Quantum Field Theory and Composite Fermions in the Fractional Quantum Hall Effect

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Vorgelegt von
Marcel Kossow
aus Ulm

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Gutachter der Dissertation:		Prof. Dr. B. Kramer Prof. Dr. P. Schupp
Gutachter der Disputation:		Prof. Dr. B. Kramer Prof. Dr. S. Kettemann
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Vorsitzender des Prüfungsausschusses:		Prof. Dr. A. Lichtenstein
Vorsitzender des Promotionsausschusses:		Prof. Dr. R. Klanner
Department Leiter:		Prof. Dr. J. Bartels
MIN-Dekan:	bis 28.02.2009 ab 01.03.2009	Prof. Dr. A. Frühwald Prof. Dr. H. Graener

Zusammenfassung

Gegenstand dieser Arbeit ist die Untersuchung der Möglichkeiten ein Composite-Fermion-Modell für den fraktionalen Quanten Hall Effekt aus der relativistischen Quantenelektrodynamik abzuleiten . Die wesentliche Motivation hierfür ist den Spin des Elektrons, betrachtet als relativistischer Effekt, in das Composite-Fermion-Modell mit einzubeziehen. Mit einfachen Argumenten wird gezeigt, dass eine spezielle Chern-Simons-Transformation der Dirac Elektronen in vier Raumzeit-Dimensionen im Niederenergielimes zu einem Einteilchen-Hamiltonoperator für Composite-Fermionen mit entsprechenden Korrekturtermen, etwa Rashba- oder Dresselhaus- Spin-Orbit Kopplung und die Zitterbewegung führt. Des weiteren stellen wir einen Mechanismus vor, der Quantenfelder, definiert auf dem vierdimensionalen Minkowskiraum, quantenmechanisch auf drei Dimensionen projiziert. Das führt zu einer relativistischen Quantenfeldtheorie in drei Dimensionen und im Speziellen zu einer relativistischen Composite-Fermion-Feldtheorie in drei Dimensionen. Relativistisch bedeutet hierbei Kovarianz unter einer Untergruppe der Poincaré-Gruppe. Die Projektionsabbildung kann mit der Projektion in ein (relativistisches) Landauniveau beziehungsweise in ein Composite-Fermion Landauniveau kombiniert werden. Das führt zu einer quasi relativistischen Quantenfeldtheorie auf einer nichtkommutativen Ebene. Quasi relativistisch bedeutet hierbei, dass die Kovarianz bezüglich der Untergruppe der Poincaré-Gruppe auf der Skala der magnetischen Länge gebrochen ist. Die von diesem Ansatz resultierenden phänomenologischen Theorien werden diskutiert und erlauben eine systematische Untersuchung der Effekte vom Spin und der Kondensation in ein Landauniveau. Wir erwarten von den relativistischen Abhandlungen Korrekturen im Sinne von Spin-Orbit-Kopplungs-Effekten. Von der Projektion in Landauniveaus erwarten wir eine Modifikation der Dispersionsrelation und ebenso eine Änderung der Composite-Fermion-Masse. Im Limes, in dem die magnetische Länge verschwindet, sollte dann die Theorie mit dem herkömmlichen Zugang zu den Composite-Fermionen übereinstimmen. Die Chern-Simons-Theorie ist ein zentraler Aspekt der Composite-Fermion-Theorie und ihre Quantisierung unumgänglich. Deshalb rekapitulieren wir die BRST-Quantisierung von Chern-Simons-Theorien mit kompakter Eichgruppe und diskutieren die phänomenologischen Konsequenzen in einem Composite-Fermion-Modell mit Spin. Die Verbindung zu Wess Zumino Witten Theorien wird aufgegriffen und eine mögliche Beziehung zwischen der Zentralladung der entsprechenden affinen Lie Algebra und dem composite Fermion Füllfaktor aufgezeigt.

Abstract

The purpose of this thesis is the investigation of the possibilities to derive a composite Fermion model for the fractional Hall effect from relativistic quantum electrodynamics. The main motivation is to incorporate the spin of the electron, considered as relativistic effect, into the composite Fermion model. With simple arguments it is shown that a special Chern Simons transformation of the Dirac electrons in four spacetime dimensions leads in the low energy limit to a single particle Hamiltonian for composite Fermions in three dimensions with correction terms such as Rashba- or Dresselhaus-spin-orbit coupling and zitterbewegung. Furthermore we provide a mechanism to quantum-mechanically project the quantum fields defined in the four dimensional Minkowski space to three dimensions. This leads to a relativistic field theory and especially a composite Fermion field theory in three dimension. Relativistic now means covariance under a subgroup of the Poincaré group. This projection map can be combined with the projection onto a (relativistic) Landau level or composite Fermion Landau level respectively. This results in a quasi relativistic quantum field theory on a noncommutative plane. Quasi relativistic means that covariance under the subgroup of the Poincaré group is broken at the scale of the magnetic length. The phenomenological models resulting from this approach are discussed and allow a systematical exploration of the effects of the spin and the condensation in a Landau level. We expect from the relativistic approach corrections in terms of spin-orbit coupling effects. From the projection onto Landau levels we expect a modification of the dispersion relation and a modified composite Fermion mass. In the limit where the magnetic length vanishes the low energy theory should correspond to the common approach to composite Fermions. The Chern Simons theory is a central aspect of the composite Fermion theory and its quantization indispensable. Therefore the BRST quantization for Chern Simons theories with compact gauge group is reviewed and the phenomenological consequences within a composite Fermion model with spin are discussed. The connection to Wess Zumino Witten theories is recalled and a possible link between the corresponding central charge of the related affine Lie algebra and the composite Fermion filling factor is pointed out.

Contents

Chapter 1

Introduction

In 1879 E. Hall [Hal79] discovered that a magnetic field perpendicular to a thin conducting plate leads to a voltage drop perpendicular to an applied current and magnetic field, the so called Hall Voltage. The Hall resistance defined as the ratio between the Hall voltage and Hall current is observed to be proportional to the applied magnetic field and zero if the field is turned off. Already classical electrodynamics covers all effects of the classical Hall effect. The model tells us that charge carriers are deflected by the magnetic field which leads to an accumulation of charges at one side of the conducting plate. The result of the annihilation of the generated electric force and the magnetic force is a linear dependence of the Hall resistance from the magnetic field B:

$$R_H = \frac{B}{n_e e}$$

and the antiproportionality of the total number n_e of the electric charge carries for instance electrons with charge e. It is therefore possible to determine the number of charge carriers by measuring the Hall resistance. Important for this effect is that the conducting plate is very thin, therefore we would like to have at best a two dimensional charge carrier system. The industrial development of semiconductors made it possible to realize quasi two dimensional electron system in Metal-Oxide-Silicon-Field-Effect-Transistors (MOSFET). In 1980 von Klitzing then observed a quantization of the Hall resistance in such a structure at temperatures of about 1.5K and below [KDP80]. The high accuracy of the quantized Hall resistance in terms of the von Klitzing constant R_K,

$$R_H = \frac{R_K}{j} \quad \text{with} \quad j = 1, 2, 3, \ldots \quad \text{and} \quad R_\mathrm{K} = \frac{h}{e^2} \approx 25.812807\mathrm{k}\Omega,$$

defines not only a standard resistor but is also suitable to determine the fine structure constant $\alpha = \mu_0 c_0 e^2/(2h)$ with high accuracy. For the discovery of this integral quantum Hall effect von Klitzing was awarded the Nobel price in 1985.

The investigation of new techniques such as molecular beam epitaxy (MBE) made it possible to produce cleaner two dimensional electron systems where the mobility of the electrons is significantly improved. The realization of high mobility electron systems between GaAs and $Al_xGa_{1-x}As$ crystals provide more insight into the physics of low dimensional electron systems at low temperature. Already in 1938 Wigner proposed a crystal like configuration of an electron gas if the Coulomb repulsion exceeds the kinetic energy. Being in the search of such a Wigner crystal in two dimensional electron systems Tsui, Stormer and Gossard made the observation of Hall plateaus in magnetotransport experiments in between the plateaus found by von Klitzing [SCT+83]. However, these plateaus are not related to the Wigner crystallization but were linked by Laughlin in 1983 to the condensation of electrons into quasi particle excitations obeying fractional statistics [Lau83]. For their discovery of the fractional quantum Hall effect and for the explanation Tsui, Stormer, Gossard and Laughlin were awarded the Nobel price in 1998.

Since then the fractional quantum Hall effect belongs to the most active research fields and nearly every theoretical community has contributed some ideas and often deeper insight not only in the fractional Hall effect but also into fundamental configurations of matter concerning for instance the spin and statistics in low dimensions and the connection to topology. For example the concepts of topological gauge field theories [Wit89] and supersymmetry [DGS89], originally discussed in the context of quantum chromodynamics and (super) string theory [Wit95], arise naturally in quantum Hall systems. Also the field of noncommutative field theory, discussed in terms of the quantum structure of spacetimes at the Planck scale [DFR95], has a special but natural realization in the physics of the quantum Hall effect. Indeed the mathematically rigorous treatment of the integer quantum Hall effect is realized within the concept of noncommutative geometry [BvS94, Con90] and there are attemps to apply the techniques of noncommutative geometry also to the fractional quantum Hall effect [MM05].

On the other hand the progress made in production techniques, especially the improvement of molecular beam epitaxy in nano engineering, lead to cleaner electron systems and improved mobility. In high mobility samples combined with a cooling well below 1K there emerge finer and finer Hall plateaus and with special prepared (doped) samples new effects can be measured in particular what concerns the spin of the charge carriers [MFHac+00, KSvKE00, TEPW07].

The quality of being a macroscopic quantum effect and its connection to topological quantum numbers and braid group or fractional statistics is the basis for the idea that the fractional Hall system is a possible candidate for the realization of a quantum computer [Wan06].

One promising approach to the theory of the fractional quantum Hall effect was introduced by J.K. Jain in 1989 [Jai89]. He transformed the electrons in a quasi two dimensional system with strong magnetic field into composite objects consisting of an electron with an even number of flux quanta attached to it. This idea came from the observation that the longitudinal resistance shows up a rather symmetric magneto oscillation around the filling factor 1/2. This motivates the postulation of a condensation into quasi particles, the composite Fermions. At filling factor 1/2 the composite Fermions are exposed to a zero total magnetic field at least at mean field approximation while at filling factor 1/3 there is one further flux quantum per electron and leads to a filling factor for composite Fermions of one. Therefore the fractional Hall effect for electrons is roughly speaking mapped to the integer Hall effect for composite Fermions. Being able to describe rather successfully many effects in the fractional Hall effect, there are many open questions left especially what concerns the impact of the spin of the electron. There are some phenomenological approaches to incorporate the spin of the electron in terms of spin pairing to describe for instance polarization effects [MMN+02, KMM+02b].

From quantum electrodynamics it is well known that the anomalous magnetic moment of the electron is a relativistic effect. In the spirit of Wigner particles are irreducible representations of the covering group of the Poincaré group and the electron is a representation of a relativistic particle with intrinsic $SU(2)$ symmetry. For electrons in the low energy regime there appear for this reason correction terms such as spin orbit coupling or zitterbewegung. It is well known that such effects have influence of the behaviour of electrons in two dimensional electron systems. For instance the Rashba spin orbit coupling is discussed in the context of spin Hall effects [Win03]. Having realized that the spin of the electron is a relativistic effect we may wonder how the spin of a composite Fermion should be understood. This is the origin of the motivation of this thesis. We will discuss the possibility to derive composite Fermions with spin from relativistic quantum electrodynamics. The main problem is that quantum electrodynamics is defined in four rather than in three dimensions. Furthermore we have to explain how to attach flux quanta in this regime. We will provide an answer to both how to attach flux quanta and how to project the quantum fields to three dimensions. This leads to a relativistic composite Fermion theory with spin in three dimensions, however relativistic then means covariance under a subgroup of the Poincaré group. Then we push the analysis forward to quasi relativistic composite Fermions in a lowest Landau level or lowest composite Fermion Landau level and quasi relativistic now means that the covariance is violated at the scale of the magnetic length. This quasi covariant model is a realization of a quantum field theory on a noncommutative plane and therefore highly nonlocal which should lead to a distortion of the dispersion relation.

From an experimental point of view it is then interesting what the influences of such theories are for instance on the composite Fermion mass. From the relativistic approach we expect at least correction terms coming from the $SU(2)$ spin. The noncommutative extension we expect to correspond to the commutative model in the limit where the magnetic length vanishes, this might happen in the scaling limit where the correlation length diverges.

The quantum field theory of composite Fermions require a Chern Simons theory in three dimensions, this is a topological field in the sense that it does not depend on a metric it is therefore a general covariant theory. The approaches we introduce in this theses also require a quantization of the Chern Simons fields and therefore we review the BRST quantization of Chern Simons theories and point out the corresponding effects in a phenomenological $SU(2)$ Chern Simons/composite Fermion model. Furthermore the Chern Simons theory corresponds to a Wess Zumino Witten theory on Manifolds with boundary and we give a possible link between the central charge of the corresponding affine Lie algebra and the composite Fermion filling factor.

This thesis is organized as follows: In chapter two we review the theory of quantum Hall systems. We introduce the main concepts and models to describe the integer quantum Hall effect and explain the difference to the fractional Hall effect and motivate why new concepts are required in this regime. Therefore we introduce the phenomenological quantum field theory of composite Fermions especially with incorporation of spin effects. Since the spin is considered as a relativistic effect we propose in the next chapters a mechanism to formulate a relativistic composite Fermion theory based on usual quantum electrodynamics.

In chapter three we discuss the connection of relativistic quantum electrodynamics in four dimensions with the theory of three dimensional composite Fermions by a simple gedanken-experiment. This model will be replaced in chapter four by a concrete mechanism. Chern Simons theories play a central role in the theory of composite Fermions furthermore it is a topological gauge field theory which has to be quantized. Therefore we review the mathematical framework of topological field theories and a modern quantization procedure, the BRST quantization for Chern Simons theories. The connection of the mathematical treatment of pure Chern Simons gauge theories and the phenomenological approach of $SU(2)$ composite Fermions is pointed out. Furthermore the BRST method might be important when we discuss the field theory of composite Fermions projected to the lowest Landau level since there we obtain a noncommutative Chern Simons theory which has to be quantized and the BRST quantization should provide the correct framework. Furthermore we recall how the Chern Simons theory on a three dimensional manifold with nonempty boundary is connected to a chiral conformal field theory on the boundary, more precisely to a Wess Zumino Witten model and how one can derive the corresponding affine Lie algebra. A possible connection between the central charge and the composite Fermion filling factor is explained.

In chapter four we introduce a formal method to project quantum-mechanically the relativistic quantum electrodynamic theory in four dimensions onto three dimensions by freezing out the third spacial component. This projection can be combined with the projection onto the lowest (relativistic) Landau level or with the projection onto the lowest composite Fermion Landau level respectively. The completely projected fields are a special realization of a field theory on a three dimensional noncommutative spacetime where the time commutes with the spacial coordinates. The connection to quantum field theory on noncommutative spacetime is pointed out, especially to noncommutative Chern Simon theories. The phenomenological theories of composite Fermions resulting from these mechanisms are discussed and further impacts for experimentally observable data are pointed out. It seems that this method opens a rather large playground for constructing field theories in lower dimensions from higher dimensional theories and allows also the treatment of more general theories defined by Wightman distribution rather than by action principles. The last chapter provides a conclusion and outlook.

Chapter 2

Introduction to Quantum Hall Systems and Composite Fermions

2.1 Quantum Hall Effect

In the classical Hall effect [Hal79], a strong magnetic field $|\mathbf{B}| \equiv B_z$ is applied perpendicularly to a thin metallic plate figure 2.1. Classical electrodynamics tells us then that charged particles with mass m and charge e move in an orbit with cyclotron frequency

$$\omega_c = eB_z/m.$$

In a first approximation of that system the thin metallic plate is replaced by a classical two dimensional electron system. Let n_e be the number of classical charged particles with charge e. If we apply an electric field E the circular orbits drift perpendicular to $\mathbf{E}$ with velocity $|\mathbf{E}|/|\mathbf{B}|$. The resulting current density $\mathbf{j}$ is such that the electromagnetic forces vanish:

$$\mathbf{j} \wedge \mathbf{B} = n_e e\mathbf{E}.$$

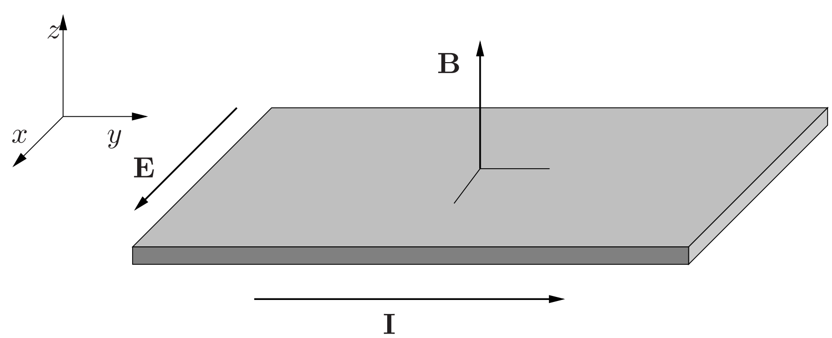


Figure 2.1: A schematic view of a Hall System. A magnetic field is applied perpendicularly to a thin metallic plate and an electric field in x direction leads to a Hall current in y direction.

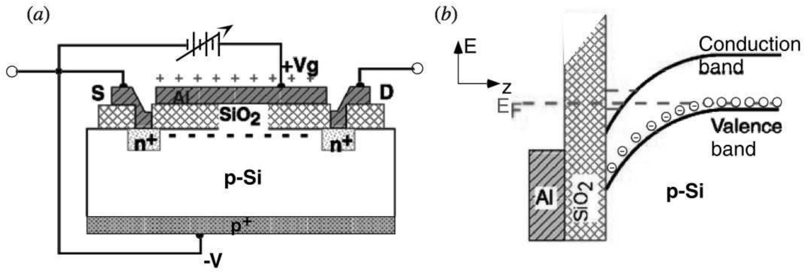


Figure 2.2: Schematic view of a silicon MOSFET a) and the corresponding band structure with the confinement potential [JJ01].

The scattering-time is short compared to the cyclotron frequency, which leads to a small component perpendicular to the field E, the Hall current:

$$\mathbf{j} = n_e e \mathbf{B} \wedge \mathbf{E}/|\mathbf{B}|^2.$$

This leads to the definition of the Hall current, being a linear function of the number of charge carriers:

$$\sigma_H := n_e e/|\mathbf{B}|.$$

The current is either positive or negative depending on the charge e and thus on the material. In the quantum Hall effect 1981 [KDP80], von Klitzing, Dorada and Pepper explored the behaviour of a two dimensional electron system at temperatures around 1K, see figure 2.3. By applying the Hall setup to a Si-MOSFET they observed a quantization of the Hall resistivity. The observation of plateaus in the Hall resistivity shows a quantum effect, which is used to determine the fine structure constant $\alpha = e^2/hc$ with high accuracy. In a first naive approach the two dimensional electron system consists of non-interacting electrons with no disorder. The corresponding one-particle Hamiltonian (Landau Hamiltonian) is given by

$$H = \frac{1}{2m}(\mathbf{p} - \mathbf{A})^2. \tag{2.1.1}$$

Then the operators $K_j = p_j - A_j$ satisfy the commutation relation

$$[K_1, K_2] = i\hbar eB$$

and the Hamiltonian can be rewritten to

$$H = \frac{\mathbf{K}^2}{2m}.$$

The corresponding energy levels are called Landau levels:

$$E_n = (n + 1/2)\hbar\omega_c. \tag{2.1.2}$$

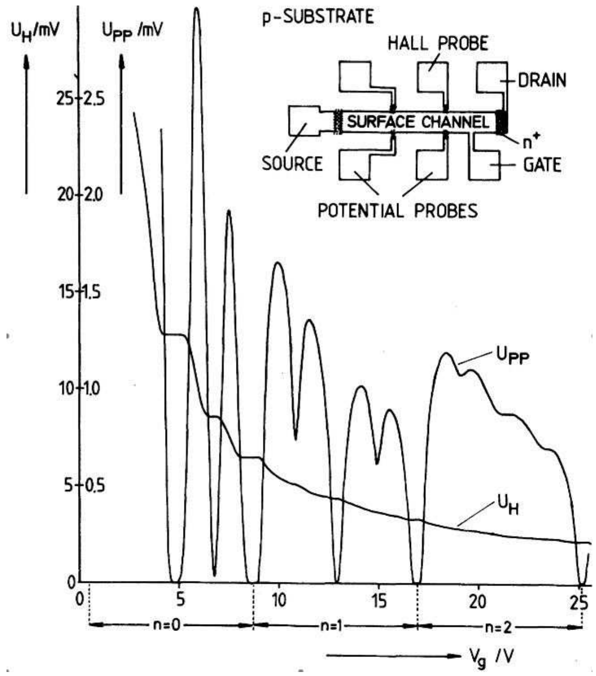


Figure 2.3: Observation of the quantum Hall effect [KDP80]. The Hall voltage forms plateaus while the longitudinal voltage drops at these plateaus.

They are highly degenerated since the system is translation invariant. It can be filled by $n_B = eB/h$ charge carriers per unit area or of flux quanta $\phi_0 = h/e$ per unit area respectively. The ration between the number of electrons n_e per number of flux quanta n_B

$$\nu = \frac{n_e}{n_B} \tag{2.1.3}$$

is called filling factor. The filling of Landau levels can be combined with the drift velocity. The Hall current is then expected to be

$$|\mathbf{I}| = n(e^2/h)|\mathbf{E}|, \tag{2.1.4}$$

with n being the number of filled Landau levels and j is in direction perpendicular to E. It follows that the Hall conductivity is an integer multiple of (e^2/h):

$$\sigma_H = n(e^2/h), \tag{2.1.5}$$

when the Fermi level is in between two Landau levels. These rather simple formulas connect the Hall conductivity linearly with the quantum number n, however it does not explain the occurrence of plateaus.

The explanation of the integrality of the Hall conductivity on the plateaus with simultaneously vanishing direct conductivity was given by Laughlin in 1981 [Lau81] . He used gauge invariance of the Hamiltonian together with a special topology of the sample see figure 2.4.

The Hamiltonian (2.1.1) in the Landau model $A = (0, Bx, 0)$ with the Landau levels (2.1.2) is solved by the Landau states

$$\psi_{nk}(x,y) = \frac{1}{\sqrt{L}} e^{iky} \phi_n(x - X_k), \tag{2.1.6}$$

where the normalized eigenfunctions

$$\phi_n(x) \frac{1}{\sqrt{\sqrt{\pi} l_B n!}} e^{-\frac{x^2}{2l_B^2}} H_n(\frac{x}{l_B}) \tag{2.1.7}$$

depend on the Hermite polynomials $H_n,\ n \in \mathbb{N}$. $X_k = -k^2 l_B^2$ and $l_B := \sqrt{\hbar/eB}$ is called magnetic lenght. Applying adiabatically a magnetic flux ϕ_1 along the center of a cylinder with lenght L changes the wave number in azimuthal direction by $e\phi_1/\hbar$ and a change by $\Delta\phi = \phi_0$ leads to a shift of $2\pi/L$ and a shift in X_k. This however is the same as transporting one electron from one edge of the cylinder to the other and in the n'th Landau level a shift of one flux quantum corresponds to a transport of n electrons. When the lenght L is finite and the Hall voltage $|\mathbf{E}|$ is fixed, the Landau levels are no more degenerated due to the presence of edge states [Hal82] and form a straight line by $|\mathbf{E}|/L$. A shift by one flux quantum $\Delta\phi = \phi_0$ gives then an energy shift of $\Delta E = ne|\mathbf{E}|$ and thus the Hall current in equation (2.1.4). The Laughlin argument can also be described in flat $\mathbb{R}^2$ in form of a singular gauge transformation [BvS94]. On the two dimensional space $\mathbb{R}^2$ an exterior magnetic filed $|B| = B_z$ is applied perpendicularly to the plane. A flux $\phi(t)$ is then pierced adiabatically through the origin, see the right side of figure 2.4. By slowly varying a flux $\phi(t)$ an electromotive force is created. In symmetric gauge and polar coordinates $\mathbf{A}(r,\theta) = (-\frac{1}{2}Br\sin\theta - \frac{1}{2\pi r}\sin\theta\phi(t), \frac{1}{2}Br\cos\theta + \frac{1}{2\pi r}\cos\theta\pi(t))$ the Landau Hamiltonian is given by

$$H = \frac{1}{2m}\Big[-\hbar^2\partial_r^2 - \hbar^2\frac{1}{r}\partial_r + \Big(\frac{-i\hbar}{r}\partial_\theta + \frac{erB}{2} + \frac{e\phi(t)}{2\pi}\Big)^2\Big] \tag{2.1.8}$$

for an adiabatic varying flux and the corresponding eigenstates

$$\psi_{n,m}(r,\theta,t) = c_{n,m,\phi} e^{-im\theta} (r/l_B)^{m+\frac{2\pi e\phi(t)}{h}} e^{-\frac{r^2}{4l_B^2}} L_n^{m+\frac{2\pi e\phi(t)}{h}}(r^2/l_B^2) \tag{2.1.9}$$

depend on the Laguerre polynomials

$$L_n^\alpha = \frac{1}{n!} e^x x^{-\alpha} \partial_x^n (e^{-x} x^{n+\alpha}), \quad n \in \mathbb{N},\ \alpha \in \mathbb{R}.$$

The Landau levels are labeled by n, the orbital angular momentum by $m \in \mathbb{Z}$, and $c_{n,m}$ is the normalization constant. A change of the flux $\phi(t) = ht/e\tau$ by one flux quanta for example from $t = 0$ to $t = \tau$ transfers a state $\psi_{n,m}$ to a state $\psi_{n,m+1}$ up to a phase $e^{-i\theta}$. Assuming the filling factor ν is an integer, the change in the flux corresponds to the transport of the state with lowest angular momentum of each Landau level to infinity. On a large circle C around the origin with radius R the current density is given by $j = -\nu e/(2\pi R\tau)$ and the strength of the electric field on the circle is $|\mathbf{E}| = -\partial_t(\phi(t)/(2\pi R)) = -\hbar/(eR\tau)$ and

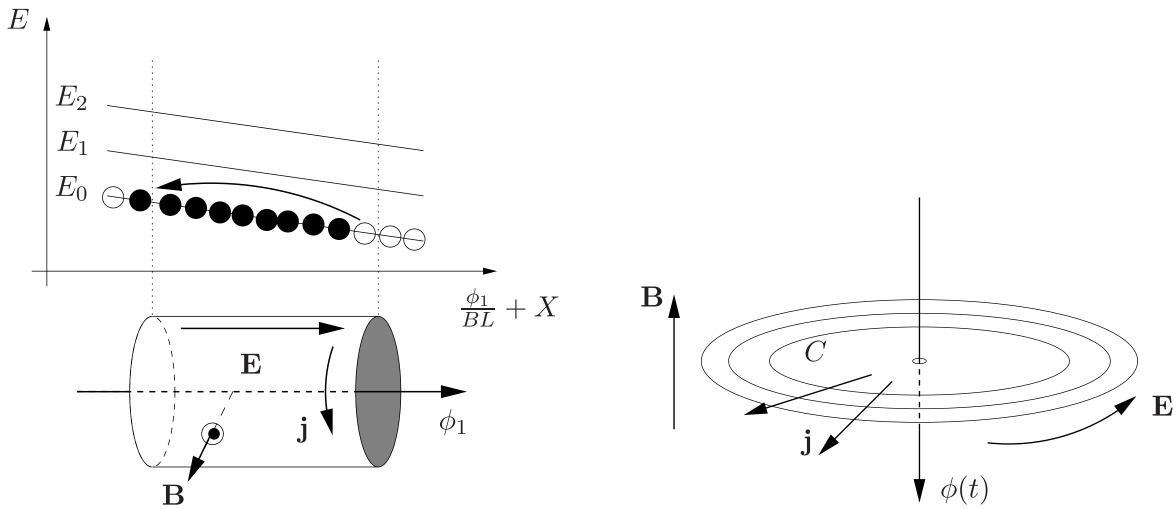


Figure 2.4: The illustration of Laughlins argument in the cylinder geometry on the left [Lau81] and in flat $\mathbb{R}^2$ on the right [BvS94].

the Hall conductivity is $\sigma = j/|\mathbf{E}| = \nu(e^2/h)$ and corresponds therefore to the statement of the cylinder geometry. A rigorous mathematical framework was found by Avron and Seiler based on the ideas of Laughlin and also Kohmoto, den Nijs, Nightingale and Thouless [TKNdN82], which we will see in the next section. However the theory depends on the sample topology and this contradicts experiments. It also gives no answer of the crucial impact of localized electrons due to impurities leading to the fact, that the occurrence of plateaus cannot belong to gaps in the spectrum of a one particle Hamiltonian. Bellissard proposed the mathematical framework of noncommutative geometry [Con90, JMGB01] to extend the arguments of Thouless et al. to disordered crystals and showed that the occurrence of plateaus is due to the finiteness of the localization length near the Fermi level. For a detailed overview see [BvS94] or in the context of noncommutative geometry also [Con90, IV.6]. Avron and Seiler then showed the connection to charge transport and experiments within this mathematical framework [JEA94].

2.1.1 Topological Origin of the Quantum Hall Effect

We will sketch the arguments of Avron and Seiler [ASS83] for the quantized Hall conductance, constructed on the combination of general gauge invariance and periodicity of wave functions in a nontrivial topology. The model is based on the works of Laughlin [Lau81] and also Kohmoto, den Nijs, Nightingale and Thouless [TKNdN82], who relate minimally coupled many body systems to topological invariants such that the systems remain invariant under small variations of i.e. impurities. At zero temperature and a strong, fixed magnetic field perpendicular to the two dimensional electron system the phenomenological effective

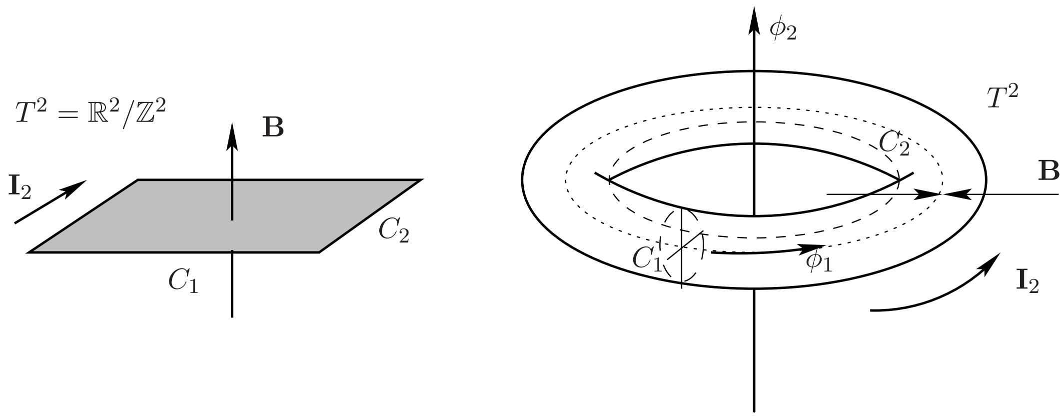


Figure 2.5: Topological origin of the Hall effect: The periodic boundary conditions are incorporated via the flat Torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, by cluing (identifying) together the opposite boundaries.

