14).

Substitution of the distinguished limit (2.14) into the nondimensional governing equations (2.9) yields

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \vec{\nabla}_h \cdot (\rho\, \vec{v}_h) + \frac{\partial (\rho w)}{\partial z} &= 0 \\
\frac{\partial (\rho \vec{v}_h)}{\partial t} + \vec{\nabla}_h \cdot (\rho \vec{v}_h \circ \vec{v}_h) + \frac{\partial (\rho \vec{v}_h w)}{\partial z} + \frac{\vec{\nabla}_h p}{\varepsilon^4} + \varepsilon (\vec{\Omega} \times \rho \vec{v})_h &= 0 \\
\frac{\partial (\rho w)}{\partial t} + \vec{\nabla}_h \cdot (\rho \vec{v}_h w) + \frac{\partial (\rho w^2)}{\partial z} + \frac{1}{\varepsilon^4} \frac{\partial p}{\partial z} + \varepsilon \vec{\Omega} \times \rho \vec{v})_\perp + \frac{\rho}{\varepsilon^4} &= 0 \\
\frac{\partial p}{\partial t} + \vec{v}_h \cdot \vec{\nabla}_h p + w \frac{\partial p}{\partial z} + \gamma p\, (\vec{\nabla}_h \cdot \vec{v}_h + \frac{\partial w}{\partial z}) &= \tilde{Q}
\end{aligned} \tag{2.15}$$

with $\tilde{Q} = (\gamma - 1)\rho Q$. Note that the primes denoting dimensionless variables have been dropped.

With the aid of the mass continuity $(2.15)_1$, the state equation (2.2) in its nondimensional form, and the nondimensionalized definition of a potential temperature[2], i.e.

$$p = \rho\, T \qquad \text{and} \qquad \Theta = T p^{-\left(\frac{\gamma-1}{\gamma}\right)} \quad , \tag{2.16}$$

the pressure equation $(2.15)_4$ can be rewritten into an equation for the potential temperature equation

$$\left(\frac{\partial}{\partial t} + \vec{v}_h \cdot \vec{\nabla}_h + w \frac{\partial}{\partial z} \right) \Theta = \frac{\gamma - 1}{\gamma} \frac{\rho\, \Theta}{p} Q \tag{2.17}$$

As pointed out by Klein (2004), the asymptotic treatment of stratified fluids can be simplified by introducing the "Newtonian limit" for the isentropic exponent. Such a limit is given by

$$\frac{\gamma - 1}{\gamma} = \varepsilon\, \Gamma^{\star\star} \qquad \text{with} \qquad \Gamma^{\star\star} = \mathcal{O}(1) \qquad \text{as} \qquad \varepsilon \to \infty \tag{2.18}$$

Thus, together with that limit, a replacement of the pressure equation $(2.15)_4$

[2] The temperature a volume of dry air at pressure P and temperature T would have if compressed adiabatically to a reference level P_{ref}, i.e. $\Theta = T(P_{ref}/P)^{(\gamma-1)/\gamma}$, where γ is the heat capacity ratio of the gas.

through the potential temperature equation (2.17) yields the following set of governing equations that can be used for an asymptotic analysis of different atmospheric flow phenomena

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \vec{\nabla}_h \cdot (\rho\, \vec{v}_h) + \frac{\partial(\rho w)}{\partial z} &= 0 \\
\frac{\partial \vec{v}_h}{\partial t} + \vec{v}_h \cdot \vec{\nabla}_h \vec{v}_h + w\frac{\partial \vec{v}_h}{\partial z} + \frac{1}{\varepsilon^4}\frac{\vec{\nabla}_h p}{\rho} + \varepsilon(\vec{\Omega} \times \vec{v})_h &= 0 \\
\frac{\partial w}{\partial t} + \vec{v}_h \cdot \vec{\nabla}_h w + w\frac{\partial w}{\partial z} + \frac{1}{\varepsilon^4}\frac{1}{\rho}\frac{\partial p}{\partial z} + \varepsilon(\vec{\Omega} \times \vec{v})_\perp + \frac{1}{\varepsilon^4} &= 0 \\
\left(\frac{\partial}{\partial t} + \vec{v}_h \cdot \vec{\nabla}_h + w\frac{\partial}{\partial z}\right)\Theta &= S
\end{aligned}
\tag{2.19}
$$

where the diabatic source term S is given by

$$
S = \varepsilon\, \Gamma^{\star\star} \frac{\rho\, \Theta}{p} Q \tag{2.20}
$$

Note that with the aid of $\vec{\nabla}_h \cdot (\rho\vec{v}_h \circ \vec{v}_h) = \rho\vec{v}_h \cdot (\vec{\nabla}_h \vec{v}_h) + \vec{v}_h \vec{\nabla}_h \cdot (\rho\vec{v}_h)$ and the mass continuity $(2.15)_1$, the momentum equations $(2.15)_{3,4}$ have been rewritten into $(2.19)_{3,4}$. Moreover, on account of the Newtonian limit (2.18) the state equation $(2.16)_2$ takes in terms of the potential temperature the following form

$$
\rho\, \Theta = p^{1-\Gamma^{\star\star}\varepsilon} \tag{2.21}
$$

2.4 Multiple-scales techniques

It is well known, that the dynamical behaviour of a particular atmospheric flow phenomena is as a result of processes acting on different length and time scales. For such multiple-scale problems, the theory of perturbation methods provides multiple-scales techniques for finding an asymptotic approximations to the solutions of the problem (Holmes, 1995). The two frequently used techniques are known as (1) multiple-scale expansions and (2) matched asymptotic expansions. The use of these methods is a further key step of the unified approach to meteorological modelling.

2.4.1 General multiple scale expansions

If multiple-scale expansions are used, it is generally assumed that the solutions of the multi-dimensional compressible flow equations can be expressed as

$$\mathcal{U}(t,\vec{x},z;\varepsilon) = \sum_{i\in N} \phi_i(\varepsilon)\, \mathcal{U}^{(i)}\left(\frac{t}{\varepsilon}, \varepsilon t, \varepsilon^2 t, ..., \frac{\vec{x}}{\varepsilon}, \varepsilon\vec{x}, \varepsilon^2\vec{x}, ..., \frac{z}{\varepsilon}, z\right) \tag{2.22}$$

where $\mathcal{U}$ denotes a shortcut for a solution component of the governing equations (2.19), i.e. $\mathcal{U} \in \{\vec{v}_h, w, p, \rho, \Theta\}$. The functions ϕ_i form an asymptotic sequence (see **Section 2.2.1**).

With such a general expansion ansatz the flow variables are considered to be functions of a number of independent time and space coordinates which are differently scaled in terms of the expansion parameter ε. Within the framework of the unified approach the above expansion can be used in different ways. Depending on the particular flow phenomena under consideration, ansatz (2.22) can be specialized in the sense that only relevant length and time scales are included. This includes all the scales that are necessary to study certain interactions between phenomena acting on separate scales. On the other hand, if the study of interactions between different scales are not focus of interest, the expansion (2.22) can be reduced to a single scale expansion with only one time, one horizontal, and one vertical coordinate.

It is important to point out that with the application of (2.22) to construct approximate solutions for a particular atmospheric flow phenomena (which is characterized by particular length and time scales) appropriate choice of scaled coordinates are used that were not used in the nondimensionalization of the three-dimensional Euler equations (see **Section 2.2**). For the sake of clarity the following example shall be given. In studying synoptic scale phenomena typical horizontal length scales are of order ~ 1000 km. However, for nondimensionalization of the Euler equations both the horizontal and vertical coordinates have been made dimensionless using the pressure scale height $h_{\mathrm{sc}} = 10^4$ m, i.e.

$$(x', y') = \left(\frac{x}{h_{\mathrm{sc}}}, \frac{y}{h_{\mathrm{sc}}}\right) \quad \text{and} \quad z' = \frac{z}{h_{\mathrm{sc}}}$$

To study phenomena acting on different scales than h_{sc} a so called stretching transformation has to be introduced that re-scales the dimensionless horizontal coordinates from (x', y') to the claimed scale $(\bar{x}, \bar{y})$. Such a transformation reads in general

$$(\bar{x}, \bar{y}) = \left(\frac{x'}{\varepsilon^\alpha}, \frac{y'}{\varepsilon^\alpha}\right) \tag{2.23}$$

Hence, with $\varepsilon = 1/8...1/6$ and the particular choice of $\alpha = -2$ the re-scaled new

Coordinate scalings	Simplified model obtained
$\mathcal{U}^{(i)}(t, \mathbf{x}, z)$	Anelastic & pseudo-incompressible models
$\mathcal{U}^{(i)}(t, \varepsilon\mathbf{x}, z)$	Linear large scale internal gravity waves
$\mathcal{U}^{(i)}(\frac{t}{\varepsilon}, \mathbf{x}, \frac{z}{\varepsilon})$	Linear small scale internal gravity waves
$\mathcal{U}^{(i)}(\varepsilon^2 t, \varepsilon^2\mathbf{x}, z)$	Mid-latitude Quasi-Geostrophic model
$\mathcal{U}^{(i)}(\varepsilon^2 t, \varepsilon^2\mathbf{x}, z)$	Equatorial Weak Temperature Gradient models
$\mathcal{U}^{(i)}(\varepsilon^2 t, \varepsilon^{-1}\xi(\varepsilon^2\mathbf{x}), z)$	Semi-geostrophic model
$\mathcal{U}^{(i)}(\varepsilon^{\frac{5}{2}} t, \varepsilon^{\frac{7}{2}} x, \varepsilon^{\frac{5}{2}} y, z)$	Equatorial Kelvin, Yanai & Rossby Waves

Table 2.1: Overview about coordinate scalings and associated classical models. (adopted from Klein (2004))

coordinates $(\bar{x}, \bar{y}) = (\varepsilon^2 x', \varepsilon^2 y')$ resolve synoptic length scales.

Note, the expansion ansatz (2.22) not only allows for specializations with respect to particular length and time scales. Furthermore the amplitudes of the variables may be varied by starting the expansions for different i's in the asymptotic sequence $\phi_i(\varepsilon)$.

It has been shown by Klein (2004) that certain specializations of the general expansion ansatz (2.22) yield upon substitution into the governing equations (2.15), and a subsequent formal asymptotic analysis, a number of well-known reduced models in theoretical meteorology. A list of the models and the accompanied expansions is given in **Table 2.1**. It is the success of these re-derivations that motivates use of the unified approach to meteorological modelling for investigations of derivation of simplified models for arbitrary atmospheric phenomena.

2.4.2 Matched asymptotic expansions

A second technique that makes it possible to study multiple-scale problems is traditionally known as matched asymptotic expansions.

The method of matched asymptotic expansions is frequently used if one is concerned with boundary layer problems, e.g. the fluid flow past a solid body. The most prominent example for such a problem is the air flow past an aeroplane wing. Here it is known that due to the effects of viscosity the physical situation in the thin layer of fluid in direct contact with the airplane (inner layer) is different from the physical situation outside this layer where such viscous effects can be neglected (outer layer). As a consequence, the difference of the physical situation in the different layers must be reflected in the mathematical behaviour of the solutions of equations modelling the flow in the inner and outer layer, respectively. This can be shown by a systematic analysis of the boundary layer problem using two asymptotic expansions, whereas

a) the inner expansion approximates flow solutions which are valid in the inner layer

b) and the outer expansion approximates flow solutions which are valid in the outer layer

With a) and b), however, the description of the solution describing the air flow past an air wing consist of two pieces. Since the main interest in studying the flow past an air wing is to get solutions of the whole flow problem, so called 'matching conditions' are derived which satisfy the inner and outer solutions within an overlapping domain. This is the main idea underlying the method of matched asymptotic expansions and which is eventually necessary to combine the inner and outer solutions to form a composite expansion.

Mathematical details on how the method of matched asymptotic expansions can be used within the framework of the unified approach to meteorological modelling are explained next. Let's assume that a two-scale problem with respect to the horizontal has to be solved, whereas the stretching transformation (see (2.23)) for the horizontal coordinates resolving the smaller scale reads $(\xi_1, \xi_2) = (\varepsilon^\alpha x, \varepsilon^\alpha y)$ and the stretching transformation for the horizontal coordinates resolving the larger scale reads $(\eta_1, \eta_2) = (\varepsilon^\beta x, \varepsilon^\beta y)$. Note, the coordinates x and y are dimensionless and that $\alpha < \beta$. Then, the construction of approximate solutions proceeds in four steps.

Step 1-2: The first two steps are related to the construction of inner ($\mathcal{U}^{(i)}$) and outer ($\check{\mathcal{U}}^{(i)}$) solutions valid in the different regions of the problem to be studied. Here the inner and outer solutions are derived by means of single scale expansions, which are defined through

$$\mathcal{U}(t, \vec{x}, z; \varepsilon) = \sum_{i \in N} \phi_i(\varepsilon)\, \mathcal{U}^{(i)}\left(t, \varepsilon^\alpha \vec{x}, z\right) = \sum_{i \in N} \phi_i(\varepsilon)\, \mathcal{U}^{(i)}\left(t, \vec{\xi}, z\right) \tag{2.24}$$

and

$$\check{\mathcal{U}}(t, \vec{x}, z; \varepsilon) = \sum_{i \in N} \check{\phi}_i(\varepsilon)\, \check{\mathcal{U}}^{(i)}\left(t, \varepsilon^\beta \vec{x}, z\right) = \sum_{i \in N} \check{\phi}_i(\varepsilon)\, \check{\mathcal{U}}^{(i)}\left(t, \vec{\eta}, z\right) \tag{2.25}$$

Step 3: The actual matching of the inner and outer expansions happens in a third step. Here it is important to have in mind that the inner (see (2.24)) and outer expansions (see (2.25)) are valid in different regions, but nevertheless have to be considered as approximations for the same function. That is why one should expect that inner and outer expansions give in their transition region the same results. Thus, the idea is to introduce an intermediate horizontal coordinate $(\chi_1, \chi_2) = (\varepsilon^\lambda x, \varepsilon^\lambda y)$ with $\alpha < \lambda < \beta$. In particular, $\vec{\chi}$ describes an 'overlap' lengthscale on which both inner and outer expansions should be valid. Note that for a fixed $\vec{\chi}$ and $\varepsilon \to 0$ one obtains that $\vec{\eta} \to 0$ and $\vec{\xi} \to \infty$. Then, changing variables (i) in the inner expansion (2.24) from $\vec{\xi}$ to $\vec{\chi}$ and (ii) in the outer expansion (2.25) from $\vec{\eta}$ to $\vec{\chi}$, a matching criterion between the inner and outer expansions that has to be satisfied, reads

$$\sum_{i \in N} \phi_i(\varepsilon)\, \mathcal{U}^{(i)}\left(t, \varepsilon^{\alpha-\lambda}\vec{x}, z\right) = \sum_{i \in N} \check{\phi}_i(\varepsilon)\, \check{\mathcal{U}}^{(i)}\left(t, \varepsilon^{\beta-\lambda}\vec{x}, z\right) \tag{2.26}$$

The matching condition (2.26) states that the solution $\mathcal{U}$ as one moves out of the smaller scale region (i.e. $\vec{\xi} \to \infty$) has to be equal to the solution $\check{\mathcal{U}}$ as one moves into the smaller region (i.e. $\vec{\eta} \to 0$). Note, the matching procedure can also be regarded as a technique to find outer boundary conditions for the inner solutions $\mathcal{U}^{(i)}$, and vice versa. In doing so the matching procedure makes it possible to elucidate the role of scale-interactions between the flow in the inner and outer layer.

Step 4: The fourth step combines the inner and outer solutions to find a composite expansion. This is done by adding the expansions and then subtracting the part that is common to both (Holmes, 1995).

Summing up, one may say that the technique of matched asymptotic expansions differs from multiple-scale expansions in that it starts with the construction of solutions in different regions that are then patched together to form a composite expansion (Holmes, 1995).

Chapter 3

Asymptotic formulation of the vortex problem

The purpose of this chapter is to provide an overview from a technical perspective about how the unified approach to meteorological modelling is used to derive simplified model equations that are suitable to study the motion and three-dimensional structure of concentrated atmospheric vortices. Regarding the mechanisms influencing these vortex features, the primary goal of the present work is to employ the unified approach in such a way so that the role of scale interactions between the mesoscale flow of the vortex itself and an large scale vertically sheared environmental flow in which the vortex is embedded in, can be explored. **Figure 3.1** and **3.2** is a schematic diagram of this situation. Additional interest is on the modifying effect brought about by moisture effects on these scale interactions compared to scale interactions in a pure dry atmosphere. It is expected that solutions of such a multiple scale problem may help to understand how an environmental forcing and diabatic processes affect the mesoscale vortex structure and the large scale vortex motion. Employing asymptotic methods, the work on this issue can be regarded as an extension of the work of Callegari & Ting (1978), who studied the motion and two-dimensional, synoptic-scale core structure of a geostrophic vortex in a dry atmosphere.

Two different techniques have been proposed in **Section 2.4** to study complex interactions of processes acting on different length and time scales. In analogy to the work of Callageri & Ting (1987) the technique of matched asymptotic expansions is used in the present work. Thus, the derivation of approximate solutions for the motion and structure of concentrated vortices that account for scale interactions between the vortex scale flow and an large scale environmental flow, is based on the construction of vortex solutions valid on vortex scales

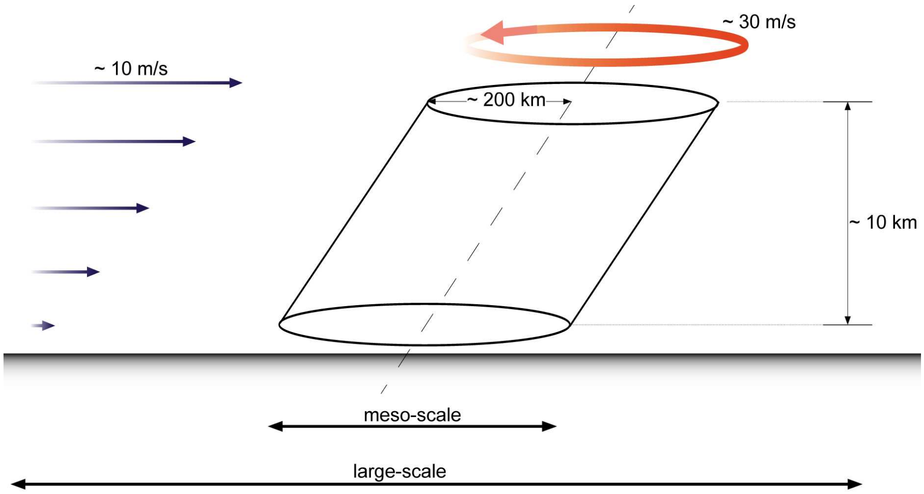

Figure 3.1: Schematic diagram showing mesoscale vortex embedded in an large scale vertically sheared environmental flow; for further explanations see the text

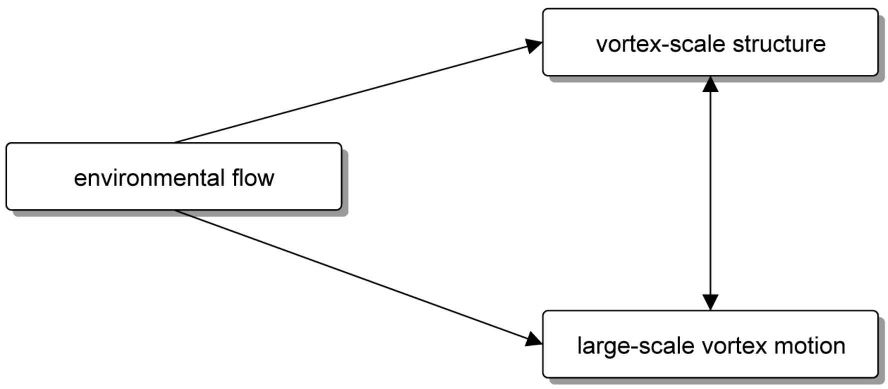

Figure 3.2: Schematic diagram showing scale interactions influencing the motion and structure of atmospheric vortices

and large scales, respectively, which are then matched together. An expansion ansatz that accounts for typical scales of concentrated vortices such as tropical-cyclone like vortices, is discussed in **Section 3.1**. Moreover, reduced model equations related to such an expansion ansatz are derived, whose solutions are refered to as *inner* solutions of the multi-scale vortex problem. An expansion ansatz and the corresponding reduced model equations suitable to find vortex solutions w.r.t to synoptic-scales and which are refered to as *outer* solutions, are derived in **Section 3.2**. Matching conditions between *inner* and *outer* solutions are discussed in **Section 3.3.1**. To account additionally for diabatic effects, an appropriate expansion for a diabatic source term is given in **Section 3.4**.

As noted in the introductory paragraph, in this chapter theoretical basics such as the choice of asymptotic expansions, asymptotic equations, matching conditions etc. are discussed. This are necessary preliminaries on which a derivation of solutions for the vortex motion and structure of adiabatic vortices in **Chapter 4** and diabatic vortices in **Chapter 5** are built. Hence, for the reader who is primarily interested how the reduced model equations for adiabatic and diabatic vortices look like, its discussion (interpretation) and further manipulations in order to derive equations for the vortex motion, it is possible to start reading with **Chapter 4** and to use **Chapter 3** only as a reference for theoretical details.

3.1 Meso-scale Regime

The atmospheric vortices considered in the present work are schematically shown in **Figure 3.1**. They are approximately 400 km in diameter and extend throughout the whole troposhere. Furthermore it is assumed that the winds within the vortex region are approximately 30 m/s. These values are characteristic for hurricanes. They measure on average 550 km in diameter and if they reach sustained winds of about 33 m/s they belong to the hurricane category I of the Saffir-Simpson scale. Due to the horizontal scales the vortices can be regarded as mesoscale flow phenomena.

3.1.1 Stretching transformations

An asymptotic expansion ansatz that accounts for the vortex scales defined above, is derived next.

Due to the circular geometry of the vortex, an asymptotic analysis of the flow field using cylindrical coordinates is convenient. This requires a transformation of the flow equations discussed in **Section 2.3**, from a frame of reference fixed at the earth into a frame of reference whose origin is attached to the centre of

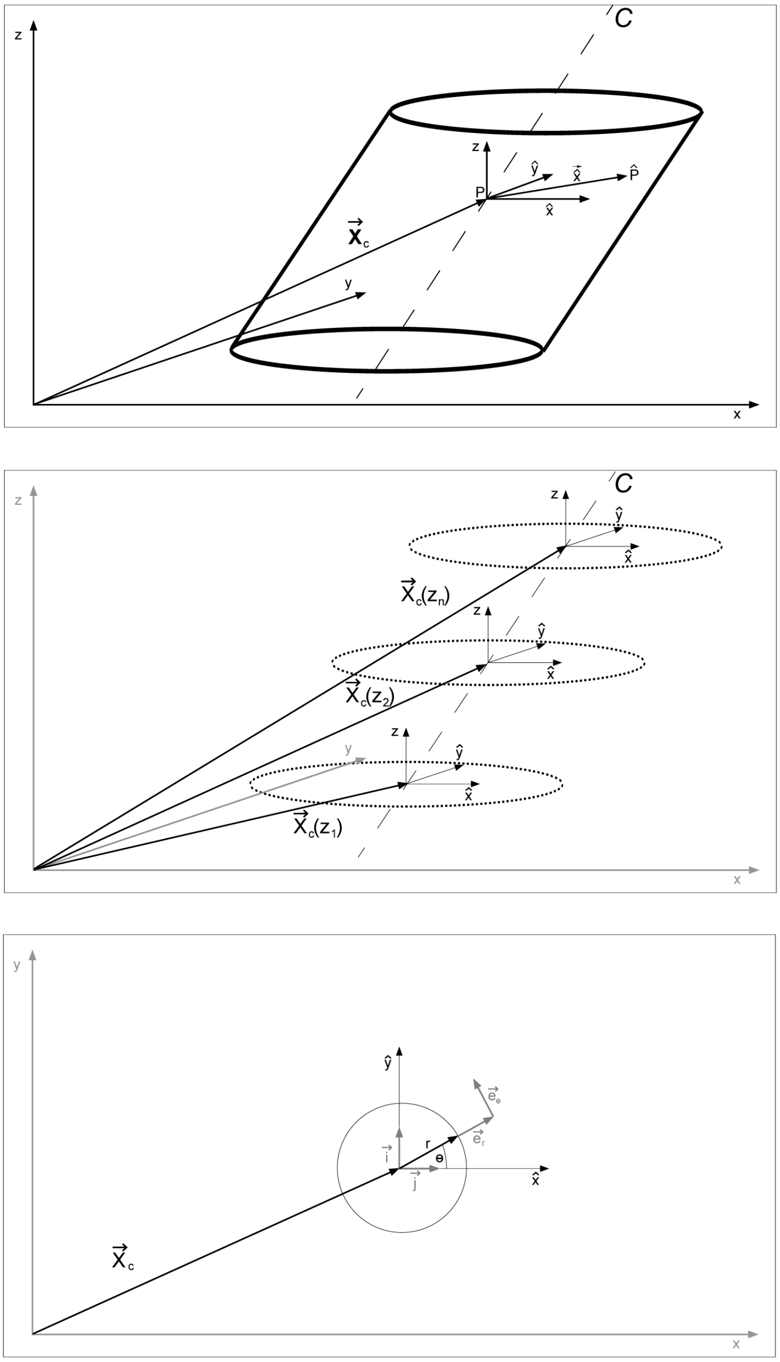

Figure 3.3: top: Schematic diagram showing frame of references, middle: Parameterization of vortex-centreline, bottom: coordinate transformation in 2D

the moving vortex (see **Figure 3.3**). For this a vector $\vec{\mathrm{X}}_C = (\mathrm{X}_C, \mathrm{Y}_C) = \vec{\mathrm{X}}_C(z,t)$ is defined, that denotes a position vector of the point P on the vortex-centreline C in the fixed frame of reference. Then, the transformation equation into the moving frame reads

$$\vec{x} = \vec{\mathrm{X}}_C(z,t) + \vec{\hat{\mathrm{x}}}\ , \qquad z = z \tag{3.1}$$

whereas the horizontal vector $\vec{\hat{\mathrm{x}}} = (\hat{\mathrm{x}}, \hat{\mathrm{y}})$ denotes a position vector of the point $\hat{P}$ in the moving frame of reference. Note, in the subsequent analysis the temporal evolution of the vortex centreline shall be used to describe the vortex motion. Thus, considering the vector $\vec{X}_C(z,t)$ not only as a function on the temporal coordinate t but also as a function on the vertical coordinate z, allows to account for a differential motion of the three-dimensional vortex with respect to the vertical, resulting in a vortex tilt. As illustrated in **Figure 3.3**, due to the vertical dependency of $\vec{X}_C(z,t)$ on the vertical coordinate z, the three-dimensional vortex may be now regarded as a stack of two-dimensional vortices in the vertical.

Using the relations $\vec{x}' = \vec{x}/h_{\mathrm{sc}}$, $\vec{\mathrm{X}}_C' = \vec{\mathrm{X}}_C/h_{\mathrm{sc}}$, $\vec{\hat{\mathrm{x}}}' = \vec{\hat{\mathrm{x}}}/h_{\mathrm{sc}}$ and $z' = z/h_{\mathrm{sc}}$ with h_{sc} defined through $(2.8)_1$, the dimensionless form of (3.1) reads

$$\vec{x}' = \vec{\mathrm{X}}_C' + \vec{\hat{\mathrm{x}}}'\ , \qquad z' = z' \tag{3.2}$$

As noted earlier, solutions for the vortex motion expressed by the motion of the vortex centreline on synoptic-scales, i.e. $\mathrm{L_S} = 1000$ km $\sim \varepsilon^{-2}\ h_{\mathrm{sc}}$, and the vortex structure on mesoscale, i.e. $\mathrm{L_M} = 400$ km $\sim \varepsilon^{-\frac{3}{2}}\ h_{\mathrm{sc}}$, are sought. Hence, a so called stretching transformation has to be introduced that re-scales the dimensionless horizontal coordinates (3.2) to the claimed scales. With the aid of (2.23) the following stretched coordinates $\vec{X}_C$ and $\vec{\hat{x}}$ are defined

$$\vec{X}_C = \frac{\vec{\mathrm{X}}_C'}{\varepsilon^{\alpha}} \quad \text{for} \quad \alpha = -2 \qquad\qquad \text{and} \qquad\qquad \vec{\hat{x}} = \frac{\vec{\hat{\mathrm{x}}}'}{\varepsilon^{\alpha}} \quad \text{for} \quad \alpha = -\frac{3}{2} \tag{3.3}$$

with $\vec{X}_C = (X_C, Y_C)$ and $\vec{\hat{x}} = (\hat{x}, \hat{y})$. Here, $\vec{X}_C$ denotes the position vector for the vortex centreline resolved on synoptic scale and $\vec{\hat{x}}$ denotes a position vector resolving the mesoscale vortex region. Because of $u_{\mathrm{ref}} = h_{\mathrm{sc}}/t_{\mathrm{ref}}$ with u_{ref} given through $(2.7)_3$, rescaled advection times with respect to synoptic- and mesoscales read

$$\tau_1 = \varepsilon^2 t' \qquad\qquad \text{and} \qquad\qquad \tau_2 = \varepsilon^{\frac{3}{2}} t' \tag{3.4}$$

where $t' = t/t_{\mathrm{ref}}$ denotes the dimensionless time coordinate. In the subsequent analysis we will restrict our interest on the vortex motion and it's structural change on the synoptic time scale τ_1. Thus, the following transformation equation for the velocity field from the frame of reference fixed at the earth into the co-moving frame is used

$$\vec{v}_h = \vec{V}_C + \vec{v}_{rel} \tag{3.5}$$

whereas the velocity vector $\vec{V}_C$ for the centreline motion, and the velocity vector $\vec{v}_{rel}$ for the relative flow in the co-moving frame of reference are given by

$$\vec{V}_C = \frac{\partial \vec{X}_C}{\partial \tau_1} = \vec{i}\ \ U_C + \vec{j}\ \ V_C\ , \qquad\qquad \vec{v}_{rel} = \vec{e}_r\ u_r + \vec{e}_\theta\ u_\theta \tag{3.6}$$

With the scalings choosen above and using the notations of **Section 2.4.1**, an expansion ansatz suitable to derive reduced model equations in order to describe concentrated atmospheric vortices with typical diameters of 400 km reads

$$\mathcal{U}(t, \vec{\hat{x}}, z; \varepsilon) = \sum_{i \in N} \varepsilon^{\frac{i}{2}}\ \mathcal{U}^{(i)}\left(\varepsilon^2 t, \varepsilon^{\frac{3}{2}} \vec{\hat{\mathrm{x}}}, z\right) \tag{3.7}$$

Note that the expansion ansatz (3.7) is written with respect to the co-moving frame of reference and the primes indicating dimensionless variables have been dropped.

Next, transformation equations for the derivative operators $\vec{\nabla}_h$, $\partial/\partial z$ and $\partial/\partial t$ appearing in the complete three-dimensional compressible flow equations (2.15) have to be derived, that account for the change of variables in (3.3) and $(3.4)_1$. Upon substitution of (3.3) into (3.2), the transformation equation into the moving frame reads

$$\vec{x} = \varepsilon^{-2} \vec{X}_C + \varepsilon^{-\frac{3}{2}} \vec{\hat{x}} \tag{3.8}$$

with $\vec{X}_C = \vec{X}_C(z, \tau_1)$ and where the primes denoting dimensionless variables have been dropped, again. Due to the circular geometry of the vortex, the relative coordinates $\vec{\hat{x}} = (\hat{x}, \hat{y})$ will be expressed in terms of cylindrical coordinates, i.e.

$$\hat{x} = r\cos\theta\ , \quad \hat{y} = r\sin\theta \tag{3.9}$$

With $\vec{\hat{x}} = \varepsilon^{\frac{3}{2}}(\vec{x} - \varepsilon^{-2}\vec{X}_C)$ from (3.8), the radius r and the azimuthal angle θ in (3.9) are defined through

$$\begin{aligned} r &= \varepsilon^{\frac{3}{2}} \left((x - \varepsilon^{-2} X_C) \cos\theta + (y - \varepsilon^{-2} Y_C) \sin\theta \right) \\ \theta &= \arctan\left(\frac{y - \varepsilon^{-2} Y_C}{x - \varepsilon^{-2} X_C} \right) \\ z &= z \end{aligned} \tag{3.10}$$

Then, from the transformations (3.3), $(3.4)_1$, (3.8)-(3.10) together with $(3.6)_1$ and the chain rule (see **Appendix A.1**), we have that

$$\begin{aligned}
\vec{\nabla}_h &= \varepsilon^{\frac{3}{2}}\left(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)=\varepsilon^{\frac{3}{2}}\hat{\vec{\nabla}}_h\\
\left.\frac{\partial}{\partial z}\right|_{x,y} &= \left.\frac{\partial}{\partial z}\right|_{r,\theta}-\varepsilon^{-\frac{1}{2}}\frac{\partial\vec{X}_C}{\partial z}\cdot\hat{\vec{\nabla}}_h\\
\frac{\partial}{\partial t} &= \varepsilon^2\frac{\partial}{\partial\tau}-\varepsilon^{\frac{3}{2}}\vec{V}_C\cdot\hat{\vec{\nabla}}_h
\end{aligned}\tag{3.11}$$

with $\tau=\tau_1$ and where $\vec{e}_r$ and $\vec{e}_\theta$ denote the unit radial and the unit tangential vector in the horizontal plane (see **Figure 3.3**). Note that we write the position vector $\vec{X}_C$ in the fixed frame of reference in cartesian coordinates, i.e. $\vec{X}_C=\vec{i}\;X_C+\vec{j}\;Y_C$, whereas the unit vectors $\vec{i},\vec{j}$ and $\vec{e}_r,\vec{e}_\theta$ are related to each other via

$$\vec{i}=(\vec{e}_r\cos\theta-\vec{e}_\theta\sin\theta)\;,\qquad \vec{j}=(\vec{e}_r\sin\theta+\vec{e}_\theta\cos\theta)\tag{3.12}$$

Eventually, substitution of the transformations (3.11) and (3.5) into the equations (2.19) yields the starting equations to study concentrated, mesoscale vortices from an asymptotic perspective. In particular one obtains

$$\begin{aligned}
&\varepsilon^2\frac{\partial\rho}{\partial\tau}+\varepsilon^{\frac{3}{2}}\hat{\vec{\nabla}}_h\cdot(\rho\;\vec{v}_{rel})+\frac{\partial(\rho w)}{\partial z}-\frac{1}{\varepsilon^{\frac{1}{2}}}\frac{\partial\vec{X}_C}{\partial z}\cdot\hat{\vec{\nabla}}_h(\rho w) &&= 0\\
&\varepsilon^2\frac{\partial(\vec{V}_C+\vec{v}_{rel})}{\partial\tau}+\varepsilon^{\frac{3}{2}}\vec{v}_{rel}\cdot\hat{\vec{\nabla}}_h\vec{v}_{rel}+w\frac{\partial(\vec{V}_C+\vec{v}_{rel})}{\partial z}-\\
&\qquad\frac{w}{\varepsilon^{\frac{1}{2}}}\frac{\partial\vec{X}_C}{\partial z}\cdot\hat{\vec{\nabla}}_h\vec{v}_{rel}+\frac{1}{\varepsilon^{\frac{5}{2}}}\frac{\hat{\vec{\nabla}}_hp}{\rho}+\varepsilon(\vec{\Omega}\times(\vec{V}_C+\vec{v}_{rel}+w\vec{k}))_h &&= 0\\
&\varepsilon^2\frac{\partial w}{\partial\tau}+\varepsilon^{\frac{3}{2}}\vec{v}_{rel}\cdot\hat{\vec{\nabla}}_hw+w\frac{\partial w}{\partial z}-\frac{w}{\varepsilon^{\frac{1}{2}}}\frac{\partial\vec{X}_C}{\partial z}\cdot\hat{\vec{\nabla}}_hw\;+\frac{1}{\varepsilon^4}\frac{1}{\rho}\frac{\partial p}{\partial z}-\\
&\qquad\frac{1}{\varepsilon^{\frac{9}{2}}}\frac{1}{\rho}\frac{\partial\vec{X}_C}{\partial z}\cdot\hat{\vec{\nabla}}_hp+\varepsilon(\vec{\Omega}\times(\vec{V}_C+\vec{v}_{rel}+w\vec{k}))_\perp+\frac{1}{\varepsilon^4} &&= 0\\
&\left(\varepsilon^2\frac{\partial}{\partial\tau}+\varepsilon^{\frac{3}{2}}\;\vec{v}_{rel}\cdot\hat{\vec{\nabla}}_h+w\frac{\partial}{\partial z}-\frac{w}{\varepsilon^{\frac{1}{2}}}\frac{\partial\vec{X}_C}{\partial z}\cdot\hat{\vec{\nabla}}_h\right)\Theta &&= S
\end{aligned}\tag{3.13}$$

Here $\hat{\vec{\nabla}}_h\cdot\vec{V}_C=0$ and $\hat{\vec{\nabla}}_h\vec{V}_C=0$ is used. We shall show in **Chapter 4** and **5** that solutions for $\vec{X}_C$ can be derived with the aid of matched asymptotics. Note that the above equations (3.13) are closed with the state equation (2.21), i.e.

$$\rho\;\Theta=p^{1-\Gamma^{\star\star}\varepsilon}\qquad.\tag{3.14}$$

3.1.2 Inner expansion schemes

Using the notation of the stretched coordinates $(3.3)_2$ and $(3.4)_1$, and its transformations into cyclindrical coordinates (see (3.9)), the expansion ansatz (3.7) takes the form

$$\mathcal{U} = \sum_{i \in N} \varepsilon^{\frac{i}{2}} \, \mathcal{U}^{(i)} \, (r, \theta, z, \tau) \tag{3.15}$$

with $\mathcal{U} \in \{\vec{v}_h, w, p, \rho, \Theta\}$. Since the method of matched asymptotics is used to derive equations for the vortex motion and structure, the expansions based on ansatz (3.15) are refered to as *inner* solutions (see **Section 2.4.2**). However, further specializations of the variables $\vec{v}_h, w$ and Θ are needed.

Regarding the potential temperature Θ, Klein & Majda (2004) points out that order-of-magnitudes estimates based on the Brunt-Väisalä frequency or buoyancy frequency $N^2 = (g/\Theta)(\partial\Theta/\partial z)$, yield a dimensionless quantity

$$N^2 \frac{h_{\text{sc}}}{g} = \frac{1}{\Theta} \frac{\partial \Theta}{\partial z} \sim \frac{1}{10} \sim \varepsilon^2 \tag{3.16}$$

It can be shown (see **Appendix A.2**), that upon substitution of the expansion (3.15) into (3.16) the following conclusions about the atmospheric stability conditions in leading orders can be drawn

$$\frac{\partial \Theta^{(\frac{i}{2})}}{\partial z} = 0 \;, \qquad i = 0, 1, 2, 3 \tag{3.17}$$

Hence, it is assumed that a solution for Θ has the expansion

$$\begin{aligned} \Theta \;=\; & \Theta^{(0)}(r, \Theta, \tau) + \varepsilon^{\frac{1}{2}} \Theta^{(\frac{1}{2})}(r, \Theta, \tau) + \varepsilon^{\frac{2}{2}} \Theta^{(\frac{2}{2})}(r, \Theta, \tau) + \\ & \varepsilon^{\frac{3}{2}} \Theta^{(\frac{3}{2})}(r, \Theta, \tau) + \varepsilon^{\frac{4}{2}} \Theta^{(\frac{4}{2})}(r, \Theta, z, \tau) + \mathcal{O}(\varepsilon^{\frac{5}{2}}) \end{aligned} \tag{3.18}$$

Here $\mathcal{O}$ denotes the Landau[1] symbol 'big O'.

The expansions for the velocity components u_θ, u_r and w will be addressed in the following way. Recall, that the governing equations (2.19) have been nondimensionalized using $u_{\text{ref}} = 10$ m/s as a typical reference value for the velocity of atmospheric flows. We are interested, however, in intensely rotating vortices characterized by circumferential velocities with magnitudes of about $u_\theta \sim 30$ m/s $\sim \varepsilon^{-\frac{1}{2}} \, u_{\text{ref}}$. Taking this into account we change the magnitude of the horizontal velocity field by assuming following asymptotic expansions

[1]The Landau symbol $\mathcal{O}$ is defined by $f(\varepsilon) = \mathcal{O}(g(\varepsilon)) : \lim_{\varepsilon \to \varepsilon_0} f(\varepsilon)/g(\varepsilon) = const. \neq 0$.

$$
\begin{aligned}
u_\theta &= \varepsilon^{-\frac{1}{2}} u_\theta^{(0)}(r,z,\tau) + u_\theta^{(\frac{1}{2})}(r,\theta,z,\tau) + \varepsilon^{\frac{1}{2}} u_\theta^{(\frac{2}{2})}(r,\theta,z,\tau) + \mathcal{O}(\varepsilon^{\frac{2}{2}}) \\
u_r &= u_r^{(\frac{1}{2})}(r,\theta,z,\tau) + \varepsilon^{\frac{1}{2}} u_r^{(\frac{2}{2})}(r,\theta,z,\tau) + \mathcal{O}(\varepsilon^{\frac{2}{2}}) \\
w &= w^{(\frac{1}{2})}(r,\theta,z,\tau) + \varepsilon^{\frac{1}{2}} w^{(\frac{2}{2})}(r,\theta,z,\tau) + \mathcal{O}(\varepsilon^{\frac{2}{2}})
\end{aligned}
\tag{3.19}
$$

Please note that the leading order flow $u_\theta^{(0)}$ is assumed to be rotationally symmetric about the vertical axis.

The position vector $\vec{X}_C = \vec{X}_C(z,\tau)$ describing the location of the vortex centreline C also depends on the small parameter ε. Recall that $\vec{X}_C$ is scaled with respect to synoptic scales (see $(3.3)_1$). However, supported by observations one can act on the assumption that the horizontal displacement between the upper and lower part of a coherent vortex is at least smaller than ~ 1000 km (i.e. smaller than synoptic scales). Taking this into account, it is assumed that $\vec{X}_C$ has the following expansion

$$
\vec{X}_C = \vec{X}_C^{(0)}(\tau) + \varepsilon^{\frac{1}{2}} \vec{X}_C^{(\frac{1}{2})}(z,\tau) + \varepsilon^{\frac{2}{2}} \vec{X}_C^{(\frac{2}{2})}(z,\tau) + \mathcal{O}(\varepsilon^{\frac{3}{2}}) \tag{3.20}
$$

Note that the leading order term in (3.20) is independent of z, such that vertical variations of the centreline have to be described by higher order corrections. Due to $\partial\vec{X}_C/\partial\tau_1 = \vec{V}_C$ (see $(3.6)_1$ with $\tau = \tau_1$), the expansion (3.20) implies immediately that an expansion for the centreline velocity $\vec{V}_C$ has to be of the form

$$
\vec{V}_C = \vec{V}_C^{(0)}(\tau) + \varepsilon^{\frac{1}{2}} \vec{V}_C^{(\frac{1}{2})}(z,\tau) + \varepsilon^{\frac{2}{2}} \vec{V}_C^{(\frac{2}{2})}(z,\tau) + \mathcal{O}(\varepsilon^{\frac{3}{2}}) \tag{3.21}
$$

Taking into account that the vector of earth rotation $\vec{\Omega} = \vec{j}\,\Omega_h + \vec{k}\Omega_\perp$ varies with latitude φ, we now turn to an asymptotic description of the vector of earth rotation $\vec{\Omega}$. The Coriolis parameter Ω_h and $\Omega_\perp$ are defined through

$$
\Omega_h = \mid \vec{\Omega} \mid \ \cos\varphi\ , \qquad \Omega_\perp = \mid \vec{\Omega} \mid \ \sin\varphi \tag{3.22}
$$

Ignoring curvature effects, the earth's surface at the patch of flow under consideration can be approximated by a plane, i.e.

$$
\varphi = \varphi_0 + \frac{y}{a} \tag{3.23}
$$

where $a \approx 6000$ km denotes the radius of the earth. With the transformation (3.1) we also can write

$$
\varphi = \varphi_0 + \frac{(\mathrm{Y}_C + \hat{\mathrm{y}})}{a} \quad . \tag{3.24}
$$

After nondimensionalization (see eqn. (3.2)) and together with the stretched variables (3.3) and the centreline expansion (3.20), one obtains

$$
\begin{aligned}
\varphi &= \varphi_0 + \frac{h_{ref}\ \mathrm{Y}'_C}{a} + \frac{h_{ref}\ \hat{\mathrm{y}}'}{a} = \varphi_0 + \frac{h_{ref}\ Y_C}{\varepsilon^2\ a} + \frac{h_{ref}\ \hat{y}}{\varepsilon^{\frac{3}{2}}\ a} \\
&= \varphi_0 + \varepsilon Y_C + \varepsilon^{\frac{3}{2}}\hat{y} \ = \ \varphi_0 + \underbrace{\varepsilon Y_C^{(0)} + \varepsilon^{\frac{3}{2}}(Y_C^{(\frac{1}{2})} + \hat{y}) + \mathcal{O}(\varepsilon^{\frac{4}{2}})}_{\tilde{\varphi}}
\end{aligned}
\tag{3.25}
$$

Then, Taylor expansion of $\sin\varphi$ and $\cos\varphi$ around φ_0 yields

$$
\begin{aligned}
\sin(\varphi_0 + \tilde{\varphi}) &\approx \sin\varphi_0 + \tilde{\varphi}\cos\varphi_0 \\
&\approx \sin\varphi_0 + \varepsilon Y_C^{(0)}\cos\varphi_0 + \mathcal{O}(\varepsilon^{\frac{3}{2}}) \\
\cos(\varphi_0 + \tilde{\varphi}) &\approx \cos\varphi_0 - \tilde{\varphi}\sin\varphi_0 \\
&\approx \cos\varphi_0 - \varepsilon Y_C^{(0)}\sin\varphi_0 + \mathcal{O}(\varepsilon^{\frac{3}{2}})
\end{aligned}
\tag{3.26}
$$

Hence, upon substitution of (3.26) into (3.22) the following asymptotic expansion for the Coriolis parameter (3.22) can be derived

$$
\begin{aligned}
\Omega_h &= |\,\vec{\Omega}\,|\ \cos\varphi_0\ -\ \varepsilon\,|\,\vec{\Omega}\,|\ \sin\varphi_0\ Y_C^{(0)} + \mathcal{O}(\varepsilon^{\frac{3}{2}}) \\
\Omega_\perp &= \underbrace{|\,\vec{\Omega}\,|\ \sin\varphi_0}_{\Omega_0}\ +\ \varepsilon\underbrace{|\,\vec{\Omega}\,|\ \cos\varphi_0}_{\beta}\ Y_C^{(0)} + \mathcal{O}(\varepsilon^{\frac{3}{2}})
\end{aligned}
\tag{3.27}
$$

This approximation is also known as beta - plane approximation. Thus, an asymptotic expansion of $\vec{\Omega}$ takes the form

$$
\vec{\Omega} = \vec{\Omega}^{(0)} + \varepsilon\vec{\Omega}^{(1)} + \mathcal{O}(\varepsilon^{\frac{3}{2}})
\tag{3.28}
$$

with $\vec{\Omega}^{(0)} = \vec{j}\ \Omega_h^{(0)} + \vec{k}\ \Omega_0$ and $\vec{\Omega}^{(1)} = \vec{j}\ \Omega_h^{(1)} + \vec{k}\ \beta Y_C^{(0)}$, whereas $\Omega_h^{(i)}$ $(i = 0, 1)$ is the projection of the earth rotation vector $\vec{\Omega}$ onto the horizontal unit vector $\vec{j}$ and Ω_0 onto the vertical unit vector $\vec{k}$.

We close this subsection with an asymptotic expansion for the source term S, reading

$$
S = S^{(0)}(r,\theta,z,\tau) + \varepsilon^{\frac{1}{2}}S^{(\frac{1}{2})}(r,\theta,z,\tau) + \varepsilon^{\frac{2}{2}}S_C^{(\frac{2}{2})}(r,\theta,z,\tau) + \mathcal{O}(\varepsilon^{\frac{3}{2}})
\tag{3.29}
$$

A discussion of further specializations of this expansion is postponed into **Section 3.4**.

3.1.3 Asymptotic equations

The construction of asymptotic approximations of the solutions of the problem (3.13) starts with a substitution of the asymptotic expansions (3.15) (for ρ and p

only), and the expansions (3.18), (3.19), (3.20)-(3.21) and (3.28)-(3.29) into the governing equations (3.13). Then equating like powers of ε, reduced or so called asymptotic equations are derived. The asymptotic equations denote different problems for solutions in different orders of ε. Solving the problems sequentially, allows one to construct solutions to the whole problem (3.13) in the form of n-term approximations (see notation in ansatz (3.15), $i = 0, 1, ..., n$, $i \in N$). This section provides an overview about the reduced equations arranged in different orders of the small parameter. It should be noted that the asymptotic equations will serve as the starting equations for the different vortex solutions to be studied for adiabatic vortices in **Chapter 4** and diabatic vortices in **Chapter 5**.

First few trivial equations It turns out that starting with the lowest order of ε, the reduced equations are trivial equations in the sense that they only give informations about the spatial and the temporal dependencies of the dependent flow variables p, ρ, Θ, etc. in leading orders.

Continuity equation:

$$\mathcal{O}(1): \qquad \frac{\partial(\rho^{(0)} w^{(\frac{1}{2})})}{\partial z} - \frac{\partial \vec{X}_C^{(\frac{1}{2})}}{\partial z} \cdot \hat{\vec{\nabla}}_h (\rho^{(0)} w^{(\frac{1}{2})}) = 0 \tag{3.30}$$

From first principles, (3.30) describes the change of mass flux experienced by a material element displaced vertically. Due to the transformation from a frame of reference fixed at the earth into a co-moving frame of reference whose origin is located at a tilted vortex-centreline, this change of mass flux appears in the equations as a sum of two contributions, i.e. the change of vertical mass flux experienced by a material element displaced along any line parallel to the first correction of the vortex centreline ($\partial/\partial z$ for a fixed r, θ, τ) and a change of mass flux from a horizontal displacement ($\hat{\vec{\nabla}}_h$ for a fixed z, τ). Due to the zero right hand side of (3.30) the mass flux ($\rho^{(0)} w^{(\frac{1}{2})}$) remains conserved with respect to vertical displacements. Thus by choosing the boundary conditions $w^{(\frac{1}{2})} = 0$ at $z = 0$ it follows immediately that

$$w^{(\frac{1}{2})} = 0 \,, \qquad \forall \, z \tag{3.31}$$

From the next two orders of the mass continuity equation, same reasoning yields

$$\mathcal{O}(\varepsilon^{\frac{j}{2}}): \qquad w^{(\frac{j+1}{2})} = 0, \qquad j = 1, 2 \tag{3.32}$$

Horizontal momentum equations:

$$\mathcal{O}(\varepsilon^{-\frac{i}{2}}): \qquad \hat{\vec{\nabla}}_h p^{(\frac{i}{2})} = 0, \qquad i = 0, 1, 2, ..., 5 \tag{3.33}$$

Vertical momentum equation:

$$\mathcal{O}(\varepsilon^{-\frac{8}{2}\cdots-\frac{3}{2}}): \qquad \frac{\partial p^{(\frac{j}{2})}}{\partial z} \;=\; -\rho^{(\frac{j}{2})}, \qquad j = 0, 1, 2, ..., 5 \tag{3.34}$$

Note, from the leading order hydrostatic conditions (3.34) together with (3.33) it follows immediately

$$\vec{\hat{\nabla}}\rho^{(\frac{i}{2})} \;=\; 0, \qquad i = 0, 1, 2, ..., 5 \tag{3.35}$$

State equation:

An overview about the $\mathcal{O}(\varepsilon^{\frac{i}{2}})$ state equations is given in **Appendix A.3**. It can easily be checked that with the aid of (3.35) together with (3.33) it follows immediately that

$$\mathcal{O}(\varepsilon^{\frac{i}{2}}): \qquad \vec{\hat{\nabla}}\Theta^{(\frac{i}{2})} \;=\; 0, \qquad i = 0, 1, 2, ..., 5 \tag{3.36}$$

Potential temperature equation:

From (3.31), (3.32) and (3.36) one obtains

$$\mathcal{O}(\varepsilon^{\frac{i}{2}}): \qquad S^{(\frac{i}{2})} \;=\; 0, \qquad i = 0, 1, 2, ..., 6 \tag{3.37}$$

Upon substitution of (3.37) into the diabatic source term expansion (3.29) it turns out that $\mathcal{O}(\varepsilon^{\frac{7}{2}})$ source terms become important if mesoscale vortices are studied. In particular, it is shown in **Section 3.4.1** that $S^{(\frac{7}{2})}$ describes heating of about 20 K / day.

First few non-trivial equations We shall next present higher order asymptotic equations of (3.13). Here we already use the matching results $\rho^{(\frac{1}{2})} = 0$, $\partial\rho^{(0)}/\partial\tau = 0$ and $\partial\Theta^{(\frac{4}{2})}/\partial\tau = 0$, $\partial\Theta^{(\frac{5}{2})}/\partial z = 0$ which will be discussed in more details in **Subsection 3.3.1**.

Horizontal momentum equations:

$$\mathcal{O}(\varepsilon^{\frac{1}{2}}): \qquad \vec{v}^{(0)}_{rel}\cdot\vec{\hat{\nabla}}_h\vec{v}^{(0)}_{rel} + \frac{\vec{\hat{\nabla}}_h p^{(\frac{6}{2})}}{\rho^{(0)}} + (\vec{\Omega}^{(0)}\times\vec{v}^{(0)}_{rel})_h \;=\; 0 \tag{3.38}$$

With $\vec{v}^{(0)}_{rel} = \vec{e}_\theta\ u^{(0)}_\theta(r, z, \tau)$ (see (3.19)), $\vec{\Omega}^{(0)} = \Omega^{(0)}_h\vec{j} + \Omega_0\vec{k}$ (see (3.28)), and $\vec{\hat{\nabla}}_h$ given by $(3.11)_1$, the component equations in $\vec{e}_r$ and $\vec{e}_\theta$ direction read respectively

$$\begin{aligned}\vec{e}_r: \quad & \frac{1}{\rho^{(0)}}\frac{\partial p^{(\frac{6}{2})}}{\partial r} - \frac{{u_\theta^{(0)}}^2}{r} - \Omega_0 u_\theta^{(0)} &=& \quad 0 \\ \vec{e}_\theta: \quad & \frac{\partial p^{(\frac{6}{2})}}{\partial \theta} &=& \quad 0\end{aligned} \tag{3.39}$$

$$\begin{aligned}\mathcal{O}(\varepsilon^{\frac{2}{2}}): \quad & w^{(\frac{4}{2})}\frac{\partial \vec{v}_{rel}^{(0)}}{\partial z} - w^{(\frac{4}{2})}\frac{\partial \vec{X}_C^{(\frac{1}{2})}}{\partial z}\cdot \tilde{\vec{\nabla}}_h \vec{v}_{rel}^{(0)} + \vec{v}_{rel}^{(0)}\cdot \tilde{\vec{\nabla}}_h \vec{v}_{rel}^{(\frac{1}{2})} + \\ & \vec{v}_{rel}^{(\frac{1}{2})}\cdot \tilde{\vec{\nabla}}_h \vec{v}_{rel}^{(0)} + \frac{\tilde{\vec{\nabla}}_h p^{(\frac{7}{2})}}{\rho^{(0)}} + (\vec{\Omega}^{(0)} \times (\vec{V}_C^{(0)} + \vec{v}_{rel}^{(\frac{1}{2})}))_h \;=\; 0\end{aligned} \tag{3.40}$$

Here, the horizontal velocity vectors are defined through $\vec{v}_{rel}^{(0)} = \vec{e}_\theta\ u_\theta^{(0)}(r,z,\tau)$ and $\vec{v}_{rel}^{(\frac{1}{2})} = \vec{e}_r\ u_r^{(\frac{1}{2})} + \vec{e}_\theta\ u_\theta^{(\frac{1}{2})}$. Recall that the centreline vector $\vec{X}_C = \vec{X}_C(z,\tau)$ is given in cartesian coordinates, i.e. $\vec{X}_C = \vec{i}\ X_C + \vec{j}\ Y_C$. Then, together with $\tau_1 = \tau$, the definition for the centreline motion $(3.6)_1$, the relation (3.12), the leading order earth rotation vector $\vec{\Omega}^{(0)} = \Omega_h^{(0)}\vec{j} + \Omega_0\vec{k}$ (see (3.28)), the horizontal operator $\tilde{\vec{\nabla}}_h$ given through $(3.11)_1$, and the following abbreviations

$$\begin{aligned}\Lambda_a^j &= -\frac{\partial X_C^{(\frac{j}{2})}}{\partial z}\sin\theta + \frac{\partial Y_C^{(\frac{j}{2})}}{\partial z}\cos\theta \\ \Lambda_b^j &= +\frac{\partial X_C^{(\frac{j}{2})}}{\partial z}\cos\theta + \frac{\partial Y_C^{(\frac{j}{2})}}{\partial z}\sin\theta\end{aligned} \tag{3.41}$$

for $j = 0, 1, 2, ..., n$ and

$$\begin{aligned}\Pi_a^j &= U_C^{(\frac{j}{2})}\sin\theta - V_C^{(\frac{j}{2})}\cos\theta \\ \Pi_b^j &= U_C^{(\frac{j}{2})}\cos\theta + V_C^{(\frac{j}{2})}\sin\theta \quad ,\end{aligned} \tag{3.42}$$

one obtains from the above $\mathcal{O}(\varepsilon^{\frac{2}{2}})$ horizontal momentum equation in $\vec{e}_r$ and $\vec{e}_\theta$ direction, respectively

$$\begin{aligned}\vec{e}_r: \quad & \Lambda_a^1\frac{w^{(\frac{4}{2})}\ u_\theta^{(0)}}{r} + \frac{u_\theta^{(0)}}{r}\frac{\partial u_r^{(\frac{1}{2})}}{\partial \theta} - \frac{2u_\theta^{(0)}u_\theta^{(\frac{1}{2})}}{r} + \frac{1}{\rho^{(0)}}\frac{\partial p^{(\frac{7}{2})}}{\partial r} + \\ & \qquad \Omega_0\Pi_a^0 - \Omega_0 u_\theta^{(\frac{1}{2})} = 0 \\ \vec{e}_\theta: \quad & w^{(\frac{4}{2})}\frac{\partial u_\theta^{(0)}}{\partial z} - \Lambda_b^1\ w^{(\frac{4}{2})}\frac{\partial u_\theta^{(0)}}{\partial r} + u_r^{(\frac{1}{2})}\frac{\partial u_\theta^{(0)}}{\partial r} + u_r^{(\frac{1}{2})}\frac{u_\theta^{(0)}}{r} + \\ & \frac{u_\theta^{(0)}}{r}\frac{\partial u_\theta^{(\frac{1}{2})}}{\partial \theta} + \frac{1}{r\ \rho^{(0)}}\frac{\partial p^{(\frac{7}{2})}}{\partial \theta} + \Omega_0\Pi_b^0 + \Omega_0 u_r^{(\frac{1}{2})} = 0\end{aligned} \tag{3.43}$$

$$\mathcal{O}(\varepsilon^{\frac{3}{2}}): \quad \frac{\partial \vec{v}_{rel}^{(0)}}{\partial \tau} + w^{(\frac{4}{2})} \frac{\partial \vec{v}_{rel}^{(\frac{1}{2})}}{\partial z} + w^{(\frac{5}{2})} \frac{\partial \vec{v}_{rel}^{(0)}}{\partial z} - w^{(\frac{5}{2})} \frac{\partial \vec{X}_C^{(\frac{1}{2})}}{\partial z} \cdot \hat{\vec{\nabla}}_h \vec{v}_{rel}^{(0)} -$$
$$w^{(\frac{4}{2})} \frac{\partial \vec{X}_C^{(\frac{2}{2})}}{\partial z} \cdot \hat{\vec{\nabla}}_h \vec{v}_{rel}^{(0)} - w^{(\frac{4}{2})} \frac{\partial \vec{X}_C^{(\frac{1}{2})}}{\partial z} \cdot \hat{\vec{\nabla}}_h \vec{v}_{rel}^{(\frac{1}{2})} + \vec{v}_{rel}^{(0)} \cdot \hat{\vec{\nabla}}_h \vec{v}_{rel}^{(\frac{2}{2})} +$$
$$\vec{v}_{rel}^{(\frac{1}{2})} \cdot \hat{\vec{\nabla}}_h \vec{v}_{rel}^{(\frac{1}{2})} + \vec{v}_{rel}^{(\frac{2}{2})} \cdot \hat{\vec{\nabla}}_h \vec{v}_{rel}^{(0)} + \frac{\hat{\vec{\nabla}}_h p^{(\frac{8}{2})}}{\rho^{(0)}} - \frac{\rho^{(\frac{2}{2})} \hat{\vec{\nabla}}_h p^{(\frac{6}{2})}}{\rho^{(0)^2}} +$$
$$(\vec{\Omega}^{(0)} \times (\vec{V}_C^{(\frac{1}{2})} + \vec{v}_{rel}^{(\frac{2}{2})}))_h + (\vec{\Omega}^{(1)} \times \vec{v}_{rel}^{(0)})_h \;=\; 0$$

Here, the horizontal velocity vectors are defined through $\vec{v}_{rel}^{(0)} = \vec{e}_\theta \; u_\theta^{(0)}(r, z, \tau)$ and $\vec{v}_{rel}^{(\frac{i}{2})} = \vec{e}_r \; u_r^{(\frac{1}{2})} + \vec{e}_\theta \; u_\theta^{(\frac{i}{2})}$ with $i = 1, 2$. Taking into account that $\vec{\Omega}^{(1)} = \Omega_h^{(1)} \vec{j} + \beta Y_C^{(0)} \vec{k}$ (see (3.28)) and that $p^{(\frac{6}{2})} = p^{(\frac{6}{2})}(r, z, \tau)$ (see $(3.39)_2$), together with the abbreviations (3.41) - (3.42) one obtains

$$\vec{e}_r: \quad w^{(\frac{4}{2})} \frac{\partial u_r^{(\frac{1}{2})}}{\partial z} - \Lambda_b^1 \; w^{(\frac{4}{2})} \frac{\partial u_r^{(\frac{1}{2})}}{\partial r} - \Lambda_a^1 \; w^{(\frac{4}{2})} \frac{1}{r} \frac{\partial u_r^{(\frac{1}{2})}}{\partial \theta} +$$
$$\Lambda_a^1 \; w^{(\frac{5}{2})} \frac{u_\theta^{(0)}}{r} + \Lambda_a^1 \; w^{(\frac{4}{2})} \frac{u_\theta^{(\frac{1}{2})}}{r} + \Lambda_a^2 \; w^{(\frac{4}{2})} \frac{u_\theta^{(0)}}{r} + u_r^{(\frac{1}{2})} \frac{\partial u_r^{(\frac{1}{2})}}{\partial r} +$$
$$\frac{u_\theta^{(\frac{1}{2})}}{r} \frac{\partial u_r^{(\frac{1}{2})}}{\partial \theta} + \frac{u_\theta^{(0)}}{r} \frac{\partial u_r^{(\frac{2}{2})}}{\partial \theta} - \frac{u_\theta^{(\frac{1}{2})^2}}{r} - \frac{2 u_\theta^{(0)} u_\theta^{(\frac{2}{2})}}{r} + \frac{1}{\rho^{(0)}} \frac{\partial p^{(\frac{8}{2})}}{\partial r} -$$
$$\frac{\rho^{(\frac{2}{2})}}{\rho^{(0)^2}} \frac{\partial p^{(\frac{6}{2})}}{\partial r} + \Omega_0 \Pi_a^1 - \Omega_0 u_\theta^{(\frac{2}{2})} - u_\theta^{(0)} \beta Y^{(0)} = 0 \qquad (3.44)$$

$$\vec{e}_\theta: \quad \frac{\partial u_\theta^{(0)}}{\partial \tau} + w^{(\frac{5}{2})} \frac{\partial u_\theta^{(0)}}{\partial z} + w^{(\frac{4}{2})} \frac{\partial u_\theta^{(\frac{1}{2})}}{\partial z} - \Lambda_b^1 \; w^{(\frac{5}{2})} \frac{\partial u_\theta^{(0)}}{\partial r} -$$
$$\Lambda_b^1 w^{(\frac{4}{2})} \frac{\partial u_\theta^{(\frac{1}{2})}}{\partial r} - \Lambda_b^2 w^{(\frac{4}{2})} \frac{\partial u_\theta^{(0)}}{\partial r} - \Lambda_a^1 \; w^{(\frac{4}{2})} \frac{u_r^{(\frac{1}{2})}}{r} - \Lambda_a^1 \; w^{(\frac{4}{2})} \frac{1}{r} \frac{\partial u_\theta^{(\frac{1}{2})}}{\partial \theta} +$$
$$u_r^{(\frac{1}{2})} \frac{\partial u_\theta^{(\frac{1}{2})}}{\partial r} + u_r^{(\frac{1}{2})} \frac{u_\theta^{(\frac{1}{2})}}{r} + u_r^{(\frac{2}{2})} \frac{\partial u_\theta^{(0)}}{\partial r} + u_r^{(\frac{2}{2})} \frac{u_\theta^{(0)}}{r} +$$
$$\frac{u_\theta^{(0)}}{r} \frac{\partial u_\theta^{(\frac{2}{2})}}{\partial \theta} + \frac{u_\theta^{(\frac{1}{2})}}{r} \frac{\partial u_\theta^{(\frac{1}{2})}}{\partial \theta} + \frac{1}{r \rho^{(0)}} \frac{\partial p^{(\frac{8}{2})}}{\partial \theta} + \Omega_0 \Pi_b^1 + \Omega_0 u_r^{(\frac{2}{2})} = 0 \qquad (3.45)$$

Vertical momentum equations:

$\mathcal{O}(\varepsilon^{-\frac{2}{2}})$:

$$\frac{\partial p^{(\frac{6}{2})}}{\partial z} - \Lambda_b^1 \frac{\partial p^{(\frac{6}{2})}}{\partial r} = -\rho^{(\frac{6}{2})} \qquad (3.46)$$

$\mathcal{O}(\varepsilon^{-\frac{1}{2}})$:

$$\frac{\partial p^{(\frac{7}{2})}}{\partial z} - \Lambda_b^1 \frac{\partial p^{(\frac{7}{2})}}{\partial r} - \Lambda_a^1 \frac{1}{r} \frac{\partial p^{(\frac{7}{2})}}{\partial \theta} - \Lambda_b^2 \frac{\partial p^{(\frac{6}{2})}}{\partial r} = -\rho^{(\frac{7}{2})} \tag{3.47}$$

Mass continuity:

$$\mathcal{O}(\varepsilon^{\frac{3}{2}}) : \quad \vec{\hat{\nabla}}_h \cdot (\rho^{(0)} \vec{v}_{rel}^{(\frac{1}{2})}) + \frac{\partial(\rho^{(0)} w^{(\frac{4}{2})})}{\partial z} - \frac{\partial \vec{X}_C^{(\frac{1}{2})}}{\partial z} \cdot \vec{\hat{\nabla}}_h(\rho^{(0)} w^{(\frac{4}{2})}) \;=\; 0 \tag{3.48}$$

Note that $\rho^{(0)} = \rho^{(0)}(z)$ (see (3.35) and matching conditions), and that in general the horizontal divergence $\vec{\hat{\nabla}}_h \cdot \vec{v}_{rel}^{(\frac{i}{2})}$ in cylindrical coordinates reads

$$\vec{\hat{\nabla}}_h \cdot \vec{v}_{rel}^{(\frac{i}{2})} = \left(\frac{\partial u_r^{(\frac{i}{2})}}{\partial r} + \frac{u_r^{(\frac{i}{2})}}{r} + \frac{1}{r} \frac{\partial u_\theta^{(\frac{i}{2})}}{\partial \theta} \right) , \qquad i = 1, 2, ..., n \tag{3.49}$$

Hence, together with the abbreviations (3.41) the above $\mathcal{O}(\varepsilon^{\frac{3}{2}})$ mass continuity takes the form

$$\rho^{(0)} \left(\frac{\partial u_r^{(\frac{1}{2})}}{\partial r} + \frac{u_r^{(\frac{1}{2})}}{r} + \frac{1}{r} \frac{\partial u_\theta^{(\frac{1}{2})}}{\partial \theta} \right) + \frac{\partial(\rho^{(0)} w^{(\frac{4}{2})})}{\partial z} - \Lambda_b^1 \frac{\partial(\rho^{(0)} w^{(\frac{4}{2})})}{\partial r} - \Lambda_a^1 \frac{1}{r} \frac{\partial(\rho^{(0)} w^{(\frac{4}{2})})}{\partial \theta} = 0 \tag{3.50}$$

$$\mathcal{O}(\varepsilon^{\frac{4}{2}}) : \quad \vec{\hat{\nabla}}_h \cdot (\rho^{(0)} \vec{v}_{rel}^{(\frac{2}{2})}) + \frac{\partial(\rho^{(0)} w^{(\frac{5}{2})})}{\partial z} - \frac{\partial \vec{X}_C^{(\frac{2}{2})}}{\partial z} \cdot \vec{\hat{\nabla}}_h(\rho^{(0)} w^{(\frac{4}{2})}) - \frac{\partial \vec{X}_C^{(\frac{1}{2})}}{\partial z} \cdot \vec{\hat{\nabla}}_h(\rho^{(0)} w^{(\frac{5}{2})}) \;=\; 0 \tag{3.51}$$

Same reasonings as for the $\mathcal{O}(\varepsilon^{\frac{3}{2}})$ mass continuity yield

$$\rho^{(0)} \left(\frac{\partial u_r^{(\frac{2}{2})}}{\partial r} + \frac{u_r^{(\frac{2}{2})}}{r} + \frac{1}{r} \frac{\partial u_\theta^{(\frac{2}{2})}}{\partial \theta} \right) + \frac{\partial(\rho^{(0)} w^{(\frac{5}{2})})}{\partial z} - \Lambda_b^2 \frac{\partial(\rho^{(0)} w^{(\frac{4}{2})})}{\partial r} - \Lambda_a^2 \frac{1}{r} \frac{\partial(\rho^{(0)} w^{(\frac{4}{2})})}{\partial \theta} - \Lambda_b^1 \frac{\partial(\rho^{(0)} w^{(\frac{5}{2})})}{\partial r} - \Lambda_a^1 \frac{1}{r} \frac{\partial(\rho^{(0)} w^{(\frac{5}{2})})}{\partial \theta} \;=\; 0 \tag{3.52}$$

Potential temperature equations:

$\mathcal{O}(\varepsilon^{\frac{7}{2}})$:

$$w^{(\frac{4}{2})} \frac{\partial \Theta^{(\frac{4}{2})}}{\partial z} = S^{(\frac{7}{2})} \tag{3.53}$$

$$\mathcal{O}(\varepsilon^{\frac{8}{2}}): \qquad \underbrace{\vec{v}_{rel}^{(0)} \cdot \vec{\tilde{\nabla}}_h \Theta^{(\frac{6}{2})}}_{\frac{u_\theta^{(0)}}{r}\frac{\partial \Theta^{(\frac{6}{2})}}{\partial \theta}} + w^{(\frac{5}{2})} \frac{\partial \Theta^{(\frac{4}{2})}}{\partial z} = S^{(\frac{8}{2})} \tag{3.54}$$

$$\begin{aligned} \mathcal{O}(\varepsilon^{\frac{9}{2}}): \qquad & \vec{v}_{rel}^{(0)} \cdot \vec{\tilde{\nabla}}_h \Theta^{(\frac{7}{2})} + \vec{v}_{rel}^{(\frac{1}{2})} \cdot \vec{\tilde{\nabla}}_h \Theta^{(\frac{6}{2})} + w^{(\frac{4}{2})} \frac{\partial \Theta^{(\frac{6}{2})}}{\partial z} + \\ & w^{(\frac{6}{2})} \frac{\partial \Theta^{(\frac{4}{2})}}{\partial z} - w^{(\frac{4}{2})} \frac{\partial \vec{X}_C^{(\frac{1}{2})}}{\partial z} \cdot \vec{\tilde{\nabla}}_h \Theta^{(\frac{6}{2})} \;=\; S^{(\frac{9}{2})} \end{aligned} \tag{3.55}$$

or

$$\begin{aligned} & u_\theta^{(0)} \frac{1}{r} \frac{\partial \Theta^{(\frac{7}{2})}}{\partial \theta} + u_r^{(\frac{1}{2})} \frac{\partial \Theta^{(\frac{6}{2})}}{\partial r} + u_\theta^{(\frac{1}{2})} \frac{1}{r} \frac{\partial \Theta^{(\frac{6}{2})}}{\partial \theta} + w^{(\frac{4}{2})} \frac{\partial \Theta^{(\frac{6}{2})}}{\partial z} + \\ & w^{(\frac{6}{2})} \frac{\partial \Theta^{(\frac{4}{2})}}{\partial z} - \Lambda_b^1 w^{(\frac{4}{2})} \frac{\partial \Theta^{(\frac{6}{2})}}{\partial r} - \Lambda_a^1 w^{(\frac{4}{2})} \frac{1}{r} \frac{\partial \Theta^{(\frac{6}{2})}}{\partial \theta} \;=\; S^{(\frac{9}{2})} \end{aligned} \tag{3.56}$$

3.1.4 Further mathematical tools

As noted earlier, a detailed discussion of the asymptotic equations specified in the previous subsection will be carried in the **Chapters 4** and **5** for adiabatic and diabatic vortices, respectively. In doing so further mathematical analysis techniques shall be employed. A short introduction into them is given next.

