# Quantum groups and Field Theory

Dissertation

zur Erlangung des Doktorgrades
der Fakultät für Mathematik, Informatik
und Naturwissenschaften
der Universität Hamburg

vorgelegt
im Department Mathematik
von

EFROSSINI TSOUCHNIKA

aus Athen

Hamburg
2008

Als Dissertation angenommen vom Department
Mathematik der Universität Hamburg

auf Grund der Gutachten von Prof. Dr. Christoph Schweigert
und Prof. Dr. Dorothea Bahns

Hamburg, den 04.06.2008

Prof. Dr. Reinhard Diestel
Leiter des Departments Mathematik

# Contents

# Introduction

Symmetries give essential insight into the structure of physical systems. A system which is covariant under symmetry transformations exhibits conserved quantities. It is possible to compute conservation laws explicitly by means of Noether's theorem if a Lagrangian action can be defined. Many successful physical theories such as the standard model for elementary particles are based on the concept of symmetries. Mathematically, many symmetry structures can be described by means of groups, Lie groups and Lie algebras.

The notion of a quantum group as a more general symmetry structure arose in the investigation of low-dimensional integrable quantum systems [Fad84]. In [Dri85, Jim85] quantum enveloping algebras were used in order to construct solutions of the quantum Yang-Baxter equation. Following the terminology of Drinfeld, we identify the concept of a quantum group with the concept of a Hopf algebra.

In the meantime quantum groups play an important role in algebra with many applications to other branches of mathematics and to physics. For example, new isotopy invariants of knots and 3-manifolds can be constructed via quantum groups: since the $R$-matrix of a quantum group satisfies the quantum Yang-Baxter equation, one can associate a representation of the braid group to it. By computing the trace of this representation a knot invariant is obtained. In particular, the Jones polynomial [Jon85] can be constructed in this way (see also the textbook [Kas]).

Quantum groups can also be used in order to find a canonical basis for any finite-dimensional representation of complex simple Lie algebras [Lus90, Lus90b]. An important application of quantum groups to physics arises in conformal field theory (CFT): the modular tensor category encoding the data of a chiral CFT is equivalent to the tensor category of representations of a (weak) quantum group [ENO02]. Via the Knizhnik-Zamolodchikov equation, which is satisfied by the $m$-point functions of Wess-Zumino-Witten theories on $\mathbb{C}P^1$ [KnZa84, Ma95], one finds again a representation of the braid group and hence a connection to links.

Quantum groups play a significant role in physical theories on noncommutative spaces, so-called quantum spaces: they arise as the symmetry structures on noncommutative spaces. By means of the deformation quantization [BFFLS78, Kon03] symmetry structures on commutative spaces can be deformed in the category of associative algebras to obtain quantum symmetry structures on noncommutative spaces.

The interest in quantum spaces results mainly from physical questions: Heisenberg [Hei30] formulated the idea of a noncommutative spacetime hoping to regularize the divergent electron self-energy. Snyder gave a first mathematical description of quantum spaces in terms of a noncommutative coordinate algebra [Sny47], and

Connes and Rieffel were able to define gauge theories on a quantum space [CoRi87].

A further motivation to discuss quantum spaces is the idea that spacetime should change its properties at small distances. It is expected that a theory for quantum gravity will not be local and that the uncertainty of energy and momentum will imply a modification of spacetime geometry at distances comparable to the Planck scale [DWi].

We work with the notion of quantum spaces in the following sense: the philosophy is that any space is determined by the algebra of functions on it with the usual product (for a review of this philosophy see [Ca01]). In this spirit we consider the noncommutative associative algebra of functions on a quantum space instead of the quantum space itself and the Hopf algebra of functions on a quantum group instead of the quantum group.

In the following, we give an explicit introduction to a mathematical description of quantum spaces and their symmetry structures and consider the $q$-deformation of the affine plane as an example. All statements made in this section are proved in the textbook [Kas].

**The commutative case: the affine plane**

Let $k$ be a field, $k\{x, y\}$ the free associative algebra in two variables and $I$ the two-sided ideal in $k\{x, y\}$ generated by the element $xy - yx$. The affine plane is defined to be the polynomial algebra $k[x, y] \cong k\{x, y\}/I$.

A symmetry group of a vector space $V$ is a group $G$ acting on $V$. Via the group action the vector space $V$ can be endowed with the structure of a module over the group algebra. The algebra of functions on the vector space $V$ is a comodule over the dual space group algebra.

We define the $k$-algebra $M(2)$ to be the polynomial algebra $k[a, b, c, d]$. For any algebra $A$ there is a natural bijection from the algebra of morphisms $\mathrm{Hom}_{alg}(M(2), A)$ to the algebra $M_2(A)$ of $2 \times 2$-matrices with entries in $A$. The algebra $M(2)$ can be endowed with the structure of a bialgebra: we write the coproduct $\Delta : M(2) \to M(2) \otimes M(2)$ and the counit $\epsilon : M(2) \to k$ in the matrix form

$$\begin{aligned} \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \otimes \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \\ \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Let $I_t$ be the ideal in the polynomial algebra $M(2)[t]$ generated by $(ad - bc)t - 1$, then we define the infinite-dimensional commutative algebras $GL(2) := M(2)[t]/I_t$ and $SL(2) := M(2)/I_{t=1}$. The algebra $SL(2)$ inherits the bialgebra structure of $M(2)$, for $GL(2)$ to be a bialgebra we must set $\epsilon(t) = 1$. The algebras $GL(2)$ and $SL(2)$ are Hopf algebras with antipode $S$

$$\begin{aligned} S \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ S(t) &= t^{-1} = ad - bc\,. \end{aligned}$$

For a bialgebra $H$ and an algebra $A$, we call $A$ an $H$-comodule algebra if the vector space $A$ is an $H$-comodule and the multiplication morphism $A\otimes A \to A$ and the unit morphism $k \to A$ are morphisms of $H$-comodules. The notion of an $H$-comodule algebra is equivalent to the notion of an algebra in the tensor category of $H$-comodules [Par77, FRS02b].

The polynomial algebra $k[x,y]$ can be equipped with the structure of an $M(2)$-comodule by

$$\begin{array}{rcl} k[x,y] & \longrightarrow & M(2)\otimes k[x,y] \\ x & \longmapsto & a\otimes x + b\otimes y \\ y & \longmapsto & c\otimes x + d\otimes y\,, \end{array} \tag{1}$$

and via the same action $k[x,y]$ becomes an $SL(2)$-comodule. In particular, the affine plane $k[x,y]$ is an $M(2)$- and an $SL(2)$-comodule algebra.

The subspace $k[x,y]_n$ of homogeneous polynomials of total degree $n$ is a subcomodule of the affine plane $k[x,y]$. The $M(2)$- (respectively $SL(2)$-) comodule $k[x,y]$ is isomorphic to the direct sum of the comodules $k[x,y]_n$.

For reasons of simplicity assume now that $k = \mathbb{C}$. Denote by $U(\mathfrak{sl}(2))$ the universal enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$ of traceless $2\times 2$-matrices with entries in $\mathbb{C}$. We choose generators $X, Y, H$ that obey the conditions

$$[X,Y] = H \qquad [H,X] = 2X \qquad [H,Y] = -2Y\,.$$

The universal enveloping algebra $U(\mathfrak{sl}(2))$ is a Hopf algebra, and there is a duality between the Hopf algebras $SL(2)$ and $U(\mathfrak{sl}(2))$: there exists a nondegenerate, bilinear pairing $<,>$ on $U(\mathfrak{sl}(2)) \times SL(2)$, such that the conditions

$$\begin{array}{rcl} < uv, x > & = & \displaystyle\sum_{(x)} < u, x' >< v, x'' > \\ < u, xy > & = & \displaystyle\sum_{(u)} < u', x >< u'', y > \\ < 1, x > & = & \epsilon(x) \\ < u, 1 > & = & \epsilon(u) \\ < S(u), x > & = & < u, S(x) > \end{array}$$

hold for by any $u, v \in U(\mathfrak{sl}(2))$, $x, y \in SL(2)$. We use the Sweedler notation $\sum_{(x)} x'\otimes x''$ for the coproduct and denote $\epsilon$ for the counit and $S$ for the antipode of the respective Hopf algebra.

For a bialgebra $H$ we can define the dual notion of an $H$-comodule algebra: an algebra $A$ is an $H$-module algebra if the vector space $A$ is an $H$-module and the multiplication morphism and the unit morphism of $A$ are morphisms of $H$-modules. The notion of an $H$-module algebra is equivalent to the notion of an algebra in the tensor category of $H$-modules [Par77, FRS02b].

The duality between the Hopf algebras $SL(2)$ and $U(\mathfrak{sl}(2))$ implies a correspondence between $SL(2)$-comodules and $U(\mathfrak{sl}(2))$-modules. In particular, the vector space $k[x,y]$ can be endowed with the structure of a $U(\mathfrak{sl}(2))$-module via

$$\begin{aligned} U(\mathfrak{sl}(2)) \otimes k[x,y] &\longrightarrow k[x,y] \\ X \otimes P &\longmapsto x\frac{\partial P}{\partial y} \\ Y \otimes P &\longmapsto y\frac{\partial P}{\partial x} \\ H \otimes P &\longmapsto x\frac{\partial P}{\partial x} - y\frac{\partial P}{\partial y} \end{aligned}$$

for any polynomial $P \in k[x,y]$. In particular, the affine plane $k[x,y]$ is a $U(\mathfrak{sl}(2))$-module algebra.

The submodule $k[x,y]_n$ of homogeneous polynomials of degree $n$ is isomorphic to the simple $U(\mathfrak{sl}(2))$-module $V(n)$ which is generated by a highest weight vector of weight $n$.

### The deformed case: the quantum plane

Now we discuss a quantum version of the structures introduced above. The main idea is to consider a space for which the variables $x, y$ do not commute, and then to define (deformed) symmetry structures.

We choose a parameter $q \in k^\times$ and consider the two-sided ideal $I_q$ in the free associative algebra $k\{x,y\}$ generated by the element $yx - qxy$. In analogy to the affine plane we define the quantum plane with deformation parameter $q$ to be the quotient algebra

$$k[x,y]_q = k\{x,y\}/I_q \,.$$

It has a grading such that the generators $x, y$ are of degree 1. In the special case $q = 1$, the quantum plane is the classical affine plane.

Next we define a deformed version of the symmetry structures. We consider the two-sided ideal $J_q$ in the free associative algebra $k\{a,b,c,d\}$ generated by the relations

$$\begin{array}{lll} ba = qab & ca = qac & ad - da = (q^{-1} - q)bc \\ db = qbd & dc = qcd & cb = bc \,. \end{array}$$

We introduce the deformation of the algebra $M(2)$ as the algebra $M_q(2) := k\{a,b,c,d\}/J_q$ that is isomorphic to $M(2)$ for $q = 1$ and not commutative for $q \neq 1$. In addition, the algebra $M_q(2)$ can be endowed with the structure of a bialgebra; as a coalgebra it is isomorphic to $M(2)$.

We introduce the quantum determinant

$$\det_q = ad - q^{-1}bc = da - qbc$$

which is a central element in the algebra $M_q(2)$ and has coproduct and counit

$$\Delta(\det_q) = \det_q \otimes \det_q \qquad \epsilon(\det_q) = 1 \,.$$

We define the algebras $GL_q(2) = M_q(2)[t]/(t\det_q - 1)$ and $GL_q(2) = M_q(2)/(\det_q - 1)$ and endow them with the structure of a Hopf algebra by setting

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det_q^{-1}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

In analogy to the commutative case, the quantum plane $k_q[x,y]$ is an $M_q(2)$- and an $SL_q(2)$-comodule algebra via (1).

The subspace $k_q[x,y]_n$ of homogeneous polynomials in $k_q[x,y]$ with degree $n$ is a subcomodule of the quantum plane and $k_q[x,y]$ is isomorphic to the direct sum of the comodules $k_q[x,y]_n$.

Again, there is a duality between Hopf algebras for $SL(2)_q$ and the quantum enveloping algebra $U_q(\mathfrak{sl}(2))$ of the Lie algebra $\mathfrak{sl}(2)$. The generators $E, F, K, K^{-1}$ of the quantum universal enveloping algebra satisfy the relations

$$\begin{aligned} KK^{-1} = K^{-1}K = 1 \qquad & [E,F] = \frac{K - K^{-1}}{q - q^{-1}} \\ KEK^{-1} = q^2 E \qquad & KFK^{-1} = q^{-2}F\,. \end{aligned}$$

It is helpful to introduce formal partial derivatives for $q \neq 1$ by

$$\frac{\partial_q(x^m y^n)}{\partial x} := \frac{q^m - q^{-m}}{q - q^{-1}} x^{m-1} y^n \qquad \frac{\partial_q(x^m y^n)}{\partial y} := \frac{q^n - q^{-n}}{q - q^{-1}} x^m y^{n-1}$$

and algebra automorphisms $\sigma_x$ and $\sigma_y$ of the quantum plane $k_q[x,y]$

$$\sigma_x(x) := qx\,, \qquad \sigma_x(y) := y\,, \qquad \sigma_y(x) := x\,, \qquad \sigma_y(y) := qy\,.$$

Again, $SL(2)_q$-comodules correspond to $U_q(\mathfrak{sl}(2))$-modules, and $k_q[x,y]$ is a $U_q(\mathfrak{sl}(2))$-module by

$$\begin{aligned} U_q(\mathfrak{sl}(2)) \otimes k_q[x,y] &\longrightarrow k_q[x,y] \\ E \otimes P &\longmapsto x\frac{\partial_q P}{\partial y} \\ F \otimes P &\longmapsto \frac{\partial_q P}{\partial x} y \\ K \otimes P &\longmapsto (\sigma_x \sigma_y^{-1})(P) \\ K^{-1} \otimes P &\longmapsto (\sigma_y \sigma_x^{-1})(P)\,. \end{aligned}$$

The quantum plane is a $U_q(\mathfrak{sl}(2))$-module algebra.

The submodule $k_q[x,y]_n$ of homogeneous polynomials of degree $n$ is isomorphic to the simple $\mathfrak{sl}(2)$-module $V_{1,n}$ which is generated by a highest weight vector of weight $q^n$.

This thesis consists of three parts: in the first part the quantum symmetries of two given quantum spaces were determined, in the second part we computed dualities of Wess-Zumino-Witten theories and (super-)minimal models and in the third part we reconstructed a weak quantum group from the data of a conformal field theory. In the following we give a short introduction to these three subjects.

## Examples for quantum spaces

In this thesis, we will work with two examples of quantum spaces. By $k\{x^1, \dots x^n\}$ we denote the free associative algebra in $n$ variables. In the following, we choose $k = \mathbb{C}$. The models we consider are relatively simple deformations of the affine space:

### $\theta$-deformation

Let $\theta^{\mu\nu}$ be an antisymmetric matrix with entries in $\mathbb{R}$. The $\theta$-space [ChHo99, Scho99] is defined to be the universal enveloping algebra of the finite-dimensional complex Lie algebra with basis $\{\hat{x}^1, \dots, \hat{x}^n\}$ and Lie brackets

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu} \qquad \mu, \nu \in \{1, \dots n\}\,.$$

It plays an important role in string theory as it arises in the low-energy limit of the correlation functions of D-branes in an antisymmetrical constant $B$-field [SeWi99]. The $\theta$-space was also obtained by introducing the nonlocality by general relativistic arguments, where the involved constructions transform covariantly under Lorentz transformations by imposing additional quantum conditions on the deformation parameter $\theta^{\mu\nu}$ [DFR94].

Seiberg-Witten map [SeWi99] and deformation quantization [BFFLS78, Kon03] have enabled the formulation of gauge field theories [JSSW00, MSSW00], the standard model for elementary particles [CJSWW02] and gravity theory [ABDMSW03] on the $\theta$-space. The main drawback in this latest approach is the absence of quantum symmetries other than translation invariance.

### $\kappa$-deformation

Let $\mathbf{g}$ be a finite-dimensional complex Lie algebra with basis $\{x^1, \dots, x^n\}$ and Lie brackets $[x^\mu, x^\nu] = \sum_\lambda C^{\mu\nu}_\lambda x^\lambda$ with structure constants $C^{\mu\nu}_\lambda = \mathrm{i}a\delta^\mu_n\delta^\nu_\lambda - \mathrm{i}a\delta^\nu_n\delta^\mu_\lambda$ and $a \in \mathbb{R}$. In this case, there is a single variable $x^n$ which does not commute with any other variable. i.e.

$$[x^n, x^j] = \mathrm{i}ax^j \qquad [x^i, x^j] = 0 \qquad i, j = 1, 2, \dots, n-1\,.$$

The $\kappa$-space[1] is defined to be the universal enveloping algebra of $\mathfrak{g}$ [LNRT91].

[1] In the original definition of this noncommutative space, the deformation parameter is $\kappa := \frac{1}{a}$, therefore the name $\kappa$-space.

## Dualities

The notion of an algebra in a tensor category introduced above arises in the context of conformal field theories as well: full rational conformal field theories can be classified using a Frobenius algebra $A$ in a modular tensor category $\mathcal{C}$ [FRS02a]. Two-dimensional conformal field theories have been an essential tool when studying universal properties of critical phenomena. They capture surprisingly many aspects of statistical models with critical points, even away from criticality. A fascinating aspect of some of these models are Kramers-Wannier like dualities, relating for example the high-temperature and low-temperature regimes. A remarkable property of order-disorder dualities of Kramers-Wannier type is that they enable the discussion of a strong coupling regime (low temperature) by perturbation theory. The critical point of these statistical models is typically self-dual. While this has been known for more than sixty years [KrWa41], the obvious question whether such dualities can be deduced from properties at the critical point has only been addressed recently.

In [FFRS04] it has been shown that high/low temperature dualities can be read off from the universality class of a statistical model, that is its description by conformal field theory: order/disorder dualities are related to specific topological defects in the CFT. In the TFT approach to rational conformal field theory, types of topological defect lines correspond to isomorphism classes of $A$-bimodules [FFRS04, FFRS07]. Given two bimodules $B_1$ and $B_2$, their tensor product $B_1 \otimes_A B_2$ is again a bimodule; this tensor product encodes the fusion of topological defects. In the same way the modular tensor category $\mathcal{C}$ of chiral data gives rise to a fusion ring $K_0(\mathcal{C})$ of chiral data, the tensor category $\mathcal{C}_{AA}$ of $A$-bimodules gives rise to a fusion ring $K_0(\mathcal{C}_{AA})$ of topological defects. Both fusion rings are semisimple. The fusion ring of defects is, however, not necessarily commutative since the category of bimodules is typically not braided.

The essential insight of [FFRS04, FFRS07] is that the fusion ring $K_0(\mathcal{C}_{AA})$ of topological defects determines both symmetries and Kramers-Wannier dualities of the full conformal field theory described by the pair $(\mathcal{C}, A)$. The fact that Kramers-Wannier dualities relate bulk fields to disorder fields located at the end points of topological defect lines [KaCe71] might have suggested a relation between Kramers-Wannier dualities and topological defects. However, the topological defect lines relevant for the dualities are of a different type than the ones created by the dual disorder fields.

Duality bimodules [FFRS04] are defined to be simple $A$-bimodules $B$, such that all simple subobjects of the tensor product $B^\vee \otimes_A B$ are invertible bimodules. For any duality bimodule $B$ the isomorphism classes of simple bimodules $B_\lambda$, such that $\dim_{\mathbb{C}} \operatorname{Hom}(B_\lambda, B^\vee \otimes_A B) > 0$, form a subgroup $H$ of the fusion ring $K_0(\mathcal{C}_{AA})$. In this case $\dim_{\mathbb{C}} \operatorname{Hom}(B_\lambda, B^\vee \otimes_A B) = 1$.

In the following, we restrict ourselves to the case where the simple symmetric special Frobenius algebra in $\mathcal{C}$ is the tensor unit $\mathbb{1}$; this situation is usually referred to as the Cardy case. A complete set of correlation functions for the full conformal field theory in the Cardy case has been constructed in [FFFS00]. In the Cardy case, the bulk partition function is given by charge conjugation. Many more simplifications occur: the category of $A$-bimodules is equivalent to the original category, $\mathcal{C}_{AA} \simeq \mathcal{C}$;

as a consequence, isomorphism classes of invertible bimodules are simple currents [ScYa90b].

As a special case of the results of [FFRS04, FFRS07], we see that the full conformal field theory in the Cardy case has Kramers-Wannier dualities if and only if the underlying modular tensor category has duality classes.

## Reconstruction of a weak Hopf algebra

The Tannaka-Krein reconstruction theorem [DeMi82, Del90] states that a quantum group can be reconstructed from a $k$-linear abelian rigid tensor category which is essentially small and admits a $k$-linear exact faithful tensor functor into the category of finite-dimensional $k$-vector spaces. But the notion of a quantum group is too restrictive to be applied in many interesting models of string theory and statistical mechanics. Weak quantum groups (or weak Hopf algebras) arise as a generalization of the concept of quantum groups [BoSz96]. In particular, they play an important role in the description of symmetries in conformal field theories [PeZu01, CoSc01]. Usually, the chiral data of rational conformal two-dimensional quantum field theories [FRS02b, FRS04a, FRS04b, FRS05, FRS06] are encoded in modular tensor categories [Tur]. Thus, a generalization of the Tannaka-Krein reconstruction theorem for weak Hopf algebras relates the characterization of a CFT by a modular tensor category to the characterization by weak Hopf algebras.

Apart from describing a chiral symmetry, a weak quantum group has an additional property, it associates a distinguished space to this symmetry: The representation category $H$-mod of a weak Hopf algebra $H$ is a tensor category. The finite-dimensional counital subalgebra $H_t$ of a weak Hopf algebra is an $H$-module and in particular, an $H$-module algebra. In the special case of a Hopf algebra, the counital subalgebra consists of multiples of the unit element and its spectrum is the one-point space. In the general case of a weak Hopf algebra, we assume that the counital subalgebra $H_t$ is commutative of dimension $n$. Then it is isomorphic to the direct sum of $n$ copies of the ground field. Thus, the spectrum of $H_t$ is the $n$-point space. The fiber over every point is the ground field. In physics, this is exactly the space of $D$-branes for the rational conformal field theory.

# Outline

In Chapter 1 we present a diagrammatical representation for algebras and coalgebras that simplifies many subsequent proofs. We introduce the notion of a separable algebra which plays an important role in the Generalized Reconstruction Theorem that we will prove in Chapter 5. We define quantum groups and weak quantum groups; we list some fundamental properties and present the important proofs. In particular, we prove that the counital subalgebras of a weak quantum group are separable.

In Chapter 2 we review the fundamental categorical notions that arise when considering the category of modules over a (weak) quantum group: we present the definitions of tensor categories and modular tensor categories. We show that one must consider a truncated tensor product in order to endow the category of modules over a weak quantum group with the structure of a tensor category. We present the definition of a module category and show that the category of modules over a counital subalgebra of a weak quantum group is a module category over the category of modules over the weak quantum group. The concept of a module category is a basic ingredient for our proof of the generalized Reconstruction Theorem. Furthermore, we introduce the notion of a duality class in order to study Kramers-Wannier dualities in Chapter 4.

In Chapter 3 we construct the quantum symmetries of two given quantum spaces, namely the $\theta$- and the $\kappa$-space. For this we introduce the notion of deformed partial derivatives and deformed rotations as endomorphisms of the quantum space that form a Hopf algebra and obey some additional consistency conditions. We derive the deformed Poincaré algebras by demanding that the noncommutative spacetime algebra is a module over it and that the deformations are again Hopf algebras. For the $\theta$-space we find a unique two-parameter family of solutions, for the $\kappa$-space we restrict ourselves to the simplest deformation. In both cases the action on the quantum space remains undeformed. Furthermore, we compute invariants of the deformed Poincaré algebra such as a space invariant, the Laplace operator and the Pauli-Lubanski vector and introduce a set of so-called Dirac derivatives that transform vector-like. Next we present the $\star$-product formalism and describe the symmetry generators of the $\kappa$-space in terms of $\star$-derivatives for three different $\star$-products, namely for the symmetric $\star$-product and for the normal ordered $\star$-products. Finally we define vector fields for the $\kappa$-space by generalizing the transformation behaviour of derivatives under the $\kappa$-deformed rotations.

In Chapter 4 we determine the Kramers-Wannier dualities for an explicit class of modular tensor categories, namely for Wess-Zumino-Witten models and (super-) minimal models. First we present model independent results: we analyze properties of duality classes; in Theorem 4.1.1 we give a necessary condition for the existence of duality classes that is used in the analysis of the $A$-series of Wess-Zumino-Witten models and the minimal models. For the $B$-, $C$-, $D$-series and the exceptional algebras $E_6$ and $E_7$, there are no simple objects meeting the conditions of Theorem 4.1.1. A different strategy involving lower bounds of quantum dimensions is developed; it is based on the results of [GRW96] on second-lowest quantum dimensions in Wess-Zumino-Witten fusion rules. For WZW-models we find the following result:

duality classes appear only for the algebras $E_7$ and $A_1$ and the $B$- and $D$-series at low level. Unitary Virasoro minimal models and superconformal minimal models admit duality classes only at low level, while duality classes for a super-Virasoro unitary minimal model only exist for odd levels.

The generalized Tannaka-Krein Reconstruction Theorem for weak quantum groups is merely sketched in the existing literature [Szl01, Ostr03]. In Chapter 5 we prove the Tannaka-Krein Reconstruction Theorem for weak quantum groups. We start by generalizing a theorem of Ostrik [Ostr03] stating that for a tensor category $\mathcal{C}$ and a separable algebra $R$ the set of $R$-fiber functors $\mathcal{C} \to R$-bimod is in bijection to the structures of a module category on $R$-mod over $\mathcal{C}$. After choosing a generator of the module category we derive an $R$-fiber functor encoding the data of the module category. We endow the natural endotransformations of this functor with the structure of a weak Hopf algebra. As an example we compute the weak Hopf algebra associated with the theory of a compactified, massless, free boson with boundary conditions.

## Acknowledgements

I am grateful to Christoph Schweigert for drawing my interest to weak quantum groups and conformal field theories and for his wise support during the course of this work. In particular, I thank him for his commitment to help me find financial support. Parts of the work presented here were developed with Marija Dimitrijevic, Florian Koch and Lutz Möller. I wish to thank them for the fruitful collaboration. Furthermore, I am indebted to Stephan Mohrdieck, Jürgen Fuchs, Jens Fjelstad and Terry Gannon for inspiring discussions.
This thesis was financially supported by a scholarship of the University of Hamburg (HmbNFG), the German-Israeli Foundation (GIF) and the Collaborative Research Centre 676 "Particles, Strings and the Early Universe - the Structure of Matter and Space-Time".

This thesis is based on the following publications:

- M. Dimitrijevic, L. Möller and E. Tsouchnika, *Derivatives, forms and vector fields on the kappa-deformed Euclidean space*, J.Phys. A 37 (2004) 9749
- F. Koch and E. Tsouchnika, *Construction of $\theta$-Poincaré Algebras and their Invariants on $\mathcal{M}_\theta$*, Nucl.Phys. B 717 (2005) 387
- C. Schweigert and E. Tsouchnika, *Kramers-Wannier dualities for WZW theories and minimal models*, arXiv:0710.0783v1 [hep-th]

# Chapter 1

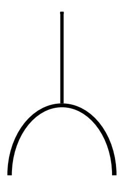
# Algebraic preliminaries

## 1.1 Diagrammatical representation of algebras and coalgebras

In the following, we will present basic properties of algebras and coalgebras by means of diagrams. The pictures are read from bottom to top, morphisms are depicted by means of boxes, composition of morphisms amounts to concatenation of lines and the tensor product to juxtaposition.

**Definition 1.1.1.**
*A unital $\mathbb{K}$-algebra $(A, \mu, \eta)$ is a vector space $A$ over $\mathbb{K}$ with two $\mathbb{K}$-linear maps*

$$\mu : A \otimes A \longrightarrow A \qquad \textit{(multiplication)}$$

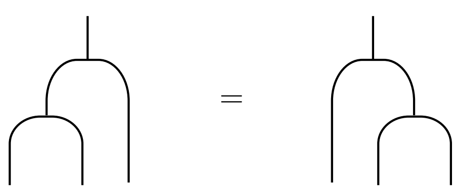
$\mu =$ 

$$\eta : \mathbb{K} \longrightarrow A \qquad \textit{(unit)}$$

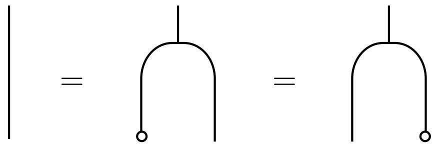
$\eta =$ 

*with the properties*

*(a)* $\mu \circ (\mu \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \mu)$ *(associativity)*



*(b)* $\mu \circ (\eta \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \eta) = \mathrm{id}_A\,.$ *(unitality)*



*Hereby we have identified*

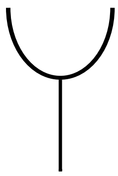
$$(A \otimes_{\mathbb{K}} A) \otimes_{\mathbb{K}} A \cong A \otimes_{\mathbb{K}} (A \otimes_{\mathbb{K}} A) \qquad \mathbb{K} \otimes_{\mathbb{K}} A \cong A \cong A \otimes_{\mathbb{K}} \mathbb{K}\,.$$

*Notice that the element* $1 := \eta(1) \in A$ *is a right and left unit.*

In order to define the notion of a coalgebra, all directions of maps are exchanged:

**Definition 1.1.2.**
*A (counital)* <u>*coalgebra*</u> $(C, \Delta, \epsilon)$ *consists of a vector space* $C$ *over* $\mathbb{K}$ *and two* $\mathbb{K}$*-linear maps*

$$\Delta : C \longrightarrow C \otimes C \qquad \textit{(coproduct)}$$

$\Delta =$ 

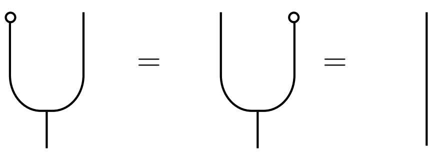
$$\epsilon : C \longrightarrow \mathbb{K} \qquad \textit{(counit)}$$

$\epsilon =$ 

*with the properties*

*(a)* $(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta$ *(coassociativity)*

*(b)* $(\epsilon \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \epsilon) \circ \Delta = \mathrm{id}_C\,.$ *(counitality)*



**Definition 1.1.3.**
*An element $x$ in a coalgebra $(C, \Delta, \epsilon)$ is called* <u>*primitive*</u> *if*

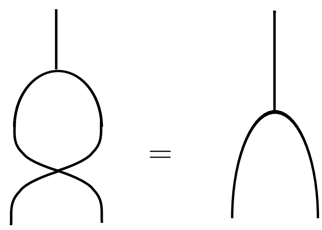
$$\Delta(x) = 1 \otimes x + x \otimes 1 \,.$$

*We denote the subspace of all primitive elements by* $\mathrm{Prim}(C)$.

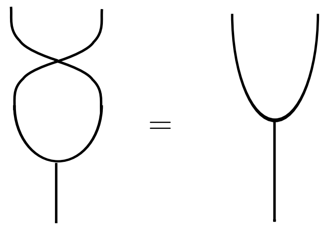
**Definition 1.1.4.**
*Let*

$$\begin{aligned} \tau_{A,A} : \quad & A \otimes A \to A \otimes A \\ & u \otimes v \mapsto v \otimes u \end{aligned}$$

*be the flip exchanging two components in the tensor product. The* <u>*opposite algebra*</u> *is the triple $(A, \mu^{\mathrm{opp}} := \mu \circ \tau_{A,A}, \eta)$, the* <u>*opposite coalgebra*</u> *is $(C, \Delta^{\mathrm{opp}} := \tau_{C,C} \circ \Delta, \epsilon)$. For a* <u>*commutative*</u> *algebra one has $\mu^{\mathrm{opp}} = \mu$, graphically*



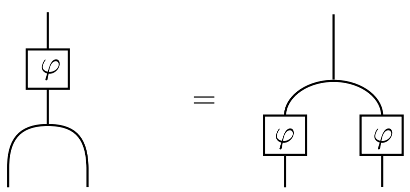
*and for a* <u>*cocommutative*</u> *coalgebra $\Delta^{\mathrm{opp}} = \Delta$ must be satisfied.*

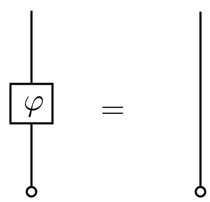


**Definition 1.1.5.**
*A* <u>*morphism of (unital) algebras*</u> *is a linear map*

$$\varphi : A \to A' \,,$$

*such that $\varphi \circ m = m' \circ (\varphi \otimes \varphi)$* *and $\varphi \circ \eta = \eta'$.*

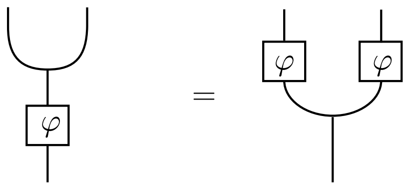


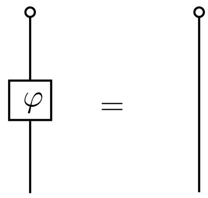


*A* <u>*morphism of coalgebras*</u> *is a linear map*

$$\varphi : C \to C' \,,$$

*such that* $\Delta' \circ \phi = (\varphi \otimes \varphi) \circ \Delta$ *and* $\epsilon' \circ \varphi = \epsilon$.




We will use the notation for the coproduct introduced by Sweedler

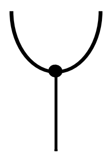
$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \,.$$

## 1.2 Separable algebras

Separable algebras are a basic ingredient for the generalized reconstruction theorem in Chapter 5. We give the definition and list some important properties of bimodules over a separable algebra.

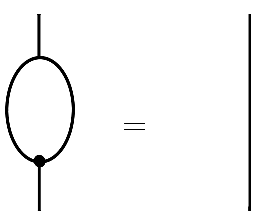
**Definition 1.2.1.**
*A $\mathbb{K}$-algebra $R$ is called* <u>*separable*</u> *if there is a right inverse for the multiplication as a morphism of bimodules, i.e. there exists a morphism of $R$-bimodules*

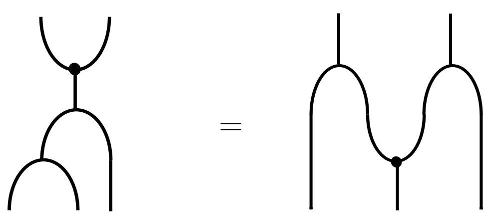
$$\varphi : R \to R \otimes_{\mathbb{K}} R$$

$\varphi =$ 

*obeying for any $a, b, c \in R$*

$$\mu \circ \varphi = \mathrm{id}_R \qquad \textit{and} \qquad \varphi(abc) = a\,\varphi(b)c\,.$$




**Remark 1.2.2.**
*A separability structure is equivalent to the existence of an idempotent $e$ in the algebra $R^{\mathrm{opp}} \otimes R$, the so-called* <u>*separability idempotent*</u>*, with the properties $\mu(e) = 1$ and $(r \otimes 1)e = e(1 \otimes r)$ for any $r \in R$: set $e := \varphi(1)$ or $\varphi(r) := (r \otimes 1)e$. It is easy to see that if the algebra $R$ is abelian the separability idempotent is unique.*

**Proposition 1.2.3.**
*A $\mathbb{K}$-algebra is separable if and only if*

$$R \cong R_1 \oplus \cdots \oplus R_r \,,$$

*where $R_i$ are finite-dimensional simple $\mathbb{K}$-algebras with $Z(R_i)/\mathbb{K}$ a separable field extension.*

For a proof see for example [Pier].

**Example 1.2.4.**
*Consider the full matrix algebra $Mat_n(S)$, where $S$ is a commutative ring and $\epsilon_{ij}$ the matrix units. An arbitrary element in $Mat_n(S)^{op} \otimes Mat_n(S)$ is of the form*

$$\sum_{i,j,k,l=1}^{n} \lambda_{ijkl} \epsilon_{ij} \otimes \epsilon_{kl}$$

*with coefficients $\lambda_{ijkl} \in \mathbb{K}$ for $1 \leq i, j, k, l \leq n$ and for it to be a separability idempotent the following conditions must be satisfied for any element $a := \sum_{p,q=1}^{n} a_{pq} e_{pq}$*

$$\begin{aligned} \sum_{i,j,k,l=1}^{n} \lambda_{ijkl} \epsilon_{ij} \epsilon_{kl} &= 1 \\ \sum_{i,j,k,l,p,q=1}^{n} \lambda_{ijkl} a_{pq} e_{pq} \epsilon_{ij} \otimes \epsilon_{kl} &= \sum_{i,j,k,l,p,q=1}^{n} \lambda_{ijkl} a_{pq} \epsilon_{ij} \otimes \epsilon_{kl} e_{pq} \,. \end{aligned}$$

*The second condition implies that $\lambda_{ijkl}$ depends only on two indices $\lambda_{ijkl} = \lambda_{jk}$ and the first condition gives $\sum_{i=1}^{n} \lambda_{ii} = 1$. Hence, the separability idempotents of the full matrix algebra $Mat_n(S)$ are of the form*

$$E = \sum_{i,j,k=1}^{n} \lambda_{jk} \epsilon_{ij} \otimes \epsilon_{ki} \in Mat_n(S)^{\mathrm{opp}} \otimes Mat_n(S)$$

*with $\sum_i \lambda_{ii} = 1$.*

## 1.3 Weak Hopf algebras

In this section we will introduce the notion of a weak bialgebra that was defined originally in [BoSz96]. We list some important properties following [BoNS99, Niks02, Scha02].

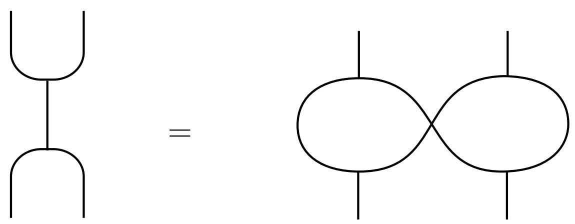
**Definition 1.3.1.**
*We endow $A \otimes A$ with the structure of a unital coassociative algebra by means of the multiplication*

$$\mu(a_1 \otimes a_2 \otimes b_1 \otimes b_2) := a_1 b_1 \otimes a_2 b_2\,.$$

*A weak bialgebra is a quintupel $(A, m, \eta, \Delta, \epsilon)$, where $(A, \mu, \eta)$ is a unital associative algebra and $(A, \Delta, \epsilon)$ is a counital coassociative coalgebra, such that*

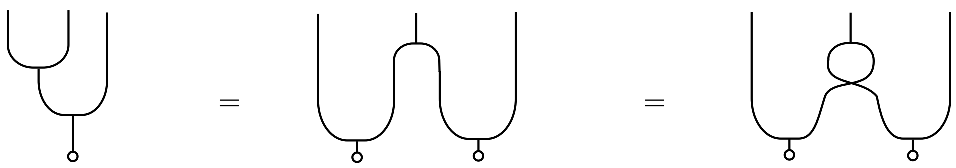
*(a) $\Delta$ is a (not necessarily unital) morphism of algebras* [1]

$$\Delta(hg) = \Delta(h)\Delta(g)\,, \tag{1.1}$$



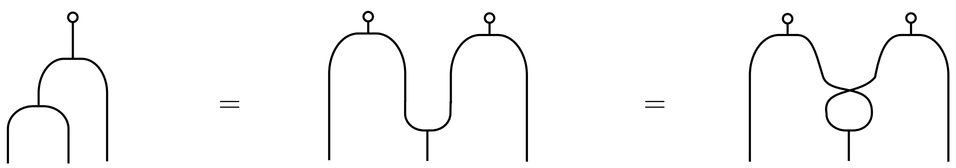
*(b) the unit satisfies the relation*

$$(\Delta \otimes \mathrm{id}) \circ \Delta(1) \quad = \quad (\Delta(1) \otimes 1) \cdot (1 \otimes \Delta(1)) = (1 \otimes \Delta(1)) \cdot (\Delta(1) \otimes 1), \tag{1.2}$$



*(c) the counit satisfies the relation*

$$\epsilon(fgh) = \epsilon(fg_{(1)})\epsilon(g_{(2)}h) = \epsilon(fg_{(2)})\epsilon(g_{(1)}h) \quad \textit{for any } f, g, h \in A\,. \tag{1.3}$$



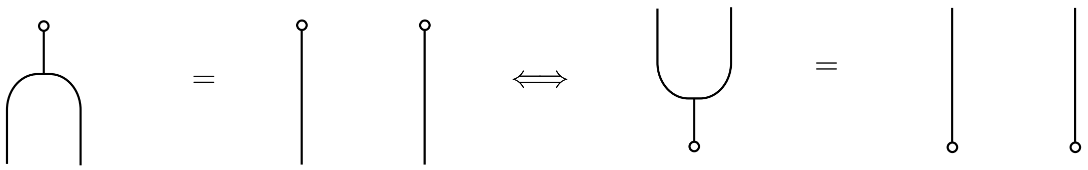
[1] This is equivalent to $\mu$ being a (not necessarily unital) morphism of coalgebras.

**Definition 1.3.2.**
*A weak bialgebra $(A, \mu, \Delta, \epsilon, \eta)$ is called* bialgebra *if the counit is a morphism of (unital) algebras. Then, identities (1.2) and (1.3) reduce to*

$$\Delta(1) = 1 \otimes 1 \quad \textit{and} \quad \epsilon(fg) = \epsilon(f)\epsilon(g)\,.$$

*In a bialgebra without zero divisor (1.3) implies $\epsilon(1) = 1$.*

**Proposition 1.3.3.**
*The counit of a weak bialgebra is a morphism of algebras if and only if the coproduct is unital:*



*This is exactly the case of a bialgebra. In a weak bialgebra one has*

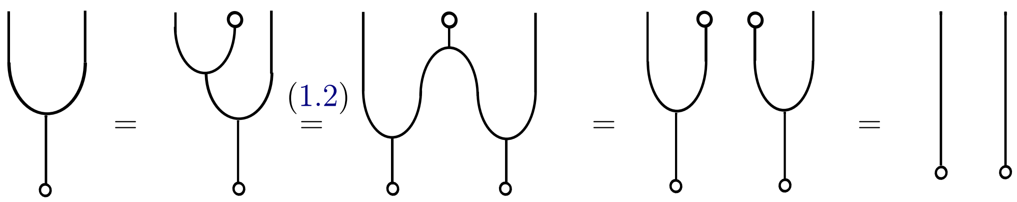
$$\Delta(1)\Delta(a) = \Delta(a) = \Delta(a)\Delta(1) \tag{1.4}$$

*and by choosing a=1*

$$\Delta(1) = \Delta(1)\Delta(1), \tag{1.5}$$

*such that $\Delta(1)$ is an idempotent in the algebra $A \otimes A$.*

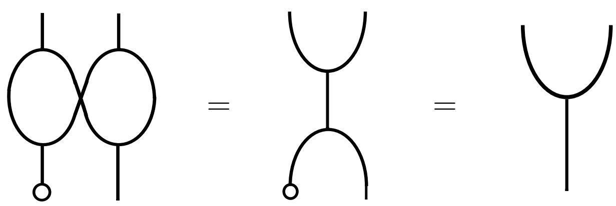
**Proof.** Assume that $\epsilon$ is a morphism of algebras and use unitality to obtain:




Hence, the coproduct $\Delta$ is unital.

The statement that the unit is a morphism of algebras can be obtained by reflection of the above diagram using (1.3) and counitality.

We prove the statement for weak bialgebras diagrammatically by means of (1.1):



$\square$

From any (weak) bialgebra $(A, \mu, \eta, \Delta, \epsilon)$ one can construct three more (weak) bialgebras:

$$\begin{aligned} A^{\rm opp} &= (A, \mu^{\rm opp}, \eta, \Delta, \epsilon) \qquad A^{\rm opp,copp} = (A, \mu^{\rm opp}, \eta, \Delta^{\rm copp}, \epsilon) \\ A^{\rm copp} &= (A, \mu, \eta, \Delta^{\rm opp}, \epsilon) \end{aligned}$$

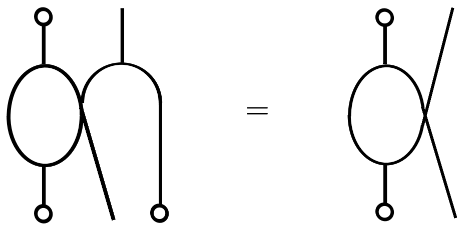
**Definition 1.3.4.**
*Grouplike elements of a weak bialgebra satisfy*

$$\Delta(x) = (x \otimes x)\Delta(1) \quad \textit{and} \quad \epsilon(x) = 1$$

*and form a monoid.*

**Definition 1.3.5.**
*The endomorphism* $\epsilon_t(a) := (\epsilon \otimes \mathrm{id})(\Delta(1)(a \otimes 1)) = \epsilon(\underline{1}_{(1)}a)\underline{1}_{(2)}$

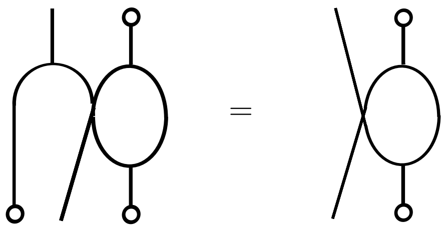
$\epsilon_t$ =



*of a weak bialgebra A is called target counital map, the endomorphism*

$$\epsilon_s(a) := (\mathrm{id} \otimes \epsilon)((1 \otimes a)\Delta(1)) = \underline{1}_{(1)}\epsilon(a\underline{1}_{(2)})$$

$\epsilon_t$ =

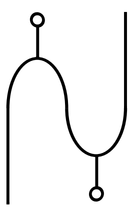


*source counital map. For a bialgebra one has* $\epsilon_t = \epsilon_s = \eta \circ \epsilon$.