Hamiltonian is given by

$$H(t) = \sum_p \frac{1}{2m}(-i\hbar\nabla_p - e\mathbf{A}(\mathbf{x}_p))^2 + V(\mathbf{x}) \ ,$$

where $\mathbf{A} = \mathbf{A}_0 + \phi_1\mathbf{A}_1 + \phi_2\mathbf{A}_2$, with the fields $\mathbf{A}_i$ normalized such that

$$\int_{C_j} \langle d\mathbf{x}, \mathbf{A}_i\rangle = \delta_{ij} \quad \text{and} \quad \int_{C_j} \langle d\mathbf{x}, \mathbf{A}\rangle = \phi_j \ .$$

The potential $V(\mathbf{x})$ is some essentially self adjoined operator which plays the rôle of a periodic potential for example in a tight binding model [TKNdN82]. The system is assumed to be in a non degenerated state. The periodic boundary conditions are Incorporated by taking the flat torus T^2 as base manifold, see figure 2.5. We can now remove two cuts C_i in order to get a star shaped region in which we can represent the $\mathbf{A}_i$ as pure gauge fields $\partial_i\Lambda(\mathbf{x})$. The battery voltage V_1 is then replaced by a time dependent flux

$$\dot{\phi}_1 = -\int_{C_1} \langle d\mathbf{x}, \mathbf{E}\rangle = -V_1 \ .$$

With a gauge transformation the vector potential $\mathbf{A}$ is reduced to A_0 while the wave functions get a phase $\psi = \exp\{ie/\hbar(\phi_1\mathbf{A}_1 + \phi_2\mathbf{A}_2)\}$. The system is periodic as ϕ_i increases with a quantum $\phi_0 = h/e$, since Λ_i changes by unity if we winds around C_1, see figure 2.5. Furthermore the system has to be in a normalized, non degenerated state ω. The current density operator is given by

$$\mathbf{j}_p = e\mathbf{v}_p = \frac{e}{m}(-i\hbar\nabla_p - e\mathbf{A}(\mathbf{x}_p))$$

and the variation with respect to ϕ_i is then

$$\mathbf{j}_i = -\partial_{\phi_i} H = \tfrac{1}{2}\sum_p \{\mathbf{j}_p, \mathbf{A}_i\}.$$

Then we have

$$I_i = \omega(\mathbf{j}_i) = -\int d\mathbf{x}_1 \dots d\mathbf{x}_n \sum_p \mathbf{A}_i(\mathbf{x}_p)\psi^* \mathbf{j}_p \psi.$$

Thus the current I_i is a response function of ϕ_i. We simplify our argumentation and use $\phi_i = tV_1$ to get the Schroedinger equation

$$-i\hbar V_i \partial_{\phi_i}\psi = H(\phi_1, \phi_2)\psi.$$

By differentiating with respect to $\phi_j,\ j \neq i$ we obtain for the evaluation in a state

$$-i\hbar V_1(\psi, \partial_{\phi_1}\partial_{\phi_2}\psi) = -I_2 + i\hbar V_1(\partial_{\phi_1}\psi, \partial_{\phi_2}\psi)$$

and we can define the Hall conductance by $I_2 = \sigma_H V_1$ by

$$\sigma_H = i\hbar(\partial_{\phi_2}(\psi, \partial_{\phi_1}\psi) + (\partial_{\phi_1}\psi, \partial_{\phi_2}\psi)) - (\partial_{\phi_2}\psi, \partial_{\phi_1}\psi). \tag{2.1.10}$$

From these rather naive considerations we obtained an explicit formula for the Hall conductance and we will see that it is related to a topological invariant. This formula however can also not describe the occurrence of plateaus. To explain this fact more carefully we have to include disorder in a thermodynamic framework of Fermions in two dimensions. We introduce a new parameter μ, the chemical potential denoted as Fermi level in the zero temperature limit. The value of an observable O in the thermal average of a free Fermi gas at temperature $\beta = 1/k_{\mathrm{B}}T$, is defined in the Gibb's grand canonical ensemble via the trace over a finite volume V for example a square box centered at the origin:

$$\langle O\rangle_{\beta,\mu} = \lim_{V\to\infty} \mathrm{Tr}_V[f_{\beta,\mu}(H)O]. \tag{2.1.11}$$

The Fermi distribution or weight function is defined by

$$f_{\beta,\mu} = (1 + e^{\beta(H-\mu)})^{-1} \tag{2.1.12}$$

and gives the charge carrier density for a suitable chemical potential μ

$$N_{\beta,\mu} = \lim_{V\to\infty} \mathrm{Tr}_V[f_{\beta,\mu}(H)]. \tag{2.1.13}$$

The charge carrier density depends only on the spectral projection E_μ of the Hamiltonian in the zero temperature limit and is thus invariant under variations of μ in a spectral gap. The current can be defined in terms of the position operator $\mathbf{x}$ and the Landau Hamiltonian

$$\mathbf{I} = \frac{ie}{\hbar}[H, \mathbf{x}] \tag{2.1.14}$$

If we switch on an uniform electric field $\mathbf{E}$ the current has a time evolution given by the Heisenberg equations of motion

$$\dot{\mathbf{I}} = \frac{i}{\hbar}[(H - e\mathbf{E}\mathbf{x}), \mathbf{x}] \tag{2.1.15}$$

and its solution is given by the current $I = I_1 + iI_2$ in complex notation

$$I(t) = \frac{ie}{B}E + e^{-i\omega_c t} I_0. \tag{2.1.16}$$

When we take the thermal and the time average of the current the initial data I_0 vanishes and we observe the equation for the current (2.1.4) in complex notation $I = I_1 + iI_2$:

$$\begin{aligned} I &= \lim_{t\to\infty} \int_0^t \frac{dt'}{t} \langle I(t')\rangle_{\beta,\mu} \\ &= \frac{ie}{B} \lim_{V\to\infty} \mathrm{Tr}_V [f_{\beta,\mu}(H)] \\ &= n\frac{ie}{B}E \end{aligned} \tag{2.1.17}$$

We proceed with the definition of the trace per unit volume to be able to calculate the average of an observable say A in a unit volume:

$$\mathcal{T}(A) := \lim_{V\to\infty} \frac{\mathrm{Tr}_V[A_V]}{|V|} \tag{2.1.18}$$

with $|V|$ being the volume and Tr_V, A_V denote the restriction to that volume. V can be considered as the square centered in the middle of the origin. The average over quasi momenta of an observable in a periodic crystal can be described by such a trace. It is a positive linear functional and the cyclicity of the trace is preserved.

Transport and the Kubo-Chern Formula

The following considerations follow the works of Bellissard et al. however for the mathematical exact proves of this analysis we refer to [BvS94, Con90]. In linear transport theory the Greenwood-Kubo formula is widely accepted and in good agreement with many experiments despite the fact that one does not really know the precise domain of validity of the linear response approximation [BvS94]. To derive the Kubo formula we assume again the charge carriers to be spin-less Fermions described by the Landau Hamiltonian H. When an electric field is turned on the time evolution of the current I can be represented by the one parameter automorphism group $\eta_t^{\mathbf{E}}$:

$$\mathbf{I}(t) = \eta_t^{\mathbf{E}}(I) = e^{i\frac{t(H-e\mathbf{E}\mathbf{x})}{\hbar}} \mathbf{I} e^{-i\frac{t(H-e\mathbf{E}\mathbf{x})}{\hbar}} \tag{2.1.19}$$

To calculate the response we need to calculate the time average. It is proved that the projection along the electric field $\mathbf{E}$ of the thermal and timely averaged current vanishes. More precisely the projection defined via the Heisenberg equation of motion

$$\mathbf{E}\cdot\mathbf{I}(t) = \frac{d}{dt}\eta_t^{\mathbf{E}}(H) \tag{2.1.20}$$

vanishes in the limit $T \to \infty$ in the time average:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt\ \mathbf{E}\cdot\mathbf{I}(t) = \frac{\eta_t^{\mathbf{E}}(H) - H}{T}. \tag{2.1.21}$$

We have to include collisions effects within the relaxation time approximation and this can be done by adding a collision term to the Hamiltonian for example with delta like potentials

$$W(t) = \sum_{n\in\mathbb{Z}} W_n \delta(t - t_n) \tag{2.1.22}$$

with t_n being the collision times $0 < t_1 < \cdots < t_n$ and the collision operators W_n commute with the Hamiltonian H. We now want to average the current taking collision effects into account. With the abbreviations $\mathcal{L}_H(I) = (i/\hbar)[H, I]$ and $\mathcal{L}_{W_j}(I) = (i/\hbar)[H, I]$ the time evolution for random variables $\xi = (\tau_n = t_n - t_{n+1}, W_n)$ of the current is again given by a one parameter automorphism group

$$\eta^{\mathbf{E}}_{\xi,t} = e^{(t-t_{n-1})(\mathcal{L}_H - \frac{e\mathbf{E}\cdot\nabla}{\hbar})} \prod_{j=1}^{n-1} e^{\mathcal{L}_{W_j}} \; e^{(t_j - t_{j-1})(\mathcal{L}_H - \frac{e\mathbf{E}\cdot\nabla}{\hbar})} \tag{2.1.23}$$

and the time average of this operator is given by

$$\hat{\eta}^{\mathbf{E}}_{\delta} = \delta \int_0^\infty dt \; e^{-t\delta} \langle \eta^{\mathbf{E}}_{\xi,t} \rangle_\xi . \tag{2.1.24}$$

The average over ξ is denoted by the brackets $\langle\rangle_\xi$ and $\delta > 0$. The random operators W_n can be implemented as an average

$$\hat{\kappa}(I) = \langle e^{iW_n/\hbar} I e^{-iW_n/\hbar} \rangle_\xi \tag{2.1.25}$$

and the time evolution operator is then given by:

$$\hat{\eta}^{\mathbf{E}}_{\delta} = \frac{\delta}{\delta + \frac{1-\hat{\kappa}}{\tau} - \mathcal{L}_H + \mathbf{E}\cdot\nabla}. \tag{2.1.26}$$

The average of the current can be expressed via the trace per volume and by using the relation

$$\mathbf{I}_{\beta,\mu,\mathbf{E}}(\delta) = -\frac{e}{\hbar} \mathcal{T}\big(f_{\beta,\mu}(H) \hat{\eta}^{\mathbf{E}}_{\delta}(i[H, \mathbf{x}]) \big) \tag{2.1.27}$$

and the definition of the inner product via the trace $\langle A|B\rangle = \mathcal{T}(A^* B)$ this can be rewritten

$$\mathbf{I}_{\beta,\mu,\mathbf{E}}(\delta) = \frac{e^2}{\hbar^2} \sum_{i=1,2} E_i \langle \partial_i f_{\beta,\mu}(H) | \frac{1}{\delta + \frac{1-\hat{\kappa}}{\tau} - \mathcal{L}_H + e/\hbar \mathbf{E}\cdot\nabla} (i[H, \mathbf{x}]) \rangle. \tag{2.1.28}$$

The Kubo formula we obtain if we consider only linear terms in $\mathbf{E}$. Then we can perform the limit $\delta \to 0$. Furthermore it is enough to consider the operator $(1 - \hat{\kappa})/\tau - \mathcal{L}_H$. The commutator of the Hamiltonian with the $\mathbf{x}$-operator is a derivative an can be expressed by

$$\nabla H = i[H, \mathbf{x}]. \tag{2.1.29}$$

In terms of quasi momenta $k_i$the nabla operator is given by $\nabla = (\partial_{k_1}, \partial_{k_2})$ and is translated in the case of formula (2.1.10) to $\nabla = (\partial_{\phi_1}, \partial_{\phi_2})$. The current in linear response is given by

$$\mathbf{I}_{\beta,\mu,\mathbf{E}} = \hat{\sigma}\mathbf{E}, \quad (\hat{\sigma}_{ij}) = \begin{pmatrix} \sigma_L & \sigma_H \\ -\sigma_H & \sigma_L \end{pmatrix} \tag{2.1.30}$$

with $\hat{\sigma}$ being the conductivity tensor, obtained via the Kubo formular

$$\sigma_{ij} = \frac{e^2}{\hbar} \langle \partial_j f_{\beta,\mu}(H) | \frac{1}{\hbar\frac{1-\hat{\kappa}}{\tau} - \hbar\mathcal{L}_H} \partial_i H \rangle. \tag{2.1.31}$$

This is a valid formula for the conductivity of the integer quantum Hall effect when the electric field is vanishingly small, the temperature is zero and the relaxation time τ is finite. In this limit we define the Fermi projection which projects the spectrum onto levels lower than the Fermi energy:

$$P_F := \lim_{\beta\to\infty} f_{\beta,\mu}(H) \tag{2.1.32}$$

Then the Kubo formula for the integer quantum Hall effect can be rewritten in the limit of zero temperature and infinite collision times if the Fermi level is not a discontinuity point of the density of states of H and the direct conductivity vanishes [BvS94]:

$$\sigma_{ij} = \frac{e^2}{h^2} (2\pi i)\mathcal{T}(P_F[\partial_i P_F, \partial_j P_F]) \tag{2.1.33}$$

On the two torus T^2 the Hall conductance is related to the first Chern character [ASS83] and is an integer multiple of (e^2/h):

$$\sigma_H = \frac{e^2}{h}\mathrm{Ch}(P_F) \tag{2.1.34}$$

and the Chern character is given by the Kubo formula

$$\mathrm{Ch}(P_F) = \int_{T^2} \frac{d^2k}{4\pi^2}\big[\frac{2\pi i}{q}\mathrm{Tr}(P_F(k_1,k_2)[\partial_{k_1}P_F(k_1,k_2), \partial_{k_2}P(k_1,k_2)])\big], \tag{2.1.35}$$

wherein the trace per volume is replaced by the integral over k space divided by $q \in \mathbb{N}$ denoting the q-periodicity of the system. This formula is a thermodynamic derivation of the naive approach (2.1.10). In the mathematical language the integrand corresponds to a complex two form on the torus and the integral corresponds to the first Chern class or Chern number, which is characterized by an integer number. We will introduce Chern classes and the theory of (principal) fiber bundles in the next chapter since they appear also in the quantum field theory of the fractional quantum Hall effect, here we just comment the connection to mathematics. Chern classes classify complex fibre bundles. A complex fiber bundle can be constructed by the map $k = (k_1, k_2) \in T^2 \mapsto P_F(k_1, k_2)$ the base manifold is the two torus and each fiber is isomorphic to $\mathbb{C}^q$ so the space $T^2 \times \mathbb{C}^q$ with the map from above defines a smooth fibre bundle if the Fermi level is in a gap. $\mathrm{Ch}(P_F)$ is then a topological invariant and insensitive to perturbations added to the Hamiltonian, in particular it is an integer number. We may comment that mathematics tells us that the Chern character of the projection P_F is equal to a densely defined cyclic cocycle on a C^*-algebra defined by the time evolution operators $\eta_\xi^{\mathbf{E}}(f(H))$ [Con90, III.6]. This is the subject of the noncommutative extension of cohomology. The corresponding formalism provide the robustness of the integrality of the Hall conductance. Indeed we ignored the mathematical subtleties and difficulties completely in our derivation since this would exceed the intention of this theses, so we recommend the works of Bellissard and Connes.

Having shown the connection between a topological invariant and the Hall conductance, we have not jet explained the occurrence of plateaus. For this reason we have to include disorder.

2.1.2 Incompressibility and Disorder in the IQHE

In the experimental observation of the quantum Hall effect the Hall conductivity σ_H forms plateaus and in addition the longitudinal (direct) conductivity σ_L vanishes at these plateaus. This effect is rather successfully described by including disorder effects leading to localized (bounded) states in the bulk of the sample. In hetero junctions there are different possibilities for impurities for example the influence of the doped ions having long range order Coulomb potential or density fluctuations in the compounds i.e. aluminium concentration is usually not homogeneous and can vary [GG87]. At low temperature phonon scattering can be neglected and also photo-emission and the disorder scattering should dominate the system. Therefore non-dissipative effects like Anderson localization [And58] may be considered.

The disorder effects can be incorporated by adding random potentials to the Hamiltonian. This will create new states with energies in between Landau levels which can be measured by the density of states (DOS). Roughly speaking it broadens the Landau levels to Landau bands and removes its degeneracy. Therefore let $N(E)$ be the number of eigenstates of the Hamiltonian per unit volume below the energy E:

$$N(E) = \lim_{V\to\infty} \frac{\#\{(\text{eigenvalues of } H|_V) \leq E\}}{|V|}. \tag{2.1.36}$$

Degenerated eigenvalues have to be counted by their multiplicity. The derivative with respect to the energy defines the density of states $\mathrm{DOS}(E)$:

$$\mathrm{DOS}(E) := \frac{d}{dE} N(E) \tag{2.1.37}$$

In the absence of disorder this gives a sum of delta functions at the Landau levels. We want to argue that the direct conductivity vanishes when the Fermi energy is in the region of localized states. The Hall current is then carried by the extended states, the edge states. If the spectrum has no localized states the Hall conductance changes with the change of the filling factor. In the free Fermion gas there can be no quantum Hall effect for that reason. Within the plateaus the Fermi energy should thus vary continuously while the conductivity value should not change when the number of charge carriers is varied. We may introduce a random potential

$$H_D = \sum_{i\neq j} V(\mathbf{x}_i - \mathbf{y}_j) \tag{2.1.38}$$

wherein the disorder potential $V(\mathbf{x}_i - \mathbf{y}_j)$ describes the potential of an impurity sitting at $\mathbf{y}_j$. For the potential a Gaussian shape is usually used to simulate a random potential landscape.

The Fermi energy is fixed by the localized states at their corresponding eigenvalues, see figure 2.6. Since correlator functions of the localized states decay exponentially these states can not contribute to the transport [KM93, LR85]. It is generally believed that at zero temperature the localization lenght diverges at specific energies close to the center of the Landau bands with an universal exponent being independent of the band index [HK90, Huc95, KOK05]. This divergence occurs if the impurities become electrically neutral in the average, therefore the influence of the impurities decreases close to the band centers. For the discussion on the criticallity, the connection to the universality of the Hall effect and the relation to universal quantum phase transitions we refer to [KOK05].

Laughlins gedankenexperiment from the previous section can now be extended to the disordered system. Therefore the Corbino disc geometry may she'd light on the mechanism, see figure 2.7. In principle the sample consists of three concentric areas: The bulk area with a weak random disorder between $r_1' < r_2'$ and the impurity free, clean edge areas between r_i and r_i'. Thus the delta like Landau levels are broadened in the bulk region. We assume that the broadening is smaller than the inter Landau level spacing. The difference in the gedankenexperiment of Laughlin from the previous section is that we have to distinguish between extended and localized states. Indeed while only the extended states can be affected by the adiabatic variation of a flux sitting in the center of the sample, the localized states

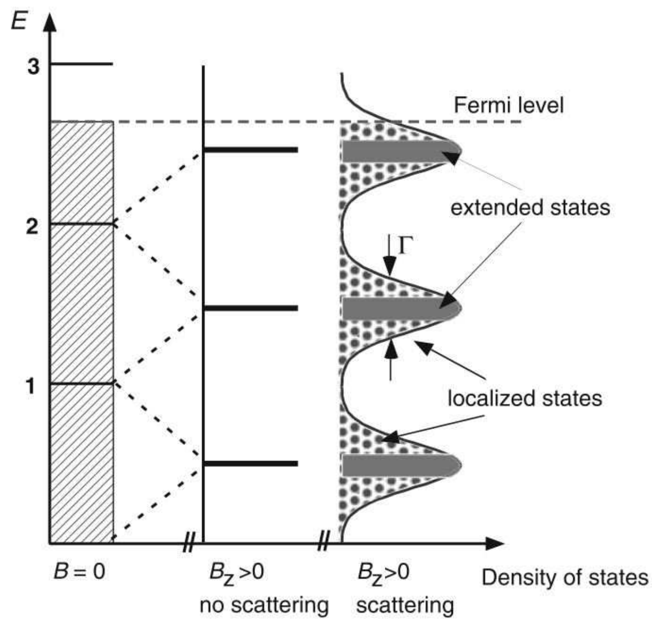


Figure 2.6: A schematic plot of the energy in terms of the density of states [JJ01]. The static random potential leads to a broadening of the Landau levels into Landau bands. The localized states corresponds to the incompressible region of the spectrum while the extended or delocalized states at the band centers correspond to compressible regions where the localization length diverges and exceeds the system size.

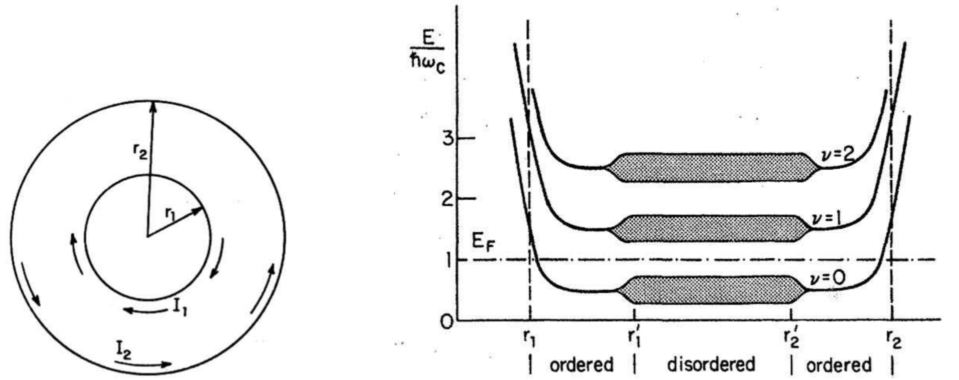


Figure 2.7: In the Corbino disc geometry (left) the energy bands (right) contain a weak random potential in the region $r_1' < r < r_2'$ while the edge regions are clean [Hal82]. The Fermi level is in a gap in between the Landau bands.

can not change especially their occupation remains unchanged. If in the disordered region all states below the Fermi level are localized there is no possibility for a charge transport by varying the flux since the localized states are not affected. This leads to the observation that extended states do exist within localized regions. In particular if the Hall conductance jumps from one integer to an another the localization length must diverge in between. The incorporation of the disorder to the Kubo Chern formula in a mathematically satisfactory way is highly nontrivial. In particular the property of the Hall conductivity being related to a topological invariant, the Chern number, needs further concepts if we introduce impurities. To introduce these concepts here would exceed the intention of this theses so we refer the interested reader to the works of Bellissard et al. [BvS94, Con90].

2.2 FQHE in the Wavefunction Picture

In 1982 Tsui, Stormer and Gossard made the observation of a quantized Hall plateaus of $\rho_{xy} = 3h/e^2$ with simultanious minimum in ρ_{xx} at $T < 5\text{K}$ [SCT+83]. In figure 2.8 the Hall resistivity ond longitudinal resistivity are plotted in the regime of the fractional Hall effect.

2.2.1 Interaction and the FQHE

Laughlin Wave Functions

In the seminal paper of Laughlin [Lau83] it is shown that the interaction in the two dimensional electron system can be explained by a condensation in a new state of matter in the lowest Landau level – at least for filling factor 1/3 by numerically diagonalizing the Hamiltonian for three and four electrons. The solutions ψ of the Hamiltonian of a single spin-less electron coupled to an external, constant electromagnetic background field perpendicular to the two dimensional electron system is modified in the presence of electron-electron inter-

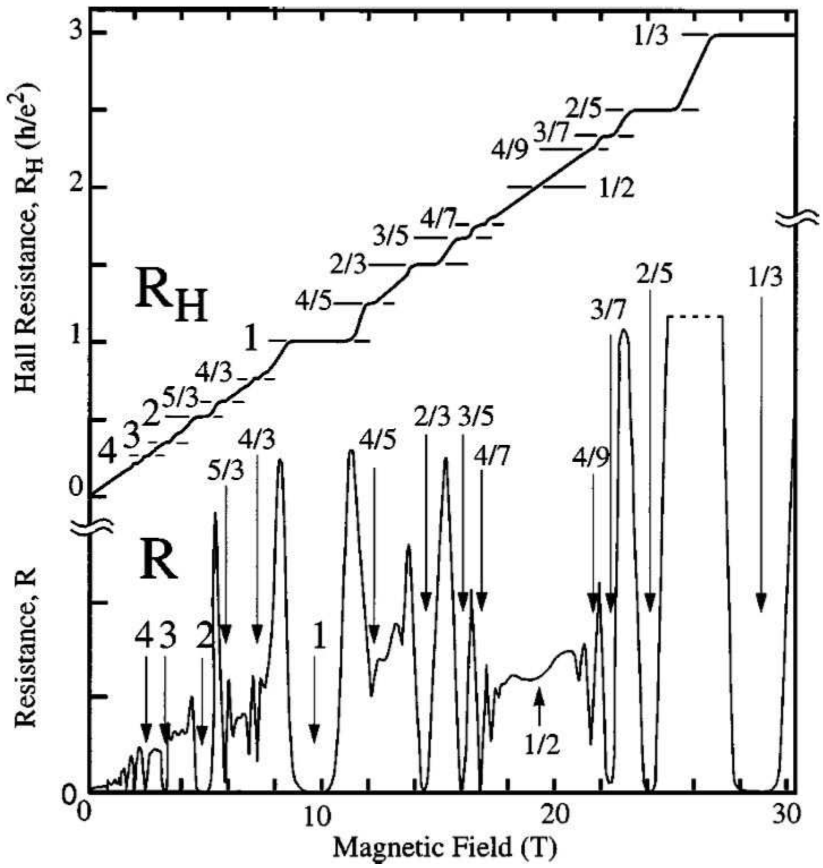


Figure 2.8: Hall resistivity and longitudinal resistivity in the fractional Hall effect [Sto99].

action. In the lowest Landau level the ground state is replaced by a product of Jastrow functions

$$\psi := \prod_{i<k} f(z_i - z_k)\ e^{-\frac{1}{4}\sum_l |z_l|^2},$$

with $f(z)$ being a polynomial in z with odd degree. Due to conservation of the angular momentum the wave function is given by

$$\psi_p := \prod_{i<k} (z_i - z_k)^p\ e^{-\frac{1}{4}\sum_l |z_l|^2} \tag{2.2.1}$$

and excitations are generated by piercing infinitely thin solenoids in z_k and passing flux quanta adiabatically through. This would result in the transformation $\psi_p \to \psi_{p+1}$.

2.2.2 Jains Wave Function Picture of Composite Fermions

In [Jai89] it is proposed that the electrons in a fractional Hall state condense in quasi particles consisting of an electron, binding an even number of flux quanta. The wave functions describing such composite Fermions are trial wave functions, constructed analogously to the Laughlin wave functions (2.2.1). Roughly speaking the strong interacting electron system is mapped to a quasi non-interacting system of composite Fermions. At least at mean field level. However, the Laughlin wave functions are derived by approximating the electron interaction for example from a Haldane pseudo-potential [HR85], while here it is just proposed to 'shift' the interaction into the wave functions. There are also attempts to construct Jains wave functions from a rational chiral conformal field theory (RCFT) [HCJV07b, HCJV07a] thus by universality criteria derived from conformal field theory. We will comment on this later on. The meaning of Jains wave functions is so far not completely understood. Especially there are fractions where in this picture the composite Fermions build generations of quasi-particle excitations where the composite Fermions start to interact and form themselves Landau levels and a second generation of composite fermions can be constructed. We will return to that subject later.

In the discussion in the next section we will see how to attach flux quanta to a particle in a mathematically satisfactory way and how cohomology plays a central rôle. Jain chooses a pragmatically approach and starts from the Hamiltonian for N non interacting, spin-less electrons in a two dimensional, non-relativistic system with constant background magnetic field in z-direction. The magnetic field is considered to produce an average flux ϕ_0/p per electron, with p being the filling factor. The flux attachment is represented via the Chern Simons fields $\mathcal{A}_i := -2m\phi_0/(2\pi)\sum_{k\neq i}\nabla_i\theta(z_i - z_k)$, with $\theta(z_i - z_k)$ defined by $(z_i - z_k) = |z_i - z_k|\exp\{i\theta(z_i - z_k)\}$. Thus $\mathcal{A}_i$ are analytic functions on the space $\mathbb{C} - \{z_i \in \mathbb{C} |\, |z_i - z_k| = 0, i, k = 0, 1, 2, \ldots, N; i < k\}$. On this space the resulting wave functions are well defined

$$\phi_{+p}^{2m} = \prod_{i<k}\left(\frac{z_i - z_k}{|z_i - z_k|}\right)^{2m}\psi_{+p}.$$

Roughly speaking we have attached so-called zeros or vortices to the particle (electron), which are simply topological defects.

2.3 FQHE in the Field Theoretical Picture

We start with the following Low energy Lagrangian of the System:

$$\mathcal{L}(\mathbf{x}, t) \quad = \quad \mathcal{L}_0(\mathbf{x}, t) + \mathcal{L}_C(\mathbf{x}, t), \tag{2.3.1}$$

wherein the free part is given by

$$\mathcal{L}_0(\mathbf{x}, t) \quad = \quad \psi^+(\mathbf{x}, t)\big[-\frac{1}{2m}\big(\mathbf{p} - e\mathbf{A}(\mathbf{x})\big)^2 + \tag{2.3.2}$$

$$+i\partial_t + \mu + eA_0(\mathbf{x})\big]\psi(\mathbf{x}, t) \tag{2.3.3}$$

where μ is the chemical potential and we assume that the spins of the electrons are completely polarized due to the strong external magnetic field. We may propose a Coulomb interaction $V(\mathbf{x} - \mathbf{y}) = e^2/\epsilon|\mathbf{x} - \mathbf{y}|$

$$\mathcal{L}_{\mathrm{C}}(\mathbf{x}, t) \quad = \quad -\frac{1}{2}\int d\mathbf{y}\rho(\mathbf{x}, t)V(\mathbf{x} - \mathbf{y})\rho(\mathbf{y}, t). \tag{2.3.4}$$

However, this will be important later. In the following we want to understand the mechanism, which transforms electrons into composite Fermions. This is the so-called statistical transmutation and is related to the Chern Simons transformation.

2.3.1 Statistical Transmutation

From the Lagrangian formalism we know that we can add a total derivative to the Lagrangian without changing the equations of motion. We will see that by changing the topology this total derivative term (exact form) has to be replaced by a close form and this mechanism is then known as statistical transmutation.

The following arguments are based on Stokes theorem and Poincarés lemma on star shaped sets, which manifests itself in classical electrodynamics. The difference to standard electrodynamics is only the origin of the topology, which is produced not only by the sample topology but by finite size effects in combination with repulsive interaction of electrons, forming an incompressible state, an incompressible Hall fluid respectively. The flux is then quantized and from a large scale point of view concentrated in one point. The crucial argument is not that the flux is concentrated at one point thus described by a singular gauge field. The crucial fact is that there exists a topological defect, which is from a more physical point of view a finite area rather than a point. It turns out that cohomlogy arguments, more precisely de Rahm cohomology, describe these effects. This is the discussion of differential forms which are closed but not exact.

De Rham Cohomology

The de Rham cohomology is constructed on differential manifolds, where the key point is Stokes theorem for differential forms

$$\int_M d\omega = \int_{\partial M} \omega,$$

where M is a topological space of $\dim M = n$, ∂M its boundary and ω is a $(n-1)$-form. We can therefore transform an integral on M to an integral on a subset on M and the class of possible subsets is provided by homology theory. The elements $[C]$ of a homology class belong to the space $Z_p(M)/B_p(M)$, where $Z_p(M)$ are all p-chains C for which $\partial C = \phi$, and $B_p(M)$ are all p-chains C for which $C = \partial\tilde{C}$ $(\partial C = \partial\partial\tilde{C} = 0)$, for some $(p+1)$-chain $\tilde{C}$. Due to the (adjoint) relationship between ∂ and d we can identify the cohomology class $[\omega]$, and for the dual of $Z_p(M)/B_p(M)$ we write $[\omega] \in Z^p(M)/B^p(M)$. So Z^p are all co-chains or p-forms ω for which $d\omega = 0$ (closed), and B^p are all p-forms ω for which $\omega = d\eta$ (exact), for some $(p-1)$-form η. So we are interested for example in forms which are closed $d\omega = 0$ but not exact $\omega \neq d\eta$, this measures whether a space M is contractible to a point (trivial cohomology) or not (nontrivial cohomology), which is a known result of Poincaré 's lemma, in other words it measures whether a space is simply connected or not. A simple example for a two dimensional space, which is not contractible is the space $\mathbb{R}^2 - \{0\}$. The de Rham cohomology $H^p(M, R) := Z^p(M)/B^p(M)$ is defined as the p-th-cohomology group of M with real coefficients.

In principle exactly this standard textbook example [CN88] or [MG89] enters our framework. A closed form which is not exact is given by the one-form

$$w = \frac{-y\ dx + x\ dy}{x^2+y^2}, \tag{2.3.5}$$

defined on the space $\mathbb{R}^2 - \{0\}$. We observe that it is closed

$$dw = \frac{(y^2-x^2)(dy\wedge dx - dx\wedge dy)}{(x^2+y^2)^2} = 0,$$

but it is not exact $w \neq d\theta$, θ being a zero form actually a scalar function. In this case we are tempted to use the angle function $\theta = \arctan(y/x)$ since $\partial_x\theta = -y/(x^2+y^2)$ and $\partial_y\theta = x/(x^2+y^2)$. However this function is only defined on $\mathbb{R}^2 - \mathbb{R}_+$, $\mathbb{R}_+ := \{x \in \mathbb{R} | x \geq 0\}$ being the nonnegative x-axis since it has to be single valued. For being a total derivative θ has to be a smooth function on all of $\mathbb{R}^2 - \{0\}$. In this sense w is only exact on $\mathbb{R}^2 - \mathbb{R}_+$ and there exists no total derivative on $\mathbb{R}^2 - \{0\}$ of $w \neq d\theta$.

First we have a look at the Lagrangian (2.3.1). Here we want to add the total derivative $\psi^+\psi d\theta$ of the polar angular function $\theta(\mathbf{x}_1 - \mathbf{x}_2)$, which is defined as the angle between the x^1-axis with the relative vector $(\mathbf{x}_1 - \mathbf{x}_2)$ between two particles. As we know now this is only possible on $\mathbb{R}^2 - \mathbb{R}_+$. So we cannot describe closed loops within this approach. From a path integral point of view this angular function gives rise to a phase φ by

$$e^{iS} = e^{i(\alpha+\pi)\int_{\theta_0}^{\theta_1} d\theta} e^{iS_0},$$

with S_0 being the action from the Lagrangian (2.3.7) and $\theta_1 - \theta_0 \leq \pi$. The factor α is due to the fact that the total derivative term in the Lagrangian is defined up to a fixed number. If we want to describe closed loops we would prefer the closed form w from above instead of $d\theta$. The situation changes to

$$e^{iS} = e^{i(\alpha+\pi)\int_{\theta_0}^{\theta_1} w} e^{iS_0}.$$

If $\theta_0 = n\theta_1$ then we obtain the usual Fermi statistics. What happens now if there is an magnetic field applied? If we consider at first the space $\mathbb{R}^2 - \{x, y \in \mathbb{R} | x^2 + y^2 > l_B^2, l_B > 0\}$, then closed forms around the circle are not exact since they cannot be shrunk to a point. The phase is then given by the integral

$$\varphi = \oint_{\partial\mathcal{O}} \mathcal{A} = \int_{\mathcal{O}} d\mathcal{A}.$$

Where now $\mathcal{A} = \mathcal{B}/2(-y, x)$ $\mathcal{B}$ is the (Chern Simons) gauge field in symmetric gauge, which coinsides on the closed path C with $w|_C \equiv \mathcal{A}|_C$, C being for example the unit circle. $\mathcal{A}$ is in this sense the real analytic continuation of w. $\mathcal{B}$ is the magnetic field entering the surrounded area $\mathcal{O}$. In the large distant limit where $l_B \to 0$ the magnetic flux (the phase) is then concentrated in the origin. This means that the flux sits directly on the surrounded electron, in the picture of two electrons from above. We chose here l_B as radius since electrons perform cyclotron motion with this radius. In this sense we do not need two electrons for generating this so called Ahrahnov Bohm or Berry phase. If the magnetic field is turned on, the phase is determined by the flux through $\mathcal{O}$ and can be any if the flux is not quantized, so we obtain anyon statistics. But if the flux is quantized the phase is fractional. This mechanism is called statistical transmutation and does not depend on whether we have a relativistic or a non-relativistic theory. It is a topological feature and as such general covariant. This means it does not depend on the metric. In particular it defines a nontrivial de Rahm cohomology since there are loops, which cannot be continuously transformed to a point, as denoted above. If we have closed loops then n counts the windings on how often one particle moves around the other and is called the winding number. The field $\mathcal{A}$ is then determined only by the topology of the system and by evaluating the topology we can identify the phase φ, which we may attach to the particle. The upper 'local' gauge transformation is called Chern Simons transformation if we attach the topological defect to the particle and the field $\mathcal{A}$ generates a nonzero electromagnetic field $\mathcal{F} = d\mathcal{A} \neq 0$.