Harmonic analysis To find solutions from the reduced equations specified in **Section 3.1.3** the technique of harmonic analysis is utilized. Recall that the dependent variables $\rho^{(\frac{i}{2})}, p^{(\frac{i}{2})}, \Theta^{(\frac{i}{2})}$, etc. are described in a (r, θ, z, τ) space. Thus its perodicity in θ allows to replace each variable by its Fourier series, which means in general

$$a(r, \theta, z, \tau) = a_0 + \sum_{j=1}^{n} \left(a_{j1} \sin(j\theta) + a_{j2} \cos(j\theta)\right) \tag{3.57}$$

with $a \in \{\rho^{(\frac{i}{2})}, p^{(\frac{i}{2})}, \Theta^{(\frac{i}{2})}, w^{(\frac{i}{2})}, \vec{v}_{rel}^{(\frac{i}{2})}\}$. The coefficients a_0, a_{j1} and a_{j2} are the Fourier coefficients (or harmonics) of a and can be determined through

$$\begin{aligned} a_0(r) &= \frac{1}{2\pi} \int_0^{2\pi} a(r, \theta, z, \tau)\; d\theta \\ a_{j1}(r) &= \frac{1}{2\pi} \int_0^{2\pi} a(r, \theta, z, \tau)\; \sin(j\theta)\; d\theta \\ a_{j2}(r) &= \frac{1}{2\pi} \int_0^{2\pi} a(r, \theta, z, \tau)\; \cos(j\theta)\; d\theta \end{aligned} \tag{3.58}$$

Helmholtz's Theorem Taking advantage of Helmholtz's theorem the expansion terms of the horizontal velocity vector $\vec{v}_{rel}^{(\frac{j}{2})} = (u_r^{(\frac{j}{2})}, u_\theta^{(\frac{j}{2})})$ will be considered as a sum of two vectors, i.e.

$$\vec{v}_{rel}^{(\frac{j}{2})} = \vec{v}_{rel}^{\,nd(\frac{j}{2})} + \vec{v}_{rel}^{\,d(\frac{j}{2})} = (u_r^{nd(\frac{j}{2})}, u_\theta^{nd(\frac{j}{2})}) + (u_r^{d(\frac{j}{2})}, u_\theta^{d(\frac{j}{2})}) \tag{3.59}$$

The superscript (nd) denotes a solenoidal vector satisfying $\vec{\tilde{\nabla}} \cdot \vec{v}_{rel}^{\,nd(\frac{j}{2})} = 0$ which allows the definition of a stream function, i.e.

$$(u_r^{nd(\frac{j}{2})}, u_\theta^{nd(\frac{1}{2})}) = \left(\frac{1}{r}\frac{\partial \psi^{(\frac{j}{2})}}{\partial \theta}, -\frac{\partial \psi^{(\frac{j}{2})}}{\partial r}\right) \tag{3.60}$$

The superscript (d) denotes an irrotational vector satisfying $\vec{\tilde{\nabla}} \times \vec{v}_{rel}^{\,d(\frac{j}{2})} = 0$ that allows the introduction of a velocity potential, i.e.

$$(u_r^{d(\frac{j}{2})}, u_\theta^{d(\frac{1}{2})}) = \left(\frac{\partial \phi^{(\frac{j}{2})}}{\partial r}, \frac{1}{r}\frac{\partial \phi^{(\frac{j}{2})}}{\partial \theta}\right) \tag{3.61}$$

Applying (3.58) to the Fourier decomposition of $u_r^{nd(\frac{j}{2})}$ and $u_\theta^{d(\frac{j}{2})}$ yields for the zeroth modes

$$u_{r,0}^{nd(\frac{j}{2})} = 0, \qquad u_{\theta,0}^{d(\frac{j}{2})} = 0 \tag{3.62}$$

and the asymmetric components of $u_r^{(\frac{j}{2})}$ and $u_\theta^{(\frac{j}{2})}$ in terms of the stream function $\psi^{(\frac{j}{2})}$ and velocity potential $\phi^{(\frac{j}{2})}$ read

$$\begin{aligned}
u_{\theta,11}^{(\frac{j}{2})} &= u_{\theta,11}^{nd(\frac{j}{2})} + u_{\theta,11}^{d(\frac{j}{2})} = -\frac{\partial \psi_{11}^{(\frac{j}{2})}}{\partial r} - \frac{\phi_{12}^{(\frac{j}{2})}}{r} \\
u_{\theta,12}^{(\frac{j}{2})} &= u_{\theta,12}^{nd(\frac{j}{2})} + u_{\theta,12}^{d(\frac{j}{2})} = -\frac{\partial \psi_{12}^{(\frac{j}{2})}}{\partial r} + \frac{\phi_{11}^{(\frac{j}{2})}}{r} \\
u_{r,11}^{(\frac{j}{2})} &= u_{r,11}^{nd(\frac{j}{2})} + u_{r,11}^{d(\frac{j}{2})} = -\frac{\psi_{12}^{(\frac{j}{2})}}{r} + \frac{\partial \phi_{11}^{(\frac{j}{2})}}{\partial r} \\
u_{r,12}^{(\frac{j}{2})} &= u_{r,12}^{nd(\frac{j}{2})} + u_{r,12}^{d(\frac{j}{2})} = +\frac{\psi_{11}^{(\frac{j}{2})}}{r} + \frac{\partial \phi_{12}^{(\frac{j}{2})}}{\partial r}
\end{aligned} \tag{3.63}$$

3.2 Synoptic-scale regime

As shown in **Section 2.4.1** first applications of the unified approach to meteorological modelling were based on a re-derivation of well-known reduced models in theoretical meteorology. Here the quasi-geostrophic (QG) theory is included, which is known as a relatively accurate approximation for synoptic-scale atmospheric motions in which the Rossby number[2] is less than unity (Pedlosky, 1987). A brief review about the most important results yielding QG-theory from an asymptotic perspective will be given in **Section 3.2.1**. In **Section 3.2.2** it is assumed that in synoptic scales, three-dimensional mesoscale vortices studied in the present work can be treated as line vortices denoting an anomaly from the synoptic scale three-dimensional quasi-geostrophic potential vorticity. Appropriate singular vortex solutions will be derived, which will serve as *outer* solutions for matching. Moreover, the inclusion of a regular flow will account for the impact of meridional variations of the Coriolis parameter Ω_0 (β effect) on the ambient vortex flow. Note, that the discussions in **Section 3.2.2** are strongly related to the works of Reznik (1992) and Callegari & Ting (1978).

3.2.1 Unified Approach and QG-theory

Considering synoptic-scale flows the characteristic length scale of motion is given by the internal deformation radius $\mathrm{L_S} = 1000$ km $\sim \varepsilon^{-2}\ h_{\mathrm{sc}}$. Hence, on account of the stretching transformations (2.23) an appropriate expansion Ansatz for an asymptotic analysis of the governing equations (2.19) with respect to synoptic scales, reads

$$\check{\mathcal{U}} = \sum_{i \in N} \varepsilon^i \check{\mathcal{U}}^{(i)}(\varepsilon^2 t, \varepsilon^2 \vec{x}, z) = \sum_{i \in N} \varepsilon^i \check{\mathcal{U}}^{(i)}(\tau, \vec{\eta}, z) \tag{3.64}$$

with $\check{\mathcal{U}} \in \{\check{\vec{v}}_h,\ \check{w},\ \check{\rho},\ \check{p},\ \check{\Theta}\}$. Here, $\vec{\eta} = (\eta_1, \eta_2) = (\varepsilon^2 x, \varepsilon^2 y)$ denote the new stretched coordinates. Note that the notation $\check{(.)}$ is used hereafter in order to indicate dependent variables resolved with respect to synoptic-scales.

It has been shown by Klein (2004) that inserting Ansatz (3.64) into the Euler equations (2.19) yields the QG-theory, which in absence of heating (i.e. $S = 0$) describes conservation of the quasigeostrophic potential vorticity $\check{q}$ along parcel trajectories, i.e.

$$\frac{d\check{q}}{d\tau} = 0 \quad \text{with} \quad \check{q} = \check{\zeta}^{(0)} + \beta\eta_2 + \frac{\Omega_0}{\check{\rho}^{(0)}} \frac{\partial}{\partial z} \left(\frac{\check{\rho}^{(0)} \check{\Theta}^{(3)}}{\frac{d\check{\Theta}^{(2)}}{dz}} \right) \tag{3.65}$$

Here, the total time derivative is given by

$$\frac{d}{d\tau} = \frac{\partial}{\partial \tau} + \check{\vec{v}}_g^{(0)} \cdot \check{\vec{\nabla}}_h \qquad . \tag{3.66}$$

[2]In order to study large scale motions which are significantly influenced by the earth's rotation, Pedlosky defines the Rossby number by $\mathrm{Ro}_L = U/(2\Omega L_S)$ with $U = 10$ m/s and $L_S = 1000$ km, such that $\mathrm{Ro}_L \leq 1$. Note, in the framework of the unified approach to meteorological modelling the Rossby number $\mathrm{Ro}_{h_{sc}}$ is defined by $(2.10)_3$.

In terms of the stretched coordinate $\vec{\eta} = (\eta_1, \eta_2)$, the Nabla operator reads $\check{\vec{\nabla}}_h = \vec{i}\, \partial/\partial\eta_1 + \vec{j}\, \partial/\partial\eta_2$. Note, in 3D the quasigeostrophic potential vorticity $\check{q}$ is comprised of three terms: a) the vertical component of the relative vorticity given by $\check{\zeta}^{(0)} = \partial\check{v}_g^{(0)}/\partial\eta_1 - \partial\check{u}_g^{(0)}/\partial\eta_2$, b) the vorticity due to the earth rotation, i.e. $\beta\eta_2$, and c) a vorticity stretching term associated with a nonconstant stratification on the vertical given by $(\Omega_0/\check{\rho}^{(0)})(\partial_z(\check{\rho}^{(0)}\check{\Theta}^{(3)}/\partial_z\check{\Theta}^{(2)}))$. Leading order advection in (3.66) is realized by the geostrophic wind $\check{\vec{v}}_g^{(0)} = (\check{u}_g^{(0)}, \check{v}_g^{(0)})$ which is a result of a two way balance between the Coriolis force and the horizontal pressure gradient force

$$\Omega_0 \vec{k} \times \check{\vec{v}}_g^{(0)} = -\frac{1}{\check{\rho}^{(0)}} \check{\vec{\nabla}}_h \check{p}^{(3)} \tag{3.67}$$

Note, that based on Ansatz (3.65) the equation (3.67) is an $\mathcal{O}(\varepsilon^{-1})$ equation of the horizontal momentum equation $(2.19)_2$.

Further results primarily obtained from the $\mathcal{O}(\varepsilon^{-i})$ equations (with $i = 0, 1, 2$) of the mass continuity $(2.19)_1$, are that the vertical velocity can be expanded as

$$\check{w} = \varepsilon^3 \check{w}^{(3)} + \varepsilon^4 \check{w}^{(4)} + \mathcal{O}(\varepsilon^4) \tag{3.68}$$

and that $\check{\vec{v}}_g^{(0)}$ satisfies the incompressibility condition, i.e.

$$\check{\vec{\nabla}} \cdot \check{\vec{v}}_g^{(0)} = 0 \tag{3.69}$$

An asymptotic expansion of the vertical momentum equation yields hydrostatic up to the fourth order, i.e.

$$\frac{\partial \check{p}^{(i)}}{\partial z} = -\check{\rho}^{(i)}, \qquad i = 0, 1, ..., 4 \tag{3.70}$$

with $\check{p}^{(i)} = \check{p}^{(i)}(z)$ and $\check{\rho}^{(i)} = \check{\rho}^{(i)}(z)$. By means of the state equation one gets $\check{\Theta}^{(3)}$ satisfying the QG hydrostatic equation

$$\frac{\check{\Theta}^{(3)}}{\check{\Theta}_\infty} = \frac{\partial}{\partial z}\left(\frac{\check{p}^{(3)}}{\check{\rho}^{(0)}}\right) \tag{3.71}$$

Here it has been assumed that the potential temperature $\check{\Theta}$ has the asymptotic expansion

$$\check{\Theta} = \Theta_\infty + \varepsilon^2 \check{\Theta}^{(2)}(z) + \varepsilon^3 \check{\Theta}^{(3)}(\vec{\eta}, z, \tau) + \mathcal{O}(\varepsilon^4) \tag{3.72}$$

In particular, Klein (2004) has shown that this is an appropriate temperature scaling with respect to synoptic scales. Taking the incompressibility condition (3.69) and the hydrostatic condition (3.71) into account a stream function can be introduced, which is defined by

$$\check{\vec{v}}_g^{(0)} = \left(\frac{\partial \check{\psi}_g^{(0)}}{\partial \eta_2}, -\frac{\partial \check{\psi}_g^{(0)}}{\partial \eta_1} \right) \quad , \qquad \check{\psi}_g^{(0)} = -\frac{\check{p}^{(3)}}{\Omega_0 \check{\rho}^{(0)}} \tag{3.73}$$

Using (3.73), the quasigeostrophic vorticity equation (3.65) can be written as

$$\frac{\partial \check{q}_1}{\partial \tau} - J(\check{\psi}_g^{(0)}, \check{q}_1) - \frac{\partial \check{\psi}_g^{(0)}}{\partial \eta_1}\beta = 0 \tag{3.74}$$

with

$$\check{q}_1 = -\vec{\tilde{\nabla}}^2 \check{\psi}_g^{(0)} - \frac{\Omega_0^2 \Theta_\infty}{\check{\rho}^{(0)}} \frac{\partial}{\partial z} \left(\frac{\check{\rho}^{(0)} \frac{\partial \check{\psi}_g^{(0)}}{\partial z}}{\frac{d\check{\Theta}^{(2)}}{dz}} \right) \tag{3.75}$$

The Jacobian in (3.74) is given by $J(\check{\psi}_g^{(0)}, \check{q}_1) = \left(\frac{\partial \check{\psi}_g^{(0)}}{\partial \eta_1} \frac{\partial \check{q}_1}{\partial \eta_2} - \frac{\partial \check{\psi}_g^{(0)}}{\partial \eta_2} \frac{\partial \check{q}_1}{\partial \eta_1} \right)$. The horizontal Laplacian of the geostrophic stream function gives the relative vorticity, i.e. $\vec{\tilde{\nabla}}^2 \check{\psi}_g^{(0)} = -\check{\zeta}^{(0)}$. For more details regarding the asymptotic analysis based on Ansatz (3.64) the reader is refered to (Klein & Vater, 2004).

3.2.2 Singular Vortex theory

A number of studies regarding interactions of concentrated small-scale vortices and large scale environmental flows under the simplifying assumption of barotropic atmospheric conditions (two-dimensional flows), are based on the idea of representing distributed small-scale vortices by point vortices that denote an anomaly from the background quasigeostrophic vorticity (Morikawa, 1960; Egger 1992; Reznik, 1992). A justification for such an approach is given by the fact that the vorticity of real atmospheric vortices often exceeds the background vorticity. Moreover, an important advantage of studying point vortices rather than distributed regular vortices lies in the ease with which many mathematical operations can be handled.

One purpose of this work is to show, however, that the point vortex approach alone has the disadvantage that effects produced by the small scale vortex flow are partially excluded, but which may be of central importance to the entire flow evolution including the vortex motion. In **Chapters 4** and **5** it will be shown that the method of matched asymptotics is a useful tool that allows one to include these effects. Using this method, however, does not mean that we reject the theory of point vortices. Indeed we use the concept of point vortices and derive in **Subsection 3.2.2.1** so called outer vortex solutions valid on synoptic scales. Later on, matching the outer vortex solutions with the inner vortex solutions obtained from the mesoscale asymptotic Ansatz (3.7), yields approximate vortex solutions that give insight into the manner in which scale-interactions between the mesoscale vortex flow itself and a synoptic scale environmental flow govern the entire flow evolution.

For the derivation of outer vortex solutions this thesis resorts to a singular-vortex theory for geostrophic, beta-plane dynamics which is proposed in Reznik (1992). Considering two-dimensional flows governed by the equivalent barotropic quasigeostrophic vorticity equation[3] Reznik managed to show that owing to the β-effect, the redistribution of the background potential vorticity induced by the vortex flow itself, generates a regular field in addition to the velocity field induced by the vortices themselves. Furthermore he found analytically that for an individual vortex this regular field affects the vortex trajectory. A short introduction of this topic is given in **Subsection 3.2.2.3**. Note that it is shown in this thesis that matched asymptotics gives an opportunity to include the net effects of the mesoscale vortex structure in addition to the effect of a regular flow on the vortex motion as shown by Reznik (1992).

In analogy to Reznik's approach and using the principle of superposition, the total geostrophic stream function $\check{\psi}_g^{(0)}$ in (3.74) will be given by a sum of three contributions

$$\check{\psi}_g^{(0)} \;=\; \check{\Psi}_B^{(0)} + \check{\psi}' \;=\; \check{\Psi}_B^{(0)} + \check{\psi}_s^{(0)} + \check{\psi}_r^{(0)} \tag{3.76}$$

Here, $\check{\Psi}_B^{(0)}$ denotes a background flow and $\check{\psi}'$ a localized vortex flow which is again regarded as a sum of two terms, namely a singular component $\check{\psi}_s^{(0)}$ and a regular component $\check{\psi}_r^{(0)}$. Then the quasigeostrophic potential vorticity $\check{q}$ in (3.65) is given by

$$\check{q} = \check{q}_B + \check{q}_s + \check{q}_r \tag{3.77}$$

Note, that $\check{\Psi}_B^{(0)}$ in (3.76) denotes the leading order term of an asymptotic expansion of the background flow in which the vortex is embedded in. In general it is assumed that the background flow has the following asymptotic expansion

$$\check{\Psi}_B(\vec{\eta}, z, \tau) = \check{\Psi}_B^{(0)}(\vec{\eta}, z, \tau) + \varepsilon^{\frac{1}{2}}\check{\Psi}_B^{(\frac{1}{2})}(\vec{\eta}, z, \tau) + \mathcal{O}(\varepsilon^{\frac{2}{2}}) \;, \tag{3.78}$$

whereas each single expansion term $\check{\Psi}_B^{(\frac{i}{2})}$ (i = 0,1,2,...) is assumed to be given.

[3] The atmosphere is refered to be in an equivalent barotropic state if the temperature gradients are such that the isotherms are parallel to the isotherms. On a β-plane the equation of barotropic flow under zero forcing is given by $\partial_\tau q + \beta\partial_x\psi + J(\psi, q) = 0$ with $q = \nabla^2\psi$.

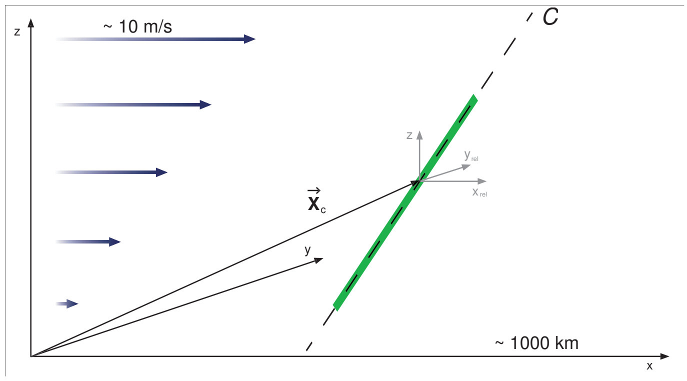

Figure 3.4: Schematic diagram showing the idealized representation of a mesoscale vortex by a line vortex embedded in an synoptic scale environmental flow; see the text for further explanations

3.2.2.1 Leading order line vortex solutions

By similar arguments that support the concept of representing mesoscale vortices by point vortices in two-dimensional and large scale flows, localized mesoscale vortices in three dimensions can be represented by the so called line vortices. In particular it is assumed that the distributed vortex (or vortex-tube) with respect to mesoscales contracts on to a curve with the strength of the vortex-tube remaining constant. Then, the line denotes a line singularity of the whole vorticity distribution. **Figure 3.4** is a schematic diagram of this situation. Note, the position of the line vortex described by the position vector $\vec{X}_C = (X_C, Y_C)$ (see **Section 3.1.1**) coincides with the position of the vortex-centreline displayed in **Figure 3.3**. Then, singular vortex solutions $\check{\psi}_s^{(0)}$ describing the induced flow in the neighbourhood of the line vortex, satisfy the nonhomogeneous equation

$$-\vec{\nabla}^2 \check{\psi}_s^{(0)} - \frac{\Omega_0^2 \Theta_\infty}{\check{\rho}^{(0)}} \frac{\partial}{\partial z} \left(\frac{\check{\rho}^{(0)} \frac{\partial \check{\psi}_s^{(0)}}{\partial z}}{\frac{d\check{\Theta}^{(2)}}{dz}} \right) = \check{q}_s \quad , \tag{3.79}$$

with $\vec{\nabla}^2 = \partial^2/\partial\eta_1^2 + \partial^2/\partial\eta_2^2$ and where the concentration of the source $\check{q}_s$ is located at the vortex centreline $\vec{X}_C = \vec{X}_C(z,\tau)$, i.e.

$$\begin{aligned} \check{q}_s &= \frac{\Gamma}{2\pi}\, \delta(\vec{\eta} - \vec{X}_C(z,\tau)) \\ &= \frac{\Gamma}{2\pi}\, \delta(\eta_1 - X_C(z,\tau))\, \delta(\eta_2 - Y_C(z,\tau)) \end{aligned} \tag{3.80}$$

Here, Γ denotes the circulation of the velocity field $\check{u}_\theta = -\partial\check{\psi}_s^{(0)}/\partial\check{r}$ along a closed curve C_Γ surrounding the line vortex, i.e. $\Gamma = \oint_{C_\Gamma} \check{u}_\theta \, ds$. The Kronecker delta defines a two dimensional Dirac delta function with $\delta(\vec{\eta} - \vec{X}_C) = 0$ for $\vec{\eta} \neq \vec{X}_C$. Introducing the relative vector $\vec{\check{r}} = \vec{\eta} - \vec{X}_C(z,\tau)$ as a new coordinate with $|\,\vec{\check{r}}\,| = \check{r}$ representing a synoptic scale radial distance from $\vec{\eta} = \vec{X}_C(z,\tau)$, solutions for (3.79) in the vicinity of the line source ($\check{r} \to 0$), to a first approximation, are

$$\begin{aligned} \check{\psi}^{(0)}_{s,0}\,(\check{r},z,\tau) &= -\frac{\Gamma(z,\tau)}{2\pi}\ln\check{r} && \text{as} \quad \check{r}\to 0 \\ \check{\psi}^{(0)}_{s,1k}\,(\check{r},z,\tau) &= \frac{c_1}{4}\,\frac{1}{\check{r}}\,Z_{s,1k}(z,\tau) && \text{as} \quad \check{r}\to 0 \end{aligned} \tag{3.81}$$

Here, $\check{\psi}^{(0)}_{s,0}$ denotes the axissymmetric contribution of $\check{\psi}^{(0)}_s$ and $\check{\psi}^{(0)}_{s,1k}$ its first Fourier coefficients (see **Subsection 3.1.4**). Note, the sign of Γ gives the sense of the circulation: for $\Gamma < 0$ it is clockwise (anticyclonic) and for $\Gamma > 0$ it is anticlockwise (cyclonic). See **Appendix A.4** for details on how (3.81) is obtained from (3.79) and the meaning of $Z_{s,1k}(z,\tau)$ and c_1.

3.2.2.2 Regular flow

The regular flow $\check{\psi}^{(0)}_r$ satisfies the nonhomogeneous problem

$$-\vec{\check{\nabla}}^2\check{\psi}^{(0)}_r - \frac{\Omega_0^2\Theta_\infty}{\check{\rho}^{(0)}}\frac{\partial}{\partial z}\left(\frac{\check{\rho}^{(0)}}{\check{\Theta}^{(2)}_z}\frac{\partial\check{\psi}^{(0)}_r}{\partial z}\right) - \beta\eta_2 = \check{q}_r \tag{3.82}$$

For a given PV distribution $\check{q}_r$ and boundary conditions, solutions for $\check{\psi}_r$ can be obtained. Since $\check{\psi}_r$ defined by (3.73) satisfies the balance conditions (3.67) and (3.71), solutions for $\check{\psi}_r$ can be used to retrieve associated pressure and temperature fields. Such a solution algorithm is referred to as the invertibility principle.

3.2.2.3 Equations for the vortex motion: Reznik's approach

As noted earlier, using the theory of singular vortices on the equivalent barotropic quasigeostrophic vorticity equation, Reznik (1992) managed to show that the large scale vortex motion is nontrivially affected by a regular flow, which is due to the β - effect generated by the vortex flow itself. A brief introduction into Reznik's technique for the derivation of equations for the vortex motion is now given. Later on, this turns out to be helpful for a comparison of the results for the vortex motion obtained by Reznik's technique and the technique of matched asymptotics used in the present work.

According to Reznik's approach, equations for the vortex motion can be derived upon substituting $\check{\psi}' = \check{\psi}_r + \check{\psi}_s$ (see (3.76)) into (3.74) and equating to zero the regular part and the parts proportional to $\delta'_\kappa(\eta_1 - X_C^{(0)}(\tau))\,\delta(\eta_2 - Y_C^{(0)}(\tau))$ and $\delta(\eta_1 - X_C^{(0)}(\tau))\,\delta'_\kappa(\eta_2 - Y_C^{(0)}(\tau))$, where δ'_κ is the derivative of the Dirac delta function with respect to $\kappa \in \{\tau, \eta_1, \eta_2\}$. Additionally, imposing a prescribed background flow $\check{\Psi}_B^{(0)}$, then the following equations for the vortex motion can be derived (see **Appendix A.5**)

$$\begin{aligned} \frac{dX_C^{(0)}}{d\tau} &= U_C^{(0)} &= -\frac{\partial(\check{\Psi}_B^{(0)} + \check{\psi}_r^{(0)})}{\partial \eta_2} \\ \frac{dY_C^{(0)}}{d\tau} &= V_C^{(0)} &= +\frac{\partial(\check{\Psi}_B^{(0)} + \check{\psi}_r^{(0)})}{\partial \eta_1} \end{aligned} \tag{3.83}$$

Moreover, an equation for the evolution of the regular flow can be derived

$$\frac{\partial \check{q}_r}{\partial \tau} - \beta \frac{\partial \check{\psi}_r^{(0)}}{\partial \eta_1} - J(\check{\psi}_r^{(0)}, \check{q}_r) + J((\check{q}_r + \beta\eta_2), \check{\psi}_s^{(0)}) \;=\; 0 \tag{3.84}$$

where solutions for $\check{\psi}_s^{(0)}$ are given through (3.81). Solving (3.84) together with (3.82) and appropriate initial/boundary conditions, yields solutions for $\check{\psi}_r^{(0)}$ as functions on β and $\check{\psi}_s^{(0)}$, which in turn yield via (3.83) solutions for the vortex motion $(U_C^{(0)}, V_C^{(0)})$. In **Chapters 4** and **5** it is shown that the inclusion of mesoscale vortex solutions via matching, yields a third contribution on the right hand side of (3.83) that affects the vortex motion.

3.3 Matching conditions

The underlying idea of the method of matched asymptotic expansions has already been explained in **Section 2.4.2**. In this section the general matching criterion (2.26) is used to obtain particular matching conditions for the velocity, potential temperature and pressure fields describing the vortex under consideration from an meso- and synoptic scale perspective, respectively. In the previous **Sections 3.1** and **3.2** asymptotic expansions and the necessary stretching transformations to obtain vortex solutions valid on meso- and synoptic scale have been discussed. The general matching criterion defined with respect to the co-moving frame of reference that guarantees that the *inner* (see (3.7)) mesoscale and *outer* (see (3.64)) synoptic scale solutions $\mathcal{U} \in \{\vec{v}_h, w, p, \rho, \Theta\}$ give in their transition region the same results, is given by

$$\sum_{i\in N} \varepsilon^{\frac{i}{2}}\, \mathcal{U}^{(i)}\left(\varepsilon^2 t, \varepsilon^{\frac{3}{2}-\lambda}\vec{\chi}, z\right) = \sum_{i\in N} \varepsilon^{i}\, \check{\mathcal{U}}^{(i)}\left(\varepsilon^2 t, \vec{X}_C(z, \varepsilon^2 t) + \varepsilon^{2-\lambda}\vec{\chi}, z\right) \tag{3.85}$$

Here $\vec{X}_C = \varepsilon^2 \vec{\mathrm{X}}_C'$ denotes the synoptic scale centreline coordinate (see $(3.3)_1$) and $\vec{\chi} = (\chi_1, \chi_2) = (\varepsilon^\lambda \hat{\mathrm{x}}', \varepsilon^\lambda \hat{\mathrm{y}}')$ with $3/2 < \lambda < 2$ denotes an intermediate coor-

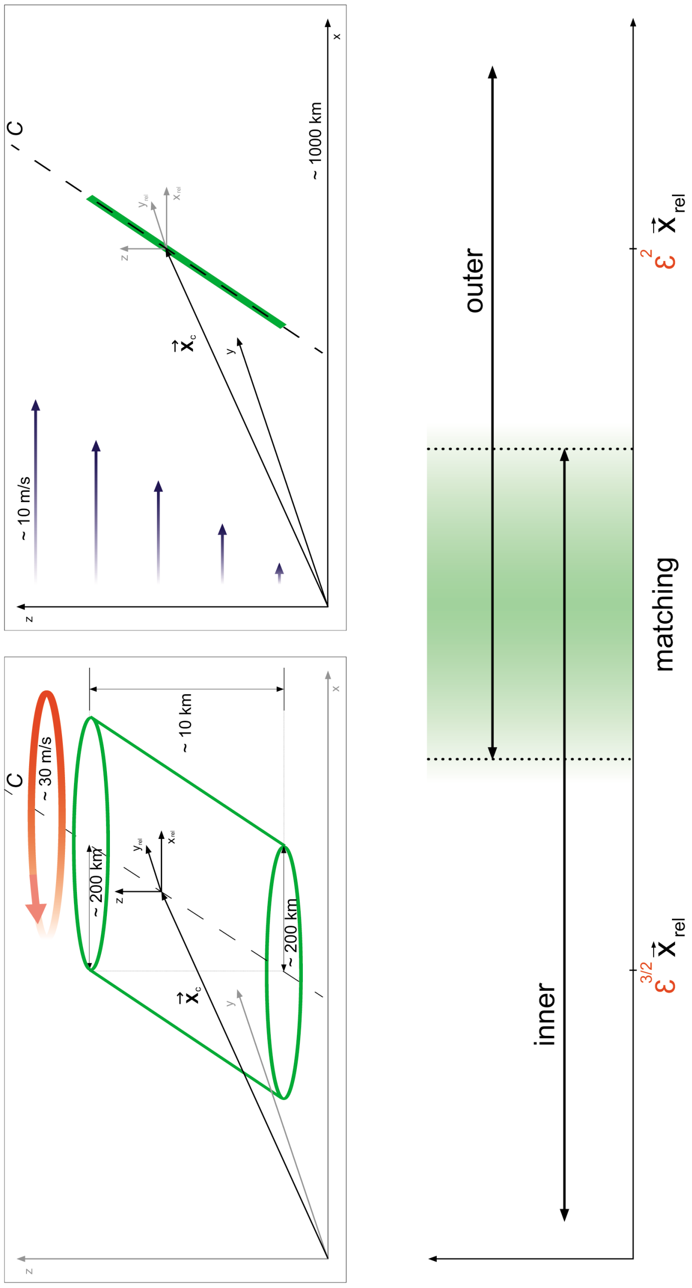

Figure 3.5: top left: distributed vortex w.r.t. mesoscale whose *inner* solutions are assumed to have the expansion (3.7), top right: line vortex w.r.t. synoptic scale whose *outer* solutions are assumed to have the expansion (3.64)), bottom: Schematic representation of the domains of validity of the *inner* and *outer* expansions, where the highlighted region marks the transition region resolved by the 'overlap' lengthscale $\vec{\chi} = \varepsilon^{\lambda}\vec{x}_{rel}$ (with $\vec{x}_{rel} = \vec{\hat{x}}$) and which is located between meso- and synoptic scales.

dinate describing an 'overlap' lengthscale between the meso- and synoptic scale. Note that to change the coordinates in the outer expansion from $\vec{\eta}$ to $\vec{\chi}$ the transformation equation (3.2) into the co-moving frame of reference has been used. To simplify things we choose a crude way to do the matching by setting $\lambda = 3/2$, which implies immediately that $\vec{\hat{x}} = \vec{\chi}$ (see the stretching transformation $(3.3)_2$), i.e. the intermediate coordinate is equal to the inner coordinates resolving the mesoscale. In doing so (3.85) becomes

$$\sum_{i \in N} \varepsilon^{\frac{i}{2}} \, \mathcal{U}^{(i)} \left(\varepsilon^2 t, \vec{\hat{x}}, z \right) = \sum_{i \in N} \varepsilon^{i} \, \check{\mathcal{U}}^{(i)} \left(\varepsilon^2 t, \vec{X}_C(z, \varepsilon^2 t) + \varepsilon^{\frac{1}{2}} \vec{\hat{x}}, z \right) \tag{3.86}$$

3.3.1 Horizontal velocity field

In **Section 3.2.1** it has been shown that the leading order outer velocity $\check{\vec{v}}^{(0)}$ satisfies the geostrophic wind balance (3.67) such that $\check{\vec{v}}^{(0)} = \check{\vec{v}}_g^{(0)}$. Moreover it has been shown that the incompressibility condition (3.69) allows us to introduce a so called geostrophic stream function $\check{\psi}_g^{(0)} = \check{\psi}_g^{(0)}(\vec{\eta}, z, \tau)$ which in turn is decomposed into three contributions representing the background flow ($\check{\Psi}_B^{(0)}$), the regular flow ($\check{\psi}_r^{(0)}$), and the singular flow ($\check{\psi}_s^{(0)}$) (see (3.76)), where $\check{\psi}_s^{(0)}$ is induced by the vortex itself that appears as a line vortex with respect to synoptic scales.

Recall that for matching purposes the outer solutions are expressed as functions on the intermediate coordinate $\vec{\chi} = \vec{\hat{x}}$ (see (3.86)). This means that a general outer stream function $\check{\psi}_d = \check{\psi}_d(\vec{\eta}, z, \tau)$ with $d \in \{B, r\}$ is considered in the following way

$$\check{\psi}_d(\vec{\eta}, z, \tau) = \check{\psi}_d \left(\vec{X}_C(z, \tau) + \varepsilon^{\frac{1}{2}} \vec{\hat{x}}, z, \tau \right) \tag{3.87}$$

with $\tau = \varepsilon^2 t$ (see $(3.4)_1$) and $\vec{\eta} = (\eta_1, \eta_2)$. Taking the centreline expansion (3.20) into account and using the transformation equations (3.9) for cartesian $(\hat{x}_1, \hat{x}_2)$ into cylindrical coordinates (r, θ), equation (3.87) takes the form

$$\check{\psi}_d(\vec{\eta}, z, \tau) = \check{\psi}_d(X_C^{(0)} + \delta_1, Y_C^{(0)} + \delta_2, z, \tau) \tag{3.88}$$

with

$$\begin{aligned} \delta_1 &= \varepsilon^{\frac{1}{2}} \left(X_C^{(\frac{1}{2})} + r \cos\theta \right) + \mathcal{O}(\varepsilon^{\frac{2}{2}}) \\ \delta_2 &= \varepsilon^{\frac{1}{2}} \left(Y_C^{(\frac{1}{2})} + r \sin\theta \right) + \mathcal{O}(\varepsilon^{\frac{2}{2}}) \end{aligned} \tag{3.89}$$

Then, Taylor expansion of (3.88) around $\vec{X}_C^{(0)} = (X_C^{(0)}, Y_C^{(0)})$ yields

$$\check{\psi}_d(\eta_1, \eta_2, z) = \check{\psi}_d(\vec{X}_C^{(0)}, z) + \delta_1 \left. \frac{\partial \check{\psi}_d}{\partial \eta_1} \right|_{\vec{\eta} = \vec{X}_C^{(0)}} + \delta_2 \left. \frac{\partial \psi_d}{\partial \eta_2} \right|_{\vec{\eta} = \vec{X}_C^{(0)}} + \mathcal{O}(\varepsilon^{\frac{2}{2}}) \tag{3.90}$$

With the aid of (3.73) the velocity of the background and regular flow are given by

$$\begin{aligned}
\vec{V}_B^{(\frac{i}{2})} &= (U_B^{(\frac{i}{2})}, V_B^{(\frac{i}{2})}) = \left(\frac{\partial \check{\Psi}_B^{(\frac{i}{2})}}{\partial \eta_2}, -\frac{\partial \check{\Psi}_B^{(\frac{i}{2})}}{\partial \eta_1}\right) \quad , \quad i = 0, 1, 2, ... \\
\vec{V}_R^{(0)} &= (U_R^{(0)}, V_R^{(0)}) = \left(\frac{\partial \check{\psi}_r^{(0)}}{\partial \eta_2}, -\frac{\partial \check{\psi}_r^{(0)}}{\partial \eta_1}\right)
\end{aligned} \tag{3.91}$$

Note that for the background flow the expansion (3.78) has been assumed. Thus, using (3.90) expansions for $\vec{V}_B^{(\frac{i}{2})} = (U_B^{(\frac{i}{2})}, V_B^{(\frac{i}{2})})$ and $\vec{V}_R^{(0)} = (U_R^{(0)}, V_R^{(0)})$ around the leading order centreline $\vec{X}_C^{(0)}$ yields

$$\begin{aligned}
U_B^{(\frac{i}{2})} &= U_B^{(\frac{i}{2})}(\vec{X}_C^{(0)}, z, \tau) + \sum_{j=1}^{2} \delta_j \left.\frac{\partial U_B^{(\frac{i}{2})}}{\partial \eta_j}\right|_{\vec{\eta}=\vec{X}_C^{(0)}} + \mathcal{O}(\varepsilon^{\frac{2}{2}}) \\
V_B^{(\frac{i}{2})} &= V_B^{(\frac{i}{2})}(\vec{X}_C^{(0)}, z, \tau) + \sum_{j=1}^{2} \delta_j \left.\frac{\partial V_B^{(\frac{i}{2})}}{\partial \eta_j}\right|_{\vec{\eta}=\vec{X}_C^{(0)}} + \mathcal{O}(\varepsilon^{\frac{2}{2}}) \\
U_R^{(0)} &= U_R^{(0)}(\vec{X}_C^{(0)}, z, \tau) + \sum_{j=1}^{2} \delta_j \left.\frac{\partial U_R^{(0)}}{\partial \eta_j}\right|_{\vec{\eta}=\vec{X}_C^{(0)}} + \mathcal{O}(\varepsilon^{\frac{2}{2}}) \\
V_R^{(0)} &= V_R^{(0)}(\vec{X}_C^{(0)}, z, \tau) + \sum_{j=1}^{2} \delta_j \left.\frac{\partial V_R^{(0)}}{\partial \eta_j}\right|_{\vec{\eta}=\vec{X}_C^{(0)}} + \mathcal{O}(\varepsilon^{\frac{2}{2}})
\end{aligned} \tag{3.92}$$

with $i = 0, 1, 2, ...$.

Solutions for the singular stream function $\check{\psi}_s^{(0)}$ as function on the radius $\check{r}$ have been derived in **Section 3.2.2.1**. Note that $\check{r}$ resolves synoptic scales (see $(A-41)_1$). Comparing the definition $(3.10)_1$ for r and the definition $(A-41)_1$ for $\check{r}$, and on account of $\vec{\eta} = (\eta_1, \eta_2) = (\varepsilon^2 x, \varepsilon^2 y)$ (see (3.64)), the transformation equation (3.2) into the co-moving frame of reference, and the centreline stretching transformation $(3.3)_1$, it can be shown that r and $\check{r}$ are related to each other in the following way

$$\check{r} = \varepsilon^{\frac{1}{2}} r \tag{3.93}$$

Thus, from (3.93) and the singular solutions (3.81) the leading order first Fourier modes of the outer tangential velocity field $\check{u}_\theta = -\partial \check{\psi}_s^{(0)} / \partial \check{r}$ induced by the line vortex are given by

$$\begin{aligned}
\check{u}_{\theta,0} &= \frac{1}{\varepsilon^{\frac{1}{2}}} \frac{\Gamma}{2\pi r} \\
\check{u}_{\theta,1k} &= -\frac{1}{\varepsilon} \frac{c_1 \; Z_{s,1k}}{4} \frac{1}{r^2}
\end{aligned} \tag{3.94}$$

Finally, taking (3.12), (3.92), (3.94) and the expansion (3.78) of the background flow into account, it turns out that within the transition region between mesoscales and synoptic scales the leading order solution for the outer velocity field $\check{\vec{v}}$ can be written as

$$
\begin{aligned}
\check{\vec{v}}_g^{(0)} \quad = \quad & \varepsilon^{-\frac{1}{2}} \frac{\Gamma}{2\pi r} \, \vec{e}_\theta + \varepsilon^{-1} \frac{c_1 Z_{s,11}}{4r^2} \sin\theta \, \vec{e}_\theta + \varepsilon^{-1} \frac{c_1 Z_{s,12}}{4r^2} \cos\theta \, \vec{e}_\theta \ + \\
& \varepsilon^0 \left(\left[(U_{B,C}^{(0)} + U_{R,C}^{(0)}) \vec{i} + (V_{B,C}^{(0)} + V_{R,C}^{(0)}) \vec{j} \right] \right) \ + \\
& \varepsilon^{\frac{1}{2}} [(U_{B,C}^{(\frac{1}{2})} + F_1) \vec{i} + (V_{B,C}^{(\frac{1}{2})} + F_2) \vec{j}] \ + \mathcal{O}(\varepsilon^{\frac{2}{2}})
\end{aligned}
\tag{3.95}
$$

where $\vec{V}_{B,C}^{(0)} = \vec{V}_B^{(0)}(\vec{X}_C^{(0)}, z, \tau)$, $\vec{V}_{R,C}^{(0)} = \vec{V}_R^{(0)}(\vec{X}_C^{(0)}, z, \tau)$ and F_s (s = 1,2) is given by

$$
F_s = -M \left. \frac{\partial (V_B^{(0)} + V_R^{(0)})}{\partial \eta_s} \right|_{\vec{\eta} = \vec{X}_C^{(0)}} + N \left. \frac{\partial (U_B^{(0)} + U_R^{(0)})}{\partial \eta_s} \right|_{\vec{\eta} = \vec{X}_C^{(0)}}
\tag{3.96}
$$

where $M = (X_C^{(\frac{1}{2})} + r\cos\theta)$ and $N = (Y_C^{(\frac{1}{2})} + r\sin\theta)$. Then, a specification of the general matching condition (3.86) for the inner velocity field $\vec{v}_h$ (see (3.5), $(3.19)_{1,2}$ and (3.21)) and the outer velocity field $\check{\vec{v}}_h$ (see eqn. (3.64)), yields

$$
\begin{aligned}
& \varepsilon^{-\frac{1}{2}} u_\theta^{(0)} \vec{e}_\theta + \varepsilon^0 \left([U_C^{(0)} \vec{i} + V_C^{(0)} \vec{j}] + [u_r^{(\frac{1}{2})} \vec{e}_r + u_\theta^{(\frac{1}{2})} \vec{e}_\theta] \right) + \\
& \varepsilon^{\frac{1}{2}} \left([U_C^{(\frac{1}{2})} \vec{i} + V_C^{(\frac{1}{2})} \vec{j}] + [u_r^{(\frac{2}{2})} \vec{e}_r + u_\theta^{(\frac{2}{2})} \vec{e}_\theta] \right) + \mathcal{O}(\varepsilon^{\frac{2}{2}}) \ = \\
& \varepsilon^{-\frac{1}{2}} \frac{\Gamma}{2\pi r} \, \vec{e}_\theta + \varepsilon^{-1} \frac{c_1 Z_{s,11}}{4r^2} \sin\theta \, \vec{e}_\theta + \varepsilon^{-1} \frac{c_1 Z_{s,12}}{4r^2} \cos\theta \, \vec{e}_\theta + \\
& \varepsilon^0 \left(\left[(U_{B,C}^{(0)} + U_{R,C}^{(0)}) \vec{i} + (V_{B,C}^{(0)} + V_{R,C}^{(0)}) \vec{j} \right] \right) + \\
& \varepsilon^{\frac{1}{2}} [(U_{B,C}^{(\frac{1}{2})} + F_1) \vec{i} + (V_{B,C}^{(\frac{1}{2})} + F_2) \vec{j}] + \mathcal{O}(\varepsilon^{\frac{2}{2}})
\end{aligned}
\tag{3.97}
$$

It has been mentioned in **Section 2.4.2** that the matching condition (3.97) requires for $\varepsilon \to 0$ that solutions for the velocity field as one moves out of the smaller scale, i.e. $r \to \infty$, have to be equal to the solutions for the velocity field as one moves into the smaller region, i.e. $\check{r} \to 0$ with $\check{r}$ given through (3.93). Since the velocity fields in (3.97) have been expressed in terms of inner coordinates r, collecting terms in corresponding powers of ε in (3.97) gives

$$
\varepsilon^{-1}: \qquad Z_{s,1k} \ = \ 0 \, , \qquad k = 1,2
\tag{3.98}
$$

$$\varepsilon^{-\frac{1}{2}}: \qquad u_\theta^{(0)} = \frac{\Gamma(z)}{2\pi r} \qquad \text{as} \quad r \to \infty \tag{3.99}$$

$$\begin{aligned}\varepsilon^{0}: \quad (u_r^{(\frac{1}{2})}\vec{e}_r + u_\theta^{(\frac{1}{2})}\vec{e}_\theta) &= \left(U_{B,C}^{(0)} + U_{R,C}^{(0)} - U_C^{(0)}\right)\vec{i} \; + \\ &\quad \left(V_{B,C}^{(0)} + V_{R,C}^{(0)} - V_C^{(0)}\right)\vec{j}\end{aligned} \tag{3.100}$$

$$\begin{aligned}\varepsilon^{\frac{1}{2}}: \quad (u_r^{(\frac{2}{2})}\vec{e}_r + u_\theta^{(\frac{2}{2})}\vec{e}_\theta) &= \left(U_{B,C}^{(\frac{1}{2})} - U_C^{(\frac{1}{2})} + F_2\right)\vec{i} \; + \\ &\quad \left(V_{B,C}^{(\frac{1}{2})} - V_C^{(\frac{1}{2})} + F_1\right)\vec{j}\end{aligned} \tag{3.101}$$

as $r \to \infty$. Using the relations $\vec{i} = (\vec{e}_r \cos\theta - \vec{e}_\theta \sin\theta)$ and $\vec{j} = (\vec{e}_r \sin\theta + \vec{e}_\theta \cos\theta)$ the matching condition (3.100) becomes

$$\begin{aligned}u_r^{(\frac{1}{2})} &= +\left(U_{B,C}^{(0)} + U_{R,C}^{(0)} - U_C^{(0)}\right)\cos\theta + \left(V_{B,C}^{(0)} + V_{R,C}^{(0)} - V_C^{(0)}\right)\sin\theta \\ u_\theta^{(\frac{1}{2})} &= -\left(U_{B,C}^{(0)} + U_{R,C}^{(0)} - U_C^{(0)}\right)\sin\theta + \left(V_{B,C}^{(0)} + V_{R,C}^{(0)} - V_C^{(0)}\right)\cos\theta\end{aligned} \tag{3.102}$$

as $r \to \infty$. Harmonic analysis (see **Section 3.1.4**) of (3.102) yields for the first harmonics

$$\begin{aligned}u_{\theta,11}^{(\frac{1}{2})} &= -\, u_{r,12}^{(\frac{1}{2})} = -\left(U_{B,C}^{(0)} + U_{R,C}^{(0)} - U_C^{(0)}\right) \\ u_{\theta,12}^{(\frac{1}{2})} &= +\, u_{r,11}^{(\frac{1}{2})} = +\left(V_{B,C}^{(0)} + V_{R,C}^{(0)} - V_C^{(0)}\right)\end{aligned} \tag{3.103}$$

as $r \to \infty$. Same procedure for the next higher order flow yields

$$\begin{aligned}u_{\theta,11}^{(\frac{2}{2})} &= -\, u_{r,12}^{(\frac{2}{2})} = -(U_{B,C}^{(\frac{1}{2})} - U_C^{(\frac{1}{2})}) \; + \\ X_C^{(\frac{1}{2})} &\left.\frac{\partial(V_B^{(0)} + V_R^{(0)})}{\partial\eta_2}\right|_{\vec{\eta}=\vec{X}_C^{(0)}} - Y_C^{(\frac{1}{2})} \left.\frac{\partial(U_B^{(0)} + U_R^{(0)})}{\partial\eta_2}\right|_{\vec{\eta}=\vec{X}_C^{(0)}}\end{aligned} \tag{3.104}$$

and

$$\begin{aligned}u_{\theta,12}^{(\frac{2}{2})} &= +\, u_{r,11}^{(\frac{2}{2})} = +(V_{B,C}^{(\frac{1}{2})} - V_C^{(\frac{1}{2})}) \; - \\ X_C^{(\frac{1}{2})} &\left.\frac{\partial(V_B^{(0)} + V_R^{(0)})}{\partial\eta_1}\right|_{\vec{\eta}=\vec{X}_C^{(0)}} + Y_C^{(\frac{1}{2})} \left.\frac{\partial(U_B^{(0)} + U_R^{(0)})}{\partial\eta_1}\right|_{\vec{\eta}=\vec{X}_C^{(0)}}\end{aligned} \tag{3.105}$$

as $r \to \infty$.

Remark: Rearranging terms in (3.103) one obtains equations for the leading order vortex motion in terms of its leading order centreline motion, i.e.

$$
\begin{aligned}
U_C^{(0)}(\tau) &= U_{B,C}^{(0)}(z) + U_{R,C}^{(0)}(z) + u_{\theta,11}^{(\frac{1}{2})}(r,z,\tau) \quad \text{as} \quad r \to \infty \\
V_C^{(0)}(\tau) &= V_{B,C}^{(0)}(z) + V_{R,C}^{(0)}(z) - u_{\theta,12}^{(\frac{1}{2})}(r,z,\tau) \quad \text{as} \quad r \to \infty
\end{aligned}
\tag{3.106}
$$

From (3.106) it is observed that the vortex motion in leading order is composed of three contributions, namely the background flow, the regular flow and asymmetric components of the leading order vortex flow. One of the objectives of the asymptotic analysis in **Chapter 4** and **5** is to show that there are solutions for $u_{\theta,1k}^{(\frac{1}{2})}$ with $k = 1, 2$ that have contributions that do not disappear for large r. The advantage of using matched asymptotics techniques is seen by comparing the centreline velocity obtained by this method (see (3.106)) and the results obtained by Reznik's (1992) approach (see (3.83)). The difference lies in such kind of asymmetric contributions of the next higher order flow characterising the mesoscale vortex structure. Thus, it turns out that the method of matched asymptotic expansions allows us to account for additional effects on the vortex motion that are due to the mesoscale flow of the vortex itself. It is further shown in **Chapter 4** and **5** that (3.106) is not only an equation for the vortex motion but also it imposes additional constraints on both the vertical structure of the vortex and the nature of the background flow for concentrated vortices to exist.

3.3.2 Potential temperature field

Using the outer expansion (3.72) and the fact that in the inner expansion the potential temperature is only a function on the vertical coordinate z up to the 5/2th order (see (3.36)), a specification of the general matching condition (3.86) for the potential temperature yields

$$
\begin{aligned}
&\Theta^{(0)}(z) + \varepsilon^{\frac{1}{2}}\Theta^{(\frac{1}{2})}(z) + ... + \varepsilon^{\frac{6}{2}}\Theta^{(\frac{6}{2})}(r,\theta,z) + \varepsilon^{\frac{7}{2}}\Theta^{(\frac{7}{2})}(r,\theta,z) + \mathcal{O}(\varepsilon^{\frac{8}{2}}) = \\
&\qquad \check{\Theta}_\infty + \varepsilon^2\check{\Theta}^{(2)}(z) + \varepsilon^3\check{\Theta}^{(3)}(\vec{X}_C(z,\tau) + \varepsilon^{\frac{1}{2}}\vec{\tilde{x}}, z, \tau) + \mathcal{O}(\varepsilon^4)
\end{aligned}
\tag{3.107}
$$

with $\tau = \varepsilon^2 t$. Note that here the inner expansions have already been expressed in terms of cylindrical coordinates (see (3.9)). Due to the horizontal homogeneity in leading orders, collecting terms in corresponding powers of ε yields $\Theta^{(0)} = \Theta_\infty = \check{\Theta}_\infty$, $\Theta^{(\frac{i}{2})} = 0$ with $i = 1, 3, 5$ and $\Theta^{(\frac{4}{2})}(z) = \check{\Theta}^{(2)}(z)$ which denotes a background stratification. Next we use the hydrostatic equation (3.71) and the geostrophic stream function defined by (3.73) in order to derive the matching conditions for $\Theta^{(\frac{6}{2})}$ and $\Theta^{(\frac{7}{2})}$. Because of (3.71) and $(3.73)_2$ the potential temperature $\check{\Theta}^{(3)}$ is related to the geostrophic stream function $\check{\psi}_g^{(0)}$ through

$$\frac{\check{\Theta}^{(3)}}{\Omega_0} = -\frac{\partial \check{\psi}_g^{(0)}}{\partial z} \tag{3.108}$$

where $\check{\Theta}^{(3)} = \check{\Theta}^{(3)}(\vec{X}_C(z,\tau) + \varepsilon^{\frac{1}{2}}\hat{\vec{x}}, z, \tau)$ and $\check{\psi}_g^{(0)} = \check{\psi}_g^{(0)}(\vec{X}_C(z,\tau) + \varepsilon^{\frac{1}{2}}\hat{\vec{x}}, z, \tau)$. With the superposition $\check{\psi}_g^{(0)} = \check{\Psi}_B^{(0)} + \check{\psi}_s^{(0)} + \check{\psi}_r^{(0)}$ (see eqn. (3.76)) one finds by use of the singular vortex solution (3.81) together with the matching result (3.98) that

$$\frac{\check{\Theta}^{(3)}}{\Omega_0} = -\frac{\partial \check{\Psi}_B^{(0)}}{\partial z} + \frac{\ln \check{r}}{2\pi}\frac{\partial \Gamma}{\partial z} - \frac{\partial \check{\psi}_r^{(0)}}{\partial z} \tag{3.109}$$

Changing outer coordinates $\check{r}$ into inner coordinates r via (3.93) and with the aid of the Taylor expansion (3.90) of the background and regular flow around the leading order centreline position $(X_C^{(0)}, Y_C^{(0)})$, one obtains

$$\frac{\check{\Theta}^{(3)}}{\Omega_0} = \frac{\partial \Gamma}{\partial z}\frac{\ln r}{2\pi} + \frac{\partial \Gamma}{\partial z}\frac{\ln \varepsilon}{4\pi} - \frac{\partial \check{\Psi}_{B,C}^{(0)}}{\partial z} - \frac{\partial \check{\psi}_{r,C}^{(0)}}{\partial z} + \mathcal{O}(\varepsilon^{\frac{1}{2}}) \tag{3.110}$$

where $\check{\Psi}_{B,C}^{(0)} = \check{\Psi}_B^{(0)}(\vec{X}_C^{(0)}, z, \tau)$ and $\check{\psi}_{r,C}^{(0)} = \check{\psi}_r^{(0)}(\vec{X}_C^{(0)}, z, \tau)$. Thus, substituting (3.110) into the matching condition (3.107) and collecting terms in corresponding powers of ε gives in the limit $r \to \infty$

$$\begin{aligned} \varepsilon^{\frac{6}{2}} \quad &: \quad \Theta^{(\frac{6}{2})}(r,\theta,z) = \Omega_0 \frac{\partial \Gamma}{\partial z}\frac{\ln r}{2\pi} - \Omega_0 \frac{\partial \check{\Psi}_{B,C}^{(0)}}{\partial z} - \Omega_0 \frac{\partial \check{\psi}_{r,C}^{(0)}}{\partial z} \\ \varepsilon^{\frac{7}{2}} \quad &: \quad \Theta^{(\frac{7}{2})}(r,\theta,z) = \Omega_0 \frac{\partial}{\partial z}\Big(M\ (V_{B,C}^{(0)} + V_{R,C}^{(0)}) - N\ (U_{B,C}^{(0)} + U_{R,C}^{(0)})\Big) \end{aligned} \tag{3.111}$$

with $M = (X_C^{(\frac{1}{2})} + r\cos\theta)$, $N = (Y_C^{(\frac{1}{2})} + r\sin\theta)$ and $\check{\Psi}_{B,C}^{(0)} = \check{\Psi}_B^{(0)}(\vec{X}_C^{(0)}, z, \tau)$, $\check{\psi}_{r,C}^{(0)} = \check{\psi}_r^{(0)}(\vec{X}_C^{(0)}, z, \tau)$ evaluated at the leading order vortex centreline position.

3.3.3 Pressure field

Taking into account that the inner ad outer pressure solutions are horizontally homogeneous up to the 5/2th order (see (3.33) and (3.70)), a specification of the general matching condition (3.86) for the pressure fields yields

$$\begin{aligned} p^{(0)}(z) + \varepsilon^{\frac{1}{2}} p^{(\frac{1}{2})}(z) + ... + \varepsilon^{\frac{6}{2}} p^{(\frac{6}{2})}(r,z) + \varepsilon^{\frac{7}{2}} p^{(\frac{7}{2})}(r,\theta,z) + \mathcal{O}(\varepsilon^{\frac{8}{2}}) = \\ \check{p}^{(0)}(z) + \varepsilon^2 \check{p}^{(2)}(z) + \varepsilon^3 \check{p}^{(3)}(\vec{X}_C(z,\tau) + \varepsilon^{\frac{1}{2}}\hat{\vec{x}}, z, \tau) + \mathcal{O}(\varepsilon^4) \end{aligned}$$

Collecting terms in corresponding powers of ε gives $p^{(0)}(z) = \check{p}^{(0)}(z)$, $p^{(\frac{i}{2})} = 0$ with $i = 1,3,5$ and $p^{(\frac{4}{2})}(z) = \check{p}^{(2)}(z)$. Moreover, using the procedure as in **Section 3.3.2**, together with the relation $(3.73)_2$ yields

$$
\begin{aligned}
\varepsilon^{\frac{6}{2}} \quad &: \quad p^{(\frac{6}{2})}(r,z) = \Omega_0 \; \rho^{(0)} \left(\frac{\Gamma}{2\pi} \ln r - \check{\Psi}^{(0)}_{B,C} - \check{\psi}^{(0)}_{r,C} \right) \\
\varepsilon^{\frac{7}{2}} \quad &: \quad p^{(\frac{7}{2})}(r,\theta,z) = \Omega_0 \; \rho^{(0)} \left(M(V^{(0)}_{B,C} + V^{(0)}_{R,C}) - N(U^{(0)}_{B,C} + U^{(0)}_{R,C}) \right)
\end{aligned}
\tag{3.112}
$$

as $r \to \infty$.

3.3.4 Vertical velocity field

Using the asymptotic results (3.31) - (3.32) derived so far, together with the outer expansion (3.68) the matching condition for the vertical velocity can be written as

$$
\begin{aligned}
\varepsilon^{\frac{3}{2}} w^{(\frac{4}{2})}(r,\theta,z,\tau) + \varepsilon^{\frac{4}{2}} w^{(\frac{5}{2})}(r,\theta,z,\tau) + \mathcal{O}(\varepsilon^{\frac{5}{2}}) = \\
\varepsilon^{3} \check{w}^{(3)}(\vec{X}_C(z,\tau) + \varepsilon^{\frac{1}{2}} \vec{x}, z, \tau) + \mathcal{O}(\varepsilon^{4})
\end{aligned}
$$

Collecting terms in corresponding powers of ε gives

$$
\varepsilon^{\frac{i}{2}} : \quad w^{(\frac{i+1}{2})}(r,\theta,z,\tau) = 0 \quad , \; i = 3,4,5 \qquad \text{as} \quad r \to \infty \tag{3.113}
$$

3.4 Formulation of diabatic source terms and its expansions

The potential temperature source term S on the right side of $(2.19)_4$ includes heating sources due to different kind of diabatic effects such as radiative heating and condensation heating. Moreover, heating effects due to turbulent heat fluxes and molecular transport are included. The manner in which the sources operate in a fluid depends strongly upon the fields of motion and temperature of the fluid itself. Unfortunately, details of these processes are poorly understood due to the complex interactions of processes covering a range of scales from microscale to large scales. That is why parameterizations describing net heating effects of unresolved physical processes in terms of resolvable variables are a popular tool to overcome these difficulties.

One of the issues addressed in this thesis is the impact that moisture related processes have on the vortex motion and structure. Thus, it is assumed that

$$
S(r,\theta,z,\tau) = S_L(r,\theta,z,\tau) \tag{3.114}
$$

where S_L denotes a heat source describing heating effects due to phase changes of water such as latent heat release due to condensation. In classical meteorological modelling one differentiates between the nature of the condensation processes while including the heating effects due to moisture conversion in large scale atmospheric models (Holton, 1992). That is why substantial differences have to be accounted for in the mathematical formulation of heating processes due to latent heat release by large scale vertical motion (hereafter indicated by S_{rs}) on the one hand, and net large scale heating effects resulting from cooperative action of many small scale convecting cumulus cloud (hereafter indicated by S_{us}) on the other hand. From that point of view (3.114) can be decomposed into

$$S = S_L = S_{rs} + S_{us} \tag{3.115}$$

Two different perceptions are possible in implementing (3.115) in the asymptotic analysis in the present work. In analogy to the work of Hack & Schubert (1986) the first possibility is to assume that (3.115) is given in terms of an externally prescribed source term. This is discussed in **Section 3.4.1**. Another possibility is to describe (3.115) via an explicit inclusion of moisture parameter which is discussed in **Section 3.4.2**.

Remark: Interactions between processes acting on different scales are captured in a systematic way by use of appropriate multi-scale asymptotic expansions in an unified approach framework. Thus, net heating effects of cumulus convection (small scale) in terms of S_{us} are actually *not* included in the source (3.114), since the choosen asymptotic expansion ansatz in this work does not resolve small cumulus scales. Note that the expansion ansatz (3.7) used in the present work resolves only vortex scales. But it is expected that with the aid of sublinear growth conditions[4] a systematic multi-scale asymptotic analysis that accounts for both large vortex scales and small cumulus scales, would yield asymptotic equations that look similar to the equations resulting from a single-scale expansion ansatz for the dependent thermodynamic variables. The difference that would occur are additive expressions denoting horizontal averages over the small cumulus scales appearing within the equations which in summary represent heating effects S_{us}. It is beyond the scope of the present work to study the interactions between vortex scales and cumulus scales in large detail using multi-scale expansions, but a crude way of studying the potential impact of the smaller scale heating effects due to small scale cumulus convection on the vortex motion and structure on the basis of single scale expansions is to assume that $S_{\mathrm{us}} \neq 0$ without knowing the details regarding its representation.

[4]A standard technique in multi-scale asymptotics to separate long wave and short wave components. For further details see Klein (2004).

3.4.1 External forcing

The heat source on the right and side of S_{us} can be regarded as an external or internal forcing. In this subsection we treat the diabatic source term externally. This allows one to use dry thermodynamics instead of complex moist thermodynamics as discussed in the next subsection.

It is important to point out that the external forcing method is only suited to applications when the latent heat release is a product of cumulus convection rather than explicitly resolved vertical motion (Nielson-Gammon & Keyser, 1999). Thus, considering the case of an external forcing it is assumed that the heating effect S_{rs} due to explicitly resolved vertical motions is negligible compared to S_{us}. This means that (3.115) simplifies to

$$S = S_L = S_{us} \tag{3.116}$$

An external forcing as suggested by Hack & Schubert (1986) is used and which in dimensional form is specified by

$$\tilde{Q} = a\,\beta(z)\,\frac{r}{r_0}\ \exp\left(d\left[1-\left(\frac{r}{r_0}\right)^2\right]\right) \tag{3.117}$$

with $\beta(z) = \tilde{Q}_1\ \sin(\pi\ z/h_{sc})\exp(-\alpha\ z/h_{sc})$ denoting a vertical heating profile. In particular, the function $\beta(z)$ is an analytic approximation to the apparent heat source obtained by Yanai et. al. (1973). Note that the nondimensional expression of (3.117) is equivalent to S_{us} with regard to the vortex setting used in the present work. The following values for the constants $\tilde{Q}_1, \alpha, a$ etc. as suggested by Hack & Schubert (1986) are choosen. The heating rate is $\tilde{Q}_1 = 7.87$ K /day, $d = 1/2$ and $\alpha = 0.554$ places the maximum of heating in the middle of the troposphere. The constant a is a normalization coefficient determined by

$$a = (r_1/r_0)^2\left[1-\exp\left(-(r_1/r_0)^2\right)\right]^{-1}\ , \tag{3.118}$$

that enforces the horizontally averaged heating inside radius r_1 to be equal to $\beta(z)$. Note, we choose $r_1 = 300$ km and $r_0 = \varepsilon^{-\frac{3}{2}}h_{\text{sc}}$, which yields $a = 2.5$. Recall that for the nondimensionalization of the diabatic source term $\tilde{Q}$ in (2.4) the reference quantities $T_{\text{ref}} = p_{\text{ref}}/(R\rho_{\text{ref}}) \sim 300$ K and $t_{\text{ref}} = h_{\text{sc}}/u_{\text{ref}} \sim 20$ min have been used (see **Section 2.2**). Thus, typical heating rates ranging around $\sim 20'000$ K/day have been assumed. However, order of magnitude estimates of (3.117) yield $\tilde{Q} \sim a\ \tilde{Q}_1 \sim 20$ K/day. It can easily be verified that the two heating rates mentioned above differ by a factor $\varepsilon^{\frac{7}{2}}$. Hence, according to the source term expansion (3.29) the external heating source specified by (3.117) is only suitable to represent diabatic heating sources given by $S^{(\frac{7}{2})}$. Note that for the vortex setting used in the present work the source term $S^{(\frac{7}{2})}$ is the first nonzero source term (see eqn. (3.37)). In particular it turns out that the nondimensional form of the source (3.117) takes the form

$$S^{(\frac{7}{2})} \;=\; S_L^{(\frac{7}{2})} \;=\; \sin(\pi z)\;\exp\left(-\alpha\; z\right)\; r\;\; \exp\left(d\;\;\left[1-r^2\right]\right) \tag{3.119}$$

where r and z denote dimensionless coordinates of $\mathcal{O}(1)$.

