# Orbitale Integrale für Schleifenalgebren

Dissertation
zur Erlangung des Doktorgrades
des Fachbereichs Mathematik
der Universität Hamburg

vorgelegt von
Robert Wendt
aus Frankfurt am Main

Hamburg
2001

Als Dissertation angenommen vom Fachbereich
Mathematik der Universität Hamburg

auf Grund der Gutachten von Prof. Dr. P. Slodowy
und Prof. Dr. R. Berndt

Hamburg, den 20. Dezember 2000
Prof. Dr. U. Eckhardt
Dekan des Fachbereichs Mathematik

# Contents

# Vorwort (Preface)

## Hintergrund und Motivation

Die vorliegende Arbeit liefert Beiträge zur orbitalen Theorie gewisser unendlichdimensionaler Lie Gruppen.

Die "orbitale Theorie" oder "Methode der koadjungierten Orbiten" ist eine Sammlung von Sätzen und Vermutungen, die das Ziel haben, der Darstellungstheorie von Lie Gruppen einen einheitlichen Rahmen zu geben [Ki3]. Die zugrundeliegende Idee ist, möglichst viel Information über die Darstellungstheorie einer Gruppe $G$ aus einer einzelnen kanonisch gegebenen Darstellung herauszulesen. Diese Darstellung ist die koadjungierte Darstellung von $G$ auf $\mathfrak{g}^*$, dem Dual der zu $G$ gehörigen Lie Algebra $\mathfrak{g}$. Die für die orbitale Theorie fundamentale Beobachtung ist, daß auf jedem $G$-Orbit in $\mathfrak{g}^*$ eine kanonische symplektische Form $\omega$, die sogenannte Kirillov/Kostant-Form, existiert.

Ursprünglich wurde die orbitale Theorie von Kirillov [Ki1] eingeführt, um das unitäre Dual $\hat{N}$ (d.h. die Menge der Äquivalenzklassen der irreduziblen unitären Darstellungen) einer endlichdimensionalen, nilpotenten Lie Gruppe $N$ zu beschreiben. In diesem Fall existiert eine Bijektion zwischen der Menge $\hat{N}$ und der Menge der Orbiten in der koadjungierten Darstellung von $N$. Ähnliche Ergebnisse, wenn auch nicht mehr ganz so stark, gelten für andere Klassen von endlichdimensionalen Lie Gruppen wie zum Beispiel für die kompakten Lie Gruppen.

Eines der Hauptresultate der orbitalen Theorie für kompakte Lie Gruppen ist die Kirillovsche Charakterformel, die es erlaubt, die irreduziblen Charaktere einer kompakten, zusammenhängenden Lie Gruppe $G$ als Integrale über gewisse koadjungierte Orbiten von $G$ auszudrücken: Im folgenden sei $G$ eine kompakte, halbeinfache, zusammenhängende Lie Gruppe mit Lie Algebra $\mathfrak{g}$ und $T \subset G$ ein maximaler Torus von $G$ mit Lie Algebra $\mathfrak{h} \subset \mathfrak{g}$. Desweiteren sei $\lambda \in \mathfrak{h}^*$ ein höchstes Gewicht von $G$, und $\phi_\lambda : G \to Aut(V_\lambda)$ sei die zugehörige irreduzible Darstellung von $G$ mit höchstem Gewicht $\lambda$. Bezeichnet $\mathcal{O}_\nu \subset \mathfrak{g}^*$ den koadjungierten Orbit von $G$, der $\nu \in \mathfrak{h}^*$ enthält, so hat Kirillovs Charakterformel für $x \in \mathfrak{g}$ die Form

$$\mathrm{tr}(\phi_\lambda(\exp(x))) = p(x)^{-1} \int_{\mathcal{O}_{\lambda+\rho}} e^{-\langle y,x\rangle} d\mu_{\lambda+\rho}(y). \qquad (1)$$

Hierbei bezeichnet $d\mu_{\lambda+\rho}$ ein invariantes Maß auf $\mathcal{O}_{\lambda+\rho}$, und $p$ ist eine universelle Funktion auf $\mathfrak{g}$, die nicht von $\lambda$ abhängt. Außerdem ist wie üblich $\rho = \frac{1}{2}\sum_{\alpha\in R_+}\alpha$ die halbe Summe der positiven Wurzeln von $G$ (wobei $R \subset \mathfrak{h}^*$ das Wurzelsystem von $G$ bezeichnet). Diese Formel impliziert wieder eine Korrespondenz zwischen gewissen koadjungierten Orbiten von $G$ und der Menge der Äquivalenzklassen irreduzibler Darstellungen von $G$. Der wesentliche Unterschied zum Fall nilpotenter Gruppen ist, daß hier nicht mehr jedem koadjungierten Orbit von $G$ eine irreduzible Darstellung von $G$ zugeordnet werden kann.

Die nächsten unendlichdimensionalen Verwandten der kompakten, endlichdimensionalen Lie Gruppen sind die sogenannten Stromgruppen, die auch in der theoretischen Physik von großem Interesse sind. Ist $G$ wie zuvor eine halbeinfache, kompakte, einfach zusammenhängende Lie Gruppe, und ist $M$ eine kompakte Mannigfaltigkeit, so ist die entsprechende Stromgruppe $G^M = C^\infty(M, G)$. Dies ist eine unendlichdimensionale Lie Gruppe mit Lie Algebra $\mathfrak{g}^M = C^\infty(M, G)$, wobei die Gruppenstruktur auf $G^M$ durch punktweise Multiplikation gegeben ist. Für Anwendungen in der Physik ist man allerdings weniger an den Stromgruppen $G^M$ selbst interessiert, sondern an gewissen zentralen Erweiterungen derselben. Pressley und Segal [PS] haben eine zentrale Erweiterung $\hat{G}^M$ von $G^M$ angegeben, deren Lie Algebra

$$\{0\} \to \mathfrak{k} \to \hat{\mathfrak{g}}^M \to \mathfrak{g}^M \to \{0\}$$

die universelle zentrale Erweiterung von $\mathfrak{g}^M$ ist. Im allgemeinen ist das Zentrum $\mathfrak{k}$ von $\hat{\mathfrak{g}}^M$ unendlichdimensional.

Es ist eine natürliche Frage, inwieweit sich die orbitale Theorie kompakter Lie Gruppen auf den Fall der zentral erweiterten Stromgruppen ausdehnen läßt. Die koadjungierten Orbiten der Gruppen $\hat{G}^M$ wurden von J. Brylinski [Br] studiert. Bisher ist die Situation nur in den Fällen $\dim(M) = 1$ und $\dim(M) = 2$ handhabbar. Im ersten Fall ist $M$ diffeomorph zu $S^1$, und die Dimension des Zentrums ist $\dim(\mathfrak{k}) = 1$. In diesem Fall konnten I. Frenkel ([F]) und G. Segal ([S], [PS]) die koadjungierten Orbiten von $\hat{G}^M$ mittels Konjugationsklassen der zugrundeliegenden kompakten Lie Gruppe $G$ klassifizieren (s.u.). Im Fall $\dim(M) = 2$ ist die Situation besonders klar, wenn man die koadjungierten Orbiten der komplexifizierten Lie Gruppe $\hat{G}^M_{\mathbb{C}}$ betrachtet. Dann definiert die Wahl einer komplexen Struktur auf $M$ auf natürliche Weise eine Unterklasse in der Menge aller koadjungierter Orbiten. Falls nun $M$ eine Riemannsche Fläche vom Geschlecht 1 ist (also eine elliptische Kurve), können die Orbiten in dieser Unterklasse durch Äquivalenzklassen holomorpher $G_{\mathbb{C}}$-Bündel auf $M$ klassifiziert werden ([EF], [Br]). Tatsächlich bedeutet die Einschränkung auf die oben beschriebene Unterklasse der koadjungierten Orbiten von $\hat{G}^M_{\mathbb{C}}$ allerdings, daß man sich im wesentlichen auf die koadjungierte Darstellung einer anderen Gruppe einschränkt. Dies ist eine der von Frenkel und Etingof in [EF] eingeführten, komplex eindimensionalen, zentralen Erweiterungen $\tilde{G}^M_{\mathbb{C}}$ von $G^M_{\mathbb{C}}$. Ihr Zentrum ist nun durch die elliptische Kurve $M$ selbst gegeben.

Die Fälle $\dim(M) = 1$ und $\dim(M) = 2$ sind auch aus anderen Gründen besonders interresant und betrachtenswert: Im Fall $M = S^1$ ist die Lie Algebra der Gruppe $\hat{G}^{S^1}$ gerade eine reelle Form der zu $G$ gehörigen ungetwisteten affinen Kac-Moody Algebra (modulo Derivation). Diese Algebren sind die einfachsten unendlichdimensionalen Verallgemeinerungen der endlichdimensionalen halbeinfachen Lie Algebren, und ihre Struktur und Darstellungstheorie verläuft in wesentlichen Teilen parallel zum endlichdimensionalen Fall. Auch die Hauptergebnisse der orbitalen Theorie für kompakte Lie Gruppen lassen sich auf diesen Fall verallgemeinern. Diese Ergebnisse werden im nächsten Abschnitt kurz dargestellt werden.

Der Fall $\dim(M) = 2$ wurde zuerst in der konformen Feldtheorie betrachtet. Hier beschreibt das sogenannte Wess Zumino Witten Modell einen geschlossenen "String", der sich auf einer halbeinfachen kompakten Lie Gruppe $G$ oder ihrer Komplexifizierung $G_{\mathbb{C}}$ bewegt. In der konformen Feldtheorie läßt man auch Mannigfaltigkeiten $M$ zu, die mehrere, jeweils durch $S^1$ parametrisierte Randkomponenten haben können. Das Bild der Mannigfaltigkeit $M$ unter einer Abbildung $g \in G^M$ wird dann als die Fläche interpretiert, die von dem String bei seiner Bewegung in $G$ überstrichen wird. In diesem Modell ist die Wirkung, der die Bewegung des Strings unterliegt, durch die sogenannte Wess Zumino Witten Wirkung $S : G^M \to \mathbb{R}$ gegeben (siehe Kapitel II.4 für die genaue Definition der Wirkung).

Für jede konforme Feldtheorie ist eine Partitionsfunktion definiert, die eine Invariante der Theorie ist. Für das WZW Modell auf einer geschlossenen Riemannschen Fläche $M$ ist die Partitionsfunktion formal durch das Integral

$$Z = \int_{G^M} e^{-S(g)} \mathcal{D}(g) \tag{2}$$

gegeben, wobei $\mathcal{D}$ als "formales Maß" auf der Menge $G^M$ gelesen werden sollte. Das Integral in Gleichung (2) ist ein Beispiel für ein sogenanntes Funktionalintegral. Obwohl Funktionalintegrale häufig in der physikalischen Literatur auftauchen und gerade in der Quantenfeldtheorie wesentlich sind, konnte für sie bis heute noch keine allgemeine Theorie entwickelt werden. In der physikalischen Literatur werden solche Funktionalintegrale oft mithilfe von Methoden "berechnet," die durch ihre Ähnlichkeit mit klassischen Integrationsmethoden gerechtfertigt werden.

Für das WZW Modell auf einer elliptischen Kurve $M$ stellt sich heraus, daß das Wirkungsfunktional $S$ eine Funktion auf einem koadjungierten Orbit der Gruppe $\tilde{G}^M$ definiert. Daher besteht die Hoffnung, daß ein besseres Verständnis der koadjungierten Orbiten von $\tilde{G}^M$ auch zu einem besseren Verständnis der dem WZW Modell und auch allgemeinerer konformer Feldtheorien zugrunde liegenden Mathematik mit sich führt.

## Grundlagen zu Schleifengruppen und affinen Lie Algebren

In diesem Abschnitt wird eine kurze Einführung in die Theorie der affinen Lie Algebren gegeben, und es wird dargestellt, wie sich die orbitale Theorie für kompakte Lie Gruppen auf diesen Fall verallgemeinert. Allerdings wird, anstatt die abstrakte Definition affiner Lie Algebren mittels verallgemeinerter Cartan Matrizen zu wiederholen, direkt die konkrete Realisierung der affinen Lie Algebren via Schleifenalgebren eingeführt. Eine ausführlichere Einführung in die Theorie der affinen Lie Algebren findet sich z.B. in [K].

Wie zuvor sei $G$ eine kompakte, halbeinfache Lie Gruppe mit Lie Algebra $\mathfrak{g}$, und $T \subset G$ sei ein maximaler Torus von $G$ mit Lie Algebra $\mathfrak{h} \subset \mathfrak{g}$. Es seien $\mathfrak{g}_{\mathbb{C}}$ und $\mathfrak{h}_{\mathbb{C}}$ die Komplexifizierungen von $\mathfrak{g}$ und $\mathfrak{h}$, und $\langle .,. \rangle$ bezeichne die Killing Form auf $\mathfrak{g}_{\mathbb{C}}$. Nimmt man zusätzlich an, daß $\mathfrak{g}_{\mathbb{C}}$ eine einfache Lie Algebra ist, so ist die zu $G$ gehörige ungetwistet affine Lie Algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$ durch

$$\tilde{\mathfrak{g}}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

definiert. Der erste Summand kann durch $z^m \mapsto e^{2\pi imt}$ als Menge der polynomialen Abbildungen $S^1 \to \mathfrak{g}_{\mathbb{C}}$ interpretiert werden. (Hier und im folgenden wird $S^1$ mit $\mathbb{R}/\mathbb{Z}$ identifiziert.) Die Lie Klammer auf $\tilde{g}_{\mathbb{C}}$ ist dann durch

$$[x(\cdot) + ac + bd, x_1(\cdot) + a_1c + b_1d] = [x, x_1](\cdot) + bx_1'(\cdot) - b_1x'(\cdot) + \langle x'(\cdot), x_1(\cdot)\rangle c$$

gegeben, wobei $x'(t) = \frac{dx(t)}{dt}$, $[x, x_1](t) = [x(t), x_1(t)]$ und

$$\langle x(\cdot), x_1(\cdot)\rangle = \int_0^1 \langle x(t), x_1(t)\rangle dt$$

gesetzt wurde.

Ein äußerer Automorphismus $\psi$ von $\mathfrak{g}_{\mathbb{C}}$ der Ordnung $r$ definiert einen Automorphismus $\tilde{\psi}$ der affinen Lie Algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$ durch $\tilde{\psi} : x(t) \mapsto \psi(x(t - \frac{1}{r}))$ für alle $t \in S^1$. Zusätzlich setzt man $\tilde{\psi}(d) = d$ und $\tilde{\psi}(c) = c$. Nun ist die zu $G$ und $\psi$ gehörige getwistet affine Lie Algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\tilde{\psi}} \subset \tilde{\mathfrak{g}}_{\mathbb{C}}$ die Fixpunktalgebra unter $\tilde{\psi}$. Hierbei ist zu beachten, daß die zu verschiedenen Automorphismen $\psi$ und $\psi_1$ von $\mathfrak{g}_{\mathbb{C}}$ gehörenden getwistet affinen Lie Algebren durchaus isomorph sein können. Tatsächlich sind $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\tilde{\psi}}$ und $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\tilde{\psi}_1}$ isomorph, sobald $\mathrm{ord}(\psi) = \mathrm{ord}(\psi_1)$ gilt.

Ist nun $\mathfrak{g}_{\mathbb{C}}$ eine einfache Lie Algebra von Typ $X_n$, und ist $\psi$ ein äußerer Automorphismus von $\mathfrak{g}_{\mathbb{C}}$ der Ordnung $\mathrm{ord}(\psi) = r$, so ist $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\tilde{\psi}}$ eine affine Lie Algebra vom Typ $X_n^{(r)}$ (wobei die Notation von [K] benutzt wurde). Die ungetwisteten und getwisteten affinen Lie Algebren erschöpfen zusammen die Menge aller affinen Lie Algebren. Die Struktur- und Darstellungstheorie dieser Lie Algebren ist sehr ähnlich zu der der endlichdimensionalen halbeinfachen Lie Algebren. Dies sei im folgenden kurz am Beispiel der ungetwistet affinen Lie Algebren erläutert:

Es sei wie zuvor $\mathfrak{g}_{\mathbb{C}}$ eine endlichdimensionale einfache Lie Algebra und $\mathfrak{h}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ sei eine Cartansche Unteralgebra. Dann läßt $\mathfrak{g}_{\mathbb{C}}$ eine Zerlegung

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha}$$

zu, wobei $R \subset \mathfrak{h}_{\mathbb{C}}^*$ das Wurzelsysten von $\mathfrak{g}_{\mathbb{C}}$ ist, und und $\mathfrak{h}_{\mathbb{C}}$ diagonal auf den (eindimensionalen) Wurzelräumen $\mathfrak{g}_{\alpha}$ operiert: $[h, X_{\alpha}] = \alpha(h)X_{\alpha}$ für alle $X_{\alpha} \in \mathfrak{g}_{\alpha}$. Nun sei $\langle .,. \rangle$ eine normierte invariante Bilinearform auf $\mathfrak{g}_{\mathbb{C}}$, d.h. ein Vielfaches der Killing-Form, die so normiert sein soll, daß die langen Wurzeln in $R$ Quadratlänge 2 haben (wobei $\mathfrak{h}_{\mathbb{C}}$ und $\mathfrak{h}_{\mathbb{C}}^*$ via $\langle .,. \rangle$ miteinander identifiziert wurden). Nun sei $P \subset \mathfrak{h}_{\mathbb{C}}^*$ das Gewichtsgitter von $R$, also das Gitter $P = \{\lambda \in \mathfrak{h}_{\mathbb{C}}^* \mid \langle \lambda, \alpha \rangle \in \mathbb{Z} \text{ für alle } \alpha \in R\}$. Wählt man nun eine Basis $\alpha_1, \dots, \alpha_l$ von $R$, so existiert eine Bijektion zwischen der Menge der dominanten Gewichte $P_+ = \{\lambda \in P \mid \langle \lambda, \alpha_i \rangle \geq 0 \text{ für alle } i = 1, \dots, l\}$ und der Menge der irreduziblen Darstellungen von $\mathfrak{g}_{\mathbb{C}}$. Bei dieser Bijektion wird jede irreduzible Darstellung von $\mathfrak{g}_{\mathbb{C}}$ auf ihr höchstes Gewicht $\lambda \in P_+$ abgebildet. Die Charaktere der entsprechenden Darstellungen können mittels der Weylschen Charakterformel berechnet werden.

Für die ungetwistet affine Lie Algebra $\tilde{\mathfrak{g}}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ sieht die Situation wie folgt aus: $\tilde{\mathfrak{g}}_{\mathbb{C}}$ hat eine Zerlegung in Wurzelräume

$$\tilde{\mathfrak{g}}_{\mathbb{C}} = (\mathfrak{h}_{\mathbb{C}} \oplus \mathbb{C}c \oplus \mathbb{C}d) \oplus \bigoplus_{\substack{\alpha \in R \\ n \in \mathbb{Z}}} (\mathfrak{g}_{\alpha} \otimes z^n) \oplus \bigoplus_{n \in \mathbb{Z}} (\mathfrak{h}_{\mathbb{C}} \otimes z^n).$$

Definiert man nun ein Element $\delta \in (\mathfrak{h}_{\mathbb{C}} \oplus \mathbb{C}c \oplus \mathbb{C}d)^*$ durch $\delta(h) = \delta(c) = 0$ für alle $h \in \mathfrak{h}_{\mathbb{C}}$, und $\delta(d) = 1$, so schreibt sich die Wurzelraumzerlegung von $\tilde{\mathfrak{g}}_{\mathbb{C}}$ als

$$\tilde{\mathfrak{g}}_{\mathbb{C}} = (\mathfrak{h}_{\mathbb{C}} \oplus \mathbb{C}c \oplus \mathbb{C}d) \oplus \bigoplus_{\tilde{\alpha} \in \tilde{\mathbb{R}}} \mathfrak{g}_{\tilde{\alpha}}.$$

Hierbei ist $\tilde{R} = \{\alpha + n\delta \mid \alpha \in R,\ n \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{Z} \setminus \{0\}\}$ das affine Wurzelsystem von $\tilde{\mathfrak{g}}_{\mathbb{C}}$. Die normierte invariante Bilinearform $\langle .,. \rangle$ auf $\mathfrak{g}_{\mathbb{C}}$ definiert mittels

$$\langle x(\cdot) + ac + bd, x_1(\cdot) + a_1c + b_1d \rangle = \int_0^1 \langle x(t), x_1(t) \rangle dt + ab_1 + ba_1$$

eine invariante Bilinearform auf $\tilde{\mathfrak{g}}_{\mathbb{C}}$, die auf $\mathfrak{h}_{\mathbb{C}} \oplus \mathbb{C}c \oplus \mathbb{C}d$ nicht ausgeartet ist. Nun wählt man wieder eine Basis $\alpha_0, \dots, \alpha_l$ von $\tilde{R}$ und normiert $\langle .,. \rangle$ wieder so, daß die langen Wurzeln von $\tilde{R}$ die Quadratlänge 2 haben. Definiert man nun das Gitter $\tilde{P} \subset (\tilde{\mathfrak{h}}_{\mathbb{C}} \oplus \mathbb{C}c)^*$ durch $\tilde{P} = \{\Lambda \in (\tilde{\mathfrak{h}}_{\mathbb{C}} \oplus \mathbb{C}c)^* \mid \langle \Lambda, \alpha_i \in \mathbb{Z} \text{ für alle } i = 0, \dots, n\}$, so existiert eine Bijektion zwischen der Menge der dominanten Gewichte $\tilde{P}_+ = \{\Lambda \in \tilde{P} \mid \langle \Lambda, \alpha_i \rangle \geq 0 \text{ für alle } i = 0, \dots, n\}$ und der Menge der integrablen irreduziblen

Höchstgewichtsdarstellungen $L(\Lambda)$ von $\tilde{\mathfrak{g}}_{\mathbb{C}}$, auf deren Höchstgewichtsvektor $v_\Lambda \in L(\Lambda)$ das Element $\in \tilde{\mathfrak{g}}_{\mathbb{C}}$ trivial operiert. Analoge Ergebnisse gelten für die getwistet affinen Lie Algebren.

Angesichts der Analogie zwischen der Darstellungstheorie endlichdimensionaler einfacher Lie Algebren und den entsprechenden affinen Lie Algebren ist es eine natürliche Frage, inwieweit sich die Ergebnisse der orbitalen Theorie für kompakte Lie Gruppen auf den affinen Fall übertragen lassen. Da die auftauchenden Räume nun unendlichdimensional sind, wird eine direkte Verallgemeinerung der Methoden der orbitalen Theorie für kompakte Lie Gruppen nicht möglich sein. I. Frenkel hat in [F] eine orbitale Theorie für ungetwistet affine Lie Algebren entwickelt, deren Hauptergebnis ein Analogon zu Kirillovs Charakterformel ist. Diese Theorie soll nun kurz beschrieben werden:

Wie zuvor bezeiche $G^{S^1}$ die Schleifengruppe einer halbeinfachen, einfach zusammenhängenden kompakten Lie Gruppe $G$ (d.h. $G^{S^1} = C^\infty(S^1, G)$). Der klassischen Notation folgend werden wir von nun an $\mathcal{L}G$ für $G^{S^1}$ schreiben (wobei "$\mathcal{L}$" für "Loop" steht). $\hat{G}$ sei die zentrale Erweiterung von $\mathcal{L}G$. Auf $\hat{G}$ existiert eine $S^1$-Operation, die die natürliche $S^1$-Operation auf $\mathcal{L}G$ überlagert. Somit kann man das semidirekte Produkt $\tilde{G} = \hat{G} \rtimes S^1$ bilden. Die Lie Algebra von $\tilde{G}$ ist durch $\tilde{\mathfrak{g}} = \mathcal{L}\mathfrak{g} \oplus \mathbb{R}c \oplus \mathbb{R}d$ gegeben, und ihre Komplexifizierung $\tilde{\mathfrak{g}} \otimes \mathbb{C}$ kann als Vervollständigung der ungetwistet affinen Lie Algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$ bezüglich der Topologie der gleichmäßigen Konvergenz angesehen werden. Somit spielt $\tilde{G}$ die Rolle der kompakten Lie Gruppe in der klassischen orbitalen Theorie. Mittels der invarianten Bilinearform $\langle .,. \rangle$ auf $\tilde{\mathfrak{g}}$, kann $\tilde{\mathfrak{g}}$ in ihr Dual $\tilde{\mathfrak{g}}^*$ eingebettet werden. Das Bild von $\tilde{\mathfrak{g}}$ in $\tilde{\mathfrak{g}}^*$ nennt man den glatten Anteil von $\tilde{\mathfrak{g}}^*$. Auf diesem operiert $\tilde{G}$ mittels der koadjungierten Darstellung. Offensichtlich operiert das Zentrum von $\tilde{G}$ trivial in der koadjungierten Darstellung, und der $S^1$-Anteil von $\tilde{G}$ operiert (nach der Identifizierung des glatten Anteils von $\tilde{\mathfrak{g}}^*$ mit $\tilde{\mathfrak{g}}$) durch "Rotation der Schleifen". Somit läßt sich die koadjungierte Operation von $\tilde{G}$ auf $\tilde{\mathfrak{g}}$ im wesentlichen auf eine Operation von $\mathcal{L}G$ auf $\mathcal{L}\mathfrak{g} \oplus \mathbb{R}$ zurückführen (siehe §I.3 für mehr Details). Betrachtet man nun $G$ als Matrixgruppe (d.h. fixiert man die Wahl einer treuen Darstellung von $G$), so ist die affine koadjungierte Operation von $\mathcal{L}G$ auf $\mathcal{L}\mathfrak{g} \oplus \mathbb{R}$ durch

$$\begin{aligned} &\mathcal{L}G \times \mathcal{L}\mathfrak{g} \oplus \mathbb{R} \to \mathcal{L}\mathfrak{g} \oplus \mathbb{R}, \\ &\quad (\gamma, (X, a)) \mapsto (\gamma X \gamma^{-1} - a\gamma'\gamma^{-1}, a) \end{aligned}$$

gegeben. I. Frenkel und G. Segal haben unabhängig voneinander beobachtet, daß für $X \in \mathcal{L}\mathfrak{g}$ und $\gamma \in \mathcal{L}G$ die Monodromien der gewöhnlichen Differentialgleichungen

$$\begin{aligned} z_1' &= a z_1 x \qquad \text{und} \\ z_2' &= a z_2 (\gamma X \gamma^{-1} - \gamma'\gamma^{-1}) \end{aligned}$$

konjugiert sind. D.h. bezeichnen $z_{(X,a)}$, $z_{(\gamma X\gamma^{-1}-\gamma'\gamma^{-1},a)} : \mathbb{R}_{\geq 0} \to G$ die Fundamentallösungen der entsprechenden gewöhnlichen Differentialgleichungen, so gilt

$$z_{(\gamma X\gamma^{-1}-\gamma'\gamma^{-1},a)}(1) = \gamma(0)z_{(X,b)}(1)\gamma(0)^{-1}.$$

Diese Beobachtung erlaubt es für $a \neq 0$, eine Bijektion zwischen der Menge der $\mathcal{L}G$-Orbiten in $\mathcal{L}\mathfrak{g} \times \{a\}$ und der Menge der Konjugationsklassen in $G$ zu konstruieren.

Frenkel hat diese Beobachtung benutzt, um ein Analogon zu Kirillovs Charakterformel herzuleiten: Es sei $\mu \in (\tilde{\mathfrak{g}})^*$ ein Element des glatten Anteils von $(\tilde{\mathfrak{g}})^*$. Mittels der oben angedeuteten Identifizierungen definiert $\mu$ ein Element $(X,a)_\mu \in \mathcal{L}\mathfrak{g} \oplus \mathbb{R}$. Es bezeichne $\mathcal{O}_\mu \subset \mathcal{L}\mathfrak{g} \oplus \mathbb{R}$ den $\mathcal{L}G$-Orbit, der $(X,a)_\mu$ enthält. Bezeichnet außerdem $\mathcal{O}_g$ die Konjugationsklasse in $G$, die $g \in G$ enthält, so definiert die Abbildung $(X,a) \mapsto z_{(X,a)}$ eine Einbettung

$$\phi : \mathcal{O}_\mu \to \{y \in C^\infty([0,1],G) \mid y(0) = e,\ y(1) \in \mathcal{O}_{z_{(X,a)}(1)}\}.$$

Die Menge $\{y \in C^\infty([0,1],G) \mid y(0) = e,\ y(1) \in \mathcal{O}_{z_{(X,a)}(1)}\}$ läßt sich auf natürliche Weise in

$$\bar{\mathcal{O}}_\mu = \{y \in C([0,1],G) \mid y(0) = e,\ y(1) \in \mathcal{O}_{z_{(X,a)}(1)}\}$$

einbetten. Nun ist $\bar{\mathcal{O}}_\mu$ eine abgeschlossene Teilmenge von $(\tilde{\mathfrak{g}})^*_0$, wobei $(\tilde{\mathfrak{g}})^*_0 \subset (\tilde{\mathfrak{g}})^*$ den sogenannten "stetigen Anteil" von $(\tilde{\mathfrak{g}})^*$ bezeichnet. Mithilfe dieser Einbettungen kann man $\bar{\mathcal{O}}_\mu$ als den Abschluß von $\mathcal{O}_\mu$ in $(\tilde{\mathfrak{g}})^*_0$ betrachten. (Siehe §I.4.5 für eine genauere Beschreibung der Einbettungen).

Auf $\bar{\mathcal{O}}_\mu$ existiert ein natürliches Maß, das sogenannte bedingte Wiener-Maß $\varpi_\mu$. Ist nun $\Lambda \in \tilde{P}$ ein höchstes Gewicht von $\tilde{\mathfrak{g}}_{\mathbb{C}}$, und $\tilde{\rho}$ das affine Analogon zum zuvor definierten Element $\rho$ (siehe §4.1 für eine genaue Definition) so hat Frenkels Charakterformel die Form

$$\chi_\Lambda(\exp(bd+K)) = p^{-1}(bd+K)\int_{\mathcal{O}_{\Lambda+\tilde{\rho}}} f_{bd+K}(z)d\varpi_{\Lambda+\tilde{\rho}}(z). \tag{3}$$

Hierbei ist $f_{bd+K} : \bar{\mathcal{O}}_{\Lambda+\tilde{\rho}} \to \mathbb{C}$ eine Funktion auf $\bar{\mathcal{O}}_{\Lambda+\tilde{\rho}}$, und $p : \mathfrak{h} \oplus \mathbb{R}d \to \mathbb{C}$ ist eine universelle Funktion, die nicht von $\Lambda$ abhängt. Für die genaue Definition der Funktionen $f_{bd+K}$ und $p$ sei auch hier wieder auf §I.4.5 (bzw. [F], §7.2) verwiesen. Bei Formel (3) ist zu beachten, daß der Charakter $\chi_\Lambda(\exp(bd+K))$ nach Definition im allgemeinen eine unendliche Summe ist, die nur für bestimmte Werte von $b$ konvergiert (vgl. §I.4.1). Formel (3) macht also nur für diese Werte Sinn. Die Tatsache, daß der Charakter $\chi_\Lambda$ in Formel (3) nur auf $\mathfrak{h} \oplus \mathbb{R}b$ ausgewertet wird und die zentrale Variable "vergessen" wird, ist keine wirkliche Einschränkung, da das Zentrum von $\tilde{\mathfrak{g}}$ auf einer irreduziblen Darstellung nach Schurs Lemma durch Skalarmultiplikation operiert. Für eine integrable Höchstgewichtsdarstellung von $\tilde{\mathfrak{g}}_{\mathbb{C}}$ mit höchstem Gewicht $\Lambda \in \tilde{P}_+$ ist dieser Skalar gerade durch das sogenannte Niveau ("Level") von $\Lambda$ gegeben.

Insgesamt ist Formel (3) also ein nahezu perfektes Analogon zur klassischen Kirillovschen Charakterformel (1) für kompakte Lie Gruppen. Wie auch im Falle kompakter Gruppen impliziert Formel (3) eine Korrespondenz zwischen einer Teilmenge der Menge der koadjungierten Orbiten von $\tilde{G}$ und den integrablen Höchstgewichtsdarstellungen der zu $\tilde{G}$ gehörigen Lie Algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$.

## Zu dieser Arbeit

Die vorliegende Arbeit beschäftigt sich mit zwei Ansätzen zur orbitalen Theorie für Schleifengruppen. Im ersten Teil der Arbeit ("Weyl's character formula for non-connected Lie groups and orbital theory for twisted affine Lie algebras") wird Frenkels orbitale Theorie für ungetwistet affine Lie Algebren auf den Fall getwistet affiner Lie Algebren verallgemeinert: Für einen äußeren Automorphismus $\psi$ von $G$ der Ordnung $\mathrm{ord}(\psi) = r$ sei $\mathcal{L}(G,\psi)$ die getwistete Schleifengruppe. D.h. $\mathcal{L}(G,\psi) = \{\gamma \in \mathcal{L}G \mid \psi(\gamma(t)) = \gamma(t+\frac{1}{r})$ für alle $t \in S^1\}$. Wie im Falle ungetwisteter Schleifengruppen betrachtet man wieder eine zentrale Erweiterung

$$1 \to S^1 \to \hat{G}^\psi \to \mathcal{L}(G,\psi) \to 1$$

und das semidirekte Produkt $\tilde{G}^\psi = \hat{G}^\psi \rtimes S^1$. Die Lie Algebra von $\tilde{G}^\psi$ ist durch $\tilde{\mathfrak{g}}^\psi = \mathcal{L}(\mathfrak{g},\psi) \oplus \mathbb{R}c \oplus \mathbb{R}d$ gegeben, wobei $\mathcal{L}(\mathfrak{g},\psi)$ die getwistete Schleifenalgebra von $\mathfrak{g}$ ist. Die Komplexifizierung $\tilde{\mathfrak{g}}^\psi \otimes \mathbb{C}$ kann wieder als Vervollständigung der getwistet affinen Lie Algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}^\psi$ betrachtet werden. Die koadjungierte Darstellung von $\tilde{G}^\psi$ auf dem glatten Anteil des Duals von $\tilde{\mathfrak{g}}^\psi$ führt sich im wesentlichen wieder auf eine Operation von $\mathcal{L}(G,\psi)$ auf $\mathcal{L}(\mathfrak{g},\psi) \oplus \mathbb{R}$ zurück. Diese Operation ist durch Einschränkung der entsprechenden Operation von $\mathcal{L}G$ auf $\mathcal{L}\mathfrak{g} \oplus \mathbb{R}$ gegeben. Im Gegensatz zum Fall ungetwisteter Schleifengruppen ist die Monodromieabbildung von der Menge der $\mathcal{L}(G,\psi)$-Orbiten in $\mathcal{L}(\mathfrak{g},\psi) \times \{a\}$ in die Menge der Konjugationsklassen von $G$ in diesem Fall aber weder surjektiv noch injektiv (vgl. [Kl]). In Kapitel I.3 wird gezeigt, daß man hier die Monodromie durch die $\frac{1}{r}$-te Monodromie der entsprechenden Differentialgleichung ersetzen muß, um eine Klassifikation der Orbiten zu erhalten:

Es sei $(X,a) \in \mathcal{L}(\mathfrak{g},\psi) \oplus \mathbb{R}$. Wie im Falle ungetwisteter Schleifengruppen betrachten wir die Lösung der gewöhnlichen Differentialgleichung $z'_{(X,a)} = a z_{(X,a)} X$ mit Anfangsbedingung $z_{(X,a)}(0) = e$. Ist nun $\gamma \in \mathcal{L}(G,\psi)$, und ist

$$z_{(\gamma X \gamma^{-1} - \gamma' \gamma^{-1}, a)} : \mathbb{R}_{\geq 0} \to G$$

die Lösung der Differentialgleichung

$$z'_{(\gamma X \gamma^{-1} - \gamma' \gamma^{-1}, a)} = a z_{(\gamma X \gamma^{-1} - \gamma' \gamma^{-1}, a)} (\gamma X \gamma^{-1} - \gamma' \gamma^{-1}),$$

so sieht man leicht, daß $z_{(\gamma X \gamma^{-1} - \gamma' \gamma^{-1}, a)}(\frac{1}{r}) = \gamma(0) z_{(X,a)}(\frac{1}{r}) \psi(\gamma(0)^{-1})$ gilt. Ähnliche Argumente wie bei Frenkel [F] und Segal [PS] liefern nun eine Bijektion zwischen der Menge der $\mathcal{L}(G,\psi)$-Orbiten in $\mathcal{L}(\mathfrak{g},\psi)\times\{a\}$ für $a\neq 0$ und der Menge der "$\psi$-getwisteten" Konjugationsklassen in $G$ (oder äquivalent, den $G$-Konjugationsklassen in der Zusammenhangskomponente $G\psi$ der nicht-zusammenhängenden kompakten Lie Gruppe $G\rtimes\langle\psi\rangle$, die $\psi$ enthält). Es bezeichne $\mathcal{O}_{g\psi}$ die $G$-Konjugationsklasse in $G\psi$, die $g\psi$ enthält. Analog zum ungetwisteten Fall können wir nun den $\mathcal{L}(G,\psi)$-Orbit in $\mathcal{L}(\mathfrak{g},\psi)\oplus\mathbb{R}$, der $(X,a)$ enthält, durch $(X,a)\mapsto z_{(X,a)}\psi$ in die Menge $\bar{\mathcal{O}}(X,a)=\{y\in C([0,\frac{1}{r}],G\psi)\mid y(0)=\psi,\ y(\frac{1}{r})\in\mathcal{O}_{z_{(X,a)}(\frac{1}{r})}\}$ einbetten.