There are different aspects to consider if we want to achieve the equations of motions for the Chern Simons fields . At first of course there are the inhomogeneous $d*\mathcal{F} = -*j$ and the homogeneous (structure equation) $d\mathcal{F} = 0$ Maxwell equations. Electric transport properties are rather described by Ohm's law $\mathbf{j} = \sigma\mathbf{E}$ leading to the diffusion equation for the fields $\mathcal{A}$. Therefore we have to consider a quasi static system. This means that particles react instantaneously on the fields. It should therefore be mentioned that Ohm's law intrinsically violates causality. We are interested in the case $\mathbf{j} = \sigma\mathbf{E}$, where the conductivity tensor is given by $\sigma = \sigma_H i\sigma_2 + \sigma_L \mathbb{1}_{2\times 2}$ and will denote it as the Ohm-Hall law. The more general case $\mathbf{j} = \sigma(\mathbf{E} - \boldsymbol{\alpha} \times \mathbf{B})$ might be interesting since it includes spin dynamics but this should be discussed elsewhere. For $\mathcal{A}_0 \equiv 0$ being a pure gauge, set to zero we derive from Ampére's law the diffusion equation $\nabla^2 \mathcal{A}_i = \sigma_l \partial_t \mathcal{A}_i$. This means the gauge fields $\mathcal{A}_i$ have a imaginary 'mass' σ_l. In a fractional Hall state we require incompressibility, which means that the longitudinal conductivity σ_l vanishes. Therefore we consider in the following only the case where $\sigma_l \equiv 0$. The fields $\mathcal{A}_i$ become then massive if we move away from a Hall state thus from incompressibility. From the structure equation (Faraday's induction law) and the continuity equation it follows that

$$j^0 = \sigma_H/2\ \varepsilon^{ij}\mathcal{F}_{ij}.$$

We may also introduce the current two-form $J = 1/2 J_{\mu\nu} dx^\mu \wedge dx^\nu$ with

$$(J_{\mu\nu}) = \begin{pmatrix} 0 & j_y & -j_x \\ -j_y & 0 & j_0 \\ j_x & -j_0 & 0 \end{pmatrix}$$

and the Hodge star operator $*(\cdot)$ in three dimensions. The equation of motion for the Chern Simons current is then given by

$$j = -*J = -*\sigma_H\mathcal{F}, \quad \Leftrightarrow \quad j^\mu = \sigma_H\ \varepsilon^{\mu\nu\rho}\mathcal{F}_{\nu\rho}.$$

If we want to implement these equations of motion then we get an additional term in the Lagrangian

$$\mathcal{L}_{\text{CS}} = \sigma_H\ \mathcal{A} \wedge \mathcal{F} = \sigma_H \mathcal{A} \wedge d\mathcal{A}\ = \sigma_H\ \varepsilon^{\mu\nu\rho}\mathcal{A}_\mu \partial_\nu \mathcal{A}_\rho\ d^3x \tag{2.3.6}$$

the so called Chern Simons term. In the case of a finite sample with boundary, we also have to include a boundary term, but this we will comment later on in terms of edge currents. Actually these are phenomenological equations and we may ask whether we can replace them by a microscopic picture. At this point it is then more constructive and systematic to follow the Yang-Mills construction to derive the field strength and then the current. So we define the field strength via the covariant derivative $D_\mu = \{\partial_\mu - eA_\mu + \mathcal{A}\}$ by

$$f_{\mu\nu} = [D_\mu, D_\nu] \;\Rightarrow\; f = F - \mathcal{F} = d(A - \mathcal{A}).$$

This means that the electromagnetic field generated by $\mathcal{A}$ reduce the external field generated by A. Indeed we may prefer this point of view when we introduce $\mathcal{A} = a + \langle a \rangle$ as a mean field $\langle a \rangle$ and some fluctuations a and propose Ohm's law for the fluctuations only by the assumption that $eA \equiv \langle a \rangle$ and thus a Lagrangian term of $\sigma_H a \wedge f$.

2.3.2 Low Energy Effective Theory

The Low energy Lagrangian of the System is proposed to be

$$\mathcal{L}(\mathbf{x},t) = \mathcal{L}_0(\mathbf{x},t) + \mathcal{L}_{\mathrm{CS}}(\mathbf{x},t) + \mathcal{L}_C(\mathbf{x},t), \tag{2.3.7}$$

wherein the Fermionic free part is given by

$$\mathcal{L}_0(\mathbf{x},t) = \psi(\mathbf{x},t)^+\big[-\frac{1}{2m}\big(\mathbf{p} - e(\mathbf{A}(\mathbf{x}) - \boldsymbol{\mathcal{A}}(\mathbf{x},t))\big)^2 + \tag{2.3.8}$$

$$+i\partial_t + \mu + e(A_0(\mathbf{x}) - \mathcal{A}_0(\mathbf{x},t))\big]\psi(\mathbf{x},t) \tag{2.3.9}$$

and for the Chern Simons action we have

$$\mathcal{L}_{\mathrm{CS}}(\mathbf{x},t) = \frac{e}{\tilde{\varphi}\phi_0}\varepsilon^{\mu\nu\rho}\mathcal{A}_\mu(\mathbf{x},t)\partial_\nu\mathcal{A}_\rho. \tag{2.3.10}$$

When we later quantize the Chern Simons fields we will see that if we take Coulomb gauge as gauge fixing, $\mathcal{A}_0$ becomes a Lagrange multiplier field while the other fields remain dynamical (otherwise not!) and we can restrict this Lagrangian term to

$$\mathcal{L}^{\mathrm{Cgf}}_{\mathrm{CS}}(\mathbf{x},t) = \frac{e}{\tilde{\varphi}\phi_0}\varepsilon^{ij}\mathcal{A}_0(\mathbf{x},t)\partial_i\mathcal{A}_j. \tag{2.3.11}$$

We will discuss the the quantization procedure in more detail in the next section. The interaction is proposed to be Coulombian $V(\mathbf{x}-\mathbf{y}) = e^2/\epsilon|\mathbf{x}-\mathbf{y}|$:

$$\mathcal{L}_{\mathrm{C}}(\mathbf{x},t) = -\frac{1}{2}\int d\mathbf{y}\rho(\mathbf{x},t)V(\mathbf{x}-\mathbf{y})\rho(\mathbf{y},t). \tag{2.3.12}$$

The equations of motions for the Chern Simons fields are obtained by varying the action with respect to the fields $\mathcal{A}_\mu$

$$\frac{\delta S}{\delta\mathcal{A}_\mu} = 0$$

the zero component leads to the relation

$$\varepsilon^{ij}\partial_i\mathcal{A}_j(\mathbf{x},t) = \tilde{\varphi}\phi_0\;\rho(\mathbf{x},t). \tag{2.3.13}$$

This means that we can replace the charge density $\rho(\mathbf{x},t)$ in the Coulomb part of the action by $(\tilde{\varphi}\phi_0)^{-1}\varepsilon^{ij}\partial_i\mathcal{A}_j(\mathbf{x},t)$.

2.3.3 Mean Field and Random Phase Approximation

The effective magnetic field acting on the charged particles is given by

$$b(\mathbf{x},t) = B - \mathcal{B} = B - \tilde{\varphi}\phi_0\rho(\mathbf{x},t) \tag{2.3.14}$$

and we can divide the total electromagnetic field or Chern Simons field in a mean field and a dynamical field respecting the fluctuations:

$$a_\mu = \mathcal{A}_\mu - \langle\mathcal{A}_\mu\rangle. \tag{2.3.15}$$

The average $\langle\mathcal{A}_\mu\rangle$ is not a dynamical field and gives no interesting contribution to the equation of motions. The only effect is that it reduces the external field and in some cases, at even fraction, it completely eliminates the external field. Since $\partial_i\langle\mathcal{A}_j\rangle = 0$ only the dynamical part a_μ contributes to the equation of motions and this leads to the relation:

$$\rho(\mathbf{x},t) = \frac{1}{\tilde{\varphi}\phi_0}\ \varepsilon^{ij}\partial_i a_j(\mathbf{x},t). \tag{2.3.16}$$

We may now calculate the free propagator of the low energy free massive charge carriers. The spin-less fields are given by

$$\psi(\mathbf{x},t) = \int \frac{d\mathbf{k}dt}{(2\pi)^3}\big(a^+(\mathbf{k})e^{-i(\mathbf{kx}-\omega t)} + a(\mathbf{k})e^{-i(\mathbf{kx}-\omega t)}\big).$$

The free part of the action is

$$S_\mathrm{F} = \int d\mathbf{x}dt\ \psi^+(\mathbf{x},t)\big[-\frac{\mathbf{p}^2}{2m} + i\partial_t + \mu\big]\psi(\mathbf{x},t). \tag{2.3.17}$$

In Fourier space this is exactly

$$S_\mathrm{F} = \int d\mathbf{k}dt\ \psi^+(\mathbf{k},t)\,\underbrace{[\omega - \frac{k^2}{2m} - \mu]}_{[G^0(\mathbf{k},\omega)]^{-1}}\,\psi(\mathbf{k},t) \tag{2.3.18}$$

and the low energy Fermionic Greens function is ($\epsilon > 0$):

$$G^0(\mathbf{k},\omega) = [\omega - \frac{k^2}{2m} - \mu + i\epsilon\mathrm{sign}(\omega)]^{-1}. \tag{2.3.19}$$

Random Phase Approximation

In the random phase approximation (RPA) the two-point function of the propagator of the gauge field is calculated up to second order time dependent perturbation theory. Therefore the Chern Simons fields have to be quantized first. Here we face the problem that the Chern Simons Theory is a gauge theory, usually a $U(1)$ but also $U(1)\otimes SU(2)$ or $U(N)$ fields are discussed. In gauge theories we have to incorporate gauge invariance and the uniqueness of the Cauchy data. To satisfy the Cauchy problem we have to fix the gauge. So far the quantization in the Chern Simons composite Fermion picture is performed in Coulomb gauge fixing. From usual quantum electrodynamics it is well known that the advantage of Coulomb gauge is, that there are only physical degrees of freedom left in the theory, only the transverse modes are included. As an effect of Coulomb gauge the zero component of the gauge field A_0 becomes a Lagrangian multiplier resulting in a constrained equation, the Poisson equation. However, there is also a disadvantage namely there appear terms violating intrinsically causality and this terms have to be eliminated in the propagator by counter-terms. For this reason the Gupta Bleuler Method in Lorentz gauge is used to circumvent the handling with counter-terms. Furthermore in the case of more complicated gauge theories especially nonabelian gauge theories, but also gauge theories on noncommutative spaces, the method of Becchi, Rouet, Stora [BRS76] and Tyutin [Tyu] called BRST quantization is preferred. It can be viewed as a generalization of the Gubta Bleuler method and is a central aspect of the Batalin Vitkovsky formalism [BV81] in the geometric quantization procedure [BV81], [AKSZ97] and for an introduction see [Fio03]. Now that we have a low energy theory, which violates causality at a fundamental level we may think that this fact might be ignored but this we should not do. The theory should always be thought as a low energy limit of a relativistic theory, like in the situation of the hydrogen atom. Then we have to perform in the same way as in the covariant formalism. More concrete this means the counter-terms required in the Coulomb gauge have to be included also in a low energy theory. The impact for the Chern Simons gauge theory is similar. Either we choose Coulomb gauge and evaluate suitable counter-terms or we choose Lorentz gauge and the Goupta Bleuler method or the BRST method respectively. This will be discussed in the next chapter in this section we perform the Coulomb gauge method.

Let us now turn to the free gauge field propagator. The quantization procedure requires a unique Cauchy problem so we fix the gauge. The Coulomb gauge $\partial^i a_i(x) = 0$ is also called transverse gauge since only transverse modes, the physical modes are left and the longitudinal modes are eliminated from the beginning. This can be seen best in momentum space. The Coulomb gauge condition is here $k^i a_i(k) = 0$. In the two dimensional plane it is clear that we can only have one longitudinal mode and one transverse mode. If we choose for example the x-direction as the direction of the momentum then clearly $a_x = a_{\mathrm{L}} = 0$ and $a_y = a_{\mathrm{T}}$ is the physical mode. Without fixing a coordinate system we have

$$\boldsymbol{a} = (a_{\mathrm{T}}, a_{\mathrm{L}}) =: a_1 \frac{i\sigma_2 \mathbf{k}}{k} + a_2 \frac{\mathbf{k}}{k}.$$

We now quantize the theory and define the operators to evaluate the free gauge field propagator and the S-matrix defined by the interaction Hamiltonian. The quantum fields in momentum space obey the following relations:

$$a_0^+(\mathbf{k}) = a_0(-\mathbf{k}), \quad a_1^+(\mathbf{k}) = -a_1(-\mathbf{k}). \tag{2.3.20}$$

The part of the action with Chern Simon fields can be written as

$$\begin{aligned} S_{\mathrm{G}} &= \frac{\varepsilon^{ij}}{\tilde{\varphi}\phi_0} \int d\mathbf{x}dt \; a_0 \partial_i a_j \qquad (2.3.21) \\ &+ \frac{\varepsilon^{ij}\varepsilon^{mn}}{2(\tilde{\varphi}\phi_0)^2} \int d\mathbf{y} \; \partial_i a_j(\mathbf{x},t) \; V(\mathbf{x}-\mathbf{y}) \; \partial_m a_n(\mathbf{y},t). \end{aligned}$$

Instead of the Coulomb potential we may consider here also a screened potential, for example a Yukawa like potential. Then in Fourier space the action is given by

$$S_{\mathrm{G}} = \int \frac{d\mathbf{k}d\Omega}{(2\pi)^3}\frac{iek}{\tilde{\varphi}\phi_0}a_0(\mathbf{k},\Omega)a_1^+(\mathbf{k},\Omega) + \frac{\mathbf{k}^2V(k)}{2(\tilde{\varphi}\phi_0)^2}a_1(\mathbf{k},\Omega)a_1^+(\mathbf{k},\Omega). \tag{2.3.22}$$

We rewrite it in terms of the free Greens function

$$\left(D^0_{\mu\nu}(\mathbf{k},\Omega)\right) = \begin{pmatrix} \frac{V(k)}{e^2} & \frac{iek}{\tilde{\varphi}\phi_0} \\ -\frac{iek}{\tilde{\varphi}\phi_0} & -\frac{\mathbf{k}^2V(k)}{(\tilde{\varphi}\phi_0)^2} \end{pmatrix} \tag{2.3.23}$$

and obtain:

$$S_{\mathrm{G}} = \int \frac{d\mathbf{k}d\Omega}{(2\pi)^3}\ a_\mu^+(\mathbf{k},\Omega)[D^0_{\mu\nu}(\mathbf{k},\Omega)]^{-1}a_\nu(\mathbf{k},\Omega). \tag{2.3.24}$$

The Fermionic fiels interact with the Chern Simons gauge fields through the interaction part of the Lagrangian (2.3.7)

$$\begin{aligned} S_{\mathrm{Int}} &= -\int dt \int \frac{d\mathbf{k}d\mathbf{q}}{(2\pi)^4}\sum_{\mu\nu}\Big\{v_\mu(\mathbf{kq})\psi^+(\mathbf{k}+q,t)a_\mu(\mathbf{q},t)\psi(\mathbf{k},t) \\ &\quad +\frac{1}{2}\int\frac{d\mathbf{q}'}{(2\pi)^2}w_{\mu\nu}(\mathbf{q},\mathbf{q}')\psi^+(\mathbf{k}+\mathbf{q},t)a_\mu(\mathbf{q},t)a_\nu(\mathbf{q}',t)\psi(\mathbf{k}-\mathbf{q}',t)\Big\} \end{aligned} \tag{2.3.25}$$

Were v_μ and $w_{\mu\nu}$ are the vertices with contribution

$$v_\mu = \begin{cases} -e, & \mu=0 \\ -\frac{e}{mq}\ \varepsilon^{ij}k_iq_j, & \mu=1 \end{cases} \tag{2.3.26}$$

$$w_{\mu\nu} = -\frac{e^2}{mqq'}q^iq_i'\delta_{\mu1}\delta_{\nu1} \tag{2.3.27}$$

The propagators are given by the expectation value of the time ordered product:

$$\langle T^{[t,0]}[a_\mu(\mathbf{q},t)a_\nu^+(\mathbf{q}',0)]\rangle = iD_{\mu\nu}(\mathbf{q},t) \tag{2.3.28}$$

$$\langle T^{[t,0]}[\psi(\mathbf{k},t)\psi^+(\mathbf{k}',0)]\rangle = iG(\mathbf{k},t) \tag{2.3.29}$$

$T^{[t_1,\ldots,t_n]}$ denotes the time ordering with respect to the times $t_1,\ldots,t_n$. The interaction term can be treated in a perturbative S-matrix formulation. The formal S-matrix is defined through the Dyson series

$$S_{[\lambda]} = T\exp\{i\int dt\int dx\ \delta(x^0-t)\lambda(x)\mathcal{L}_{\mathrm{Int}}(x)\} \tag{2.3.30}$$

$$= I + \sum_{N=1}^{\infty}\frac{(-i)^N}{N!}S^{(N)}_{[\lambda]} \tag{2.3.31}$$

T denotes the time ordering and I is the identity. We introduced for the time being formally an adiabatic switching function from Schwartz space $\lambda(x) \in \mathcal{S}(\mathbb{R}^3)$ to get a well defined expression. The N-th coefficient $S^{(N)}_{[\lambda]}$ of the Dyson series is given by

$$S^{(N)}_{[\lambda]} = \int dx_1 \dots dx_n \lambda(x_1) \mathbf{:} \mathcal{L}_{\mathrm{Int}}(x_1) \mathbf{:} \dots \lambda(x_n) \mathbf{:} \mathcal{L}_{\mathrm{Int}}(x_n) \mathbf{:} \tag{2.3.32}$$

We may remark that a S-matrix is formally unitary $S^+_{[\lambda]} S_{[\lambda]} = S_{[\lambda]} S^+_{[\lambda]} = \mathbb{1}$ if the interaction Hamiltonian

$$H^{\lambda}_{\mathrm{Int}}(t) = \int dx \; \delta(x^0 - t) \lambda(x) \mathcal{L}_{\mathrm{Int}}(x) \tag{2.3.33}$$

is symmetric: $H^{\lambda}_{\mathrm{Int}}(t) = H^{\lambda}_{\mathrm{Int}}(t)^+$. Then we only have to show that the interaction Hamiltonian is symmetric but this is given by construction.

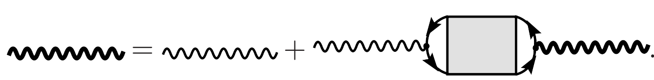
The Dyson equation for the gauge field propagators $D(\mathbf{q}, \Omega)$ is formally given by

$$D_{\mu\nu} = D^0_{\mu\nu} + D^0_{\mu\rho} \Pi^{\rho\sigma} D_{\sigma\nu} \tag{2.3.34}$$

The self-energy part Π is called the exact irreducible polarization function and D^0 is the free propagator

$$\langle a_\mu(\mathbf{q}, t) a_\nu(\mathbf{q}', t) \rangle = i(-1)^\nu D^{(0)}(\mathbf{q}, t - t') \delta(\mathbf{q} - \mathbf{q}'). \tag{2.3.35}$$

We will call the lowest order approximation Π^0 of the polarization function Π the random phase approximation together with the free Fermionic Green's functions. In diagrammatic terms the Dyson equation can be drawn as:



(2.3.36)

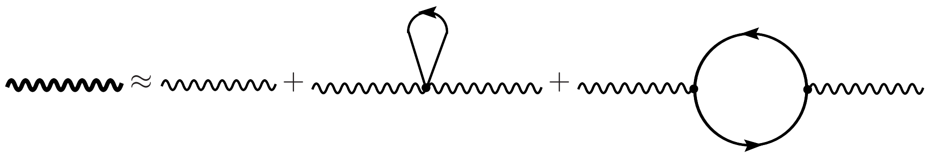
The thick wiggled line corresponds to the exact propagator, the thin wiggled line to the free propagator and the shaded area represents the exact polarization. The lowest order corrections to the free gauge field propagator is given by ($\epsilon > 0$)

$$\begin{aligned} S^1 &= -D^0_{\mu\rho}(\mathbf{q}, \Omega) w^{\rho\sigma}(\mathbf{q}, \mathbf{q}) \int \frac{d\mathbf{k} d\omega}{(2\pi)^3} \; G^0(\mathbf{k}, \omega) \; e^{i\omega\epsilon} \; D^0_{\sigma\nu}(\mathbf{q}, \Omega) \\ &+ D^0_{\mu\rho}(\mathbf{q}, \Omega) \int \frac{d\mathbf{k} d\omega}{(2\pi)^3} \; v^\rho(\mathbf{k}, \mathbf{q}) \; G^0(\mathbf{k} + \mathbf{q}, \omega + \Omega) G^0(\mathbf{k}, \omega) \; v^\sigma(\mathbf{k}, \mathbf{q}) D^0_{\sigma\nu}(\mathbf{q}, \Omega). \end{aligned} \tag{2.3.37}$$

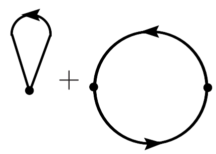
The self-energy part gives the free polarization function to lowest order perturbation theory

$$\begin{aligned} \Pi^0_{\mu\nu} &= -i w^{\rho\sigma}(\mathbf{q}, \mathbf{q}) \int \frac{d\mathbf{k} d\omega}{(2\pi)^3} \; G^0(\mathbf{k}, \omega) \; e^{i\omega\epsilon} \\ &+ \; i \int \frac{d\mathbf{k} d\omega}{(2\pi)^3} \; v^\rho(\mathbf{k}, \mathbf{q}) \; G^0(\mathbf{k} + \mathbf{q}, \omega + \Omega) G^0(\mathbf{k}, \omega) \; v^\sigma(\mathbf{k}, \mathbf{q}). \end{aligned} \tag{2.3.38}$$

In term of Feynman graphs this corresponds to the approximation to lowest order in the vertices:


(2.3.39)

wherein the contribution of the free polarization function corresponds to the loops:

$\Pi^0_{\mu\nu} =$ 
(2.3.40)

The polarization function is analyzed in [HLR93] and [SH95] and it renormalizes the composite Fermions effective mass in the small energy sector in the Fermi Liquids theory. The relevant regime is defined by $|\Omega| \ll v_{\mathrm{F}} q \ll v_{\mathrm{F}} k_{\mathrm{F}}$. The polarization function in this regime is calculated to

$$\Pi^0(\mathbf{q}, \Omega) \approx \begin{pmatrix} -\frac{me^2}{2\pi}(1 + i\frac{|\Omega|}{v_{\mathrm{F}}q}) & 0 \\ 0 & \frac{q^2e^2}{24\pi m} - i\frac{2\rho e^2|\Omega|}{mv_{\mathrm{F}}q} \end{pmatrix} \tag{2.3.41}$$

Then the approximated gauge field propagator is

$$D(\mathbf{q}, \Omega) \approx \frac{1}{\xi_q + i\gamma\frac{|\Omega|}{v_{\mathrm{F}}q}} \begin{pmatrix} -\tilde{\xi}_q + i\tilde{\gamma}_q|\Omega| & -i\beta q \\ i\beta q & \frac{m}{2\pi} \end{pmatrix} \tag{2.3.42}$$

with the abbreviations $\beta = e/\tilde{\phi}\Phi_0$ and the potential like terms $\tilde{\xi}_q = -q^2V(q)/\tilde{\phi}\Phi^2$ and $\xi_q = \tilde{\xi}_q m/2\pi$ and $\tilde{\gamma}_q = 2\rho/k_{\mathrm{F}}q$ and $\gamma = \rho/\pi$. The dominant component is D_{11}. The imaginary pole

$$\Omega = -i\frac{2\pi e^2}{k_{\mathrm{F}}\varepsilon\tilde{\phi}\Phi_0^2}q^2 \tag{2.3.43}$$

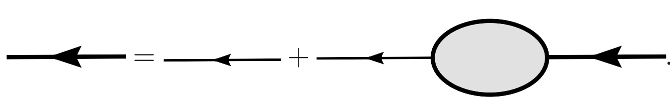
gives a slowly decaying mode where the decay time diverges for very small momenta.

Self-energy Correction for the Fermionic Two Point Function

The Dyson equation for the Fermion propagator is given by

$$G(\mathbf{k}, \omega) = G^0(\mathbf{k}, \Omega) + G^0(\mathbf{k}, \Omega)\Sigma(\mathbf{k}, \omega)G(\mathbf{k}, \omega) \tag{2.3.44}$$

and can also be understood in terms of irreducible Feynman graphs:


(2.3.45)

The thick line corresponds to the exact propagator, the thin line to the free propagator and the exact self-energy part is denoted by the shaded loop. The renormalized spectrum can by calculated via the self-consistent equation

$$\varepsilon^*(\mathbf{k}) = \varepsilon(\mathbf{k}) + \mathrm{Re}(\Sigma(\mathbf{k}, \varepsilon(\mathbf{k}))) \tag{2.3.46}$$

with the free $\varepsilon(\mathbf{k})$ and renormalized ε^* Fermionic dispersion. Then the quasi-particle lifetime is

$$\tau = \mathrm{Im}(\Sigma(\mathbf{k}, \varepsilon^*(\mathbf{k})))^{-1}. \tag{2.3.47}$$

The effective quasi-particle mass m^* is defined via the relation

$$\varepsilon^*(\mathbf{k}) := \frac{k^2}{2m^*} - \mu. \tag{2.3.48}$$

If we take the derivative of (2.3.46) with respect to k at the Fermi level we obtain

$$\frac{k_\mathrm{F}}{m^*} = \frac{k_\mathrm{F}}{m} + \partial_\varepsilon \Sigma|_{k=k_\mathrm{F}} \frac{k_\mathrm{F}}{m} + \partial_\omega \Sigma|_{\omega=\varepsilon^*} \frac{k_\mathrm{F}}{m^*} \tag{2.3.49}$$

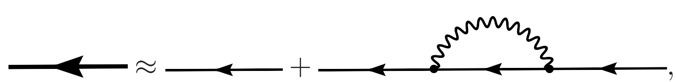
which gives the relation

$$\frac{m^*}{m} = \frac{1 - \partial_\omega \Sigma|_{\omega=\varepsilon^*}}{1 + \partial_\varepsilon \Sigma|_{k=k_\mathrm{F}}}. \tag{2.3.50}$$

The Fermionic propagator in the random phase approximation is considered only up to the first order in the gauge field propagator:

$$\begin{aligned} iG(\mathbf{k},\omega) \quad &\approx \quad iG^0(\mathbf{k},\omega) \\ &-iG^0(\mathbf{k},\omega) \int \frac{d\mathbf{k}' d\Omega}{(2\pi)^3} v_1(\mathbf{k},\mathbf{k}') G^0(\mathbf{k}'.\omega - \Omega) D_{11}(\mathbf{k}-\mathbf{k}',\Omega) iG^0(\mathbf{k},\omega). \end{aligned} \tag{2.3.51}$$

The Hartree correction and the first order contribution are neglected since they are only uninteresting constants. In diagrammatic terms this approximation corresponds to



(2.3.52)

wherein the gauge field loop corresponds to the Fock type self-energy contribution. The self-energy part in terms of the propagators is

$$\Sigma(\mathbf{k},\omega) \approx i \int \frac{d\mathbf{k}' d\Omega}{(2\pi)^3} v_1^2(\mathbf{k},\mathbf{k}') G^0(\mathbf{k}',\omega - \Omega) D_{11}(\mathbf{k}-\mathbf{k}',\Omega) \tag{2.3.53}$$

and it is convenient to introduce the difference

$$\delta\Sigma(kb,\omega) = \Sigma(\mathbf{k},\omega) - \Sigma(\mathbf{k},0) \tag{2.3.54}$$

explicitly given by:

$$\delta\Sigma(\mathbf{k},\omega) \approx i\int \frac{d\mathbf{k}' d\Omega}{(2\pi)^3} v_1^2(\mathbf{k},\mathbf{k}') D_{11}(\mathbf{k}-\mathbf{k}',\Omega)(G^0(\mathbf{k}'.\omega-\omega) - G^0(\mathbf{k}'.\omega-\Omega)). \quad (2.3.55)$$

We introduce spherical coordinates $q = |\mathbf{k}-\mathbf{k}'|$ so v_1 becomes

$$v_1^2 = \frac{e^2 k^2 k'^2}{m^2 q^2}\sin^2\theta \quad \text{and} \quad \int d\mathbf{k}' = 2\int_0^\infty \int_{|\mathbf{k}-\mathbf{k}'|}^{\mathbf{k}+\mathbf{k}'} dk' dq \frac{q}{k\sin\theta}. \quad (2.3.56)$$

The angle θ is between the two momenta and

$$\sin\theta = \left(1 - (\frac{k^2 + k'^2 - q^2}{2kk'})^2\right)^{1/2}. \quad (2.3.57)$$

At the Fermi level $q \ll k_\mathrm{F}$ $q \in [0, 2k_\mathrm{F}]$ this gives $\sin\theta \approx q/k$ and $v_1^2 \approx e^2 k_\mathrm{F}^2/m^2$. The measure can be approximated to one and the self-energy difference is

$$\delta\Sigma(\mathbf{k},\omega) \approx i\frac{2e^2 k_\mathrm{F}^2}{(2\pi)^3 m^2}\int_{-\infty}^{\infty} d\Omega \int_0^{2k_\mathrm{F}} \int_0^\infty D_{11}(q,\Omega)(G^0(k'.\omega-\omega) - G^0(k'.\omega-\Omega)). \quad (2.3.58)$$

The evaluation of these integrals is rather technical and is done only approximately. We start with the integral over k' with small ω, which we can approximate

$$\begin{aligned} &\int_0^\infty dk' \ (G^0(k'.\omega-\omega) - G^0(k'.\omega-\Omega)) \approx \quad (2.3.59)\\ \approx \ & \frac{1}{2}\int_{-\infty}^{\infty} dk' \ \frac{\omega}{(\omega-\Omega-\eta+i\epsilon\mathrm{sign}(\omega-\Omega))(\Omega+\varepsilon_{k'}+i\epsilon\mathrm{sign}(\Omega))} \end{aligned}$$

Near the Fermi energy the dispersion relation may be approximated by $\varepsilon_{k'} \approx v_\mathrm{F} = \eta$ and this simplifies the integral to

$$\begin{aligned} &\frac{\omega}{2v_\mathrm{F}}\int_{-\infty}^{\infty} d\eta \ \frac{\omega}{(\omega-\Omega-\eta+i\epsilon\mathrm{sign}(\omega-\Omega))(\Omega+\eta+i\epsilon\mathrm{sign}(\Omega))} = \quad (2.3.60)\\ = \ & -\frac{i\pi}{v_\mathrm{F}}\Theta(\Omega)\Theta(\omega-\Omega). \end{aligned}$$

For positive ω (quasi-particle properties) the integral is calculated by closing the contour in the upper half-space on the complex plane. Next we calculate

$$\begin{aligned} \int_0^{2k_\mathrm{F}} dq \ D_{11} &= \int_0^{2k_\mathrm{F}} dq \ \big(-\frac{qe^2}{\varepsilon\tilde{\varphi}^2\phi_0^2} + i\frac{2\rho|\Omega|}{k_\mathrm{F} q}\big) \quad (2.3.61)\\ &= \frac{\varepsilon\tilde{\varphi}^2\phi_0^2}{2e^2}\big(\ln\big(i\frac{2\rho|\Omega|}{k_\mathrm{F}}\big) - \ln\big(i\frac{2\rho|\Omega|}{k_\mathrm{F}} - 4k_\mathrm{F}^2\frac{e^2}{\varepsilon\tilde{\varphi}^2\phi_0^2}\big)\big). \end{aligned}$$

Being in the regime $\omega \ll v_\mathrm{F} q$, we are allowed to expand the integral with respect to $\Omega/v_\mathrm{F} q$ and this gives

$$\frac{\varepsilon\tilde{\varphi}^2\phi_0^2}{2e^2}\big(\ln\big(i\frac{2\rho|\Omega|}{k_\mathrm{F}}\big) - \ln\big(-4k_\mathrm{F}^2\frac{e^2}{\varepsilon\tilde{\varphi}^2\phi_0^2}\big)\big) \quad (2.3.62)$$

and the self-energy difference is calculated to

$$\delta\Sigma \approx \frac{k_{\mathrm{F}}\varepsilon\tilde{\varphi}^2\phi_0^2}{8\pi^2 m}\omega \ln\left(\frac{\varepsilon\tilde{\varphi}^2\phi_0^2}{8\pi e^2 k_{\mathrm{F}}}\omega\right) - i\frac{k_{\mathrm{F}}\varepsilon\tilde{\varphi}^2\phi_0^2}{16\pi m}\omega. \tag{2.3.63}$$

Having calculated the self-energy we can perform a mass renormalization

$$m^* \propto \frac{\varepsilon\tilde{\varphi}^2\phi_0^2}{8\pi e^2 m}\ln\omega. \tag{2.3.64}$$

This quasi-particle mass diverges due to the $1/q$ singularity leading to infrared divergences. It seems that the correction $\delta\Sigma$ is exact in the limit $\omega \ll v_{\mathrm{F}}q$, which is shown in [SH95] by evaluating Ward identities. These singularities of the single particle Green's function of the composite Fermions allows for a derivation of an energy gap [SH95]:

$$\Delta(p) \approx \frac{e^2 k_{\mathrm{F}}\pi}{\varepsilon\tilde{\varphi}(\tilde{\varphi}+1)}\frac{1}{\ln(2p+1)} \tag{2.3.65}$$

with p being the composite Fermion filling. In the regime where p is large the approximations $\ln(2p+1) \approx \ln p$ and $k_{\mathrm{F}} \approx \sqrt{2/p}/l$ leads to the energy gap:

$$\Delta(p) \approx \sqrt{\frac{2}{\tilde{\varphi}}}\frac{e^2\pi}{\varepsilon l\tilde{\varphi}\ln(2p)}. \tag{2.3.66}$$

The interaction with the gauge field leads to incompressibility, which scales in the large p limit with the suggested potential, interacting between the charge densities. It is in this case the Coulomb potential. This scaling is expected also in terms of dimensional estimations [HLR93]. Some experiments [DST$^+$94][PXS$^+$99] [YST$^+$99] near half and quarter filling factor suggests a large effective composite Fermion mass.