3.4.2 Explicit treatment of moisture

A more physical alternative to account for diabatic effects is based on an explicit inclusion of moisture. Motivated by their interests in convective clouds, Klein & Majda (2006) developed an extended version of the unified approach to meteorological modelling that accounts for moist physics. Here, the bulk microphysics parameterizations of moist processes from Grabowski (1998) provided the basis for such an extension. The implementation of the latter into the asymptotic framework as described in **Chapter 2** was accompanied by a careful nondimensionalization of the bulk microphysics equations and the choice of appropriate distinguished limits for the small parameters that appear.

For the explicit treatment of moisture in the present work, the asymptotic framework accounting for moist physics from Klein & Majda (2006) is used. For the sake of simplicity, the studies are restricted to vortices in an idealized, saturated model atmosphere. Under that assumption the resolved contribution S_{rs} of the nondimensional diabatic source term (3.115) describes diabatic heating due to condensation-evaporation of cloud water (no evaporation of precipitation) and takes the form

$$S_{\mathrm{rs}} = -\frac{\gamma-1}{\gamma}\frac{\rho\;\Theta}{p}L^{\star}q_{vs}^{\star}\frac{dq_{vs}}{dt} \tag{3.120}$$

Here $q_{vs} = q_{vs}(\Theta, p)$ denotes a scaled saturated water vapor mixing ratio[5] and $q_{vs}^{\star}$ is a constant denoting the saturation value for the water vapor mixing ratio at some reference conditions. The constant $L^{\star} = L/(p_{\mathrm{ref}}/\rho_{\mathrm{ref}})$ is a value for the nondimensionalized latent heat per unit mass of water vapor. The superscripts $(.)^{\star}$ indicate that the dimensionless constants are still unscaled, i.e. they are generally functions of ε. With the knowledge of typical orders of magnitude for the constant parameters gained from different textbooks, distinguished limits have been introduced by Klein & Majda (2006) which relate $L^{\star}$, $q_{vs}^{\star}$ (and further constant parameters that appear in the nondimensionalized bulk microphysics parameterizations) via the small expansion parameter ε to each other. The

[5] The water vapor mixing ratio is defined as the ratio of the water vapor density versus that of the dry air.

limits for $L^\star$ and $q_{vs}^\star$ read

$$q_{vs}^\star = \varepsilon^2 q_{vs}^{\star\star} \qquad \text{and} \qquad L^\star = \frac{1}{\varepsilon} L^{\star\star} \tag{3.121}$$

Here, $q_{vs}^{\star\star}$ and $L^{\star\star}$ denote scaled dimensionless constants so that $q_{vs}^{\star\star}, L^{\star\star} = \mathcal{O}(1)$ as $\varepsilon \to 0$. Then, with the aid of (3.121) and together with the Newtonian limit (2.18) the source term (3.120) can be rewritten into

$$S_{\rm rs} = -\varepsilon^2\ \Gamma^{\star\star} L^{\star\star} q_{vs}^{\star\star} \frac{\rho\ \Theta}{p} \frac{dq_{vs}}{dt} \tag{3.122}$$

For an asymptotic analysis that accounts for heating effects determined through (3.122), an asymptotic expansion for $S_{\rm rs}$ is required. Based on a careful nondimensionalization of the equation for the saturation vapor mixing ratio and Boltons formula that approximates the saturation vapor pressure, Klein & Majda (2006) suggest the following formulation of the saturation mixing ratio appropriated for the purposes of asymptotic analysis

$$q_{vs}(\Theta, p) = \frac{1}{p} \exp\left(\frac{A^{\star\star}}{\varepsilon} \frac{T(\theta,p) - 1}{1 + (T(\theta,p) - 1 - \varepsilon T_1^{\star\star(1)})} \right) \tag{3.123}$$

Here, $A^{\star\star}$ and $T_1^{\star\star(1)}$ are constants of $\mathcal{O}(1)$. With the aid of the potential temperature definition (2.16) and the Newtonian limit (2.18) an equation for the temperature $T(\theta, p)$ reads

$$T(\theta, p) = \theta\ p^{\frac{\gamma-1}{\gamma}} = \theta\ p^{\varepsilon\Gamma} \tag{3.124}$$

Upon substitution of (3.124) into (3.123) and after a number of manipulations (see **Appendix A.6**), the right hand side of (3.123) can be rewritten to obtain

$$q_{vs} = q_{vs}^{(0)}(z) + \varepsilon q_{vs}^{(1)}(z) + \mathcal{O}(\varepsilon^2) \tag{3.125}$$

where

$$q_{vs}^{(0)}(z) = \frac{1}{p^{(0)}} \exp\left(A^{\star\star}\Gamma^{\star\star} \ln p^{(0)}\right) \tag{3.126}$$

and

$$q_{vs}^{(1)}(z) = -\left((1 + A^{\star\star}\Gamma^{\star\star}) \frac{p^{(1)}}{p^{(0)2}} + \frac{\tilde{\mu}^{(0)}}{p^{(0)}} \right) \exp\left(A^{\star\star}\Gamma^{\star\star} \ln p^{(0)}\right) \tag{3.127}$$

with $\tilde{\mu}^{(0)} = A^{\star\star}\{\Gamma^{\star\star} \ln p^{(0)}\ (\Gamma^{\star\star} \ln p^{(0)} + T_1^{\star\star(1)}) - \frac{\Gamma^{\star\star 2}}{2}(\ln p^{(0)})^2 - \Theta^{(2)}\}$ and where $p^{(0)} = p^{(0)}(z),\ p^{(1)} = p^{(1)}(z)$ and $\Theta^{(2)} = \Theta^{(2)}(z)$ (see (3.33) and (3.36)). Note, until $\mathcal{O}(\varepsilon^2)$ there appear no square roots of ε within the expansion (A-99).

Thus, it is concluded that

$$q_{vs}^{(\frac{i}{2})} = 0 \qquad \text{for} \qquad i = 1, 3 \tag{3.128}$$

With the above expansion for the saturation water vapor mixing ratio asymptotic expansions for $S_{\rm rs}$ with respect to the co-moving frame of reference used in the present work can be derived. Recall that within the co-moving frame of reference whose origin is located at the vortex centre $\vec{X}_C$ and whose expansion is of the form $\vec{X}_C = \vec{X}_C^{(0)}(\tau) + \varepsilon \vec{X}_C^{(\frac{1}{2})}(z,\tau) + \mathcal{O}(\varepsilon^{\frac{2}{2}})$, the substantial derivative $d/dt = \partial/\partial t + \vec{v}_h \cdot \vec{\nabla}_h + w\partial/\partial z$ in (3.122) takes the form (see the transformations (3.11) and (3.5))

$$\frac{d}{dt} = \varepsilon^2 \frac{\partial}{\partial \tau} + \vec{v}_{rel} \cdot \varepsilon^{\frac{3}{2}} \tilde{\vec{\nabla}}_h + w \frac{\partial}{\partial z} - \varepsilon^{-\frac{1}{2}} w \frac{\partial \vec{X}_C}{\partial z} \cdot \tilde{\vec{\nabla}}_h \tag{3.129}$$

with $\tilde{\vec{\nabla}}_h = \vec{e}_r \; \partial/\partial r + \vec{e}_\theta \; r^{-1}\partial/\partial\theta$. Due to (3.31) and (3.32) the asymptotic expansion $(3.19)_3$ for the vertical velocity w simplifies to $w = \varepsilon^{\frac{3}{2}} w^{(\frac{4}{2})} + \mathcal{O}(\varepsilon^{\frac{4}{2}})$. Thus, the substantial derivative (3.129) applied to (3.128) simplifies in leading orders to

$$\begin{aligned} \frac{dq_{vs}}{dt} &= \varepsilon^{\frac{3}{2}} w^{(\frac{4}{2})} \frac{dq_{vs}^{(0)}}{dz} + \varepsilon^{\frac{4}{2}} w^{(\frac{5}{2})} \frac{dq_{vs}^{(0)}}{dz} + \\ & \quad \varepsilon^{\frac{5}{2}} \left(w^{(\frac{6}{2})} \frac{dq_{vs}^{(0)}}{dz} + w^{(\frac{4}{2})} \frac{dq_{vs}^{(1)}}{dz} \right) + \mathcal{O}(\varepsilon^{\frac{6}{2}}) \end{aligned} \tag{3.130}$$

Based on the inner expansions (3.15) and (3.18) together with the matching results $\rho^{(\frac{1}{2})} = 0$, $p^{(\frac{1}{2})} = 0$ (see **Section 3.3.1**), Taylor series can be used to expand $\Theta\rho/p$, yielding

$$\begin{aligned} \frac{\Theta\rho}{p} &= \frac{\rho^{(0)}}{p^{(0)}} + \varepsilon \left(-\frac{\rho^{(0)} p^{(1)}}{p^{(0)2}} + \frac{\rho^{(1)}}{p^{(0)}} \right) + \mathcal{O}(\varepsilon^2) \\ &= 1 + \varepsilon \Gamma^{\star\star} z + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{3.131}$$

Note that the last equality can be derived by use of (A-97)-(A-98). Thus, substituting (3.131) and (3.130) into (3.122), an expansion for $S_{\rm rs}$ reads

$$S_{\rm rs} = \varepsilon^{\frac{7}{2}} S_{\rm rs}^{(\frac{7}{2})} + \varepsilon^{\frac{8}{2}} S_{\rm rs}^{(\frac{8}{2})} + \varepsilon^{\frac{9}{2}} S_{\rm rs}^{(\frac{9}{2})} + \mathcal{O}(\varepsilon^5) \tag{3.132}$$

where

$$
\begin{aligned}
S_{\mathrm{rs}}^{(\frac{7}{2})} &= -\Gamma^{\star\star} L^{\star\star} q_{vs}^{\star\star}\ w^{(\frac{4}{2})} \frac{dq_{vs}^{(0)}}{dz} \\
S_{\mathrm{rs}}^{(\frac{8}{2})} &= -\Gamma^{\star\star} L^{\star\star} q_{vs}^{\star\star}\ w^{(\frac{5}{2})} \frac{dq_{vs}^{(0)}}{dz} \\
S_{\mathrm{rs}}^{(\frac{9}{2})} &= -\Gamma^{\star\star} L^{\star\star} q_{vs}^{\star\star} \left(w^{(\frac{6}{2})} \frac{dq_{vs}^{(0)}}{dz} + (\Gamma^{\star\star} z)\ w^{(\frac{4}{2})} \frac{dq_{vs}^{(0)}}{dz} \right)
\end{aligned}
\tag{3.133}
$$

It can be observed, that source terms are proportional to the upward motion. This is what Nielsen-Gammon & Keyser (1999) call the Effective Stratification approach. Note that so far it has only been shown how heating effects due to vortex-scale forced uplift can be included in the asymptotic analysis. Since in the present work only single-scale expansion for vortex scales are used, the unresolved heating contribution S_{us} in (3.115) needs to be parameterized in terms of resolved variables. However, if multi-scale expansion for both the vortex scale and cumulus scales would have been employed, same procedure as described above would also give explicit expressions for S_{us} in terms of flow and moisture related variables resolving cumulus scales.

Finally, without going into details, Klein & Majda (2006) concluded from the estimates for typical CAPE[6] values that the background stratification of potential temperature which is nontrivially affected by moist processes, must satisfy

$$
\frac{d\Theta^{(\frac{4}{2})}}{dz} = -\Gamma^{\star\star} L^{\star\star} \frac{dq_{vs}^{(0)}}{dz} \tag{3.134}
$$

with $\Theta^{(4/2)} = \Theta^{(4/2)}(z)$ and $q_{vs}^{(0)} = q_{vs}^{(0)}(z)$.

3.5 General balance conditions

So far, the discussions in the **Sections 3.1 - 3.4** provide a complete basis for a detailed asymptotic analysis of adiabatic vortices ($S = 0$) and diabatic vortices ($S \neq 0$) which follows in **Chapter 4** and **Chapter 5**, respectively. There are, however, vortex flow conditions that are valid for both adiabatic and diabatic vortices which is discussed in the section below.

[6]Convective available potential energy (CAPE) provides a measure of the maximum possible kinetic energy that a statically unstable parcel can acquire (neglecting effects of water vapor and condensed water on the buoyancy), assuming that the parcel ascends without mixing with the enviroment and instantaneously adjusts to the local environmental pressure (Holton, 1992).

Gradient wind relation One of the first balance condition coming out from the asymptotic approach is the gradient wind relation $(3.39)_1$, i.e.

$$\frac{1}{\rho^{(0)}}\frac{\partial p^{(\frac{6}{2})}}{\partial r} - \frac{u_\theta^{(0)^2}}{r} - \Omega_0 u_\theta^{(0)} = 0 \tag{3.135}$$

This relation describes a three-way balance between the pressure gradient force $\partial p^{(\frac{6}{2})}/\partial r$, the centrifugal force $u_\theta^{(0)^2}/r$ and the Coriolis force $\Omega_0 u_\theta^{(0)}$. The gradient wind is a wind that blows parallel to curved isobars $p^{(\frac{6}{2})} = \mathit{const}$.

Hydrostatics for the first few pressure terms With $p^{(\frac{6}{2})} = p^{(\frac{6}{2})}(r, z, \tau)$ (see $(3.39)_2$) and $\rho^{(\frac{6}{2})} = \rho^{(\frac{6}{2})}(r, \theta, z, \tau)$ integration of the vertical momentum equation (3.46) with respect to θ from 0 to 2π yields

$$\frac{\partial p^{(\frac{6}{2})}}{\partial z} = -\rho_0^{(\frac{6}{2})} \tag{3.136}$$

Thus, the pressure term $p^{(\frac{6}{2})}$ which is relevant for the leading order vortex flow satisfies the hydrostatic balance condition. Note, as already shown in **Section 3.1.3** same holds for $p^{(\frac{i}{2})}$ with $i = 0, 2, 4$.

Leading order vortex tilt and background flow Taking into account that $p^{(\frac{7}{2})} = p^{(\frac{7}{2})}(r, \theta, z, \tau)$ the zeroth mode equation of the vertical momentum equation (3.47) reads

$$\frac{\partial p_0^{(\frac{7}{2})}}{\partial z} - \frac{\tilde{P}_{12}}{2}\frac{\partial X_C^{(\frac{1}{2})}}{\partial z} - \frac{\tilde{P}_{11}}{2}\frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} = -\rho_0^{(\frac{7}{2})} \tag{3.137}$$

with

$$\tilde{P}_{1k} = \left(\frac{\partial p_{1k}^{(\frac{7}{2})}}{\partial r} + \frac{p_{1k}^{(\frac{7}{2})}}{r}\right) \qquad k = 1, 2$$

Contrary to $p^{(\frac{6}{2})}$ it turns out that due to asymmetric pressure terms $p_{1k}^{(\frac{7}{2})}$ the axissymmetric pressure component $p_0^{(\frac{7}{2})}$ is no more a hydrostatic one. In particular it can be observed that the strength of the hydrostatic imbalance depends on both $p_{1k}^{(\frac{7}{2})}$ and the vortex tilt $\partial \vec{X}_C^{(\frac{1}{2})}/\partial z = (\partial X_C^{(\frac{1}{2})}/\partial z,\ \partial Y_C^{(\frac{1}{2})}/\partial z)$. By means of matched asymptotics it will be shown next, that the strength of the hydrostatic imbalance governed by the vortex tilt can be related to environmental flow conditions. With the aid of the zeroth mode $\mathcal{O}(\varepsilon^{\frac{7}{2}})$ state equation of (A-39), i.e. $\rho_0^{(\frac{7}{2})}\Theta^{(0)} + \rho^{(0)}\Theta_0^{(\frac{7}{2})} = p_0^{(\frac{7}{2})}$ (with $\Theta^{(0)} = \Theta_\infty$ (see (3.107))), elimination of $\rho_0^{(\frac{7}{2})}$ in (3.137) yields

$$\frac{\partial p_0^{(\frac{7}{2})}}{\partial z} - \frac{\tilde{P}_{12}}{2}\frac{\partial X_C^{(\frac{1}{2})}}{\partial z} - \frac{\tilde{P}_{11}}{2}\frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} + \frac{p_0^{(\frac{7}{2})}}{\Theta_\infty} = \frac{\rho^{(0)}}{\Theta_\infty}\Theta_0^{(\frac{7}{2})} \tag{3.138}$$

The matching conditions $(3.111)_2$ and $(3.112)_2$ yield for the axissymmetric and first Fourier components of $p^{(\frac{7}{2})}$ and $\Theta^{(\frac{7}{2})}$ (see **Section 3.1.4**) in the limit as r approaches ∞

$$\begin{aligned}
p_0^{(\frac{7}{2})} &= \Omega_0\rho^{(0)}\left[X_C^{(\frac{1}{2})}\left(V_{B,C}^{(0)}+V_{R,C}^{(0)}\right) - Y_C^{(\frac{1}{2})}\left(U_{B,C}^{(0)}+U_{R,C}^{(0)}\right)\right] \\
p_{11}^{(\frac{7}{2})} &= -\frac{\Omega_0}{2}\,\rho^{(0)}\;r\;\left(U_{B,C}^{(0)}+U_{R,C}^{(0)}\right) \\
p_{12}^{(\frac{7}{2})} &= +\frac{\Omega_0}{2}\,\rho^{(0)}\;r\;\left(V_{B,C}^{(0)}+V_{R,C}^{(0)}\right) \\
\Theta_0^{(\frac{7}{2})} &= +\,\Omega_0\frac{\partial}{\partial z}\left[X_C^{(\frac{1}{2})}\left(V_{B,C}^{(0)}+V_{R,C}^{(0)}\right) - Y_C^{(\frac{1}{2})}\left(U_{B,C}^{(0)}+U_{R,C}^{(0)}\right)\right]
\end{aligned} \tag{3.139}$$

Substituting (3.139) into (3.138) for large r, yields by additional use of the leading order zeroth mode state equation $\rho^{(0)}\Theta_\infty = p^{(0)}$ (see **Appendix A.3**) and with $\Theta_\infty = 1$

$$\frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\left(U_{B,C}^{(0)}+U_{R,C}^{(0)}\right) = \frac{\partial X_C^{(\frac{1}{2})}}{\partial z}\left(V_{B,C}^{(0)}+V_{R,C}^{(0)}\right) \tag{3.140}$$

which can be rewritten as

$$\vec{X}_z^{(\frac{1}{2})} \times \;(\vec{V}_{B,C}^{(0)}+\vec{V}_{R,C}^{(0)}) \;=\; 0\;, \tag{3.141}$$

implying $\partial\vec{X}_C^{(\frac{1}{2})}/\partial z \parallel (\vec{V}_{B,C}^{(0)}+\vec{V}_{R,C}^{(0)})$. It can be observed that the direction of the leading order vortex tilt coincides with the direction of the background and regular flow. Note that for a pure zonal background flow, i.e. $\vec{V}_B = (U_B, 0)$ and in absence of any regular flow, i.e. $\vec{V}_R = 0$, it would follow immediately that $\partial Y_C^{(\frac{1}{2})}/\partial z = 0$ or $Y_C^{(\frac{1}{2})} = Y_C^{(\frac{1}{2})}(\tau)$. Hence, the result (3.140) indicates that the existence of a background (regular) flow may have a nontrivial effect on the vortex tilt. More detailed discussions on this issue follow in **Chapter 4** and **5**.

Chapter 4

Adiabatic Vortex

Attention of this chapter is focused on the leading and next higher order motion and structure of concentrated *adiabatic* vortices which are embedded within a background flow with vertical shear. The indication *adiabatic* in that context means that the influence of moisture effects on the vortex motion and structure will be neglected by equating diabatic sources equal to zero. More complex studies accounting for diabatic effects are carried out in **Chapter 5**.

The study of isolated effects of an environmental flow on the vortex motion and structure is motivated by works of Jones (1994), Wang & Holland (1996), Frank & Ritchie (1999), Schecter et al. (2002), Reasor & Montgomery (2001; 2004), and others. In particular, one major objective in their studies is to describe and to explain the influence of an environmental flow on the vortex tilt. Once a vertically sheared background flow is imposed, one would expect that the vortex becomes tilted and eventually shears away due to the differential advection. Numerical simulations for mature tropical cyclones carried out by Frank & Ritchie (1999), however, have shown that after 48 h and in the presence of a 5 m/s environmental shear throughout the troposphere, the vortex remained in its vertically upright position while a strongly asymmetric, quasi-steady vertical motion pattern was observed with maximum upward motion downshear left of the centre. The mechanism behind the generation of such asymmetries is refered to as the adiabatic lifting mechanism. A detailed discussion of this mechanism follows in **Section 4.2.1**.

The maintenance of a coherent vortex structure in absence of any diabatic effects has also been observed by Wang & Holland (1996). In particular a quasi-steady tilt to the downshear left was found after a 72 h period of simulation.

On an f-plane, Reasor & Montgomery (2001) observed a free[1] alignment of a

[1]Free alignment means that during the alignment phase the environmental shear has been turned off.

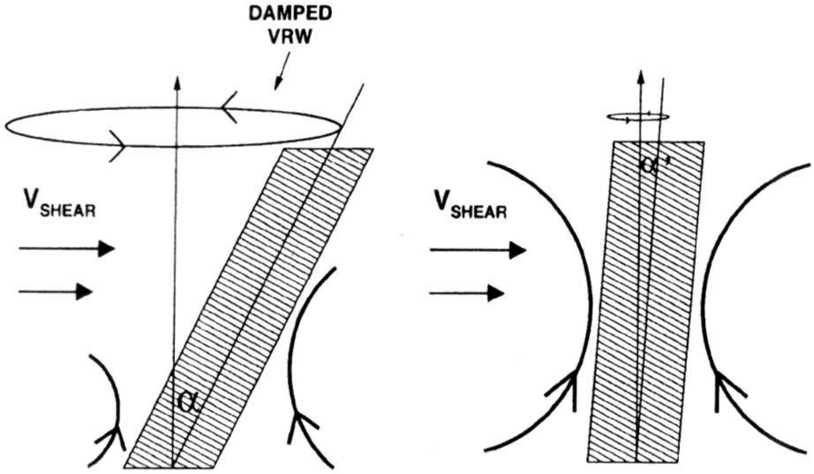

Figure 4.1: Schematic showing the resonant vortex Rossby wave damping mechanims (Reasor & Montgomery 2004, Fig.1)

quasigeostrophic vortex which initially has been tilted by a sheared background flow. Their observations were based upon numerical simulations valid for small Rossby number satisfying the use of quasigeostrophic dynamics. The simulations in turn were carried out using the concept of vortex Rossby wave (VRW) theory. Here a tilt perturbation is defined as a departure from a vertically averaged azimuthal mean component of a tilted potential vorticity (PV) column. Assuming the tilted PV column is vertically bounded by rigid lids the tilt perturbation is described in terms of barotropic and internal baroclinic modes. Depending on internal Rossby deformation radii larger or smaller than the horizontal vortex scale two different alignment mechanisms have been found. The manner in which an initially tilted vortex reached its upright position for internal Rossby deformation radii larger than the horizontal vortex scale is described by Schecter et al. (2002) in the following way: "In time, the orientation of the tilt rotates, while the amplitude of the vortex tilt decays ". Based on that observations Schecter et al. (2002) managed to derive a theory that explains this relaxation to an upright position by a resonant damping mode. Considering the vortex tilt perturbation in terms of an excited discrete vortex Rossby mode Schecter et al. (2002) have shown that the rotation frequency of this mode "is resonant with the flow rotation frequency at a critical radius r_c in the outer skirt of the vortex". Eventually that resonance has been identified to be responsible for an exponential decay with time of the vortex tilt "provided that the radial PV gradient is negative at r_c" (Schecter et al. 2002). **Figure 4.1** illustrates schematically the alignment mechanism described above, which is also called resonant VRW damping. Later on, Reasor & Montgomery (2004) extended their work to finite Rossby numbers Ro to study vortex regimes more characteristic for real tropical cyclones. Then, depending on Ro and the ratio of the horizontal scale of the vortex to the global internal Rossby deformation radius, i.e. $L/l_{r,G}$, they found that the tilt decay occured either via outward propagating sheared VRW disturbances or again via an inviscid damping mechanism intrinsic to the dry adiabatic dynamics (Mallen et al. 2004). Here, the deformation radius $l_{r,G}$ is defined as

$$l_{r,G} = \frac{Nh}{m\ \pi\ f} \tag{4.1}$$

whereas h denotes the height of the vortex, f is the Coriolis parameter, N the Brunt-Väisälä frequency and m is a vertical core mode number that dominates the vortex tilt. Note, the graphs in **Figure 4.2** taken from Reasor & Montgomery (2004), give an overview about which vortex regimes have been found to realign via sheared VRW's (labeled by S) or the VRW damping mechanism (labeled by Q). Based on the above findings Reasor & Montgomery (2004) argue that the diabatically driven secondary circulation observed in real tropical cyclones is not directly responsible for maintaining the vertical alignment of the vortex in presence of a vertically sheared background flow. Note that the background flow choosen in their studies was a weak to moderate unidirectional vertical shear flow with ambient vertical shear between 0 and 4 m/s per 10 km.

There are, however, numerical studies that yield results which are completely contrary to the results described right above. Comparing dry and moist numerical simulations Frank & Ritchie (1999) observed that diabatic vortices have a greater chance to withstand an imposed weak background shear than adiabatic ones. Considering initially barotropic vortice, Jones (1994) observed a rotation of the upper and lower-level vortex centres about the mid level centre, shortly after the vortex was tilted in the plane of the shear. The rotation, however, decreased in time, while the magnitude of the vortex tilt increased in time. Although the first observed rotation of the vortex centres about the mid level coincides with the observations made by Reasor & Montgomery (2004), the increase of the vortex tilt in time shows a completely contradictory vortex behaviour. Studies undertaken by Mallen et al (2004) to find the sources of the discrepancy between the simulations of Jones and Reasor & Montgomery, point out that a vortex realignment depends strongly on the initial radial structure of the vortex profile. In particular they showed that the idealized radial vortex velocity profile used by Jones does not exhibit the negative gradient at the critical radius r_c necessary for a resonant damping mechanism.

The primary goal of this chapter is to find out whether from an asymptotic

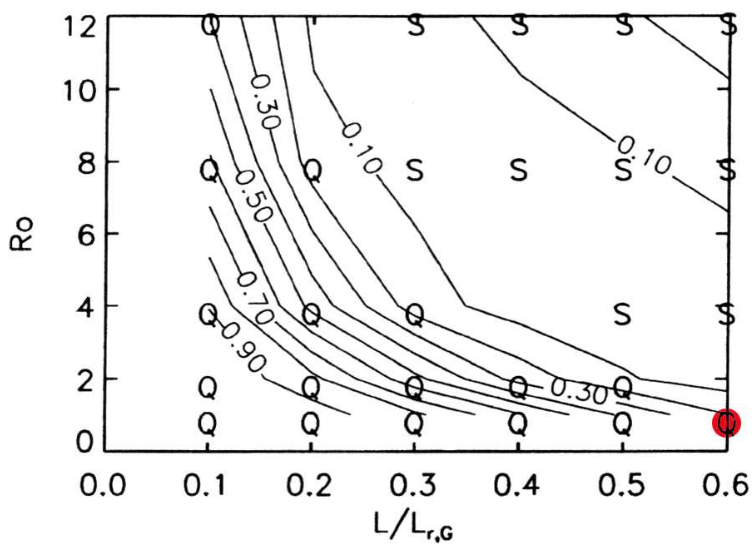

Figure 4.2: Schematic diagram showing the vortex regimes aligning either via resonant VRW damping denoted by "Q" or sheared VRW's denoted by "S" (Reasor & Montgomery 2004, Fig.3)

perspective results regarding the vertical vortex alignment mechanisms can be obtained, similar to those of Reasor & Montgomery. For the vortex regime studied in the present work with the horizontal vortex scale $\mathrm{L} = \varepsilon^{-\frac{3}{2}} h_{\mathrm{sc}} \sim 200$ km and the tangential velocity $U = \varepsilon^{-\frac{1}{2}} u_{\mathrm{ref}} \sim 30$ m/s, the Rossby number is $\mathrm{Ro} = \mathrm{Ro}_L = 0.625$. With $N = 10^{-2}\ \mathrm{s}^{-1}$, $h = h_{\mathrm{sc}} = 10$ km, $f \equiv \Omega_0 = 10^{-4}\ \mathrm{s}^{-1}$ and $m = 1$ the internal Rossby deformation radius is $\mathrm{l}_{r,G} \sim 300$ km, giving a typical value for the ratio between L and $\mathrm{l}_{r,G}$ that is $\mathrm{L}/\mathrm{l}_{r,G} \sim 0.67$. Thus, according to the graphs in **Figure 4.2** vortices studied in the present work should realign via an inviscid damping mechanism (Q). To elucidate the role of an environmental flow for the vortex tilt, the present work uses the definition of the vortex centreline $\vec{X}_C = (X_C, Y_C)$ to determine a vortex tilt. Because of the centreline expansion (3.20) vertical variations of the next higher order vortex centreline $(X_C^{(\frac{1}{2})}, Y_C^{(\frac{1}{2})})$ give a first approximation to the vortex tilt. Thus, in the rest of the work we refer to the vertical gradients $\partial X_C^{(\frac{1}{2})}/\partial z$ and $\partial Y_C^{(\frac{1}{2})}/\partial z$ as the vortex tilt. In particular it will be shown that by means of matched asymptotics, equations describing the temporal evolution of $X_C^{(\frac{1}{2})}$ and $Y_C^{(\frac{1}{2})}$ can be derived. Assuming a weak background flow, solutions of these equations describe a precession motion of a tilted vortex-centreline, which is in agreement with the observations made by Reasor & Montgomery on an f-plane. However, solutions describing a simultaneous realignment of an initially tilted vortex via resonant VRW waves can not be derived due to the slow time scales choosen in Ansatz (3.7). An asymptotic analysis including both fast and slow times scales is subject for the future work.

The outline of this chapter is as follows. In **Section 4.1** an overview about the relevant equations is given which constitute the basis for the derivation of solutions describing the structure and motion of adiabatic vortices up to the first orders. **Section 4.2** explains how wavenumber-one vertical velocities fields are related to the first order vortex tilt. Together with an analysis of appropriate potential temperature fields a mechanism known as the adiabatic lifting mechanism is recovered. In addition solutions describing the leading order vortex motion are derived by means of matched asymptotics. Provided that such solutions exist, the equation describing the leading order vortex motion can be used to explore the necessary environmental conditions for a concentrated vortex to survive. In **Section 4.3** solutions describing the second order horizontal asymmetric velocity fields are used to derive via matched asymptotics an evolution equation for the first order vortex centreline. Depending on the initial conditions different solutions are derived and discussed in comparison with the results obtained by Reasor & Montgomery (2001).

4.1 Governing equations

Unless otherwise stated a constant and statically stable background stratification is assumed for the derivations below, i.e. $\partial\Theta^{(2)}/\partial z = \sigma$ with $\sigma > 0$. Then, in absence of any heating from the $\mathcal{O}(\varepsilon^{\frac{7}{2}})$ thermodynamic equation (3.53) it follows immediately that $w^{(\frac{4}{2})} = 0$ and the relevant equations describing the leading and next higher order vortex structure are (see **Section 3.1.3**) listed below. Note that leading order vertical momentum and potential temperature equations does not change if $w^{(\frac{4}{2})} = 0$.

Vertical momentum equations:

$$\begin{aligned} \frac{\partial p^{(\frac{6}{2})}}{\partial z} - \Lambda_b^1 \frac{\partial p^{(\frac{6}{2})}}{\partial r} &= -\rho^{(\frac{6}{2})} \\ \frac{\partial p^{(\frac{7}{2})}}{\partial z} - \Lambda_b^1 \frac{\partial p^{(\frac{7}{2})}}{\partial r} - \Lambda_a^1 \frac{1}{r}\frac{\partial p^{(\frac{7}{2})}}{\partial \theta} - \Lambda_b^2 \frac{\partial p^{(\frac{6}{2})}}{\partial r} &= -\rho^{(\frac{7}{2})} \end{aligned} \tag{4.2}$$

Potential temperature equation:

$$\frac{u_\theta^{(0)}}{r}\frac{\partial \Theta^{(\frac{6}{2})}}{\partial \theta} + w^{(\frac{5}{2})}\frac{\partial \Theta^{(\frac{4}{2})}}{\partial z} = 0 \tag{4.3}$$

Horizontal momentum equations:

With $w^{(\frac{4}{2})} = 0$ one obtains from (3.43)

$$\begin{aligned}
\vec{e}_r : \qquad & \frac{u_\theta^{(0)}}{r}\frac{\partial u_r^{(\frac{1}{2})}}{\partial \theta} - \frac{2u_\theta^{(0)}u_\theta^{(\frac{1}{2})}}{r} + \frac{1}{\rho^{(0)}}\frac{\partial p^{(\frac{7}{2})}}{\partial r} + \Omega_0 \Pi_a^0 - \Omega_0 u_\theta^{(\frac{1}{2})} \;=\; 0 \\
\vec{e}_\theta : \qquad & u_r^{(\frac{1}{2})}\zeta^{(0)} + \frac{u_\theta^{(0)}}{r}\frac{\partial u_\theta^{(\frac{1}{2})}}{\partial \theta} + \frac{1}{r\,\rho^{(0)}}\frac{\partial p^{(\frac{7}{2})}}{\partial \theta} + \Omega_0 \Pi_b^0 + \Omega_0 u_r^{(\frac{1}{2})} \;=\; 0
\end{aligned} \tag{4.4}$$

where $\zeta^{(\frac{j}{2})} = (\partial u_\theta^{(\frac{j}{2})}/\partial r + u_\theta^{(\frac{j}{2})}/r)$ with i = 0,1,2,... denotes the relative vorticity with respect to the vertical. The next higher order momentum equations (3.44) and (3.45) simplify to

$$\begin{aligned}
\vec{e}_r : \quad & \Lambda_a^1 \; w^{(\frac{5}{2})}\frac{u_\theta^{(0)}}{r} + u_r^{(\frac{1}{2})}\frac{\partial u_r^{(\frac{1}{2})}}{\partial r} + \frac{u_\theta^{(\frac{1}{2})}}{r}\frac{\partial u_r^{(\frac{1}{2})}}{\partial \theta} + \frac{u_\theta^{(0)}}{r}\frac{\partial u_r^{(\frac{2}{2})}}{\partial \theta} - \frac{{u_\theta^{(\frac{1}{2})}}^2}{r} - \\
& \frac{2u_\theta^{(0)}u_\theta^{(\frac{2}{2})}}{r} + \frac{1}{\rho^{(0)}}\frac{\partial p^{(\frac{8}{2})}}{\partial r} - \frac{\rho^{(\frac{2}{2})}}{{\rho^{(0)}}^2}\frac{\partial p^{(\frac{6}{2})}}{\partial r} + \Omega_0 \Pi_a^1 - \Omega_0 u_\theta^{(\frac{2}{2})} - u_\theta^{(0)}\beta Y^{(0)} \;=\; 0
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
\vec{e}_\theta : \quad & \frac{\partial u_\theta^{(0)}}{\partial \tau} + w^{(\frac{5}{2})}\frac{\partial u_\theta^{(0)}}{\partial z} - \Lambda_b^1 \; w^{(\frac{5}{2})}\frac{\partial u_\theta^{(0)}}{\partial r} + u_r^{(\frac{1}{2})}\zeta^{(\frac{1}{2})} + u_r^{(\frac{2}{2})}\zeta^{(0)} + \\
& \frac{u_\theta^{(0)}}{r}\frac{\partial u_\theta^{(\frac{2}{2})}}{\partial \theta} + \frac{u_\theta^{(\frac{1}{2})}}{r}\frac{\partial u_\theta^{(\frac{1}{2})}}{\partial \theta} + \frac{1}{r\rho^{(0)}}\frac{\partial p^{(\frac{8}{2})}}{\partial \theta} + \Omega_0 \Pi_b^1 + \Omega_0 u_r^{(\frac{2}{2})} \;=\; 0
\end{aligned} \tag{4.6}$$

with Π_a^j, Π_b^j and Λ_a^j, Λ_b^j (j = 0,1,2,...) given through (3.42) and (3.41).

Continuity equations:

With $w^{(\frac{4}{2})} = 0$ the mass continuity equations (3.50) and (3.52) take the form

$$\frac{\partial u_r^{(\frac{1}{2})}}{\partial r} + \frac{u_r^{(\frac{1}{2})}}{r} + \frac{1}{r}\frac{\partial u_\theta^{(\frac{1}{2})}}{\partial \theta} \;=\; 0 \tag{4.7}$$

$$\begin{aligned}
\rho^{(0)}\left(\frac{\partial u_r^{(\frac{2}{2})}}{\partial r} + \frac{u_r^{(\frac{2}{2})}}{r} + \frac{1}{r}\frac{\partial u_\theta^{(\frac{2}{2})}}{\partial \theta}\right) + \frac{\partial(\rho^{(0)}w^{(\frac{5}{2})})}{\partial z} \;=\; & \\
\Lambda_b^1 \frac{\partial(\rho^{(0)}w^{(\frac{5}{2})})}{\partial r} + \Lambda_a^1 \frac{1}{r}\frac{\partial(\rho^{(0)}w^{(\frac{5}{2})})}{\partial \theta} &
\end{aligned} \tag{4.8}$$

4.2 Asymptotic solutions of the first order equations

The purpose of this section is to find asymptotic solutions for the leading order vertical velocity $w^{(\frac{5}{2})}$ and the associated thermodynamic fields $\Theta^{(\frac{6}{2})}$ and $\rho^{(\frac{6}{2})}$ (**Section 4.2.1**). Furthermore solutions for the first order horizontal velocities $u_r^{(\frac{1}{2})}$ and $u_\theta^{(\frac{1}{2})}$ shall be derived (**Section 4.2.2**). In doing so it is assumed that the leading order circumferential velocity $u_\theta^{(0)}$, satisfying the matching condition (3.99), is given. Finally, matching the horizontal velocity fields with the environmental vortex flow gives equations for the leading order vortex centreline motion $\vec{V}_C^{(0)} = \partial \vec{X}_C^{(0)} / \partial \tau$ (**Section 4.2.3**).

4.2.1 Wavenumber-one leading order vertical velocity patterns and the adiabatic lifting mechanism

Dry simulations were carried out by Frank & Ritchie (1999) "to determine the patterns of forced ascent that occur as the vortex responds to imposed vertical wind shear and translational flow". They revealed a mechanism modulating vertical velocity fields and which has been called the adiabatic lifting mechanism. Via a three-stage sequence of events caused by vertical wind shear the lifting mechanism can be summarized as follows:

[1] The first stage describes a downshear tilting of the vortex in response to an environmentally sheared background flow.

[2] In a second stage the tilt causes an unbalanced wind and mass field of the vortex flow itself, resulting in weak ascent downshear and descent upshear of the low-level vortex centre. This in turn gives rise to a bulging of the cold/warm isentropes downshear/upshear of the surface vortex.

[3] The third stage includes a further development of the secondary circulation forcing the primary vortex circulation to flow up and down the tilted isentropes. The resulting vertical circulation pattern was determined by maximum upward motion in the downshear right quadrant and subsidence in the upshear left quadrant.

Similar observations as described above have been done by Jones (1994) studying the evolution of vortices in vertical shear using a primitive equation numerical model on an f-plane. A schematic summary of Jones results is illustrated in **Figure 4.3**.

In the following paragraph it is shown that a systematic analysis of the governing equations for an adiabatic vortex yields a set of equations that describes the same mechanisms as observed in the simulations mentioned above.

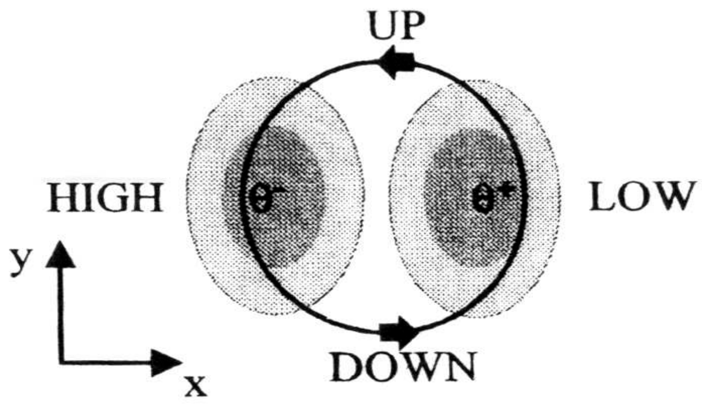

Figure 4.3: Adiabatic lifting mechanism: - plan view of asymmetric potential temperature patterns after the vortex has been tilted downshear in response to an imposed westerly environmental shear. The environmental flow is 4 m/s near the surface and zero at the upper boundary, - bold circle: relative motion through the anomaly (Jones 1994, Fig. 4 c)

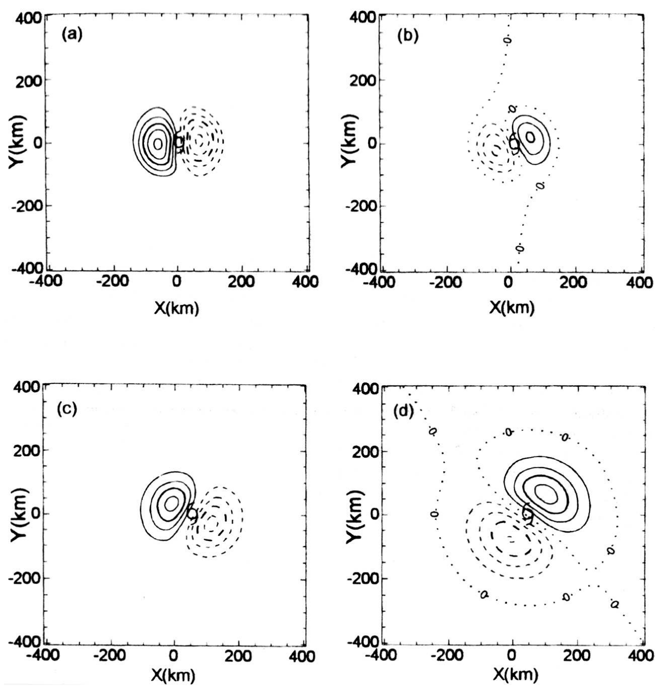

Figure 4.4: Horizontal cross-sections showing wavenumber-one vertical velocity (a) and potential temperature fields (b) after 30 min simulation; (c), (d) show the same after 6 h simulation (taken from Jones (1994), Fig. 3)

[1] Taking the general balance condition (3.140) between the leading order background flow and the vortex tilt into account, the mechanisms that cause a vortex tilt will be discussed in **Section 4.3.3** at length. In particular, an equation is derived that describes how the evolution of $(X_C^{(\frac{1}{2})}, Y_C^{(\frac{1}{2})})$ is forced by a vertically sheared background flow. Hence, a tilted vortex in response to an environmental vertical background shear can be identified, similar to that described by Frank & Ritchie (1999) in the first stage of the adiabatic lifting mechanism.

[2] The essence of the second stage described by Frank & Ritchie (1999) is the occurence of asymmetric potential temperature anomalies once the vortex has been tilted. From an asymptotic perspective the following derivation confirms the existence of such anomalies. Applying $(3.58)_2$ and $(3.58)_3$ on the vertical momentum (4.2) gives

$$\frac{\partial X_C^{(\frac{1}{2})}}{\partial z}\frac{\partial p^{(\frac{6}{2})}}{\partial r} = \rho_{12}^{(\frac{6}{2})} \qquad \text{and} \qquad \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\frac{\partial p^{(\frac{6}{2})}}{\partial r} = \rho_{11}^{(\frac{6}{2})} \tag{4.9}$$

In addition, the harmonic analysis of the $\mathcal{O}(\varepsilon^{\frac{6}{2}})$ state equation (A-37) yields a direct relation between asymmetric density fields $\rho_{1k}^{(\frac{6}{2})}$ and asymmetric potential temperature fields $\Theta_{1k}^{(\frac{6}{2})}$, reading

$$\rho^{(0)}\Theta_{1k}^{(\frac{6}{2})} + \rho_{1k}^{(\frac{6}{2})}\Theta_\infty = 0, \qquad k = 1, 2 \tag{4.10}$$

Here we have used the fact that $\Theta^{(0)} = \Theta_\infty$ and that $p^{(\frac{i}{2})} = p^{(\frac{i}{2})}(r, z, \tau)$ for $i = 2, 4, 6$. Thus, substitution of (4.10) into (4.9) gives

$$\frac{\partial X_C^{(\frac{1}{2})}}{\partial z}\frac{1}{\rho^{(0)}}\frac{\partial p^{(\frac{6}{2})}}{\partial r} = -\frac{\Theta_{12}^{(\frac{6}{2})}}{\Theta_\infty} \qquad \text{and} \qquad \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\frac{1}{\rho^{(0)}}\frac{\partial p^{(\frac{6}{2})}}{\partial r} = -\frac{\Theta_{11}^{(\frac{6}{2})}}{\Theta_\infty} \tag{4.11}$$

It turns out that a tilted vortex with $\partial X_C^{(\frac{1}{2})}/\partial z \neq 0$ and $\partial Y_C^{(\frac{1}{2})}/\partial z \neq 0$ requires asymmetric potential temperature patterns to achieve a balanced state. From the general balance condition (3.140) we know that on an f-plane (i.e. $\vec{V}_R^{(0)} = 0$) and for a pure westerly environmental flow (i.e. $V_B^{(0)} = 0$) we have $\partial Y_C^{(\frac{1}{2})}/\partial z = 0$ but $\partial X_C^{(\frac{1}{2})}/\partial z > 0$. Then, the above equations state that cold temperature anomalies occur downshear and warm potential temperature anomalies occur upshear which is in agreement with Jones schematic illustration in **Figure 4.3**. Note that further observations can be made regarding the location of the centres of the anomalies. Since the pressure gradient $\partial p^{(\frac{6}{2})}/\partial r$ in (4.11) can be viewed as a weighting factor the location of the centre of the anomalies will correspond with the location of strongest pressure gradients and hence strongest circumferential winds $u_\theta^{(0)}$.

[3] Finally it is shown how an asymmetric secondary circulation can be enforced by vertically tilted isentropes, i.e. potential temperature anomalies. For a stable background stratification, i.e. $\partial\Theta^{(\frac{4}{2})}/\partial z > 0$, the zeroth mode equation of (4.3) implies immediately that

$$w_0^{(\frac{5}{2})} = 0 \tag{4.12}$$

From the first sine and cosine modes of (4.3) one obtains a direct relation between tilt induced asymmetric potential temperature patterns $\Theta_{1k}^{(\frac{6}{2})}$ and an asymmetric vertical velocity field $w_{1k}^{(\frac{5}{2})}$

$$w_{11}^{(\frac{5}{2})}\frac{\partial\Theta^{(\frac{4}{2})}}{\partial z} = +\frac{u_\theta^{(0)}}{r}\Theta_{12}^{(\frac{6}{2})} \qquad \text{and} \qquad w_{12}^{(\frac{5}{2})}\frac{\partial\Theta^{(\frac{4}{2})}}{\partial z} = -\frac{u_\theta^{(0)}}{r}\Theta_{11}^{(\frac{6}{2})} \tag{4.13}$$

Thus, together with the observations from (4.11) it follows that within a pure zonal background shear flow maximum ascent appears downshear right of the vortex centre and maximum descent downshear left of the vortex centre which in turn results in a 90° phase shift between the vertical velocity and potential temperature anomalies. This is in good agreement with the observations made by Jones (1994) (see **Figure 4.3**).

After an initial adjustment time of 6 hours Jones (1994) further observed that the potential temperature anomaly started to rotate due to the rotation of the vortex tilt. In the present work, solutions describing a precession motion of the vortex centreline $\vec{X}_C^{(\frac{1}{2})} = (X_C^{(\frac{1}{2})}, Y_C^{(\frac{1}{2})})$ are discussed in more detail in **Section 4.3**. Taking such solutions into account, a precession motion of $\vec{X}_C^{(\frac{1}{2})}$ implies via (4.11) and (4.13) a rotation of the asymmetric velocity and potential temperature patterns $w_{1k}^{(\frac{5}{2})}$ and $\Theta_{1k}^{(\frac{6}{2})}$, which is in agreement with the observations made by Jones (1994) (see **Figure 4.4**).

Summarizing the results discussed under point [1] - [3] the following cause-effect can be found

$$\boxed{\frac{\partial U_B^{(\frac{1}{2})}}{\partial z} \Longrightarrow \frac{\partial X_C^{(\frac{1}{2})}}{\partial z} \Longrightarrow w_{11}^{(\frac{5}{2})} \Longrightarrow \Theta_{12}^{(\frac{6}{2})}}$$

and

$$\boxed{\frac{\partial V_B^{(\frac{1}{2})}}{\partial z} \Longrightarrow \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} \Longrightarrow w_{12}^{(\frac{5}{2})} \Longrightarrow \Theta_{11}^{(\frac{6}{2})}} \tag{4.14}$$

The equations (4.14) state that a vertically sheared environmental flow induces a vortex tilt which in turn generates asymmetric patterns of the vertical velocity and potential temperature fields within the mesoscale vortex region.

Inner and outer boundary limits For subsequent analysis in this chapter, knowledge about the behaviour of $w_{1k}^{(\frac{5}{2})}$ for both the far field region ($r \to \infty$) and regions near the vortex centre ($r \to 0$) is needed. By combining (4.13) with (4.11) one obtains

$$T_k \, \frac{u_\theta^{(0)}}{r} \frac{\partial \pi}{\partial r} \;=\; w_{1k}^{(\frac{5}{2})} \frac{\partial \Theta^{(\frac{4}{2})}}{\partial z} \,, \quad k = 1, 2 \tag{4.15}$$

with $\pi = p^{(\frac{6}{2})}/\rho^{(0)}$ and

$$T_1 = -\frac{\partial X_C^{(\frac{1}{2})}}{\partial z} \quad \text{and} \quad T_2 = +\frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} \tag{4.16}$$

Elimination of $\partial\pi/\partial r$ with the aid of the gradient wind relation (3.135) one obtains

$$w_{1k}^{(\frac{5}{2})} = \Theta_\infty \left(\frac{\partial \Theta^{(\frac{4}{2})}}{\partial z} \right)^{-1} T_k \, \frac{u_\theta^{(0)}}{r} \left(\frac{{u_\theta^{(0)}}^2}{r} + \Omega_0 u_\theta^{(0)} \right) \qquad k = 1, 2 \tag{4.17}$$

From (4.17) it is observed that the radial behaviour of $w_{1k}^{(\frac{5}{2})}$ is determined uniquely by the radial profile of the leading order vortex flow $u_\theta^{(0)}$. Thus, with the aid of the matching condition (3.99) one obtains

$$w_{1k}^{(\frac{5}{2})} = \tilde{g} \; T_k \, \frac{\Gamma^2}{r^3} \qquad \text{as} \qquad r \to \infty \tag{4.18}$$

with $\tilde{g} = \tilde{g}(z) = \Theta_\infty \Omega_0 (4\pi^2 \partial \Theta^{(\frac{4}{2})}/\partial z)^{-1}$ and the circulation $\Gamma = \Gamma(z, \tau)$. A further observation that can be made from (4.17) is that the asymmetric vertical velocities $w_{11}^{(\frac{5}{2})}$ and $w_{12}^{(\frac{5}{2})}$ differ only in the tilt components T_1 and T_2. Thus, from (4.17) the following relation is derived

$$w_{11}^{(\frac{5}{2})} \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} + w_{12}^{(\frac{5}{2})} \frac{\partial X_C^{(\frac{1}{2})}}{\partial z} = 0 \tag{4.19}$$

Next, solutions for $w_{1k}^{(\frac{5}{2})}$ in the limit $r \to 0$ are derived. Solving the gradient wind relation (3.135) for $u_\theta^{(0)}$ gives

$$u_\theta^{(0)} = -\frac{\Omega_0 r}{2} \pm \sqrt{\frac{\Omega_0^2 r^2}{4} + \frac{r}{\rho^{(0)}} \frac{\partial p^{(\frac{6}{2})}}{\partial r}} \tag{4.20}$$

The minimum of the pressure in cyclones is located at the centre of the vortex, which implies $\partial p^{(\frac{6}{2})}/\partial r = 0$ at $r = 0$. Hence, considering the limit $r \to 0$ of equation (4.17) one obtains with the aid of L' Hospitals rule

$$w_{1k}^{(\frac{5}{2})} \sim \frac{u_\theta^{(0)^3}}{r^2} + \Omega_0 \frac{u_\theta^{(0)^2}}{r} = 0 \qquad \text{as} \qquad r \to 0 \tag{4.21}$$

4.2.2 Wavenumber-one first order horizontal velocity fields

The purpose of this section is to derive solutions for the asymmetric contributions of the next higher order vortex flow described by $u_r^{(\frac{1}{2})}$ and $u_\theta^{(\frac{1}{2})}$. For one thing the solutions are used to get a first order correction on the leading order vortex field given by $u_\theta^{(0)}$. For another, the solutions serve as a basis of deriving equations for the leading order vortex motion $\vec{V}_C^{(0)} = (U_C^{(0)}, V_C^{(0)})$ by means of matched asymptotics, which is discussed in **Section 4.2.3**.

Incompressible first order horizontal flow The first non-trivial mass continuity (4.7) implies a nondivergent first order horizontal flow $(u_r^{(\frac{1}{2})}, u_\theta^{(\frac{1}{2})})$, i.e.

$$\frac{\partial u_r^{(\frac{1}{2})}}{\partial r} + \frac{u_r^{(\frac{1}{2})}}{r} + \frac{1}{r}\frac{\partial u_\theta^{(\frac{1}{2})}}{\partial \theta} = 0 \tag{4.22}$$

Thus, based on Helmholtz's Theorem (see **Section 3.1.4**) a stream function $\psi^{(\frac{1}{2})}$ can be introduced which is defined by

$$(u_r^{nd(\frac{1}{2})}, u_\theta^{nd(\frac{1}{2})}) = \left(\frac{1}{r}\frac{\partial \psi^{(\frac{1}{2})}}{\partial \theta}, -\frac{\partial \psi^{(\frac{1}{2})}}{\partial r}\right) \tag{4.23}$$

Since $u_r^{(\frac{1}{2})}$ and $u_\theta^{(\frac{1}{2})}$ have no divergent contributions, i.e. $u_r^{d(\frac{1}{2})} = 0$ and $u_\theta^{d(\frac{1}{2})} = 0$, the 'nd' superscripts denoting nondivergent flows are dropped in subsequent analysis. Note also that (3.62) implies

$$u_{r,0}^{(\frac{1}{2})} = 0 \tag{4.24}$$

Solutions for the stream function Elimination of $p^{(\frac{7}{2})}$ from $(4.4)_1$ and $(4.4)_2$ by cross-differentiation, i.e. $\partial_\theta (4.4)_1 - \partial_r (r\ (4.4)_2)$, gives

$$\frac{u_\theta^{(0)}}{r}\frac{\partial^2 u_r^{(\frac{1}{2})}}{\partial \theta^2} - \frac{2u_\theta^{(0)}}{r}\frac{\partial u_\theta^{(\frac{1}{2})}}{\partial \theta} - \Omega_0 \frac{\partial u_\theta^{(\frac{1}{2})}}{\partial \theta} - \frac{\partial}{\partial r}\left(r u_r^{(\frac{1}{2})}\frac{\partial u_\theta^{(0)}}{\partial r}\right) -$$
$$\frac{\partial}{\partial r}(u_r^{(\frac{1}{2})} u_\theta^{(0)}) - \frac{\partial}{\partial r}\left(u_\theta^{(0)}\frac{\partial u_\theta^{(\frac{1}{2})}}{\partial \theta}\right) - \Omega_0 \frac{\partial}{\partial r}(r u_r^{(\frac{1}{2})}) = 0 \tag{4.25}$$

We substitute (4.23) into (4.25) to obtain an equation for $\psi^{(\frac{1}{2})}$. Then, a harmonic analysis (see **Section 3.1.4**) yields homogeneous linear second-order ordinary differential equations for the first Fourier modes $\psi_{1k}^{(\frac{1}{2})}$, i.e.

$$u_\theta^{(0)} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left[\frac{\zeta_r^{(0)}}{u_\theta^{(0)}} + \frac{1}{r^2} \right] \right) \psi_{1k}^{(\frac{1}{2})} = 0 \ , \qquad k = 1, 2 \tag{4.26}$$

Here the symbol $\zeta_r^{(0)} = \partial \zeta^{(0)}/\partial r$ is used. When constructing asymptotic approximations, it is not always immediately clear which boundary conditions the solutions should satisfy (Holmes, 1995). Motivated studies of Wang & Holland (1996) which show that a non-zero relative flow at $r = 0$ was responsible for the deflection of the vortex motion from the steering flow, the following boundary conditions (BC's hereafter) are choosen

$$\psi_{1k}^{(\frac{1}{2})} = 0 \ , \qquad \frac{\partial \psi_{1k}^{(\frac{1}{2})}}{\partial r} = A_{1k} \qquad \text{at} \quad r = 0 \tag{4.27}$$

Here, $A_{1k} = A_{1k}(z, \tau)$ is a constant accounting for the possibility of fluid parcels flowing through the vortex centre. But note, $A_{1k} = A_{1k}(z, \tau)$ is presently unknown and, in fact, it could turn out to be zero. To find solutions for $\psi_{1k}^{(\frac{1}{2})}$ it is helpful to make a transformation of (4.26) and (4.27) into an initial value problem with homogeneous BC's. For this purpose a stream function $\overline{\psi_{1k}^{(\frac{1}{2})}}$ is introduced, defined through

$$\overline{\psi_{1k}^{(\frac{1}{2})}} = \psi_{1k}^{(\frac{1}{2})} - A_{1k} r \tag{4.28}$$

Upon substitution of (4.28) into (4.26) one obtains the following equation

$$u_\theta^{(0)} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left[\frac{\zeta_r^{(0)}}{u_\theta^{(0)}} + \frac{1}{r^2} \right] \right) \overline{\psi_{1k}^{(\frac{1}{2})}} = \zeta_r^{(0)} A_{1k} r \ , \tag{4.29}$$

which we solve subject to the homogeneous BC's

$$\overline{\psi_{1k}^{(\frac{1}{2})}} = 0, \qquad \frac{\partial \overline{\psi_{1k}^{(\frac{1}{2})}}}{\partial r} = 0 \qquad \text{at} \quad r = 0 \tag{4.30}$$

Integration of (4.29) yields (see **Appendix B.1**)

$$\overline{\psi_{1k}^{(\frac{1}{2})}} = u_\theta^{(0)} \int_0^{\bar{\bar{r}}} \frac{1}{\bar{r} u_\theta^{(0)2}} \left[\int_0^{\bar{r}} A_{1k} \, r^2 \, \zeta_r^{(0)} \, dr \right] d\bar{r} \tag{4.31}$$

Eventually, solving the integrals in (4.31) (see **Appendix B.2**) and use of (4.28) yields

$$\psi_{1k}^{(\frac{1}{2})} = u_\theta^{(0)} \frac{2A_{1k}}{\zeta_\star^{(0)}} \tag{4.32}$$

Here, the subscript $(.)_\star$ denotes values evaluated at the vortex centre, i.e. $\zeta_\star^{(0)} = \zeta^{(0)}(r = 0, z, \tau)$. Using the notation in terms of the asymmetric velocity fields $u_{\theta,1k}^{(\frac{1}{2})}$ and $u_{r,1k}^{(\frac{1}{2})}$ (k = 1, 2), from (4.32) and together with (3.63) one obtains

$$\begin{aligned} u_{\theta,1k}^{(\frac{1}{2})} &= -\frac{\partial u_\theta^{(0)}}{\partial r} \frac{2A_{1k}}{\zeta_\star^{(0)}} , \qquad k = 1, 2 \\ u_{r,11}^{(\frac{1}{2})} &= -\frac{u_\theta^{(0)}}{r} \frac{2A_{12}}{\zeta_\star^{(0)}} \\ u_{r,12}^{(\frac{1}{2})} &= +\frac{u_\theta^{(0)}}{r} \frac{2A_{11}}{\zeta_\star^{(0)}} \end{aligned} \tag{4.33}$$

Here, however, a contradiction appears by comparing the far field conditions of the asymmetric velocity components (4.33) with the matching conditions (3.103). Taking into account that $u_\theta^{(0)} = \Gamma/2\pi r$ as r approaches ∞ (see eqn. (3.99)), one finds from (4.33)

$$u_{\theta,11}^{(\frac{1}{2})} = +\, u_{r,12}^{(\frac{1}{2})} \qquad \text{and} \qquad u_{\theta,12}^{(\frac{1}{2})} = -\, u_{r,11}^{(\frac{1}{2})} \qquad \text{as} \qquad r \to \infty \tag{4.34}$$

By contrast, the matching conditions (3.103) yield

$$u_{\theta,11}^{(\frac{1}{2})} = -\, u_{r,12}^{(\frac{1}{2})} \qquad \text{and} \qquad u_{\theta,12}^{(\frac{1}{2})} = +\, u_{r,11}^{(\frac{1}{2})} \qquad \text{as} \qquad r \to \infty \tag{4.35}$$

The only possibility to avoid the contradiction appearing between (4.34) and (4.35) is to require $A_{1k} = 0$ which implies immediately that

$$\boxed{\psi_{1k}^{(\frac{1}{2})} = 0} \qquad \text{or} \qquad \boxed{u_{\theta,1k}^{(\frac{1}{2})} \equiv u_{r,1k}^{(\frac{1}{2})} = 0} \tag{4.36}$$

4.2.3 Leading order vortex motion and constraints on the environmental flow

Equations for the leading order vortex motion $\vec{V}_C^{(0)}$ shall be derived by means of matched asymptotic techniques. As noted earlier, the idea of this approach stems from the work of Callagari & Ting (1978) who studied the motion and decay of Vortex filaments, and also from the work of Ling & Ting (1987) who studied the motion and core structure of a two dimensional geostrophic vortex. Furthermore it is worth noting that with the scalings used in the present work centreline velocities of order $\vec{V}_C^{(0)}$ are a measure for dimensional velocities of ~ 10 m/s. From observations it is known that most hurricanes may move along at 5 - 10 m/s, in extreme cases they even could be moving as fast as 20 m/s. But there are also situations where hurricanes go much more slowly or even become quasi-stationary. The winter storm 'Lothar'[2] (24-26 December 1999) is an example for hurricane-like vortices in the mid-latitudes which reached translations speeds of ~ 30 m/s and which were distinctively larger than the ambient background flow of ~ 10 m/s (Wernli, 2002). The numbers above provide a wide variety of how fast hurricane-like vortices may move. From the solutions for $\vec{V}_C^{(0)}$ a discussion is given on environmental conditions necessary for a concentrated vortex to reach translation speeds of ~ 10 m/s.

For the present case, matching the inner velocities $u_{\theta,1k}^{(\frac{1}{2})}$ and $u_{r,1k}^{(\frac{1}{2})}$ with the environmental flow becomes easy. Since $u_{\theta,1k}^{(\frac{1}{2})}$ and $u_{r,1k}^{(\frac{1}{2})}$ are zero for all r (see (4.36)), it follows from the matching condition (3.103) that

$$\boxed{\begin{array}{rcl} U_C^{(0)}(\tau) & = & U_{B,C}^{(0)}(z,\tau) \ + \ U_{R,C}^{(0)}(z,\tau) \\ V_C^{(0)}(\tau) & = & V_{B,C}^{(0)}(z,\tau) \ + \ V_{R,C}^{(0)}(z,\tau) \end{array}} \tag{4.37}$$

where $U_C^{(0)}(\tau)$ and $V_C^{(0)}(\tau)$ denote the zonal and meridional velocity components of the leading order vortex centreline with $U_C^{(0)}(\tau) = \partial X_C^{(0)}/\partial\tau$ and $V_C^{(0)}(\tau) = \partial Y_C^{(0)}/\partial\tau$ (see $(3.6)_1$). Recall that the leading order centreline expansion term $\vec{X}_C^{(0)}(\tau) = (X_C^{(0)}(\tau), Y_C^{(0)}(\tau))$ has been assumed to be independent on the vertical coordinate z (see (3.20)). That is why vortices with a horizontal displacement of about 1000 km between the upper und lower vortex part can not be viewed as concentrated vortices. This, however, sets specific constraints on the right hand sides of (4.37). In particular it turns out that (4.37) is only satisfied if the z dependence of the background flow $\vec{V}_B^{(0)}(\vec{\eta}, z, \tau)$ compensates the z dependence of the regular flow $\vec{V}_R^{(0)}(\vec{\eta}, z, \tau)$ at the leading order centreline position $\vec{\eta} = \vec{X}_C^{(0)}$ such that the totals $(U_{B,C}^{(0)} + U_{R,C}^{(0)})$ and $(V_{B,C}^{(0)} + V_{R,C}^{(0)})$ are constant with respect to the vertical z, respectively. In the present work the background flow at the leading order centreline, i.e. $\vec{V}_{B,C}^{(0)} = \vec{V}_B^{(0)}(X_C^{(0)}, Y_C^{(0)})$, is assumed to be given. Equations determining the regular flow $\vec{V}_R^{(0)}$ in terms of a regular stream function $\check{\psi}_r^{(0)}$ have been derived in **Sections 3.2.2.2 - 3.2.2.3**. Unfortunately, due to the nonlinearity of (3.84) it turns out to be difficult to find analytical solutions for $\check{\psi}_r^{(0)}$. Due to the lack of exact analytical solutions for $\check{\psi}_r^{(0)}$, three different cases are considered below.

[2] 'Lothar' was a winter storm originating in the North Atlantic, which after its rapid intensification devastated regions in northern France and Switzerland, and southern Germany.

Case A: This case is based on the assumption that solutions for $\check{\psi}_r^{(0)}$ (or $\vec{V}_{R,C}^{(0)}$) exist such that the right hand side of (4.37) is constant with respect to the vertical. Then (4.37) state that the concentrated vortex is in leading order steered by the background flow $\vec{V}_{B,C}^{(0)} = (U_{B,C}^{(0)}, V_{B,C}^{(0)})$ evaluated at the leading order centreline, but its path is modified by the regular flow field $\vec{V}_{R,C}^{(0)} = (U_{R,C}^{(0)}, V_{R,C}^{(0)})$. Here it is worth pointing out that a partial cancellation of the effects of environmental vertical wind shear by a β-plane-induced shear vector has been observed by Frank & Ritchie (2002), provided that the imposed environmental wind shear was easterly.

Case B: Now if no physically reasonable solutions for $\check{\psi}_r^{(0)}$ (or $\vec{V}_{R,C}^{(0)}$) exist such that for a given vertically sheared background flow $\vec{V}_{B,C}^{(0)}$ the right hand side of (4.37) is constant with respect to the vertical, then one has to draw a conclusion different from Case A. In this case the choosen background flow conditions are not adequate for a concentrated vortex to maintain its coherence to leading order. This in turn would imply that not only a strong sheared background flow can inhibit the maintenance of concentrated vortices, but also the β effect.

Case C: Since no exact solutions for $\check{\psi}_r^{(0)}$ are available, a more simplified case describing the leading order vortex motion on a f plane is considered by assuming that $\beta = 0$. Considering a 2D vortex, Reznik (1992) found that for a singular point vortex and $\beta = 0$ the generation of a non-zero regular field is not possible. Assuming that this is true for the 3D vortex case studied in the present work, the impact of a regular flow on the leading order vortex motion would disappear. This, however, would restrict the environmental conditions necessary for a concentrated vortex to survive. In particular, (at least at this stage of analysis), favourable conditions are either a leading order background flow constant throughout the whole troposphere or a background flow is of 1/2 th order, i.e. $\vec{V}_B^{(0)} = 0$. Considering the first possibility, an appropriate asymptotic expansion (see (3.78)) for the background flow $\vec{V}_B = (U_B, V_B)$ with $U_B = -\partial\check{\psi}_B/\partial\eta_2$ and $V_B = \partial\check{\psi}_B/\partial\eta_1$ at the leading order centreline position $\vec{\eta} = \vec{X}_C^{(0)}$ would read

$$\vec{V}_{B,C} \;=\; \vec{V}_{B,C}^{(0)}(\tau) + \varepsilon^{\frac{1}{2}}\vec{V}_{B,C}^{(\frac{1}{2})}(z,\tau) + \mathcal{O}(\varepsilon^{\frac{2}{2}}) \qquad . \tag{4.38}$$

4.3 Asymptotic analysis of the second order equations

The primary goal of this section is to elucidate the role of a vertically sheared background flow, i.e. $\partial\vec{V}_B/\partial z \neq 0$, on the leading order vortex tilt $\partial\vec{X}_C^{(\frac{1}{2})}/\partial z$ and therefore the vertical distribution of next higher order vortex centreline motion $\vec{V}_C^{(\frac{1}{2})} = \partial\vec{X}_C^{(\frac{1}{2})}/\partial\tau$. The relevant equations for the analysis are the mass-conservation equation (4.8) and the horizontal momentum equations (4.5) and (4.6). The derivations are similar to the procedure in the previous section.

4.3.1 Wavenumber-one second order horizontal velocity fields for non-zero first order vortex tilt

In numerical simulations carried out by Wu & Wang (2001) focusing on vertical coupling and movement of adiabatic baroclinic tropical cyclones (TC) affected either by a vertical environmental shear or a differential beta drift, the following observations were made: A three-dimensional asymmetric circulation with a typical radius of 100 km developed within the TC core region, after the vortex has been tilted in the vertical in response to the environmental forcing. It was shown in **Section 4.2.1** that asymmetric vertical velocity patterns $w_{1k}^{(\frac{5}{2})}$ only occur in presence of a vortex tilt $T_k^{(\frac{1}{2})}$. It will be shown below that the same holds for second order asymmetric contributions $u_{r,1k}^{(\frac{2}{2})}$ and $u_{\theta,1k}^{(\frac{2}{2})}$. Therefore, from an asymptotic perspective, a tilt induced three-dimensional asymmetric flow given by $w_{1k}^{(\frac{5}{2})}$, $u_{r,1k}^{(\frac{2}{2})}$ and $u_{\theta,1k}^{(\frac{2}{2})}$ can be derived that is in agreement with the observations made by Wu & Wang (2001).