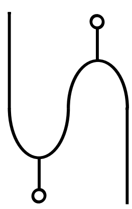
We introduce the target and source counital maps of the opposite bialgebra $A^{opp}$

$$\epsilon_t{}'(a) := (\mathrm{id} \otimes \epsilon)(\Delta(1)(1 \otimes a)) \qquad \text{and} \qquad \epsilon_s{}'(a) := (\mathrm{id} \otimes \epsilon)(\Delta(1)(1 \otimes a)).$$

$\epsilon_t{}' =$



$\epsilon_s{}' =$



**Theorem 1.3.6.**
*Denote by $A_t = \epsilon_t(A)$ and $A_s = \epsilon_s(A)$ the images of the counital maps.*

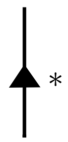
*(i) The subalgebra $A_t$ is generated by the right tensorands of $\Delta(1)$, the subalgebra $A_s$ by the left tensorands of $\Delta(1)$*

$$\begin{array}{lclclcl} A_t &=& \{a \in A \mid \Delta(a) &=& \underline{1}_{(1)}a \otimes \underline{1}_{(2)}\} &=& \{(\phi \otimes \mathrm{id})\Delta(1) \mid \phi \in A^*\} \\ A_s &=& \{a \in A \mid \Delta(a) &=& \underline{1}_{(1)} \otimes a\underline{1}_{(2)}\} &=& \{(\mathrm{id} \otimes \phi)\Delta(1) \mid \phi \in A^*\}\,. \end{array}$$

*(ii) $A_s$ and $A_t$ are left- and respectively right coideals of $A$ and unital subalgebras of $A$. They commute as subalgebras of $A$.*

For a proof of the Theorem see section 2.2 in [BoNS99].

We call $A_s$ and $A_t$ source and target counital subalgebras. In the following, we denote



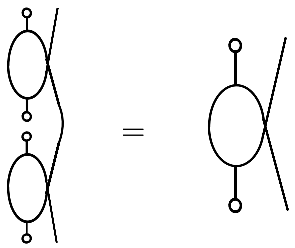
for elements in $A_*$ with $* \in \{s, t\}$.

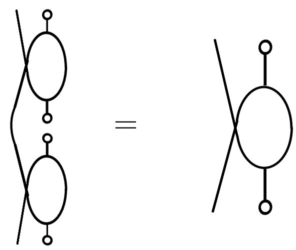
The counital maps have many interesting properties that we list in the following:

**Lemma 1.3.7.**

*(i) The counital maps are idempotent*

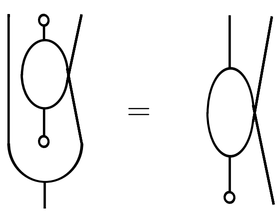
$$\epsilon_t \circ \epsilon_t = \epsilon_t \qquad\qquad \epsilon_s \circ \epsilon_s = \epsilon_s\,. \tag{1.6}$$

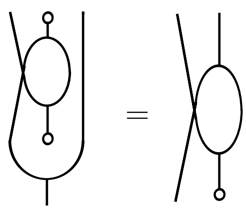




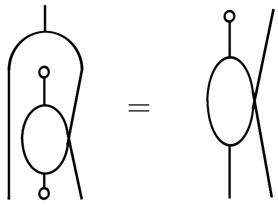
*(ii) For all $a, b \in A$ one has*

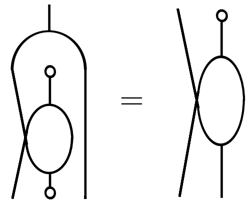
$$(\mathrm{id} \otimes \epsilon_t)\Delta(h) = \underline{1}_{(1)}h \otimes \underline{1}_{(2)} \qquad (\epsilon_s \otimes \mathrm{id})\Delta(h) = \underline{1}_{(1)} \otimes h\underline{1}_{(2)} \tag{1.7}$$



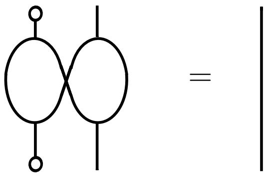


*and* $\quad a\epsilon_t(b) = \epsilon(a_{(1)}b)a_{(2)} \qquad\qquad \epsilon_s(a)b = a_{(1)}\epsilon(ba_{(2)})$.



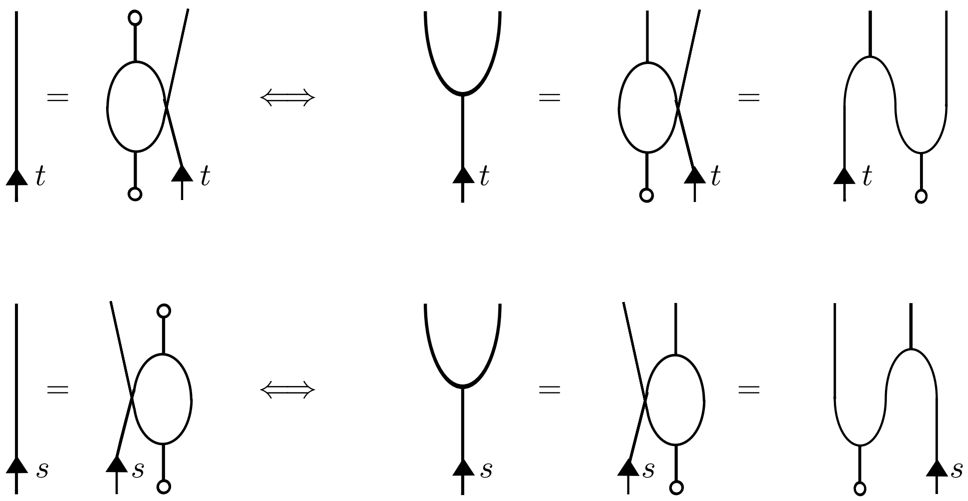


*(iii)* $\mu \circ (\epsilon_t \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$

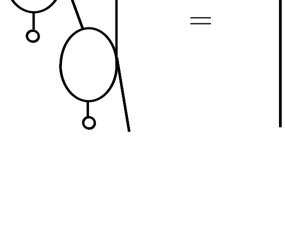


*(iv) For all $a \in A$ one has*

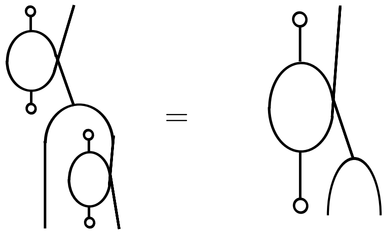
$$a = \epsilon_t(a) \iff \Delta(a) = \underline{1}_{(1)}a \otimes \underline{1}_{(2)} = a\underline{1}_{(1)} \otimes \underline{1}_{(2)} \tag{1.8}$$
$$a = \epsilon_s(a) \iff \Delta(a) = \underline{1}_{(1)} \otimes a\underline{1}_{(2)} = \underline{1}_{(1)} \otimes \underline{1}_{(2)}a\,. \tag{1.9}$$



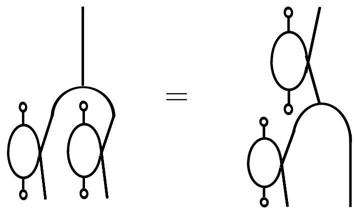

*(v)* $\mu \circ (\epsilon_t \otimes \mathrm{id}) \circ (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta(1) \otimes \mathrm{id}) = \mathrm{id}$



*(vi)* $\epsilon_t \circ \mu \circ (\mathrm{id} \otimes \epsilon_t) = \epsilon_t \circ \mu$ (1.10)



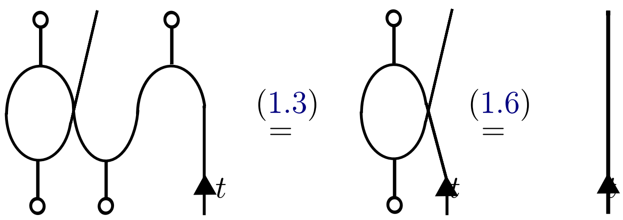
*(vii)* $\mu \circ (\epsilon_t \otimes \epsilon_t) = \epsilon_t \circ \mu \circ (\epsilon_t \otimes \mathrm{id})$ (1.11)



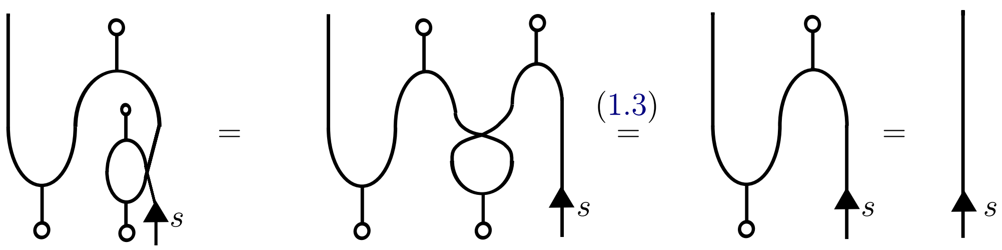
For a proof of (i), (ii), (vi), (vii) see section 2 in [BoNS99] and for (iii), (iv), (v) in [Scha02].

**Theorem 1.3.8.**
*The counital maps are antiisomorphisms of algebras* $\epsilon_t : A_s \xrightarrow{\sim} A_t$ *and* $\epsilon_s : A_t \xrightarrow{\sim} A_s$ *with the inverses* $\epsilon_t^{-1} = \epsilon_s{}'$ *and* $\epsilon_s^{-1} = \epsilon_t{}'$.

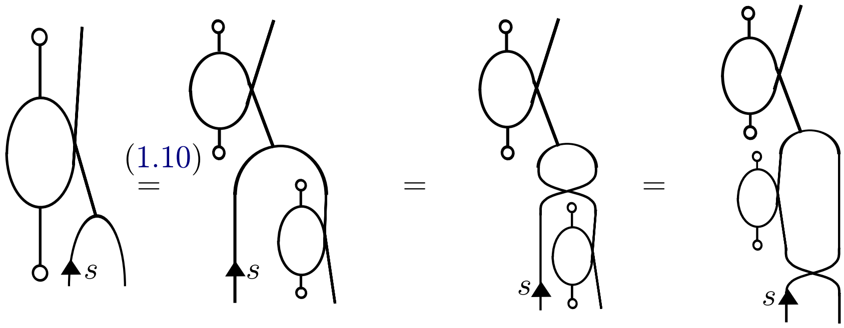
**Proof.** We consider $\epsilon_t \circ \epsilon_s'$ and $\epsilon_s' \circ \epsilon_t$ in order to prove that $\epsilon_t$ is a bijection:

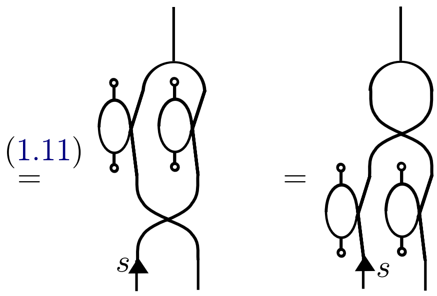



Hence, $\epsilon_t \circ \epsilon_s' = \mathrm{id}_{A_t}$ and by




$\epsilon_s' \circ \epsilon_t = \mathrm{id}_{A_s}$ holds. The last equality in the diagram results from the idempotency of $\epsilon_s'$. Next we show that $\epsilon_t$ is an antiisomorphism of algebras




The second equality results from Theorem 1.3.6 (ii), since an element in $A_s$ is multiplied by one in $A_t$.

The statement for $\epsilon_s$ follows similarly. □

In the proof of the Theorem we have shown the more general relation

$$\epsilon_t(a_s b) = \epsilon_t(b)\epsilon_t(a_s) \tag{1.12}$$

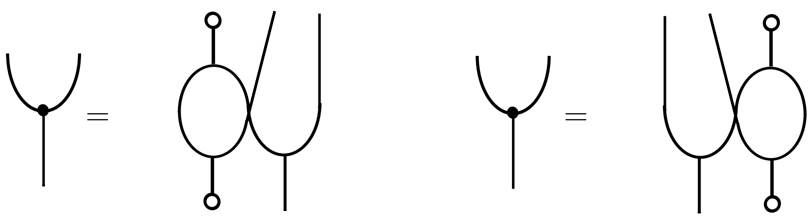
for any $a_s \in A_s$ and any $b \in A$. Clearly, the source counital map $\epsilon_s$ obeys a similar relation

$$\epsilon_s(b a_t) = \epsilon_s(a_t)\epsilon_s(b)$$

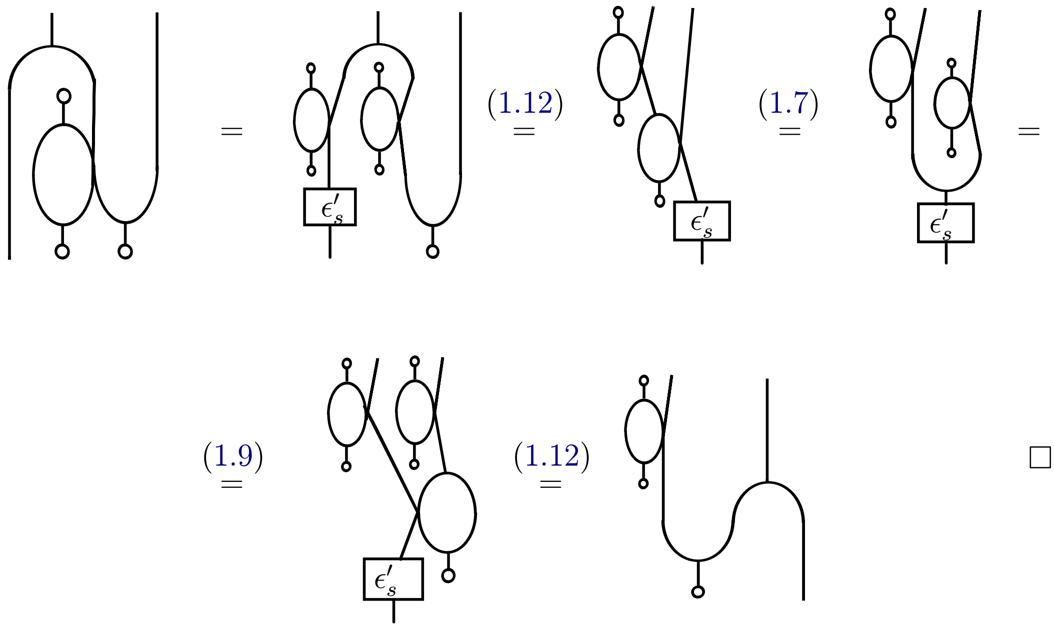
for any $a_t \in A_t$ and any $b \in A$.

**Theorem 1.3.9.**
*The counital algebras $A_t$ and $A_s$ are separable. One choice for the separability idempotents is*

$$e_t = (\epsilon_t \otimes \mathrm{id})\Delta(1) \quad \textit{and} \quad e_s = (\mathrm{id}\otimes\epsilon_s)\Delta(1)\,.$$



**Proof.** Relation (1.1) implies $\mu(e_t) = 1$. We will prove $(a \otimes 1)e_t = e_t(1 \otimes a)$ for any $a \in A_t$ diagrammatically. We denote $\epsilon_t^{-1} = \epsilon_s'$ as a morphism in a box.




□

**Definition 1.3.10.**
*Let $(H, \mu, \eta, \Delta, \epsilon)$ be a weak bialgebra over a field $\mathbb{K}$. We consider the $\mathbb{K}$-vector space $End(H)$ of linear maps in $H$ and define the <u>convolution</u> as the product of endomorphisms $f, g \in End(H)$*

$$f * g : H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{\mu} H\,.$$

Obviously this endomorphism is $k$-bilinear. The unit for the convolution is given by

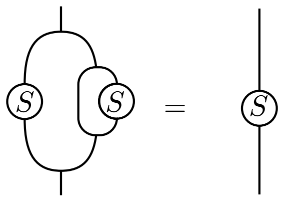
$$H \xrightarrow{\epsilon} k \xrightarrow{\eta} H\,.$$

The Sweedler notation for the convolution is $(f * g)(x) = \sum_{(x)} f(x_{(1)})g(x_{(2)})\,.$

**Definition 1.3.11.**
*A <u>weak quantum group</u> or <u>weak Hopf algebra</u> is a weak bialgebra $H$ with a map $S : H \to H$ called antipode obeying*

$$id * S = \epsilon_t \qquad\qquad S * id = \epsilon_s$$

*and* $\qquad S * id * S = S\,.$



**Theorem 1.3.12.**
*If a weak bialgebra admits an antipode, then the antipode is unique and bijective. It is an algebra antihomomorphism and a coalgebra antihomomorphism.*

For a proof of the Theorem see section 2.3 in [BoNS99].

**Theorem 1.3.13.**
*One has $S \circ \epsilon_s = \epsilon_t \circ S$ and $\epsilon_s \circ S = S \circ \epsilon_t$. If $\dim H_t < \infty$ and $\dim H_s < \infty$ the restriction of $S$ to $H_s$, respectively to $H_t$, is an algebra antiisomorphism from $H_s$ to $H_t$, respectively from $H_t$ to $H_s$.*

**Proof.** Similar to the Proof of Theorem 1.3.8.

**Lemma 1.3.14.**
*If $(H, \mu, \eta, \Delta, \epsilon, S)$ is a weak Hopf algebra, then the dual vector space $H^*$ can also be endowed with the structure of a weak Hopf algebra via*

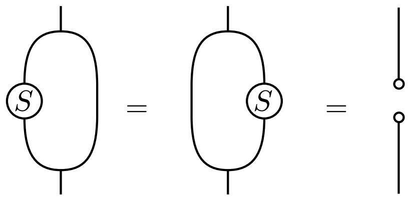
$$\begin{aligned}
\Delta^* &: (H \otimes H)^* \cong H^* \otimes H^* \longrightarrow H^* \\
\epsilon^* &: k \longrightarrow H^* \\
\mu^* &: H^* \longrightarrow (H \otimes H)^* = H^* \otimes H^* \\
\eta^* &: H^* \to k \\
S^* &: H^* \to H^*\,.
\end{aligned}$$

**Proof.** The axioms for a weak Hopf algebra can be checked straightforwardly.

## 1.4 Hopf algebras

**Definition 1.4.1.**
*A quantum group or Hopf algebra is a bialgebra with antipode $S : H \to H$. The antipode satisfies*

$$\mu \circ (\mathrm{id} \otimes S) \circ \Delta = \mu \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon \tag{1.13}$$



*i.e. $S$ is a left and right inverse for the identity $\mathrm{id}_H$.*

*A morphism of Hopf algebras is a morphism of bialgebras that commutes with the antipode.*

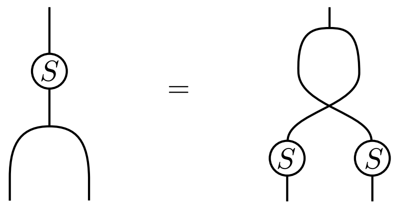
If the antipode exists, it is unique

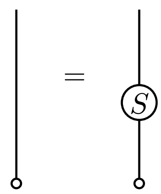
$$S \;=\; S * (\eta\epsilon) = S * (\mathrm{id}_H * S') = (S * \mathrm{id}_H) * S' = \eta\epsilon * S' = S' \;.$$

**Theorem 1.4.2.**
*Let $(H, \mu, \eta, \Delta, \epsilon)$ be a Hopf algebra.*

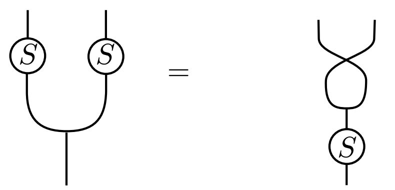
*(i) The antipode $S$ is a morphism of bialgebras from $H$ to $H^{\mathrm{opp,copp}}$, that is*

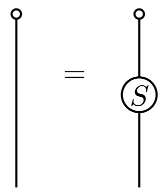
$$S(xy) = S(y)S(x) \qquad\qquad S(1) = 1$$





$$(S \otimes S) \circ \Delta = \Delta^{\mathrm{opp}} \circ S \qquad\qquad \epsilon \circ S = \epsilon\,.$$





*(ii) The following statements are equivalent:*

*(a)* $S^2 = \mathrm{id}_H$

*(b)* $\sum_x S(x_{(2)})x_{(1)} = \epsilon(x)1$ *for all* $x \in H$

*(c)* $\sum_x x_{(2)}S(x_{(1)}) = \epsilon(x)1$ *for all* $x \in H$

*(iii)* $H$ *being commutative or cocommutative implies* $S^2 = \mathrm{id}_H$.

For a proof see for example section III.3 in [Kas].

**Corollary 1.4.3.**
*In a Hopf algebra every grouplike element* $x \in H$ *has* $S(x)$ *as an inverse.*

# Chapter 2

# Categorical preliminaries

In this chapter we give the basic definitions needed for the representation theory of quantum groups. For more details compare with the reviews [BaKi, CaEt04].

In this thesis we consider only small categories. Let $\mathbb{K}$ be a field.

## 2.1 Categories and functors

**Definition 2.1.1.**
*A category $\mathcal{C}$ is $\mathbb{K}$-linear or enriched over $\mathrm{Vect}_{\mathbb{K}}$ if all sets of morphisms $Mor_{\mathcal{C}}(U,V)$ are $\mathbb{K}$-vector spaces and the composition of morphisms is $\mathbb{K}$-bilinear. We denote the morphism spaces by $\mathrm{Hom}_{\mathcal{C}}(U,V)$.*

**Definition 2.1.2.**
*A $\mathbb{K}$-linear category is called additive if there exist finite direct sums in $\mathcal{C}$ and there exists a zero object $0 \in Ob(\mathcal{C})$, such that $\mathrm{Hom}_{\mathcal{C}}(X,0) = 0 = \mathrm{Hom}_{\mathcal{C}}(0,X)$ for all objects $X \in Ob(\mathcal{C})$.*

**Definition 2.1.3.**
*Consider an additive category $\mathcal{C}$.*

*(i) The morphism $f \in \mathrm{Hom}_{\mathcal{C}}(U,V)$ is a monomorphism (epimorphism) if for any $g \in \mathrm{Hom}_{\mathcal{C}}(W,U)$ $fg = 0$ implies $g = 0$ (for $g \in \mathrm{Hom}_{\mathcal{C}}(V,W)$ $gf = 0$ implies $g = 0$).*

*(ii) The kernel of a morphism $f : U \to V$ is an object $ker(f)$ and a morphism $\kappa : ker(f) \to U$, such that $f\kappa = 0$ and if for $g \in \mathrm{Hom}_{\mathcal{C}}(W,U)$ we have $fg = 0$ then $g = \kappa h$ for a unique $h \in \mathrm{Hom}_{\mathcal{C}}(W, ker(f))$. The cokernel is the dual notion to the kernel.*

*(iii) An exact sequence consists of a set of morphisms $f_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, A_{i+1})$ and a set of objects $A_i \in Ob(\mathcal{C})$ labelled by an index set $I$, such that the image of $f_{i-1}$ is equal to the kernel of $f_i$ for any $i \in I$.*

**Definition 2.1.4.**
*An additive category is called abelian if*

*(i) every morphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(U,V)$ has a kernel and a cokernel,*

*(ii) every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel, and*

*(iii) every morphism can be written as the composition of an epimorphism with a monomorphism.*

**Definition 2.1.5.**
*An object $M$ of an abelian category $\mathcal{M}$ is a* <u>*generator*</u> *if for any morphisms $f, g \in \mathrm{Hom}_{\mathcal{M}}(M_1, M_2)$ and any morphism $m \in \mathrm{Hom}_{\mathcal{M}}(M, M_1)$ satisfying $f \circ m = g \circ m$ one has $f = g$.*

**Definition 2.1.6.**
*Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories. An* <u>*exact*</u> *functor maps exact sequences to exact sequences.*

**Definition 2.1.7.**
*An object $U$ of an abelian category is* <u>*simple*</u> *if every nontrivial monomorphism $V \to U$ and every nontrivial epimorphism $U \to W$ is an isomorphism.*
*A* <u>*semisimple*</u> *object is the finite direct sum of simple objects. A* <u>*semisimple*</u> *category is a category in which every object is semisimple.*

**Definition 2.1.8.**
*For an abelian category $\mathcal{C}$ let $\mathcal{F}(\mathcal{C})$ be the free abelian group generated by the isomorphism classes of the objects of $\mathcal{C}$ and $[U]$ the element in $\mathcal{F}(\mathcal{C})$ corresponding to the object $U$. The* <u>*Grothendieck group*</u> *$Gr(\mathcal{C})$ is the quotient of $\mathcal{F}(\mathcal{C})$ modulo the subgroup generated by elements of the form $[W] - [U] - [V]$ for all short exact sequences $U \to W \to V$. If the category is semisimple, the Grothendieck group $Gr(\mathcal{C})$ is generated as an abelian group by the simple objects of the category.*

**Definition 2.1.9.**
*A category is called* <u>*artinian*</u> *if it is semisimple and has finitely many isomorphism classes of simple objects.*

**Example 2.1.10.**
*Consider a separable $\mathbb{K}$-algebra $R$, the category of $R$-modules $R$-*mod *that are finite $\mathbb{K}$-vector spaces and the category of $R$-bimodules $R$-*bimod *that are finite $\mathbb{K}$-vector spaces. From Proposition 1.2.3 it follows that the categories $R$-*mod *and $R$-*bimod *are abelian and artinian.*

## 2.2 Tensor categories

In the following, we will extend certain set theoretic concepts to category theory. Therefore, sets are replaced by categories, functions by functors and equations between functions by natural transformations between functors. This process is called categorification.

We firstly define the notion of a tensor category which is the categorification of the notion of a ring.

**Definition 2.2.1.**
*A tensor category is a category $\mathcal{C}$ including*

*(i)* *a bifunctor* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *(tensor product),*

*(ii)* *a natural isomorphism* $a : \otimes(\otimes \times \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes)$ *(associativity rule),*

*(iii)* *an object* $\mathbb{1} \in Ob(\mathcal{C})$ *(tensor unit) and natural isomorphisms*

$$\lambda : \otimes(\mathbb{1} \times \mathrm{id}) \to \mathrm{id} \quad \textit{and} \quad \mu : \otimes(\mathrm{id} \times \mathbb{1}) \to \mathrm{id} \qquad (\textit{unit constraints}),$$

*such that the following axioms are satisfied:*

*(a)* *Pentagon Axiom*
*The diagram*

$$\begin{tikzcd}
 & ((U \otimes V) \otimes W) \otimes X \arrow[dl, "a_{U,V,W} \otimes \mathrm{id}_X"'] \arrow[dr, "a_{U \otimes V,W,X}"] & \\
(U \otimes (V \otimes W)) \otimes X \arrow[d, "a_{U,V \otimes W,X}"] & & (U \otimes V) \otimes (W \otimes X) \arrow[d, "a_{U,V,W \otimes X}"] \\
U \otimes ((V \otimes W) \otimes X) \arrow[rr, "\mathrm{id}_U \otimes a_{V,W,X}"] & & U \otimes (V \otimes (W \otimes X))
\end{tikzcd}$$

*commutes for all* $U, V, W, X \in \mathrm{Obj}(\mathcal{C})$*.*

*(b)* *Triangle Axiom*
*The triangle diagram*

$$\begin{tikzcd}
(V \otimes \mathbb{1}) \otimes W \arrow[rr, "a_{V,\mathbb{1},W}"] \arrow[dr, "r_V \otimes \mathrm{id}_W"'] & & V \otimes (\mathbb{1} \otimes W) \arrow[dl, "\mathrm{id}_V \otimes l_W"] \\
 & V \otimes W &
\end{tikzcd}$$

*commutes for all pairs* $V, W \in \mathrm{Obj}(\mathcal{C})$*.*

*For a strict tensor category* $a, l$ *and* $r$ *must be the identity.*

**Remark 2.2.2.**
*It can be shown that any tensor category is (tensor-) equivalent to a strict tensor category (see for example [Kas], Theorem XI.5.3). This implies Mac Lane's coherence theorem which states that any diagram built from associativity and unit constraints commutes.*

In a $\mathbb{K}$-linear tensor category one demands $\mathbb{K}$-bilinearity for the tensor product.

**Proposition 2.2.3.**
*For any unital associative $\mathbb{K}$-algebra $A$ the category of $A$-bimodules can be equipped with the structure of a tensor category by means of the tensor product of $A$-bimodules*

$$B_1 \otimes_A B_2 := \mathit{coker}\ \varphi_{B_1,B_2}\,,$$

*where*

$$\begin{array}{rcccc} \varphi_{B_1,B_2} & : & B_1 \otimes A \otimes B_2 & \longrightarrow & B_1 \otimes_k B_2 \\ & & b_1 \otimes a \otimes b_2 & \longmapsto & b_1 a \otimes b_2 - b_1 \otimes a b_2 \end{array}$$

*for any bimodules $B_1, B_2$. The tensor unit is $A$ itself.*

**Proof.** A straightforward calculation yields that the Pentagon and the Triangle Axiom are satisfied.

**Corollary 2.2.4.**
*The category of bimodules* $R$-bimod *over a separable $\mathbb{K}$-algebra $R$ has the structure of a tensor category.*

**Remark 2.2.5.**
*In general the category of left modules of a separable algebra is not a tensor category.*

**Definition 2.2.6.**
*An object $V$ in a tensor category is absolutely simple if the morphism*

$$\begin{array}{rcl} \mathbb{K} & \to & \mathrm{End}(V) \\ \lambda & \mapsto & \lambda\ \mathrm{id}_V \end{array}$$

*is an isomorphism of $\mathbb{K}$-vector spaces.*

**Definition 2.2.7.**
*Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1\!\!1_{\mathcal{C}}, a, l, r)$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1\!\!1_{\mathcal{D}}, \tilde{a}, \tilde{l}, \tilde{r})$ be tensor categories.*

*(i) A tensor functor $(F, \varphi_0, \varphi_2)$ from $\mathcal{C}$ to $\mathcal{D}$ consists of*

$$\begin{array}{rrcl} \textit{a functor} & F & : & \mathcal{C} \to \mathcal{D} \\ \textit{an isomorphism} & \varphi_0 & : & 1\!\!1_{\mathcal{D}} \xrightarrow{\sim} F(1\!\!1_{\mathcal{C}}) \\ \textit{and} & \varphi_2 & : & \otimes_{\mathcal{D}} \circ (F \times F) \to F \circ \otimes_{\mathcal{C}} \end{array}$$

*a natural isomorphism of functors from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{D}$, i.e. isomorphisms*

$$\varphi_2(U, V) : F(U) \otimes_{\mathcal{D}} F(V) \xrightarrow{\sim} F(U \otimes_{\mathcal{C}} V)\,,$$

*such that the diagrams*

$$\begin{array}{ccc} (F(U) \otimes F(V)) \otimes F(W) & \xrightarrow{\tilde{a}_{F(U),F(V),F(W)}} & F(U) \otimes (F(V) \otimes F(W)) \\ \downarrow \varphi_2(U,V) \otimes \mathrm{id}_{F(W)} & & \downarrow \mathrm{id}_{F(U)} \otimes \varphi_2(V,W) \\ F(U \otimes V) \otimes F(W) & & F(U) \otimes F(V \otimes W) \\ \downarrow \varphi_2(U \otimes V, W) & & \downarrow \varphi_2(U, V \otimes W) \\ F((U \otimes V) \otimes W) & \xrightarrow{F(a_{UVW})} & F(U \otimes (V \otimes W)) \end{array} \tag{2.1}$$

*and*

$$\begin{array}{ccc} \mathbb{1}\otimes F(U) & \xrightarrow{\tilde{l}_{F(U)}} & F(U) \\ \downarrow \varphi_0 \otimes \mathrm{id}_{F(U)} & & \uparrow F(l_U) \\ F(\mathbb{1})\otimes F(U) & \xrightarrow{\varphi_2(\mathbb{1},U)} & F(\mathbb{1}\otimes U) \end{array} \tag{2.2}$$

$$\begin{array}{ccc} F(U)\otimes \mathbb{1} & \xrightarrow{\tilde{r}_{F(U)}} & F(U) \\ \downarrow \mathrm{id}_{F(U)}\otimes\varphi_0 & & \uparrow F(r_U) \\ F(U)\otimes F(\mathbb{1}) & \xrightarrow{\varphi_2(U,\mathbb{1})} & F(U\otimes \mathbb{1}) \end{array} \tag{2.3}$$

*are commutative.*

*(ii) A tensor functor is strict if $\varphi_0$ and $\varphi_2$ are identities on $\mathcal{D}$.*

*(iii) A natural transformation of tensor functors*

$$\eta : (F, \varphi_0, \varphi_2) \to (F', \varphi_0', \varphi_2')$$

*is a natural transformation, such that the diagrams*

$$\begin{array}{ccc} & & F(\mathbb{1}) \\ & \varphi_0 \nearrow & \\ \mathbb{1} & & \downarrow \eta_{\mathbb{1}} \\ & \varphi_0' \searrow & \\ & & F'(\mathbb{1}) \end{array}$$

*and*

$$\begin{array}{ccc} F(U)\otimes F(V) & \xrightarrow{\varphi_2(U,V)} & F(U\otimes V) \\ \downarrow \eta_U \otimes \eta_V & & \downarrow \eta_{U\otimes V} \\ F'(U)\otimes F'(V) & \xrightarrow{\varphi_2'(U,V)} & F'(U\otimes V) \end{array}$$

*are commutative for all objects $U, V$. A natural tensor isomorphism is a natural tensor transformation that is a natural isomorphism. An equivalence of tensor categories is a tensor functor $F : \mathcal{C} \to \mathcal{D}$, such that there are a tensor functor $G : \mathcal{D} \to \mathcal{C}$ and natural tensor isomorphisms $\mathrm{id}_{\mathcal{D}} \cong FG$ and $GF \cong \mathrm{id}_{\mathcal{C}}$.*

**Definition 2.2.8.**
*A fiber functor for a tensor category $\mathcal{C}$ is a tensor functor into the category of finite-dimensional $\mathbb{K}$-vector spaces $\mathcal{C} \to \mathrm{vect}_{\mathbb{K}}$.*
*For a tensor category $\mathcal{C}$ and a separable algebra $R$ an R-fiber functor is defined to be a tensor functor $\mathcal{C} \to R$-bimod.*

**Definition 2.2.9.**
*Let $\mathcal{C}$ be a* strict *tensor category.*

*(i) A* <u>*commutativity constraint*</u> *in $\mathcal{C}$ is a natural isomorphism*

$$c : \otimes \to \otimes^{\mathrm{opp}}$$

*of functors from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$. That means an isomorphism*

$$c_{V,W} : V \otimes W \to W \otimes V$$

*for every pair of objects $(V, W)$ in $\mathcal{C}$, such that the diagram*

$$\begin{tikzcd} V \otimes W \arrow[r, "c_{V,W}"] \arrow[d, "f \otimes g"'] & W \otimes V \arrow[d, "g \otimes f"] \\ V' \otimes W' \arrow[r, "c_{V',W'}"'] & W' \otimes V' \end{tikzcd}$$

*commutes for all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(V, V')$, $g \in \mathrm{Hom}_{\mathcal{C}}(W, W')$.*

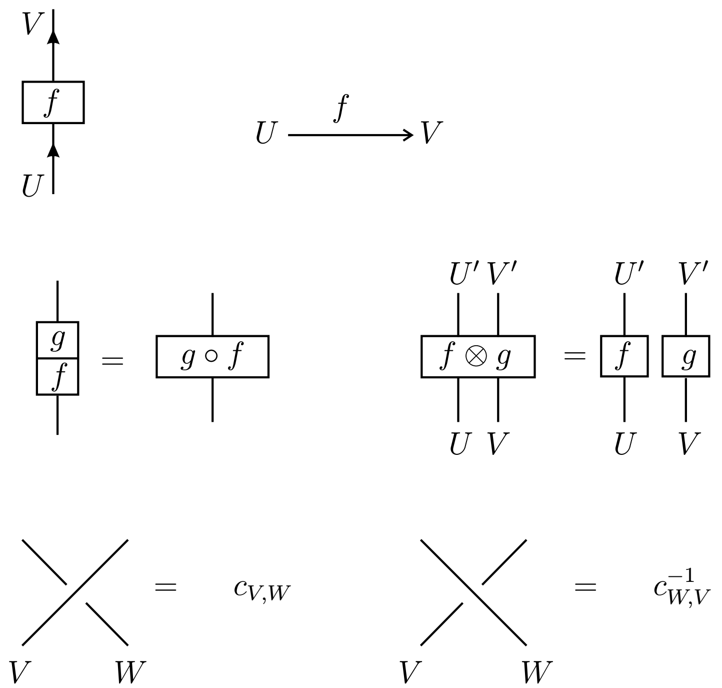
*(ii) Let $\mathcal{C}$ be a* strict *tensor category. A* <u>*braiding*</u> *is a commutativity constraint satisfying the relations*

$$\begin{aligned} c_{U \otimes V, W} &= (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W}) \\ c_{U, V \otimes W} &= (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W) \end{aligned}$$

*for all objects $U, V, W$ of $\mathcal{C}$.*

*(iii) A* <u>*braided tensor category*</u> *is a tensor category with a braiding.*

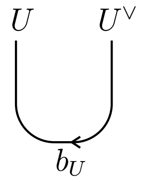
In any *strict* tensor category morphisms can be visualized by graphs. We will use diagrams of Joyal-Street type [JoSt91]. The conventions used here are the same as the ones introduced for algebras. The pictures are read from bottom to top, morphisms are depicted by means of boxes, braidings by crossing of lines, composition of morphisms amounts to concatenation of lines and the tensor product to juxtaposition.

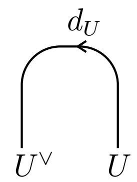


**Definition 2.2.10.**
*Let $\mathcal{C}$ be a strict tensor category.*

*(i) Assume that for each object $U \in \mathrm{Obj}(\mathcal{C})$ there exists an object $U^\vee$ in $\mathcal{C}$ and two morphisms:*

$$b_U : \mathbb{1} \to U \otimes U^\vee \qquad\qquad d_U : U^\vee \otimes U \to \mathbb{1}\,.$$




*This structure is called a (right) duality if for all objects $U$ one has:*

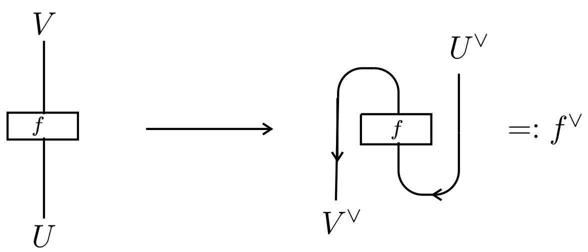
$$(\mathrm{id}_U \otimes d_U) \circ (b_U \otimes \mathrm{id}_U) \;=\; \mathrm{id}_U \qquad \textit{and} \qquad (d_U \otimes \mathrm{id}_{U^\vee}) \circ (\mathrm{id}_{U^\vee} \otimes b_U) = \mathrm{id}_{U^\vee}\,.$$


*The morphism $d_U$ is called evaluation and the morphism $b_U$ coevaluation.*

*(ii) A left duality is defined in an analogous way.*

By definition a duality gives a map for objects $U \mapsto U^\vee$, and by defining

$$f \mapsto f^\vee = (d_V \otimes \mathrm{id}_{U^\vee}) \circ (\mathrm{id}_{V^\vee} \otimes f \otimes \mathrm{id}_{U^\vee}) \circ (\mathrm{id}_{V^\vee} \otimes b_U)$$




for morphisms we obtain a contravariant functor from the category into itself.

**Definition 2.2.11.**
*In a rigid tensor category $\mathcal{C}$ every object has a left and a right dual.*

**Proposition 2.2.12.**
*In an abelian rigid tensor category $\mathcal{C}$ the bifunctor $\otimes$ is exact, that means that for any object $C \in \mathcal{C}$ the functors $C \otimes -$ and $- \otimes C$ are exact.*

**Proof.** See for example [BaKi], Proposition 2.1.8

**Remark 2.2.13.**
*Hence, when considering an abelian rigid tensor category $\mathcal{C}$ the Grothendieck group $Gr(\mathcal{C})$ becomes a ring, also called the* <u>*fusion ring*</u>

$$[X] * [Y] = \sum_{[Z]\,simple} \mathcal{N}_{XY}^{Z}[Z]\,,$$

*where $\mathcal{N}_{XY}^{Z} = \dim \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \in \mathbb{N}_0$ are the* <u>*fusion coefficients*</u>*. The fusion ring is commutative if the category $\mathcal{C}$ has a braiding.*

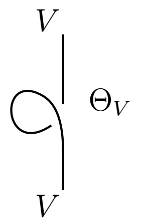
**Definition 2.2.14.**

*(i) A* <u>*twist*</u> *in a braided tensor category is a natural family of isomorphisms $\Theta_V : V \to V$ satisfying*

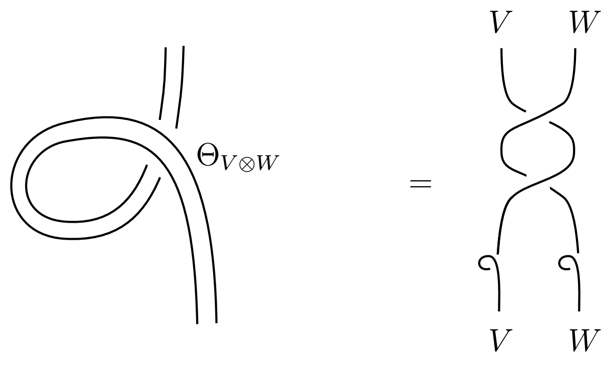
$$\Theta_{V\otimes W} = c_{W,V} \circ c_{V,W} \circ (\Theta_V \otimes \Theta_W)$$

*for all $V \in \mathrm{Obj}(\mathcal{C})$. Naturality means that the relation $\Theta_V \circ f = f \circ \Theta_U$ holds for any morphism $f : U \to V$.*

*Graphically we depict the twist by:*

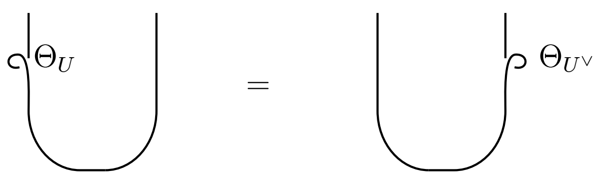



*Then, the defining relation reads:*



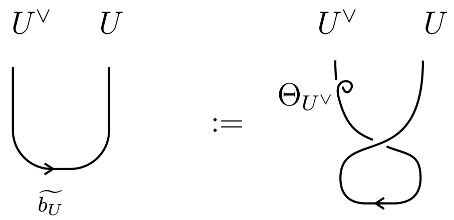

*(ii) A* <u>*ribbon category*</u> *is a (strict) tensor category with braiding, twist and (right) duality, such that twist and duality satisfy*

$$(\Theta_U \otimes \mathrm{id}_{U^\vee}) \circ b_U = (\mathrm{id}_U \otimes \Theta_{U^\vee}) \circ b_U\,.$$

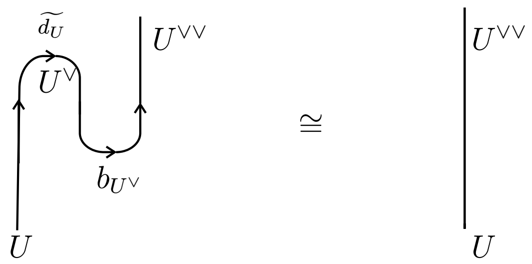


**Remark 2.2.15.**
*A ribbon category is automatically equipped with a left duality by setting*

$$\widetilde{b}_U := (\Theta_{U^\vee} \otimes \mathrm{id}_U) \circ c_{U,U^\vee} \circ b_U$$



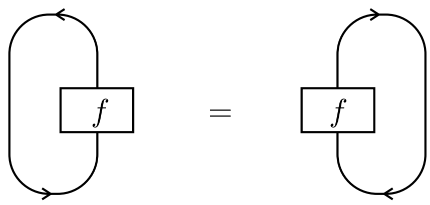

*We set $U^\vee =: {}^\vee U$, then one checks ${}^\vee f = f^\vee$. Hence, the bidual $U^{\vee\vee}$ is canonically isomorphic to $U$, i.e. the bidual functor $?^{\vee\vee}$ is naturally isomorphic to the identity:*




Note that in a tensor category the monoid $\mathrm{End}(\mathbb{1})$ is commutative.

**Definition 2.2.16.**
*We define the* <u>trace</u> *of an endomorphism $f : V \to V$ in a ribbon category as*

$$\mathrm{tr}(f) := d_U \circ (\mathrm{id}_U \otimes f) \circ b_{U^\vee} = d_{U^\vee} \circ (f \otimes \mathrm{id}_{U^\vee}) \circ b_U$$

$\mathrm{tr}(f) :=$  $\in \mathrm{End}(\mathbb{1})$.