2.3.4 Low Energy Effective Theory With Spin

So far we considered in the whole framework only systems where the spin of the electrons and composite Fermions respectively are completely polarized in a Landau band. It was long time believed, that the strong magnetic field allows for such assumption, wherein for example the Zeeman splitting is large enough [Hal84]. However, non fully spin polarized systems may exist due to the special physical environment provided by the material of the sample. The electron effective mass for example in GaAs and the small g-factor and leads in an estimation to a Zeeman energy, which is about 70 times lower than the cyclotron energy [Hal83]:

$$m^*_{\mathrm{GaAs}} \approx 0.067 m_0 \tag{2.3.67}$$
$$g_{\mathrm{GaAs}} \approx -0.44 \tag{2.3.68}$$
$$\hbar\omega_{c\mathrm{GaAs}} \approx 20\mathrm{TK} \tag{2.3.69}$$
$$E_{\mathrm{ZGaAs}} \approx 0.29\mathrm{TK}. \tag{2.3.70}$$

The Zeeman energy is known to be a relativistic correction, which appears when we perform the low energy limit of the Dirac equation. The naive considerations about the Zeeman energy gives then a clear indication that the relativistic effects might become important at least in form of the spin and the corresponding correction terms to the energy.

So far only a phenomenological effective approach is present, trying to incorporate the spin in the quantum Hall regime and in the composite Fermion model. A precise derivation from a relativistic theory analog to the Hydrogen Atom is still an open question. We will discuss this issue in the next chapters in more detail. First we will have a peek on the phenomenological approaches [KMM+02a], [MMSK05b], [MMN+02] and [KMM+02b].

Lagrangian Formulation of the Effective Low Energy Theory With Spin

The proposed ansatz for the composite Fermion Lagrangian including spin degrees of freedom is the following:

$$\mathcal{L}(\mathbf{x},t) = \mathcal{L}_0(\mathbf{x},t) + \mathcal{L}_{\mathrm{CS}}(\mathbf{x},t) + \mathcal{L}_C(\mathbf{x},t), \tag{2.3.71}$$

Wherein the Fermionic free part with spin is given by

$$\mathcal{L}_0(\mathbf{x},t) = \sum_{s=\uparrow,\downarrow} \psi_s^+(\mathbf{x},t)\big[-\frac{1}{2m}\big(\mathbf{p} - e(\mathbf{A}(\mathbf{x}) - \boldsymbol{\mathcal{A}}_s(\mathbf{x},t))\big)^2 + \tag{2.3.72}$$

$$+i\partial_t + \mu + e(A_0(\mathbf{x}) - \mathcal{A}_{0s}(\mathbf{x},t))\big]\psi_s(\mathbf{x},t) \tag{2.3.73}$$

where s stands for the spin degrees of freedom. The Chern Simons action changes in this setting to

$$\mathcal{L}_{\mathrm{CS}}(\mathbf{x},t) = \frac{e}{\tilde{\varphi}\phi_0}\sum_{s=\uparrow,\downarrow} \varepsilon^{\mu\nu\rho}\mathcal{A}_{\mu s}(\mathbf{x},t)\partial_\nu \mathcal{A}_{\mu s}. \tag{2.3.74}$$

The quantization of the Chern Simons fields is later done for each spin degree separately in the Coulomb gauge fixing assuming a pure Chern Simons theory. $\mathcal{A}_{0s}$ becomes a Lagrange multiplier field for each degree of freedom and we can restrict this Lagrangian term to

$$\mathcal{L}_{\mathrm{CS}}^{\mathrm{Cgf}}(\mathbf{x},t) = \frac{e}{\tilde{\varphi}\phi_0}\sum_{s=\uparrow,\downarrow} \varepsilon^{ij}\mathcal{A}_{0s}(\mathbf{x},t)\partial_i \mathcal{A}_{js}. \tag{2.3.75}$$

This is a pragmatically approach to handle the Chern Simons term, including spin, but we will discuss a more constructive and general formalism in the next chapter especially what concerns the quantization procedure. Since it will turn out to become a problem of quantizing nonabelian Chern Simons theories, we will then introduce the BRST quantization method. The interaction is proposed to be Coulombian: $V(\mathbf{x}-\mathbf{y}) = e^2/\epsilon|\mathbf{x}-\mathbf{y}|$:

$$\mathcal{L}_{\mathrm{C}}(\mathbf{x},t) = -\sum_{ss'}\frac{1}{2}\int d\mathbf{y}\rho_s(\mathbf{x},t)V_{ss'}(\mathbf{x}-\mathbf{y})\rho_{s'}(\mathbf{y},t). \tag{2.3.76}$$

The equations of motion for the Chern Simons fields with spin are obtained by varying the action with respect to the fields $\mathcal{A}_{\mu s}$ where now each degree of spin gives its own equation of motion:

$$\frac{\delta S}{\delta \mathcal{A}_{\mu s}} = 0$$

and the zero component leads to the relation

$$\varepsilon^{ij}\partial_i \mathcal{A}_{js}(\mathbf{x},t) = \tilde{\varphi}\phi_0 \ \rho_s(\mathbf{x},t). \tag{2.3.77}$$

This means that we can replace the charge density $\rho_s(\mathbf{x},t)$ in the Coulomb part of the action by $(\tilde{\varphi}\phi_0)^{-1}\varepsilon^{ij}\partial_i \mathcal{A}_{js}(\mathbf{x},t)$. Due to the spin degrees of freedom the effective magnetic field acting on the charged particles gets an index s:

$$b_s(\mathbf{x},t) = B - \mathcal{B}_s = B - \tilde{\varphi}\phi_0\rho_s(\mathbf{x},t). \tag{2.3.78}$$

We can also divide the total electromagnetic field or Chern Simons field with spin in a mean field and a dynamical field respecting the fluctuations:

$$a_{\mu s} = \mathcal{A}_{\mu s} - \langle \mathcal{A}_{\mu s}\rangle. \tag{2.3.79}$$

Again the average field $\langle \mathcal{A}_{\mu s}\rangle$ is not a dynamical field and gives no interesting contribution to the equation of motions. Thus only the dynamical part $a_{\mu s}$ contributes separately to the equation of motion and this leads to the relation:

$$\rho_s(\mathbf{x},t) = \frac{1}{\tilde{\varphi}\phi_0}\ \varepsilon^{ij}\partial_i a_{js}(\mathbf{x},t). \tag{2.3.80}$$

The free Fermionic propagator is evaluated in the same way as in the spin-less case since we work in the low energy regime where the Hilbert space $\mathcal{H}_s$ of particles with spin is divided in two sectors, labeled by the index s. The free field operator acting on the vacuum $|\Omega_s\rangle$ is defined by

$$\psi_s(\mathbf{x},t) = \int \frac{d\mathbf{k}dt}{(2\pi)^3}\psi_s(\mathbf{k},t)\ e^{i(\mathbf{k}\mathbf{x}-\omega t)}. \tag{2.3.81}$$

Then the free part of the action is given by

$$S_{\mathrm{F}} = \sum_s \int d\mathbf{x}dt\ \psi_s^+(\mathbf{x},t)\big[-\frac{\mathbf{p}^2}{2m} + i\partial_t + \mu\big]\psi_s(\mathbf{x},t). \tag{2.3.82}$$

and in Fourier space by

$$S_{\mathrm{F}} = \sum_s \int d\mathbf{k}dt\ \psi_S^+(\mathbf{k},t)\underbrace{[\omega - \frac{k^2}{2m} - \mu]}_{[G^0(\mathbf{k},\omega)]^{-1}}\psi_s(\mathbf{k},t) \tag{2.3.83}$$

with the same low energy Fermionic Greens function as in the spin-less case:

$$G^0(\mathbf{k},\omega) = [\omega - \frac{k^2}{2m} - \mu + i\epsilon\mathrm{sign}(\omega)]^{-1}. \tag{2.3.84}$$

Random Phase Approximation

The difference in the random phase approximation to the case without spin comes only from the spin mixing part in the Coulomb interaction term of the Lagrangian. The quantization of the gauge field is done separately for each spin degree of freedom, which means that also here the Hilbert space of Chern Simons particles is divided into two sectors labeled by the index s with vacuum $\left|\Omega^{\mathrm{Cs_s}}\right\rangle$. In Coulomb gauge the fields $\boldsymbol{a}(\mathbf{k},\Omega)=(a_1,a_2)=(a_{\mathrm{T}},a_{\mathrm{L}})$ satisfy the same relations as in the case without spin:

$$a^+_{0s}(\mathbf{k})=a_{0s}(-\mathbf{k}),\quad a^+_{1s}(\mathbf{k})=-a_{1s}(-\mathbf{k}). \tag{2.3.85}$$

and therefore the gauge field part of the action is given by

$$\begin{aligned} S_{\mathrm{G}} &= \frac{\varepsilon^{ij}}{\tilde{\varphi}\phi_0}\sum_{s,s'}\int d\mathbf{x}dt\ a_{0s}\partial_i a_{js} \\ &+\frac{\varepsilon^{ij}\varepsilon^{mn}}{2(\tilde{\varphi}\phi_0)^2}\int d\mathbf{y}\ \partial_i a_{js}(\mathbf{x},t)\ V_{ss'}(\mathbf{x}-\mathbf{y})\ \partial_m a_{ns'}(\mathbf{y},t). \end{aligned} \tag{2.3.86}$$

Since the action is non diagonal in the fields $a_{\mu s}$ due to the Fourier transformed interaction potential we may introduce gauge field combinations diagonalizing the action part:

$$a_{\mu\pm}:=\tfrac{1}{2}[a_{\mu\uparrow}\pm a_{\mu\downarrow}]\quad \text{and vice versa} : a_{\mu\uparrow\downarrow}=a_{\mu+}\pm a_{\mu-}. \tag{2.3.87}$$

However, then the free gauge field part changes to

$$S_{\mathrm{G}}=-\frac{1}{2}\sum_{\alpha}\int\frac{d\mathbf{k}d\Omega}{(2\pi)^3}\ a^+_{\alpha}(\mathbf{k},\Omega)[D^0_{\mu\nu\alpha}(\mathbf{k},\Omega)]^{-1}a_{\nu\alpha}(\mathbf{k},\Omega). \tag{2.3.88}$$

with the free gauge field propagator

$$\left(D^0_{\mu\nu\alpha}(\mathbf{k},\Omega)\right)=\begin{pmatrix} -\frac{V(k)}{e^2}\delta_{\alpha,+} & -\frac{i\tilde{\varphi}\phi_0}{ek} \\ \frac{i\tilde{\varphi}\phi_0}{ek} & 0\end{pmatrix} \tag{2.3.89}$$

or respectively the inverse propagator

$$\left(D^0_{\mu\nu\alpha}(\mathbf{k},\Omega)\right)^{-1}=\begin{pmatrix} 0 & \frac{iek}{\tilde{\varphi}\phi_0} \\ -\frac{iek}{\tilde{\varphi}\phi_0} & \frac{k^2V(k)}{\tilde{\varphi}^2\phi_o^2}\delta_{\alpha,+}\end{pmatrix}. \tag{2.3.90}$$

The S-Matrix contribution from the interaction part of the Lagrangian changes only in such a way that we have an index s labeling the spin degrees of freedom but since there is no mixing assumed it gives no effect:

$$\begin{aligned} S_{\mathrm{Int}} &= -\sum_s\int dt\int\frac{d\mathbf{k}d\mathbf{q}}{(2\pi)^4}\sum_{\mu\nu}\Big\{v_{\mu s}(\mathbf{kq})\psi^+_s(\mathbf{k}+q,t)a_{\mu s}(\mathbf{q},t)\psi_s(\mathbf{k},t) \\ &+\frac{1}{2}\int\frac{d\mathbf{q}'}{(2\pi)^2}w_{\mu\nu s}(\mathbf{q},\mathbf{q}')\psi^+_s(\mathbf{k}+\mathbf{q},t)a_{\mu s}(\mathbf{q},t)a_{\nu s}(\mathbf{q}',t)\psi_s(\mathbf{k}-\mathbf{q}',t)\Big\}. \end{aligned} \tag{2.3.91}$$

The $v_{\mu s}$ and $w_{\mu\nu s}$ are the known vertices which do not depend on the spin degrees of freedom:

$$v_{\mu s} = \begin{cases} -e, & \mu = 0 \\ -\frac{e}{mq}\,\varepsilon^{ij}k_i q_j, & \mu = 1 \end{cases} \tag{2.3.92}$$

$$w_{\mu\nu s} = -\frac{e^2}{mqq'}q^i q'_i \delta_{\mu 1}\delta_{\nu 1}. \tag{2.3.93}$$

This transformation in our approach leads to a simultanious contribution of D^+ and D^- in a fixed combination of s with s'.

The interaction term can be treated in the same way as in the spin-less case so we can define the corresponding S-matrix via the Dyson series. The two point function defines the propagators:

$$\langle T^{[t,0]}[a_{\mu\alpha}(\mathbf{q},t)a^+_{\nu\alpha'}(\mathbf{q}',0)]\rangle = iD_{\mu\nu\alpha}(\mathbf{q},t)\delta_{\alpha\alpha'}\delta(\mathbf{q}-\mathbf{q}') \tag{2.3.94}$$

$$\langle T^{[t,0]}[\psi_s(\mathbf{k},t)\psi^+_{s'}(\mathbf{k},0)]\rangle = iG(\mathbf{k},t)\delta_{ss'}. \tag{2.3.95}$$

The free polarization function to lowest order in the Dyson equation is with $\epsilon > 0$

$$\begin{aligned} \Pi^0_{\mu\nu} &= -i\sum_s w_s^{\rho\sigma}(\mathbf{q},\mathbf{q})\int \frac{d\mathbf{k}d\omega}{(2\pi)^3}\;G_s^0(\mathbf{k},\omega)\;e^{i\omega\epsilon} \\ &+ i\int\frac{d\mathbf{k}d\omega}{(2\pi)^3}\;v_s^\rho(\mathbf{k},\mathbf{q})\;G_s^0(\mathbf{k}+\mathbf{q},\omega+\Omega)G_s^0(\mathbf{k},\omega)\;v_s^\sigma(\mathbf{k},\mathbf{q}). \end{aligned} \tag{2.3.96}$$

Close to the Fermi energy in the regime $|\Omega| \ll v_{\mathrm{F}}q \ll v_{\mathrm{F}}k_{\mathrm{F}}$ the polarization function is

$$\Pi^0(\mathbf{q},\Omega) \approx \begin{pmatrix} -\frac{me^2}{2\pi}(1+i\frac{|\Omega|}{v_{\mathrm{F}}q}) & 0 \\ 0 & \frac{q^2e^2}{24\pi m} - i\frac{2\rho e^2|\Omega|}{mv_{\mathrm{F}}q} \end{pmatrix}. \tag{2.3.97}$$

The polarization function does not depend on the spin therefore it corresponds to the case without spin. The approximated gauge field propagator is then

$$D_\alpha(\mathbf{q},\Omega) \approx \frac{1}{\zeta(q)(\gamma_+(q)\delta_{\alpha,+}+\gamma_-(q)-\eta|\Omega|/q)-\beta^2q^2} \tag{2.3.98}$$

$$\times\begin{pmatrix} \gamma_+(q)\delta_{\alpha,+}+\gamma_-(q)-\eta\frac{|\Omega|}{q} & -i\beta q \\ i\beta q & \zeta(q) \end{pmatrix}. \tag{2.3.99}$$

We use the abbreviations $\zeta(q) = e^2m(1-|\Omega|/qv_{\mathrm{F}})/\pi$, $\gamma_+ = -4q^2V(q)/\tilde{\varphi}\phi_0$, $\gamma_- = -q^2e^2/12\pi m$ and $\eta = 2e^2\rho/mv_{\mathrm{f}}$. The dominant matrix elements for small momenta and small frequencies are $D_{11\alpha}$ with $\alpha = \pm$, $\alpha_+ = 4qV(q)/\tilde{\varphi}^2\phi_0^2$ and $\alpha_- = (e^2/12\pi+4\pi/\tilde{\varphi}^2\phi_0^2)/m$. For the Coulomb interaction we neglect the term α_- and from analytic continuation we obtain the retarded propagator

$$D^R_{11+} \approx \frac{-q}{\alpha_+(q)q^2+\alpha_-q^3-i\eta\Omega} \tag{2.3.100}$$

$$D^R_{11-} \approx \frac{-q}{\alpha_-q^3-i\eta\Omega} \tag{2.3.101}$$

with the complex poles at

$$\Omega_+ = -i\frac{\alpha_+(q)q^2 - \alpha_- q^3}{\eta} \quad (2.3.102)$$

$$\Omega_- = -i\frac{\alpha_- q^3}{\eta}. \quad (2.3.103)$$

The effective interaction expressed in terms of the propagator in an analysis in [Bon93] is:

$$V_{ss'\mathrm{eff}} := -v_{1s}(\mathbf{k},\mathbf{q})v_{1s'}(\mathbf{k}',-\mathbf{q})(D_{11+}(\mathbf{q},\Omega) + (2\delta_{ss'} - 1)D_{11-}(\mathbf{q},\Omega)). \quad (2.3.104)$$

The Fermionic self-energy in terms of the dominant symmetric configuration of the propagators is calculated in [MMN+02]:

$$\delta\Sigma(\mathbf{k},\omega) \approx i\int \frac{d\mathbf{k}'d\Omega}{(2\pi)^3}\ v_1(\mathbf{k},\mathbf{k}')^2 D_{11-}(\mathbf{k}-\mathbf{k}',\Omega)\ (G^0(\mathbf{k}',\omega-\Omega) - G^0(\mathbf{k},-\Omega)). \quad (2.3.105)$$

With the relation

$$\int_0^{2k_{\mathrm{F}}} dqD_{11-} = \int_0^{2k_{\mathrm{F}}} dq\ \frac{-q}{\alpha_- q^3 - i\eta}\Omega \approx -\frac{2\pi\sqrt{3}}{9}\left(\frac{i}{\alpha_\eta^2\Omega}\right)^{1/3} \quad (2.3.106)$$

the self-energy difference part is approximated to:

$$\delta\Sigma(\mathbf{k},\omega) \approx \frac{\pi}{\sqrt{3}}\left(\frac{i}{\alpha_\eta^2\Omega}\right)^{1/3}\omega^{2/3} \quad (2.3.107)$$

and has the effect that the mass diverges near the Fermi level with $\omega^{2/3}$ instead of the logarithmic divergence in the case without spin.

2.3.5 Modified Low Energy Theory with Spin

In the Lagrangian (2.3.71) also a modification in the Chern Simons term is discussed [MR96], being also relevant in the second generation composite Fermion model [MMSK05a]. The coupling constant of the Chern Simons term

$$\mathcal{L}_{\mathrm{CS}}(\mathbf{x},t) = \frac{e}{\tilde{\varphi}\phi_0}\sum_{s=\uparrow,\downarrow}\varepsilon^{\mu\nu\rho}\mathcal{A}_{\mu s}(\mathbf{x},t)\partial_\nu\mathcal{A}_{\mu s} \quad (2.3.108)$$

is turned into a real valued symmetric matrix $(\Theta_{ss'})$ allowing to tune the filling factor for each spin degree of freedom.

$$\mathcal{L}_{\mathrm{CS}}(\mathbf{x},t) = \frac{e}{\tilde{\varphi}\phi_0}\sum_{s=\uparrow,\downarrow}\Theta_{ss'}\mathcal{A}_{0s}(\mathbf{x},t)\varepsilon^{ij}\partial_i\mathcal{A}_{js'} \quad (2.3.109)$$

The task of a proper quantization of such an expression we shift to the next chapter in the sense that this approach is just a special case of a minimal coupled $SU(2)$ nonabelian Gauge field with Chern Simons action. A $SU(2)$ Chern Simons action is also discussed in terms of bi-layer Hall effects [Bon93], where both layers are coupled via the $su(2)$ gauge fields, denoting the occurrence of an isospin or pseudospin respectively. When we include the spin degrees of freedom in both layers then the Θ-matrix is the special realization where the spins in the layers are converse.

Mean Field and Random Phase Approximation in the Modified Spin Model

We take the same approach as in the previous section and quantize the Chern Simons field for each spin degree of freedom separately or in the diagonalized fields respectively, within the Coulomb gauge method. The classical equations of motion for the fluctuating fields

$$a_{\mu s} = \mathcal{A}_{\mu s} - \langle \mathcal{A}_{\mu s} \rangle \tag{2.3.110}$$

give the relation for the particle densities:

$$\phi_0 \rho_s = \sum_{s'} \Theta_{ss'} \, \varepsilon^{ij} \partial_i a_{js'}. \tag{2.3.111}$$

The matrix can be diagonalized by introducing the new fields [MR96],[MR97]:

$$a_{\mu\pm} := a_{\mu\uparrow} \pm a_{\mu\downarrow} \tag{2.3.112}$$

$$\theta_\pm := \theta_1 \pm \theta_2 \tag{2.3.113}$$

$$\rho_\pm(\mathbf{x},t) := \rho_\uparrow(\mathbf{x},t) \pm \rho(\mathbf{x},t)_\downarrow \tag{2.3.114}$$

The densities can then be rewritten in the form:

$$\phi_0 \rho_\pm(\mathbf{x},t) := \theta_\pm \, \varepsilon^{ij} \partial a_{j\pm}(\mathbf{x},t). \tag{2.3.115}$$

Different combinations of the θ_i can be fixed. So we may choose $\theta_- = 0$ and $\theta_i = \theta_j \neq 0$, which results in the relation $\rho_\uparrow = \rho_\downarrow$ and we may consider cooper pair formations, bosons with charge $2e$. This is the situation where the inverse of the Θ-matrix diverges since $(\Theta^{-1})_{1i} = \theta_i/(\theta_1^2 - \theta_2^2)$. The mean field can be defined for each spin pairing or spin polarization separately. This is a main feature of this approach it allows for example different filling factors for different spin polarizations. If we require Fermionic statistics then the θ's are not independent and fulfill the equation

$$\frac{1}{\theta_+} + \frac{1}{\theta_-} = 2n, \quad n \in \mathbb{N}. \tag{2.3.116}$$

The mean magnetic field of the mean gauge field in 2.3.110 can be defined in the diagonal representation with the mean densities:

$$\langle \mathcal{B}_\pm \rangle := \frac{\langle \rho_\pm \rangle}{\theta_\pm} \tag{2.3.117}$$

or in terms of the spins respectively

$$\langle \mathcal{B}_{\uparrow\downarrow} \rangle := \frac{\langle \rho_{\uparrow\downarrow} \rangle}{\theta_{\uparrow\downarrow}}. \tag{2.3.118}$$

The corresponding filling factor can be expressed in terms of the filling factors corresponding to the different spin polarization. The spin resolved filling factors of the composite Fermions are defined via the equation:

$$p_{\uparrow\downarrow} := \frac{\phi_0 \langle \rho_{\uparrow\downarrow} \rangle}{B}. \tag{2.3.119}$$

tThis gives the relation for the total filling factor:

$$\nu = \nu_\uparrow + \nu_\downarrow = \frac{\phi_0(\langle\rho_\uparrow\rangle + \langle\rho_\downarrow\rangle)}{B}. \tag{2.3.120}$$

When we move away from an even denominator filling then the fluctuating fields a_s become larger and the composite Fermions form themselves Landau levels so called composite Fermion Landau Levels (CFLL). How we can handle composite Fermions in (composite Fermion) Landau levels we will discuss in chapter 4. The filling factor of composite Fermions can be described in terms of the spin resolved magnetic field.

$$\nu_{\uparrow\downarrow} := \frac{\phi_0\langle\rho_{\uparrow\downarrow}\rangle}{b_{\uparrow\downarrow}}. \tag{2.3.121}$$

Where the magnetic field with respect to the spin $b_{\uparrow\downarrow}$ can be recovered from the diagonal fields $b_\pm$ via:

$$b_{\uparrow\downarrow} = \frac{1}{2}(b_+ \pm b_-)\ . \tag{2.3.122}$$

In terms of the diagonal representations of the θ matrix we may also define the filling factors

$$p_\pm := \frac{\langle\rho_\pm\rangle}{b_\pm}. \tag{2.3.123}$$

These equations give an expression of the total filling factor in terms of the spin resolved composite Fermion filling factors:

$$\nu = \frac{p_\uparrow + p_\downarrow + 4p_\uparrow p_\downarrow/\theta_-}{1 + (1/\theta_+ + 1/\theta_-)(p_\uparrow + p_\downarrow) + 4p_\uparrow p_\downarrow/(\theta_+\theta_-)}. \tag{2.3.124}$$

We see immediately that we recover the Jain's sequence if we take the limit of completely spin polarized particles:

$$\nu = \frac{p_{\uparrow\downarrow}}{2np_{\uparrow\downarrow} + 1}, \quad n,\ p_{\uparrow\downarrow} \in \mathbb{N}. \tag{2.3.125}$$

In this sense we can discuss within this approach non fully spin polarized Hall states. The z-component of the spin has always to be a good quantum number in this setup. This is true for stationary states, however in terms of transport the z-component might not remain a good quantum number due to spin orbit interaction. We will comment on this issue in the next chapter. With the parameters $\theta_\pm$ we can now tune the systems polarization and may fit spin resolved experiments. For example if we assume that spin up and spin down composite Fermions experience the same effective magnetic field, this corresponds to the limit $\theta_- \to \infty$ and the total filling factor is given by:

$$\nu = \frac{p_\uparrow + p_\downarrow}{2n(p_\uparrow + p_\downarrow) + 1}. \tag{2.3.126}$$

This gives again the Jain's principal sequence. The Chern Simons term can also be expressed in terms of the diagonal representation:

$$\mathcal{L}_{\mathrm{CS}} = -\frac{e}{\phi_0} \sum_{\alpha=\pm} \theta_\alpha a_{0\alpha}\, \varepsilon^{ij} \partial_i a_{j\alpha}. \tag{2.3.127}$$

However, in terms of the Lagrangian, the limit $\theta_- \to \infty$ has to be taken carefully since this leads to an undefined singular Lagrangian term. Thus in that limit the field $a_{\mu-}$ has to vanish simultaneously. Otherwise the Chern Simons term is no more defined and meaningless. It corresponds to the theories for partially polarized Hall states described in [MR96]. The incorporation of $SU(2)$ Chern Simons fields [LF95] and [BF91] respectively differs from that approach since an additional cubic term is needed in the nonabelian Chern Simons Lagrangian, which is missing in this approach. Furthermore, if we perform a Chern Simons transformation with local spin 1/2 fluxes (flux-attachment to electrons or composite Fermions) we can not systematically obtain the Θ-matrix approach since for the attachment for $SU(2)$ fields i.e. spins we have to perform a local $SU(2)$-Chern Simons transformation which automatically generates a corresponding coupling to a $Lie(SU(2))$ valued gauge field. The corresponding Lagrangian is a nonabelian Chern Simons Lagrangian. In this sense the Θ-matrix approach is a pure phenomenological approach which can only be justified by heuristic arguments rather than fundamental gauge principles. The implementation of a $SU(2)$ symmetry need some care what concerns a quantization in a perturbative treatment. In [MR96] the quantization is performed in terms of Witten's approach to invariants [Wit89], however a perturbative treatment requires a different approach [AS91]. In the next chapter we introduce a proper quantization procedure for general Chern Simons theories with semi simple compact gauge group as symmetry group in the composite Fermion picture and discuss the $SU(2)$ example.

2.3.6 Spin Resolved Experiments

In the previous section we developed some possibilities to implement the spin degrees of freedom in the theory of composite Fermions in fractional Hall states and in the next chapters we will extend these models in the sense that the spin is treated as a real $SU(2)$ symmetry in a relativistic Dirac equation. The spin can be interpreted as a relativistic effect in this manner starting from a relativistic equation may shed more light on the physics of a Hall system. However, we should also clarify what physical impacts of such theories are accessible to experiments. For the hydrogen atom the relativistic effects are known and can be perfectly described by quantum electrodynamics. A similar approach for Hall systems however does not exist so far but might be crucial for experimental predictions.

In the late nineties the interests on spin resolved experiments emerged following some discrepancies of the composite Fermion model. For the effective mass $m^* \approx 0.6 m_e$ at a magnetic field of about ten Tesla $B \approx 10\mathrm{T}$ as published in [DYS+95] and [LNFH94] the Fermi energy of composite Fermions is estimated to about ten Kelvin $E_{\mathrm{F}}^{\mathrm{CF}} \approx 10K$. Taking the small g factor of GaAs into account the Zeeman energy is estimated to about three Kelvin $E_{\mathrm{Z}} \approx 3\mathrm{K}$ at $B \approx 10\mathrm{T}$. The fact that $E_{\mathrm{Z}} \ll E_{\mathrm{F}}^{\mathrm{CF}}$ then suggests that the system is completely spin unpolarized. Experiments where they measured the composite Fermion wave vector [GPS+96] suggest that the system is spin polarized at a magnetic field of ten Tesla. The investigation of

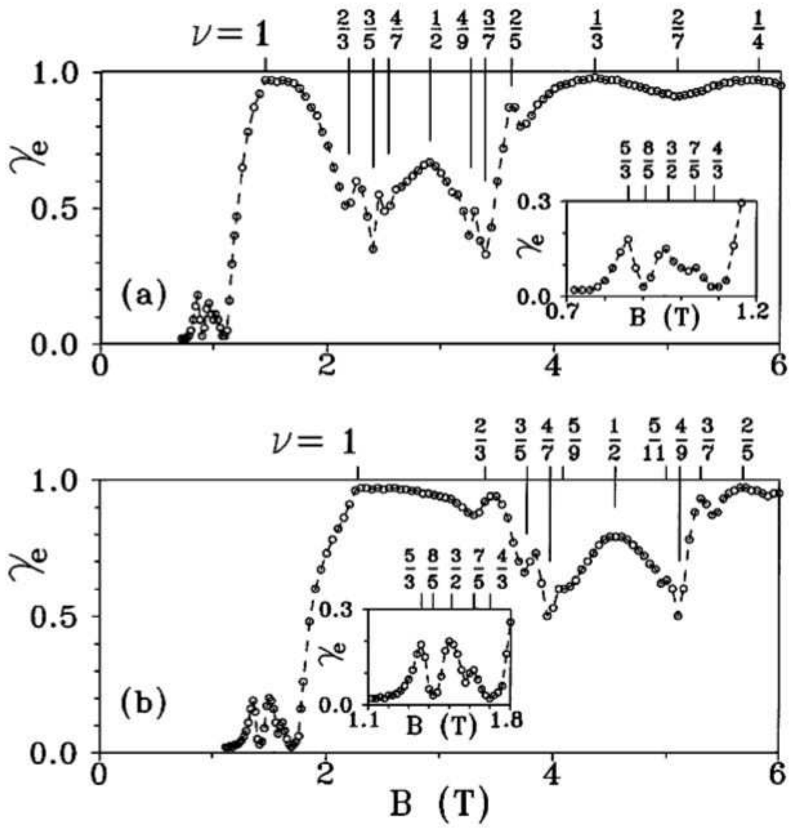


Figure 2.9: The Spin polarization function γ_e in terms of the perpendicular magnetic field B in Tesla with details around $\nu = 3/2$, for (a): $n_s = 3.5 \times 10^{0}\text{cm}^{-2}$ and (b): $n_s = 5.5 \times 10^{0}\text{cm}^{-2}$ [KvKE99].

the magnetic field dependence of the spin polarization of a two dimensional Fermion system allows for the measuring of spin transitions between spin polarized, partially polarized and unpolarized systems [KvKE99]. The whole spectrum of the system can be observed by magnetoluminescense measurements, where holes where induced by laser pulses in a separated δ-doped Be layer parallel to the Hall system. The measurement of the circular polarization of the light, emitted by the recombination of the holes with the electrons from the Hall bar, gives then information about the spin polarization. The dependence of the spin polarization function γ_e in terms of the perpendicular magnetic field B at temperatures between 0.3K and 1.8K is shown in figure 2.9 for two different fixed densities in a GaAs/AlGaAs hetero-junction with δ-doped mono-layer of Be acceptors. The magnetoluminescence signals shows that the Hall state at $\nu = 1$ is always completely spin polarized in contrast to the many other fractional filling factors, where the polarization differs with the densities. With a back-gate-density modulation it is possible to keep the filling factor constant, which leads to more precise results figure 2.10. Here we see wide plateaus where the polarization is nearly constant and a smooth transition between different plateaus with a generic "shoulder" in-

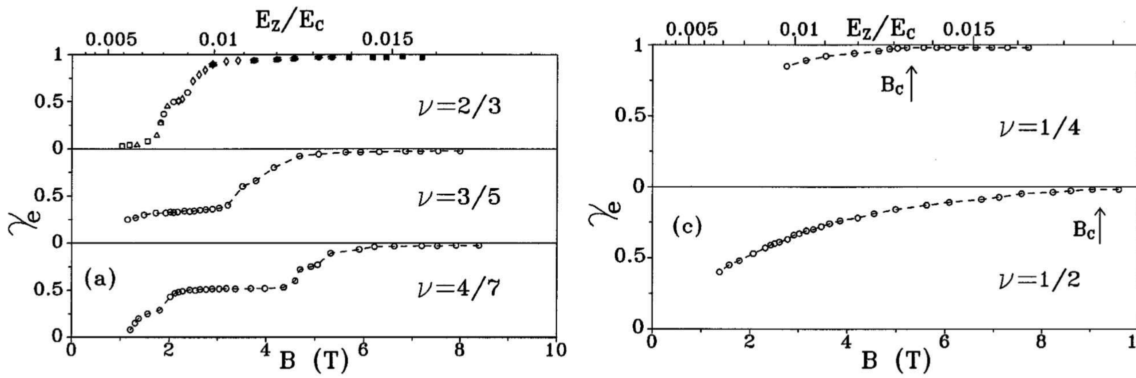


Figure 2.10: The spin polarization γ_e plotted in terms of the magnetic field B: (a) the sequence $\nu = n/(2n-1)$ and (c) $\nu = 1/2,\ 1/4$. The filling factor is fixed with a back-gate density modulation method [KvKE99].

between, indicating a stabilization of the polarization. Since the temperature is very low we may discuss a model where the transition is smoothed even at zero temperature. This is the so called zero temperature smoothing (ZTS). These experiments show that the simplest model of spin polarised composite Fermions is insufficient for a realistic description of the fractional Hall state. Furthermore there is strong evidence that at half filling a spin transition occurs from partially spin polarization at low densities and low field configuration to a complete spin polarization at high densities and high field configuration observed in many experiments: [DKK+99], [KvKLE99], [MFHac+00], [FHacB+02], [SFF+04] and [DPK+05]. This fact is strongly supported by recent experiments [TEPW07] of resistively detected nuclear magnetic resonance (RDNMR) methods [DMP+02]. In this method the coupling between the two dimensional electron gas and the spin of nuclei by hyperfine interaction modifies the electronic Zeeman energy E_Z. This modification is due to a finite spin polarization ξ_N creating an effective magnetic field B_N and gives a Zeeman energy

$$E_\mathrm{Z} = g\mu_B(B + B_N)$$

with the GaAs g-factor of $g = -0.44$ and $B_N = -5.3$T if the nuclear spins of $({}^{69}\mathrm{Ga}, s_N = 3/2)$, $({}^{71}\mathrm{Ga}, s_N = 3/2)$ and $({}^{75}\mathrm{As}, s_N = 3/2)$ are completely polarized. In the nuclear magnetic resonance radio frequency method magnetic fields of about $H_1 \approx 0.1\mu$T are applied parallel to the long axis of the Hall bar. This induces excitations, which reduces the polarization of the nuclear spin and the electronic Zeeman energy E_Z increases and modifies the longitudinal resistivity if $\partial\rho_{xx}/\partial E_\mathrm{Z} \neq 0$. In figure 2.11 (a) the resistivity is plotted versus the magnetic field and shows no evidence for a transition and (b)-(d) shows typical data of RDMNR. In figure 2.11 (e) the differential resistivity $(\partial\rho_{xx}/\partial E_\mathrm{Z})/\rho_{xx}$ is plotted in terms of the magnetic field at $\nu = 1/2$ and $T = 45\mathrm{mK}, 100\mathrm{mK}$. The slope of the $T = 45$mK data shows a strong evidence for a transition of the polarization form low field/density to high field/density due to a peak of $\partial\rho_{xx}/\partial E_\mathrm{Z}$ in the transition region.