Auf der Menge $\bar{\mathcal{O}}(X,a)$ existiert wieder ein natürliches Maß, nämlich das bedingte Wiener-Maß. Mit dessen Hilfe können wir eine Formel für die Charaktere der irreduziblen integrablen Höchstgewichtsmoduln der getwistet affinen Lie Algera $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\psi}$ herleiten, die eine Verallgemeinerung von Formel (3) darstellt (siehe Satz I.4.9). Die verallgemeinerte Formel gibt wieder eine Korrespondenz zwischen einer Teilmenge der Menge der koadjungierten Orbiten von $\tilde{G}^{\psi}$ und der Menge der irreduziblen integrablen Höchstgewichtsmoduln der getwistet affinen Lie Algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\psi}$ und kann somit als Analogon der klassischen Kirillovschen Charakterformel angesehen werden. In der Herleitung der Charakterformel spielen die irreduziblen Charaktere der nicht-zusammenhängenden kompakten Lie Gruppe $G\rtimes\langle\psi\rangle$ eine wesentliche Rolle. Daher werden in Kapitel I.2 die Ergebnisse der in [W1] entwickelten Charaktertheorie für diese Gruppen zusammengetragen. An dieser Stelle sei erwähnt, daß M. Kleinfeld [Kl] bereits einen ersten Schritt zur Verallgemeinerung von Frenkels Charakterformel auf den Fall getwistet affiner Lie Algebren geleistet hat. Mangels einer Klassifikation der koadjungierten Orbiten der zugrundeliegenden getwisteten Schleifengruppe kann seine Formel allerdings nicht als "orbitales Integral", d.h. als Integral über den Abschluß eines koadjungierten Orbits von $\tilde{G}^{\psi}$, interpretiert werden.

Im zweiten Teil der Arbeit ("A symplectic approach to certain functional integrals and partition functions") wird ein neuer Ansatz zur orbitalen Theorie affiner Lie Algebren entwickelt. Ziel ist wieder, ein Analogon zu Kirillovs Charakterformel für kompakte Lie Gruppen herzuleiten. Allerdings soll nun, anstatt über einen Abschluß $\bar{\mathcal{O}}_{\mu}$ eines koadjungierten Orbits $\mathcal{O}_{\mu}$ von $\tilde{G}$, bzw. $\tilde{G}^{\psi}$ zu integrieren (wie bei [F] und im ersten Teil der Arbeit), die Integration über den Orbit $\mathcal{O}_{\mu}$ selbst erfolgen. Das wesentliche Problem hierbei ist, daß auf $\mathcal{O}_{\mu}$ bisher keine Maßtheorie entwickelt wurde. Um dieses Problem zu umgehen, werden Ideen benutzt, die in der physikalischen Literatur entwickelt wurden (siehe z.B. [A], [P]): Die Existenz der kanonischen symplektischen Struktur auf $\mathcal{O}_{\mu}$ wird ausgenutzt, um ein formales Analogon zur Integration bezüglich des Liouville-Maßes auf endlichdimensionalen symplektischen Mannigfaltigkeiten zu definieren.

Ist $(M,\omega)$ eine endlichdimensionale symplektische Mannigfaltigkeit der Dimension $\dim(M)=2n$, so ist das Liouville-Maß auf $M$ durch die Volumenform $\frac{\omega^n}{n!}$ gegeben. Ist weiterhin $M$ kompakt, und ist $T\times M\to M$ eine symplektische Torusoperation auf $M$ mit diskreter Fixpunktmenge $P$, so reduziert für eine Hamiltonsche

Funktion $J_H$ dieser Torusoperation die Duistermaat-Heckmansche Integralformel das Integral

$$\int_M e^{-tJ_H} \frac{\omega^n}{n!}$$

zu einer (endlichen) Summe über die Fixpunkte der $T$-Operation auf $M$ (siehe §II.2.1 für eine genaue Formulierung der Duistermaat-Heckmanschen Integralformel).

Falls nun $M$ unendlichdimensional ist, macht das Liouville-Maß keinen Sinn. In diesem Fall gehen wir umgekehrt vor und benutzen ein Analogon der Duistermaat-Heckmannschen Integralformel um ein Funktional auf der Menge der Funktionen auf $M$ zu definieren, die Hamiltonsch bezüglich einer symplektischen Torusoperation auf $M$ sind. Aufgrund seiner formalen Ähnlichkeit mit dem Liouville-Maß wird das oben erwähnte Funktional das "Liouville-Funktional" genannt und mit $L$ bezeichnet.

Die koadjungierten Orbiten $\mathcal{O}_\mu$ der zentral erweiterten Schleifengruppen $\tilde{G}$ und $\tilde{G}^\psi$ tragen eine natürliche symplektische Struktur, die Kirillov/Kostant-Form. Bezüglich dieser Form operieren ihre maximalen Tori symplektisch und mit einer diskreten Fixpunktmenge auf ihnen. Daher läßt sich das Liouville-Funktional auf die Hamiltonschen Funktionen dieser Torusaktion anwenden. Mithilfe des Liouville-Funktionals definieren wir ein zweites Funktional auf der Menge der Funktionen, die Hamiltonsch bezüglich der oben beschriebenen Torusoperation sind. Dieses Funktional ist ein Analogon zur Integration über die Riemannsche Volumenform auf einer endlichdimensionalen symplektischen Mannigfaltigkeit. Daher wird es als "formale Integration bezüglich der Riemannschen Volumenform" bezeichnet.

Mittels dieser formalen Integration können wir nun einige Funktionen auf den koadjungierten Orbiten der Gruppen $\tilde{G}$ und $\tilde{G}^\psi$ integrieren. Insbesondere können wir in diesem Rahmen ein formales Analogon zur Kirillovschen Charakterformel herleiten, die wieder eine Korrespondenz zwischen einer Menge der koadjungierten Orbiten von $\tilde{G}$ (bzw. $\tilde{G}^\psi$) und der Menge der integrablen Höchstgewichtsmoduln von $\tilde{\mathfrak{g}}_{\mathbb{C}}$ (bzw. $\tilde{\mathfrak{g}}_{\mathbb{C}}^\psi$) impliziert. Dies ist die gleiche Korrespondenz, die sich aus Frenkels orbitaler Theorie ergibt. Tatsächlich kann gezeigt werden, daß die formale Riemannsche Volumenform auf dem koadjungierten Orbit $\mathcal{O}_\mu$ im wesentlichen der "Pullback" des Wiener-Maßes auf dem Abschluß $\bar{\mathcal{O}}_\mu$ ist (vgl. §II3.4).

Von diesem Standpunkt aus ergibt der Ansatz zur orbitalen Theorie affiner Lie Algebren via formaler Integration also nichts wirklich Neues. Der Vorteil der formalen Integration mittels des Liouville-Funktionals oder der formalen Riemannschen Volumenform ist, daß sie auch in Situationen angewendet werden kann, in denen kein natürliches Maß existiert. Als Beispiel für eine solche Situation berechnen wir in Kapitel II.4 die Partitionsfunktion des geeichten WZW Modells auf einer elliptischen Kurve $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ (wobei $\tau \in \mathbb{C}$, $im(\tau) > 0$ der elliptische Parameter der Kurve $\Sigma_\tau$ ist). Die Wirkung $S_{H,\tau} : G_{\mathbb{C}}^{\Sigma_\tau}/T_{\mathbb{C}} \to \mathbb{C}$ des geeichten WZW Modells hängt vom modularen Parameter $\tau$ der elliptischen Kurve $\Sigma_\tau$ und von einem Element $H \in \mathfrak{h}$ ab. Hierbei ist $T$ wieder ein maximaler Torus von $G$ mit Lie Algebra $\mathfrak{h}$, und $T_{\mathbb{C}} \subset G_{\mathbb{C}}$ ist seine Komplexifizierung. Wie oben beschrieben ist für $\kappa \in \mathbb{Z}$ die Partitionsfunktion des geeichten WZW Modells im Level $\kappa$ formal als Integral

$$\int_{G_{\mathbb{C}}^{\Sigma_\tau}/T_{\mathbb{C}}} e^{-\kappa S_{H,\tau}}$$

gegeben. Nach Ergebnissen von Frenkel und Etingof [EF] kann $G_{\mathbb{C}}^{\Sigma_\tau}/T_{\mathbb{C}}$ mit einem generischen koadjungierten Orbit vin $\tilde{G}^{\Sigma_\tau}$ identifiziert werden. Somit existiert eine (komplexwertige) symplektische Form auf $G_{\mathbb{C}}^{\Sigma_\tau}/T_{\mathbb{C}}$. Außerdem operiert der Torus $S^1 \times S^1 \times T$ auf $G_{\mathbb{C}}^{\Sigma_\tau}/T_{\mathbb{C}}$, wobei die ersten beiden Faktoren durch Rotation der Schleifen operieren und $T$ durch Linksmultiplikation. Es wird gezeigt, daß die Wirkung $S_H$ Hamiltonsch bezüglich dieser Torusoperation ist. Somit läßt sich der Formalismus des Liouville-Funktionals (bzw. der formalen Integration über die Riemannsche Volumenform) auf diese Situation anwenden und die Partitionsfunktion des geeichten WZW Modells berechnen. Es sei $h^\vee$ die duale Coxeter-Zahl von $G$, und $k \in \mathbb{N}$ sei beliebig. Dann gilt

$$\int_{G_{\mathbb{C}}^{S^1\times S^1}/T_{\mathbb{C}}} e^{-(k+h^\vee)S_H} = c \sum_{\lambda \in \tilde{P}_+^k} |\chi_\lambda(\tau, H)|^2, \tag{4}$$

wobei mit dem Integralzeichen die formale Integration bezüglich der Riemannschen Volumenform gemeint ist. Außerdem bezeichnet $\tilde{P}_+^k$ die Menge der dominanten Gewichte von Level $k$ der zu $G$ gehörigen ungetwistet affinen Lie Algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$, $\chi_\lambda$ ist der entsprechende (normierte) Charakter ([K], Ch.12), und $c \in \mathbb{R}$ ist eine Konstante.

In [KP] wurde mit rein darstellungstheoretischen Methoden gezeigt, daß die Summe auf der rechten Seite von Formel (4) invariant unter einer gewissen Operation von $SL(2,\mathbb{Z})$ auf dem Raum der $(\tau, H)$ ist. Formel (4) erklärt das Erscheinen dieser $SL(2,\mathbb{Z})$ Operation von einem geometrischen Standpunkt: $SL(2,\mathbb{Z})$ operiert auf natürliche Weise auf dem modularen Parameter der elliptischen Kurve, auf der das WZW Modell definiert ist. Die komplexe Struktur der elliptischen Kurve ist unter dieser $SL(2,\mathbb{Z})$ Operation invariant.

Die Arbeit schließt mit einer Bemerkung über eine getwistete Version des WZW Modells und einigen Spekulationen über eine mögliche maßtheoretische Interpretation des Integrals in Gleichung (4). Bei der Berechnung der Partitionsfunktion des getwisteten WZW Modells tauchen "Dualitäten" von Wurzelsystemen auf, die ähnlich auch bei der Berechnung von Charakteren gewisser nicht-zusammenhängender Lie Gruppen in Erscheinung treten (vgl.[W1] und Teil I der vorliegenden Arbeit).

Die beiden Teile der vorliegenden Arbeit sind jeweils in sich abgeschlossen und können unabhängig voneinander gelesen werden. Dies hat zur Folge, daß einige Überschneidungen nicht ausgeschlossen werden konnten. Zusätzlich ist eine Bemerkung zu den verwendeten Notationen notwendig: In Teil I der Arbeit bezeichnet $\tau$ einen äußeren Automorphismus einer endlichdimensionalen kompakten Lie

Gruppe $G$. Diese Notation wurde aus [W1] und [Mo] übernommen. In der Theorie der elliptischen Kurven steht $\tau$ normalerweise für den modularen Parameter einer elliptischen Kurve $\Sigma_\tau$, also für eine komplexe Zahl. Da im zweiten Teil der vorliegenden Arbeit elliptische Kurven eine wesentliche Rolle spielen, wurde diese Tradition übernommen. Wie in der Einleitung werden äußere Automorphismen, die in Teil II nur selten vorkommen, mit $\psi$ bezeichnet.

## Danksagung

Die vorliegende Arbeit enstand während meiner Tätigkeit als wissenschaftlicher Mitarbeiter am Fachbereich Mathematik der Universität Hamburg. Ich möchte den Mitgliedern des Schwerpunkts Algebra und Zahlentheorie, insbesondere Herrn Berndt, für ihr Interesse an meiner Arbeit und für die angenehme Atmosphäre in der Arbeitsgruppe danken. Meine Stelle wurde durch Drittmittel der Deutschen Forschungsgemeinschaft finanziert, bei der ich mich dafür bedanken möchte. Der zweite Teil der Arbeit entstand im wesentlichen während eines ausgedehnten Aufenthaltes an der Yale University im akademischen Jahr 1999/2000. Mein Dank gilt meinen Gastgebern in Yale, insbesondere Prof. I. Frenkel. Er hat mich auf das im zweiten Teil der Arbeit behandelte Thema aufmerksam gemacht, und die vielen hilfreichen Gespräche mit ihm haben zu meinem Verständnis beigetragen. Mein größter Dank gilt meinem Doktorvater, Prof. P. Slodowy, auf dessen Anregung der erste Teil dieser Arbeit beruht. Unter seiner Anleitung - oft aus großer Ferne - habe ich mich stets wohlbehütet gefühlt, und die Diskussionen mit ihm haben meine Einsicht in die Problematik meiner Arbeit und meine Freude an der Mathematik immer wieder vertieft.

Zu guter Letzt möchte ich mich bei meiner Freundin Britt für ihre Geduld bedanken ...

# Chapter I

# Weyl's character formula for non-connected Lie groups and Orbital theory for twisted affine Lie algebras

## 1 Introduction

In [F], Frenkel develops a generalization of Kirillov's orbit theory for finite dimensional Lie groups [K2] in the case of untwisted affine Lie algebras. In particular, he classifies affine (co-)adjoint orbits of the underlying loop groups in terms of conjugacy classes of connected compact Lie groups. Using this classification and the theory of Wiener Integration on compact Lie groups, Frenkel obtains an interpretation of the character of a highest weight representation as an integral over an associated orbit in the coadjoint representation.

The aim of this paper is to generalize Frenkel's orbital theory to the case of twisted affine Lie algebras. Kleinfeld ([Kl]) already made a first step towards the adaptation of Frenkel's theory to this more general case, but he was unable to interpret the character formula as an orbital integral. In order to obtain an interpretation of the formulas as orbital integrals we shall introduce certain non-connected compact Lie groups, so called principal extensions, into the geometrical picture. It turns out that the affine orbits of the adjoint representation of a twisted loop group are parametrized by the conjugacy classes in the "outer component" of such a group. With this result and a character formula for non-connected Lie groups at hand we are able to translate Frenkel's program to the twisted case. I now give a brief description of the contents of this paper.

§2 contains some facts about non-connected compact Lie groups, some of them well known. After constructing the principal extension $\tilde{G}$ of a semisimple compact group $G$, we derive an analogue of the Weyl character formula for the connected components of $\tilde{G}$ not containing the identity. The proof of this formula involves an analogue of the Weyl integral formula for $G$-invariant functions on $\tilde{G}$, and the character itself is governed by the dual of a certain "folded" root system. That is, if $R$ is the root system of $G$ and $\tau$ is a diagram automorphism of $R$ then $\tau$ acts as an outer automorphism on $G$, and the characters on the connected components of $\tilde{G}$ are governed by the root system $R^{\tau\vee}$, the dual to the "folded" root system $R^\tau$ of the fixed point group $G^\tau$. It is worthwhile to note that no group belonging to the root system $R^{\tau\vee}$ can, in general at least, be realized as a subgroup of $G$ in contrast to $G^\tau$. The group $G$ acting on $\tilde{G}$ by conjugation, we shall view each connected component of $\tilde{G}$ as a $G$-manifold. As another direct application of the integral formula, we compute the radial component of the Laplacian on the connected components of $\tilde{G}$ with respect to that $G$-action.

In §3, we study affine orbits of the adjoint representation of a twisted loop group $\mathcal{L}(G,\tau)$. By a slight alteration of Frenkel's original methods we see that for $G$ compact, every such orbit contains a constant loop and the orbits in certain affine "shells" are parametrized by the $G$-orbits in the connected component of $\tilde{G}$ containing $\tau$. To be more precise and using different terminology, Frenkel regards a loop into the Lie algebra $\mathfrak{g}$ as a connection on a principal, trivial fibre bundle over the circle $S^1$ with structure group $G$ and he associates to this loop the monodromy of this connection which is an element of $G$. The action of $\mathcal{L}(G)$ on an affine shell in the affine Lie algebra is then given by gauge transformations and it is compatible with the $G$-conjugation on the monodromies in $G$. That way he obtains a well defined bijection from the adjoint orbits of $\mathcal{L}(G)$ (of some fixed affine shell) to the set of conjugacy classes in $G$. In the case of a twisted loop group $\mathcal{L}(G,\tau)$ where $\tau$ is a diagram automorphism of $G$ the corresponding map will neither be surjective nor injective (cf. [Kl]). In this case, it is appropriate to replace the monodromy with the "$\frac{1}{r}$-th monodromy" (i.e. "monodromy" after $\frac{1}{r}$-th of the full circle) multiplied by $\tau$. Here $r$ is the order of $\tau$. The gauge action of $\mathcal{L}(G,\tau)$ is then compatible with conjugation in the component $G\tau$, and we are able to classify the affine adjoint orbits of $\mathcal{L}(G,\tau)$ in terms of conjugacy classes in $G\tau$.

In §4, we use the theory of Wiener integration on a non-connected compact Lie group to rewrite the irreducible highest weight characters of a twisted affine Lie algebra as an integral over the space of paths inside the connected component of $\tilde{G}$ containing the element $\tau$. After a brief summary of some results about affine Lie algebras and their representations in §4.1, we shall show in §4.2 and in §4.3 how the irreducible highest weight characters of a twisted affine Lie algebra are linked to the fundamental solution of the heat equation on a non-connected compact Lie group. At this point, the characters on the connected component $G\tau$ enter the picture. The fundamental solution of the heat equation is used in §4.4 to define the Wiener measure on the space of paths in the connected component of a compact Lie group.

Computing a certain integral with respect to this measure, we can rewrite the affine characters as an integral over a path space. In §4.5 we then show, adopting the original procedure of [F], how this integral can be interpreted as an integral over a coadjoint orbit of the corresponding twisted loop group thus completing Frenkel's program for twisted affine algebras as well.

# 2 Integration and character formulas for non-connected Lie groups

## 2.1 Principal extensions and conjugacy classes

Let $G$ be a simply connected semisimple compact Lie group, $T$ a maximal torus in $G$, and $R$ the root system of $G$ with respect to $T$. If $W$ is the Weyl group of $R$ (and $G$) and $\Gamma$ is the group of diagram automorphisms of the Dynkin diagram of $G$ then we have $Aut(R) = W \rtimes \Gamma$. Every $\tau \in \Gamma$ can be lifted to an automorphism of $G$ in the following way. Let $\Pi$ be a basis of $R$. If $\mathfrak{g}$ is the Lie algebra of $G$ and $\mathfrak{g}_{\mathbb{C}}$ its complexification, we choose a set $\{e_\alpha, f_\alpha, h_\alpha\}_{\alpha\in\Pi}$ of Chevalley generators of $\mathfrak{g}_{\mathbb{C}}$ and set $\tau(e_\alpha) = e_{\tau(\alpha)}$. This extends to a Lie algebra automorphism of $\mathfrak{g}_{\mathbb{C}}$ which leaves $\mathfrak{g}$ invariant. Since we chose $G$ to be simply connected, $\tau$ can again be lifted to an automorphism of $G$ leaving $T$ invariant. Thus we have defined a homomorphism $\varphi : \Gamma \to Aut(G)$, and we can set $\tilde{G} = G \rtimes_\varphi \Gamma$, and call $\tilde{G}$ the principal extension of $G$. Obviously we have $\tilde{G}/G = \Gamma$.

If $G$ is not simply connected we have $G = \bar{G}/K$, where $\bar{G}$ denotes the universal covering group of $G$, and $K$ is some subgroup of the center of $\bar{G}$. Let $\Gamma_K$ be the subgroup of $\Gamma$ which leaves $K$ fixed. Then the group $G \rtimes_\varphi \Gamma_K$ is called the principal extension of $G$. The principal extensions of compact Lie groups are, in general, central extensions of the automorphism groups of the compact groups and play a crucial role in the structure theory of the non-connected compact groups (cf. [dS]).

$G$ acts on the components of $\tilde{G}$ by conjugation. The orbits of this action on the component $G\tau$, the connected component of $\tilde{G}$ containing $\tau$, are parametrized by a component of the space $S/W(S)$, where $S$ is a Cartan subgroup of $\tilde{G}$ in the sense of [BtD] such that $\tau \in S$. That is in our cases $S = T^\tau \times \langle\tau\rangle$ where $T^\tau$ is the connected component containing the identity of the $\tau$-invariant part of the maximal torus $T$. The group $W(S) = N(S)/S$ is a finite group and is called the Weyl group belonging to $S$. In particular, if $S_0 = T^\tau$ is the connected component of $S$ containing $e$, then every element of $G\tau$ is conjugate under $G$ to an element of $S_0\tau$, and two elements of $S_0\tau$ are conjugate under $G$ if and only if they are conjugate in $N(S)$. Furthermore, $S_0$ is regular in $G$, i.e. there is a unique $\tau$-invariant maximal torus of $G$ containing $S_0$, cf. e.g. [BtD].

## 2.2 A 'Weyl integral formula' for $G\tau$

Let $G$ be a connected semisimple Lie group of type $\mathrm{A}_n$, $\mathrm{D}_n$, or $\mathrm{E}_6$ and $\tilde{G}$ its principal extension. Since the other connected Dynkin diagrams do not admit any diagram automorphisms, the principal extensions of the corresponding compact Lie groups are trivial. We want to derive an analogue of the Weyl integral formula for $G$-invariant functions on the component $G\tau$ of $\tilde{G}$. Let $S \subset \tilde{G}$ be a Cartan subgroup of $G$ containing $\tau$ such that $S/S_0$ is generated by $S_0\tau$, and let $(.,.)$ be the negative of the Killing form on $\mathfrak{g}$. Since $G$ is semisimple, this gives an $\mathrm{Ad}(G)$ invariant scalar product on $\mathfrak{g}$, and $\mathfrak{g}$ decomposes into a direct sum $\mathfrak{g} = LT \oplus L(G/T)$, where $LT$ is the Lie algebra of $T$ and $L(G/T)$ its orthogonal complement. In the same way, $LT$ decomposes into $LT = LS_0 \oplus L(T/S_0)$, hence we have $\mathfrak{g} = LS_0 \oplus L(T/S_0) \oplus L(G/T)$.

There are normalized left invariant volume forms $d(gS_0)$, $ds$, and $dg$ on $G/S_0$, $S_0$ and $G$ which are unique up to sign (i.e. orientation). The form $dg$ defines a volume form on $G\tau$ by right translation with $\tau$, which will be called $dg$ as well. Hence, on $G$, we have $|\Gamma_K| \cdot d\tilde{g} = dg$, where $d\tilde{g}$ is the normalized left invariant volume form on $\tilde{G}$.

The projection $\pi : G \to G/S_0$ induces a map $D\pi : \mathfrak{g} \to T_{eS_0}G/S_0$ which maps $L(G/S_0) := L(G/T) \oplus L(T/S_0)$ isomorphically to the tangent space of $G/S_0$ at the point $eS_0$. Hence we can identify these spaces via $D\pi$. Let $n = \dim G$ and $k = \dim S_0$. Then $\pi^*d(gS_0)$ is a left invariant $(n-k)$-form on $G$. Using the $k$-form $ds_e \in \mathrm{Alt}^k(LS_0)$ we get $\mathrm{pr}_2^*ds_e \in \mathrm{Alt}^k\mathfrak{g}$ where $\mathrm{pr}_2 : \mathfrak{g} = L(G/S_0) \oplus LS_0 \to LS_0$ is the second projection. The form $\mathrm{pr}_2^*ds_e$ defines a left-invariant $k$-form $\beta$ on $G$ by left translation, so $\pi^*d(gS_0) \wedge \beta$ is a volume form on $G$. Hence we have $\pi^*d(gS_0) \wedge \beta = c dg$. We may chose the signs so that $c > 0$, and it is not hard to see that in this case $c = 1$ (cf. [BtD]).

There is a volume form $\alpha = \mathrm{pr}_1^*d(gS_0) \wedge \mathrm{pr}_2^*ds$ on $G/S_0 \times S_0$. Identifying $\mathfrak{g}$ with $L(G/S_0) \oplus LS_0$ and evaluating the forms at the unit element, one finds $\alpha_{(eS_0,e)} = dg_{(e)}$.

Since every element of $G\tau$ is conjugate under $G$ to an element of $S_0\tau$, the map

$$q : G/S_0 \times S_0 \to G\tau,$$
$$(gS_0, s) \mapsto gs\tau g^{-1}$$

is surjective, and by the above we get $q^*dg = \det(q)\alpha$.

**Lemma 2.1** *The functional determinant of the conjugation map $q$ is given by*

$$\det(q)(gS_0, s) = \det(\mathrm{Ad}|_{L(G/S_0)}(s\tau)^{-1} - I|_{L(G/S_0)}).$$

PROOF. Similar to the proof of Prop. IV, 1.8 in [BtD]. (For more details on the proofs in this section see [W1].) □

**Lemma 2.2** *Let $z$ be a generator of $S$. Then*

(i) $|q^{-1}(z)| = |W(S)|$.

(ii) *For $(g_1S_0, s_1), (g_2S_0, s_2) \in q^{-1}(z)$ we have*

$$\det(q)(g_1S_0, s_1) = \det(q)(g_2S_0, s_2).$$

(iii) *There exists a generator $z$ of $S$ such that $q$ is regular in each $(gS_0, s) \in q^{-1}(z)$.*

Proof. Similar to the proof of Prop. IV, 1.9 in [BtD]. □

From this we obtain the mapping degree of $q$, $\deg(q) = \operatorname{sign}(\det(q)) \cdot |W(S)|$. Thus using Fubini's theorem, one gets

**Proposition 2.3** *Let $f : G\tau \to \mathbb{R}$ be a $G$-invariant function. Then*

$$\int_{G\tau} f(g)dg = \frac{1}{|W(S)|} \cdot \int_{S_0} f(s\tau) \cdot |\det(\mathrm{Ad}|_{L(G/S_0)}(s\tau)^{-1} - I|_{L(G/S_0)})|ds.$$

In order to obtain an analogue of the classical Weyl integral formula, one has to calculate the functional determinant of $q$ in terms of the root system of $G$. We adapt the notation used in [BtD]. That is, if $\alpha \in LT^*$ is an infinitesimal root of $G$, then $\vartheta_\alpha$ denotes the corresponding global root $T \to S^1$. So for $H \in LT$ one has $\vartheta_\alpha \circ \exp(H) = e^{2\pi i\alpha(H)}$. Setting $e(x) = e^{2\pi ix}$, we get $\vartheta_\alpha = e(\alpha)$.

Now we consider the action of $(\mathrm{Ad}(s\tau) - I)$ on $L(G/S_0)_\mathbb{C} = \mathbb{C} \otimes L(G/S_0)$. As before, $L(G/S_0)_\mathbb{C}$ decomposes into two orthogonal subspaces

$$L(G/S_0)_\mathbb{C} = L(T/S_0)_\mathbb{C} \oplus L(G/T)_\mathbb{C}.$$

Let $X \in L(T/S_0)$ be an eigenvector of $\mathrm{Ad}(\tau)$. A short calculation shows

$$(\mathrm{Ad}(s\tau) - I)(X) = -X + \gamma X,$$

where $\gamma = \pm 1$ if $\tau^2 = e$, and $\gamma$ is a third root of unity if $\tau^3 = e$. We can choose $(gS_0, s)$ to be a regular point of $q$, so $\gamma \neq 1$ and $(\mathrm{Ad}(s\tau) - I)(X) = -2X$ if $t^2 = e$, and $(\mathrm{Ad}(s\tau) - I)(X) = (\gamma - 1)X$ if $\tau^3 = e$.

Now $L(G/T)_\mathbb{C}$ decomposes into the direct sum of root spaces

$$L(G/T)_\mathbb{C} = \bigoplus_{\alpha \in R} L_\alpha,$$

and for $X \in L_\alpha$, $s \in S_0$, one has

$$\mathrm{Ad}(s)(X) = \vartheta_\alpha(s)(X).$$

Let $\bar{\alpha} = \alpha|_{LS_0}$ for $\alpha \in R$. It is a well known fact that the set $R^\tau = \{\bar{\alpha}|\alpha \in R\}$ is a (not necessarily reduced) root system. The relation between the type of $R$ and the type of $R^\tau$ is shown in the following table:

| $R$ | $\mathrm{A}_{2n-1}$ | $\mathrm{A}_{2n}$ | $\mathrm{D}_n (n \geq 4)$ | $\mathrm{D}_4$ | $E_6$ |
|---|---|---|---|---|---|
| $\mathrm{ord}(\tau)$ | 2 | 2 | 2 | 3 | 2 |
| $R^\tau$ | $\mathrm{C}_n$ | $\mathrm{BC}_n$ | $\mathrm{B}_{n-1}$ | $\mathrm{G}_2$ | $\mathrm{F}_4$ |

So if $R$ is of type $\mathrm{A}_{2n-1}$, $\mathrm{D}_n$, or $E_6$, then $R^\tau$ is a reduced root system. It is easy to see that in this case, $\bar{\alpha}$ is a long root of $R^\tau$ if and only if $\tau(\alpha) = \alpha$. Otherwise $\bar{\alpha}$ is a short root of $R^\tau$. If $R$ is of type $\mathrm{A}_{2n}$, then the root system $R^\tau$ is not reduced and three distinct root lengths occur. In this case, $\bar{\alpha}$ is a long root in $R^\tau$ if $\alpha$ is invariant under $\tau$. If $\alpha$ and $\tau(\alpha)$ are orthogonal to each other, then $\bar{\alpha}$ is a root of medium length in $R^\tau$, and otherwise $\bar{\alpha}$ is a short root in $R^\tau$. (Remember that the root system $\mathrm{BC}_n$ is the union of two root systems of types $B_n$ and $C_n$ such that the long roots of $B_n$ coincide with the short roots of $C_n$.)

Now let $G$ be of type $\mathrm{A}_{2n-1}$, $\mathrm{D}_n$, or $E_6$, and consider the case $\tau^2 = e$. We have seen that $\tau$ defines a Lie algebra automorphism via $\tau(X_\alpha) = X_{\tau(\alpha)}$ for $\alpha \in \Pi$ and $X_\alpha \in L_\alpha$. Extending this to the entire Lie algebra, one gets $\tau(X_\alpha) = X_{\tau(\alpha)}$ for all $\alpha \in R$. The eigenvectors of $\mathrm{Ad}(\tau)$ are the following: If $\alpha$ is invariant under $\tau$ then $X_\alpha$ is an eigenvector with eigenvalue 1. If $\alpha$ is not invariant under $\tau$, there are two eigenvectors $X_\alpha \pm X_{\tau(\alpha)}$ of eigenvalue $\pm 1$. Thus for $\tau(\alpha) = \alpha$ and $X = X_\alpha$ we have

$$(\mathrm{Ad}(s\tau)^{-1} - I)(X) = (\vartheta_{\bar{\alpha}}(s^{-1}) - 1)X,$$

and

$$(\mathrm{Ad}(s\tau)^{-1} - I)(X) = (\pm\vartheta_{\bar{\alpha}}(s^{-1}) - 1)X$$

for $\tau(\alpha) \neq \alpha$ and $X = X_\alpha \pm X_{\tau(\alpha)}$ respectively. This yields

$$\begin{aligned} \det(\mathrm{Ad}|_{L(G/S_0)}(s\tau)^{-1} - I|_{L(G/S_0)}) = \\ (-2)^{\dim(T/S_0)} \cdot \prod_{\substack{\bar{\alpha} \in R^\tau \\ \tau(\alpha) = \alpha}} (\vartheta_{\bar{\alpha}}(s^{-1}) - 1) \\ \cdot \prod_{\substack{\bar{\alpha} \in R^\tau \\ \tau(\alpha) \neq \alpha}} (\vartheta_{\bar{\alpha}}(s^{-1}) - 1)(-\vartheta_{\bar{\alpha}}(s^{-1}) - 1) \end{aligned}$$

Multiplying each factor by $-1$, using the remark above on the relative length of the $\bar{\alpha}$ as well as the equality

$$(1 - \vartheta_{\bar{\alpha}})(1 + \vartheta_{\bar{\alpha}}) = (1 - \vartheta_{2\bar{\alpha}}),$$

this becomes

$$= (-2)^{\dim(T/S_0)} \cdot \prod_{\substack{\bar{\alpha}\in R^\tau \\ \bar{\alpha}\text{long}}} (1-\vartheta_{\bar{\alpha}}(s^{-1})) \prod_{\substack{\bar{\alpha}\in R^\tau \\ \bar{\alpha}\text{short}}} (1-\vartheta_{\bar{2}\alpha}(s^{-1}))$$
$$= (-2)^{\dim(T/S_0)} \cdot \Delta(s^{-1})\bar{\Delta}(s^{-1}),$$

with

$$\Delta(s) = \prod_{\bar{\alpha}\in R_+^{\tau\vee}} (1-\vartheta_{\bar{\alpha}}(s)).$$

Here $R^{\tau\vee}$ denotes the dual root system of $R^\tau$ which is given by $\bar{\alpha}^\vee = \frac{2\bar{\alpha}}{\langle\bar{\alpha},\bar{\alpha}\rangle}$ for $\bar{\alpha}\in R^\tau$ and $\langle .,.\rangle$ is a multiple of the Killing form such that $\langle\bar{\alpha},\bar{\alpha}\rangle = 2$ for a long root $\bar{\alpha}\in R^\tau$.

If $G$ is of type $\mathrm{D}_4$ and $\tau^3 = e$, a similar calculation gives

$$\det(\mathrm{Ad}|_{L(G/S_0)}(s\tau)^{-1} - I|_{L(G/S_0)}) = 3\cdot\Delta(s^{-1})\bar{\Delta}(s^{-1}),$$

with $\Delta(s)$ as above. Observe that in this case $R^\tau$ is of type $\mathrm{G}_2$, so $\dim T/S_0 = 2$.

If $G$ is of type $A_{2n}$ we have to be more careful since $R^\tau$ is not reduced and three different root lengths occur. Also, in this case the Lie algebra automorphism $\tau$ is slightly more complicated. For $X_\alpha \in L_\alpha$ we have

$$\tau(X_\alpha) = (-1)^{1+\mathrm{ht}(\alpha)} X_{\tau(\alpha)}.$$

Now $\tau(\alpha)=\alpha$ implies that $\mathrm{ht}(\alpha)$ is even and a similar calculation yields

$$\det(\mathrm{Ad}|_{L(G/S_0)}(s\tau)^{-1} - I|_{L(G/S_0)}) = (-2)^{\dim(T/S_0)} \cdot \prod_{\substack{\bar{\alpha}\in R^\tau \\ \bar{\alpha}\text{ long}}} (1+\vartheta_{\bar{\alpha}}(s^{-1})) \cdot \prod_{\substack{\bar{\alpha}\in R^\tau \\ \bar{\alpha}\text{ middle}}} (1-\vartheta_{2\bar{\alpha}}(s^{-1})) \cdot \prod_{\substack{\bar{\alpha}\in R^\tau \\ \bar{\alpha}\text{ short}}} (1-\vartheta_{2\bar{\alpha}}(s^{-1}))$$

But the length of the long roots in $\mathrm{BC}_n$ is twice the length of the short roots. So we can put these together to obtain

$$\det(\mathrm{Ad}|_{L(G/S_0)}(s\tau)^{-1} - I|_{L(G/S_0)}) = (-2)^{\dim(T/S_0)} \cdot \Delta(s^{-1})\bar{\Delta}(s^{-1}),$$

with

$$\Delta(s) = \prod_{\bar{\alpha}\in R_+^1} (1-\vartheta_{\bar{\alpha}}(s)).$$

Here $R^1 = \{2\bar{\alpha}|\bar{\alpha}\in \mathrm{BC}_n, \bar{\alpha}\text{ long}\} \cup \{2\bar{\alpha}|\bar{\alpha}\in\mathrm{BC}_n, \bar{\alpha}\text{ middle}\}$. This is a root system of type $\mathrm{C}_n$.

Before stating the integral formula for $G\tau$, we have to compare the different Weyl groups involved. Let $T$ be the maximal torus of $G$ such that $S_0 \subset T$ and let $W(T) = N_G(T)/T$ be the usual Weyl group of $G$. If we set $W^\tau = \{w \in W(T) | \tau w \tau^{-1} = w\}$, then $W^\tau$ is the Weyl group of the root system $R^\tau$ (and also of course of its dual $R^{\tau\vee}$).