Compressible second order horizontal flow Unlike the mass continuity equation (4.7), the mass-conservation equation (4.8) describes a divergent second order vortex flow given by $u_r^{(\frac{2}{2})}$ and $u_\theta^{(\frac{2}{2})}$. This is due to the tilt induced asymmetric vertical velocity patterns $w_{1k}^{(\frac{5}{2})}$ on the right hand side of (4.8). Thus, the velocity components can be decomposed according to Helmholtz's Theorem

$$(u_r^{(\frac{2}{2})}, u_\theta^{(\frac{2}{2})}) = (u_r^{nd(\frac{2}{2})}, u_\theta^{nd(\frac{2}{2})}) + (u_r^{d(\frac{2}{2})}, u_\theta^{d(\frac{2}{2})}) \qquad , \tag{4.39}$$

where the divergent flow component has to satisfy

$$\rho^{(0)}\left(\frac{\partial u_r^{d(\frac{2}{2})}}{\partial r} + \frac{u_r^{d(\frac{2}{2})}}{r} + \frac{1}{r}\frac{\partial u_\theta^{d(\frac{2}{2})}}{\partial\theta}\right) = -\frac{\partial(\rho^{(0)}w^{(\frac{5}{2})})}{\partial z} + \Lambda_b^1\frac{\partial(\rho^{(0)}w^{(\frac{5}{2})})}{\partial r} + \Lambda_a^1\frac{1}{r}\frac{\partial(\rho^{(0)}w^{(\frac{5}{2})})}{\partial\theta} \tag{4.40}$$

Here, Λ_a^1 and Λ_b^1 are given by (3.41). Since $w_0^{(\frac{5}{2})} = 0$ (see (4.12)) and $\rho^{(0)} = \rho^{(0)}(z)$, equations for the first two harmonics of $u_r^{d(\frac{2}{2})}$ are

$$2\,\frac{\partial}{\partial r}\left(r u_{r,0}^{d(\frac{2}{2})}\right) = \frac{\partial X_C^{(\frac{1}{2})}}{\partial z}\frac{\partial}{\partial r}\left(r w_{12}^{(\frac{5}{2})}\right) + \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\frac{\partial}{\partial r}\left(r w_{11}^{(\frac{5}{2})}\right) \tag{4.41}$$

and

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(r u_{r,11}^{d(\frac{2}{2})}\right) - \frac{u_{\theta,12}^{d(\frac{2}{2})}}{r} &= -\frac{1}{\rho^{(0)}}\frac{\partial(\rho^{(0)} w_{11}^{(\frac{5}{2})})}{\partial z} \\
\frac{1}{r}\frac{\partial}{\partial r}\left(r u_{r,12}^{d(\frac{2}{2})}\right) + \frac{u_{\theta,11}^{d(\frac{2}{2})}}{r} &= -\frac{1}{\rho^{(0)}}\frac{\partial(\rho^{(0)} w_{12}^{(\frac{5}{2})})}{\partial z}
\end{aligned} \tag{4.42}$$

Note that radial integration of (4.41) from 0 to r' together with (4.21) and (4.19) yields

$$2u_{r,0}^{d(\frac{2}{2})} = \frac{\partial X_C^{(\frac{1}{2})}}{\partial z} w_{12}^{(\frac{5}{2})} + \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} w_{11}^{(\frac{5}{2})} = 0 \tag{4.43}$$

which implies immediately $u_{r,0}^{d(\frac{2}{2})} = 0$. This in turn yields via (3.59) and (3.62) that

$$u_{r,0}^{(\frac{2}{2})} = 0 \tag{4.44}$$

Solutions for the velocity potential Rewriting (4.42) in terms of a velocity potential $\phi^{(\frac{2}{2})}$ (see **Section 3.1.4**) one obtains a linear second order partial differential equation for the asymmetric components $\phi_{1k}^{(\frac{2}{2})}$ $(k = 1, 2)$

$$\nabla_1^2 \phi_{1k}^{(\frac{2}{2})} = -\frac{1}{\rho^{(0)}}\frac{\partial(\rho^{(0)} w_{1k}^{(\frac{5}{2})})}{\partial z} \quad , \tag{4.45}$$

where the operator ∇_1^2 is defined by

$$\nabla_1^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) \tag{4.46}$$

In a similar manner as in **Section 4.2.2** we will describe the flow through the vortex centre in terms of nondivergent components of the velocity field given by $u_r^{(\frac{2}{2})}$ and $u_\theta^{(\frac{2}{2})}$. Hence, we assume the following BC's for the velocity potential $\phi_{1k}^{(\frac{2}{2})}$

$$\phi_{1k}^{(\frac{2}{2})} = 0, \qquad \frac{\partial \phi_{1k}^{(\frac{2}{2})}}{\partial r} = 0 \qquad \text{at} \qquad r = 0 \tag{4.47}$$

Integrating (4.45) using the identity

$$\nabla_1^2 \phi_{1k}^{(\frac{i}{2})} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\left(r\frac{\partial \phi_{1k}^{(\frac{i}{2})}}{\partial r} - \phi_{1k}^{(\frac{i}{2})}\right)\right) \qquad k = 1,2; \quad i = 0,1,... \tag{4.48}$$

solutions of (4.45) satisfying the BC's (4.47) are (see **Appendix B.3**)

$$\phi_{1k}^{(\frac{2}{2})} = -r\int_0^r \frac{1}{\bar{r}^3}\left[\int_0^{\bar{r}} \frac{\bar{\bar{r}}^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)} w_{1k}^{(\frac{5}{2})})}{\partial z}\, d\bar{\bar{r}}\right] d\bar{r} \tag{4.49}$$

Further manipulations can be made using integration by parts and taking into account that $w_{1k}^{(\frac{5}{2})} = 0$ at $r = 0$ (see (4.21)). This gives

$$\boxed{\phi_{1k}^{(\frac{2}{2})} = \frac{1}{2}\left(\frac{1}{r}\int_0^r \frac{\bar{r}^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)} w_{1k}^{(\frac{5}{2})})}{\partial z} d\bar{r} - r\int_0^r \frac{1}{\rho^{(0)}}\frac{\partial(\rho^{(0)} w_{1k}^{(\frac{5}{2})})}{\partial z} d\bar{r}\right)} \tag{4.50}$$

Note that solutions for $\phi_{1k}^{(\frac{2}{2})}$ vanish for a zero vortex tilt in leading order, i.e. $T_k = 0$, since in such a case $w_{1k}^{(\frac{5}{2})} = 0$ from (4.17).

Solutions for the stream function Applying the same procedure as in **Section 4.2.2**, a linear second order differential equation for the higher order sreamfunction $\psi_{1k}^{(\frac{2}{2})}$ can be obtained from (4.5)-(4.6), (3.59)-(3.61) together with $(4.36)_2$. However, contrary to (4.26) this is an inhomogeneous equation

$$\begin{aligned}
-u_\theta^{(0)}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left[\frac{\zeta_r^{(0)}}{u_\theta^{(0)}} + \frac{1}{r^2}\right]\right)\psi_{12}^{(\frac{2}{2})} &= \mathcal{H}_{11} + \mathcal{I}_{11} \\
u_\theta^{(0)}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left[\frac{\zeta_r^{(0)}}{u_\theta^{(0)}} + \frac{1}{r^2}\right]\right)\psi_{11}^{(\frac{2}{2})} &= \mathcal{H}_{12} + \mathcal{I}_{12}
\end{aligned} \tag{4.51}$$

with

$$\begin{aligned}
\mathcal{H}_{1k} &= \frac{\partial}{\partial r}\left(r w_{1k}^{(\frac{5}{2})}\frac{\partial u_\theta^{(0)}}{\partial z}\right) \\
\mathcal{I}_{1k} &= r[\zeta^{(0)} + \Omega_0]\nabla_1^2\phi_{1k}^{(\frac{2}{2})} + \frac{\partial \phi_{1k}^{(\frac{2}{2})}}{\partial r}\left(r\frac{\partial \zeta^{(0)}}{\partial r}\right)
\end{aligned} \tag{4.52}$$

Again, to allow for non-zero relative flow at the vortex centre (see **Section 4.2.2**) inhomogeneous BC's are assumed

$$\psi_{1k}^{(\frac{2}{2})} = 0\ , \qquad \frac{\partial \psi_{1k}^{(\frac{2}{2})}}{\partial r} = B_{1k} \qquad \text{at} \quad r = 0 \qquad , \tag{4.53}$$

where $B_{1k} = B_{1k}(z,\tau)$ is still unknown at this stage of analysis. Using the transformation

$$\overline{\psi_{1k}^{(\frac{2}{2})}} = \psi_{1k}^{(\frac{2}{2})} - B_{1k} r \tag{4.54}$$

the initial value problem (4.51) - (4.53) can be rewritten as

$$-u_\theta^{(0)} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left[\frac{\zeta_r^{(0)}}{u_\theta^{(0)}} + \frac{1}{r^2} \right] \right) \overline{\psi_{1k}^{(\frac{2}{2})}} = \mathcal{K}_{1k} \tag{4.55}$$

where

$$\begin{aligned} \mathcal{K}_{11} &= -\mathcal{H}_{12} - \mathcal{I}_{12} - \zeta_r^{(0)} B_{11} r \\ \mathcal{K}_{12} &= +\mathcal{H}_{11} + \mathcal{I}_{11} - \zeta_r^{(0)} B_{12} r \end{aligned}$$

and the BC's are

$$\overline{\psi_{1k}^{(\frac{1}{2})}} = 0, \qquad \frac{\partial \overline{\psi_{1k}^{(\frac{1}{2})}}}{\partial r} = 0 \qquad \text{at} \quad r = 0 \qquad . \tag{4.56}$$

Integration of (4.55) (see **Appendix B.1** and **Appendix B.2**) and using (4.54) yields

$$\boxed{\begin{aligned} \psi_{11}^{(\frac{2}{2})} &= +u_\theta^{(0)} \int_0^{\bar{\bar{r}}} \frac{1}{\bar{r} u_\theta^{(0)^2}} \left[\int_0^{\bar{r}} r \, (\mathcal{H}_{12} + \mathcal{I}_{12}) \, dr \right] d\bar{r} + u_\theta^{(0)} \frac{2B_{11}}{\zeta_*^{(0)}} \\ \psi_{12}^{(\frac{2}{2})} &= -u_\theta^{(0)} \int_0^{\bar{\bar{r}}} \frac{1}{\bar{r} u_\theta^{(0)^2}} \left[\int_0^{\bar{r}} r \, (\mathcal{H}_{11} + \mathcal{I}_{11}) \, dr \right] d\bar{r} + u_\theta^{(0)} \frac{2B_{12}}{\zeta_*^{(0)}} \end{aligned}} \tag{4.57}$$

Since the existence of $\mathcal{H}_{1k}$ and $\mathcal{I}_{1k}$ depends primarily on vertical velocities $w_{1k}^{(\frac{5}{2})}$ induced by the vortex tilt T_k, the first sum on the right of (4.57) disappears for $T_k = 0$ and the resulting solutions for $\psi_{1k}^{(\frac{2}{2})}$ does not differ from solutions for $\psi_{1k}^{(\frac{1}{2})}$ (see (4.36)).

Boundary conditions Recall that in **Section 4.2.2** solutions for the stream functions $\psi_{1k}^{(\frac{1}{2})}$ become trivial, because far field conditions for the higher order velocity components $u_{\theta,1k}^{(\frac{1}{2})} = -\partial\psi_{1k}^{(\frac{1}{2})}/\partial r$ and $u_{r,1k}^{(\frac{1}{2})} = (-1)^k \psi_{1(\delta_{1k}+1)}^{(\frac{1}{2})}/r$ have not been satisfied. However, with respect to the next higher order, non-zero components $u_{\theta,1k}^{(\frac{2}{2})} = -\partial\psi_{1k}^{(\frac{2}{2})}/\partial r = B_{1k}$ at $r = 0$ can be derived. This can be done by substituting the far field solutions for $\psi_{1k}^{(\frac{2}{2})}$ and $\phi_{1k}^{(\frac{2}{2})}$ given by (B-48) and (B-27)-(B-31) into the matching conditions (3.104) and (3.105). In doing so it can be shown that

$$
\begin{aligned}
B_{11} &= +\frac{\zeta_\star^{(0)}\,\pi}{2\Gamma}\int_0^\infty \frac{r^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{12}^{(\frac{5}{2})})}{\partial z}\,dr \\
B_{12} &= -\frac{\zeta_\star^{(0)}\,\pi}{2\Gamma}\int_0^\infty \frac{r^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{11}^{(\frac{5}{2})})}{\partial z}\,dr
\end{aligned}
\tag{4.58}
$$

for $\Gamma \neq 0$. Hence, the flow through the vortex centre is determined by the horizontally averaged leading order mass flux over the mesoscale vortex region. The existence of the integrals in (4.58) can be proved using the far field behaviour of $w_{1k}^{(\frac{5}{2})}$ (see (4.18)). To show this we write the integrals in the following way

$$
\begin{aligned}
\int_0^\infty \frac{r^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{1k}^{(\frac{5}{2})})}{\partial z}\,dr &= \int_0^R \frac{r^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{1k}^{(\frac{5}{2})})}{\partial z}\,dr + \int_R^\infty \frac{r^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{1k}^{(\frac{5}{2})})}{\partial z}\,dr \\
&= \int_0^R \frac{r^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{1k}^{(\frac{5}{2})})}{\partial z}\,dr + h(z)\int_R^\infty \frac{1}{r}\,dr
\end{aligned}
\tag{4.59}
$$

where $0 < R < r$ and

$$
h(z) = \frac{1}{\rho^{(0)}}\frac{\partial(\rho^{(0)}\tilde{g}\;T_k\;\Gamma^2)}{\partial z}
\tag{4.60}
$$

Taking $w_{1k}^{(\frac{5}{2})} = 0$ at $r = 0$ into account (see (4.21)), from (4.59) it is observed that the integrals in (4.58) give only finite values if $h(z) = 0$ as shown in **Section 4.3.3**.

4.3.2 Leading order vortex intensity changes

Before discussing how the next higher order vortex motion can be determined by means of matched asymptotics, it is helpful to have a knowledge about the temporal evolution of $u_\theta^{(0)}$. Applying $(3.58)_1$ to the $\mathcal{O}(\varepsilon^{\frac{3}{2}})$ horizontal momentum equation (4.6) an evolution equation for $u_\theta^{(0)}$ can be derived

$$
\begin{aligned}
\frac{\partial u_\theta^{(0)}}{\partial \tau} + w_0^{(\frac{5}{2})}\frac{\partial u_\theta^{(0)}}{\partial z} - \frac{1}{2}\left(\frac{\partial X_C^{(\frac{1}{2})}}{\partial z}w_{12}^{(\frac{5}{2})} + \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}w_{11}^{(\frac{5}{2})}\right)\frac{\partial u_\theta^{(0)}}{\partial r} + \\
u_{r,0}^{(\frac{1}{2})}\zeta_0^{(\frac{1}{2})} + \frac{1}{2}\sum_{k=1}^{2} u_{r,1k}^{(\frac{1}{2})}\zeta_{1k}^{(\frac{1}{2})} + u_{r,0}^{(\frac{2}{2})}[\zeta^{(0)} + \Omega_0] \;=\; 0
\end{aligned}
\tag{4.61}
$$

However, using the results (4.12), (4.19), (4.24), (4.44) and (4.36) as derived in the previous sections, the above equation reduces to

$$
\frac{\partial u_\theta^{(0)}}{\partial \tau} = 0
\tag{4.62}
$$

Thus the leading order circumferential flow can be viewed as a steady flow, i.e. $u_\theta^{(0)} = u_\theta^{(0)}(r,z)$ which also implies that $\Gamma = \Gamma(z)$. This seems to be a reasonable result, since so far neither diabatic effects nor frictional effects have been accounted for.

4.3.3 Higher order vortex motion - equations for the higher order vortex centreline correction

Having obtained the solutions for $\psi^{(\frac{2}{2})}$ and $\phi^{(\frac{2}{2})}$, equations for the higher order vortex motion $\vec{V}_C^{(\frac{1}{2})}$ and therefore the vortex centreline determined through $\vec{X}_C^{(\frac{1}{2})}$ (see (3.6)) can be derived. Rewriting the matching conditions (3.104) and (3.105) with the aid of (3.59)-(3.61) in terms of $\psi_{1k}^{(\frac{2}{2})}$ and $\phi_{1k}^{(\frac{2}{2})}$ one obtains

$$\begin{aligned}
-\frac{\psi_{12}^{(\frac{2}{2})}}{r} + \frac{\partial \phi_{11}^{(\frac{2}{2})}}{\partial r} &= V_{B,C}^{(\frac{1}{2})} - V_C^{(\frac{1}{2})} - R_1 \qquad \text{as} \qquad r \to \infty \\
+\frac{\psi_{11}^{(\frac{2}{2})}}{r} + \frac{\partial \phi_{12}^{(\frac{2}{2})}}{\partial r} &= U_{B,C}^{(\frac{1}{2})} - U_C^{(\frac{1}{2})} - R_2 \qquad \text{as} \qquad r \to \infty
\end{aligned} \tag{4.63}$$

with

$$R_s = X_C^{(\frac{1}{2})} \left. \frac{\partial (V_B^{(0)} + V_R^{(0)})}{\partial \eta_s} \right|_{\vec{\eta}=\vec{X}_C^{(0)}} - Y_C^{(\frac{1}{2})} \left. \frac{\partial (U_B^{(0)} + U_R^{(0)})}{\partial \eta_s} \right|_{\vec{\eta}=\vec{X}_C^{(0)}} \tag{4.64}$$

for $s = 1, 2$.

Far field solutions The far field solutions for $\psi_{1k}^{(\frac{2}{2})}$ and $\partial \phi_{1k}^{(\frac{2}{2})}/\partial r$ can be derived from (4.45) and (4.51) with the aid of the matching result $u_\theta^{(0)} = \Gamma(z)/2\pi r$ as $r \to \infty$ (see (3.99)), since the radial behaviour of both the source terms $\mathcal{H}_{1k}$ and $\mathcal{I}_{1k}$ (see (4.52)) and the asymmetric vertical velocities $w_{1k}^{(\frac{4}{2})}$ (see (4.17)) only depends on the radial behaviour of $u_\theta^{(0)} = u_\theta^{(0)}(r,z)$. In particular, as shown in **Appendix B.4**, in the limit $r \to \infty$ one obtains

$$\begin{aligned}
\psi_{12}^{(\frac{2}{2})} &\sim C_{12}^2\, r - \frac{\pi}{\Gamma} \left(b\, T_2^\sharp + c \frac{\partial T_2^\sharp}{\partial z} \right) r\, \ln r - \frac{C_{12}^1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \\
\psi_{11}^{(\frac{2}{2})} &\sim C_{11}^2\, r - \frac{\pi}{\Gamma} \left(b\, T_1^\sharp + c \frac{\partial T_1^\sharp}{\partial z} \right) r\, \ln r - \frac{C_{11}^1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)
\end{aligned} \tag{4.65}$$

where

$$T_1^\sharp = \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} \quad \text{and} \quad T_2^\sharp = \frac{\partial X_C^{(\frac{1}{2})}}{\partial z} \tag{4.66}$$

and

$$b(z) = \frac{\Omega_0}{\rho^{(0)}} \frac{\partial(\rho^{(0)} \tilde{g}\; \Gamma^2)}{\partial z} \;, \quad c(z) = \Omega_0 \; \tilde{g} \; \Gamma^2 \;, \quad \tilde{g}(z) = \frac{\Theta_\infty \Omega_0}{4\pi^2 \partial \Theta^{(\frac{4}{2})}/\partial z} \tag{4.67}$$

The integration constants C^1_{1k} and C^2_{1k} that may depend on the vertical coordinate z are given by (B-29), (B-31) and (B-36). The far field solutions for the gradient of the velocity potential, as shown in **Appendix B.6**, are given by

$$\frac{\partial \phi^{(\frac{2}{2})}_{1k}}{\partial r} \;\sim\; \bar{C}^2_{1k} + \mathcal{O}(r^{-2}) \;, \qquad k = 1, 2 \tag{4.68}$$

as $r \to \infty$ with $\bar{C}^2_{1k}$ independent on r.

Finally, upon substitution of (4.68) and (4.65) into the matching conditions (4.63) one obtains in the limit $r \to \infty$

$$\begin{aligned} V_C^{(\frac{1}{2})} &= V_{B,C}^{(\frac{1}{2})} - R_1 + C^2_{12} - \bar{C}^2_{11} - \frac{\pi}{\Gamma} \left(b \; T_2^\sharp + c \frac{\partial T_2^\sharp}{\partial z} \right) \ln r \\ U_C^{(\frac{1}{2})} &= U_{B,C}^{(\frac{1}{2})} - R_2 - C^2_{11} - \bar{C}^2_{12} + \frac{\pi}{\Gamma} \left(b \; T_1^\sharp + c \frac{\partial T_1^\sharp}{\partial z} \right) \ln r \end{aligned} \tag{4.69}$$

for $\Gamma = \Gamma(z) \neq 0$.

4.3.3.1 Eigenmode of the first order vortex centreline

From (4.69) it is observed that in the limit $r \to \infty$ bounded solutions for both $V_C^{(\frac{1}{2})}$ and $U_C^{(\frac{1}{2})}$ only exist when

$$\left(b \; T_k^\sharp + c \; \partial T_k^\sharp / \partial z \right) = 0 \quad , \qquad k = 1, 2 \tag{4.70}$$

With the aid of the expressions for b and c (see (4.67)), equation (4.70) can be written as $\partial/\partial z \left(\sigma^{-1} \rho^{(0)} T_k^\sharp \; \Gamma^2 \right) = 0$ with $\sigma = \partial \Theta^{(\frac{4}{2})}/\partial z$ which implies that $\sigma^{-1} \rho^{(0)} T_k^\sharp \; \Gamma^2 = C_k^\sharp$, By use of (4.66) this can be written as

$$\sigma^{-1} \rho^{(0)} \frac{\partial \xi_k}{\partial z} \; \Gamma^2 = C_k^\sharp \tag{4.71}$$

with $k = 1, 2$ and where $\xi_1 = Y_C^{(\frac{1}{2})}(z, \tau)$ and $\xi_2 = X_C^{(\frac{1}{2})}(z, \tau)$. Note that because of $\rho^{(0)} = \rho^{(0)}(z)$ and $\Gamma = \Gamma(z)$ the integration constant $C_k^\sharp = C_k^\sharp(\tau)$ is a function of the time coordinate τ. Dividing (4.71) through by $\sigma^{-1} \rho^{(0)} \Gamma^2$ and integrating again yields

$$\xi_k(z, \tau) = C_k^\sharp(\tau) \int_{z_0}^{z} \frac{1}{\rho^{(0)}(z) \; \Gamma^2(z)} \frac{\partial \Theta^{(\frac{4}{2})}}{\partial z} \; dz + C_k^{\sharp\sharp}(\tau) \tag{4.72}$$

where z_0 denotes an arbitrary lower boundary. $C_k^{\sharp\sharp} = C_k^{\sharp\sharp}(\tau)$ is a second constant of integration which may depend on τ again. From (4.72) it is observed that centreline solutions belong to a certain class of separation of variables solutions

$$\xi_k = C_k^{\sharp}(\tau)\ J(z) + C_k^{\sharp\sharp}(\tau) \qquad \text{with} \qquad J(z) = \int_{z_0}^{z} \frac{1}{\rho^{(0)}\ \Gamma^2}\ \frac{\partial \Theta^{(\frac{4}{2})}}{\partial z}\ dz \tag{4.73}$$

Here, the function $J(z)$ determines an Eigenmode of the vortex centreline's vertical structure, depending on the background density $\rho^{(0)}(z)$, the background stratification $\partial\Theta^{(\frac{4}{2})}/\partial z$ and the vertical structure of the vortex circulation $\Gamma(z)$. Note that this Eigenmode also determines the vortex tilt $T_k^{\sharp} = \partial\xi_k/\partial z$. The amplitude of the Eigenmode is controlled by $C_k^{\sharp} = C_k^{\sharp}(\tau)$. Equations for the latter and some solutions under certain simplifying assumptions are derived in **Section 4.3.4**.

It is worth pointing out that Reasor & Montgomery (2004) studied the re-alignment phenomena of tilted vortices embedded in vertically sheared background flows. In their studies the vortex tilt is described in terms of a PV departure $q' = q'(r, z, \lambda, t)$ from an azimuthally ($\int d\lambda$) and vertically ($\int dz$) averaged tilted PV column $q = q(r, z, \lambda, t)$, i.e. $q = \bar{q}(r, t) + q'(r, z, \lambda, t)$. To mimic a simple tilt they expressed the tilt perturbation in terms of an $m = 1$ baroclinic mode[3], i.e.

$$q' \sim \cos\left(\frac{\pi z}{H}\right) \tag{4.74}$$

Moreover they used the same vertical distribution to represent an environmental vertical shear forcing, i.e.

$$(u_s, v_s) \sim \left(\cos\left(\frac{\pi z}{H}\right), -\cos\left(\frac{\pi z}{H}\right)\right) \tag{4.75}$$

Then, for an initially barotropic and vertically upright vortex they observed the development of a vortex tilt (4.74) after the environmental shear flow (4.75) was imposed. The observations based on (4.74) and (4.75) raises the question, whether the asymptotic solutions (4.73) with the vertical Eigenmode $J(z)$ also sets out certain requirements on the vertical structure of the imposed background flow. For instance, this background flow might also be compatible with the vortex tilt $\partial\xi_k/\partial z \sim \partial J(z)/\partial z$. This is discussed in detail in **Section 4.3.4**.

Remark Recall that so far the matching conditions (3.104) and (3.105) does not account for analytic solutions of the regular stream function $\check{\psi}_r^{(0)}$ satisfying (3.84). It proves to be difficult to derive analytic solutions for $\check{\psi}_r^{(0)}$, due to the

[3]Using the Boussinesq approximation, Reasor & Montgomery (2004) describe the vertical structure of q' and (u_s, v_s) in terms of $\cos(m\ \pi z/H)$ modes, where m is the vertical mode number and H is the physical depth of the vortex.

nonlinearity of (3.84). However, it is conceivable that taking into account an analytical expression of the regular stream function would yield a non-zero left hand side of (4.70) while matching in the radial direction. As a consequence the vertical Eigenmodes $J(z)$ determining the vertical structure of ξ_k would look different than those given in (4.72). In principle, a solution of the form $\check{\psi}_r^{(0)} \sim \check{r}\;\ln\check{r}$ can lead to such a non-zero left hand side. This can be seen as follows. Recall that the inner and outer expansions for the velocity fields $(\check{u}, \check{v})$ and (u_r, u_θ), i.e. (3.64) and $(3.19)_{1,2}$, in terms of the respective inner and outer stream functions read

$$\begin{aligned} \psi(r,\theta,z,\tau) &= \varepsilon^{-\frac{1}{2}}\psi^{(0)} + \psi^{(\frac{1}{2})} + \varepsilon^{\frac{1}{2}}\psi^{(\frac{2}{2})} + \mathcal{O}(\varepsilon) \\ \check{\psi}(\check{r},\theta,z,\tau) &= \varepsilon^{0}\;\check{\psi}_g^{(0)} + \varepsilon\check{\psi}^{(1)} + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{4.76}$$

Seeking outer solutions $\check{\psi}^{(\alpha)} \sim \check{r}\ln\check{r}$ as $\check{r} \to 0$ that match with inner solutions $\psi^{(\frac{2}{2})} \sim r\ln r$ as $r \to \infty$, using the transformation $\check{r} = \varepsilon^{\frac{1}{2}} r$ (see (3.93)) one finds

$$\begin{aligned} \varepsilon^{\frac{1}{2}} \lim_{r\to\infty} \psi^{(\frac{2}{2})} \sim \varepsilon^{\frac{1}{2}} \lim_{r\to\infty} (r\;\ln r) &= \lim_{\check{r}\to 0} \varepsilon^{\frac{1}{2}} \left(\varepsilon^{-\frac{1}{2}}\check{r}\;\ln(\varepsilon^{-\frac{1}{2}}\check{r})\right) \\ &= \varepsilon^{0} \lim_{\check{r}\to 0}(\check{r}\;\ln\check{r}) - \ln\varepsilon^{\frac{1}{2}} \lim_{\check{r}\to 0}\check{r} \\ &= \varepsilon^{0} \lim_{\check{r}\to 0}\check{\psi}_g^{(0)} - \ln\varepsilon^{\frac{1}{2}} \lim_{\check{r}\to 0}\check{r} \end{aligned} \tag{4.77}$$

By comparing the right hand side of the (4.77) with $(4.76)_2$ it is observed that the leading order outer $\check{\psi}_g^{(0)}$ terms would in principle match the inner solutions $\psi^{(\frac{2}{2})}$. Note that the singular stream function $\check{\psi}_s^{(0)}$ as one of the three contributions of $\check{\psi}_g^{(0)}$ (see (3.76)) behaves like $\check{\psi}_s^{(0)} \sim \ln\check{r}$ as $\check{r} \to 0$ (see (3.81)). Thus, if contributions that match $\psi^{(\frac{2}{2})}$ exist they have to be sought in regular solutions $\check{\psi}_r^{(0)}$.

4.3.3.2 Evolution equation for the first order centreline on a β-plane

With the condition (4.70) and the expressions for C_{1k}^2 and $\bar{C}_{1k}^2$ (see (B-36) and (B-49)), the equations (4.69) for the next higher order vortex motion take the form

$$\boxed{\begin{aligned} V_C^{(\frac{1}{2})} &= V_{B,C}^{(\frac{1}{2})} - R_1 + \frac{\pi}{\Gamma}\int_0^\infty \mathcal{W}_1\;dr \\ U_C^{(\frac{1}{2})} &= U_{B,C}^{(\frac{1}{2})} - R_2 + \frac{\pi}{\Gamma}\int_0^\infty \mathcal{W}_2\;dr \end{aligned}} \tag{4.78}$$

with

$$\mathcal{W}_k = r w_{1k}^{(\frac{5}{2})} \frac{\partial u_\theta^{(0)}}{\partial z} + \left[\Omega_0 + \frac{u_\theta^{(0)}}{r}\right] \frac{r^2}{\rho^{(0)}} \frac{\partial(\rho^{(0)} w_{1k}^{(\frac{5}{2})})}{\partial z} \tag{4.79}$$

and

$$R_k \;=\; X_C^{(\frac{1}{2})}\, \frac{\partial(V_B^{(0)}+V_R^{(0)})}{\partial \eta_k}\Bigg|_{\vec{\eta}=\vec{X}_C^{(0)}} - Y_C^{(\frac{1}{2})}\, \frac{\partial(U_B^{(0)}+U_R^{(0)})}{\partial \eta_k}\Bigg|_{\vec{\eta}=\vec{X}_C^{(0)}} \tag{4.80}$$

where $k = 1, 2$. The existence of the integrals in (4.78) can be proved using the far field behaviour of $w_{1k}^{(\frac{5}{2})}$ (see (4.18)) and $u_\theta^{(0)}$ (see (3.99)). For this we write the integrals in the following way

$$\begin{aligned}
\int_0^\infty \mathcal{W}_1\; dr &= \int_0^R \mathcal{W}_1\; dr + \int_R^\infty \mathcal{W}_1\; dr \\
&= \int_0^R \mathcal{W}_1\; dr + h_1(z)\int_R^\infty \frac{1}{r^3}\; dr + h_2(z)\int_R^\infty (\Omega_0 + \frac{\Gamma}{2\pi r^2})\frac{1}{r}\; dr
\end{aligned}$$

with

$$h_1(z) = \frac{\tilde{g}\; T_k\; \Gamma^2}{2\pi}\frac{\partial \Gamma}{\partial z} \quad , \qquad h_2(z) = \frac{1}{\rho^{(0)}}\frac{\partial(\rho^{(0)}\tilde{g}\; T_k\; \Gamma^2)}{\partial z} \tag{4.81}$$

However, taking the matching result (4.66) into account it follows that $h_2 = 0$. Hence, the last integral in the above equation disappears which in turn allows for finite values for the integrals in (4.78). Two conclusions can be made from (4.78) as regards the modification to the background flow. Firstly, horizontal gradients of the leading order background and regular flow evaluated at the leading order vortex centreline position $\vec{\eta} = \vec{X}_C^{(0)}$ enforce a modification of the higher order vortex motion from the first order background flow $\vec{V}_{B,C}^{(\frac{1}{2})}$. Secondly, the integral terms in (4.78) describe a net-effect of processes acting on the mesoscale vortex region on the synoptic scale vortex motion. Please note that this net-effect is primarily due to the existence of asymmetric vertical velocities $w_{1k}^{(\frac{5}{2})}$ which in turn are strongly related to non-zero vortex tilt components T_k (see (4.17)). Note that $\mathcal{W}_k = 0$ for $w_{1k}^{(\frac{5}{2})} = 0$.

It is shown next that the relation between $w_{1k}^{(\frac{5}{2})}$ and T_k allows us to obtain from (4.78) partial differential equations (PDEs) for the first order centreline components $X_C^{(\frac{1}{2})}$ and $Y_C^{(\frac{1}{2})}$. Upon substitution of (4.17) into (4.79) and taking into account that due to (4.70) it is possible to replace $\partial T_k/\partial z$ by $-(b/c)\; T_k$, one finds

$$\begin{aligned}
\mathcal{W}_1 &= -\frac{\partial X_C^{(\frac{1}{2})}}{\partial z}\;\tilde{\mathcal{W}} \\
\mathcal{W}_2 &= +\frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\;\tilde{\mathcal{W}}
\end{aligned} \tag{4.82}$$

where

$$\tilde{\mathcal{W}} = \frac{\partial u_\theta^{(0)}}{\partial z} r f^\star + r^2 \left[\Omega_0 + \frac{u_\theta^{(0)}}{r}\right] \left(g^\star - \frac{b}{c} f^\star\right) \tag{4.83}$$

with

$$\begin{aligned} f^\star &= \Theta_\infty \left(\frac{\partial \Theta^{(\frac{4}{2})}}{\partial z}\right)^{-1} \frac{u_\theta^{(0)}}{r} \left(\frac{{u_\theta^{(0)}}^2}{r} + \Omega_0 u_\theta^{(0)}\right) \\ g^\star &= \frac{1}{\rho^{(0)}} \frac{\partial(\rho^{(0)} f^\star)}{\partial z} \end{aligned} \tag{4.84}$$

Recall that b and c are given by (4.67). If one accounts further that $V_C^{(\frac{1}{2})} = \partial Y_C^{(\frac{1}{2})}/\partial\tau$ and $U_C^{(\frac{1}{2})} = \partial X_C^{(\frac{1}{2})}/\partial\tau$ (see $(3.6)_1$) the first order vortex motion equations (4.78) can be written as

$$\boxed{\begin{aligned} \frac{\partial Y_C^{(\frac{1}{2})}}{\partial\tau} &= V_{B,C}^{(\frac{1}{2})} - R_1 - \frac{\partial X_C^{(\frac{1}{2})}}{\partial z} \frac{\pi}{\Gamma} \int_0^\infty \tilde{\mathcal{W}}\, dr \\ \frac{\partial X_C^{(\frac{1}{2})}}{\partial\tau} &= U_{B,C}^{(\frac{1}{2})} - R_2 + \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} \frac{\pi}{\Gamma} \int_0^\infty \tilde{\mathcal{W}}\, dr \end{aligned}} \tag{4.85}$$

It has been found in **Section 4.3.3.1** that centreline solutions belong to a certain class of separation of variables solutions given by (4.73), i.e.

$$\begin{aligned} Y_C^{(\frac{1}{2})}(z,\tau) &= C_1^\sharp(\tau)\; J(z) + C_1^{\sharp\sharp}(\tau) \\ X_C^{(\frac{1}{2})}(z,\tau) &= C_2^\sharp(\tau)\; J(z) + C_2^{\sharp\sharp}(\tau) \end{aligned} \tag{4.86}$$

where $J(z)$ denotes an vertical Eigenmode determined by the background density $\rho^{(0)}(z)$, the background stratification $\partial\Theta^{(\frac{4}{2})}/\partial z$, and the vortex circulation $\Gamma(z)$. Thus, upon substitution of the above equations into (4.85), ordinary differential equations (ODE) for $C_k^\sharp = C_k^\sharp(\tau)$ are obtained whose solutions eventually determine the temporal evolution of $X_C^{(\frac{1}{2})}$ and $Y_C^{(\frac{1}{2})}$. Here, however it should be noted that solutions for $X_C^{(\frac{1}{2})}$ and $Y_C^{(\frac{1}{2})}$ should also satisfy at any time τ the general balance condition (3.140), i.e.

$$\frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} \left(U_{B,C}^{(0)} + U_{R,C}^{(0)}\right) = \frac{\partial X_C^{(\frac{1}{2})}}{\partial z} \left(V_{B,C}^{(0)} + V_{R,C}^{(0)}\right) \tag{4.87}$$

Remark So far, equations for the leading and first order vortex motions have been derived (see (4.37) and (4.78)). Thus, based on the asymptotic expansion (3.21) for the synoptic scale vortex motion $\vec{V}_C = (U_C, V_C)$ we find the following two-term approximation for the synoptic scale vortex motion

$$\begin{aligned} U_C &= U^{(0)}_{B,C}(z,\tau) + U^{(0)}_{R,C}(z,\tau) + \varepsilon \left(U^{(\frac{1}{2})}_{B,C} - R_2 + \frac{\pi}{\Gamma} \int_0^\infty \mathcal{W}_2 \, dr \right) \\ V_C &= V^{(0)}_{B,C}(z,\tau) + V^{(0)}_{R,C}(z,\tau) + \varepsilon \left(V^{(\frac{1}{2})}_{B,C} - R_1 + \frac{\pi}{\Gamma} \int_0^\infty \mathcal{W}_1 \, dr \right) \end{aligned} \tag{4.88}$$

4.3.4 Centerline solutions on an f-plane for an initially baroclinic vortex embedded within a spatial uniform background flow in leading order

Solutions describing the temporal evolution of the first order centreline components $X^{(\frac{1}{2})}_C$ and $Y^{(\frac{1}{2})}_C$ on an f-plane are now studied. Particular attention is focused on the question, whether an alignment mechanism similar to a resonant VRW damping can be explored from an asymptotic perspective. Recall that the theory of VRW damping has been described in detail at the beginning of this chapter.

The temporal evolution of the first order centreline components $X^{(\frac{1}{2})}_C$ and $Y^{(\frac{1}{2})}_C$ are studied, based on the assumption that no regular fields are generated by the vortex flow itself as long as $\beta = 0$. Thus the studies below are carried out for a zero regular flow, i.e. $\vec{V}^{(0)}_R = 0$. Recall that in this case the leading order background flow $\vec{V}^{(0)}_B$ is indepenent of height throughout whole troposphere in order to avoid any contradictions with the equations which describe the leading order vortex motion (see (4.37)). Moreover, a horizontally uniform leading order background flow is assumed, so that $\vec{V}^{(0)}_B = \vec{V}^{(0)}_B(\tau)$. Note that with $\vec{V}^{(0)}_R = 0$ and $\vec{V}^{(0)}_B = \vec{V}^{(0)}_B(\tau)$ it follows immediately that $R_k = 0$ (see (4.80)). Hence the equations (4.85) simplify to

$$\begin{aligned} \frac{\partial Y^{(\frac{1}{2})}_C}{\partial \tau} &= V^{(\frac{1}{2})}_{B,C} - \frac{\partial X^{(\frac{1}{2})}_C}{\partial z} \frac{\pi}{\Gamma} \int_0^\infty \tilde{\mathcal{W}} \, dr \\ \frac{\partial X^{(\frac{1}{2})}_C}{\partial \tau} &= U^{(\frac{1}{2})}_{B,C} + \frac{\partial Y^{(\frac{1}{2})}_C}{\partial z} \frac{\pi}{\Gamma} \int_0^\infty \tilde{\mathcal{W}} \, dr \end{aligned} \tag{4.89}$$

with $\Gamma = \Gamma(z)$, $\tilde{\mathcal{W}} = \tilde{\mathcal{W}}(r,z)$ and $\vec{V}^{(\frac{1}{2})}_{B,C} = \vec{V}^{(\frac{1}{2})}_B(\vec{\eta} = \vec{X}^{(\frac{1}{2})}_C, z, \tau)$. The general balance condition (4.87) takes the form

$$\frac{\partial Y^{(\frac{1}{2})}_C}{\partial z} \, U^{(0)}_B = \frac{\partial X^{(\frac{1}{2})}_C}{\partial z} \, V^{(0)}_B \tag{4.90}$$

It has been discussed in **Section 4.3.3.1** that the vertical Eigenmode $J(z)$ of the first order centreline components $X^{(\frac{1}{2})}_C$ and $Y^{(\frac{1}{2})}_C$ might change if the regular flow field $\check{\psi}_r^{(0)}$ behaves like $\check{r}\ln\check{r}$ in the limit $\check{r}\to 0$. Since it is assumed that there are no regular fields on an f-plane the only possible vertical Eigenmode of $X_C^{(\frac{1}{2})}$ and $Y_C^{(\frac{1}{2})}$ is determined by $J(z)$ which is given by $J(z)=\int 1/(\rho^{(0)}\,\Gamma^2)\;d\Theta^{(\frac{4}{2})}/dz\;dz$.

In the following subsections two different cases based on the assumptions that either $\vec{V}_B^{(0)}=\text{const.}$ (steady leading order background flow) or $\vec{V}_B^{(0)}=0$ are analysed. Note that regarding the latter assumption the additional constraint (4.90) on first order centreline solutions disappears.

4.3.4.1 Impact of a strong background flow $\vec{V}_B^{(0)}=\text{const.}$ with weak vertical shear $\vec{V}_B^{(\frac{1}{2})}=\vec{V}_B^{(\frac{1}{2})}(z,\tau)$

In the case of vortices embedded in strong background flow (i.e. $\vec{V}_{B,C}^{(0)}=\text{const.}$) with weak vertical shear (i.e. $\vec{V}_{B,C}^{(\frac{1}{2})}=\vec{V}_{B,C}^{(\frac{1}{2})}(z,\tau)$), substitution of (4.90) into (4.89) yields centreline transport equations

$$\boxed{\frac{\partial\xi_k}{\partial\tau}+c_k(z)\frac{\partial\xi_k}{\partial z}=F_k(z,\tau)\;,\qquad k=1,2} \tag{4.91}$$

where

$$\xi_1=Y_C^{(\frac{1}{2})},\quad \xi_2=X_C^{(\frac{1}{2})}\quad\text{and}\quad F_1=V_{B,C}^{(\frac{1}{2})},\quad F_2=U_{B,C}^{(\frac{1}{2})} \tag{4.92}$$

The advection velocity c_k is given by

$$c_k(z)=\tilde{\lambda}_k\,\frac{\pi}{\Gamma(z)}\int_0^\infty \tilde{\mathcal{W}}(z,r)\;dr \tag{4.93}$$

with

$$\tilde{\lambda}_1=U_{B,C}^{(0)}/V_{B,C}^{(0)}\quad,\quad\tilde{\lambda}_2=-V_{B,C}^{(0)}/U_{B,C}^{(0)} \tag{4.94}$$

It has been shown in **Section 4.3.3.1** that centreline solutions ξ_k belong to a certain class of separation of variable solutions given by $\xi_k(z,\tau)=C_k^\sharp(\tau)\;J(z)+C_k^{\sharp\sharp}(\tau)$ (see (4.73)), where the time-dependent functions $C_k^\sharp$ and $C_k^{\sharp\sharp}$ are the only unknowns. Linear first order non-homogeneous ordinary differential equations (ODEs) for $C_k^\sharp$ can be derived by substituting (4.73) into (4.91). In so doing one obtains

$$\frac{dC_k^\sharp}{d\tau}+\Pi_k C_k^\sharp=\frac{1}{J}\left[F_k(z,\tau)-\frac{dC_k^{\sharp\sharp}}{d\tau}\right] \tag{4.95}$$

where

$$\Pi_k=c_k\frac{1}{J}\frac{dJ}{dz}=\tilde{\lambda}_k\frac{\pi}{\rho^{(0)}\Gamma^3 J}\int_0^\infty\tilde{\mathcal{W}}\;dr=\tilde{\lambda}_k\;\Pi^+ \tag{4.96}$$

Recalling the general form of a first order ODE, i.e. $g_1(x)\;y'+g_0(x)\;y=f(x)$, the variable Π_k in (4.95) may be in principle a function on the time τ. For the present case, however, this is not allowed since $c_k = c_k(z)$ and $J = J(z)$. For mathematical convenience it is assumed that $\Pi_k =$ const.. Thus it is important to note that this sets together with $\rho^{(0)}(z)$ and $\Gamma(z)$ a certain constraint on the vertical structure of $\tilde{\mathcal{W}} = \tilde{\mathcal{W}}(r,z)$. Moreover, because of $\Pi_k =$ const. a right hand side of (4.95) independent on z is only guaranteed if $dC_k^{\sharp\sharp}/d\tau = 0$, and if the first order background flow (see $(4.92)_{3,4}$) can be written as

$$F_k(z,\tau) \;=\; J(z)\; B_k(\tau) \tag{4.97}$$

There are two different ways for interpretating (4.97). One possibility is to consider $J(z)$ (i.e. $\rho(z)$, $\Gamma(z)$ and $d\Theta^{(\frac{4}{2})}/dz$ (see $(4.73)_2$)) as to be given. This would mean that there is no free choice for the type of vertical shear for an imposed first order background flow. In particular, the prescribed vortex circulation $\Gamma(z)$, background stratification $d\Theta^{(\frac{4}{2})}/dz$ and the background density $\rho^{(0)}(z)$ would fix the vertical distribution of the background flow in which the vortices studied in the present work may exist. However, things can also be viewed in the opposite sense. A prescribed background flow $F_k(z,\tau)$ could also act in the sense that it determines the vertical structure of the vortex circulation $\Gamma(z)$.

Based on the necessary assumptions made above (4.95) simplifies to

$$\frac{dC_k^\sharp}{d\tau} \;+\; \Pi_k C_k^\sharp \;=\; B_k(\tau) \tag{4.98}$$

Homogeneous solutions $C_{k,h}^\sharp$ of (4.98) describe an exponential damping/growing process determined by $C_{k,h}^\sharp = \tilde{C}_{k,h}^\sharp \exp\left(-\Pi_k\tau\right)$ where the damping/growing rate Π_k is given by (4.96). Particular solutions can be derived using the method of variation of constants[4]. In so doing it can be shown that general solutions of (4.98) read

$$C_k^\sharp = \exp\left(-\Pi_k\tau\right)\left(\int B_k(\tau)\exp\left(+\Pi_k\tau\right)\,d\tau + C_k^+\right) \tag{4.99}$$

where C_k^+ denotes a constant of integration. To simplify matters a steady first order background flow, i.e. $dB_k/d\tau = 0$, is assumed. Then, the above solution simplifies to

$$C_k^\sharp = \mu_k + C_k^+ \exp\left(-\Pi_k\tau\right) \tag{4.100}$$

[4]Given a homogeneous solution a particular solution of the inhomogeneous equation can be found by considering the constants of integration as a function on the independent variables. For the case considered in the present work that means by treating $\tilde{C}_{k,h}^\sharp \equiv \alpha$ as a function on τ, i.e. $\alpha = \alpha(\tau)$. Then, substituting this ansatz into the inhomogeneous equations yields equations that have to be solved for α.

where $\mu_k = B_k/\Pi_k$. Thus, upon substitution of (4.100) into (4.71) one obtains general centreline solutions for the first order centreline components ξ_k

$$\xi_k(z,\tau) = \big(\mu_k + C_k^+ \exp\left(-\Pi_k \tau\right)\big) \int_{z_0}^{z} \frac{1}{\rho^{(0)}\Gamma^2} \frac{\partial \Theta^{(\frac{4}{2})}}{\partial z}\, dz + C_k^{\sharp\sharp} \tag{4.101}$$

Accordingly one obtains for the centreline tilt

$$\frac{\partial \xi_k}{\partial z} = \big(\mu_k + C_k^+ \exp\left(-\Pi_k \tau\right)\big) \frac{1}{\rho^{(0)}\Gamma^2} \frac{\partial \Theta^{(\frac{4}{2})}}{\partial z} \tag{4.102}$$

Note that the difference between Π_1 and Π_2 relies in the difference between $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ (see (4.96)). Since $\tilde{\lambda}_k$ with $(k = 1, 2)$ are of opposite sign (see (4.94)) this results in an exponential decrease for the tilt $\partial\xi_2/\partial z$ in zonal direction but an exponential increase for the tilt $\partial\xi_1/\partial z$ in meridional direction. Unfortunately, such solutions does not satisfy the general balance condition (4.90) which is valid as long as cases with strong background flow ($\vec{V}^{(0)}_{B,C} \neq 0$) are considered. It can easily be verified that only centreline solutions (4.101) with $C_k^+ = 0$ can satisfy (4.90). Therefore the solutions (4.101) become stationary, i.e.

$$\boxed{\begin{array}{rcl} \xi_k(z) & = & \mu_k \displaystyle\int_{z_0}^{z} \frac{1}{\rho^{(0)}\Gamma^2} \frac{\partial \Theta^{(\frac{4}{2})}}{\partial z}\, dz \;+\; C_k^{\sharp\sharp} \\ \text{or} & & \\ \xi_k(z) & = & \dfrac{1}{\Pi_k}\, F_k(z) \;+\; C_k^{\sharp\sharp} \end{array}} \tag{4.103}$$

with $\xi_1 = Y_C^{(\frac{1}{2})}$, $\xi_2 = X_C^{(\frac{1}{2})}$, $\mu_k = B_k/\Pi_k$ and F_k given through (4.92). From $(4.103)_2$ it is observed that for a prescribed background flow F_k it follows that

$$\frac{d\xi_k}{dz} \sim \frac{dF_k}{dz} \qquad \text{or} \qquad \frac{d\vec{X}_C^{(\frac{1}{2})}}{dz} \sim \frac{d\vec{V}_B^{(\frac{1}{2})}}{dz} \tag{4.104}$$

Thus, we find a vortex tilt that is induced by the vertical shear of the background flow. Recall that a non-zero vortex tilt was strongly related to asymmetric patterns in the vertical velocity $w^{(\frac{5}{2})}$ and potential temperature $\Theta^{(\frac{6}{2})}$ fields (adiabatic lifting mechanism; see **Section 4.2.1**). Note that, however, that choice of the higher order background flow F_k cannot be arbitrary since the centreline solutions (4.103) must satisfy the general balance condition (4.90). In particular, it can be shown that upon substitution of $(4.103)_2$ into (4.90) one obtains

$$V_B^{(0)} \frac{\partial V_{B,C}^{(\frac{1}{2})}}{\partial z} \;+\; U_B^{(0)} \frac{\partial U_{B,C}^{(\frac{1}{2})}}{\partial z} \;=\; \vec{V}_B^{(0)} \cdot \frac{\partial \vec{V}_{B,C}^{(\frac{1}{2})}}{\partial z} \;=\; 0 \tag{4.105}$$

which implies that

$$\vec{V}_B^{(0)} \perp \frac{\partial \vec{V}_{B,C}^{(\frac{1}{2})}}{\partial z} \tag{4.106}$$

Hence, the vertical shear of the next higher order background flow $\vec{V}_{B,C}^{(\frac{1}{2})}$ has to be orthogonal to the leading order background flow $\vec{V}_{B,C}^{(0)}$.

A further observation that can be made is that the relation (4.104) coincides with the assumptions (4.74) and (4.75) made by Reasor & Montgomery (2004). As noted earlier, Reasor & Montgomery (2004) found a realignment mechanism for tilted adiabatic vortices in sheared environmental flow, that could be explained by means of VRW damping. Unfortunately, solutions describing a realignment could not be found for the vortex case considered here. However, this does not mean that such a realignment mechanism does not exist. Since in the asymptotic analysis only the synoptic time scale $\tau = \varepsilon^2 t$ is used, it is expected that an additional inclusion of faster time scales into the asymptotic analysis may lead to different equations describing a vortex behaviour that may be similar to the observations made by Reasor & Montgomery (2004).

Summing up, (4.37) and (4.103)-(4.104) show that on an f-plane ($\vec{V}_R^{(0)} = 0$) the vortex moves with the leading order background flow $\vec{V}_B^{(0)} = \text{const.}$ while having a vortex tilt $d\vec{X}_C^{(\frac{1}{2})}/dz$ induced by the vertical shear of the higher order background flow corrections.

4.3.4.2 Impact of a weak background flow $\vec{V}_B^{(0)} = 0$ with weak vertical shear $\vec{V}_B^{(\frac{1}{2})} = \vec{V}_B^{(\frac{1}{2})}(z, \tau)$

The general balance condition (4.90) disappears, when one considers the case of a zero leading order background flow, i.e. $\vec{V}_B^{(0)} = 0$. Unlike the previous section it will be shown that this allows for non-stationary centreline solutions. Please note that, however, the general solution (4.73) for the first order centreline coordinates has to be satisfied, regardless whether the leading order background flow is zero or not.

In absence of the general balance condition (4.90), substitution of (4.73) into (4.89) yields a system of two first order ODEs for $C_1^\sharp$ and $C_2^\sharp$

$$\begin{aligned} \frac{dC_1^\sharp}{d\tau} &= \frac{V_{B,C}^{(\frac{1}{2})}}{J} - \frac{1}{J}\frac{dC_1^{\sharp\sharp}}{d\tau} - C_2^\sharp \frac{\pi}{\rho^{(0)}\Gamma^3 J}\int_0^\infty \tilde{\mathcal{W}}\, dr \\ \frac{dC_2^\sharp}{d\tau} &= \frac{U_{B,C}^{(\frac{1}{2})}}{J} - \frac{1}{J}\frac{dC_2^{\sharp\sharp}}{d\tau} + C_1^\sharp \frac{\pi}{\rho^{(0)}\Gamma^3 J}\int_0^\infty \tilde{\mathcal{W}}\, dr \end{aligned} \tag{4.107}$$

For same reasons as discussed in the previous subsection, solutions for $C_k^\sharp$ only exist if $F_k(z,\tau) = J(z)\; B_k(\tau)$ (see (4.97)) with $(F_2, F_1) = (U_{B,C}^{(\frac{1}{2})}, V_{B,C}^{(\frac{1}{2})})$ and

$dC_k^{\sharp\sharp}/d\tau = 0$ in order to satisfy $C_k^\sharp = C_k^\sharp(\tau)$. Moreover, the vertical structure of $\tilde{\mathcal{W}}(z,r)$ has to be in such a way that

$$\Pi^+ \;=\; \frac{\pi}{\rho^{(0)}(z)\;\Gamma^3(z)\;J(z)} \int_0^\infty \tilde{\mathcal{W}}(z,r)\;dr \tag{4.108}$$

is satisfied, where for mathematical convenience Π^+ is a constant independent on z and τ. It is assumed that the vertical structure of the background flow $\vec{V}_{B,C}^{(\frac{1}{2})}$ is prescribed. Then, this fixes the vertical structure of the vortex circulation $\Gamma(z)$ (see **Section 4.3.4.1**), i.e. for a given vertical structure of $\rho^{(0)}$, Γ and J there is no free choice for the vertical structure $\tilde{W}$.

Taking into account the necessary conditions discussed above, (4.107) simplify to

$$\begin{aligned} \frac{dC_1^\sharp}{d\tau} &= B_1(\tau) - \Pi^+\;C_2^\sharp \\ \frac{dC_2^\sharp}{d\tau} &= B_2(\tau) + \Pi^+\;C_1^\sharp \end{aligned} \tag{4.109}$$

The unknown $C_2^\sharp$ can be eliminated from $(4.109)_1$ first by differentiating the latter one with respect to τ and a subsequent replacement of the occuring temporal derivative of $C_2^\sharp$ by $(4.109)_2$. Similar procedure can be used in order to eliminate $C_1^\sharp$ from $(4.109)_2$. In doing so one obtains the following second order ODEs for $C_k^\sharp$

$$\begin{aligned} \frac{d^2C_1^\sharp}{d\tau^2} + {\Pi^+}^2 C_1^\sharp &= \frac{dB_1}{d\tau} - \Pi^+ B_2 \\ \frac{d^2C_2^\sharp}{d\tau^2} + {\Pi^+}^2 C_2^\sharp &= \frac{dB_2}{d\tau} + \Pi^+ B_1 \end{aligned} \tag{4.110}$$

Note that the above equations are characteristic equations for an harmonic oscillator without damping but with an external force. Assuming a stationary background flow again, i.e. $dB_k/d\tau = 0$, general solutions of (4.110) are

$$\begin{aligned} C_1^\sharp(z,\tau) &= a_1\cos(\Pi^+\tau) + b_1\sin(\Pi^+\tau) - \Pi^+ B_2 \\ C_2^\sharp(z,\tau) &= a_2\cos(\Pi^+\tau) + b_2\sin(\Pi^+\tau) + \Pi^+ B_1 \end{aligned} \tag{4.111}$$

Note that the free parameters of general solutions of (4.111) have to be choosen in such a way that (4.109) is satisfied. For simplification solutions with $a_2 = 0$ and $b_1 = 0$ are considered. Then it can easily be verified that general solutions (4.111) with $a_1 = b_2 \equiv a$ and $\Pi^+ = 1$ satisfy (4.109). Therefore, general centreline solutions are

$$
\boxed{
\begin{aligned}
X_C^{(\frac{1}{2})} &= (a\cos(\tau)+B_1)\int_{z_0}^{z}\frac{1}{\rho^{(0)}\Gamma^2}\frac{\partial\Theta^{(\frac{4}{2})}}{\partial z}\,dz\\
Y_C^{(\frac{1}{2})} &= (a\sin(\tau)-B_2)\int_{z_0}^{z}\frac{1}{\rho^{(0)}\Gamma^2}\frac{\partial\Theta^{(\frac{4}{2})}}{\partial z}\,dz\\
\text{or} \qquad & \\
X_C^{(\frac{1}{2})} &= F_1(z)\ (\tilde{a}_1\cos(\tau)+1)\\
Y_C^{(\frac{1}{2})} &= F_2(z)\ (\tilde{a}_2\sin(\tau)-1)
\end{aligned}
}
\tag{4.112}
$$

with $\tilde{a}_k = a/B_k$ and F_k determined by (4.92) and (4.97). Two things are observed from (4.112). As in the previous section, the tilt of the vortex centreline is determined by the vertical shear of the given background flow, i.e. $\partial\vec{X}_C^{(\frac{1}{2})}/\partial z \sim d\vec{V}_B^{(\frac{1}{2})}/dz$. Moreover it turns out that for an initially, zonally tilted vortex ($B_2 = 0$ at $\tau = 0$) the trajectory of the tilted first order centreline $(X_C^{(\frac{1}{2})}, Y_C^{(\frac{1}{2})})$ describes a circle around the stationary leading order vortex centreline position $(X_C^{(0)}, Y_C^{(0)})$. This means that the tilted vortex makes a precession motion. Such steadily oscillating solutions are similar to those of an undamped harmonic oscillator. Note that the coordinates $(X_C^{(0)}, Y_C^{(0)})$ are stationary because of the assumption $\vec{V}_{B,C}^{(0)} = 0$. Comparing the above result (4.112) with the centreline solutions (4.103) of the previous subsection, one can conclude that a strong, but vertically uniform background flow $\vec{V}_{B,C}^{(0)}$ 'holds the vortex tilt tight' such that a rotation is no longer possible.

Recall that a precession motion of a tilted vortex has also been observed by Reasor & Montgomery (2001), although they considered an initiallly barotropic vortex. Refering to the work of Reasor & Montgomery, in Schecter et al. (2002) the initial conditions and subsequent temporal evolution of the vortex tilt in absence of any environmental shear has been described as follows: "At $t = 0$, the vortex is tilted by an episode of external vertical shear, and then the shear is turned off. In time, the orientation of the tilt rotates, while the amplitude of the vortex tilt decays. Eventually the vortex relaxes to an upright position." For the case considered in this section a precession of a tilted vortex but no alignment can be observed. Note, however, that in Reasor & Montgomery (2004) the damping was attributed to resonance of the tilt rotation frequency with the 'ambient flow rotation frequency at a critical radius'. This implies that their observed tilt rotation must have been on time scale of the ambient rotational flow. Since in the present work the temporal evolution of $\vec{X}_C^{(\frac{1}{2})}$ is described on time scales slower than the ambient rotational vortex flow, it remains to be seen whether an asymptotic analysis on faster time scales yields solutions including the resonant damping effect described above.

It has been shown in **Section 4.2.1** that a first order vortex tilt is strongly related to asymmetric vertical velocities $w_{1k}^{(\frac{5}{2})} \neq 0$ (see (4.17)), whose generation can be explained by means of an adiabtic lifting mechanism. Note that based on the solutions derived in this section the rotation of the $w_{1k}^{(\frac{5}{2})}$ patterns as described by Jones (1994) can be attributed to the precession of the vortex tilt. Moreover it is important to point out, that centreline solutions similar to those of an undamped harmonic oscillator could only be derived for $\int_0^\infty \tilde{\mathcal{W}}\ dr \neq 0$ (see (4.89)). It will be shown in the next subsection that the integral disappears for barotropic ($du_\theta^{(0)}/dz = 0$) vortex conditions with the consequence that $\vec{X}_C^{(\frac{1}{2})} = 0$. Another possibility of getting rid of $\int_0^\infty \tilde{\mathcal{W}}$ is to set $w_{1k}^{(\frac{5}{2})} = 0$ resulting in the same centreline solutions as for the barotropic vortex case. Hence, considering baroclinic vortices it turns out that the maintenance of the vortex tilt in a sheared background flow with the same vertical Eigenmode as those of the vortex centreline is strongly related to the balancing impact of asymmetric vertical velocities $w_{1k}^{(\frac{5}{2})} \neq 0$ which is generated immediately after the vortex has been tilted. This is in agreement with observations made by Flatau et al. (1994) and Wang & Li (1992).

4.3.5 Centerline solutions on an f-plane for an initially barotropic vortex embedded within a horizontally uniform background flow

To study the evolution of a vortex tilt, both Jones (1994) and Reasor & Montgomery (2004) assumed initially barotropic conditions for the vortex flow. While Jones observed an increasing tilt with time, Reasor observed vortices becoming more and more upright until a quasi steady tilt has been reached. Using same initial conditions for the leading order vortex flow in the present work, because of $\partial u_\theta^{(0)}/\partial\tau = 0$ (see (4.62)) it follows for an initially barotropic vortex ($\partial u_\theta^{(0)}/\partial z = 0$) at $\tau = 0$, that

$$u_\theta^{(0)} = u_\theta^{(0)}(r) \qquad \text{for all} \quad \tau \tag{4.113}$$

which in turn yields $\Gamma =$ const. for all τ. For such a condition together with a constant background stratification $\partial\Theta^{(\frac{4}{2})}/\partial z =$ const. the last term on the right and side of (4.83) becomes zero such that (4.89) simplifies to

$$\frac{\partial Y_C^{(\frac{1}{2})}}{\partial\tau} = V_{B,C}^{(\frac{1}{2})}(z,\tau) \quad \text{and} \quad \frac{\partial X_C^{(\frac{1}{2})}}{\partial\tau} = U_{B,C}^{(\frac{1}{2})}(z,\tau) \tag{4.114}$$

Substitution of (4.73) into (4.114) yields again two 1st order ODEs for $C_k^\sharp = C_k^\sharp(\tau)$

$$\frac{dC_k^\sharp}{d\tau} \;=\; \frac{F_k}{J} - \frac{1}{J}\frac{dC_k^{\sharp\sharp}}{d\tau} \tag{4.115}$$

with F_k given by $(4.92)_2$. Note that with $\rho^{(0)} = \rho_0 \exp(-z)$ (see (A-94)) in a barotropic vortex case the vertical Eigenmode J can be written as

$$J \;=\; \frac{1}{\Gamma^2}\int_{z_0}^{z} \frac{1}{\rho^{(0)}}\frac{\partial \Theta^{(\frac{4}{2})}}{\partial z}\, dz \tag{4.116}$$

As in the previous section it turns out that $C_k^\sharp = C_k^\sharp(\tau)$ is only satisfied if $dC_k^{\sharp\sharp}/d\tau = 0$ and if the background flow satisfies

$$F_k(z,\tau) \;=\; \tilde{\mu}_k(\tau)\; J(z) \tag{4.117}$$

Hence, it turns out that for the barotropic vortex case there is no free choice for the vertical structure of the first order background flow F_k. Taking the above constraints into account and assuming a steady first order background flow, i.e. $F_k = F_k(z)$, integration of (4.115) with respect to the time yields

$$C_k^\sharp(\tau) \;=\; \tilde{\mu}_k\; \tau + C_k^\sharp(\tau_0) \tag{4.118}$$

with $\tilde{\mu}_k = const.$ Finally, upon substitution of (4.116)-(4.118) into (4.73), general solutions describing the temporal evolution of the first order vortex centreline of a barotropic vortex are

$$\xi_k(z,\tau) \;=\; (\tau + \kappa_k)\; F_k(z) + C_k^{\sharp\sharp} \tag{4.119}$$

with $\kappa_k = C_k^\sharp(\tau_0)/\tilde{\mu}_k$ and where $C_k^{\sharp\sharp}$ is a constant. The centreline solutions (4.119) state that the vortex is sheared away with increasing time since upper portions of the vortex centreline are advected faster by larger background flow velocities than lower portions of the vortex due to smaller background velocities (see (4.117)). Hence, neither a precession motion nor a realignment of the tilted vortex is observed using Reasor & Montgomery's (2001) initial conditions.

Chapter 5

Diabatic Vortex

This chapter examines the motion and structure of concentrated atmospheric vortices if they are affected by diabatic heat sources, i.e. $S \neq 0$. Then, unlike **Chapter 4** stronger vertical velocities of order $w^{(\frac{4}{2})}$ become important. The aim of this chapter is to find out the manner in which non-zero diabatically induced vertical velocities may lead to a modification of the vortex structure and its motion.

There are studies based on observation that give evidence on the importance of diabatic effects on the vortex motion. For instance, Willoughby (1990) and Holland & Lander (1992) have shown that there is a consistent relationship between steady spiral rainband[1] asymmetries in the core region of tropical cyclones and a tendency for tropical cyclones to meander[2] about a longer term-track with periods of several day and amplitudes around 100 km. Wang (1995a) and Wang & Holland (1995) used a three-dimensional primitive equation model with simple physical parameterizations to study the potential impacts of convective asymmetries on tropical cyclone motion. They found that "the convective asymmetries developed in the vortex core region influence the vortex motion through development of asymmetric divergent flow crossing the vortex centre, which tends to deflect the vortex toward the region with maximum convection."

The observations described above raise the question what kind of mechanisms generate convective asymmetries. A number of studies attribute the occurrence of convective asymmetries in tropical cyclones to the vertically sheared environmental flow. Simulations carried out by Wang & Holland (1996) and

[1] Spiral rainbands are a unique feature of tropical cyclones and play an important role in tropical cyclone structure and intensity changes. They are made up of organized intense convective cells embedded in widespread stratiform precipitation (Chen & Yau, 2001). Willoughby et. al (1984) also use the terminology stationary band complexes (SBCs) to describe such phenomena.

[2] Note, from observation it is known that the meandering (oscillation) of tropical cyclones covers a wide range of scales and take on several forms. Holland & Lander (1992)

later on by Frank & Ritchie (1999; 2001) show pronounced shear induced asymmetric patterns. In particular, enhanced upward motion and convection occurred to the left for an observer facing downshear and upward motion and convection was suppressed to the right for an observer facing upshear. As noted in **Chapter 4**, Jones (1995) and Frank & Ritchie (1999) found using dry simulations for tropical cyclone-like vortices, that wavenumber one asymmetries in the vertical motion field occured (adiabatic lifting mechanism) when the vortex tilted away from the vertical in response to an imposed vertical background shear. Based on these results it was postulated that these vertical motion patterns would modulate convection in real tropical cyclones. However, the results of subsequent simulations including moist physics carried out by Frank & Ritchie (1999) have shown that asymmetries developed in response to imbalances caused by vertical shear, but which differed significantly from the adiabatic simulations. Based on these observations they concluded that the adiabatic lifting mechanism vanishs with a set up of saturated conditions.

Observational studies carried out by Corbosiero & Molinari (2002) verify the existence of convective asymmetries in tropical cyclones due to vertical shear. Using cloud-to-ground lightning data they found a strong correlation between the azimuthal distribution of flashes and the direction of the vertical wind shear in the environment. While differentating between the inner core region (r $<$ 100 km) and the outer band region ($100 \leq r \leq 300$ km) they found for a vertical shear throughout the troposphere exceeding 5 m s^{-1}, that 90 % of the flashes occurred downshear, where a slight preference for downshear left occurred in the storm core, and a strong preference for downshear right in the outer rainbands (see **Figure 5.1**). Moreover, based on their observations of a huge number of tropical cyclones in vertical shear flows ranging from weak shear (0-2 m s^{-1}) over medium shear (6-8 m s^{-1}) up to strong shear (10-24 m s^{-1}), Corbosiero & Molinari (2002) argue that in convectively active tropical cyclones, deep divergent circulations may oppose vertical shear up to about 13 m s^{-1} and act to minimize the vortex tilt. Similar observations have been made by Zehr (1992). This might give one possible answer to a frequently asked question: How does real tropical cyclone-like vortex sustain its coherent vertical structure in vertical shear? Focusing on this issue Frank & Ritchie (1999) used numerical simulations to study the impact of an environmental shear flow on the vortex structure. They compared dry with moist numerical simulations and found that there was a greater chance for a moist vortex to survive in environmental vertical shear flow than for a dry vortex.