Having elaborated the detection of the spin transitions we may ask for the spin-flip gap

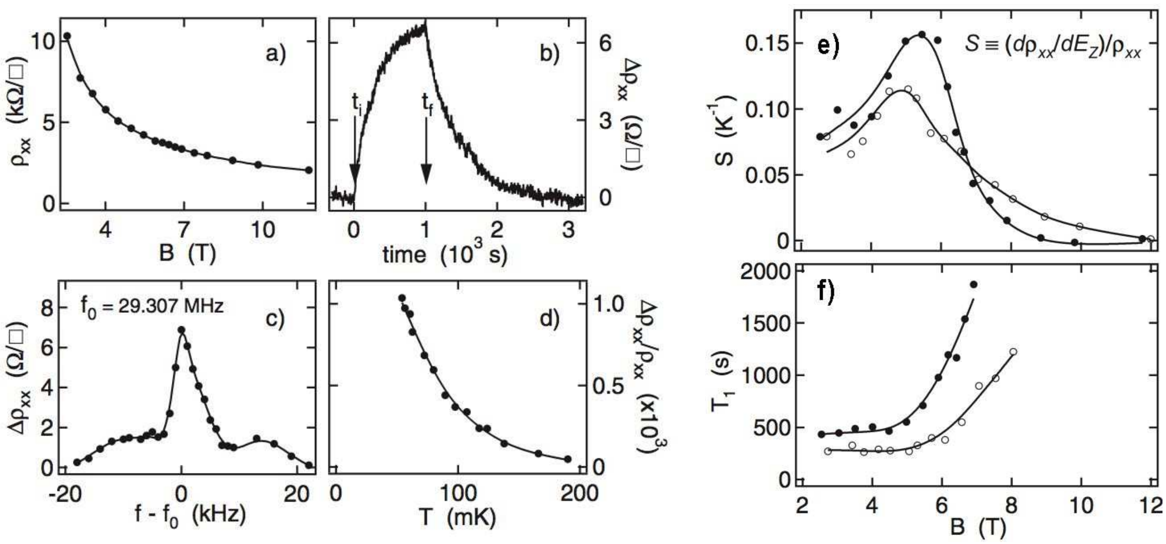


Figure 2.11: In the left figure (a): Resistivity at $\nu = 1/2$ against the magnetic field B at $T =$ 45mK. (b) Typical response of the resistivity around the NMR Line. (c) Typical RDNMR line. (d) Temperature dependence of the RDNMR signal with peak at $\nu = 1/2$ and $B = 4.01$T. In the right figure: (e): The differential $\partial \rho_{xx}/\partial E_{\mathrm{Z}}$ is plotted against the perpendicular magnetic field at filling factor $\nu = 1/2$: the $T = 45$mK slope corresponds to the dots, and the $T = 100$mK slope to the circles. The peak shows a strong evidence for a transition in the spin polarization. The nuclear spin relaxation time $T_1 =$ (b) supports this evidence [DMP$^+$02].

in fractional Hall states. This was first measured in experiments that take the temperature dependence of the spin polarization into account [KSvKE00] at $\nu = 1/3$ and $\nu = 2/3$. Furthermore the spin-flip gaps allow for the determination of the composite Fermion interaction energy. These experiments where first done with magnetoluminescence as described above. In figure 2.12 the temperature dependence of the spin polarization function is plotted in terms of the temperature. On the left $\gamma_e(T)$ is plotted for $\nu = 1/3$ and can be described at low temperature via an exponential dependence

$$\gamma_e = 1 - 2e^{-\frac{\Delta}{2k_{\mathrm{B}}T}}.$$

This makes it possible to experimentally determine the energy gap in the spin-flip transitions. A recent experiment on resonant inelastic light scattering shows that neutral spin textures emerges at filling factor $\nu = 1/3$ [GDG$^+$08]. A finite magnetic field parallel to the sample is achieved by tilting the sample figure left of figure 2.13.

In figure 2.13 on the right the excitation spectra are plotted for different laser energies ω_l at $T = 70$mK for $\nu = 1/3$ ($B = 8$T). The resonant light scattering spectra shows the possibility to selectively excite the magneto roton mode and the spin texture mode by varying ω_l.

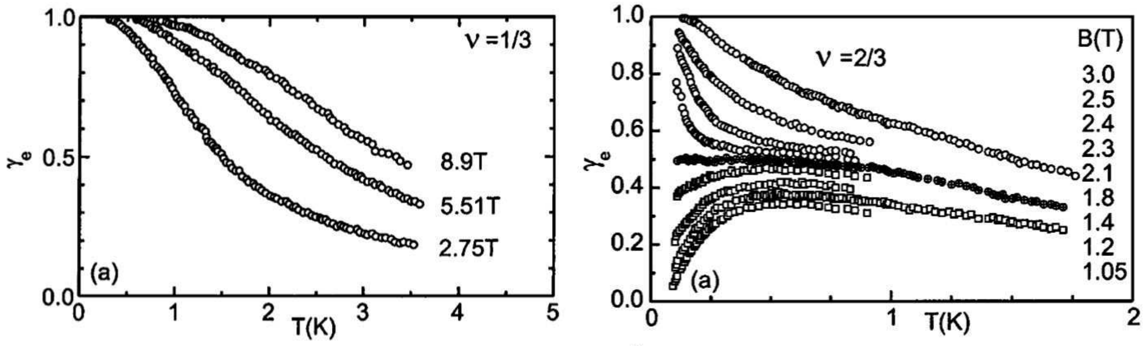


Figure 2.12: In the left figure the temperature dependence of the spin polarization is plotted for $\nu = 1/3$. In the right picture for $\nu = 2/3$ [KSvKE00].

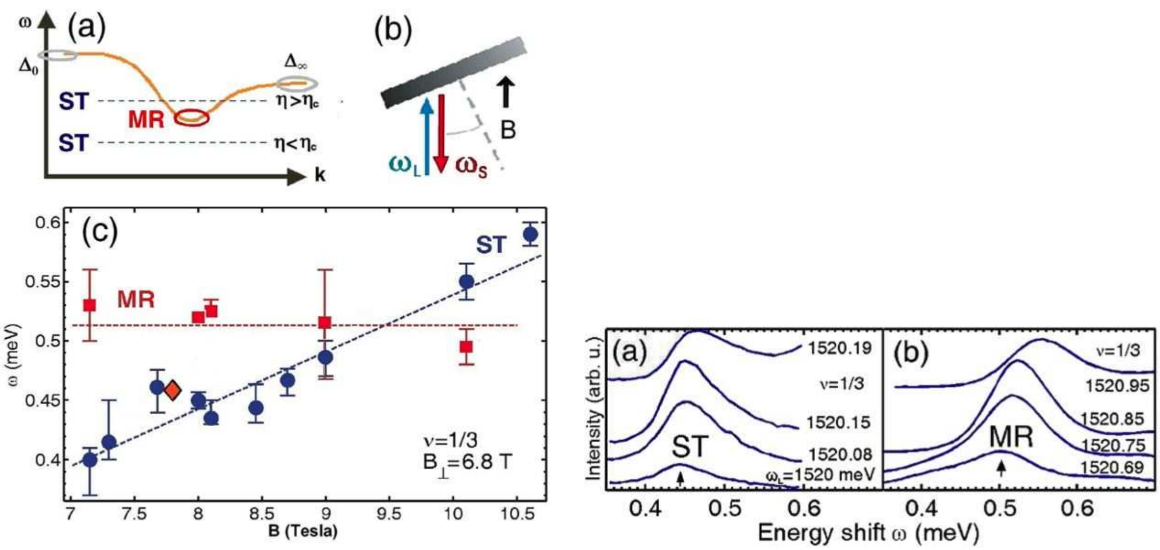


Figure 2.13: On the left: (a) Dispersion of the lowest magneto roton branch at $\nu = 1/3$.(b) The sample ist tilted against the magnetic field. (c) Energies of the spin texture and magneto roton modes at constant $B_\perp$ and varying B. Circles (squares) are the resonant peaks of the spin texture (magneto roton) modes and the diamond is the transport activation gap [GDG$^+$08]. On the right: The different spectra of the spin texture mode on the right (a) and the magneto roton mode on the right (b) are plotted in terms of the energy shift ω for different laser energies ω_l [GDG$^+$08].

Chapter 3

QED and Chern Simons theory

The appearance of spin effects in a quantum Hall especially in a fractional quantum Hall regime rises the question, how we could incorporate the $SU(2)$ symmetry of charged electrons. It has a long time been believed that at high classical magnetic external fields all spin moments of two dimensional electrons are polarized. Now, there is the effect, that at very low temperatures the system forms an Abrikosov lattice, consisting of flux tubes, which are considered as being topological defects in the sample and they are described by singularities of the gauge field. In such a situation the electron may feel a reduced external field while binding flux-tubes to it and the polarization of the spin moment is reduced or vanishes completely.

3.1 Quantum Electrodynamics on Background Electromagnetic Fields

It is a matter of fact, that relativistic QED provides the most precise, accepted explanation of the spectrum of the hydrogen atom and thus it is reasonable to ask for an explanation in a Hall system, too. There are approaches which consider effects of QED indirectly from a phenomenological point of view for example as Rashba spin orbit interaction or Darwin terms and Zeeman terms, which provide in some Hall systems some new effects for example Abrikosov or spin Hall effects. The appearance of charged mass-less Dirac particles in graphene also motivates the question to properly incorporate QED in Hall systems. The question of a formulation based on QED for the fractional Hall effect remains an open one. More precise we would like to know whether a composite particle theory can be formulated derived from a QED action and if so, does it give new insight for Hall systems? This is the question we would like to address in this and the next chapter.

Though in QED there are many open questions especially in a description with background external fields concerning for example the uniqueness of the vacuum, we try to formulate the problem for Hall systems and remark the difficulties of a more consistent theory. Also we address the question how to project quantum mechanically the relativistic QED (action) on a 2+1 dimensional system in the next chapter, which is not completely understood so far.

The most general QED action for electrons in an external magnetic field with a parity violating term is

$$S = \int d^4x \left[\tfrac{i}{2}[\bar{\psi}\overleftrightarrow{\not{\partial}}\psi] - m\bar{\psi}\psi - e\bar{\psi}\not{A}\,\psi - \left(\tfrac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}\right) - \left(\tfrac{1}{16\pi}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right) + J^{\mu}A_{\mu}\right]$$

where ψ is a Dirac spinor field, A_μ the classical external electromagnetic field, $F^{\mu\nu}$ the corresponding field strength tensor and J^μ the external electromagnetic current considered as a constant function, which may change adiabatically with time. The total antisymmetric term then gives just a contribution on a suitable boundary, usually pushed to infinity where all fields are zero, so it is usually not considered.

The equations of motion for the massive Fermi fields coupled to an external magnetic field are given by the Dirac equation in the 4-dimensional Minkowski space:

$$i(\gamma^\mu D_\mu - m)\psi_D = 0$$

with $D_\mu = \partial_\mu - ieA_\mu$. Now we perform a 'local'Chern Simons gauge transformation

$$\psi_D \to \psi = e^{i\Theta(x)}\psi_D$$

and the covariant derivative is changed to

$$D_\mu \to \mathcal{D}_\mu = \partial_\mu - ieA_\mu - i\mathcal{A}_\mu$$

and define the total gauge field

$$a_\mu := eA_\mu + \mathcal{A}_\mu.$$

The standard Yang-Mills construction leads to the field strength tensor. This is a tensor field, a two form on a hermitian line bundle, which is related to the curvature two form:

$$f = [\mathcal{D}, \mathcal{D}] = da - a \wedge a$$

or in local coordinates:

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - [a_\mu, a_\nu]$$

where the last term can be dropped for abelian gauge fields. The Maxwell action is now given in a certain spacetime region $\mathcal{O}$ as usual by:

$$S_{\mathrm{M}} = \tfrac{1}{4}\int_{\mathcal{O}} f \wedge * f \ .$$

In our situation we may have the further total antisymmetric term sometimes also called the Pontrjagin action

$$S_{\mathrm{PJ}} = \tfrac{1}{4}\int_{\mathcal{O}} f \wedge f$$

leading to the famous Chern Simons action on the boundary $\partial\mathcal{O}$ of the spacetime region $\mathcal{O}$. It is a straight forward calculation to show that for any local $U(N)$ gauge theory $a_\mu \in Lie(U(N))$

$$\mathrm{tr} f \wedge f = d(\mathrm{tr} W)$$

is a total derivative of the Chern Simons three form W

$$W = a \wedge da + \tfrac{2}{3} a \wedge a \wedge a$$

and by making use of the Stokes theorem the Pontrjagin action on $\mathcal{O}$ is turned to the Chern Simons action on the boundary $\partial\mathcal{O}$:

$$\int_{\mathcal{O}} \mathrm{tr} f \wedge f = \int_{\partial\mathcal{O}} \mathrm{tr}[a \wedge da + \tfrac{2}{3} a \wedge a \wedge a].$$

In the specific case of a $U(1)$ theory this is reduced to

$$S_{\mathrm{CS}} = \int_{\partial\mathcal{O}} a \wedge da.$$

The cubic term vanishes, which means that the field connection forms commute in the $U(1)$ case. Later we will see that this changes if we project the fields onto the lowest Landau level. In local coordinates the action is:

$$S_{\mathrm{CS}} = \int_{\partial\mathcal{O}} d^3\sigma \ \varepsilon^{\mu\nu\rho} \ a_\mu \partial_\nu a_\rho.$$

It is reasonable to consider not only the abelian $U(1)$ gauge theory on a commutative Minkowski space but also in the lowest Landau level projection, which would result in an $U(1)$ gauge theory on a special version of the noncommutative Minkowski space and also it is reasonable to consider local $SU(N) \subset U(N)$ gauge symmetries for multilayer Hall Systems or second generation of composite Fermions. We will comment on this in the next chapter. For general non-abelian theories the gauge transformation can be performed by

$$U = e^{-\Theta(x)^\alpha T^\alpha} \in SU(N), \quad \Theta^\alpha \in \mathcal{C}_0(\mathbb{R}) \text{ and } \alpha = 1, \ldots, N_c^2 - 1. \tag{3.1.1}$$

The traceless anti-hermitian operators T^α generate the corresponding Lie algebra $su(N)$ with the usual commutation relation:

$$[T^\alpha, T^\beta] = f^{\alpha\beta}{}_\gamma T^\gamma \tag{3.1.2}$$

with normalization

$$\mathrm{tr}(T^\alpha T^\beta) = -\frac{1}{2}\delta^{\alpha\beta} \tag{3.1.3}$$

and in the case $N = 2$ the total antisymmetric structure constant is given by the epsilon tensor $f^{\alpha\beta\gamma} = \varepsilon^{\alpha\beta\gamma}$. In this sense the gauge potential is a Lie algebra valued one form on the corresponding principal bundle. The covariant derivative and the curvature (field strength) is obtained via the usual YM-construction. The purely topological part of the action is given by

$$S_{\mathrm{PJ}} = -\tfrac{1}{2} \int_{\mathcal{O}} \mathrm{tr}(f \wedge f).$$

We can define the generalized electric and magnetic fields by $e_i := f_{0i}$ and $b^i = \frac{1}{2}\epsilon^{ijk} f_{jk}$, thus the action S_{PJ} has only terms proportional to $e_i b^i$, which results in a violation of CP-symmetry. More precisely $C(e_i b_i) = +e_i b_i$=even and $P(e_i b_i) = -e_i b_i$=negative parity thus $CP(e_i b_i) = -e_i b_i$ is not a good quantum number.

In the absence of matter fields the Euler Lagrange equations of motions of the action S_{YM}

$$S_{\mathrm{YM}} = \tfrac{1}{4}\int_{\mathcal{O}} \mathrm{tr} f \wedge *f$$

then gives the generalized Maxwell equations for non-abelian gauge fields

$$*D * f = 0, \quad D_\nu f^{\mu\nu} = 0$$

for the homogeneous equations and via the second Bianch identity

$$Df = 0, \quad D_\mu(\varepsilon^{\mu\nu\rho\sigma} f_{\rho\sigma}) = 0$$

the inhomogeneous equations. While the first part of the Maxwell equations are results of the action principle the appearance of the Bianchi identity is purely geometrical. The action S_{PJ} would give a deviation of the upper equations only at the boundary of the system.

How can we now understand this Chern Simons term on the boundary and what is the boundary of our system? In mathematics it is well known, that the value of the Chern Simons action is a topological invariant, the Chern number or second Chern class, which counts the windings around singularities on a closed manifold, a manifold without boundary. This is a fact, which is due to a nontrivial cohomology there are closed forms which are not exact. It was due to Eduard Witten, who related the Chern number of closed manifolds (actually the S^3) to the Jones-polynomial, which gives the linking number, the invariants of knots on that manifold. We will comment in the end of this chapter on the connection between Chern Simons theories and Wess Zumino Witten theories.

We may perform the following gedankenexperiment. In the Fermion Theory an even number of fluxes are attached to non-relativistic electrons in three dimensions, producing a Chern Simons term in three dimensions. On the other hand a Chern Simons term on a boundary of a four dimensional manifold can be obtained via a Pontryagin functional, as shown above. We will now try to incorporate these both derivations, at least on a heuristic level. It is clear, that if the fluxes are assumed to be solenoids pinning the closed manifold $\partial\mathcal{O}$ twice, the Chern Simons action cancels to zero if there is no other suitable source (A_0), which gives rise to a non vanishing contribution. It is exactly the statement, that we do not have magnetic monopoles. However, in two dimensions we have vortices which emerge in a Hall system and the fluxes are concentrated therein. To obtain a non-vanishing Chern Simons term we try to transform the closed three dimensional manifold without boundary $\partial\mathcal{O}$ topologically to a 2+1 dimensional space $C \times I$, C being a cylinder with cylinder barrel ∂C and in a next step we argue that only the contribution of the cylinder bottom and head are left. Let us look at our Lagrangian for one particle in an external, for the moment classical field with Ahranov-Bohm fluxes attached to it:

$$\mathcal{L} = \tfrac{i}{2}[\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi] - m\bar{\psi}\psi - j \wedge a - \tfrac{1}{4} f \wedge *f - \tfrac{1}{4} f \wedge f,$$

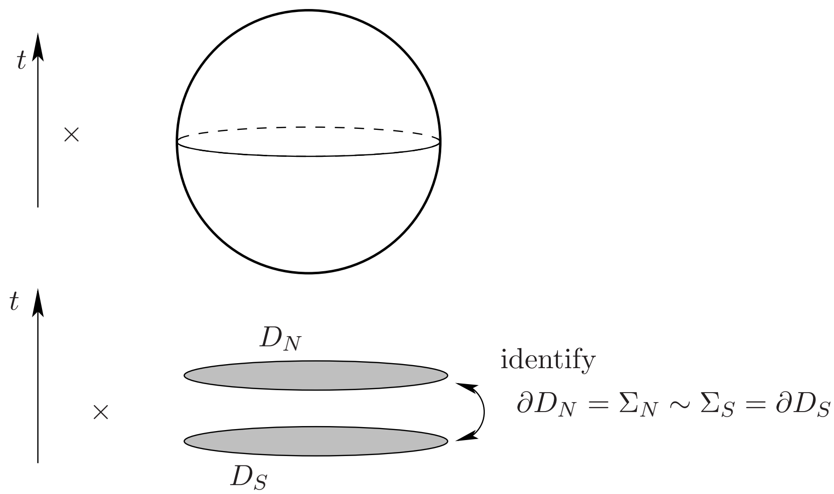


Figure 3.1: A possible choice of the spacetime.

where the term with the massive current $j^\mu = \bar\psi\gamma^\mu\psi$ couples the particle to the gauge field. It is now clear, that the last term can be written as a total derivative:

$$\mathcal{L} = \tfrac{i}{2}[\bar\psi \overleftrightarrow{\partial\!\!\!/}\psi] - m\bar\psi\psi - j\wedge a - \tfrac{1}{4} f \wedge * f - dW$$

where

$$W = a \wedge da = a_\mu \partial_\nu a_\rho \; dx^\mu \wedge dx^\nu \wedge dx^\rho .$$

This means that the Chern Simons form W affects the equation of motion only at the boundary of the system. If we use the usual gauge-convention for the $\mathcal{A}_\mu$:

$$\mathcal{A}_i(x) = \phi_0 \int dy \; (\partial_i \theta(x-y)) \mathbf{:}\psi^+(y)\psi(y)\mathbf{:} \quad \text{for } i = 1,2$$

and $\mathcal{A}_3 = 0$, $\mathcal{A}_0$ not yet determined, the question arises when this action is nonzero. Indeed we have to choose suitable spacetime region and take the integral on the boundary. For example take a four-ball in $\mathbb{R}^4$, thus let the boundary to be the S^3. For small time intervals we may consider the space $\mathbb{R}\times S^2$ otherwise take a time-like interval figure 3.1 and furthermore to be more realistic the two sphere is jolted such that we may consider two discs: D_N associated to the northern hemisphere (with boundary Σ_N) and D_S associated to the southern hemisphere also with boundary Σ_S. By gluing the boundaries together, identify Σ_N with Σ_S, we will reobtain topologically the S^2. Instead of the two sphere we can also choose a cylinder, the result remains the same. We have now two integrals, which we require not to cancel each other but in fact they would if we consider the above gauge and $\mathcal{A}_0|_{D_N} \equiv \mathcal{A}_0|_{D_S}$ due to the orientation of the two-sphere.

$$S_{\mathrm{CS}} = \int_{\mathbb{R}\times D_N} dx_0 dx_1 dx_2 \; [\mathcal{A}_0 \epsilon^{ij}\partial_i \mathcal{A}_j]_{D_N} - \int_{\mathbb{R}\times D_S} dx_0 dx_1 dx_2 \; [\mathcal{A}_0 \epsilon^{ij}\partial_i \mathcal{A}_j]_{D_S} .$$

One way out of this trouble is to assume $\mathcal{A}_0|_{D_N} \neq \mathcal{A}_0|_{D_S}$ on the discs. But what happens at the boundary, the edge of the discs by gluing them together? Actually there are two possibilities. The first is that we may assume the $\mathcal{A}_0|_{(\cdot)}$ to vanish at the edge. The second is that we may consider a chiral field theory, which determines both the northern an the southern field theory, if this is possible. It is known since Chern Simons' seminal paper [CS74] that the Chern Simons theory is related to a conformal field theory on the boundary on the classical level and since the seminal paper of Witten [Wit89] it is known that this remains true on the quantum level. This is for the Chern Simons term only, the electrons or composite Fermions might then be considered as modes or string-like objects with for example Dirichlet boundary conditions on each disk (or brane). In the next chapter we will discuss the quantum-mechanically projection of four dimensional electrons, in this setup the projection of the Chern Simons term can be replaced by performing the Chern Simons transformation, the flux attachment, after the projection onto three dimensions.

Though this gauge choice seems to be somehow artificial the whole approach is useful to derive quickly a low energy effective theory with spin and relativistic corrections. This we will study in more detail in the next section.

3.2 Effective Theory and the Low Energy Limit

We start with the single particle relativistic Lagrangian of an electron coupled to an external field. In the low energy limit we are not interested in the Maxwell term since it gives only a photon background. We start with the following Lagrangian in four dimensional Minkowski space

$$\mathcal{L} = \tfrac{i}{2}[\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi] - m\bar{\psi}\psi - j \wedge A,$$

where the term with the massive current $j^\mu = \bar{\psi}\gamma^\mu\psi$ couples the particle to the electromagnetic gauge field. Since we do not have so far a two dimensional system and no quantized fluxes, we have to implement these quantize fluxes artificially. In the next chapter we propose a more constructive approach. In two dimensional systems the quantized fluxes appear as holes in a plane and are attached by the use of cohomology. Instead of holes we take cylinders or lines which we subtract from the Minkowski space. Consider for example the space $\mathbb{R}^{3,1} - R_z$, where R_z denotes the z-axis, this space has a topological one dimensional defect, which corresponds to a flux concentrated in R_z. In this way we can construct composite Fermions by attaching an even number of such fluxes or lines to an electron by a Chern Simons transformation. Then the discussion from the previous chapter can be adapted for the Chern Simons term since we can gather the paths with nontrivial homology in equivalence classes $[c(t)]$, where all paths are identified with the paths in the x-y-plane. We can then perform a 'local' Chern Simons gauge transformation

$$\psi_D \to \psi = e^{i\Theta(x)}\psi_D$$

and obtain the covariant derivative

$$D_\mu \to \mathcal{D}_\mu = \partial_\mu - ieA_\mu - i\mathcal{A}_\mu$$

and define the total gauge field

$$ea_\mu := eA_\mu + \mathcal{A}_\mu.$$

We will now concentrate on the composite Fermions for which the action principle leads to the Euler Lagrange equation of motion, the Dirac equation for electrons with an even number of fluxes attached to it:

$$i(\gamma^\mu D_\mu - m)\psi = 0$$

with the covariant derivative $D_\mu = \partial_\mu - iea_\mu$. It can then be used in the usual way to get the low energy effective Hamiltonian. For example we can apply the Foldy-Wouthuysen transformation to the relativistic equations of motion, see for example [CI06]. Therefore we define in the same way as in the usual case a unitary transformation

$$\psi \to e^{-iS}\psi'$$

and transform the Dirac equations for 'composite Fermions' in the following sense

$$i\partial_t\psi' = [e^{iS}(H - i\partial_t)e^{-iS}]\psi' =: H'\psi'.$$

The Hamiltonian has the same structure as the usual we just replaced the electromagnetic fields A_μ by the overall field a_μ,

$$H = [\boldsymbol{\alpha} \cdot (\mathbf{p} - e\boldsymbol{a}) + ea^0 + \beta m]$$

and the unitary transformation changes similarly to $S = \boldsymbol{\alpha} \cdot (\mathbf{p} - e\boldsymbol{a})$. Performing the transformations thrice we end up with the required low energy Hamiltonian

$$H_{\mathrm{LE}} = \beta\left(\frac{1}{2m}(\mathbf{p} - e\boldsymbol{a})^2 - \frac{1}{8m^3}\mathbf{p}^4 + m\right) + ea^0 \tag{3.2.1}$$

$$-\frac{e}{2m}\beta\boldsymbol{\sigma} \cdot \boldsymbol{b} - \frac{ie}{8m}\boldsymbol{\sigma} \cdot (\nabla \times \mathbf{e}) - \frac{e}{4m^2}\boldsymbol{\sigma} \cdot (\mathbf{e} \times \mathbf{p}) - \frac{e}{8m^2}(\nabla \cdot \mathbf{e}). \tag{3.2.2}$$

The term inside the parentheses corresponds to the expansion of $\sqrt{(\mathbf{p} - e\boldsymbol{a}) + m^2}$, the term ea^0 corresponds to the electrostatic energy of a point-like charge, the term $-e/(2m)\beta\boldsymbol{\sigma} \cdot \boldsymbol{b}$ corresponds to a magnetic dipole for $g = 2$ and next terms are identified with the spin orbit interaction

$$H_{\mathrm{SO}} = -\frac{ie}{8m}\boldsymbol{\sigma} \cdot (\nabla \times \mathbf{e}) - \frac{e}{4m^2}\boldsymbol{\sigma} \cdot (\mathbf{e} \times \mathbf{p}), \tag{3.2.3}$$

which reduces for $\nabla \times \mathbf{e} = 0$ and $\mathbf{e} = -\nabla a^0$ to

$$H_{\mathrm{SO}} = -\frac{e}{4m^2}\boldsymbol{\sigma} \cdot (\mathbf{e} \times \mathbf{p}) = -\frac{e}{2m}\boldsymbol{\sigma} \cdot \boldsymbol{b}' \tag{3.2.4}$$

with the magnetic field $\boldsymbol{b}' = -\mathbf{v} \times \mathbf{e}$ acting on the particle. The last term $-e/(8m^2)\nabla \cdot \mathbf{e}$ is the Darwin term and corresponds to the zitterbewegung.

Once we have derived the low energy Hamiltonian in three dimensions we can directly write down the corresponding two dimensional Hamiltonian with its relativistic corrections.

Usually we would take $ea^0 \equiv eA^0$ since the Chern Simons transformation has no influence on the static electric field in our case.