**Proposition 2.4** *Let $W^\tau$ be as above. Then*

(i) *There exists a split exact sequence*

$$e \to (T/S_0)^\tau \to W(S) \to W^\tau \to e.$$

*Here $(T/S_0)^\tau$ denotes the fixed point set under conjugation with $\tau$.*

(ii) *We have*

$$|W^\tau| = \begin{cases} \frac{1}{2^{\dim(T/S_0)}} \cdot |W(S)| & \text{if } \tau^2 = e \\ \frac{1}{3} \cdot |W(S)| & \text{if } \tau^3 = e \end{cases}$$

PROOF. Observe that in our cases we have $W(S) = (N(S) \cap G)/G$ (this is not true for general non-connected Lie groups), so one can define a map $\varphi : W(S) \to W(T)$, $gS_0 \mapsto gT$. This map is well defined and one has $Im(\varphi) \subset W^\tau$. The rest is done by a calculation in the Lie algebra of $T$. For details see [W1] (also cf. 3.9). To see that the sequence splits, observe that $S_0$ is by construction a maximal torus in the connected component of $G^\tau$ containing $e$, and $W^\tau$ is the corresponding Weyl group. □

Putting everything together, one gets

**Theorem 2.5 ('Weyl integral formula' for $G\tau$)** *Let $f : G\tau \to \mathbb{R}$ be a function which is integrable and invariant under conjugation by $G$. Then*

$$\int_{G\tau} f(g)dg = \frac{1}{|W^\tau|} \int_{S_0} f(s\tau)\Delta(s)\bar{\Delta}(s)ds,$$

*with*

$$\Delta(s) = \prod_{\bar{\alpha} \in R^1_+} (1 - \vartheta_{\bar{\alpha}}(s)).$$

*Here $R^1$ denotes the root system $R^{\tau\vee}$ if $R^\tau$ is reduced and*

$$R^1 = \{2\bar{\alpha} | \bar{\alpha} \in \mathrm{BC}_n, \bar{\alpha} \text{ long}\} \cup \{2\bar{\alpha} | \bar{\alpha} \in \mathrm{BC}_n, \bar{\alpha} \text{ middle}\}$$

*is of type $\mathrm{C}_n$ if $R$ is of type $A_{2n}$.*

## 2.3 Applications of the integral formula

Let $G$ be a compact connected semisimple Lie group of type $\mathrm{A}_n$, $D_n$, or $E_6$ as above, and let $\bar{G} \subset \tilde{G}$ be any non trivial subextension of $G$. As a first application of the integral formula we compute the irreducible characters of $\bar{G}$ on the component $G\tau$ for $\tau \in \Gamma$. Let $T$ be the $\tau$-invariant maximal torus of $G$, and let $S$ be a Cartan subgroup of $\bar{G}$ such that $S/S_0$ is generated by $\tau S_0$. In the case $\mathrm{D}_4$, this notation is not quite unique since in general more than one diagram automorphism occurs, but it will always be clear which respective Cartan subgroup is being used at the moment.

Since $S_0$ is regular in $G$, we can choose a Weyl chamber $K \subset LT^*$ such that $K \cap LS_0^*$ is not empty and we let $\bar{K}$ denote its closure. Then the set $K^\tau = K \cap LS_0^*$ is a Weyl chamber in $LS_0^*$ with respect to the root system $R^\tau$. Furthermore, let $W = W(T)$ be the Weyl group of $G$, let $I$ denote the lattice $I = ker(\exp) \cap LT$, and $I^* \subset LT^*$ its dual.

For linear forms $\lambda \in LT^*$ and $\mu \in LS_0^*$ we define the alternating sums

$$A(\lambda) = \sum_{w \in W(T)} \epsilon(w) \cdot e(w\lambda)$$

and

$$A^\tau(\mu) = \sum_{w \in W^\tau} \epsilon(w) \cdot e(w\mu)$$

respectively. Here we have set $\epsilon(w) = (-1)^{\mathrm{length}(w)}$. Note that for $w \in W^\tau$ the two $\epsilon(w)$ in the equations above do not necessarily coincide since they come from the presentations of $w$ as an element of two different Weyl groups. In this notation, $A^\tau$ is a complex valued function on $LS_0$ which is alternating with respect to $W^\tau$.

Now let $R^1$ be the root system used in Theorem 2.5. As usual, we set

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha,$$

and

$$\rho^\tau = \frac{1}{2} \sum_{\bar{\alpha} \in R^1_+} \bar{\alpha}.$$

Now we can define functions $\delta : LT \to \mathbb{C}$, resp. $\delta^\tau : LS_0 \to \mathbb{C}$ via

$$\delta = e(\rho) \cdot \prod_{\alpha \in R_+} (1 - e(-\alpha))$$

and

$$\delta^\tau = e(\rho^\tau) \cdot \prod_{\bar{\alpha} \in R^1_+} (1 - e(-\bar{\alpha})).$$

The function $\delta^\tau \cdot \bar{\delta}^\tau$ factorizes through exp, and we have $\delta^\tau \cdot \bar{\delta}^\tau = (\Delta \cdot \bar{\Delta}) \circ \exp$, where $\Delta$ is as in Theorem 2.5. With this notation the classical Weyl character formula for the irreducible character $\chi_\lambda$ of $G$ belonging to the highest weight $\lambda \in I^* \cap \bar{K}$ reads

$$\chi_\lambda = A(\lambda + \rho)/\delta.$$

Now if $\mu \in I^* \cap LS_0^*$ then $A^\tau(\mu + \rho^\tau)/\delta^\tau$ can be extended uniquely to a function on $LS_0$. This function factors through exp (cf. [BtD]). In this way $A^\tau(\mu + \rho^\tau)/\delta^\tau$ can be considered as a function on $S_0$.

For an arbitrary character $\chi$ of $\bar{G}$ we define the function $\chi^\tau : S_0 \to \mathbb{C}$ via $\chi^\tau(s) = \chi(s\tau)$. So on the connected component $G\tau$ of $\bar{G}$, the character $\chi$ is determined by $\chi^\tau$. Now we can state the analogue of the Weyl character formula for the subextensions $\bar{G} \subset \tilde{G}$. First we consider the case when $\bar{G}$ consists of two connected components. So only one diagram automorphism $\tau$ is involved, and $\tau^2 = e$.

**Theorem 2.6** *There exists an irreducible character $\tilde{\chi}_\lambda$ of $\bar{G}$ for each $\lambda \in I^* \cap \bar{K}$. If $\lambda \notin LS_0^*$, then*

$$\tilde{\chi}_\lambda|_G = \chi_\lambda + \chi_{\tau(\lambda)},$$

*and*

$$\chi_\lambda|_{G\tau} \equiv 0.$$

*Here $\chi_\lambda$ denotes the irreducible character of $G$ of highest weight $\lambda$*

*For each $\lambda \in LS_0^*$, there exist two irreducible characters of $\bar{G}$ associated to $\lambda$, and we have*

$$\tilde{\chi}_\lambda|_G = \chi_\lambda,$$

*and*

$$\tilde{\chi}_\lambda^\tau = \pm A^\tau(\mu + \rho^\tau)/\delta^\tau.$$

If $\bar{G}$ consists of three connected components, the character formula is essentially the same, except that there are three irreducible characters for each $\lambda \in LS_0^*$. If $G$ is of type $D_4$ and $\bar{G} = \tilde{G}$, then one can use the character formulas above together with some information about the conjugacy classes in $S_3$ to determine all irreducible characters. Since this will not be needed in the sequel, we will omit the statement of the result.

REMARKS ON THE PROOF. The proof of theorem 2.6 is essentially the same as the proof of the classical Weyl character formula in [BtD]. The integral formula is used along with the orthogonality relations for irreducible characters to show that the irreducible characters of $\bar{G}$ must have the given form. Then one can apply the Peter-Weyl theorem to see that each of the functions above must be an irreducible character of $\bar{G}$. For more details on this see [W1]. □

**Remark 2.7** Kostant [Ko] states a character formula for non-connected complex Lie groups which is not quite as explicit as theorem 2.6. In particular, he gives the character on the component $G\tau$ as a function on $T\tau$, where $T$ is a maximal torus in $G$. So the different root systems do not appear explicitly (although it is not hard to derive theorem 2.6 from his formula).

The formula itself was discovered before by Jantzen [J], who calculated the trace of the outer automorphism $\tau$ on the weight spaces of an irreducible representation with invariant highest weight of a semisimple algebraic group, and by Fuchs et al. [FuS] who studied the characters of "$\tau$-twisted" representations of a generalized Kac-Moody algebra.

As in the classical case, there is a Weyl denominator formula:

**Corollary 2.8** *With the same notation as above we have*

$$\delta^\tau = A^\tau(\rho^\tau).$$

**Remark 2.9** It is interesting to note that the group belonging to the root system $R^1$ can, in general, not be realized as a subgroup of $G$. For example, $SO(2n+1)$, or its covering group $Spin(2n+1)$, which are the groups with root system $\mathrm{B}_n$ can not be realized as subgroup of $SU(2n)$ which belongs to the root system $\mathrm{A}_{2n-1}$.

As a second application of theorem 2.5 we can derive a formula for the radial part of the Laplacian on $G\tau$ with respect to the $G$-action by conjugation. The negative of the Killing form on $\mathfrak{g}_\mathbb{C}$ defines a positive definite scalar product on $\mathfrak{g}$ which defines a biinvariant Riemannian metric on $G$ by left translation. We can use right multiplication by $\tau^{-1}$ to pull back this Riemannian metric to $G\tau$. So $G$ and $G\tau$ are isomorphic as Riemannian manifolds. This metric is invariant under the $G$-action on $G\tau$. Let $\Delta_{G\tau}$ be the Laplacian on $G\tau$ with respect to this metric, and let $\Delta_{S_0}$ be the Laplacian on $S_0$. Now we can use the general theory of radial parts of invariant differential operators [He] Part III, Theorem 3.7 (also cf. Prop. 3.12) to get the following Proposition.

**Proposition 2.10** *Let $f : G\tau \to \mathbb{R}$ be a $G$-invariant function, and $f^\tau : G \to \mathbb{R}$ be given by $f^\tau(g) = f(g\tau)$. Let $\delta^\tau$ and $\rho^\tau$ be as above. Then we have*

$$\delta^\tau \cdot \Delta_{G\tau}(f) = (\Delta_{S_0} + \|\rho^\tau\|^2)(\delta^\tau \cdot f^\tau),$$

*as functions restricted to $LS_0$. Here $\|.\|$ is the metric on $LS_0^*$ induced by the negative of the Killing form.*

# 3 The affine adjoint representation of a twisted loop group

## 3.1 Affine Lie algebras

Let $\mathfrak{g}_\mathbb{C}$ be a complex semi simple Lie algebra, with compact involution $\omega$ and let $\mathfrak{g}$ be a compact form of $\mathfrak{g}_\mathbb{C}$. That is, $i\mathfrak{g} = \{x \in \mathfrak{g}_\mathbb{C} \mid \omega(x) = x\}$. The loop algebra $\mathcal{L}\mathfrak{g}_\mathbb{C}$ (resp. $\mathcal{L}\mathfrak{g}$) is the algebra of $C^\infty$-maps from $S^1$ to $\mathfrak{g}_\mathbb{C}$ (resp. $\mathfrak{g}$). It is a Lie algebra under pointwise Lie bracket. If $\mathfrak{g}_\mathbb{C}$ is a complex simple Lie algebra of type $X_n$ and if the circle $S^1$ is parametrized by the real line $\mathbb{R}$ via the exponential $e(t) = e^{2\pi it}$ then the untwisted affine Lie algebra $\tilde{\mathfrak{g}}_\mathbb{C}$ of type $X_n^{(1)}$ is given by

$$\tilde{\mathfrak{g}}_\mathbb{C} = \mathcal{L}\mathfrak{g}_\mathbb{C} \oplus \mathbb{C}C \oplus \mathbb{C}D,$$

with Lie bracket

$$[x(\cdot) + aC + bD, x_1(\cdot) + a_1C + b_1D] \\ = [x, x_1](\cdot) + bx_1'(\cdot) - b_1x'(\cdot) + (x'(\cdot), x_1(\cdot))C.$$

Here $x'(t) = \frac{dx(t)}{dt}$, $[x, x_1](t) = [x(t), x_1(t)]$, and

$$(x(\cdot), x_1(\cdot)) = \int_0^1 (x(t), x_1(t))dt,$$

where $(.,.)$ under the integral sign denotes the Killing form on $\mathfrak{g}_\mathbb{C}$.

**Remark 3.1** Remark We shall adhere to some slight abuse of terminology, here and in the sequel. The affine algebras in the sense of [K] are based on algebraic loops, with finite Fourier expansion. Our algebras may be viewed as completions of these algebraic ones, cf. below. The notation for types will be that of [K].

An invariant bilinear form $(.,.)$ on $\tilde{\mathfrak{g}}$ is given by

$$(x(\cdot) + aC + bD, x_1(\cdot) + a_1b + b_1D) = \int_0^1 (x(t), x_1(t))dt + ab_1 + a_1b.$$

We obtain a so-called compact form $\tilde{\mathfrak{g}}$ of $\tilde{\mathfrak{g}}_\mathbb{C}$ by considering $\tilde{\mathfrak{g}} = \mathcal{L}\mathfrak{g} \oplus \mathbb{R}C \oplus \mathbb{R}D$.

If $\tau$ is the outer automorphism of $\mathfrak{g}_\mathbb{C}$ considered in §2.1 and $\mathrm{ord}(\tau) = r$, then the twisted loop algebras $\mathcal{L}(\mathfrak{g}_\mathbb{C}, \tau)$ and $\mathcal{L}(\mathfrak{g}, \tau)$ are given by

$$\mathcal{L}(\mathfrak{g}_{(\mathbb{C})}, \tau) = \{x \in \mathcal{L}\mathfrak{g}_{(\mathbb{C})} | \tau(x(t)) = x(t + 1/r), \text{for all } t \in [0, 1]\}.$$

Now if $\mathfrak{g}_\mathbb{C}$ is a simple Lie algebra of type $X_n$ with $X_n = \mathrm{A}_n$, $\mathrm{D}_n$ or $E_6$, then the twisted affine Lie algebra $\tilde{\mathfrak{g}}_\mathbb{C}$ of type $X_n^{(r)}$ is given by

$$\tilde{\mathfrak{g}}_\mathbb{C} = \mathcal{L}(\mathfrak{g}_\mathbb{C}, \tau) \oplus \mathbb{C}C \oplus \mathbb{C}D$$

with the same Lie bracket as above. The invariant bilinear form $(.,.)$ on the corresponding untwisted affine Lie algebra yields an invariant bilinear form on the twisted affine Lie algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$ by restriction and it is denoted by the same symbol. The compact form $\tilde{\mathfrak{g}}$ of a twisted affine Lie algebra is obtained in the same way as in the untwisted case.

We now define the $C^\infty$-topology on the Lie algebras $\tilde{\mathfrak{g}}$. The topology on $\tilde{\mathfrak{g}}_{\mathbb{C}}$ is obtained by viewing it as a direct sum of two copies of $\tilde{\mathfrak{g}}$. If $\mathcal{L}\mathfrak{g}$ is an untwisted loop algebra, one defines the $C^\infty$-topology on $\mathcal{L}\mathfrak{g}$ via the set of semi-norms

$$p_n(x) = \sup_{\substack{s\leq n\\ t\in[0,1]}} |\frac{d^s x(t)}{dt^s}|, \quad n = 1, 2, \dots , \quad x \in \mathcal{L}\mathfrak{g}.$$

With respect to this topology $\mathcal{L}\mathfrak{g}$ is complete. It extends immediately to the enlarged $\tilde{\mathfrak{g}}$ as well as to the twisted subalgebras.We may view these topological algebras as $C^\infty$-completions of the algebraic loop algebras as e.g. in [K].

For later applications, we introduce a second topology on the spaces $\mathcal{L}(\mathfrak{g})$ and $\mathcal{L}(\mathfrak{g}, \tau)$. The completions of these spaces with respect to the new topology will not carry a Lie algebra structure any more (cf. [F]). As before, let $(.,.)$ denote the Killing form on $\mathfrak{g}_{\mathbb{C}}$, and let $(.,.)_{\mathfrak{g}}$ denote the negative of the Killing form on $\mathfrak{g}$. Then $(x, y)_{\mathfrak{g}} = \int_0^1 (x(t), y(t))_{\mathfrak{g}} dt$ gives a scalar product on $\mathcal{L}(\mathfrak{g})$, and by restriction we get a scalar product on $\mathcal{L}(\mathfrak{g}, \tau)$ as well. Let $\mathcal{L}(\mathfrak{g}, \tau)(L_2)$ denote the completion of $\mathcal{L}(\mathfrak{g}, \tau)$ with respect to the metric induced by the scalar product. After extending the scalar product to the completion, $\mathcal{L}(\mathfrak{g}, \tau)(L_2)$ is a Hilbert space.

## 3.2 Loop groups and their affine adjoint representation

From now on, we will write $G_{(\mathbb{C})}$ if we mean either a compact group $G$ or the corresponding complex group $G_{\mathbb{C}}$, and analogously we write $\tilde{\mathfrak{g}}_{(\mathbb{C})}$ for the associated Lie algebras.

If $G_{(\mathbb{C})}$ is a simply connected compact (complex) Lie group then the corresponding untwisted loop group $\mathcal{L}G_{(\mathbb{C})}$ is defined to be the topological group of $C^\infty$ mappings from $S^1$ to $G_{(\mathbb{C})}$ with pointwise multiplication and the usual $C^\infty$-topology. If $\tau$ is one of the outer automorphisms of $G_{(\mathbb{C})}$ considered in §2.1 and $\mathrm{ord}(\tau) = r$ then the twisted loop group $\mathcal{L}(G_{(\mathbb{C})}, \tau)$ is the subgroup

$$\mathcal{L}(G_{(\mathbb{C})}, \tau) = \{x \in \mathcal{L}G_{(\mathbb{C})} | \tau(x(t)) = x(t + 1/r) \text{ for all } t \in [0, 1]\}.$$

Then $\mathcal{L}\mathfrak{g}_{(\mathbb{C})}$ (resp. $\mathcal{L}(\mathfrak{g}_{(\mathbb{C})}, \tau)$) may be viewed as the Lie algebra of $\mathcal{L}G_{(\mathbb{C})}$ (resp. $\mathcal{L}(G_{(\mathbb{C})}, \tau)$) and there are natural adjoint actions of these groups on their Lie algebras (by pointwise finite-dimensional adjoint action). However, for purposes of representation theory it is essential to consider not this action but the adjoint action of the affine Kac-Moody groups on the affine Lie algebras. These groups are given, similar to the Lie algebra case, as central extensions of semidirect products

$$e \to S^1 \to \hat{G} \to \mathcal{L}G \rtimes S^1 \to e,$$

where the circle in the semidirect product acts on $\mathcal{L}G$ by rotation on the loops (in the complex case the circles $S^1$ are usually replaced by $\mathbb{C}^*$), and accordingly in the twisted cases. See [PS] for a construction of these groups. Obviously the center of $\hat{G}$ acts trivially in the adjoint representation, thus it is sufficient to consider the action of the quotient group $\mathcal{L}G \rtimes S^1$. It will turn out that the adjoint action of the rotation group $S^1$ is without relevance for our purposes (i.e. in the case that an $\mathcal{L}G$-orbit contains a constant loop , see below, this action preserves the $\mathcal{L}G$-orbit). Therefore, it is sufficient to look at the adjoint action of only the loop group $\mathcal{L}G$ on $\tilde{\mathfrak{g}}$. We call that action, which significantly differs from the usual adjoint action of $\mathcal{L}G$ on its Lie algebra $\mathcal{L}g$, the affine adjoint action and denote it by $\widetilde{\mathrm{Ad}}$. Exploiting the natural exponential mapping from $\mathcal{L}g$ to $\mathcal{L}G$ and working with a fixed faithful matrix representation of $G$, Frenkel was able to determine the exact form of the affine adjoint action in the untwisted case ([F]).

**Proposition 3.2** *Let $g \in \mathcal{L}G_{(\mathbb{C})}$, $y \in \mathcal{L}\mathfrak{g}_{(\mathbb{C})}$ and $a, b \in \mathbb{C}$ (resp. $\mathbb{R}$), then*

$$\widetilde{\mathrm{Ad}}\ g(aC + bD + y) = \tilde{a}C + bD + gyg^{-1} - bg'g^{-1}$$

*with $\tilde{a} = a + (g^{-1}g', y) - \frac{b}{2}(g'g^{-1}, g'g^{-1})$ and $g'(t) = \frac{dg(t)}{dt}$.*

The affine adjoint representation of a twisted loop group $\mathcal{L}(G_{(\mathbb{C})}, \tau)$ on the corresponding twisted affine Lie algebra $\tilde{\mathfrak{g}}_{(\mathbb{C})}$ is obtained by restriction of the adjoint representation of $\mathcal{L}G_{(\mathbb{C})}$ on the corresponding untwisted affine Lie algebra $\tilde{\mathfrak{g}}'_{(\mathbb{C})}$ to $\mathcal{L}(G_{(\mathbb{C})}, \tau)$ and $\tilde{\mathfrak{g}}_{(\mathbb{C})}$. Hence the formula of Proposition 3.2 remains true for the twisted cases, as well.

## 3.3 Classification of affine orbits

In the case of an untwisted affine Lie algebra Frenkel and Segal have classified affine adjoint orbits of the loop group $\mathcal{L}G_{(\mathbb{C})}$ on the affine Lie algebra $\tilde{\mathfrak{g}}_{(\mathbb{C})}$ in terms of conjugacy classes of the group $G_{(\mathbb{C})}$ ([F], [PS]). By a slight alteration of Frenkel's original methods, we can extend this classification to the twisted loop groups $\mathcal{L}(G_{(\mathbb{C})}, \tau)$. Technically, the study of the affine adjoint action (on elements with $b \neq 0$) is equivalent to that of transformations of ordinary differential equations on the real line with periodic coefficients or, in more advanced terminology, to that of the action of the gauge group on connections in a trivial fibre bundle over the circle $S^1$. In the context of twisted loops we shall, in addition, have to work with differential equations with 'twisted periodic' coefficients. To obtain a unified statement of the results, we shall allow the twisting automorphism $\tau$ to be the identity on $G_{(\mathbb{C})}$. In this case, the group $\mathcal{L}(G_{(\mathbb{C})}, \tau)$ is just the untwisted group $\mathcal{L}G_{(\mathbb{C})}$.

Consider first the system of linear differential equations

$$z'(t) = z(t)x(t),$$

where $x(t), z(t) \in M_n(\mathbb{C})$ for all $t \geq 0$, and $x(t)$ is continuous in $t$. A fundamental result from the theory of differential equations secures the existence of a unique solution $z(t)$ of the above equation with $z(0) = I_n$. This solution $z$ is usually called the fundamental solution of the differential equation.

Now let $x$ be twisted periodic, that is, $x(t + 1/r) = \tau x \tau^{-1}$ for some invertible matrix $\tau$ with $\tau^r = I_n$ and all $t \geq 0$. If $z$ is the fundamental solution of $z' = zx$, then obviously $z_1(t) = \tau^{-1} z(t + 1/r)\tau$ is another solution. Hence there exists a matrix $\widetilde{M}(x)$ such that $z_1(t) = \widetilde{M}(x)z(t)$. Since we have chosen $z(0) = I_n$, we get $\widetilde{M}(x) = z_1(0) = \tau^{-1} z(1/r)\tau$. Now let $M(x) := z(1/r)$ be the "$\frac{1}{r}$-th monodromy" of the differential equation $z' = zx$. We then obtain

$$z(t + 1/r) = M(x)\tau z(t)\tau^{-1}$$

for all $t \geq 0$.

For a twisted periodic continuously differentiable $g$ with $g(t) \in Gl_n(\mathbb{C})$ for all $t \geq 0$ let us denote

$$\begin{aligned} z_g(t) &= g(0)z(t)g^{-1}(t) \\ x_g(t) &= g(t)x(t)g^{-1}(t) - g'(t)g^{-1}(t). \end{aligned}$$

Then we have the following proposition.

**Proposition 3.3** *Let $x$ be a twisted periodic, continuous, matrix valued function, and let $z$ be the fundamental solution of $z' = zx$. Then*

(i) *$z_g(t)$ is the fundamental solution of $z_g' = z_g x_g$.*

(ii) $M(x_g) = g(0)M(x)\tau g^{-1}(0)\tau^{-1}$.

(iii) *If $x_1$ is twisted periodic and there exists a $g_0$ such that*

$$M(x_1) = g_0 M(x)\tau g_0^{-1}\tau^{-1},$$

*then there exists a twisted periodic Matrix $g(t)$ such that $g(0) = g_0$ and $x_g(t) = x_1(t)$ for all $t \geq 0$.*

Proof. (i) This is a direct calculation using $(g^{-1})' = -g^{-1}g'g^{-1}$.

(ii) Since $g$ is twisted periodic, we have

$$M(x_g) = z_g(1/r) = g(0)z(1/r)g^{-1}(1/r) = g(0)M(x)\tau g^{-1}(0)\tau^{-1}.$$

(iii) Put $g(t) = z_1^{-1}(t)g_0 z(t)$, where $z$ and $z_1$ are fundamental solutions of $z' = zx$ and $z_1' = z_1 x_1$ respectively. Then the same calculation as in the proof of [F], Prop.(3.2.5) yields $x_g = x_1$. Again, a similar explicit calculation as in [F] gives $g(t+1/r) = \tau g(t)\tau^{-1}$ for all $t \geq 0$. □

Before we can use the results above to classify the affine adjoint orbits for arbitrary $\mathfrak{g}$ we need a general fact from differential geometry, which is stated as follows in [F].

**Proposition 3.4** *Let $\mathfrak{g}_{\mathbb{C}} \subset M_n(\mathbb{C})$ be a matrix Lie algebra, and $G_{\mathbb{C}} \subset Gl_n(\mathbb{C})$ the corresponding Lie group. If $z$ is a solution of the linear differential equation $z' = zx$, then $z(t) \in G_{\mathbb{C}}$ for all $t \geq 0$ if and only if $x(t) \in \mathfrak{g}_{\mathbb{C}}$ for all $t \geq 0$.*

Now let $\tilde{\mathfrak{g}}_{\mathbb{C}}$ be an affine Lie algebra of type $X_n^{(r)}$, and let $\tau$ be the corresponding diagram automorphism of the underlying finite dimensional Lie algebra $\mathfrak{g}_{\mathbb{C}}$ used in the loop realization of $\tilde{\mathfrak{g}}_{\mathbb{C}}$. In the case of an untwisted affine Lie algebra, $\tau$ is just the identity on $\mathfrak{g}_{\mathbb{C}}$. Let $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ denote the corresponding compact forms. Following [F], we define an affine shell ("standard paraboloid" in [F]) to be the following submanifold of codimension 2 in $\tilde{\mathfrak{g}}_{\mathbb{C}}$.

$$\mathcal{P}_{\mathbb{C}}^{a,b} = \{x(\cdot) + a_1 C + b_1 D \in \tilde{\mathfrak{g}}_{\mathbb{C}} | 2a_1 b_1 + (x,x) = a, b_1 = b\},$$

where $a, b \in \mathbb{C}$ and $b \neq 0$. The zero-hyperplane in $\tilde{\mathfrak{g}}_{\mathbb{C}}$ is defined to be the subspace

$$\hat{\mathfrak{g}}_{\mathbb{C}} = \{x(\cdot) + aC + bD \in \tilde{\mathfrak{g}}_{\mathbb{C}} | b = 0\}.$$

By $\mathcal{P}^{a,b}$ with $a, b \in \mathbb{R}$, $a \neq 0$ and $\hat{\mathfrak{g}}$ we shall denote the corresponding submanifolds of $\tilde{\mathfrak{g}}$. Let $\mathcal{O}_X$ be the $\mathcal{L}G_{(\mathbb{C})}$-Orbit of $X$ in $\tilde{\mathfrak{g}}_{(\mathbb{C})}$, and let $\mathcal{O}_{g\tau}$ be the $G_{(\mathbb{C})}$-Orbit of $g\tau$ in $G_{(\mathbb{C})}\tau$. Here $G\tau$ is the connected component of the principal extension $\tilde{G}$ of the compact group $G$ as constructed in §2.1, and $\tilde{G}_{\mathbb{C}}$ is the corresponding complexification.

**Theorem 3.5** (i) *Each $\mathcal{L}(G_{\mathbb{C}}, \tau)$ (resp. $\mathcal{L}(G, \tau)$)-Orbit in the complex (resp. compact) affine Lie algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$ (resp. $\tilde{\mathfrak{g}}$) lies either in one of the affine shells $\mathcal{P}_{\mathbb{C}}^{a,b}$ (resp. $\mathcal{P}^{a,b}$) or in the zero-hyperplane.*

(ii) *For a fixed affine shell, the monodromy map*

$$\mathcal{O}_{x(\cdot)+aC+bD} \mapsto \mathcal{O}_{M(\frac{1}{b}x)\tau}$$

*is well defined and injective.*

(iii) *For a fixed affine shell, the map defined in* (ii) *gives a bijection between the* $\mathcal{L}(G_{\mathbb{C}}, \tau)$ *(resp.* $\mathcal{L}(G, \tau)$*)-Orbits in* $\tilde{\mathfrak{g}}_{\mathbb{C}}$ *(resp.* $\tilde{\mathfrak{g}}$*) which contain a constant loop and the* $G_{\mathbb{C}}$ *(resp.* $G$*)-Orbits in* $G_{\mathbb{C}}\tau$ *(resp.* $G\tau$*) which contain an element that is invariant under conjugation with* $\tau$.

Proof. (i) Follows from the formula in lemma 3.2. Note that the affine shells in a twisted affine Lie algebra are just the $\tau$-invariant parts of the affine shells in the corresponding untwisted algebras.

(ii) We look at the map $\mathcal{O}_{x(\cdot)+aC+bD} \mapsto \mathcal{O}_{M(\frac{1}{b}x)\tau}$ where, as above, $M(\frac{1}{b}x) = z(1/r)$, and $z$ is the fundamental solution of $z' = z \cdot \frac{1}{b} \cdot x$. Now, by (3.2), we have

$$\widetilde{\mathrm{Ad}}(g)(aC + bD + x) = \tilde{a}C + dD + gxg^{-1} - bg'g^{-1}.$$

But using 3.3 (i), we see that $\mathcal{O}_{\tilde{a}C+bD+gxg^{-1}-bg'g^{-1}}$ is being mapped to $\mathcal{O}_{z_g(1/r)\tau}$, and 3.3 (ii) yields

$$z_g(1/r) = M(\frac{1}{b}x_g) = g(0)M(\frac{1}{b}x)\tau g(0)^{-1}\tau^{-1},$$

hence

$$z_g(1/r)\tau = g(0)z(1/r)\tau g(0)^{-1} \in \mathcal{O}_{z(1/r)\tau}.$$

So the map is well defined and injectivity follows with 3.3 (iii).

(iii) If $s\tau \in G_{(\mathbb{C})}\tau$ is invariant under conjugation with $\tau$ then so is $r \cdot b \cdot \log(s)$ and $\mathcal{O}_{a_1C+bD+r\cdot b\cdot \log(s)}$ is a preimage of $\mathcal{O}_{s\tau}$ whenever it belongs to $\mathcal{P}^{a,b}$. On the other hand, if the orbit $\mathcal{O}_{a_2C+bD+x(\cdot)}$ contains a constant loop $aC + bD + x_0$, then $x_0$ has to be invariant under conjugation with $\tau$. Now the fundamental solution of the differential equation $z' = z \cdot \frac{1}{b}x_0$ is given by $z(t) = \exp(t\frac{1}{b}x_0)$. Hence $z(1/r)$ is invariant under conjugation with $\tau$ as well. □

**Corollary 3.6** *If* $\mathrm{ord}(\tau) = 1$*, or* $G$ *is compact, then the monodromy map in Theorem 3.5 is surjective and hence induces a bijection between the* $\mathcal{L}G$*-orbits in a fixed affine shell* $\mathcal{P}^{a,b}$ *and the* $G$*-orbits in* $G\tau$.

Proof. If $\tau = id$ then the statement is trivial. For compact $G$ we use the fact that every $G$-orbit in $G\tau$ intersects $S\tau$ in at least one point. Here $S$ is a Cartan subgroup of $\tilde{G}$ containing $\tau$. □

**Remark 3.7** In the case of complex groups the classification of $\mathcal{L}(G_{\mathbb{C}}, \tau)$-orbits in $\tilde{\mathfrak{g}}_{\mathbb{C}}$ remains open. In this case it is no longer true that every $G_{\mathbb{C}}$-orbit in $G_{\mathbb{C}}\tau$ contains a $\tau$-invariant element (cf. [Mo] for an example), so different arguments may have

have to be applied. In any case, it is easy to see that the image under the monodromy map defined in 3.5 (ii) are the $G_{(\mathbb{C})}$-Orbits in $G_{(\mathbb{C})}\tau$ for which there exists a $C^\infty$ path $z : [0,1] \to G_{(\mathbb{C})}$ such that $z(0) = e$, $z(1/r)\tau \in \mathcal{O}_{g\tau}$ and $z(t + 1/r) = z(1/r)\tau z(t)\tau^{-1}$ for all $t \geq 0$.

**Remark 3.8** We have not dealt with the $\mathcal{L}(G,\tau)$-orbits in the zero-hyperplane, which are basically the orbits of the adjoint representation of $\mathcal{L}(G,\tau)$ on its Lie algebra. As we shall see later, the orbits relevant for representation theory are the $\mathcal{L}(G,\tau)$-orbits in some fixed affine shell with $b \neq 0$. Also, the classification of $\mathcal{L}(G,\tau)$-orbits in $\hat{\mathfrak{g}}$ is presumably not manageable, i.e. it certainly yields an infinite dimensional “moduli space”.

In a simply connected, semi-simple, compact Lie group, the conjugacy classes are in one-to-one correspondence with a fundamental domain of the affine Weyl group $\tilde{W}$ acting on $LT$ in the notation of §2. The affine Weyl group acts like the finite Weyl group of the root system $R$, belonging to the group $G$, extended by the group of translations generated by the dual roots $R^* \in LT$. Here, the set of dual roots is given as $R^* = \{\alpha^* \mid \alpha \in R\}$, and $(\alpha^*, .) = \alpha$ where $(.,.)$ is the Killing form. Hence the the orbits in a given affine shell $\mathcal{P}^{a,b}$ are in one-to-one correspondence with a fundamental domain of $\tilde{W}$.

The $G$-orbits in $G\tau$ are in one-to-one correspondence with the set $S_0\tau/W(S)$, where $W(S) = N(S)/S$ acts by conjugation on $S_0\tau$. We can pull back this action to $S_0$ by right multiplication with $\tau^{-1}$. Then $tS_0 \in W(S)$ acts on $S_0$ via $s \mapsto ts\tau t^{-1}\tau^{-1}$. By Proposition 2.4, we have $W(S) = (T/S_0)^\tau \rtimes W^\tau$. In this way, the action of $W^\tau$ is the usual Weyl group action of the Weyl group of $G^\tau$ on the maximal torus $S_0$. Hence the orbits of this action are parametrized by a fundamental domain of the affine Weyl group $\tilde{W}^\tau$, where the translation part of $\tilde{W}^\tau$ is given by the dual roots $R^{\tau*} \in LS_0$ of the root system $R^\tau$ belonging to the group $G^\tau$.

An element $tS_0 \in (T/S_0)^\tau$ acts on $S_0$ via $s \mapsto ts\tau t^{-1}\tau^{-1} = ss_{t^{-1}}\tau$, where $\tau t\tau^{-1} = ts_t$ with $s_t \in S_0$. Viewing $LS_0$ as the universal covering of $S_0$ via exp, we see that $(T/S_0)^\tau$ acts on $LS_0$ as a group of translations. A direct calculation shows that a set of generators of this group is given by the set

$$\{\frac{1}{r}\sum_{i=1}^{r}\tau^i(\alpha^*) \mid \alpha^* \in R^*, \tau(\alpha^*) \neq \alpha^*\}.$$

So together these two groups yield exactly the action of the affine Weyl group $\tilde{W}^1$ belonging to the root system $R^1$ from §2. Hence we have proved

**Proposition 3.9** *In the twisted compact case, the set of orbits in a given affine shell $\mathcal{P}^{a,b}$ is in one-to-one correspondence with a simplex in $LS_0$ which is a fundamental domain for the action of the affine Weyl group $\tilde{W}^1$ belonging to the root system $R^1$.*

# 4 Orbital integrals

## 4.1 Affine Lie algebras and the Kac-Weyl character formula

Before we start deriving the analogue of Frenkel's character formula for the twisted affine Lie algebras, let us briefly review some facts from the structure and representation theory of affine Lie algebras. Let $\tilde{\mathfrak{g}}_{\mathbb{C}}$ be an affine Lie algebra of type $X_n^{(r)}$ with Weyl group $\tilde{W}$. Let $\tilde{\mathfrak{h}}_{\mathbb{C}}$ be a Cartan subalgebra and $\Pi = \{\alpha_0, \dots, \alpha_n\} \subset \tilde{\mathfrak{h}}_{\mathbb{C}}^*$ be a set of simple roots of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ where the roots are labeled in the usual way (cf.[K]). $\tilde{R}$ denotes the set of roots of $\tilde{\mathfrak{g}}_{\mathbb{C}}$, and define $R^\circ$ to be the root system which is obtained by deleting the 0-th vertex from the extended Dynkin diagram of $\tilde{R}$. Also, let $W^\circ$ denote the Weyl group belonging to $R^\circ$. Let $\alpha_i^\vee \in \tilde{\mathfrak{h}}_{\mathbb{C}}$ be the dual simple roots such that $\langle \alpha_i, \alpha_j^\vee \rangle = (A)_{i,j}$, where $A$ is the generalized Cartan matrix belonging to $\tilde{\mathfrak{g}}_{\mathbb{C}}$. Let $a_i$ be the "minimal" integers such that $A(a_0, \dots, a_n) = 0$, set $\delta = \sum_{i=0}^n a_i \alpha_i$ and define $d = \frac{1}{2\pi i} D$ with $D$ as in §3.1. Then we have $\langle \alpha_i, d \rangle = 0$ and $\langle \delta, d \rangle = 1$. Now, we can define an element $\theta = \delta - a_0 \alpha_0 \in \mathfrak{h}_{\mathbb{C}}^{\circ *}$ and the lattice $M = \mathbb{Z} W^\circ \theta^\vee \subset \mathfrak{h}_{\mathbb{C}}^\circ$. Here $\mathfrak{h}_{\mathbb{C}}^\circ \subset \tilde{\mathfrak{h}}_{\mathbb{C}}$ denotes the subspace generated by $\{\alpha_1^\vee, \dots \alpha_n^\vee\}$, and $\theta^\vee$ means the element dual to $\theta$ in the sense of [K]. A fundamental result in the theory of affine Lie algebras states that $\tilde{W} = W^\circ \ltimes M$. Observe that in [K] Kac uses a lattice $M' \subset \mathfrak{h}_{\mathbb{C}}^{\circ *}$ after identifying $\mathfrak{h}_{\mathbb{C}}^{\circ *}$ with $\mathfrak{h}_{\mathbb{C}}^\circ$ via an invariant bilinear form.