*The quantum dimension for any object $U$ of a ribbon category is defined to be $\mathrm{tr}(\mathrm{id}_U)$. It depends only on the isomorphism class of an object. For simple objects, we introduce the abbreviation*

$$\mathcal{D}_i := \dim(U_i)\,.$$

The trace has the following properties that can be derived straightforwardly from the definition:

- $\mathrm{tr}(fg) \quad = \mathrm{tr}(gf)$

- $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f)\mathrm{tr}(g)$
- For $k \in \mathrm{End}(\mathbb{1})$ : $\quad \mathrm{tr}(k) = k$

**Lemma 2.2.17.**
*In a ribbon category one has the following identities for the fusion coefficients*

$$\mathcal{N}_{XY}^{\ \ Z} = \mathcal{N}_{Z^\vee X}^{\ \ \ Y^\vee} = \mathcal{N}_{YZ^\vee}^{\ \ \ X^\vee} .$$

**Proof.** Clearly the maps

$$\begin{aligned} \chi_{X,Y,Z} : \mathrm{Hom}_{\mathcal{C}}(X\otimes Y, Z) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z\otimes Y^\vee) \\ f &\longmapsto (f\otimes \mathrm{id}_{Y^\vee}) \circ (\mathrm{id}_X \otimes b_Y) \end{aligned}$$

and

$$\begin{aligned} \omega_{X,Y,Z} : \mathrm{Hom}_{\mathcal{C}}(X\otimes Y, Z) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, X^\vee \otimes Z) \\ f &\longmapsto (\mathrm{id}_{X^\vee}\otimes f) \circ (b_X \otimes \mathrm{id}_Y) \end{aligned}$$

are isomorphisms. Then $\omega^{-1}_{X,Z,Y^\vee} \circ \chi_{X,Y,Z}$ and $\chi^{-1}_{X,Z,Y^\vee} \circ \omega_{X,Y,Z}$ are the required isomorphisms. □

Modular tensor categories encode the data of a chiral conformal field theory. In the following, we introduce the basic notions that will be needed in order to derive Kramers-Wannier dualities in Chapter 4.

**Definition 2.2.18.**
*Let $\mathcal{C}$ be an abelian, artinian ribbon category with simple tensor unit $\mathbb{1}$ and $I$ the index set parameterizing the isomorphism classes of simple objects. In particular, given $i \in I$, we find a unique $i^\vee$, such that $U_i^\vee \cong U_{i^\vee}$. We choose the representatives, such that $U_0 = \mathbb{1}$ and $0 \in I$.*
*We call $\mathcal{C}$ a* <u>*modular tensor category*</u> *if the $|I| \times |I|$- matrix with entries in $\mathbb{K}$ defined by*

$$s_{ij} = \mathrm{tr}(c_{U_j,U_i} \circ c_{U_i,U_j}) \in \mathrm{End}(\mathbb{1}) \cong \mathbb{K}$$

*for simple objects $U_i, U_j$ is invertible over $\mathbb{K}$.* [1]

For the models considered in Chapter 4 there exists a positive real factor, such that the matrix $s$ rescaled by this factor is unitary. We call this unitary matrix $S$. The fusion coefficients can be expressed in terms of the modular matrix $S$ by the Verlinde formula

$$\mathcal{N}_{ij}^{\ \ k} = \sum_{l\in I} \frac{S_{il}S_{jl}\bar{S}_{kl}}{S_{0l}} . \tag{2.4}$$

Another property of the modular matrix $S$ in the models of our interest are the inequalities $S_{0\kappa} \geq S_{00} > 0$ for any simple object $U_\kappa$. Entries for dual objects are

[1] This definition is slightly more restrictive than the original one in [Tur].

related by complex conjugation, $S_{\lambda\overline{\mu}} = \overline{S_{\lambda\mu}}$. We finally note the following easy consequence of the Verlinde formula (2.4) and the unitarity of $S$

$$\frac{S_{\rho\kappa}}{S_{0\kappa}}\frac{S_{\sigma\kappa}}{S_{0\kappa}} = \sum_{\tau\in I} \mathcal{N}_{\rho\sigma}^{\ \ \tau}\, \frac{S_{\tau\kappa}}{S_{0\kappa}}\ . \tag{2.5}$$

The quantum dimension is related to the modular $S$-matrix via

$$\mathcal{D}_i = \frac{S_{0i}}{S_{00}}\ .$$

It is a ring homomorphism from the fusion ring $Gr(\mathcal{C})$ to the ring of algebraic integers over the rational numbers. Dual objects have identical dimension $\mathcal{D}_{\overline{i}} = \mathcal{D}_i$.

In a modular tensor category one can define invertible objects and fixed points:

**Definition 2.2.19.**

*(i)* *A simple current in a modular tensor category $\mathcal{C}$ is an isomorphism class $[J]$ of simple objects $J$ satisfying*

$$J \otimes J^\vee \cong 1\!\!1\,.$$

*(ii)* *A fixed point $[U_f]$ of a simple current $[J]$ is an isomorphism class of simple objects of $\mathcal{C}$ satisfying*

$$J \otimes U_f \cong U_f\,.$$

*(iii)* *A duality class is an isomorphism class $[U_\phi]$ of simple objects, such that the tensor product $U_\phi^\vee \otimes U_\phi$ is isomorphic to a direct sum of invertible objects containing at least two non-isomorphic invertible objects.*

The tensor product gives rise to a finite abelian group on the set of simple currents, the so-called Picard group $\mathrm{Pic}(\mathcal{C})$. It turns out that every isomorphism class $[U_l]$ of simple objects of $\mathcal{C}$ gives rise to a character on the group $\mathrm{Pic}(\mathcal{C})$

$$\chi_l([U_k]) := \frac{S_{kl}}{S_{0\,l}}$$

which we call the monodromy character of the object $U_k$ with respect to the simple current $U_l$.

## 2.3 Representation theory for weak Hopf algebras

In this thesis we consider only modules and bimodules that are finite-dimensional vector spaces.

The tensor product $U \otimes V$ of two modules $U, V$ of a bialgebra $A$ can be defined by means of the coproduct

$$a(u \otimes v) := \Delta(a)(u \otimes v) = \sum_{(a)} a_{(1)} u \otimes a_{(2)} v$$

for $a \in A, u \in U, v \in V$. For weak bialgebras the coproduct is not a morphism of algebras, therefore

$$1(u \otimes v) := \Delta(1)(u \otimes v) = \sum_{(1)} 1_{(1)} u \otimes 1_{(2)} v$$

which is not equal to $u \otimes v$. Thus, the unitality of the action is not satisfied. In order to obtain a unital module we restrict the tensor product of modules over weak bialgebras to the subspace

$$\mathrm{Im}\ \rho_U \otimes \rho_V(\Delta(1)) \subseteq U \otimes V\,.$$

**Lemma 2.3.1.**
*The target counital algebra $A_t$ is a left module over the weak bialgebra $A$ by*

$$a \cdot z = \epsilon_t(az) \qquad \text{for } a \in A, z \in A_t$$

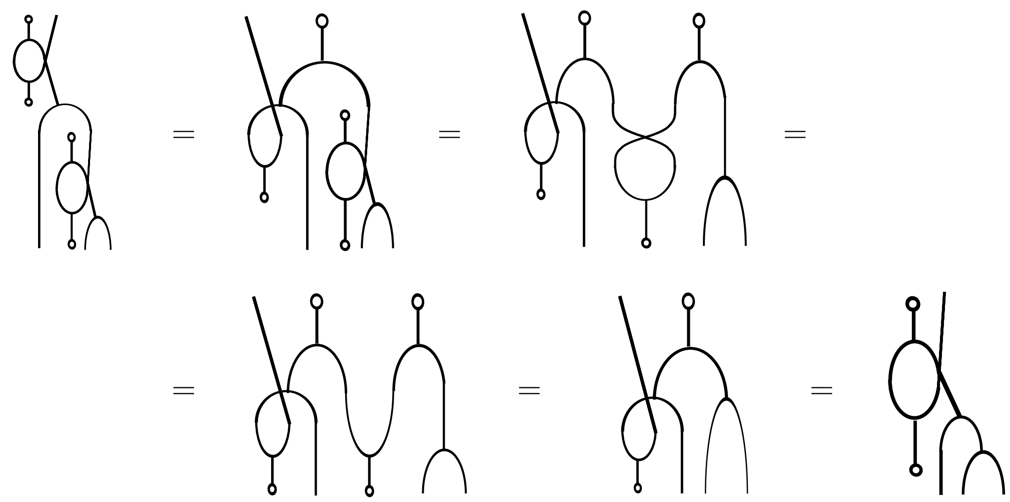
*and the source counital algebra $A_s$ is a right module over $A$ by*

$$z \cdot a = \epsilon_s(za) \qquad \text{for } a \in A, z \in A_s\,.$$

**Proof.** For any $a, b \in A$ and $z \in A_t$ one has

$$\epsilon_t(a\epsilon_t(bz)) = \epsilon(\underline{1}'_{(1)}a\underline{1}_{(2)})\epsilon(\underline{1}_{(1)}bz)\underline{1}'_{(2)} = \epsilon(\underline{1}'_{(1)}a\underline{1}bz)\underline{1}'_{(2)} = \epsilon_t(abz)\,.$$

Pictorially,



$\square$

For a weak Hopf algebra $H$ the target counital algebra $H_t$ is a left module over $H$ by

$$h \cdot z = S(hz) \qquad \text{for } h \in H, z \in H_t$$

and the source counital algebra $H_s$ is a right module over $H$ by

$$z \cdot h = S(zh) \qquad \text{for } h \in H, z \in H_s\,.$$

In general the counital subalgebra $A_t$ of a weak bialgebra is not simple as a left module. In [Niks02] it has been shown that the tensor unit in $A$-mod is simple if and only if $Z(A) \cap A_t = \mathbb{K}$, where $Z(A)$ denotes the center of $A$.

**Lemma 2.3.2.**
*Every left module $V$ over a weak bialgebra $A$ can be equipped with the structure of a bimodule over $A_t$ by:*

$$\begin{array}{rcl} A_t \times V & \longrightarrow & V \\ (a, v) & \longmapsto & a \cdot v \\ V \times A_t & \longrightarrow & V \\ (v, a) & \longmapsto & \epsilon_s(a) \cdot v \end{array}$$

**Proof.** It is clear that $V$ is a left $A_t$-module. The right module structure comes from the fact that $\epsilon_s$ is an algebra antihomomorphism and for the bimodule structure we use that elements of the counital subalgebras commute.

For a weak Hopf algebra $H$ one endows every left module $V$ over $H$ with the structure of a bimodule over $H_t$ by:

$$\begin{array}{rcl} H_t \times V & \longrightarrow & V \\ (h, v) & \longmapsto & h \cdot v \\ V \times H_t & \longrightarrow & V \\ (v, h) & \longmapsto & S(h) \cdot v\,. \end{array}$$

**Definition 2.3.3.**
*The <u>truncated</u> tensor product for two modules $U, V$ over a weak bialgebra $A$ in $A$-mod is defined by*

$$U \otimes V := \{x \in U \otimes_{\mathbb{K}} V | x = \Delta(1) \cdot x\}$$

*with the action via the comultiplication.*

**Theorem 2.3.4.** *[NiVa00]*
*The category $A$-mod of finite-dimensional modules over a weak bialgebra $A$ is a tensor category with tensor unit $A_t$.*

**Proof.** Consider the truncated tensor product; the tensor product of morphisms is the restriction of the tensor product in the tensor category of vector spaces. Coassociativity of the coproduct implies that the associator satisfies the Pentagon Axiom.

By Lemma 2.3.1 the unit constraints for the tensor unit are

$$\begin{aligned} l_V : A_t \otimes V \to V\,, \qquad & l_V(\underline{1}_{(1)} z \otimes \underline{1}_{(2)} v) = z \cdot v \\ r_V : V \otimes A_t \to V\,, \qquad & r_V(\underline{1}_{(1)} v \otimes \underline{1}_{(2)} z) = \epsilon_s(z) \cdot v \end{aligned}$$

with the inverses $l_V^{-1}(v) = \epsilon_t(\underline{1}_{(1)}) \otimes \underline{1}_{(2)} v$ and $l_V^{-1}(v) = \underline{1}_{(1)} v \otimes \underline{1}_{(2)}$. □

For a weak Hopf algebra $H$ the unit constraints are chosen to be

$$\begin{aligned} l_V : A_t \otimes V \to V\,, \qquad & l_V(\underline{1}_{(1)} z \otimes \underline{1}_{(2)} v) = z \cdot v \\ r_V : V \otimes A_t \to V\,, \qquad & r_V(\underline{1}_{(1)} v \otimes \underline{1}_{(2)} z) = S(z) \cdot v \end{aligned}$$

with the inverses $l_V^{-1}(v) = S(\underline{1}_{(1)}) \otimes \underline{1}_{(2)} v$ and $l_V^{-1}(v) = \underline{1}_{(1)} v \otimes \underline{1}_{(2)}$.

**Remark 2.3.5.**
*In the special case of a bialgebra $A$, the ground field $\mathbb{K}$ becomes an $A$-module by means of the counit*

$$ax := \epsilon(a)x$$

*for $a \in A$ and $x \in \mathbb{K}$.*

**Corollary 2.3.6.**
*Lemma 2.3.2 and Theorem 2.3.4 imply the existence of an $H_t$-fiber functor of $H$-mod.*

**Theorem 2.3.7.** *[NiVa00]*
*The category of finite-dimensional modules $H$-mod over a weak Hopf algebra $H$ is a ribbon tensor category.*

## 2.4 Module categories

In this section we introduce the concept of a module category over a tensor category which is the categorification of the notion of a module over a ring.

**Definition 2.4.1.**
*A (right) module category over a tensor category $(\mathcal{C}, a, \lambda)$ is an abelian category $\mathcal{M}$ with an exact bifunctor*

$$\otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$$

*and functorial isomorphisms*

$$\begin{array}{rcrcl} \psi & : & \bullet \otimes (- \otimes -) & \to & (\bullet \otimes -) \otimes - \\ \eta & : & \bullet \otimes \mathbb{1} & \to & \bullet \end{array}$$

*with $-$ in $\mathcal{C}$ and $\bullet$ in $\mathcal{M}$, such that the diagrams*

$$
\begin{tikzcd}
 & M\otimes(X\otimes(Y\otimes Z)) \arrow[dl, "\psi(M,X,Y\otimes Z)"'] \arrow[dr, "\mathrm{id}\otimes a(X,Y,Z)"] & \\
(M\otimes X)\otimes(Y\otimes Z) \arrow[d, "\psi(M\otimes X,Y,Z)"'] & & M\otimes((X\otimes Y)\otimes Z) \arrow[d, "\psi(M,X\otimes Y,Z)"] \\
((M\otimes X)\otimes Y)\otimes Z & & (M\otimes(X\otimes Y))\otimes Z \arrow[ll, "\psi(M,X,Y)\otimes\mathrm{id}"']
\end{tikzcd}
$$

*and*

$$
\begin{tikzcd}
M\otimes(1\!\!1\otimes X) \arrow[rr, "\psi(M,1\!\!1,X)"] \arrow[dr, "\mathrm{id}\otimes\lambda_X"'] & & (M\otimes 1\!\!1)\otimes X \arrow[dl, "\eta_M\otimes\mathrm{id}"] \\
 & M\otimes X &
\end{tikzcd}
$$

*commute for all* $X, Y, Z \in \mathrm{Obj}(\mathcal{C}), M \in \mathrm{Obj}(\mathcal{M})$.

**Example 2.4.2.**
*For any separable algebra* $R$ *the category of* $R$*-modules is a module category over the tensor category bimod-*$R$*, where the bifunctor*

$$\text{mod-}R \times \text{bimod-}R \longrightarrow \text{mod-}R$$

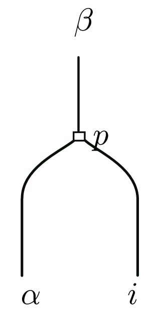
*is the tensor product over* $R$*. There is a canonical choice for the associativity isomorphism and the unit isomorphism.*

**Proposition 2.4.3.**
*The category* $H_t$-mod *of modules over the counital subalgebra of a weak Hopf algebra* $H$ *is a module category over the tensor category* $H$-mod.

**Proof.** By Lemma 2.3.2 any $H$-module can be endowed with the structure of an $H_t$-bimodule, thus there exists a functor $S : H\text{-mod} \to H_t\text{-bimod}$. Since the counital subalgebra $H_t$ is separable, the category of $H_t$-modules is a module category over the tensor category $H_t$-bimod. The composition $\otimes_{H_t} \circ (S \times \mathrm{id})$ yields the structure of a module category on $H$-mod over $H_t$-bimod.

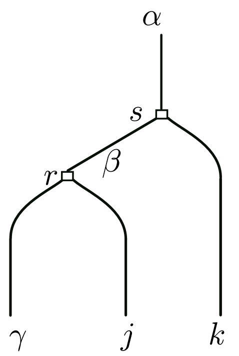
**Remark 2.4.4.**
*The Grothendieck group* $Gr(\mathcal{M})$ *of a module category* $\mathcal{M}$ *over a tensor category* $\mathcal{C}$ *is a module over the Grothendieck ring* $Gr(\mathcal{C})$*.*

Consider a strict semisimple module category $\mathcal{M}$ over a strict semisimple tensor category $\mathcal{C}$ with simple objects labelled by $M_\alpha$, respectively $U_i$, and let $\{\lambda^p_{(\alpha,j)\beta}\}$ be a basis in $\mathrm{Hom}_{\mathcal{C}}(M_\alpha \otimes U_i, M_\beta)$ that we denote pictorially by:

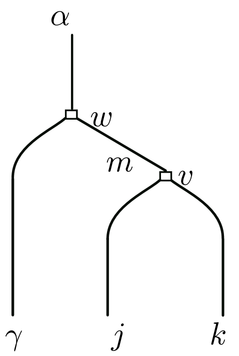



The space $\mathrm{Hom}(M_\alpha \otimes U_j \otimes U_k, M_\beta)$ has two distinguished bases that correspond to its decompositions $\oplus_\beta \mathrm{Hom}(M_\beta \otimes U_k, M_\alpha) \otimes \mathrm{Hom}(M_\gamma \otimes U_j, M_\beta)$ and $\oplus_m \mathrm{Hom}(M_\gamma \otimes U_m, M_\alpha) \otimes \mathrm{Hom}(U_j \otimes U_k, U_m)$. The 6j-symbols (or fusing matrices) $F$ are the components of the isomorphism

$$\begin{aligned} \oplus_m \mathrm{Hom}(M_\gamma \otimes U_m, M_\alpha) \quad \otimes \quad & \mathrm{Hom}(U_j \otimes U_k, U_m) \longrightarrow \\ & \oplus_\beta \mathrm{Hom}(M_\beta \otimes U_k, M_\alpha) \otimes \mathrm{Hom}(M_\gamma \otimes U_j, M_\beta)\,. \end{aligned} \tag{2.6}$$




$$= \sum_\beta \sum_{v,w} F^{(\gamma j k)\alpha}_{r\beta s, v m w}$$




They have the following properties:

$$F^{(\alpha 0 l)\gamma}_{r\epsilon s, viw} = \delta_{li}\delta_{\alpha\epsilon}\delta_{r1}\delta_{v1}\delta_{sw} \quad = \quad F^{(\gamma l 0)\alpha}_{s\epsilon r, viw} \tag{2.7}$$

$$\sum_{v,i,w} F^{(\alpha j l)\gamma}_{r\epsilon s, viw} \; (F^{-1})^{(\alpha j l)\gamma}_{r'\epsilon' s', viw} \quad = \quad \delta_{\epsilon\epsilon'}\delta_{ss'}\delta_{rr'} \tag{2.8}$$

$$\sum_{s,\epsilon,r} F^{(j l \alpha)\gamma}_{r\epsilon s, viw} \; (F^{-1})^{(j l \alpha)\gamma}_{r\epsilon s, v'i'w'} \quad = \quad \delta_{vv'}\delta_{ii'}\delta_{ww'} \tag{2.9}$$

By definition the fusing matrices satisfy the Pentagon Axiom:

$$\sum_t F^{(\alpha i l)\beta}_{rmq, s\gamma t} \; F^{(\gamma j k)\beta}_{tlp, u\epsilon v} = \sum_{w,n,y,x} F^{(ijk)m}_{qlp, wny} \; F^{(\alpha n k)\beta}_{rmy, x\epsilon v} \; F^{(\alpha i j)\epsilon}_{xnw, s\gamma u} \tag{2.10}$$

**Definition 2.4.5.**
*Let $(\mathcal{M}_1, \otimes, \psi, \eta)$ and $(\mathcal{M}_2, \otimes, \tilde{\psi}, \tilde{\eta})$ be module categories over a tensor category $\mathcal{C}$.*

*(i) A module functor is defined as a functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ with functorial morphisms*

$$\gamma_{M,X} : F(M \otimes X) \to F(M) \otimes X,$$

*such that the diagrams*

$$\begin{tikzcd}
 & F(M\otimes(X\otimes Y)) \arrow[dl, "F(\psi_{M,X,Y})"'] \arrow[dr, "\gamma_{M,X\otimes Y}"] & \\
F((M\otimes X)\otimes Y) \arrow[d, "\gamma_{M\otimes X,Y}"] & & F(M)\otimes(X\otimes Y) \arrow[d, "\tilde{\psi}_{F(M),X,Y}"] \\
F(M\otimes X)\otimes Y \arrow[rr, "\gamma_{M,X}\otimes\mathrm{id}"] & & (F(M)\otimes X)\otimes Y
\end{tikzcd}$$

*and*

$$\begin{tikzcd}
F(M\otimes\mathbb{1}) \arrow[rr, "F(\eta_M)"] \arrow[dr, "\gamma_{M,\mathbb{1}}"'] & & F(M) \\
 & F(M)\otimes\mathbb{1} \arrow[ur, "\tilde{\eta}_{F(M)}"'] &
\end{tikzcd}$$

*commute for all $X, Y \in Ob(\mathcal{C})$, $M \in Ob(\mathcal{M}_1)$. If all morphisms $\gamma_{M,X}$ are isomorphisms, the module functor is called strict.*

*(ii) A natural transformation of module functors*

$$\eta : (F, \gamma) \to (F', \gamma')$$

*is a natural transformation, such that the diagram*

$$\begin{tikzcd}
F(M\otimes X) \arrow[r, "\gamma_{M,X}"] \arrow[d, "\eta_{M\otimes X}"] & F(M)\otimes X \arrow[d, "\eta_M\otimes\mathrm{id}_X"] \\
F'(M\otimes X) \arrow[r, "\gamma'_{M,X}"] & F'(M)\otimes X
\end{tikzcd}$$

*is commutative for all $X \in Ob(\mathcal{C})$, $M \in Ob(\mathcal{M})$. A natural tensor isomorphism is a natural module transformation that is a natural isomorphism. An equivalence of module categories is a module functor $F : \mathcal{C} \to \mathcal{D}$, such that there are a module functor $G : \mathcal{D} \to \mathcal{C}$ and natural module isomorphisms $\mathrm{id}_{\mathcal{D}} \cong FG$ and $GF \cong \mathrm{id}_{\mathcal{C}}$.*

Any module functor for a rigid tensor category is strict (see [Ostr03], Remark 4).

**Definition 2.4.6.**
*The direct sum of two semisimple module categories $\mathcal{M}_1, \mathcal{M}_2$ over a tensor category $\mathcal{C}$ is defined to be the category $\mathcal{M}_1 \times \mathcal{M}_2$ with objects $(M_1, M_2)$ and morphisms $(f, g)$. Additive and modular structure are defined componentwise*

$$\begin{aligned}
(M_1, M_2) \oplus (M_3, M_4) &:= (M_1 \oplus M_3, M_2 \oplus M_4) \\
(f_1, g_1) \oplus (f_2, g_2) &:= (f_1 \oplus f_2, g_1 \oplus g_2) \\
C\otimes(M_1, M_2) &:= (C\otimes M_1, C\otimes M_2) \\
\gamma\otimes(f, g) &:= (\gamma\otimes f, \gamma\otimes g)
\end{aligned}$$

*for all* $M_1, M_3 \in \mathrm{Obj}(\mathcal{M}_1)$, $M_2, M_4 \in \mathrm{Obj}(\mathcal{M}_2)$, $f \in \mathrm{Hom}_{\mathcal{M}_1}(M_1, M_3)$, $g \in \mathrm{Hom}_{\mathcal{M}_2}(M_2, M_4)$.

*A module category is called* *indecomposable* *if it is not the direct sum of nontrivial module categories.*

**Definition 2.4.7.**
*Let* $\mathcal{C}$ *be a tensor category.*

*(i) An* *algebra in* $\mathcal{C}$ *is an object* $A$ *with a multiplication morphism* $m \in \mathrm{Hom}_{\mathcal{C}}(A \otimes A, A)$ *and a unit morphism* $\eta \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$*, such that*

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m) \qquad m \circ (\eta \otimes \mathrm{id}) = \mathrm{id} = m \circ (\mathrm{id} \otimes \eta)\,.$$

*(ii) Dually, a* *coalgebra in* $\mathcal{C}$ *is an object* $C$ *with a comultiplication morphism* $\Delta \in \mathrm{Hom}_{\mathcal{C}}(C, C \otimes C)$ *and a counit morphism* $\epsilon \in \mathrm{Hom}_{\mathcal{C}}(C, \mathbb{1})$*, such that*

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \qquad (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta\,.$$

*(iii) A* *Frobenius algebra in* $\mathcal{C}$ *is an object in* $\mathcal{C}$ *which is both an algebra and a coalgebra and satisfies*

$$(\mathrm{id} \otimes m) \circ (\Delta \otimes \mathrm{id}) = \Delta \circ m = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta)\,.$$

*It is called* *special* *if*

$$\epsilon \circ \eta = c_1 \mathrm{id}_{\mathbb{1}} \qquad m \circ \Delta = c_2 \mathrm{id}_A\,.$$

*Assume that* $\mathcal{C}$ *is rigid with coevaluation morphism* $b_U$ *for any object* $U$ *of* $\mathcal{C}$*. A Frobenius algebra* $A$ *is called* *symmetric* *if*

$$((\epsilon \circ m) \otimes \mathrm{id}_{A^\vee}) \circ (\mathrm{id}_A \otimes b_A) = (\mathrm{id}_{A^\vee} \otimes (\epsilon \circ m)) \circ (b_{A^\vee} \otimes \mathrm{id}_A)\,.$$

*(iv) An algebra* $A$ *in a category* $\mathcal{C}$ *is called* *haploid* *if* $\dim \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$.

**Conjecture 2.4.8.** *[ENO02]*
*For an artinian rigid tensor category there exist only finitely many non equivalent indecomposable module categories.*

In [FRS04b] it has been shown that in a modular tensor category there are only finitely many inequivalent haploid special Frobenius algebras. Haploid special Frobenius algebras give rise to most of the known module categories, namely the ones of D-type in the A-D-E-classification.

# Chapter 3

# Construction of quantum symmetries

By $k\{\hat{x}^1, \dots \hat{x}^n\}$ we denote the free associative algebra in $n$ variables. In the following, we choose $k = \mathbb{C}$. We consider two examples of quantum spaces that are relatively simple deformations of the affine space:

- Let $\theta^{\mu\nu}$ be an antisymmetric matrix with entries in $\mathbb{R}$. The $\theta$-space [ChHo99, Scho99] is defined to be the universal enveloping algebra of the finite-dimensional complex Lie algebra with basis $\{\hat{x}^1, \dots, \hat{x}^n\}$ and Lie brackets

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu} \qquad \mu, \nu \in \{1, \dots n\}\,. \tag{3.1}$$

  We denote it by $k_\theta[\hat{x}^1, \dots \hat{x}^n]$.

- For $\kappa \in \mathbb{R}^*$ the $\kappa$-space [LNRT91] is defined to be the universal enveloping algebra of the finite-dimensional complex Lie algebra with basis $\{\hat{x}^1, \dots, \hat{x}^n\}$ and Lie brackets[1]

$$\begin{aligned} [\hat{x}^n, \hat{x}^i] &= \mathrm{i}a\hat{x}^i \\ \left[\hat{x}^i, \hat{x}^j\right] &= 0 \end{aligned} \tag{3.2}$$

  for $a := \frac{1}{\kappa}$ and $i, j = 1, 2, \dots, n-1$. We denote it by $k_\kappa[\hat{x}^1, \dots \hat{x}^n]$.

We consider an involution $\dagger$ on the algebra $\mathbb{C}\{\hat{x}^1, \dots \hat{x}^n\}$ and demand additionally that it is an antihermitian algebra antihomomorphism. Set $(\hat{x}^\mu)^\dagger = \hat{x}^\mu$ for all variables $\hat{x}^\mu$ and extend to any polynomial. Then the ideal $< [\hat{x}^\mu, \hat{x}^\nu] - \mathrm{i}\theta^{\mu\nu} >$ (respectively $< [\hat{x}^\mu, \hat{x}^\nu] - \mathrm{i}a\delta^\mu_n\delta^\nu_\lambda - \mathrm{i}a\delta^\nu_n\delta^\mu_\lambda >$) is invariant under $\dagger$ and therefore $\dagger$ can be defined on the quotient $k_\theta[\hat{x}^1, \dots \hat{x}^n]$ (respectively $k_\kappa[\hat{x}^1, \dots \hat{x}^n]$).

In this chapter we construct quantum symmetries, invariants, differential forms and vector fields for these two quantum spaces. We demand that these structures are a deformation of the corresponding structures of the commutative space: setting the deformation parameter $\theta$ (respectively $a$) to 0 must yield the structures in the commutative setting.

[1] The corresponding structure constants are $C^{\mu\nu}_\lambda = \mathrm{i}a\delta^\mu_n\delta^\nu_\lambda - \mathrm{i}a\delta^\nu_n\delta^\mu_\lambda$ with $a \in \mathbb{R}^*$.

We derive quantum versions of the universal enveloping algebras of the Lie algebra of derivatives and of rotations for the $\theta$- and the $\kappa$-space. As in the commutative case, we demand that they are Hopf algebras. We compute invariants of the quantum symmetries for the $\kappa$- and the $\theta$-space: we find a deformed space invariant, a deformed Laplace operator and a deformed Pauli-Lubanski vector. We associate an algebra of functions on a commutative space to the $\kappa$-space by means of deformation quantization. Finally, we define vector fields.

We introduce the following notation: Greek indices take values $1, \ldots, n$, Roman indices run from $1, \ldots, n-1$. Indices can arbitrarily be lowered or raised via the metric $g^{\mu\nu} = \delta^{\mu\nu}$. We use Einstein's summation convention. We denote the commutative space by $k[x^1, \ldots x^n]$ with variables $x^\mu$, the $\theta$-space algebra by $k_\theta[\hat{x}^1, \ldots \hat{x}^n]$ and the $\kappa$-space algebra by $k_\kappa[\hat{x}^1, \ldots \hat{x}^n]$ with variables $\hat{x}^\mu$. By $\lim_{\theta\to 0}$ (respectively $\lim_{a\to 0}$) we denote the evaluation of a formal power series at deformation parameter equal to 0.

## 3.1 The classical Poincaré algebra

We recall the definition of a derivation and of a universal enveloping algebra.

**Definition 3.1.1.**
*A* <u>*derivation*</u> *of an associative* $k$*-algebra* $A$ *is a* $k$*-linear map* $D : A \to A$ *satisfying* $D(a_1 a_2) = (Da_1)a_2 + a_1(Da_2)$.

**Lemma 3.1.2.**
*Derivations are determined on generators.*

**Proof.** Arbitrary elements in the $k$-algebra $A$ can be written as the finite sum of finite products of the generators. By $k$-linearity it suffices to consider finite products of the generators. The Lemma is proved by induction on the number of factors in the product.

**Definition 3.1.3.**
*Let* $\mathfrak{g}$ *be a Lie algebra with Lie bracket* $[\ ,\ ]$, $\mathcal{I}$ *the ideal of the tensor algebra* $\mathcal{T}(\mathfrak{g})$ *generated by relations of the form* $x \otimes y - y \otimes x - [x, y] = 0$ *for any* $x, y \in \mathfrak{g}$. *The associative unital algebra*

$$U(\mathfrak{g}) := \mathcal{T}(\mathfrak{g})/\mathcal{I}$$

*with a Lie algebra morphism* $i : \mathfrak{g} \to U(\mathfrak{g})$ *is called* <u>*universal enveloping algebra*</u> *of the Lie algebra* $\mathfrak{g}$.

The universal enveloping algebra of $\mathfrak{g}$ can be endowed with the structure of a Hopf algebra by setting and for the unit

$$\begin{aligned} \Delta 1 &= 1 \otimes 1 \\ \epsilon(1) &= 1 \\ S(1) &= 1 \end{aligned}$$

and for $g, g_1 \dots g_m \in \mathfrak{g}$

$$\begin{aligned}
\Delta(g_1 \dots g_m) &= g_1 \dots g_m \otimes 1 + 1 \otimes g_1 \dots g_m \\
&\quad + \sum_{l=1}^{m-1} \sum_{\sigma} g_{\sigma(1)} \dots g_{\sigma(l)} \otimes g_{\sigma(l+1)} \dots g_{\sigma(m)} \\
S(g_1 \dots g_m) &= (-1)^m g_m \dots g_1 \\
\epsilon(g) &= 0
\end{aligned}$$

with $\sigma$ in the coproduct formula running over all $(p, m-p)$-shuffles of the symmetric group $S_m$.

**Theorem 3.1.4.** *[Bour]*
*Let $k$ be a field of characteristic 0, $\mathfrak{g}$ a Lie algebra over $k$. Then the Lie algebra of primitive elements of $U(\mathfrak{g})$ is equal to the Lie algebra $\mathfrak{g}$*

$$\mathrm{Prim}(U(\mathfrak{g})) = \mathfrak{g}\,.$$

**Proof.** Define $U^n(\mathfrak{g}) := \mathrm{span}_k\{x^n | x \in \mathfrak{g}\}$. Then one has $U(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} U^n(\mathfrak{g})$, and the coproduct of $U(\mathfrak{g})$ preserves the grading because of

$$\Delta((x)^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}$$

for all $x \in \mathfrak{g}$. For an element $u = \sum \lambda_x x^n \in U^n(\mathfrak{g})$ one has

$$\Delta(u) = \sum_{x \in \mathfrak{g}} \lambda_x \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}\,,$$

therefore $u$ is primitive if and only if all components of bidegree $(k, n-k)$ vanish for all $1 \leq k \leq n-1$,

$$\sum_x \lambda_x \binom{n}{k} x^k \otimes x^{n-k} = 0\,.$$

In particular, applying multiplication to the previous expression yields

$$\binom{n}{k} \sum_x \lambda_x x^n = 0\,.$$

Since $k$ is a field of characteristic 0, the element $u$ must be 0. □

In the commutative case, partial derivatives $\partial_\mu := \frac{\partial}{\partial x^\mu}$ are defined to be derivations of the algebra $k[x^1, \dots x^n]$ with action on the variables $x^\mu$

$$\partial_\mu x^\nu = \delta_\mu^\nu\,.$$

Partial derivatives form the abelian Lie algebra of translations on $k[x^1, \dots x^n]$ with Lie brackets $[\partial_\mu, \partial_\nu] = 0$ for all $\mu, \nu \in \{1, \dots n\}$. The universal enveloping algebra of the Lie algebra of partial derivatives is a Hopf algebra. The vector space $k[x^1, \dots x^n]$ is a module over the universal enveloping algebra of the Lie algebra of derivatives.

For the Lie algebra $\mathfrak{so}(n)$ of rotations we choose generators $M^{\mu\nu}$ that are antisymmetric $M^{\mu\nu} = -M^{\nu\mu}$ with Lie brackets

$$[M^{\mu\nu}, M^{\sigma\tau}] \;=\; \delta^{\mu\sigma} M^{\nu\tau} + \delta^{\nu\tau} M^{\mu\sigma} - \delta^{\mu\tau} M^{\nu\sigma} - \delta^{\nu\sigma} M^{\mu\tau} \, .$$

In physical terminology $\mathfrak{so}(n)$ is the complexification of the Lorentz algebra. The semidirect sum of the Lie algebra of partial derivatives and the Lie algebra of rotations is again a Lie algebra, the so-called Poincaré algebra $\mathfrak{p}$. Its Lie brackets are given by

$$[M^{\mu\nu}, \partial_\sigma] \;=\; \delta^\mu_\sigma \partial_\nu - \delta^\nu_\sigma \partial_\mu \, .$$

The universal enveloping algebras of the algebra of rotations and of the Poincaré algebra are Hopf algebras. The affine space $k[x^1, \dots x^n]$ is a module over the universal enveloping algebra of the algebra of rotations and a module over the universal enveloping algebra of the Poincaré algebra via the action of rotations and partial derivatives on $k[x^1, \dots x^n]$

$$M^{\mu\nu} x^\sigma = \delta^{\mu\sigma} x^\nu - \delta^{\nu\sigma} x^\mu \qquad \partial_\mu x^\nu = \delta^\nu_\mu \, .$$

## 3.2 The deformed partial derivatives

Also in the quantum case, partial derivatives should be algebra endomorphisms $\hat{\partial}_\mu : k_\theta[\hat{x}^1, \dots \hat{x}^n] \to k_\theta[\hat{x}^1, \dots \hat{x}^n]$ (respectively $\hat{\partial}_\mu : k_\kappa[\hat{x}^1, \dots \hat{x}^n] \to k_\kappa[\hat{x}^1, \dots \hat{x}^n]$). Additionally, we demand that the following conditions hold:

1. the action on variables $\hat{x}^\mu$ is a deformation of the action of ordinary partial derivatives in the following sense:
   $\lim_{\theta \to 0} \hat{\partial}_\mu(\hat{x}^\nu) = \delta^\nu_\mu$ for the $\theta$-space and
   $\lim_{a \to 0} \hat{\partial}_\mu(\hat{x}^\nu) = \delta^\nu_\mu$ for the $\kappa$-space,

2. partial derivatives $\hat{\partial}_\mu$ form an abelian Lie algebra isomorphic to the classical one, that is $[\hat{\partial}_\mu, \hat{\partial}_\nu]$=0. This fixes the algebra structure on the deformed universal enveloping algebra of $k_\theta[\hat{x}^1, \dots \hat{x}^n]$ (respectively $k_\kappa[\hat{x}^1, \dots \hat{x}^n]$).

3. the coalgebra structure of the deformed universal enveloping algebra of $k_\theta[\hat{x}^1, \dots \hat{x}^n]$ (respectively $k_\kappa[\hat{x}^1, \dots \hat{x}^n]$) is deformed, such that the Hopf algebra axioms hold,

4. the quantum space $k_\theta[\hat{x}^1, \dots \hat{x}^n]$ (respectively $k_\kappa[\hat{x}^1, \dots \hat{x}^n]$) is a module over the deformed universal enveloping algebra of the Lie algebra of partial derivatives $\hat{\partial}_\mu$.

As we shall see in the following, the partial derivatives for the $\kappa$-space are not derivations, since their coproduct is deformed. Hence, partial derivatives are not uniquely determined by conditions 1-4. Additionally, the deformed universal enveloping algebra of the Lie algebra of partial derivatives $\hat{\partial}_\mu$ is not necessarily isomorphic as a

Hopf algebra to the universal enveloping algebra of the Lie algebra of classical partial derivatives, although the corresponding Lie algebras are isomorphic by definition.

We define a grading in the algebra $k_\theta[\hat{x}^1, \dots \hat{x}^n]$ (respectively $k_\kappa[\hat{x}^1, \dots \hat{x}^n]$) by setting any variable $\hat{x}^\mu$ to be of degree 1. In the Lie algebra of derivatives we set derivatives to be of degree -1. Because of conditions (3.1) and (3.2) the deformation parameter $\theta^{\mu\nu}$ is of degree 2 and the deformation parameter $a$ is of degree 1. Since the action of partial derivatives on variables $\hat{\partial}_\mu(\hat{x}^\nu)$ is of degree 0 both for the $\theta$-space and for the $\kappa$-space, it cannot depend on the deformation parameter $\theta$ (respectively $a$). Hence, we have an undeformed action of partial derivatives on variables

$$\hat{\partial}_\mu(\hat{x}^\nu) = \delta^\nu_\mu$$

both for the $\theta$-space and for the $\kappa$-space.

In physical terminology partial derivatives are considered as the generators of infinitesimal translations, in particular the impulsion operator is defined to be $\mathrm{i}\hat{\partial}_\mu$. Motivated by the commutative case we extend the involution $\dagger$ to the Lie algebra of partial derivatives by demanding that $\mathrm{i}\hat{\partial}_\mu$ is invariant. Thus, $\hat{\partial}^\dagger_\mu = -\hat{\partial}_\mu$.

## Partial derivatives for the $\theta$-space

**Theorem 3.2.1.**
*There exists a unique deformed universal enveloping algebra of the Lie algebra of partial derivatives on the $\theta$-space satisfying condition 1-4. The unique Hopf algebra structure is*

$$\begin{aligned} \Delta\hat{\partial}_\mu &= \hat{\partial}_\mu \otimes 1 + 1 \otimes \hat{\partial}_\mu \\ \epsilon(\hat{\partial}_\mu) &= 0 \\ S(\hat{\partial}_\mu) &= -\hat{\partial}_\mu\,. \end{aligned} \tag{3.3}$$

**Proof.** For the coproduct of partial derivatives we make the following most general ansatz

$$\begin{aligned} \Delta\hat{\partial}_\mu = {} & \hat{\partial}_\mu \otimes 1 + 1 \otimes \hat{\partial}_\mu + c_1\theta^{\mu\sigma}\hat{\partial}_\sigma\hat{\partial}_\rho\hat{\partial}_\rho \otimes 1 + c_2\theta^{\mu\sigma}\hat{\partial}_\rho\hat{\partial}_\rho \otimes \hat{\partial}_\sigma + c_3\theta^{\mu\sigma}\hat{\partial}_\sigma\hat{\partial}_\rho \otimes \hat{\partial}_\rho \\ & + c_4\theta^{\mu\sigma}\hat{\partial}_\sigma \otimes \hat{\partial}_\rho\hat{\partial}_\rho + c_5\theta^{\mu\sigma}\hat{\partial}_\rho \otimes \hat{\partial}_\sigma\hat{\partial}_\rho + c_6\theta^{\mu\sigma} 1 \otimes \hat{\partial}_\sigma\hat{\partial}_\rho\hat{\partial}_\rho \end{aligned}$$

with coefficients $c_i \in \mathbb{C}$ which is compatible with the grading and yields the classical coproduct when setting $\theta = 0$. Higher order terms in $\theta$ are not admitted in this ansatz because $\theta$ is an antisymmetric matrix and the partial derivatives $\hat{\partial}_\mu$ commute.

We apply the ansatz from above to relation (3.1)

$$\hat{\partial}_\mu(\hat{x}^\rho\hat{x}^\sigma - \hat{x}^\sigma\hat{x}^\rho - \mathrm{i}\theta^{\rho\sigma}) = \delta^\rho_\mu\hat{x}^\sigma + \delta^\sigma_\mu\hat{x}^\rho - \delta^\sigma_\mu\hat{x}^\rho - \delta^\rho_\mu\hat{x}^\sigma = 0,$$

this does not imply any restriction on the coefficients $c_i$. Application on monomials of degree 3 yields

$$\begin{aligned} \hat{\partial}_\mu(\hat{x}^\alpha(\hat{x}^\beta\hat{x}^\gamma)) &= \delta^\alpha_\mu\hat{x}^\beta\hat{x}^\gamma + \delta^\beta_\mu\hat{x}^\alpha\hat{x}^\gamma + \delta^\gamma_\mu\hat{x}^\alpha\hat{x}^\beta + 2c_4\theta^{\mu\alpha}\delta^{\beta\gamma} + c_5\theta^{\mu\beta}\delta^{\alpha\gamma} \\ &\quad + c_5\theta^{\mu\gamma}\delta^{\alpha\beta} \\ \hat{\partial}_\mu((\hat{x}^\alpha\hat{x}^\beta)\hat{x}^\gamma) &= \delta^\alpha_\mu\hat{x}^\beta\hat{x}^\gamma + \delta^\beta_\mu\hat{x}^\alpha\hat{x}^\gamma + \delta^\gamma_\mu\hat{x}^\alpha\hat{x}^\beta + c_3\theta^{\mu\alpha}\delta^{\beta\gamma} + c_3\theta^{\mu\beta}\delta^{\alpha\gamma} \\ &\quad + 2c_2\theta^{\mu\gamma}\delta^{\alpha\beta}\,. \end{aligned}$$

The condition $\hat{\partial}_\mu(\hat{x}^\alpha(\hat{x}^\beta\hat{x}^\gamma)) = \hat{\partial}_\mu((\hat{x}^\alpha\hat{x}^\beta)\hat{x}^\gamma)$ implies $2c_2 = c_3 = 2c_4 = c_5$. We proceed analogously for monomials of degree 4 and find $c_1 = c_6 = 0$. It is easy to see that monomials of higher degree give no additional conditions.