The Hamiltonian (3.2.1) can therefore directly be used to derive the Rashba spin orbit coupling term [BR84] for two dimensional systems in the case of the combined effect of spatial inversion symmetry and time inversion symmetry for example in GaAs/AlGaAs heterojunctions and also the Dresselhaus term [Dre55] for bulk inversion asymmetry of the crystal structure. This would generate a spin splitting of the composite Fermion states for example at half filling where the mean magnetic field is zero. The advantage of our considerations is that we don't have to perform the whole derivation from band structure analysis (Kane Model) since the only difference to the electron spin orbit Hamiltonian is the field $\boldsymbol{a}$, which only replaces the field $\mathbf{A}$ in the common approaches. The Rashba spin orbit Hamiltonian may be written as

$$H_{\mathrm{RSO}} = \frac{\alpha}{\hbar}(\sigma_x p_y - \sigma_y p_x) \tag{3.2.5}$$

with the Rashba spin orbit coupling constant

$$\alpha = \langle (a_{46}/e)\partial_z a^0(z) \rangle \tag{3.2.6}$$

with the parameter a_{46} derived in the Kane model (kp-model) from band structure analysis [Win03]. In [MMSK02] a Rashba spin orbit coupling term was proposed for composite Fermions leading to a contribution to the so-called zero temperature smoothing of the spin transition. Here we performed a possible derivation from a relativistic theory, which leads not only to a Rashba spin orbit term. It also motivates the Dresselhaus term in a suitable sample and additional relativistic corrections like zitterbewegung. The Dresselhaus Hamiltonian may be written as

$$H_{\mathrm{DSO}} = \frac{\beta}{\hbar}(\sigma_x p_x - \sigma_y p_y) \tag{3.2.7}$$

with the Dresselhaus coupling constant

$$\beta = -a_{42}\langle k_z \rangle \tag{3.2.8}$$

where the constant a_{42} is determined by the band structure in the (14×14) Kane model for multi-band analysis. The parameters for different III-V semiconductors are shown in table 3.1. If we assume $\langle \partial_z a^0 \rangle \approx 1\mathrm{mV/Å}$ and $\langle k_z \rangle \approx 3.6 \times 10^{-4}\mathrm{Å}^{-2}$ then the Rashba coupling in GaAs is about $\alpha_{\mathrm{Ga}} \approx 5.206\mathrm{m}e\mathrm{VÅ}$ and the Dresselhaus coupling $\beta_{\mathrm{GaAs}} \approx -9.93\mathrm{m}e\mathrm{VÅ}$. It should be mentioned that contrary to the Dresselhaus term the strength of the Rashba term can be controlled by a gate voltage, modifying the potential a^0. We may also write down the so-called spin transverse force induced by spin orbit coupling [ZRS06]. The meaning of such a force for electrons is controversially discussed [Zaw07, Bli05] and has basically heuristic meaning for a possible spin Hall effect. In the Dirac equation such a force is always zero while in the low energy limit it is argued via the Heisenberg equations of motion that a quantum force exists. This discrepancy is argued to depend on the possible electron positron coupling in the Dirac equation via external fields, since in the low energy Hamiltonian a

	GaAs	InAs	InSb	vacuum
a_{46}	$5.206e\text{Å}$	$117.1e\text{Å}$	$523.0e\text{Å}$	$-3.7 \times 10^{-6}e\text{Å}$
a_{42}	$27.58e\text{Å}^3$	$27.18e\text{Å}^3$	$760.1e\text{Å}^3$	0

Table 3.1: Band structure parameters for electrons in typical III-V semiconductors. From our analysis they are also valid for composite Fermions. The Rashba spin orbit coupling is six orders of magnitude higher in a crystal than in the vacuum and the Dresselhaus spin orbit coupling is zero in the vacuum.

positron solution is not included. However a satisfying proof of the existence/nonexistence of a force acting only on the electron solution in the Dirac equation is still missing even perturbatively. The same discussion might appear also in the case of composite Fermions. It would be interesting to detect effects or contributions to the Hall current in terms of spin currents or even a spin Hall effect for composite Fermions at least at even denominator filling for example by varying a suitable gate voltage.

3.3 Chern Simons Gauge Theory and BRST Cohomology

The construction of the QED (Tomaga, Schwinger, Feynman and Dyson) as a quantized gauge theory has successfully been developed in the last century, using for example the path integral approach. For the generalization to non-abelian gauge theories, the Yang-Mills theories, however some new concept were needed and have been formulated by Faddeev and Popov [FP67]. By introducing new fields, so called Faddeev-Popov ghost fields, in the Lagrangian they were able to define a unitary S-Matrix and thus Feynman rules. But this construction breaks gauge covariance and this problem then solved Becchi, Rouet, Stora [BRS76] and Tyutin [Tyu]. They constructed a Lagrangian with BRST ghost fields, where the new fields have to satisfy a new global and nilpotent symmetry, later called BRST-symmetry. Now the problem of the breaking of gauge invariance due to renormalization [PS95] is then controlled by the BRST-cohomology classes of the BRST-operator [BBH00].

In a field theory the equations of motion require that the solution depends uniquely on the initial data, in other words the Cauchy problem has to be unique. To obtain a unique Cauchy problem the gauge is fixed usually. The problem now is that the equations of motions are no more invariant under gauge transformations. The way out of this trouble is introducing the BRST ghost fields, where now the Lagrangian satisfies a new global symmetry, the BRST symmetry. The new BRST symmetry replaces in this sense the gauge symmetry.

In this section we will recall the BRST quantization of Chern Simons theories with semi simple compact gauge group as gauge group on a suitable differential three manifold. Since this mechanism requires some mathematics we introduce also the mathematical description of gauge theories. Since on the mathematical side the Chern Simons theory is strongly connected to the theory of Chern numbers or Pontryagin numbers respectively we also introduce the framework of characteristic classes. In the quantum Hall effect the topological invariants i.e. Chern numbers play a central role. For example in the integral quantum Hall effect the conductivity is directly connected to the first Chern number, while the appearance of a Chern Simons functional in the fractional Hall effect suggests for the consideration of the second Chern number or Pontryagin number respectively. At the end of this section we apply this quantization method in the perturbative Chern Simons theory and examine the connection with the phenomenology of composite Fermions with spin in a fractional Hall state at least at even denominator filling.

3.3.1 Mathematical Description of Topological Gauge Theories

We will recall briefly the basic background for the mathematical framework of gauge theories, for more details we refer the interested reader to [MG89] and [CN88].

Principal Fiber Bundles

A principal G bundle P is a fiber bundle where each fiber is isomorphic to a Lie group G, this means that P is a differential manifold, which is locally the product of an open set $U \subset M$ in the base manifold M and a Lie group G, $U \times G \subset P$. Roughly speaking at each point p on the manifold M there is a fiber F_p attached, which is isomorphic to the gauge group G,

$F_p \cong G$. There exists a surjective projection map $\pi : P \to M$, such that

$$\pi^{-1}(p) = F_p \cong G.$$

The fiber F_p is the orbit of the left (or right) action of the gauge group at a point p. From $p = (\pi(u))$ it follows that $F_p = \{ug | g \in G\}$. The manifold M can be covered with topologically trivial open sets $\{U_i\}$. The bundle restricted to a topological trivial subset is trivial and each $u = \pi^{-1}(p)$ can be described by the coordinates $(x \in U_i, g \in G)$ in the direct product $U_i \times G$. If U_i covers the whole manifold M then P is trivial and it is local trivial otherwise. There is a further definition of triviality in terms of sections. A local section s_i is a smooth mapping $s_i : U_i \to \pi^{-}(U_i)$ such that

$$(s^{-1} \circ \pi)(p) = id_{F_p},$$

the choice of the local sections s is a choice of a local coordinate system. The transition from one local section to another has the meaning of a gauge transition. Two different local sections s and $\tilde{s}$ are connected by the map $g : U \to G$ via the equation

$$s(p) = \tilde{s}(p)g(p).$$

Thus the change of a local section corresponds to a gauge transformation and therefore the choice of a local section corresponds to the choice of a local gauge. Two local sets (s_i, U_i), (s_j, U_j) with $p \in U_i \cap U_j$ are connected by the equation:

$$s_j(p) = s_i(p){t^i}_j(p), \quad {t^i}_j \in G$$

however now with the transition function ${t^i}_j : U_i \cap U_j \to G$ satisfying the compatibility relations:

$$\begin{aligned} t_{ii}(p) &= id_G, \quad p \in U_i && (3.3.1)\\ t_{ij}(p) &= t_{ji}^{-1}(p), \quad p \in U_i \cap U_j && (3.3.2)\\ t_{ij}(p)t_{jk}(p) &= t_{ik}(p), \quad p \in U_i \cap U_j \cap U_k. && (3.3.3) \end{aligned}$$

The last line is also called the cocycle, fixing the cocycle corresponds then to the choice of a gauge. If the bundle is (global) trivial then all transition functions can be chosen to be the identity.

Matter Fields

Formally matter fields are mathematically defined as maps from the bundle P to a vector space V:

$$\begin{aligned} \phi : \quad & P \to V && (3.3.4)\\ & \phi \mapsto \phi(ug) = \pi_V(g^{-1})\phi(u) && (3.3.5) \end{aligned}$$

with π_V being a representation of the gauge group G in the vector space V. A local representation we obtain by a local section

$$\phi_i(x) = (s_i^* \phi)(p) = \phi(s_i(p)).$$

Parallel Transport, Covariant Derivative and Field Strength

We want to compare the matter fields at different points on our manifold therefore we have to define a parallel transport (connection) on our principal G bundle. So let T_uP be the tangent space at $u \in P$, we divide the tangent space in a so called horizontal and vertical part:

$$T_uP = V_uP \oplus H_uP$$

with V_uP being tangent to the fibre $\pi^{-1}(\pi(u)) \cong G$ and thus being isomorphic to the Lie algebra $Lie(G)$ of the structure group G. With the mapping $u \mapsto ug,\ g \in G$ the horizontal subspace H_uP is identified with the horizontal subspace at $H_{ug}P$. We may define the vertical space V_uP as a projection from the tangent space T_uP. The projection map is given by a Lie algebra valued connection one form ω, which is smooth on all of P. The horizontal subspace can then be defined by

$$H_uP := \{\xi \in T_uP | \omega(\xi) = 0\}.$$

In physics the gauge fields are called gauge potentials and correspond to the connection one form, usually given in local coordinates. The connection one form ω however is globally defined on P. Pulling back the connection one form ω with a local section s (chose a coordinate system) gives the local connection one form A_i on each open covering set $U_i \subset M$:

$$A_i := s_i^* \omega,$$

in this sense A_i is a $Lie(G)$ valued one form on $U_i \times G$. If s is a global section then the bundle is globally trivial and A becomes a global connection. With the compatibility relations (3.3.1) we can transform the local connection A_i and A_j on $U_i \cap U_j$ by

$$A_j = t_{ij}^{-1} A_i t_{ij} + t_{ij}^{-1} dt_{ij}. \tag{3.3.6}$$

This is known to be a gauge transformation on the other hand a gauge transformation is also given by the choice of a different section $\tilde{s}_i = s_i g,\ \tilde{s}_i, s_i \in U_i$ and $g \in G$ by

$$\tilde{A}_i = g^{-1} A_i g + g^{-1} dg. \tag{3.3.7}$$

The exterior derivative d gives the variation of a field on M, however the variation on the fiber bundle P is given by the covariant derivative

$$D := d + \omega \tag{3.3.8}$$

and the curvature form Ω can be obtained by applying the covariant derivative to the connection one form:

$$\Omega = D\omega = d\omega + \omega \wedge \omega. \tag{3.3.9}$$

In physics the local curvature is called field strength and is a Lie algebra valued two form, which can be recovered with the pull-back of the curvature

$$F_i = s_i^* \Omega$$

with the local section (s_i, U_i) we have

$$F_i = dA_i + A_i \wedge A_i$$

and may define the local covariant derivative

$$D_i = d + A_i.$$

A gauge transformation of the field strength is given by

$$\begin{aligned} F_j &= t_{ij}^{-1} F_i t_{ij}, \text{ on } U_i \cap U_j && (3.3.10) \\ \tilde{F}_i &= g^{-1} F_i g, \text{ with } s, \tilde{s} \in U_i. && (3.3.11) \end{aligned}$$

We now want to define a mapping Γ from one fiber $F_p = \pi^{-1}(p)$ to another fiber $F'_{p'} = \pi^{-1}$, $\Gamma : F_p \to F'_{p'}$. Let $c : [0,1] \to M$ be a smooth path on M with $c(0) = p$ and $c(1) = p'$. The path γ in the principal bundle P is defined by $\gamma = \pi^{-1}(c)$ and is called horizontal lift of the path c. The tangent vectors $[\dot{\gamma}](u) \in H_u P$ are horizontal. The parallel transport is defined as the map

$$\Gamma(\gamma) : \pi^{-1}(p) \to \pi^{-1}(p'). \tag{3.3.12}$$

More precisely given a curve c and a fixed $u \in \pi^{-1}$ there exists only one horizontal lift γ, which fixes then also $u' \in \pi^{-1}(p')$. Locally the parallel transport is given via the path-integral along the path c.

$$u' = s_i(p') \mathcal{P} e^{-\int_{c(0)}^{c(1)} A_i}.$$

The curvature can be recovered from the parallel transport along a closed path c. The difference between initial and end point of the corresponding horizontal lift γ is a measure for the curvature. The horizontal lift is only a closed curve for vanishing curvature thus the parallel transport called Wilson-loop

$$W_c = \mathcal{P} e^{-\int_c A} \tag{3.3.13}$$

is then zero in this case.

Topology and Fiber Bundles

The construction of a principal G fiber bundle from a given manifold M and a given structure group G is not unique. Given an open covering $\{U_i\}$ of the manifold M the fibers $\pi^{-1}(U_i)$ are glued together corresponding to the choice of the transition functions t_{ij}. Since the choice of the t_{ij} is not fixed the bundles are not uniquely defined. This issue is addressed by the classification of bundles by their topology, which characterizes the bundle structure. The task is to divide the bundles into equivalence classes or characteristic classes. The theory of characteristic classes is due to Chern and Weil, they generalized the Gauss-Bonnet theorem to principal bundles.

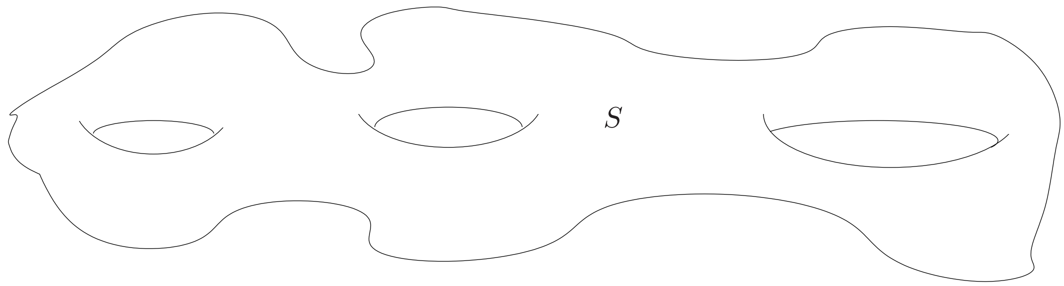


Figure 3.2: A closed manifold S of genus three $g = 3$.

Gauss-Bonnet Theorem

Let S be a closed oriented Riemannian manifold and let K be the Gauss curvature and $\chi(S)$ the Euler number of the manifold S, then the following relation holds:

$$\frac{1}{2\pi}\int_S K\ dA = \chi(S) \tag{3.3.14}$$

for example consider the case in figure 3.2 here we have the Euler number $\chi(S) = 2-2g = -4$ with g being the genus of the oriented closed manifold S. If the manifold $\tilde{S}$ is only compact and oriented then the formula (3.3.14) changes to

$$\frac{1}{2\pi}\int_{\tilde{S}} K\ dA - \int_{\partial\tilde{S}} k_g\ ds = \chi(S) \tag{3.3.15}$$

which means that the geodesic curvature contributes on the boundary and can be defined for any loop on the surface $\partial\tilde{S}$ see figure 3.3 . This can be viewed as some intrinsic geometric property which we can integrate out and are left with a pure topological number, the Euler number. In the Chern Simons Theory we have some extrinsic geometric property reflected in a compact gauge group G say $SU(N)$ which we may integrate out and are then also left with some pure topological properties.

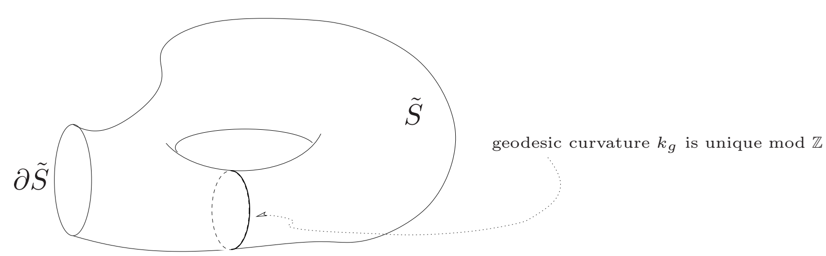


Figure 3.3: A compact manifold $\tilde{S}$ with boundary $\partial\tilde{S}$.

3.3.2 Characteristic Classes

The topology of a fiber bundle plays a crucial role in topological gauge field theories, therefore we need a tool to quantify and qualify the bundles by their topology. The Chern Weil theory extends the Gauss-Bonnet theorem to vector bundles where roughly speaking the Gaussian curvature is replaced by the Pfaffian of the corresponding curvature.

Characteristic classes are a suitable tool to identify with each vector bundle $E \to M$ over a differential manifold M a cohomology class in the de Rham cohomology $H^*(M, \mathbb{R})$, which measures the non-triviality of the vector bundle. In order to fix the mathematical framework some formal definitions and remarks are needed for a more detailed discussion we refer to [Gil95] and there exist also a chrash course [Bae] .

Chern Classes

We consider only complex vector bundles $E \to M$ and set the Lie group to $G = GL(n, \mathbb{C})$ and its Lie algebra to $\mathfrak{g} = Mat(n, \mathbb{C})$. A polynomial function $P : \mathfrak{g} \to \mathbb{C}$ is called invariant polynomial or G-invariant, if for all $X \in \mathfrak{g}$ and all $T \in G$

$$P(TXT^{-1}) = P(X). \tag{3.3.16}$$

For example the trace over finite matrices is invariant under unitary transformations of the matrices. Another example is the determinant. To (3.3.16) equivalent is

$$P(XYZ) = P(ZXY)$$

for all $X, Y, Z \in \mathfrak{g}$.

Let ∇ be the connection (covariant derivative) on E and R the corresponding (Riemannian) curvature tensor. On subsets $U \subset M$ one can always find trivialisations, n smooth sections $s_1, \dots, s_n$ of E on U, such that they are linearly independent on U. Since $R(X, Y)$ is antisymmetric in X, Y we can define a $\mathbb{C}$-valued 2-form Ω_i^j on U such that

$$R(X, Y)s_i =: \sum_j \Omega(X, Y)s_j.$$

Let then Ω be a $\mathfrak{g}$-valued 2-form, or a matrix-representation of a commutative $\mathbb{C}$-algebra of the even $\mathbb{C}$-valued forms. Ω is denoted as the curvature matrix of ∇ with respect to $s_1, \dots, s_n$. It can be shown, that $P_\nabla := P(\Omega)$ is a well defined form on M. Let ω be the connection matrix of ∇ with respect to the frame $s_1, \dots s_n$ with $\nabla_X s_i := \sum_j \omega(X)_i^j s_j$. Then

$$\Omega = d\omega - \omega \wedge \omega \quad \text{and} \quad d\Omega = \omega \wedge \Omega - \Omega \wedge \omega \text{ (Bianchi-identity)}$$

and P_∇ is closed; $dP_\nabla = 0$. This determines a cohomology class:

$$[P_\nabla] \in H^{\text{even}}(M, \mathbb{C}) \ .$$

For a matrix $(A_j^i) \in \mathfrak{g}$ we write for derivative $P'(A)_j^i := \frac{\partial P}{\partial A_j^i}$ simply $P'(A) := P'(A)_j^i$. Let $P(A) := \det(1 + \frac{1}{2\pi i}A)$ then

$$c(E) := [P(E)] \in H^{\text{even}}(M, \mathbb{C})$$

is called the total Chern class. Is A diagonal $A = \text{diag}\,(\lambda_1, \dots, \lambda_n)$, then

$$P(A) = \det(1 + \tfrac{1}{2\pi i}A) = \prod_{j=1}^{n}(1 + \tfrac{1}{2\pi i}\lambda_j) \tag{3.3.17}$$

$$= \sum_{k=0}^{n} \sigma_k(1 + \tfrac{1}{2\pi i}\lambda_1, \dots, 1 + \tfrac{1}{2\pi i}\lambda_n). \tag{3.3.18}$$

Here σ_k is the kth elementary-symmetric function. Is A diagonalizable and λ_j are the eigenvalues we obtain the same result. Diagonalizable matrices are a dense subset in $\mathfrak{g}$ due to Jordan normal-form and furthermore P is continuous. It follows then that this formula is valid for all $A \in \mathfrak{g}$. We set

$$P_k(A) := \sigma_k(1 + \tfrac{1}{2\pi i}\lambda_1, \dots, 1 + \tfrac{1}{2\pi i}\lambda_n) = \left(\frac{1}{2\pi i}\right)^k \sigma_k(\lambda_1, \dots, \lambda_n).$$

To be more explicit let $F = s^*\Omega$ be the curvature two form or field strength and $F^i_j = s^*\Omega^i_j$ the corresponding matrix representation then

$$\begin{aligned}
P_0 &= 1 \\
P_1 &= \tfrac{1}{2\pi i}\mathrm{tr}(F) \\
P_2 &= \tfrac{-1}{8\pi^2}(\mathrm{tr}(F \wedge F) - \underbrace{(\mathrm{tr}F) \wedge (\mathrm{tr}F)}_{=0 \text{ for } G=SU(N)} \\
&\vdots \\
P_n &= \left(\frac{1}{2\pi i}\right)^n \det(F)
\end{aligned}$$

and call $c_k(E) = [P_k(\Omega^E)] \in H^{2k}(M, \mathbb{C})$ the k-th Chern class of E and $c(E) = c_0(E) + \dots + c_n(E)$ the total Chern class. Furthermore it can be shown that the Chern class is real valued:

$$c(E) \in H^{\text{even}}(M, \mathbb{R}) \subset H^{\text{even}}(M, \mathbb{C})$$

and also that; if E is trivial, then $c(E) = 1 \in H^0(M, \mathbb{R})$.

Pontryagin Classes

To define characteristic classes for real vector bundles we first let $E = F \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of a real vector-bundle F. Then the dual vector bundles are isomorphic $F^* \cong F \Rightarrow E^* \cong E$. From this it follows that $c_k(E) = c_k(E^*) = (-1)^k c_k(E)$ and thus $c_k = 0$ for k odd.

Definition 3.3.1. Let $F \to M$ be a real vector-bundle. Then

$$p_j(F) := (-1)^j c_{2j}(F \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4j}(M, \mathbb{R})$$

is called the j-th Pontrjagin class of F and

$$p(F) = \sum_j p_j(F) \in H^{4j}(M, \mathbb{R})$$

the total Pontryagin-class of F.

Pontryagin Index

We now consider the case of $SU(N)$ bundles on a four dimensional manifold M. In this case it turns out that all Chern classes are trivial except the second one:

$$c_2[F] = -\frac{1}{8\pi^2}\mathrm{tr}(F \wedge F). \tag{3.3.19}$$

Since the second Chern class is closed on a four manifold it can be locally written as a total derivative

$$c_2[F] = -\frac{1}{8\pi^2}dK[A, F] \tag{3.3.20}$$

and the three form K we call local Chern Simons form of c_2 and in physics we call it topological current:

$$K[A, F] = \mathrm{tr}(F \wedge A - \frac{1}{3}A \wedge A \wedge A). \tag{3.3.21}$$

The Pontryagin index is defined as the integral over the second Chern class

$$\nu[F] = \int_M c_2[F] = -\frac{1}{8\pi^2}\int_M \mathrm{tr}(F \wedge F). \tag{3.3.22}$$

On $M = \mathbb{R}^4$ the Pontryagin index is related to the winding number. Assume that the field strength F vanishes at infinity, thus at the boundary ∂M. In this case the connection or gauge potential becomes a pure gauge

$$A \to A|_{\partial M} = gdg^+. \tag{3.3.23}$$

This defines a map from the boundary of the manifold to the gauge group $g : \partial M \cong S^3 \to G = SU(N)$ and is therefore classified by the homotopy group $\pi_3(SU(N)) = \mathbb{Z}$. In the case of $N = 2$ the homotopy group $\pi_3(SU(2))$ corresponds to the winding number w of the map g since $SU(2) \cong S^3$. In the case of $N > 2$ we can deform g such that it maps the boundary only in the subgroup $SU(2) \subset SU(N)$ and the winding number remains well defined. The winding number is defined by the map

$$w[g] := -\frac{1}{24\pi^2}\int_{S^3} \mathrm{tr}(gdg^+ \wedge gdg^+ \wedge gdg^+) \tag{3.3.24}$$

and correspond to the Pontryagin index $\nu[F] = -w[g]$ if the Gauge field becomes a pure gauge on S^3.

3.3.3 Classical Chern Simons Theory

In the seminal paper of Shing-Shen Chern and James Simons [CS74] the nowadays called Chern Simons term arises as a nontrivial boundary term. The expression of the Pontrjagin index of a four manifold is turned via Stokes theorem into an integral over a simplex, the boundary. It emerges that the integral over the second Chern character gives precisely the

Pontyagin index, which is an expression for the winding number. Later we also consider the Chern Simons term on three manifolds which are not the boundary of a four manifold.

Since the discovery of Chern and Simons the now called Chern Simons theory became an interesting research field not only in mathematics but also in physics for instance in terms of nonabelian gauge theories in three dimensions or in term of instantons [Hoo74]. It has a rather rich mathematical structure with connections to various other fields of mathematics even on the classical level [Fre95].

Let M be a compact, oriented three dimensional manifold without boundary. We consider global trivializeable principal G bundles P over M, where G is a compact, simply connected Lie group and $\mathfrak{g}$ the corresponding Lie algebra, for example for $G = SU(2)$ each principal-G bundle is trivial. Furthermore let ω be the $\mathfrak{g}$-valued connection form on P. We choose global trivializations $s : M \to P$ and identify with $A := s^*\omega \in \mathscr{A} := \Omega^1(M, \mathfrak{g})$ with $A = A_\mu(x)dx^\mu$ and $A_\mu(x)$ being the gauge field. The covariant derivative is then given by

$$D^A := d + \frac{1}{2}[A, \cdot]_\wedge : \Omega^k(M, \mathfrak{g}) \to \Omega^{k+1}(M, \mathfrak{g})$$

and the curvature is given by

$$F(A) := s^*\Omega = dA + A \wedge A.$$

We call a connection α flat if $F(\alpha) = 0$ and for flat connections it follows that $(D^\alpha)^2 = 0$. The group of all gauge transformations is defined by $\mathcal{G} := \mathcal{C}^\infty(M, G)$ acting from the left on a connection

$$g^*A := g^{-1}Ag + g^*\theta = g^{-1}Ag + g^{-1}dg$$

where θ is the left-invariant Maurer-Cartan Form. From (3.3.7) and (3.3.10) the gauge transformation for the field strength or curvature is given by

$$\tilde{F}(\tilde{A}) = g^{-1}F(A)g = F(g^*A)$$

The Chern Simons form (3.3.21) $L_{\text{CS}} \in \Omega^3(M, \mathfrak{g})$ on the principal G bundle can be defined by

$$\mathcal{L}_{\text{CS}} := s^*L_{\text{CS}} = s^*(\text{tr}[\omega \wedge \Omega - \tfrac{1}{6}\omega \wedge \omega \wedge \omega]) \tag{3.3.25}$$

$$= \text{tr}[A \wedge dA + \tfrac{2}{3}A \wedge A \wedge A] \tag{3.3.26}$$

and the Chern Simons functional can be defined by

$$S_{\text{CS}} := \frac{k}{8\pi^2}\int_M \text{tr}[A \wedge dA + \tfrac{2}{3}A \wedge A \wedge A],$$

where the normalization factor in front of the integral is due to the volume of the S^3 [L.H96]. Under global gauge transformations the action transforms on the manifold M with $\partial M \neq \emptyset$

$$S_{\text{CS}}(g^*A) = S_{\text{CS}} + \frac{1}{8\pi}\int_{\partial M} \text{tr}(a \wedge dgg^{-1}) - \int_M g^*\sigma.$$

In the case where $\partial M = \emptyset$ the boundary term in the midle vanisches. $\sigma = 2/3\ \mathrm{tr}\theta \wedge \theta \wedge \theta \in \Omega^3(M, G)$ is the canonical three-form of G and its integral

$$\int_M g^*\sigma$$

is an integer and corresponds to the Pontryagin index (3.3.22), or the windingnumber (3.3.24) respectively, and its cohomology class is integral $[\sigma] \in H^3_{\mathrm{dR}}(G)$.

Infinitesimal gauge transformations with $g = \exp\{tf\}$ for any $f \in \omega(M, \mathfrak{g})$ can be obtained by a power expansion:

$$g^*A = A + t(D^A f) + \mathcal{O}(t^2)$$

The critical points of the Chern Simons functional are exactly those A where the curvature is flat. Let $A_t = A + ta$ be a family of connections then the Chern Simons functional

$$\begin{aligned} S_{\mathrm{CS}}(A_t) &= S_{\mathrm{CS}}(A) + \frac{1}{8\pi^2}\int_M t\ d(\mathrm{tr}(A \wedge a)) + \frac{1}{8\pi^2}\int_M 2t\ d\mathrm{tr}(F(A) \wedge a) + \mathcal{O}(t^2) \\ &= S_{\mathrm{CS}}(A) + \frac{t}{4\pi^2}\int_M \mathrm{tr}(F(A) \wedge a) + \mathcal{O}(t^2) \end{aligned} \tag{3.3.27}$$

is critical on $(M, \partial M = \emptyset)$, $\delta_t S_{\mathrm{CS}}(A_t) = 0$ iff $F_A = dA + A \wedge A = 0$ for any connection a and $t \in \mathbb{R}$. If $a = D^A f$ then we immediately see that the Chern Simons functional on a closed manifold is invariant under infinitesimal gauge transformations for flat connections $A = \alpha$ and under global gauge transformation it is invariant modulus $\mathbb{Z}$, compare with (3.3.28). The flat connections α are thus the classical solutions of the Euler Lagrange equations of motions.

$$\frac{\delta S_{\mathrm{CS}}}{\delta \alpha} - D^\alpha \frac{\delta S_{\mathrm{CS}}}{\delta(D^\alpha \alpha)} = 0 \Leftrightarrow F_\alpha(\alpha) = 0$$

We fixed a specific trivialization, in the sense that the equations depend on the choice of the s_i. It means that we have chosen a specific flat connection for gauge fixing. This breaks gauge invariance. Physically this would be a choice of a constant background field, a one form, which we can set to zero $s^*\alpha = 0$.

3.3.4 Quantizing Chern Simons Theories

There are several ways to quantize Chern-Simons theories, E. Witten for example related the partition function

$$Z_k(M) = \int_{\mathscr{A}_M} \mathcal{D}A\ e^{2\pi i k\ S_{\mathrm{CS}}(a)}$$

which is apriori not well defined on the infinite-dimensional Space $\mathscr{A}_M$, to the Jones theory of knots [Wit89]. Beside this theory there exist perturbative approaches for example in flat $\mathbb{R}^3$ by Birmingham et.al.[BRT90] and Deluc [DGS89] and for compact manifolds there exists the works by Axelrod and Singer [AS91]. In this section we are interested in the perturbative treatment of Chern Simons theories since these arise precisely in the perturbative treatment of composite Fermions in a fractional Hall state at least at even denominator filling in the

Random phase approximation. We will follow mainly [AS91] and [Koh02] respectively. There are many motivations to consider not only $U(1)$ fields but also more general $U(N)$ or $SU(N)$ fields. In terms of bi-layer Hall Systems there arises a further $SU(2)$ symmetry also in terms of the Θ-matrix approach (2.3.109) for composite Fermions with spin and it may also play a role in the formalism of second generation composite Fermions. On the other hand the composite Fermion theory projected in the lowest Landau level leads to a Chern Simons theory on a noncommutative space which may be treated with the Seiberg Witten map [SW99] and may be mapped onto a nonabelian $U(N)$ Chern Simons theory an a commutative spacetime. We will comment on that in the next chapter.