Although derived from different perspectives, the above observations make it clear that diabatic effects play an important role for the motion and structure of concentrated atmospheric vortices. It was shown in **Chapter 4** that the use of asymptotic methods in the context of an unified approach to meteorological modelling, proved to be useful in deriving approximate solutions describing adiabatic processes induced by an environmentally vertical shear flow and its consequences for the vortex motion and structure. It is shown in this chapter that a similar method accounting for non-zero diabatic source terms yields reduced sets of model equations that can be used to describe some aspects of the behaviour of diabatic vortices in vertical shear flows as observed in the above studies described right above. Additionally it is shown that the reduced model equations provide insights into the mechanism determining the vortex motion. Here, the influence of the vortex tilt will play an important role. Since the asymptotic model equations that account for diabatic source terms look familiar with the Eliassen balanced vortex model, a short introduction into the governing equations of this model is given in **Section 5.1**. In **Section 5.2** a new version of an Eliassen balanced vortex model is derived and different solutions for an externally prescribed diabatic source term are discussed. Modifications to this version are studied, based on an explicit inclusion of moisture in **Section 5.3**. Additionally, equations for the leading order vortex motion are derived that describe specific effects on the vortex motion related to diabatic processes, such as latent heat release due to condensation.

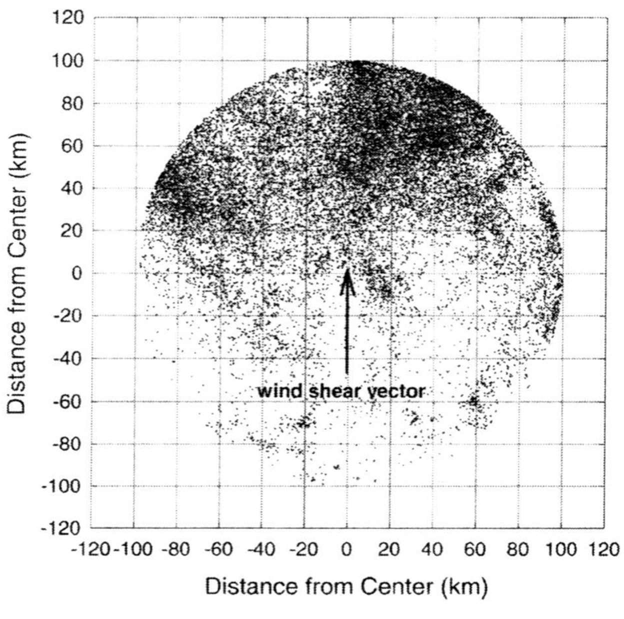

Figure 5.1: Locations of flashes occuring within 100 km of the storm centre which have been observed in 35 Atlantic basin tropical cyclones from 1985-99 and for medium shear (5 - 10 m/s) time periods. The flashes have been rotated around the centre so that the vertical wind vector is pointing due north. (Graphical illustration from Corbosiero, 2002)

5.1 The Eliassen balanced vortex model (EbVM)

The Eliassen balanced vortex model is an idealized two-dimensional model that was originally derived by Eliassen (1952) in order to investigate the response of an arbitrary axially symmetric vortex in gradient wind balance to sources of heat and angular momentum. If the sources are acting on the fluid, the balance of the vortex will be disturbed and a secondary circulation[3] superimposed upon the vortex motion will develope. Realizing that axially symmetric meridional circulations of the type described by this model also occur in atmospheric flows as for example in form of the general circulation, Eliassen was the first who made an attempt to apply this theory to suitable meridional currents in the earth's atmosphere. Later on, many authors discovered that this model is also of considerable value to study certain aspects causing tropical cyclone development (e.g. Charney & Eliassen, (1964); Hack & Schubert, (1982, 1983) and others). For instance, Hack & Schubert considered the axissymmetric balanced flow occuring in a thermally forced vortex in which the frictional inflow was confined to a thin boundary layer. For that case sources of momentum can be omitted and the governing equations of the EbVM take the form

$$\begin{aligned} M^2 - \frac{1}{4} f^2 r^4 &= r^3 \frac{\partial \phi}{\partial r} \\ \frac{DM}{Dt} &= 0 \\ \frac{\partial \phi}{\partial z} &= \frac{g}{\Theta_0}\Theta \\ \frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{\rho}\frac{\partial (\rho w)}{\partial z} &= 0 \\ c_p \frac{D \ln \Theta}{Dt} &= \frac{Q}{T} \end{aligned} \tag{5.1}$$

The notation of the equation set (5.1) uses the pseudoheight coordinate (Hoskins & Bretherton, 1972)

$$z = \left[1 - \frac{p}{p_0}^{\kappa}\right] \frac{c_p \Theta_0}{g} \tag{5.2}$$

and the definition of the absolute angular momentum per unit mass, i.e.

$$M = \frac{1}{2} f R^2 = rv + \frac{1}{2} f r^2 \quad . \tag{5.3}$$

The equations in (5.1) describe sequentially gradient wind balance, conservation of absolute angular momentum along the air parcel trajectory, hydrostatic ba-

[3]Other terms used for secondary ciculation are meridional circulation or transverse circulation.

lance, mass continuity and the evolution equation of the potential temperature. The dependent variables u, v, w are the radial, tangential and vertical components of velocity, T is the temperature, ϕ is the geopotential, ρ is the pseudo-density (known), $D/Dt = \partial/\partial t + u\ \partial/\partial r + w\ \partial/\partial z$ is the material derivative in the $r - z$ plane, f is the Coriolis parameter and Q is a specified heating function. The subscript zero denote values evaluated at the top of the boundary layer. Based on the assumption that the source term Q is given, the equations (5.1) are closed for the unknowns M, ϕ, θ, u, w. Note that Eliassen assumes the sources of heat (and angular momentum) are and are distributed symmetrically with respect to the axis of the vortex. This assumption shall ensure the axissymmetry of the vortex. Then, the assumption of a weak source allows to study the secondary circulation necessary for a vortex that is in its balanced state all the time. Three steps are necessary to derive a diagnostic equation for the secondary circulation; **(i)** the derivation of the thermal wind equation from $(5.1)_1$ and $(5.1)_3$

$$\frac{1}{r^3}\frac{\partial M^2}{\partial z} = \frac{g}{\Theta_0}\frac{\partial \Theta}{\partial r} \quad ,$$

(ii) the definition of a stream function ($u = -\partial\psi/\partial z,\ w = \partial(r\psi)/\partial r$) satisfying the mass continuity $(5.1)_4$, and finally **(iii)** the elimination of the local time changes between $(5.1)_2$ and $(5.1)_5$. Combining all these steps one obtains

$$\frac{\partial}{\partial r}\left(\mathcal{A}\frac{1}{r}\frac{\partial(r\psi)}{\partial r} + \mathcal{B}\frac{\partial\psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\mathcal{B}\frac{1}{r}\frac{\partial(r\psi)}{\partial r} + \mathcal{C}\frac{\partial\psi}{\partial z}\right) = \frac{g}{\Theta_0}\frac{\partial Q}{\partial r} \tag{5.4}$$

where

$$\begin{aligned}
\mathcal{A} &= \frac{g}{\Theta_0}\frac{\partial\Theta}{\partial z} \\
\mathcal{B} &= -\frac{g}{\Theta_0}\frac{\partial\Theta}{\partial r} = -\frac{1}{r^3}\frac{\partial M^2}{\partial z} \\
\mathcal{C} &= \frac{1}{r^3}\frac{\partial M^2}{\partial r} = \left(f + \frac{1}{r}\frac{\partial(rv)}{\partial r}\right)\left(f + \frac{2v}{r}\right)
\end{aligned} \tag{5.5}$$

The coefficients $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ denote three stabilizing factors acting against diabatically driven air parcels. Static stabiliy ($\mathcal{A}$) and inertial stability ($\mathcal{C}$) provide resistance to vertical and radial displacements, respectively. Baroclinicity ($\mathcal{B}$) determines the outward tilt with height of the ascending branch of the transverse circulation. Boundary conditions for (5.4) that have been used by Hack & Schubert (1983) are

$$\psi(0, z) = \psi(0, z_T) = 0 \quad \text{and} \quad r\psi \to 0 \quad \text{as} \quad r \to \infty \tag{5.6}$$

Note that in order to solve (5.4) for ψ (and hence for the transverse circulation given by u and w) the coefficients $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as functions on Θ and M must be known. A temporal evolution of Θ and M is obtained by solving the equations $(5.1)_2$ and $(5.1)_5$.

As noted earlier, balanced models such as the EbVM have proven to be useful to study certain aspects causing tropical cyclone development. Charney & Eliassen (1964) used the concept of balance to derive the well-known CISK theory (Convective Instability of the Second Kind). The linear theory describes cyclone development by a kind of secondary instability, in which the interaction between small-scale cumulus convection and convective circulations of cyclone scale leads to a large-scale self-amplification of a pre-hurricane depression. Charney & Eliassen (1964) suggest that this requires that the small-scale cumulus convection and the convective circulation of the cyclone support one another - "the cumuls cell by supplying the heat energy for driving the depression, and the depression by producing the low-level covergence of moisture into the cumulus cell". Later on Hack & Schubert (1982) employed the EbVM to study nonlinear effects on intensity changes of hurricane-like vortices, since it was believed that CISK alone can't be responsible for cyclone development. In particular, Hack & Schubert (1982) noted that the linear CISK process is inefficient, if adiabatic cooling of rising air parcels is balanced by heating due to latent heat release resulting in zero net warming of the air column. Therefore no reservoir of convective energy would be available to drive the cyclone intensification. Hack & Schubert (1982) argue that such a condition is favoured by linear CISK due to the approximation of the inertial stability C (see $(5.5)_3$) by f^2. Including the full effect of inertial instability on the cyclone development, Hack & Schubert (1982) found that an increased inertial stability within regions of deep convection yields an imbalance between adiabatic cooling of rising air parcels and heating due to latent heat release such that a net warming of an air column is realized to enhance the vortex intensity.

One shortcoming of the EbVM, however, is its limitation to the investigation of symmetric dynamics and thermodynamics in hurricane-like vortices, since real tropical cyclones are often highly asymmetric, particularly in the upper troposphere outside of the core (Molinari et.al, 1993). Attempts have been made to design extensions of the EbVM that make it possible to study the influence of azimuthal eddies on tropical cyclone development (e.g. Pfeffer & Challa, 1981; Molinari & Vollaro, 1990). Numerical solutions of such extended model versions show, for instance, that the inclusion of lateral fluxes of angular momentum by azimuthal eddies results in an enhanced secondary circulation that deepened the model storm to hurricane intensity. Based on the findings described above it is also possible that asymmetries in the vortex structure not only have an influnece on the vortex intensification but also may have a non-trivial influence on the vortex trajectory. Hence, in awareness of the importance of asymmetric vortex features, attempts are made in **Section 5.2** and **Section 5.3** to derive extended versions of the Eliassen balanced vortex model from an asymptotic perspective, that account for the influence of diabatically induced asymmetries in the velocity and potential temperature fields on the leading order secondary circulation. Among others it will be shown, that solutions for the secondary circulation obtained from such a model have a non-trivial effect on the vortex trajectory.

5.2 Modified EbVM with externally prescribed source term

For the vortices under consideration the relevant equations are the general balance conditions (3.135), (3.136), the horizontal momentum equations (3.43), the mass continuity (3.50), the thermodynamic equations (3.53) and the state equation (A-37). Since Eliassen's theory is developed within an axissymmetric framework, equations describing the axissymmetric thermodynamic fields are obtained by applying $(3.58)_1$ to the equations mentioned above. To be comparable with Eliassen's balanced vortex model a leading order absolute angular momentum $M^{(0)}$ is defined by

$$M^{(0)} = r u_\theta^{(0)} + \frac{\Omega_0}{2} r^2 \tag{5.7}$$

Then, the following general balanced vortex model can be derived

$$\begin{aligned}
M^{(0)^2} - \frac{1}{4}\Omega_0^2 r^4 &= r^3 \frac{\partial \pi}{\partial r} \\
\left(w_0^{(\frac{4}{2})} \frac{\partial}{\partial z} + u_{r,0}^{(\frac{1}{2})} \frac{\partial}{\partial r} \right) M^{(0)} &= K(r,z,\tau) \\
\frac{\partial \pi}{\partial z} &= \Theta_0^{(\frac{6}{2})} \\
\frac{1}{r}\frac{\partial (r u_{r,0}^{(\frac{1}{2})})}{\partial r} + \frac{1}{\rho^{(0)}} \frac{\partial (\rho^{(0)} w_0^{(\frac{4}{2})})}{\partial z} &= L(r,z,\tau) \\
w_0^{(\frac{4}{2})} \frac{\partial \Theta^{(\frac{4}{2})}}{\partial z} &= S_0^{(\frac{7}{2})}
\end{aligned} \tag{5.8}$$

with $\pi = p^{(\frac{6}{2})}/\rho^{(0)}$ and the sources K and L are given by

$$
\begin{aligned}
K(r,z,\tau) &= \frac{1}{2}\frac{\partial u_\theta^{(0)}}{\partial r}\left(r w_{12}^{(\frac{4}{2})}\frac{\partial X_C^{(\frac{1}{2})}}{\partial z} + r w_{11}^{(\frac{4}{2})}\frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\right) \\
L(r,z,\tau) &= \frac{1}{2}\left(\frac{\partial X_C^{(\frac{1}{2})}}{\partial z}\frac{1}{r}\frac{\partial (r w_{12}^{(\frac{4}{2})})}{\partial r} + \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\frac{1}{r}\frac{\partial (r w_{11}^{(\frac{4}{2})})}{\partial r}\right)
\end{aligned}
\tag{5.9}
$$

The asymmetric vertical velocities $w_{1k}^{(\frac{4}{2})}$ appearing in (5.9) are determined through the first sine and cosine modes of the $\mathcal{O}(\varepsilon^{8/2})$ thermodynamic equation

$$
w_{1k}^{(\frac{4}{2})}\frac{\partial \Theta^{(\frac{4}{2})}}{\partial z} = S_{1k}^{(\frac{7}{2})} \tag{5.10}
$$

The equations (5.8) - (5.10) are closed, provided the vortex tilt $\partial \vec{X}_C^{(\frac{1}{2})}/\partial z$, the background stratification $\partial \Theta^{(\frac{4}{2})}/\partial z$ and the diabatic source term $S^{(\frac{7}{2})}$ are known.

The following observations are made when comparing the asymptotically derived equation set (5.8) - (5.10) with the original governing equations (5.1) of the EbVM:

1. The temporal derivatives appearing in the original equations (5.1) are missing in set (5.8), which leads to a steady version of a Eliassen kind of balanced vortex model. This is attributed to the temporal scaling $\tau = \varepsilon^2 t$ used in expansion ansatz (3.15). Note that equations for the temporal evolution are determined by the next order $\mathcal{O}(\varepsilon^{\frac{3}{2}})$ horizontal momentum equation (3.45).

2. Equations $(5.8)_5$ and (5.10) describe a balance between adiabatic cooling $w^{(\frac{4}{2})}\partial_z \Theta^{(\frac{4}{2})}$ and diabatic heating $S^{(\frac{7}{2})}$.

3. A diabatic source term including asymmetric contributions must not necessarily violate the leading order axissymmetry of the circumferential flow. Thus the assumption by Eliassen that sources have to be axissymmeric in order to ensure the axissymmetry of the vortex circumferential flow is not necessary.

4. The equation set (5.8) includes with $\Theta^{(\frac{6}{2})}$ in the hydrostatic balance $(5.8)_3$ and $\Theta^{(\frac{4}{2})}$ in the $\mathcal{O}(\varepsilon^{\frac{7}{2}})$ potential temperature equation $(5.8)_5$ two asymptotically different potential temperature terms. Thus, with $\partial\Theta^{(\frac{4}{2})}/\partial z > 0$ as a given background stratification and $S^{(\frac{7}{2})}$ as a given source term, equation $(5.8)_5$ is decoupled from the rest of equations and can be solved for $w^{(\frac{4}{2})}$ independently.

5. Transverse circulation determined through $u_{r,0}^{(1/2)}$ and $w^{(\frac{4}{2})}$ is not divergence free for $\partial\vec{X}_C^{(\frac{1}{2})}/\partial z \neq 0$, $w_{1k}^{(\frac{4}{2})} \neq 0$.

6. The momentum equation $(5.8)_3$ has a non-zero source term owing to both $\partial\vec{X}_C^{(\frac{1}{2})}/\partial z \neq 0$ and $w_{1k}^{(\frac{4}{2})} \neq 0$ although frictional effects are neglected.

Note that a closer agreement of the asymptotically derived equations (5.8) - (5.10) with the governing equations (5.1) of the original EbVM can be achieved if one either assumes that the leading order diabatic heating rate is symmetric with respect to the vortex axis, which implies immediately that $w_{1k}^{(\frac{4}{2})} = 0$, or if one assumes that the leading order vortex tilt is zero i.e. $\partial\vec{X}_C^{(\frac{1}{2})}/\partial z = 0$. Based on such assumptions different versions of a balanced vortex model can be derived. Some aspects of the solutions for the secondary circulation of those versions are studied in the following subsections.

5.2.1 Far field solutions for the secondary circulation

In this section far field solutions for the secondary circulation $(u_{r,0}^{(\frac{1}{2})}, w_0^{(\frac{4}{2})})$ are derived. To simplify things lets first assume axissymmetric vertical velocities, i.e. $w^{(\frac{4}{2})} = w^{(\frac{4}{2})}(r,z,\tau)$ which implies that $w_{1k}^{(\frac{4}{2})} = 0$ and $S_{1k}^{(\frac{7}{2})} = 0$ (see (5.10)). Such an assumption simplifies (5.8). In particular, it turns out that the source terms $K = 0$ and $L = 0$ vanish such that (5.8) take the form of

$$
\begin{aligned}
M^{(0)^2} - \frac{1}{4}\Omega_0^2 r^4 &= r^3 \frac{\partial \pi}{\partial r} \\
\left(w^{(\frac{4}{2})} \frac{\partial}{\partial z} + u_{r,0}^{(\frac{1}{2})} \frac{\partial}{\partial r} \right) M^{(0)} &= 0 \\
\frac{\partial \pi}{\partial z} &= \Theta_0^{(\frac{6}{2})} \\
\frac{1}{r}\frac{\partial (r u_{r,0}^{(\frac{1}{2})})}{\partial r} + \frac{1}{\rho^{(0)}} \frac{\partial(\rho^{(0)} w^{(\frac{4}{2})})}{\partial z} &= 0 \\
w^{(\frac{4}{2})} \frac{\partial \Theta^{(\frac{4}{2})}}{\partial z} &= S^{(\frac{7}{2})}
\end{aligned}
\tag{5.11}
$$

Comparing $(5.11)_2$ with $(5.1)_2$ it is observed that in absence of asymmetric diabatic source terms and on time scales of about 1-3 days the leading order absolute angular momentum $M^{(0)}$ may be considered as being constant along streamlines instead as conserved along particle trajectories. With the aid of the far field behaviour of the leading order circumferential flow $u_\theta^{(0)}$ it will be shown that this allows us to deduce some fundamental features about the secondary circulation within the vortex region.

To begin with a kind of stream function $\acute{\psi}$ is introduced which is defined by

$$(\rho^{(0)} u_{r,0}^{(\frac{1}{2})}, \rho^{(0)} w^{(\frac{4}{2})}) = \left(-\frac{\partial \acute{\psi}}{\partial z}, \frac{1}{r} \frac{\partial (r\acute{\psi})}{\partial r} \right) \tag{5.12}$$

Based on this definition the corresponding streamline tangential to the velocity field described by $u_{r,0}^{(\frac{1}{2})}$ and $w^{(\frac{4}{2})}$ at every point in the $r-z$ plane is given by

$$\chi = r\acute{\psi} = \text{const.} \tag{5.13}$$

which is a solution of $(u_r^{(\frac{1}{2})}, w^{(\frac{4}{2})}) \times (dr, dz) = 0$. Using the matching condition (3.99), i.e. $u_\theta^{(0)} = \Gamma(z)/2\pi r$ as $r \to \infty$ the leading order angular momentum (5.7) takes the form

$$M^{(0)} = \frac{\Gamma(z)}{2\pi} + \frac{\Omega_0}{2} r^2 \qquad \text{as} \qquad r \to \infty \tag{5.14}$$

Upon substitution of (5.14) and (5.12) into $(5.11)_2$ a partial differential equation (PDE) satisfying far field solutions of χ is obtained

$$-\frac{\partial \chi}{\partial z} + \frac{1}{2\pi\Omega_0} \frac{\partial \Gamma(z)}{\partial z} \frac{1}{r} \frac{\partial \chi}{\partial r} = 0 \qquad \text{as} \qquad r \to \infty \qquad . \tag{5.15}$$

A method that can be used to solve first order PDE's is the method of characteristics. Applying this method to (5.15) the first step is to change coordinates from (r, z) space to a new coordinate system (z_0, s) in which the PDE becomes an ordinary differential equation along characteristic curves s. Note, the new variable s will vary, and the new variable z_0 will be constant along the characteristics. In so doing one assumes that $\chi = \chi(r(s), z(s))$ such that

$$\frac{d\chi}{ds} = \frac{\partial \chi}{\partial r}\frac{dr}{ds} + \frac{\partial \chi}{\partial z}\frac{dz}{ds} \tag{5.16}$$

By comparing (5.16) with (5.15) one obtains the following set of characteristic equations

$$\frac{d\chi}{ds} = 0, \qquad \frac{dz}{ds} = -1, \qquad \frac{dr}{ds} = \frac{1}{2\pi\Omega_0}\frac{\partial \Gamma(z)}{\partial z}\frac{1}{r} \tag{5.17}$$

The boundary conditions for (5.17) are

$$z(s=0) = z_0, \quad r(s=0) = r_0 \qquad \text{and} \qquad \chi(r, z_0) = r\ \acute{\psi}(r, z_0) \tag{5.18}$$

From $(5.17)_1$ it is observed that χ is conserved along characteristic curves s. This means that χ and s surfaces coincide and the characteristic curves can be comparably regarded as streamlines of the transversal circulation. Solutions of $(5.17)_2$ is $z = -s + z_0$ which can be used to rewrite $(5.17)_3$ as

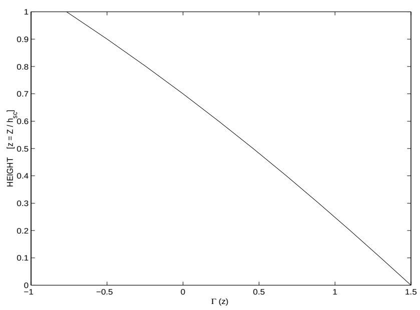

Figure 5.2: Vortex circulation $\Gamma(z)$ typically for hurricane-like vortices

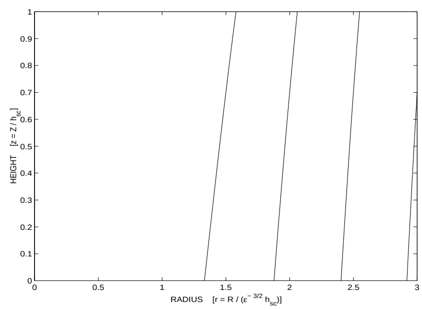

Figure 5.3: Characteristic curves denoting the streamlines for large r.

$$\frac{dr}{dz} = -\frac{1}{r}\frac{1}{2\pi\Omega_0}\frac{\partial\Gamma(z)}{\partial z} \tag{5.19}$$

Solving the equation (5.19) gives the characteristic curves, i.e.

$$r = \pm\sqrt{\frac{\Gamma(z_0) - \Gamma(z)}{\pi\Omega_0} + r_0^2} \qquad \text{as} \qquad r \to \infty \tag{5.20}$$

Assuming a circulation profile $\Gamma(z)$ that is typically for a hurricane, i.e. maximum values near the surfaces and a slow decrease upward becoming anticyclonic near the top of the storm (see **Figure 5.2**), characteristic curves determined by (5.20) are displayed in **Figure 5.3**. It is observed that the characteristics run mainly vertically with a slight slope. Since the characteristics can be similarly identified as streamlines, a conclusion can be drawn that the curves mark a branch of a leading order secondary circulation with mainly vertically moving air parcels. Hence, the shape of the streamlines for large r indicates a clear separation of inner vortex air masses from environmental air masses, leading to the picture of a closed leading order secondary circulation. Note that the direction (upward/downward) of the vertically moving air parcels depends on the boundary conditions choosen for χ at the reference level $z = z_0$.

The question comes up, however, whether far field solutions as derived above exist only because of the assumption of axissymmetric vertical velocities, i.e. $w^{(\frac{4}{2})} = w^{(\frac{4}{2})}(r, z, \tau)$ which implies that $w_{1k}^{(\frac{4}{2})} = 0$. Taking the matching condition into account that $w^{(\frac{4}{2})}(r, \theta, z, \tau) = 0$ as r approaches ∞ (see (3.113)), it can easily be verified that the source terms K and L in (5.8) disappear for large r even if asymmetric contributions $w_{1k}^{(\frac{4}{2})} \neq 0$ exist in the near core region. From that a conclusion can be drawn that the asymptotically derived picture of a closed leading order secondary circulation with respect to the far field region also holds in a more general case $w^{(\frac{4}{2})} = w^{(\frac{4}{2})}(r, \theta, z, \tau)$ for all r.

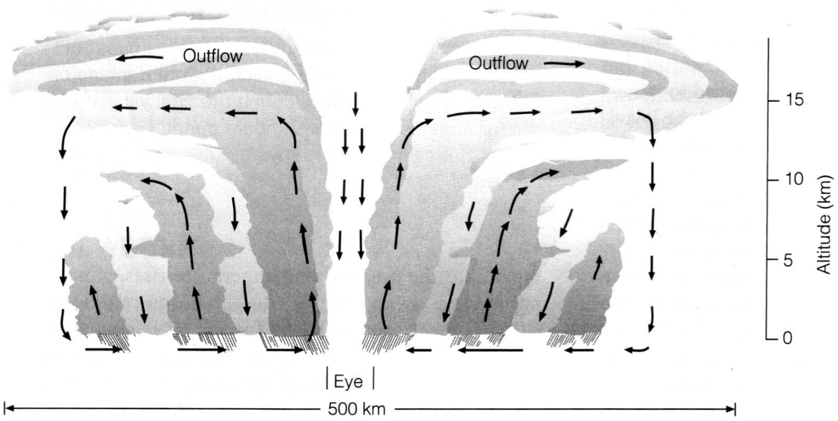

Figure 5.4: A model that shows a vertical view of air motions, clouds, and precipitation in a typical hurricane. (Graphical illustration from Ahrens (1999).)

A question that naturally comes up in connection with the derived far field results for the secondary circulation, is whether such solutions are typical for real hurricane like vortices. According to Emanuel (1991) real hurricanes are open systems that continually exchange mass with their environments. This wouldn't be in agreement with the just derived leading order results describing a closed secondary circulation where at least to a first approximation an exchange of air masses with the environment is not possible. As opposed to Emanuel's view, however, other textbooks (e.g. Ahrens, 1999) explain the clear weather conditions occuring immediately outside the storm area by sinking and warming air masses at the storms periphery (see **Figure 5.4**). Such arguments, in turn, would support the picture of a closed secondary circulation.

We conclude this section with the derivation of a certain class of solutions for the streamline χ applying the method of separation of variables. The solutions are needed for later discussions on the derivation of an equation for the leading order vortex trajectory. Assuming special product solutions, i.e. $\chi(r,z) = H(r)\ h(z)$, equation (5.15) can be separated into equations for $H(r)$ and $h(z)$, respectively. In particular one obtains

$$-2\pi\Omega_0 \left(\frac{\partial \Gamma}{\partial z}\right)^{-1} \frac{1}{h}\frac{\partial h}{\partial z} = -\frac{1}{r}\frac{1}{H}\frac{\partial H}{\partial r} = +\lambda^2, \quad \text{as} \quad r \to \infty \tag{5.21}$$

It turns out that general solutions describe exponential behaviour in r and $\Gamma(z)$ respectively, i.e.

$$\begin{aligned} h(z) &= C_h\ \exp\left(-\lambda^2/(2\pi\Omega_0)\ \ \Gamma(z)\right) \\ H(r) &= C_H\ \exp\left(-\lambda^2/2\ \ r^2\right) \end{aligned} \tag{5.22}$$

From (5.12) one obtains

$$\rho^{(0)} w^{(\frac{4}{2})} = C_H \; h(z) \; \exp\left(-\lambda^2/2 \;\; r^2\right) \qquad \text{as} \quad r \to \infty \tag{5.23}$$

Note that to guarantee a closed transverse circulation with upward moving flow in the vicinity of the core and downward moving flow far away from the vortex centre the z dependent solution $h(z)$ has to satisfy $h(z) < 0$ if $C_H > 0$.

5.2.2 Constraints on the nature of an externally prescribed diabatic source

Two different routes for the treatment of diabatic heat sources S_L due to latent heat release have been proposed in **Section 3.4**. In this section the leading order diabatic source is externally prescribed by means of (3.119). For such a case it is shown that solutions of $(5.11)_5$ cannot fulfill the constraints on the far field behaviour of the streamlines χ denoting the transverse circulation as discussed in **Section 5.2.1**. It turns out that physically meaningful results are only obtainable if the prescribed heating function meets certain conditions. Moreover it is shown that in this context certain conclusion can be drawn about the nature of the condensation process producing heating rates in the order $S_L^{(\frac{7}{2})}$.

Solutions for a transverse circulation Because of the axissymmetry of the prescribed heat source (3.119), from (5.10) it follows immediately that asymmetric vertical velocities disappear, i.e. $w_{1k}^{(\frac{4}{2})} = 0$. Hence, (5.11) is used in the subsequent analysis. Substitution of (5.12) and (3.119) into $(5.11)_5$ yields an equation for $\acute{\psi}$, reading

$$\frac{1}{r}\frac{\partial(r\acute{\psi})}{\partial r} \; \frac{d\Theta^{(\frac{4}{2})}}{dz} \;=\; \rho^{(0)}(z) \; \tilde{\beta}(z) \; r \; \exp\left(d \; \left[1 - r^2\right]\right) \tag{5.24}$$

with $\tilde{\beta}(z) = \sin(\pi z)\exp\left(-\alpha \; z\right)$. Assuming a constant background stratification, i.e. $d\Theta^{(\frac{4}{2})}/dz = \text{const.}$, and the boundary condition $(r\acute{\psi}) = \chi = 0$ at $r = 0$ (i.e. no vertical mass flux at the vortex centre), integration of (5.24) in radial direction from 0 to r yields

$$\acute{\psi} r = \mu(z) \; b^\star \; \text{erf}\left(\sqrt{d}\; r\right) - \mu(z) \; b^{\star\star} \; r \exp(-d \; r^2) \tag{5.25}$$

Here, erf(r) is the 'error function'. The vertical distribution of $\acute{\psi}$ is determined through $\mu(z) = \rho^{(0)}\tilde{\beta} \; (\partial\Theta^{(4/2)}/\partial z)^{-1}$ and the remaining coefficients are $b^{\star\star} = \exp\left(d\right)/(2d)$ and $b^\star = \exp(d) \; \sqrt{\pi}/4 \; d^{3/2}$. Considering the limit for large r,

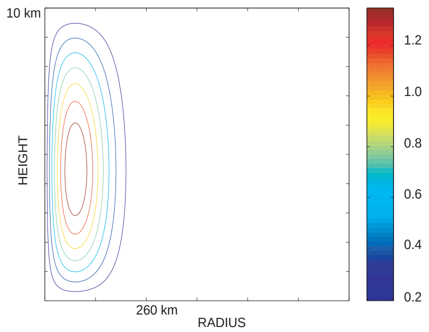

Figure 5.5: Heat Source $S^{(\frac{7}{2})}$

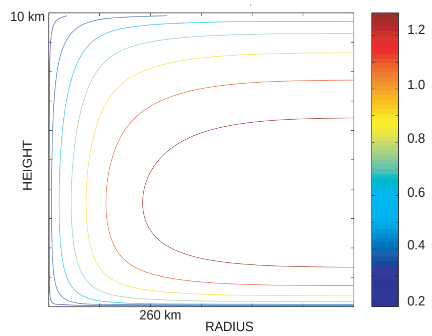

Figure 5.6: Streamlines χ

the far field solution of (5.25) is

$$\acute{\psi} r = \mu(z)\ b^{\star} \qquad \text{as} \quad r \to \infty \tag{5.26}$$

The graphs of the heating field used and the streamlines $\chi = \acute{\psi} r$ of the corresponding enforced transverse circulation are shown in **Figure 5.5** and **5.6**. It is observed that the heating source induces an ascending branch with vertically upward moving air parcels in the vicinity of the vortex core. In the region far away from the bulk of latent heat release, however, it is observed that the streamlines $\chi = r\acute{\psi}$ run horizontally, which leads to the picture of an open secondary circulation with respect to the surrounding. Unfortunately, such solutions for the stream function do not conform with the constraints on the far field solutions discussed in **Section 5.2.1**. Recall that the far field constraints on χ have been derived using the matching condition (3.99) for circumferential velocity in leading order, i.e. $u_\theta^{(0)} = \Gamma/(2\pi r)$ as $r \to \infty$. Note that with the solutions for the stream function $\chi = r\acute{\psi}$ shown in **Figure 5.6**, this condition can not be satisfied. That is why conservation of $M^{(0)}$ along χ (see $(5.11)_2$ and (5.7)) together with the horizontal run of the streamlines in the far field region yields unphysically, large negative circumferential velocities $u_\theta^{(0)}$ for fluid particles approaching ∞.

Finally, the only way out of the above dilemma is to conclude that the given forcing function (3.119) can not be used to describe the leading order thermal forcing $S_L^{(\frac{7}{2})}$.

External heat source requirements From the discussions above it is obvious that an appropriate heat source in leading order should include the net effects of mechanisms that force the streamlines to become vertically oriented far away from the bulk of heating as shown in **Figure 5.3**. Possible mechanisms, for instance, can be studied by adopting Emanuel's view about hurricanes as a natural Carnot engine that converts heat energy extracted from the ocean to mechanical energy (Emanuel, 1991). In his idealized picture (see **Figure 5.7**) Emanuel relates the first three steps of the Carnot-process to the secondary circulation as follows:

1. *isothermal expansion* between point a and C as air flows along the sea surface radially inward toward regions of much lower surface pressure; here so much heat is added from the warm ocean surface to the air that the surface air temperature remains nearly constant

2. *adiabatic expansion* between point C and o as air ascends within deep convective clouds in the eyewall of the storm and then flows out to large radii (adiabatic cooling due to ascent = heating due to latent hat release)

3. *isothermal compression* between point o and o' as air descends slowly in the lower stratosphere; a nearly constant temperature is retained while loosing heat by electromagnetic radiation to space

As noted earlier, Emanuel (1991) considers hurricanes as open systems that continually exchange mass with the environment. For that reason he points out that in his idealized picture the fourth branch of a Carnot cycle does not really exists, since this would require the downward movement of air parcels to close the cycle. Please note that only the first three branches in **Figure 5.7** are streamlines denoting the path of an air parcel under steady state conditions. The fourth branch is an absolute vortex line which has nothing to do with the path taken by the moving air parcels. Returning to the results of the present work, however, it is observed that the leading order secondary circulation given through $(u_{r,0}^{(\frac{1}{2})}, w^{(\frac{4}{2})})$ can be regarded as a closed circulation with respect to the far field region. That means in our picture about a secondary circulation all branches denote streamlines, which means that we have indeed descending air masses in the far field vortex region. Hence staying with the idea of a hurricane Carnot cycle, there must be an additional cooling mechanism along the fourth branch that balances the heating due to adiabatic descent. Here, however, the question arises on what causes such cooling effects. One possibility, for instance, is to put the descending branch in the far field region down to radiative cooling effects. Recall that in **Figure 5.7** the existence of a third descending branch confined to the upper troposphere has already been explained through a loss of heat due to electromagnetic radiation to space. One might assume that this cooling effect works throughout the whole troposphere in order to bring the air masses in the far field region down from the top of the troposphere to the surface. Another mechanism that also might explain how air masses come down to the surface is strongly related to moisture conversion processes. At

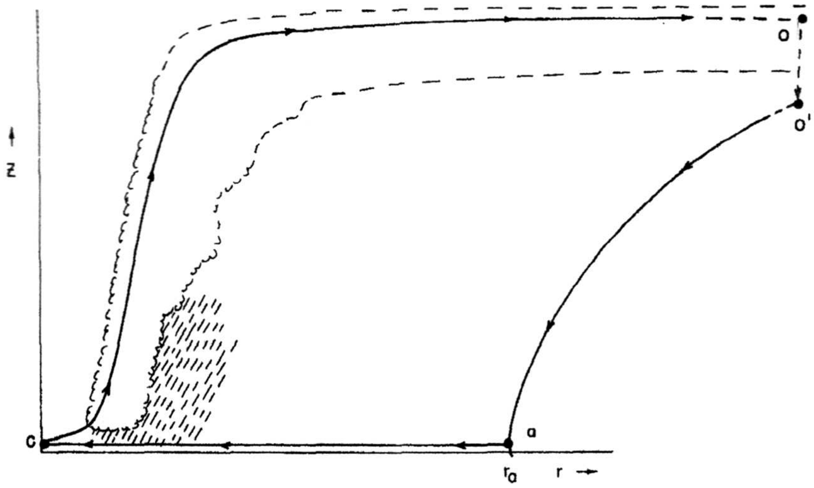

Figure 5.7: Schematic diagram for the hurricane Carnot cycle. (see the text for explanations; graphical illustration from Emanuel (1991))

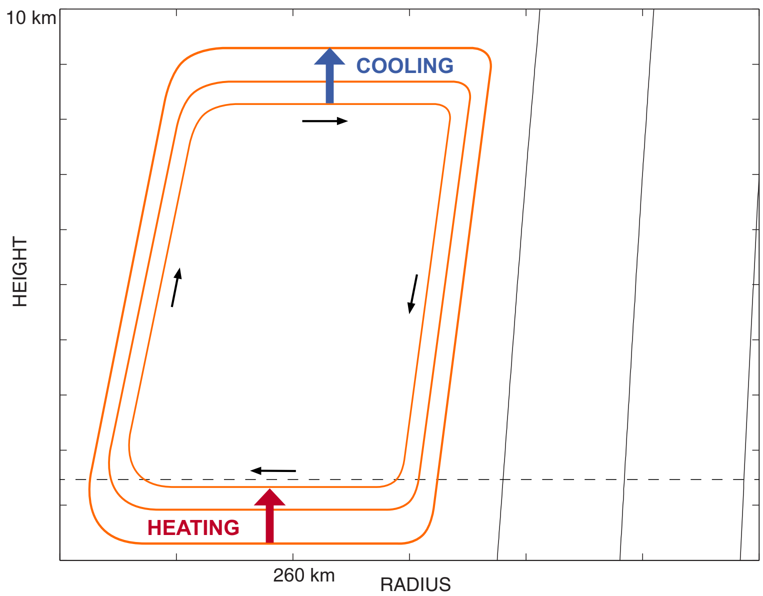

Figure 5.8: Schematic diagram for radial and vertical motions defining the secondary circulation from an asymptotic perspective (see text for further explanations)

least from theoretical point of view it is conceivable that cooling effects may be caused by evaporation processes. This may occur if dry air parcels from the outer edge of the outflow branch move downward along vertically and slightly inward directed streamlines as indicated in **Figure 5.3**. In this way dry air reaches regions of rich precipitation falling out from the upper outflow branch and rain may evaporate within the dry air parcels. The latter is accompanied by evaporative cooling which in turn forces the air masses downward towards the surface. Hence, in the context of the present work a fourth descending branch of a secondary vortex circulation can be described by

4. *adiabatic compression* as air descends through rainy regions (adiabatic heating due to descent = adiabatic cooling due to evaporation of rain)

But note that it is the subject of future work to investigate whether the above mentioned cooling effect due to evaporative cooling offers an adequate explanation for mechanisms that force the air masses down from the top of the troposphere to the surface. Thus an asymptotic analysis should be carried out based on multiple scale expansions that also resolve smaller cumulus scales including moisture parameters explicitly.

The diagram in **Figure 5.8** summarizes the results from the thought experiments described above. With such a picture in mind a specified heating function for $S_L^{(\frac{7}{2})}$ has to account for both latent heat release due to condensation near the storm centre and latent heat removal due to evaporation. This implies that a heating function used to describe $S_L^{(\frac{7}{2})}$ must not only mirror the net effect of convective heating but also additional cooling processes. Representing the latter in an appropriate way may eliminate the unphysical results for the leading order circumferential velocity.

Based on the discussion above one may also question the treatment of diabatic source terms as an externalized force in question. Recall that the heating function applied above was designed to describe the net effect of heating due to cumulus convection. In section **Section 3.4.2**, however, it has been pointed out that it is important to differentiate between the nature of the condensation processes. Thus, the lack of success of the externally pescribed heat source in generating a secondary circulation may be attributed to the fact that heating rates $S_L^{(\frac{7}{2})}$ may be regarded as a result of large-scale vertical motion rather than as a result of deep cumulus convection acting on smaller scales. Whether this could be the case will be studied in the section below. This, however, calls for an explicit treatment of moisture. The importance of an explicit treatment of moisture has already been discussed by Emanuel (1991). In particular, in step two of the Carnot cycle he points out that modelling hurricane-like vortices, adiabatic conditions within the ascending branch of the secondary circulation can only be reached if water vapor is properly included. That is why the diabatic source is purely a function of the flow itself which makes an external treatment of a diabatic source difficult.

5.3 Modified EbVM with explicit inclusion of moisture

In the analysis below moisture shall be included explicitly. Then, retracing the route suggested in **Section 3.4.2**, the diabatic source S_L (see (3.115)) is considered as a sum of a resolved contribution S_{rs} and an unresolved contribution S_{us}. Using the notations (3.133) for S_{rs} under the idealized assumption of a saturated atmosphere for the vortex region, an asymptotic expansion for S_L is

$$S_L = \varepsilon^{\frac{7}{2}} S_L^{(\frac{7}{2})} + \varepsilon^{\frac{8}{2}} S_L^{(\frac{8}{2})} + \varepsilon^{\frac{9}{2}} S_L^{(\frac{9}{2})} + \mathcal{O}(\varepsilon^5)$$

where

$$\begin{aligned}
S_L^{(\frac{7}{2})} &= -\Gamma^{\star\star} L^{\star\star} q_{vs}^{\star\star}\ w^{(\frac{4}{2})} \frac{dq_{vs}^{(0)}}{dz} + S_{us}^{(\frac{7}{2})} \\
S_L^{(\frac{8}{2})} &= -\Gamma^{\star\star} L^{\star\star} q_{vs}^{\star\star}\ w^{(\frac{5}{2})} \frac{dq_{vs}^{(0)}}{dz} + S_{us}^{(\frac{8}{2})} \\
S_L^{(\frac{9}{2})} &= -\Gamma^{\star\star} L^{\star\star} q_{vs}^{\star\star} \left(w^{(\frac{6}{2})} \frac{dq_{vs}^{(0)}}{dz} + (\Gamma^{\star\star} z)\ w^{(\frac{4}{2})} \frac{dq_{vs}^{(1)}}{dz} \right) + S_{us}^{(\frac{9}{2})}
\end{aligned} \tag{5.27}$$

Recall that S_{us} denotes a net-heating effect of convective processes due to small-scale vertical motions upon the vortex-scale flow. Because of the fact that single-scale expansions resolving vertical motions on vortex scales are used, only heat sources S_{rs} can be regarded as a function of the vortex flow itself, whereas S_{us} remains to be parameterized. In order to elucidate the role of S_{us} on both the three-dimensional vortex-scale flow and the synoptic scale vortex motion, two different vortex models are discussed below. The derivation of a vortex model based on the assumption $S_{\text{us}} = 0$ (hereafter refered to as `Model A`) is presented in **Subsection 5.3.1** and in **Subsection 5.3.2** a model based on $S_{\text{us}} \neq 0$ (hereafter refered to as `Model B`) is discussed.

5.3.1 Model A - no net-heating due to small-scale convective processes

For the subsequent asymptotic analysis of this section it is assumed that $S_{\text{us}} = 0$, which in turn implies that $S_{\text{us}}^{(\frac{i}{2})} = 0$ for $i \in N$. Then, substitution of the diabatic source term $(5.27)_1$ into the $\mathcal{O}(\varepsilon^{\frac{7}{2}})$ thermodynamic equation $(5.11)_5$ yields

$$w^{(\frac{4}{2})}\left(\frac{d\Theta^{(\frac{4}{2})}}{dz}+\Gamma L^{\star\star}\frac{dq_{vs}^{(0)}}{dz}\right)=0 \tag{5.28}$$

The above equation is satisfied either if $w^{(\frac{4}{2})}=0$ or if the sum in the bracket is zero. Recall that from asymptotic analysis of moist atmospheric processes, Klein & Majda (2005) found solutions that allow for deep convection that the background potential temperature $\Theta^{(\frac{4}{2})}=\Theta^{(\frac{4}{2})}(z)$ satisfies the moist adiabatic equation (3.134). In such a case the term in brackets in (5.28) disappears. As a consequence, next higher order potential temperature equations have to be considered in order to find equations that can be used to determine $w^{(\frac{4}{2})}$.

The $\mathcal{O}(\varepsilon^{\frac{8}{2}})$ thermodynamic equation (3.54) taken together with the source term $(5.27)_2$ and the assumption that $S_{us}^{(\frac{8}{2})}=0$ results in

$$\frac{u_\theta^{(0)}}{r}\frac{\partial\Theta^{(\frac{6}{2})}}{\partial\theta}+w^{(\frac{5}{2})}\frac{d\Theta^{(\frac{4}{2})}}{dz}=-\Gamma L^{\star\star}w^{(\frac{5}{2})}\frac{dq_{vs}^{(0)}}{dz} \tag{5.29}$$

Note that the higher order advection terms $w^{(\frac{4}{2})}d\Theta^{(\frac{5}{2})}/dz$ and $w^{(\frac{4}{2})}dq_{vs}^{(\frac{1}{2})}/dz$ do not appear in (5.29) due to $\Theta^{(\frac{5}{2})}=0$ and $q_{vs}^{(\frac{1}{2})}=0$ (see (3.107) and (3.125)). It can easily be verified that rearranging terms in (5.29) and use of the moist adiabatic equation (3.134), yields

$$\frac{u_\theta^{(0)}}{r}\frac{\partial\Theta^{(\frac{6}{2})}}{\partial\theta}=0 \quad ,$$

For $u_\theta^{(0)}\neq 0$ this implies that

$$\Theta^{(\frac{6}{2})}=\Theta^{(\frac{6}{2})}(r,z,\tau)\quad\text{or}\quad\Theta_{ik}^{(\frac{6}{2})}=0\qquad i=1,2,...,n;\quad k=1,2 \tag{5.30}$$

The two terms in (5.29) cancel only identically provided that sufficient amount of liquid water is available along the downward moving branches within the secondary circulation, i.e. in the regions where $w^{(\frac{5}{2})}$ is negative. Since for $w^{(\frac{5}{2})}<0$ the term on the right hand side of (5.29) reflects an evaporation rate, the term would disappear if there is no liquid water available for evaporation. Here, however, we assume that sufficient liquid water is available through the presence of cloud water and rain droplets.

Using (5.30) a harmonic analysis of the vertical momentum equation (3.46) and the state equation (A-37) yields

$$\frac{\partial X_C^{(\frac{1}{2})}}{\partial z}\frac{1}{\rho^{(0)}}\frac{\partial p^{(\frac{6}{2})}}{\partial r}=0\qquad\text{and}\qquad\frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\frac{1}{\rho^{(0)}}\frac{\partial p^{(\frac{6}{2})}}{\partial r}=0 \tag{5.31}$$

Two possible conclusions can be drawn from (5.31). Either that the radial pressure gradient is zero or that the vortex tilt is zero. A zero pressure gradient together with the gradient wind relation (3.135) would yield $u_\theta^{(0)} = \Omega_0\, r$. Such a solution for the leading order tangential velocity, however, does not satisfy the matching condition (3.99). Thus, for reasons of consistency we have to conclude that

$$\boxed{\frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial X_C^{(\frac{1}{2})}}{\partial z} = 0} \tag{5.32}$$

Summarizing the findings (5.30) - (5.32) it is observed that unlike the adiabatic vortex case (see **Section 4.2.1**), zero asymmetric potential temperature fields $\Theta_{1k}^{(\frac{6}{2})}$ in the diabatic vortex case exclude the existence of vertically varying centreline solutions $\vec{X}_C^{(\frac{1}{2})}$. This motivates the following thought experiment. It is assumed, as for adiabatic vortices, that a matching of the higher order velocity fields yields for diabatic vortices a direct relation between the vertical background shear and the vortex tilt in next higher order, i.e. $\partial\vec{X}_C^{(\frac{1}{2})}/\partial z \sim \partial\vec{V}_{B,C}^{(\frac{1}{2})}/\partial z$ (see (4.104)). Because of (5.32), however, this would set a constraint $\partial\vec{V}_{B,C}^{(\frac{1}{2})}/\partial z = 0$ on the vertical shear of a given background flow, i.e. there would be no degree of freedom in the choice of a vertical distribution for the higher order background flow. This conclusion together with (5.30) - (5.32) would mean that vertically varying centreline solutions $\vec{X}_C^{(\frac{1}{2})}$ do not exist in a vertically sheared background flow $\partial\vec{V}_{B,C}^{(\frac{1}{2})}/\partial z \neq 0$. Please note, however, that a verification of the validity of $\partial\vec{X}_C^{(\frac{1}{2})}/\partial z \sim \partial\vec{V}_{B,C}^{(\frac{1}{2})}/\partial z$ for the diabatic vortex is difficult due to the high complexity of the asymptotic equations for $u_r^{(\frac{2}{2})}$ and $u_\theta^{(\frac{2}{2})}$ if moisture is included.

Since the vertical velocity $w^{(\frac{4}{2})}$ does not appear in the $\mathcal{O}(\varepsilon^{\frac{8}{2}})$ thermodynamic equation (5.29), the next higher order thermodynamic equation is considered such that the equations $(5.8)_1$ - $(5.8)_4$ are closed if moisture is included explicitly. Using the above derived results (5.30), (5.32), the moist adiabatic equation (3.134), and the diabatic source term $(5.27)_3$ (where $S_{\text{us}}^{(\frac{9}{2})} = 0$), the $\mathcal{O}(\varepsilon^{\frac{9}{2}})$ thermodynamic equation (3.56) takes the form

$$u_r^{(\frac{1}{2})} \frac{\partial\Theta^{(\frac{6}{2})}}{\partial r} + u_\theta^{(0)} \frac{1}{r} \frac{\partial\Theta^{(\frac{7}{2})}}{\partial\theta} + w^{(\frac{4}{2})} \left(\frac{\partial\Theta^{(\frac{6}{2})}}{\partial z} + \mathcal{J}(z) \right) = 0 \tag{5.33}$$

with

$$\mathcal{J}(z) = {\Gamma^{\star\star}}^2 L^{\star\star} q_{vs}^{\star\star}\; z\; \frac{dq_{vs}^{(1)}}{dz} \tag{5.34}$$

Note that due to the explicit treatment of moisture the process of latent heat release affects the stratification in the vertical advection term of (5.33). Refer to **Section 5.3.2.2** for further comments on this issue. Finally, taking into account that $q_{vs}^{(1)} = q_{vs}^{(1)}(z)$ and $\Theta^{(\frac{6}{2})} = \Theta^{(\frac{6}{2})}(z, r, \tau)$, the zeroth mode equation of (5.33) is

$$u_{r,0}^{(\frac{1}{2})} \frac{\partial \Theta^{(\frac{6}{2})}}{\partial r} + w_0^{(\frac{4}{2})} \left(\frac{\partial \Theta^{(\frac{6}{2})}}{\partial z} + \mathcal{J} \right) = 0 \qquad (5.35)$$

Now that a closed set of equations have been obtained, we summarize and discuss the axissymmetric vortex structure equations in **Section 5.3.1.1** and the asymmetric equations in **Section 5.3.1.3**. Equations for the leading order vortex motion are derived in **Section 5.3.1.2**. Please note that the equations discussed in the mentioned subsection constitute a closed set of equations that can be used to study the isolated effect of diabatic heating sources on the motion and structure of hurricane-like vortices, whereas it has been taken into account that the diabatic source is a function of the vortex-scale flow itself.

5.3.1.1 Model A - Part I

Vortex features related to the *axissymmetric* vortex structure are discussed in this subsection. The system of equations determining the axissymmetric vortex structure is refered to as `Model A - Part I` and is summarized in **Table 5.1**. It turns out that with the replacement of $(5.8)_5$ through (5.35) and taking into account a zero tilt (5.32), the modified EbVM (5.8) valid for an externally given diabatic source term becomes a modified EbVM that explicitly accounts for condensation heating due to vortex-scale forced uplift. Note that together with (5.37)-(5.38), i.e. asymptotic solutions for the leading and next higher order saturation mixing ratio $q_{vs}^{(0)}$, $q_{vs}^{(\frac{2}{2})}$, density $\rho^{(0)}$ and pressure $p^{(\frac{2}{2})}$ (see **Appendix A.6**), the modified EbVM (5.36) is closed in the unknowns $u_\theta^{(0)}$, $u_{r,0}^{(\frac{1}{2})}$, $w_0^{(\frac{4}{2})}$, $\Theta^{(\frac{6}{2})}$ and $p^{(\frac{6}{2})}$.

Far field constraints It is observed that the equation set (5.36) with explicit treatment of moisture and the equation set (5.11) based on an externally prescribed diabatic source term differ only in the potential temperature equation. It is shown next, that based on the new potential temperature equation $(5.36)_5$ and the matching conditions for $\Theta^{(\frac{6}{2})}$ with the environmental temperature field $\hat{\Theta}^{(3)}$, stronger constraints on the secondary circulation in the far field region can be derived, than obtained from the horizontal momentum equation $(5.36)_2$ alone which has been discussed in **Section 5.2.1**.

It can easily be verified that with the aid of the definitions for the stream function (see (5.12)-(5.13)), the matching condition (3.111) for $\Theta^{(\frac{6}{2})}$ and using

`Model A - Part I`

$$
\begin{aligned}
M^{(0)^2} - \frac{1}{4}\Omega_0^2 r^4 &= r^3 \frac{\partial \pi}{\partial r} \\
\left(w_0^{(\frac{4}{2})} \frac{\partial}{\partial z} + u_{r,0}^{(\frac{1}{2})} \frac{\partial}{\partial r} \right) M^{(0)} &= 0 \\
\frac{\partial \pi}{\partial z} &= \Theta^{(\frac{6}{2})} \\
\frac{1}{r}\frac{\partial (r u_{r,0}^{(\frac{1}{2})})}{\partial r} + \frac{1}{\rho^{(0)}} \frac{\partial (\rho^{(0)} w_0^{(\frac{4}{2})})}{\partial z} &= 0 \\
w_0^{(\frac{4}{2})} \left(\frac{\partial \Theta^{(\frac{6}{2})}}{\partial z} + \mathcal{J}(z) \right) + u_{r,0}^{(\frac{1}{2})} \frac{\partial \Theta^{(\frac{6}{2})}}{\partial r} &= 0
\end{aligned}
\tag{5.36}
$$

where

$$
\begin{aligned}
M^{(0)} &= r u_\theta^{(0)} + \frac{\Omega_0}{2} r^2 \\
\pi &= \frac{p^{(\frac{6}{2})}}{\rho^{(0)}} \\
\mathcal{J}(z) &= \Gamma^{\star\star^2} L^{\star\star} q_{vs}^{\star\star} \; z \; \frac{d q_{vs}^{(1)}}{dz}
\end{aligned}
\tag{5.37}
$$

and

$$
\begin{aligned}
q_{vs}^{(0)} &= \exp\left(-[A^{\star\star}\Gamma^{\star\star} - 1] z\right) \\
q_{vs}^{(1)} &= q_{vs}^{(0)} \left[\left(A^{\star\star} \frac{\partial \Theta^{(\frac{4}{2})}}{\partial z} - \frac{1}{2} A^{\star\star} \Gamma^{\star\star^2} z^2 \right) + \right. \\
&\qquad \left. \exp(-z)(A^{\star\star}\Gamma^{\star\star} - 1) \; p^{(1)}(z) \right] \\
\rho^{(0)} &= p_0 \exp(-z) \\
p^{(1)} &= p_0 \Gamma^{\star\star} \left(-\frac{1}{2} z^2 \right) \exp(-z)
\end{aligned}
\tag{5.38}
$$

Table 5.1: `Model A - Part I`, see the text for explanations

the fact that $\rho^{(i)} = \rho^{(i)}(z)$, $p^{(i)} = p^{(i)}(z)$ and $q_s^{(i)} = q_s^{(i)}(z)$ (with $i = 0, 1$), $(5.36)_5$ can be written as

$$\frac{1}{r}\frac{\partial \chi}{\partial r}\left(\frac{\Omega_0}{2\pi}\frac{\partial^2 \Gamma}{\partial z^2}\ln r + \tilde{\mathcal{J}}(z)\right) - \frac{\partial \chi}{\partial z}\frac{\Omega_0}{2\pi}\frac{\partial \Gamma}{\partial z}\frac{1}{r^2} = 0 \qquad \text{as} \quad r \to \infty \tag{5.39}$$

with

$$\tilde{\mathcal{J}}(z) = \mathcal{J}(z) - \Omega_0\left(\frac{\partial^2 \check{\Psi}^{(0)}_{B,C}}{\partial z^2} + \frac{\partial^2 \check{\psi}^{(0)}_{R,C}}{\partial z^2}\right)$$

and where $\mathcal{J}(z)$ is given by (5.34). Elimination of $\partial\chi/\partial z$ from (5.39) and the horizontal momentum equation (5.15) yields

$$\frac{1}{r}\frac{\partial \chi}{\partial r}\left(\frac{\Omega_0}{2\pi}\frac{\partial^2 \Gamma}{\partial z^2}\ln r + \tilde{\mathcal{J}}(z) - \frac{1}{4\pi^2}\left(\frac{\partial \Gamma}{\partial z}\right)^2\frac{1}{r^2}\right) = 0 \qquad \text{as} \quad r \to \infty \tag{5.40}$$

Note that (5.40) is only satisfied if

$$\frac{\partial \chi}{\partial r} = o\left((\ln r)^{-1}\right) \qquad \text{as} \quad r \to \infty \tag{5.41}$$

Taking (5.41) into account, one obtains from (5.15) together with (5.12), that

$$-\frac{\partial \chi}{\partial z} \;=\; \rho^{(0)} u_{r,0}^{(\frac{1}{2})} \;=\; 0 \qquad \text{as} \quad r \to \infty \tag{5.42}$$

This implies that the radial inflow $u_{r,0}^{(\frac{1}{2})}$ vanishs rapidly for large r, since $\rho^{(0)} = \rho^{(0)}(z)$. Hence, (5.42) implies that for the vortex case studied in the present section the location of the descending branch of the secondary circulation must be closer to the vortex centre than indicated by the asymptotically derived characteristic curves shown in **Figure 5.8**.

General remarks Recall that in **Section 5.2.2** an externally imposed heating source with maximum heating in near the vortex core but no cooling in the far field region was not suitable to generate a descending branch of a secondary circulation in the far field region. Recall that the far field cooling constraint was as a result of the matching condition $u_\theta^{(0)} = \Gamma/(2\pi r)$ for large r (see (3.99)). Unfortunately, even in the present case where moisture effects resolved on vortex-scales are explicitly included, there is no guarantee that the solutions of the equations summarized in **Table 5.1** characterise such a closed secondary circulation. Since `Model A - Part I` does not know that the matching condition (3.99) has to be satisfied, principially it might be possible that solutions of `Model A - Part I` describe a secondary circulation that does not satisfy the feature of a closed circulation. This, however, can only be verified by solving the equations numerically which is beyond the scope of this work. Failure of `Model A - Part I` to satisfy the far field constraints may be as a consequence of assuming that $S_{\mathrm{us}}^{(\frac{9}{2})} = 0$. This assumption may lead to an exclusion of mechanisms necessary to generate a descending branch in the far field region of the secondary vortex circulation. This possibility is studied in more detail on the basis of `Model B` in **Section 5.3.2**. However, even if in the currently discussed `Model A - Part I` effects caused by $S_{\mathrm{us}}^{(\frac{9}{2})}$ are excluded, there is a possibility of avoiding solutions for the secondary circulations that are not compatible with the far field constraints. Recall, that unphysically large tangential velocities $u_\theta^{(0)}$ are a consequence of solutions for the stream function χ that describes an open circulation (see **Section 5.2.2**). To avoid such unphysical results which are not compatible with the matching condition (3.99), i.e. $u_\theta^{(0)} = \Gamma(z)/(2\pi r)$ for large r, momentum sinks in the horizontal momentum equation $(5.36)_2$ can be introduced. This decelerate unphysically large tangential velocities $u_\theta^{(0)}$ towards resonable results, which in turn would also enforce a closed secondary circulation. Please keep in mind, however, that such an approach is somewhat arbitrary, especially when it is not clear whether in the order considered dissipative effects may appear in the interior of the vortex at all. Though, such a simplified approach may be fruitful if `Model A - Part I` together with `Model A - Part II` is used to study both the impact of the leading order primary vortex circulation given by $u_\theta^{(0)}$ and the impact an asymmetric secondary circulation given by $w_{1k}^{(\frac{1}{2})}$, on the vortex trajectory. This is addressed in the next two subsections.

Note that other alternatives that may result in simplified model equations without the deficiencies described above are based on the inclusion of radiative heating and a weakening of the idealized picture of a fully saturated model atmosphere within the vortex region. These alternatives have not been addressed in this thesis.

5.3.1.2 Leading order vortex motion

In this section equations for the leading order motion of diabatic vortices are derived by means of matched asymptotics. While studying adiabatic vortices in **Section 4.2.2** it has been shown that in absence of diabatic effects the leading order vortex trajectory is determined by a superposition of a prescribed background field and regular field caused by the β-effect. It is the purpose of this section to show how the inclusion of diabatic source terms leads to a modification of the results for the leading order vortex motion.

Starting point of derivation is the $\mathcal{O}(\varepsilon^{\frac{3}{2}})$ mass continuity equation (3.50).

For the moist vortex setting considered in the present section with $w^{(\frac{4}{2})} \neq 0$ but $\partial \vec{X}_C^{(\frac{1}{2})}/\partial z = 0$ (see (5.32)), the mass continuity takes the form

$$\rho^{(0)} \left(\frac{\partial u_r^{(\frac{1}{2})}}{\partial r} + \frac{u_r^{(\frac{1}{2})}}{r} + \frac{1}{r} \frac{\partial u_\theta^{(\frac{1}{2})}}{\partial \theta} \right) + \frac{\partial (\rho^{(0)} w^{(\frac{4}{2})})}{\partial z} = 0 \tag{5.43}$$

It is observed from this equation that due to non-zero vertical velocities $w^{(\frac{4}{2})} \neq 0$ the next higher order horizontal velocity field $(u_r^{(\frac{1}{2})}, u_\theta^{(\frac{1}{2})})$ of diabatic vortices is divergent unlike the next higher order flow of adiabatic vortices. It is shown in the subsequent analysis that this already causes a departure of the diabatic vortex trajectory from the background flow in leading order.

Velocity potential Using Helmholtz's theorem (see **Section 3.1.4**) solutions for the divergent velocity components $u_r^{d(\frac{1}{2})}$ and $u_\theta^{d(\frac{1}{2})}$ have to satisfy

$$\frac{\partial u_r^{d(\frac{1}{2})}}{\partial r} + \frac{u_r^{d(\frac{1}{2})}}{r} + \frac{1}{r} \frac{\partial u_\theta^{d(\frac{1}{2})}}{\partial \theta} = -\frac{1}{\rho^{(0)}} \frac{\partial \rho^{(0)} w^{(\frac{4}{2})}}{\partial z} \tag{5.44}$$

A harmonic analysis for wavenumber-one asymmetric contributions $u_r^{d(\frac{1}{2})}$,$u_\theta^{d(\frac{1}{2})}$ and $w_{1k}^{(\frac{4}{2})}$ with $k = 1, 2$ yields

$$\begin{aligned} \frac{\partial u_{r,11}^{d(\frac{1}{2})}}{\partial r} + \frac{u_{r,11}^{d(\frac{1}{2})}}{r} - \frac{u_{\theta,12}^{d(\frac{1}{2})}}{r} &= -\frac{1}{\rho^{(0)}} \frac{\partial (\rho^{(0)} w_{11}^{(\frac{4}{2})})}{\partial z} \\ \frac{\partial u_{r,12}^{d(\frac{1}{2})}}{\partial r} + \frac{u_{r,12}^{d(\frac{1}{2})}}{r} + \frac{u_{\theta,11}^{d(\frac{1}{2})}}{r} &= -\frac{1}{\rho^{(0)}} \frac{\partial (\rho^{(0)} w_{12}^{(\frac{4}{2})})}{\partial z} \end{aligned} \tag{5.45}$$

Note, that the zeroth mode equation of (5.43) has already been discussed in `Model A - Part I`. With the aid of (3.63) and the definition of an operator

$$\nabla_1^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) \quad , \tag{5.46}$$

the terms on the left of (5.45) can be rewritten to obtain a second order inhomogeneous ordinary equation for the velocity potential $\phi_{1k}^{(\frac{1}{2})}$, i.e.

$$\nabla_1^2 \phi_{1k}^{(\frac{1}{2})} = -\frac{1}{\rho^{(0)}} \frac{\partial (\rho^{(0)} w_{1k}^{(\frac{4}{2})})}{\partial z} \,, \qquad k = 1, 2 \tag{5.47}$$

By determining the flow through the vortex centre in terms of $\psi_{1k}^{(\frac{1}{2})}$, boundary conditions for the first harmonics $\phi_{1k}^{(\frac{1}{2})}$ are assumed to be homogeneous, i.e.

$$\phi_{1k}^{(\frac{1}{2})} = 0 \qquad \text{and} \qquad \frac{\partial \phi_{1k}^{(\frac{1}{2})}}{\partial r} = 0 \qquad \text{at} \quad r = 0 \tag{5.48}$$

Using the identity (4.48) integration of (5.47) in radial direction yields

$$\frac{\partial \phi_{1k}^{(\frac{1}{2})}}{\partial \bar{r}} - \frac{\phi_{1k}^{(\frac{1}{2})}}{\bar{r}} = \bar{r}\frac{\partial}{\partial \bar{r}}\left(\frac{\phi_{1k}^{(\frac{1}{2})}}{\bar{r}}\right) = -\frac{1}{\bar{r}^2}\int_0^{\bar{r}} \frac{\bar{\bar{r}}^2}{\rho^{(0)}} \frac{\partial(\rho^{(0)} w_{1k}^{(\frac{4}{2})})}{\partial z} d\bar{\bar{r}} \tag{5.49}$$

Integrating once again gives

$$\phi_{1k}^{(\frac{1}{2})} = -\int_0^r \frac{1}{\bar{r}^3}\left[\int_0^{\bar{r}} \frac{\bar{\bar{r}}^2}{\rho^{(0)}} \frac{\partial(\rho^{(0)} w_{1k}^{(\frac{4}{2})})}{\partial z} d\bar{\bar{r}}\right] d\bar{r} \tag{5.50}$$

In the calculations that follow, far field solutions for $\phi_{1k}^{(\frac{1}{2})}$ are derived which are required for matching with the environmental flow. For this, precise knowledge about the behaviour of $w_{1k}^{(\frac{4}{2})}$ if r becomes large is needed. As long as the environmental flow is regarded as a dry flow such that the QG-theory describes conservation of quasi-geostrophic vorticity along parcel trajectories (see **Section 3.2.1**), the matching conditions (3.113) on the vertical velocity field are valid so that

$$w^{(\frac{4}{2})} \to 0 \qquad \text{as} \quad r \to \infty \tag{5.51}$$

Recall, that the axissymmetric contribution $w_0^{(\frac{4}{2})}$ decays exponentially (see (5.23)). For the subsequent analysis it is assumed that asymmetric contributions $w_{1k}^{(\frac{4}{2})}$ behave like

$$w_{1k}^{(\frac{4}{2})} \sim r^{-m} \qquad \text{as} \quad r \to \infty\ , \quad \text{for} \quad m \geq 0 \tag{5.52}$$

Then, using the identity (4.48) and (5.52), equation (5.47) can be written as

$$\frac{\partial}{\partial r}\left(r\left(r\frac{\partial \phi_{1k}^{(\frac{1}{2})}}{\partial r} - \phi_{1k}^{(\frac{1}{2})}\right)\right) = \frac{\alpha_{1k}}{r^{(m-2)}} \qquad \text{as} \quad r \to \infty, \quad \text{for} \quad m \geq 0 \tag{5.53}$$

Here, $\alpha_{1k}(z)$ is a function that may depend on z. Integrating the above equation twice yields for

$m = 1$:

$$\phi_{1k}^{(\frac{1}{2})} = \frac{\alpha_{1k}}{2} r \ln r - \frac{C_{1k_1}^1}{2r} + C_{1k_1}^2 r \qquad \text{as} \quad r \to \infty \tag{5.54}$$

$m = 2$:

$$\phi_{1k}^{(\frac{1}{2})} = -\frac{\alpha_{1k}}{3}\frac{1}{r^2} - \frac{C_{1k_2}^1}{4r^3} + C_{1k_2}^2 r \qquad \text{as} \quad r \to \infty \tag{5.55}$$

$m \geq 3$:

$$\phi_{1k}^{(\frac{1}{2})} = \mathcal{O}(r^{-n}) + C_{1k_m}^2 r \qquad \text{as} \quad r \to \infty\ , \quad n \geq 1 \tag{5.56}$$

Here, $C_{1k_m}^1$ and $C_{1k_m}^2$ denote constants of integration.