We insert the previous ansatz into the coassociativity condition $(\Delta\otimes\mathrm{id})\circ\Delta \stackrel{!}{=} (\mathrm{id}\otimes\Delta)\circ\Delta$. In the lengthy expression terms up to third order in $\theta$ appear. Comparing coefficients for example of terms of the form $\hat{\partial}\hat{\partial}\otimes\hat{\partial}\hat{\partial}\otimes\hat{\partial}$ yields that the coefficient $c_3$ must be zero.

In order to endow the $\theta$-deformed universal enveloping algebra of partial derivatives with the structure of a Hopf algebra we must define a counit and an antipode. The counit $\epsilon$ must be a morphism of algebras that obeys

$$(\epsilon\otimes\mathrm{id})(\Delta\hat{\partial}_\mu) = (\mathrm{id}\otimes\epsilon)(\Delta\hat{\partial}_\mu) = \hat{\partial}_\mu\,.$$

We assume that $\epsilon(1) = 1$ as in the commutative case and compute $\hat{\partial}_\mu \stackrel{!}{=} \epsilon(\hat{\partial}_\mu) + \hat{\partial}_\mu$, hence the unique solution for the counit is

$$\epsilon(\hat{\partial}_\mu) = 0\,.$$

The antipode $S$ must be an antihomomorphism of algebras that obeys

$$\mu\circ(S\otimes\mathrm{id})\circ\Delta = \mu\circ(\mathrm{id}\otimes S)\circ\Delta = \epsilon\circ\eta\,.$$

We assume that $S(1) = 1$ as in the commutative case and compute $0 \stackrel{!}{=} S(\hat{\partial}_\mu) + \hat{\partial}_\mu$, therefore, the unique solution for the antipode is

$$S(\hat{\partial}_\mu) = -\hat{\partial}_\mu\,. \qquad \square$$

**Proposition 3.2.2.**
*The Hopf algebra of deformed partial derivatives is isomorphic to the Hopf algebra of partial derivatives on the commutative space $k[x^1,\dots x^n]$.*

**Proof.** The map $\hat{\partial}_\mu \mapsto \partial_\mu$ is clearly a Hopf algebra isomorphism. $\square$

## Partial derivatives for the $\kappa$-space

There exists a wide range of solutions for the $\kappa$-deformed partial derivatives (compare with our publication [DMT04]). For a distinguished solution we demand additionally that

5. the coproduct of $\hat{\partial}_n$ remains undeformed
6. any additional terms in the coproduct of $\hat{\partial}_i$ are of the form $(a\hat{\partial}_n)^m$.

In the following, we derive the simplest solution for a deformed universal enveloping algebra of the Lie algebra of partial derivatives on the $\kappa$-space. It is a completion of the universal enveloping algebra of the Lie algebra of partial derivatives for which terms of the form $\exp(cia\hat{\partial}_n)$ for $c\in\mathbb{R}$ exist.

**Theorem 3.2.3.**
*The simplest solution for a deformed universal enveloping algebra of the Lie algebra of partial derivatives on the $\kappa$-space satisfying conditions 1-6 has the following Hopf algebra structure depending on the parameter $b \in \mathbb{C}$*

$$\begin{aligned} \Delta\hat{\partial}_n &= \hat{\partial}_n \otimes 1 + 1 \otimes \hat{\partial}_n \\ \Delta\hat{\partial}_j &= \hat{\partial}_j \otimes \exp(\mathrm{i}ab\hat{\partial}_n) + \exp(\mathrm{i}a(b+1)\hat{\partial}_n) \otimes \hat{\partial}_j \\ \epsilon(\hat{\partial}_\mu) &= 0 \\ S(\hat{\partial}_n) &= -\hat{\partial}_n \\ S(\hat{\partial}_j) &= -\hat{\partial}_j \exp(-\mathrm{i}a(2b+1)\hat{\partial}_n)\,. \end{aligned} \tag{3.4}$$

**Proof.** Proceeding as for the $\theta$-space, we compute the action of partial derivatives on monomials in $k_\kappa[\hat{x}^1, \dots \hat{x}^n]$. One sees that it is not sufficient to use only finite sums of partial derivatives in an ansatz for the coproduct. Hence, we consider the algebra of formal power series in the partial derivatives. In order to find the simplest solution we make the following ansatz which is compatible with the grading

$$\begin{aligned} \Delta\hat{\partial}_n &= \hat{\partial}_n \otimes 1 + 1 \otimes \hat{\partial}_n \\ \Delta\hat{\partial}_i &= \sum_{m=0}^{\infty} b_m \hat{\partial}_i \otimes (\mathrm{i}a\hat{\partial}_n)^m + c_m (\mathrm{i}a\hat{\partial}_n)^m \otimes \hat{\partial}_i \end{aligned}$$

with coefficients $b_m, c_m \in \mathbb{C}$. It yields the classical coproduct when setting $a = 0$ if and only if $b_0 = c_0 = 1$.

We apply the ansatz from above to relation (3.2)

$$\hat{\partial}_i(\hat{x}^n\hat{x}^s - \hat{x}^s\hat{x}^n - \mathrm{i}a\hat{x}^s) = (c_1 - b_1 - 1)\mathrm{i}a\delta_i^s = 0\,,$$

this implies $c_1 - b_1 = 1$. As for the $\theta$-space we consider the action on monomials. Application on monomials of degree 3 yields

$$\begin{aligned} \hat{\partial}_i(\hat{x}^n(\hat{x}^n\hat{x}^j)) &= \delta_i^j(\hat{x}^n + \mathrm{i}ac_1)^2 \\ \hat{\partial}_i((\hat{x}^n\hat{x}^n)\hat{x}^j) &= \delta_i^j(\hat{x}^n\hat{x}^n + 2\mathrm{i}ac_1\hat{x}^n + 2(\mathrm{i}a)^2c_2)\,. \end{aligned}$$

The condition $\hat{\partial}_\mu(\hat{x}^\alpha(\hat{x}^\beta\hat{x}^\gamma)) = \hat{\partial}_\mu((\hat{x}^\alpha\hat{x}^\beta)\hat{x}^\gamma)$ implies $2c_2 = c_1^2$. We proceed analogously for monomials of degree 4 and find $6c_3 = c_1^3$. By induction one can show

$$c_m = \frac{1}{m!}c_1^m\,.$$

By considering monomials with $\hat{x}^n$ on the furthest right one finds $b_m = \frac{1}{m!}b_1^m$. With the notation $b := b_0$ there exists a one-parameter family of deformed universal enveloping algebras of partial derivatives with coproduct

$$\begin{aligned} \Delta\hat{\partial}_n &= \hat{\partial}_n \otimes 1 + 1 \otimes \hat{\partial}_n \\ \Delta\hat{\partial}_j &= \hat{\partial}_j \otimes \exp(\mathrm{i}ab\hat{\partial}_n) + \exp(\mathrm{i}a(b+1)\hat{\partial}_n) \otimes \hat{\partial}_j\,. \end{aligned}$$

Coassociativity can be checked straightforwardly using

$$\begin{aligned}
\Delta \exp(\mathrm{i}ab\hat{\partial}_n) &= \sum_m \frac{(\mathrm{i}ab)^m}{m!}\Delta(\hat{\partial}_n)^m \\
&= \sum_m \sum_{l=0}^{m} \frac{(\mathrm{i}ab)^m}{m!} \begin{pmatrix} m \\ l \end{pmatrix} (\hat{\partial}_n)^l \otimes (\hat{\partial}_n)^{m-l} \\
&= \exp(\mathrm{i}ab\hat{\partial}_n) \otimes \exp(\mathrm{i}ab\hat{\partial}_n)\,. \qquad (3.5)
\end{aligned}$$

It is easy to see that the counit and the antipode are

$$\epsilon(\hat{\partial}_\mu) = 0 \qquad S(\hat{\partial}_n) = -\hat{\partial}_n \qquad S(\hat{\partial}_j) = -\hat{\partial}_j \exp(-\mathrm{i}a(2b+1)\hat{\partial}_n)\,.$$

□

**Proposition 3.2.4.**
*The Hopf algebra of deformed partial derivatives for the $\kappa$-space is not isomorphic to the Hopf algebra of partial derivatives on the commutative space $k[x^1, \dots x^n]$.*

**Proof.** Assume that there exists a Hopf algebra isomorphism $i$ from the Hopf algebra of partial derivatives on the commutative space $k[x^1, \dots x^n]$ to the Hopf algebra of deformed partial derivatives. In particular, it is an algebra morphism $i(\partial_\mu \partial_\nu) = i(\partial_\mu) i(\partial_\nu)$ and $i(1) = 1$ and obeys

$$\Delta(i(\partial_j)) \stackrel{!}{=} i(\Delta \partial_j) \;=\; i(\partial_j)\otimes 1 + 1 \otimes i(\partial_j).$$

From the coproduct formula (3.4) it is obvious that no element in the $\kappa$-deformed universal enveloping algebra can have such a coproduct. □

## 3.3 The deformed Poincaré algebra

We want to find a quantum version of the universal enveloping algebra of the Poincaré algebra $\mathfrak{p}$ for a given quantum space. Therefore, deformed rotations $\hat{M}^{\mu\nu}$ are regarded as endomorphisms of the quantum space $k_\theta[\hat{x}^1, \dots \hat{x}^n]$ (respectively $k_\kappa[\hat{x}^1, \dots \hat{x}^n]$). We demand that:

1. the action on variables $\hat{x}^\mu$ is a deformation of the action of ordinary rotations in the following sense:
   $\lim_{\theta \to 0} \hat{M}^{\mu\nu}(\hat{x}^\rho) = \delta^{\mu\rho}\hat{x}^\nu - \delta^{\nu\rho}\hat{x}^\mu$ for the $\theta$-space and
   $\lim_{a \to 0} \hat{M}^{\mu\nu}(\hat{x}^\rho) = \delta^{\mu\rho}\hat{x}^\nu - \delta^{\nu\rho}\hat{x}^\mu$ for the $\kappa$-space,

2. the deformed universal enveloping algebra forms a Lie algebra with the commutator as a Lie bracket. It shall be a deformation of the classical Poincaré algebra $\mathfrak{p}$

$$\begin{aligned}
\lim_{\theta\to 0}[\hat{M}^{\mu\nu}, \hat{M}^{\sigma\tau}] &= \delta^{\mu\sigma}\hat{M}^{\nu\tau} + \delta^{\nu\tau}\hat{M}^{\mu\sigma} - \delta^{\mu\tau}\hat{M}^{\nu\sigma} - \delta^{\nu\sigma}\hat{M}^{\mu\tau} \\
\lim_{\theta\to 0}[\hat{M}^{\mu\nu}, \hat{\partial}_\sigma] &= \delta^\mu_\sigma \hat{\partial}_\nu - \delta^\nu_\sigma \hat{\partial}_\mu
\end{aligned}$$

and respectively for $\kappa \to 0$,

3. the coalgebra structure is deformed, such that the Hopf algebra axioms hold,
4. the quantum space $k_\theta[\hat{x}^1, \dots \hat{x}^n]$ (respectively $k_\kappa[\hat{x}^1, \dots \hat{x}^n]$) is a module over the deformed universal enveloping algebra of the Lie algebra of rotations.

In the semidirect sum of $k_\theta[\hat{x}^1, \dots \hat{x}^n]$ (respectively $k_\kappa[\hat{x}^1, \dots \hat{x}^n]$) and the Poincaré algebra we set the rotations to be of degree 0. Since the action of rotations on variables $\hat{M}^{\rho\sigma}(\hat{x}^\nu)$ is of degree 0, both for the $\theta$-space and for the $\kappa$-space, it cannot depend on the deformation parameter $\theta$ (respectively $\kappa$). Hence, we have an undeformed action of rotations on variables

$$\hat{M}^{\mu\nu}(\hat{x}^\rho) = \delta^{\mu\rho}\hat{x}^\nu - \delta^{\nu\rho}\hat{x}^\mu$$

both for the $\theta$-space and for the $\kappa$-space.

As for the partial derivatives we extend the involution $\dagger$ to the Lie algebra of rotations by demanding $(\hat{M}^{\mu\nu})^\dagger = \hat{M}^{\mu\nu}$.

## The $\theta$-Poincaré algebra

We restrict ourselves to the simplest solutions for the deformed universal enveloping algebra of the Poincaré algebra:

5a. additional terms are linear in $\theta^{\mu\nu}$ and do not contain rotations.

**Theorem 3.3.1.**
*For the $\theta$-space there exists a unique two-parameter family of deformed universal enveloping algebras of the Poincaré algebra satisfying conditions 1-5a with Lie brackets of the form*

$$\begin{aligned}
[\hat{M}^{\mu\nu}, \hat{\partial}_\rho] &= \delta^{\mu\rho}\hat{\partial}_\nu - \delta^{\nu\rho}\hat{\partial}_\mu \\
[\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}] &= \delta^{\mu\rho}\hat{M}^{\nu\sigma} + \delta^{\nu\sigma}\hat{M}^{\mu\rho} - \delta^{\mu\sigma}\hat{M}^{\nu\rho} - \delta^{\nu\rho}\hat{M}^{\mu\sigma} \\
&\quad + \mathrm{i}\,(1-2\lambda_1)\Big(\theta^{\mu\rho}\hat{\partial}_\nu\hat{\partial}_\sigma - \theta^{\nu\rho}\hat{\partial}_\mu\hat{\partial}_\sigma - \theta^{\mu\sigma}\hat{\partial}_\nu\hat{\partial}_\rho + \theta^{\nu\sigma}\hat{\partial}_\mu\hat{\partial}_\rho\Big) \\
&\quad - \mathrm{i}\lambda_2\,(\theta^{\mu\rho}\delta^{\nu\sigma} - \theta^{\nu\rho}\delta^{\mu\sigma} - \theta^{\mu\sigma}\delta^{\nu\rho} + \theta^{\nu\sigma}\delta^{\mu\rho})\,\hat{\partial}_\alpha\hat{\partial}_\alpha
\end{aligned} \tag{3.6}$$

*with parameters $\lambda_1, \lambda_2 \in \mathbb{R}$.*

**Proof.** The most general ansatz for the Lie brackets compatible with the grading is of the form

$$\begin{aligned}
[\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}] &= \delta^{\mu\rho}\hat{M}^{\nu\sigma} + \delta^{\nu\sigma}\hat{M}^{\mu\rho} - \delta^{\mu\sigma}\hat{M}^{\nu\rho} - \delta^{\nu\rho}\hat{M}^{\mu\sigma} \\
&\quad + e_1(\theta^{\mu\rho}\hat{\partial}_\nu\hat{\partial}_\sigma - \theta^{\nu\rho}\hat{\partial}_\mu\hat{\partial}_\sigma + \theta^{\nu\sigma}\hat{\partial}_\mu\hat{\partial}_\rho - \theta^{\mu\sigma}\hat{\partial}_\nu\hat{\partial}_\rho) \\
&\quad + e_2(\theta^{\mu\rho}\delta^{\nu\sigma} - \theta^{\nu\rho}\delta^{\mu\sigma} + \theta^{\nu\sigma}\delta^{\mu\rho} - \theta^{\mu\sigma}\delta^{\nu\rho})\hat{\partial}_\tau\hat{\partial}_\tau \\
&\quad + e_3(\theta^{\mu\alpha}\delta^{\nu\sigma}\hat{\partial}_\alpha\hat{\partial}_\rho - \theta^{\mu\alpha}\delta^{\nu\rho}\hat{\partial}_\alpha\hat{\partial}_\sigma + \theta^{\nu\alpha}\delta^{\mu\rho}\hat{\partial}_\alpha\hat{\partial}_\sigma - \theta^{\nu\alpha}\delta^{\mu\sigma}\hat{\partial}_\alpha\hat{\partial}_\rho \\
&\qquad + \theta^{\rho\alpha}\delta^{\mu\sigma}\hat{\partial}_\alpha\hat{\partial}_\nu - \theta^{\rho\alpha}\delta^{\nu\sigma}\hat{\partial}_\alpha\hat{\partial}_\mu + \theta^{\sigma\alpha}\delta^{\nu\rho}\hat{\partial}_\alpha\hat{\partial}_\mu - \theta^{\sigma\alpha}\delta^{\mu\rho}\hat{\partial}_\alpha\hat{\partial}_\nu) \\
[\hat{M}^{\mu\nu}, \hat{\partial}_\rho] &= \delta^\mu_\rho\hat{\partial}_\nu - \delta^\nu_\rho\hat{\partial}_\mu + d_1\theta^{\mu\nu}\hat{\partial}_\rho\hat{\partial}_\sigma\hat{\partial}_\sigma + d_2(\theta^{\mu\sigma}\hat{\partial}_\nu - \theta^{\nu\sigma}\hat{\partial}_\mu)\hat{\partial}_\sigma\hat{\partial}_\rho \\
&\quad + d_3(\theta^{\mu\rho}\hat{\partial}_\nu - \theta^{\nu\rho}\hat{\partial}_\mu)\hat{\partial}_\sigma\hat{\partial}_\sigma
\end{aligned}$$

with coefficients $e_i, d_i \in \mathbb{C}$. For the Jacobi identity for partial derivatives and rotations

$$0 = [[\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}], \hat{\partial}_\lambda] + [[\hat{M}^{\rho\sigma}, \hat{\partial}_\lambda], \hat{M}^{\mu\nu}] + [[\hat{\partial}_\lambda, \hat{M}^{\mu\nu}], \hat{M}^{\rho\sigma}]$$

to be satisfied, the coefficients $d_1, d_2, d_3$ must vanish. Thus, the Lie brackets with the partial derivatives are undeformed

$$[\hat{M}^{\mu\nu}, \hat{\partial}_\rho] = \delta^{\mu\rho}\hat{\partial}_\nu - \delta^{\nu\rho}\hat{\partial}_\mu \,.$$

A lengthy, but straightforward calculation for the Jacobi identity for rotations

$$0 = [[\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}], \hat{M}^{\alpha\beta}] + [[\hat{M}^{\rho\sigma}, \hat{M}^{\alpha\beta}], \hat{M}^{\mu\nu}] + [[\hat{M}^{\alpha\beta}, \hat{M}^{\mu\nu}], \hat{M}^{\rho\sigma}]$$

yields the unique set of solutions labelled by the parameters $\lambda_1, \lambda_2$

$$\begin{aligned} [\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}] &= \delta^{\mu\rho}\hat{M}^{\nu\sigma} + \delta^{\nu\sigma}\hat{M}^{\mu\rho} - \delta^{\mu\sigma}\hat{M}^{\nu\rho} - \delta^{\nu\rho}\hat{M}^{\mu\sigma} \\ &\quad + \mathrm{i}\,(1-2\lambda_1)\left(\theta^{\mu\rho}\hat{\partial}_\nu\hat{\partial}_\sigma - \theta^{\nu\rho}\hat{\partial}_\mu\hat{\partial}_\sigma - \theta^{\mu\sigma}\hat{\partial}_\nu\hat{\partial}_\rho + \theta^{\nu\sigma}\hat{\partial}_\mu\hat{\partial}_\rho\right) \\ &\quad - \mathrm{i}\lambda_2\left(\theta^{\mu\rho}\delta^{\nu\sigma} - \theta^{\nu\rho}\delta^{\mu\sigma} - \theta^{\mu\sigma}\delta^{\nu\rho} + \theta^{\nu\sigma}\delta^{\mu\rho}\right)\hat{\partial}_\alpha\hat{\partial}_\alpha \,. \end{aligned}$$

□

The solution $(\lambda_1 = \frac{1}{2}, \lambda_2 = 0)$ gives the classical relations for the Lie algebra of rotations, but with deformed coproduct for the $\theta$-deformed universal enveloping algebra - as we shall see later. This result was also obtained by alternative considerations [CKNT04].

For the $\theta$-space the action of rotations on the variables $\hat{x}^\mu$ cannot be deformed because $\hat{M}^{\mu\nu}$ has grading 0.

**Theorem 3.3.2.**
*For the $\theta$-space the two-parameter family of deformed universal enveloping algebras of the Poincaré algebra given in the previous theorem is a Hopf algebra with coproduct*

$$\begin{aligned} \Delta\hat{\partial}_\mu &= \hat{\partial}_\mu \otimes 1 + 1 \otimes \hat{\partial}_\mu \\ \Delta\hat{M}^{\mu\nu} &= \hat{M}^{\mu\nu} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{M}^{\mu\nu} - \mathrm{i}(1-\lambda_1)\hat{\partial}_\alpha \otimes \left(\theta^{\mu\alpha}\hat{\partial}_\nu - \theta^{\nu\alpha}\hat{\partial}_\mu\right) \\ &\quad + \mathrm{i}\lambda_1\left(\theta^{\mu\alpha}\hat{\partial}_\nu - \theta^{\nu\alpha}\hat{\partial}_\mu\right) \otimes \hat{\partial}_\alpha + 2\mathrm{i}\lambda_2\theta^{\mu\nu}\hat{\partial}_\alpha \otimes \hat{\partial}_\alpha \,, \end{aligned}$$

*unique counit and antipode*

$$\begin{aligned} \epsilon(\hat{M}^{\mu\nu}) &= 0 \\ S(\hat{M}^{\mu\nu}) &= -\hat{M}^{\mu\nu} - \mathrm{i}(1-2\lambda_1)(\theta^{\mu\alpha}\hat{\partial}_\nu - \theta^{\nu\alpha}\hat{\partial}_\mu)\hat{\partial}_\alpha + 2\mathrm{i}\lambda_2\theta^{\mu\nu}\hat{\partial}_\alpha\hat{\partial}_\alpha \,. \end{aligned}$$

**Proof.** For the coproduct of rotations we make the following ansatz

$$\begin{aligned} \Delta\hat{M}^{\mu\nu} &= \hat{M}^{\mu\nu} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{M}^{\mu\nu} + b_1(\theta^{\mu\rho}\hat{\partial}_\nu - \theta^{\nu\rho}\hat{\partial}_\mu)\partial_\rho \otimes \mathbb{1} \\ &\quad + b_2(\theta^{\mu\rho}\hat{\partial}_\nu - \theta^{\nu\rho}\hat{\partial}_\mu) \otimes \partial_\rho + b_3\partial_\rho \otimes (\theta^{\mu\rho}\hat{\partial}_\nu - \theta^{\nu\rho}\hat{\partial}_\mu) \\ &\quad + b_4\mathbb{1} \otimes (\theta^{\mu\rho}\hat{\partial}_\rho\hat{\partial}_\nu - \theta^{\nu\rho}\hat{\partial}_\rho\hat{\partial}_\mu) + b_5\theta^{\mu\nu}\hat{\partial}_\rho\hat{\partial}_\rho \otimes \mathbb{1} \\ &\quad + b_6\theta^{\mu\nu}\hat{\partial}_\rho \otimes \hat{\partial}_\rho + b_7\theta^{\mu\nu}\mathbb{1} \otimes \hat{\partial}_\rho\hat{\partial}_\rho \end{aligned}$$

with coefficients $b_i \in \mathbb{C}$. In the limit $\theta \to 0$ we obtain the classical coproduct. Higher order terms in $\theta$ are not admitted in this ansatz because $\theta$ is an antisymmetric matrix and the partial derivatives $\hat{\partial}_\mu$ commute.

Applying the ansatz from above to (3.1) yields

$$\hat{M}^{\mu\nu}([\hat{x}^\alpha, \hat{x}^\beta - \mathrm{i}\theta^{\alpha\beta}]) = (b_2 - b_3 - \mathrm{i})(\theta^{\mu\beta}\delta^{\mu\alpha} - \theta^{\nu\beta}\delta^{\mu\alpha} + \theta^{\nu\alpha}\delta^{\mu\beta} - \theta^{\mu\alpha}\delta^{\nu\beta}) \stackrel{!}{=} 0\,.$$

Thus $b_2 - b_3 = \mathrm{i}$. Application on monomials of degree 3 implies that all other coefficients except $b_5$ are 0. It is easy to see that monomials of higher degree give no additional conditions.

By definition the coproduct is an algebra morphism, in particular, in the case of a universal enveloping algebra of a Lie algebra $\mathfrak{g}$ the condition

$$\Delta([i(x), i(y)]) = [\Delta i(x), \Delta i(y)]$$

must hold for any $x, y \in \mathfrak{g}$. For the $\theta$-deformed universal enveloping algebra of the Poincaré algebra we apply the coproduct on (3.6) and obtain $b_2 = \mathrm{i}\lambda_1$, $b_3 = \mathrm{i}(\lambda_1 - 1)$, $b_5 = 2\mathrm{i}\lambda_2$. Thus, the coproducts of the $\theta$-deformed universal enveloping algebra of the Poincaré algebra are restricted to be

$$\begin{aligned}
\Delta\hat{\partial}_\mu &= \hat{\partial}_\mu \otimes 1 + 1 \otimes \hat{\partial}_\mu \\
\Delta\hat{M}^{\mu\nu} &= \hat{M}^{\mu\nu} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{M}^{\mu\nu} - \mathrm{i}(1-\lambda_1)\hat{\partial}_\alpha \otimes \left(\theta^{\mu\alpha}\hat{\partial}_\nu - \theta^{\nu\alpha}\hat{\partial}_\mu\right) \\
&\quad + \mathrm{i}\lambda_1 \left(\theta^{\mu\alpha}\hat{\partial}_\nu - \theta^{\nu\alpha}\hat{\partial}_\mu\right) \otimes \hat{\partial}_\alpha + 2\mathrm{i}\lambda_2\theta^{\mu\nu}\hat{\partial}_\alpha \otimes \hat{\partial}_\alpha\,.
\end{aligned}$$

We demanded that the $\theta$-deformed universal enveloping algebra of the Poincaré algebra is a Hopf algebra as in the commutative case. Similarly to the case of the partial derivatives, it is easy to compute the unique solution for the counit and the antipode

$$\begin{aligned}
\epsilon(\hat{M}^{\mu\nu}) &= 0 \\
\mathrm{S}(\hat{M}^{\mu\nu}) &= -\hat{M}^{\mu\nu} - \mathrm{i}(1-2\lambda_1)(\theta^{\mu\alpha}\hat{\partial}_\nu - \theta^{\nu\alpha}\hat{\partial}_\mu)\hat{\partial}_\alpha + 2\mathrm{i}\lambda_2\theta^{\mu\nu}\hat{\partial}_\alpha\hat{\partial}_\alpha\,.
\end{aligned}$$

□

We remark that $\mathrm{S}^2 = \mathrm{id}$ for the $\theta$-deformed universal enveloping algebras of the Poincaré algebra.

Similarly to Proposition 3.2.4, one can show that the Hopf algebra of deformed rotations is not isomorphic to the Hopf algebra of rotations on the commutative space $k[x^1, \ldots x^n]$.

In order to endow the algebra $k_\theta[\hat{x}^1, \ldots \hat{x}^n]$ with the structure of a module over the Lie algebra of rotations we must find an algebra homomorphism $\phi$ from the Lie algebra of rotations into $\mathrm{End}_k(k_\theta[\hat{x}^1, \ldots \hat{x}^n])$. The most general ansatz linear in $\theta$ and compatible with the grading is of the form

$$\phi(\hat{M}^{\mu\nu}) = \hat{x}^\nu\hat{\partial}_\mu - \hat{x}^\mu\hat{\partial}_\nu + \mathrm{i}f_1(\theta^{\mu\alpha}\hat{\partial}_\nu - \theta^{\nu\alpha}\hat{\partial}_\mu)\hat{\partial}_\alpha + \mathrm{i}f_2\theta^{\mu\nu}\hat{\partial}_\alpha\hat{\partial}_\alpha$$

with coefficients $f_1, f_2 \in \mathbb{R}$ and it is easy to compute that $\phi$ is an algebra homomorphism if and only if $f_1 = \lambda_1$ and $f_2 = \lambda_2$.

## The $\kappa$-Poincaré algebra

In the following, we denote the generators of the Lie algebra of rotations by $\hat{M}^{rs}$ and $\hat{N}^l := \hat{M}^{nl}$ for $r, s, l \in \{1, \ldots n-1\}$. The $\kappa$-deformed universal enveloping algebra of the Poincaré algebra is a completion of the universal enveloping algebra of the Poincaré algebra for which terms of the form $\exp(cia\hat{\partial}_n)$ for $c \in \mathbb{R}$ exist. We consider only the simplest $\kappa$-deformed universal enveloping algebra of the Poincaré algebra for which

5b. the Lie brackets of rotations are undeformed

$$\begin{aligned} [\hat{M}^{rs}, \hat{M}^{tu}] &= \delta^{rt}\hat{M}^{su} + \delta^{su}\hat{M}^{rt} - \delta^{ru}\hat{M}^{st} - \delta^{st}\hat{M}^{ru} \\ [\hat{M}^{rs}, \hat{N}^{i}] &= \delta^{ri}\hat{N}^{s} - \delta^{si}\hat{N}^{r} \\ [\hat{N}^{i}, \hat{N}^{j}] &= \hat{M}^{ij} \end{aligned} \tag{3.7}$$

6. only the mixed Lie brackets $[\hat{N}^l, \hat{\partial}_i]$ are deformed

$$\begin{aligned} [\hat{M}^{rs}, \hat{\partial}_n] &= 0 \\ [\hat{M}^{rs}, \hat{\partial}_j] &= \delta^r_j\hat{\partial}_s - \delta^s_j\hat{\partial}_r \\ [\hat{N}^{l}, \hat{\partial}_n] &= \hat{\partial}_l \end{aligned}$$

and additional terms including $\hat{\partial}_s$ appear only up to first order in $a$

7. only the coproduct of $\hat{N}^r$ is deformed

$$\Delta\hat{M}^{rs} = \hat{M}^{rs} \otimes 1 + 1 \otimes \hat{M}^{rs}$$

and additional terms include $\hat{\partial}_s$ and $\hat{M}^{rs}$ linearly.

**Proposition 3.3.3.**
*There exist two unique one-parameter families of Lie algebras satisfying conditions 2 and 5b and 6. The mixed Lie brackets are*

$$[\hat{N}^l, \hat{\partial}_j] = \delta^l_j \frac{1 - \exp(\mathrm{i}a(2e-f)\hat{\partial}_n)}{\mathrm{i}a(2e-f)} + \delta^l_j \mathrm{i}ae\hat{\partial}_k\hat{\partial}_k + \mathrm{i}af\hat{\partial}_l\hat{\partial}_j$$

*with $e = 0$ or $e = -\frac{1}{2}f$ and parameter $f \in \mathbb{R}$.*

**Proof.** We make the following ansatz which is compatible with the grading and obeys condition 2

$$[\hat{N}^l, \hat{\partial}_j] = -\delta^l_j\hat{\partial}_n \sum_{m=0}^{\infty} d_m(\mathrm{i}a\hat{\partial}_n)^m + \delta^l_j\mathrm{i}ae\hat{\partial}_k\hat{\partial}_k + \mathrm{i}af\hat{\partial}_l\hat{\partial}_j$$

with parameters $d_m, e, f \in \mathbb{R}$ and $d_0 = 1$. It is easy to see that the Jacobi identity for $\hat{N}^l, \hat{\partial}_i, \hat{\partial}_n$, for $\hat{N}^l, \hat{\partial}_i, \hat{\partial}_j$ and for $\hat{M}^{rs}, \hat{N}^l, \hat{\partial}_i$ is satisfied for any choice

of $d_m, e, f$.
We compute the brackets $[\hat{N}^l, \hat{\partial}_j\hat{\partial}_j]$ and $[\hat{N}^l, \hat{\partial}_j\hat{\partial}_i]$

$$\begin{aligned}
[\hat{N}^l, \hat{\partial}_j\hat{\partial}_j] &= -2\hat{\partial}_l\hat{\partial}_n \sum_{m=0}^{\infty} d_m(\mathrm{i}a\hat{\partial}_n)^m + 2\mathrm{i}a(e+f)\hat{\partial}_k\hat{\partial}_k\hat{\partial}_l \qquad (3.8)\\
[\hat{N}^l, \hat{\partial}_i\hat{\partial}_j] &= -(\delta^l_j\hat{\partial}_i + \delta^l_i\hat{\partial}_j)\hat{\partial}_n \sum_{m=0}^{\infty} d_m(\mathrm{i}a\hat{\partial}_n)^m + (\delta^l_j\hat{\partial}_i + \delta^l_i\hat{\partial}_j)\mathrm{i}ae\hat{\partial}_k\hat{\partial}_k\\
&\quad +2\mathrm{i}af\hat{\partial}_l\hat{\partial}_j\hat{\partial}_i\,.
\end{aligned}$$

By induction one shows

$$[\hat{N}^l, (\hat{\partial}_n)^m] = m\hat{\partial}_l(\hat{\partial}_n)^{m-1} \qquad (3.9)$$

and hence

$$[\hat{N}^l, \exp(\mathrm{i}a\hat{\partial}_n)] = \mathrm{i}a\hat{\partial}_l \exp(\mathrm{i}a\hat{\partial}_n) \qquad (3.10)$$

We insert the ansatz into the Jacobi identity for $\hat{N}^i, \hat{N}^j, \hat{\partial}_l$

$$[[\hat{N}^i, \hat{N}^j], \hat{\partial}_l] - [[\hat{N}^i, \hat{\partial}_l], \hat{N}^j] + [[\hat{N}^j, \hat{\partial}_l], \hat{N}^i] = 0$$

and compute using the formulae (3.8), (3.9), (3.9) and (3.10)

$$\begin{aligned}
0 &\overset{!}{=} \left[\hat{M}^{ij}, \hat{\partial}_l\right] - \left[-\delta^i_l\hat{\partial}_n \sum_{m=0}^{\infty} d_m(\mathrm{i}a\hat{\partial}_n)^m + \delta^i_l\mathrm{i}ae\hat{\partial}_k\hat{\partial}_k + \mathrm{i}af\hat{\partial}_l\hat{\partial}_i, \hat{N}^j\right]\\
&\quad + \left[-\delta^j_l\hat{\partial}_n \sum_{m=0}^{\infty} d_m(\mathrm{i}a\hat{\partial}_n)^m + \delta^j_l\mathrm{i}ae\hat{\partial}_k\hat{\partial}_k + \mathrm{i}af\hat{\partial}_l\hat{\partial}_j, \hat{N}^i\right]\\
&= \delta^i_l\hat{\partial}_j - \delta^j_l\hat{\partial}_i - (\delta^l_i\hat{\partial}_j - \delta^l_j\hat{\partial}_i)\sum_{m=0}^{\infty} d_m(\mathrm{i}a)^m(m+1)\hat{\partial}_n^m\\
&\quad +(\delta^l_i\hat{\partial}_j - \delta^l_j\hat{\partial}_i)\mathrm{i}ae(-2\hat{\partial}_n \sum_{m=0}^{\infty} d_m(\mathrm{i}a\hat{\partial}_n)^m + 2\mathrm{i}a(e+f)\hat{\partial}_k\hat{\partial}_k)\\
&\quad +\mathrm{i}af(-(\delta^l_j\hat{\partial}_i - \delta^l_i\hat{\partial}_j)\hat{\partial}_n \sum_{m=0}^{\infty} d_m(\mathrm{i}a\hat{\partial}_n)^m + (\delta^l_j\hat{\partial}_i - \delta^l_i\hat{\partial}_j)\mathrm{i}ae\hat{\partial}_k\hat{\partial}_k)\,.
\end{aligned}$$

We consider separately terms including $\delta^l_j\hat{\partial}_i$ and $\delta^l_i\hat{\partial}_j$

$$\begin{aligned}
0 &\overset{!}{=} 1 - \sum_{m=0}^{\infty} d_m(\mathrm{i}a)^m(m+1)\hat{\partial}_n^m + (f-2e)\sum_{m=0}^{\infty} d_m(\mathrm{i}a\hat{\partial}_n)^{m+1}\\
&\quad +(\mathrm{i}a)^2 e(2e+f)\hat{\partial}_k\hat{\partial}_k\,.
\end{aligned}$$

Thus, $e = 0$ or $e = -\frac{1}{2}f$ and for any $m > 0$

$$d_m = \frac{f-2e}{m+1}d_{m-1} = \frac{(f-2e)^m}{(m+1)!}d_0\,.$$

Inserting the solution for $d_m$ into the ansatz yields

$$[\hat{N}^l, \hat{\partial}_j] = \delta^l_j \frac{1 - \exp(\mathrm{i}a(f-2e)\hat{\partial}_n)}{\mathrm{i}a(f-2e)} + \delta^l_j \mathrm{i}ae\hat{\partial}_k\hat{\partial}_k + \mathrm{i}af\hat{\partial}_l\hat{\partial}_j\,.$$

with $e = 0$ or $e = -\frac{1}{2}f$.

□

**Theorem 3.3.4.**
*There are exactly two $\kappa$-deformed universal enveloping algebra of the Poincaré algebra obeying conditions 1-7. The first one is a Hopf algebra with coproduct*

$$\begin{aligned}
\Delta\hat{M}^{rs} &= \hat{M}^{rs}\otimes 1 + 1\otimes\hat{M}^{rs} \\
\Delta\hat{N}^l &= \hat{N}^l\otimes 1 + \exp(\mathrm{i}a\hat{\partial}_n)\otimes\hat{N}^l - \mathrm{i}a\hat{\partial}_j\otimes\hat{M}^{lj} \\
\Delta\hat{\partial}_n &= \hat{\partial}_n\otimes 1 + 1\otimes\hat{\partial}_n \\
\Delta\hat{\partial}_j &= \hat{\partial}_j\otimes 1 + \exp(\mathrm{i}a\hat{\partial}_n)\otimes\hat{\partial}_j\,,
\end{aligned} \tag{3.11}$$

*counit*

$$\epsilon(\hat{M}^{rs}) = 0 \qquad \epsilon(\hat{N}^l) = 0 \qquad \epsilon(\hat{\partial}_\mu) = 0\,,$$

*and antipode*

$$\begin{aligned}
S(\hat{M}^{rs}) &= -\hat{M}^{rs} \\
S(\hat{N}^l) &= -\hat{N}^l\exp(-\mathrm{i}a\hat{\partial}_n) - \mathrm{i}a\hat{M}^{lk}\hat{\partial}_k\exp(-\mathrm{i}a\hat{\partial}_n) - \mathrm{i}a(n-1)\hat{\partial}_l\exp(-\mathrm{i}a\hat{\partial}_n) \\
S(\hat{\partial}_n) &= -\hat{\partial}_n \\
S(\hat{\partial}_j) &= -\hat{\partial}_j\exp(-\mathrm{i}a\hat{\partial}_n)\,,
\end{aligned}$$

*the second one is a Hopf algebra with coproduct*

$$\begin{aligned}
\Delta\hat{M}^{rs} &= \hat{M}^{rs}\otimes 1 + 1\otimes\hat{M}^{rs} \\
\Delta\hat{N}^l &= \hat{N}^l\otimes\exp(-\mathrm{i}a\hat{\partial}_n) + 1\otimes\hat{N}^l + \mathrm{i}a\hat{M}^{lj}\otimes\hat{\partial}_j \\
\Delta\hat{\partial}_n &= \hat{\partial}_n\otimes 1 + 1\otimes\hat{\partial}_n \\
\Delta\hat{\partial}_j &= \hat{\partial}_j\otimes\exp(-\mathrm{i}a\hat{\partial}_n) + 1\otimes\hat{\partial}_j
\end{aligned}$$

*counit*

$$\epsilon(\hat{M}^{rs}) = 0 \qquad \epsilon(\hat{N}^l) = 0 \qquad \epsilon(\hat{\partial}_\mu) = 0\,,$$

*and antipode*

$$\begin{aligned}
S(\hat{M}^{rs}) &= -\hat{M}^{rs} \\
S(\hat{N}^l) &= -\hat{N}^l\exp(\mathrm{i}a\hat{\partial}_n) + \mathrm{i}a\hat{M}^{lk}\hat{\partial}_k\exp(\mathrm{i}a\hat{\partial}_n) \\
S(\hat{\partial}_n) &= -\hat{\partial}_n \\
S(\hat{\partial}_j) &= -\hat{\partial}_j\exp(\mathrm{i}a\hat{\partial}_n)\,.
\end{aligned}$$

*The mixed Lie brackets of the first solution are*

$$[\hat{N}^l, \hat{\partial}_j] = \delta^l_j \frac{1-\exp(2\mathrm{i}a\hat{\partial}_n)}{2\mathrm{i}a} - \delta^l_j \frac{\mathrm{i}a}{2}\hat{\partial}_k\hat{\partial}_k + \mathrm{i}a\hat{\partial}_l\hat{\partial}_j\,,$$

*for the second solution the mixed Lie brackets are given by*

$$[\hat{N}^l, \hat{\partial}_j] = -\delta^l_i \frac{1-\exp(-2\mathrm{i}a\hat{\partial}_n)}{2\mathrm{i}a} + \delta^l_i \frac{\mathrm{i}a}{2}\hat{\partial}_k\hat{\partial}_k - \mathrm{i}a\hat{\partial}_l\hat{\partial}_i\,.$$

**Proof.** For the coproduct of $\hat{N}^l$ we make the most general ansatz

$$\begin{aligned}\Delta\hat{N}^l &= \hat{N}^l \otimes \sum_{m=0}^{\infty} \alpha_m(\mathrm{i}a\hat{\partial}_n)^m + \sum_{m=0}^{\infty}\beta_m(\mathrm{i}a\hat{\partial}_n)^m \otimes \hat{N}^l \\ &\quad + \mathrm{i}a\gamma\hat{\partial}_j \otimes \hat{M}^{lj}\sum_{m=0}^{\infty}\epsilon_m(\mathrm{i}a\hat{\partial}_n)^m + \mathrm{i}a\zeta\hat{M}^{lj}\sum_{m=0}^{\infty}\eta_m(\mathrm{i}a\hat{\partial}_n)^m \otimes \hat{\partial}_j\end{aligned}$$

with coefficients $\alpha_m, \beta_m, \gamma, \epsilon_m, \zeta, \eta_m \in \mathbb{R}$ and $\eta_0 = \epsilon_0 = \beta_0 = \alpha_0 = 1$. In the limit $a \to 0$ we obtain the classical coproduct. Applying this ansatz to (3.2) yields

$$\begin{aligned}\hat{N}^l([\hat{x}^n, \hat{x}^i] - \mathrm{i}a\hat{x}^i) &= \mathrm{i}a(\beta_1 - \alpha_1 - 1)\delta^{li}\hat{x}^n \stackrel{!}{=} 0 \\ \hat{N}^l([\hat{x}^i, \hat{x}^j]) &= \mathrm{i}a(\zeta - \gamma - 1)(\delta^{li}\hat{x}^j - \delta^{lj}\hat{x}^i) \stackrel{!}{=} 0\,.\end{aligned}$$

Thus $\beta_1 = \alpha_1 + 1$ and $\zeta = \gamma + 1$. Application on monomials of degree 3 yields

$$\begin{aligned}\hat{N}^l((\hat{x}^n\hat{x}^n)\hat{x}^i)) &= \hat{x}^l(\hat{x}^n + \mathrm{i}a\alpha_1)\hat{x}^i + (\hat{x}^n + \mathrm{i}a\beta_1)\hat{x}^l\hat{x}^i \\ &\quad -\delta^l_i(\hat{x}^n\hat{x}^n + 2a\beta_1\hat{x}^n + 2(\mathrm{i}a)^2\beta_2) \\ \hat{N}^l(\hat{x}^n(\hat{x}^n\hat{x}^i)) &= \hat{x}^l(\hat{x}^n + \mathrm{i}a\alpha_1)\hat{x}^i + (\hat{x}^n + \mathrm{i}a\beta_1)(\hat{x}^l\hat{x}^i - \delta^l_i(\hat{x}^n + \mathrm{i}a\beta_1)\hat{x}^n)\,.\end{aligned}$$

The condition $\hat{N}^l(\hat{x}^n(\hat{x}^n\hat{x}^i)) = \hat{N}^l((\hat{x}^n\hat{x}^n)\hat{x}^i)$ implies $2\beta_2 = \beta_1^2$. We proceed analogously for monomials of degree 4 and find $6\beta_3 = \beta_1^3$. By induction one can show

$$\beta_m = \frac{1}{m!}\beta_1^m\,.$$

Similarly one finds $\alpha_m = \frac{1}{m!}\alpha_1^m$ by considering monomials of type $\hat{x}^i\hat{x}^n\ldots\hat{x}^n$. The condition $\hat{N}^l(\hat{x}^n(\hat{x}^j\hat{x}^i)) = \hat{N}^l((\hat{x}^n\hat{x}^j)\hat{x}^i)$ implies $\alpha_0 = b$ and $\eta_1 = (b+1)$. Monomials of higher degree yield

$$\eta_m = \frac{1}{m!}(b+1)^m\,.$$

Similarly one finds $\epsilon_m = \frac{1}{m!}b^m$ by considering monomials of type $\hat{x}^i\hat{x}^j\hat{x}^n\ldots\hat{x}^n$. Thus, the coproduct of $\hat{N}^l$ reads

$$\begin{aligned}\Delta\hat{N}^l &= \hat{N}^l \otimes \exp(\mathrm{i}ab\hat{\partial}_n) + \exp(\mathrm{i}a(b+1)\hat{\partial}_n) \otimes \hat{N}^l + \mathrm{i}a\gamma\hat{\partial}_j \otimes \hat{M}^{lj}\exp(\mathrm{i}ab\hat{\partial}_n) \\ &\quad + \mathrm{i}a(\gamma+1)\hat{M}^{lj}\exp(\mathrm{i}a(b+1)\hat{\partial}_n) \otimes \hat{\partial}_j\,.\end{aligned}$$

Coassociativity can be checked straightforwardly.