Turning to the representation theory of affine Lie algebras we define as usual

$$
\begin{aligned}
\tilde{P} &= \{\lambda \in \tilde{\mathfrak{h}}_{\mathbb{C}}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for all } i = 0, \dots, n\}, \\
\tilde{P}_+ &= \{\lambda \in \tilde{P} \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i = 0, \dots, n\}, \text{ and} \\
\tilde{P}_{++} &= \{\lambda \in \tilde{P} \mid \langle \lambda, \alpha_i^\vee \rangle > 0 \text{ for all } i = 0, \dots, n\}.
\end{aligned}
$$

Then there exists a bijection between the irreducible integrable highest weight modules of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ and the dominant integral weights $\Lambda \in \tilde{P}_+$. If $L(\Lambda)$ is an integrable irreducible highest weight module with highest weight $\Lambda \in \tilde{P}_+$ then the formal character of $L(\Lambda)$ is the formal sum

$$
\text{ch } L(\Lambda) = \sum_{\lambda \in \tilde{\mathfrak{h}}^*} \dim L(\Lambda)_\lambda e(\lambda).
$$

Here $L(\Lambda)_\lambda$ is the weight space corresponding to the weight $\lambda$, and $e(\lambda)$ is a formal exponential. The Kac-Weyl character formula now reads (cf. [K])

$$
\text{ch } L(\Lambda) = \frac{\sum_{w \in \tilde{W}} \epsilon(w) e(w(\Lambda + \tilde{\rho}))}{e(\tilde{\rho}) \prod_{\alpha \in R} (1 - e(-\alpha))^{mult(\alpha)}}.
$$

Here $\tilde{\rho}$ is defined by $\langle \tilde{\rho}, \alpha_i^\vee \rangle = 1$ for $i = 0, \dots, n$ and $\langle \tilde{\rho}, d \rangle = 0$. As usual we have set $\epsilon(w) = (-1)^{lenght(w)}$.

So far, the character was considered as a formal sum involving the formal exponentials $e(\lambda)$. Now we set $e^{\lambda}(h) = e^{\langle \lambda, h\rangle}$ for $h \in \tilde{\mathfrak{h}}_{\mathbb{C}}$. In this way one can consider the character of a highest weight $\tilde{\mathfrak{g}}_{\mathbb{C}}$-module $V$ as an infinite series. Let us set $Y(V) = \{h \in \tilde{\mathfrak{h}}_{\mathbb{C}} \mid \mathrm{ch}_V(h) \text{ converges absolutely}\}$. Then $ch_V$ defines a holomorphic function on $Y(V)$ and the following result holds [K]:

**Proposition 4.1** *Let $V(\Lambda)$ be the irreducible highest weight module with highest weight $\Lambda \in P_+$. Then*

$$Y(V(\Lambda)) = \{h \in \tilde{\mathfrak{h}}_{\mathbb{C}} \mid Re\langle \delta, h\rangle > 0\},$$

*where $\delta = \sum_{i=0}^{n} a_i \alpha_i$ and the $a_i$ are the labels of the vertices of the affine Dynkin diagram (cf. [K]).*

*In this setting the Kac-Weyl character formula gives an identity of holomorphic functions on $Y(V(\Lambda))$.*

Now we have $\tilde{\mathfrak{h}}_{\mathbb{C}} = \mathfrak{h}^{\circ}_{\mathbb{C}} \oplus \mathbb{C}C \oplus \mathbb{C}D$ with $C$ and $D$ as in §3.1. With this notation one gets

$$Y(V(\Lambda)) = \{h + aC + bD \mid h \in \mathfrak{h}^{\circ}_{\mathbb{C}},\ \ a, b \in \mathbb{C},\ \ Im\ \ b < 0\}.$$

## 4.2 Poisson transformation of the numerator of affine characters

In this section we will start do derive an analogue of Frenkel's character formula for twisted affine Lie algebras by deriving a formula for the numerator of the character formula in terms of the underlying non-connected Lie group. To do this, we need to introduce some more notation. So let $\tilde{\mathfrak{g}}_{\mathbb{C}}$ be an arbitrary affine Lie algebra of type $X_n^{(r)}$, and let $\tilde{\mathfrak{g}}'_{\mathbb{C}}$ be the untwisted affine Lie algebra of type $X_n^{(1)}$ such that $\tilde{\mathfrak{g}}_{\mathbb{C}} \subset \tilde{\mathfrak{g}}'_{\mathbb{C}}$ and let $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}'$ be the corresponding compact forms. If $\tilde{\mathfrak{g}}_{\mathbb{C}}$ is untwisted, we have $\tilde{\mathfrak{g}}_{\mathbb{C}} = \tilde{\mathfrak{g}}'_{\mathbb{C}}$. Furthermore, let $R$ be the root system of the finite dimensional Lie algebra $\mathfrak{g}_{\mathbb{C}}$ used to construct $\mathcal{L}(\mathfrak{g}_{\mathbb{C}}, \tau)$ in §3.1, and for a diagram automorphism $\tau$ of $\mathfrak{g}_{\mathbb{C}}$ let $R^{\tau}$ be the "folded" root system introduced in §2.2. If $\tilde{\mathfrak{g}}_{\mathbb{C}}$ is a twisted affine Lie algebra with root system $\tilde{R}$, then $R^{\tau} = R^{\circ}$. Also, let $(.,.)$ denote the Killing form on $\tilde{\mathfrak{g}}'_{\mathbb{C}}$, and let $(.,.)_r$ denote its restriction to $\tilde{\mathfrak{g}}_{\mathbb{C}}$.

We now turn to the analytic Kac-Weyl character formula from §4.1. Let $\Lambda$ be a highest weight of $\tilde{\mathfrak{g}}_{\mathbb{C}}$. There is no essential loss in generality by assuming $\langle \Lambda, d\rangle = 0$. Then, after identifying $\tilde{\mathfrak{h}}_{\mathbb{C}} \cong \tilde{\mathfrak{h}}^*_{\mathbb{C}}$ via $(.,.)_r$ we can choose $a \in \mathbb{C}$ and $H \in \mathfrak{h}^{\circ}_{\mathbb{C}}$ such that $\Lambda + \tilde{\rho} = aD + H$. The condition $\Lambda \in \tilde{P}_+$ implies $a \in i\mathbb{R}$, $Im\ \ a < 0$ and $H \in i\mathfrak{h}^{\circ}$. The numerator of the Kac-Weyl character formula evaluated at $bD + K$ with $b \in \mathbb{C}$, $Im\ \ b < 0$ and $K \in \mathfrak{h}^{\circ}_{\mathbb{C}}$ now reads

$$\sum_{w \in \tilde{W}} \epsilon(w) e^{(w(aD+H), bD+K)_r}.$$

Let us assume in the following $b \in i\mathbb{R}$ and $K \in i\mathfrak{h}^\circ$. We then set $t = -1/ab$, $h = H/a$ and $k = K/b$ yielding $t \in \mathbb{R}_+$ and $h, k \in \mathfrak{h}$. With $2\pi i d = D$, the sum above reads

$$\sum_{w \in \tilde{W}} \epsilon(w) e^{-\frac{1}{t}(w(2\pi i d + h), 2\pi i d + k)_r}.$$

Let us set $c = 2\pi i C$. Then the lattice $M$ operates on on $\tilde{h}_{\mathbb{C}}$ via

$$\gamma(h + ac + bd) = h + ac + bd - \left( (h, \gamma)_r + \frac{ba_0}{2} \|\gamma\|_r^2 \right) c.$$

Now a short calculation (cf. [F]) shows for $w \in W^\circ$, $\gamma \in M$, and $w^{-1}\gamma \in \tilde{W}$

$$\left(w^{-1}\gamma(2\pi i d + h), 2\pi i d + k\right)_r = -\frac{1}{2}\|2\pi i a_0 \gamma + h - wk\|_r^2 + \frac{1}{2}\|h\|_r^2 + \frac{1}{2}\|k\|_r^2.$$

Inserting this into the sum above, we get

$$\sum_{w \in \tilde{W}} \epsilon(w) e^{-\frac{1}{t}(w(2\pi i d + h), 2\pi i d + k)_r} = $$
$$e^{-\frac{1}{2t}\|h\|_r^2 - \frac{1}{2t}\|k\|_r^2} \sum_{\gamma \in 2\pi i a_0 M} \sum_{w \in W^\circ} \epsilon(w) e^{\frac{1}{2t}\|\gamma + h - wk\|_r^2}.$$

We now need to apply the Poisson transformation formula: For a Euclidean vector space $V$, a lattice $Q \in V$ and a Schwartz function $f : V \to \mathbb{C}$ one has

$$\sum_{\mu \in Q^\vee} \hat{f}(\mu) = vol \;\; Q \sum_{\gamma \in Q} f(\gamma)$$

with

$$\hat{f}(\mu) = \int_V e^{2\pi i (\gamma, \mu)} f(\gamma) d\gamma.$$

For a fixed $x \in \mathfrak{h}^\circ$ set $f(\mu) = e^{(x,\mu)_r - \frac{t}{2}\|\mu\|_r^2}$. Then we get

$$\hat{f}(\gamma) = \left(\frac{2\pi}{t}\right)^{\frac{l}{2}} e^{\frac{1}{2t}\|x + 2\pi i \gamma\|_r^2}.$$

with $l = \dim_{\mathbb{R}} \mathfrak{h}^\circ$. So for $x = h - wk$ with $h, k \in \mathfrak{h}^\circ$ and $w \in W^\circ$ we obtain the identity

$$\sum_{\gamma \in a_0 M} e^{\frac{1}{2t}\|2\pi i \gamma + h - wk\|_r^2} = vol(a_0 M)^{-1} \left(\frac{2\pi}{t}\right)^{-\frac{l}{2}} \sum_{\mu \in (a_0 M)^\vee} e^{(\mu, h - wk)_r - \frac{t}{2}\|\mu\|_r^2}.$$

If $\tilde{R}$ is of type Aff1, then $\theta$ is a long root in $R^\circ$, and if $\tilde{R}$ is a root system of type Aff2 or Aff3, but not of type $\mathrm{A}_{2n}^{(2)}$, then $\theta$ is a short root in $R^\circ$. In case $\tilde{R}$ is of type $\mathrm{A}_{2n}^{(2)}$ then $R^\circ$ is a root system of type $\mathrm{BC}_n$, and $\theta$ is a root of medium length in $R^\circ$.

So if $\tilde{R}$ is of type $X_n^{(r)}$ with $r = 2, 3$ and $\tilde{R} \neq A_{2n}^{(2)}$, then $\theta^\vee$ is a long root in $R^{\circ\vee}$, and hence $M$ is the lattice which is generated by the long roots in $R^{\circ\vee}$. If $\tilde{R}$ is of type $A_{2n}^{(2)}$, then $\theta^\vee$ is a root of medium length in $R^{\circ\vee}$ and in this case we have $a_0 = 2$. Thus, in all cases, $M$ is the lattice which is generated by the root system $R^1$ from §2.2.

For an arbitrary root system $S$ let $P^\circ(S)$ denote the weight lattice of $S$. Then the above implies

$$\sum_{\gamma \in a_0 M} e^{\frac{1}{2t}\|2\pi i\gamma + h - wk\|_r^2} = \frac{1}{vol\ \ a_0 M}\left(\frac{2\pi}{t}\right)^{-\frac{l}{2}} \sum_{\mu \in P^\circ(R^1)} e^{(\mu, h-wk)_r - \frac{t}{2}\|\mu\|_r^2}.$$

Putting the above formulas together, we get

$$\sum_{w \in \tilde{W}} \epsilon(w) e^{-\frac{1}{t}(w(2\pi i d + h), 2\pi i d + k)_r} =$$

$$\frac{e^{-\frac{1}{2t}\|h\|_r^2 - \frac{1}{2t}\|k\|_r^2}}{vol(\mathbb{Z}\ R^1)\left(\frac{2\pi}{t}\right)^{\frac{l}{2}}} \sum_{w \in W^\circ} \sum_{\mu \in P^\circ(R^1)} \epsilon(w) e^{(\mu, h-wk)_r - \frac{t}{2}\|\mu\|_r^2}.$$

Now let $W(R^1)$ denote the Weyl group of the root system $R^1$. It is a well known fact that (after the choice of a basis of $R^1$) every weight $\lambda \in P^\circ(R^1)$ is conjugate under $W(R^1)$ to some dominant weight $\lambda' \in P_+^\circ(R^1)$. Since the root systems $R^1$ and $R^\circ$ are dual to each other, we have $W^\circ = W(R^1)$. So we get

$$\sum_{w \in W^\circ} \sum_{\mu \in P^\circ(R^1)} \epsilon(w) e^{(\mu, h-wk)_r - \frac{t}{2}\|\mu\|_r^2} =$$

$$\sum_{\mu \in P_+^\circ(R^1)} \sum_{w \in W^\circ} \sum_{w' \in W^\circ} \epsilon(ww')\epsilon(w') e^{\langle w'\mu, h-wk\rangle} e^{-\frac{t}{2}\|\mu\|_r^2}$$

Here we have identified $\mathfrak{h}^\circ$ and $\mathfrak{h}^{\circ *}$ via $(.,.)_r$. In the equation above, the singular weights cancel out, so it is enough to sum over the strictly dominant weights, or equivalently to replace $\lambda$ by $\lambda + \rho^\tau$ with $\rho^\tau = \frac{1}{2}\sum_{\bar{\alpha} \in R_+^1} \bar{\alpha}$ as in §2.2. Hence

$$\begin{aligned}
\sum_{w \in W^\circ} \sum_{\mu \in P^\circ(R^1)} & \epsilon(w) e^{(\mu, h-wk)_r - \frac{t}{2}\|\mu\|_r^2} = \\
&= \sum_{\lambda \in P_+^\circ(R^1)} \sum_{w \in W^\circ} \sum_{w' \in W^\circ} \epsilon(ww')\epsilon(w') e^{<w'(\lambda+\rho^\tau), h-wk>} e^{-\frac{t}{2}\|\lambda+\rho^\tau\|_r^2} \\
&= \sum_{\lambda \in P_+^\circ(R^1)} \delta^\tau(h)\delta^\tau(-k)\chi_\lambda^\tau(h)\chi_\lambda^\tau(-k) e^{-\frac{t}{2}\|\lambda+\rho^\tau\|_r^2}
\end{aligned}$$

with $A^\tau(\lambda)$, $\delta^\tau$ and $\chi^\tau_\lambda = A^\tau(\lambda+\rho^\tau)/\delta^\tau$ as in §2.3. As before, let $\mathfrak{g}_\mathbb{C}$ be the finite dimensional complex Lie algebra used to construct $\mathcal{L}(\mathfrak{g}_\mathbb{C},\tau)$ with root system $R$ and compact form $\mathfrak{g}$. Let $G$ be the simply connected compact Lie group belonging to $\mathfrak{g}$ and let $G\tau$ denote the connected component of the non-connected Lie group $G \ltimes \langle\tau\rangle$ containing $\tau$. In §2.3 we have seen that $\chi^\tau_\lambda(h) = \tilde{\chi}_\lambda(e^h\tau)$ for $h \in \mathfrak{h}^\circ$. Here $\tilde{\chi}_\lambda$ denotes the character of $G \rtimes \langle\tau\rangle$ belonging to the highest weight $\lambda$ (cf. Theorem 2.6 and observe that in this notation we have $\mathfrak{h}^\circ = LS_0$ with $LS_0$ as in §2.2).

So putting everything together, we have proved the following theorem (which is the analogue of Theorem (4.3.4) in [F]).

**Theorem 4.2** *For $h,k \in \mathfrak{h}^\circ$ one has*

$$e^{\frac{1}{2t}\|h\|_r^2} e^{\frac{1}{2t}\|k\|_r^2} \sum_{w\in\tilde{W}} \epsilon(w) e^{-\frac{1}{t}(w(2\pi i d+h),2\pi i d+k)_r} =$$

$$\frac{\delta^\tau(h)\delta^\tau(-k)}{vol(\mathbb{Z}R^1)(\frac{2\pi}{t})^{\frac{l}{2}}} \sum_{\lambda\in P^\circ_+(R^1)} \chi_\lambda(e^h\tau)\chi_\lambda(e^{-k}\tau) e^{-\frac{t}{2}\|\lambda+\rho^\tau\|_r^2}$$

**Remark 4.3** This section is basically a reformulation of of the analogous results of [F] in the non-twisted cases and of [Kl] in the twisted cases. Kleinfeld did his calculations for the twisted affine Lie algebras in concrete realizations of the corresponding root systems. Not realizing the appearance of the characters of the non-connected Lie group $G \rtimes \langle\tau\rangle$, he had to work with the irreducible characters of the different Lie groups corresponding to the root systems $R$, $R^\tau$ and $R^{\tau\vee}$.

## 4.3 The heat equation

In this section we will see how the expression for the numerator in Theorem 4.2 is connected to the fundamental solution of the heat equation on the component $G\tau$ of the non-connected group $G \rtimes \langle\tau\rangle$. To this end, let $\Delta_G$ denote the Laplacian on the compact simply connected group $G$ with respect to the Riemannian metric on $G$ induced by the negative of the Killing form on $\mathfrak{g}_\mathbb{C}$. We can pull back this metric to $G\tau$ such that right multiplication with $\tau$ induces an isometry between the Riemannian manifolds $G$ and $G\tau$. The Laplacian on $G\tau$ shall be denoted with $\Delta_{G\tau}$.

Now for a fixed parameter $T > 0$, the heat equation on $G\tau$ reads

$$\frac{\partial f(g\tau,t)}{\partial t} = \frac{sT}{2}\Delta_{G\tau} f(g\tau,t)$$

with $g \in G$, $s \in \mathbb{R}$, $s > 0$ and where $f : G\tau \to \mathbb{R}$ is continuous in both variables, $C^2$ in the first and $C^1$ in the second variable. The fundamental solution of the heat equation is defined by the initial data

$$f(g\tau,t)|_{t=+0} = \delta_\tau(g\tau),$$

where $\delta_\tau$ is the Dirac delta distribution centered at $\tau \in G\tau$.

For a highest weight $\lambda \in P_+^\circ(R)$ of $G$ let $d(\lambda)$ denote the dimension of the corresponding irreducible representation of $G$ and $\chi_\lambda$ its character. Then the fundamental solution of the heat equation on $G$ is given by

$$u_s(g,t) = \sum_{\lambda \in P_+^\circ(R)} d(\lambda) x_\lambda(g) e^{-\frac{stT}{2}(\|\lambda+\rho\|^2 - \|\rho\|^2)}$$

(see [Fe]). Now $G$ and $G\tau$ are isometric as Riemannian manifolds, hence the fundamental solutions of the respective heat equations coincide. That is, the fundamental solution of the heat equation on $G\tau$ is given by

$$v_s(g\tau,t) = \sum_{\lambda \in P_+^\circ(R)} d(\lambda) x_\lambda(g) e^{-\frac{stT}{2}(\|\lambda+\rho\|^2 - \|\rho\|^2)}.$$

There is a well known identity for the characters of a compact group which can be derived using the orthogonality relations for irreducible characters:

$$d(\lambda)\int_G \chi_\lambda(g_1 g g_2^{-1} g^{-1}) dg = \chi_\lambda(g_1)\chi_\lambda(g_2^{-1}),$$

where $dg$ denotes the normalized Haar measure on $G$. Using a version of the orthogonality relations for non-connected groups, it is easy to proof an analogous formula for the characters on the outer components:

$$d(\lambda)\int_G \chi_\lambda(g_1 \tau g \tau^{-1} g_2^{-1} g^{-1}) dg = \chi_\lambda(g_1\tau)\chi_\lambda(\tau^{-1} g_2^{-1}).$$

Hence we obtain

$$\int_G v_s(g g_1 \tau g^{-1} \tau^{-1} g_2^{-1} \tau, t) dg = \sum_{\lambda \in P_+^\circ(R)} \chi_\lambda(g_1\tau)\chi_\lambda(\tau^{-1} g_2^{-1}) e^{-\frac{stT}{2}(\|\lambda+\rho\|^2 - \|\rho\|^2)}.$$

By Theorem 2.6 we see that $\chi_\lambda(g\tau) = 0$ if $\lambda$ is not $\tau$-invariant. Furthermore, we have $\|\lambda+\rho\|^2 = \|\lambda+\rho^\tau\|_r^2$ if $\lambda$ is $\tau$-invariant. Thus, using Theorem 4.2, exchanging the role of $s$ and $t$, and fixing the parameter value $s = T$, we have proved the following proposition.

**Proposition 4.4** *For $h, k \in \mathfrak{h}^\circ$ one has*

$$\sum_{w \in \tilde{W}} \epsilon(w) e^{-\frac{1}{t}(w(2\pi i d+h), 2\pi i d+k)_r} =$$

$$\frac{e^{-\frac{1}{2t}\|h\|_r^2} e^{-\frac{1}{2t}\|k\|_r^2} e^{-\frac{t}{2}\|\rho^\tau\|_r^2} \delta^\tau(h)\delta^\tau(-k)}{vol(\mathbb{Z}R^1)(\frac{2\pi}{t})^{\frac{l}{2}}} \cdot$$

$$\int_G v_{\frac{t}{T^2}}(g e^h \tau g^{-1} \tau^{-1} e^{-k} \tau, T) dg$$

## 4.4 Wiener measures and a path integral

The main result of this subsection will be a further reformulation of the numerator of the character formula as an integral over a certain path space on the connected component $G\tau$. This is based on Proposition 4.4, above, and the theory of Wiener measure on $G\tau$ which we will study first. (Compare with [F] and consult e.g. [Ku] for a comprehensive treatment of the theory of Wiener measures on a vectorspace.)

The Wiener measure on an Euclidean vectorspace $V$ of variance $s > 0$ is a measure $\omega_V^s$ on the Banach space of paths

$$C_V = \{x : [0,T] \to V \mid x(0) = 0, \ x \text{ continuous}\}$$

(with the supremum norm) and is defined using the fundamental solution $w_s(x,t)$ of the heat equation

$$\frac{\partial f(x,t)}{\partial t} = \frac{sT}{2} \Delta_V f(x,t).$$

on $V$ as follows: First, one defines cylinder sets in $C_V$ to be the following subsets of $C_V$:

$$\{x \in C_V : (x(t_1) \in A_1, \cdots, x(t_m) \in A_m)\},$$

with $0 < t_1 \leq t_2 \leq \ldots \leq t_m \leq T$, $m \in \mathbb{N}$, and where $A_1, \ldots A_m$ are Borel sets in $V$. Then the Wiener measure $\omega_V^s$ of variance $s > 0$ is defined on the cylinder sets of $C_V$ via

$$\omega_V^s(x(t_1) \in A_1, \cdots, x(t_m) \in A_m) = \int_{A_1} \cdots \int_{A_m} w_s(\Delta x_1, \Delta t_1) \cdots w_s(\Delta x_m, \Delta t_m) dx_1 \cdots dx_m,$$

where $dx$ is a Lebesgue measure on $V$ and we have set $x_k = x(t_k)$, $\Delta x_k = x_k - x_{k-1}$, $\Delta t_k = t_k - t_{k-1}$ and $x_0 = 0$.

The conditional Wiener measure $\omega_{V,Z}^s$ of variance $s > 0$ is defined on the closed subspace $C_{V,X} \subset C_V$ with fixed endpoint $x(T) = X$ on the cylinder sets via

$$\omega_{V,X}^s(x(t_1) \in A_1, \cdots, x(t_{m-1}) \in A_{m-1}) = \int_{A_1} \cdots \int_{A_{m-1}} w_s(\Delta x t_1, \Delta t_1) \cdots w_s(\Delta x_m, \Delta t_m) dx_1 \cdots dx_{m-1},$$

where additionally $x_m = X$ and $t_m = T$.

Now a classical result in the theory of Wiener measures states that the measures $\omega_V^s$ and $\omega_{V,X}^s$ are $\sigma$-additive on the $\sigma$-field generated by the cylinder sets in $C_V$ and $C_{V,X}$ respectively. Furthermore, the $\sigma$-fields generated by the cylinder sets are exactly the Borel fields of the respective Banach spaces (cf. [Ku]). As another result, we have

$$\omega_V^s(C_V) = 1$$

and

$$\omega_{V,X}^s(C_{V,X}) = w_s(X,T),$$

where $w_s(x,t)$ is the fundamental solution of the heat equation on $V$.

Using the fundamental solution $u_s(g,t)$ of the heat equation on the compact Lie group $G$, one can define the Wiener measure $\omega_G^s$ and the conditional Wiener measure $\omega_{G,Z}^s$ on the complete metric spaces

$$C_G = \{z : [0,T] \to G \mid z(0) = e, \ \ z \text{ continuous}\}$$

and $C_{G,Z} = \{z \in C_G, \ z(T) = Z)\}$ in exactly the same fashion as the Wiener measure on the vectorspace $V$ (cf. [F]). The metric $\varrho$ on $C_G$ is given by $\varrho(z,z_1) = \sup_{t\in[0,T]} \varrho_0(z(t), z_1(t))$, where $\varrho_0(g,g_1)$ denotes the length of a shortest geodesic in $G$ connecting two given points $g$ and $g_1$ (the metric on $G$ still being given by the negative of the Killing form on $\mathfrak{g}$).

There is an important connection between the Wiener measure on $G$ and the Wiener measure on the Lie algebra $\mathfrak{g}$ which was discovered by Ito [I] and explicitly constructed by McKean [McK]: Let $y \in C_{\mathfrak{g}}$ be a continuous path. Then for $n \in \mathbb{N}$ and $k = 0, \dots, 2^n - 1$, we define a path $z_n : [0,T] \to G$ by $z_n(0) = e$ and

$$z_n(t) = z_n(\frac{k}{2^n}T) \exp\left(y(t) - y(\frac{k}{2^n}T)\right) \quad \text{for } \frac{k}{2^n}T < t \le \frac{k+1}{2^n}T.$$

Note that if $y$ is a differentiable path, then $\lim_{n\to\infty} z_n$ is the fundamental solution of the differential equation of $z' = zy'$ and hence a well defined path in $G$. Let us define a map

$$i : C_{\mathfrak{g}} \to C_G$$

via

$$y \mapsto \begin{cases} \lim_{n\to\infty} z_n & \text{if the limit exists,} \\ e & \text{else.} \end{cases}$$

The fundamental result of Ito and McKean states that the series $z_n$, converges with $n \to \infty$ in the topology of $C_G$ almost everywhere with respect to the measure $\omega_{\mathfrak{g}}^s$. Furthermore, the measure on $C_G$ induced by the map $i$ coincides with the Wiener measure $\omega_G^s$ on $G$. Hence $i$ induces an isomorphism

$$I : L_1(C_{\mathfrak{g}}, \omega_{\mathfrak{g}}^s) \to L_1(C_G, \omega_G^s)$$

via $If(iy) = f(y)$.

This isomorphism is called Ito's isomorphism in the literature. With its help it is easy to translate most of the results about the Wiener measure on a vectorspace to a corresponding result about Wiener measure on a compact Lie group. For example we have

$$\omega_G^s(C_G) = 1$$

and

$$\omega_{G,Z}^s(C_{G,Z}) = u_s(Z,T),$$

where $u_s(g,t)$ denotes the fundamental solution of the heat equation on $G$ (cf. [F]).

We will denote the integrals with respect to the Wiener measure and the conditional Wiener measure on $V$ as

$$\int_{C_V} f(x) d\omega_V^s(x),$$

resp.

$$\int_{C_{V,X}} f(x) d\omega_{V,X}^s(x),$$

and accordingly for the Wiener measures on $G$.

One of the most important properties of these integrals is its translation qasi-invariance (cf. [Ku]): Let $f : C_V \to \mathbb{R}$ be an integrable function and let $y \in C_V$ be a $C^\infty$-path. Then

$$\int_{C_V} f(x) d\omega_V^s(x) = \int_{C_V} f(x+y) e^{-\frac{1}{s}(x',y') - \frac{1}{2s}(y',y')} d\omega_V^s(x).$$

For a function $f : C_{V,X} \to \mathbb{R}$ the translation quasi-invariance of the conditional Wiener measure reads

$$\int_{C_{V,X}} f(x) d\omega_{V,X}^s(x) = \int_{C_{V,X+Y}} f(x+y) e^{-\frac{1}{s}(x',y') - \frac{1}{2s}(y',y')} d\omega_{V,X+Y}^s(x),$$

with $Y = y(T)$. In the above formulas, $(x', y')$ denotes the Stieltjes integral

$$\frac{1}{T} \int_0^T (y'(t), dx(t)),$$

and $(.,.)$ denotes the scalar product on $V$.

Translated to the Wiener integral on $G$, the translation quasi-invariance looks as follows (see [F]):

**Proposition 4.5** (i) *Let $f : C_G \to \mathbb{R}$ be an integrable function, and $g \in C_G$ be a $C^\infty$-path. Then*

$$\int_{C_G} f(z) d\omega_G^s(z) = \int_{C_G} f(zg) e^{-\frac{1}{s}(z^{-1}z', g'g^{-1})_{\mathfrak{g}} - \frac{1}{2s}(g^{-1}g', g^{-1}g')_{\mathfrak{g}}} d\omega_G^s(z).$$

(ii) *Let $f : C_{G,Z} \to \mathbb{R}$ be an integrable function, and $g \in C_G$ be a $C^\infty$-path. Then*

$$\int_{C_{G,Z}} f(z) d\omega^s_{G,Z}(z) =$$

$$\int_{C_{G,Zg(T)^{-1}}} f(zg) e^{-\frac{1}{s}(z^{-1}z', g'g^{-1})_{\mathfrak{g}}} \cdot e^{-\frac{1}{2s}(g^{-1}g', g^{-1}g')_{\mathfrak{g}}} d\omega^s_{G,Zg(T)^{-1}}(z).$$

Here the term $(z^{-1}z', g'g^{-1})_{\mathfrak{g}}$ should be interpreted as the Stieltjes integral

$$\frac{1}{T}\int_0^T (g'g^{-1}, d(i^{-1}(z)))_{\mathfrak{g}},$$

where $(.,.)_{\mathfrak{g}}$ denotes the negative of the Killing form on $\mathfrak{g}$ (we add the subscript $\mathfrak{g}$ here to avoid possible confusions in later calculations). Note that $i^{-1}$ is a well defined map almost everywhere on $C_G$ with respect to $\omega^s_G$.

Using the translation quasi-invariance of the Wiener measure on $G$, Frenkel computes the following integral with respect to this measure ([F], Prop. 5.2.12):

**Lemma 4.6** *Let $Y \in \mathcal{L}(\mathfrak{g}, \tau)$ and let $g \in C_G$ be a $C^\infty$- path such that $g' = gY$. Then*

$$e^{-\frac{\|Y\|^2_{\mathfrak{g}}}{2s}} \int_{C_{G,Z}} e^{\frac{1}{s}(z^{-1}z', Y)_{\mathfrak{g}}} d\omega^s_{G,Z}(z) = u_s(Zg(T)^{-1}, T),$$

*where $u_s(z,t)$ is the fundamental solution of the heat equation on $G$.*

Now let $\mathcal{O}_{g\tau}$ denote the $G$-orbit in $G\tau$ containing the element $g\tau$ (cf. §3.3). Multiplying each element of $G\tau$ with $\tau^{-1}$, we can identify $\mathcal{O}_{g\tau}$ with a $G$-orbit in $G$, where $G$ acts on itself by twisted conjugation: $(h,g) \mapsto hg\tau h^{-1}\tau^{-1}$. This orbit will be denoted with $\mathcal{O}_{g\tau}$ as well. We can now define $C_{G,\mathcal{O}_{g\tau}} \subset C_G$ to be the space of continuous paths with $z : [0,T] \to G$ with $z(T) \in \mathcal{O}_{g\tau}$. A conditional Wiener measure $\omega^s_{G,\mathcal{O}_{g\tau}}$ on $C_{G,\mathcal{O}_{g\tau}}$ is defined via

$$\int_{C_{G,\mathcal{O}_{g\tau}}} f(z) d\omega^s_{G,\mathcal{O}_{g\tau}}(z) = \int_G \left( \int_{C_{G,g_1 g\tau g_1^{-1}\tau^{-1}}} f(z) d\omega^s_{G,g_1 g\tau g_1^{-1}\tau^{-1}}(z) \right) dg_1,$$

where $f$ is integrable on $C_{G,g_1 g\tau g_1^{-1}\tau^{-1}}$ for almost all $g_1 \in G$. Inserting this definition into Lemma 4.6, one gets:

**Corollary 4.7** *Let $Y \in \mathcal{L}(\mathfrak{g}, \tau)$ and let $g \in C_G$ be a $C^\infty$- path such that $g' = gY$. Then*

$$e^{-\frac{\|Y\|^2_{\mathfrak{g}}}{2s}} \int_{C_{G,\mathcal{O}_{Z\tau}}} e^{\frac{1}{s}(z^{-1}z', Y)_{\mathfrak{g}}} d\omega^s_{G,\mathcal{O}_{Z\tau}}(z) = \int_G u_s(g_1 Z\tau g_1^{-1}\tau^{-1}g(T)^{-1}, T) dg_1,$$

*where $u_s(z,t)$ is the fundamental solution of the heat equation on $G$.*

We want to interpret the numerator of the Kac-Weyl character formula as an integral in a space of paths in $G\tau$, so we need to define a Wiener measure on the space

$$C_{G\tau} = \{\tilde{z} : [0,T] \to G\tau \mid \tilde{z}\tau^{-1} \in C_G\}.$$

This can be done with the help of the fundamental solution $v_s(g\tau, t)$ on $G\tau$ in exactly the same way as on $G$ and on $V$. But as we have seen before, we have $v_s(g\tau, t) = u_s(g, t)$ for all $g \in G$ and $t > 0$. So the integrals on $G$ and $G\tau$ will not differ, and we can define the measure $\omega^s_{G\tau}$ directly by

$$\int_{C_{G\tau}} f(\tilde{z}) d\omega^s_{G\tau}(\tilde{z}) = \int_{C_G} \hat{f}(\tilde{z}\tau^{-1}) d\omega^s_G(\tilde{z}\tau^{-1}),$$

where $\hat{f}$ is a function on $C_G$ which is given by $\hat{f}(z) = f(z\tau)$. The conditional Wiener measures on $C_{G\tau, Z\tau}$ and $C_{G\tau, \mathcal{O}_{g\tau}}$ are defined analogously.

So the formula in Corollary 4.7 now reads

$$e^{-\frac{\|Y\|^2_{\mathfrak{g}} T^2}{2t}} \int_{C_{G\tau, \mathcal{O}_{Z\tau}}} e^{\frac{T^2}{t}(z^{-1}z', Y)_{\mathfrak{g}}} d\omega^{\frac{t}{T^2}}_{G\tau, \mathcal{O}_{Z\tau}}(z\tau) = \int_G v_{\frac{t}{T^2}}(g_1 Z\tau g_1^{-1}\tau^{-1} g(T)^{-1}\tau, T) dg_1,$$

where $v_s(g\tau, t)$ is the fundamental solution of the heat equation on $G\tau$. Note, that we also have set $s = \frac{t}{T^2}$ in the above calculation.

Now we fix the parameter value $T = \frac{1}{r}$. Observe that for $Y \in \mathcal{L}(\mathfrak{g}, \tau)$ we then have $\|Y\|_r = -\|Y\|_{\mathfrak{g}}$. So for $h, k \in \mathfrak{h}^\circ$, we can set $Y = \frac{1}{T}k = rk$ and $Z = e^h$. Then we have $g(T)^{-1} = e^{-k}$. Hence the integral formula above and Proposition 4.4 yield:

**Proposition 4.8** *Let $h, k \in \mathfrak{h}^\circ$. Then*

$$\sum_{w \in \tilde{W}} \epsilon(w) e^{-\frac{1}{t}(w(2\pi i d + h), 2\pi i d + k)_r} =$$

$$\frac{\delta^\tau(h)\delta^\tau(-k) e^{-\frac{1}{2t}\|h\|_r - \frac{t}{2}\|\rho^\tau\|_r}}{vol(\mathbb{Z}R^1)\left(\frac{2\pi}{t}\right)^{\frac{l}{2}}} \int_{C_{G\tau, \mathcal{O}_{e^h\tau}}} e^{\frac{1}{tr^2}(z^{-1}z', k)_{\mathfrak{g}}} d\omega^{tr^2}_{G\tau, \mathcal{O}_{e^h\tau}}(z\tau)$$

## 4.5 Affine characters and orbital integrals

In this section we will indicate, how the integral in Proposition 4.8 can be interpreted as an integral over an affine coadjoint orbit of $\mathcal{L}(G, \tau)$. This interpretation and the analytic Kac-Weyl character formula from §4.1 will then yield an analogue of the Kirillov character formula for compact semisimple Lie groups. For precise details we have to refer to Frenkel's work [F].

For fixed $a, b \in \mathbb{R}$ and $b \neq 0$ let $\mathcal{P}^{a,b}$ be the affine shell defined in §3.3. We can identify $\mathcal{P}^{a,b}$ with $\mathcal{L}(\mathfrak{g}, \tau)$ via the projection $p : C \mapsto 0$ and $D \mapsto 0$. Under this projection, the affine adjoint $\mathcal{L}(G, \tau)$-action on $\mathcal{L}(\mathfrak{g}, \tau)$ is given by $(g, y) \mapsto gyg^{-1} - bg'g^{-1}$ (cf. Prop. 3.2).