Stream function Solutions for $\psi_{1k}^{(\frac{1}{2})}$ are derived next. In doing so an approach similar to that in **Section 4.2.2** is used. First, elimination of $p^{(\frac{7}{2})}$ from the horizontal momentum equations $(3.43)_1$ and $(3.43)_2$ yields

$$
\begin{aligned}
&\frac{u_\theta^{(0)}}{r}\frac{\partial^2 u_r^{(\frac{1}{2})}}{\partial \theta^2} - \frac{2u_\theta^{(0)}}{r}\frac{\partial u_\theta^{(\frac{1}{2})}}{\partial \theta} - \Omega_0 \frac{\partial u_\theta^{(\frac{1}{2})}}{\partial \theta} - \frac{\partial}{\partial r}\left(r u_r^{(\frac{1}{2})} \frac{\partial u_\theta^{(0)}}{\partial r}\right) - \\
&\qquad \frac{\partial}{\partial r}(u_r^{(\frac{1}{2})} u_\theta^{(0)}) - \frac{\partial}{\partial r}\left(u_\theta^{(0)} \frac{\partial u_\theta^{(\frac{1}{2})}}{\partial \theta}\right) - \Omega_0 \frac{\partial}{\partial r}(r u_r^{(\frac{1}{2})}) = \mathcal{F}
\end{aligned}
\tag{5.57}
$$

where

$$
\mathcal{F} = \frac{\partial}{\partial r}\left(r w^{(\frac{4}{2})} \frac{\partial u_\theta^{(0)}}{\partial z}\right)
$$

Note that unlike (4.25) the non-zero right hand side of (5.57) is due to diabatic effects causing non-zero $w^{(\frac{4}{2})}$. By use of Helmholtz's decomposition (3.63), Fourier decomposition yields for the first asymmetric modes reading

$$
\begin{aligned}
-u_\theta^{(0)} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left[\frac{\zeta_r^{(0)}}{u_\theta^{(0)}} + \frac{1}{r^2}\right]\right) \psi_{12}^{(\frac{1}{2})} &= \mathcal{F}_{11} + \mathcal{B}_{11} \\
u_\theta^{(0)} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left[\frac{\zeta_r^{(0)}}{u_\theta^{(0)}} + \frac{1}{r^2}\right]\right) \psi_{11}^{(\frac{1}{2})} &= \mathcal{F}_{12} + \mathcal{B}_{12}
\end{aligned}
\tag{5.58}
$$

with

$$
\begin{aligned}
\mathcal{F}_{1k} &= \frac{\partial}{\partial r}\left(r w_{1k}^{(\frac{4}{2})} \frac{\partial u_\theta^{(0)}}{\partial z}\right) \\
\mathcal{B}_{1k} &= r[\zeta^{(0)} + \Omega_0]\nabla_1^2 \phi_{1k}^{(\frac{1}{2})} + \frac{\partial \phi_{1k}^{(\frac{1}{2})}}{\partial r}(r\zeta_r^{(0)}) \qquad k = 1, 2
\end{aligned}
\tag{5.59}
$$

and $\zeta_r^{(0)} = \partial \zeta^{(0)}/\partial r$. Using the same boundary conditions as for the dry case, i.e. (4.27), and the stream function transformation (4.28) the following PDE can be obtained

$$
\begin{aligned}
&-u_\theta^{(0)} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left[\frac{\zeta_r^{(0)}}{u_\theta^{(0)}} + \frac{1}{r^2}\right]\right) \overline{\psi_{1k}^{(\frac{1}{2})}} = \mathcal{M}_{1k} \\
&\text{with} \quad \overline{\psi_{1k}^{(\frac{1}{2})}} = 0, \qquad \frac{\partial \overline{\psi_{1k}^{(\frac{1}{2})}}}{\partial r} = 0 \qquad \text{at} \quad r = 0
\end{aligned}
\tag{5.60}
$$

Note, the right hand side of (5.60) denotes a sum of

$$
\begin{aligned}
\mathcal{M}_{11} &= -\mathcal{F}_{12} - \mathcal{B}_{12} - A_{11}\; r\; \zeta_r^{(0)} \\
\mathcal{M}_{12} &= +\mathcal{F}_{11} + \mathcal{B}_{11} - A_{12}\; r\; \zeta_r^{(0)}
\end{aligned}
\tag{5.61}
$$

Integration of (5.60) (see **Appendix B.1 and B.2**) and reverse-transformation yields

$$
\begin{aligned}
\psi_{11}^{(\frac{1}{2})} &= +u_\theta^{(0)} \int_0^{\bar{\bar{r}}} \frac{1}{\bar{r} u_\theta^{(0)^2}} \left[\int_0^{\bar{r}} r\; (\mathcal{F}_{12} + \mathcal{B}_{12})\; dr \right] d\bar{r} + u_\theta^{(0)} \frac{2A_{11}}{\zeta_*^{(0)}} \\
\psi_{12}^{(\frac{1}{2})} &= -u_\theta^{(0)} \int_0^{\bar{\bar{r}}} \frac{1}{\bar{r} u_\theta^{(0)^2}} \left[\int_0^{\bar{r}} r\; (\mathcal{F}_{11} + \mathcal{B}_{11})\; dr \right] d\bar{r} + u_\theta^{(0)} \frac{2A_{12}}{\zeta_*^{(0)}}
\end{aligned}
\tag{5.62}
$$

Far field solutions for $\psi_{1k}^{(\frac{1}{2})}$ with $k = 1, 2$ are (see **Appendix C.1**)

$m = 1$:

$$\psi_{1k}^{(\frac{1}{2})} = \Omega_0 \tilde{\alpha}_{1k} r^2 - \frac{1}{2r} D_{1k_1}^1 + r D_{1k_1}^2 \qquad \text{as} \quad r \to \infty \tag{5.63}$$

$m = 2$:

$$\psi_{1k}^{(\frac{1}{2})} = \frac{\Omega_0 \tilde{\alpha}_{1k}}{2} r \ln r - \frac{1}{2r} D_{1k_2}^1 + r D_{1k_2}^2 \qquad \text{as} \quad r \to \infty \tag{5.64}$$

$m = 3$:

$$\psi_{1k}^{(\frac{1}{2})} = -\Omega_0 \tilde{\alpha}_{1k} - \frac{1}{r} D_{1k_3}^1 + r D_{1k_3}^2 \qquad \text{as} \quad r \to \infty \tag{5.65}$$

$m \geq 4$:

$$\psi_{1k}^{(\frac{1}{2})} = \mathcal{O}(r^{-n}) + r D_{1k_m}^2 \qquad \text{as} \quad r \to \infty\;, \quad n \geq 1 \tag{5.66}$$

Here, $D_{1k_m}^1$ and $D_{1k_m}^2$ denote constants of integration.

Matching Once solutions for $\psi_{1k}^{(\frac{1}{2})}$ and $\phi_{1k}^{(\frac{1}{2})}$ $(k = 1, 2)$ have been derived, equations for the leading order vortex motion $(U_C^{(0)}, V_C^{(0)})$ can be derived by means of matched asymptotics. With $u_r^{(\frac{1}{2})}$ and $u_\theta^{(\frac{1}{2})}$ expressed in terms of $\phi^{(\frac{1}{2})}$ and $\psi^{(\frac{1}{2})}$ (see (3.63)) and with the aid of (3.12) the matching condition (3.100) of the inner and outer velocity fields takes the form

$$
\begin{aligned}
-\frac{\psi_{12}^{(\frac{1}{2})}}{r} + \frac{\partial \phi_{11}^{(\frac{1}{2})}}{\partial r} &= V_{B,C}^{(0)} + V_{R,C}^{(0)} - V_C^{(0)} \qquad \text{as} \qquad r \to \infty \\
+\frac{\psi_{11}^{(\frac{1}{2})}}{r} + \frac{\partial \phi_{12}^{(\frac{1}{2})}}{\partial r} &= U_{B,C}^{(0)} + U_{R,C}^{(0)} - U_C^{(0)} \qquad \text{as} \qquad r \to \infty
\end{aligned}
\tag{5.67}
$$

For simplification, it is assumed that the asymmetric vertical velocities $w_{1k}^{(\frac{4}{2})}$ decay faster than $1/r^2$ for large r. This means that it is assumed that $m \geq 3$ (see (5.52)). Considering that case, one obtains from the matching condition (5.67) together with (5.56) and (5.65)-(5.66) that

$$\begin{aligned} U_C^{(0)} &= U_{B,C}^{(0)} + U_{R,C}^{(0)} - D_{11} - C_{11} \\ V_C^{(0)} &= V_{B,C}^{(0)} + V_{R,C}^{(0)} + D_{12} - C_{12} \end{aligned} \tag{5.68}$$

Note that the superscripts on the constants have been dropped. By equating (5.56) to (5.50) in the limit as r approaches ∞, it turns out that the constants C_{1k} are determined by

$$C_{1k} = -\lim_{r\to\infty}\left(\frac{1}{r}\int_0^r \frac{1}{\bar{r}^3}\left[\int_0^{\bar{r}} \frac{\bar{\bar{r}}^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{1k}^{(\frac{4}{2})})}{\partial z}d\bar{\bar{r}}\right]d\bar{r}\right) \tag{5.69}$$

If the limit exists, i.e. $C_{1k} \neq 0$, using the L' Hospitals rule twice one obtains

$$\begin{aligned} C_{1k} &= -\lim_{r\to\infty}\left(\frac{1}{r^3}\int_0^r \frac{\bar{r}^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{1k}^{(\frac{4}{2})})}{\partial z}d\bar{r}\right) \\ &= -\lim_{r\to\infty}\frac{1}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{1k}^{(\frac{4}{2})})}{\partial z} \end{aligned} \tag{5.70}$$

It follows from $\rho^{(0)} = \rho^{(0)}(z)$ and the matching condition that

$$C_{1k} = 0 \tag{5.71}$$

Equating (5.62) equal to (5.65) in the limit $r \to \infty$, together with (3.99) similar procedure yields for D_{1k} that

$$\begin{aligned} D_{11} &= +\frac{\pi}{\Gamma}\int_0^\infty r(\mathcal{F}_{12} + \mathcal{B}_{12})\ dr \\ D_{12} &= -\frac{\pi}{\Gamma}\int_0^\infty r(\mathcal{F}_{11} + \mathcal{B}_{11})\ dr \end{aligned} \tag{5.72}$$

Thus, upon substitution of (5.71) and (5.72) into (5.68) the equations for the leading order vortex motion of diabatic vortices take the following form

$$\boxed{\begin{aligned} U_C^{(0)} &= U_{B,C}^{(0)} + U_{R,C}^{(0)} - \frac{\pi}{\Gamma}\int_0^\infty r(\mathcal{F}_{12} + \mathcal{B}_{12})\ dr \\ V_C^{(0)} &= V_{B,C}^{(0)} + V_{R,C}^{(0)} - \frac{\pi}{\Gamma}\int_0^\infty r(\mathcal{F}_{11} + \mathcal{B}_{11})\ dr \end{aligned}} \tag{5.73}$$

The functions $\mathcal{F}_{1k}$ and $\mathcal{B}_{1k}$ can be regarded as $\mathcal{F}_{1k} = f(u_\theta^{(0)}, w_{1k}^{(\frac{4}{2})})$ and $\mathcal{B}_{1k} = g(u_\theta^{(0)}, w_{1k}^{(\frac{4}{2})}, \Omega_0)$ (see (5.59)). It is observed that additional terms appear in the equation for the leading order vortex motion for diabatic vortices (5.73) compared to the equations for adiabatic vortices (4.37). These terms describe net effects primarily caused by diabatically induced asymmetries in the vertical velocity field $w^{(\frac{4}{2})} = w^{(\frac{4}{2})}(r, \theta, z, \tau)$ resolved with respect to mesoscales. A system of equations that can be used to find solutions for $w_{1k}^{(\frac{4}{2})}$ is discussed in **Section 5.3.1.3**. Recall that in the next higher order vortex motion equation for adiabatic vortices (see (4.78)) the appearence of the integral term describing net effects of mesoscale processes was due to $w_{1k}^{(\frac{5}{2})} \neq 0$ (see (4.17)) which was strongly related to a non-zero vortex tilt $\partial \vec{X}_C^{(\frac{1}{2})}/\partial z$. However, for the diabatic vortex case studied here, asymmetric vertical velocities $w_{1k}^{(\frac{4}{2})}$ are independent on the vortex tilt, which in turn makes the appearence of the integral term in (5.73) possible, even for zero next higher order perturbations in the vortex tilt, i.e. $\partial \vec{X}_C^{(\frac{1}{2})}/\partial z = 0$. For this reason (5.73) is suitable to describe the motion of vortices having small tilts with horizontal displacements between the upper and lower vortex part equal or smaller than 80 km (i.e. $\partial \vec{X}_C/\partial z = \varepsilon^{\frac{2}{2}} \partial \vec{X}_C^{(\frac{2}{2})}/\partial z + \mathcal{O}(\varepsilon^{\frac{3}{2}})$). Mature hurricanes are examples for vortices having such a small tilt. Another comparison between the leading order motion of adiabatic and diabatic vortices is that the requirement of a vertically constant background flow $\vec{V}_{B,C}^{(0)}$ on an f-plane for the constraint $\vec{V}_C^{(0)} = \vec{V}_C^{(0)}(\tau)$ to be satisfied is not necessary if the background shear satisfies the following

$$\begin{aligned} \frac{\partial U_{B,C}^{(0)}}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{\pi}{\Gamma}\int_0^\infty r(\mathcal{F}_{12} + \mathcal{B}_{12})\, dr\right) \\ \frac{\partial V_{B,C}^{(0)}}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{\pi}{\Gamma}\int_0^\infty r(\mathcal{F}_{11} + \mathcal{B}_{11})\, dr\right) \end{aligned} \tag{5.74}$$

Regarding an interpretation of the above constraint, two different perspectives are possible. One possibility is to say that the vertical distribution of the integral on the right hand side of (5.74) determines the vertical distribution of a background flow within a concentrated vortex may exist. Another possibility is that a given background flow determines the vertical distributions of the integrals and therefore indirectly the vertical distribution of $w_{1k}^{(\frac{4}{2})}$ and u_θ. Please note that such a constraint must be taken into account when solving the governing equations of the entire `Model A` numerically. Note, however, that if the initial conditions for `Model A` are not compatible with such a constraint, then this may be interpreted as a hint that certain processes must take place on faster time scale that eventually would generate the balance (5.74) with respect to the slower time scale (synoptic time scale).

5.3.1.3 Model A - Part II

The equations of `Model A - Part II` (see **Table 5.2** and **5.3**) constitute a closed set of equations describing *asymmetric* contributions of (i) a next higher order correction for the primary circulation given by $u_{\theta,1k}^{(\frac{1}{2})}$, (ii) a diabatically driven secondary circulation given by $u_{r,1k}^{(\frac{1}{2})}$ and $w_{1k}^{(\frac{4}{2})}$, and (iii) appropriate potential temperature and pressure fields given by $p_{1k}^{(\frac{7}{2})}$, $\Theta_{1k}^{(\frac{7}{2})}$ and $\rho_{1k}^{(\frac{7}{2})}$ with $k = 1, 2$. Note that solutions for $u_\theta^{(0)}, \rho^{(0)}$ etc. can be obtained from solving the equations summarized `Model A - Part I`. Furthermore, `Model A - Part II` includes equations for the leading order vortex motion components in meridional and zonal direction, i.e. $U_C^{(0)}$ and $V_C^{(0)}$. It is argued that using the Fredholm Alternative Theorem (see for instance Holmes (1995) and Werner (2000)) the equations for the vortex motion can be used to find solvability conditions for the higher order vortex tilt components $\partial X_C^{(\frac{2}{2})}/\partial z$ and $\partial Y_C^{(\frac{2}{2})}/\partial z$ as well.

The equations (5.75) in **Table 5.2** have been derived first by eliminating the pressure variable $p^{(\frac{7}{2})}$ from the $\mathcal{O}(\varepsilon^{\frac{2}{2}})$ horizontal momentum equations $(3.43)_1$ and $(3.43)_2$ via cross-differentiation and a subsequent harmonic analysis using $(3.58)_2$ and $(3.58)_3$ with $j = 1$. Note that an elimination of $p^{(\frac{7}{2})}$ comes along with an elimination of the leading order meridional and zonal velocity components $V_C^{(0)}$ and $U_C^{(0)}$ determining the leading order vortex motion. The equations (5.76) - (5.79) emanate sequentially from a harmonic analysis of the $\mathcal{O}(\varepsilon^{-\frac{1}{2}})$ vertical momentum equation (3.47), the $\mathcal{O}(\varepsilon^{\frac{3}{2}})$ mass continuity (3.50), the $\mathcal{O}(\varepsilon^{\frac{9}{2}})$ potential temperature equation (5.33) and the $\mathcal{O}(\varepsilon^{\frac{7}{2}})$ state equation (A-39). The equations (5.80) in **Table 5.3** are the equations for the vortex motion that have been derived in the previous section with the aid matched asymptotics . The equations (5.82) emanate from a harmonic analysis of the $\mathcal{O}(\varepsilon^{\frac{2}{2}})$ horizontal momentum equations (3.43) and a subsequent combination of the equations including $p_{11}^{(\frac{7}{2})}$ and $p_{12}^{(\frac{7}{2})}$, respectively.

As noted earlier, the Fredholm Alternative Theorem can be used to solve the equations summarized in `Model A - Part II` for the 18 unknowns mentioned at the beginning of this section. For this, the inhomogeneous system of equations consisting of the twelve equations given through (5.75) - (5.79) and (5.82) has to be solved for $u_{r,1k}^{(\frac{1}{2})}, u_{\theta,1k}^{(\frac{1}{2})}, p_{1k}^{(\frac{7}{2})}, w_{1k}^{(\frac{7}{2})}, \theta_{1k}^{(\frac{7}{2})}$ and $\rho_{1k}^{(\frac{7}{2})}$, whereas the solutions still depend on the unknowns $U_C^{(0)}$, $V_C^{(0)}$ and $\partial X_C^{(\frac{2}{2})}/\partial z$, $\partial Y_C^{(\frac{2}{2})}/\partial z$. Assuming the case that the homogeneous part of the linear system has only the trivial solution, then the solutions of the inhomogeneous system are unique for every $U_C^{(0)}$, $V_C^{(0)}$ and $\partial X_C^{(\frac{2}{2})}/\partial z$, $\partial Y_C^{(\frac{2}{2})}/\partial z$. Then, the solutions for $u_{r,1k}^{(\frac{1}{2})}, u_{\theta,1k}^{(\frac{1}{2})}$ etc. as functions on $U_C^{(0)}$, $V_C^{(0)}$ and $\partial X_C^{(\frac{2}{2})}/\partial z$, $\partial Y_C^{(\frac{2}{2})}/\partial z$ can be used in order to solve the integrals in the equations for the vortex motion, i.e. (5.80). Because of $\partial U_C^{(0)}/\partial z = 0$ and $\partial V_C^{(0)}/\partial z = 0$, however, the equations for the vortex motion will set certain con-

horizontal momentum ($p^{(\frac{7}{2})}$ eliminated):

$$
\begin{aligned}
u_{r,11}^{(\frac{1}{2})}\frac{u_\theta^{(0)}}{r} - u_{\theta,12}^{(\frac{1}{2})}\left(\frac{2u_\theta^{(0)}}{r}+\Omega_0\right) + \frac{\partial}{\partial r}\left(u_{r,11}^{(\frac{1}{2})} r\ \zeta_{abs}^{(0)} - u_{\theta,12}^{(\frac{1}{2})}u_\theta^{(0)}\right) &= \mathcal{F}_{11} \\
u_{r,12}^{(\frac{1}{2})}\frac{u_\theta^{(0)}}{r} + u_{\theta,11}^{(\frac{1}{2})}\left(\frac{2u_\theta^{(0)}}{r}+\Omega_0\right) + \frac{\partial}{\partial r}\left(u_{r,12}^{(\frac{1}{2})} r\ \zeta_{abs}^{(0)} - u_{\theta,11}^{(\frac{1}{2})}u_\theta^{(0)}\right) &= \mathcal{F}_{12}
\end{aligned}
\tag{5.75}
$$

$$
\text{where}\quad \mathcal{F}_{1k} = -\frac{\partial}{\partial r}\left(r w_{1k}^{(\frac{4}{2})}\frac{\partial u_\theta^{(0)}}{\partial z}\right)\ ,\quad \zeta_{abs}^{(0)} = \zeta^{(0)}+\Omega_0\ ,\quad \zeta^{(0)} = \frac{1}{r}\frac{\partial(r u_\theta^{(0)})}{\partial r}
$$

vertical momentum :

$$
\begin{aligned}
\frac{\partial p_{11}^{(\frac{7}{2})}}{\partial z} - \frac{\partial Y_C^{(\frac{2}{2})}}{\partial z}\frac{\partial p^{(\frac{6}{2})}}{\partial r} &= -\rho_{11}^{(\frac{7}{2})} \\
\frac{\partial p_{12}^{(\frac{7}{2})}}{\partial z} - \frac{\partial X_C^{(\frac{2}{2})}}{\partial z}\frac{\partial p^{(\frac{6}{2})}}{\partial r} &= -\rho_{12}^{(\frac{7}{2})}
\end{aligned}
\tag{5.76}
$$

mass continuity :

$$
\begin{aligned}
\frac{\partial u_{r,11}^{(\frac{1}{2})}}{\partial r} + \frac{u_{r,11}^{(\frac{1}{2})}}{r} - \frac{u_{\theta,12}^{(\frac{1}{2})}}{r} &= -\frac{1}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{11}^{(\frac{4}{2})})}{\partial z} \\
\frac{\partial u_{r,12}^{(\frac{1}{2})}}{\partial r} + \frac{u_{r,12}^{(\frac{1}{2})}}{r} + \frac{u_{\theta,11}^{(\frac{1}{2})}}{r} &= -\frac{1}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{12}^{(\frac{4}{2})})}{\partial z}
\end{aligned}
\tag{5.77}
$$

thermodynamic equation :

$$
\begin{aligned}
u_{r,11}^{(\frac{1}{2})}\frac{\partial\Theta^{(\frac{6}{2})}}{\partial r} - \frac{u_\theta^{(0)}}{r}\Theta_{12}^{(\frac{7}{2})} + w_{11}^{(\frac{4}{2})}\left(\frac{\partial\Theta^{(\frac{6}{2})}}{\partial z}+\mathcal{J}(z)\right) &= 0 \\
u_{r,12}^{(\frac{1}{2})}\frac{\partial\Theta^{(\frac{6}{2})}}{\partial r} + \frac{u_\theta^{(0)}}{r}\Theta_{11}^{(\frac{7}{2})} + w_{12}^{(\frac{4}{2})}\left(\frac{\partial\Theta^{(\frac{6}{2})}}{\partial z}+\mathcal{J}(z)\right) &= 0
\end{aligned}
\tag{5.78}
$$

state equation :

$$
\rho^{(0)}\Theta_{1k}^{(\frac{7}{2})} + \rho_{1k}^{(\frac{7}{2})}\Theta_\infty = p_{1k}^{(\frac{7}{2})}
\tag{5.79}
$$

Table 5.2: Model A - Part IIa, see the text for explanations

vortex motion:

$$\begin{aligned} U_C^{(0)} &= U_{B,C}^{(0)} + U_{R,C}^{(0)} - \frac{\pi}{\Gamma}\int_0^\infty r(\mathcal{F}_{12} + \mathcal{B}_{12})\ dr \\ V_C^{(0)} &= V_{B,C}^{(0)} + V_{R,C}^{(0)} - \frac{\pi}{\Gamma}\int_0^\infty r(\mathcal{F}_{11} + \mathcal{B}_{11})\ dr \end{aligned} \tag{5.80}$$

with

$$\begin{aligned} \mathcal{F}_{1k} &= \frac{\partial}{\partial r}\left(r w_{1k}^{(\frac{4}{2})} \frac{\partial u_\theta^{(0)}}{\partial z} \right) \\ \mathcal{B}_{1k} &= r[\zeta^{(0)} + \Omega_0]\nabla_1^2 \phi_{1k}^{(\frac{1}{2})} + \frac{\partial \phi_{1k}^{(\frac{1}{2})}}{\partial r}\left(r\frac{\partial \zeta^{(0)}}{\partial r} \right)\ , \qquad k = 1,2 \\ \nabla_1^2 \phi_{1k}^{(\frac{1}{2})} &= -\frac{1}{\rho^{(0)}} \frac{\partial(\rho^{(0)} w_{1k}^{(\frac{4}{2})})}{\partial z} \end{aligned} \tag{5.81}$$

horizontal momentum:

$$\begin{aligned} \frac{1}{\rho^{(0)}}\frac{1}{r}\frac{\partial(rp_{12}^{(\frac{7}{2})})}{\partial r} &= +u_{\theta,12}^{(\frac{1}{2})}\frac{u_\theta^{(0)}}{r} + \Omega_0(2V_C^{(0)} + u_{\theta,12}^{(\frac{1}{2})} + u_{r,11}^{(\frac{1}{2})}) + \\ & \quad w_{11}^{(\frac{4}{2})}\frac{\partial u_\theta^{(0)}}{\partial z} + u_{r,11}^{(\frac{1}{2})}\frac{\partial u_\theta^{(0)}}{\partial r} \\ \frac{1}{\rho^{(0)}}\frac{1}{r}\frac{\partial(rp_{11}^{(\frac{7}{2})})}{\partial r} &= +u_{\theta,11}^{(\frac{1}{2})}\frac{u_\theta^{(0)}}{r} - \Omega_0(2U_C^{(0)} - u_{\theta,12}^{(\frac{1}{2})} + u_{r,11}^{(\frac{1}{2})}) - \\ & \quad w_{12}^{(\frac{4}{2})}\frac{\partial u_\theta^{(0)}}{\partial z} - u_{r,12}^{(\frac{1}{2})}\frac{\partial u_\theta^{(0)}}{\partial r} \end{aligned} \tag{5.82}$$

Table 5.3: Model A - Part IIb, see the text for explanations

straints on the vertical variations of the higher order vortex tilt components $\partial Y_C^{(\frac{2}{2})}/\partial z$ and $\partial X_C^{(\frac{2}{2})}/\partial z$. It is beyond the scope of the present work to solve this problem.

5.3.2 Model B - with net-heating due to small-scale convective processes

The derivation of `Model B` is based on the assumption that $S_{us} \neq 0$. It is shown in the subsequent sections that net-heating sources caused by small-scale convective flows, have a nontrivial impact on the vortex tilt, the secondary vortex circulation and the leading order vortex motion. In analogy with the derivation of

`Model A` in **Section 5.3.1**, it is distinguished between `Model B - Part I` and `Model B - Part II` summarizing equations for the axissymmetric and asymmetric vortex structure, respectively.

At the beginning of the derivation of `Model B - Part I`, the source term $S^{(\frac{7}{2})}$ in $(5.8)_5$ is replaced by $(5.27)_1$ to obtain

$$w^{(\frac{4}{2})}\left(\frac{d\Theta^{(\frac{4}{2})}}{dz} + \Gamma L^{\star\star}\frac{dq_{vs}^{(0)}}{dz}\right) = S_{\text{us}}^{(\frac{7}{2})} \tag{5.83}$$

Due to the moist adiabatic equation (3.134) the bracket on the left hand side of (5.83) becomes zero, which implies that

$$S_{\text{us}}^{(\frac{7}{2})} = 0 \qquad . \tag{5.84}$$

Thus, unlike the discussions in **Section 5.3.1**, zero net heating effects caused by small-scale cumulus convection in leading order are no longer an assumption, but it follows directly from an asymptotic analysis. For this reason leading order heating effects given by $S_L^{(\frac{7}{2})}$ can basically be regarded as a consequence of convective flows on vortex-scale in a conditionally neutral[4] environment with respect to leading order.

Also as in **Section 5.3.1** it turns out that with the explicit inclusion of moisture the $\mathcal{O}(\varepsilon^{\frac{7}{2}})$ thermodynamic equation becomes trivial, and the next higher order equation becomes important in the determination of $w^{(\frac{4}{2})}$. It is shown below, that based on the next higher order thermodynamic equation a nontrivial relation between net-heating effects $S_{\text{us}}^{(\frac{8}{2})}$ caused by mesoscale processes and the leading order vortex tilt $\partial\vec{X}_C^{(\frac{1}{2})}/\partial z$ can be found.

5.3.2.1 Vortex tilt and small-scale convection

With the aid of the $\mathcal{O}(\varepsilon^{\frac{8}{2}})$ source term $(5.27)_2$, the $\mathcal{O}(\varepsilon^{\frac{8}{2}})$ thermodynamic equation (3.54) takes the form

$$\frac{u_\theta^{(0)}}{r}\frac{\partial\Theta^{(\frac{6}{2})}}{\partial\theta} + w^{(\frac{5}{2})}\frac{d\Theta^{(\frac{4}{2})}}{dz} = -\Gamma^{\star\star}L^{\star\star}q_{vs}^{\star\star}w^{(\frac{5}{2})}\frac{dq_s^{(0)}}{dz} + S_{\text{us}}^{(\frac{8}{2})} \tag{5.85}$$

On account of the moist adiabatic equation (3.134), equation (5.85) simplifies to

$$\frac{u_\theta^{(0)}}{r}\frac{\partial\Theta^{(\frac{6}{2})}}{\partial\theta} = S_{\text{us}}^{(\frac{8}{2})} \tag{5.86}$$

Because of $u_\theta^{(0)} = u_\theta^{(0)}(r,z,\tau)$, integration in circumferential direction from 0 to 2π yields for the axissymmetric contribution of $S_{\text{us}}^{(\frac{8}{2})}$ that $S_{\text{us},0}^{(\frac{8}{2})} = 0$. A

[4] In a conditionally neutral atmosphere is the lapse rate exactly equal to the dry adiabatic rate, i.e. rising or sinking saturated air will cool or warm at the same rate as the air around it.

harmonic analysis of (5.86) for the sine and cosine modes yields

$$\frac{u_\theta^{(0)}}{r}\Theta_{11}^{(\frac{6}{2})} = S_{\mathrm{us},12}^{(\frac{8}{2})} \qquad \text{and} \qquad -\frac{u_\theta^{(0)}}{r}\Theta_{12}^{(\frac{6}{2})} = S_{\mathrm{us},11}^{(\frac{8}{2})} \tag{5.87}$$

In **Section 4.2.1** it has been shown that a combination of the first harmonics of the vertical momentum equation (4.2) and state equation (A-37) gives

$$\frac{\partial X_C^{(\frac{1}{2})}}{\partial z}\frac{1}{\rho^{(0)}}\frac{\partial p^{(\frac{6}{2})}}{\partial r} = -\frac{\Theta_{12}^{(\frac{6}{2})}}{\Theta_\infty} \qquad \text{and} \qquad \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\frac{1}{\rho^{(0)}}\frac{\partial p^{(\frac{6}{2})}}{\partial r} = -\frac{\Theta_{11}^{(\frac{6}{2})}}{\Theta_\infty} \tag{5.88}$$

Then, elimination of $\Theta_{1k}^{(\frac{6}{2})}$ in the above equation by use of (5.87) yields together with $\pi = p^{(\frac{6}{2})}/\rho^{(0)}$, that

$$-\frac{u_\theta^{(0)}}{r}\frac{\partial \pi}{\partial r}\frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} = S_{\mathrm{us},12}^{(\frac{8}{2})} \qquad \text{and} \qquad +\frac{u_\theta^{(0)}}{r}\frac{\partial \pi}{\partial r}\frac{\partial X_C^{(\frac{1}{2})}}{\partial z} = S_{\mathrm{us},11}^{(\frac{8}{2})} \tag{5.89}$$

Using the gradient wind relation (3.135) the gradient $\partial\pi/\partial r$ can be replaced by

$$\frac{\partial \pi}{\partial r} = \frac{1}{\rho^{(0)}}\frac{\partial p^{(\frac{6}{2})}}{\partial r} = \frac{{u_\theta^{(0)}}^2}{r} + \Omega_0 u_\theta^{(0)} \tag{5.90}$$

The equations (5.89) provide a direct relationship between asymmetric fields of active convection reflected by $S_{\mathrm{us},1k}^{(\frac{8}{2})}$ and the direction of the vortex tilt determined by vertical gradients of the vortex centreline components, i.e. $\partial X_C^{(\frac{1}{2})}/\partial z$ and $\partial Y_C^{(\frac{1}{2})}/\partial z$ in zonal and meridional direction, respectively. Difficulties arise in separating cause and effect in the physical interpretation of (5.89). **[i]** One possible interpretation is that net-heating effects due to small-scale cumulus convection force the vortex tilt. Then, for a given leading order velocity field $u_\theta^{(0)}$ and an appropriate parameterization for $S_{\mathrm{us},1k}^{(\frac{8}{2})}$ the equations (5.89) may be regarded as equations for the vortex tilt. Note that a parameterization for $S_{\mathrm{us},1k}^{(\frac{8}{2})}$ in terms of resolved variables should be in such a way that $\vec{X}_C^{(\frac{1}{2})} = \vec{X}_C^{(\frac{1}{2})}(z,\tau)$ is satisfied, which sets a certain constraint on the radial distribution of $S_{\mathrm{us},1k}^{(\frac{8}{2})}$. **[ii]** Another possible interpretation is that convection and therefore convective heating can also be seen as response to a vortex tilt which in turn is initiated by some other mechanisms. Note that unlike the adiabatic vortex case it is not easy to show within the asymptotic framework, that there might be a relation between the background shear and the votex tilt. One of the difficulty is to apply the same matching strategy as for adiabatic vortices. However, there are studies from Frank & Ritchie (1999) and Corbosiero & Molinari (2003) that would support the argument [ii]. Frank & Ritchie (1999) designed a series of numerical simulations to study the effects of an imposed external circulation upon the structure of tropical cyclone like vortices. In doing so they found evidence for their hypothesis that sheared zonal winds and boundary layer processes modulate dynamically the vertical velocity field within the storm which in turn forces regions of ascent within the cyclone that organize and control the amount and distribution of latent heat release. Moreover, based on a statistical analysis of lightning data which were measured in convective rainbands in hurricane-like vortices, Corbosiero & Molinari (2003) showed that there are prefered regions for lightnings depending on the direction in which the vertical shear vector points. In particular they found enhanced flash activity downshear left within the core region ($r < 100$ km) and enhanced flash activity downshear right in the outer rainbands ($100 \leq r \leq 300$) km (see **Figure 5.1**). Note that under the assumption of a pure zonally tilted vortex (i.e. $\partial Y_C^{(\frac{1}{2})}/\partial z = 0$), from (5.89) one obtains enhanced convective heating ($S_{\mathrm{us},11}^{(\frac{8}{2})} > 0$) downshear left which is in agreement with Corbosiero & Molinaris observations within the core region.

Comparing the equations (5.89) with the corresponding equations (4.15) for adiabatic vortices, it is observed that for the dry case the right hand side of (5.89), i.e. the asymmetric diabatic source term due to small-scale processes, is replaced by a term denoting vertical advection of the background potential temperature $\Theta^{(\frac{4}{2})}$ via asymmetric vertical velocities $w_{1k}^{(\frac{5}{2})}$. Recall that the existence of $w_{1k}^{(\frac{5}{2})}$ is due an adiabatic lifting mechanism that attributes the existence of the vertical velocity fields to a shear induced vortex tilt. Thus, comparing the dry and moist (saturated) vortex case studied here it turns out that the effects caused by a vortex tilt are dependent on whether moisture is included or not. This finding is in agreement with numerical simlations by Frank & Ritchie (1999). Comparing moist and dry runs they observe that vertical motion patterns in both simulations are dominated by similar adiabatic lifting mechanism prior to the development of partial eyewall saturation. However, the adiabatic lifting mechanism vanishes with a set up of saturated conditions within the simulations that account for moist physics. Based on these findings Frank & Ritchie (1999) argue that the patterns of forced ascent in the dry runs should be relevant for understanding patterns of convection in loosely organized systems such as tropical depressions, but not in mature hurricanes. In light of the present work, it is important to point out again that the results derived in this section are based on the idealized assumption of a completely saturated atmosphere (see **Section 3.4.2**). Thus, motivated by the above cited argumentation of Frank & Ritchie (1999), an interesting topic for future research would be to elucidate the role of an adiabatic lifting mechanism on the transition from an undersaturated air regime into a saturated regime. For this a multi-scale expansion ansatz that accounts for both vortex-scales and small cumulus-scales should be applied within the framework of an unified approach to meteorological modelling that accounts for bulk microphysics parameterization of moist processes. See Klein & Majda (2005) for details regarding such parameterizations within an unified approach. Studies on this issue may contribute to a better understanding of conditions that favour an intensification of tropical depressions towards hurricane intensity. Moreover it might be helpul to find a detailed answer to the following question asked by Reasor & Montgomery (2004): 'Do the details of cumulus convection determine whether a given storm shears apart or remains vertically aligned, or can the effect of convection meaningfully parameterized and still yield a reasonably accurate forecast ?'

Another interesting observation that is made from (5.89) is that the maximum of latent heat release given by $S_{\text{us},1k}^{(\frac{8}{2})}$ and therefore the maximum of convection corresponds roughly with the location of strongest pressure gradients $\partial p^{(\frac{6}{2})}/\partial r$ and thus strongest circumferential winds $u_\theta^{(0)}$. It is known from observational studies that hurricanes may extend 1000 km from its centre. At the core there is an eye of 5-50 km with calm winds and little convection. The eye, however, is surrounded by an 10-20 km eyewall cloud with extremely strong tangential wind flow and intense convection. Based on these observations it is argued, that the convective clouds causing the source terms $S_{\text{us},1k}^{(\frac{8}{2})}$ must be an integral part of an eyewall cloud.

5.3.2.2 Model B - Part I

In the previous section it has been shown that the $\mathcal{O}(\varepsilon^{\frac{8}{2}})$ thermodynamic equation is useful to study interactions between the vortex tilt and small-scale convective fields. Similar to **Section 5.3.1** it is shown below that the $\mathcal{O}(\varepsilon^{\frac{9}{2}})$ thermodynamic equation can be used in order to reshape the general balanced vortex model (5.8)-(5.10) into a modified EbVM named `Model B - Part I` that unlike `Model A - Part I` not only accounts for condensation heating due to vortex-scale forced uplift, but also accounts for net heating effects due to small-scale cumulus convection.

Because of $\partial \vec{X}_C^{(\frac{1}{2})}/\partial z \neq 0$ due to $\partial \Theta^{(\frac{6}{2})}/\partial \theta \neq 0$, substitution of the diabatic source term (5.27) into the $\mathcal{O}(\varepsilon^{\frac{9}{2}})$ thermodynamic equation (3.56) yields

$$
\begin{aligned}
u_r^{(\frac{1}{2})}\frac{\partial \Theta^{(\frac{6}{2})}}{\partial r} + u_\theta^{(\frac{1}{2})}\frac{1}{r}\frac{\partial \Theta^{(\frac{6}{2})}}{\partial \theta} + u_\theta^{(0)}\frac{1}{r}\frac{\partial \Theta^{(\frac{7}{2})}}{\partial \theta} + w^{(\frac{4}{2})}\left(\frac{\partial \Theta^{(\frac{6}{2})}}{\partial z} + \mathcal{J}(z)\right) - \\
w^{(\frac{4}{2})}\Lambda_b^1\frac{\partial \Theta^{(\frac{6}{2})}}{\partial r} - w^{(\frac{4}{2})}\Lambda_a^1\frac{1}{r}\frac{\partial \Theta^{(\frac{6}{2})}}{\partial \theta} = S_{us}^{(\frac{9}{2})} \qquad (5.91)
\end{aligned}
$$

with

$$
\begin{aligned}
\Lambda_b^1 &= +\partial X_C^{(\frac{1}{2})}/\partial z\ \cos\theta\ +\ \partial Y_C^{(\frac{1}{2})}/\partial z\ \sin\theta \\
\Lambda_a^1 &= -\partial X_C^{(\frac{1}{2})}/\partial z\ \sin\theta\ +\ \partial Y_C^{(\frac{1}{2})}/\partial z\ \cos\theta
\end{aligned} \qquad (5.92)
$$

and

$$\mathcal{J}(z) = \Gamma^{\star\star^2} L^{\star\star} q_{vs}^{\star\star}\ z\ \frac{dq_{vs}^{(1)}}{dz} \qquad . \tag{5.93}$$

To simplify things, in the following it is assumed that $w^{(\frac{4}{2})}$ is purely axissymmetric, i.e. $w^{(\frac{4}{2})} = w^{(\frac{4}{2})}(r, z, \tau)$. It is shown in the following subsections, that this assumption allows one to study the isolated effect of a vortex tilt on the secondary vortex circulation and the leading order vortex motion. Then, a harmonic analysis for the zeroth mode equation of (5.91) yields

$$u_{r,0}^{(\frac{1}{2})} \frac{\partial \Theta_0^{(\frac{6}{2})}}{\partial r} + w_0^{(\frac{4}{2})} \left(\frac{\partial \Theta_0^{(\frac{6}{2})}}{\partial z} + \tilde{\mathcal{J}}(r, z, \tau) \right) = \tilde{S} \tag{5.94}$$

with

$$\begin{aligned}
\tilde{\mathcal{J}}(r, z, \tau) &= \Gamma^{\star\star^2} L^{\star\star} q_{vs}^{\star\star}\ z\ \frac{dq_{vs}^{(1)}}{dz} - \frac{\mathcal{D}}{2} \\
\tilde{S}(r, z, \tau) &= S_{\text{us},0}^{(\frac{9}{2})} - \frac{1}{2} \left(\sum_{k=1}^{2} u_{r,1k}^{(\frac{1}{2})} \frac{\partial \Theta_{1k}^{(\frac{6}{2})}}{\partial r} \right) - \\
&\quad \frac{1}{2} \left(u_{\theta,12}^{(\frac{1}{2})} \frac{\Theta_{11}^{(\frac{6}{2})}}{r} - u_{\theta,11}^{(\frac{1}{2})} \frac{\Theta_{12}^{(\frac{6}{2})}}{r} \right)
\end{aligned} \tag{5.95}$$

and

$$\begin{aligned}
\mathcal{D} &= \left(\frac{\partial X_C^{(\frac{1}{2})}}{\partial z} \frac{1}{r} \frac{\partial (r\ \Theta_{12}^{(\frac{6}{2})})}{\partial r} + \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} \frac{1}{r} \frac{\partial (r\ \Theta_{11}^{(\frac{6}{2})})}{\partial r} \right) \\
&= -\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \pi}{\partial r} \right) \left| \frac{\partial \vec{X}_C^{(\frac{1}{2})}}{\partial z} \right|^2
\end{aligned} \tag{5.96}$$

The simplified form of $\mathcal{D}$ is obtained by eliminating the asymmetric contributions of $\Theta^{(\frac{6}{2})}$ with the aid of (4.10) and (4.9). Eventually, together with the assumption that $w^{(\frac{4}{2})} = w^{(\frac{4}{2})}(r, z, \tau)$, a replacement of the potential temperature equation $(5.7)_5$ through (5.94) yields a second version of a modified EbVM which is referred to as `Model B - Part I` and is summarized in **Table 5.4**. While the net-heat source $S_{us,0}^{(\frac{9}{2})}$ has to be parameterized, it can be observed that unlike `Model A - Part I`, the equations in `Model B - Part I` are not closed in the variables describing the leading order axissymmetric vortex structure, i.e. the unknowns $u_\theta^{(0)}$, $u_{r,0}^{(\frac{1}{2})}$, $w_0^{(\frac{4}{2})}$, $\Theta_0^{(\frac{6}{2})}$ and $p^{(\frac{6}{2})}$. This is due to the occurence of additional terms including the vortex tilt components $\partial X_C^{(\frac{1}{2})}/\partial z$ and $\partial Y_C^{(\frac{1}{2})}/\partial z$, asymmetric potential temperature contributions $\Theta_{1k}^{(\frac{6}{2})}$, and asymmetric velocity components $u_{r,1k}^{(\frac{1}{2})}$ and $u_{\theta,1k}^{(\frac{1}{2})}$ in next higher order. The equations necessary to close the equations in `Model B - Part I` are summarized and discussed in `Mo-`

Model B - Part I

$$
\begin{aligned}
M^{(0)^2} - \frac{1}{4}\Omega_0^2 r^4 &= r^3 \frac{\partial \pi}{\partial r} \\
\left(w_0^{(\frac{4}{2})} \frac{\partial}{\partial z} + u_{r,0}^{(\frac{1}{2})} \frac{\partial}{\partial r} \right) M^{(0)} &= 0 \\
\frac{\partial \pi}{\partial z} &= \Theta_0^{(\frac{6}{2})} \\
\frac{1}{r}\frac{\partial (r u_{r,0}^{(\frac{1}{2})})}{\partial r} + \frac{1}{\rho^{(0)}} \frac{\partial (\rho^{(0)} w_0^{(\frac{4}{2})})}{\partial z} &= 0 \\
u_{r,0}^{(\frac{1}{2})} \frac{\partial \Theta_0^{(\frac{6}{2})}}{\partial r} + w_0^{(\frac{4}{2})} \left(\frac{\partial \Theta_0^{(\frac{6}{2})}}{\partial z} + \tilde{\mathcal{J}}(r,z,\tau) \right) &= \tilde{S}
\end{aligned}
\tag{5.97}
$$

where

$$
\begin{aligned}
M^{(0)} &= r u_\theta^{(0)} + \frac{\Omega_0}{2} r^2 \,, \qquad \pi = \frac{p^{(\frac{6}{2})}}{\rho^{(0)}} \\
\tilde{\mathcal{J}}(r,z,\tau) &= \Gamma^{\star\star^2} L^{\star\star} q_{vs}^{\star\star}\, z\, \frac{dq_{vs}^{(1)}}{dz} - \frac{\mathcal{D}}{2} \\
\mathcal{D} &= -\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \pi}{\partial r} \right) \left| \frac{\partial \vec{X}_C^{(\frac{1}{2})}}{\partial z} \right|^2 \\
\tilde{S}(r,z,\tau) &= S_{\mathrm{us},0}^{(\frac{9}{2})} - \frac{1}{2} \left(\sum_{k=1}^{2} u_{r,1k}^{(\frac{1}{2})} \frac{\partial \Theta_{1k}^{(\frac{6}{2})}}{\partial r} \right) - \\
& \frac{1}{2} \left(u_{\theta,12}^{(\frac{1}{2})} \frac{\Theta_{11}^{(\frac{6}{2})}}{r} - u_{\theta,11}^{(\frac{1}{2})} \frac{\Theta_{12}^{(\frac{6}{2})}}{r} \right)
\end{aligned}
\tag{5.98}
$$

and

$$
\begin{aligned}
q_{vs}^{(0)} &= \exp\left(-[A^{\star\star}\Gamma^{\star\star} - 1] z\right) \\
q_{vs}^{(1)} &= q_{vs}^{(0)} \left[\left(A^{\star\star} \frac{\partial \Theta^{(\frac{4}{2})}}{\partial z} - \frac{1}{2} A^{\star\star} \Gamma^{\star\star^2} z^2 \right) + \right. \\
& \left. \exp(-z)(A^{\star\star}\Gamma^{\star\star} - 1)\; p^{(1)}(z) \right] \\
\rho^{(0)} &= p_0 \exp(-z) \\
p^{(1)} &= p_0 \Gamma^{\star\star} \left(-\frac{1}{2} z^2 \right) \exp(-z)
\end{aligned}
\tag{5.99}
$$

Table 5.4: Model B - Part I, see the text for explanations

`del B - Part II`, in **Section 5.3.2.4**. Note that the solutions $(5.99)_3$ and $(5.99)_4$ in **Table 5.4**, i.e. solutions for the leading order density $\rho^{(0)}$ and higher order pressure $p^{(1)}$, are obtained from the leading order vertical momentum equations (3.34) and the state equation (A-31). Based on these results the equations (3.126) and (3.127) for the leading order saturation water vaper mixing ratio $q_{vs}^{(0)}$ and $q_{vs}^{(\frac{1}{2})}$ can be simplified to obtain the equations $(5.99)_1$ and $(5.99)_2$ shown in **Table 5.4** (see **Appendix A.6**).

Total effective stability Recalling Eliassen's original transverse circulation equation (5.4) in terms of a kind of stream function ψ, three stability criteria turned out to be important for the final establishment of an externally forced transverse circulation. The criteria were called static stability, baroclinicity and inertial stability. Unfortunately, it becomes difficult to derive an analogous equation to Eliassen's original equation (5.4) using equation set (5.97). That is why the asymmetric advection terms in $(5.97)_5$ turn out to be functions on the axissymmetric vertical velocity $w^{(\frac{4}{2})}$ itself. This is discussed in detail in **Section 5.3.2.4**. Though, studying the thermodynamic equation $(5.97)_5$ alone allows one to get some deeper insights into the mechanism that influence the static stability and therefore the establishment of a secondary circulation.

Recall that in Eliassen's original model (5.4) a mechanism called the static stability has an influence on the secondary circulation. A simlar mechanism can be derived here, which is refered to as total effective stability (see the term in the bracket of (5.94))

$$\left(\frac{\partial\Theta}{\partial z}\right)_{\text{teff}} = \frac{\partial\Theta_0^{(\frac{6}{2})}}{\partial z} + \underbrace{\Gamma^{\star\star^2} L^{\star\star} q_{vs}^{\star\star}\, z\, \frac{dq_{vs}^{(1)}}{dz}}_{\mathcal{J}_1} + \underbrace{\frac{1}{2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\pi}{\partial r}\right)\left|\frac{\partial\vec{X}_C^{(\frac{1}{2})}}{\partial z}\right|^2}_{\mathcal{J}_2} \tag{5.100}$$

Let's assume that net heating effects $S_{us,1k}^{(\frac{8}{2})}$ caused by small-scale convective flows, are primarily stimulated by an environmentally shear induced vortex tilt (see (5.89)). Then, the contributions $\mathcal{J}_1$ and $\mathcal{J}_2$ in (5.100) control the stratification through two different physical processes. The first process is related to phase changes from gaseous water into liquid water and vice versa. The accompanied latent heat release affects the potential temperature as though the stratification is reduced by an amount equal to $\mathcal{J}_1$ (note that $dq_{vs}^{(1)}/dz < 0$). The second process is only efficient if $\partial\vec{X}_C^{(\frac{1}{2})}/\partial z \neq 0$. If it is assumed that there is a nontrivial relation between the vortex tilt and the background flow, the appearance of $\mathcal{J}_2$ in (5.100) can be regarded as a direct indication of an impact of the environmental flow on the smaller scale vortex structure. Due to $\partial\pi/\partial r \geq 0$ for cyclonically rotating flows, it turns out that a non-zero vortex tilt has a stabilizing effect on the stratification and therefore vertically upward moving air parcels. Moreover, due to the weighting factor $\partial\pi/\partial r \sim \partial p^{(\frac{6}{2})}/\partial r$ it can be observed, that this effect will be more/less prominent in regions of strong/weak pressure gradients. It is known from observations that approaching a hurricane from the outer edge, the barometric pressure drops slowly at first, but then more rapidly as one moves closer to the centre until the so called eyewall[5] has reached. Then moving further into the eye[6] the winds slacken again, as a result of weak pressure gradients. Motivated by these observations it is hypothesized that the influence of the vortex tilt on the stratification as described above, might play a nontrivial role for the formation of an eye and eyewall during the transition process from tropical cyclones into mature hurricanes. The verification of this is beyond the scope of this work.

5.3.2.3 Leading order vortex trajectory

In the context of `Model A` it has been shown in **Section 5.3.1.2** that diabatically induced asymmetric contributions $w_{1k}^{(\frac{4}{2})}$ of the vertical velocity field $w^{(\frac{4}{2})}$ are responsible for a modfication of the leading order vortex trajectory from an environmental steering flow in leading order. While asymmetric contributions $w_{1k}^{(\frac{4}{2})}$ are neglected in the derivation of `Model B`, it is the purpose of this section to show how next higher order effects of the background flow on the vortex tilt $\partial\vec{X}_C^{(\frac{1}{2})}/\partial z$ may affect the leading order vortex trajectory. Since the derivation of equations for the leading order vortex motion is similar to those in **Section 5.3.1.2**, only key steps of derivation are presented below.

Velocity potential Using Helmholtz's theorem (see **Section 3.1.4**) to the $\mathcal{O}(\varepsilon^{\frac{3}{2}})$ mass continuity (3.50), divergent velocity components $u_r^{d(\frac{1}{2})}$ and $u_\theta^{d(\frac{1}{2})}$ have to satisfy

$$\frac{\partial u_r^{d(\frac{1}{2})}}{\partial r} + \frac{u_r^{d(\frac{1}{2})}}{r} + \frac{1}{r}\frac{\partial u_\theta^{d(\frac{1}{2})}}{\partial\theta} = \mathcal{G} \tag{5.101}$$

with

$$\mathcal{G} = -\frac{1}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w^{(\frac{4}{2})})}{\partial z} + \left(\frac{\partial X_C^{(\frac{1}{2})}}{\partial z}\cos\theta + \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\sin\theta\right)\frac{\partial w^{(\frac{4}{2})}}{\partial r} \quad . \tag{5.102}$$

Based on the assumption that $w^{(\frac{4}{2})} = w^{(\frac{4}{2})}(r,z,\tau)$ which implies immediately

[5] A ring of intense thunderstorms that whirl around the storm's centre and extend upward to almost 15 km above sea level (Ahrens, 1999)

[6] A region in the centre of a hurricane (tropical storm) where the winds are light and skies are clear to partly cloudy. (Ahrens, 1999)

that $w^{(\frac{4}{2})} = w_0^{(\frac{4}{2})}$, equations for the first asymmetric horizontal velocity contributions $u_r^{d(\frac{1}{2})}$ and $u_\theta^{d(\frac{1}{2})}$ are

$$\begin{aligned}
\frac{\partial u_{r,11}^{d(\frac{1}{2})}}{\partial r} + \frac{u_{r,11}^{d(\frac{1}{2})}}{r} - \frac{u_{\theta,12}^{d(\frac{1}{2})}}{r} &= \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\frac{\partial w_0^{(\frac{4}{2})}}{\partial r} \\
\frac{\partial u_{r,12}^{d(\frac{1}{2})}}{\partial r} + \frac{u_{r,12}^{d(\frac{1}{2})}}{r} + \frac{u_{\theta,11}^{d(\frac{1}{2})}}{r} &= \frac{\partial X_C^{(\frac{1}{2})}}{\partial z}\frac{\partial w_0^{(\frac{4}{2})}}{\partial r}
\end{aligned} \tag{5.103}$$

Note that unlike (5.45) in `Model A`, in `Model B` the asymmetric divergent velocity field given by $u_{r,1k}^{d(\frac{1}{2})}$ and $u_{\theta,1k}^{d(\frac{1}{2})}$ is not caused by asymmetric vertical velocities $w_{1k}^{(\frac{4}{2})}$, but by non-zero tilt components $\partial X_C^{(\frac{1}{2})}/\partial z$ and $\partial Y_C^{(\frac{1}{2})}/\partial z$. Writing the asymmetric contributions $u_{r,1k}^{d(\frac{1}{2})}$ and $u_{\theta,1k}^{d(\frac{1}{2})}$ in terms of a velocity potential (see (3.63)) the equations (5.103) take the form

$$\nabla_1^2 \phi_{1k}^{(\frac{1}{2})} = T_k^\sharp \frac{\partial w_0^{(\frac{4}{2})}}{\partial r} \qquad k = 1, 2 \tag{5.104}$$

The operator ∇_1^2 is defined by (5.46) and $T_k^\sharp$ denotes a shortcut for the vortex tilt components

$$T_1^\sharp = \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} \quad \text{and} \quad T_2^\sharp = \frac{\partial X_C^{(\frac{1}{2})}}{\partial z} \tag{5.105}$$

Using the identity (4.48) and the boundary conditions (5.48), integrating (5.104) twice, yields

$$\phi_{1k}^{(\frac{1}{2})} = T_k^\sharp \; r \int_0^r \frac{1}{\bar{r}^3} \left[\int_0^{\bar{r}} \bar{\bar{r}}^2 \frac{\partial w_0^{(\frac{4}{2})}}{\partial \bar{\bar{r}}} \; d\bar{\bar{r}} \right] d\bar{r} \tag{5.106}$$

Taking into account that the radial behaviour of $w_0^{(\frac{4}{2})}$ for large r is given by (5.23), far field solutions of (5.106) take the form

$$\phi_{1k}^{(\frac{1}{2})} = \mathcal{O}\left(\frac{1}{r}\right) + r\bar{D}_{1k}^2 \qquad \text{as} \qquad r \to \infty \tag{5.107}$$

where

$$\bar{D}_{1k}^2 = T_k^\sharp \int_0^\infty \frac{1}{\bar{r}^3} \left[\int_0^{\bar{r}} \bar{\bar{r}}^2 \frac{\partial w_0^{(\frac{4}{2})}}{\partial \bar{\bar{r}}} \; d\bar{\bar{r}} \right] d\bar{r} \tag{5.108}$$

Refer to **Appendix C.2** for a detailed derivation of (5.107) - (5.108) these equations.

Stream function Elimination of $p^{(\frac{7}{2})}$ from the horizontal momentum equations $(3.43)_1$ and $(3.43)_2$ via cross-differentiation yields

$$\frac{u_\theta^{(0)}}{r}\frac{\partial^2 u_r^{(\frac{1}{2})}}{\partial\theta^2} - \frac{2u_\theta^{(0)}}{r}\frac{\partial u_\theta^{(\frac{1}{2})}}{\partial\theta} - \Omega_0\frac{\partial u_\theta^{(\frac{1}{2})}}{\partial\theta} - \frac{\partial}{\partial r}\left(r u_r^{(\frac{1}{2})}\frac{\partial u_\theta^{(0)}}{\partial r}\right) -$$
$$\frac{\partial}{\partial r}(u_r^{(\frac{1}{2})}u_\theta^{(0)}) - \frac{\partial}{\partial r}\left(u_\theta^{(0)}\frac{\partial u_\theta^{(\frac{1}{2})}}{\partial\theta}\right) - \Omega_0\frac{\partial}{\partial r}(r u_r^{(\frac{1}{2})}) \;=\; \mathcal{P} \qquad (5.109)$$

with

$$\mathcal{P} = \left(\frac{\partial X_C^{(\frac{1}{2})}}{\partial z}\cos\theta + \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\sin\theta\right)\frac{w^{(\frac{4}{2})}\,u_\theta^{(0)}}{r} + \frac{\partial}{\partial r}\left(r w^{(\frac{4}{2})}\frac{\partial u_\theta^{(0)}}{\partial z}\right) -$$
$$\frac{\partial}{\partial r}\left(r w^{(\frac{4}{2})}\left(\frac{\partial X_C^{(\frac{1}{2})}}{\partial z}\cos\theta + \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\sin\theta\right)\frac{\partial u_\theta^{(0)}}{\partial r}\right)$$

Note the difference between (5.109) and the analogous equation (5.57) in `Model A` because of a non-zero vortex tilt, i.e. $\partial\vec{X}_C^{(\frac{1}{2})}/\partial z \neq 0$. With the aid of Helmholtz's decomposition (see (3.63)), a harmonic analysis of (5.109) yields equations for the first Fourier modes of the stream function $\psi^{(\frac{1}{2})}$

$$\begin{aligned}
-u_\theta^{(0)}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left[\frac{\zeta_r^{(0)}}{u_\theta^{(0)}} + \frac{1}{r^2}\right]\right)\psi_{12}^{(\frac{1}{2})} &= \mathcal{P}_{11} + \mathcal{R}_{11} \\
u_\theta^{(0)}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left[\frac{\zeta_r^{(0)}}{u_\theta^{(0)}} + \frac{1}{r^2}\right]\right)\psi_{11}^{(\frac{1}{2})} &= \mathcal{P}_{12} + \mathcal{R}_{12}
\end{aligned} \qquad (5.110)$$

where $\zeta_r^{(0)} = \partial\zeta^{(0)}/\partial r$ and

$$\begin{aligned}
\mathcal{P}_{1k} &= T_k^\sharp\left(w_0^{(\frac{4}{2})}\frac{u_\theta^{(0)}}{r} - \frac{\partial}{\partial r}\left(r w_0^{(\frac{4}{2})}\frac{\partial u_\theta^{(0)}}{\partial r}\right)\right) \\
\mathcal{R}_{1k} &= r[\zeta^{(0)} + \Omega_0]\nabla_1^2\phi_{1k}^{(\frac{1}{2})} + \frac{\partial\phi_{1k}^{(\frac{1}{2})}}{\partial r}(r\zeta_r^{(0)}) \qquad k = 1,2
\end{aligned} \qquad (5.111)$$

Here, $T_k^\sharp$ with $k = 1,2$ represents the vortex tilt components $\partial X_C^{(\frac{1}{2})}/\partial z$ and $\partial Y_C^{(\frac{1}{2})}/\partial z$ (see (5.105)). The asymmetric contributions of the velocity potential $\phi_{1k}^{(\frac{1}{2})}$ are given by (5.106). Using the same boundary conditions as used in order to derive equations for the leading order motion of adiabatic vortices (see (4.27)), i.e.

$$\psi_{1k}^{(\frac{1}{2})} = 0\ , \qquad \frac{\partial\psi_{1k}^{(\frac{1}{2})}}{\partial r} = A_{1k} \qquad \text{at} \quad r = 0 \qquad , \qquad (5.112)$$

the same strategy as in **Section 4.2.2** and **Section 5.3.1.2** is applied to solve (5.110). Then, one obtains

$$
\begin{aligned}
\psi_{11}^{(\frac{1}{2})} &= +u_\theta^{(0)} \int_0^{\bar{\bar{r}}} \frac{1}{\bar{r} u_\theta^{(0)^2}} \left[\int_0^{\bar{r}} r\ (\mathcal{P}_{12} + \mathcal{R}_{12})\ dr \right] d\bar{r} + u_\theta^{(0)} \frac{2A_{11}}{\zeta_*^{(0)}} \\
\psi_{12}^{(\frac{1}{2})} &= -u_\theta^{(0)} \int_0^{\bar{\bar{r}}} \frac{1}{\bar{r} u_\theta^{(0)^2}} \left[\int_0^{\bar{r}} r\ (\mathcal{P}_{11} + \mathcal{R}_{11})\ dr \right] d\bar{r} + u_\theta^{(0)} \frac{2A_{12}}{\zeta_*^{(0)}}
\end{aligned}
\tag{5.113}
$$

To derive far field solutions for $\psi_{1k}^{(\frac{1}{2})}$, far field constraints for $u_\theta^{(0)}$, i.e. (3.99), and for $w_0^{(\frac{4}{2})}$, i.e. (5.23), have been used to obtain (see **Appendix C.3**)

$$
\begin{aligned}
\psi_{12}^{(\frac{1}{2})} &= \mathcal{O}\left(\frac{1}{r}\right) + r D_{12}^2 \qquad \text{as} \quad r \to \infty \\
\psi_{11}^{(\frac{1}{2})} &= \mathcal{O}\left(\frac{1}{r}\right) + r D_{11}^2 \qquad \text{as} \quad r \to \infty
\end{aligned}
\tag{5.114}
$$

with

$$
\begin{aligned}
D_{12}^2 &= -\frac{\pi}{\Gamma} \int_0^\infty r\ (\mathcal{P}_{11} + \mathcal{R}_{11})\ dr \\
D_{11}^2 &= +\frac{\pi}{\Gamma} \int_0^\infty r\ (\mathcal{P}_{12} + \mathcal{R}_{12})\ dr
\end{aligned}
\tag{5.115}
$$

Matching Upon substitution of the far field solutions (5.107) for the velocity potential $\phi_{1k}^{(\frac{1}{2})}$ and far field solutions (5.114) for the stream function $\psi_{1k}^{(\frac{1}{2})}$ into the matching conditions (5.67), the leading order vortex motion in the context of `Model B` (with $S_{us} \neq 0$) is determined through

$$
\begin{aligned}
V_C^{(0)} &= V_{B,C}^{(0)} + V_{R,C}^{(0)} + D_{12}^2 - \bar{D}_{11}^2 \\
U_C^{(0)} &= U_{B,C}^{(0)} + U_{R,C}^{(0)} - D_{11}^2 - \bar{D}_{12}^2
\end{aligned}
\tag{5.116}
$$

Here, $D_{1k}^2 = D_{1k}^2(z,\tau)$ and $\bar{D}_{1k}^2 = \bar{D}_{1k}^2(z,\tau)$ denote functions of the vertical coordinate z and the time coordinate τ and are given through (5.108) and (5.115), respectively. After some further manipulations it turns that the sums $D_{12}^2 - \bar{D}_{11}^2$ and $-D_{11}^2 - \bar{D}_{12}^2$ can be written as a product of the vortex tilt components $T_k^\sharp$ in meridional/latitudinal direction and an improper integral over the mesoscale vortex region, i.e.

$$
\begin{aligned}
D_{12}^2 - \bar{D}_{11}^2 &= -T_1^\sharp \frac{\pi}{\Gamma(z)} \int_0^\infty \mathcal{C}(u_\theta^{(0)}, w^{(\frac{4}{2})}, \Omega_0)\ dr \\
-D_{11}^2 - \bar{D}_{12}^2 &= -T_2^\sharp \frac{\pi}{\Gamma(z)} \int_0^\infty \mathcal{C}(u_\theta^{(0)}, w^{(\frac{4}{2})}, \Omega_0)\ dr
\end{aligned}
\tag{5.117}
$$

with

$$\mathcal{C} = w^{(\frac{4}{2})} u_\theta^{(0)} - r \frac{\partial}{\partial r} \left(r w^{(\frac{4}{2})} \frac{\partial u_\theta^{(0)}}{\partial r} \right) + r^2 [\zeta^{(0)} + \Omega_0] \frac{\partial w^{(\frac{4}{2})}}{\partial r} + r^2 \frac{\partial \zeta^{(0)}}{\partial r} \frac{\partial \tilde{\phi}^{(\frac{1}{2})}}{\partial r} - \frac{\Gamma(z)}{\pi} \frac{1}{r^3} \int_0^r \bar{r}^2 \frac{\partial w^{(\frac{4}{2})}}{\partial \bar{r}} \, d\,\bar{r} \tag{5.118}$$

where $\tilde{\phi}^{(\frac{1}{2})} = \phi_{1k}^{(\frac{1}{2})} / T_k^\sharp$ (see (5.106)). Finally, upon substitution of (5.105) and (5.117) into (5.116) the equations for the leading order vortex motion take the following form

$$\boxed{\begin{aligned} V_C^{(0)} &= V_{B,C}^{(0)} + V_{R,C}^{(0)} - \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} \frac{\pi}{\Gamma} \int_0^\infty \mathcal{C}(u_\theta^{(0)}, w^{(\frac{4}{2})}, \Omega_0) \, dr \\ U_C^{(0)} &= U_{B,C}^{(0)} + U_{R,C}^{(0)} - \frac{\partial X_C^{(\frac{1}{2})}}{\partial z} \frac{\pi}{\Gamma} \int_0^\infty \mathcal{C}(u_\theta^{(0)}, w^{(\frac{4}{2})}, \Omega_0) \, dr \end{aligned}} \tag{5.119}$$

It has been shown in **Section 5.3.1.2**, that diabatic effects caused by convective flows with respect to vortex scales have a nontrivial impact on the vortex motion in leading order. In particular, it was found that a deviation of the vortex trajectory from the environmental mean flow was primarily caused by diabatically induced asymmetries in the vertical velocity field $w^{(\frac{4}{2})}$. Even though such asymmetries are neglected in the present case, it is observed from (5.119) that a non-zero vortex tilt $\partial \vec{X}_C^{(\frac{1}{2})} / \partial z$ has a similar modifying effect on the vortex trajectory. Please keep in mind that $\partial \vec{X}_C^{(\frac{1}{2})} / \partial z \neq 0$ is only satisfied as long as net effects of small-scale cumulus convection, i.e. $S_{us,1k}^{(\frac{8}{2})} \neq 0$ are accounted for, since they contribute to a stabilization of a tilted vortex (see (5.89)). These effects were neglected in the derivation of the vortex motion equations in the context of `Model A`.

5.3.2.4 Model B - Part II

With the introduction of `Model B - Part II` it is the purpose in this section to provide an overview about the equations that determine the *asymmetric* vortex structure. Studying the equations describing the *axissymmetric* vortex structure, i.e. `Model B - Part I`, it has been noted that the equations are only closed together with the equations describing the *asymmetric* vortex structure. Hence the equations summarized in this section are subdivided in the following way. `Model B - Part IIa` includes those asymmetric vortex structure equations that are necessary to close the equations in `Model B - Part I`.