For the coproduct to respect the Lie algebra structure of the deformed Poincaré algebra we must have

$$\begin{aligned}\Delta([\hat{M}^{\mu\nu},\hat{M}^{\rho\sigma}]) &= [\Delta\hat{M}^{\mu\nu},\Delta\hat{M}^{\rho\sigma}]\\ \Delta([\hat{M}^{\mu\nu},\hat{\partial}_\rho]) &= [\Delta\hat{M}^{\mu\nu},\Delta\hat{\partial}_\rho]\,.\end{aligned}$$

We compute

$$\begin{aligned}[\Delta\hat{N}^l,\Delta\hat{\partial}_i] = & [(\hat{N}^l\otimes\exp(\mathrm{i}ab\hat{\partial}_n)+\exp(\mathrm{i}a(b+1)\hat{\partial}_n)\otimes\hat{N}^l),\hat{\partial}_i\otimes\exp(\mathrm{i}ab\hat{\partial}_n)]\\ &+[\hat{N}^l\otimes\exp(\mathrm{i}ab\hat{\partial}_n),\exp(\mathrm{i}a(b+1)\hat{\partial}_n)\otimes\hat{\partial}_i)]\\ &+[\exp(\mathrm{i}a(b+1)\hat{\partial}_n)\otimes\hat{N}^l,\exp(\mathrm{i}a(b+1)\hat{\partial}_n)\otimes\hat{\partial}_i)]\\ &+[\mathrm{i}a\gamma\hat{\partial}_j\otimes\exp(\mathrm{i}ab\hat{\partial}_n)\hat{M}^{lj},\hat{\partial}_i\otimes\exp(\mathrm{i}ab\hat{\partial}_n)]\\ &+[\mathrm{i}a\gamma\hat{\partial}_j\otimes\exp(\mathrm{i}ab\hat{\partial}_n)\hat{M}^{lj},\exp(\mathrm{i}a(b+1)\hat{\partial}_n)\otimes\hat{\partial}_i)]\\ &+[\mathrm{i}a(\gamma+1)\exp(\mathrm{i}a(b+1)\hat{\partial}_n)\hat{M}^{lj}\otimes\hat{\partial}_j,\hat{\partial}_i\otimes\exp(\mathrm{i}ab\hat{\partial}_n)]\\ &+[\mathrm{i}a(\gamma+1)\exp(\mathrm{i}a(b+1)\hat{\partial}_n)\hat{M}^{lj}\otimes\hat{\partial}_j,\exp(\mathrm{i}a(b+1)\hat{\partial}_n)\otimes\hat{\partial}_i].\end{aligned}$$

Using the formulae (3.8), (3.9), (3.9) and (3.10) the previous expression reads

$$\begin{aligned}[\Delta\hat{N}^l,\Delta\hat{\partial}_i] = & \left(\delta_i^l\,\frac{1-\exp(\mathrm{i}a(f-2e)\hat{\partial}_n)}{\mathrm{i}a(f-2e)}\right)\otimes\exp(2\mathrm{i}ab\hat{\partial}_n)\\ &+(\delta_i^l\mathrm{i}ae\hat{\partial}_k\hat{\partial}_k+\mathrm{i}af\hat{\partial}_l\hat{\partial}_i)\otimes\exp(2\mathrm{i}ab\hat{\partial}_n)\\ &+\mathrm{i}a(b+1)\hat{\partial}_l\exp(\mathrm{i}a(b+1)\hat{\partial}_n)\otimes\hat{\partial}_i\exp(\mathrm{i}ab\hat{\partial}_n)\\ &+\mathrm{i}ab\hat{\partial}_i\exp(\mathrm{i}a(b+1)\hat{\partial}_n)\otimes\hat{\partial}_l\exp(\mathrm{i}ab\hat{\partial}_i)\\ &+\exp(2\mathrm{i}a(b+1)\hat{\partial}_n)\otimes\left(\delta_i^l\,\frac{1-\exp(\mathrm{i}a(f-2e)\hat{\partial}_n)}{\mathrm{i}a(f-2e)}\right)\\ &+\exp(2\mathrm{i}a(b+1)\hat{\partial}_n)\otimes(\delta_i^l\mathrm{i}ae\hat{\partial}_k\hat{\partial}_k+\mathrm{i}af\hat{\partial}_l\hat{\partial}_i)\\ &+\mathrm{i}a\gamma\hat{\partial}_j\exp(\mathrm{i}a(b+1)\hat{\partial}_n)\otimes\exp(\mathrm{i}ab\hat{\partial}_n)(\delta^{li}\hat{\partial}_j-\delta^{ji}\hat{\partial}_l)\\ &+\mathrm{i}a(\gamma+1)\exp(\mathrm{i}a(b+1)\hat{\partial}_n)(\delta^{li}\hat{\partial}_j-\delta^{ji}\hat{\partial}_l)\otimes\hat{\partial}_j\exp(\mathrm{i}ab\hat{\partial}_n)\,.\end{aligned}$$

Comparing with

$$\begin{aligned}\Delta[\hat{N}^l,\hat{\partial}_i] = & \delta_i^l\,\frac{1\otimes 1-\exp(\mathrm{i}a(f-2e)\hat{\partial}_n)\otimes\exp(\mathrm{i}a(f-2e)\hat{\partial}_n)}{\mathrm{i}a(f-2e)}\\ &+\delta_i^l\mathrm{i}ae(\hat{\partial}_k\hat{\partial}_k\otimes\exp(2\mathrm{i}ab\hat{\partial}_n)+\exp(2\mathrm{i}a(b+1)\hat{\partial}_n)\otimes\hat{\partial}_k\hat{\partial}_k)\\ &+2\hat{\partial}_k\exp(\mathrm{i}a(b+1)\hat{\partial}_n)\otimes\hat{\partial}_k\exp(\mathrm{i}ab\hat{\partial}_n)+\mathrm{i}af\hat{\partial}_l\hat{\partial}_i\otimes\exp(2\mathrm{i}ab\hat{\partial}_n)\\ &+\mathrm{i}af\hat{\partial}_l\exp(\mathrm{i}a(b+1)\hat{\partial}_n)\otimes\hat{\partial}_i\exp(\mathrm{i}ab\hat{\partial}_n)\\ &+\mathrm{i}af\hat{\partial}_i\exp(\mathrm{i}a(b+1)\hat{\partial}_n)\otimes\hat{\partial}_l\exp(\mathrm{i}ab\hat{\partial}_n))\\ &+\mathrm{i}af\exp(2\mathrm{i}a(b+1)\hat{\partial}_n)\otimes\hat{\partial}_l\hat{\partial}_i\end{aligned}$$

implies $2e=2\gamma+1$ and $f=b-\gamma$ and

$$\begin{aligned}1\otimes 1-\exp(\mathrm{i}a(f-2e)\hat{\partial}_n) = & (1-\exp(\mathrm{i}a(f-2e)\hat{\partial}_n))\otimes\exp(2\mathrm{i}ab\hat{\partial}_n)\\ &+\exp(2\mathrm{i}a(b+1)\hat{\partial}_n)\otimes(1-\exp(\mathrm{i}a(f-2e)\hat{\partial}_n))\,.\end{aligned}$$

This last condition holds only if $b = 0$ and $f - 2e = 2$ or if $b = -1$ and $f - 2e = -2$. Recall that $e = 0$ or $2e = -f$. Hence, there are four possible solutions:

1. $b = 0$ $\quad f = 1$ $\quad e = -\frac{1}{2}$
2. $b = -1$ $\quad f = -1$ $\quad e = \frac{1}{2}$
3. $b = 0$ $\quad f = 2$ $\quad e = 0$
4. $b = -1$ $\quad f = -2$ $\quad e = 0$

The conditions $2e = 2\gamma + 1$ and $f = b - \gamma$ hold only for solution 1 (where $\gamma = -1$) and solution 2 (where $\gamma = 0$). Thus, there are two $\kappa$-deformed universal enveloping algebra of the Poincaré algebra obeying conditions 1-7: one with coproducts

$$\begin{aligned}\Delta\hat{N}^l &= \hat{N}^l \otimes 1 + \exp(\mathrm{i}a\hat{\partial}_n) \otimes \hat{N}^l - \mathrm{i}a\hat{\partial}_j \otimes \hat{M}^{lj} \\ \Delta\hat{\partial}_j &= \hat{\partial}_j \otimes 1 + \exp(\mathrm{i}a\hat{\partial}_n) \otimes \hat{\partial}_j\end{aligned}$$

and Lie algebra structure

$$[\hat{N}^l, \hat{\partial}_j] = \delta^l_j \frac{1 - \exp(2\mathrm{i}a\hat{\partial}_n)}{2\mathrm{i}a} - \delta^l_j \frac{\mathrm{i}a}{2}\hat{\partial}_k\hat{\partial}_k + \mathrm{i}a\hat{\partial}_l\hat{\partial}_j$$

and one with coproducts

$$\begin{aligned}\Delta\hat{N}^l &= \hat{N}^l \otimes \exp(-\mathrm{i}a\hat{\partial}_n) + 1 \otimes \hat{N}^l + \mathrm{i}a\hat{M}^{lj} \otimes \hat{\partial}_j \\ \Delta\hat{\partial}_j &= \hat{\partial}_j \otimes \exp(-\mathrm{i}a\hat{\partial}_n) + 1 \otimes \hat{\partial}_j\end{aligned}$$

and Lie algebra structure

$$[\hat{N}^l, \hat{\partial}_j] = -\delta^l_i \frac{1 - \exp(-2\mathrm{i}a\hat{\partial}_n)}{2\mathrm{i}a} + \delta^l_i \frac{\mathrm{i}a}{2}\hat{\partial}_k\hat{\partial}_k - \mathrm{i}a\hat{\partial}_l\hat{\partial}_i \,.$$

Since the coproduct of $\hat{M}^{rs}$ is undeformed, its antipode and its counit are as in the commutative case

$$\epsilon(\hat{M}^{rs}) = 0 \qquad S(\hat{M}^{rs}) = -\hat{M}^{rs}$$

The counit of $\hat{N}^l$ for solution 1 can quickly computed to be $\epsilon(\hat{N}^l) = 0$. The antipode must obey (1.13)

$$S(\hat{N}^l) + \exp(-\mathrm{i}a\hat{\partial}_n)\hat{N}^l + \mathrm{i}a\hat{\partial}_k \exp(-\mathrm{i}a\hat{\partial}_n)\hat{M}^{lk} = 0\,.$$

We write the derivatives on the right side of the rotations

$$\begin{aligned}S(\hat{N}^l) &= -\hat{N}^l \exp(-\mathrm{i}a\hat{\partial}_n) - \mathrm{i}a\hat{\partial}_l \exp(-\mathrm{i}a\hat{\partial}_n) - \mathrm{i}a\hat{M}^{lk}\hat{\partial}_k \exp(-\mathrm{i}a\hat{\partial}_n) \\ &\quad + \mathrm{i}a(\delta^{lk}\hat{\partial}_k - \delta^{kk}\hat{\partial}_l)\exp(-\mathrm{i}a\hat{\partial}_n) \\ &= -\hat{N}^l \exp(-\mathrm{i}a\hat{\partial}_n) - \mathrm{i}a\hat{M}^{lk}\hat{\partial}_k \exp(-\mathrm{i}a\hat{\partial}_n) - \mathrm{i}a(n-1)\hat{\partial}_l \exp(-\mathrm{i}a\hat{\partial}_n)\,.\end{aligned}$$

Similarly we find the counit and the antipode for solution 2 to be

$$\epsilon(\hat{N}^l) = 0 \qquad S(\hat{N}^l) = -\hat{N}^l \exp(\mathrm{i}a\hat{\partial}_n) + \mathrm{i}a\hat{M}^{lk}\hat{\partial}_k \exp(\mathrm{i}a\hat{\partial}_n)\,.$$

□

Similarly to Proposition 3.2.4, one can show that the Hopf algebra of deformed rotations is not isomorphic to the Hopf algebra of rotations on the commutative space $k[x^1, \ldots x^n]$.

In the following sections we work with the first solution of Theorem 3.3.4.

## 3.4 Invariants of the deformed Poincaré algebra

In order to prepare future works on field theories on the $\theta$- and the $\kappa$-space we first consider central elements and space invariants of the deformed Poincaré algebra. We compute a deformed space invariant, a deformed Laplace operator and a deformed Pauli-Lubanski vector. Additionally, we introduce new partial derivatives, the Dirac derivatives, by demanding that they transform linearly under rotations.

### Space Invariants

By space invariant we refer to the lowest order polynomial $\hat{I}_1$ in the variables $\hat{x}^\mu$ invariant under rotations which is equal to $x^\mu x^\mu$ when setting the deformation parameter to zero. Since the coproduct of $\hat{M}^{\mu\nu}$ is deformed for the $\theta$- and for the $\kappa$-space, we expect that the space invariant depends on the deformation parameter.

For the $\theta$-space the action of $\hat{M}^{\mu\nu}$ on $\hat{I}_\theta = \hat{x}^\rho \hat{x}^\rho$ is

$$\hat{M}^{\mu\nu}(\hat{x}^\rho \hat{x}^\rho) = \theta^{\mu\nu}(2\lambda_2 n + 4\lambda_1 - 2),$$

where $n$ denotes the dimension of spacetime. Hence, a space invariant of degree 2 appears only if one considers the $\theta$-deformed universal enveloping algebra of the Poincaré algebra with $\lambda_1 = \frac{1}{2}(1 - n\lambda_2)$. The next possible invariant will be of degree 4, therefore of the form $\theta^{\rho\sigma}\hat{x}^\rho\hat{x}^\sigma$. This is indeed an invariant for any choice of the parameters $\lambda_1, \lambda_2$

$$\hat{M}^{\mu\nu}(\theta^{\rho\sigma}\hat{x}^\rho\hat{x}^\sigma) = \theta^{\mu\sigma}\hat{x}^\nu\hat{x}^\sigma - \theta^{\nu\sigma}\hat{x}^\mu\hat{x}^\sigma + \theta^{\rho\mu}\hat{x}^\rho\hat{x}^\nu - \theta^{\rho\nu}\hat{x}^\rho\hat{x}^\mu = 0\,.$$

For the $\kappa$-space a familiar result [MaRu94] is that the lowest order polynomial in the variables $\hat{x}^\mu$ invariant under rotations is not $\hat{x}^\mu\hat{x}^\mu$, but

$$\hat{I}_\kappa = \hat{x}^\mu\hat{x}^\mu - \mathrm{i}a(n-1)\hat{x}^n\,.$$

This is the lowest order invariant in the variables $\hat{x}^\mu$.

### Laplace operator

The Laplace operator $\hat{\Box}$ is defined to be the lowest order invariant under rotations built out of partial derivatives. Since for the $\theta$-space the Lie brackets of partial derivatives are undeformed, the Laplace operator $\hat{\Box} = \hat{\partial}_\mu\hat{\partial}_\mu$ is the one of the commutative case.

For the $\kappa$-space the Laplace operator is known [LNRT91] to be of the form

$$\hat{\Box} = \hat{\partial}_k \hat{\partial}_k \exp(-\mathrm{i}a\hat{\partial}_n) + \frac{2}{a^2}(1 - \cos(a\hat{\partial}_n)) \,, \tag{3.12}$$

all operators $\mathcal{G}(a^2\hat{\Box})\hat{\Box}$ are invariants as well.

## Dirac derivatives

Going beyond the conditions on partial derivatives from the last section, we will demand that the Lie brackets of the Poincaré algebra should remain undeformed. Therefore, we define other partial derivatives which we will call Dirac derivatives[2] by demanding that they transform linearly under rotations

$$[\hat{M}^{\rho\sigma}, \hat{D}_\mu] = \delta^\rho_\mu \hat{D}_\sigma - \delta^\sigma_\mu \hat{D}_\rho \,. \tag{3.13}$$

In the $\theta$-space these conditions hold for the partial derivatives $\hat{\partial}_\mu$.

For the $\kappa$-space we drop the linearity condition (condition 5 in subsection 3.2) on partial derivatives and admit terms depending on $a^2\hat{\partial}_k\hat{\partial}_k$. Again we consider only the simplest solution for the Dirac derivative.

**Theorem 3.4.1.**
*The simplest Dirac derivative transforming under rotations as demanded in (3.13) is of the form*

$$\begin{aligned} \hat{D}_n &= \frac{1}{a}\sin(a\hat{\partial}_n) + \frac{\mathrm{i}a}{2}\hat{\partial}_k\hat{\partial}_k \exp(-\mathrm{i}a\hat{\partial}_n) \\ \hat{D}_j &= \hat{\partial}_j \exp(-\mathrm{i}a\hat{\partial}_n) \,. \end{aligned} \tag{3.14}$$

*The Dirac derivatives form an abelian Lie algebra. The $\kappa$-deformed universal enveloping algebra of Dirac derivatives is a Hopf algebra with unique coproduct*

$$\begin{aligned} \Delta\hat{D}_n &= \hat{D}_n \otimes \Big( -\mathrm{i}a\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}\Big) + \frac{\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}}{1 - a^2\hat{D}_k\hat{D}_k} \otimes \hat{D}_n \\ &\quad + \mathrm{i}a\hat{D}_i \frac{\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}}{1 - a^2\hat{D}_k\hat{D}_k} \otimes \hat{D}_i \\ \Delta\hat{D}_j &= \hat{D}_j \otimes \Big( -\mathrm{i}a\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}\Big) + 1 \otimes \hat{D}_j \,, \end{aligned} \tag{3.15}$$

*unique counit*

$$\begin{aligned} \epsilon(\hat{D}_n) &= 0 \\ \epsilon(\hat{D}_j) &= 0 \end{aligned}$$

[2] In future work we will study the Dirac equation for the $\kappa$-space in the sense of [NST93].

*and unique antipode*

$$\begin{aligned} S(\hat{D}_n) &= -\hat{D}_n + \mathrm{i}a\hat{D}_l\hat{D}_l \frac{\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}}{1-a^2\hat{D}_k\hat{D}_k} \\ S(\hat{D}_j) &= -\hat{D}_j \frac{\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}}{1-a^2\hat{D}_k\hat{D}_k} \,. \end{aligned} \tag{3.16}$$

**Proof.** We insert the simplest ansatz for the Dirac derivative

$$\begin{aligned} \hat{D}_n &= \hat{\partial}_n + \sum_m (b_m\hat{\partial}_k\hat{\partial}_k + c_m)(a\hat{\partial}_n)^m \\ \hat{D}_i &= \hat{\partial}_i \sum_m d_m (a\hat{\partial}_n)^m \end{aligned}$$

with coefficients $b_m, c_m, d_m \in \mathbb{C}$ and $b_0 = 0, d_0 = 1$ in the transformation law under rotations (3.13). It is easy to see that the condition $[\hat{M}^{rs}, \hat{D}_\mu] = \delta^r_\mu \hat{D}_s - \delta^s_\mu \hat{D}_r$ holds for any choice of the coefficients. For $[\hat{N}^i, \hat{D}_n]$ we compute using (3.8) and (3.9)

$$\begin{aligned} [\hat{N}^i, \hat{D}_n] &= \hat{\partial}_i + \sum_m b_m [\hat{N}^i, \hat{\partial}_k\hat{\partial}_k](a\hat{\partial}_n)^m + \sum_m (b_m\hat{\partial}_k\hat{\partial}_k + c_m)[\hat{N}^i, (a\hat{\partial}_n)^m] \\ &= \hat{\partial}_i + \left( \frac{1-\exp(2\mathrm{i}a\hat{\partial}_n)}{\mathrm{i}a} + \mathrm{i}a\hat{\partial}_j\hat{\partial}_j \right) \hat{\partial}_l \sum_m b_m (a\hat{\partial}_n)^m \\ &\quad + \sum_{m=1} (b_m\hat{\partial}_k\hat{\partial}_k + c_m) a m \hat{\partial}_i (a\hat{\partial}_n)^{m-1} \\ &= \hat{\partial}_i + \frac{1-\exp(2\mathrm{i}a\hat{\partial}_n)}{\mathrm{i}a} \hat{\partial}_i \sum_m b_m (a\hat{\partial}_n)^m + \sum_{m=1} c_m a m \hat{\partial}_i (a\hat{\partial}_n)^{m-1} \\ &\quad + \sum_{m=0} (b_{m+1}(m+1) + \mathrm{i}b_m) a \hat{\partial}_k\hat{\partial}_k\hat{\partial}_i (a\hat{\partial}_n)^m \\ &\stackrel{!}{=} \hat{\partial}_i \sum_m d_m (a\hat{\partial}_n)^m \,. \end{aligned}$$

Therefore, $b_{m+1} = -\frac{\mathrm{i}}{m+1} b_m$ and by induction we obtain $b_m = \frac{1}{m!}(-\mathrm{i})^m$ and $\hat{D}_j = \hat{\partial}_j \exp(-\mathrm{i}a\hat{\partial}_n)$. Next we calculate

$$\begin{aligned} [\hat{N}^l, \hat{D}_j] &= \delta^l_j \frac{1-\exp(2\mathrm{i}a\hat{\partial}_n)}{2\mathrm{i}a} \exp(-\mathrm{i}a\hat{\partial}_n) - \delta^l_j \frac{\mathrm{i}a}{2} \hat{\partial}_k\hat{\partial}_k \exp(-\mathrm{i}a\hat{\partial}_n) \\ &= -\delta^l_j \frac{1}{a} \sin(a\hat{\partial}_n) - \delta^l_j \frac{\mathrm{i}a}{2} \hat{\partial}_k\hat{\partial}_k \exp(-\mathrm{i}a\hat{\partial}_n) \,, \end{aligned}$$

hence $\hat{D}_n = \frac{1}{a}\sin(a\hat{\partial}_n) + \frac{\mathrm{i}a}{2}\hat{\partial}_k\hat{\partial}_k \exp(-\mathrm{i}a\hat{\partial}_n)$. Note that the term $\hat{D}_\mu\, g(a^2\hat{\Box})$ also obeys transformation law (3.13).

The coproduct of the Dirac derivative can be computed as for the partial derivatives $\hat{\partial}_\mu$ in Section 3.2 by applying $\hat{D}_\mu$ on monomials and demanding coassociativity. We omit this tedious, but straightforward calculation (for details see our work [DMT04] and [Moe04]). Alternatively one can derive the coproduct of the Dirac derivative by applying $\Delta$ to (3.14).

It is easy to see that the unique counit of the Dirac derivative is $\epsilon(\hat{D}_\mu) = 0$; the unique result for the antipode follows from the Hopf algebra axioms. $\square$

Additionally, we find $S^2(\hat{D}_\mu) = \hat{D}_\mu$. The $\kappa$-deformed universal enveloping algebra of the Poincaré algebra with the Dirac derivatives is undeformed in the algebra sector (cf. (3.7) and (3.13)) and deformed in the coalgebra sector (cf. (3.11) and (3.15)).

The square of the Dirac derivative can easily be calculated to be $\hat{D}_\mu\hat{D}_\mu = \hat{\Box}(1 - \frac{a^2}{4}\hat{\Box})$. We could rescale the Dirac derivative $\hat{D}'_\mu = \frac{\hat{D}_\mu}{\sqrt{1-\frac{a^2}{4}\hat{\Box}}}$, such that $\hat{D}'_\mu\hat{D}'_\mu = \hat{\Box}$ by choosing factors $f(a^2\hat{\Box})$ and $g(a^2\hat{\Box})$ in relations (3.12) and (3.14) appropriately. However, we do not follow this reasoning in the following.

Using relations (3.15) and (3.16), the identities

$$
\begin{aligned}
\exp(-\mathrm{i}a\hat{\partial}_n) &= -\mathrm{i}a\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu} = 1 - \mathrm{i}a\hat{D}_n - \frac{a^2}{2}\hat{\Box} \\
\exp(\mathrm{i}a\hat{\partial}_n) &= \frac{\mathrm{i}a\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}}{1 - a^2\hat{D}_j\hat{D}_j}
\end{aligned} \tag{3.17}
$$

and the fact that the coproduct is an algebra morphism (in particular $\Delta(\hat{D}_\mu\hat{D}_\mu) = \Delta(\hat{D}_\mu)\Delta(\hat{D}_\mu)$) we compute the coproduct, the counit and the antipode of the Laplace operator to be

$$
\begin{aligned}
\Delta\hat{\Box} &= \hat{\Box}\otimes\exp(-\mathrm{i}a\hat{\partial}_n) + \exp(\mathrm{i}a\hat{\partial}_n)\otimes\hat{\Box} + 2\hat{D}_i\exp(\mathrm{i}a\hat{\partial}_n)\otimes\hat{D}_i \\
&\quad + \frac{2}{a^2}(1 - \exp(\mathrm{i}a\hat{\partial}_n))\otimes(1 - \exp(-\mathrm{i}a\hat{\partial}_n)) \\
\epsilon(\hat{\Box}) &= 0 \\
S(\hat{\Box}) &= \hat{\Box}\,.
\end{aligned}
$$

## Pauli-Lubanski vector

The Pauli-Lubanski vector $W^\lambda$ is defined to be built out of partial derivatives and rotations and to transform as a classical vector under the action of rotations $M^{\mu\nu}$, such that its square is invariant under these operations. In the commutative case, the Pauli-Lubanski vector is of the form $\epsilon^{\lambda\kappa\rho\sigma}\partial_\kappa M^{\rho\sigma}$.

Since for the $\theta$-space the Lie brackets $[M^{\mu\nu}, M^{\alpha\beta}]$ of rotations are deformed, the $\theta$-Pauli-Lubanski vector must include additional terms depending on the deformation parameter $\theta$. We make the simplest ansatz linear in $\theta$ and compatible with the grading in which additional terms do not contain rotations

$$\hat{W}^\nu = \epsilon^{\nu\kappa\rho\sigma}\hat{\partial}_\kappa\hat{M}^{\rho\sigma} + f\epsilon^{\nu\kappa\rho\sigma}\hat{\partial}_\kappa\theta^{\rho\sigma}\hat{\partial}_\alpha\hat{\partial}_\alpha$$

and quickly compute $f = \mathrm{i}\lambda_2$. Thus, we obtain a one-parameter family of $\theta$-deformed Pauli-Lubanski vectors $W^\lambda$.

Since for the $\kappa$-space the Lie algebra of rotations is isomorphic to the classical Lie algebra $\mathfrak{so}(n)$, the Pauli-Lubanski vector is undeformed.

## 3.5 $\star$-products for the $\kappa$-space

The framework of deformation quantization [BFFLS78, Kon03] allows to associate an algebra of functions on a commutative space to the algebra of functions on a noncommutative space. We give here the basic definitions. For more details see for example the reviews [DMZ07, Ster98].

**Definition 3.5.1.**
*A formal deformation of an associative $k$-algebra $A$ is given by a family of $k$-bilinear maps $\{\mu_i : A \otimes A \to A | i \in \mathbb{N}\}$ satisfying*

- $\mu_0(a, b) = ab$ *for any* $a, b \in A$
- *for all* $a, b, c \in A$ *and each* $l \geq 1$

$$\sum_{i+j=l,\ i,j\geq 0} \mu_i(\mu_j(a,b),c) = \sum_{i+j=l,\ i,j\geq 0} \mu_i(a, \mu_j(b,c))\,.$$

For a Poisson algebra $A$ with bracket $\{\cdot,\cdot\}$ one demands additionally

$$\{a,b\} = \mu_1(a,b) - \mu_1(b,a)$$

for all $a, b \in A$. Thus, we have a new associative product, the so-called $\star$-product, for the associative $k$-algebra $A$

$$a \star b = ab + \sum_{i\geq 1} t^i \mu_i(a,b)\,,$$

where $t$ is a formal variable.

We want to describe the $\kappa$-space via a formal deformation: first we define a Poisson structure on $k[x^1, \ldots x^n]$ by setting

$$\{f(x), g(x)\} = C^{\mu\nu}_\lambda x^\lambda \partial_\mu f(x) \partial_\nu g(x)\,,$$

where $C^{\mu\nu}_\lambda = \mathrm{i}a\delta^\mu_n\delta^\nu_\lambda - \mathrm{i}a\delta^\nu_n\delta^\mu_\lambda$ with $a \in \mathbb{R}$ are the structure constants of the Lie algebra we consider. Second we fix an order relation for monomials in $k_\kappa[\hat{x}^1, \ldots \hat{x}^n]$, for example the symmetric order $\frac{1}{m}\sum_{\sigma\in S_m} \hat{x}_{\sigma(1)} \ldots \hat{x}_{\sigma(m)}$. Third we deform the Poisson algebra $k[x^1, \ldots x^n]$, such that there exists an algebra homomorphism from the algebra $k_\kappa[\hat{x}^1, \ldots \hat{x}^n]$ to the deformation of $k[x^1, \ldots x^n]$ which maps $\hat{x}^\mu$ to $x^\mu$ and multiplication in $k_\kappa[\hat{x}^1, \ldots \hat{x}^n]$ to the $\star$-multiplication in the deformation of $k[x^1, \ldots x^n]$

$$\hat{x}^\mu \hat{x}^\nu \mapsto x^\mu \star x^\nu\,.$$

In order to be well-defined the $\star$-product for the $\kappa$-space must obey the two conditions

$$
\begin{aligned}
[x^l \stackrel{\star}{,} x^j] := x^l \star x^j - x^j \star x^l &= 0 \\
[x^n \stackrel{\star}{,} x^j] := x^n \star x^j - x^j \star x^n &= \mathrm{i}ax^j \,.
\end{aligned}
$$

Obviously the $\star$-product depends on the deformation parameter $a$ and on the ordering for monomials in $k_\kappa[\hat{x}^1, \ldots \hat{x}^n]$.

## Different $\star$-products

A $\star$-product allows a presentation of the generators $\hat{D}_\mu$ and $\hat{M}^{\mu\nu}$ of the $\kappa$-Poincaré algebra in terms of differential operators of ordinary, commuting partial derivatives and variables $x^\mu$. The description by means of $\star$-products is particularly suitable, since it allows an expansion order by order in the deformation parameter.

The $\star$-product replaces the point-wise product of commutative space. For a given algebra, there are generically several non isomorphic $\star$-products.

Many interesting $\star$-products simply reproduce different ordering prescriptions imposed on the noncommutative space. For example for the $\kappa$-space the order of the non commuting variables $\hat{x}^n$ and $\hat{x}^j$ in a monomial has to be specified. After multiplying two polynomials in $k_\kappa[\hat{x}^1, \ldots \hat{x}^n]$, the product has to be reordered according to the chosen prescription. The $\star$-products of interest here perform this reordering for commuting variables $x^\mu$.

In the case of $\kappa$-deformed space with only one noncommuting variable $\hat{x}^n$, three ordering prescriptions can be chosen in a generic way: all factors of $\hat{x}^n$ to the furthest left, all factors of $\hat{x}^n$ to the furthest right or complete symmetrisation of all variables. We are especially interested in the symmetric $\star$-product $\star_S$ because of its hermiticity

$$
(f \star_S g)^\dagger = g^\dagger \star_S f^\dagger \,,
$$

where $f^\dagger$ denotes the hermitian conjugate of $f \in k_\kappa[\hat{x}^1, \ldots \hat{x}^n]$.

The associative symmetric $\star$-product for the $\kappa$-space is given by [KLM00]

$$
\begin{aligned}
f(x) \star_S g(x) \;=\; \lim_{\substack{y \to x \\ z \to x}} \exp \Bigg( & x^j \partial_{y^j} \frac{\partial_{y^n} + \partial_{z^n}}{\partial_{y^n}} \;\exp(-\mathrm{i}a\partial_{z^n}) \;\frac{1 - \exp(-\mathrm{i}a\partial_{y^n})}{1 - \exp(-\mathrm{i}a(\partial_{y^n} + \partial_{z^n}))} \\
& + x^j \partial_{z^j} \frac{\partial_{y^n} + \partial_{z^n}}{\partial_{z^n}} \;\frac{1 - \exp(-\mathrm{i}a\partial_{z^n})}{1 - \exp(-\mathrm{i}a(\partial_{y^n} + \partial_{z^n}))} \\
& - x^j \partial_{y^j} - x^j \partial_{z^j} \Bigg) f(y) g(z) \,.
\end{aligned}
$$

The normal ordered $\star$-products can be derived via a Weyl quantisation procedure [MSSW00]

$$
\begin{aligned}
f(x) \star_L g(x) &= \lim_{\substack{y \to x \\ z \to x}} \;\exp\left( x^j \frac{\partial}{\partial y^j} (\exp(-\mathrm{i}a\partial_{z^n}) - 1) \right) \; f(y) g(z) \\
f(x) \star_R g(x) &= \lim_{\substack{y \to x \\ z \to x}} \;\exp\left( x^j \frac{\partial}{\partial z^j} (\exp(\mathrm{i}a\partial_{y^n}) - 1) \right) \; f(y) g(z) \,. \qquad (3.18)
\end{aligned}
$$

The $\star$-product $\star_L$ reproduces an ordering for which all $\hat{x}^n$ stand on the left hand side in any monomial; $\star_R$ reproduces the opposite ordering prescription.

## Partial derivatives and rotations in the $\star$-formalism

The algebra $k_\kappa[x^1, \dots x^n]$ with a $\star$-product as multiplication can be endowed with the structure of a module over the $\kappa$-Poincaré algebra $\mathfrak{p}_\kappa$; we denote for any generator $G$ of $\mathfrak{p}_\kappa$

$$\begin{array}{ccc} \mathfrak{p}_\kappa & \longrightarrow & \mathrm{End}(k_\kappa[x^1, \dots x^n]) \\ G & \longmapsto & G^\star\,. \end{array}$$

The module structure can be derived in a perturbation expansion on symmetrized monomials multiplied with the $\star$-product. Alternatively one can use the expressions $x^\mu \star f(x)$. For details of this lengthy calculation see our work [DMT04]. We present the results for the symmetric $\star$-product:

$$\begin{aligned} \partial_n^{\star_S} &= \partial_n \\ \partial_j^{\star_S} &= \partial_j\Big(\frac{\exp(\mathrm{i}a\partial_n)-1}{\mathrm{i}a\partial_n}\Big) \\ M^{\star_S\, rs} &= x^s\partial_r - x^r\partial_s \\ N^{\star_S\, l} &= x^l\partial_n - x^n\partial_l + x^l\partial_\mu\partial_\mu\frac{\exp(\mathrm{i}a\partial_n)-1}{2\partial_n} - x^\mu\partial_\mu\partial_l\frac{\exp(\mathrm{i}a\partial_n)-1-\mathrm{i}a\partial_n}{\mathrm{i}a\partial_n^2}\,. \end{aligned} \tag{3.19}$$

Similarly one can compute $\star$-analoga of the Dirac derivatives and the Laplace operator

$$\begin{aligned} D_n^{\star_S} &= \frac{1}{a}\sin(a\partial_n) - \frac{1}{\mathrm{i}a\partial_n\partial_n}\partial_k\partial_k(\cos(a\partial_n)-1) \\ D_j^{\star_S} &= \partial_j\;\frac{\exp(-\mathrm{i}a\partial_n)-1}{-\mathrm{i}a\partial_n} \\ \Box^{\star_S} &= \partial_\mu\partial_\mu\;\frac{2(1-\cos(a\partial_n))}{a^2\partial_n\partial_n}\,. \end{aligned}$$

Similar descriptions can be derived for the normal ordered $\star$-products; they will differ from (3.19). For the left ordered $\star$-product ($\star_L$) we obtain

$$\begin{aligned} \partial_j^{\star_L} &= \partial_j\exp(\mathrm{i}a\partial_n) \\ \partial_n^{\star_L} &= \partial_n \\ D_j^{\star_L} &= \partial_j \\ D_n^{\star_L} &= \frac{1}{a}\sin(a\partial_n) + \frac{\mathrm{i}a}{2}\partial_k\partial_k\exp(\mathrm{i}a\partial_n) \\ N^{\star_L\, l} &= x^l\frac{1}{a}\sin(a\partial_n) - x^n\partial_l\exp(\mathrm{i}a\partial_n) + \frac{\mathrm{i}a}{2}x^l\partial_k\partial_k\exp(\mathrm{i}a\partial_n) \\ M^{\star_L\, rs} &= x^s\partial_r - x^r\partial_s \\ \Box^{\star_L} &= -\frac{2}{a^2}(\cos(a\partial_n)-1) + \partial_k\partial_k\exp(\mathrm{i}a\partial_n)\,. \end{aligned}$$

The result for the right ordered $\star$-product ($\star_R$) is

$$
\begin{aligned}
\partial_j^{\star_R} &= \partial_j \\
\partial_n^{\star_R} &= \partial_n \\
D_j^{\star_R} &= \partial_j \exp(-\mathrm{i}a\partial_n) \\
D_n^{\star_R} &= \frac{1}{a}\sin(a\partial_n) + \frac{\mathrm{i}a}{2}\partial_k\partial_k \exp(-\mathrm{i}a\partial_n) \\
N^{\star_R\, l} &= x^l \frac{1}{2\mathrm{i}a}(\exp(2\mathrm{i}a\partial_n) - 1) - x^n\partial_l - \mathrm{i}a x^k\partial_k\partial_l + \frac{\mathrm{i}a}{2}x^l\partial_k\partial_k \\
M^{\star_R\, rs} &= x^s\partial_r - x^r\partial_s \\
\Box^{\star_R} &= -\frac{2}{a^2}(\cos(a\partial_n) - 1) + \partial_k\partial_k \exp(-\mathrm{i}a\partial_n)\,.
\end{aligned}
$$

## 3.6 Vector fields for the $\kappa$-space

The aim of future work is to define field theories. Vector fields are a necessary ingredient for the definition of field theories and in particular gauge theories. For simplicity we consider global vector fields that have the same transformation properties as partial derivatives under rotations. The central assumption is that the transformation behaviour of vector fields is such that the vector fields appear linearly on the right hand side of the Lie brackets with the generators of rotations.

### Linearly transforming vector fields

We define vector fields $\hat{V}_\mu$ associated to the Dirac derivative by demanding that they transform vector-like under rotations. As in (3.13) we denote the action of rotations on vector fields by [,] and write

$$[\hat{M}^{\rho\sigma}, \hat{V}_\mu] \;=\; \delta^\rho_\mu \hat{V}_\sigma - \delta^\sigma_\mu \hat{V}^\rho\,. \tag{3.20}$$

The square $\hat{V}_\mu\hat{V}_\mu$ of the vector field $\hat{V}_\mu$ corresponding to the Dirac derivative is an invariant under rotations.

### Vector fields related to the derivatives $\hat{\partial}_\mu$

It is more difficult to find vector fields with transformation properties analogous to the partial derivatives $\hat{\partial}_\mu$. Although we have argued that the partial derivatives $\hat{\partial}_\mu$ are in a sense irrelevant for the geometric construction of $\kappa$-deformed space, they play an important role for making contact with the commutative regime. Since $\partial_n^\star = \partial_n$ for all three $\star$-products, the partial derivatives $\hat{\partial}_\mu$ provide information on the connection between the algebra $k_\kappa[\hat{x}^1, \dots \hat{x}^n]$ and its description by the $\star$-product.

Therefore, we now investigate vector fields $\hat{A}_\mu$ analogous to $\hat{\partial}_\mu$: we demand that the transformation law of $\hat{A}_\mu$ under rotations coincides with (3.8) when $\hat{A}_\mu$ is substituted with $\hat{\partial}_\mu$. We make the choice that partial derivatives are always to the left of the vector field $\hat{A}_\mu$ in nonlinear expressions such as the vector field analog of (3.8). We do not evaluate partial derivatives on $\hat{A}_\mu$ and assume that vector fields appear only linearly in the transformation law.

**Theorem 3.6.1.**
*The vector fields $\hat{A}_\mu$ transform under rotations as*[3]

$$
\begin{aligned}
[\hat{M}^{rs}, \hat{A}_n] &= 0 \\
[\hat{M}^{rs}, \hat{A}_i] &= \delta_i^r \hat{A}_s - \delta_i^s \hat{A}_r \\
[\hat{N}^l, \hat{A}_n] &= \hat{A}_l \\
[\hat{N}^l, \hat{A}_i] &= \delta_i^l \frac{1-\exp(2\mathrm{i}a\hat{\partial}_n)}{2\mathrm{i}a\hat{\partial}_n}\hat{A}_n - \frac{\mathrm{i}a}{2}\delta_i^l \hat{\partial}_j \hat{A}_j + \frac{\mathrm{i}a}{2}\big(\hat{\partial}_l \hat{A}_i + \hat{\partial}_i \hat{A}_l\big) \\
&\quad -\delta_i^l \frac{a}{2\hat{\partial}_n}\tan\big(\frac{a\hat{\partial}_n}{2}\big)\big(\hat{\partial}_n \hat{\partial}_j \hat{A}_j - \hat{\partial}_j \hat{\partial}_j \hat{A}_n\big) \\
&\quad +\Big(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n}\cot\big(\frac{a\hat{\partial}_n}{2}\big)\Big)\big(\hat{\partial}_n \hat{\partial}_i \hat{A}_l + \hat{\partial}_n \hat{\partial}_l \hat{A}_i - 2\hat{\partial}_l \hat{\partial}_i \hat{A}_n\big) .
\end{aligned}
\tag{3.21}
$$

**Proof.** It is straightforward to compute that the above action is well-defined and coincides with (3.8) when $\hat{A}_\mu$ is substituted with $\hat{\partial}_\mu$.

To form an invariant out of the vector field $\hat{A}_\mu$, we have to define a vector field $\breve{A}_\mu$ with transformation laws in which the partial derivatives are to the right of the vector field $\breve{A}_\mu$. Demanding that $\breve{A}_\mu \hat{A}_\mu$ is an invariant of the Lie algebra of rotations

$$[\hat{M}^{\rho\sigma}, \breve{A}_\mu \hat{A}_\mu] = 0,$$

we straightforwardly find the transformation laws for $\breve{A}_\mu$ to be

$$
\begin{aligned}
[\hat{M}^{rs}, \breve{A}_i] &= \delta_i^r \breve{A}_s - \delta_i^s \breve{A}_r \\
[\hat{M}^{rs}, \breve{A}_n] &= 0 \\
[\hat{N}^l, \breve{A}_i] &= -\delta_i^l \breve{A}_n + \frac{\mathrm{i}a}{2}\breve{A}_l \hat{\partial}_i - \frac{\mathrm{i}a}{2}\breve{A}_i \hat{\partial}_l - \frac{\mathrm{i}a}{2}\delta_i^l \breve{A}_j \hat{\partial}_j \\
&\quad +\frac{a}{2}\breve{A}_l \tan\big(\frac{a\hat{\partial}_n}{2}\big)\hat{\partial}_i - (\delta_i^l \breve{A}_j \hat{\partial}_j + \breve{A}_i \hat{\partial}_l)\Big(\frac{1}{\hat{\partial}_n} - \frac{a}{2}\cot\big(\frac{a\hat{\partial}_n}{2}\big)\Big) \\
[\hat{N}^l, \breve{A}_n] &= \breve{A}_l \frac{\exp(2\mathrm{i}a\hat{\partial}_n) - 1}{2\mathrm{i}a\hat{\partial}_n} - \breve{A}_l \frac{a}{2\hat{\partial}_n}\tan\big(\frac{a\hat{\partial}_n}{2}\big)\hat{\partial}_j \hat{\partial}_j \\
&\quad +2\breve{A}_j \Big(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n}\cot\big(\frac{a\hat{\partial}_n}{2}\big)\Big)\hat{\partial}_l \hat{\partial}_j \ .
\end{aligned}
\tag{3.22}
$$

Comparing (3.21) and (3.22), we see that $\breve{A}_\mu$ transforms under rotations with the partial derivatives on the right hand side, but it is not equal to the hermitian conjugate $\hat{A}_\mu^\dagger$ of the vector field $\hat{A}_\mu$. The transformation for $\hat{A}_\mu^\dagger$ is simply (3.21), with all $\hat{A}_\mu$ standing to the furthest right in any expression replaced by $\hat{A}_\mu^\dagger$ standing to the furthest left. The vector field $\breve{A}_\mu^\dagger$ transforms as in (3.22), with all $\breve{A}_\mu$ shifted to the right and replaced by $\breve{A}_\mu^\dagger$.

[3] This solution is not unique. If the symmetrisation in the third term of $[N^l, \hat{A}_i]$ is not performed, the last term of $[N^l, \hat{A}_i]$ vanishes.