We have a series of maps

$$\mathcal{P}^{a,b} \xrightarrow{p} \mathcal{L}(\mathfrak{g}, \tau) \xrightarrow{s} C^\infty_{\mathfrak{g}} \xrightarrow{i} C^\infty_G \xrightarrow{e_\tau} G\tau,$$

with $s(x)(t) = \int_0^t x(\kappa) d\kappa$, and where $i$ maps a path $y \in C^\infty_{\mathfrak{g}}$ to the fundamental solution of the differential equation $z' = \frac{1}{b} z y'$. The map $e_\tau$ is given by $e_\tau(z) = z(1/r)\tau$. From Ito's isomorphism we have a map $\tilde{i} : C_{\mathfrak{g}} \to C_G$, which is the extension of the map $i : C^\infty_{\mathfrak{g}} \to C^\infty_G$ above to the corresponding completions $C_{\mathfrak{g}}$ and $C_G$.

Now every element $y \in C_{\mathfrak{g}}$ defines an element $dy \in \mathcal{L}(\mathfrak{g}, \tau)^*$ via the Stieltjes integral

$$\langle x, dy \rangle = r \int_0^{\frac{1}{r}} (x(\kappa), dy(\kappa)),$$

where $(.,.)$ denotes the Killing form on $\mathfrak{g}$ as in §3.1 and §4.1. Note that for $y \in \mathcal{L}(\mathfrak{g}, \tau)$, we have $\langle x, d(s(y)) \rangle = (x, y)_r$, where $(.,.)_r$ is the bilinear form on $\mathcal{L}(\mathfrak{g}, \tau)$ defined in §4.1. Let $\mathcal{L}(\mathfrak{g}, \tau)^*_0$ denote the image of $C_{\mathfrak{g}}$ under this map with the topology induced from $C_{\mathfrak{g}}$, and let $\tilde{s} : \mathcal{L}(\mathfrak{g}, \tau)^*_0 \to C_{\mathfrak{g}}$ denote the inverse map. In this notation, $\tilde{s}$ is the extension of the map $s$ to $\mathcal{L}(\mathfrak{g}, \tau)^*_0$. Putting the above remarks together, we get the following commutative diagram:

$$\begin{array}{ccccccccc}
\mathcal{P}^{a,b} & \xrightarrow{p} & \mathcal{L}(\mathfrak{g}, \tau) & \xrightarrow{s} & C^\infty_{\mathfrak{g}} & \xrightarrow{i} & C^\infty_G & \xrightarrow{e_\tau} & G\tau \\
 & & \bigcap & & \bigcap & & \bigcap & & \\
 & & \mathcal{L}(\mathfrak{g}, \tau)^*_0 & \xrightarrow{\tilde{s}} & C_{\mathfrak{g}} & \xrightarrow{\tilde{i}} & C_G & \xrightarrow{\tilde{e}_\tau} & G\tau
\end{array}$$

Here $\tilde{e}_\tau$ denotes the extension of $e_\tau$ to $C_G$. We have seen in §3.3 that under the composition $e_\tau \circ i \circ s \circ p$, a $\mathcal{L}(G, \tau)$-orbit $\mathcal{O}_{x(\cdot)+a_1 C + b_1 D} \subset \mathcal{P}^{a,b}$ is mapped to the $G$-orbit $\mathcal{O}_{e_\tau \circ i \circ s(x)} \subset G\tau$.

On the Hilbert space $\mathcal{L}(\mathfrak{g}, \tau)(L_2)$ introduced in §3.1, we can define a norm by

$$|x(\cdot)| = \sup_{t \in [0, \frac{1}{r}]} |\int_0^t x(\kappa) d\kappa|.$$

The completion of $\mathcal{L}(\mathfrak{g}, \tau)(L_2)$ with respect to this norm will be $\mathcal{L}(\mathfrak{g}, \tau)^*_0$. So we have a series of completions

$$\mathcal{L}(\mathfrak{g}, \tau) \subset \mathcal{L}(\mathfrak{g}, \tau)(L_2) \subset \mathcal{L}(\mathfrak{g}, \tau)^*_0,$$

with respect to the $L_2$-topology on $\mathcal{L}(\mathfrak{g}, \tau)$ and the norm on $\mathcal{L}(\mathfrak{g}, \tau)(L_2)$ introduced above.

In this picture, the set $\tilde{s}^{-1} \circ \tilde{i}^{-1} \circ \tilde{e}_\tau^{-1}(\mathcal{O}_{e_\tau \circ i \circ s(x)}) \subset \mathcal{L}(\mathfrak{g}, \tau)_0^*$ can be viewed as the closure of the affine adjoint orbit $\mathcal{O}_{x(\cdot)+a_1 C + b_1 D}$ in $\mathcal{L}(\mathfrak{g}, \tau)_0^*$ and is mapped to $C_{G, \mathcal{O}_{i \circ s(x)}}$ under the map $\tilde{i} \circ \tilde{s}$. Accordingly, the integral

$$\int_{C_{G\tau, \mathcal{O}_{Z\tau}}} f(z) d\omega^s_{G\tau, \mathcal{O}_{Z\tau}}(z\tau)$$

can be viewed as an integral over the closure of the corresponding affine adjoint orbit in $\mathcal{L}(\mathfrak{g}, \tau)_0^*$. For more details, which involve the construction of a Gaussian measure on $\mathcal{L}(\mathfrak{g}, \tau)_0^*$, cf. [F]

The discussion above allows us to interpret the integral appearing in the formula for the numerator of the Kac-Weyl character formula in Prop. 4.8 as an integral over the closure in $\mathcal{L}(\mathfrak{g}, \tau)_0^*$ of the affine adjoint orbit containing $aD + H = \Lambda + \tilde{\rho}$. The denominator of the Kac-Weyl character formula is a function $p$ not depending on the highest weight $\Lambda$ of the corresponding representation and hence can be seen as the analogue of the universal function appearing in the Kirillov character formula for compact Lie groups (in the context of compact Lie groups, the universal function is given by the denominator of the Weyl character formula as well). Hence our character formula for affine Lie algebras can be written in the following way:

**Theorem 4.9** *Let $bD + K \in \tilde{h}$ with $K \in \mathfrak{h}^\circ$ and $b \in i\mathbb{R}$, $im(b) < 0$. Furthermore, for $\Lambda \in \tilde{P}_+$ let $\Lambda + \tilde{\rho} = aD + H$. Then the character of the highest weight representation corresponding to $\Lambda$ evaluated at $bD + K$ is given by*

$$\text{ch } L(\Lambda)(bD+K) = p^{-1}(bD+K) \cdot \frac{\delta^\tau(\frac{H}{a})\delta^\tau(-\frac{K}{b}) e^{\frac{ab}{2}\|\frac{H}{a}\|_r - \frac{1}{2ab}\|\rho^\tau\|_r}}{vol(\mathbb{Z}R^1)\,(-2ab\pi)^{\frac{l}{2}}} \cdot \int_{C_{G\tau, \mathcal{O}_{(e^{\frac{H}{a}})\tau}}} e^{-\frac{ab}{r^2}(z^{-1}z', \frac{K}{b})_{\mathfrak{g}}} d\omega^{-\frac{r^2}{ab}}_{G\tau, \mathcal{O}_{(e^{\frac{H}{a}})\tau}}(z\tau).$$

Following [F], the path integral above can be interpreted as a Gaussian integral over the closure of the affine orbit containing $aD+H$ in $\mathcal{L}(\mathfrak{g}, \tau)_0^*$. In that setup, the above formula may be seen as an exact analogue of Kirillov's classical character formula.

# Chapter II

# A symplectic approach to certain functional integrals and partition functions

## 1 Introduction

Functional integration methods play an important role in modern quantum field theory (cf. e.g. [W], [G]). Despite this fact, the precise mathematical meaning of the appearing integrals remains, at least at this time, rather mysterious. Only in certain situations, one can rely on well developed theories such as the Wiener measure, but usually functional integrals are "calculated" by ad hoc methods which are justified merely by their analogy with finite dimensional integration methods. Surprisingly enough, quite often these calculations yield results which can also be derived without using functional integration. Two examples of such "calculations" are the papers [A] and [P] in which the Duistermaat Heckman exact integration formula is applied to integrals on certain infinite dimensional symplectic manifolds.

Let us briefly review these results: The Duistermaat Heckman formula applies to the computation of integrals of the form $\int_M e^{-tf}\frac{\omega^n}{n!}$, where $(M,\omega)$ is a finite dimensional compact symplectic manifold of dimension $2n$, and $\frac{\omega^n}{n!}$ is the associated Liouville measure. Let us assume that the circle $S^1$ acts on $M$ preserving $\omega$ and that $f$ is a Hamiltonian function corresponding to the vector field generated by the $S^1$-action. If the fixed point set of the $S^1$-action is discrete, the Duistermaat Heckman formula [DH1] allows to reduce the integral $\int_M e^{-tf}\frac{\omega^n}{n!}$ to a sum over the fixed points of the circle action (see §2.1 for an exact statement of the formula). Following ideas of Witten, Atiyah [A] indicated how applying the Duistermaat Heckman formula to the loop space of a Riemannian manifold which has a (degenerate) two-form and a natural $S^1$-action, one can formally derive the index theorem for the Dirac operator. Perret [P] used the Duistermaat Heckman theorem on the loop space of a coadjoint orbit of a compact Lie group, resp. a loop group to give "physical proofs" of the Weyl and the Kac-Weyl character formulas respectively.

In the present paper we will use similar ideas to calculate certain integrals over infinite dimensional symplectic manifolds naturally arising in the theory of loop groups and double loop groups. But our approach to these "integrals" will be somewhat more conceptual than the one employed in the physics literature. In particular, instead of "calculating" the integrals using the Duistermaat Heckman formula, we will use an analogue of the Duistermaat Heckman formula to define a functional on the Hamiltonian functions corresponding to some symplectic torus action on $M$. We will call this functional the Liouville functional because of its analogy with the Liouville measure on a finite dimensional manifold. By comparing the symplectic form $\omega$ with a Riemannian structure $\sigma$ on $M$, we define a second functional. In the finite-dimensional case, this second functional is equal to the integration over the Riemannian volume form. Thus, by analogy, we will refer to this functional as "integration" over the Riemannian volume form $d\sigma$. This will be done in section 2.

In section 3, we will "integrate" several functions on the coadjoint orbit of the centrally extended loop group $\hat{G}$ of a compact Lie group $G$ with respect to the Riemannian volume form. The resulting functions on $G$ arise naturally in the representation theory of $G$ and its associated affine Lie algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$ and can be interpreted as the partition function of a quantum mechanical particle moving on the group $G$. In [F], Frenkel gave an interpretation of the same functions in terms of integrals with respect to the Wiener measure on a completion of the corresponding coadjoint orbit of $\hat{G}$. We will compare these two approaches in §3.4 and see that they are, in a sense, equivalent. Therefore, at least in these cases the name "integral" for our functionals is justified not only by analogy.

Section 4 is devoted to the calculation of the partition function of the gauged Wess-Zumino-Witten model on an elliptic curve using the Liouville functional approach. The WZW model is a quantum field theory on a compact Riemann surface $\Sigma$ with values in a compact Lie group $G$, or more generally in its complexification $G_{\mathbb{C}}$. We will only consider the case in which $\Sigma$ is the elliptic curve $\Sigma_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, where $\tau \in \mathbb{C}$, $im(\tau) > 0$ is the modular parameter of the elliptic curve. The partition function of the gauged WZW model at level $\kappa \in \mathbb{N}$ is formally given by the functional integral

$$\int_{C^\infty(\Sigma_\tau, G_{\mathbb{C}})} e^{-S_{G,H,\kappa}(g)} \mathcal{D}(g),$$

where $H$ is an element of the Lie algebra of $G$, and $S_{G,H,\kappa} : C^\infty(\Sigma_\tau, G_{\mathbb{C}}) \to \mathbb{R}$ is the so called gauged Wess-Zumino-Witten action (see §4.1 for more details and references). At this time, the measure theoretic meaning of this integral is not clear (but cf. §4.4 for some speculations). In any case, let $T$ be a maximal torus of $G$ such that $\exp(H) \in T$ and let $T_{\mathbb{C}} \subset G_{\mathbb{C}}$ be its complexification. Assume that

$\exp(H)$ is a regular element of $G$ (i.e. the maximal torus $T$ such that $\exp(H) \in T$ is unique). The action function $S_{G,H,\kappa}$ factors through $LLG_{\mathbb{C}}/T_{\mathbb{C}}$, where $LLG_{\mathbb{C}}$ denotes the double loop group of $G_{\mathbb{C}}$ (i.e. the space of all smooth maps from $S^1 \times S^1$ to $G_{\mathbb{C}}$). Now $LLG_{\mathbb{C}}/T_{\mathbb{C}}$ has a (complex valued) symplectic form and a natural action of the torus $S^1 \times S^1 \times T$, where the first two factors act by rotating the loops and $T$ acts by left multiplication. We will show that the gauged WZW action is the Hamiltonian of a vector field defined by this torus action. Since the torus action has a discrete fixed point set, we can calculate the "integral"

$$\int_{LLG_{\mathbb{C}}/T_{\mathbb{C}}} e^{-S_{G,H,\kappa}} = c \sum_{\lambda \in \tilde{P}^k_+} |\chi_\lambda(\tau, H)|^2,$$

with some $c \in \mathbb{R}$. Here $P^k_+$ denotes the set of highest weights at some level $k$ of the untwisted affine Lie algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$ corresponding to $G$ and $\chi_\lambda$ denotes the corresponding affine character (see theorem 4.5).

It is well known that the sum above is invariant under a certain $SL(2,\mathbb{Z})$-action (cf. [KP]). Since the holomorphic structure on the torus $S^1 \times S^1$ defining the elliptic curve $\Sigma_\tau$ stays invariant under the natural $SL(2,\mathbb{Z})$-action on $S^1 \times S^1$, we can use the functional integral approach developed in this paper to deduce the modular invariance of the function

$$\sum_{\lambda \in \tilde{P}^k_+} |\chi_\lambda(\tau, 0)|^2.$$

We close this exposition with a remark on a "twisted" version of the WZW model and some speculations about a measure theoretic interpretation of the calculations leading to the partition function of the WZW model.

# 2 The Liouville functional

## 2.1 The Liouville functional

Let $(M,\omega)$ be a finite dimensional compact symplectic manifold of dimension $2n$. The Liouville volume form associated with the symplectic form $\omega$ is the $2n$-form $\frac{\omega^n}{n!}$. Let us assume that we have an action of some torus $T = \mathbb{R}^l/\mathbb{Z}^l$ on $M$ which preserves $\omega$. The Lie algebra of $T$ will be denoted by $\mathfrak{h}$. We will call $H \in \mathfrak{h}$ a generic element of $\mathfrak{h}$ if the group generated by $\exp(H)$ is a dense subgroup of $T$. Any $H$ defines an $\mathbb{R}$-action and thereby a vector field $\widetilde{H}$ on $M$. Assume that $J_H$ is a Hamiltonian function corresponding to this $\mathbb{R}$-action. That is, we have the identity $dJ_H = \iota_{\widetilde{H}}\omega$. Furthermore, assume that $T$ acts effectively on $M$ and that the fixed point set $P$ consists of isolated points $p \in P$. Then $T$ acts linearly on the tangent spaces $T_pM$. After picking an almost complex structure on $M$ which is compatible with the symplectic form $\omega$ and which commutes with the $T$-action, we have a decomposition $T_pM = \bigoplus_{j=1}^n V_j^p$ into (complex) one-dimensional representations $V_j^p$ of $T$. Here $T$ acts on $V_j$ via the complex character $t \mapsto \exp(2\pi i \alpha_j^p(H))$, where $\exp(H) = t$.

Now we can state the Duistermaat Heckman exact integration formula ([DH1]):

**Theorem 2.1** *Let $M, \omega, H, P$ be as above. Then*

$$\int_M e^{-tJ_H} \frac{\omega^n}{n!} = \sum_{p \in P} \frac{e^{-tJ_H(p)}}{t^n \prod_{j=1}^n \alpha_j^p(H)},$$

*where $t$ can be a real or complex parameter.*

This theorem can be easily extended to the case when the fixed point set of the $T$-action consists of sub-manifolds instead of isolated points (cf. [DH1], [DH2], [A]).

In the case that $M$ has a Riemannian metric $\sigma$ the associated Riemannian volume form $d\sigma$ and the Liouville form on $M$ are related via

$$\frac{\omega^n}{n!} = \mathrm{Pf}(B_\sigma) d\sigma,$$

where $B_\sigma$ is the skew symmetric endomorphism of the tangent bundle associated to $\omega$ by the metric $\sigma$ (i.e. $\omega_x(X, Y) = \sigma_x(B_{\sigma,x}(X), Y)$ for $X, Y \in T_xM$), and Pf is the Pfaffian.

If the manifold $M$ is infinite dimensional the Liouville measure does not make sense. Ignoring this fact, physicists use the Duistermaat Heckman formula to "calculate" certain integrals over the Liouville measure (see e.g. [A], [P] for examples). We will proceed in the opposite direction and use an analogue of the Duistermaat Heckman formula to define a functional on the set of functions on $M$ which are Hamiltonian with respect to the $T$-action on $M$. We will call this functional the Liouville functional because of its analogy with the Liouville measure on a finite dimensional manifold.

To get started, let $(M, \omega)$ be an infinite dimensional symplectic manifold. That is, $M$ is a Frechet-manifold together with a closed two form $\omega$ which is non-degenerate in the sense that the map $T_mM \to T_m^*M$, $X_m \mapsto \omega_m(X_m, \cdot)$ is injective at each $m \in M$. Furthermore, we assume the tangent spaces $T_mM$ to have a countable basis for all $m \in M$. Finally, we have to make the further assumption that $(M, \omega)$ admits a compatible almost complex structure, that is an endomorphism $I$ of $TM$ such that $I^2 = -1$, $I^*\omega = \omega$ and $\omega(\cdot, I\cdot)$ is positive definite.

Now suppose there is an effective action of a torus $T$ on $M$ which preserves $\omega$ and $I$. Let us assume that the fixed point set $P$ of the $T$-action consists of (possibly infinitely many) isolated points $p \in P$. Then we have a $T$-action on the tangent spaces $T_pM$ which again decompose into the direct sum of complex one dimensional representations of $T$. That is, $T_pM = \bigoplus_{j\in\mathbb{N}} V_j^p$ where, as before, $T$ acts on $V_j^p$ via a complex character $\exp(H) \mapsto \exp(2\pi i \alpha_j^p(H))$ for $H \in \mathfrak{h}$. Now we have $\alpha_j^p(H) \in \mathbb{R}$ for all $p \in P$ and $j \in \mathbb{N}$. Let us assume that the series $\{|\alpha_j^p(H)|\}_{j\in\mathbb{N}}$ is zeta-multipliable for each $p \in P$ and denote the corresponding zeta-function by $\zeta$. (See the appendix for a brief introduction to the theory of zeta-regularized products.) Then we set

$$Z_p(H) = \left(\prod_{j\in\mathbb{N},} |\alpha_j^p(H)|\right)_\zeta .$$

Up to sign, this is the infinite dimensional analogue of the denominator in the Duistermaat Heckman formula for finite dimensional compact manifolds. To take care of the sign, we define $\#p$ to be the number of rotation planes $V_j^p$ for which $\alpha_j^p(H) < 0$. We have to make the assumption that $\#p$ is finite for all $p \in P$. Then $(-1)^{\#p}$ will be the desired sign.

We have collected all the necessary structures to define the Liouville functional:

**Definition 2.2** *Let $(M,\omega)$ be an infinite dimensional symplectic manifold with a $T$-action for which all the assumptions above are satisfied. For $H \in \mathfrak{h}$ as above, let $J_H$ be a Hamiltonian function of the $\mathbb{R}$-action on $M$ defined by $H$. Then for $t \in \mathbb{R}_{>0}$ we define the Liouville functional $L_t(J_H)$ via*

$$L_t(J_H) = \sum_{p\in P} (-1)^{\#p} \frac{e^{-tJ_H(p)}}{t^{\zeta(0)} Z_p(H)}$$

*whenever this sum makes sense.*

Note that $L_t$ is a non-linear functional.

Let us assume, that additionally to being symplectic, $M$ is a Riemannian manifold with Riemannian metric $\sigma$. That is, we have a non-degenerate symmetric bilinear form $\sigma_m = \langle .,. \rangle_m$ on each tangent space $T_mM$ which varies smoothly with $m$. As before, by non-degeneracy we mean that the induced map $T_mM \to T_m^*M$ is injective. In this case, we can define an analogue to the integration with respect to the Riemannian volume form on $M$ provided that $\omega$ and $\sigma$ are compatible in the following sense:

Let us assume, there exists a skew symmetric automorphism $B_{\sigma,x}$ of $T_xMx \in M$ for each $x \in M$ such that

$$\omega_x(X,Y) = \sigma_x(B_{\sigma,x}(X),Y).$$

Furthermore, let us assume that the zeta-regularized determinant $\det_\zeta(B_{\sigma,x})$ (i.e. the zeta-regularized product of the eigenvalues of $B_{\sigma x}$) exists. Now, if the zeta-regularized determinant defines a nowhere vanishing positive function on $M$, we can define a function $\mathrm{Pf}_\zeta(B_\sigma) : M \to \mathbb{R}_+$ such that $\mathrm{Pf}_\zeta(B_\sigma)(x)^2 = \det_\zeta(B_{\sigma,x})$. This function will be called the zeta-regularized Pfaffian of $B_\sigma$. We will call the symplectic form $\omega$ and the Riemannian metric $\sigma$ zeta-compatible if such $\mathrm{Pf}_\zeta(B_\sigma)$ exists. On a finite dimensional manifold, $\omega$ and $\sigma$ are compatible, exactly if they define the same orientation of $M$, and $\mathrm{Pf}(B_\sigma)$ is the function relating the two $2n$-forms.

In analogy with the finite dimensional case, we can now set

$$\int_M e^{-tJ_H}\mathrm{Pf}_\zeta(B_\sigma)d\sigma = L_t(J_H).$$

In the cases we will be considering, $M$ will be a homogeneous space $M = G/G'$ and $\omega$ and $\sigma$ can be chosen invariant under the canonical $G$-action on $M$. In this case, if $\mathrm{Pf}_\zeta(B_\sigma)$ exists, it will be a constant. Therefore such an $M$ is always orientable and we have

$$\int_M e^{-tJ_H}d\sigma = \frac{L_t(J_H)}{\mathrm{Pf}_\zeta(B_\sigma)(x_0)}.$$

## 2.2 The Liouville functional on a complex manifold

In this section we will describe how one can extend the formalism developed in the last section to the case when the manifold $M$ is complex and $\omega = \omega_1 + i\omega_2$ is a closed non-degenerate complex valued $\mathbb{C}$-linear two-form. In this case $\omega$ will not be compatible with the natural complex structure $I$ on $TM$ which is given by multiplication with $i$, since we have $\omega(IX, IY) = -\omega(X, Y)$. So we have to assume that $TM$ admits a second complex structure $J$ which anti-commutes with $I$ and is compatible with $\omega$ in the following sense: $J^*\omega = \omega$, and $\omega(\cdot, J\cdot)$ is "positive definite" in the sense that for all $m \in M$, $X_m \in T_mM$ we have either $\omega_2(X_m, J_m(X_m)) > 0$, or $\omega_2(X_m, J_m(X_m)) = 0$ and $\omega_1(X_m, J_m(X_m)) > 0$.

As in the preceding section, let us assume that we have an action of some torus $T$ on $M$ which leaves the symplectic form $\omega$ and the two complex structures $I$ and $J$ invariant, and which has a discrete fixed point set $P$. Thus, $J$ gives the tangent spaces $T_pM$ the structure of quaternionic representations of $T$ (see [BtD]). Now we can decompose each $T_pM = T_pM^+ \oplus T_pM^-$, where $T_pM^\pm$ denotes the $\pm i$ eigenspace of $J$. The spaces $T_pM^+$ and $T_pM^-$ are isomorphic as vector spaces.

We can decompose $T_pM^+ = \bigoplus_{j\in\mathbb{N}} V_j^p$, into a direct sum of one dimensional complex representations (with respect to the complex structure $J$), such that $T$ acts on $V_j^p$ via the character $\exp(2\pi i\alpha_j^p)$. With these choices made, we can define $Z_p(H)$, $\#p$, and the Liouville functional $L_t(J_H)$ of some Hamiltonian function $J_H$ exactly as in §2.1.

Finally, in the complex setting it is natural and in fact necessary for applications, to allow the element $H$ considered above to be in the complexified Lie algebra $\mathfrak{h}_\mathbb{C}$ of $T$ rather than in the real Lie algebra $\mathfrak{h}$. Since $M$ is complex, such $H$ defines a vector field $\tilde{H}$ on $M$, and we shall call a function $J_H : M \to \mathbb{C}$ Hamiltonian with respect to $\tilde{H}$ if $dJ_H = \iota_{\tilde{H}}\omega$. The formalism of the Liouville functional can be generalized to this setting without major changes. We get a decomposition of the tangent spaces $T_pM = \bigoplus_{j\in\mathbb{N}} V_j^p$ into complex (with respect to $I$) one dimensional spaces $V_j^p$ on which the Abelian Lie algebra $\mathfrak{h}_{\mathbb{C}}$ acts via the character $2\pi i\alpha_j^p$. The only difference to the real case is that now the $\alpha_j^p(H)$ might not be in $\mathbb{R}$, which causes problems with the zeta-regularized products and the definition of the number $\#p$ appearing in the definition of the Liouville functional. In the examples we will consider, we will be able calculate the zeta-regularized products using a trick. So the only thing, we have to take care about is the definition of the number $\#p$ in this more general setting.

To generalize the definition of $\#p$, let us first take a closer look at what happened in the case that $H \in \mathfrak{h}$: We have decompositions of the tangent spaces of $M$ at $p$ into 4-dimensional real representations $T_pM = \bigoplus_j (V_{\alpha_j}^p \oplus V_{-\alpha_j}^p)$ of $T$, such that if one diagonalizes the $T$-action with respect to the complex structure $I$, the torus acts on $V_{\alpha_j}^p$ via the character $\exp(2\pi i\alpha_j^p)$. In this setting, the complex structure $J$ defines an $\mathbb{R}$-linear map $J : V_\alpha^p \to V_{-\alpha}^p$. Restricting the $T$-action to the $+i$ eigenspace of $J$ amounts to picking one character out of each pair $\pm\alpha_j^p$ appearing in the decomposition of $T_pM$. Now the choice of some regular $H \in \mathfrak{h}$ (i.e. $\beta(H) \notin \mathbb{Z}$ for all characters $\exp(2\pi i\beta)$ of $T$), gives a decomposition of the character lattice $Q$ of $T$ into positive and negative characters via declaring $\beta \in Q$ positive if $\beta(H) > 0$. In this picture, $\#p$ is exactly the number of negative characters appearing in the series $\{\alpha_j^p\}_{j\in\mathbb{N}}$.

Now it is straight forward to generalize our definition of $\#p$ to the complex case: The element $H \in \mathfrak{h}_{\mathbb{C}}$ comes from an infinitesimal action of the complexified torus $T_{\mathbb{C}}$ on $M$. So we have to pick a decomposition of the character lattice $Q\setminus\{0\} = Q_+ \cup Q_-$ of $T_{\mathbb{C}}$ into positive and negative characters. In analogy with the real case, this decomposition should be defined by the element $H \in \mathfrak{h}_{\mathbb{C}}$: We define $\alpha \in Q$ to be positive, if either $Im(\alpha(H)) > 0$, or $Im(\alpha(H)) = 0$ and $Re(\alpha(H)) > 0$. This choice of decomposition of the character lattice $Q$ into a positive and negative part agrees with the definition of "positive definiteness" of the $\mathbb{C}$-valued symmetric bilinear form $\omega(\cdot, J\cdot)$ above. Then, as before, we can set $\#p$ to be the number of negative characters appearing in the series $\{\alpha_j^p\}_{j\in\mathbb{N}}$.

# 3 The partition function for compact Lie Groups

## 3.1 An infinite dimensional flag manifold

Let us look at a first example in which the Liouville functional can be calculated and gives rise to an interesting function: Consider the infinite dimensional manifold $LG/T$, where $G$ is a compact semi-simple simply connected Lie group with maximal torus $T$ and $LG$ denotes the loop group of $G$ in the sense of [PS]. That is, $LG = C^\infty(S^1, G)$ with pointwise multiplication. The Lie algebra of $LG$ is $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of $G$. Let $\langle .,. \rangle$ denote the negative of the Killing form on $\mathfrak{g} \otimes \mathbb{C}$. This gives a positive definite $G$-invariant bilinear form on $\mathfrak{g}$. We can use $\langle .,. \rangle$ to define an antisymmetric bilinear form $\omega$ on $L\mathfrak{g}$ via

$$\omega(X,Y) = \int_{t=0}^{1} \langle X'(t), Y(t) \rangle dt,$$

where we have parametrized the circle via $t \in [0,1]$. Obviously, this form is degenerate exactly in the space of constant loops. Thus, using the $G$-invariance of $\langle .,. \rangle$, it gives rise to a symplectic form $\omega$ on $LG/G$ via left translation. Also, $G/T$ is a generic coadjoint orbit of $G$ and hence has a symplectic structure for each generic $H \in \mathfrak{h}$ given by the Kirillov form $\omega_0^H$ . On the tangent space at $eT$, this form is given by

$$\omega_0^H(X,Y) = \langle H, [Y,X] \rangle.$$

As a manifold, $LG/T$ is isomorphic to $LG/G \times G/T$. So for each generic $H \in \mathfrak{h}$, we have a symplectic form on $LG/T$ given by $\omega^H = \mathrm{pr}_1^* \omega + \mathrm{pr}_2^* \omega_0^H$.

There is a second symplectic structure on $LG/T$ which comes from the fact that $LG/T$ is isomorphic to a coadjoint orbit of the group $LG$. The coadjoint action of $\gamma \in LG$ on $L\mathfrak{g} \oplus \mathbb{R}$ is given by (cf. [F], [PS])

$$\gamma(X, \lambda) = (\mathrm{Ad}\gamma(X) + \lambda \gamma' \gamma^{-1}, \lambda).$$

Here $L\mathfrak{g}$ is identified with the smooth part of $(L\mathfrak{g})^*$ via the non degenerate symmetric bilinear form $\langle .,. \rangle$ on $L\mathfrak{g}$ defined by

$$\langle X, Y \rangle = \int_0^1 \langle X(t), Y(t) \rangle dt.$$

The orbits of this action can be classified in terms of conjugacy classes of the corresponding compact Lie group $G$ via the following construction: For each $(X, \lambda) \in L\mathfrak{g} \oplus \mathbb{R}$ with $\lambda \neq 0$ one can solve the differential equation

$$z' = -\lambda^{-1} X z$$

with initial condition $z(0) = 1$. Since $X$ is periodic in $t$, we have $z(t+1) = z(t).M_X$, where $M_X = z(1) \in G$ is the monodromy of the differential equation. Now the theory of differential equations with periodic coefficients (cf. [F], [PS]) implies:

**Proposition 3.1**

(i) *For $\lambda \neq 0$, the orbits of $LG$ on $L\mathfrak{g} \times \{\lambda\}$ correspond precisely to the conjugacy classes of $G$ under the map $(X, \lambda) \mapsto M_X$.*

(ii) *The stabilizer of $(X, \lambda)$ in $LG$ is isomorphic to the centralizer $Z_X$ of $M_X$ in $G$ under the map $\gamma \mapsto \gamma(0)$.*

Let $H$ be a generic element of $\mathfrak{h}$ and $\lambda \neq 0$. From Proposition (3.1) we see that the orbit of $LG$ through $(H, \lambda)$ is isomorphic to $LG/T$. The Kirillov form $\widetilde{\omega}^H$ on $LG/T$ is defined exactly as in the finite dimensional case. Now we can compare the two symplectic forms on $LG/T$:

**Lemma 3.2** *The 2-forms $\omega^H$ and $-\widetilde{\omega}^H$ lie in the same cohomology class of $LG/T$.*

PROOF. This is a direct generalization of [PS], Prop 4.4.4. □

It was shown in [PS], Ch. 8.9 that the symplectic manifold $LG/G$ admits a complex structure which makes it into a Kähler manifold. Furthermore, we can pick a complex structure on $G/T$ which is compatible with the symplectic form $\omega_0^H$. Putting these two structures together, we get a complex structure on $LG/T$ which is compatible with $\omega^H$.

Our next goal is to show that the symplectic form $\omega^H$ and the Riemannian metric $\sigma$ on $M$ are compatible in the sense of §2.1. The Riemannian metric on $LG/T$ is given by

$$\sigma_{eT}(X, Y) = \int_0^1 \langle X(t), Y(t) \rangle dt,$$

while the symplectic form $\omega^H$ is given by

$$\begin{aligned} \omega^H_{eT}(X, Y) = &\int_0^1 \langle X'(t), Y(t) \rangle + \int_0^1 \langle H, [Y(t), X(t)] \rangle dt \\ = &\sigma_{eT}(X', Y) - \frac{1}{2\pi} \int_0^{2\pi} \langle [H, X(t)], Y(t) \rangle dt \end{aligned}$$

So the endomorphism $B_{\sigma,eT}$ is given by

$$B_{\sigma,eT} = \frac{\partial}{\partial t} - \mathrm{ad}(H).$$

Since both the Riemannian metric and the symplectic form are defined on $LG/T$ via left translation, the zeta-regularized Pfaffian of $B_\sigma$ - if it exists - will be a constant. So to check the compatibility of $\omega$ and $\sigma$, we have to show that the zeta-regularized Pfaffian $\mathrm{Pf}_\zeta(B_\sigma)(eT)$ indeed exists. Let us identify the root system $\Delta$ of $\mathfrak{g} \otimes \mathbb{C}$ with a subset of the character lattice $Q$ of $T$ defined above (see e.g.[BtD]).

**Lemma 3.3** *The Pfaffian* $\mathrm{Pf}_\zeta(B_\sigma)(eT)$ *exists and is given by*

$$\mathrm{Pf}_\zeta(B_\sigma)(eT) = (2\pi)^{\dim \mathfrak{g}} \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H)),$$

*where* $\Delta_+$ *is the set of positive roots with respect to the Weyl chamber* $K \subset \mathfrak{h}$ *such that* $H \in K$.

**Remark 3.4** Note that $\prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))$ is the denominator in the Weyl character formula for the compact Lie group $G$.

PROOF. Using the root space decomposition of $\mathfrak{g}\otimes\mathbb{C}$, one sees that the eigenvalues of $B_{\sigma,eT}$ are $\{\pm 2\pi i n\}_{n\in\mathbb{N}} \cup \{\pm 2\pi i n \pm 2\pi i\alpha(H)\}_{n\in\mathbb{N}_0,\alpha\in\Delta_+}$. The multiplicity of the eigenvalues is 1 if $\alpha \neq 0$ and $l = \dim T$ if $\alpha = 0$. Thus, the zeta-regularized determinant of $B_{\sigma,eT}$ is

$$\begin{aligned}
\det_\zeta(B_{\sigma,eT}) &= \prod_{\alpha\in\Delta_+}\left((2\pi\alpha(H))^2\prod_{n=1}^{\infty}(2\pi)^4(n^2-\alpha(H)^2)^2\right)_\zeta \cdot \left(\prod_{n=1}^{\infty}(2\pi n)^2\right)_\zeta^l \\
&= \prod_{\alpha\in\Delta_+}\left(2\pi\alpha(H)\prod_{n=1}^{\infty}(2\pi n)^2(1-\frac{\alpha(H)^2}{n^2})\right)_\zeta^2 \cdot \left(\prod_{n=1}^{\infty}2\pi n\right)_\zeta^{2l} \\
&= \prod_{\alpha\in\Delta_+} 4\sin^2(\pi\alpha(H))\left(\prod_{n=1}^{\infty}2\pi n\right)_\zeta^{2\dim\mathfrak{g}} \\
&=(2\pi)^{2\dim\mathfrak{g}}\prod_{\alpha\in\Delta_+}4\sin^2(\pi\alpha(H)),
\end{aligned}$$

where we have used the identity

$$\prod_{n=1}^{\infty}(1-\frac{x^2}{\pi^2n^2}) = \frac{\sin(x)}{x}$$

and the first example for zeta-regularized products from the appendix. Since the Pfaffian $\mathrm{Pf}_\zeta(B_\sigma)(eT)$ is defined as the square root of $\det_\zeta(B_{\sigma,eT})$, the lemma follows. □

## 3.2 A torus-action on $LG/T$

Let us identify $LG/G$ with the space of based loops $\Omega G = \{\gamma \in LG \mid \gamma(0) = e\}$. The circle $S^1$ acts on $\Omega G$ by rotations $R_t$, where

$$R_t\gamma(u) = \gamma(u+t)\gamma(t)^{-1}.$$

The maximal torus $T \subset G$ acts by conjugation on $LG/G$, so together we get an $S^1 \times T$ action: An element $(t, \exp(H)) \in S^1 \times T$ acts on $LG/G$ via

$$(t, \exp(H)) : \gamma \mapsto \exp(tH) R_t(\gamma) \exp(-tH).$$

The fixed points of this action are precisely the homomorphisms $\gamma : S^1 \to T$ ([PS], Ch.8.9). Furthermore, $T$ acts by left multiplication on $G/T$ with fixed point set $N(T)/T$ where $N(T)$ is the normalizer of $T$ in $G$. Letting $S^1$ act trivially on $G/T$, we get an $S^1 \times T$-action on $LG/T$ which has fixed point set $Q^\vee \times W$, where $Q^\vee$ denotes the lattice of homomorphisms $S^1 \to T$ and $W = N(T)/T$ is the Weyl group of $G$. It is straight forward to check, that this torus action leaves the symplectic form $\omega^H$ as well as the complex structure $I$ invariant.

Take a generic $H \in \mathfrak{h}$. Such $H$ defines an $\mathbb{R}$-action on $LG/T$ by the construction outlined above. If $tH \in ker(\exp)$ for some $t \in \mathbb{R}^*$, we get in fact an $S^1$-action but this will be of no concern for us. In both cases, the fixed point set of this $\mathbb{R}$-action is still $Q^\vee \times W$.