Model B - Part IIa

vertical momentum equation : $(\mathcal{O}(\varepsilon^{\frac{6}{2}}))$

$$T_k^\sharp \frac{\partial \pi}{\partial r} = -\Theta_{1k}^{(\frac{6}{2})} \qquad \text{with} \quad T_1^\sharp = \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} \quad , \; T_2^\sharp = \frac{\partial X_C^{(\frac{1}{2})}}{\partial z} \tag{5.120}$$

thermodynamic equation $(\mathcal{O}(\varepsilon^{\frac{8}{2}}))$:

$$\frac{u_\theta^{(0)}}{r}\Theta_{1k}^{(\frac{6}{2})} = \tilde{S}_k^{(\frac{8}{2})} \qquad \text{with} \quad \tilde{S}_1^{(\frac{8}{2})} = -S_{us,12}^{(\frac{8}{2})} \quad , \; \tilde{S}_2^{(\frac{8}{2})} = S_{us,11}^{(\frac{8}{2})} \tag{5.121}$$

horizontal momentum ($p^{(\frac{7}{2})}$ eliminated):

$$\begin{aligned}
u_{r,11}^{(\frac{1}{2})}\frac{u_\theta^{(0)}}{r} - u_{\theta,12}^{(\frac{1}{2})}\left(\frac{2u_\theta^{(0)}}{r} + \Omega_0\right) + \frac{\partial}{\partial r}\left(u_{r,11}^{(\frac{1}{2})} r\, \zeta_{abs}^{(0)} - u_{\theta,12}^{(\frac{1}{2})} u_\theta^{(0)}\right) &= \mathcal{P}_{11} \\
u_{r,12}^{(\frac{1}{2})}\frac{u_\theta^{(0)}}{r} + u_{\theta,11}^{(\frac{1}{2})}\left(\frac{2u_\theta^{(0)}}{r} + \Omega_0\right) + \frac{\partial}{\partial r}\left(u_{r,12}^{(\frac{1}{2})} r\, \zeta_{abs}^{(0)} - u_{\theta,11}^{(\frac{1}{2})} u_\theta^{(0)}\right) &= \mathcal{P}_{12}
\end{aligned} \tag{5.122}$$

$$\begin{aligned}
\text{with} \quad \zeta_{abs}^{(0)} &= \zeta^{(0)} + \Omega_0 \; , \quad \zeta^{(0)} = \frac{1}{r}\frac{\partial (r u_\theta^{(0)})}{\partial r} \\
\mathcal{P}_{1k} &= T_k^\sharp \left(w_0^{(\frac{4}{2})}\frac{u_\theta^{(0)}}{r} - \frac{\partial}{\partial r}\left(r w_0^{(\frac{4}{2})} \frac{\partial u_\theta^{(0)}}{\partial r}\right)\right)
\end{aligned}$$

mass continuity:

$$\begin{aligned}
\frac{\partial u_{r,11}^{(\frac{1}{2})}}{\partial r} + \frac{u_{r,11}^{(\frac{1}{2})}}{r} - \frac{u_{\theta,12}^{(\frac{1}{2})}}{r} &= \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\frac{\partial w_0^{(\frac{4}{2})}}{\partial r} \\
\frac{\partial u_{r,12}^{(\frac{1}{2})}}{\partial r} + \frac{u_{r,12}^{(\frac{1}{2})}}{r} + \frac{u_{\theta,11}^{(\frac{1}{2})}}{r} &= \frac{\partial X_C^{(\frac{1}{2})}}{\partial z}\frac{\partial w_0^{(\frac{4}{2})}}{\partial r}
\end{aligned} \tag{5.123}$$

general balance condition:

$$\frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\left(U_{B,C}^{(0)} + U_{R,C}^{(0)}\right) = \frac{\partial X_C^{(\frac{1}{2})}}{\partial z}\left(V_{B,C}^{(0)} + V_{R,C}^{(0)}\right) \tag{5.124}$$

Table 5.5: Model B - Part IIa, see the text for explanations

horizontal momentum :

$$
\begin{aligned}
\frac{1}{\rho^{(0)}}\frac{1}{r}\frac{\partial(rp_{12}^{(\frac{7}{2})})}{\partial r} &= -\frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}w^{(\frac{4}{2})}\zeta^{(0)} + \Omega_0(2V_C^{(0)} + u_{\theta,12}^{(\frac{1}{2})} + u_{r,11}^{(\frac{1}{2})}) + \\
&\quad u_{\theta,12}^{(\frac{1}{2})}\frac{u_\theta^{(0)}}{r} + u_{r,11}^{(\frac{1}{2})}\frac{\partial u_\theta^{(0)}}{\partial r} \\
\frac{1}{\rho^{(0)}}\frac{1}{r}\frac{\partial(rp_{11}^{(\frac{7}{2})})}{\partial r} &= \frac{\partial X_C^{(\frac{1}{2})}}{\partial z}w^{(\frac{4}{2})}\zeta^{(0)} - \Omega_0(2U_C^{(0)} - u_{\theta,11}^{(\frac{1}{2})} + u_{r,12}^{(\frac{1}{2})}) + \\
&\quad u_{\theta,11}^{(\frac{1}{2})}\frac{u_\theta^{(0)}}{r} - u_{r,12}^{(\frac{1}{2})}\frac{\partial u_\theta^{(0)}}{\partial r}
\end{aligned}
\tag{5.125}
$$

vertical momentum $(\mathcal{O}(\varepsilon^{\frac{7}{2}}))$:

$$
\begin{aligned}
\frac{\partial p_{11}^{(\frac{7}{2})}}{\partial z} - \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\frac{\partial p_0^{(\frac{7}{2})}}{\partial r} - \frac{\partial Y_C^{(\frac{2}{2})}}{\partial z}\frac{\partial p^{(\frac{6}{2})}}{\partial r} &= -\rho_{11}^{(\frac{7}{2})} \\
\frac{\partial p_{12}^{(\frac{7}{2})}}{\partial z} - \frac{\partial X_C^{(\frac{1}{2})}}{\partial z}\frac{\partial p_0^{(\frac{7}{2})}}{\partial r} - \frac{\partial X_C^{(\frac{2}{2})}}{\partial z}\frac{\partial p^{(\frac{6}{2})}}{\partial r} &= -\rho_{12}^{(\frac{7}{2})}
\end{aligned}
\tag{5.126}
$$

vortex motion :

$$
\begin{aligned}
V_C^{(0)} &= V_{B,C}^{(0)} + V_{R,C}^{(0)} - \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z}\frac{\pi}{\Gamma}\int_0^\infty \mathcal{C}(u_\theta^{(0)}, w^{(\frac{4}{2})}, \Omega_0)\, dr \\
U_C^{(0)} &= U_{B,C}^{(0)} + U_{R,C}^{(0)} - \frac{\partial X_C^{(\frac{1}{2})}}{\partial z}\frac{\pi}{\Gamma}\int_0^\infty \mathcal{C}(u_\theta^{(0)}, w^{(\frac{4}{2})}, \Omega_0)\, dr
\end{aligned}
\tag{5.127}
$$

with

$$
\begin{aligned}
\mathcal{C} &= w^{(\frac{4}{2})}u_\theta^{(0)} - r\frac{\partial}{\partial r}\left(rw^{(\frac{4}{2})}\frac{\partial u_\theta^{(0)}}{\partial r}\right) + r^2[\zeta^{(0)} + \Omega_0]\frac{\partial w^{(\frac{4}{2})}}{\partial r} + \\
&\quad r^2\frac{\partial \zeta^{(0)}}{\partial r}\frac{\partial \tilde{\phi}^{(\frac{1}{2})}}{\partial r} - \frac{\Gamma(z)}{\pi}\frac{1}{r^3}\int_0^r \bar{r}^2\frac{\partial w^{(\frac{4}{2})}}{\partial \bar{r}}\, d\,\bar{r}
\end{aligned}
\tag{5.128}
$$

$$
\phi_{1k}^{(\frac{1}{2})} = T_k^\sharp\, r\int_0^r \frac{1}{\bar{r}^3}\left[\int_0^{\bar{r}} \bar{\bar{r}}^2\frac{\partial w^{(\frac{4}{2})}}{\partial \bar{\bar{r}}}\, d\bar{\bar{r}}\right] d\bar{r}\ , \qquad \tilde{\phi}^{(\frac{1}{2})} = \phi_{1k}^{(\frac{1}{2})}/T_k^\sharp\ , \qquad k = 1,2
$$

equation of state:

$$
\rho^{(0)}\Theta_{1k}^{(\frac{7}{2})} + \rho_{1k}^{(\frac{7}{2})}\Theta_\infty = p_{1k}^{(\frac{7}{2})} \tag{5.129}
$$

Table 5.6: Model B - Part I, see the text for explanations

In `Model B - Part IIb` the rest of the asymmetric structure equations are summarized.

Let's assume the asymmetric source terms $S_{us,1k}^{(\frac{8}{2})}$, $S_{us,0}^{(\frac{9}{2})}$, the background flow $\vec{U}_{B,C} = (U_{B,C}, V_{B,C})$ and the regular flow $\vec{U}_{R,C} = (U_{R,C}, V_{R,C})$ are given. Then, the equations in `Model B - Part I/IIa` which are shown in **Table 5.4** and **Table 5.5** constitute a closed set of equations in the unknown variables $u_\theta^{(0)}, u_{r,0}^{(\frac{1}{2})}, w_0^{(\frac{4}{2})}, p^{(\frac{6}{2})}, \Theta_0^{(\frac{6}{2})}$ and $u_{r,1k}^{(\frac{1}{2})}, u_{\theta,1k}^{(\frac{1}{2})}, \Theta_{1k}^{(\frac{6}{2})}$. The derivations of the first two equations (5.120) and (5.121) have already been discussed in **Section 5.3.2.1**. Both (5.122) and (5.123) result from a harmonic analysis of the horizontal momentum equations (5.109) (with $p^{(\frac{7}{2})}$ eliminated via cross- differentiation) and the mass continuity (3.50), using $(3.58)_2$ and $(3.58)_3$ with $j = 1$. It is important to point out, that for non-zero vortex tilt components $\partial X_C^{(\frac{1}{2})}/\partial z$ and $\partial Y_C^{(\frac{1}{2})}/\partial z$ the general balance condition (5.124) has to be satisfied. Its derivation is discussed in **Section 3.5**. But note that the general balance condition (5.124) becomes trivial if vortices on an f-plane ($\vec{V}_{R,C} = 0$) embedded in a weak background flow ($\vec{V}_{B,C}^{(0)} = 0$) are studied (see **Section 4.104**).

Having once solved the equations in `Model B - Part I/IIa`, the mass $\rho_{1k}^{(\frac{7}{2})}$, temperature $\Theta_{1k}^{(\frac{7}{2})}$, and pressure $p_{1k}^{(\frac{7}{2})}$ fields that are related to the asymmetric, horizontal velocity field components $u_{r,1k}^{(\frac{1}{2})}$ and $u_{\theta,1k}^{(\frac{1}{2})}$ are obtained by solving the equations in `Model B - Part IIb` which are shown in **table 5.6**. Here, the equations (5.125) emanate from a harmonic analysis of the $\mathcal{O}(\varepsilon^{\frac{2}{2}})$ horizontal momentum equations (3.43) and a subsequent combination of the equations including $p_{11}^{(\frac{7}{2})}$ and $p_{12}^{(\frac{7}{2})}$, respectively. A harmonic analysis of the vertical momentum equations (3.47) yields the equations (5.126). Details on the derivation of the equations (5.127) for the leading order vortex motion are given in **Section 5.3.2.3**. Note, due to $\partial X_C^{(\frac{1}{2})}/\partial z \neq 0$ and $\partial Y_C^{(\frac{1}{2})}/\partial z \neq 0$ the equations are closed in the unknowns $\rho_{1k}^{(\frac{7}{2})}, \Theta_{1k}^{(\frac{7}{2})}, p_{1k}^{(\frac{7}{2})}$, if $\partial X_C^{(\frac{2}{2})}/\partial z$, $\partial Y_C^{(\frac{2}{2})}/\partial z$ and $p_0^{(\frac{7}{2})}$ together with the solutions obtained from `Model B - Part I/IIa`, are given.

Note that the equations of `Model B` are complex as those in `Model A`. Hence, to get a better understanding about the mechanisms that determine the motion and structure of diabatic votices, the equations have to be solved numerically which is beyond the scope of this work.

Chapter 6

Summary

In this thesis a derivation and discussion of reduced model equations describing the motion and structure of three-dimensional, concentrated atmospheric vortices has been presented. The manner in which the derivations have been carried out were aimed at constructing reduced model equations that are suited to study complex scale interactions between the inner core, mesoscale (~ 200 km) structure of hurricane-like vortices with typical wind speeds of ~ 30 m s^{-1} and and the vortex motion over synoptic scale (~ 1000 km) distances. Focusing on possible mechanisms that may generate and/or influence such kind of scale interactions, two different vortex settings have been choosen. In a so-called adiabatic vortex case reduced model equations have been derived that describe scale interactions between the mesoscale vortex structure and the synoptic scale vortex motion influenced by an environmental flow with a vertical wind shear up to ~ 10 m s^{-1} over the depth of the troposphere (~ 10 km). In a so-called diabatic vortex case the additional influence of diabatic effects caused by moisture conversion processes occuring within the inner core vortex region have been taken into account. Among others, one of the main objectives in the derivation of such reduced model equations was to provide a set of equations whose solutions may help to explore a manner in which processes acting on smaller scales may lead to a modification of the synoptic scale vortex motion from its environmental steering. Knowledge about such scale interactions can be used to find information about favourable structurally vortex features with respect to mesoscales, which allow a vortex to sustain its coherence in the presence of a vertically sheared environmental flow.

The derivations of reduced model equations for both the adiabatic and diabatic vortex case were based on matched asymptotic methods that have been applied within the framework of an unified approach to meteorological modelling developed by Klein (2004) and Klein & Majda (2005). The approach is based on a set of carefully chosen distinguished limits for several small non-dimensional parameters, and on specializations of a very general multiple-scales asymptotic ansatz which is applied to the full three-dimensional compressible flow equations. The concept of using matched asymptotic methods to derive equations for the vortex motion goes back to Ling & Ting (1988). They were the first who applied this concept successfully in order to derive solutions for the motion of two-dimensional geostrophic vortices in a dry atmosphere.

Using matched asymptotic methods for the derivation of equations for the motion and structure of both adiabatic and diabatic vortices, the following sequence of steps have been necessary in the present work: (i) construction of reduced model equations for the vortex structure with respect to mesoscales and synoptic scales on the basis of appropriate single scale inner and outer expansions, (ii) derivation of analytical solutions for the inner and outer velocity fields from the leading order reduced model equations, and (iii) matching of the velocity fields on the basis of suitable matching conditions. For the different vortex settings choosen in the present work, the matching procedure came with a variety of results. On the one hand equations for the vortex motion have been derived that account for net effects of mesoscale processes acting within the inner core vortex region. Depending on whether the adiabatic or diabatic vortex case has been considered, these mesoscale processes differ in the mechanisms that cause them. On the other hand, the matching procedure gave a number of additional constraints that reflect the impact of an environmental flow on the mesoscale vortex structure. An overview of the main results obtained for the adiabtic and diabatic vortex case is given below.

Adiabatic vortex Considering the adiabatic vortex case, leading order solutions for the synoptic scale vortex motion on a β-plane have been derived that describe how the vortex motion is determined by the background flow and a regular flow field which is due to the β-effect induced by the vortex flow itself. Here the vortex motion has been expressed in terms of the temporal evolution of a vortex centreline. Basically, the solutions are in agreement with those obtained by Morikawa (1960) and Reznik (1992) who considered two-dimensional geostrophic vortices. However, since three-dimensional vortices have been studied in the present work, the leading order equations for the vortex motion additionally allow one to make some statements about the vertical shear of the environmental flow in which vortex solutions describing coherent vortices may exist. Moreover, next to the solutions for the leading order vortex motion, solutions denoting higher order corrections have been derived. They describe how the vortex motion in higher order is determined by higher order corrections of the background flow and a net effect caused by the mesoscale structure of the vortex itself. This net effect, however, is only efficient as long as baroclinic vortices are considered and/or the vortex has a nonzero tilt. Here the tilt has been expressed in terms of vertical derivatives of the higher order vortex centreline.

A relation between the vortex tilt and the background flow has been found using the equations for the higher order vortex motion and an vertical Eigenmode constraint for the next higher order vortex centreline. The latter was an additional result of the matching procedure with respect to the inner and outer velocity fields. In particular it has been shown that the vortex tilt components in meridional and zonal direction are proportional to the vertical shear of the respective flow components denoting higher order corrections of a prescribed background flow. Therefore, a vortex tilt caused by the vertical shear of an environmental flow was identified. Considering baroclinic vortices it has been shown that both the direction in which the vortex is tilted and the magnitude of the vortex tilt are fixed as long as a steady and strong background flow is assumed. For baroclinic vortices embedded in a steady but weak background flow, however, the higher order vortex centreline makes a precession motion, i.e. the direction in which the vortex is tilted changes with time, while the magnitude of the tilt remains constant. Regarding the precession motion of the tilted vortex, a similar vortex behaviour has been observed by Reasor & Montgomery (2001, 2004) and Jones (1994) on the basis of numerical simulations. In view of the temporal changes of the magnitude of the vortex tilt, different results were obtained for barotropic vortices. As noted earlier, the influence of the mesoscale vortex structure on the higher corrections for the vortex motion vanishes if barotropic vortices are considered. As a consequence, the solutions derived for the vortex centreline describe how the vortex is sheared away with increasing time (i.e. the tilt becomes infinite large) due to the differential advection by the vertically sheared environmental flow.

Finally, with the understanding obtained on the impact of an vertically sheared environmental flow on the vortex tilt, it was possible to find solutions describing the influence of the evironmental flow on the mesoscale vortex structure. In particular, a harmonic analysis of the reduced model equations valid for the mesoscale vortex region revealed direct relationships between the vortex tilt and both wavenumber-one asymmetries in the potential temperature patterns and vertical velocity fields. Since these mesoscale asymmetries would only exist if the vortex has been tilted by the background flow, the derived relations can be used to gain insights into the manner in which an environmental flow affects the mesoscale vortex structure. From a comparison of the asymptotically derived relationships between the background flow induced vortex tilt and wavenumber-one asymmetries with the observations made by Frank & Ritchie (1999) on the basis of numerical simulations, it has been found that the asymptotically derived relations describe a mechanism known as the adiabatic lifting mechanism. This terminology was introduced by Frank & Ritchie (1999) in order to summarize a sequence of events that explains the generation of patterns of forced ascent that occur as the vortex responds to imposed vertical wind shear.

It is envisaged that with an introduction of a faster time scale, asymptotically solutions describing a realignment of a tilted vortex in an environmental flow with vertical shear, similar to the observations made by Reasor & Montgomery (2001, 2004), may be derived.

Diabatic vortex Considering the diabatic vortex case two different models, `Model A` and `Model B`, for the motion and structure of atmospheric vortices have been derived under the simplifying assumption of an saturated atmosphere.

Although the difference between the two models is in the physical treatment of the diabatic source term, both models can be regarded as extendend versions of the Eliassen balanced vortex model (Eliassen, 1952). The original Eliassen balanced vortex model is an idealized two-dimensional model that can be used to investigate the response of an axially symmetric vortex in gradient wind balance to sources and sinks of heat and angular momentum, where the sources have to be prescribed externally. Compared to the original Eliassen balanced vortex model, the modifications of the two models derived in the present work consists of four points. The first two are related to an additional provision for asymmetries in the thermodynamic fields describing the vortex structure and an explicit description of the diabatic source term which is possible due to an explicit inclusion of moisture parameters. Dropping the simplifying assumption of an axissymmetric vortex structure makes it possible to get closer to the description of real atmospheric cyclones which are highly asymmetric. The explicit inclusion of moisture parameters is meaningful since the diabatic source term has to be regarded as a pure function of the flow itself which makes an external treatment of the diabatic source difficult. The third point regards the influence of a vertically sheared environmental flow on the vortex structure which has been taken into account through matched asymptotic methods. The fourth modification relies in a restriction of the description of the temporal evolution of the vortex structure to the synoptic scale advection time. Although the models have not been solved in this thesis, an interpretation of the equations allows for some discussions concerning the influence of an environmental flow and diabatic effects on the vortex motion and structure.

`Model A:` Regarding the formulation of the diabatic source in `Model A`, only latent heat release due to condensation-evaporation of cloud water in vertically moving air parcels has been taken into account, whereas the vertical motion of the air parcels has been resolved with respect to the scales choosen in the asympotic ansatz. Net mesoscale heating effects resulting from a cooperative action of smaller scale convective clouds have been neglected. It has been shown that the latter assumption yields reduced model equations for the mesoscale vortex structure that exclude vortex solutions describing vortices having a relatively large tilt. That means the reduced model equations summarized in `Model A` are only suitable to study the motion and structure of atmospheric vortices with a horizontal displacement between the upper and lower vortex part equal or smaller than $\sim$ 80 km, to which for example mature hurricanes belong to. Moreover and unlike the adiabatic vortex case, equations for the vortex motion have been derived, that include in leading order a net effect on the synoptic scale vortex motion that are related to mesoscale processes. This net effect is effective as long as diabatically induced asymmetries in the vertical velocity fields, which are one order larger than in the adiabatic vortex case, determine the leading order vortex structure.

`Model B:` In `Model B` both latent heat release due to evaporation-condensation of cloud water in vertically moving air parcels which are resolved with respect to mesoscales and the possibility of net heating effects caused by small-scale cumulus convection have been taken into account. However, because of the single scale asymptotic ansatz used here, an explicit description of the latter has not been possible. Nonetheless, it has been shown that the additional inclusion of net heating effects due to small scale convection allows the existence of a vortex tilt one order larger than in `Model A`. In particular, reduced model equations have been derived that describe a relationship between a vortex tilt and a net heating effect within the mesoscale vortex region due to smaller scale cumulus convection. It has been argued that this heating effect has to be identified as a direct response to a vortex tilt which in turn is initiated by some other mechanisms, as for instance a vertically sheared environmental flow. This argument is supported by observational studies on real hurricanes made by Corbosiero & Molinari (2003). Furthermore, it has been observed that the relationship between the vortex tilt and a net heating effect due to smaller scale cumulus convection derived in the diabatic vortex case, replaces the relationship between a background shear induced vortex tilt and wavenumber-one asymmetries in the vertical velocity fields derived in the adiabatic vortex case. This means that the adiabatic lifting mechanism that modulates the vertical velocity fields in a dry vortex vanishes if moisture effects are included. This finding is in agreement with numerical simulations by Frank & Ritchie (1999). Since Frank & Ritchie (1999) argue that the adiabatic lifting mechanism might be relevant for an understanding of patterns of convection in loosely organized systems such as tropical depressions, it would be interesting to connect the dry and moist (saturated) vortex case studied here in order to understand the role of an adiabatic lifting mechanism on the transition from an undersaturated dry air regime into a saturated regime. Although asymmetric vertical velocities in leading order have been neglected in the derivation of `Model B`, as in `Model A` the leading order equations for the vortex motion include a net effect caused by the mesoscale vortex structure. This net effect, however, is only effective as long as the vortex tilt is nonzero.

In this thesis reduced model equations have been derived without attempting to solve them numerically. However, to enable one to gain a better insight into the mechanisms describing the vortex motion and structure, the equations have to be solved numerically. Moreover, a possible extension of these models is to resolve the smaller cumulus scales in addition to the mesoscale region of the vortex itself via a multiple-scales expansion ansatz to study the interactions between these scales.

In summing up it can be said that the unified approach to meteorological modelling used in the present work is a useful tool to derive systematically reduced model equations for the motion and structure of hurricane-like atmospheric vortices under the influence of an environmental flow with vertical shear and diabatic effects. However, research is required on how reduced equations derived in this thesis can be used for practical applications.

Appendix A

Auxiliary calculations for Chapter 3

A.1

Transformations for the derivative operators $\vec{\nabla}_h, \partial/\partial z, \partial/\partial t$ appearing in (2.19), from an (x, y, z, t) space into an (r, θ, z, τ) space are derived, on account of the coordinate and stretching transformations (3.3), $(3.4)_1$, (3.8)-(3.10). For further manipulations it is instructive to rewrite the latter into

$$\begin{aligned} \hat{x} &= \varepsilon^{\frac{3}{2}} \left(x - \varepsilon^{-2} X_C \left(z, \varepsilon^2 t\right)\right) = r \cos\theta \\ \hat{y} &= \varepsilon^{\frac{3}{2}} \left(y - \varepsilon^{-2} Y_C \left(z, \varepsilon^2 t\right)\right) = r \sin\theta \end{aligned} \qquad \text{(A-1)}$$

where the radius r and the azimuthal angle θ satisfy

$$r = (\hat{x} \cos\theta + \hat{y} \sin\theta) , \qquad \theta = \arctan\left(\frac{\hat{y}}{\hat{x}}\right) \qquad \text{(A-2)}$$

Moreover we have

$$z = z , \qquad \tau = \varepsilon^2 t \qquad \text{(A-3)}$$

Note, the primes denoting dimensionless variables have been dropped.

a) Horizontal Nabla Operator With the above transformations and with the aid of the chain rule one obtains

$$
\begin{aligned}
\vec{\nabla}_h &= \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} && \text{(A-4)}\\
&= \vec{i}\,\left(\frac{\partial}{\partial r}\,\frac{\partial r}{\partial \hat{x}}\,\frac{\partial \hat{x}}{\partial x} + \frac{\partial}{\partial \theta}\,\frac{\partial \theta}{\partial \hat{x}}\,\frac{\partial \hat{x}}{\partial x}\right) + \vec{j}\,\left(\frac{\partial}{\partial r}\,\frac{\partial r}{\partial \hat{y}}\,\frac{\partial \hat{y}}{\partial y} + \frac{\partial}{\partial \theta}\,\frac{\partial \theta}{\partial \hat{y}}\,\frac{\partial \hat{y}}{\partial y}\right)
\end{aligned}
$$

where

$$
\frac{\partial r}{\partial \hat{x}} = \cos\theta\,,\ \frac{\partial r}{\partial \hat{y}} = \sin\theta\,,\ \frac{\partial \hat{x}}{\partial x} = \frac{\partial \hat{y}}{\partial y} = \varepsilon^{\frac{3}{2}}\,,\ \frac{\partial \theta}{\partial \hat{x}} = -\frac{\sin\theta}{r}\,,\ \frac{\partial \theta}{\partial \hat{y}} = +\frac{\cos\theta}{r} \quad \text{(A-5)}
$$

Thus, together with the transformation (3.12) for the unit vectors $\vec{i}$ and $\vec{j}$, the right hand side of (A-4) takes the form

$$
\begin{aligned}
\vec{\nabla}_h &= \epsilon^{3/2}\left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)(\cos\theta\;\vec{e}_r - \sin\theta\;\vec{e}_\theta)\;+\\
&\quad \epsilon^{3/2}\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial \theta}\right)(\sin\theta\;\vec{e}_r + \cos\theta\;\vec{e}_\theta)\\
&= \epsilon^{3/2}\left(\frac{\partial}{\partial r}\;\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\;\vec{e}_\theta\right) = \epsilon^{3/2}\hat{\vec{\nabla}}_h && \text{(A-6)}
\end{aligned}
$$

b) Vertical Derivative Operator With the transformations (A-1) - (A-2) and with the aid of the chain rule one obtains

$$
\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\frac{\partial r}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial z} + \frac{\partial}{\partial r}\frac{\partial r}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial z} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial z} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial z} + \frac{\partial}{\partial z} \quad \text{(A-7)}
$$

where

$$
\frac{\partial \hat{x}}{\partial z} = -\varepsilon^{-\frac{1}{2}}\frac{\partial X_C}{\partial z}\,,\ \frac{\partial \hat{y}}{\partial z} = -\varepsilon^{-\frac{1}{2}}\frac{\partial Y_C}{\partial z} \quad \text{(A-8)}
$$

Hence, upon substitution of (A-8) and (A-5) into (A-7), one obtains

$$
\begin{aligned}
\frac{\partial}{\partial z} &= -\varepsilon^{-\frac{1}{2}}\left(\frac{\partial X_C}{\partial z}\cos\theta + \frac{\partial Y_C}{\partial z}\sin\theta\right)\frac{\partial}{\partial r} -\\
&\quad \varepsilon^{-\frac{1}{2}}\left(-\frac{\partial X_C}{\partial z}\frac{\sin\theta}{r} + \frac{\partial Y_C}{\partial z}\frac{\cos\theta}{r}\right)\frac{\partial}{\partial \theta} + \frac{\partial}{\partial z} && \text{(A-9)}
\end{aligned}
$$

With $\vec{X}_C = \vec{i}\,X_C + \vec{j}\,Y_C$, (A-6) and the relation (3.12), equation (A-9) can be rewritten into

$$
\frac{\partial}{\partial z} = \frac{\partial}{\partial z} - \varepsilon^{-\frac{1}{2}}\frac{\partial \vec{X}_C}{\partial z}\cdot\hat{\vec{\nabla}}_h \quad \text{(A-10)}
$$

c) Temporal Derivative Operator With the transformations (A-1) - (A-2) and with the aid of the chain rule one obtains

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial r}\frac{\partial r}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial}{\partial r}\frac{\partial r}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial t} \tag{A-11}$$

where

$$\frac{\partial \tau}{\partial t} = \epsilon^2 \ , \ \frac{\partial \hat{x}}{\partial \tau} = -\varepsilon^{-\frac{1}{2}}\frac{\partial X_C}{\partial \tau} \ , \ \frac{\partial \hat{y}}{\partial \tau} = -\varepsilon^{-\frac{1}{2}}\frac{\partial Y_C}{\partial \tau} \tag{A-12}$$

Upon substitution of (A-12) and (A-5) into (A-11) yields

$$\begin{aligned}\frac{\partial}{\partial t} &= -\,\epsilon^{3/2}\left(\frac{\partial X_C}{\partial \tau}\cos\theta + \frac{\partial Y_C}{\partial \tau}\sin\theta\right)\frac{\partial}{\partial r} - \\ &\quad \epsilon^{\frac{3}{2}}\left(-\frac{\partial X_C}{\partial \tau}\frac{\sin\theta}{r} + \frac{\partial Y_C}{\partial \tau}\frac{\cos\theta}{r}\right)\frac{\partial}{\partial \theta} + \epsilon^2\frac{\partial}{\partial \tau} \\ &= \epsilon^2\frac{\partial}{\partial \tau} - \frac{\partial \vec{X}_C}{\partial \tau}\cdot\vec{\nabla}_h\end{aligned} \tag{A-13}$$

A.2

An asymptotic expansion of (3.16) based on expansion ansatz (3.15) reads

$$\frac{1}{\Theta}\frac{\partial \Theta}{\partial z} = \frac{1}{(\Theta^{(0)}+\delta)}\left(\frac{\partial \Theta^{(0)}}{\partial z} + \varepsilon^{\frac{1}{2}}\frac{\partial \Theta^{(\frac{1}{2})}}{\partial z} + \varepsilon^{\frac{2}{2}}\frac{\partial \Theta^{(\frac{2}{2})}}{\partial z} + \mathcal{O}(\varepsilon^{\frac{3}{2}})\right) \tag{A-14}$$

with $\delta = \varepsilon^{\frac{1}{2}}\Theta^{(\frac{1}{2})} + \varepsilon^{\frac{2}{2}}\Theta^{(\frac{2}{2})} + \varepsilon^{\frac{3}{2}}\Theta^{(\frac{3}{2})} + \mathcal{O}(\varepsilon^{\frac{4}{2}})$. With the aid of Taylor's theorem it can be shown, that asymptotic approximations of $1/\Theta$ read

$$\begin{aligned}\frac{1}{\Theta} &= \frac{1}{\Theta^{(0)}} - \varepsilon^{\frac{1}{2}}\frac{\Theta^{(\frac{1}{2})}}{\Theta^{(0)^2}} + \varepsilon^{\frac{2}{2}}\left(-\frac{\Theta^{(\frac{2}{2})}}{\Theta^{(0)^2}} + \frac{1}{4}\frac{\Theta^{(\frac{1}{2})^2}}{\Theta^{(0)^3}}\right) + \\ &\quad \varepsilon^{\frac{3}{2}}\left(-\frac{\Theta^{(\frac{3}{2})}}{\Theta^{(0)^2}} + \frac{1}{2}\frac{\Theta^{(\frac{1}{2})}\Theta^{(\frac{2}{2})}}{\Theta^{(0)^3}}\right) + \mathcal{O}(\varepsilon^{\frac{4}{2}})\end{aligned} \tag{A-15}$$

Hence, (A-14) can be written as

$$\frac{1}{\Theta}\frac{\partial\Theta}{\partial z} = \frac{1}{\Theta^{(0)}}\frac{\partial\Theta^{(0)}}{\partial z} + \varepsilon^{\frac{1}{2}}\left(\frac{1}{\Theta^{(0)}}\frac{\partial\Theta^{(\frac{1}{2})}}{\partial z} - \frac{\partial\Theta^{(0)}}{\partial z}\frac{\Theta^{(\frac{1}{2})}}{\Theta^{(0)^2}}\right) +$$
$$\varepsilon^{\frac{2}{2}}\left(\frac{1}{\Theta^{(0)}}\frac{\partial\Theta^{(\frac{2}{2})}}{\partial z} + \frac{\partial\Theta^{(0)}}{\partial z}\left[-\frac{\Theta^{(\frac{2}{2})}}{\Theta^{(0)^2}} + \frac{1}{4}\frac{\Theta^{(\frac{1}{2})^2}}{\Theta^{(0)^3}}\right]\right) +$$
$$\varepsilon^{\frac{3}{2}}\left(\frac{1}{\Theta^{(0)}}\frac{\partial\Theta^{(\frac{3}{2})}}{\partial z} + \frac{\partial\Theta^{(0)}}{\partial z}\left[-\frac{\Theta^{(\frac{3}{2})}}{\Theta^{(0)^2}} + \frac{1}{2}\frac{\Theta^{(\frac{1}{2})}\Theta^{(\frac{2}{2})}}{\Theta^{(0)^3}}\right]\right) + \mathcal{O}(\varepsilon^{\frac{4}{2}}) \qquad \text{(A-16)}$$

Since order of magnitude estimates of the Bruint-Väisalä frequency yield $(1/\Theta)\ \partial\Theta/\partial z = \mathcal{O}(\varepsilon^2)$ (see (3.16)), from (A-16) one has to conclude that

$$\frac{\partial\Theta^{(\frac{i}{2})}}{\partial z} = 0\ , \qquad i = 0, 1, 2, 3 \qquad \text{(A-17)}$$

A.3

Expansion of the equation of state $\rho\Theta = p^{1-\Gamma^{\star\star}\varepsilon}$. For an asymptotic expansion the state equation can be rewritten into

$$p^{\Gamma^{\star\star}\varepsilon}\rho\Theta = p \qquad \text{(A-18)}$$

Taylor's theorem is used to find an approximation of $f(\varepsilon\Gamma) = p^{\varepsilon\Gamma}$ by expanding about $\varepsilon_0 = 0$. With $f'(\varepsilon\Gamma) = p^{\varepsilon\Gamma}\ln p$, $f''(\varepsilon\Gamma) = p^{\varepsilon\Gamma}(\ln p)^2$ and $f'''(\varepsilon\Gamma) = p^{\varepsilon\Gamma}(\ln p)^3$ one obtains

$$p^{\varepsilon\Gamma} = 1 + \varepsilon\Gamma\ln p + \frac{(\varepsilon\Gamma)^2}{2!}(\ln p)^2 + \frac{(\varepsilon\Gamma)^3}{3!}(\ln p)^3 + \mathcal{O}(\varepsilon^4) \qquad \text{(A-19)}$$

With the asymptotic expansion $p = p^{(0)} + \tilde{\delta}$ where $\tilde{\delta} = \varepsilon^{\frac{1}{2}}p^{(\frac{1}{2})} + \varepsilon^{\frac{2}{2}}p^{(\frac{2}{2})} + \varepsilon^{\frac{3}{2}}p^{(\frac{3}{2})} + \varepsilon^{\frac{4}{2}}p^{(\frac{4}{2})} + \mathcal{O}(\varepsilon^{(\frac{5}{2})})$, Taylor series expansion of $\ln p$ around $p^{(0)}$ takes the form

$$\ln p = a + \varepsilon^{\frac{1}{2}}b + \varepsilon^{\frac{2}{2}}c + ... + \varepsilon^{\frac{7}{2}}h + \mathcal{O}(\varepsilon^{(\frac{8}{2})}) \qquad \text{(A-20)}$$

where

$$a = \ln p^{(0)}\ , \qquad b = \frac{p^{(\frac{1}{2})}}{p^{(0)}}\ , \qquad c = \left(\frac{p^{(\frac{2}{2})}}{p^{(0)}} - \frac{A}{2p^{(0)^2}}\right) \qquad \text{(A-21)}$$

and

$$\begin{aligned}
d &= \left(\frac{p^{(\frac{3}{2})}}{p^{(0)}} - \frac{B}{2p^{(0)^2}} + \frac{A'}{3!p^{(0)^3}}\right) \\
e &= \left(\frac{p^{(\frac{4}{2})}}{p^{(0)}} - \frac{C}{2p^{(0)^2}} + \frac{B'}{3!p^{(0)^3}} - \frac{A''}{4!p^{(0)^4}}\right) \\
f &= \left(\frac{p^{(\frac{5}{2})}}{p^{(0)}} - \frac{D}{2p^{(0)^2}} + \frac{C'}{3!p^{(0)^3}} - \frac{B''}{4!p^{(0)^4}} + \frac{A'''}{5!p^{(0)^5}}\right) \\
g &= \left(\frac{p^{(\frac{6}{2})}}{p^{(0)}} - \frac{E}{2p^{(0)^2}} + \frac{D'}{3!p^{(0)^3}} - \frac{C''}{4!p^{(0)^4}} + \frac{B'''}{5!p^{(0)^5}} - \frac{A''''}{6!p^{(0)^6}}\right) \\
h &= \left(\frac{p^{(\frac{7}{2})}}{p^{(0)}} - \frac{F}{2p^{(0)^2}} + \frac{E'}{3!p^{(0)^3}} - \frac{D''}{4!p^{(0)^4}} + \frac{C'''}{5!p^{(0)^5}} - \frac{B''''}{6!p^{(0)^6}} + \right. \\
&\left. \frac{A'''''}{7!p^{(0)^7}}\right)
\end{aligned} \tag{A-22}$$

with

$$\begin{aligned}
A &= p^{(\frac{1}{2})^2} \\
B &= 2p^{(\frac{1}{2})}p^{(\frac{2}{2})} \\
C &= 2p^{(\frac{1}{2})}p^{(\frac{3}{2})} + p^{(\frac{2}{2})^2} \\
D &= 2p^{(\frac{1}{2})}p^{(\frac{4}{2})} + 2p^{(\frac{2}{2})}p^{(\frac{3}{2})} \\
E &= 2p^{(\frac{1}{2})}p^{(\frac{5}{2})} + 2p^{(\frac{2}{2})}p^{(\frac{4}{2})} + p^{(\frac{3}{2})^2} \\
F &= 2p^{(\frac{1}{2})}p^{(\frac{6}{2})} + 2p^{(\frac{2}{2})}p^{(\frac{5}{2})} + 2p^{(\frac{3}{2})}p^{(\frac{4}{2})}
\end{aligned} \tag{A-23}$$

and

$$\begin{aligned}
A' &= p^{(\frac{1}{2})^3} \\
B' &= Ap^{(\frac{2}{2})} + Bp^{(\frac{1}{2})} \\
C' &= Ap^{(\frac{3}{2})} + Bp^{(\frac{2}{2})} + Cp^{(\frac{1}{2})} \\
D' &= Ap^{(\frac{4}{2})} + Bp^{(\frac{3}{2})} + Cp^{(\frac{2}{2})} + Dp^{(\frac{1}{2})} \\
E' &= Ap^{(\frac{5}{2})} + Bp^{(\frac{4}{2})} + Cp^{(\frac{3}{2})} + Dp^{(\frac{2}{2})} + Ep^{(\frac{1}{2})}
\end{aligned} \tag{A-24}$$

and

$$
\begin{aligned}
A'' &= A'p^{(\frac{1}{2})} \\
B'' &= A'p^{(\frac{2}{2})} + B'p^{(\frac{1}{2})} \\
C'' &= A'p^{(\frac{3}{2})} + B'p^{(\frac{2}{2})} + C'p^{(\frac{1}{2})} \\
D'' &= A'p^{(\frac{4}{2})} + B'p^{(\frac{3}{2})} + C'p^{(\frac{1}{2})} + C'p^{(\frac{1}{2})} \\
A''' &= A''p^{(\frac{1}{2})} \\
B''' &= A''p^{(\frac{2}{2})} + B''p^{(\frac{1}{2})} \\
C''' &= A''p^{(\frac{3}{2})} + B''p^{(\frac{2}{2})} + C''p^{(\frac{1}{2})} \\
A'''' &= A'''p^{(\frac{1}{2})}
\end{aligned}
\tag{A-25}
$$

With the aid of (A-20) an asymptotic expansion for $(\ln p)^2$ and $(\ln p)^3$ is

$$
\begin{aligned}
(\ln p)^2 &= a^2 + \varepsilon^{\frac{1}{2}} 2ab + \varepsilon^{\frac{2}{2}}(2ac + b^2) + \varepsilon^{\frac{3}{2}}(2ad + 2bc) + \mathcal{O}(\varepsilon^{\frac{4}{2}}) \\
(\ln p)^3 &= a^3 + \varepsilon^{\frac{1}{2}}(2a^2b + ba^2) + \mathcal{O}(\varepsilon^{\frac{2}{2}})
\end{aligned}
\tag{A-26}
$$

Using (A-20)-(A-26), then (A-19) becomes

$$
p^{\varepsilon\Gamma} = 1 + \varepsilon a' + \varepsilon^{\frac{3}{2}} b' + \varepsilon^{\frac{4}{2}} c' + ... + \varepsilon^{\frac{7}{2}} f' \tag{A-27}
$$

where

$$
\begin{aligned}
a' &= \Gamma a \\
b' &= \Gamma b \\
c' &= \Gamma c + \frac{\Gamma^2 a^2}{2} \\
d' &= \Gamma d + \Gamma^2 ab \\
e' &= \Gamma e + \frac{\Gamma^2}{2}(2ac + b^2) + \frac{\Gamma^3}{6} a^2 \\
f' &= \Gamma f + \frac{\Gamma^2}{2}(2ad + 2bc) + \frac{\Gamma^3}{6}(2a^2b + ba^2)
\end{aligned}
\tag{A-28}
$$

The expansion of the product $\rho\Theta$ on the left hand side of (A-19) can be written as

$$\begin{aligned}
\rho\Theta &= \sum_{j=0}^{\infty}(\rho^{(0)}\Theta^{(j)} + \rho^{(\frac{1}{2})}b^{(j-\frac{1}{2})} + a^{(j)}b^{(0)})\varepsilon^{j} \\
&= \underbrace{\rho^{(0)}\Theta^{(0)}}_{(\rho\Theta)^{(0)}} + \varepsilon^{\frac{1}{2}}\underbrace{(\rho^{(0)}\Theta^{(\frac{1}{2})} + \rho^{(\frac{1}{2})}\Theta^{(0)})}_{(\rho\Theta)^{(\frac{1}{2})}} + \\
&\quad \varepsilon^{\frac{2}{2}}\underbrace{(\rho^{(0)}\Theta^{(\frac{2}{2})} + \rho^{(\frac{1}{2})}\Theta^{(\frac{1}{2})} + \rho^{(\frac{2}{2})}\Theta^{(0)})}_{(\rho\Theta)^{(\frac{2}{2})}} + ... \qquad \text{(A-29)}
\end{aligned}$$

where $(j = 0, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, ...)$. Thus, with the aid of (A-29) and (A-27) an asymptotic expansion of the left hand side of (A-18) reads

$$\begin{aligned}
p^{\Gamma^{\star\star}\varepsilon}\rho\Theta = (\rho\Theta)^{(0)} + \varepsilon^{\frac{1}{2}}(\rho\Theta)^{(\frac{1}{2})} + \varepsilon^{\frac{2}{2}}[(\rho\Theta)^{(\frac{2}{2})} + a'(\rho\Theta)^{(0)}] + \\
\varepsilon^{\frac{3}{2}}[(\rho\Theta)^{(\frac{3}{2})} + a'(\rho\Theta)^{(\frac{1}{2})} + b'(\rho\Theta)^{(0)}] + \\
\varepsilon^{\frac{4}{2}}[(\rho\Theta)^{(\frac{4}{2})} + a'(\rho\Theta)^{(\frac{2}{2})} + b'(\rho\Theta)^{(\frac{1}{2})} + c'(\rho\Theta)^{(0)}] + \\
\varepsilon^{\frac{5}{2}}[(\rho\Theta)^{(\frac{5}{2})} + a'(\rho\Theta)^{(\frac{3}{2})} + b'(\rho\Theta)^{(\frac{2}{2})} + c'(\rho\Theta)^{(\frac{1}{2})} + d'(\rho\Theta)^{(0)}] + \\
\varepsilon^{\frac{6}{2}}[(\rho\Theta)^{(\frac{6}{2})} + a'(\rho\Theta)^{(\frac{4}{2})} + b'(\rho\Theta)^{(\frac{3}{2})} + c'(\rho\Theta)^{(\frac{2}{2})} + d'(\rho\Theta)^{(\frac{1}{2})} + e'(\rho\Theta)^{(0)}] + \\
\varepsilon^{\frac{7}{2}}\Big[(\rho\Theta)^{(\frac{7}{2})} + a'(\rho\Theta)^{(\frac{5}{2})} + b'(\rho\Theta)^{(\frac{4}{2})} + c'(\rho\Theta)^{(\frac{3}{2})} + d'(\rho\Theta)^{(\frac{2}{2})} + \\
e'(\rho\Theta)^{(\frac{1}{2})} + f'(\rho\Theta)^{(0)}\Big] + \mathcal{O}(\varepsilon^{\frac{8}{2}}) \qquad \text{(A-30)}
\end{aligned}$$

Taking additionally the matching results $\Theta^{(\frac{1}{2})} = 0$, $\Theta^{(\frac{2}{2})} = 0$ and $\Theta^{(\frac{3}{2})} = 0$ into account (see the matching condition (3.107)), collecting same powers of ε, yields

$\mathcal{O}(1)$:

$$\rho^{(0)}\Theta^{(0)} = p^{(0)} \qquad \text{(A-31)}$$

$\mathcal{O}(\varepsilon^{\frac{1}{2}})$:

$$\rho^{(\frac{1}{2})}\Theta^{(0)} = p^{(\frac{1}{2})} \qquad \text{(A-32)}$$

$\mathcal{O}(\varepsilon^{\frac{2}{2}})$:

$$\rho^{(\frac{2}{2})}\Theta^{(0)} + \Gamma\rho^{(0)}\Theta^{(0)}\ln p^{(0)} = p^{(\frac{2}{2})} \qquad \text{(A-33)}$$

$\mathcal{O}(\varepsilon^{\frac{3}{2}})$:

$$\rho^{(\frac{3}{2})}\Theta^{(0)} + \rho^{(\frac{1}{2})}\Gamma\ln p^{(0)} + \Gamma\frac{p^{(\frac{1}{2})}}{p^{(0)}} = p^{(\frac{3}{2})} \qquad \text{(A-34)}$$

$\mathcal{O}(\varepsilon^{\frac{4}{2}})$:

$$\rho^{(\frac{4}{2})}\Theta^{(0)} + \rho^{(0)}\Theta^{(\frac{4}{2})} + \rho^{(\frac{2}{2})}\Gamma\ln p^{(0)} + \rho^{(\frac{1}{2})}\Gamma\frac{p^{(\frac{1}{2})}}{p^{(0)}} + = p^{(\frac{4}{2})} \qquad \text{(A-35)}$$

$\mathcal{O}(\varepsilon^{\frac{5}{2}})$:

$$\rho^{(\frac{5}{2})}\Theta^{(0)} + \rho^{(\frac{1}{2})}\Theta^{(\frac{4}{2})} + \rho^{(\frac{3}{2})}\Theta^{(0)}\Gamma\ln p^{(0)} + \rho^{(\frac{2}{2})}\Theta_\infty\Gamma\frac{p^{(\frac{1}{2})}}{p^{(0)}} + \left(\Gamma\left[\frac{p^{(\frac{2}{2})}}{p^{(0)}} - \frac{p^{(\frac{1}{2})^2}}{2p^{(0)^2}}\right] + \frac{\Gamma^2(\ln p^{(0)^2})}{2}\right)\rho^{(\frac{1}{2})}\Theta^{(0)} + \left(\Gamma\left[\frac{p^{(\frac{3}{2})}}{p^{(0)}} - \frac{p^{(\frac{1}{2})}p^{(\frac{2}{2})}}{2p^{(0)^2}} + \frac{p^{(\frac{1}{2})^3}}{3!p^{(0)^3}}\right] + \Gamma^2\ln p^{(0)}\frac{p^{(\frac{1}{2})}}{p^{(0)}}\right)\rho^{(0)}\Theta^{(0)} = p^{(\frac{5}{2})} \quad \text{(A-36)}$$

$\mathcal{O}(\varepsilon^{\frac{6}{2}})$:

$$\rho^{(\frac{6}{2})}\Theta^{(0)} + \rho^{(\frac{1}{2})}\Theta^{(\frac{5}{2})} + \rho^{(\frac{2}{2})}\Theta^{(\frac{4}{2})} + \Gamma\ln p^{(0)}(\rho^{(0)}\Theta^{(\frac{4}{2})} + \rho^{(\frac{4}{2})}\Theta^{(0)}) + \rho^{(\frac{3}{2})}\Theta^{(0)}\Gamma\frac{p^{(\frac{1}{2})}}{p^{(0)}} + \left(\Gamma\left[\frac{p^{(\frac{2}{2})}}{p^{(0)}} - \frac{p^{(\frac{1}{2})^2}}{2p^{(0)^2}}\right] + \frac{\Gamma^2(\ln p^{(0)^2})}{2}\right)\rho^{(\frac{2}{2})}\Theta^{(0)} + \left(\Gamma\left[\frac{p^{(\frac{3}{2})}}{p^{(0)}} - \frac{p^{(\frac{1}{2})}p^{(\frac{2}{2})}}{2p^{(0)^2}} + \frac{p^{(\frac{1}{2})^3}}{3!p^{(0)^3}}\right] + \Gamma^2\ln p^{(0)}\frac{p^{(\frac{1}{2})}}{p^{(0)}}\right)\rho^{(\frac{1}{2})}\Theta^{(0)} + e'\rho^{(0)}\Theta^{(0)} = p^{(\frac{6}{2})} \quad \text{(A-37)}$$

where

$$e' = \Gamma\left(\frac{p^{(\frac{4}{2})}}{p^{(0)}} - \frac{2p^{(\frac{1}{2})}p^{(\frac{3}{2})} + p^{(\frac{2}{2})^2}}{2p^{(0)^2}} + \frac{3p^{(\frac{1}{2})^2}p^{(\frac{2}{2})}}{3!p^{(0)^3}} - \frac{p^{(\frac{1}{2})^4}}{4!p^{(0)^4}}\right) + \frac{\Gamma^2}{2}\left(2\ln p^{(0)}\left(\frac{p^{(\frac{2}{2})}}{p^{(0)}} - \frac{p^{(\frac{1}{2})^2}}{2p^{(0)^2}}\right) + \frac{p^{(\frac{1}{2})^2}}{p^{(0)^2}}\right) + \frac{\Gamma^3}{6}(\ln p^{(0)})^2 \quad \text{(A-38)}$$

$\mathcal{O}(\varepsilon^{\frac{7}{2}})$:

$$\rho^{(\frac{7}{2})}\Theta^{(0)} + \rho^{(\frac{3}{2})}\Theta^{(\frac{4}{2})} + \rho^{(\frac{2}{2})}\Theta^{(\frac{5}{2})} + \rho^{(\frac{1}{2})}\Theta^{(\frac{6}{2})} + \rho^{(0)}\Theta^{(\frac{7}{2})} + \Gamma\ln p^{(0)}(\rho^{(0)}\Theta^{(\frac{5}{2})} + \rho^{(\frac{1}{2})}\Theta^{(\frac{4}{2})} + \rho^{(\frac{5}{2})}\Theta^{(0)}) + (\rho^{(\frac{4}{2})}\Theta^{(0)} + \rho^{(0)}\Theta^{(\frac{4}{2})})\Gamma\frac{p^{(\frac{1}{2})}}{p^{(0)}} + \left(\Gamma\left[\frac{p^{(\frac{2}{2})}}{p^{(0)}} - \frac{p^{(\frac{1}{2})^2}}{2p^{(0)^2}}\right] + \frac{\Gamma^2(\ln p^{(0)^2})}{2}\right)\rho^{(\frac{3}{2})}\Theta^{(0)} + \left(\Gamma\left[\frac{p^{(\frac{3}{2})}}{p^{(0)}} - \frac{p^{(\frac{1}{2})}p^{(\frac{2}{2})}}{2p^{(0)^2}} + \frac{p^{(\frac{1}{2})^3}}{3!p^{(0)^3}}\right] + \Gamma^2\ln p^{(0)}\frac{p^{(\frac{1}{2})}}{p^{(0)}}\right)\rho^{(\frac{2}{2})}\Theta^{(0)} + e'\rho^{(\frac{1}{2})}\Theta^{(0)} + f'\rho^{(0)}\Theta_\infty = p^{(\frac{7}{2})} \quad \text{(A-39)}$$

where

$$f' = \Gamma\left(\frac{p^{(\frac{5}{2})}}{p^{(0)}} - \frac{2(p^{(\frac{1}{2})}p^{(\frac{4}{2})} + p^{(\frac{2}{2})}p^{(\frac{3}{2})})}{2p^{(0)^2}} + \frac{3(p^{(\frac{1}{2})^2}p^{(\frac{3}{2})} + p^{(\frac{1}{2})}p^{(\frac{2}{2})^2})}{3!p^{(0)^3}}\right) +$$
$$\Gamma\left(-\frac{4p^{(\frac{1}{2})^3}p^{(\frac{2}{2})}}{4!p^{(0)^4}} + \frac{p^{(\frac{1}{2})^5}}{5!p^{(0)^5}}\right) +$$
$$\Gamma^2\left(\ln p^{(0)}\left(\frac{p^{(\frac{3}{2})}}{p^{(0)}} - \frac{2p^{(\frac{1}{2})}p^{(\frac{2}{2})}}{2p^{(0)^2}} + \frac{p^{(\frac{1}{2})^3}}{3!p^{(0)^3}}\right)\right) +$$
$$\Gamma^2\left(\frac{p^{(\frac{1}{2})}}{p^{(0)}}\left(\frac{p^{(\frac{2}{2})}}{p^{(0)}} - \frac{p^{(\frac{1}{2})^2}}{2p^{(0)^2}}\right)\right) + \frac{\Gamma^3}{6}\left(3(\ln p^{(0)})^2\frac{p^{(\frac{1}{2})}}{p^{(0)}}\right) \quad \text{(A-40)}$$

A.4

To study the effect of the point source $\check{q}_s$ in the vicinity of $\vec{X}_C(z,\tau)$ the relative vector $\check{\vec{x}} = \vec{\eta} - \vec{X}_C(z,\tau)$ is introduced as a new coordinate with $\mid \check{\vec{x}} \mid = \check{r}$ representing a synoptic-scale radial distance from $\vec{\eta} = \vec{X}_C$. Note that $\vec{\eta} = (\eta_1, \eta_2)$, $\vec{X}_C(z,\tau) = (X_C(z,\tau)), Y_C(z,\tau))$ and $\check{\vec{x}} = (\check{x}, \check{y})$. We write $\check{\vec{x}} = (\check{x}, \check{y})$ in terms of cylindrical coordinates, i.e. $\check{x} = \check{r}\cos\theta$ and $\check{y} = \check{r}\sin\theta$, with

$$\begin{aligned} \check{r} &= (\eta_1 - X_C(z,\tau))\cos\theta + (\eta_2 - Y_C(z,\tau))\sin\theta \\ \theta &= \arctan\left(\frac{\eta_2 - Y_C(z,\tau)}{\eta_1 - X_C(z,\tau)}\right) \end{aligned} \quad \text{(A-41)}$$

Then, spatial derivatives in (3.79) take the form

$$\begin{aligned} \check{\vec{\nabla}}^2 &= \frac{1}{\check{r}}\frac{\partial}{\partial\check{r}}\left(\check{r}\,\frac{\partial}{\partial\check{r}}\right) + \frac{1}{\check{r}^2}\frac{\partial^2}{\partial\theta^2} \\ \left.\frac{\partial}{\partial z}\right|_{\eta_1,\eta_2} &= \left.\frac{\partial}{\partial z}\right|_{\check{r},\theta} - \tilde{\Lambda}_b\frac{\partial}{\partial\check{r}} - \tilde{\Lambda}_a\frac{1}{\check{r}}\frac{\partial}{\partial\theta} \end{aligned} \quad \text{(A-42)}$$

with

$$\begin{aligned} \tilde{\Lambda}_a &= -\frac{\partial X_C}{\partial z}\,\sin\theta + \frac{\partial Y_C}{\partial z}\,\cos\theta \\ \tilde{\Lambda}_b &= +\frac{\partial X_C}{\partial z}\,\cos\theta + \frac{\partial Y_C}{\partial z}\,\sin\theta \end{aligned} \quad \text{(A-43)}$$

Note, the derivation for the transformation $(A-42)_2$ is similar to the derivation of $(3.11)_2$. Furthermore, with the transformations made, equation (3.80) can be written as

$$\check{q}_s = \frac{\Gamma}{2\pi}\,\frac{1}{\check{r}}\;\delta(\check{r} - \check{r}_0)\;\delta(\theta - \theta_0) \quad \text{(A-44)}$$

Then, upon substitution of (A-42) and (A-44) into (3.79) singular solutions $\check{\psi}_s^{(0)}$ have to satisfy

$$
\begin{aligned}
\frac{1}{\check{r}}\ \delta(\check{r}-\check{r}_0)\ \delta(\theta-\theta_0) \ &= \ -\frac{1}{\check{r}}\frac{\partial}{\partial \check{r}}\left(\check{r}\ \frac{\partial \check{\psi}_s^{(0)}}{\partial \check{r}}\right) - \frac{1}{\check{r}^2}\frac{\partial^2 \check{\psi}_s^{(0)}}{\partial \theta^2} - \\
&\frac{\check{\rho}^{(0)}}{\check{\Theta}_z^{(2)}}\left(\frac{\partial^2 \check{\psi}_s^{(0)}}{\partial z^2} - \frac{\partial}{\partial z}\left[\tilde{\Lambda}_b \frac{\partial \check{\psi}_s^{(0)}}{\partial \check{r}} + \tilde{\Lambda}_a \frac{1}{\check{r}}\frac{\partial \check{\psi}_s^{(0)}}{\partial \theta}\right]\right) - \\
&\frac{\partial}{\partial z}\left(\frac{\check{\rho}^{(0)}}{\check{\Theta}_z^{(2)}}\right)\left(\frac{\partial \check{\psi}_s^{(0)}}{\partial z} - \tilde{\Lambda}_b \frac{\partial \check{\psi}_s^{(0)}}{\partial \check{r}} - \tilde{\Lambda}_a \frac{1}{\check{r}}\frac{\partial \check{\psi}_s^{(0)}}{\partial \theta}\right) \qquad \text{(A-45)}
\end{aligned}
$$

with $\check{\Theta}_z^{(2)} = \partial \check{\Theta}^{(2)}/\partial z$. A complete analysis of (A-45) is difficult. However, recalling the centreline expansion (3.20) we obtain approximate solutions for the singular vortex flow $\check{\psi}_s^{(0)}$ by assuming that $\vec{X}_C = \vec{X}_C^{(0)}(\tau)$. Then, equation (A-45) simplifies to

$$
\begin{aligned}
&\frac{1}{\check{r}}\ \delta(\check{r}-\check{r}_0)\ \delta(\theta-\theta_0) = \\
-\frac{1}{\check{r}}\frac{\partial}{\partial \check{r}}\left(\check{r}\ \frac{\partial \check{\psi}_s^{(0)}}{\partial \check{r}}\right) &- \frac{1}{\check{r}^2}\frac{\partial^2 \check{\psi}_s^{(0)}}{\partial \theta^2} - \frac{\Omega_0^2 \Theta_\infty}{\check{\rho}^{(0)}}\frac{\partial}{\partial z}\left(\frac{\check{\rho}^{(0)}}{\check{\Theta}_z^{(2)}}\frac{\partial \check{\psi}_s^{(0)}}{\partial z}\right) \qquad \text{(A-46)}
\end{aligned}
$$

A harmonic analysis (see **Subsection 3.1.4**) of (A-46) for the first Fourier modes of the singular streamfunction $\check{\psi}_s^{(0)}$, i.e. $\check{\psi}_{s,0}^{(0)}$ and $\check{\psi}_{s,1k}^{(0)}$ where $k = 1, 2$, together with $\check{\rho}^{(0)} = \check{\rho}^{(0)}(z)$ and $\check{\Theta}^{(2)} = \check{\Theta}^{(2)}(z)$ (see (3.72)), yields

$$
\begin{aligned}
\frac{1}{2\pi\check{r}}\delta(\check{r}-\check{r}_0) &= -\frac{1}{\check{r}}\frac{\partial}{\partial \check{r}}\left(\check{r}\frac{\partial \check{\psi}_{s,0}^{(0)}}{\partial \check{r}}\right) - \frac{\Omega_0^2 \Theta_\infty}{\check{\rho}^{(0)}}\frac{\partial}{\partial z}\left(\frac{\check{\rho}^{(0)}}{\check{\Theta}_z^{(2)}}\frac{\partial \check{\psi}_{s,0}^{(0)}}{\partial z}\right) \\
\frac{\nu_{1k}}{\pi\check{r}}\delta(\check{r}-\check{r}_0) &= -\frac{1}{\check{r}}\frac{\partial}{\partial \check{r}}\left(\check{r}\frac{\partial \check{\psi}_{s,1k}^{(0)}}{\partial \check{r}}\right) + \frac{\check{\psi}_{s,1k}^{(0)}}{\check{r}^2} - \frac{\Omega_0^2 \Theta_\infty}{\check{\rho}^{(0)}}\frac{\partial}{\partial z}\left(\frac{\check{\rho}^{(0)}}{\check{\Theta}_z^{(2)}}\frac{\partial \check{\psi}_{s,1k}^{(0)}}{\partial z}\right)
\end{aligned}
\qquad \text{(A-47)}
$$

with $\nu_{11} = \cos(\theta_0)$ and $\nu_{12} = \sin(\theta_0)$. Note that for the derivation of the left hand side of (A-47) the Fourier series expansions $\delta(\theta - \theta_0) = 1/(2\pi) + (1/\pi)\sum_{n=1}^{\infty}\cos(n(\theta-\theta_0))$ of the Dirac delta function $\delta(\theta-\theta_0)$ has been used.

Axissymmetric solutions For $\check{r} \neq \check{r}_0$ (where $\check{r}_0 = 0$), the left hand side in $(A-47)_1$ disappears and one obtains

$$
\frac{1}{\check{r}}\frac{\partial}{\partial \check{r}}\left(\check{r}\ \frac{\partial \check{\psi}_{s,0}^{(0)}}{\partial \check{r}}\right) + \frac{\Omega_0^2 \Theta_\infty}{\check{\rho}^{(0)}}\frac{\partial}{\partial z}\left(\frac{\check{\rho}^{(0)}}{\check{\Theta}_z^{(2)}}\frac{\partial \check{\psi}_{s,0}^{(0)}}{\partial z}\right) = 0 \qquad \text{(A-48)}
$$

Assuming the singular component $\hat{\psi}_{s,0}$ is separable, i.e.

$$\check{\psi}^{(0)}_{s,0} = R_{s,0}(\check{r})\ Z_{s,0}(z)\ , \tag{A-49}$$

then the following two equations are obtained

$$\begin{aligned} \frac{1}{R_{s,0}}\frac{1}{\check{r}}\frac{d}{d\check{r}}\left(\check{r}\frac{dR_{s,0}}{d\check{r}}\right) &= +\lambda_0^2 \\ \frac{\Omega_0^2\Theta_\infty}{\check{\Theta}_z^{(2)}}\frac{1}{Z_{s,0}}\frac{d^2Z_{s,0}}{dz^2} + \frac{\Omega_0^2\Theta_\infty}{\check{\rho}^{(0)}}\frac{1}{Z_{s,0}}\frac{d}{dz}\left(\frac{\check{\rho}^{(0)}}{\check{\Theta}_z^{(2)}}\right)\frac{dZ_{s,0}}{dz} &= -\lambda_0^2 \end{aligned} \tag{A-50}$$

where λ_0^2 is the separation constant. Equation $(A-50)_1$ is known as a modified Bessel's differential equation[1], of order $m = 0$, i.e.

$$\frac{d^2R_{s,0}}{d\check{r}^2} + \frac{1}{\check{r}}\frac{dR_{s,0}}{d\check{r}} - \lambda_0^2 R_{s,0} = 0 \qquad . \tag{A-51}$$

and solutions are the zeroth order modified Bessel functions of the first kind $K_0(\lambda_0^2\ \check{r})$ and of the second kind $I_0(\lambda_0^2\ \check{r})$ of zeroth order, i.e.

$$R_{s,0}(\check{r}) = c_1\ K_0(\lambda_0^2\check{r}) + c_2\ I_0(\lambda_0^2\check{r}) \qquad . \tag{A-52}$$

For bounded solution one has to set $c_2 = 0$, therefore

$$R_{s,0}(\check{r}) = c_1 K_0(\lambda_0^2\check{r}) \tag{A-53}$$

A solution for $(A-50)_2$ is obtained by assuming a constant background stratification, i.e. $\check{\Theta}_z^{(2)} = \text{const}$. Then $(A-50)_2$ becomes

$$\frac{d^2Z_{s,0}}{dz^2} + \frac{1}{\check{\rho}^{(0)}}\frac{d\check{\rho}^{(0)}}{dz}\frac{dZ_{s,0}}{dz} + \left(\frac{\lambda_0}{\alpha}\right)^2 Z_{s,0} = 0 \tag{A-54}$$

where $\alpha^2 = (\Omega_0^2\Theta_\infty)/\Theta_z^{(2)}$. From the hydrostatic relation (3.70) and the gas law (2.21) (where the potential temperature Θ has the expansion (3.72)) it is known that

$$\frac{d\check{p}^{(0)}}{dz} = -\check{\rho}^{(0)} \quad \text{and} \quad \check{\rho}^{(0)}\Theta_\infty = \check{p}^{(0)} \tag{A-55}$$

which implies $(1/\check{\rho}^{(0)})(\partial\check{\rho}^{(0)}/\partial z) = -1/\Theta_\infty$. Hence, (A-54) can be written as a second order ODE with constant coefficients, i.e.

$$\frac{d^2Z_{s,0}}{dz^2} - \frac{1}{\Theta_\infty}\frac{dZ_{s,0}}{dz} + \left(\frac{\lambda_0}{\alpha}\right)^2 Z_{s,0} = 0 \tag{A-56}$$

[1]The modified Bessel's differential equation, of order m, in general is $w^2d^2f/dw^2 + zdf/dw + (-w^2 - m^2)f = 0$, where different values of m denote different differential equations. Refer to Habermann (1983) for further details.

The corresponding characteristic equation is

$$s^2 - \frac{1}{\Theta_\infty} s + \left(\frac{\lambda_0}{\alpha}\right)^2 = 0 \tag{A-57}$$

whose solutions are given through

$$s_{1,2} = 1/(2\Theta_\infty) \pm \sqrt{D} \qquad \text{with} \qquad D = \frac{1}{4\Theta_\infty^2} - \left(\frac{\lambda_0}{\alpha}\right)^2 \tag{A-58}$$

There are three cases to consider:

A: For non-complex roots (i.e. $D > 0$) general solutions of (A-56) read

$$Z_{s,0}(z) = \exp\left(\frac{1}{2\Theta_\infty}\right)\left[c_3 \exp\left(-D\ z\right) + c_4 \exp\left(+D\ z\right)\right] \tag{A-59}$$

Note, here the separation constant λ has to satisfy

$$\lambda_0^2 < \frac{\alpha^2}{4\Theta_\infty^2} = \frac{\Omega_0^2}{4\Theta_\infty \check{\Theta}_z^{(2)}} \tag{A-60}$$

B: For complex roots (i.e. $D < 0$) of the characteristic equation (A-57) real-valued solutions are

$$Z_{s,0}(z) = \exp\left(\frac{1}{2\Theta_\infty} z\right)\left[c_3 \cos\left(\sqrt{\mid D \mid\ z}\right) + c_4 \sin\left(\sqrt{\mid D \mid\ z}\right)\right] \tag{A-61}$$

C: If $D = 0$, then we have

$$Z_{s,0}(z) = c_5 \exp\left(\frac{1}{2\Theta_\infty} z\right) \tag{A-62}$$

Thus, upon substitution of (A-53) into the separation Ansatz (A-69) and having (A-59) - (A-62) in mind, singular solutions $\check{\psi}_{s,0}^{(0)}$ take the form

$$\check{\psi}_{s,0} = c_1\ K_0(\lambda_0^2 \check{r})\ Z_{s,0}(z) \tag{A-63}$$

We determine the constant c_1 as follows. In polar coordinates the circumferential velocity is defined by

$$\check{u}_{\theta,0} = -\frac{\partial \check{\psi}_{s,0}}{\partial \check{r}} = -c_1\ \lambda_0\ K_1(\lambda_0^2 \check{r})\ Z_{s,0}(z) \tag{A-64}$$

where $K_1 = \partial K_0 / \partial \check{r}$ is the modified Bessel function of the first kind, first order.

The circulation for small ($\check{r} \to 0$) and concentrated vortices is

$$\Gamma = \lim_{\check{r}\to 0} \left(\int_0^{2\pi} \check{u}_{\theta,0}\ \check{r}\ d\theta \right) = -\lim_{\check{r}\to 0} \left(c_1 \lambda_0\ K_1(\lambda_0^2 \check{r})\ Z_{s,0}(z)\ \check{r} \int_0^{2\pi}\ d\theta \right) \quad \text{(A-65)}$$

Note, as $\check{r} \to 0$ one can use the approximations $K_1(\lambda_0^2 \check{r}) \sim (\lambda_0^2 \check{r})^{-1}$ and $K_0(\lambda_0^2 \check{r}) \sim \ln(\lambda_0^2 \check{r})$. It follows

$$\Gamma(z) = -c_1\ 2\pi\ Z_{s,0}(z) \quad \text{or} \quad c_1 = -\frac{\Gamma(z)}{2\pi} \frac{1}{Z_{s,0}(z)} \quad \text{as} \quad \check{r} \to 0 \qquad \text{(A-66)}$$

and the streamfunction for a concentrated single vortex of strength Γ at the origin is

$$\check{\psi}_{s,0}(\check{r}, z) = -\frac{\Gamma(z)}{2\pi}\ \ln(\lambda_0^2 \check{r}) \qquad \text{as} \qquad \check{r} \to 0 \qquad \text{(A-67)}$$

Note that Γ can, in principle, depend on the temporal coordinate τ.