The Dirac derivatives can be expressed in terms of the partial derivatives $\hat{\partial}_\mu$ as shown in Theorem 3.4.1. We make a similar statement for the vector fields $\hat{V}_\mu$ and $\hat{A}_\mu$:

**Theorem 3.6.2.**
*There exists an invertible derivative-valued map $\hat{e}_{\mu\nu}$, such that $\hat{V}_\mu = \hat{e}_{\mu\nu}\hat{A}_\nu$.*

**Proof.** To zeroth order we assume that $\hat{V}_\mu\big|_{\mathcal{O}(a^0)} = \hat{A}_\mu\big|_{\mathcal{O}(a^0)}$ are the same vector field, hence $\hat{e}_{\mu\nu}\big|_{\mathcal{O}(a^0)} = 1$. Since the vector fields $\hat{V}_\mu, \hat{A}_\mu$ correspond to the derivatives $\hat{D}_\mu, \hat{\partial}_\mu$, the map $\hat{e}_{\mu\nu}$ must satisfy $\hat{D}_\mu = \hat{e}_{\mu\nu}\hat{\partial}_\nu$. Therefore, we consider the following ansatz for $\hat{e}_{\mu\nu}$

$$
\begin{aligned}
\hat{e}_{nn} &= \frac{1}{a}\sin(a\hat{\partial}_n) + \frac{\mathrm{i}a}{2}\hat{\partial}_k\hat{\partial}_k\exp(-\mathrm{i}a\hat{\partial}_n)P(a\hat{\partial}_n) + \frac{1}{\hat{\partial}_n}Q_j(a\hat{\partial}_n)\hat{\partial}_j \\
\hat{e}_{ni} &= \frac{1}{a}\sin(a\hat{\partial}_n) + \frac{\mathrm{i}a}{2}\hat{\partial}_i\exp(-\mathrm{i}a\hat{\partial}_n)(1 - P(a\hat{\partial}_n)) - \frac{1}{\hat{\partial}_n}Q_i(a\hat{\partial}_n) \\
\hat{e}_{in} &= \frac{1}{\hat{\partial}_n}\hat{\partial}_i\exp(-\mathrm{i}a\hat{\partial}_n)R(a\hat{\partial}_n) + \frac{1}{\hat{\partial}_n}\hat{\partial}_i S(a\hat{\partial}_n) \\
\hat{e}_{ij} &= \delta_{ij}\frac{1}{\hat{\partial}_n}\exp(-\mathrm{i}a\hat{\partial}_n)(1 - R(a\hat{\partial}_n)) - \delta_{ij}\frac{1}{\hat{\partial}_n}S(a\hat{\partial}_n)\,,
\end{aligned}
$$

where $P, Q_j, R, S$ are functions of $a\hat{\partial}_n$. Inserting this ansatz into the transformation law of $\hat{V}_\rho$ (3.20) yields the solution

$$
\begin{aligned}
\hat{e}_{nn} &= \frac{1}{a\hat{\partial}_n}\sin(a\hat{\partial}_n) + \exp(-\mathrm{i}a\hat{\partial}_n)\Big(\frac{\mathrm{i}a}{2} - \frac{\mathrm{i}}{\hat{\partial}_n}\tan\big(\frac{a\hat{\partial}_n}{2}\big)\Big)\frac{\hat{\partial}_k\hat{\partial}_k}{\hat{\partial}_n} \\
\hat{e}_{nj} &= \frac{\mathrm{i}}{\hat{\partial}_n}\exp(-\mathrm{i}a\hat{\partial}_n)\tan\big(\frac{a\hat{\partial}_n}{2}\big)\hat{\partial}_j \\
\hat{e}_{ln} &= \Big(\exp(-\mathrm{i}a\hat{\partial}_n) - \frac{1 - \exp(-\mathrm{i}a\hat{\partial}_n)}{\mathrm{i}a\hat{\partial}_n}\Big)\frac{\hat{\partial}_l}{\hat{\partial}_n} \\
\hat{e}_{lj} &= \frac{1 - \exp(-\mathrm{i}a\hat{\partial}_n)}{\mathrm{i}a\hat{\partial}_n}\delta_{lj}
\end{aligned}
$$

in a straightforward way. Using the abbreviation

$$
\begin{aligned}
F(\hat{\partial}_\mu) &= \frac{1}{\mathrm{i}a^2\hat{\partial}_n^2}\sin(a\hat{\partial}_n)\Big(1 - \exp(-\mathrm{i}a\hat{\partial}_n)\Big) - \\
&\quad -\frac{\hat{\partial}_k\hat{\partial}_k}{2\mathrm{i}\hat{\partial}_n^2}\tan(\frac{a\hat{\partial}_n}{2})\exp(-\mathrm{i}a\hat{\partial}_n)\Big(1 - \exp(-\mathrm{i}a\hat{\partial}_n)\Big)\,.
\end{aligned}
$$

the inverse of the matrix $\hat{e}_{\mu\nu}$ can easily be computed to be

$$
\begin{aligned}
(\hat{e}^{-1})_{nn} &= F(\hat{\partial}_\mu)^{-1}\,\frac{\exp(-\mathrm{i}a\hat{\partial}_n)-1}{-\mathrm{i}a\hat{\partial}_n}\\
(\hat{e}^{-1})_{nj} &= F(\hat{\partial}_\mu)^{-1}\left(-\frac{\mathrm{i}}{\hat{\partial}_n}\exp(-\mathrm{i}a\hat{\partial}_n)\tan(\frac{a\hat{\partial}_n}{2})\right)\hat{\partial}_j\\
(\hat{e}^{-1})_{ln} &= F(\hat{\partial}_\mu)^{-1}\left(\frac{\exp(-\mathrm{i}a\hat{\partial}_n)-1}{-\mathrm{i}a\hat{\partial}_n}-\exp(-\mathrm{i}a\hat{\partial}_n)\right)\hat{\partial}_l\\
(\hat{e}^{-1})_{lj} &= \frac{\frac{\mathrm{i}}{\hat{\partial}_n^2}\exp(-\mathrm{i}a\hat{\partial}_n)\tan(\frac{a\hat{\partial}_n}{2})\left(\exp(-\mathrm{i}a\hat{\partial}_n)-\frac{\exp(-\mathrm{i}a\hat{\partial}_n)-1}{-\mathrm{i}a\hat{\partial}_n}\right)}{F(\hat{\partial}_\mu)\left(\frac{\exp(-\mathrm{i}a\hat{\partial}_n)-1}{-\mathrm{i}a\hat{\partial}_n}\right)}\hat{\partial}_l\hat{\partial}_j\\
&\quad+\frac{-\mathrm{i}a\hat{\partial}_n}{\exp(-\mathrm{i}a\hat{\partial}_n)-1}\delta_{lj}
\end{aligned}
$$

□

# Chapter 4

# Kramers-Wannier dualities in CFT

In this chapter we will answer the question whether Kramers-Wannier dualities can be deduced from properties of the conformal field theory at the critical point. The – affirmative – answer uses[1] an algebraic approach to full (rational) conformal field theories [FRS02a, FRS02b] that describes correlation functions of these theories in terms of two pieces of data:

- the chiral data of the conformal field theory, which are encoded in a modular tensor category $\mathcal{C}$
- a (symmetric special) Frobenius algebra $A$ in the tensor category $\mathcal{C}$.

In the topological field theory (TFT) approach to rational conformal field theory, types of topological defect lines correspond to isomorphism classes of $A$-bimodules [FFRS04, FFRS07]. Given two $A$-bimodules $B_1$ and $B_2$, their tensor product $B_1 \otimes_A B_2$ is again a bimodule because of the Frobenius properties of $A$. The bimodule $B_1 \otimes_A B_2$ encodes the fusion of topological defects. In the same way that the modular tensor category $\mathcal{C}$ of chiral data gives rise to a fusion ring $K_0(\mathcal{C})$ of chiral data, the tensor category $\mathcal{C}_{AA}$ of $A$-bimodules gives rise to a fusion ring $K_0(\mathcal{C}_{AA})$ of topological defects. Both fusion rings are semisimple. The fusion ring of defects is, however, not necessarily commutative, since the category of bimodules is typically not braided.[2]

The following insight of [FFRS04, FFRS07] is essential for our considerations: the fusion ring $K_0(\mathcal{C}_{AA})$ of topological defects determines Kramers-Wannier dualities of the full conformal field theory described by the pair $(\mathcal{C}, A)$.

We explain the pertinent results of [FFRS04, FFRS07] in more detail: symmetries of the full conformal field theory $(\mathcal{C}, A)$ correspond to isomorphism classes of invertible objects in $\mathcal{C}_{AA}$. These are objects $B$ satisfying

$$B \otimes_A B^\vee \cong A \qquad \text{and} \qquad B^\vee \otimes_A B \cong A\,, \tag{4.1}$$

where $B^\vee$ is the bimodule dual to $B$. The isomorphism classes of the invertible bimodules form a group, the Picard group $\mathrm{Pic}(\mathcal{C}_{AA})$. This group is not necessarily

[1] See also [Ruel05] for a different approach.

[2] For a more detailed discussion of the fusion ring $K_0(\mathcal{C}_{AA})$ see e.g. [FRS07].

commutative: for the three state Potts model, for example, it turns out to be the symmetric group $S_3$ on three letters, see [FFRS04].

To describe dualities, we need the following

**Definition 4.0.3.** *[FFRS04]*
*Given a modular tensor category $\mathcal{C}$ and a simple symmetric special Frobenius algebra $A$ in $\mathcal{C}$, a simple $A$-bimodule $B$ is called a duality bimodule if and only if all simple subobjects of the tensor product $B^\vee \otimes_A B$ are invertible bimodules.*

It is easy to see that for any duality bimodule $B$ the isomorphism classes of simple bimodules $B_\lambda$, such that $\dim_{\mathbb{C}} \mathrm{Hom}(B_\lambda, B^\vee \otimes_A B) > 0$ form a subgroup $H$ of the Picard group $\mathrm{Pic}(\mathcal{C}_{AA})$ and that in this case $\dim_{\mathbb{C}} \mathrm{Hom}(B_\lambda, B^\vee \otimes_A B) = 1$. We call $H$ the stabilizer of the duality bimodule $B$.

In the following, we restrict ourselves to the case where the simple symmetric special Frobenius algebra in $\mathcal{C}$ is the tensor unit $\mathbb{1}$; this situation is usually referred to as the Cardy case. In the Cardy case, many simplifications occur: the category of $A$-bimodules is equivalent to the original category, $\mathcal{C}_{AA} \simeq \mathcal{C}$; as a consequence, isomorphism classes of invertible bimodules are simple currents [ScYa90b].

As a special case of the results of [FFRS04, FFRS07], we see that the full conformal field theory in the Cardy case has Kramers-Wannier dualities if and only if the underlying modular tensor category has duality classes.

In the following, we study two classes of unitary rational conformal field theories: (super-)Virasoro minimal models and Wess-Zumino-Witten (WZW) theories. There is a WZW theory for every reductive finite-dimensional complex Lie algebra; here, we limit ourselves to simple Lie algebras. Thus, both classes of conformal field theories come in families that are parameterized by a positive integer, called the level.

Simple currents in Virasoro minimal models (and thus their symmetries) are well known; for WZW theories, simple currents have been classified in [Fu91]: with the exception of the WZW theory based on $E_8$ at level 2, they are in bijection with the center of the corresponding simple, connected, simply-connected compact Lie group. It might be surprising at first sight that only the center – rather than at least the full Lie group – shows up as the symmetry group. One should keep in mind, however, that the symmetries we discuss are required to preserve all chiral symmetries, i.e. the complete current algebra. Relaxing this requirement – which amounts to working with the representation category of a subalgebra of the current algebra and a nontrivial symmetric special Frobenius algebra in this category – leads to larger symmetry groups. We will present a classification of Kramers-Wannier dualities in these models.

## 4.1 Model-independent considerations

We start by introducing some notation: we choose a set $(U_\lambda)_{\lambda\in I}$ of representatives for the isomorphism classes of simple objects of the modular tensor category $\mathcal{C}$. In particular, given $\mu \in I$, we find a unique $\mu^\vee$, such that $U_\mu^\vee \cong U_{\mu^\vee}$. The tensor unit $\mathbb{1}$ of a modular tensor category is simple; we choose the representatives, such that $U_0 = \mathbb{1}$ and $0 \in I$.

The classes $[U_\lambda]$ form the distinguished basis of the fusion ring $K_0(\mathcal{C})$. The fusion coefficients

$$\mathcal{N}_{\lambda\mu}^{\ \ \rho} := \dim_{\mathbb{C}} \mathrm{Hom}(U_\lambda \otimes U_\mu, U_\rho)$$

are the structure constants of the multiplication on $K_0(\mathcal{C})$.

For simple objects, we introduce the abbreviation for the quantum dimension

$$\mathcal{D}_\lambda := \dim(U_\lambda)\,.$$

Since the quantum dimensions of the modular tensor categories we consider are the Frobenius-Perron eigenvalues of their fusion matrices, they are real and obey $\mathcal{D}_\lambda \geq 1$. Equality is achieved precisely for simple currents.

It follows immediately from the properties of the quantum dimension that the quantum dimension of a duality class $[U_f]$ of $\mathcal{C}$ is the square root of an integer $\mathcal{D}_f = \sqrt{|H|}$, with $|H|$ the order of the stabilizer $H \leq \mathrm{Pic}(\mathcal{C})$.

In WZW theories the chiral algebra is generated by an untwisted affine Lie algebra $X_r^{(1)}$ of rank $r$; moreover, one has to fix a positive integral value $k$ of the level. We denote the corresponding modular tensor category by $\mathcal{C}[(X_r)_k]$, the fundamental weights by $\Lambda^i$ and the dual Coxeter labels by $a_i^\vee$. The sum of the latter equals the dual Coxeter number, $g^\vee = \sum_{i=0}^r a_i^\vee$. A labelling of the nodes of the Dynkin diagram provides a labelling of simple roots and fundamental weights; we use the conventions of [Kac].

At level $k$, there are finitely many integrable highest weights $\lambda$

$$\lambda \in P_+^k = \left\{ (\lambda_0, \lambda_1, \ldots \lambda_r) := \sum_{i=0}^r \lambda_i \Lambda^i \mid \lambda_i \in \mathbb{Z}_{\geq 0},\ \sum_{i=0}^r a_i^\vee \lambda_i = k \right\}.$$

Isomorphism classes of simple objects of the modular tensor category $\mathcal{C}[(X_r)_k]$ are in bijection to elements of $P_+^k$; in particular, the irreducible highest weight representation with highest weight $k\Lambda_0$ is the tensor unit $U_0$.

For fixed level $k$, the zeroth component of a highest weight is redundant. We therefore work with the finite-dimensional simple Lie algebra $\bar{X}_r$, called the horizontal subalgebra, and the horizontal part $\bar{\lambda} = \sum_{i=1}^r \lambda_i \Lambda^i$ of the weight $\lambda$. The quantum dimension of the simple object $U_\lambda$ can be expressed by a deformed version of Weyl's dimension formula in terms of a product over a set of positive roots of the horizontal subalgebra $\bar{X}_r$

$$\mathcal{D}_\lambda^{(k)} = \prod_{\bar{\alpha}>0} \frac{\lfloor(\bar{\lambda}+\bar{\rho}, \bar{\alpha})\rfloor_k}{\lfloor(\bar{\rho}, \bar{\alpha})\rfloor_k}\,. \tag{4.2}$$

Here $\bar{\rho} = \sum_{i=1}^{r} \Lambda^i$ is the Weyl vector of $\bar{X}_r$ and for given level $k \in \mathbb{N}$, the bracket $\lfloor x \rfloor_k$ of a rational number $x$ is the real number

$$\lfloor x \rfloor_k := \sin\left(\frac{\pi x}{k + g^\vee}\right) .$$

The identity $\lfloor x \rfloor_k = \lfloor k + g^\vee - x \rfloor_k$ is immediate.

We recall that the monodromy character of the object $U_l$ on the group $\mathrm{Pic}(\mathcal{C})$ is defined to be

$$\chi_l([U_k]) := \frac{S_{kl}}{S_{0\,l}}. \tag{4.3}$$

## A criterion for the existence of dualities

We start with the following useful criterion:

**Theorem 4.1.1.**
*Let $H \leq \mathrm{Pic}(\mathcal{C})$ be a subgroup of simple currents. Suppose that there is a simple object $U_\mu$ with the following two properties:*

1. *The restriction of the monodromy character $\chi_\mu$ of $U_\mu$ to $H$ is nontrivial.*
2. *The tensor product of $U_\mu$ with its dual object contains, apart from the tensor unit, just one more simple object:*

$$U_\mu \otimes U_\mu^\vee \cong \mathbb{1} \oplus U_\nu \ .$$

*If r a duality class with stabilizer $H$ can only exist if the simple object $U_\nu$ appearing in the tensor product $U_\mu \otimes U_\mu^\vee$ is a non-trivial simple current.*

**Proof.** Suppose $U_\phi$ is a duality class; then formula (2.5) gives for any $\kappa \in I$:

$$\left|\frac{S_{\phi\kappa}}{S_{0\kappa}}\right|^2 = \frac{S_{\phi\kappa}}{S_{0\kappa}} \frac{S_{\phi^\vee\kappa}}{S_{0\kappa}} = \sum_{J \in H} \frac{S_{J\kappa}}{S_{0\kappa}} \quad ,$$

where $H$ is the stabilizer of $U_\phi$.

The definition of the monodromy character (4.3) together with standard properties of characters of finite abelian groups implies

$$\left|\frac{S_{\phi\kappa}}{S_{0\kappa}}\right|^2 = \sum_{J \in H} \chi_\kappa(J) \ . \tag{4.4}$$

This expression is nonvanishing if and only if the restriction of the monodromy character of $U_\kappa$ to $H$ is trivial. In this case, it equals the order $|H|$ of the group $H$. Thus, $S_{\phi\kappa} = 0$ whenever $U_\phi$ is a duality class and $U_\kappa$ a simple object whose monodromy character restricted to $H$ is nontrivial.

The second property of the simple object $U_\mu$ together with (2.5) immediately gives

$$\left|\frac{S_{\mu\phi}}{S_{0\phi}}\right|^2 = 1 + \frac{S_{\nu\phi}}{S_{0\phi}} \ .$$

By the first assumption, $U_\mu$ has nontrivial monodromy character; therefore $S_{\mu\phi} = 0$. As a consequence,

$$1 + \frac{S_{\nu\phi}}{S_{0\phi}} = 0 \,. \tag{4.5}$$

Next, we notice that due to its appearance in the tensor product $U_\mu \otimes U_\mu^\vee$, the object $U_\nu$ has necessarily trivial monodromy character. Therefore, relation (4.4) yields

$$\left|\frac{S_{\phi\nu}}{S_{0\nu}}\right|^2 = |H| \text{ for } U_\nu \text{ and } \left|\frac{S_{\phi 0}}{S_{00}}\right|^2 = |H| \text{ for the tensor unit } U_0 \,.$$

Taking the quotient of the last two relations gives

$$\left|\frac{S_{\phi\nu}}{S_{\phi 0}}\right|^2 = \left|\frac{S_{0\nu}}{S_{00}}\right|^2 = \mathcal{D}_\nu{}^2 \,.$$

Equation (4.5) now implies that the left hand side of this equation equals one; hence $\nu$ has to be a simple current. □

Since $[U_\nu]$ is a simple current, the simple object $[U_\mu]$ is itself a duality class for the cyclic stabilizer generated by $[U_\nu]$. The simple current $[U_\nu]$ has order two.

The criterion of Theorem 4.1.1 will be applied to the Virasoro minimal models and to WZW models based on untwisted affine Lie algebras in the $A$-series.

## Monotonicity of quantum dimensions

Let $[J]$ be a simple current and $\phi \in I$. Due to the relation

$$\mathcal{N}_{\phi\phi^\vee}^{\ J} = \mathcal{N}_{J\phi}^{\ \phi}$$

the simple current $J$ appears in the decomposition of the tensor product $U_\phi \otimes U_\phi^\vee$, if and only if $[U_\phi]$ is a fixed point of $[J]$. Duality classes are thus, in particular, fixed points under a subgroup $H$ of the Picard group. A fixed point under a subgroup $H$ of the Picard group is a duality class with stabilizer $H$ if and only if its quantum dimension equals $\sqrt{|H|}$.

To obtain constraints on the existence of duality classes, we study the growth of the quantum dimension of fixed points of subgroups of the Picard group as a function of the level. The following Lemma (cf. also [Fu92]) stating the monotonicity of quantum dimensions plays a key role in order to find duality classes for WZW theories based on untwisted affine Lie algebras in the $B$-, $C$-, $D$- series and exceptional Lie algebras.

**Lemma 4.1.2.**
*Let $X_r^{(1)}$ be an untwisted affine Lie algebra. Let $\lambda(k)$ be a family of integral highest weights of $X_r^{(1)}$ at level $k$ of the form*

$$\lambda(k) = \sum_{i=1}^{r} \lambda_i \Lambda^i \ + (k - \sum_{i=1}^{r} a_i^\vee \lambda_i)\Lambda^0 \ ,$$

*i.e. where only the zeroth component of $\lambda(k)$ depends on the level. Assume that not all $\lambda_i$ vanish. Denote by $\mathcal{D}_\lambda(k)$ the quantum dimension of $\lambda(k)$ at level $k \in \mathbb{Z}_{\geq 0}$. Then the expression of equation (4.2) for $\mathcal{D}_\lambda(k)$ defines an analytic function on $\mathbb{R}_{\geq 0}$ which is strictly monotonically increasing.*

**Proof.** Differentiating equation (4.2) with respect to $k$ yields

$$\frac{\partial}{\partial k}\mathcal{D}_\lambda(k) = \mathcal{D}_\lambda(k) \sum_{\bar\alpha > 0} (f_k((\bar\rho, \bar\alpha)) - f_k((\bar\lambda + \bar\rho, \bar\alpha)))\,, \tag{4.6}$$

where the function

$$f_k(x) := \frac{\pi x}{(k+g^\vee)^2} \cot(\frac{\pi x}{k+g^\vee})$$

is strictly monotonically decreasing as a function of $x$ for all fixed levels $k$. Thus, all summands are nonnegative; moreover, $(\bar\lambda, \bar\alpha) > 0$ for at least one positive root $\bar\alpha$. Hence, the expression (4.6) is positive. □

# 4.2 Dualities for WZW theories

In the sequel we will use the slightly redundant notation $\mathcal{D}_\lambda^{(k,r)}$ for the quantum dimension of the weight $\lambda$ of the algebra $(X_r)_k$ at level $k$. Simple objects will be referred to by the horizontal part of their weight; for reasons of simplicity, we will drop overlines over horizontal weights.

The modular tensor categories based on the Lie algebras $E_8$ for level greater or equal to three, $F_4$ and $G_2$ do not have non-trivial simple currents [Fu91]. As a consequence, no duality classes exist. At level two, $E_8$ has Ising fusion rules and thus has a duality class with cyclic stabilizer of order two.

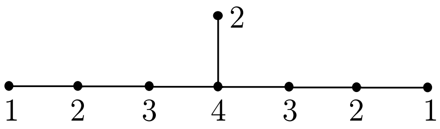
We establish the following results:

**Theorem 4.2.1.**
*The only Wess-Zumino-Witten theories $(\mathcal{C}[(X_r)_k], \mathbb{1})$ with duality classes are $(E_7)_2, (E_8)_2, (A_1)_2, (B_r)_1$ and $(D_{2r})_2$ with $r \geq 2$.*

Since the finite-dimensional complex simple Lie algebras $B_2$ and $C_2$ are isomorphic, we do not list $(C_2)_1$ separately. It should be noted that dualities only appear at low level. We are, unfortunately, not aware of any a priori argument for this finding. For all cases except $(D_{2r})_2$, the stabilizer is cyclic of order two. For most of the cases, the existence of the dualities does not come as a surprise: the modular tensor categories for $(A_1)_2$ and $(B_r)_1$ have the same fusion rules as the Ising model. For $(E_7)_2$ the fusion rules are isomorphic to those of the tricritical Ising model and hence the category contains a tensor subcategory with Ising fusion rules.

## The affine Lie algebra $E_7^{(1)}$

We depict the Dynkin diagram of $E_7^{(1)}$ with the Coxeter labels:




The Picard groups of the tensor categories based on $E_7^{(1)}$ are cyclic of order two, generated by the irreducible highest weight representation with highest weight $k\Lambda^6$. We are thus led to classify fixed points of quantum dimension $\sqrt{2}$. The action of a simple current on highest weights corresponds to a symmetry of the Dynkin diagram; the nodes invariant under the symmetry corresponding to the nontrivial simple current all have even Coxeter labels. Since fixed points correspond to highest weights invariant under this symmetry, they – and hence duality classes – only occur at even level.

According to [GRW96], the second smallest quantum dimension for given level $k \geq 5$ occurs for horizontal weight $\Lambda^6$, $\mathcal{D}_\lambda^{(k)} \geq \mathcal{D}_{\Lambda^6}^{(k)}$. The monotonicity lemma 4.1.2 yields

$$\mathcal{D}_{\Lambda^6}^{(k)} \geq \mathcal{D}_{\Lambda^6}^{(5)} = \frac{\lfloor 10 \rfloor_5}{\lfloor 1 \rfloor_5} > \sqrt{2}.$$

Thus, there are no duality classes for level $k \geq 5$.

The simple objects of second lowest quantum dimension at level $k = 2$ and at level $k = 4$ have been listed in Table 3 of [GRW96]. At level $k = 2$, one finds $\Lambda^7$ which is a fixed point of quantum dimension

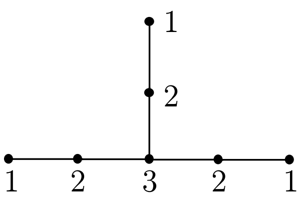
$$\mathcal{D}_{\Lambda^7}^{(2)} = \frac{\lfloor 2 \rfloor_2 \lfloor 6 \rfloor_2 \lfloor 8 \rfloor_2}{\lfloor 3 \rfloor_2 \lfloor 4 \rfloor_2 \lfloor 5 \rfloor_2 \lfloor 7 \rfloor_2} = \sqrt{2}$$

and thus a duality class. Indeed, the fusion rules of $E_7^{(1)}$ at level 2 are isomorphic to the fusion rules of the tricritical Ising model which is known to exhibit a Kramers-Wannier duality.

At level 4, according to the same table, the second smallest quantum dimension is assumed for $2\Lambda^7$. The quantum dimension is larger than $\sqrt{2}$ (this can be shown e.g. by using the computer program KAC [Schell]). We conclude that WZW theories based on the untwisted affine Lie algebra $E_7^{(1)}$ exhibit a duality class only at level two.

## The affine Lie algebra $E_6^{(1)}$

The Dynkin diagram of $E_6^{(1)}$ with the Coxeter labels is:



The Picard groups of the tensor categories based on $E_6^{(1)}$ are cyclic of order three. Duality objects are therefore precisely the fixed points of quantum dimension $\sqrt{3}$. It follows from the values of the Coxeter labels that they can only occur at levels divisible by three.

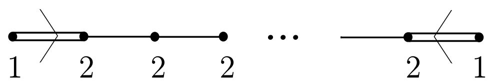
According to [GRW96], for $k \geq 3$, the second smallest quantum dimension occurs for the weight $\Lambda^1$, $\mathcal{D}_\lambda^{(k)} \geq \mathcal{D}_{\Lambda^1}^{(k)}$. By monotonicity in $k$, we derive a lower bound for the quantum dimensions of simple objects, provided the level $k$ is not smaller than three,

$$\mathcal{D}_\lambda^{(k)} \geq \mathcal{D}_{\Lambda^1}^{(3)} = \frac{\lfloor 3 \rfloor_3 \lfloor 6 \rfloor_3}{\lfloor 1 \rfloor_3 \lfloor 4 \rfloor_3} > \sqrt{3}$$

and deduce that there are no Kramers-Wannier dualities for WZW theories based on the untwisted affine Lie algebra $E_6^{(1)}$.

## The series $C_r^{(1)}$

We depict the Dynkin diagram of $C_r^{(1)}$ and the Coxeter labels:




The Picard groups of WZW theories based on the untwisted affine Lie algebra $C_r^{(1)}$ are cyclic of order two. A duality class must be a fixed point of quantum dimension $\sqrt{2}$. It follows from the values of the Coxeter labels that the level $k$ must be even for odd rank $r$; for even rank there is no restriction on the level.

The isomorphism of complex simple Lie algebras $C_1^{(1)} \cong A_1^{(1)}$ allows us to assume that $r \geq 2$. For $r \geq 2$ and $k = 1$ or $k+r \geq 6$ the second minimal quantum dimension is given [GRW96] by

$$\mathcal{D}_\lambda^{(k,r)} \geq \mathcal{D}_{\Lambda^1}^{(k,r)} = \frac{\lfloor \frac{r}{2} \rfloor_k \lfloor r+1 \rfloor_k}{\lfloor \frac{1}{2} \rfloor_k \lfloor \frac{r+1}{2} \rfloor_k} \,.$$

Lemma 4.1.2 states $\mathcal{D}_{\Lambda^1}^{(k,r)} \geq \mathcal{D}_{\Lambda^1}^{(1,r)}$ with equality of $k = 1$ and we compute

$$\mathcal{D}_{\Lambda^1}^{(1,r)} = \frac{\lfloor 1 \rfloor_1 \lfloor \frac{r}{2} \rfloor_1}{\lfloor \frac{1}{2} \rfloor_1 \lfloor \frac{r+1}{2} \rfloor_1} \geq \sqrt{2}\,,$$

with equality for $r = 2$. Indeed, the tensor category for $(C_2^{(1)})_1 \cong (B_2^{(1)})_1$ has Ising fusion rules and therefore displays a Kramers-Wannier duality.

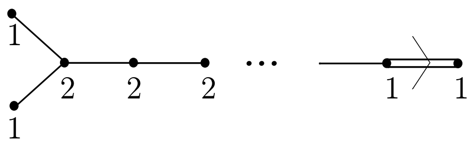
For $k + r < 6$ second minimal quantum dimensions are

$$\begin{aligned}
\mathcal{D}_{2\Lambda^1}^{(2,2)} &= \frac{\lfloor \frac{5}{2} \rfloor_2 \lfloor 1 \rfloor_2}{\lfloor 1 \rfloor_2 \lfloor 2 \rfloor_2} = 2 > \sqrt{2} \\
\mathcal{D}_{\Lambda^1}^{(2,3)} &= \frac{\lfloor \frac{3}{2} \rfloor_2 \lfloor 4 \rfloor_2}{\lfloor \frac{1}{2} \rfloor_2 \lfloor 2 \rfloor_2} = 1 + \sqrt{3} > \sqrt{2} \\
\mathcal{D}_{\Lambda^1}^{(3,2)} &= \frac{\lfloor 1 \rfloor_3 \lfloor 3 \rfloor_3}{\lfloor \frac{1}{2} \rfloor_3 \lfloor \frac{3}{2} \rfloor_3} = 1 + \sqrt{3} > \sqrt{2}\,.
\end{aligned}$$

As a consequence, the only duality class occurs for WZW theories based on the untwisted affine Lie algebra $C_2^{(1)}$ at level 1.

### The series $B_r^{(1)}$

We depict the Dynkin diagram of $B_r^{(1)}$ and the Coxeter labels:




Again the Picard group is cyclic of order two so that we are led to classify fixed points of quantum dimension $\sqrt{2}$.

Because of the isomorphisms $C_1^{(1)} \cong A_1^{(1)}$ and $C_2^{(1)} \cong B_2^{(1)}$ of finite-dimensional complex Lie algebras, we restrict ourselves to $r \geq 3$. For $r \geq 3$ and $k \geq 4$ or $k = 2$ the second minimal quantum dimension is given [GRW96] by

$$\mathcal{D}_{\Lambda^1}^{(k,r)} = \frac{\lfloor r+\frac{1}{2}\rfloor_k \lfloor 2r-1\rfloor_k}{\lfloor 1\rfloor_k \lfloor r-\frac{1}{2}\rfloor_k}$$

and with the monotonicity lemma 4.1.2 we obtain

$$\mathcal{D}_{\Lambda^1}^{(k,r)} > \mathcal{D}_{\Lambda^1}^{(2,r)} = \frac{\lfloor 2\rfloor_2 \lfloor r+\frac{1}{2}\rfloor_2}{\lfloor 1\rfloor_2 \lfloor r-\frac{1}{2}\rfloor_2} = 2\,.$$

This excludes duality classes for all levels $k \geq 4$ and $k = 2$.

At level $k = 1$, we find Ising fusion rules and thus the unique duality class $\Lambda^r$. At level $k = 3$, the second lowest quantum dimension is assumed for the weight $3\Lambda^r$ [GRW96]. For its quantum dimension, we find

$$\mathcal{D}_{3\Lambda^r}^{(3,r)} = \frac{\lfloor 2\rfloor_3}{\lfloor r\rfloor_3} \prod_{l=1}^{r} \frac{\lfloor 2l-1\rfloor_3}{\lfloor l-\frac{1}{2}\rfloor_3}$$

which is strictly larger than $\sqrt{2}$ for all ranks $r \geq 2$: we consider the case of even and odd rank separately; for even rank we find

$$\mathcal{D}_{3\Lambda^r}^{(3,r)} = \frac{\lfloor 1\rfloor_3 \lfloor 2\rfloor_3}{\lfloor \frac{1}{2}\rfloor_3 \lfloor \frac{3}{2}\rfloor_3} \prod_{l=2}^{\frac{r}{2}} \frac{\lfloor 2l-1\rfloor_3 \lfloor 2l-1\rfloor_3}{\lfloor 2l-\frac{1}{2}\rfloor_3 \lfloor 2l-\frac{3}{2}\rfloor_3} \; \frac{\lfloor r+1\rfloor_3}{\lfloor r\rfloor_3}\;;$$

with the notation $\lceil x\rceil_k := \cos\left(\frac{\pi x}{k+g^\vee}\right)$ this is equal to

$$\mathcal{D}_{3\Lambda^r}^{(3,r)} = \frac{\lfloor 1\rfloor_3}{\lfloor \frac{1}{2}\rfloor_3} \frac{\lfloor 2\rfloor_3}{\lfloor \frac{3}{2}\rfloor_3} \prod_{l=2}^{\frac{r}{2}} \frac{1-\lceil 4l-2\rceil_3}{\lceil 1\rceil_3 - \lceil 4l-2\rceil_3} \; \frac{\lfloor r+1\rfloor_3}{\lfloor r\rfloor_3}\,.$$

Since the arguments of the sine- and cosine-functions are all smaller than $\pi/2$, all quotients are bigger than one. Because of $\frac{\frac{\pi}{2}}{2r+2} < \frac{\pi}{6}$ for any value of $r$, one has

$\lfloor 1 \rfloor_3 \geq \sqrt{3} \lfloor \frac{1}{2} \rfloor_3$. Therefore, the quantum dimension $\mathcal{D}^{(3,r)}_{3\Lambda^r}$ is strictly larger than $\sqrt{2}$ for even rank.

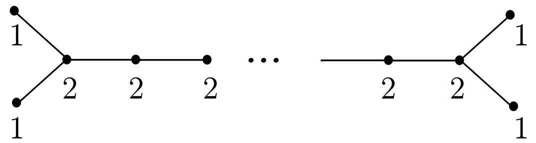
We proceed in an analogous way for odd rank to find

$$\mathcal{D}^{(3,r)}_{3\Lambda^r} = \frac{\lfloor 1 \rfloor_3}{\lfloor \frac{1}{2} \rfloor_3} \frac{\lfloor 2 \rfloor_3}{\lfloor \frac{3}{2} \rfloor_3} \prod_{l=2}^{\frac{r-1}{2}} \frac{1 - \lceil 4l-2 \rceil_3}{\lceil 1 \rceil_3 - \lceil 4l-2 \rceil_3} \frac{\lfloor r \rfloor_3}{\lfloor r - \frac{1}{2} \rfloor_3} \geq \sqrt{3}\,.$$

We conclude that WZW theories based on the untwisted affine Lie algebra $B^{(1)}_r$ do not exhibit dualities for levels $k \geq 2$.

## The series $D_r$

We depict the Dynkin diagram of $D^{(1)}_r$ and the Coxeter labels:




The structure of the Picard group depends on whether the rank is odd or even. For even rank, it is isomorphic to the Kleinian four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ with generators $k\Lambda^{r-1}$ and $k\Lambda^r$. Accordingly, we have to consider this group as well as its three cyclic subgroups of order two as possible stabilizers. For odd rank, the Picard group is cyclic of order four with $k\Lambda^{r-1}$ or $k\Lambda^r$ as possible generators. The possible stabilizers are then the full Picard group and the cyclic group of order two generated by $k\Lambda^1$.

For $r \geq 4$ and $k \geq 3$ the second minimal quantum dimension is given by

$$\mathcal{D}^{(k,r)}_{\Lambda^1} = \frac{\lfloor r \rfloor_k \lfloor 2r-2 \rfloor_k}{\lfloor 1 \rfloor_k \lfloor r-1 \rfloor_k}$$

and a lower bound is by Lemma 4.1.2

$$\mathcal{D}^{(k,r)}_{\Lambda^1} > \mathcal{D}^{(2,r)}_{\Lambda^1} = \frac{\lfloor 2 \rfloor_2 \lfloor r \rfloor_2}{\lfloor 1 \rfloor_2 \lfloor r-1 \rfloor_2} = 2\,.$$

We deduce that for $k \geq 3$ no duality classes for any stabilizer exist: for cyclic stabilizers of order two this is immediate. Duality objects whose stabilizer is the full Picard group must be fixed points under all four simple currents. They can only occur at even levels; but for level equal to four and higher, by the monotonicity properties of the quantum dimensions, no simple object of quantum dimension two exists.

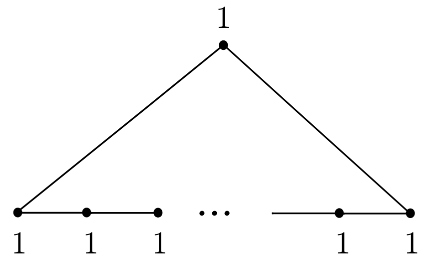
The remaining case is level $k = 2$. It is convenient to treat $D_4$ separately: in this case, simple objects have quantum dimension equal to one or two. There is a single duality class with highest weight $\Lambda^2$ which is a fixed point under the whole Picard group.

For rank greater or equal to five and level $k = 2$, there are four simple currents, four simple objects of quantum dimension strictly bigger than two and all simple objects with weights $\Lambda^1$, $\Lambda^{r-1} + \Lambda^r$ and $\Lambda^i$ with $i = 2, \ldots r-2$ have quantum dimension 2. Among these weights, however, only $\Lambda^{r/2}$ for even rank is a fixed point under the whole Picard group and provides a duality class with the Picard group $\mathbb{Z}_2 \times \mathbb{Z}_2$ as the stabilizer.

Thus, WZW theories based on the untwisted affine Lie algebra $D_r^{(1)}$ exhibit dualities only for even rank at level $k = 2$.

## The series $A_r$

We depict the Dynkin diagram of $A_r^{(1)}$ and the Coxeter labels:




The Picard group is isomorphic to $\mathbb{Z}_{r+1}$ for rank $r$. In order to find dualities for the $A$-series the monotonicity argument is not sufficient: for $k \geq 2$ the second minimal quantum dimension is given [GRW96] by

$$\mathcal{D}_{\Lambda^1}^{(k,r)} = \frac{\lfloor r+1 \rfloor_k}{\lfloor 1 \rfloor_k}$$

and with the monotonicity lemma 4.1.2 we obtain

$$\mathcal{D}_{\Lambda^1}^{(k,r)} > \mathcal{D}_{\Lambda^1}^{(2,r)} = \frac{\lfloor 2 \rfloor_2}{\lfloor 1 \rfloor_2} = 2\cos(\frac{\pi}{r+3}) \geq \sqrt{2}\,,$$

which does not excludes duality classes for levels $k \geq 2$.

Thus, we must follow a different strategy. We can use Theorem 4.1.1 with $U_\mu$ equal to the defining $(r+1)$-dimensional irreducible representation with highest weight $\Lambda^1$. Monodromy classes of WZW theories are in correspondence with conjugacy classes of representations. For $A_r$ the defining representation is a generator of the group of conjugacy classes of representations and hence has nontrivial monodromy character. For the tensor product with the dual simple object we find

$$\Lambda^1 \otimes (\Lambda^1)^\vee = \Lambda^1 \otimes \Lambda^{r-1} \cong 0 \oplus (\Lambda^1 + \Lambda^r).$$

The second simple object in the direct sum is an invertible object only for $r = 1$ and $k = 2$. Thus, the only duality class appears for $(A_1)_2$ which is known to have Ising fusion rules.

## 4.3 Dualities for (super-)minimal models

Concerning Virasoro minimal models and their superconformal counterparts, we find the following situation.

**Theorem 4.3.1.**

*(i) Duality classes only exist for the unitary Virasoro minimal models at level $k = 1$ and $k = 2$, i.e. for the Ising model and the tricritical Ising model.*

*(ii) Duality classes for an $N = 1$ super-Virasoro unitary minimal model only exist for odd levels. If they exist, they are unique.*

*(iii) $N = 2$ superconformal minimal models have duality classes only for level $k = 2$.*

Our findings agree, for the nonsupersymmetric theories, with the ones of [Ruel05]; our methods, however, are different. Again we find a close relation to Ising fusion rules: all models with a Kramers-Wannier duality have a realization by a coset construction involving $(A_1)_2$.

### Virasoro minimal models

Nonsupersymmetric Virasoro minimal models can be obtained by the coset construction [GKO86] from the diagonal embedding

$$(A_1)_{k+1} \hookrightarrow (A_1)_k \oplus (A_1)_1 \ .$$

To describe isomorphism classes of simple objects, consider triples $\Phi_t^{l\ s}$, where $l \in \{0, \dots k\}$, $s \in \{0, 1\}$ and $t \in \{0, \dots k+1\}$, subject to the condition

$$l + s - t = 0 \bmod 2 \, .$$

On such triples, simple objects are equivalence classes of the relation

$$\Phi_t^{l\ s} \sim \Phi_{k+1-t}^{k-l\ \ 1-s} \, .$$

The decomposition of the tensor product of simple objects into a direct sum of simple objects is given by

$$\Phi_{t_1}^{l_1\ s_1} \otimes \Phi_{t_2}^{l_2\ s_2} \cong \bigoplus_{l_3=|l_1-l_2|}^{l_{\max}} \ \bigoplus_{t_3=|t_1-t_2|}^{t_{\max}} \Phi_{t_3}^{l_3\ s_3}$$

with $l_{\max} = \min(l_1 + l_2, 2k - l_1 - l_2)$ and $t_{\max} = \min(t_1 + t_2, 2k + 2 - t_1 - t_2)$, and where the indices are required to fulfill

$$l_1 + l_2 + l_3 = 0 \bmod 2 \qquad \text{and} \qquad t_1 + t_2 + t_3 = 0 \bmod 2 \, .$$

The selection rule $l + s - t = 0 \bmod 2$ fixes the value of $s_3$ in terms of $l_3$ and $t_3$.

The simple currents apart from the isomorphism class of the tensor unit are

$$\begin{array}{lcll} \Phi_{k+1}^{0\ \ 0} & \sim & \Phi_0^{k\ 1} & \text{for } k \text{ odd} \\ \Phi_{k+1}^{0\ \ 1} & \sim & \Phi_0^{k\ 0} & \text{for } k \text{ even,} \end{array}$$

hence $\mathrm{Pic}(\mathcal{C})$ is cyclic of order two. The monodromy characters are products of monodromy characters for WZW theories based on $A_1$.

The only simple objects that can be used to apply Theorem 4.1.1 are $\Phi_0^{1\ 1}, \Phi_1^{0\ 1}$ and $\Phi_k^{0\ 1}$ for odd level $k$ and $\Phi_{k+1}^{1\ \ 0}$ for even level $k$. The tensor product with the dual object is

$$\begin{array}{rcll} \Phi_0^{1\ 1} \otimes \Phi_0^{1\ 1} & \cong & \Phi_0^{0\ 0} \oplus \Phi_0^{2\ 0} & \text{for level } k \geq 2 \\ \Phi_1^{0\ 1} \otimes \Phi_1^{0\ 1} & \cong & \Phi_0^{0\ 0} \oplus \Phi_2^{0\ 0} & \\ \Phi_k^{0\ 1} \otimes \Phi_k^{0\ 1} & \cong & \Phi_0^{0\ 0} \oplus \Phi_2^{0\ 0} & \text{for odd level} \\ \Phi_{k+1}^{1\ \ 0} \otimes \Phi_{k+1}^{1\ \ 0} & \cong & \Phi_0^{0\ 0} \oplus \Phi_0^{2\ 0} & \text{for even level.} \end{array}$$

$\Phi_2^{0\ 0}$ is a simple current only for $k = 1$; this case is indeed the Ising model with its well-known Kramers-Wannier duality. The primary field $\Phi_0^{2\ 0}$ is a simple current only for $k = 2$. In this case, we recover the known Kramers-Wannier duality of the tricritical Ising model.

## $N = 1$ super-Virasoro minimal models

The $N = 1$ superconformal minimal models have the following coset description [GKO86] based on the diagonal embedding

$$(A_1)_{k+2} \hookrightarrow (A_1)_k \oplus (A_1)_2\ .$$

To describe isomorphism classes of simple objects, consider triples $\Phi_t^{l\ s}$ with $l \in \{0, \dots k\}$, $s \in \{0, 1, 2\}$ and $t \in \{0, \dots k + 2\}$, subject to the requirement

$$l + s - t = 0 \bmod 2.$$

For odd level $k$, representatives for simple objects are labelled by equivalence classes of triples under the equivalence relation

$$\Phi_t^{l\ s} \sim \Phi_{k+2-t}^{k-l\ \ 2-s}.$$

For even level, the same holds with the exception that there are two simple objects corresponding to the triple $\Phi_{\frac{k}{2}+1}^{\frac{k}{2}\ \ 1}$. This phenomenon is called "fixed point resolution" in the physics literature (see e.g. [ScYa90a] or [ScYa90b] for a review).