Now let us compute the denominator $Z_p(H)$ of the Liouville functional: The tangent space of $LG/T$ at the point $eT$ is isomorphic to $L\mathfrak{g}/\mathfrak{h}$. Its decomposition into rotation planes is exactly the decomposition of $L\mathfrak{g}/\mathfrak{h}$ into eigenspaces of the endomorphism $B_\sigma(eT)$ used in lemma 3.3. As we saw in the proof of lemma 3.3, the eigenvalues of the torus-action are given by the two series $\{2\pi i(\pm\alpha(H) \pm n)\}_{\alpha\in\Delta_+, n\geq 0}$ and and $\{\pm 2\pi i n\}_{n>0}$ again with the multiplicity 1 if $\alpha = 0$ and $l$ if $\alpha \neq 0$. Remember that in the definition of $Z_{eT}(H)$, the eigenvalues of the torus-action were multiplied by $\frac{1}{2\pi}$. Therefore, we have to calculate the regularized product

$$Z_{eT}(H) = \prod_{\alpha\in\Delta_+} \left( (\alpha(H))^2 \prod_{n=1}^{\infty} (n^2 - \alpha(H)^2)^2 \right)_\zeta \cdot \left( \prod_{n=1}^{\infty} n^2 \right)_\zeta^l .$$

Now the same calculation as in the proof of lemma 3.3 yields

**Lemma 3.5**

$$Z_{eT}(H) = (\sqrt{2\pi})^l \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H)),$$

So the series defining $Z_{eT}(H)$ is zeta-multipliable and we have checked all the necessary premises to calculate the Liouville functional of a Hamiltonian of our $\mathbb{R}$-action on $M$.

Finally, we have to calculate the number $\#(\beta, w)$ for the fixed points $(\beta, w) \in Q^\vee \times W$ of the torus action. To do this, we will identify the fixed point set $Q^\vee \times W$ with the Weyl group $\widetilde{W}$ of the untwisted affine Lie algebra $\tilde{\mathfrak{g}}_\mathbb{C}$ corresponding to the Lie algebra G. Furthermore, the set $\{\pm\alpha\pm n \mid \alpha \in \Delta_+, n \geq 0\} \cup \{\pm n \mid n > 0\}$ can be identified with the root system $\tilde{\Delta}$ of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ (cf. [K]). Let us assume, that $H \in \mathfrak{h}$ lies in a fundamental alcove of the $\widetilde{W}$-action on $\mathfrak{h}$. Then the set $\tilde{\Delta}_+ = \{\alpha \in \tilde{\Delta} \mid \alpha(H) > 0\}$ defines a decomposition of $\tilde{\Delta}$ into positive and negative roots. Now one can see (for example by identifying the tangent spaces $T_{(\beta,w)}LG/T$ with $T_e LG/T$ via left multiplication by a representative of $(\beta,w)^{-1}$ in $LG$) that $\#(\beta,w)$ is exactly the number of positive roots of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ which are mapped to negative roots by the action of $(\beta,w)$ on the root system of $\tilde{\mathfrak{g}}_{\mathbb{C}}$. By definition, this is the length $l(\beta,w)$ of $(\beta,w)$ in $\widetilde{W}$.

## 3.3 Calculation of a Liouville functional

Given $H \in \mathfrak{g}$ and $\gamma \in LG$, we can define a vector field $\mathrm{ad}H(\gamma)$ along $\gamma$ via

$$\mathrm{ad}H(\gamma) = \frac{\partial}{\partial s}|_{s=0} \exp(sH)\gamma\exp(-sH).$$

For generic $H \in \mathfrak{h}$ let us define a function $J_H : LG \to \mathbb{R}$ via

$$\gamma \mapsto \frac{1}{2}\int_0^1 \|(\frac{\partial}{\partial t}\gamma(t) - \mathrm{ad}H(\gamma(t)))\gamma^{-1}(t)\|^2 dt.$$

Since the scalar product $\langle .,. \rangle$ is $G$ invariant, we have $J_H(\gamma) = J_H(\gamma h)$ for all $h \in T$. Therefore, $J_H$ defines a function on $LG/T$ which will be denoted with the same symbol. Suppose we have fixed a finite dimensional faithful representation of the group $G$. Then we can write $J_H(\gamma) = \frac{1}{2}\int_0^1 \|\gamma'(t)\gamma^{-1}(t) + \gamma(t)H\gamma^{-1}(t) - H\|^2 dt$.

**Lemma 3.6** *The Hamiltonian vector field on $LG/T$ corresponding to $J_H$ is exactly the vector field on $LG/T$ coming from the $\mathbb{R}$-action defined by $H$.*

Proof. This is a calculation similar to the proof of Prop. 8.9.3 in [PS]: Choose a representative $\gamma$ for $\gamma T \in LG/T$ and let $\delta\gamma$ be an infinitesimal variation of $\gamma$. That is, $\delta\gamma = \gamma Y$ with $Y \in L\mathfrak{g}$. The vector field generated by the $\mathbb{R}$-action on $LG/T$ is given at the point $\gamma T$ by $\gamma' + \mathrm{ad}H(\gamma)$ (mod $\mathfrak{h}$). So we have to show that

$$(dJ_H)_\gamma(\delta\gamma) = \omega^H_\gamma(\delta\gamma, \gamma' + \mathrm{ad}H(\gamma)).$$

But $dJ_{H\gamma}(\delta\gamma)$ is given by

$$\begin{aligned} dJ_{H\gamma}(\delta\gamma) &= \int_0^1 \langle \delta(\gamma'\gamma^{-1} + \gamma H\gamma^{-1} - H), \gamma'\gamma^{-1} + \gamma H\gamma^{-1} - H\rangle dt \\ &= \int_0^1 \big(\langle \delta(\gamma'\gamma^{-1}), \gamma'\gamma^{-1}\rangle + \langle \delta(\gamma'\gamma^{-1}), \gamma H\gamma^{-1}\rangle \\ &\qquad - \langle \delta(\gamma'\gamma^{-1}), H\rangle + \langle \delta(\gamma^{-1}H\gamma), \gamma'\gamma^{-1}\rangle \\ &\qquad + \langle \delta(\gamma^{-1}H\gamma), \gamma H\gamma^{-1}\rangle - \langle \delta(\gamma^{-1}H\gamma), H\rangle\big) dt \end{aligned}$$

A direct calculation shows

$$\delta(\gamma'\gamma^{-1}) = (\delta\gamma\gamma^{-1})' + [\delta\gamma\gamma^{-1}, \gamma'\gamma^{-1}]$$

and

$$\delta(\gamma H\gamma^{-1}) = [\delta\gamma\gamma^{-1}, \gamma H\gamma^{-1}].$$

Furthermore, pointwise $G$-invariance of the scalar product $\langle .,. \rangle$ implies

$$\langle [\delta\gamma\gamma^{-1}, \gamma'\gamma^{-1}], \gamma'\gamma^{-1}\rangle = 0 = \langle [\delta\gamma\gamma^{-1}, \gamma H\gamma^{-1}], \gamma H\gamma^{-1}\rangle.$$

Partial integration yields

$$\int_0^1 \langle (\delta\gamma\gamma^{-1})', H\rangle = 0,$$

so that we get

$$dJ_{H\gamma}(\delta\gamma) = \int_0^1 \langle (\delta\gamma\gamma^{-1})', \gamma'\gamma^{-1} + \gamma H\gamma^{-1}\rangle dt + \int_0^1 \langle H, [\gamma'\gamma^{-1} + \gamma H\gamma^{-1}, \delta\gamma\gamma^{-1}]\rangle.$$

This is the assertion. □

The next step is to calculate $J_H$ in the fixed points of the $\mathbb{R}$-action: Let $g_w \in N(T)$ be a representative for $w \in W$, and let $\beta \in Q^\vee$. Then for $\gamma(t) = g_w \exp(t\beta)$ we have

$$\begin{aligned} J_H(\gamma) =& \frac{1}{2}\int_0^1 \|\beta + w(H) - H\|^2 dt \\ =& \frac{1}{2}\|\beta + w(H) - H\|^2 \end{aligned}$$

Plugging this into the definition of the Liouville functional yields

$$L_1(J_H) = \frac{1}{(\sqrt{2\pi})^l \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))} \sum_{w\in W}\sum_{\beta\in Q^\vee} (-1)^{l(w)} e^{-\frac{1}{2}\|\beta+w(H)-H\|^2}$$

Let $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$ denote the half sum of positive roots. A calculation due to I. Frenkel ([F], Theorem 4.3.4) involving Poisson re-summation of the sum above, gives

$$\sum_{w\in W}\sum_{\beta\in Q^\vee} (-1)^{l(w)} e^{-\frac{1}{2}\|\beta+w(H)-H\|^2} = \frac{\prod_{\alpha\in\Delta_+} 4\sin^2(\pi\alpha(H))}{(\sqrt{2\pi})^l \cdot \mathrm{vol}(Q^\vee)} \sum_{\lambda\in P_+} |\chi_\lambda(\exp(H))|^2 e^{-\frac{1}{2}\|\lambda+\rho\|^2},$$

where $P_+$ denotes the set of dominant weights of $G$ and $\chi_\lambda$ is the irreducible character of $G$ corresponding to $\lambda \in P_+$.

Putting the above calculations together with the definition of the Riemannian volume $d\sigma$ in §2.1 yields the main theorem for this section:

**Theorem 3.7** *The following identity is valid:*

$$\int_{LG/T} e^{-J_H} d\sigma = \frac{1}{(2\pi)^{l+\dim \mathfrak{g}} \mathrm{vol}(Q^\vee)} \sum_{\lambda \in P_+} |\chi_\lambda(\exp(H))|^2 e^{-\frac{1}{2}\|\lambda+\rho\|^2}.$$

**Remark 3.8** We can give a physical interpretation of our calculations leading to theorem 3.7: Consider a quantum mechanical particle moving on the compact Lie group $G$ with classical action $J_H$. Then according to Feynman's path integral formulation of quantum mechanics, the trace or partition function of this quantum mechanical system is formally given by $\int_{C(S^1,G)} e^{-J_H(\gamma)} d\gamma$, where the integration is over all closed loops in $G$. But this is basically the same as the "integral" $\int_{LG/T} e^{-J_H} d\sigma$ we calculated in theorem 3.7. Indeed, our definition of $\int_M e^{-J_H} d\sigma$ for a symplectic manifold $M$ is merely a formalization of the heuristic techniques employed in the physics literature in calculating such integrals.

**Remark 3.9** In our calculations leading to the partition function of the Lie group $G$, we always chose the element $H \in \mathfrak{h}$ which defines the symplectic structure on $LG/T$ to be the same as the element $K \in \mathfrak{h}$ which defines the $\mathbb{R}$-action. Of course this is not necessary and was only done to emphasize the similarity of the calculations with those leading to the partition function of the WZW-model in §4.3. Indeed, for certain choices of $H$ and $K$, the resulting function will have a very natural interpretation as we shall see momentarily:

Choose $a \in \mathbb{R}_{>0}$ and $H \in \mathfrak{h}$ such that $aH$ is generic in $\mathfrak{h}$. Then we can define a non-degenerate closed 2-form $\omega^{H,a}$ on $LG/T$ via $\omega^{H,a} = \mathrm{pr}_1^* \omega^a + \mathrm{pr}_2^* \omega_0^{aH}$, where we have identified $LG/T$ with $LG/G \times G/T$ as before, and $\omega^a_{eT}(X,Y) = \int_0^1 \langle aX'(t), y(t)\rangle dt$. Furthermore, let us choose $b \in \mathbb{R}_{>0}$ and $K \in \mathfrak{h}$. Then we can define an $\mathbb{R}$-action on $LG/T$ via

$$u : \gamma \mapsto \exp(buK) R_{bu}(\gamma) \exp(-buK).$$

If $bK$ is generic, the fixed point set of this action is $Q^\vee \times W$ as before. The vector field defined by this $\mathbb{R}$-action at a point $\gamma T \in LG/T$ is $b\gamma' + \mathrm{ad}(bK)(\gamma)$, and as in the proof of lemma 3.6 we can deduce that this vector field is exactly the Hamiltonian vector field on $LG/T$ corresponding to the function $J_{H,K,a,b}$, where

$$J_{H,K,a,b}(\gamma) = \frac{ab}{2} \int_0^1 \|\gamma'(t)\gamma^{-1}(t) + \gamma(t)K\gamma^{-1}(t) - H\|^2 dt.$$

The skew symmetric automorphism relating the Riemannian and the new symplectic structure on $LG/T$ is now $B_\sigma(eT) = a\frac{\partial}{\partial t} + \mathrm{ad}(aH)$ and its zeta-regularized Pfaffian is given by

$$\mathrm{Pf}_\zeta(B_\sigma)(eT) = a^{\dim \mathfrak{g}}(2\pi)^{\dim \mathfrak{g}} \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H)).$$

Accordingly, we get

$$Z_{eT}(K) = b^{\dim \mathfrak{g}}(2\pi)^{\dim \mathfrak{g}}(\sqrt{2\pi})^l \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(K)).$$

Set $c = \frac{1}{ab}$. Then the same calculation as the one leading to theorem 3.7 gives

$$\sum_{w\in W}\sum_{\beta\in Q^\vee}(-1)^{l(w)}e^{-\frac{1}{2c}\|\beta+w(K)-H\|^2} = \frac{\prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(-H))2\sin(\pi\alpha(K))}{(\sqrt{2\pi})^l\cdot \mathrm{vol}(Q^\vee)} \cdot \sum_{\lambda\in P_{++}} \chi_\lambda(\exp(-H))\chi_\lambda(\exp(K))e^{-\frac{c}{2}\|\lambda+\rho\|^2}.$$

So putting everything together gives the following generalization of theorem 3.7:

**Theorem 3.10** *Let $a, b, K, H$ as above. Then the following identity is valid:*

$$\int_{LG/T} e^{-J_{H,K,a,b}}d\sigma = \frac{1}{(ab)^{\dim \mathfrak{g}}(2\pi)^{l+\dim \mathfrak{g}}\mathrm{vol}(Q^\vee)} \sum_{\lambda\in P_+} \chi_\lambda(\exp(-H))\chi_\lambda(\exp(K))e^{-\frac{1}{2ab}\|\lambda+\rho\|^2}.$$

In [F], Frenkel has shown how for certain $H$ and $b$, the numerator of the right hand side of the equation in theorem 3.10 can be interpreted as the numerator of the Kac-Weyl character formula for highest weight representations of the untwisted affine Lie algebra corresponding to $\mathfrak{g}$ evaluated at $K, b$. So theorem 3.10 gives a realization of the affine characters as integrals over a coadjoint orbit. This is one of the main features of Kirillov's "method of orbits" in the representation theory of Lie groups [K2]. In the next paragraph we will compare our approach to these orbital integrals via the Liouville functional to the analytic approach using the Wiener measure on a compact Lie group developed in [F].

## 3.4 Comparison with Wiener measure

In his heuristic deduction of the index theorem for the Dirac operator on a Riemannian manifold, Witten (cf. [A]) has suggested that the Wiener measure on a

Riemannian manifold $M$ should be closely connected to the "Riemannian measure" on the loop space of $M$. (Of course, the loop space of $M$ is not a symplectic manifold in our sense, but one can extend the definition of the "Riemannian volume form" $d\sigma$ to this case.) In the case of the homogeneous space $LG/T$ we consider, we can make this connection between $d\sigma$ and the Wiener measure $d\varpi$ on the compact group $G$ explicit. In fact, one can embed $LG/T$ into a space of continuous maps $[0,1] \to G$ on which the Wiener measure is defined. So the first guess would be that after possibly some identifications one has $d\sigma = d\varpi$. But by construction, $d\sigma$ is invariant under left translations, whereas the Wiener measure is only quasi invariant: Set

$$C_G = \{z : [0,1] \to G \mid z(0) = e,\ z \text{ continuous}\}$$

and let $f : C_G \to \mathbb{R}$ be integrable with respect to the Wiener measure $\varpi$ on $C_G$. Then

$$\int_{C_G} f(z) d\varpi(z) = \int_{C_G} f(gz) e^{-\langle z'z^{-1}, g^{-1}g'\rangle - \frac{1}{2}\langle g'g^{-1}, g'g^{-1}\rangle} d\varpi(z),$$

where $g \in C_G$ and $\langle X, Y\rangle = \int_0^1 \langle X(t), Y(t)\rangle dt$ for $X, Y \in C([0,1], \mathfrak{g})$. See [F] for more details. To get rid of this defect, we will replace $d\varpi(z)$ with $d\widetilde{\varpi} = e^{\frac{1}{2}\|z'z^{-1}\|^2} d\varpi(z)$. The new "measure" $d\widetilde{\varpi}$ is indeed invariant under left translations and we will formally have $d\sigma = d\widetilde{\varpi}$ as desired.

To be more concrete, remember the classification of the $LG$-orbits on $L\mathfrak{g} \times \{1\}$ from Proposition 3.1: Let $\mathcal{O}_g$ denote the conjugacy class of $G$ containing the element $g$ and set

$$C_{G,\mathcal{O}_g} = \{z \in C_G \text{ such that } z(1) \in \mathcal{O}_g\}.$$

Let us identify $L\mathfrak{g} \times \{1\}$ with $L\mathfrak{g}$. Then $LG$ acts via $\gamma : X \mapsto \gamma X \gamma^{-1} + \gamma'\gamma^{-1}$. After identifying $LG/T$ with the $LG$-orbit through $H$, we can define a map

$$\phi : LG/T \to C_{G,\mathcal{O}_{\exp(H)}}$$

via

$$\gamma H \gamma^{-1} + \gamma'\gamma^{-1} \mapsto z_{\gamma H \gamma^{-1} + \gamma'\gamma^{-1}},$$

where $z_X$ denotes the fundamental solution of the differential equation $z' = -Xz$. Now one can identify $C_G$ with a subspace $(L\mathfrak{g})_0^*$ of $(L\mathfrak{g})^*$ (see [F]), and in this identification, $C_{G,\mathcal{O}_{\exp(H)}}$ can be viewed as the closure in $(L\mathfrak{g})_0^*$ of the coadjoint orbit containing $H$.

The most natural measure $C_{G,\mathcal{O}_g}$ is the conditional Wiener measure $\varpi_{G,\mathcal{O}_g}$ constructed in [F]. Let $f : C_{G,\mathcal{O}_g} \to \mathbb{R}$ be an integrable function with respect to this measure. The integral over $f$ will be denoted by

$$\int_{C_{G,\mathcal{O}_g}} f(z) d\varpi_{G,\mathcal{O}_g}(z).$$

This integral has the quasi invariance properties stated above. As outlined before, let us replace $d\varpi_{G,\mathcal{O}_g}(z)$ with $d\widetilde{\varpi}_{G,\mathcal{O}_g}(z) = e^{\frac{1}{2}\|z'z^{-1}\|^2} d\varpi_{G,\mathcal{O}_g}(z)$ such that we get

$$\int_{C_{G,\mathcal{O}_g}} f(\gamma z) d\widetilde{\varpi}_{G,\mathcal{O}_{\gamma(2\pi)g}}(z) = \int_{C_{G,\mathcal{O}_g}} f(z) d\widetilde{\varpi}_{G,\mathcal{O}_g}(z)$$

for all $\gamma \in C_G$.

Now let us define a function $\tilde{J}_H : C_{G,\mathcal{O}_{\exp(H)}} \to \mathbb{R}$ via $\tilde{J}_H(z) = \frac{1}{2}\|z'z^{-1} + H\|^2$. One checks directly that $\phi^* \tilde{J}_H = J_H$ with $\phi : LG/T \to C_{G,\mathcal{O}_{\exp(H)}}$ as before. The main result of this section is the following

**Proposition 3.11**

$$\int_{LG/T} e^{-J_H(\gamma)} d\sigma(\gamma) = c \cdot \int_{C_{G,\mathcal{O}_{\exp(H)}}} e^{-\tilde{J}_H(z)} d\widetilde{\varpi}_{G,\mathcal{O}_{\exp(H)}}(z)$$

*with* $c = e^{\frac{1}{2}\|\rho\|^2} (2\pi)^{l+\dim \mathfrak{g}} \mathrm{vol}(Q^\vee)$.

So up to a constant which does not depend on $H$, the Wiener measure on $C_{G,\mathcal{O}_{\exp(H)}}$ and the "Riemannian measure" on $LG/T$ are equal.

Proof. For $z \in C_{G,\mathcal{O}_{\exp(H)}}$ we have

$$\tilde{J}_H(z) = \frac{1}{2}\|z'z^{-1}\|^2 + \langle H, z'z^{-1}\rangle + \frac{1}{2}\|H\|^2,$$

so that we get

$$\int_{C_{G,\mathcal{O}_{\exp(H)}}} e^{-\tilde{J}_H(z)} d\widetilde{\varpi}_{G,\mathcal{O}_{\exp(H)}}(z) = e^{-\frac{1}{2}\|H\|^2} \int_{C_{G,\mathcal{O}_{\exp(H)}}} e^{-\langle H, z'z^{-1}\rangle} d\varpi_{G,\mathcal{O}_{\exp(H)}}(z).$$

The last integral was computed in [F], Theorem 5.2.15:

$$e^{-\frac{1}{2}\|H\|^2} \int_{C_{G,\mathcal{O}_{\exp(H)}}} e^{-\langle H, z'z^{-1}\rangle} d\varpi_{G,\mathcal{O}_{\exp(H)}}(z) = \sum_{\lambda \in P_+} |\chi_\lambda(H)|^2 e^{-\frac{1}{2}\|\lambda+\rho\|^2 - \|\rho\|^2}$$

Comparing this result with theorem 3.7 finishes the proof. □

**Remark 3.12** Theorem 3.11 can be easily extended to the function $J_{H,K,a,b}$ which shows that the approaches to the orbit theory of affine Lie algebras via the Wiener measure and via the Liouville functional are equivalent. Of course, following Witten's assumption that our "Riemannian volume form" does indeed have something to do with the Wiener measure, this is the result one should have expected.

## 3.5 The twisted partition function

In this section, we will "integrate" functions on the coadjoint orbits of twisted loop groups. In this case, the calculation of the zeta-regularized Pfaffian gives a "duality" between root systems which also appears in the calculation of characters of certain non-connected compact Lie groups (cf. [W2]).

Let $\psi$ be an outer automorphism of order $\mathrm{ord}(\psi) = r$ of the simply connected compact semi-simple Lie group $G$ such that $\psi$ acts as an automorphism of the Dynkin diagram on the root system of the complexified Lie algebra $\mathfrak{g} \otimes \mathbb{C}$. Let us denote by $L(G, \psi)$ the corresponding twisted loop group:

$$L(G, \psi) = \{\gamma \in LG \mid \psi(\gamma(t)) = \gamma(t + \frac{1}{r}) \text{ for all } t \in [0, 1]\}$$

The Lie algebra of $L(G, \psi)$ will be denoted by $L(\mathfrak{g}, \psi)$. By restriction, the symmetric invariant form $\langle ., . \rangle$ and the antisymmetric form $\omega$ on $L\mathfrak{g}$ give a symmetric and an antisymmetric form on $L(\mathfrak{g}, \psi)$ which will be denoted by the same symbols. The form $\langle ., . \rangle$ is non-degenerate on $L(\mathfrak{g}, \psi)$ and defines a Riemannian structure $\sigma$ on $L(G, \psi)$ by left translation. The form $\omega$ is degenerate exactly in the subspace of constant loops so that it defines a symplectic form on $L(G, \psi)/G^\psi$ where $G^\psi$ denotes the group of fixed points under the automorphism $\psi$. Since we chose $G$ to be compact and semi-simple, so will be $G^\psi$ with maximal torus $T^\psi$. The manifold $G^\psi/T^\psi$ can be viewed as a coadjoint orbit of $G^\psi$ through a generic $H \in \mathfrak{h}^\psi$. As before, the Kirillov form on such orbit will be denoted by $\omega_0^H$. After identifying $L(G, \psi)/T^\psi$ with $L(G, \psi)/G^\psi \times G^\psi/T^\psi$, we can define a symplectic structure $\omega^H$ on $L(G, \psi)/T^\psi$ via $\omega^H = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega_0^H$. As in §3.1, the skew symmetric endomorphism of the tangent space at $eT^\psi$ of $L(G, \psi)/T^\psi$ relating the Riemannian metric $\sigma$ and the symplectic form is given by $B_{\sigma, eT^\psi}) : X \mapsto X' + \mathrm{ad}H(X)$. The calculation of the zeta-regularized Pfaffian of $B_{\sigma, eT^\psi}$ is essentially the same as the calculation of the zeta-regularized Pfaffian in §3.1 but we have to be more careful with the multiplicities of the eigenvalues:

Let $\Delta$ denote the root system of $\mathfrak{g}\otimes\mathbb{C}$ and let $\Delta^\psi$ denote the "folded" root system, i.e. $\Delta^\psi = \{\bar{\alpha} \mid \alpha \in \Delta\}$, where $\bar{\alpha}$ denotes the element $\bar{\alpha} = \frac{1}{\mathrm{ord}(\psi)} \sum_{i=1}^{\mathrm{ord}(\psi)} \psi^i(\alpha)$. Let us assume for the moment that $\Delta$ is an irreducible root system of type ADE but not of type $\mathrm{A}_{2n}$. In this case $\Delta^\psi$ is a root system of type BCFG. Let $\Delta_s^\psi$ and $\Delta_l^\psi$ denote the subsets of short and long roots in $\Delta^\psi$ respectively. Now we can use [K], Prop 6.3 to see that the eigenvalues of $B_{\sigma, eT^\psi}$ are given by

$$\{\pm 2\pi i n \mid n \in \mathbb{N}_{>0}\} \;\cup\; \{2\pi i(\pm\alpha(H) + n) \mid \alpha \in \Delta_s^\psi,\ n \in \mathbb{Z}\} \\ \cup\; \{2\pi i(\pm\alpha(H) + nr) \mid \alpha \in \Delta_l^\psi,\ n \in \mathbb{Z}\}.$$

Furthermore, if $\Delta$ is of type $X_N$ in the notation of [K] then for an arbitrary eigenvalue $2\pi i\lambda$ of $B_{\sigma, eT^\psi}$, we have $\mathrm{mult}(2\pi i\lambda) = 1$ if $\lambda = \alpha(H) + n$, with $\alpha \in \Delta^\psi$. In case $\lambda = n$

we have $\mathrm{mult}(2\pi i\lambda) = l = \dim(T^\psi)$ if $r$ divides $n$ and $\mathrm{mult}(2\pi i\lambda) = (N-l)/(r-1)$ if $r \not| n$. So the zeta-regularized determinant of $B_{\sigma,eT^\psi}$ is given by

$$\begin{aligned}
\det_\zeta(B_{\sigma,eT^\psi}) = &\prod_{\alpha\in\Delta^\psi_{s+}} \left((2\pi\alpha(H))^2 \prod_{n=1}^\infty ((2\pi)^2(n^2-\alpha(H)^2))^2\right)_\zeta \\
&\times \prod_{\alpha\in\Delta^\psi_{l+}} \left((2\pi\alpha(H))^2 \prod_{n=1}^\infty ((2\pi)^2(r^2n^2-\alpha(H)^2))^2\right)_\zeta \\
&\times \left(\prod_{n=1}^\infty (2\pi n)^2\right)_\zeta^{\frac{(N-l)}{(r-1)}} \left(\prod_{n=1}^\infty r^2(2\pi n)^2\right)_\zeta^{l-\frac{(N-l)}{(r-1)}} \\
= &\prod_{\alpha\in\Delta^\psi_{s+}} 4\sin^2(\pi\alpha(H)) \prod_{\alpha\in\Delta^\psi_{l+}} 4\sin^2(\frac{\pi}{r}\alpha(H)) \\
&\times \left(\prod_{n=1}^\infty 2\pi n\right)_\zeta^{\frac{2(N-l)}{(r-1)}+4|\Delta^\psi_{s+}|} \left(\prod_{n=1}^\infty 2\pi r n\right)_\zeta^{2l-\frac{2(N-l)}{(r-1)}+4|\Delta^\psi_{l+}|} \\
= &(2\pi)^{2\dim\mathfrak{g}^\psi} r^{l-\frac{(N-l)}{(r-1)}+2|\Delta^\psi_{l+}|} \prod_{\alpha\in\Delta^{\psi\vee}_+} 4\sin^2(\pi\alpha(H)),
\end{aligned}$$

where $\Delta^{\psi\vee}$ denotes the root system dual to $\Delta^\psi$. So analogous to lemma 3.3, the zeta regularized Pfaffian of $B_{\sigma,eT^\psi}$ now reads

$$\mathrm{Pf}_\zeta(B_\sigma)(eT^\psi) = (2\pi)^{\dim\mathfrak{g}^\psi}(\sqrt{r})^{l-\frac{(N-l)}{(r-1)}+2|\Delta^\psi_{l+}|} \prod_{\alpha\in\Delta^{\psi\vee}_+} 2\sin(\pi\alpha(H)).$$

Note that up to a constant coefficient, the Pfaffian $\mathrm{Pf}_\zeta(B_\sigma)(eT^\psi)$ is exactly the denominator of the Weyl character formula for the compact Lie group with root system $\Delta^{\psi\vee}$, or equivalently, the denominator of the characters on the outer component of the principal extension of the Lie group with root system $\Delta$ (cf. [W2]). In any case, we see that the symplectic form and the Riemannian metric on $L(G,\psi)/T^\psi$ are compatible in the sense of §2.1.

The $S^1 \times T$-action on $LG/T$ considered in §3.2 defines an $S^1 \times T^\psi$-action on $L(G,\psi)/T^\psi$ by restriction. That is, $S^1 \times T^\psi$ acts on $L(G,\psi)/G^\psi$ by "twisted rotation" and $T^\psi$ acts on $G^\psi/T^\psi$ by conjugation. A similar calculation to the corresponding one for the untwisted case shows that the fixed point set of the "twisted rotation action" of $S^1 \times T^\psi$ on $L(G,\psi)/G^\psi$ is given by the lattice $M \subset \mathfrak{h}$ which is generated by the long roots in $\Delta^\psi$ (where we have identified $\mathfrak{h}$ with $\mathfrak{h}^*$ via the negative of the Killing form on $G$). The fixed point set of the $T^\psi$-action on $G^\psi/T^\psi$ is given by the Weyl group $W^\psi$ of $G^\psi$. As a set, $W^\psi \times M$ can be identified with the affine Weyl group belonging to the twisted affine Lie algebra corresponding to $L(G,\psi)$ (see e.g. [K] or [W2] for more on the theory of twisted affine Lie algebras). The number $\#(w,\beta)$ for $(w,\beta)\in W^\psi\times M$ is again given by $\#(w,\beta)=l((w,\beta))$, where $l((w,\beta))$ denotes the length of $(w,\beta)$ in $W^\psi\ltimes M$.

Let us compute $Z_{eT^\psi}(H)$: A calculation similar to the one we used for the Pfaffian now gives

$$Z_{eT^\psi}(H)=c\cdot\prod_{\alpha\in\Delta^1_+}2\sin(\pi\alpha(H))$$

with $c=(\sqrt{2\pi})^l(\sqrt{r})^{l-\frac{(N-l)}{(r-1)}+2|\Delta^\psi_{l+}|}$.

Now let us consider the function $J_H|_{L(G,\psi)/T^\psi}:L(G,\psi)/T^\psi\to\mathbb{R}$, where $J_H$ is the function we considered in §3.3. That is

$$J_H(\gamma)=\frac{1}{2}\int_0^1\|\gamma'(t)\gamma^{-1}(t)+\gamma(t)H\gamma^{-1}(t)-H\|dt.$$

Since for generic $H\in\mathfrak{h}^\psi$, the corresponding $\mathbb{R}$-action on $L(G,\psi)/T^\psi$ is just the restriction of the corresponding $\mathbb{R}$-action on $LG/T$, it follows from lemma 3.6 that the Hamiltonian vector field on $L(G,\psi)/T^\psi$ corresponding to $J_H$ is the vector field generated by the $\mathbb{R}$-action. Therefore we can calculate the Liouville functional of $J_H$: A calculation in [W2], §4.2 implies

$$\sum_{w\in W^\psi}\sum_{\beta\in a_0M}(-1)^{l(w)}e^{-\frac{1}{2}\|\beta+w(H)-H\|^2}$$
$$=\frac{\prod_{\alpha\in\Delta^1_+}4\sin^2(\pi\alpha(H))}{(2\pi)^{\frac{l}{2}}\mathrm{vol}(a_0M)}\sum_{\lambda\in P_+(\Delta^1)}|\chi_\lambda(\exp(H))|^2e^{-\frac{1}{2}\|\lambda+\rho^\psi\|^2},$$

where $P(\Delta^{\psi\vee})$ denotes the weight lattice of the root system $\Delta^1$, $P_+(\Delta^{\psi\vee})$ denotes the cone of dominant weights, $\chi_\lambda$ denotes the irreducible character of the compact simply connected semi-simple Lie group of the same type as the root system $\Delta^{\psi\vee}$, and $\rho^\psi$ is the half sum of the positive roots of $\Delta^{\psi\vee}$. Putting this together with the calculation of the Pfaffian gives:

**Proposition 3.13**

$$\int_{L(G,\psi)/T^\psi}e^{-J_H(\gamma)}d\sigma(\gamma)=c\cdot\sum_{\lambda\in P_+(\Delta^{\psi\vee})}|\chi_\lambda(\exp(H))|^2e^{-\frac{1}{2}\|\lambda+\rho^\psi\|^2}$$

*with* $c=((2\pi)^{l+\dim\mathfrak{g}^\psi}\mathrm{vol}(M)r^{l-\frac{N-l}{r-1}+2|\Delta^\psi_{l+}|})^{-1}$

**Remark 3.14** If $\Delta$ is of type $\mathrm{A}_{2n}$, the root system $\Delta^\psi$ is a non-reduced root system of type $\mathrm{BC}_n$. That is, three different root lengths occur. In this case, a similar calculation shows, that after replacing the root system $\Delta^{\psi\vee}$ with a reduced root system of type $\mathrm{C}_n$, proposition 3.13 still holds true.

As in the non twisted case, one can compare the Liouville functional with a certain Wiener measure: The space $L(G,\psi)/T^\psi$ can be embedded into space of paths in the outer component $G\psi$ of the non-connected Lie group $\mathrm{G} \rtimes \langle\psi\rangle$ on which the Wiener measure is defined (see [W2] for details). Comparison of the Wiener measure on this path space with the "Riemannian measure" on $L(G,\psi)/T^\psi$ yields the same result as §3.4. The calculations leading to theorem 3.10 can be easily adjusted to the twisted case so that we have an analogous re-interpretation of the orbital theory for the twisted loop groups developed in [W2].

# 4 The Wess-Zumino-Witten Model

## 4.1 The WZW model at level $\kappa$

The WZW model is a quantum field theory on a Riemann surface $\Sigma$ with values in a simply connected semi-simple compact Lie group $G$ (or more generally in its complexification $G_{\mathbb{C}}$). See e.g. [G] for an introduction to quantum field theory in general and the WZW model on an arbitrary Riemann surface in particular. We will be mostly interested in the case when $\Sigma$ is the elliptic curve $\Sigma_\tau$, i.e. the torus $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ together with a complex structure which is defined by $f : \Sigma_\tau \to \mathbb{C}$ is holomorphic if $\bar\partial f := (\partial_s + \tau\partial_t)f = 0$. Here $\tau = \tau_1 + i\tau_2$ with $\tau_1, \tau_2 \in \mathbb{R}$, $\tau_2 > 0$ denotes the modular parameter of the elliptic curve $\Sigma_\tau$.

As before, let $\langle .,. \rangle$ denote the Killing form on $\mathfrak{g}_{\mathbb{C}}$ normalized so that the long roots have square length 2 and set $\partial = \partial_s + \bar\tau\partial_t$. In our normalization, the action functional of the WZW model at level $\kappa$ is given by

$$S_{G,\kappa}(g) = -\frac{\kappa\pi}{2\tau_2}\int_\Sigma \langle g^{-1}\partial g, g^{-1}\bar\partial g\rangle ds dt + \frac{i\kappa\pi}{3}\int_B \mathrm{tr}(\widetilde{g}^{-1}d\widetilde{g})^{\wedge 3}$$

where $B$ is a three-dimensional manifold with boundary $\partial B = \Sigma$, and and $\widetilde{g} : B \to G$ is a map such that $\widetilde{g}|_{\partial B} = g$ and tr denotes the negative of the normalized Killing form as well. The second term in the action is the so called Wess-Zumino term. Up to the factor $i\kappa$, it is the integral over the pull back of the generator of $H^3(G,\mathbb{Z})$ to $B$ via the map $\widetilde{g}$. The action $S_{G,k}$ was first studied by Witten [Wi].

If $\widetilde{g}_1$ and $\widetilde{g}_2$ are two different extensions of $g$ they differ by a map $\widetilde{h} : B \to G$ such that $\widetilde{h}|_{\partial B} = e$. But for such $\widetilde{h}$ we have $\frac{\pi}{3}\int_B \mathrm{tr}(\widetilde{h}^{-1}\widetilde{h})^{\wedge 3} \in 2\pi\mathbb{Z}$ such that the action $e^{S_{G,\kappa}(g)}$ is well defined for $\kappa \in \mathbb{Z}$.

Some of the transformation properties of the WZW action are given by the Polyakov-Wiegmann formula (cf. [PW], [GK]):

**Proposition 4.1** *Let $g, h : \Sigma \to G$. Then the following identity is valid:*

$$S_{G,\kappa}(gh) = S_{G,\kappa}(g) + S_{G,\kappa}(h) - \frac{\kappa\pi}{\tau_2}\int_\Sigma \langle g^{-1}\partial g, \bar\partial h h^{-1}\rangle dsdt$$

Note that the imaginary part of the term $\frac{\kappa\pi}{\tau_2}\int_\Sigma \langle g^{-1}\partial g, \bar\partial h h^{-1}\rangle dsdt$ is exactly the cocycle in the explicit construction of the central extension $\hat G$ of the loop group $LG$ as a quotient (see [M], [FKh]).