The equations of motion have to be unique regarding the Cauchy data otherwise the gauge symmetry would be degenerated. Therefore we have to fix the gauge. In [BRT90] it is observed that in Landau gauge there exists an additional Abelian supersymmetry and the preservation of this symmetry fixes the Lorentz structure of the gauge propagator. We will restrict or discussion to the Landau gauge. The action

$$S_{\mathrm{CS}}(a) = \int_M \mathrm{tr}(a \wedge da + \frac{2}{3} a \wedge a \wedge a)$$

is constant along the gauge path of the flat connection α. For infinitesimal transformations

$$g^* \alpha = \alpha + t D^\alpha f + \mathcal{O}(t^2)$$

around flat connections we see from (3.3.27) that the Chern Simons action is constant

$$S_{\mathrm{CS}}(g^* \alpha) = S_{\mathrm{CS}}(\alpha) + \frac{t}{4\pi^2} \int_M \mathrm{tr}(F(\alpha) \wedge D^\alpha f) + \mathcal{O}(t^2) = S_{\mathrm{CS}}(\alpha) + \mathcal{O}(t^2) \ . \tag{3.3.28}$$

Thus the gauge symmetry leads to a degeneracy which has to be repaired by fixing the gauge. First we look at the finite dimensional case. Let Γ be a l dimensional Lie group acting free from the left on $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be invariant function $f(\gamma.x) = f(x)$. Then choose a fixing function $F : \mathbb{R}^n \to \mathbb{R}^l$ having exactly one zero on the path $\Gamma.x$. For any point x we consider a function $\phi_x : \Gamma \to \mathbb{R}^l,\ \gamma \mapsto F(\gamma.x)$ which serves as a parametrization. The volume form at a point x is then given by $\mathrm{vol}_{\Gamma.x} = \det(D\phi_x|_1)\mathrm{vol}_\Gamma$. This means that we take the integral over $\mathbb{R}^n$ of $f(x)$ along the paths of Γ and replace the integral over each path by the contributions at the zeros of F on this path multiplied by the volume of the path. This is then the same as the functional determinant of ϕ_x multiplied by the constant group volume. Picking the zeros can be described by delta functions $\delta(F(x))$ and the group volume is a constant c. The following should hold true:

$$\int_{\mathbb{R}^n} d^n x\ e^{ikf(x)} = c \int_{\mathbb{R}^n} d^n x\ e^{ikf(x)}\ \delta(F(x)) \det(D\phi_x|_1)$$

and $\delta(F(x)) \det(D\phi_x|_1)$ is independent of the choice of the gauge F. Roughly speaking a gauge transformation in $\delta(F(x))$ is compensated by $|\det(D\phi_x|_1)|$ which can be seen by the relation $\delta(h(x)) = \delta(x)/|h'(x)|_{x=0}|$. It should be mentioned that there exists no proof of these hypotheses for the infinite dimensional case. The Dirac delta function can be represented as:

$$\delta(F(x)) = \frac{1}{(2\pi)^l} \int_{\mathbb{R}^l} d^l \phi\ e^{iF(x)\phi}$$

the Jacobian can be expressed in terms of Grassmann variables, thus we introduce $\mathbb{R}^l$ valued Grassmann numbers $c, \bar{c}$ with

$$\begin{aligned} & \{c,c\} = \{\bar{c},\bar{c}\} = 0 \quad \text{and} \quad [c,\bar{c}] = 0. \\ \Leftrightarrow \quad & c_i c_j = -c_j c_i, \quad \bar{c}_i \bar{c}_j = -\bar{c}_j \bar{c}_i, \quad \text{and} \quad c_i \bar{c}_j = \bar{c}_j c_i. \end{aligned}$$

For any matrix $M \in GL(n \times n, \mathbb{R})$ the Berizin integral is given by

$$\int d^l \bar{c} d^l c \; e^{\bar{c} M c} = \det M$$

this holds also true for $M = iD\phi_x|_1$. We define the gauge fixing potential as

$$f^{\mathrm{gf}}(x, \phi, c, \bar{c}) := iF(x)\phi + i\bar{c}(D\phi_x|_1)c \; .$$

Then the following hypothesis should hold true:

$$\int d^n x \; e^{ikf(x)} = c \int_{\mathbb{R}^n} d^n x d^l \phi d^l \bar{c} d^l c \; e^{ik(f(x) + f^{\mathrm{gf}}(x,\phi,c,\bar{c}))}.$$

Now we want to perform the transition from the finite dimensional integral to the infinite dimensional integral (over the space $\mathscr{A}_M$), which is known as Fadeev-Popov-method. The Lie group Γ is replaced by the infinite dimensional Lie group G. First we choose a gauge fixing function, in favour to our problem. We choose the Lorentz gauge:

$$F(a) := d * a = \partial_\mu a^\mu \; dx^1 \dots dx^3 = 0$$

and we assume that this fixing function has only one zero on each gauge path $d*a = 0$. Then we introduce in analogy $\phi \in \Omega^0(M, Lie(G))$ and Grassmann fields $c, \bar{c} \in \Omega^0(M, Lie(G))$. The Jacobian of the function

$$\phi_a(g) = F(g^* a) = d * g^* a$$

we obtain by picking the linear term in

$$\phi_a(e^{tf}) = d * a + t(d * D^a f) + \mathcal{O}(t^2),$$

the expansion around the unity in G:

$$D\phi_a|_1 = d * D^a \; .$$

Thus we have the gauge fixing potential in Lorentz gauge

$$S^{\mathrm{gf}}(a, \phi, c, \bar{c}) = \frac{1}{8\pi^2} \int_M \mathrm{tr}\big((d * a) \wedge \phi + \bar{c} \wedge (d * D^a c)\big).$$

However there is an additional freedom to add the term $(-\alpha F(a)^2)$, with α being the Yang Mills gauge fixing parameter. Thus the gauge fixing potential is

$$S^{\mathrm{gf}}(a, \phi, c, \bar{c}) = \frac{1}{8\pi^2} \int_M \mathrm{tr}\big((d * a) \wedge \phi + \bar{c} \wedge (d * D^a c) - \alpha F(a)^2\big). \tag{3.3.29}$$

In [BRT90] it is observed that in Landau gauge $\alpha = 0$ there exists an additional abelian supersymmetry and the preservation of this symmetry fixes the Lorentz structure of the gauge propagator. In the following we will restrict our discussion to the Landau gauge since the additional term is zero in Lorentz gauge $F(a) = 0$ anyway.

Despite the fact that there is no derivative of the field ϕ, which means that it gives no contribution to physical processes, the Grassmann fields do contribute and have physical effects. For M having no boundary the Leibnitz rule gives

$$\int_M \mathrm{tr}(\bar{c} \wedge (d * D^a c)) = -\int_M \mathrm{tr}(d\bar{c} \wedge (*D^a c)) = \int_M \mathrm{tr}((*d\bar{c}) \wedge dc - d\bar{c} \wedge [*a, c])$$

and thus a kinetic term $(*d\bar{c}) \wedge dc = \partial_\mu \bar{c} \partial^\mu c dx^0 \dots dx^2$. We have the following path integral:

$$Z_k(M) = \int \mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}\phi\mathcal{D}a \; e^{2\pi i k(S_{\mathrm{CS}}(a)+S^{\mathrm{gf}}(a,\phi,\bar{c},c))}. \tag{3.3.30}$$

In this partition function there are unphysical field combinations due to the introduction of the BRST fields. They appear to have negative norm in the underlying Hilbert space of the quantized theory. Thus we can introduce an equivalence relation the so-called BRST cohomology, which identifies these states with zero.

To construct the BRST cohomology two structures are needed. First we introduce the ghost number gh with $\mathrm{gh}(a) = \mathrm{gh}(\phi) = 0$ and $\mathrm{gh}(c) = -\mathrm{gh}(\bar{c}) = 1$. The ghost number is additive under products of fields and constant under derivations. The second structure is the BRST operator $\hat{\mathrm{s}}$ which is defined by

$$\hat{\mathrm{s}}(a) = -D^a c, \quad \hat{\mathrm{s}}(\phi) = 0$$

and

$$\hat{\mathrm{s}}(c) = \tfrac{1}{2}[c, c], \quad \hat{\mathrm{s}}(\bar{c}) = \phi \; .$$

Then we can show that $\hat{\mathrm{s}}^2 = 0$ and $\{\hat{\mathrm{s}}, d\} = 0$. The physical actions have ghost number zero and so has the gauge fixing potential and can therefore be added to the action. Furthermore it is the image of the BRST operator applied to

$$\hat{\mathrm{s}}\Big(\int_M \mathrm{tr}(d * a \wedge \bar{c})\Big) = S^{\mathrm{gf}}.$$

The physical Hilbert space with no norm-negative states is now given as the cohomology $H(\hat{\mathrm{s}}) = \ker(\hat{\mathrm{s}})/\mathrm{im}(\hat{\mathrm{s}})$.

3.3.5 Perturbative Chern Simons Theory

We now want to apply the formalism of BRST quantization to the example of a $SU(2)$ Chern Simons action, for the formalism see [Koh02]. This action arises for example in the case of a bilayer system where both layers are coupled. Another scenario for such an application is a Hall system where disorder potentials separates different regions of the sample but allow for a coupling of the charge carriers. If we are interested not only in the electronic coupling but also in the coupling of the spin of the Fermions then one can minimal couple the $SU(2)$ Chern Simons field to the Lagrangian of a two dimensional Fermion system with spin. A variety realization in the composite Fermion model is the Θ-matrix (2.3.109) introduced in the previous chapter. In the next Chapter we show how one can derive a Dirac Lagrangian in three dimensions by projecting quantum-mechanically the Dirac Lagrangian in four dimensions. The local $SU(2)$ Chern Simons Transformation, in the sense of (3.1.1), attaches not only the flux quanta but also couples the spin of the two system.

In the gauge fixed partition function (3.3.30) we perform the integral over the Nakanishi Lautrup field ϕ and obtain a delta distribution with the constrained equation $d * a = 0$:

$$Z_k(M) = \int \mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}a \; e^{2\pi i k(S_{\mathrm{CS}}(a)+S_{\mathrm{gh}}(\bar{c},c))} \; \delta(d * a) \tag{3.3.31}$$

with the ghost action

$$S_{\mathrm{gh}}(\bar{c},c) = \frac{1}{8\pi^2}\int_M \; \mathrm{tr}((*d\bar{c}) \wedge dc - d\bar{c} \wedge [*a, c]). \tag{3.3.32}$$

First we observe that the Dirac Operator

$$\begin{aligned} &D : \Omega^1(m,g) \to \Omega^0(M,g) \oplus \Omega^2(M,g) \\ &D := d + d^* \end{aligned} \tag{3.3.33}$$

corresponds exactly to the exterior derivative d since the gauge fixing constrained, the Lorentz gauge $d * a = 0$ enters as follows:

$$Da = da \oplus d^*a = da \oplus *d * a = da \tag{3.3.34}$$

The hodge star operator fulfills the condition $*^2 = 1$ and the formal adjoint operator d^* is given by the map

$$\begin{aligned} &d^* : \Omega^k(M,g) \to \Omega^{k-1}(M,g) \\ &d^* = (-1)^{k-1} * d * . \end{aligned} \tag{3.3.35}$$

It is formally defined via the relation

$$\int_M \mathrm{tr}(A \wedge dB) = \int_M \mathrm{tr}(d^*A \wedge B) \tag{3.3.36}$$

with A, B being Lie algebra valued one forms. We can then rewrite the quadratic part of the Chern Simons action

$$\int_M \mathrm{tr}(a \wedge da) \quad \mapsto \quad \int_M \mathrm{tr}(a \wedge Da). \tag{3.3.37}$$

Before we start the computation we look at the finite dimensional case where we have to identify the quadratic and the cubic part of the Chern Simons action. We follow [Koh02] and start with a Gaussian integral given by

$$\int_{-\infty}^{\infty} dx \; e^{-\mu x^2} = \sqrt{\frac{\pi}{\mu}} \tag{3.3.38}$$

and generalize it with analytic continuation to

$$\int_{-\infty}^{\infty} dx\ e^{-\lambda x^2} = \sqrt{\frac{\pi}{|\lambda|}} e^{\frac{i\pi}{4}\frac{\lambda}{|\lambda|}}. \tag{3.3.39}$$

For the n dimensional case we introduce a quadratic form

$$Q(x) = \tfrac{1}{2}\sum_{i,j} q_{ij}x_i x_j, \tag{3.3.40}$$

which we choose to be non-degenerated $\det Q \neq 0$ thus the matrix (q_{ij}) is invertible. Then we can evaluate the integral

$$\int_{\mathbb{R}^n} dx_1 \dots dx_n\ e^{iQ(x_1,\dots,x_n)} = \frac{\pi^{n/2}}{\sqrt{|\det Q|}} e^{\frac{i\pi}{4}\mathrm{sign}Q}. \tag{3.3.41}$$

We need an expression with a further linear term $J^i x_i$ with $J, x \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} dx_1 \dots dx_n\ e^{iQ+iJx} = c\ e^{-\frac{i}{2}\sum_{i,j} q^{ij} J_i J_j} \tag{3.3.42}$$

and an integral with additional cubic term

$$f(x) := Q(x) + \sum_{i,j,k} f^{ijk} x_i x_j x_k. \tag{3.3.43}$$

Let $k \geq 0$ be a positive constant, then expand the integral

$$Z_k = \int_{\mathbb{R}^n} dx_1 \dots dx_n\ e^{ikf(x)} \tag{3.3.44}$$

in a power series

$$Z_k = \int_{\mathbb{R}^n} dx_1 \dots dx_n\ e^{ikQ} \sum_{m=0}^{\infty} \frac{(ik)^m}{m!}\Big(\sum_{i,j,k} f^{ijk} x_i x_j x_k\Big)^m \tag{3.3.45}$$

and with a rescale of the variables

$$x_i \ \mapsto\ \frac{x_i}{\sqrt{k}} \tag{3.3.46}$$

$$dx_i \ \mapsto\ \frac{dx_i}{\sqrt{k}} \tag{3.3.47}$$

the factor k cancels in the exponential

$$Z_k = \frac{1}{\sqrt{k^n}} \sum_{m=0}^{\infty} \frac{(i)^m}{m!(\sqrt{k})^m} \int_{\mathbb{R}^n} dx_1 \dots dx_n\ e^{iQ}\Big(\sum_{i,j,k} f^{ijk} x_i x_j x_k\Big)^m. \tag{3.3.48}$$

This integral can be rewritten in the usual form by introducing sources $J \in \mathbb{R}^n$ and we obtain a power series in $1/\sqrt{k}$:

$$Z_k = \tfrac{c}{\sqrt{k}} \sum_{m=0}^{\infty} \frac{i^m}{m!\sqrt{k^m}} \Big[\Big(\sum_{i,j,k} f_{ijk} \frac{-i\partial}{\partial J_i}\frac{-i\partial}{\partial J_j}\frac{-i\partial}{\partial J_k}\Big)^m \; e^{-\frac{i}{2}\sum_{i,j} q^{ij} J_i J_j}\Big]_{J=0}. \tag{3.3.49}$$

The quadratic term in the Chern Simons functional can be specified. The basis T^α of the Lie algebra g is given by relation (3.1.2) and the normalization condition (3.1.3). x corresponds to the connection one form and therefore also J is a one form on our manifold M. The coefficients of the inverse of the quadratic form Q are given by the q^{ij}'s. The equivalent expression in the Chern Simons term is given by the inverse of the Dirac operator map is an integral operator map:

$$\begin{aligned} &D^{-1} : \Omega^k(M,g) \to \Omega^{k+1}(M,g) \\ &D^{-1} = D \circ (D^2)^{-1} = D \circ \Delta^{-1}. \end{aligned} \tag{3.3.50}$$

We requiered $H^*(M,g) = 0$ and therefore the inverse of the Laplacian $\Delta = dd^* + d^*d = d\delta + \delta d$ should exist. The action of D^{-1} on the sources J considered as two forms is defined by the integral

$$D^{-1}(J_A)(x) = \sum_B \int_{M_y} L_{AB}(x,y) \wedge J_B(y) \tag{3.3.51}$$

$$= \sum_B \int_{M_y} d^3y \; \varepsilon^{\mu\nu\rho} L^{AB}_{\mu\nu}(x,y) J^B_\rho(y) \tag{3.3.52}$$

where we sum over the dimension of the Lie algebra. The integral kernel L_{AB} is a Lie algebra valued two form on $M_x \times M_y$, $L_{AB} \in \Omega^2(M_x \times M_y, g \otimes g)$ and thus $D^{-1}(J_A)(x) \in \Omega^2(M_x, g)$ is a Lie algebra valued two form on M_x. In the infinite dimensional case the sum over i in (3.3.49) is turned into an integral and since the fields are Lie algebra valued one forms we also have to sum over the dimension of the Lie algebra $\sum_i \mapsto \sum_A \int_M$. We can express the quadratic form of the Chern Simons action when we identify the gauge potentials with the currents in the usual way

$$a = \sum_A a^A T^A = \sum_A T^A \; D^{-1}(J_A) \tag{3.3.53}$$

and make use of the fact that $D \circ D^{-1} = id$ is the identity map. When we multiply (3.3.51) from the left the source J_A and integrate over the manifold M_x and sum up over the dimension of the Lie algebra we obtain an expression of the quadratic form of the Chern Simons action as a generalization of (3.3.40):

$$iQ(A) = i \int_{M_x} \mathrm{tr}(a \wedge Da)(x)$$

$$= -\frac{i}{2} \sum_A \int_{M_x} (D^{-1}(J_A) \wedge (D \circ D^{-1}(J_A)))(x) \tag{3.3.54}$$

$$= -\frac{i}{2} \sum_{AB} \int_{M_x \times M_y} J_A(x) \wedge L_{AB}(x,y) \wedge J_B(y). \tag{3.3.55}$$

The integral kernel L_{AB} is antisymmetric in the indices A, B

$$L_{AB}(x,y) = -L_{BA}(x,y) \tag{3.3.56}$$

and corresponds to the propagator form, or Green's form, satisfying the equation

$$D_{M_x \times M_y} L_{AB}(x,y) = -\delta_{AB}\delta(x,y) \tag{3.3.57}$$

with $D_{M_x \times M_y}$ being the covariant derivative on $M_x \times M_y$ and the delta distribution is defined as a current of degree three

$$\int_{M_x \times M_y} \delta(x,y)\ h(x,y) = \int_{M_x} h(x,x) \tag{3.3.58}$$

for a three form $h(x,y)$ on $M_x \times M_y$. In local coordinates this corresponds to the propagator, or Green's function respectively, of the gauge fields:

$$Q(A) = -\frac{1}{2}\sum_{AB}\int_{M_x \times M_y} d^3x d^3y\ J^A_\mu(x)\ L^{\mu\nu}_{AB}(x,y)\ J^B_\nu(y). \tag{3.3.59}$$

The cubic term only arises in nonabelian Chern Simons theories or in gauge theories on a noncommutative spacetime. The fields a can be expanded in the basis of the Lie algebra $a(x) = \sum_A T^A a_A(x)$ with the commutator $[T^A, T^B] = f^{AB}{}_C T^C$ of the normalized basis $\mathrm{tr}(T^A T^B) = 1/2\delta^{AB}$ there is the relation:

$$\mathrm{tr}(T^A T^B T^C) = f^{ABC} \tag{3.3.60}$$

and the trace over the cubic term can be expressed as:

$$\mathrm{tr}(a \wedge a \wedge a) = \sum_{ABC} f^{ABC} a_A \wedge a_B \wedge a_C. \tag{3.3.61}$$

The partition function for the gauge fixed Chern Simons functional can now be written down since the coefficients f^{ijk} in (3.3.49) correspond to the structure constant of the Lie algebra.

$$\begin{aligned} Z_k = \frac{c}{\sqrt{k}}\sum_{m=0}^{\infty}\frac{i^m}{m!\sqrt{k^m}} \quad & \Big[\Big(\sum_{A,B,C}\int_{M_{123}} f^{ABC}\frac{-i\delta}{\delta J_i(x_1)}\frac{-i\delta}{\delta J_j(x_2)}\frac{-i\delta}{\delta J_k(x_3)}\Big)^m \\ & \exp\Big\{-\frac{i}{2}\sum_{A,B}\int_{M_{12}} J_A(x_1)\wedge L_{AB}(x_1,x_2)\wedge J_B(x_2)\Big\}\Big]_{J=0} \\ & + \text{ghost contributions} \end{aligned} \tag{3.3.62}$$

We may express the partition function in local coordinates:

$$\begin{aligned} Z_k \ =\ & \frac{c}{\sqrt{k}}\sum_{m=0}^{\infty}\frac{i^m}{m!\sqrt{k^m}}\Big[\\ & \times\Big(\sum_{A,B,C}\int_{M_{123}} d^3x_1 d^3x_2 d^3x_3\ f^{ABC}\frac{-i\delta}{\delta J^\mu_A(x_1)}\frac{-i\delta}{\delta J^\nu_B(x_2)}\frac{-i\delta}{\delta J^\rho_C(x_3)}\Big)^m \\ & \times\exp\Big\{-\frac{i}{2}\sum_{A,B}\int_{M_{12}} d^3x_1 d^3x_2\ J^A_\mu(x_1)L^{\mu\nu}_{AB}(x_1,x_2)J^B_\nu(x_2)\Big\}\Big]_{J=0} \\ & + \text{ghost contributions.} \end{aligned} \tag{3.3.63}$$

The functional derivation

$$\frac{\delta}{\delta J_A(x)}$$

is defined on functionals in the following sense:

$$\frac{\delta}{\delta J_A(x)}\int J_A(y)\ a^A(y) = a^A(x)$$

and this leads to the distributional equation:

$$\frac{\delta}{\delta J_A(x)} J^B(y) = -\delta^{AB}\delta(x,y).$$

So far the ghost contribution have not bee taken into account in the perturbative Chern Simons theory in our framework here. However, it is known that they play a crucial role in nonabelian gauge theories in terms of the Yang-Mills action in the four dimensional Minkowski space and they are also necessary and important here. Let's see how the ghost fields appear in this setup and follow the methods in [AS91] for the superfield formulation. First we redefine the following variables

$$\begin{aligned} \mathcal{C} &:= *d\bar{c} &\quad (3.3.64)\\ \mathcal{B} &:= *d\phi &\quad (3.3.65)\end{aligned}$$

such that the Jacobian is one and the gauge fixed Chern Simons functional is given via Leibniz rule and Stokes by

$$S^{\mathrm{gf}}_{\mathrm{CS}}(a,c,\mathcal{C},\mathcal{B}) = \frac{k}{2\pi}\int_M \mathrm{tr}[\tfrac{1}{2}a\wedge da + \tfrac{1}{3}a\wedge a\wedge a - \mathcal{C}\wedge dc - \mathcal{C}\wedge[a,c] - \mathcal{B}\wedge a]. \quad (3.3.66)$$

The Lorentz gauge fixing constrained is now encoded in the field $\mathcal{B}$. We may expand the fields in the basis of the Lie algebra and perform the trace:

$$\begin{aligned} S(a,c,\mathcal{C},\mathcal{B}) &= \frac{-ik}{4\pi}\sum_{ABC}\int a^A\wedge da^A + \frac{2}{3}f_{ABC}a^A\wedge a^B\wedge a^C &\quad (3.3.67)\\ &\qquad -\mathcal{C}^A\wedge dc^A - 2f_{ABC}\mathcal{C}^A\wedge a^B\wedge c^C - \mathcal{B}^A\wedge a^A \end{aligned}$$

We introduce the superfield formalism with a redefinition of the fields as super fields. Therefore let V be a $\mathbb{C}$-vector space with two copies V_0, V_1 and $V_0\oplus V_1$. V is a super vector space or $\mathbb{Z}_2$-graded vector space with grading map

$$p : (V_0\cup V_1) - \{0\} \to \mathbb{Z}_2,\ p(v) = j \text{ for } v\in V_j.$$

For a vector space U the Grassmann algebra is defined by the wedge product $\Lambda^*U = \bigoplus_{k\geq 0}\Lambda^k U$ and becomes a superalgebra for

$$V_0 := (\Lambda^*U)_0 := \bigoplus_{j\geq 0}\Lambda^{2j}U$$

and

$$V_1 := (\Lambda^* U)_1 := \bigoplus_{j\geq 0} \Lambda^{2j+1} U.$$

We then consider the cotangent space T_p^*M of the underlying manifold equipped with the wedge product "$\wedge$". We introduce a basis in T_p^*M with the Fermionic coordinates θ^μ being determined by the coordinate system x^μ on M. We use then the relation between differential forms $\tilde{T} \in \Omega^*(M,g)$ and (super) functions $T \in \mathcal{C}^\infty(W \subset M) \otimes_{\mathbb{R}} \Lambda^* \mathbb{R}^n$ which is essentially given by the Grassmann algebra considered as superalgebra:

$$\begin{aligned} \tilde{T}(x) &= \sum_i T_{\mu_1 \ldots \mu_i}(x)\; dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_i} && (3.3.68) \\ \Leftrightarrow T(x,\theta) &= \sum_i \theta_{\mu_1} \ldots \theta_{\mu_i} T_{\mu_1 \ldots \mu_i}(x). && (3.3.69) \end{aligned}$$

In this framework the exterior derivative d on differential forms becomes a differential operator $d = \theta^\mu \partial_\mu$ on the superfields and products of superfields are defined via the components: For two superfields $T(X)$ and $R(Y)$, $X = (x, \theta)$ the product is defined by

$$T(X)R(Y) := \sum_{i,j} (-1)^{(i+\mathrm{sign}(T))j} \theta^{\mu_1} \ldots \theta^{\mu_i} \theta^{\nu_1} \ldots \theta^{\nu_j}\; T_{\mu_1 \ldots \mu_i}(x)\; R_{\nu_1 \ldots \nu_j}(y). \quad (3.3.70)$$

The Grassmann numbers commute with Bosonic fields but anticommute with Fermionic fields which is encountered here by $\mathrm{sign}(T) = +1$ for Bosonic T and $\mathrm{sign}(T) = -1$ for Fermionic T and then $\mathrm{sign}(T_{\mu_1 \ldots \mu_i}) = (-1)^{i+\mathrm{sign}(T)}$. The Chern Simons action can be rewritten in terms of these Grassmann coordinates:

$$\begin{aligned} S(a,c,\mathcal{C},\mathcal{B}) &= \frac{-ik}{4\pi} \int d^3X\; \theta^\mu \theta^\nu \theta^\rho\; (a_\mu^A \partial_\nu a_\rho^A - \mathcal{C}_{\mu\nu}^A \partial_\rho c^A - \mathcal{B}_{\mu\nu}^A a_\rho^A) && (3.3.71) \\ &\quad + \frac{2}{3} f_{ABC} \theta^\mu \theta^\nu \theta^\rho (a_\mu^A a_\nu^B a_\rho^C - 6\mathcal{C}_{\mu\nu}^A a_\rho^B c^C) \end{aligned}$$

with the integration measure changed to $d^3X := dX^1 dX^2 dX^3 := d\theta^1 d\theta^2 d\theta^3 dx^1 dx^2 dx^3$. We can now introduce $Lie(G)$ algebra valued, Fermionic superfields $\mathcal{V}^A$ [AS94]:

$$\mathcal{V}^A(x,\theta) := c^A(x) + \theta^\mu a_\mu^A(x) + \theta^\mu \theta^\nu \mathcal{C}_{\mu\nu}^A(x) + \theta^\mu \theta^\nu \theta_\rho \tilde{\mathcal{B}}_{\mu\nu\rho}^A(x). \quad (3.3.72)$$

The fields $\mathcal{V}, c, \mathcal{C}$ are Fermionic and $a, \mathcal{B}, \tilde{\mathcal{B}}$ are Bosonic. The Chern Simons action can then be formulated with these superfields and reduces to a simple form

$$\begin{aligned} S_{\mathrm{CS}}(\mathcal{V},\mathcal{B}) &:= \frac{-ik}{4\pi} \sum_{ABC} \int d^3X\; (\mathcal{V}^A\; d\mathcal{V}^A)(X) + \frac{2}{3}\; f_{ABC} (\mathcal{V}^A \mathcal{V}^B \mathcal{V}^C)(X) && (3.3.73) \\ &\quad - \theta^\mu \theta^\nu \theta^\rho \mathcal{B}_{\mu\nu}^A a_\rho^A \end{aligned}$$

and we have to calculate the partition function

$$Z_k(M) = \int_{\{\mathcal{V},\mathcal{B}\}} \mathcal{D}\mathcal{V}\; \mathcal{D}\mathcal{B}\; e^{-S_{\mathrm{CS}}(\mathcal{V},\mathcal{B})}. \quad (3.3.74)$$

The integration over $\mathcal{B}$, the redefined Nakanishi-Lautrup field ϕ, requires only the Lorentz gauge condition $d * \mathcal{V} = 0$. So performing the integral over $\mathcal{B}$ the partition function is:

$$Z_k(M) = \int_{\{\mathcal{V}, d*\mathcal{V}=0\}} \mathcal{D}\mathcal{V}\; e^{-S_{\mathrm{CS}}(\mathcal{V})}. \tag{3.3.75}$$

We may visualize the perturbation theory in terms of suitable Feynman diagrams. Since the ghost fields are unphysical fields, they can only occur inside a Feynman diagram and not as incoming or outgoing modes. They represent roughly speaking inner degrees of freedom which contribute to the self-energy part of a diagram respecting the gauge invariance of the theory.

In the superfield formulation of the Chern Simons action the ghost contributions are automatically included. So we may apply the perturbation setup from above to the superfields. We introduce sources $\mathcal{I}$ and in analogy take the inverse of the Dirac operator to express the superfields in terms of suitable propagators:

$$\mathcal{V}_A(X) := D^{-1}(\mathcal{I}_A)(X) = \sum_B \int dY\; \mathcal{W}_{AB}(X,Y)\; \mathcal{I}_B(Y). \tag{3.3.76}$$

With the antisymmetric, Fermionic propagator Green's form

$$\mathcal{W}_{AB}(X,Y) = -\mathcal{W}_{BA}(Y,X)$$

satisfying the distributional equation

$$D_{M_x \times M_y} \mathcal{W}_{AB}(X,Y) = -\delta_{AB}\delta(X,Y).$$

The delta distribution is the analog of the current three form from above but is now antisymmetric in X, Y. The quadratic term of the Chern Simons form takes a familar form, it can be expressed in terms of the propagator:

$$\begin{aligned} iQ(\mathcal{V}) &= i \sum_A \int d^3X\; (\mathcal{V}^A D \mathcal{V}^A)(X) && (3.3.77)\\ &= \frac{-i}{2} \sum_{AB} \int d^3X d^3Y\; \mathcal{I}_A(X) \mathcal{W}_{AB}(X,Y)\; \mathcal{I}_B(Y). && (3.3.78)\end{aligned}$$

The complete partition function in the superfield formulation has a similar form to (3.3.63) just that now the sources and the propagator are redefined:

$$\begin{aligned} Z_k &= \frac{c}{\sqrt{k}} \sum_{m=0}^{\infty} \frac{i^m}{m!\sqrt{k^m}} \Big[\Big(\sum_{ABC} \int d^3X_1 d^3X_2 d^3X_3\; f^{ABC} \frac{-i\delta}{\delta \mathcal{I}_A(X_1)} \frac{-i\delta}{\delta \mathcal{I}_B(X_2)} \frac{-i\delta}{\delta \mathcal{I}_C(X_3)} \Big)^m \\ &\quad \exp\Big\{ \frac{-i}{2} \sum_{AB} \int d^3X_1 d^3X_2\; \mathcal{I}_A(X_1) \mathcal{W}_{AB}(X_1,X_2)\; \mathcal{I}_B(X_2) \Big\} \Big]_{\mathcal{I}=0} \end{aligned} \tag{3.3.79}$$

The (super) derivation is in analogy defined on functionals in the following sense:

$$\frac{\delta}{\delta \mathcal{I}_A(X)} \int \mathcal{I}_A(Y)\; \mathcal{V}_A(Y) = \mathcal{V}_A(X)$$

the difference in the superfield formulation is now that the derivation $\delta/\delta\mathcal{I}$ is Fermionic while $\mathcal{I}$ is Bosonic. This leads to the distributional equation

$$\frac{\delta}{\delta\mathcal{I}_A(X)}\mathcal{I}_A = -\delta_{AB}\delta(X,Y) = \delta_{AB}\delta(Y,X)$$

which means that the delta function is antisymmetric in the coordinates X, Y, $\delta(X,Y) = -\delta(Y,X)$ in contrast to usual case. Before we turn to the $SU(2)$ example we may discuss the effect of the cubic interaction so we look at the contribution to second order $m = 2$. With the definition

$$Z_k = \frac{c}{\sqrt{k}}\sum_{m=0}^{\infty}\frac{i^m}{m!\sqrt{k^m}}Z_k^{(m)} \tag{3.3.80}$$

$$\tag{3.3.81}$$

the second order loop contribution of the Lagrangian without ghosts is given by

$$\begin{aligned} Z_k^{(2)} &= C_1\int_{M_1\times M_2}\sum_{ABC,A'B'C'} f^{ABC}f^{A'B'C'} \\ &\times L_{AA'}(x_1,x_1)\wedge L_{BB'}(x_2,x_2)\wedge L_{CC'}(x_1,x_2) \\ &+C_2\int_{M_1\times M_2}\sum_{ABC,A'B'C'} f^{ABC}f^{A'B'C'}L_{AB}(x_1,x_2)\wedge L_{CC'}(x_1,x_2)\wedge L_{A'B'}(x_1,x_2) \end{aligned} \tag{3.3.82}$$

where the Green's forms $L(x_i, x_j)$ are well defined if in the diagonal set the divergent part is removed by local counter terms, see [AS91]. The constants C_i enter the vertices of the corresponding Feynman graphs. The corresponding contribution to the graphs are shown in figure 3.4. Since the superfield formulation of the Chern Simons action has the same structure we can also calculate the partition function to second order with ghosts degrees of freedom included:

$$\begin{aligned} Z_k^{(2)} &= C_1\int d^3X d^3Y \sum_{ABC,A'B'C'} f^{ABC}f^{A'B'C'} \\ &\times \mathcal{W}_{AA'}(X_1,X_1)\mathcal{W}_{BB'}(X_2,X_2)\mathcal{W}_{CC'}(X_1,X_2) \\ &+C_2\int\sum_{ABC,A'B'C'} f^{ABC}f^{A'B'C'}\mathcal{W}_{AB}(X_1,X_2)\ \mathcal{W}_{CC'}(X_1,X_2)\ \mathcal{W}_{A'B'}(X_1,X_2) \end{aligned} \tag{3.3.83}$$

The cubic part gives thus a nontrivial contribution to the perturbative expansion of Chern Simons partition function. In addition we have to take into account the ghost contributions which is possible in the superfield formulation. It seems that it is still an open question how to include the ghost terms without superfield formulations in the perturbation series. In [Koh02] the ghost terms are ignored and also there they have not performed the superfield formulation in order to obtain the complete perturbative loop contributions. However, one may propose Feynman rules on that basis for the Chern Simons theory without the superfield formulation but with ghosts as has been done in [Wal05] (proposed but not derived). The

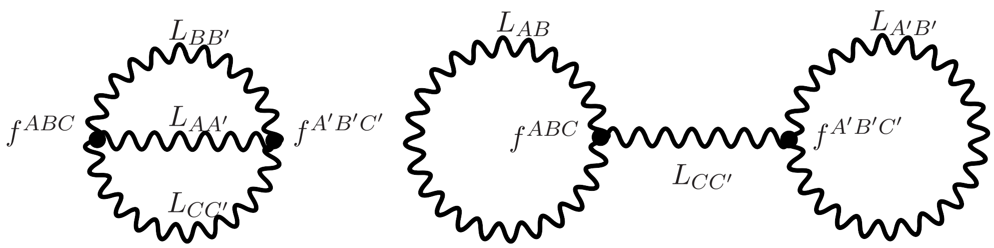


Figure 3.4: The selfinteracting loop graphs of the cubic term give a nontrivial contribution to the selfenergy [Koh02].