The tensor products with no fixed points involved are given by

$$\Phi_{t_1}^{l_1\ s_1} \otimes \Phi_{t_2}^{l_2\ s_2} \cong \bigoplus_{l_3=|l_1-l_2|}^{l_{\max}} \bigoplus_{s_3=|s_1-s_2|}^{s_{\max}} \bigoplus_{t_3=|t_1-t_2|}^{t_{\max}} \Phi_{t_3}^{l_3\ s_3},$$

where the indices are required to obey

$$\begin{array}{rcl} l_1 + l_2 + l_3 & = & 0 \bmod 2 \\ s_1 + s_2 + s_3 & = & 0 \bmod 2 \\ t_1 + t_2 + t_3 & = & 0 \bmod 2 \end{array}$$

with $l_{\max} = \min(l_1+l_2, 2k-l_1-l_2)$, $s_{\max} = \min(s_1+s_2, 4-s_1-s_2)$ and $t_{\max} = \min(t_1+t_2, 2k+4-t_1-t_2)$.

The simple currents are

$$\begin{array}{rclr} \Phi^{0\ 0}_{0} & \sim & \Phi^{k\ \ 2}_{k+2} & \\ \Phi^{0\ 2}_{0} & \sim & \Phi^{k\ \ 0}_{k+2} & \\ \Phi^{0\ \ 0}_{k+2} & \sim & \Phi^{k\ 2}_{0} & \text{for } k \text{ even} \\ \Phi^{0\ \ 2}_{k+2} & \sim & \Phi^{k\ 0}_{0} & \text{for } k \text{ even,} \end{array}$$

hence $\mathrm{Pic}(\mathcal{C}) \cong \mathbb{Z}_2$ for odd level $k$ and $\mathrm{Pic}(\mathcal{C}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ for even level $k$.

We start with the discussion of dualities with cyclic stabilizer of order two. The duality classes have to be fixed points of quantum dimension $\sqrt{2}$ under the action of a nontrivial simple current of order two. The quantum dimension of $\Phi^{l\ m}_{s}$ is a product of the $A_1$ quantum dimensions.

We consider first fixed points of the simple current $\Phi^{0\ 2}_{0}$: in this case $s=1$, so that this $A_1$ summand already contributes a multiplicative factor $\sqrt{2}$ to the quantum dimension; the other labels $l, t$ have to correspond to $A_1$-simple currents: $l \in \{0, k\}$ and $t \in \{0, k+2\}$. This is excluded by the selection rule $l+s+t = 0 \bmod 2$ for even level $k$. For odd level $k$ we find a single duality class $\Phi^{0\ \ 1}_{k+2}$ with a cyclic stabilizer of order two generated by $\Phi^{0\ 2}_{0}$.

For even $k$ we have to consider fixed points under the action of the simple current $\Phi^{0\ \ 0}_{k+2}$ as well. In this case, the relevant $A_1$ constituent has level greater or equal to three, so that already this part makes a multiplicative contribution to the quantum dimensions strictly bigger than $\sqrt{2}$. Thus, the cyclic group of order two generated by $\Phi^{0\ \ 0}_{k+2}$ for $k$ even never occurs as a stabilizer.

We finally consider fixed points of the simple current $\Phi^{k\ 0}_{0}$ for even level $k$. By the same arguments, quantum dimension $\sqrt{2}$ for a fixed point can only be achieved for $k=2$ and for $\Phi^{1\ s}_{t}$ with $s, t$ describing simple currents. Such fixed points are, however, excluded by the selection rule $l+m+s = 0 \bmod 2$.

For even level $k$, the full Picard group could appear as a stabilizer as well. We should therefore find all fixed points of quantum dimension two under the action of all four simple currents. Only the two simple objects arising in the "fixed point resolution" of $\Phi^{k/2\ \ \ 1}_{(k+2)/2}$ qualify. Indeed, a computation with KAC [Schell] shows that quantum dimension 2 is achieved for level $k=2$. Since the monotonicity lemma for WZW theories implies, by the coset construction, the same monotonicity properties in $k$, this is the only relevant case. A computation of the fusion rules, e.g. again with KAC, however shows that the two simple objects arising in the fixed resolution are not fixed points of all four simple currents. Hence, for even $k$, there is no duality object whose stabilizer is the full Picard group.

## $N = 2$ super-Virasoro minimal models

A coset realization for $N = 2$ superconformal minimal models is based on the embedding

$$u(1)_{2(k+2)} \hookrightarrow (A_1)_k \oplus u(1)_4\,,$$

where $(u_1)_N$ stands for the modular tensor category with $K_0((u(1)_N) = \mathbb{Z}\,/\,N\mathbb{Z}$ (for details, see e.g. Section 2.5.1 of [FRS02b]). To describe simple objects, consider triples ${\Phi^l}_t^s$ with $l \in \{0, \dots k\}$, $s \in \mathbb{Z}/4\mathbb{Z}$ and $t \in \mathbb{Z}/2(k+2)\mathbb{Z}$, subject to the condition

$$l + s - t = 0 \bmod 2\ .$$

Isomorphism classes of simple objects can be labelled by equivalence classes of the equivalence relation

$$\Phi_t^{l\ s} \sim \Phi_{t\pm(k+2)}^{k-l\quad s\pm 2}.$$

The tensor products read

$$\Phi_{t_1}^{l_1\ s_1} \otimes \Phi_{t_2}^{l_2\ s_2} \ \cong \ \bigoplus_{l_3=|l_1-l_2|}^{l_{\max}} \ \Phi_{(t_1+t_2)}^{l_3\quad (s_1+s_2)}$$

with $l_{\max} = \min(l_1 + l_2, 2k - l_1 - l_2)$, where the index $l$ must obey the selection rule

$$l_1 + l_2 + l_3 = 0 \bmod 2\ .$$

We can apply Theorem 4.1.1 to the isomorphism class of simple objects ${\Phi^1}_0^1$ for which the relevant tensor product reads for $k \geq 2$:

$$\Phi_0^{1\ 1} \otimes \Phi_0^{1\ -1} \ \cong \ \Phi_0^{0\ 0} \oplus \Phi_0^{2\ 0}\,.$$

Unless $k = 2$, the second simple object is not a simple current. For $k = 2$ we find 16 simple currents $\Phi_t^{0\ s} \sim \Phi_{t\pm 4}^{2\ s\pm 2}$ with $s - t$ even. All other 8 objects $\Phi_t^{1\ s}$ with $s - t$ even are duality classes with fusion with the dual

$$\Phi_t^{1\ s} \otimes \Phi_{-t}^{1\ -s} \ \cong \ \Phi_0^{0\ 0} \oplus \Phi_0^{2\ 0}\,.$$

Thus, they all have a cyclic stabilizer generated by $\Phi_0^{2\ 0}$, the worldsheet supercurrent.

The $N = 2$ superconformal theory with $k = 1$ has Virasoro central charge $c = 1$ and is equivalent to a free boson compactified on a circle of appropriate radius. As a consequence, all isomorphism classes of simple objects are simple currents in this case and there are no duality objects.

# Chapter 5

# Generalized Reconstruction

The main aim of this chapter is to provide a link between the language of tensor categories and weak Hopf algebras. We present a generalization to weak Hopf algebras of the Tannaka-Krein Reconstruction Theorem. The classical situation of Tannaka-Krein Reconstruction [DeMi82, Del90]: Let $\mathbb{K}$ be a field, $\mathcal{C}$ a $\mathbb{K}$-linear abelian artinian rigid tensor category which is essentially small and $(\omega, \omega_2, \omega_0) : \mathcal{C} \to \mathrm{vect}_{\mathbb{K}}$ an exact and faithful fiber functor[1]. Then the $\mathbb{K}$-algebra $H = \operatorname{End}\omega$ of natural endotransformations of $\omega$ is a Hopf algebra. In addition, the category of finite-dimensional $H$-left modules $H$-mod is equivalent to $\mathcal{C}$ as a tensor category and composition of the equivalence with the forgetful functor $H\text{-mod} \to \mathrm{vect}_{\mathbb{K}}$ is equal to $\omega$. The coalgebra structure of $H$ is given by

$$\Delta h = \omega_2^{-1} \circ h \circ \omega_2 \qquad\qquad \epsilon(h) = \omega_0^{-1} \circ h_{\mathbb{1}} \tag{5.1}$$

and the antipode is

$$S(h_U) = (h_{U^\vee})^\vee . \tag{5.2}$$

There exists a second formulation of the Tannaka-Krein Reconstruction Theorem: given a field $\mathbb{K}$, a $\mathbb{K}$-linear abelian artinian tensor category $\mathcal{C}$ and an exact and faithful fiber functor $\omega$, there exists a $\mathbb{K}$-algebra $\hat{H}$, such that the category of finite-dimensional $\hat{H}$-left comodules $\hat{H}$-comod is equivalent to $\mathcal{C}$ as a tensor category and composition of the equivalence with the forgetful functor is equal to $\omega$. The bialgebra $\hat{H}$ is the dual vector space of $H$.

In most cases of conformal field theory, the modular tensor category encoding the data of the CFT does not admit a fiber functor into the category of vector spaces, but a tensor functor into the category of bimodules over a separable $\mathbb{C}$-algebra $R$, that is an $R$-fiber functor. Hence, a generalization to $R$-fiber functors and weak bialgebras is needed.

The second formulation of the Tannaka-Krein Reconstruction Theorem has been generalized to weak Hopf algebras in [Hay99] and [Pfe07]. In the following we generalize the first formulation of the Tannaka-Krein Reconstruction Theorem to weak Hopf algebras. The advantage of our approach is that the weak Hopf algebra structure is obtained in a very simple way: we compose the maps (5.1), (5.2) with

[1] Compare the notation with Definition 2.2.7.

appropriate structural maps of the separable algebra $R$.

In the following all algebras we consider shall be $\mathbb{C}$-algebras. We only consider modules and bimodules that are finite-dimensional $\mathbb{C}$-vector spaces. All categories we study shall be small, enriched over $\mathrm{vect}_{\mathbb{C}}$, abelian, semisimple and artinian. By $\mathcal{M}$ we denote a right module category over a tensor category $\mathcal{C}$; for the associativity and unit isomorphisms of $\mathcal{M}$ we write $\psi$ and $\eta$, for the associativity and unit isomorphisms of $\mathcal{C}$ we write $a$, $\lambda$ and $\mu$. We introduce the following notation: we write $\{M_\alpha\}_{\alpha\in\Gamma}$, respectively $\{U_i\}_{i\in I}$, for representatives of the isomorphism classes of simple objects in $\mathcal{M}$, respectively $\mathcal{C}$. We choose the representatives, such that $U_0 = \mathbb{1}_{\mathcal{C}}$ and $0 \in I$. Since $\mathcal{M}$ is semisimple, any generator of $\mathcal{M}$ is isomorphic to $\tilde{M} := \oplus_{\alpha\in\Gamma} n_\alpha M_\alpha$ for some choice of $n_\alpha \in \mathbb{Z}_{>0}$.

## 5.1 Module categories and fiber functors

For a category $\mathcal{M}$ let $R$ be the algebra of endomorphisms of a generator $\tilde{M}$. In [Ostr03] it has been shown that there exists a bijection between the set of the structures of a module category over a tensor category $\mathcal{C}$ on $\mathcal{M}$ and the set of $R$-fiber functors from $\mathcal{C}$. In this section we will prove that this bijection of sets even lifts to an equivalence of suitably defined categories.

We recall that for a separable algebra $R$ the categories mod-$R$ and $R$-bimod are abelian, semisimple and artinian (compare Section 1.2).

**Lemma 5.1.1.**
*For a right module category $(\mathcal{M}, \otimes, \psi, \eta)$ over a tensor category $\mathcal{C}$ choose a generator $\tilde{M} := \oplus_{\alpha\in\Gamma} n_\alpha M_\alpha$ with $n_\alpha \in \mathbb{Z}_{>0}$.*

*(i) The endomorphism algebra $R := \mathrm{End}_{\mathcal{M}}(\tilde{M})$ is separable.*

*(ii) For all objects $M$ of $\mathcal{M}$ the vector space $\mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M)$ is endowed with the structure of a right $R$-module via composition*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M) \times R &\longrightarrow \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M) \\ (a, r) &\longmapsto a \circ r\,. \end{aligned} \tag{5.3}$$

*(iii) Consider the subcategory $\mathcal{A}$ of the category mod-$R$ of right $R$-modules that has objects $\mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M)$ for objects $M$ of $\mathcal{M}$ and morphisms $f\circ ? : \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M) \to \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M')$ for morphisms $f : M \to M'$ of $\mathcal{M}$. The category $\mathcal{A}$ is endowed with the structure of a right module category over the tensor category $\mathcal{C}$:*

$$\begin{array}{rlcl} \otimes_{\mathcal{C}} \; : & \mathcal{A} \times \mathcal{C} & \longrightarrow & \mathcal{A} \\ & (\mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M), X) & \longmapsto & \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M)\otimes_{\mathcal{C}} X := \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M\otimes X) \\ & (a, f) & \longmapsto & (a\otimes_{\mathcal{C}} f) := (a\otimes f)\circ ? \end{array}$$

*with functorial isomorphisms*

$$\begin{array}{lcll} \psi_{MXY}\circ ? & : & \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M\otimes(X\otimes Y)) & \longrightarrow \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, (M\otimes X)\otimes Y) \\ \eta_M\circ ? & : & \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M\otimes\mathbb{1}) & \longmapsto \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M) \end{array}$$

*for all* $X, Y \in \mathrm{Obj}(\mathcal{C})$ *and* $M \in \mathrm{Obj}(\mathcal{M})$.

*(iv) Then* $\mathcal{A} \simeq \text{mod-}R$ *as abelian categories and by identifying* mod-$R$ *with* $\mathcal{A}$ *there exists a strict equivalence of module categories over* $\mathcal{C}$

$$\begin{array}{llll} \chi & : & \mathcal{M} & \longrightarrow & \text{mod-}R \\ & & M & \longmapsto & \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M) \\ & & (f : M \mapsto M') & \longmapsto & (f\circ ? : \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M) \mapsto \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M')) \,. \end{array}$$

*with the functorial isomorphisms*

$$\gamma_{M,X} : \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M \otimes X) \to \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M) \otimes_{\mathcal{C}} X \,. \tag{5.4}$$

**Proof.** (i) By Schur's Lemma one has

$$\begin{aligned} \mathrm{End}_{\mathcal{M}}(\tilde{M}) &= \oplus_{\alpha \in \Gamma} \mathrm{End}_{\mathcal{M}}(M_\alpha) \otimes \mathrm{Mat}(n_\alpha \times n_\alpha, \mathbb{C}) \\ &= \oplus_{\alpha \in \Gamma} \mathbb{C}\mathrm{id}_{M_\alpha} \otimes \mathrm{Mat}(n_\alpha \times n_\alpha, \mathbb{C}), \end{aligned}$$

hence $R$ is isomorphic to the direct sum of $|\Gamma|$-many full matrix algebras of $n_\alpha \times n_\alpha$ matrices over $\mathbb{C}$. Therefore $R$ is separable.

(ii) Because of the associativity of the composition of morphisms of $\mathcal{M}$ the vector space $\mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M)$ with the action (5.3) satisfies the axioms of a right $R$-module.

(iii) The category $\mathcal{A}$ inherits the structure of a module category over $\mathcal{C}$ from the category $\mathcal{M}$: the properties of the functorial isomorphisms follow straightforwardly from the corresponding properties of the structural morphisms of the module category $\mathcal{M}$.

(iv) We prove the equivalence $\mathcal{A} \simeq \text{mod-}R$ as abelian categories:
Since $R$ is separable, the category of right $R$-modules mod-$R$ is semisimple with $|\Gamma|$-many isomorphism classes of simple objects. The isomorphism classes of simple objects of $\mathcal{A}$ have representatives $\mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M_\alpha)$. The category mod-$R$ has the same number of isomorphism classes of simple objects as its subcategory $\mathcal{A}$. Thus, they are equivalent as abelian categories.

Next we prove the equivalence $\mathcal{M} \simeq \text{mod-}R$ as abelian categories. The inverse of the functor $\chi$ is given by

$$\begin{array}{llll} \chi^{-1} & : & \text{mod-}R & \longrightarrow & \mathcal{M} \\ & & \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M) & \longmapsto & M \\ & & (f\circ ? : \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M) \mapsto \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M')) & \longmapsto & (f : M \mapsto M') \,. \end{array}$$

It is straightforward to compute $\chi \circ \chi^{-1} \cong \mathrm{id}_{\text{mod-}R}$ and $\chi^{-1} \circ \chi \cong \mathrm{id}_{\mathcal{M}}$.

The functor $\chi$ is a module functor over $\mathcal{C}$: By definition $\mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M \otimes X) = \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M) \otimes_{\mathcal{C}} X$, hence the functorial morphisms $\gamma_{M,X}$ from (5.4) can be omitted. The axioms of a module functor (compare definition 2.4.5) reduce to $\chi(\psi_{MXY}) = \psi_{MXY} \circ ?$ and $\chi(\eta_M) = \eta_M \circ ?$ that hold by definition.

The functor $\chi$ is an equivalence of module categories over $\mathcal{C}$: Assume that mod-$R$ is endowed with the structure of a module category over $\mathcal{C}$ as in (iii). The inverse functor $\chi^{-1}$ is a module functor: by definition $\chi^{-1}(\mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M) \otimes X) = M \otimes X = \chi^{-1}(\mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M)) \otimes X$. $\square$

For two (tensor, respectively module) categories $\mathcal{A}, \mathcal{B}$ denote by $\mathcal{F}un(\mathcal{A}, \mathcal{B})$ the category of exact (tensor, respectively module) functors with morphisms the natural (tensor, respectively module) transformations. The category of endofunctors $\mathcal{F}un(\mathcal{A}, \mathcal{A})$ is a tensor category with the composition as tensor product: $F \otimes G := G \circ F$ for all functors $F, G : \mathcal{A} \to \mathcal{A}$.

From module theory one knows that the structure of a module $M$ over a ring $S$ corresponds to a ring homomorphism $S \to \mathrm{End}_S(M)$. Recall that the categorification of a ring is a tensor category $\mathcal{C}$ and that the categorification of a module over a ring is a module category $\mathcal{M}$ over $\mathcal{C}$. The categorification of $\mathrm{End}_S(M)$ is the tensor category $\mathcal{F}un(\mathcal{M}, \mathcal{M})$ of module endofunctors. The categorification of the ring homomorphism $S \to \mathrm{End}_S(M)$ is a tensor functor $\mathcal{C} \to \mathcal{F}un(\mathcal{M}, \mathcal{M})$. Using the next lemma and the next definition we will prove a more general statement in Theorem 5.1.4.

**Lemma 5.1.2.**

*i) Let $\mathcal{M}$ be a category, $(\mathcal{C}, a, \lambda, \mu)$ a tensor category and $(\Phi, \phi_2, \phi_0) : \mathcal{C} \to \mathcal{F}un(\mathcal{M}, \mathcal{M})$ a tensor functor. Consider the bifunctor*

$$\begin{array}{rcllcl} \boxtimes_\Phi & : & \mathcal{M} \times \mathcal{C} & \longrightarrow & \mathcal{M} \\ & & (M, X) & \longmapsto & M \boxtimes_\Phi X := \Phi(X)(M) \\ & & (f : M \mapsto M', g : X \mapsto Y) & \longmapsto & f \boxtimes_\Phi g = \Phi(g)(f) : \\ & & & & \qquad \Phi(X)(M) \mapsto \Phi(Y)(M') . \end{array}$$

*for any $X, Y \in Ob(\mathcal{C}), M \in Ob(\mathcal{M})$. Then $\boxtimes_\Phi$ defines the structure of a module category over $\mathcal{C}$ on $\mathcal{M}$ with functorial isomorphisms $(\phi_2^{-1})_{\bullet,\bullet}(-)$ and $(\phi_0^{-1})(-)$ given by*

$$\begin{array}{lclcl} (\phi_2^{-1})_{X,Y}(M) & : & \Phi(X \otimes Y)(M) & \longrightarrow & (\Phi(Y) \circ \Phi(X))(M) \\ (\phi_0^{-1})(M) & : & \Phi(\mathbb{1})(M) & \longrightarrow & M . \end{array}$$

*ii) For a given the structure of a module category $(\boxtimes, \psi, \eta)$ over $\mathcal{C}$ on a category $\mathcal{M}$ and an object $X$ of $\mathcal{C}$ consider the functor*

$$\begin{array}{rcllcl} \Phi_\boxtimes & : & \mathcal{C} & \longrightarrow & \mathcal{F}un(\mathcal{M}, \mathcal{M}) \\ & & X & \longmapsto & - \boxtimes X : M \mapsto M \boxtimes X \\ & & f : X \mapsto Y & \longmapsto & - \boxtimes f : - \boxtimes X \mapsto - \boxtimes Y . \end{array}$$

*It is exact because of the exactness of $\boxtimes$. In addition it is a tensor functor with associativity and unit isomorphisms $(\psi^{-1})_{-,\bullet,\bullet}$ and $(\eta^{-1})_-$*

$$\begin{array}{lcl} (\psi^{-1})_{-,X,Y} & : & (- \boxtimes X) \circ (- \boxtimes Y) \longrightarrow - \boxtimes (X \otimes Y) \\ (\eta^{-1})_- & : & \mathbb{1}_{\mathcal{F}un(\mathcal{M},\mathcal{M})} \longrightarrow - \boxtimes \mathbb{1}_\mathcal{C} . \end{array}$$

**Proof.** i) We assume that $\mathcal{C}$ is a strict tensor category. For $(\mathcal{M}, (\phi_2^{-1})_{X,Y}, \phi_0^{-1})$ to be a module category over $\mathcal{C}$ the Pentagon Axiom

$$((\phi_2^{-1})_{X,Y} \otimes \mathrm{id}_Z) \circ (\phi_2^{-1})_{X \otimes Y, Z}(M) = (\mathrm{id}_X \otimes (\phi_2^{-1})_{Y,Z}) \circ (\phi_2^{-1})_{X, Y \otimes Z}(M)$$

and the Triangle Axiom

$$(\mathrm{id}_M \otimes \phi_0^{-1}) \circ (\phi_2^{-1})_{\mathbb{1},X}(M) = \mathrm{id}_M \otimes \mathrm{id}_X$$

must be satisfied. They hold because of relation (2.1) and (2.3) for tensor functors.

ii) Similarly, the axioms (2.1), (2.2) and (2.3) can be verified straightforwardly using the Pentagon Axiom and the Triangle Axiom of the module category $\mathcal{A}$.

**Definition 5.1.3.**
*Let $\mathcal{M}$ be a category. We define the category $\mathcal{S}(\mathcal{M},\mathcal{C})$ of structures of a right module category over $\mathcal{C}$ on $\mathcal{M}$:*
*The objects are triples $(\boxtimes,\psi,\eta)$, where $\boxtimes : \mathcal{M}\times\mathcal{C} \to \mathcal{M}$ is an exact bifunctor and $\psi$ and $\eta$ are the associativity and unit isomorphisms. The morphisms $\nu : (\boxtimes,\psi,\eta) \mapsto (\boxtimes',\psi',\eta')$ are natural transformations $\nu : \boxtimes \to \boxtimes'$, such that the diagrams*

$$\begin{array}{ccc} M \boxtimes (X \otimes Y) & \xrightarrow{\psi_{M,X,Y}} & (M \boxtimes X) \boxtimes Y \\ \downarrow \nu_{M,X\otimes Y} & & \downarrow\ \nu_{M,X} \otimes \mathrm{id}_Y \\ M \boxtimes' (X \otimes Y) & \xrightarrow{\psi'_{M,XY}} & (M \boxtimes' X) \boxtimes' Y \end{array}$$

*and*

$$\begin{array}{ccc} M \boxtimes \mathbb{1} & \xrightarrow{\eta_M} & M \\ \downarrow \nu_{M,\mathbb{1}} & & \downarrow\ \nu_M \\ M \boxtimes' \mathbb{1} & \xrightarrow{\eta'_M} & M \end{array}$$

*are commutative for all $X, Y \in Ob(\mathcal{C})$, $M \in Ob(\mathcal{M})$.*

**Theorem 5.1.4.**
*For any category $\mathcal{M}$ and any tensor category $\mathcal{C}$ there exists a strict equivalence of categories*

$$\begin{array}{rcll} J & : & \mathcal{S}(\mathcal{M},\mathcal{C}) & \longrightarrow \mathcal{F}un(\mathcal{C},\mathcal{F}un(\mathcal{M},\mathcal{M})) \\ & & (\boxtimes,\psi,\eta) & \longmapsto ((X \mapsto - \boxtimes X), (\psi^{-1})_{-,\bullet,\bullet}, (\eta^{-1})_{-}) \\ & & \nu : (\boxtimes,\psi,\eta) \mapsto (\boxtimes',\psi',\eta') & \longmapsto \nu\circ ? : ((X \mapsto - \boxtimes X), (\psi^{-1})_{-,\bullet,\bullet}, (\eta^{-1})_{-}) \\ & & & \quad \mapsto ((X \mapsto - \boxtimes' X), (\psi'^{-1})_{-,\bullet,\bullet}, (\eta'^{-1})_{-}), \end{array}$$

*where $\bullet$ is in $\mathcal{C}$ with inverse functor $G$*

$$\begin{array}{ll} \mathcal{F}un(\mathcal{C},\mathcal{F}un(\mathcal{M},\mathcal{M})) & \longrightarrow \mathcal{S}(\mathcal{M},\mathcal{C}) \\ (\Phi,\phi_2,\phi_0) & \longmapsto (\boxtimes_\Phi, (\phi_2^{-1})_{\bullet,\bullet}(-), (\phi_0^{-1})(-)) \\ \alpha : (\Phi,\phi_2,\phi_0) \mapsto (\Phi',\phi_2',\phi_0') & \longmapsto \alpha(\bullet)(-) : (\boxtimes_\Phi, (\phi_2^{-1})_{\bullet,\bullet}(-), (\phi_0^{-1})(-)) \\ & \quad \mapsto (\boxtimes_{\Phi'}, (\phi_2'^{-1})_{\bullet,\bullet}(-), (\phi_0'^{-1})(-)). \end{array}$$

**Proof.** We apply the bifunctor $GJ(\boxtimes)$ to an object $M$ of $\mathcal{M}$ and to an object $X$ of $\mathcal{C}$:

$$M \boxtimes_{\Phi_\boxtimes} X \quad = \quad \Phi_\boxtimes(X)(M) = M \boxtimes X.$$

For objects $M$ of $\mathcal{M}$ and objects $X, Y$ of $\mathcal{C}$ the isomorphisms $\psi, \eta$ are mapped to

$$\begin{aligned} (GJ(\psi))_{M,X,Y} &= (((\psi^{-1})_{-,X,Y})^{-1})_{X,Y}(M) = \psi_{M,X,Y} \\ (GJ(\eta))_M &= ((\eta^{-1})_{-})^{-1}(M) = \eta_M \,. \end{aligned}$$

Thus, we have shown $GJ = \mathrm{id}_{\mathcal{S}(\mathcal{M},\mathcal{C})}$ for objects $(\boxtimes, \psi, \eta)$ of $\mathcal{S}(\mathcal{M}, \mathcal{C})$. Similarly one can prove this identity for morphisms.

The functor $JG(\Phi)$ for a tensor functor $\Phi : \mathcal{C} \to \mathcal{F}un(\mathcal{M}, \mathcal{M})$ and objects $M$ of $\mathcal{M}$ and $X$ of $\mathcal{C}$ reads

$$JG(\Phi)(X)(M) = M \boxtimes_\Phi X = \Phi(X)(M) \,.$$

Furthermore, we compute for an object $M$ of $\mathcal{M}$ and objects $X, Y$ of $\mathcal{C}$:

$$\begin{aligned} JG(\phi_2)_{X,Y}(M) &= (((\phi_2^{-1})_{X,Y}(-))^{-1})_{M,X,Y} = (\phi_2)_{X,Y}(M) \\ JG(\phi_0)(M) &= (((\phi_0^{-1})(-))^{-1})_M = \phi_0(M) \,. \end{aligned}$$

Similarly one can prove $JG = \mathrm{id}_{\mathcal{F}un(\mathcal{C},\mathcal{F}un(\mathcal{M},\mathcal{M}))}$ for morphisms. □

By Proposition 1.2.3 a separable $\mathbb{C}$-algebra $R$ is isomorphic to the direct sum of full matrix algebras. Denote by $r$ the number of summands. The categories of right $R$-modules mod-$R$ and of $R$-bimodules $R$-bimod are semisimple and artinian. In addition, mod-$R$ is a module category over the tensor category $R$-bimod with a canonical choice for the associativity isomorphism $\psi^{bimod}$ and the unit isomorphism $\eta^{bimod}$ (see Example 2.4.2). For $i, j \in \{1, \ldots r\}$ denote by ${}_iB_j$ representatives of the isomorphism classes of $R$-bimodules: only the $i$-th ideal acts nontrivially on ${}_iB_j$ from the left and only the $j$-th ideal acts nontrivially on ${}_iB_j$ from the right. The tensor unit $R$ in $R$-bimod can be decomposed as $\oplus_{i=1}^r \, {}_iB_i$. By $L_j$ we denote representatives of the isomorphism classes of simple $R$-right modules: only the $j$-th ideal acts nontrivially on $L_j$.

For a separable $\mathbb{C}$-algebra $R$ the category $\mathcal{F}un(\text{mod-}R, \text{mod-}R)$ is semisimple and artinian. The representatives of the isomorphism classes of simple objects are functors ${}_iF_j$ that map the simple object $L_i$ to $L_j$ and all others to 0. The tensor unit in $\mathcal{F}un(\text{mod-}R, \text{mod-}R)$ is the identity functor $\oplus_{i=1}^r \, {}_iF_i$.

**Lemma 5.1.5.**
*For a separable $\mathbb{C}$-algebra $R$ the functor*

$$\begin{array}{cclcl} \mathcal{E} & : & R\text{-bimod} & \longrightarrow & \mathcal{F}un(\text{mod-}R, \text{mod-}R) \\ & & B & \longmapsto & (- \otimes B : M \mapsto M \otimes B) \\ & & (\beta : B_1 \mapsto B_2) & \longmapsto & (-\otimes\beta : - \otimes B_1 \mapsto -\otimes B_2), \end{array}$$

*which is a tensor functor with functorial isomorphisms*

$$\begin{array}{lcccl} (\psi^{bimod})^{-1}_{-B_1B_2} & : & (-\otimes B_1)\otimes(-\otimes B_2) & \longrightarrow & -\otimes(B_1 \otimes_R B_2) \\ (\eta^{bimod})^{-1}_{-} & : & \mathrm{id}_{\text{mod-}R} & \longrightarrow & -\otimes_R R \,, \end{array}$$

*is an equivalence of tensor categories.*

**Proof.** We will prove the Lemma by showing that the functor $\mathcal{E}$ is fully faithful and essentially surjective.

The functor $\mathcal{E}$ is fully faithful: for $R$-bimodules $B_1, B_2$ consider the map

$$\mathcal{E}_{B_1B_2} : \mathrm{Hom}_{R\text{-bimod}}(B_1, B_2) \to \mathrm{Hom}_{\mathcal{F}un(\text{mod-}R,\text{mod-}R)}(-\otimes B_1, -\otimes B_2)\,.$$

Since $R$-bimod is semisimple, there exist $b_1^{ij}, b_2^{ij} \in \mathbb{Z}_{>0}$, such that $B_1 \cong \oplus_{i,j} b_1^{ij}\ {}_iB_j$ and $B_2 \cong \oplus_{i,j} b_2^{ij}\ {}_iB_j$. Thus,

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{R\text{-bimod}}(B_1, B_2)) = \sum_{i,j} b_1^{ij} b_2^{ij}\,.$$

Because of $L_i \otimes {}_iB_j \cong L_j$ the functor $-\otimes {}_iB_j$ is isomorphic to the simple object ${}_iF_j$ in $\mathcal{F}un(\text{mod-}R, \text{mod-}R)$. Therefore, $-\otimes B_1 \cong b_1^{ij}\ {}_iF_j$ and $-\otimes B_2 \cong b_2^{ij}\ {}_iF_j$. We conclude that

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{F}un(\text{mod-}R,\text{mod-}R)}(-\otimes B_1, -\otimes B_2)) = \sum_{i,j} b_1^{ij} b_2^{ij}\,.$$

Hence, $\mathcal{E}_{B_1B_2}$ is a bijection and consequently $\mathcal{E}$ is fully faithful.

The functor $\mathcal{E}$ is essentially surjective: consider a functor $F : \text{mod-}R \to \text{mod-}R$. Since $\mathcal{F}un(\text{mod-}R, \text{mod-}R)$ is semisimple, there exists $f^{ij} \in \mathbb{Z}_{>0}$, such that $F \cong \oplus_{i,j} f^{ij}\ {}_iF_j$. Then $F \cong \oplus_{i,j}(-\otimes f^{ij}\ {}_iB_j)$. □

**Theorem 5.1.6.**
*Let $\mathcal{C}$ be a tensor category. For a category $\mathcal{M}$ consider the endomorphism algebra $R := \mathrm{End}_{\mathcal{M}}(\tilde{M})$ of a generator $\tilde{M}$ of $\mathcal{M}$. Then there exists an equivalence of categories*

$$\mathcal{S}(\mathcal{M}, \mathcal{C}) \simeq \mathcal{F}un(\mathcal{C}, R\text{-bimod})\,.$$

**Proof.** The equivalence $\mathcal{M} \simeq \text{mod-}R$ from Lemma 5.1.1 (iv) implies $\mathcal{F}un(\mathcal{M}, \mathcal{M}) \simeq \mathcal{F}un(\text{mod-}R, \text{mod-}R)$ and by Lemma 5.1.5 one has $\mathcal{F}un(\mathcal{M}, \mathcal{M}) \simeq R$-bimod. Inserting this equivalence into the equivalence $\mathcal{S}(\mathcal{M}, \mathcal{C}) \simeq \mathcal{F}un(\mathcal{C}, \mathcal{F}un(\mathcal{M}, \mathcal{M}))$ of Theorem 5.1.4 yields that the category of module structures over $\mathcal{C}$ on $\mathcal{M}$ is equivalent to the category of $R$-fiber functors on $\mathcal{C}$. □

## 5.2 Reconstruction of a weak Hopf algebra

For a separable algebra $R$ choose a separability idempotent $s = \sum s_{(1)}\otimes s_{(2)} \in R\otimes R$. Denote by $\pi_{B_1,B_2} : B_1\otimes_{\mathbb{C}}B_2 \to B_1\otimes_R B_2$ the natural projection of two $R$-bimodules $B_1, B_2$. For any separability idempotent $s$ the family of morphisms of $R$-bimodules

$$\begin{aligned} \rho_{B_1,B_2}(s) : B_1\otimes_R B_2 &\longrightarrow B_1\otimes_{\mathbb{C}} B_2, \\ b_1\otimes_R b_2 &\longmapsto \sum b_1 s_{(1)}\otimes_{\mathbb{C}} s_{(2)}b_2 \end{aligned}$$

is well defined because of the properties of the separability idempotent. One has

$$\pi_{B_1,B_2}\circ\rho_{B_1,B_2}(s) = \mathrm{id}_{B_1\otimes_R B_2}.$$

We introduce some more notation: Let $\nu : R\text{-bimod} \to \mathrm{vect}_{\mathbb{C}}$ be the forgetful functor. Given an exact $R$-fiber functor $\Phi : \mathcal{C} \to R$-bimod define $\bar{\Phi}$ to be the composite functor $\bar{\Phi} = \nu\circ\Phi$ which is exact and faithful. Only in the Tannaka case, that is if $R = \mathbb{C}$, $\bar{\Phi}$ is a tensor functor.

In the following we will reconstruct a weak Hopf algebra from the data of a conformal field theory. We start with a modular tensor category $\mathcal{C}$ and a Frobenius algebra $A$ in $\mathcal{C}$. The category $\mathcal{M} := \mathcal{C}_A$ of $A$-left modules is a module category over $\mathcal{C}$ with functorial isomorphisms denoted by $\psi, \eta$. We choose a generator $\tilde{M} := \oplus_{\alpha\in\Gamma} n_\alpha M_\alpha$ of $\mathcal{C}_A$ and consider the endomorphism algebra $R$ of the generator $\tilde{M}$. We write $\mathcal{N}_{\beta j}^{\alpha} = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{M}}(M_\alpha, M_\beta\otimes U_j)$ for the fusion coefficients of the module category $\mathcal{M}$.

Using Theorem 5.1.6 we construct an $R$-fiber functor explicitly: Theorem 5.1.4 yields a tensor functor

$$\begin{array}{lcl} \mathcal{C} & \longrightarrow & \mathcal{F}un(\mathcal{M},\mathcal{M}) \\ X & \longmapsto & -\otimes X : M \to M\otimes X \\ (f : X\mapsto Y) & \longmapsto & (-\otimes f : -\otimes X \mapsto -\otimes Y) \end{array}$$

with functorial isomorphisms $(\psi^{-1})_{-,\bullet,\bullet}, (\eta^{-1})_{-}$. Via the equivalence $\mathcal{M} \simeq \text{mod-}R$ of Lemma 5.1.1 we obtain a tensor functor

$$\begin{array}{lllcl} \tilde{\Phi} & : & \mathcal{C} & \longrightarrow & \mathcal{F}un(\text{mod-}R, \text{mod-}R) \\ & & X & \longmapsto & -\otimes_{\mathcal{C}} X : \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M) \\ & & (f : X\mapsto Y) & \longmapsto & (-\otimes_{\mathcal{C}} f : \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M\otimes X) \mapsto \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M\otimes Y)) \end{array}$$

with functorial isomorphisms $(\psi^{-1})_{-,\bullet,\bullet}\circ ?, (\eta^{-1})_{-}\circ ?$ and $\bullet$ in $\mathcal{C}$. Next we consider how $-\otimes_{\mathcal{C}} X$ maps the generator $\mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M})$ of mod-$R$

$$-\otimes_{\mathcal{C}} X : \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M}) \mapsto \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M}\otimes X)\,.$$

It is straightforward to see that the $\mathbb{C}$-vector space $\mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M}\otimes X)$ can be endowed with the structure of an $R$-bimodule via

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M}\otimes X)\times R & \longrightarrow & \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M\otimes X) \\ (f, r) & \longmapsto & f\circ r \\ R\times \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M}\otimes X) & \longrightarrow & \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M}\otimes X) \\ (r, g) & \longmapsto & (r\otimes \mathrm{id}_X)\circ g\,. \end{array}$$

The tensor product of $\mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M} \otimes X)$ with a right $R$-module of the form $\mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M)$ is then given by

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M) \otimes_R \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M} \otimes X) & \longrightarrow & \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M \otimes X) \\ (h, g) & \longmapsto & (h \otimes \mathrm{id}_X) \circ g \,. \end{array}$$

Thus, we obtain an $R$-fiber functor

$$\begin{array}{rlcl} \Phi : & \mathcal{C} & \longrightarrow & R\text{-bimod} \\ & X & \longmapsto & \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M} \otimes X) \\ & f : X \mapsto Y & \longmapsto & (\mathrm{id}_{\tilde{M}} \otimes f) \circ ? : \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M} \otimes X) \mapsto \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M} \otimes Y) \end{array} \tag{5.5}$$

with functorial isomorphisms

$$\begin{array}{rccl} (\psi^{-1})_{\tilde{M},X,Y} \circ ? : & \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M} \otimes X) & \otimes_R & \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M} \otimes Y) \\ & & & \longrightarrow \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M} \otimes (X \otimes Y)) \\ (\eta^{-1})_{\tilde{M}} : & \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M}) & \longrightarrow & \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M} \otimes 1\!\!1_{\mathcal{C}}) \,. \end{array}$$

We recall that the algebra $R$ is isomorphic to the direct sum of $|\Gamma|$-many full matrix algebras of $n_\alpha \times n_\alpha$ matrices over $\mathbb{C}$. Thus, the category mod-$R$ has $|\Gamma|$-many simple objects and the category $R$-bimod has $|\Gamma|^2$-many simple objects ${}_\alpha B_\beta$. As an $R$-bimodule $R$ is isomorphic to $\oplus_{\alpha \in \Gamma}\, {}_\alpha B_\alpha$. The vector space $\mathrm{Hom}_{\mathcal{M}}(\tilde{M}, M_\alpha)$ has dimension $n_\alpha$, the vector space ${}_\alpha B_\beta$ has dimension $n_\alpha n_\beta$.

**Lemma 5.2.1.**
*For a generator $\tilde{M}$ of a module category $\mathcal{M}$ over a tensor category $\mathcal{C}$ and $R$ the endomorphism algebra of $\tilde{M}$ consider the $R$-fiber functor $\Phi$ given by (5.5).*

*(i) The unital associative algebra $H := \mathrm{End}(\bar{\Phi}) = \mathrm{End}_{\mathbb{C}}(\Phi)$ of natural endotransformations of $\bar{\Phi}$ is the direct sum of $|I|$-many full matrix algebras over $\mathbb{C}$.*

*(ii) For any object $U$ in the tensor category $\mathcal{C}$ the vector space $\bar{\Phi}(U)$ carries a natural structure of a left $H$-module via*

$$\begin{array}{rrcl} \rho_{\bar{\Phi}(U)} : & \mathrm{End}(\bar{\Phi}) \times \bar{\Phi}(U) & \longrightarrow & \bar{\Phi}(U) \\ & (h_U, e) & \longmapsto & h_U(e) \,. \end{array} \tag{5.6}$$

*(iii) Any left $H$-module is isomorphic to $\bar{\Phi}(U)$ for some object $U$ of $\mathcal{C}$, simple $H$-modules are isomorphic to $\bar{\Phi}(U_i)$.*

**Proof.** (i) Since $\bar{\Phi}$ is exact, it respects direct sums. Thus, any natural transformation $h \in H$ is determined by the linear maps

$$h_{U_i} : \bar{\Phi}(U_i) \to \bar{\Phi}(U_i)$$

for all simple objects $U_i$ of $\mathcal{C}$. Thus, the algebra $H$ is semisimple and decomposes to $|I|$-many summands. Because of

$$\bar{\Phi}(U_i) = \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M} \otimes U_i) \cong \oplus_{\alpha,\beta} \mathrm{Hom}_{\mathcal{M}}(n_\beta M_\beta, n_\alpha M_\alpha \otimes U_i)$$

the summands are full matrix algebras over $\mathbb{C}$ with dimension $(\sum_{\alpha,\beta\in\Gamma}\mathcal{N}_{\alpha\, i}^{\beta}\, n_\alpha n_\beta)^2$.

(ii) $(\bar\Phi(U), \rho_{\bar\Phi(U)})$ is a left $H$-module, since the relations

$$(g\circ h)_U(e) = g_U \circ h_U(e) \qquad \text{and} \qquad \mathrm{id}_U(e) = e$$

hold for any $g, h \in H$ and any $e \in \bar\Phi(U)$.

(iii) The category of $H$-modules has $I$-many isomorphism classes of simple objects and so does the subcategory of $H$-mod with objects $\bar\Phi(U)$. Since these maps can be chosen at will, this subcategory of $H$-mod is equivalent to $H$-mod as an abelian category.

Note that

$$\dim_{\mathbb{C}} H = \sum_{i\in I}\Big(\sum_{\alpha,\beta\in\Gamma}\mathcal{N}_{\alpha\, i}^{\beta}\, n_\alpha n_\beta\Big)^2 . \tag{5.7}$$

**Lemma 5.2.2.**
*Let $\rho_{\bar\Phi(U)}$ be the $H$-module action defined in (5.6). The construction from the previous Lemma yields the following distinguished functor*

$$\begin{array}{cccc} \omega: & \mathcal{C} & \longrightarrow & H\text{-mod} \\ & U & \longmapsto & (\bar\Phi(U), \rho_{\bar\Phi(U)}) \\ & f & \longmapsto & \bar\Phi(f) \end{array} \tag{5.8}$$

*which is an equivalence of abelian categories.*

**Proof.** The functor $\omega$ is well-defined: for any objects $X, Y$ in $\mathcal{C}$, any morphism $f: X \mapsto Y$ and $h \in H$ one has $\bar\Phi(f)(h_X) = h_Y(\bar\Phi(f))$, since the natural transformation $h$ is functorial. Therefore, $\bar\Phi(f)$ is an $H$-module morphism.