Proof. One directly checks that

$$\begin{aligned} -\frac{\kappa\pi}{2\tau_2}\int_\Sigma \langle h^{-1}g^{-1}\partial(gh), h^{-1}g^{-1}\bar\partial(gh)\rangle dsdt = \\ -\frac{\kappa\pi}{2\tau_2}\int_\Sigma \langle g^{-1}\partial g, g^{-1}\bar\partial g\rangle dsdt - \frac{\kappa\pi}{2\tau_2}\int_\Sigma \langle h^{-1}\partial h, h^{-1}\bar\partial h\rangle dsdt \\ -\frac{\kappa\pi}{\tau_2}\int_\Sigma (\langle g^{-1}\partial_s g, \partial_s h h^{-1}\rangle + \tau_1\langle g^{-1}\partial_s g, \partial_t h h^{-1}\rangle \\ + \tau_1\langle g^{-1}\partial_t g, \partial_s h h^{-1}\rangle + \tau\bar\tau\langle g^{-1}\partial_t g, \partial_t h h^{-1}\rangle) dsdt \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \frac{i\kappa\pi}{3}\int_B \mathrm{tr}((gh)^{-1}d(gh))^{\wedge 3} =& \frac{i\kappa\pi}{3}\int_B \mathrm{tr}(g^{-1}dg)^{\wedge 3} + \frac{i\kappa\pi}{3}\int_B \mathrm{tr}(h^{-1}dh)^{\wedge 3} \\ &+ i\kappa\pi\int_B \mathrm{tr}((g^{-1}dg)^{\wedge 2}\wedge dhh^{-1} + g^{-1}dg\wedge(dhh^{-1})^{\wedge 2}) \\ =& \frac{i\kappa\pi}{3}\int_B \mathrm{tr}(g^{-1}dg)^{\wedge 3} + \frac{i\kappa\pi}{3}\int_B \mathrm{tr}(h^{-1}dh)^{\wedge 3} \\ &+ i\kappa\pi\int_B d\mathrm{tr}(g^{-1}dg\wedge dhh^{-1}) \end{aligned}$$

Now Stoke's theorem implies $i\kappa\pi\int_B d\mathrm{tr}(g^{-1}dg\wedge dhh^{-1}) = i\kappa\pi\int_\Sigma \mathrm{tr}(g^{-1}dg\wedge dhh^{-1})$. Writing $dg = \partial_s g ds + \partial_t g dt$ and $dh = \partial_s h ds + \partial_t h dt$ yields the assertion. □

Let $H \in \mathfrak{h}$ be generic. We will extend the WZW action slightly by adding an $H$-dependent term: Set

$$\begin{aligned} S_{G,H,\kappa}(g) = S_{G,\kappa}(g) + \frac{\kappa\pi}{\tau_2}\int_{\Sigma_\tau} (\langle g^{-1}\partial g, H\rangle - \langle \bar\partial g g^{-1}, H\rangle \\ - \langle H, g^{-1}Hg\rangle + \langle H, H\rangle) dsdt. \end{aligned}$$

$S_{G,H,\kappa}(g)$ is essentially the action of the gauged WZW model studied in [GK]. The partition function of the gauged WZW model at level $\kappa$ is formally given by the integral

$$\int_{C^\infty(\Sigma_\tau, G_\mathbb{C})} e^{S_{G,H,\kappa}(g)} \mathcal{D}(g),$$

where the integration ranges over all $C^\infty$-maps $g : \Sigma \to G_\mathbb{C}$.

The main goal of the next two paragraphs is to make sense of this integral and to calculate the partition function using the Liouville functional approach. To do this, we will have to work in a complex setting as described in §2.2.

## 4.2 Double loop groups and a torus action

From now on we will consider the group $G_\mathbb{C}$, which is the complexification of the compact semi-simple simply connected Lie group $G$. Let $LLG_\mathbb{C}$ denote the set of all $C^\infty$-maps from the torus $S^1 \times S^1$ to $G_\mathbb{C}$. Together with pointwise multiplication $LLG_\mathbb{C}$ becomes a Lie group with Lie algebra $LL\mathfrak{g}_\mathbb{C}$, the set of $C^\infty$-maps from $S^1 \times S^1$ to the Lie algebra $\mathfrak{g}_\mathbb{C}$ of $G_\mathbb{C}$. A one dimensional central extension $\widetilde{LLG}_\mathbb{C}$ and the corresponding coadjoint representation of $LLG_\mathbb{C}$ was constructed in [EF]. One of the results of [EF] is that a generic coadjoint orbit of $\widetilde{LLG}_\mathbb{C}$ is isomorphic to $LLG_\mathbb{C}/T_\mathbb{C}$, where $T$ is a maximal torus in $G$ and $T_\mathbb{C}$ denotes its complexification.

Let $H \in \mathfrak{h}$ be generic and choose a modular parameter $\tau = \tau_1 + i\tau_2 \in \mathbb{C}$ such that $\tau_2 > 0$. With these choices made, we can define a non-degenerate closed two-form and an $\mathbb{R}$-action on $LLG_\mathbb{C}/T_\mathbb{C}$ in total analogy with §3.1 and §3.2 respectively: Note that we have $LLG_\mathbb{C}/T_\mathbb{C} \cong \Omega\Omega G_\mathbb{C} \times G_\mathbb{C}/T_\mathbb{C}$, where $\Omega\Omega G_\mathbb{C}$ denotes the set of maps $g \in LLG_\mathbb{C}$ such that $g(1,1) = e$. Let $X, Y : S^1 \times S^1 \to \mathfrak{g}_\mathbb{C}$ be elements of the Lie algebra of $LLG_\mathbb{C}$. Then

$$\omega_e(X,Y) = \frac{\pi}{\tau_2} \int_{S^1 \times S^1} \langle \bar\partial X(s,t), Y(s,t)\rangle ds dt$$

defines a $\mathbb{C}$-valued skew symmetric bilinear form on $LL\mathfrak{g}_\mathbb{C}$ which is degenerate on the set of holomorphic maps. Since $\Sigma_\tau$ is compact, any holomorphic map from $\Sigma_\tau$ to $G_\mathbb{C}$ has to be constant. Hence, by left translation, $\omega$ defines a non-degenerate $\mathbb{C}$-valued two-form on $LLG_\mathbb{C}/G_\mathbb{C} \cong \Omega\Omega G_\mathbb{C}$. We can choose a $\mathbb{C}$-valued two-form $\omega_0^H$ on $G_\mathbb{C}/T_\mathbb{C}$ which is defined via $\omega^H_{0,eT_\mathbb{C}}(A,B) = \frac{\pi}{\tau_2}\langle H, [A,B]\rangle$ for $A, B \in T_{eT_\mathbb{C}}$ and extended to $G_\mathbb{C}/T_\mathbb{C}$ via left translation. Putting these two forms together, we obtain a non-degenerate $\mathbb{C}$-valued two-form $\omega^H = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega_0^H$ on $LLG_\mathbb{C}/T_\mathbb{C}$. As in the case of loop groups, one checks that $\omega^H$ is closed. Hence it can be considered as a $\mathbb{C}$-valued symplectic form in the sense of §2.2.

Our next goal is to find an almost complex structure $J$ on $LLG_\mathbb{C}/T_\mathbb{C}$ which is compatible with the complex valued symplectic form $\omega^H$ in the sense of §2.2.

Consider the decomposition $\mathfrak{g}_\mathbb{C}/\mathfrak{h}_\mathbb{C} = \bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$ into one-dimensional root spaces. Here, as before, $\Delta$ denotes the root system of $\mathfrak{g}_\mathbb{C}$. For each $\alpha \in \Delta$ choose $X_\alpha \in \mathfrak{g}_\alpha$ such that $\langle X_\alpha, X_{-\alpha}\rangle = 1$. Furthermore, choose an orthonormal basis $H_1, \dots, H_l$ of $\mathfrak{h}$. Then any $X \in LL\mathfrak{g}_\mathbb{C}/\mathfrak{h}_\mathbb{C}$ can be written as

$$X(s,t) = \sum_{(n,m)\in\mathbb{Z}^2,}\sum_{\alpha\in\Delta} c_{n,m,\alpha}X_\alpha e^{2\pi i(ns+mt)} + \sum_{\substack{(n,m)\in\mathbb{Z}^2\\(n,m)\neq(0,0)}}\sum_{j=1}^{l} c_{n,m,j}H_j e^{2\pi i(ns+mt)}$$

with $c_{n,m,\alpha}, c_{n,m,j} \in \mathbb{C}$.

As always, let $\tau = \tau_1 + i\tau_2$ denote the modular parameter of the elliptic curve. Let $\Delta_+$ be a set of positive roots of $\mathfrak{g}_\mathbb{C}$ with respect to some basis of $\Delta$. Then let us decompose the set $\tilde{\tilde{\Delta}} = \{(\alpha,n,m) \mid \alpha\in\Delta\cup\{0\},\ (n,m)\in\mathbb{Z}^2\ (\alpha,n,m)\neq(0,0,0)\}$ into $\tilde{\tilde{\Delta}}_+\cup\tilde{\tilde{\Delta}}_-$ via defining $(\alpha,n,m)$ to be positive if either $n+\tau_1 m > 0$ or $n+\tau_1 m = 0$ and $m < 0$ or $n = m = 0$ and $\alpha\in\Delta_+$. Now we can define an $\mathbb{R}$ linear anti-involution $J$ of $LL\mathfrak{g}_\mathbb{C}/\mathfrak{h}_\mathbb{C}$ which anti-commutes with multiplication by $i$ as follows: For $c \in \mathbb{C}$, set

$$J(c_{n,m,\alpha}X_\alpha e^{2\pi i(ns+mt)}) = \begin{cases} \bar{c}_{n,m,\alpha}X_{-\alpha}e^{-2\pi i(ns+mt)} \text{ if } (\alpha,n,m)\in\tilde{\tilde{\Delta}}_+ \\ -\bar{c}_{n,m,\alpha}X_{-\alpha}e^{-2\pi i(ns+mt)} \text{ if } (\alpha,n,m)\in\tilde{\tilde{\Delta}}_-. \end{cases}$$

Analogously, set

$$J(c_{n,m,\nu}H_\nu e^{2\pi i(ns+mt)}) = \begin{cases} \bar{c}_{n,m,\nu}H_\nu e^{-2\pi i(ns+mt)} \text{ if } (0,n,m)\in\tilde{\tilde{\Delta}}_+ \\ -\bar{c}_{n,m,\nu}H_\nu e^{-2\pi i(ns+mt)} \text{ if } (0,n,m)\in\tilde{\tilde{\Delta}}_-. \end{cases}$$

Now if the the set of positive roots $\Delta_+$ is chosen in such a way that $H$ lies in the fundamental Weyl chamber with respect to $\Delta_+$, it is straight forward to check that the complex structure $J$ is indeed compatible with the symplectic form $\omega^H$ in the sense of §2.2. Furthermore, $J$ commutes with the natural $T_\mathbb{C}$-action on $LL\mathfrak{g}_\mathbb{C}/\mathfrak{h}_\mathbb{C}$. So $J$ defines an automorphism of the tangent bundle and thus an almost complex structure on $LLG_\mathbb{C}/T_\mathbb{C}$ by left translation.

A bilinear form on $LL\mathfrak{g}_\mathbb{C}$ is given by

$$\sigma(X,Y) = \pi\int_{S^1\times S^1}\langle X(s,t), Y(s,t)\rangle ds dt.$$

Hence the skew-symmetric automorphism of the tangent space relating $\omega^H$ and $\sigma$ reads $B_{\sigma,eT_\mathbb{C}} = \frac{1}{\tau_2}(\bar\partial + \mathrm{ad}H)$. As before, let $\Delta$ denote the root system of $G$. Then the eigenvalues of $B_{\sigma,eT_\mathbb{C}}$ are

$$\begin{cases} \frac{2\pi i}{\tau_2}(n+\tau m+\alpha(H)) \quad \text{for } \alpha\in\Delta, \quad n,m\in\mathbb{Z}, \\ \frac{2\pi i}{\tau_2}(n+\tau m) \quad \text{for } n,m\in\mathbb{Z} \text{ and } n\neq 0 \text{ or } m\neq 0. \end{cases}$$

The multiplicity of the eigenvalues in the first series is 1 and the multiplicity of the eigenvalues in the second series is $l = \dim_{\mathbb{R}} \mathfrak{h}$. This can be seen using the root space decomposition of the semi-simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

Unfortunately, we can not calculate the zeta-regularized product of the eigenvalues of $B_{\sigma,eT_{\mathbb{C}}}$ since the product ranges over a series of complex numbers. To avoid this difficulty, we will introduce an appropriate torus action on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$ and then calculate the product $\mathrm{Pf}_{\zeta} B_{\sigma,eT_{\mathbb{C}}} \cdot Z_{eT_{\mathbb{C}}}(H)$ which will be the denominator of the partition function. Here, $Z_{eT_{\mathbb{C}}}(H)$ is the zeta-regularized product of the "positive" eigenvalues of the torus-action.

There is a natural action of the torus $S^1 \times S^1 \times T$ on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$ where the first two factors act by rotating the loops and the second factor acts by conjugation. This action is defined analogously to the $S^1 \times T$ action on $LG/T$ considered in §3.2. The fixed point set of this action is the set $Q^\vee \times Q^\vee \times W$, where, as before, $Q^\vee \times Q^\vee$ denotes the lattice of homomorphisms $S^1 \times S^1 \to T$ and $W$ is the Weyl group of $G$. This follows from exactly the same calculation which gave the fixed point set of the $S^1 \times T$-action on $LG/T$ in §3.2. Furthermore, one checks directly that the $S^1 \times S^1 \times T$-action on the tangent spaces $T_{(\beta_1,\beta_2,w)}LLG_{\mathbb{C}}/T_{\mathbb{C}}$ commutes with the complex structure $J$ for all $(\beta_1, \beta_2, w) \in Q^\vee \times Q^\vee \times W$.

The differential $\partial = \partial_s + \bar{\tau}\partial_t$ defines an element in the complexified Lie algebra of $S^1 \times S^1$. So for $H \in \mathfrak{h}$, the pair $(\partial, H)$ can be viewed as an element in the complexified Lie algebra $Lie_{\mathbb{C}}(S^1 \times S^1 \times T)$ of $S^1 \times S^1 \times T$ and hence defines a vector field on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$. The eigenvalues of the corresponding action of $(\partial, H)$ on the tangent space at $T_e T_{\mathbb{C}} LLG_{\mathbb{C}}/T_{\mathbb{C}}$ are

$$\begin{cases} 2\pi i(n + \bar{\tau}m + \alpha(H)) & \text{for } \alpha \in \Delta, \quad n, m \in \mathbb{Z}, \\ 2\pi i(n + \bar{\tau}m) & \text{for } n, m \in \mathbb{Z} \text{ and } n \neq 0 \text{ or } m \neq 0. \end{cases}$$

Again, the multiplicity of the eigenvalues is 1 if $\alpha \neq 0$ and $l$ if $\alpha = 0$.

Let us compute the product $\mathrm{Pf}_{\zeta}(B_\sigma)(eT_{\mathbb{C}}) \cdot Z(H)$. For simplicity, we will compute $(\mathrm{Pf}_{\zeta}(B_\sigma)(eT_{\mathbb{C}}) \cdot Z(H))^2$. Remember that in the definition of $Z(H)$, the eigenvalues of the infinitesimal $\mathbb{R}$-action on the tangent spaces at the fixed points of the torus action have to be divided by $2\pi$, so that we get

$$\begin{aligned} (\mathrm{Pf}_{\zeta}(B_\sigma)(eT_{\mathbb{C}}) \cdot Z(H))^2 = \prod_{\alpha \in \Delta_+} \Big( \prod_{(n,m) \in \mathbb{Z}^2} \big(\frac{2\pi}{\tau_2} |n + \tau m + \alpha(H)|^2\big)^2 \Big)_{\zeta} \\ \times \Big( \prod_{\substack{(n,m) \in \mathbb{Z}^2 \\ (n,m) \neq (0,0)}} \big(\frac{2\pi}{\tau_2} |n + \tau m|^2\big)^l \Big)_{\zeta} \end{aligned}$$

Since $H \in \mathfrak{h}$ was chosen to be generic, we have $\alpha(H) \in \mathbb{R} \setminus \mathbb{Z}$ for all $\alpha \in \Delta$. Therefore we can calculate the zeta-regularized product using the Epstein zeta-functions $\zeta_\tau(s;v)$ and $\tilde{\zeta}_\tau(s;v)$ which are defined in the appendix. Lemma A.2 implies

$$(\mathrm{Pf}_\zeta(B_\sigma)(eT_{\mathbb{C}})\cdot Z(H))^2 = \prod_{\alpha\in\Delta_+}(2\pi)^{2\zeta_\tau(0;\alpha(H))}\Big(\prod_{(n,m)\in\mathbb{Z}^2}(\frac{1}{\tau_2}|n+\tau m+\alpha(H)|^2)^2\Big)_\zeta$$
$$\times (2\pi)^{l\zeta_\tau(0;0)}\Big(\prod_{\substack{(n,m)\in\mathbb{Z}^2\\(n,m)\neq(0,0)}}(\frac{1}{\tau_2}|n+\tau m|^2)\Big)^l_\zeta$$

Now we can use lemma A.7 to obtain one of the main results of this section:

**Proposition 4.2** *The following identity is valid:*

$$\mathrm{Pf}_\zeta(B_\sigma)\cdot Z(H) = C\cdot(2\pi\sqrt{\tau_2}|\eta(\tau)|^2)^l\cdot\prod_{\alpha\in\Delta_+}|q^{\frac{1}{12}}(e(\frac{1}{2}\alpha(H))-e(-\frac{1}{2}\alpha(H)))$$
$$\times\prod_{n=1}^{\infty}(1-q^n e(\alpha(H)))(1-q^n e(-\alpha(H)))|^2$$

*with* $C=(2\pi)^{-\frac{l}{2}}$.

Note, that up to a coefficient, the right hand side of the equation in proposition 4.2 is exactly the squared absolute value of the denominator of the Kac-Weyl character formula (cf. [K]).

**Remark 4.3** Let us consider the differential operator $(\partial+\mathrm{ad}H)(\bar\partial+\mathrm{ad}H)$ acting on the space $C^\infty(S^1\times S^1,\mathfrak{g})$. This operator can be viewed as a non-Abelian generalization of the Laplacian acting on the space $C^\infty(S^1\times S^1,\mathfrak{h})$ considered in [T]. Up to a coefficient, the product $(\mathrm{Pf}_\zeta(B_\sigma)\cdot Z(H))^2$ is the zeta-regularized determinant $\det_\zeta((\partial+\mathrm{ad}H)(\bar\partial+\mathrm{ad}H))$. In the case of finite dimensional Gaussian integrals, we have the equality

$$\int_{\mathbb{R}^n} e^{-\langle x,Bx\rangle}d^n x = \frac{1}{\sqrt{\det(B)}},$$

where $B$ denotes a symmetric matrix. Thus, by analogy, the product $(\mathrm{Pf}_\zeta(B_\sigma)\cdot Z(H))^{-1}$ can be viewed as the Gaussian integral

$$\int_{C^\infty(S^1\times S^1,\mathfrak{g})/\mathfrak{h}} e^{-\langle X,(\partial+\mathrm{ad}H)(\bar\partial+\mathrm{ad}H)X\rangle}\mathcal{D}(X).$$

Maybe more interestingly, each $\alpha(H)$ defines unitary representation of the lattice $\mathbb{Z}+\tau\mathbb{Z}$ via $(m+\tau n)\mapsto e^{2\pi i(\alpha(H)m+n)}$. Such representation gives rise to a complex line bundle over the elliptic curve $\Sigma_\tau$ and our calculation of the zeta-regularized determinant of $(\partial+\mathrm{ad}H)(\bar\partial+\mathrm{ad}H)$ is exactly the calculation leading to the analytic torsion of this line bundle (see [RS]).

Finally, we have to calculate the sign $(-1)^{\#p}$ for the fixed points $p$ of the torus action on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$. We can proceed as in §3.2: Let $(\beta_1, \beta_2, w) \in Q^\vee \times Q^\vee \times W$ be a fixed point of the $S^1 \times S^1 \times T$-action on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$. If we choose a representative $g_w \in G$ for each $w \in W$, we can view $(\beta_1, \beta_2, g_w)$ as an element of $LLG_{\mathbb{C}}$. Since $LLG_{\mathbb{C}}$ acts transitively on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$, we can use left translation by $(\beta_1, \beta_2, g_w)$ to identify the tangent spaces $T_{eT_{\mathbb{C}}} LLG_{\mathbb{C}}/T_{\mathbb{C}}$ and $T_{(\beta_1,\beta_2,w)} LLG_{\mathbb{C}}/T_{\mathbb{C}}$. Since $(\beta_1, \beta_2, g_w)$ is an element of the normalizer of $T_{\mathbb{C}}$, this identification is well defined. With this identification, the corresponding infinitesimal $S^1 \times S^1 \times T$-action on $T_{(\beta_1,\beta_2,w)} LLG_{\mathbb{C}}/T_{\mathbb{C}}$ is given by $(\beta_1, \beta_2, g_w)^{-1}(\partial_s + \partial_t + H)(\beta_1, \beta_2, g_w)$ for $(\partial_s + \partial_t + H) \in Lie(S^1 \times S^1 \times T)$. This defines an action of $Q^\vee \times Q^\vee \times W$ on the set $\tilde{\tilde{\Delta}}$. According to the definition in §2.2, $\#(\beta_1, \beta_2, w)$ is the number of elements of $\tilde{\tilde{\Delta}}_+$ which are mapped to $\tilde{\tilde{\Delta}}_-$ under $(\beta_1, \beta_2, w)$. Now, as in the case of affine root systems and Weyl groups, one can see (for example by using the fact that the cardinality of the finite root system $\Delta$ is even) that we always have $(-1)^{\#(\beta_1,\beta_2,w)} = (-1)^{l(w)}$, where $l(w)$ denotes the length of $w$ in $W$.

## 4.3 Calculation of the WZW partition function

Now we will calculate the integral defining the partition function of the WZW-model at level $\kappa$. Note, that $S_{G,H,\kappa}(g)$ does not depend on the representative of $g$ modulo the complex torus $T_{\mathbb{C}}$. Hence, $S_{G,H,\kappa}$ defines a function on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$. To apply the Liouville functional approach, we have to check, that $S_{G,H,\kappa}$ is the Hamiltonian of a vector field on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$ which comes from the $S^1 \times S^1 \times T$-action considered in the last paragraph.

**Lemma 4.4** *The Hamiltonian vector field on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$ corresponding to $-S_{G,H,1}$ is exactly the vector field on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$ defined by the element $(\partial, H) \in Lie_{\mathbb{C}}(S^1 \times S^1 \times T)$ as in §4.2.*

PROOF. The proof is analogous to the proof of lemma 3.6: Let $g$ be a representative of $gT_{\mathbb{C}}$ in $LLG_{\mathbb{C}}/T_{\mathbb{C}}$ and let $\delta g$ be an infinitesimal variation of $g$ (i.e. a vector field along $g$). The vector field on $LLG_{\mathbb{C}}/T_{\mathbb{C}}$ generated be the element $(\partial, H) \in Lie_{\mathbb{C}}(S^1 \times S^1 \times T)$ is given at the point $gT_{\mathbb{C}}$ by $\partial g + \mathrm{ad}H(g)$. So we have to show that

$$-dS_{G,H,1}(\delta g) = \omega^H_{gT}(\delta g, \partial g + \mathrm{ad}H(g)).$$

Let us denote the $H$-dependent term in $S_{G,H,1}(g)$ by $\widetilde{S}_H(g)$ so that we have

$$S_{G,H,1}(g) = S_{G,1} + \widetilde{S}_H(g).$$

From the Polyakov-Wiegmann formula (prop. 4.1) one deduces that

$$-dS_{G,1}(\delta g) = \frac{\pi}{\tau_2} \int_\Sigma \langle g^{-1}\partial g, \bar{\partial}(g^{-1}\delta g)\rangle ds dt.$$

On the other hand, we have

$$-d\widetilde{S}_H(\delta g) = \frac{\pi}{\tau_2}\int_\Sigma (\langle \delta(g^{-1}\partial g), H\rangle - \langle \delta(\bar\partial g g^{-1}), H\rangle - \langle H, \delta(g^{-1}Hg)\rangle) ds dt$$

We already know that $\delta(g^{-1}\partial g) = \partial(g^{-1}\delta g) + [g^{-1}\partial g, g^{-1}\delta g]$. Thus, by partial integration, $\int_\Sigma \langle \delta(g^{-1}\partial g), H\rangle = \int_\Sigma \langle [g^{-1}\partial g, g^{-1}\delta g], H\rangle ds dt$. Furthermore, a short calculation shows $\langle \delta(\bar\partial g g^{-1}), H\rangle = \langle \bar\partial(g^{-1}\delta g), g^{-1}Hg\rangle$, and as in the proof of lemma 3.6, we have $\langle H, \delta(g^{-1}Hg)\rangle = \langle H, [g^{-1}Hg, g^{-1}\delta g]\rangle$. So putting all these terms together gives

$$\begin{aligned} -dS_{G,H,1}(\delta g) =& \frac{\pi}{\tau_2}\int_\Sigma \langle \bar\partial(g^{-1}\delta g), g^{-1}\partial g + g^{-1}Hg\rangle ds dt \\ &+ \frac{\pi}{\tau_2}\int_\Sigma \langle H, [g^{-1}\delta g, g^{-1}\partial g - g^{-1}Hg]\rangle ds dt, \end{aligned}$$

which is the assertion. □

Since we have $S_{G,H,\kappa} = \kappa S_{G,H,1}$, lemma 4.4 allows us to calculate the integral $\int_{LLG_\mathbb{C}/T_\mathbb{C}} e^{S_{G,H,\kappa}} d\sigma$ via the Liouville functional approach. Before we start with the calculation, let us briefly recall some facts from the theory of affine Lie algebras (see [K] for details). Let $\tilde{\mathfrak{g}}_\mathbb{C} = L\mathfrak{g}_\mathbb{C} \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the untwisted affine Lie algebra corresponding to $\mathfrak{g}_\mathbb{C}$ and let $A$ denote its generalized Cartan matrix. Let $\tilde\Delta$ denote the root system of $\tilde{\mathfrak{g}}_\mathbb{C}$ and choose a set $\alpha_0 \dots, \alpha_l$ of simple roots. Denote by $\alpha_0^\vee \dots, \alpha_l^\vee$ the dual simple roots, i.e. $\alpha_i \in \tilde{\mathfrak{h}}$ such that $\langle \alpha_i, \alpha_j^\vee\rangle = (A)_{i,j}$. Let $a_i$ be the "minimal" integers such that $A(a_0, \dots, a_n) = 0$ and set $\delta = \sum_{i=0}^n a_i\alpha_i$. Following [K], §12.4, we define the canonical central element of $\tilde{\mathfrak{g}}_\mathbb{C}$ as

$$K = \sum_{i=0}^l a_i \alpha_i^\vee.$$

Turning to the representation theory of affine Lie algebras we define as usual

$$\begin{aligned} \tilde P =& \{\lambda \in \tilde{\mathfrak{h}}_\mathbb{C}^* \mid \langle \lambda, \alpha_i^\vee\rangle \in \mathbb{Z} \text{ for all } i = 0, \dots, n\}, \\ \tilde P_+ =& \{\lambda \in \tilde P \mid \langle \lambda, \alpha_i^\vee\rangle \geq 0 \text{ for all } i = 0, \dots, n\}, \text{ and} \\ \tilde P_{++} =& \{\lambda \in \tilde P \mid \langle \lambda, \alpha_i^\vee\rangle > 0 \text{ for all } i = 0, \dots, n\}. \end{aligned}$$

There exists a bijection between the irreducible integrable highest weight modules of $\tilde{\mathfrak{g}}_\mathbb{C}$ and the dominant integral weights $\lambda \in \tilde P_+$. For $\lambda \in P_+$ let $L(\lambda)$ denote the corresponding irreducible integrable highest weight module of $\tilde{\mathfrak{g}}_\mathbb{C}$. There is no essential loss in generality for the representation theory of $\tilde{\mathfrak{g}}_\mathbb{C}$ if we assume $\langle \lambda, d\rangle = 0$ for a highest weight $\lambda$ of $\tilde{\mathfrak{g}}_\mathbb{C}$, so from now on, we restrict $\tilde P$ to $\{\lambda \text{ s.t. } \langle \lambda, d\rangle = 0\}$.

Since the highest weight modules $L(\lambda)$ are irreducible, $K$ operates as a scalar on $L(\lambda)$. We define the level $k$ of $\lambda$ to be the non-negative integer $k = \langle \lambda, K \rangle$ and set

$$\tilde{P}_+^k = \{\lambda \in \tilde{P}_+ \text{ s.t. level}(\lambda) = k\}.$$

Now for $\lambda \in \tilde{P}_+$ let $\mathrm{ch}(\lambda)$ be the character and $\chi_\lambda$ be the normalized character of the $\tilde{\mathfrak{g}}_{\mathbb{C}}$-module $L(\lambda)$ (see [K], Ch.10 and §12.7). Choose an element $\Lambda_0 \in \tilde{\mathfrak{h}}$ such that $\langle \Lambda_0, \alpha_0 \rangle = 1$ and $\langle \Lambda_0, \Lambda_0 \rangle = \langle \Lambda_0, \alpha_i^\vee \rangle = 0$ for all $i = 1, \ldots, l$. If we choose orthonormal coordinates $v_1, \ldots, v_l$ of $\mathfrak{h}$ (with respect to the negative Killing form on $\mathfrak{g}$), we can coordinatize $\tilde{\mathfrak{h}}_{\mathbb{C}}$ via

$$v = 2\pi i (\sum_{\nu=1}^{l} z_\nu v_\nu - \tau \Lambda_0 + u\delta),$$

and identify $v \in \tilde{\mathfrak{h}}_{\mathbb{C}}$ with the vector $(\tau, H, u)$ with $H = \sum z_\nu v_\nu \in \mathfrak{h}_{\mathbb{C}}$ and $\tau, u \in \mathbb{C}$.

It is known (see e.g. [K]), that for any $\lambda \in \tilde{P}_+$, the character $\mathrm{ch}(\lambda)$ and the normalized character $\chi_\lambda$ converge absolutely on the domain

$$Y = \{(\tau, H, u) \mid H \in \mathfrak{h}_{\mathbb{C}}; \tau, u \in \mathbb{C}, Im(\tau) > 0\}$$

Hence $\mathrm{ch}(\lambda)$ and $\chi_\lambda$ define holomorphic functions on $Y$. Since the center of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ acts on $L(\lambda)$ by scalar multiplication, we can view $\mathrm{ch}(\lambda)$ and $\chi_\lambda$ as functions $\mathrm{ch}(\lambda)(\tau, H)$ and $\chi_\lambda(\tau, H)$ of $\tau$ and $H$ and forget about the central $u$-coordinate without loss of information. For a geometric interpretation of the $\chi_\lambda(\tau, H)$ as sections of certain line bundles over certain Abelian varieties, see e.g. [EFK] and [Lo].

An explicit formula for the normalized character $\chi_\lambda(\tau, H)$ is given by the Kac-Weyl character formula: As before, let $Q^\vee \subset \mathfrak{h}$ be the dual root lattice of $\mathfrak{g}_{\mathbb{C}}$ (with the appropriate identifications) and let $\Delta_+$ and $\tilde{\Delta}_+$ be the set of positive roots of $\mathfrak{g}_{\mathbb{C}}$ and $\tilde{\mathfrak{g}}_{\mathbb{C}}$ respectively (with respect to the simple roots $\alpha_0, \ldots, \alpha_l$). Set $\rho = 1/2 \sum_{\alpha \in \Delta_+} \alpha$. For $\mu \in \tilde{P}$ define $\bar{\mu} \in \mathfrak{h}$ to be the projection of $\mu$ to $\mathfrak{h}$ and for $x, \tau \in \mathbb{C}$, set $e(x) = e^{2\pi i x}$ and $q = e^{2\pi i \tau}$. Then for $\lambda \in \tilde{P}_+^k$ define

$$\Theta_\lambda(\tau, H) = \sum_{\gamma \in Q^\vee + k^{-1}\bar{\lambda}} e(\frac{1}{2} k\tau \langle \gamma, \gamma \rangle + k \langle \gamma, H \rangle)$$

With these definitions, the Kac-Weyl character formula (cf. [K], Ch.10, 12) reads

$$\chi_\lambda(\tau, H) = \frac{\sum_{w \in W} (-1)^{l(w)} \Theta_{w(\lambda + \tilde{\rho})}(\tau, H)}{q^{\frac{\dim \mathfrak{g}}{24}} e(\langle \rho, H \rangle) \prod_{\alpha \in \tilde{\Delta}_+} (1 - e(-\alpha(\tau, H)))^{\text{mult } \alpha}},$$

where $\text{mult } \alpha = \dim \mathfrak{g}_\alpha$ denotes the dimension of the root space corresponding to $\alpha \in \tilde{\Delta}$, and $\tilde{\rho} \in \mathfrak{h}^*$ is defined via $\langle \tilde{\rho}, \alpha_i^\vee \rangle = 1$ for $i = 0, \ldots, l$ and $\langle \tilde{\rho}, d \rangle = 0$.

The squared absolute value of the denominator of this formula is, up to the coefficient $C$, exactly the product $\mathrm{Pf}_\zeta(B_\sigma)\cdot Z(H)$ calculated in proposition 4.2. This can be seen using the decomposition of $\tilde{\Delta}_+$ into real and imaginary roots

$$\tilde{\Delta}_+ = \Delta_+ \cup \bigcup_{n=1}^{\infty}\{\Delta + n\delta\} \cup \bigcup_{n=1}^{\infty} n\delta.$$

$\Delta_+^{im} = \{n\delta \mid n \in \mathbb{N}\}$ is called the set of positive imaginary roots. The multiplicities of the roots are given by mult $\alpha = 1$ for $\alpha \in \Delta_+ - \Delta_+^{im}$ and mult $\alpha = l$ for $\Delta_+^{im}$.

We can now state the main theorem of this section.