Asymmetric solutions In a similar way as shown for $(A-47)_1$, solutions $\check{\psi}^{(0)}_{s,1k}$ satisfying $(A-47)_2$ for $\check{r} \neq \check{r}_0$ (where $\check{r}_0 = 0$) is derived below. Thus, the following equation has to be solved

$$\frac{1}{\check{r}} \frac{\partial}{\partial \check{r}} \left(\check{r}\ \frac{\partial \check{\psi}^{(0)}_{s,1k}}{\partial \check{r}} \right) - \frac{\check{\psi}^{(0)}_{s,1k}}{\check{r}^2} + \frac{\Omega_0^2 \Theta_\infty}{\check{\rho}^{(0)}} \frac{\partial}{\partial z} \left(\frac{\check{\rho}^{(0)}}{\check{\Theta}^{(2)}_z} \frac{\partial \check{\psi}^{(0)}_{s,1k}}{\partial z} \right) = 0 \qquad \text{(A-68)}$$

Assuming $\check{\psi}^{(0)}_{s,1k}$ is separable, i.e.

$$\check{\psi}^{(0)}_{s,1k} = R_{s,1k}(\check{r})\ Z_{s,1k}(z)\ , \qquad \text{(A-69)}$$

from (A-68) one obtains under the assumption $\check{\Theta}^{(2)}_z = \text{const}$ and together with (A-55), that

$$\begin{aligned} \frac{d^2 R_{s,1k}}{d\check{r}^2} + \frac{1}{\check{r}} \frac{dR_{s,1k}}{d\check{r}} - R_{s,1k} \left(\frac{1}{\check{r}^2} + \lambda_{1k}^2 \right) &= 0 \\ \frac{d^2 Z_{s,1k}}{dz^2} - \frac{1}{\Theta_\infty} \frac{dZ_{s,1k}}{dz} + \left(\frac{\lambda_{1k}}{\alpha} \right)^2 Z_{s,1k} &= 0 \end{aligned} \qquad \text{(A-70)}$$

where the variable λ_{1k} denotes a constant of separation. Note that unlike $(A-50)_1$, equation $(A-70)$ denotes Bessel's differential equation of order $m = 1$ (with argument $\lambda_{1k}^2 \check{r}$), which has the general solution

$$R_{s,1k}(\check{r}) = c_1 K_1(\lambda_{1k}^2 \check{r}) + c_2 I_1(\lambda_{1k}^2 \check{r}) \qquad \text{(A-71)}$$

where K_1 and I_1 denote modified Bessel functions of the first and second kind, and of order $m = 1$. For bounded solutions one has to set $c_2 = 0$. It can be shown, that the behaviour of K_1 in the neighbourhood of $\check{r}_0 = 0$ is given by $K_1 = 1/4\ \check{r}^{-1}$ (see Habermann (1983)). Thus, one obtains for (A-71) in the limit $\check{r} \to \check{r}_0$

$$R_{s,1k} = \frac{c_1}{4}\frac{1}{\check{r}} \qquad \text{as} \qquad \check{r} \to 0 \tag{A-72}$$

The type of solutions for Z_{1k} satisfying $(A-70)$ can be derived in a similar manner as shown for (A-62) - (A-59). Hence, solutions for the first asymmetric modes $\check{\psi}^{(0)}_{s,1k}$ read

$$\check{\psi}^{(0)}_{s,1k} = \frac{c_1}{4}\ \frac{1}{\check{r}}\ Z_{s,1k}(z) \qquad \text{as} \qquad \check{r} \to 0 \tag{A-73}$$

Note that $Z_{s,1k}$ can, in principle, depend on the temporal coordinate τ.

A.5

Substituting $\psi' = \check{\psi}^{(0)}_r + \check{\psi}^{(0)}_s$ for $\check{\psi}^{(0)}_g$ into (3.74) yields

$$\begin{aligned} \frac{\partial \check{q}_r}{\partial \tau} + \frac{\partial \check{q}_s}{\partial \tau} - \beta\frac{\partial \check{\psi}^{(0)}_r}{\partial \eta_1} - \beta\frac{\partial \check{\psi}^{(0)}_s}{\partial \eta_1} - J(\check{\psi}^{(0)}_r, \check{q}_r) - J(\check{\psi}^{(0)}_r, \check{q}_s) - \\ J(\check{\psi}^{(0)}_s, \check{q}_r) - J(\check{\psi}^{(0)}_s, \check{q}_s) \ = \ 0 \end{aligned} \tag{A-74}$$

Taking (3.80) into account and using the notation $\partial\delta/\partial\kappa = \delta'_\kappa$ with $\kappa \in \{\tau, \eta_1, \eta_2\}$ one can write

$$\begin{aligned} \frac{\partial \check{q}_s}{\partial \tau} &= \delta'_\tau(\eta_1 - X^{(0)}_C(\tau))\ \delta(\eta_2 - Y^{(0)}_C(\tau))\ + \\ & \quad \delta(\eta_1 - X^{(0)}_C(\tau))\ \delta'_\tau(\eta_2 - Y^{(0)}_C(\tau)) \end{aligned} \tag{A-75}$$

$$\begin{aligned} J(\check{\psi}^{(0)}_r, \check{q}_s) &= \frac{\partial \check{\psi}^{(0)}_r}{\partial \eta_1}\frac{\partial \check{q}_s}{\partial \eta_2} - \frac{\partial \check{\psi}^{(0)}_r}{\partial \eta_2}\frac{\partial \check{q}_s}{\partial \eta_1} \\ &= \frac{\partial \check{\psi}^{(0)}_r}{\partial \eta_1}\ \delta(\eta_1 - X^{(0)}_C(\tau))\ \delta'_{\eta_2}(\eta_2 - Y^{(0)}_C(\tau)) - \\ & \quad \frac{\partial \check{\psi}^{(0)}_r}{\partial \eta_2}\ \delta'_{\eta_1}(\eta_1 - X^{(0)}_C(\tau))\ \delta(\eta_2 - Y^{(0)}_C(\tau)) \end{aligned} \tag{A-76}$$

Then, in (A-74) equating to zero the regular part and the parts proportional to $\delta'_\kappa(\eta_1 - X_C^{(0)}(\tau))\ \delta(\eta_2 - Y_C^{(0)}(\tau))$ and $\delta(\eta_1 - X_C^{(0)}(\tau))\ \delta'_\kappa(\eta_2 - Y_C^{(0)}(\tau))$, yields

$$\begin{aligned} \delta'_\tau(\eta_1 - X_C^{(0)}(\tau)) + \frac{\partial \check{\psi}_r^{(0)}}{\partial \eta_2}\ \delta'_{\eta_1}(\eta_1 - X_C^{(0)}(\tau)) &= 0 \\ \delta'_\tau(\eta_2 - Y_C^{(0)}(\tau)) - \frac{\partial \check{\psi}_r^{(0)}}{\partial \eta_1}\ \delta'_{\eta_2}(\eta_2 - Y_C^{(0)}(\tau)) &= 0 \end{aligned} \tag{A-77}$$

and

$$\frac{\partial \check{q}_r}{\partial \tau} - \beta \frac{\partial \check{\psi}_r^{(0)}}{\partial \eta_1} - J(\check{\psi}_r^{(0)}, \check{q}_r) + J((\check{q}_r + \beta\eta_2), \check{\psi}_s^{(0)}) = 0 \tag{A-78}$$

Note that the derivative of the delta function is in general $\delta'(x-a) = -(x-a)^{-1}\ \delta(x-a)$. Hence together with the chain rule the equations (A-77) can be written as

$$\frac{\partial X_C^{(0)}}{\partial \tau} = -\frac{\partial \check{\psi}_r^{(0)}}{\partial \eta_2}\ , \qquad \frac{\partial Y_C^{(0)}}{\partial \tau} = +\frac{\partial \check{\psi}_r^{(0)}}{\partial \eta_1} \tag{A-79}$$

A.6

It follows the derivation of an asymptotic expansion for q_{vs} given by

$$q_{vs}(\Theta, p) = \frac{1}{p} \exp\left(\frac{A^{\star\star}}{\varepsilon} \frac{T(\theta,p) - 1}{1 + (T(\theta,p) - 1 - \varepsilon T_1^{\star\star(1)})} \right) \tag{A-80}$$

whereas $T(\theta, p)$ is of the form

$$T(\theta, p) = \theta\ p^{\frac{\gamma-1}{\gamma}} = \theta\ p^{\varepsilon\Gamma} \tag{A-81}$$

First, to obtain an asymptotic approximation for $T(\theta, p)$ Taylor's theorem is used to find an approximation of $f(\varepsilon\Gamma) = p^{\varepsilon\Gamma}$ by expanding about $\varepsilon_0 = 0$. With $f'(\varepsilon\Gamma) = p^{\varepsilon\Gamma} \ln p$ and $f''(\varepsilon\Gamma) = p^{\varepsilon\Gamma}(\ln p)^2$ one obtains

$$p^{\varepsilon\Gamma} = 1 + \varepsilon\Gamma \ln p + \frac{(\varepsilon\Gamma)^2}{2!}(\ln p)^2 + \mathcal{O}(\varepsilon^3) \tag{A-82}$$

Together with the asymptotic expansion ansatz (3.18) for Θ an asymptotic approximation for $T(\theta, p)$ takes the form

$$T(\theta, p) = 1 + \varepsilon\Gamma \ln p + \varepsilon^2 \left(\frac{\Gamma^2}{2!}(\ln p)^2 + \Theta^{(2)} \right) + \mathcal{O}(\varepsilon^3) \tag{A-83}$$

Substitution of the above expansion into the equation (3.123) for the saturation mixing ratio q_{vs}, yields

$$q_{vs}(\theta,p) = \frac{1}{p}\exp\left(\frac{A^{\star\star}}{\varepsilon}\frac{(\varepsilon\Gamma\ln p + \varepsilon^2\mathcal{A}(\varepsilon) + \mathcal{O}(\varepsilon^3))}{1+\varepsilon(\Gamma\ln p - T_1^{\star\star^{(1)}}) + \varepsilon^2\mathcal{A}(\varepsilon) + \mathcal{O}(\varepsilon^3)}\right) \tag{A-84}$$

with $\mathcal{A}(\varepsilon) = \frac{\Gamma^2}{2}(\ln p)^2 + \Theta^{(2)}$. Next Taylor's theorem is used to find an asymptotic expansion of $f(\varepsilon) = 1/(1+\varepsilon(\Gamma\ln p - T_1^{\star\star^{(1)}}) + \varepsilon^2\mathcal{A}(\varepsilon) + \mathcal{O}(\varepsilon^3))$. Using the notation $f(\varepsilon) = 1/(1+\delta)$, whereas $\delta = \varepsilon(\Gamma\ln p - T_1^{\star\star^{(1)}}) + \varepsilon^2\mathcal{A}(\varepsilon) + \mathcal{O}(\varepsilon^3)$ an expansion around $\delta = 0$ (i.e. $\varepsilon = \varepsilon_0 = 0$) yields

$$\begin{aligned}1/(1+\delta) &= 1 + \delta + \delta^2/2 + ...\\ &= 1 - \varepsilon(\Gamma\ln p + T_1^{\star\star^{(1)}}) + \varepsilon^2\left(\mathcal{A}(\varepsilon) + (\Gamma\ln p - T_1^{\star\star^{(1)}})^2\right) + \mathcal{O}(\varepsilon^3)\end{aligned}$$

Thus the fraction appearing in the exponential in (A-84) can be written as

$$\frac{\varepsilon\Gamma\ln p + \varepsilon^2\mathcal{A}(\varepsilon) + \mathcal{O}(\varepsilon^3)}{1+\delta} = \varepsilon\Gamma\ln p - \varepsilon^2\left(\Gamma\ln p\ (\Gamma\ln p + T_1^{\star\star^{(1)}}) - \mathcal{A}(\varepsilon)\right) + \mathcal{O}(\varepsilon^3)$$

Based on the above simplification, equation (A-84) can be rewritten as

$$q_{vs}(\theta,p) = \frac{1}{p}\exp\left(A^{\star\star}(\Gamma\ln p - \varepsilon\{\Gamma\ln p\ (\Gamma\ln p + T_1^{\star\star^{(1)}}) - \mathcal{A}(\varepsilon)\} + \mathcal{O}(\varepsilon^2))\right) \tag{A-85}$$

Next, Taylor expansion is used to find an asymptotic series of the exponential function $\exp(A^{\star\star}\Gamma\ln p - \varepsilon\tilde{\mu} + \mathcal{O}(\varepsilon^2))$, with $\tilde{\mu}(\varepsilon) = A^{\star\star}\{\Gamma\ln p\ (\Gamma\ln p + T_1^{\star\star^{(1)}}) - \mathcal{A}(\varepsilon)\}$. In doing so one obtains

$$\exp(A^{\star\star}\Gamma\ln p - \varepsilon\tilde{\mu}(\varepsilon) + \mathcal{O}(\varepsilon^2)) = \exp(A^{\star\star}\Gamma\ln p) - \varepsilon\tilde{\mu}(\varepsilon)\exp(A^{\star\star}\Gamma\ln p) + \mathcal{O}(\varepsilon^2)$$

and (A-85) becomes

$$q_{vs}(\theta,p) = \frac{1}{p}\left(\exp(A^{\star\star}\Gamma\ln p) - \varepsilon\tilde{\mu}(\varepsilon)\exp(A^{\star\star}\Gamma\ln p) + \mathcal{O}(\varepsilon^2)\right) \tag{A-86}$$

Finally, one has to account for the asymptotic expansion of the pressure variable p. As shown earlier, an expansion for p takes the form $p = p^{(0)}(z) + \varepsilon p^{(1)}(z) + \varepsilon^2 p^{(2)}(z) + \varepsilon^3 p^{(3)}(r,z,\tau) + \mathcal{O}(\varepsilon^{\frac{7}{2}})$. Thus, with the aid of Taylor's theorem it can be shown, that asymptotic approximations of $1/p$ and $\ln p$ read

$$\frac{1}{p} = \frac{1}{p^{(0)}} - \varepsilon\frac{p^{(1)}}{p^{(0)^2}} + \varepsilon^2\left(\frac{p^{(2)}}{p^{(0)^2}} - \frac{1}{4}\frac{p^{(1)^2}}{p^{(0)^3}}\right) + \mathcal{O}(\varepsilon^3) \tag{A-87}$$

and

$$\ln p = \ln p^{(0)} + \varepsilon \frac{p^{(1)}}{p^{(0)}} + \varepsilon^2 \left(\frac{p^{(2)}}{p^{(0)}} - \frac{p^{(1)}}{2p^{(0)^2}} \right) + \mathcal{O}(\varepsilon^3) \tag{A-88}$$

Thus, for the exponential function $\exp(A^{\star\star}\Gamma \ln p)$ in (A-86) can be written

$$\exp(A^{\star\star}\Gamma \ln p) = \exp\left(A^{\star\star}\Gamma \left[\ln p^{(0)} + \varepsilon \frac{p^{(1)}}{p^{(0)}} + \varepsilon^2 \left(\frac{p^{(2)}}{p^{(0)}} - \frac{p^{(1)}}{2p^{(0)^2}} \right) + \mathcal{O}(\varepsilon^3) \right] \right)$$

In general holds that $\exp(a + \varepsilon b + \varepsilon^2 c) = \exp(a)\ \exp(\varepsilon b)\ \exp(\varepsilon^2 c)$. Thus, using Taylor series of $\exp(\varepsilon b)$ around $\varepsilon_0 = 0$, i.e. $\exp(\varepsilon b) = 1 + \varepsilon b + \varepsilon^2 b^2/2 + ...$, the above expression can be simplified to obtain

$$\begin{aligned} \exp(A^{\star\star}\Gamma \ln p) &= \exp(A^{\star\star}\Gamma \ln p^{(0)}) + \\ &\quad \varepsilon A^{\star\star}\Gamma \frac{p^{(1)}}{p^{(0)}} \exp(A^{\star\star}\Gamma \ln p^{(0)}) + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{A-89}$$

Upon substitution of (A-89) and (A-87) into (A-86) one obtains

$$q_{vs}(\theta, p) = q_{vs}^{(0)} + \varepsilon q_{vs}^{(1)} + \mathcal{O}(\varepsilon^2) \tag{A-90}$$

where

$$\begin{aligned} q_{vs}^{(0)} &= \frac{1}{p^{(0)}} \exp(A^{\star\star}\Gamma^{\star\star} \ln p^{(0)}) \\ q_{vs}^{(1)} &= -\left((1 + A^{\star\star}\Gamma^{\star\star}) \frac{p^{(1)}}{p^{(0)^2}} + \frac{\tilde{\mu}^{(0)}}{p^{(0)}} \right) \exp(A^{\star\star}\Gamma^{\star\star} \ln p^{(0)}) \end{aligned} \tag{A-91}$$

with $\tilde{\mu}^{(0)} = A^{\star\star}\{\Gamma^{\star\star} \ln p^{(0)}\ (\Gamma \ln p^{(0)} + T_1^{\star\star^{(1)}}) - \frac{\Gamma^{\star\star^2}}{2}(\ln p^{(0)})^2 - \Theta^{(2)}\}$.

Further simplifications can be obtained for (A-91). From the $\mathcal{O}(1)$ hydrostatic relation (3.34) and gas law (A-31) it is known that

$$\frac{dp^{(0)}}{dz} = -\rho^{(0)} \quad \text{and} \quad \rho^{(0)}\Theta_\infty = p^{(0)} \tag{A-92}$$

where $\Theta_\infty = 1$. Substitution, rearranging terms and integation gives

$$\frac{dp^{(0)}}{p^{(0)}} = -dz \qquad \Longrightarrow \qquad p^{(0)} = p_0 \exp(-z) \tag{A-93}$$

whereas $p_0 = p^{(0)}(z = 0)$. Note, from $(A - 92)_2$ it follows immediately that

$$\rho^{(0)} = \rho_0 \exp(-z) \tag{A-94}$$

with $\rho_0 = \rho^{(0)}(z = 0)$. The $\mathcal{O}(\varepsilon)$ hydrostatic relation (3.34) and gas law (A-33) read

$$\frac{dp^{(1)}}{dz} = -\rho^{(1)} \quad \text{and} \quad \Gamma^{\star\star}\rho^{(0)}\ln p^{(0)}\Theta_\infty + \Theta_\infty\rho^{(1)} = p^{(1)} \tag{A-95}$$

with $\Theta^{(0)} = \Theta_\infty$. Elimination of $\rho^{(1)}$ from the above equations gives

$$\frac{dp^{(1)}}{dz} + p^{(1)} = \Gamma^{\star\star}\ln p^{(0)} \tag{A-96}$$

It can be easily checked that a solution of (A-96) reads

$$p^{(1)} = \Gamma^{\star\star}\left(-\frac{1}{2}z^2\right)p^{(0)} \tag{A-97}$$

Substitution of (A-97) into $(A-95)_2$ gives

$$\rho^{(1)} = \Gamma^{\star\star}\left(-\frac{1}{2}z^2 + z\right)\rho^{(0)} \tag{A-98}$$

Based on the fact that the variables $p^{(0)}, p^{(1)}, \Theta^{(2)}$ are horizontally homogeneous (see **Section 3.3.1**), the asymptotic expansion (A-90) for q_{vs} can be specified in the following way

$$q_{vs} = q_{vs}^{(0)}(z) + \varepsilon q_{vs}^{(1)}(z) + \mathcal{O}(\varepsilon^2) \tag{A-99}$$

whereas solutions (A-91) for the leading and next higher order saturation water vapor mixing ratio take together with (A-93) and (A-97) the following final form

$$\begin{aligned} q_{vs}^{(0)} &= \exp\left(-[A^{\star\star}\Gamma^{\star\star} - 1]z\right) \\ q_{vs}^{(1)} &= q_{vs}^{(0)}\left[\left(A^{\star\star}\frac{\partial\Theta^{(\frac{4}{2})}}{\partial z} - \frac{1}{2}A^{\star\star}{\Gamma^{\star\star}}^2 z^2\right) + \right. \\ &\qquad \left. \exp\left(-z\right)(A^{\star\star}\Gamma^{\star\star} - 1)\ p^{(1)}(z)\right] \end{aligned} \tag{A-100}$$

Appendix B

Auxiliary calculations for Chapter 4

B.1

A general approach is given to solve ODE's of type

$$-u_\theta^{(0)}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\left[\frac{\zeta_r^{(0)}}{u_\theta^{(0)}}+\frac{1}{r^2}\right]\right)\overline{\psi_{1k}^{(\frac{i}{2})}}=\mathcal{D}_{1k}\ , \tag{B-1}$$

subject to the homogeneous boundary conditions

$$\overline{\psi_{1k}^{(\frac{i}{2})}}=0, \qquad \frac{\partial\overline{\psi_{1k}^{(\frac{i}{2})}}}{\partial r}=0 \qquad \text{at} \quad r=0 \tag{B-2}$$

with $k=1,2$ and $i=1,2,...$ and $\mathcal{D}_{1k}$ denoting an arbitrary inhomogeneity of (B-1). Let

$$\begin{aligned} u_\theta^{(0)} &= -\frac{\partial\psi^{(0)}}{\partial r} = -\psi_r^{(0)} \\ \zeta_r^{(0)} &= -\nabla_1^2\psi_r^{(0)} \end{aligned} \tag{B-3}$$

where the operator ∇_1^2 is given through

$$\nabla_1^2 = \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right) \tag{B-4}$$

Then, after some manipulations the left hand side of (B-1) can be written as

$$-u_\theta^{(0)}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\left[\frac{\zeta_r^{(0)}}{u_\theta^{(0)}}+\frac{1}{r^2}\right]\right)\overline{\psi_{1k}^{(\frac{i}{2})}}=(\psi_r^{(0)}\nabla_1^2\overline{\psi_{1k}^{(\frac{i}{2})}}-\overline{\psi_{1k}^{(\frac{i}{2})}}\nabla_1^2\psi_r^{(0)}) \tag{B-5}$$

Then, the identity

$$\psi_r^{(0)} \nabla_1^2 \overline{\psi_{1k}^{(\frac{i}{2})}} - \overline{\psi_{1k}^{(\frac{i}{2})}} \nabla_1^2 \psi_r^{(0)} = \frac{1}{r} \left[r \left(\psi_r^{(0)} \left(\overline{\psi_{1k}^{(\frac{i}{2})}} \right)_r - \overline{\psi_{1k}^{(\frac{i}{2})}} \psi_{rr}^{(0)} \right) \right]_r \tag{B-6}$$

can be used to solve (B-1) via integration. Note, the index r denotes a partial derivative with respect to r. Using the above manipulations, (B-1) can be rewritten into

$$\left[r \left(\psi_r^{(0)} \left(\overline{\psi_{1k}^{(\frac{i}{2})}} \right)_r - \overline{\psi_{1k}^{(\frac{i}{2})}} \psi_{rr}^{(0)} \right) \right]_r = r \; \mathcal{D}_{1k} \tag{B-7}$$

Integration over r from 0 to $\bar{r}$ and adjacent division through by $\bar{r}$ gives

$$\left(\psi_r^{(0)} \left(\overline{\psi_{1k}^{(\frac{i}{2})}} \right)_r - \overline{\psi_{1k}^{(\frac{i}{2})}} \psi_{rr}^{(0)} \right) = \frac{1}{\bar{r}} \int_0^{\bar{r}} r \; \mathcal{D}_{1k} \; dr \tag{B-8}$$

With $\psi_r^{(0)} = -u_\theta^{(0)}$ one can also write

$$-u_\theta^{(0)} \left(\overline{\psi_{1k}^{(\frac{i}{2})}} \right)_r + \overline{\psi_{1k}^{(\frac{i}{2})}} \left(u_\theta^{(0)} \right)_r = \frac{1}{\bar{r}} \int_0^{\bar{r}} r \; \mathcal{D}_{1k} \; dr \tag{B-9}$$

Multiplying through both sides of the equation with $-1/u_\theta^{(0)^2}$ one obtains

$$\frac{1}{u_\theta^{(0)}} \left(\overline{\psi_{1k}^{(\frac{i}{2})}} \right)_r - \frac{\overline{\psi_{1k}^{(\frac{i}{2})}}}{u_\theta^{(0)^2}} \left(u_\theta^{(0)} \right)_r = \left(\frac{\overline{\psi_{1k}^{(\frac{i}{2})}}}{u_\theta^{(0)}} \right)_r = -\frac{1}{\bar{r} u_\theta^{(0)^2}} \int_0^{\bar{r}} r \; \mathcal{D}_{1k} \; dr \tag{B-10}$$

Integration yields

$$\left[\frac{\overline{\psi_{1k}^{(\frac{i}{2})}}}{u_\theta^{(0)}} \right]_0^{\bar{\bar{r}}} = -\int_0^{\bar{\bar{r}}} \frac{1}{\bar{r} u_\theta^{(0)^2}} \left[\int_0^{\bar{r}} r \; \mathcal{D}_{1k} \; dr \right] d\bar{r} \tag{B-11}$$

From (4.20) we know that $u_\theta^{(0)} = 0$ at $r = 0$. Then, using L'Hospital's rule one finds

$$\lim_{r \to 0} \frac{u_\theta^{(0)}}{r} = \lim_{r \to 0} \frac{\partial u_\theta^{(0)}}{\partial r} \tag{B-12}$$

such that one can write for $\zeta^{(0)}$ at $r = 0$

$$\lim_{r \to 0} \zeta^{(0)} = \zeta_*^{(0)} = \lim_{r \to 0} \left(\frac{\partial u_\theta^{(0)}}{\partial r} + \frac{u_\theta^{(0)}}{r} \right) = \lim_{r \to 0} \frac{2 u_\theta^{(0)}}{r} \tag{B-13}$$

Using the boundary conditions (B-2) one can write

$$\lim_{r\to 0}\left(\frac{\overline{\psi_{1k}^{(\frac{i}{2})}}}{u_\theta^{(0)}}\right) = \lim_{r\to 0}\left(\frac{\frac{\overline{\psi_{1k}^{(\frac{i}{2})}}}{\partial r}}{\frac{\partial u_\theta^{(0)}}{\partial r}}\right) = \frac{0}{\zeta_*^{(0)}} = 0 \tag{B-14}$$

Hence (B-11) can be rewritten into

$$\overline{\psi_{1k}^{(\frac{i}{2})}} = -u_\theta^{(0)} \int_0^{\bar{\bar{r}}} \frac{1}{\bar{r} u_\theta^{(0)2}} \left[\int_0^{\bar{r}} r\ \mathcal{D}_{1k}\ dr\right] d\bar{r} \qquad k = 1, 2 \tag{B-15}$$

B.2

The following simplifications can be made using (B-13) and integration by parts

$$\begin{aligned}
u_\theta^{(0)} \int_0^{\bar{\bar{r}}} \frac{1}{\bar{r} u_\theta^{(0)2}} \left[\int_0^{\bar{r}} r^2 \zeta_r^{(0)}\ dr\right] d\bar{r} &= u_\theta^{(0)} \int_0^{\bar{\bar{r}}} \frac{1}{\bar{r} u_\theta^{(0)2}} \left[-2\bar{r} u_\theta^{(0)} + \bar{r}\frac{\partial}{\partial \bar{r}}(\bar{r} u_\theta^{(0)})\right] d\bar{r} \\
&= u_\theta^{(0)} \int_0^{\bar{\bar{r}}} \left(-\frac{1}{u_\theta^{(0)}} + \frac{\bar{r}}{u_\theta^{(0)2}} \frac{\partial u_\theta^{(0)}}{\partial \bar{r}}\right) d\bar{r} \\
&= -u_\theta^{(0)} \int_0^{\bar{\bar{r}}} \frac{\partial}{\partial \bar{r}} \left(\frac{\bar{r}}{u_\theta^{(0)}}\right) d\bar{r} \\
&= -u_\theta^{(0)} \left[\frac{\bar{r}}{u_\theta^{(0)}}\right]_0^{\bar{\bar{r}}}
\end{aligned}$$

Using L'Hospital's rule and (B-13) one finds that

$$u_\theta^{(0)} \int_0^{\bar{\bar{r}}} \frac{1}{\bar{r} u_\theta^{(0)2}} \left[\int_0^{\bar{r}} r^2 \zeta_r^{(0)}\ dr\right] d\bar{r} = -\bar{\bar{r}} + u_\theta^{(0)} \frac{2}{\zeta_*^{(0)}} \tag{B-16}$$

B.3

Here an equation of this type is solved

$$\nabla_1^2 \phi_{1k}^{(\frac{i}{2})} = h(r, z) \quad ,$$

subject to the boundary conditions $\phi_{1k}^{(\frac{1}{2})} = 0$ and $\partial \phi_{1k}^{(\frac{1}{2})} / \partial r = 0$ at $r = 0$. Using the identity (4.48), integration from 0 to $\bar{r}$ yields

$$\frac{\partial \phi_{1k}^{(\frac{i}{2})}}{\partial \bar{r}} - \frac{\phi_{1k}^{(\frac{i}{2})}}{\bar{r}} = \frac{1}{\bar{r}^2} \int_0^{\bar{r}} \bar{\bar{r}}^2 h\ d\bar{\bar{r}} \tag{B-17}$$

Rewriting the left hand side of (B-17) yields

$$\bar{r}\frac{\partial}{\partial \bar{r}}\left(\frac{\phi_{1k}^{(\frac{i}{2})}}{\bar{r}}\right) = \frac{1}{\bar{r}^2}\int_0^{\bar{r}} \bar{\bar{r}}^2 h \, d\bar{\bar{r}} \tag{B-18}$$

and again integration from 0 to r yields

$$\phi_{1k}^{(\frac{i}{2})} = r\int_0^r \frac{1}{\bar{r}^3}\left[\int_0^{\bar{\bar{r}}} \bar{\bar{r}}^2 h \, d\bar{\bar{r}}\right] d\bar{r} \tag{B-19}$$

B.4

Far field solutions for $\psi_{1k}^{(\frac{2}{2})}$ can be obtained by solving (4.51) for large r. Considering this limit, the differential equations for $\psi_{1k}^{(\frac{2}{2})}$ reduce to

$$\begin{aligned}
\nabla_1^2\psi_{12}^{(\frac{2}{2})} &= -\frac{2\pi}{\Gamma(z)}\left(b\frac{\partial X_C^{(\frac{1}{2})}}{\partial z} + c\frac{\partial^2 X_C^{(\frac{1}{2})}}{\partial z^2}\right)\frac{1}{r} \qquad \text{as} \quad r \to \infty \\
\nabla_1^2\psi_{11}^{(\frac{2}{2})} &= -\frac{2\pi}{\Gamma(z)}\left(b\frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} + c\frac{\partial^2 Y_C^{(\frac{1}{2})}}{\partial z^2}\right)\frac{1}{r} \qquad \text{as} \quad r \to \infty
\end{aligned} \tag{B-20}$$

The above equations can be derived by use of (3.99), (4.17) and (4.45), which in particular yield for the inhomogeneous terms on the right hand side of (4.51)

$$\begin{aligned}
\mathcal{H}_{1k} &= -T_k\frac{a}{r^4} \qquad \text{as} \quad r \to \infty \\
\mathcal{I}_{1k} &= -\left(bT_k + c\frac{\partial T_k}{\partial z}\right)\frac{1}{r^2} \qquad \text{as} \quad r \to \infty
\end{aligned} \tag{B-21}$$

with $a = a(z) = \frac{3\tilde{g}\Gamma^2}{2\pi}\frac{\partial \Gamma}{\partial z}$, $b = b(z) = \frac{\Omega_0}{\rho^{(0)}}\frac{\partial}{\partial z}(\rho^{(0)}\tilde{g}\Gamma^2)$, $c = c(z) = \Omega_0\tilde{g}\Gamma^2$ and whereas T_k denotes the tilt components, i.e.

$$T_1 = -\frac{\partial X_C^{(\frac{1}{2})}}{\partial z} \quad \text{and} \quad T_2 = +\frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} \tag{B-22}$$

The operator ∇_1^2 is defined through (B-4). Using the identity (4.48) integration of (B-20) yields

$$\frac{\partial \psi_{12}^{(\frac{2}{2})}}{\partial r} - \frac{\psi_{12}^{(\frac{2}{2})}}{r} = -\frac{\pi}{\Gamma(z)}\left(b\frac{\partial X_C^{(\frac{1}{2})}}{\partial z} + c\frac{\partial^2 X_C^{(\frac{1}{2})}}{\partial z^2}\right) + \frac{C_{12}^1}{r^2} \tag{B-23}$$

with $C_{12}^1 = C_{12}^1(z,\tau)$ is a constant of integration. The associated homogeneous solution reads

$$\psi_{12}^{(\frac{2}{2})} = \tilde{C}\ r \tag{B-24}$$

The method of variation of parameters is used to find the particular solution. Then, by assuming that $\tilde{C} = \tilde{C}(r,z)$ substitution of (B-24) into (B-23) yields

$$\frac{\partial \tilde{C}}{\partial r} = -\frac{\pi}{\Gamma(z)} \left(b \frac{\partial X_C^{(\frac{1}{2})}}{\partial z} + c \frac{\partial^2 X_C^{(\frac{1}{2})}}{\partial z^2} \right) \frac{1}{r} + \frac{C_{12}^1}{r^3} \tag{B-25}$$

and after integration

$$\tilde{C} = -\frac{\pi}{\Gamma(z)} \left(b \frac{\partial X_C^{(\frac{1}{2})}}{\partial z} + c \frac{\partial^2 X_C^{(\frac{1}{2})}}{\partial z^2} \right) \ln r - \frac{C_{12}^1}{r^2} + C_{12}^2 \tag{B-26}$$

where $C_{12}^2 = C_{12}^2(z,\tau)$ is a second constant of integration. Thus, the far field solution for $\psi_{12}^{(\frac{2}{2})}$ is in the limit $r \to \infty$

$$\psi_{12}^{(\frac{2}{2})} = C_{12}^2 \; r - \frac{\pi}{\Gamma(z)} \left(b \frac{\partial X_C^{(\frac{1}{2})}}{\partial z} + c \frac{\partial^2 X_C^{(\frac{1}{2})}}{\partial z^2} \right) r \;\ln r - \frac{C_{12}^1}{r} \tag{B-27}$$

The unknown function C_{12}^2 is determined by equating (B-27) to (4.57) for large r. This yields

$$\begin{aligned} C_{12}^2 = \lim_{r\to\infty} \Bigg(-\frac{u_\theta^{(0)}}{r} \int_0^{\bar{\bar{r}}} \frac{1}{\bar{r} u_\theta^{(0)^2}} \left[\int_0^{\bar{r}} r \; (\mathcal{H}_{11} + \mathcal{I}_{11}) \; dr \right] d\bar{r} + \\ \frac{\pi}{\Gamma(z)} \left(b \frac{\partial X_C^{(\frac{1}{2})}}{\partial z} + c \frac{\partial^2 X_C^{(\frac{1}{2})}}{\partial z^2} \right) \ln r \Bigg) \end{aligned} \tag{B-28}$$

Furthermore one finds, that

$$C_{12}^1 = -\frac{\Gamma(z)}{2\pi} \frac{2B_{12}}{\zeta_*^{(0)}} \tag{B-29}$$

Same procedure yields for $\psi_{11}^{(\frac{2}{2})}$

$$\psi_{11}^{(\frac{2}{2})} = C_{11}^2 \; r - \frac{\pi}{\Gamma(z)} \left(b \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} + c \frac{\partial^2 Y_C^{(\frac{1}{2})}}{\partial z^2} \right) r \;\ln r - \frac{C_{11}^1}{r} \quad \text{as} \quad r \to \infty$$

where

$$\begin{aligned} C_{11}^2 = \lim_{r\to\infty} \Bigg(+\frac{u_\theta^{(0)}}{r} \int_0^{\bar{\bar{r}}} \frac{1}{\bar{r} u_\theta^{(0)^2}} \left[\int_0^{\bar{r}} r \; (\mathcal{H}_{12} + \mathcal{I}_{12}) \; dr \right] d\bar{r} + \\ \frac{\pi}{\Gamma(z)} \left(b \frac{\partial Y_C^{(\frac{1}{2})}}{\partial z} + c \frac{\partial^2 Y_C^{(\frac{1}{2})}}{\partial z^2} \right) \ln r \Bigg) \end{aligned} \tag{B-30}$$

and

$$C_{11}^{1} = -\frac{\Gamma(z)}{2\pi}\frac{2B_{11}}{\zeta_*^{(0)}} \qquad \text{(B-31)}$$

Note that with the aid of the matching condition (3.99) for $u_\theta^{(0)}$, the expressions (B-28) and (B-30) can be written as

$$\begin{aligned}
C_{12}^{2} &= \frac{\Gamma}{2\pi}\lim_{r\to\infty}\left(-\frac{1}{r^2}\int_0^{\bar{\bar{r}}}\frac{1}{\bar{r}u_\theta^{(0)^2}}\left[\int_0^{\bar{r}} r\ (\mathcal{H}_{11}+\mathcal{I}_{11})\ dr\right]d\bar{r}\right)\\
C_{11}^{2} &= \frac{\Gamma}{2\pi}\lim_{r\to\infty}\left(+\frac{1}{r^2}\int_0^{\bar{\bar{r}}}\frac{1}{\bar{r}u_\theta^{(0)^2}}\left[\int_0^{\bar{r}} r\ (\mathcal{H}_{12}+\mathcal{I}_{12})\ dr\right]d\bar{r}\right)
\end{aligned} \qquad \text{(B-32)}$$

If the limit exists, i.e. $C_{1k}^{2} \neq 0$ for large r, L' Hospitals rule and (3.99) can be used to obtain

$$\begin{aligned}
C_{12}^{2} &= -\frac{\pi}{\Gamma}\int_0^{\infty} r\ (\mathcal{H}_{11}+\mathcal{I}_{11})\ dr\\
C_{11}^{2} &= +\frac{\pi}{\Gamma}\int_0^{\infty} r\ (\mathcal{H}_{12}+\mathcal{I}_{12})\ dr
\end{aligned} \qquad \text{(B-33)}$$

Further simplifications for the integrals can be made. Taking into account that $\mathcal{H}_{1k}$ is given through $(4.52)_1$, integration by parts yields together with the BC's (4.21) and (4.18) for the first integrals in (B-33)

$$\int_0^{\infty} r\ \mathcal{H}_{1k}\ dr = -\int_0^{\infty} r w_{1k}^{(\frac{5}{2})}\frac{\partial u_\theta^{(0)}}{\partial z}dr\ , \quad k=1,2 \qquad \text{(B-34)}$$

With $\mathcal{I}_{1k}$ given through $(4.52)_2$ it is shown in **Appendix B.5** that the second integrals in (B-33) can be written as

$$\begin{aligned}
\int_0^{\infty} r\ \mathcal{I}_{1k}\ dr \quad &= \quad -\int_0^{\infty}\left[\Omega_0+\frac{u_\theta^{(0)}}{r}\right]\frac{r^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{1k}^{(\frac{5}{2})})}{\partial z}\ dr\ +\\
&\qquad \frac{\Gamma(z)}{2\pi}\int_0^{\infty}\frac{1}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{1k}^{(\frac{5}{2})})}{\partial z}\ dr
\end{aligned} \qquad \text{(B-35)}$$

Eventually, upon substitution of (B-34) and (B-35) into (B-33) one obtains

$$\begin{aligned}
C_{12}^{2} \quad &= \quad +\frac{\pi}{\Gamma}\int_0^{\infty} r w_{11}^{(\frac{5}{2})}\frac{\partial u_\theta^{(0)}}{\partial z}+\left[\Omega_0+\frac{u_\theta^{(0)}}{r}-\frac{\Gamma}{2\pi r^2}\right]\frac{r^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{11}^{(\frac{5}{2})})}{\partial z}\ dr\\
C_{11}^{2} \quad &= \quad -\frac{\pi}{\Gamma}\int_0^{\infty} r w_{12}^{(\frac{5}{2})}\frac{\partial u_\theta^{(0)}}{\partial z}+\left[\Omega_0+\frac{u_\theta^{(0)}}{r}-\frac{\Gamma}{2\pi r^2}\right]\frac{r^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{12}^{(\frac{5}{2})})}{\partial z}\ dr
\end{aligned} \qquad \text{(B-36)}$$

B.5

Here the improper integral over $r\mathcal{I}_{1k}$ is simplified. With $(4.52)_2$ the integral is

$$\int_0^\infty r\,\mathcal{I}_{1k}\,dr = \int_0^\infty r^2\,[\zeta^{(0)}+\Omega_0]\nabla_1^2\phi_{1k}^{(\frac{2}{2})} + \frac{\partial\phi_{1k}^{(\frac{2}{2})}}{\partial r}\left(r^2\frac{\partial\zeta^{(0)}}{\partial r}\right)\,dr \qquad \text{(B-37)}$$

Using (4.48) the above equation can be rewritten into

$$\int_0^\infty r\,\mathcal{I}_{1k}\,dr = \int_0^\infty [\zeta^{(0)}+\Omega_0]\frac{\partial}{\partial r}\left(r\left(r\frac{\partial\phi_{1k}^{(\frac{i}{2})}}{\partial r} - \phi_{1k}^{(\frac{i}{2})}\right)\right) + \frac{\partial\phi_{1k}^{(\frac{2}{2})}}{\partial r}\left(r^2\frac{\partial\zeta^{(0)}}{\partial r}\right)\,dr\,,$$

and integration by parts yields

$$\begin{aligned}\int_0^\infty r\,\mathcal{I}_{1k}\,dr \;=\;& \left[\zeta^{(0)}\,r^2\left(\frac{\partial\phi_{1k}^{(\frac{2}{2})}}{\partial r} - \frac{\phi_{1k}^{(\frac{2}{2})}}{r}\right)\right]_0^\infty + \\ & \Omega_0\left[r^2\left(\frac{\partial\phi_{1k}^{(\frac{2}{2})}}{\partial r} - \frac{\phi_{1k}^{(\frac{2}{2})}}{r}\right)\right]_0^\infty + \int_0^\infty \frac{\partial\zeta^{(0)}}{\partial r}\,r\phi_{1k}^{(\frac{2}{2})}\,dr \qquad \text{(B-38)}\end{aligned}$$

Taking into account that $\zeta^{(0)}$ is finite and not singular at $r=0$, and together with the BC's (4.47), the far field behavior $\zeta^{(0)} = o(r^{-n})$ for all n as r approaches ∞ (see (3.99)), the first term on the right hand side in (B-38) disappears. With the aid of (B-17), where h is given through the right hand side of (4.45), the bracket in the second term of (B-38) can be written as

$$\Omega_0\left[r^2\left(\frac{\partial\phi_{1k}^{(\frac{2}{2})}}{\partial r} - \frac{\phi_{1k}^{(\frac{2}{2})}}{r}\right)\right]_0^\infty = -\Omega_0\int_0^\infty \frac{\bar{r}^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{1k}^{(\frac{5}{2})})}{\partial z}\,d\bar{r} \qquad \text{(B-39)}$$

With (4.50) the second integral on the right hand side of the above equation can be written as

$$\begin{aligned}\int_0^\infty \frac{\partial\zeta^{(0)}}{\partial r}\,r\phi_{1k}^{(\frac{2}{2})}\,dr \;=\;& \frac{1}{2}\int_0^\infty \frac{\partial\zeta^{(0)}}{\partial r}\left(\int_0^r \frac{\bar{r}^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{1k}^{(\frac{5}{2})})}{\partial z}d\bar{r}\right)\,dr\; - \\ & \frac{1}{2}\int_0^\infty \frac{\partial\zeta^{(0)}}{\partial r}\,r^2\left(\int_0^r \frac{1}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{1k}^{(\frac{5}{2})})}{\partial z}d\bar{r}\right)\,dr \qquad \text{(B-40)}\end{aligned}$$

Integration by parts yields for the first integral on the right hand side of (B-40)

$$\begin{aligned}&\int_0^\infty \frac{\partial\zeta^{(0)}}{\partial r}\left(\int_0^r \frac{\bar{r}^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{1k}^{(\frac{5}{2})})}{\partial z}d\bar{r}\right)\,dr \;= \\ &\left[\zeta^{(0)}\int_0^r \frac{\bar{r}^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{1k}^{(\frac{5}{2})})}{\partial z}d\bar{r}\right]_0^\infty - \int_0^\infty \zeta^{(0)}\frac{r^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)}w_{1k}^{(\frac{5}{2})})}{\partial z}\,dr\,, \qquad \text{(B-41)}\end{aligned}$$

and for the second integral on the right hand side of (B-40)

$$\int_0^\infty \frac{\partial \zeta^{(0)}}{\partial r}\, r^2 \left(\int_0^r \frac{1}{\rho^{(0)}} \frac{\partial(\rho^{(0)} w_{1k}^{(\frac{5}{2})})}{\partial z} d\bar{r} \right) dr \;=$$

$$\left[\zeta^{(0)} r^2 \int_0^r \frac{\bar{r}^2}{\rho^{(0)}} \frac{\partial(\rho^{(0)} w_{1k}^{(\frac{5}{2})})}{\partial z} d\bar{r} \right]_0^\infty - \int_0^\infty \zeta^{(0)} \frac{r^2}{\rho^{(0)}} \frac{\partial(\rho^{(0)} w_{1k}^{(\frac{5}{2})})}{\partial z}\, dr \;-$$

$$\frac{\Gamma}{\pi} \int_0^\infty \frac{1}{\rho^{(0)}} \frac{\partial(\rho^{(0)} w_{1k}^{(\frac{5}{2})})}{\partial z}\, dr + 2 \int_0^\infty u_\theta^{(0)} r \frac{1}{\rho^{(0)}} \frac{\partial(\rho^{(0)} w_{1k}^{(\frac{5}{2})})}{\partial z}\, dr \qquad \text{(B-42)}$$

Note that here we have used that $\zeta^{(0)} = r^{-1}(\partial\, r u_\theta^{(0)}/\partial r)$ and $u_\theta^{(0)} = \Gamma/2\pi r$ as r approaches ∞. Again, taking into account that $\zeta^{(0)}$ is finite and not singular at $r = 0$, the far field behavior $\zeta^{(0)} = o(r^{-n})$ for all n as r approaches ∞ (see (3.99)), the first terms on the right hand side of (B-41) and (B-42), respectively, disappear. Thus, (B-40) simplifies to

$$\int_0^\infty \frac{\partial \zeta^{(0)}}{\partial r}\, r \phi_{1k}^{(\frac{2}{2})}\, dr \;=$$

$$\frac{\Gamma}{2\pi} \int_0^\infty \frac{1}{\rho^{(0)}} \frac{\partial(\rho^{(0)} w_{1k}^{(\frac{5}{2})})}{\partial z}\, dr - \int_0^\infty u_\theta^{(0)} r \frac{1}{\rho^{(0)}} \frac{\partial(\rho^{(0)} w_{1k}^{(\frac{5}{2})})}{\partial z}\, dr \qquad \text{(B-43)}$$

Therefore, one obtains for (B-37)

$$\int_0^\infty r\, \mathcal{I}_{1k}\, dr \;=\; -\int_0^\infty \left[\Omega_0 + \frac{u_\theta^{(0)}}{r} \right] \frac{r^2}{\rho^{(0)}} \frac{\partial(\rho^{(0)} w_{1k}^{(\frac{5}{2})})}{\partial z}\, dr \;+$$

$$\frac{\Gamma(z)}{2\pi} \int_0^\infty \frac{1}{\rho^{(0)}} \frac{\partial(\rho^{(0)} w_{1k}^{(\frac{5}{2})})}{\partial z}\, dr \qquad \text{(B-44)}$$

B.6

Solutions for $\phi_{1k}^{(\frac{2}{2})}$ in the limit $r \to \infty$ are derived here. Using the identity (4.48) and the far field conditions for $w_{1k}^{(\frac{5}{2})}$ (see (4.18)), equation (4.45) becomes for large r

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r \left(r \frac{\partial \phi_{1k}^{(\frac{2}{2})}}{\partial r} - \phi_{1k}^{(\frac{2}{2})} \right) \right) = -\frac{P}{r^3} \qquad \text{as} \quad r \to \infty \qquad \text{(B-45)}$$

with $P = P(z) = {\rho^{(0)}}^{-1} \partial(\rho^{(0)} \tilde{g}\; T_k\; \Gamma^2)/\partial z$. Integration yields

$$\frac{\partial \phi_{1k}^{(\frac{2}{2})}}{\partial r} - \frac{\phi_{1k}^{(\frac{2}{2})}}{r} = r \frac{\partial}{\partial r} \left(\frac{\phi_{1k}^{(\frac{2}{2})}}{r} \right) = -\frac{P\;\ln r}{r^2} + \frac{\bar{C}_{1k}^1}{r^2} \qquad \text{as} \quad r \to \infty \qquad \text{(B-46)}$$

with $\bar{C}^1_{1k} = \bar{C}^1_{1k}(z)$ a constant of integration. Integration again gives

$$\phi^{(\frac{2}{2})}_{1k} = P\left(\frac{2\ln r + 1}{4\ r}\right) - \frac{\bar{C}^1_{1k}}{2\ r} + \bar{C}^2_{1k}\ r \qquad \text{as} \quad r \to \infty \tag{B-47}$$

and by use of L' Hospital's rule

$$\phi^{(\frac{2}{2})}_{1k} = \left(\frac{3}{4}P - \frac{1}{2}\bar{C}^1_{1k}\right)\frac{1}{r} + \bar{C}^2_{1k} r \qquad \text{as} \quad r \to \infty \tag{B-48}$$

The unknown $\bar{C}^2_{1k}$ can be determined by equating (B-48) to (4.50) for large r. In doing so, one gets

$$\begin{aligned} \bar{C}^2_{1k} &= -\frac{1}{2}\int_0^\infty \frac{1}{\rho^{(0)}}\frac{\partial(\rho^{(0)} w^{(\frac{5}{2})}_{1k})}{\partial z} d\bar{r} \\ \bar{C}^1_{1k} &= -\int_0^\infty \frac{r^2}{\rho^{(0)}}\frac{\partial(\rho^{(0)} w^{(\frac{5}{2})}_{1k})}{\partial z} d\bar{r} + \frac{3}{2}P \end{aligned} \tag{B-49}$$

Appendix C

Auxiliary calculations for Chapter 5

C.1

Using (3.99) and (5.52) for large r, (5.58) can be written as

$$\nabla_1^2 \psi_{1k}^{(\frac{1}{2})} = +\Omega_0 \frac{\tilde{\alpha}_{1k}}{r^{(m-1)}} \qquad \text{as} \quad r \to \infty \ , \quad \text{for all} \quad m \geq 0 \tag{C-1}$$

with ∇_1^2 defined through (5.46) and $\tilde{\alpha}_{12} = \alpha_{11}$ and $\tilde{\alpha}_{11} = -\alpha_{12}$ which are functions that may depend on z. Here the fact that F_{1k} decays faster than B_{1k} for large r is used, i.e

$$\begin{aligned} F_{1k} &= \frac{\partial}{\partial r}\left(r\alpha_{1k}^{\star}(z) \frac{1}{2\pi r^{|m|}} \frac{\partial \Gamma}{\partial z} \right) \sim \frac{1}{r^{(|m|+1)}} \qquad \text{as} \qquad r \to \infty \\ B_{1k} &= -r\Omega_0 \alpha_{1k} \frac{1}{r^{|m|}} \sim \frac{1}{r^{(|m|-1)}} \qquad \text{as} \qquad r \to \infty \end{aligned} \tag{C-2}$$

Using the identity (4.48), equation (C-1) can be simplified to

$$\frac{\partial}{\partial r}\left(r \left(r \frac{\partial \psi_{1k}^{(\frac{1}{2})}}{\partial r} - \psi_{1k}^{(\frac{1}{2})} \right) \right) = +\Omega_0 \frac{\tilde{\alpha}_{1k}}{r^{(m-3)}} \qquad \text{as} \quad r \to \infty$$

Then, integrating the above equation twice yields for

$m = 1$:
$$\psi_{1k}^{(\frac{1}{2})} = \Omega_0 \tilde{\alpha}_{1k} r^2 - \frac{1}{2r} D_{1k_1}^1 + r D_{1k_1}^2 \qquad \text{as} \quad r \to \infty \tag{C-3}$$

$m = 2$:
$$\psi_{1k}^{(\frac{1}{2})} = \frac{\Omega_0 \tilde{\alpha}_{1k}}{2} r \ln r - \frac{1}{2r} D_{1k_2}^1 + r D_{1k_2}^2 \qquad \text{as} \quad r \to \infty \tag{C-4}$$

$m = 3$: $$\psi_{1k}^{(\frac{1}{2})} = -\Omega_0 \tilde{\alpha}_{1k} - \frac{1}{r} D^1_{1k_3} + r D^2_{1k_3} \qquad \text{as} \quad r \to \infty \tag{C-5}$$

$m \geq 4$: $$\psi_{1k}^{(\frac{1}{2})} = \mathcal{O}(r^{-n}) + r D^2_{1k_m} \qquad \text{as} \quad r \to \infty \ , \quad n \geq 1 \tag{C-6}$$

Here, $D^1_{1k_m}$ and $D^2_{1k_m}$ denote constants of integration.

C.2

Seeking far field solutions for $\phi_{1k}^{(\frac{1}{2})}$ in the absence of asymmetric vertical velocities, equation (5.47) simplifies together with (5.23) to

$$\nabla_1^2 \phi_{1k}^{(\frac{1}{2})} = -T_k^\sharp \ \lambda^2 \ \kappa(z) \ r \ \exp\left(-\alpha \ r^2\right) \qquad \text{as} \qquad r \to \infty \tag{C-7}$$

with $\kappa(z) = C_H \ h(z)/\rho^{(0)}$ and $\alpha = \lambda^2/2$. Using the identity (4.48), integration of (C-7) yields

$$\begin{aligned} \frac{\partial \phi_{1k}^{(\frac{1}{2})}}{\partial r} - \frac{\phi_{1k}^{(\frac{1}{2})}}{r} &= -T_k^\sharp \ \frac{\lambda^2 \kappa(z)\sqrt{\pi}}{2\sqrt{\alpha}} \ \frac{\operatorname{erf}(\sqrt{\alpha} \ r)}{r^2} + \frac{\bar{D}^1_{1k}}{r^2} \qquad \text{as} \qquad r \to \infty \\ &= -T_k^\sharp \ \frac{\lambda^2 \kappa(z)\sqrt{\pi}}{2\sqrt{\alpha}} \ \frac{1}{r^2} + \frac{\bar{D}^1_{1k}}{r^2} \qquad \text{as} \qquad r \to \infty \end{aligned} \tag{C-8}$$

whereas for large r: $r^3 \ \exp\left(-\alpha \ r^2\right) \approx \exp\left(-\alpha \ r^2\right)$ and $\operatorname{erf}(\sqrt{\alpha} \ r) \approx 1$ with $\alpha = \lambda^2/2$. $\bar{D}^1_{1k}$ denotes a constant of integration that may be a function on z, i.e. $\bar{D}^1_{1k} = \bar{D}^1_{1k}(z)$. Rewriting the left hand side of (C-8) yields

$$r \frac{\partial}{\partial r} \left(\frac{\phi_{1k}^{(\frac{1}{2})}}{r} \right) = -T_k^\sharp \ \frac{\lambda^2 \kappa(z)\sqrt{\pi}}{2\sqrt{\alpha}} \ \frac{1}{r^2} + \frac{\bar{D}^1_{1k}}{r^2} \qquad \text{as} \qquad r \to \infty \tag{C-9}$$

and integration gives

$$\phi_{1k}^{(\frac{1}{2})} = \frac{1}{r} \left(T_k^\sharp \ \frac{\lambda^2 \kappa(z)\sqrt{\pi}}{4\sqrt{\alpha}} - \frac{1}{2} \bar{D}^1_{1k} \right) + r \bar{D}^2_{1k} \qquad \text{as} \qquad r \to \infty \tag{C-10}$$

The constants $\bar{D}^2_{1k}$ can be determined by equating (C-10) to (5.106) for large r. Then, one obtains

$$\bar{D}^2_{1k} = T_k^\sharp \int_0^\infty \frac{1}{\bar{r}^3} \left[\int_0^{\bar{r}} \bar{\bar{r}}^2 \frac{\partial w^{(\frac{4}{2})}}{\partial \bar{\bar{r}}} \ d\bar{\bar{r}} \right] d\bar{r} \tag{C-11}$$

C.3

Using (3.99) and (5.23) for large r, equation (5.110) can be written as

$$\begin{aligned} \nabla_1^2 \psi_{12}^{(\frac{1}{2})} &= +T_1^\sharp \ \tilde{a}(z) \ \exp\left(-\alpha r^2\right) \qquad \text{as} \quad r \to \infty \\ \nabla_1^2 \psi_{11}^{(\frac{1}{2})} &= -T_2^\sharp \ \tilde{a}(z) \ \exp\left(-\alpha \ r^2\right) \qquad \text{as} \quad r \to \infty \end{aligned} \tag{C-12}$$

with ∇_1^2 defined through (5.46), $\tilde{a} = 2\pi \ (\alpha \ \kappa(z) \ [\Gamma(z)\pi^{-1} + 2\Omega_0]) \ / \ \Gamma(z)$ and where

$$\begin{aligned} \mathcal{F}_{1k} &= -T_k^\sharp \ \alpha \ \kappa(z) \ \Gamma(z) \ \pi^{-1} \exp\left(-\alpha \ r^2\right) \qquad \text{as} \quad r \to \infty \\ \mathcal{B}_{1k} &= -T_k^\sharp \ \alpha \ \kappa(z) \ 2\Omega_0 \ \exp\left(-\alpha \ r^2\right) \qquad \text{as} \quad r \to \infty \end{aligned} \tag{C-13}$$

Using the identity (4.48), equation $(C-12)_1$ can be simplified to

$$\frac{\partial}{\partial r}\left(r\left(r\frac{\partial \psi_{12}^{(\frac{1}{2})}}{\partial r} - \psi_{12}^{(\frac{1}{2})}\right)\right) = +T_1^\sharp \tilde{\alpha}(z) \ r^2 \exp\left(-\alpha r^2\right) \quad \text{as} \quad r \to \infty \tag{C-14}$$

Note that $r^2 \exp\left(-\alpha r^2\right) \approx \exp\left(-\alpha r^2\right)$ as $r \to \infty$. Thus, integration of (C-14) leads

$$\frac{\partial \psi_{12}^{(\frac{1}{2})}}{\partial r} - \frac{\psi_{12}^{(\frac{1}{2})}}{r} \quad = \quad +T_1^\sharp \ \frac{\sqrt{\pi}\tilde{\alpha}(z)}{2\sqrt{\alpha}} \ \frac{\operatorname{erf}(\sqrt{\alpha} \ r)}{r^2} + \frac{D_{12}^1}{r^2} \quad \text{as} \quad r \to \infty \tag{C-15}$$

with D_{12}^1 a constant of integration. Homogenous solutions of (C-15) read $\psi_{12}^{(\frac{1}{2})} = r\tilde{D}_{12}$ with $\tilde{D}_{12}$ another constant of integration. Then, employing the method of variation of parameters, i.e. assuming that $\tilde{D}_{12} = \tilde{D}_{12}(r, z)$, yields

$$\frac{\partial \tilde{D}_{12}}{\partial r} \quad = \quad +T_1^\sharp \ \frac{\sqrt{\pi}\tilde{\alpha}(z)}{2\sqrt{\alpha}} \ \frac{1}{r^3} + \frac{D_{12}^1}{r^3} \qquad \text{as} \quad r \to \infty \tag{C-16}$$

Since $\operatorname{erf}(\sqrt{\alpha} \ r) \approx 1$ for large r, further integration yields

$$\tilde{D}_{12} \quad = \quad -q_2^\star T_1^\sharp \frac{1}{2r^2} + D_{12}^2 \qquad \text{as} \quad r \to \infty \tag{C-17}$$

with $q_2^\star = (\sqrt{\pi}\tilde{\alpha}(z)/2\sqrt{\alpha} \ + \ D_{12}^1)$ and D_{12}^2 the second constant of integration. Eventually, applying same procedure to $(C-12)_2$ the far field behavior of $\psi_{1k}^{(\frac{1}{2})}$ is given by

$$\begin{aligned} \psi_{12}^{(\frac{1}{2})} &\sim -q_2^\star T_1^\sharp \frac{1}{2r} + rD_{12}^2 \qquad \text{as} \quad r \to \infty \\ \psi_{11}^{(\frac{1}{2})} &\sim -q_1^\star T_2^\sharp \frac{1}{2r} + rD_{11}^2 \qquad \text{as} \quad r \to \infty \end{aligned} \tag{C-18}$$

where $q_1^\star = (-\sqrt{\pi}\tilde{\alpha}(z)/2\sqrt{\alpha} \; + \; D_{11}^1)$. The unknown functions $D_{1k}^2 = D_{1k}^2(z)$ can be determined by equating (C-18) to (5.113) for large r. This yields

$$
\begin{aligned}
D_{12}^2 &= \lim_{r\to\infty} \left(-\frac{u_\theta^{(0)}}{r} \int_0^{\bar{\bar{r}}} \frac{1}{\bar{r} u_\theta^{(0)^2}} \left[\int_0^{\bar{r}} r \; (\mathcal{F}_{11} + \mathcal{B}_{11}) \; dr \right] d\bar{r} \right) \\
D_{11}^2 &= \lim_{r\to\infty} \left(+\frac{u_\theta^{(0)}}{r} \int_0^{\bar{\bar{r}}} \frac{1}{\bar{r} u_\theta^{(0)^2}} \left[\int_0^{\bar{r}} r \; (\mathcal{F}_{12} + \mathcal{B}_{12}) \; dr \right] d\bar{r} \right)
\end{aligned} \tag{C-19}
$$

C.4

Following integrals have to be solved

$$
\begin{aligned}
\int_0^\infty r \; \tilde{\mathcal{B}}_{1k} \; dr &= \int_0^\infty r^2 [\Omega_0 + \zeta^{(0)}] \nabla_1^2 \phi_{1k}^{(\frac{1}{2})} + \frac{\partial \phi_{1k}^{(\frac{1}{2})}}{\partial r} (r^2 \zeta_r^{(0)}) \; dr \\
\int_0^\infty r \; \tilde{\mathcal{F}}_{1k} \; dr &= T_k^\sharp \int_0^\infty w^{(\frac{4}{2})} u_\theta^{(0)} - r \frac{\partial}{\partial r} \left(r w^{(\frac{4}{2})} \frac{\partial u_\theta^{(0)}}{\partial r} \right) \; dr
\end{aligned} \tag{C-20}
$$

With the aid of (4.48), integration by parts and solutions for $\phi_{1k}^{(\frac{1}{2})}$, i.e. (5.106), the integral $(C-20)_1$ can be written as

$$
\begin{aligned}
\int_0^\infty r \; \mathcal{B}_{1k} \; dr = \; & T_k^\sharp \left[[\Omega_0 + \zeta^{(0)}] T_k^\sharp \left(-2 \int_0^r w^{(\frac{4}{2})} r dr + r^2 w^{(\frac{4}{2})} \right) \right]_0^\infty + \\
& \int_0^\infty \frac{\partial \zeta^{(0)}}{\partial r} r \phi_{1k}^{(\frac{1}{2})} \; dr
\end{aligned} \tag{C-21}
$$

Assuming that $w^{(\frac{4}{2})}(r = 0, z) = 0$ one can show together with the far field solution (5.23) for $w^{(\frac{4}{2})}$, the streamfunction (5.12) and the boundary conditions $\chi(r, z) \to 0$ for large r and $\chi(r = 0, z) = 0$, that

$$
\int_0^\infty r \; \mathcal{B}_{1k} \; dr = \int_0^\infty \frac{\partial \zeta^{(0)}}{\partial r} r \phi_{1k}^{(\frac{1}{2})} \; dr \tag{C-22}
$$

Together with (3.99) and (5.106) further manipulations give

$$\begin{aligned}
\int_0^\infty r\ \mathcal{B}_{1k}\ dr &= \int_0^\infty \frac{\partial \zeta^{(0)}}{\partial r} r \phi_{1k}^{(\frac{1}{2})}\ dr = T_k^\sharp \int_0^\infty \frac{\partial \zeta^{(0)}}{\partial r} \left[\int_0^r w^{(\frac{4}{2})} \bar{r}\ d\bar{r}\right]\ dr \\
&= T_k^\sharp \left[\zeta^{(0)} \int_0^r w^{(\frac{4}{2})} \bar{r}\ d\bar{r}\right]_0^\infty - T_k^\sharp \int_0^\infty \zeta^{(0)} w^{(\frac{4}{2})} r\ dr \\
&= -T_k^\sharp \int_0^\infty r w^{(\frac{4}{2})} \frac{\partial u_\theta^{(0)}}{\partial r}\ dr - T_k^\sharp \int_0^\infty w^{(\frac{4}{2})} u_\theta^{(0)}\ dr \qquad \text{(C-23)}
\end{aligned}$$

whereas the latter equality is obtained from $\zeta^{(0)} = r^{-1}\partial(r u_\theta^0)/\partial r$. On a same way as above it can be shown that $(C-20)_2$ has the form

$$\int_0^\infty r\ \mathcal{F}_{1k}\ dr = +T_k^\sharp \int_0^\infty r w^{(\frac{4}{2})} \frac{\partial u_\theta^{(0)}}{\partial r}\ dr + T_k^\sharp \int_0^\infty w^{(\frac{4}{2})} u_\theta^{(0)}\ dr \qquad \text{(C-24)}$$

C.5

Considering the case $w_{1k}^{(\frac{4}{2})} = 0$ with $k = 1, 2$, the expressions for $\mathcal{F}_{1k}$ and $\mathcal{B}_{1k}$ together with (5.47) simplify to

$$\begin{aligned}
\mathcal{F}_{1k} &= T_k^\sharp \left(w_0^{(\frac{4}{2})} \frac{u_\theta^{(0)}}{r} - \frac{\partial}{\partial r}\left(r w_0^{(\frac{4}{2})} \frac{\partial u_\theta^{(0)}}{\partial r}\right)\right) \\
\mathcal{B}_{1k} &= r[\zeta^{(0)} + \Omega_0]\ T_k^\sharp \frac{\partial w_0^{(\frac{4}{2})}}{\partial r} + \frac{\partial \phi_{1k}^{(\frac{1}{2})}}{\partial r}(r\zeta_r^{(0)})
\end{aligned} \qquad \text{(C-25)}$$

Taking the sum $\mathcal{F}_{1k} + \mathcal{B}_{1k}$ and rearranging terms one can write

$$\begin{aligned}
\mathcal{F}_{1k} + \mathcal{B}_{1k} = T_k^\sharp \Bigg(-r \left[\frac{\partial^2 u_\theta^{(0)}}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta^{(0)}}{\partial r} - \frac{u_\theta^{(0)}}{r^2}\right] w_0^{(\frac{4}{2})} + \\
\left[u_\theta^{(0)} + r\Omega_0\right] \frac{\partial w_0^{(\frac{4}{2})}}{\partial r}\Bigg) + \frac{\partial \phi_{1k}^{(\frac{1}{2})}}{\partial r}(r\zeta_r^{(0)}) \qquad \text{(C-26)}
\end{aligned}$$

Here $\zeta^{(0)} = \partial u_\theta^{(0)}/\partial r + u_\theta^{(0)}/r$ is used. Using the definition for the operator ∇_1^2 in (B-3) further simplifications can be made, i.e.

$$\mathcal{F}_{1k} + \mathcal{B}_{1k} = T_k^\sharp \left(-r\ w_0^{(\frac{4}{2})} \nabla_1^2 u_\theta^{(0)} + \left[u_\theta^{(0)} + r\Omega_0\right] \frac{\partial w_0^{(\frac{4}{2})}}{\partial r}\right) + r\ \zeta_r^{(0)}\ \frac{\partial \phi_{1k}^{(\frac{1}{2})}}{\partial r}$$

With the aid of (B-3) and using the fact that $w_0^{(\frac{4}{2})}$ can be written in terms of a streamfunction (see (5.12)) it can be shown that

$$
\begin{aligned}
\mathcal{F}_{1k} + \mathcal{B}_{1k} &= T_k^{\sharp}\left(-r\;\zeta_r^{(0)}\;\frac{1}{\rho^{(0)}}\left(\frac{\partial\acute{\psi}}{\partial r} + \frac{\acute{\psi}}{r}\right) + \left[u_\theta^{(0)} + r\Omega_0\right]\frac{1}{\rho^{(0)}}\nabla_1^2\acute{\psi}\right) + \\
&\quad r\;\zeta_r^{(0)}\;\frac{T_k^{\sharp}}{\rho^{(0)}}\frac{\partial\acute{\psi}}{\partial r} - r\;\zeta_r^{(0)}\;\frac{T_k^{\sharp}}{\rho^{(0)}}\;\frac{\partial\acute{\psi}}{\partial r}\bigg|_{r=0} \\
&= \frac{T_k^{\sharp}}{\rho^{(0)}}\left(-\zeta_r^{(0)}\;\acute{\psi} + \left[u_\theta^{(0)} + r\Omega_0\right]\nabla_1^2\acute{\psi}\right) - r\;\zeta_r^{(0)}\;\frac{T_k^{\sharp}}{\rho^{(0)}}\;\frac{\partial\acute{\psi}}{\partial r}\bigg|_{r=0} \\
&= -u_{\mathrm{abs}}\frac{T_k}{\rho^{(0)}}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left[\frac{\zeta_r^{(0)}}{u_{\mathrm{abs}}} + \frac{1}{r^2}\right]\right)\acute{\psi} - \\
&\quad r\;\zeta_r^{(0)}\;\frac{T_k^{\sharp}}{\rho^{(0)}}\;\frac{\partial\acute{\psi}}{\partial r}\bigg|_{r=0} \qquad \text{(C-27)}
\end{aligned}
$$

where $u_{\mathrm{abs}} = u_\theta^{(0)} + r\Omega_0$.

Acknowledgements

Thanks are expressed to Prof. Dr. M. Claussen for his encouragement and interest in the work. I especially want to thank my main supervisor Prof. Dr. R. Klein whose support and guidance made this work possible. I wish to acknowledge Dr. A. Owinoh for support as well as his assistance in the translation of this work. I am also grateful to Dr. N. Botta, S. Dolaptchiev, G. Carque, S. Vater and P. Marschalik for useful discussions and the good time at PIK.

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