By Lemma 5.2.1 (iii) simple $H$-modules are isomorphic to $\bar\Phi(U_i)$. Thus, the functor $\omega$ maps simple objects in $\mathcal{M}$ to simple objects in mod-$R$. Since the categories $\mathcal{M}$ and mod-$R$ are semisimple, the functor $\omega$ is an equivalence of abelian categories. □

**Theorem 5.2.3.**
*If the generator $\tilde M$ of $\mathcal{M}$ is chosen to be $\oplus_\alpha M_\alpha$, the algebra $H = \mathrm{End}_{\mathbb{C}}\Phi$ from Lemma 5.2.1 (i) can be endowed with the structure of a coalgebra with coproduct and counit*

$$\begin{array}{cccc} \Delta & : \mathrm{End}_{\mathbb{C}}\Phi & \longrightarrow & \mathrm{End}_{\mathbb{C}}\Phi \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}\Phi \\ & h & \longmapsto & \rho\circ {\phi_2}^{-1}\circ h\circ\phi_2\circ\pi \\ \epsilon & : \mathrm{End}_{\mathbb{C}}\Phi & \longrightarrow & \mathbb{C} \\ & h & \longmapsto & \mathrm{tr}_{\mathbb{C}}(h(1\!\!1_{\mathcal{C}}))\,. \end{array}$$

**Proof.** In case of a generator without multiplicities, $\tilde M = \oplus_{\alpha\in\Gamma} M_\alpha$, we choose a basis

$$\{e_{\alpha\beta}^{j\ r}\}$$

with $r \in \{1, \ldots, \mathcal{N}_{\alpha j}^{\beta}\}$ for the vector spaces $\mathrm{Hom}_{\mathcal{M}}(M_\beta, M_\alpha \otimes U_j)$. Moreover, the set $\{\mathrm{id}_{M_\alpha}\}$ is a basis for the algebra $R$. Let

$$\{h^{i\ rr'}_{(\alpha\beta)(\alpha'\beta')} : e^{i\ r}_{\alpha\beta} \mapsto e^{i\ r'}_{\alpha'\beta'} \text{ and } 0 \text{ elsewhere}\}$$

be a basis in $\mathrm{Hom}_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{M}}(M_\beta, M_\alpha \otimes U_i), \mathrm{Hom}_{\mathcal{M}}(M_{\beta'}, M_{\alpha'} \otimes U_i))$.

We write the $R$-bimodule structure of $\Phi(U_i)$ for simple objects in the chosen basis

$$\begin{array}{rcl}
\mathrm{Hom}_{\mathcal{M}}(M_\gamma, M_\beta \otimes U_i) \times \mathrm{Hom}_{\mathcal{M}}(M_\alpha, M_\alpha) & \longrightarrow & \mathrm{Hom}_{\mathcal{M}}(M_\alpha, M_\beta \otimes U_i) \\
(e^{i\ t}_{\beta\gamma}, \mathrm{id}_{M_\alpha}) & \longmapsto & \delta_{\alpha\gamma} e^{i\ t}_{\beta\alpha} \\
\mathrm{Hom}_{\mathcal{M}}(M_\alpha, M_\alpha) \times \mathrm{Hom}_{\mathcal{M}}(M_\gamma, M_\beta \otimes U_i) & \longrightarrow & \mathrm{Hom}_{\mathcal{M}}(M_\gamma, M_\alpha \otimes U_i) \\
(\mathrm{id}_{M_\alpha}, e^{i\ t}_{\beta\gamma}) & \longmapsto & \delta_{\alpha\beta} e^{i\ t}_{\alpha\gamma} \,.
\end{array}$$

The isomorphism $\phi_2 : \Phi(U_j) \otimes_R \Phi(U_l) \to \Phi(U_j \otimes U_l)$ reads

$$\begin{array}{rcl}
\mathrm{Hom}_{\mathcal{M}}(M_\epsilon, M_\alpha \otimes U_j) \times \mathrm{Hom}_{\mathcal{M}}(M_\beta, M_\epsilon \otimes U_l) & \longrightarrow & \mathrm{Hom}_{\mathcal{M}}(M_\beta, M_\alpha \otimes (U_j \otimes U_l)) \\
e^{j\ r}_{\alpha\epsilon} \otimes_R e^{l\ r'}_{\epsilon\beta} & \longmapsto & \displaystyle\sum_{i,s,v} F^{(\alpha j l)\beta}_{r\epsilon r', siv}\ e^{i\ v}_{\alpha\beta}[j,l,s] \,,
\end{array}$$

where we denote by $F$ the fusing matrices (6j-symbols) of the module category $\mathcal{M}$ defined in (2.6) in the basis $\{e^{j\ r}_{\alpha\beta}\}$.

We compute the coproduct

$$\begin{array}{rcl}
\Delta : \mathrm{End}_{\mathbb{C}}\Phi & \longrightarrow & \mathrm{End}_{\mathbb{C}}\Phi \otimes \mathrm{End}_{\mathbb{C}}\Phi \\
h & \longmapsto & \rho \circ {\phi_2}^{-1} \circ h \circ \phi_2 \circ \pi
\end{array}$$

step by step: first consider the map

$$e^{j\ r}_{\alpha\epsilon} \otimes_k e^{l\ s}_{\zeta\beta} \xmapsto{\phi_2 \circ \pi} \delta_{\epsilon\zeta} \sum_{i,t,v} F^{(\alpha j l)\beta}_{r\epsilon s, vit}\ e^{i\ t}_{\alpha\beta}[j,l,v]$$

and second apply an element $h^{i\ tt'}_{(\alpha\beta)(\alpha'\beta')} \in H$

$$\delta_{\epsilon\zeta} \sum_{i,t,v} F^{(\alpha j l)\beta}_{r\epsilon s, vit}\ e^{i\ t}_{\alpha\beta}[j,l,v] \xmapsto{h} \delta_{\epsilon\zeta} \sum_{i,t,v} F^{(\alpha j l)\beta}_{r\epsilon s, vit}\ e^{i\ t'}_{\alpha'\beta'}[j,l,v] \,.$$

Application of $\phi_2^{-1}$ yields

$$\delta_{\epsilon\zeta} \sum_{i,t,v} F^{(\alpha j l)\beta}_{r\epsilon s, vit}\ e^{i\ t'}_{\alpha'\beta'}[j,l,v] \xmapsto{\phi_2^{-1}} \delta_{\epsilon\zeta} \sum_{i,t,v,r',s',\epsilon'} F^{(\alpha j l)\beta}_{r\epsilon s, vit}\ (F^{-1})^{(\alpha' j l)\beta'}_{r'\epsilon' s', vit'}\ e^{j\ r'}_{\alpha'\epsilon'} \otimes_R e^{l\ s'}_{\epsilon'\beta'} \,.$$

From Example 1.2.4 the unique separability idempotent is known to be

$$s = \sum_{\alpha \in \Gamma} \mathrm{id}_{M_\alpha} \otimes \mathrm{id}_{M_\alpha}$$

Thus, acting with the morphism $\rho_{B_1,B_2} : B_1 \otimes_R B_2 \to B_1 \otimes B_2$ on the previous expression implies

$$\Delta(h^{i\ tt'}_{(\alpha\beta)(\alpha'\beta')})(e^{j\ r}_{\alpha\epsilon} \otimes_k e^{l\ s}_{\zeta\beta}) = \delta_{\epsilon\zeta} \sum_{i,t,v,r',s',\epsilon'} F^{(\alpha jl)\beta}_{r\epsilon s,vit} \, (F^{-1})^{(\alpha' jl)\beta'}_{r'\epsilon' s',vit'} e^{j\ r'}_{\alpha'\epsilon'} \otimes_k e^{l\ s'}_{\epsilon'\beta'} \, .$$

Hence, the coproduct of $h^{i\ tt'}_{(\alpha\beta)(\alpha'\beta')} \in H$ is given by

$$\Delta h^{i\ tt'}_{(\alpha\beta)(\alpha'\beta')} = \sum_{j,l,\epsilon,\eta,v,r,r',s,s'} F^{(\alpha jl)\beta}_{r\epsilon s,vit} \, (F^{-1})^{(\alpha' jl)\beta'}_{r'\eta s',vit'} h^{j\ rr'}_{(\alpha\epsilon)(\alpha'\eta)} \otimes h^{l\ ss'}_{(\epsilon\beta)(\eta\beta')} \, .$$

Furthermore, we compute

$$h^{i\ tt'}_{(\alpha\beta)(\alpha'\beta')}(\mathrm{id}_{M_\gamma}) = \delta_{i0}\delta_{1t}\delta_{1t'}\delta_{\alpha\beta}\delta_{\gamma\beta}\delta_{\alpha'\beta'}\mathrm{id}_{M_{\alpha'}} \, ,$$

hence the counit $\epsilon(h) = \mathrm{tr}(h(\mathbb{1}_{\mathcal{C}}))$ is of the form

$$\epsilon(h^{i\ tt'}_{(\alpha\beta)(\alpha'\beta')}) = \delta_{i0}\delta_{1t}\delta_{1t'}\delta_{\alpha\beta}\delta_{\alpha'\beta'} \, .$$

We prove that the algebra $H$ has the counitality property: we compute

$$(\epsilon \otimes \mathrm{id})(\Delta h^{i\ tt'}_{(\alpha\beta)(\alpha'\beta')}) = \sum_{l,v,s,s'} F^{(\alpha 0 l)\beta}_{1\alpha s,vit} \, (F^{-1})^{(\alpha' 0 l)\beta'}_{1\alpha' s',vit'} h^{l\ ss'}_{(\alpha\beta)(\alpha'\beta')}$$

and using (2.7) we obtain

$$(\epsilon \otimes \mathrm{id})(\Delta h^{i\ tt'}_{(\alpha\beta)(\alpha'\beta')}) = h^{i\ tt'}_{(\alpha\beta)(\alpha'\beta')} \, .$$

Similarly one proves $(\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}$. Coassociativity can be checked straightforwardly using the Pentagon identity (2.10) for the fusing matrices $F$ and relation (2.9). Then the coalgebra structure in the basis $\{h^{i\ st}_{(\alpha\beta)(\alpha'\beta')}\}$ reads

$$\begin{aligned} \Delta h^{i\ st}_{(\alpha\beta)(\alpha'\beta')} &= \sum_{j,l,r,r',p,p',v,\epsilon,\eta} F^{(\alpha jl)\beta}_{r\epsilon r',vis} \, (F^{-1})^{(\alpha' jl)\beta'}_{p\eta p',vit} \, h^{j\ rp}_{(\alpha\epsilon)(\alpha'\eta)} \otimes h^{l\ r'p'}_{(\epsilon\beta)(\eta\beta')} \\ \epsilon(h^{i\ st}_{(\alpha\beta)(\alpha'\beta')}) &= \delta_{i0}\delta_{1t}\delta_{1s}\delta_{\alpha\beta}\delta_{\alpha'\beta'} \end{aligned} \tag{5.9}$$

□

In order to endow $H$ with the structure of a coalgebra we have used the 6j-symbols of the module category $\mathcal{M}$. Since the 6j-symbols are the components of the associativity isomorphisms of a module category, the coalgebra structure of $H$ results from the associativity constraint of the module category.

**Theorem 5.2.4.**
*The algebra and coalgebra $H = \mathrm{End}_{\mathbb{C}}\Phi$ from the previous Theorem can be endowed with the structure of a weak bialgebra for which the counital subalgebras are isomorphic to $R = \oplus_\alpha \mathrm{End}_{\mathcal{M}}(M_\alpha)$.*

**Proof.** The product in $H$ is defined via composition of morphisms. Thus, the algebra structure of $H$ in the basis $\{h^{i\ rr'}_{(\alpha\beta)(\alpha'\beta')}\}$ reads

$$\begin{aligned} h^{i\ uu'}_{(\alpha\beta)(\alpha'\beta')} \cdot h^{j\ vv'}_{(\gamma\epsilon)(\gamma'\epsilon')} &= \delta_{ij}\delta_{\alpha\gamma'}\delta_{\beta\epsilon'}\delta_{uv'}h^{i\ vu'}_{(\gamma\epsilon)(\alpha'\beta')} \\ 1_H &= \sum_{i,u,\alpha,\beta} h^{i\ uu}_{(\alpha\beta)(\alpha\beta)}\,, \end{aligned}$$

the coalgebra structure is given in (5.9). Using the property (2.8) of the fusing matrices $F$ one can check straightforwardly that the coproduct $\Delta$ is an algebra morphism. Properties (1.2) and (1.3) hold because of property (2.9), respectively (2.7) of the fusing matrices $F$.

Using (2.8) we find

$$\Delta(1_H) = \sum_{j,l,r,r',\epsilon,\alpha,\beta} h^{j\ rr}_{(\alpha\epsilon)(\alpha\epsilon)} \otimes h^{l\ r'r'}_{(\epsilon\beta)(\epsilon\beta)}$$

which is not equal to the unit in $H\otimes H$ unless $\Gamma = 1$ (Tannaka case).

We compute the target counital map to be

$$\begin{aligned} \epsilon_t(h^{i\ vv'}_{(\gamma\zeta)(\gamma'\zeta')}) &= \epsilon(1_{(1)}h^{i\ vv'}_{(\gamma\zeta)(\gamma'\zeta')})1_{(2)} = \sum_{j,l,r,r',\epsilon,\alpha,\beta} \epsilon(h^{j\ rr}_{(\alpha\epsilon)(\alpha\epsilon)}h^{i\ vv'}_{(\gamma\zeta)(\gamma'\zeta')})\otimes h^{l\ r'r'}_{(\epsilon\beta)(\epsilon\beta)} \\ &= \sum_{l,r',\beta} \epsilon(h^{i\ vv'}_{(\gamma\zeta)(\gamma'\zeta')})\otimes h^{l\ r'r'}_{(\zeta'\beta)(\zeta'\beta)} \\ &= \delta_{i0}\delta_{1v}\delta_{1v'}\delta_{\gamma\zeta}\delta_{\gamma'\zeta'} \sum_{l,r',\beta} h^{l\ r'r'}_{(\gamma'\beta)(\gamma'\beta)}\,. \end{aligned} \tag{5.10}$$

Similarly, one finds

$$\epsilon_s(h^{i\ vv'}_{(\gamma\zeta)(\gamma'\zeta')}) = \delta_{i0}\delta_{1v}\delta_{1v'}\delta_{\gamma\zeta}\delta_{\gamma'\zeta'} \sum_{l,r',\beta} h^{l\ r'r'}_{(\beta\gamma)(\beta\gamma)}\,.$$

Therefore, the counital subalgebras are of the form

$$H_t = \langle \sum_{i,t,\beta} h^{i\ tt}_{(\alpha\beta)(\alpha\beta)} \rangle \qquad \text{and} \qquad H_s = \langle \sum_{i,t,\alpha} h^{i\ tt}_{(\alpha\beta)(\alpha\beta)} \rangle$$

with multiplication in $H_t$ given by

$$\sum_{i,s,\beta} h^{i\ ss}_{(\alpha\beta)(\alpha\beta)} \cdot \sum_{j,t,\delta} h^{j\ tt}_{(\gamma\delta)(\gamma\delta)} = \delta_{\alpha\gamma} \sum_{i,s,\beta} h^{i\ ss}_{(\alpha\beta)(\alpha\beta)}$$

and multiplication in $H_s$

$$\sum_{i,s,\alpha} h^{i\ ss}_{(\alpha\beta)(\alpha\beta)} \cdot \sum_{j,t,\gamma} h^{j\ tt}_{(\gamma\delta)(\gamma\delta)} = \delta_{\beta\delta} \sum_{i,s,\beta} h^{i\ ss}_{(\alpha\beta)(\alpha\beta)}\,.$$

Since the counital algebras are commutative, one has $H_t \cong H_s$. By mapping $\sum_{i,t,\beta} h^{i\ tt}_{(\alpha\beta)(\alpha\beta)}$ to $\mathrm{id}_{M_\alpha}$ one obtains an algebra isomorphism from the counital algebra $H_t$ to $R$. $\square$

Note that because of

$$\Delta(1_H) = \sum_{j,l,r,r',\epsilon,\alpha,\beta} h^{j\ rr}_{(\alpha\epsilon)(\alpha\epsilon)} \otimes h^{l\ r'r'}_{(\epsilon\beta)(\epsilon\beta)}$$

$H$ is a bialgebra if and only if $\Gamma = \{1\}$, that is in the Tannaka case. Denote by $M_1$ a representative of the unique isomorphism class of simple objects in $\mathcal{M}$ and by $F^{jl}_{rr',vis}$ the 6j-symbols of $\mathcal{M}$. We choose a basis $\{e^{j\ r}\}$ with $r \in \{1,\ldots,\mathcal{N}_j\}$ for the vector spaces $\mathrm{Hom}_{\mathcal{M}}(M_1, M_1 \otimes U_j)$ and a basis

$$\{h^{i\ rs} : e^{i\ r} \mapsto e^{i\ s} \text{ and } 0 \text{ elsewhere}\}$$

in $\mathrm{Hom}_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{M}}(M_1, M_1 \otimes U_i), \mathrm{Hom}_{\mathcal{M}}(M_1, M_1 \otimes U_i))$. The bialgebra structure reads

$$\begin{aligned} h^{i\ uu'} \cdot h^{j\ vv'} &= \delta_{ij}\delta_{uv'} h^{i\ vu'} \\ 1_H &= \sum_{i,u} h^{i\ uu} \\ \Delta h^{i\ st} &= \sum_{j,l,r,r',p,p',v} F^{jl}_{rr',vis}\ (F^{-1})^{jl}_{pp',vit}\ h^{j\ rp} \otimes h^{l\ r'p'} \\ \epsilon(h^{i\ st}) &= \delta_{i0}\delta_{1t}\delta_{1s} \end{aligned}$$

In Theorem 2.3.4 it has been shown that the category of $H$-left modules is a tensor category with tensor unit $H_t$. The action of $H$ on a tensor product of $H$ modules is defined by means of the coproduct. We show that the equivalence from Lemma 5.2.2 is a tensor equivalence.

**Proposition 5.2.5.**
*The category of left $H$-modules is equivalent to $\mathcal{C}$ as a tensor category.*

**Proof.** In order to endow the functor $\omega : \mathcal{C} \to H$-mod from (5.8) with the structure of a tensor functor we use the isomorphisms

$$\begin{array}{lcll} (\psi^{-1})_{\tilde{M},X,Y} \circ ? & : & \bar{\Phi}(X) \otimes_{H_t} \bar{\Phi}(Y) & \longrightarrow \bar{\Phi}(X \otimes Y) \\ (\eta^{-1})_{\tilde{M}} \circ ? & : & H_t & \longrightarrow \bar{\Phi}(\mathbb{1}_{\mathcal{C}}) \end{array}$$

of the $R$ fiber functor given in (5.5). □

By Corollary 2.3.6 there exists a tensor functor $\mathcal{H} : H\text{-mod} \longrightarrow H_t\text{-bimod}$.

**Proposition 5.2.6.**
*The functor $\Phi$ is equal to $\mathcal{H} \circ \omega$.*

**Proof.** As vector spaces $\Phi(U) = \mathrm{Hom}_{\mathcal{M}}(\tilde{M}, \tilde{M} \otimes U) = \mathcal{H}(\omega(U))$.

We prove that $\Phi = \mathcal{H} \circ \omega$ as abelian functors by comparing the $R$-bimodule structure of $\Phi(U_i)$ with the $R$-bimodule structure of $\mathcal{H}(\omega(U_i))$: for $\Phi(U_i)$ one has in the chosen basis

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{M}}(M_\gamma, M_\beta \otimes U_i) \times \mathrm{Hom}_{\mathcal{M}}(M_\alpha, M_\alpha) & \longrightarrow & \mathrm{Hom}_{\mathcal{M}}(M_\alpha, M_\beta \otimes U_i) \\ (e^{i\ t}_{\beta\gamma}, \mathrm{id}_{M_\alpha}) & \longmapsto & \delta_{\alpha\gamma} e^{i\ t}_{\beta\alpha} \\ \mathrm{Hom}_{\mathcal{M}}(M_\alpha, M_\alpha) \times \mathrm{Hom}_{\mathcal{M}}(M_\gamma, M_\beta \otimes U_i) & \longrightarrow & \mathrm{Hom}_{\mathcal{M}}(M_\gamma, M_\alpha \otimes U_i) \\ (\mathrm{id}_{M_\alpha}, e^{i\ t}_{\beta\gamma}) & \longmapsto & \delta_{\alpha\beta} e^{i\ t}_{\alpha\gamma}\,. \end{array}$$

By Lemma 2.3.2 any $H$-module $V$ becomes an $H_t$-bimodule using $\epsilon_s$. We decompose $\mathcal{H}(\omega(U_i)) = \oplus_{\beta,\gamma}\mathrm{Hom}_{\mathcal{M}}(M_\gamma, M_\beta \otimes U_i)$ and write the bimodule structure for each summand

$$\begin{array}{rcl} (e^{i\ t}_{\beta\gamma}, \sum_{j,u,\epsilon} h^{j\ uu}_{(\alpha\epsilon)(\alpha\epsilon)}) & \longmapsto & \sum_{j,u,\epsilon} h^{j\ uu}_{(\epsilon\alpha)(\epsilon\alpha)}(e^{i\ t}_{\beta\gamma}) = \delta_{\alpha\gamma} e^{i\ t}_{\beta\gamma} \\ (\sum_{j,u,\epsilon} h^{j\ uu}_{(\alpha\epsilon)(\alpha\epsilon)}, e^{i\ t}_{\beta\gamma}) & \longmapsto & \sum_{j,u,\epsilon} h^{j\ uu}_{(\alpha\epsilon)(\alpha\epsilon)}(e^{i\ t}_{\beta\gamma}) = \delta_{\alpha\beta} e^{i\ t}_{\beta\gamma} \,. \end{array}$$

Thus, $\Phi = \mathcal{H} \circ \omega$ on all objects of $\mathcal{C}$. Therefore, the functors $\Phi$ and $\mathcal{H} \circ \omega$ are equal as tensor functors. □

**Proposition 5.2.7.**
*If the tensor category $\mathcal{C}$ is ribbon, the weak bialgebra $H = \mathrm{End}_{\mathbb{C}}\Phi$ from Theorem 5.2.4 can be endowed with the structure of a weak Hopf algebra.*

**Proof.** The coevaluation and the evaluation morphisms of $\Phi(U)$ in $R$-bimod in the basis $\{e^{i\ u}_{\alpha\beta}\}$ read

$$\begin{array}{rcccl} b_{\Phi(U_i)} & : & \mathrm{id}_{M_\alpha} & \longmapsto & \sum_{\beta,p} e^{i\ p}_{\alpha\beta} \otimes_R e^{i^\vee\ p}_{\beta\alpha} \\ d_{\Phi(U_i)} & : & e^{i^\vee\ p}_{\alpha\beta} \otimes_R e^{i\ q}_{\beta\gamma} & \longmapsto & \delta_{\alpha\gamma}\delta_{pq}\mathrm{id}_{M_\alpha} \,. \end{array}$$

The separable algebra $R$ can be endowed with the structure of a coalgebra

$$\Delta_R\,\mathrm{id}_{M_\alpha} = \mathrm{id}_{M_\alpha} \otimes \mathrm{id}_{M_\alpha} \qquad \text{and} \qquad \epsilon_R(\mathrm{id}_{M_\alpha}) = 1.$$

Consider the map

$$\begin{array}{rcl} S : H & \longrightarrow & H \\ h_U & \longmapsto & (h_{U^\vee})^\vee \,, \end{array}$$

where

$$\begin{array}{rcl} (h_{U^\vee})^\vee & = & (\mathrm{id}_{\Phi(U)} \otimes \epsilon_R \circ d_{\Phi(U)}) \circ (\mathrm{id}_{\Phi(U)} \otimes_R h_{\Phi(U^\vee)} \otimes_R \mathrm{id}_{\Phi(U)}) \\ & & \circ (b_{\Phi(U)} \otimes_R \mathrm{id}_{\Phi(U)}) \circ \pi \circ (1_R \otimes \mathrm{id}_{\Phi(U)}). \end{array}$$

In the basis $\{h^{i\ uu'}_{(\alpha\beta)(\alpha'\beta')}\}$ of $H$ one has $S(h^{i\ uu'}_{(\alpha\beta)(\alpha'\beta')}) = h^{i^\vee\ u'u}_{(\beta'\alpha)(\beta\alpha)}$. To be an antipode of $H$, the map $S$ must obey

$$\begin{array}{rcl} \mathrm{id} * S & = & \epsilon_t \\ S * \mathrm{id} & = & \epsilon_s \\ S * \mathrm{id} * S & = & S \,. \end{array}$$

We compute

$$\begin{array}{rcl} \mathrm{id} * S(h^{i\ st}_{(\alpha\beta)(\alpha'\beta')}) & = & \sum_{j,l,r,r',p,p',v,\epsilon,\eta} F^{(\alpha jl)\beta}_{r\epsilon r',vis} \, (F^{-1})^{(\alpha' jl)\beta'}_{p\eta p',vit} \, h^{j\ rp}_{(\alpha\epsilon)(\alpha'\eta)} S(h^{l\ r'p'}_{(\epsilon\beta)(\eta\beta')}) \\ & = & \sum_{j,l,r,r',p,p',v,\epsilon,\eta} F^{(\alpha jl)\beta}_{r\epsilon r',vis} \, (F^{-1})^{(\alpha' jl)\beta'}_{p\eta p',vit} \, h^{j\ rp}_{(\alpha\epsilon)(\alpha'\eta)} h^{l^\vee\ p'r'}_{(\beta'\eta)(\beta\epsilon)} \\ & = & \delta_{\alpha\beta} \sum_{l,r,r',p,p',v,\epsilon,\eta} F^{(\alpha l^\vee l)\alpha}_{r\epsilon r,vis} \, (F^{-1})^{(\alpha' l^\vee l)\beta'}_{p\eta p',vit} \, h^{l^\vee\ p'p}_{(\beta'\eta)(\alpha'\eta)} \,. \end{array}$$

We insert $1 = (F^{-1})^{(\alpha l^\vee l)\alpha}_{r\epsilon r,101}$ in the previous expression and use (2.9) to obtain

$$\begin{aligned}
\mathrm{id} * S(h^{i\ st}_{(\alpha\beta)(\alpha'\beta')}) &= \delta_{\alpha\beta}\delta_{i0}\delta_{s1} \sum_{l,p,p',v,\eta} (F^{-1})^{(\alpha' l^\vee l)\beta'}_{p\eta p',10t}\ h^{l^\vee\ p'p}_{(\beta'\eta)(\alpha'\eta)} \\
&= \delta_{i0}\delta_{\alpha\beta}\delta_{\alpha'\beta'}\delta_{s1}\delta_{t1} \sum_{l,p,\eta} h^{l^\vee\ pp}_{(\beta'\eta)(\alpha'\eta)} \,.
\end{aligned}$$

This is equal to $\epsilon_t(h^{i\ st}_{(\alpha\beta)(\alpha'\beta')})$ from (5.10). Hence, $\mathrm{id} * S = \epsilon_t$. It is straightforward to check that the other two conditions on an antipode hold. $\square$

In order to endow the weak bialgebra $H$ with the structure of a weak Hopf algebra we have used the ribbon structure of $\mathcal{C}$. Note that the weak Hopf algebra we reconstructed is isomorphic to the one introduced in [PeZu01].

We have the following situation for an abelian artinian ribbon category $\mathcal{C}$, a commutative, separable $\mathbb{C}$-algebra $R$:

$$\begin{tikzcd}
\begin{array}{c}\text{exact fiber functor}\\ \Phi : \mathcal{C} \to R\text{-bimod}\end{array} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \begin{array}{c}\text{mod-}R \text{ is a left module}\\ \text{category over } \mathcal{C}\end{array} \\
& \begin{array}{c}\text{weak Hopf algebra } H \text{ with}\\ \text{counital subalgebra } R\end{array} \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

## 5.3 Example: the free boson

We consider the tensor category $\mathcal{C}$ as a module category over itself. Then the 6j-symbols $F^{(\alpha j l)\beta}_{vit,p\eta q}$ of the module category $\mathcal{M}$ reduce to the associativity isomorphisms of the tensor category.

As a particular example we consider tensor categories for which the fusion ring is the group algebra of a finite group $G$. The fusion is $U_i \otimes U_j \cong U_{ij}$ for all $i, j \in G$ and the 6j-symbols reduce to the associator $a$ in the tensor category. By equation (5.7) the weak bialgebra is of dimension $|G|^3$.

We choose a basis $\{e^j_l\}$ for the one-dimensional morphism spaces $\mathrm{Hom}_{\mathcal{C}}(U_{jl}, U_j \otimes U_l)$. Let

$$\{h^i_{jl} : e^i_j \mapsto e^i_l\}$$

be a basis in $\mathrm{Hom}_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{C}}(U_{ij}, U_i \otimes U_j), \mathrm{Hom}_{\mathcal{C}}(U_{il}, U_i \otimes U_l))$. Then the weak Hopf algebra structure is

$$\begin{aligned}
h^i_{kl} * h^j_{fg} &= \delta_{ij}\delta_{ag} h^i_{lf} \\
1_H &= \sum_{i,g} h^i_{gg} \\
\Delta h^i_{jk} &= \sum_{l \in I} a_{jl(il^{-1})} a^{-1}_{kl(il^{-1})} h^l_{jk} \otimes h^{il^{-1}}_{(lj)(lk)} \\
\epsilon(h^i_{kl}) &= \delta_{i1} \\
S(h^i_{kl}) &= h^{i^{-1}}_{li\ ki} \,.
\end{aligned}$$

The counital subalgebras are $H_{t,s} = \langle \sum_i h^i_{ll} \rangle$. Note that this weak Hopf algebra is the dual vector space of the weak Hopf algebra constructed in [Hay99].

As an example we consider the theory of a compactified, massless, free boson with boundary conditions. The modular tensor category associated to a free boson field has $2N$ isomorphism classes of simple objects, the fusion rules furnish the group $\mathbb{Z}_{2N}$ (see for example [FRS02b]). Denote by $\sigma(n)$ the unique integer for which $n - 2N\sigma(n) \in \{0, 1, \ldots, 2N-1\}$. The fusion matrices of the modular tensor category associated to a free boson are

$$F^{(i\ j\ k)[i+j+k]}_{[j+k][i+j]} = (-1)^{(i+j+k)[j\sigma(i+j+k)+(j+k)(\sigma(i+j)+\sigma(j+k))]} .$$

We consider this modular tensor category as a module category over itself; the associated weak bialgebra is then

$$\begin{aligned}
h^i_{kl} * h^j_{fg} &= \delta_{ij}\delta_{ag} h^i_{lf} \\
1_H &= \sum_{i,g} h^i_{gg} \\
\Delta h^i_{jk} &= \sum_{l \in I} F^{(j\ l\ i-l)[j+i]}_{[i][j+l]} (F^{-1})^{(k\ l\ i-l)[k+i]}_{[i][k+l]} h^l_{jk} \otimes h^{i-l}_{(lj)(lk)} \\
&= \sum_{l \in I} (-1)^{(i+j)[l\sigma(i+j)+i(\sigma(j+l)+\sigma(i))]+(i+k)[l\sigma(i+k)+i(\sigma(k+l)+\sigma(i))]} h^l_{jk} \otimes h^{i-l}_{(lj)(lk)} \\
\epsilon(h^i_{kl}) &= \delta_{i1} \\
S(h^i_{kl}) &= h^{-i}_{li\ ki} .
\end{aligned}$$

# Bibliography


[ABDMSW03] P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp, J. Wess, *A Gravity Theory on Noncommutative Spaces*, Class. Quant. Grav. 22 (2005) 3511

[BaKi] B. Bakalov and A.A. Kirillov, *Lectures on Tensor Categories and Modular functors,* American Mathematical Society, Providence 2000

[BFFLS78] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, *Deformation theory and quantization. I. Deformations of symplectic structures,* Ann. Phys. 111 (1978) 61

[BoNS99] G. Böhm, F. Nill, and K. Szlachányi, *Weak hopf algebras: I. Integral theory and $C^*$-structure*, J. Algebra 221 (1999) 385

[BoSz96] G. Böhm and K. Szlachányi, *A coassociative $C^*$-quantum groupoid with non-integral dimensions*, Lett. Math. Phys. 38 (1996) 437

[Bour] N. Bourbaki, *Lie groups and Lie Algebras,* Springer, Berlin 1989

[CaEt04] D. Calaque and P.I. Etingof, *Lectures on tensor categories*, preprint math.QA/0401246

[CJSWW02] X. Calmet, B. Jurčo, P. Schupp, J. Wess, M. Wohlgenannt, *The standard model on non-commutative space-time,* Eur. Phys. J. C 23 (2002) 363

[Ca01] P. Cartier, *A mad day's work: from Grothendieck to Connes and Kontsevich, The evolution of concepts of space and symmetry*, Bulletin of the Am. Math. Society, vol. 38, no. 4 (2001) 389

[CKNT04] M. Chaichian, P.P. Kulish, K. Nishijima, A. Tureanu, *On a Lorentz-Invariant Interpretation of Noncommutative Space-Time and Its Implications on Noncommutative QFT*, Phys. Lett. B 604 (2004) 98

[ChHo99] C-S. Chu, P-M Ho, *Non-commutative open string and D-brane,* Nucl. Phys. B 550 (1999) 151

[CoRi87] A. Connes and M. Rieffel, *Yang-mills for noncommutative two-tori*, Contemp. Math. Oper. Algebra. Math. Phys. 62 (1987) 237

[CoSc01] R. Coquereaux and G. Schieber, *Twisted partition functions for ADE boundary conformal field theories and Ocneanu algebras of quantum symmetries*, J. Geom. and Phys. 42 (2002) 216

[Del90] P. Deligne, *Catégories tannakiennes.* In: The Grothendieck Festschrift, vol. II, P. Cartier et al., eds., Birkhäuser, Boston 1990

[DeMi82] P. Deligne and J.S. Milne, *Tannakian categories*, Springer Lecture Notes in Mathematics 900 (1982) 101

[DWi] B. DeWitt, in: *Gravitation: An Introduction to Current Research,* L. Witten, ed., John Wiley, New York 1962

[DMT04] M. Dimitrijević, L. Möller, and E. Tsouchnika, *Derivatives, forms and vector fields on the kappa-deformed Euclidean space,* J. Phys. A 37 (2004) 9749

[DFR94] S. Doplicher, K. Fredenhagen, and J.E. Roberts, *Spacetime quantization induced by classical gravity*, Phys. Lett. B 331 (1994) 39

[DMZ07] M. Doubek, M. Markl, P. Zima, *Deformation Theory (lecture notes)*, preprint arXiv: 0705.3719

[Dri85] V.G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, Sov. Math. Dokl. 32 (1985) 254

[ENO02] P.I. Etingof, D. Nikshych, and V. Ostrik, *On fusion categories,* Ann. Math. 162 (2005) 581

[Fad84] L.D. Faddeev, *Integrable models in (1+1)-dimensional quantum field theory.* In: Recent Advances in Field Theory and Statistical Mechanics, Springer Lectures 1984

[FFFS00] G. Felder, J. Fröhlich, J. Fuchs, and C. Schweigert, *Conformal boundary conditions and three-dimensional topological field theory*, Phys. Rev. Lett. 84 (2000) 1659

[FRS06] J. Fjelstad, J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators V: Proof of modular invariance and factorisation,* Theory and Applications of Categories 16 (2006), 342

[FFRS04] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert, *Kramers-Wannier duality from conformal defects*, Phys. Rev. Lett. 93 (2004) 070601

[FFRS07] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert, *Duality and defects in rational conformal field theory*, Nucl. Phys. B 763 (2007) 354

[Fu91] J. Fuchs, *Simple WZW currents*, Commun. Math. Phys. 136 (1991) 345

[Fu92] J. Fuchs, *WZW quantum dimensions*, Int. J. Mod. Phys. B 6 (1992) 1951

[FRS02a] J. Fuchs, I. Runkel, and C. Schweigert, *Conformal correlation functions, Frobenius algebras and triangulations*, Nucl. Phys. B 624 (2002) 452

[FRS02b] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators I: Partition functions*, Nucl. Phys. B 646 (2002) 353

[FRS04a] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators II: Unoriented world sheets*, Nucl. Phys. B 678 (2004) 511

[FRS04b] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators III: Simple currents*, Nucl. Phys. B 694 (2004) 277

[FRS05] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators IV: Structure constants and correlation functions*, Nucl. Phys. B 715 (2005) 539

[FRS07] J. Fuchs, I. Runkel, and C. Schweigert, *The fusion algebra of bimodule categories*, Applied Categorical Structures (2007)

[GRW96] T. Gannon, P. Ruelle, and M.A. Walton, *Automorphism modular invariants of current algebras*, Commun. Math. Phys. 179 (1996) 121

[GKO86] P. Goddard, A. Kent, and D.I. Olive, *Unitary representations of the Virasoro and super Virasoro algebras*, Commun. Math. Phys. 103 (1986) 105

[Hay99] T. Hayashi, *A canonical Tannaka duality for finite semimisimple tensor categories*, preprint math/9904073

[Hei30] *Letter of Heisenberg to Peierls (1930)*, in: Wolfgang Pauli, Scientific Correspondence, vol. II, 15, Ed. Karl von Meyenn, Springer Verlag 1985

[Jim85] M. Jimbo, *A q-difference analogue of* $\mathsf{u}(g)$ *and the Yang-Baxter equation*, Lett. Math. Phys. 10 (1985) 63

[Jon85] V.F.R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. 12 (1985) 103

[JoSt91] A. Joyal and R. Street, *The geometry of tensor calculus, I*, Adv. Math. 88 (1991) 55

[JSSW00] B. Jurčo, S. Schraml, P. Schupp, J. Wess, *Enveloping algebra valued gauge transformations for non-abelian gauge groups on non-commutative spaces*, Eur. Phys. J. C 17 (2000) 521

[Kac] V.G. Kac, *Infinite-dimensional Lie Algebras*, third edition, Cambridge University Press, Cambridge 1990

[KaCe71] L.P. Kadanoff and H. Ceva, *Determination of an operator algebra for the two-dimensional Ising model*, Phys. Rev. B 3 (1971) 3918

[Kas] C. Kassel, *Quantum Groups*, Springer Verlag, New York 1995

[KnZa84] V.G. Knizhnik and A.B. Zamolodchikov, *Current algebra and Wess-Zumino model in two dimensions*, Nucl. Phys. B 247 (1984) 83

[Kon03] M. Kontsevich, *Deformation Quantization of Poisson Manifolds*, Lett. Math. Phys. 66 (2003) 157

[KLM00] K. Kosiński, J. Lukierski and P. Maślanka, *Local field theory on $\kappa$-Minkowski space, $\star$-products and noncommutative translations*, Czech. J. Phys. 50 (2000) 1283

[KrWa41] H.A. Kramers and G.H. Wannier, *Statistics of the two-dimensional ferromagnet, Part I.*, Phys. Rev. 60 (1941) 252

[LNRT91] J. Lukierski, A. Nowicki, H. Ruegg and V. N. Tolstoy, *Q-deformation of Poincaré algebra*, Phys. Lett. B 264 (1991) 331

[Lus90] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. 3 (1990) 447

[Lus90b] G. Lusztig, *Canonical bases arising from quantized enveloping algebras. II*, Progr. Theor. Phys. Suppl. 102 (1990) 175

[MSSW00] J. Madore, S. Schraml, P. Schupp and J. Wess, *Gauge theory on noncommutative spaces*, Eur. Phys. J. C 16 (2000) 161

[MaRu94] S. Majid and H. Ruegg, *Bicrossproduct structure of $\kappa$-Poincaré group and noncommutative geometry*, Phys. Lett. B 334 (1994) 348

[Ma95] O. Mathieu, *Équations de Knizhnik-Zamolodchikov et théorie des représentations*, Astérisque 227 (1995) 47

[Moe04] L. Möller, *Noncommutative gauge theory and $\kappa$-deformed spacetime*, PhD thesis at the Ludwig Maximilian Universität, Munich (2004)

[Niks02] D. Nikshych, *On the structure of weak hopf algebras*, Adv. Math. 170 (2002) 257

[NiVa00] D. Nikshych and L. Vainerman, *Finite quantum groupoids and their applications*, preprint math-ph/0006057

[NST93] A. Nowicki, E. Sorace and M. Tarlini, *The quantum deformed Dirac equation from the $\kappa$-Poincaré algebra*, Phys. Lett. B 302 (1993) 419

[Ostr03] V. Ostrik, *Module categories, weak hopf algebras and modular invariants*, Transform. Groups 8 (2003) 177

[Par77] B. Pareigis, *Non-additive ring and module theory I. General theory of monoids*, Publ. Math. Debrecen 24 (1977) 189

[PeZu01] V.B. Petkova and J.-B. Zuber, *The many faces of Ocneanu cells*, Nucl. Phys. B 603 (2001) 449

[Pier] R.S. Pierce, *Associative Algebras*, Springer Verlag, New York 1980

[Pfe07] H. Pfeiffer, *Tannaka-Krein reconstruction and a characterization of modular tensor categories*, preprint 0711.1402

[Ruel05] Ph. Ruelle, *Kramers-Wannier dualities via symmetries*, Phys. Rev. Lett. 95 (2005) 225701

[Scha02] P. Schauenburg, *Weak hopf algebras and quantum groupoids*, preprint math.QA/0204180

[Schell] A.N. Schellekens, The program kac, see `http://www.nikhef.nl/~t58/kac.html`

[ScYa90a] A.N. Schellekens and S. Yankielowicz, *Field identification fixed points in the coset construction*, Nucl. Phys. B 334 (1990) 67

[ScYa90b] A.N. Schellekens and S. Yankielowicz, *Simple currents, modular invariants, and fixed points*, Int. J. Mod. Phys. A 5 (1990) 2903

[Scho99] V. Schomerus, *D-branes and deformation quantization*, J. High Energy Phys. 9906 (1999) 030

[SeWi99] N. Seiberg and E. Witten, *String Theory and Noncommutative Geometry,* JHEP 9909 (1999) 032

[Sny47] H.S. Snyder, *Quantized Space-Time,* Phys. Rev. 71 (1947) 38

[Ster98] D. Sternheimer, in: *Particles, Fields and Gravitation*, J. Rembieliński (ed.) (AIP Press, New York 1998), p. 107, *Deformation quantization: Twenty years after (Lodz 1998)*

[Szl01] K. Szlachányi, *Finite quantum groupoids and inclusions of finite type*, Fields Institute Commun. 30 (2001) 393

[Tur] V.G. Turaev, *Quantum Invariants of Knots and 3-Manifolds,* de Gruyter, New York 1994

Efrossini Tsouchnika
Quantum Groups and Field Theory
Hamburg 2008

# Zusammenfassung

In dieser Arbeit untersuchen wir drei verschiedene Aspekte von Quantengruppen und ihren Darstellungskategorien in Feldtheorien.

Für klassische Feldtheorien auf zwei verschiedenen Typen von nichtkommutativen Räumen, nämlich dem $\theta$-Raum und dem $\kappa$-Raum, konstruieren wir die Quantensymmetrien. Diese sind Quantengruppen und ergeben sich als Deformationen der Poincaréalgebra des klassischen Raums. Für den $\theta$-Raum gibt es eine eindeutige Zwei-Parameter-Familie von Lösungen, für den $\kappa$-Raum schränken wir uns auf die einfachste Lösung ein. Weiterhin berechnen wir für diese beiden nichtkommutativen Räume Invarianten und konstruieren globale Vektorfelder.

Für Wess-Zumino-Witten-Modelle und (super-)minimale Modelle im Cardy-Fall bestimmen wir mit Hilfe des TFT Zugangs zur vollen konformen Feldtheorie die Kramers-Wannier-Dualitäten. Der Grothendieckring der modularen Tensorkategorie legt die Dualitätsklassen fest. Für die $A$-Serie von Wess-Zumino-Witten-Modellen und die Virasoro minimalen Modelle benutzen wir Eigenschaften der $S$-Matrix, für die $B$-, $C$-, $D$-Serien und die exzeptionellen Algebren $E_6$ und $E_7$ schätzen wir die Quantendimension von Fixpunkten ab. Es zeigt sich, dass Dualitäten nur bei niedrigem Level auftreten. Die modularen Tensorkategorien für $(A_1)_2$ und $(B_r)_1$ haben die selben Fusionsregeln wie das Isingmodell. Für $(E_7)_2$ sind die Fusionsregeln isomorph zu denen des trikritischen Isingmodells.

Das Tannaka-Krein Rekonstruktionstheorem wird für schwache Quantengruppen verallgemeinert. Ausgehend von einem Satz von Ostrik (2003) zeigen wir die Äquivalenz zwischen der Kategorie der Strukturen einer Modulkategorie und der Kategorie der Faserfunktoren in die Kategorie der Bimoduln über einer separablen Algebra. Wir beweisen, dass die natürlichen Endotransformationen eines solchen Faserfunktors die Struktur einer schwachen Quantengruppe besitzen. Für graduierte Vektorräume bestimmen wir die schwache Quantengruppe. Diese ist dual zu der schwachen Quantengruppe, die von Hayashi (1999) bestimmt wurde.

# Lebenslauf

| | |
|---|---|
| 29.05.1979 | geboren in Athen |
| 1998 | Abitur, Rupprecht-Gymnasium in München |
| 1998-2003 | Stipendium nach dem bayerischen Begabtenförderungsgesetz |
| 2003 | Diplom in Physik, LMU München |
| 2004-2006 | Promotionsstipendium der Universität Hamburg (HmbNFG) |
| 2006-2007 | Stipendium der German-Israeli Foundation (GIF) |
| 2007-2008 | Promotionsstipendium der DFG |