**Theorem 4.5** *Let $h^\vee$ denote the dual Coxeter number of $\mathfrak{g}_\mathbb{C}$ and let $k$ be a positive level of $\tilde{\mathfrak{g}}_\mathbb{C}$. Set $\kappa = h^\vee + k$. Then the following identity is valid:*

$$\int_{LLG_\mathbb{C}/T_\mathbb{C}} e^{S_{G,H,\kappa}} d\sigma = \frac{C_0}{C_1 C}\sum_{\lambda\in\tilde{P}_+^k} |\chi_\lambda(\tau, H)|^2,$$

*with $C_0 = \kappa^l$, $C_1 = \frac{\sqrt{2\kappa\tau_2}^{\,l}}{vol\ \ \kappa Q^\vee}$ and $C$ as in Proposition 4.2.*

Proof. The equality of the denominators in the equation follows with proposition 4.2 and the remarks after the Kac-Weyl character formula. So we have to calculate $S_{G,H,\kappa}$ in the fixed points of the $S^1 \times S^1 \times T$-action which are given by $(s,t) \mapsto g_w \cdot \exp(s\beta)\cdot\exp(t\mu)$, with $\beta, \mu \in Q^\vee$ and where $g_w$ is a representative of $w \in W$. The action function $S_{G,H,\kappa}$ can be rewritten as

$$\begin{aligned} S_{G,H,\kappa}(g) = -\frac{\kappa\pi}{2\tau_2}\int_\Sigma &(\langle g^{-1}\partial g + g^{-1}Hg - H, g^{-1}\bar{\partial} g + g^{-1}Hg - H\rangle \\ &- 2i\langle g^{-1}\partial_t g, g^{-1}Hg + H\rangle)dsdt + \frac{i\kappa\pi}{3}\int_B \mathrm{tr}(\widetilde{g}^{-1}d\widetilde{g})^{\wedge 3}. \end{aligned}$$

For $g(s,t) = g_w \cdot \exp(s\beta)\cdot\exp(t\mu)$, the Wess-Zumino term is easily calculated using its translation quasi-invariance. In the proof of lemma 4.1 we saw that

$$\begin{aligned} \frac{i\kappa\pi}{3}\int_B \mathrm{tr}((gh)^{-1}d(gh))^{\wedge 3} = &\frac{i\kappa\pi}{3}\int_B \mathrm{tr}(g^{-1}dg)^{\wedge 3} + \frac{i\kappa\pi}{3}\int_B \mathrm{tr}(h^{-1}dh)^{\wedge 3} \\ &+ i\kappa\pi\int_\Sigma \mathrm{tr}(g^{-1}dg\wedge dhh^{-1}). \end{aligned}$$

Writing $g(s,t) = g_w \cdot (\exp(s\beta)\cdot\exp(t\mu))$, we see that the constant term $g_w$ does not contribute to the integral. Furthermore, we have

$$\begin{aligned}\frac{i\kappa\pi}{3}\int_B \mathrm{tr}((\exp(s\beta)\exp(t\mu))^{-1}d(\exp(s\beta)\exp(t\mu)))^{\wedge 3} = \\ \frac{i\kappa\pi}{3}\int_B \mathrm{tr}(\exp(-s\beta)d(\exp(s\beta)))^{\wedge 3} + \frac{i\kappa\pi}{3}\int_B \mathrm{tr}(\exp(-t\mu)d(\exp(t\mu)))^{\wedge 3} \\ + i\kappa\pi\int_\Sigma \mathrm{tr}(\exp(-s\beta)d(\exp(s\beta))\wedge d(\exp(t\mu))\exp(-t\mu)).\end{aligned}$$

The first two terms of the left hand side of the equation vanish and the third term is easily calculated to be $\kappa\pi i\langle\beta,\mu\rangle$. So in the fixed points of the torus action, the action function reads

$$\begin{aligned}S_{G,H,\kappa}(g_w\exp(s\beta)\exp(t\mu)) = -\frac{\pi\kappa}{2\tau_2}\langle\beta+\tau\mu+w^{-1}H-H,\beta+\bar\tau\mu+w^{-1}H-H\rangle \\ + \pi i\kappa\langle\mu,w^{-1}H+H\rangle - \pi i\kappa\langle\beta,\mu\rangle.\end{aligned}$$

Now after rearranging the order of the summation, theorem 4.5 follows with lemma 4.6. □

**Lemma 4.6** *The following identity is valid:*

$$\begin{aligned}\frac{vol\ \ \kappa Q^\vee}{\sqrt{2\kappa\tau_2}^l}\sum_{\beta,\mu\in Q^\vee}\sum_{w\in W}(-1)^{l(w)}e^{-\frac{\pi\kappa}{2\tau_2}\langle\beta+\tau\mu+H-wH,\beta+\bar\tau\mu+H-wH\rangle} \\ \times e^{\pi i\kappa\langle\beta,\mu\rangle+\pi i\kappa\langle\mu,H+wH\rangle} \\ = \sum_{\lambda\in\tilde P^k_+}\sum_{w_1\in W}(-1)^{l(w_1)}\Theta_{w_1(\lambda+\tilde\rho)}(\tau,H) \\ \times\sum_{w_2\in W}(-1)^{l(w_2)}\overline{\Theta_{w_2(\lambda+\tilde\rho)}(\tau,H)}.\end{aligned}$$

Proof. Let us denote the right hand side of the equation in lemma 4.6 with $N_k(\tau,H)$. Note that we have $\bar{\tilde\rho}=\rho$ and $\mathrm{level}(\tilde\rho)=h^\vee$ (see [K], Ch.12). Therefore

$$\begin{aligned}N_k(\tau,H) &= \sum_{\lambda\in\tilde P^k_+}\sum_{w_1\in W}(-1)^{l(w_1)}\Theta_{w_1(\lambda+\tilde\rho)}(\tau,H)\sum_{w_2\in W}(-1)^{l(w_2)}\overline{\Theta_{w_2(\lambda+\tilde\rho)}(\tau,H)} \\ &= \sum_{\lambda\in\tilde P^\kappa_{++}}\sum_{w_1\in W}\sum_{\gamma\in Q^\vee+\frac{1}{\kappa}w_1\bar\lambda}(-1)^{l(w_1)}e(\frac{1}{2}\kappa\tau\langle\gamma,\gamma\rangle+\kappa k\langle\gamma,H\rangle) \\ &\quad\times\sum_{w_2\in W}(-1)^{l(w_2)}\overline{\Theta_{w_2(\lambda+\tilde\rho)}(\tau,H)}\end{aligned}$$

$$= \sum_{\lambda \in \tilde{P}^\kappa_{++}} \sum_{\alpha \in Q^\vee} \sum_{w,w' \in W} (-1)^{l(w')}(-1)^{l(ww')} e(\frac{1}{2}\kappa\tau\langle \frac{1}{\kappa} w'\bar{\lambda} + \alpha, \frac{1}{\kappa} w'\bar{\lambda} + \alpha\rangle)$$
$$\times\, e(\kappa\langle \frac{1}{\kappa} w'\bar{\lambda} + \alpha, H\rangle) \overline{\Theta_{w'\bar{\lambda}+\alpha}(\tau, wH)}$$

The set $\{\frac{1}{\kappa}\bar{\lambda} | \lambda \in \tilde{P}^\kappa_{++}\}$ lies in a fundamental alcove of the affine Weyl group and the singular weights do not contribute to the sum below. Since $\Theta_\gamma$ only depends on the class of $\gamma$ modulo $Q^\vee$, we can sum over $\alpha \in Q^\vee$ and $w' \in W$ and get

$$= \sum_{\gamma \in \frac{1}{\kappa}P} \sum_{w \in W} (-1)^{l(w)} e(\frac{1}{2}\tau\kappa\langle\gamma,\gamma\rangle + \kappa\langle\gamma, H\rangle) \cdot \overline{\Theta_\gamma(\tau, wH)}$$
$$= \sum_{\gamma \in \frac{1}{\kappa}P} \sum_{w \in W} \sum_{\mu \in Q^\vee} (-1)^{l(w)} e(\frac{1}{2}\tau\kappa\langle\gamma,\gamma\rangle + \kappa\langle\gamma, H\rangle)$$
$$\times\, e(-\frac{1}{2}\bar{\tau}\kappa\langle\gamma+\mu, \gamma+\mu\rangle - \kappa\langle\gamma+\mu, wH\rangle)$$
$$= \sum_{\gamma \in \frac{1}{\kappa}P} \sum_{w \in W} \sum_{\mu \in Q^\vee} (-1)^{l(w)} e^{-2\pi\kappa\tau_2\langle\gamma,\gamma\rangle - 2\pi i\kappa\bar{\tau}\langle\gamma,\mu\rangle - \pi i\kappa\bar{\tau}\langle\mu,\mu\rangle}$$
$$\times\, e^{2\pi i\kappa\langle\gamma,H\rangle - 2\pi i\kappa\langle\gamma+\mu, wH\rangle}$$
$$= \sum_{\gamma \in \frac{1}{\kappa}P} \sum_{w \in W} \sum_{\mu \in Q^\vee} (-1)^{l(w)} e^{-2\pi\kappa\tau_2(\langle\gamma,\gamma\rangle + \langle\gamma,\mu\rangle) - 2\pi i\kappa\tau_1\langle\gamma,\mu\rangle + 2\pi i\kappa\langle\gamma, H - wH\rangle}$$
$$e^{-\pi i\bar{\tau}\kappa\langle\mu,\mu\rangle - 2\pi i\kappa\langle\mu, wH\rangle}$$

We now need to apply the Poisson transformation formula: For an Euclidean vector space $V$, a lattice $M \subset V$ and a Schwartz function $f : V \to \mathbb{C}$ one has

$$\sum_{\beta \in M^\vee} \hat{f}(\beta) = vol\;\; M \sum_{\gamma \in M} f(\gamma)$$

with

$$\hat{f}(\beta) = \int_V e^{2\pi i\langle\gamma,\beta\rangle} f(\gamma) d\gamma,$$

and where $M^\vee$ denotes the dual lattice of $M$ with respect to the scalar product on $V$. If we choose

$$f(\gamma) = e^{-2\pi\kappa\tau_2(\langle\gamma,\gamma\rangle + \langle\gamma,\mu\rangle) - 2\pi i\kappa\tau_1\langle\gamma,\mu\rangle + 2\pi i\kappa\langle\gamma, H-wH\rangle - \pi i\bar{\tau}\kappa\langle\mu,\mu\rangle - 2\pi i\kappa\langle\mu, wH\rangle},$$

a direct calculation yields

$$\hat{f}(\beta) = \frac{1}{\sqrt{2\kappa\tau_2}^l} e^{-\pi i\langle\mu,\beta\rangle - \frac{\pi}{2\tau_2\kappa}\langle\beta,\beta\rangle + \frac{\pi\tau_1}{\tau_2}\langle\beta,\mu\rangle - \frac{\pi}{\tau_2}\langle\beta,H-wH\rangle - \frac{\pi(\tau_1^2+\tau_2^2)\kappa}{2\tau_2}\langle\mu,\mu\rangle}$$
$$\times\, e^{\frac{\pi\tau_1\kappa}{\tau_2}\langle\mu,H-wH\rangle - \frac{\pi\kappa}{2\tau_2}\langle H-wH,H-wH\rangle - \pi i\kappa\langle\mu,H+wH\rangle}.$$

So by the Poisson summation formula, we get

$$N_k(\tau,H) = \frac{1}{vol\ \frac{1}{\kappa}P\cdot\sqrt{2\kappa\tau_2}^l} \sum_{\beta\in\kappa Q^\vee}\sum_{w\in W}\sum_{\mu\in Q^\vee}(-1)^{l(w)} e^{-\pi i\langle\mu,\beta\rangle - \frac{\pi}{2\tau_2\kappa}\langle\beta,\beta\rangle}$$
$$\times\, e^{\frac{\pi\tau_1}{\tau_2}\langle\beta,\mu\rangle - \frac{\pi}{\tau_2}\langle\beta,H-wH\rangle - \frac{\pi(\tau_1^2+\tau_2^2)\kappa}{2\tau_2}\langle\mu,\mu\rangle + \frac{\pi\tau_1\kappa}{\tau_2}\langle\mu,H-wH\rangle}$$
$$\times\, e^{-\frac{\pi\kappa}{2\tau_2}\langle H-wH,H-wH\rangle - \pi i\kappa\langle\mu,H+wH\rangle}$$

$$= \frac{vol\ \kappa Q^\vee}{\sqrt{2\kappa\tau_2}^l}\sum_{\beta,\mu\in Q^\vee}\sum_{w\in W}(-1)^{l(w)} e^{-\pi i\kappa\langle\mu,\beta\rangle - \frac{\pi\kappa}{2\tau_2}\langle\beta,\beta\rangle}$$
$$\times\, e^{\frac{\pi\tau_1\kappa}{\tau_2}\langle\beta,\mu\rangle - \frac{\pi\kappa}{\tau_2}\langle\beta,H-wH\rangle - \frac{\pi(\tau_1^2+\tau_2^2)\kappa}{2\tau_2}\langle\mu,\mu\rangle + \frac{\pi\tau_1\kappa}{\tau_2}\langle\mu,H-wH\rangle}$$
$$\times\, e^{-\frac{\pi\kappa}{2\tau_2}\langle H-wH,H-wH\rangle - \pi i\kappa\langle\mu,H+wH\rangle}$$
$$= \frac{vol\ \kappa Q^\vee}{\sqrt{2\kappa\tau_2}^l}\sum_{\beta,\mu\in Q^\vee}\sum_{w\in W}(-1)^{l(w)} e^{-\frac{\pi\kappa}{2\tau_2}\langle\beta-\tau\mu+H-wH,\beta-\bar{\tau}\mu+H-wH\rangle}$$
$$\times\, e^{-\pi i\kappa\langle\beta,\mu\rangle - \pi i\kappa\langle\mu,H+wH\rangle}$$
$$= \frac{vol\ \kappa Q^\vee}{\sqrt{2\kappa\tau_2}^l}\sum_{\beta,\mu\in Q^\vee}\sum_{w\in W}(-1)^{l(w)} e^{-\frac{\pi\kappa}{2\tau_2}\langle\beta+\tau\mu+H-wH,\beta+\bar{\tau}\mu+H-wH\rangle}$$
$$\times\, e^{\pi i\kappa\langle\beta,\mu\rangle + \pi i\kappa\langle\mu,H+wH\rangle}.$$

This finishes the proof of lemma 4.6. □

The modular group $SL(2,\mathbb{Z})$ acts naturally on the torus $S^1\times S^1$. Under this action, the modular parameter $\tau$ of the elliptic curve $\Sigma_\tau$ is transformed via

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}$$

for $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})$. This $SL(2,\mathbb{Z})$-action can be extended to the domain $Y$ via

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} : (\tau,H,u)\mapsto \left(\frac{a\tau+b}{c\tau+d}, \frac{H}{c\tau+d}, u - \frac{c\langle H,H\rangle}{2(c\tau+d)}\right).$$

It was shown in [KP] that for each $k \in \mathbb{N}$, the $SL(2,\mathbb{Z})$-action on $Y$ constructed above gives rise to an $SL(2,\mathbb{Z})$-action on the set of normalized characters of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ at level $k$. In particular, it follows from the explicit transformation properties of the characters under the $SL(2,\mathbb{Z})$-action that the sum $\sum_{\lambda\in\tilde{P}^k_+} |\chi_\lambda(\tau,H)|^2$ is $SL(2,\mathbb{Z})$-invariant. Since the $SL(2,\mathbb{Z})$-action on the modular parameter arises naturally in the functional integral setup considered in this paper, one should expect that the $SL(2,\mathbb{Z})$-invariance of the partition function at level $k$ can easily be deduced from its representation as a functional integral. In fact, the search for a derivation of the $SL(2,\mathbb{Z})$-invariance of the partition function using only functional integrals was one of the starting points of this paper. Unfortunately, this does not seem possible since for the zeta regularization to work, $\alpha(H)$ has to be a real parameter for all $\alpha \in \Delta$. But the $SL(2,\mathbb{Z})$-action on $Y$ does not leave the space $\{H \mid \alpha(H) \in \mathbb{R} \text{ for all } \alpha \in \Delta\}$ invariant. We can only deduce a slightly weaker result:

The sum $\sum_{\lambda\in\tilde{P}^k_+} |\chi_\lambda(\tau,H)|^2$ depends continuously on $H \in \mathfrak{h}$ and is well defined for $H = 0$. The functional integral in theorem 4.5 is defined for all generic $H \in \mathfrak{h}$ and from the equality proved in the theorem we can deduce that it is a continuous function of $H$ as well. Thus, we can formally write

$$\int_{LLG_{\mathbb{C}}/T_{\mathbb{C}}} e^{S_{G,0,\kappa}} d\sigma = \frac{C_0}{C_1 C_{\tau,H}} \sum_{\lambda\in\tilde{P}^k_+} |\chi_\lambda(\tau,0)|^2.$$

Now one can easily check that the denominator and the numerator of the left hand side in the equation above are $SL(2,\mathbb{Z})$-invariant, and since the $\tau_2$-dependent terms in the coefficients of the right hand side cancel, we have the following corollary of theorem 4.5:

**Corollary 4.7**

$$\sum_{\lambda\in\tilde{P}^k_+} |\chi_\lambda(\tau,0)|^2$$

*is a modular invariant function of* $\tau$.

**Remark 4.8** Let us mention a slight generalization of the WZW model: Analogously to §3.5, one can define a twisted version of the WZW model by twisting the loop directions with an outer automorphism $\psi$ of order $\mathrm{ord}(\psi) = r$ of the group $G_{\mathbb{C}}$. That is, we consider the space

$$LL(G_{\mathbb{C}},\psi) = \{g \in LLG_{\mathbb{C}} \mid \psi(g(s,t)) = g(s,t+\frac{1}{r}) = g(s+\frac{1}{r},t) \text{ for all } t \in [0,1]\}.$$

For $H \in T^\psi$, we can restrict the action function $S_{G,H,\kappa}$ and the corresponding $\mathbb{R}$-action on $LG_{\mathbb{C}}/T_{\mathbb{C}}$ to obtain an action functional and an $\mathbb{R}$-action on $LL(G_{\mathbb{C}},\psi)/T^\psi_{\mathbb{C}}$.

Now we can use the Liouville functional to calculate the formal integral

$$\int_{LL(G_{\mathbb{C}},\psi)/T^{\psi}_{\mathbb{C}}} e^{S_{G,H,\kappa}(g)} d\sigma(g).$$

For simplicity, let us exclude the case that $G_{\mathbb{C}}$ is of type $\mathrm{A}_{2l}$. Then similar calculations as in §3.5 and §4.3 show that the partition function of the twisted WZW model is given by

$$\int_{LL(G_{\mathbb{C}},\psi)/T^{\psi}_{\mathbb{C}}} e^{S_{G,H,\kappa}(g)} d\sigma(g) = C_{\psi} \cdot \sum_{\lambda \in \tilde{P}^{\psi\vee,k}_{+}} |\chi_{\lambda}(\tau, H)|^2,$$

with $\kappa = k + h^{\vee}$ as before. Here $C_{\psi} \in \mathbb{R}$ is some constant, and $P^{\psi\vee,k}_{+}$ denotes the set of highest weights of level $k$ of the twisted affine Lie algebra $\tilde{g}^{\psi,\vee}_{\mathbb{C}}$, whose finite dimensional root system is dual to the root system of the finite dimensional Lie algebra $\mathfrak{g}^{\psi}_{\mathbb{C}}$. For example, if $g_{\mathbb{C}}$ is of type $D_l$, then $\tilde{g}^{\psi,\vee}_{\mathbb{C}}$ is the twisted affine Lie algebra of type $\mathrm{A}^{(2)}_{2(l-1)-1}$, and accordingly in the other cases. This is the same "duality" between root systems which appears in the calculation of the irreducible characters of certain non-connected compact Lie groups [W2].

In any case, the largest subgroup of $SL(2,\mathbb{Z})$ acting on $LL(G_{\mathbb{C}},\psi)$ is the congruence subgroup

$$\Gamma(r) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \mid a \equiv b \equiv 1 \bmod r \text{ and } b \equiv c \equiv 0 \bmod r\}.$$

Now a similar argument as above yields the $\Gamma(r)$-invariance of the partition function of the twisted WZW-model at $H = 0$.

## 4.4 Concluding remarks

Let us close this exposition with some remarks about the measure theoretic interpretation of the calculations leading to the partition function of the WZW model at level $\kappa$. As we saw in §3.4, the partition function of the compact Lie group $G$ can be can be expressed as an integral over the space of continuous paths $\gamma : [0,1] \to G$ such that $\gamma(0) = e$ and $\gamma(1) \in \mathcal{O}_{\exp(H)}$. One of the main ingredients in this interpretation is the observation that the coadjoint orbits of the centrally extended loop group $\hat{G}$ can be classified in terms of conjugacy classes of the group $G$. This observation generalizes directly to the case of double loop groups:

Let $L_hG_{\mathbb{C}}$ be the holomorphic loop group, i.e. the set of holomorphic maps from the cylinder $\mathbb{C}/\mathbb{Z}$ to $G_{\mathbb{C}}$. The group $\mathbb{C}/\mathbb{Z}$ acts on $L_hG_{\mathbb{C}}$ by automorphisms, and we denote the semi-direct product $\mathbb{C}/\mathbb{Z} \ltimes L_hG_{\mathbb{C}}$ by $\check{G}_{\mathbb{C}}$. Let $(\tau, \delta) \in \check{G}_{\mathbb{C}}$. Then a short calculation shows that $\tau$ is invariant under conjugation with any element

$(z, \gamma_1) \in \check{G}_{\mathbb{C}}$. Thus, the coset $C_\tau = \{(\tau, \gamma) \mid \gamma \in L_h G_{\mathbb{C}}\}$ is fixed under conjugation. The group $LLG_{\mathbb{C}}$ admits a one dimensional central extension $LL\hat{\ }G_{\mathbb{C}}$. Now the corresponding coadjoint orbits of $LL\hat{\ }G_{\mathbb{C}}$ (resp. the affine coadjoint orbits of $LLG_{\mathbb{C}}$) can be classified in terms of $L_h G_{\mathbb{C}}$-conjugacy classes inside $C_\tau$ in the same way the coadjoint orbits of $\hat{G}$ are classified in terms of conjugacy classes of $G$ (see [EF]).

Thus, in order to generalize the Wiener measure approach to the calculation of the partition on a compact Lie group in §3.4 to the WZW model, one has to develop the notion of Brownian motion on the holomorphic loop group $L_h G_{\mathbb{C}}$ or an appropriate completion thereof. It does not seem unlikely that such a generalization is possible since important ingredients for the construction of the Wiener measure on the compact group $G$ like the Haar measure and the Laplacian admit generalizations to the case of Kac-Moody groups (see e.g. [Pi], [EFK]). The generalization of the Wiener measure approach to functional integrals would be very interesting since it would provide solid mathematical ground for dealing with such integrals.

Of course, even such a measure theoretic reinterpretation of our calculation of the WZW partition function would still leave open a much more fundamental question: The Duistermaat Heckman formula is a well established result for finite dimensional symplectic manifolds. As realized in the physics literature ([A], [P]) and in this paper, a straight forward generalization of the formalism to certain infinite dimensional manifolds gives interesting results which in some cases can also be derived by usual integration methods. So it would be interesting to know how a class of infinite dimensional manifolds could look like on which a stringent measure theory can be developed which includes a version of the Duistermaat Heckman formula as a theorem. As we saw in §3.4, an appropriate closure of the coadjoint orbits of the loop group $LG$ should certainly belong to such a class.

# Appendix: Zeta-regularized Products

In this section we will recall the definition and some basic properties of zeta-regularized products. See e.g. [JoL] for a more comprehensive introduction to the theory of regularized products. To motivate the following definition, recall that the product of $\lambda_1, \dots, \lambda_N \in \mathbb{R}_{>0}$ can be written as

$$\prod_{n=1}^{N} \lambda_n = \exp\left(-\frac{d}{ds}|_{s=0} \sum_{n=0}^{N} \frac{1}{\lambda_n^s}\right).$$

**Definition A.1** *Let $\Lambda = \{\lambda_1, \lambda_2, \dots\}$ be a sequence of positive real numbers. For $s \in \mathbb{C}$, define*

$$\zeta_\Lambda(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}.$$

*If $\zeta_\Lambda(s)$ converges for $Re(s)$ sufficiently large and if the function $\zeta_\Lambda$ can be analytically continued to a meromorphic function on $\mathbb{C}$ which is regular at $s = 0$, we define the zeta-regularized product of $\{\lambda_1, \lambda_2, \dots\}$ by*

$$\left(\prod_{n=1}^{\infty} \lambda_n\right)_\zeta = \exp(-\zeta'_\Lambda(0))$$

*and call such a sequence zeta multipliable.*

Note that if the usual limit $\prod_{n=1}^{\infty} \lambda_n$ exists, then $\lim_{n\to\infty} \lambda_n = 1$. Thus the corresponding zeta function does not converge anywhere. Two important properties of the zeta-regularized product are stated in the following lemma:

**Lemma A.2** *Let $\Lambda = \{\lambda_1, \lambda_2 \dots\}$ be a zeta-multipliable sequence and $a, b \in \mathbb{R}_{>0}$. Then the sequences $\{\lambda_1{}^a, \lambda_2{}^a, \dots\}$ and $\{\lambda_1 b, \lambda_2 b, \dots\}$ are zeta-multipliable and we have*

$$\left(\prod_{n=1}^{\infty} \lambda_n{}^a\right)_\zeta = \left[\left(\prod_{n=1}^{\infty} \lambda_n\right)_\zeta\right]^a,$$

*and*

$$\left(\prod_{n=1}^{\infty} \lambda_n b\right)_\zeta = \left(\prod_{n=1}^{\infty} \lambda_n\right)_\zeta \cdot b^{\zeta_\Lambda(0)}.$$

□

Let us state the examples of zeta-regularized products which were used in chapters 3 and 4: First consider the Riemann zeta-function which is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Obviously, $\zeta(s)$ converges for $Re(s) > 1$. Furthermore, $\zeta$ can be analytically continued to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$ and $\zeta(0) = \frac{1}{2}$. Standard theory of the Riemann zeta-function (cf. [L1], [L2]) implies $\zeta'(0) = -\log\sqrt{2\pi}$. Thus, the sequence $\{1, 2, 3, \dots\}$ is zeta-multipliable and we have

$$\left(\prod_{n=1}^{\infty} n\right)_\zeta = \sqrt{2\pi}.$$

As a second example, we consider a class of zeta-functions, the so called Epstein zeta-functions:

**Definition A.3** *Let $\tau_1, \tau_2, v \in \mathbb{R}$ and $\tau = \tau_1 + i\tau_2 \in \mathbb{C}$ such that $\tau_2 > 0$. The Epstein zeta-functions are given by*

$$\zeta_\tau(s;v) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)+(v,0)\neq 0}} \frac{\tau_2^s}{|m+\tau n+v|^{2s}}.$$

*and*

$$\tilde{\zeta}_\tau(s;v) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \frac{\tau_2^s \cdot e^{2\pi i m v}}{|m+\tau n|^{2s}}.$$

The series defining $\zeta_\tau$ and $\tilde{\zeta}_\tau$ converge absolutely for $s > 1$ and define analytic functions on $\{s \in \mathbb{R}, s > 1\}$. The most important properties of $\zeta_\tau$ and $\tilde{\zeta}_\tau$ are stated in the following theorem:

**Theorem A.4** *The functions $\zeta_\tau(s;v)$ and $\tilde{\zeta}_\tau(s;v)$ have analytic continuations to the entire $s$-plane. If $v \notin \mathbb{Z}$, the continuations are entire functions of $s$. If $v \in \mathbb{Z}$, then $\zeta_\tau(s;v)$ and $\tilde{\zeta}_\tau(s;v)$ are meromorphic in the entire $s$-plane with the only singularity at $s = 1$. In all cases, $\zeta_\tau$ and $\tilde{\zeta}_\tau$ satisfy the functional equation*

$$\pi^{-s}\Gamma(s)\zeta_\tau(s;v) = \pi^{-(1-s)}\Gamma(1-s)\tilde{\zeta}_\tau(1-s;v).$$

*where $\Gamma(s)$ is the usual $\Gamma$-function.*

For a proof of theorem A.4 as well as an exposition of the theory of much more general Epstein zeta-functions, see [Si], Ch.1 §5.

Using the functional equation satisfied by $\zeta_\tau$ and $\tilde{\zeta}_\tau$, one can prove the Kronecker limit formulas (see [Si], [L2]):

**Theorem A.5 (First Kronecker limit formula)** *Let $\gamma$ be the Euler constant. Then, in a neighborhood of $s = 1$, we have*

$$\zeta_\tau(s;0) = \frac{\pi}{s-1} + 2\pi(\gamma - \log 2 - \log(\sqrt{\tau_2}|\eta(\tau)|^2)) + \mathcal{O}(s-1),$$

*where $\eta(\tau)$ is the Dedekind eta-function.*

**Theorem A.6 (Second Kronecker limit formula)** *For $v \in \mathbb{R}\setminus\mathbb{Z}$, we have*

$$\tilde{\zeta}_\tau(1,v) = -\pi \log |q^{\frac{1}{12}}(e(\frac{1}{2}v) - e(-\frac{1}{2}v)) \prod_{n=1}^{\infty}(1-q^n e(v))(1-q^n e(-v))|^2,$$

*with $e(v) = e^{2\pi i v}$ and $q = e^{2\pi i \tau}$.*

From theorems (A.4), (A.5) and (A.6), one can directly deduce the following formulas:

**Lemma A.7** *Let $\tau$ and $v$ be as above. Then*

$$\frac{\partial}{\partial s}\zeta_\tau(s;0)|_{s=0} = -\log(4\pi^2\tau_2|\eta(\tau)|^4)$$

*and*

$$\frac{\partial}{\partial s}\zeta_\tau(s;v)|_{s=0} = -\log|q^{\frac{1}{12}}(e(\frac{1}{2}v) - e(-\frac{1}{2}v)) \times \prod_{n=1}^{\infty}(1 - q^n e(v))(1 - q^n e(-v))|^2$$

*for $v \in \mathbb{R} \setminus \mathbb{Z}$. Furthermore, we have $\zeta_\tau(0;0) = -1$ and $\zeta_\tau(0;v) = 0$.* □

# Bibliography


[A] M.F. Atiyah, *Spherical Symmetry and Stationary-Phase Approximation*, Asterisque **131** (1985), 43-59.

[B] M. Berger, P. Gaudechon, E. Mazet, *Le Spectre d'une Variété Riemannienne*, LMN **194**, Springer, Berlin, Heidelberg, New York, 1971.

[BtD] T. Bröcker, T. tomDieck, *Representations of Compact Lie Groups*, Springer, Heidelberg, New York, 1985.

[Br] J. Brylinski, *Coadjoint Orbits of Central Extensions of Gauge Groups*, Comm. Math. Phys. **188** (1997), 351-365.

[DH1] J.J. Duistermaat, G.J. Heckman, *On the Variation in the Cohomology of the Symplectic Form of the Reduced Phase Space*, Inv. Math. **69** (1982), 259-268.

[DH2] J.J. Duistermaat, G.J. Heckman, *Addendum to :"On the Variation in the Cohomology of the Symplectic Form of the Reduced Phase Space"*, Inv. Math. **72** (1983), 153-158.

[EF] P. Etingof, I.B. Frenkel, *Central extensions of current groups in two dimensions*, Comm. Math. Phys. **165** (1994), 429-444.

[EFK] P. Etingof, I.B. Frenkel, A. Kirillov, *Spherical functions on affine Lie groups*, Duke. Math. J. **80** (1995), 59-90.

[Fe] H.G. Fegan, *The heat equation on a compact Lie Group*, Trans. Amer. Math. Soc., **246** (1978), 339-357.

[F] I.B. Frenkel, *Orbital theory for affine Lie algebras*, Invent. Math. **77** (1984), 301-352.

[FKh] I.B. Frenkel, B.A. Khesin, *Four Dimensional Realization of Two Dimensional Current Groups*, Comm. Math. Phys. **178** (1996), 541-562.

[FuS] J. Fuchs, U. Ray, C. Schweigert, *Some Automorphisms of Generalized Kac-Moody Algebras*, J. Algebra, **191** (1997), 518-540.

[G] K. Gawedzki, *Lectures on Conformal Field Theory*, in: Quantum Fields and Strings: A Course For Mathematicians, P. Deligne et al. eds., American Mathematical Society, Providence, 1999.

[GK] K. Gawedzki, A. Kupiainen, *Coset Construction from Functional Integrals*, Nucl. Phys. B **320** (1989), 625-668.

[He] S. Helgason, *Groups and Geometric Analysis*, Academic Press, Orlando, San Diego, New York, London, Toronto, Montreal, Sidney, Tokyo, Sao Paulo, 1984.

[H] J. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer, Heidelberg, New York, 1971.

[I] K. Ito, *Brownian Motion in a Lie Group*, Proc. Japan. Acad., **26** (1950), 4-10.

[J] J.C. Jantzen, *Darstellungen halbeinfacher algebraischer Gruppen und zugeordnete kontravariante Formen*, Bonner Mathematische Schriften, Nr. 67, (1973).

[JoL] J. Jorgenson, S. Lang *Basic Analysis of Regularized Series and Products*, LNM **1564**, Springer, Heidelberg, New York, 1993.

[K] V.G. Kac, *Infinite-dimensional Lie Algebras*, 3rd edition, Cambridge University Press, Cambridge, 1990.

[KP] V.G. Kac, D.H. Peterson, *Infinite-dimensional Lie algebras, theta functions and modular forms*, Adv. Math. **53** (1984), 125-264.

[Ki1] A.A. Kirillov, *Unitary representations of nilpotent Lie groups*, Russian Math. Surveys **17** (1962), 57-110.

[K2] A.A. Kirillov, *Elements of the Theory of Representations*, Springer, Heidelberg, New York, 1976.

[Ki3] A.A. Kirillov, *Merits and Demerits of the Orbit Method*, Bull. of the AMS **36** (1999), 433-488.

[Kl] M. Kleinfeld, *Orbitaltheorie für getwistet affine Kac-Moody-Algebren*, Inaugural-Dissertation, Westfälische Wilhelms-Universität Münster, 1996.

[Ko] B. Kostant, *Lie Algebra Cohomology and the Generalized Borel-Weil Theorem*, Ann. Math. **74** (1961), 320-387.

[Ku] H.H. Kuo, *Gaussian Measures in Banach Spaces*, LNM **463**, Springer, Heidelberg, New York, 1975.

[L1] S. Lang, *Complex analysis*, 3rd edition, Springer, Heidelberg, New York, 1993.

[L2] S. Lang, *Elliptic functions*, Addison-Wesley, Reading, 1973.

[Lo] E. Looijenga, *Root Systems and Elliptic Curves*, Invent. Math. **38** (1976), 17-32.

[McK] H.P. McKean, *Stochastic Integrals*, Academic Press, New York, 1969.

[M] J. Mickelsson, *Kac-Moody groups, topology of the Dirac determinant bundle and fermionization*, Comm. Math. Phys. **110** (1986), 173-183.

[Mo] S. Mohrdieck, *Conjugacy classes of non-connected semi-simple algebraic groups*, PhD thesis, Universität Hamburg, 2000.

[P] R.E. Perret, *Path Integral Derivation of Character for Kac-Moody Groups*, Nucl. Phys. B **356** (1991), 229-244.

[Pi] D. Pickrell, *Invariant measures for unitary groups associated to Kac-Moody Lie algebras*, Mem. Amer. Math. Soc. **146** (2000).

[PW] A.M. Polyakov, P.B. Wiegmann, *Goldstone fields in two dimensions with multivalued actions*, Phys. Lett. B **141** (1984), 223-228.

[PS] A. Pressley, G. Segal, *Loop Groups*, Oxford University Press, Oxford, 1986.

[RS] D.B. Ray, I.M. Singer, *Analytic torsion for complex manifolds*, Ann. Math. **98** (1971), 154-177.

[S] G. Segal, *Unitary Representations of some Infinite Dimensional Groups*, Comm. Math. Phys. **80** (1981), 301-342.

[dS] J. deSiebenthal, *Sur le groups de Lie compactes non connexes*, Comm. Math. Helv. **31** (1956), 41-89.

[Si] C.L. Siegel, *Lectures on Advanced Analytic Number Theory*, Tata Institute, 1961.

[T] H. Tsukada, *String Path Integral Realization of Vertex Operator Algebras*, Mem. Am. Math. Soc. **91** (1991).

[W] S. Weinberg, *The Quantum Theory of Fields*, two volumes, Cambridge University Press, Cambridge, 1995.

[W1] R. Wendt, *Eine Verallgemeinerung der Weylschen Charakterformel*, Diplomarbeit, Mathematisches Seminar der Universität Hamburg, 1998.

[W2] R. Wendt, *Weyl's character formula for non-connected Lie groups and orbital theory for twisted affine Lie algebras*, J. Funct. Anal. to appear (also Chapter I of this thesis).

[Wi] E. Witten, *Non-abelian bozonization in two dimensions*, Comm. Math. Phys. **92** (1984), 455-472.

# Zusammenfassung (Abstract)

Die vorliegende Arbeit beschäftigt sich mit geometrischen und analytischen Ansätzen zur Charakter-Theorie affiner Lie Algebren und deren Anwendungen in der mathematischen Physik.

Im ersten Teil wird die in [F] entwickelte orbitale Theorie für ungetwistet affine Lie Algebren auf den Fall getwistet affiner Lie Algebren verallgemeinert: Es werden die koadjungierten Orbiten einer getwisteten Schleifengruppe mittels Konjugationsklassen gewisser nicht-zusammenhängender Lie Gruppen klassifiziert. Wir geben eine Formel für die irreduziblen Charaktere der auftauchenden nicht-zusammenhängenden Lie Gruppen an. Mit diesem Ergebnis und der Theorie der Wärmeleitung auf kompakten Lie Gruppen können die Charaktere der integrablen Höchstgewichtsmoduln der getwistet affinen Lie Algebren als Integrale über einen Raum von Wegen in einer Zusammenhangskomponente einer nicht-zusammenhängenden Lie Gruppe interpretiert werden. Das der Integration zugrundeliegende Maß ist das Wiener-Maß. Mithilfe der Klassifikation der koadjungierten Orbiten kann dieses Integral als ein Integral über den Abschluß eines koadjungierten Orbits der zugrundeliegenden Schleifengruppe interpretiert und somit als ein Analogon zu Kirillovs Charakterformel für kompakte Lie Gruppen angesehen werden.

Im zweiten Teil der Arbeit wird das Liouville-Funktional auf der Menge der Funktionen auf einer symplektischen Mannigfaltigkeit $M$ eingeführt, die Hamiltonsch bezüglich einer symplektischen Torus-Operation auf $M$ sind. Für endlichdimensionale Mannigfaltigkeiten hängt dieses Funktional nach einem Satz von Duistermaat und Heckmann [DH1] eng mit der Integration über das Liouville-Maß zusammen. Einer Idee Wittens [A] folgend, benutzen wir das Liouville-Funktional, um ein formales Analogon zur Integration über die Riemannsche Volumenform auf $M$ zu definieren. Wir benutzen diesen Ansatz, um Funktionen auf koadjungierten Orbiten bestimmter unendlichdimensionaler Lie Gruppen über die "Riemannsche Volumenform" zu "integrieren". Im Fall von Schleifen-, bzw. getwisteten Schleifengruppen können wir mit dieser formalen Integration ein weiteres Analogon zur Kirillovschen Charakterformel herleiten, das, wie gezeigt wird, in gewissem Sinne äquivalent zu der Charakterformel aus [F], bzw. dem ersten Teil dieser Arbeit ist.

Die Wirkung der geeichten Wess Zumino Witten Quantenfeldtheorie definiert eine Funktion auf den generischen koadjungierten Orbiten der von Etingof und Frenkel [EF] eingeführten zentral erweiterten Stromgruppen. Wir zeigen, daß die WZW Wirkung Hamiltonsch bezüglich einer natürlichen Torus-Operation auf diesen Orbiten ist und benutzen die oben beschriebene Integration über die Riemannsche Volumenform auf diesen Orbiten, um die Partitionsfunktion des geeichten WZW Modells zu berechnen und ein Ergebnis zu deren modularer Invarianz herzuleiten.

# Lebenslauf

| | |
|---|---|
| 20.9.1971 | Geboren in Frankfurt am Main |
| Aug. 78 - Juni 82 | Grundschule in Frankfurt am Main |
| Aug. 82 - Juni 91 | Wöhlergymnasium in Frankfurt am Main mit Abschluß Abitur |
| Sept. 91 - Nov. 92 | Zivildienst bei der Deutschen Multiple Sklerose Gesellschaft |
| Okt. 92 - Mai 98 | Studium der Mathematik mit Nebenfach Physik an der Universität Hamburg mit Abschluß Diplom. Titel der Arbeit: Eine Verallgemeinerung der Weylschen Charakterformel |
| Sept. 95 - Juli 96 | Studium am Queens' College der Universität Cambridge, England. Certificate of Advanced Studies in Mathematics |
| Seit Okt. 98 | Promotionsstudent und Wissenschaftlicher Mitarbeiter am Fachbereich Mathematik der Universität Hamburg |
| Juni 99 | Forschungsaufenthalt am RIMS in Kyoto, Japan |
| Sept. 99 - Juni 00 | Forschungsaufenthalt an der Yale University, New Haven, USA |
| Seit Sept. 00 | Graduate Fellow am Fields Institut, Toronto, Kanada |