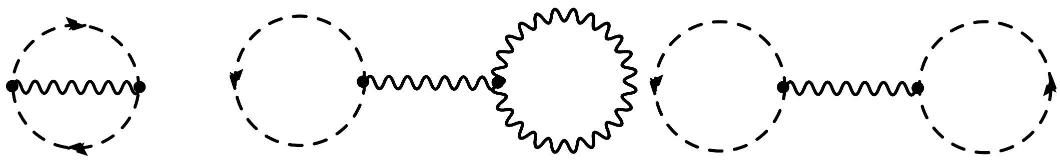


Figure 3.5: The ghost fields denoted by the dashed lines contribute non-trivially to the self-energy. Here at second order perturbation theory [Wal05].

contribution of the ghost fields can then be implemented via the proposed graphs shown if figure 3.5. In a proper treatment of the Chern Simons perturbation setup the contribution of ghost fields should be taken into account. We may end this section with the comment that even in this perturbative treatment of Chern Simons theories Axelrod and Singer [AS91] recovered topological invariants of the closed oriented manifold M for $SU(2)$ flat connections. This means precisely the expression:

$$I = Z_k^{(2)} - \frac{1}{4\pi} S_{\rm CS} \tag{3.3.84}$$

does not depend on the choice of the metric. There the ghost contributions are included via the superfield formulation.

3.3.6 Phenomenology and the SU(2) Chern Simons Theory

So far we developed a general quantization and perturbation framework for pure Chern Simons theories with compact semi simple Lie groups on three dimensional closed manifolds. In this section we want to connect this abstract formalism to the phenomenology of the quantum Hall effect in terms of composite Fermions. In the previous chapter we introduced the non relativistic phenomenological field theory of composite Fermions where a Chern Simons term naturally appears in the Lagrangian. The random phase approximation is defined via a perturbative expansion of the two point function, therefore a quantization is required for the Fermionic fields as well as for the Chern Simons gauge fields. The direct application of the perturbative formalism, we provided in this chapter, is not straight forward since we made the requirements on the manifold to be oriented and compact and without boundary. In the common phenomenology of composite Fermions one always chose the manifold to be $\mathbb{R}^3$, thus non compact. Furthermore the 'real' manifold of a Hall system seems to be neither closed nor $\mathbb{R}^3$. It is rather a direct product of a spacial disc with boundary and a time-like real line. Therefore we should discuss the Chern Simon theory on compact manifolds with non empty boundary. Here the connection between Wess Zumino Witten theories and Chern Simons theories may play the crucial role. In this section however we will treat the Fermions as free particles in $\mathbb{R}^3$ and the Chern Simons theory on a closed manifold. To justify the ladder we may heuristically argue that the Chern Simons term does not depend on the metric so we may rescale the fields and map the one point compactification $\mathbb{R}^3 \cup \{\infty\}$ of $\mathbb{R}^3$ to the three sphere S^3 via stereographic projection.

As in the previous chapter we will quantize the non relativistic free Fermionic fields and the free Chern Simons fields separately. In the next chapter we will generalize this setup to relativistic systems and to systems projected into the lowest Landau level. The low energy Lagrangian we will consider in this section is proposed for systems where particles with spin s and particles with spin s' contribute to the current and may couple. The case where only one part for instance spin s particles contribute to the current but couple to a particles with spin s' which do not contribute to the current is noted in the next section. Let us start with the following Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{CS}}^{\text{gf}} + \mathcal{L}_{\text{C}} \tag{3.3.85}$$

where in the free part

$$\begin{aligned} \mathcal{L}_0(\mathbf{x},t) &= \frac{1}{2}\sum_{s,s'} \psi^+_{s,s'}(\mathbf{x},t)\big[-\frac{1}{2m}\big(\mathbf{p}-\boldsymbol{a}(\mathbf{x},t)\big)^2 + \\ &\quad +i\partial_t + \mu + e(a_0(\mathbf{x},t))\big]\psi_{s,s'}(\mathbf{x},t) \end{aligned} \tag{3.3.86}$$

we defined the Chern Simons field $\mathcal{A}$ as $su(2)$ valued one form and introduced the overall field $a = eA - \mathcal{A}$ as a $su(2)$-matrix valued one form

$$a = \begin{pmatrix} a^{ss} & a^{ss'} \\ -\bar{a}^{ss'} & a^{s's'} \end{pmatrix} \tag{3.3.87}$$

which can be expanded in the basis of this Lie algebra $a = \sum_{B=1}^3 T^B a^B$ where $\sigma^B = i\sigma^i/2$ and σ^i being the usual Pauli matrices. We define the spinors

$$\psi^+_{s,s'} = (\varphi^+_s, \chi^+_{s'}) \tag{3.3.88}$$

such that φ_s represent a particle with spin s and $\chi_{s'}$ a particle with spin s' and the $su(2)$ field couples then both fields. The Lorentz gauge fixed Chern Simons action is

$$\begin{aligned} \mathcal{L}_{\text{CS}}^{\text{gf}} &= \text{tr}(a\wedge da + \frac{2}{3}a\wedge a\wedge a) \\ &\quad +\text{tr}(d*a\wedge\phi + (*d\bar{c})\wedge dc - d\bar{c}[*a,c]) \end{aligned} \tag{3.3.89}$$

this emerges in local coordinates to

$$\mathcal{L}_{\mathrm{CS}}^{\mathrm{gf}} = \frac{e}{\tilde{\varphi}\phi_o}\varepsilon^{\mu\nu\rho}\mathrm{tr}(a_\mu\partial_\nu a_\rho + \frac{2e}{3\tilde{\varphi}\phi_0}a_\mu a_\nu a_\rho) \tag{3.3.90}$$

$$+\frac{e}{\tilde{\varphi}\phi_o}\mathrm{tr}(\partial^\mu a_\mu\phi + \partial_\mu\bar{c}\partial^\mu c - (\partial_\mu\bar{c})a_\mu c). \tag{3.3.91}$$

When we compare this Chern Simons term with the usual one of the effective theories in the previous chapter we immediately see that neither the cubic term

$$\varepsilon^{\mu\nu\rho}\frac{2e}{3\tilde{\varphi}\phi_0}\mathrm{tr}(a_\mu a_\nu a_\rho)$$

nor the ghost term are considered. This is now a crucial difference between the constructive BRST method and the phenomenological approach. This difference should emerge in the calculation of the self-energy of the gauge field propagator and the Fermion propagator. Therefore we expect in the end a different effective composite Fermion mass and also a different energy gap. However it is difficult to incorporate the interaction term for example the Coulomb interaction:

$$\mathcal{L}_{\mathrm{C}} = \sum_{ss'}\int_{M_y} d^2\mathbf{y}\ \rho_{ss'}(x)\ V_{ss'}(\mathbf{x},\mathbf{y})\ \rho_{ss'}(y). \tag{3.3.92}$$

The charge densities can be replaced in the same way by the constrained equation, coming from the equations of motion for the gauge fields. The zero component of the current can be obtained by identifying

$$\sum_A a_0^A \underbrace{\psi_A^+ T^A \psi_A}_{=:\rho^A} = \psi_{s,s'}^+ \begin{pmatrix} a_0^{ss} & a_0^{ss'} \\ a_0^{s's} & a_0^{s's'} \end{pmatrix} \psi_{s,s'}$$

where ψ_A is the eigenspinor representation with respect to the corresponding basis. By the variation of the Lagrangian (without ghosts terms) we obtain

$$\begin{aligned}\rho := \sum_A \rho^A &= \frac{e}{\tilde{\varphi}\phi_0}\varepsilon^{ij}\sum_{AB}\partial_i a_i^B \mathrm{tr}(T^A T^B) + \frac{e^2}{\tilde{\varphi}^2\phi_0^2}\varepsilon^{ij}\sum_{BC} a_i^B a_j^C \mathrm{tr}(T^A T^B T^C) \\ &= \frac{e}{2\tilde{\varphi}\phi_0}\varepsilon^{ij}\sum_{AB}\partial_i a_i^B \delta^{AB} + \frac{e^2}{2\tilde{\varphi}^2\phi_0^2}\varepsilon^{ij}\sum_{ABC} a_i^B a_j^C f^{ABC}.\end{aligned}$$

The index A is just an expansion in a suitable basis for the calculation of the trace. Then the interaction is rewritten as

$$\begin{aligned}\mathcal{L}_{\mathrm{C}} = \frac{e^2}{(\tilde{\varphi}\phi_0)^2}\int_{M_y} d^2\mathbf{y}\quad & \varepsilon^{ij}\varepsilon^{mn}\mathrm{tr}[\hat{e}(\partial_i a_i + \frac{2e}{3\tilde{\varphi}\phi_0}a_m a_n)](x) \\ & \times V(\mathbf{x},\mathbf{y}) \\ & \times\mathrm{tr}[\hat{e}(\partial_m a_n + \frac{2e}{3\tilde{\varphi}\phi_0}a_i a_j)(y)]\end{aligned} \tag{3.3.93}$$

with $\hat{e} = \sum_A T^A$. We realize that we have a far more complicated expression for the interaction. However, the complicated terms are cubic or higher ordered in the coupling constant and may be neglected in a first approach and anyway we have $\sum_{ABC} a^B a^C f^{ABC} = 0$ due to the Jacobi identity $f^{ABC} f^{DEF} + cyclic = 0$. So we reduce (3.3.93) to

$$\begin{aligned}\mathcal{L}_{\mathrm{C}} = \frac{e^2}{(\tilde{\varphi}\phi_0)^2}\int_{M_y} \quad d^2\mathbf{y} \quad & \varepsilon^{ij}\varepsilon^{mn}\mathrm{tr}[(\hat{e}\partial_i a_j)(x) \\ & \times V(\mathbf{x},\mathbf{y}) \\ & \times \mathrm{tr}(\hat{e}\partial_m a_n)(y)]\end{aligned} \tag{3.3.94}$$

Since the Coulomb potential is purely phenomenologically motivated we may also take different components if required

$$V \mapsto V_{AB} \quad \text{with } A, B = 1, 2, 3 \tag{3.3.95}$$

We expand then the fields in the basis of the Lie algebra and may use as interaction the expression

$$\begin{aligned}\mathcal{L}_{\mathrm{C}} = \frac{e^2}{(2\tilde{\varphi}\phi_0)^2}\int_{M_y} \quad d^2\mathbf{y} \quad & \sum_{AB}\varepsilon^{ij}\varepsilon^{mn}(\partial_i a_j^A)(x) \\ & \times V_{AB}(\mathbf{x},\mathbf{y}) \\ & \times (\partial_m a_n^B)(y)].\end{aligned} \tag{3.3.96}$$

The total gauge field Lagrangian is given in local coordinates

$$\begin{aligned}\mathcal{L}_{\mathrm{G}}^{\mathrm{gf}} \quad = \quad & \frac{e}{\tilde{\varphi}\phi_o}\varepsilon^{\mu\nu\rho}\mathrm{tr}(a_\mu\partial_\nu a_\rho + \frac{2e}{3\tilde{\varphi}\phi_0}a_\mu a_\nu a_\rho) \\ & + \frac{e}{\tilde{\varphi}\phi_o}\mathrm{tr}(\partial^\mu a_\mu \phi + \partial_\mu\bar{c}\partial^\mu c - (\partial_\mu\bar{c})a_\mu c)] \\ & + \frac{e^2}{(2\tilde{\varphi}\phi_0)^2}\sum_A\int_{M_y} d^2\mathbf{y}\sum_{AB}\varepsilon^{ij}\varepsilon^{mn}(\partial_i a_j^A)(x) \times V_{AB}(\mathbf{x},\mathbf{y}) \ \times (\partial_m a_n^B)(y)].\end{aligned} \tag{3.3.97}$$

From this Lagrangian we immediately see, that we have a self-interacting cubic part entering the gauge field propagator and ghost terms contributing to the self-energy part. We should also include the ghost part in the charge density by including the ghost terms in the variation. Nevertheless, by comparing the order of the coupling constant we also recognize that the correction to the phenomenological approach may not be neglected. It is very interesting how this affects for example the effective mass of the composite Fermions or the energy gap.

It should be noted that the $\Theta_{ss'}$-matrix approach (2.3.109) can only be reconstructed within the $SU(2)$ approach for a pure diagonal theta matrix. The reason is that the theta matrix approach is purely phenomenological and can only be implemented by a minimal coupling of a gauge field if the theta matrix is diagonal. However, for non zero off-diagonal terms this is no more possible. The inconsistency has its origin in the couplings of the spins: In the free part of the Lagrangian there exists no coupling between the spins while in the interaction term and Chern Simons term there is a coupling for non-zero off-diagonal elements. This means that the composite Fermions with different spins do not couple but the Chern Simons action mimic a coupling. However, the motivation for the introduction of the theta matrix are the nonzero off-diagonal terms. This inconsistency can be repaired with the introduction of a real $SU(2)$ symmetry which is a minimal coupling of a matrix valued two form and as such consistent. Furthermore it gives a coupling with nonzero off-diagonal terms in a consistent way. However, the off-diagonal terms are not the same as in the theta matrix approach. In this sense the $\Theta_{ss'}$-matrix approach destroys the intrinsic connection between the minimal coupling and the Chern Simons action and is therefore a purely phenomenological variety of the $SU(2)$ approach.

Composite Fermions Coupled to Electrically Isolated Spins

In the case where the composite Fermions in a given Hall state couple to spins of particles which do not contribute to the current we have to slightly modify the theory. This scenario we may observe when some region in a given Hall sample is isolated by disorder such that the charge carriers inside can not contribute to the current of the sample or when there are other Layers of charge carriers are present (multilayer samples). In this case a suitable Chern Simon transformation gives an almost similar Lagrangian. There are just marginal changes in the structure required. The free part of (3.3.85) changes to

$$\begin{aligned} \mathcal{L}_0(\mathbf{x},t) &= \frac{1}{2}\sum_{s,s'} \psi^+_{ss'}(\mathbf{x},t)\big[-\frac{1}{2m}\big(\mathbf{p}-\boldsymbol{a}^{s'}(\mathbf{x},t)\big)^2 + \\ &\quad +i\partial_t+\mu+e(a_0^{s'}(\mathbf{x},t))\big]\psi_{ss'}(\mathbf{x},t) \end{aligned} \tag{3.3.98}$$

where now the Chern Simons field represents the isolated spin s' and the charge carriers contributing to the current have spin s. This can be recovered in the usual way if we perform a G-Chern Simons transformation in the Lagrangian in the following sence:

$$(\mathbf{p}-eA)^2 \mapsto (\mathbf{p}-\boldsymbol{a}^{s'})^2 = g_{\mathrm{CS}}(\mathbf{p}-eA)^2 g_{\mathrm{CS}}^{-1}, \tag{3.3.99}$$

$$\psi_s \mapsto \psi_{ss'} = g_{\mathrm{CS}}\;\psi_s \text{ with } g_{\mathrm{CS}} = e^{-i\Theta^{s'}(x)}. \tag{3.3.100}$$

So ψ_s are single spinors in contrast to the case from above. The Chern Simons field can again be expanded in the basis of the underlying Lie algebra which is $su(2)$ for spin one half but can also be $u(N)$ if the isolated spin is more complicated. We may also introduce the overall field a_μ and write down the corresponding Chern Simons Lagrangian. It is then the Chern Simons Lagrangian of $su(2)$ or $u(N)$ valued gauge fields. The charge density is then given by

$$\rho_s^A = \psi_A^+ T^A \psi_A \quad \text{with} \quad \psi_s \mapsto \psi_A = \exp\{a^A T^A\}\psi_s$$

and can be replaced in the same way as above in terms of the gauge fields. This is a slight modification of the theory from above and describes the coupling of composite Fermions to isolated spins.

3.3.7 Connection Between Chern Simons Theories and WZNW Nonlinear Sigma Models

The connection between three dimensional topological field theory and two dimensional conformal field theory was discovered early [DW90]. Chern Simon theories as topological field theories are therefore also motivated from the context of (edge) current algebras (Kac-Moody algebras being the current algebra of WZNW models). This is also known under the holographic principle which enters here in the sense that a topological field theory on a spactime region can be dermined by a field theory on the boundary.

It is a well known fact that the Chern Simons Lagrangian on closed manifolds transforms covariantly under gauge transformations mod $\mathbb{Z}$ and on compact manifolds with nonempty boundary it transforms covariantly mod $\mathbb{Z}$ under a subgroup, where all gauge transformations are unity at the boundary. Let a' be the gauge transformed field of a and let $g \in U(N)$ be the gauge transformation $a' = g^*a = g^{-1}ag + g^{-1}dg$. By cyclicity of the trace the Chern Simons Lagrangian transforms as

$$\mathcal{L}_{\text{CS}}(a') = \mathcal{L}_{\text{CS}}(a) + \text{tr}[d(a \wedge dgg^{-1})] - \tfrac{1}{3}\text{tr}[(g^{-1}dg) \wedge (g^{-1}dg) \wedge (g^{-1}dg)].$$

The total derivative term in the middle gives only a contribution at the boundary say Σ of the system manifold M. By shifting the boundary to infinity where all fields are zero this term then vanishes. In the special case where we consider a $U(1)$ gauge field only (usual electrodynamics) the cubic term disappears automatically. The gauge group can be restricted to a subgroup of gauge transformations which are the identity map on Σ. However, we may consider the full gauge group and then we obtain automatically a WZNW nonlinear sigma model at level k by $\sigma_H = k/(4\pi)$ with the WZNW action:

$$S_{\text{WZNW}} = \frac{k}{8\pi} \int_\Sigma \text{tr}[a \wedge dgg^{-1}] - \frac{k}{24\pi^2} \int_M \text{tr}[(g^{-1}dg)^3].$$

Usually if a WZNW model is fixed the starting point is a nonlinear sigma model in two dimensions with a group manifold as target space. Conformal invariance then requires the introduction of a WZNW term [Wit84] and then k must take integer values.

3.3.8 BRST Quantization and WZNW Models

In the quantum Hall regime the classical Ohm-Hall law should be turned into a quantum version. The matter current is quantized and therefore the Chern Simons fields should also be quantized. Since the Chern Simons theory is a (here massive) gauge theory we have to satisfy simultaneously gauge covariance and uniqueness of the Cauchy data. In the previous section we motivated the use of the Batalin-Vilkovisky superfield formalism and the BRST Cohomology to extract the physical states from the unphysical ghosts. While we discussed there Chern Simons theories on closed (without boundary) manifolds the situation in a realistic quantum Hall system might change due to the existence of a boundary. In the previous section we examined a classical correspondence between Chern Simons theories and WZNW models. This correspondence holds in the quantum limit [Wit89]. The BRST method provides some remarkable features in this context. It seems that the Chern Simons theory on a three dimensional manifold with boundary is equivalent to a WZNW model on that boundary using the BRST symmetry without constraining the theory or specifying boundary conditions [FH99]. It might then be interesting to combine WZNW models in quantum Hall states for specific BRST symmetries and incorporate these symmetries within the perturbative approach. The correspondence between chiral conformal field theories on the boundary and Chern Simon theories in the bulk is a well known fact [FPSW00], however this correspondence is not one to one. Using BRST symmetry arguments might shed more light into that problem and may determine also the perturbative treatment of composite Fermion models.

To be more precise lets consider the BRST gauge fixed Chern Simons action on a manifold M with boundary $\partial M = \Sigma$. We use the Leibniz rule and Stokes theorem to extract boundary terms in the action

$$S = S_{\mathrm{CS}}(a) + S^{\mathrm{gf}}(a, c, \bar{c}, \phi) \tag{3.3.101}$$

with the usual Chern Simons part $S_{\mathrm{CS}}(a)$ and the gauge fixing part

$$S^{\mathrm{gf}} = \int_M \mathrm{tr}[d * a \wedge \phi + \bar{c} \wedge d * D^a c]. \tag{3.3.102}$$

The extraction of pure boundary contributions in the gauge fixing potential

$$S^{\mathrm{gf}} = S^{\mathrm{gf}}|_\Sigma + S^{\mathrm{gf}}|_M \tag{3.3.103}$$

is obtained via Stokes theorem and gives

$$S^{\mathrm{gf}}|_\Sigma = \int_\Sigma \mathrm{tr}[*a \wedge \phi + \bar{c} \wedge *D^a c] \tag{3.3.104}$$

and the bulk part is then

$$S^{\mathrm{gf}}|_M = -\int_M \mathrm{tr}[*a \wedge d\phi + d\bar{c} \wedge *D^a c]. \tag{3.3.105}$$

As already noted above the gauge fixed action can be obtained by applying the BRST operator $\hat{\mathrm{s}}$ to the action

$$S^c = \int \mathrm{tr}[d * a \wedge \bar{c}] \tag{3.3.106}$$

from this action we extract a pure boundary part in the same way

$$S^c = S^c|_\Sigma + S^c|_M \tag{3.3.107}$$

with

$$S^c|_\Sigma = \int_\Sigma \mathrm{tr}[*a \wedge \bar{c}] \tag{3.3.108}$$

and the bulk part

$$S^c|_M = \int_M \mathrm{tr}[*a \wedge d\bar{c}]. \tag{3.3.109}$$

The gauge fixing action for the boundary and the bulk can be then obtained by applying the BRST operator

$$S^{\mathrm{gf}} = \hat{\mathrm{s}}(S^c|_\Sigma) + \hat{\mathrm{s}}(S^c|_M). \tag{3.3.110}$$

From the Chern Simons action we can also separate a pure boundary term applying the Leibniz rule and Stokes theorem to the non cubic term:

$$\begin{aligned} S_{\mathrm{CS}} &= \int_M \mathrm{tr}[a \wedge da] + \tfrac{2}{3}\int_M \mathrm{tr}[a \wedge a \wedge a] & (3.3.111)\\ &= -\int_\Sigma \mathrm{tr}[a \wedge a] + \int_M \mathrm{tr}[da \wedge a] + \tfrac{2}{3}\int_M \mathrm{tr}[a \wedge a \wedge a]. & (3.3.112)\end{aligned}$$

The gauge fixed pure boundary part of the action

$$S|_\Sigma = \int_\Sigma \mathrm{tr}[a \wedge a + *a \wedge \phi + \bar{c} \wedge *D^a c] \tag{3.3.113}$$

together with the BRST operator $\hat{\mathrm{s}}$ determines the physical states, the boundary modes of the underlying Hilbert space in the usual sense:

$$\mathcal{H}^\Sigma_{\mathrm{phys}} := \ker(\hat{\mathrm{s}})/\mathrm{im}(\hat{\mathrm{s}}). \tag{3.3.114}$$

An infinitesimal gauge transformation $a' = g^{-1}ag + g^{-1}dg$ leads to the action

$$S|_\Sigma(a') = S|_\Sigma(a) + \int_\Sigma \mathrm{tr}[2a \wedge g^{-1}dg + *g^{-1}dg \wedge \phi + \bar{c} \wedge *D^a c + \mathcal{O}^2]. \tag{3.3.115}$$

Now let $f : \Sigma \to G$ be a smooth map from the boundary Σ to the semi simple Lie group G of the Chern Simons field theory for instance $G = SU(2)$ and $\tilde{f} : M \to G$ its extension to M. For a being a pure gauge field we can then rewrite the action. We may also introduce light cone coordinates $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$. That is, we are interested in pure chiral (edge) representation then the Wess Zumino Witten model is described by the action

$$S_{\mathrm{WZW}} = \frac{-ik}{4\pi}\int_\Sigma \mathrm{tr}[f^{-1}\partial f \wedge f^{-1}\bar{\partial} f] + \frac{-ik}{12\pi}\int_M \mathrm{tr}[\tilde{f}^{-1}d\tilde{f} \wedge \tilde{f}^{-1}d\tilde{f} \wedge \tilde{f}^{-1}d\tilde{f}] \tag{3.3.116}$$

with a fixed integer k called the level. We introduce the set $Map(\Sigma, G_{\mathbb{C}})$ of smooth maps from the boundary Σ to the complexification $G_{\mathbb{C}}$ of the gauge group G for instance the complexification of $SU(2)$ is $SL(2,\mathbb{C})$. This set has a group structure by the pointwise multiplication $(f * g)(z) = f(z)g(z)$ and can be identified with a topological trivial principal $G_{\mathbb{C}}$ bundle over Σ since $\pi \circ f = id_\Sigma$ with $\pi^{-1}(z)$ being the fibers of the bundle and $Map(\Sigma, G_{\mathbb{C}})$ contains n linearly independent sections f_i.

The current algebra of the WZW model is an affine Lie algebra. We will reconsider how we can recover the current algebra from the action. Therefore we will focus on the loop groups and their representations [Koh02].

Let $S^1 = \{z \in \mathbb{C}|\ |z| = 1\}$ then the space of smooth maps $LG := \mathcal{C}^\infty(S^1, G)$ has a group structure by the pointwise multiplication and is an infinite dimensional Lie group which we denote as the loop group LG. There exists a corresponding Lie algebra to the loop group and its complexification is denoted by $L\mathfrak{g}$ which we call loop algebra. The Laurent series

$$f(t) = \sum_{n=-m}^{\infty} a_n t^n \tag{3.3.117}$$

together with the pointwise product gives a C-algebra $C(t)$. The loop algebra $L\mathfrak{g}$ is defined by the tensor product

$$L\mathfrak{g} = \mathfrak{g} \otimes C(t) \tag{3.3.118}$$

and is again a complex Lie algebra with Lie bracket

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg. \tag{3.3.119}$$

The central extension of $L\mathfrak{g}$ is denoted by $\hat{\mathfrak{g}}$ and defined by the direct sum of the loop algebra with a complex vector space

$$\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}. \tag{3.3.120}$$

The Lie algebra structure we obtain by introducing a bilinear form $\omega : L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{C}$ with the properties

$$\omega(\xi, \eta) = -\omega(\eta, \xi) \tag{3.3.121}$$

$$\omega([\xi, \zeta], \eta) + cyclic = 0 \tag{3.3.122}$$

together with the bracket

$$[\xi + \alpha c, \eta + \beta c] = [\xi, \zeta] + \omega(\xi, \zeta)c \quad \text{with} \quad \xi, \zeta \in L\mathfrak{g},\ c \in \mathcal{Z}(L\mathfrak{g}) \text{ and } \alpha, \beta \in \mathbb{C}. \tag{3.3.123}$$

The Lie bracket for $\hat{\mathfrak{g}}$ is then defined by

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \omega(X \otimes f, Y \otimes g)c. \tag{3.3.124}$$

The introduction of such a bilinear form, appearing in the central extension, is some times called affinization of the Lie algebra $\mathfrak{g}$ [Sch08]. We introduce the Cartan-Killing form $\mathcal{K}(\cdot, \cdot) = \langle \cdot, \cdot \rangle$, a non degenerated symmetric bilinear form which is invariant under the action of the adjoint representation of the Lie algebra $\mathfrak{g}$

$$\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \tag{3.3.125}$$

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle \tag{3.3.126}$$

In the case of $G = SU(N)$ or respectively $SL(2,\mathbb{C})$ we can represent the Killing form by the trace on the Lie algebra of G

$$\mathcal{K}(X,Y) := \mathrm{tr}(XY) \tag{3.3.127}$$

A 2-cocycle can be constructed with the Killing form which then defines a central extension of the Lie algebra. The 2-cocycle

$$\omega(X\otimes f, Y\otimes g) = \langle X, Y\rangle \ Res_{t=0}(df g) \tag{3.3.128}$$

with $Res_{t=0}(\sum_n c_n t^n dt) = c_{-1}$ defines the central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}$. $\hat{\mathfrak{g}}$ is an affine Lie algebra with the Lie bracket

$$[X\otimes t^m, Y\otimes t^n] = [X,Y]\otimes t^{m+n} + c\langle X,Y\rangle m\delta_{m+n,0}. \tag{3.3.129}$$

Now lets turn back to representations of the gauge group $Map(\Sigma, G_{\mathbb{C}})$ defined in the WZNW model. We define the following operators:

$$X_{n,\epsilon}(z) = e^{\epsilon X z^n} \quad , z\in D,\ \epsilon\in\mathbb{R},\ n\in\mathbb{N}_0 \tag{3.3.130}$$

and

$$X_{n,\epsilon}(\bar{z}) = e^{\epsilon X \bar{z}^{-n}} \quad , z\in D,\ \epsilon\in\mathbb{R},\ n\in\mathbb{N}_0. \tag{3.3.131}$$

The Lie bracket defines the product on the central extended algebra

$$[X_m, Y_n] = [X,Y]_{m+n} + mk\delta_{m+n,0}\langle X, Y\rangle \tag{3.3.132}$$

where we set for

$$\delta_{m+n,0}mk\langle X,Y\rangle = \lim_{\epsilon_1,\epsilon_2\to 0}\Gamma_D(g,f) = \frac{-ik}{2\pi}\int_D \mathrm{tr}[(mz^{m-1}Xdz)\wedge(n\bar{z}^{-n-1}Yd\bar{z})]. \tag{3.3.133}$$

We may now identify the currents

$$\begin{aligned} J_m(z) &= k\partial X_m X_m^{-1}(z) = -mkz^{m-1}Jdz & (3.3.134)\\ \bar{J}_n(\bar{z}) &= -kX_n^{-1}\bar{\partial}X_n(\bar{z}) = nk\bar{z}^{-n-1}\bar{J}d\bar{z} & (3.3.135)\end{aligned}$$

as representations of the affine Lie algebra $\hat{\mathfrak{g}}$:

$$[J_m, \bar{J}_n] = [J,\bar{J}]_{m+n} + mk\delta_{m+n,0}\langle J,\bar{J}\rangle. \tag{3.3.136}$$

This is the affine Lie algebra of the Wess Zumino theory [Koh02]. The task is then to find the composite Fermion filling factor, being related to the central charge, for the required composite Fermion model in the corresponding Hall state. Consider as an example a $SU(2)$ Chern Simons theory and introduce a mean field $\langle\mathcal{A}_\mu\rangle$ and a fluctuating field $\tilde{\mathcal{A}}_\mu$ then $a_\mu = eA_\mu - \tilde{\mathcal{A}}_\mu - \langle\mathcal{A}_\mu\rangle$ and the relevant equations of motion are

$$\begin{aligned} j &= \sigma_H^{CF} * \mathrm{tr}[\hat{e}\ da + \frac{2}{3}\hat{e}\ a\wedge a], \quad \hat{e} = \sum_A T^A, & (3.3.137)\\ j^\mu &= \frac{e^2}{h}\nu_{CF}\varepsilon^{\mu\nu\rho}\mathrm{tr}(\hat{e}\ \partial_\nu a_\rho + \frac{1}{3}\hat{e}\ [a_\nu, a_\rho]). & (3.3.138)\end{aligned}$$

If we divide the Hall conductivity $\sigma_H = \sigma_H^{MF} + \sigma_H^{CF}$ into a mean field part σ_H^{MF} and a composite Fermion part σ_H^{CF} then the relevant dynamics is fully encountered by the fields a_μ. Furthermore the central charge c of the affine Lie algebra of the $SU(2)$ valued fields in the WZNW model gives the composite Fermion filling factor ν_{CF} via $c = k = \nu_{CF}$. It is now possible to determine the composite Fermion filling factor by a conformal field theory of the chiral edge currents of the Chern Simons field. More precisely we may now start to consider different WZW models, determine the composite Fermion filling factor and the corresponding Chern Simons gauge field gives the spin and charge couplings of the composite Fermion or simply speaking the corresponding symmetry. As an example we may consider a WZW model with a gauge group $U(1)\otimes SU(2)$. The $U(1)$ is isomorphic to $O(2)$ and the group $SU(2)$ is locally isomorphic to $SO(3)$ then $O(2)\otimes SO(3) \subset SO(5)$. So we may define a WZW model for $Lie(SO(5))$ valued gauge fields $a \in \Omega^1(M, Lie(SO(5)))$. The central charge of the corresponding affine Lie algebra then determines the composite Fermion filling factor. This model can obviously simply be extended to filling factors being described by a next generation of composite Fermions. As mentioned in the previous chapter this is for example the filling factor 4/11 which corresponds to 4/3 composite Fermion filling factor. Performing again a mean field ansatz we may then determine the filling factor ν_{2CF} for the second generation composite Fermions via the corresponding WZNW model. This can be done for each generation giving a natural hierarchy for filling factors.

Within the connection between Chern Simons theories and WZW models and composite Fermion filling factors it is interesting and important for consistency what exactly the implication of the BRST cohomology is.

In the next chapter we consider the composite Fermion model in the lowest Landau level which gives rise to a noncommutative version of the Chern Simons theory. Then it would be interesting to search for a similar connection on the noncommutative level. This might then be a noncommutative version of the Wess Zumino Witten theory.

Chapter 4

Projection of Quantum Fields and Statistics

In quantum field theory, after theorems of Wigner [NW49] and Bargman [Bar47] particles are described as irreducible, continuous and unitary representations of the covering group $\mathcal{P}^c$ of the proper orthocronous Poincaré group $\mathcal{P}_+^\uparrow$ on a suitable separable Hilbert space. In low energy physics we may choose instead of the Poincaré group the Euclidean version, however we may prefer the more general case since a low energy approximation can always be done at the end of a theory and from a more conceptual basis the covariant formalism is more natural. In the four dimensional Minkowski space $\mathcal{P}^c$ can be represented by the group $SL(2,\mathbb{C})$. The spin statistics connection tells us then, that the possible particle representations obey either Bose-Einstein statistics in the case of integer spin or Fermi-Dirac statistic in the case of half integer spin. In a more restrictive version the spins and statistics are connected via the spin statistics theorem [Pau40]. This results in the fact, that fields with integer (half integer) spin are necessarily trivial, if they obey Fermi-Dirac (Bose-Einstein) statistics. The Bose-Fermi alternative excludes other statistics like para-statistics [GM65] (occurring naturally in algebraic quantum field theory [DHR70, DHR71]) or braid statistics [FRS89, FM91] important for lower dimensions, which was first realized by Leinaas and Myrheim [LM88] and discussed in an Abelian physical model by Wilczek[Wil82]. The spin statistics theorem can be regarded in this sense as a no-go theorem. If we consider now usual massive Fermion fields (electrons) as unitary irreducible representations of the Poincaré group in four dimensional Minkowski space and we want to confine them to three dimensions, we have to ask if this is the same as to start directly from irreducible representations of the Poincaré group in three dimensions and if the statistics is different. In this section we want to go into more detail to that question.

In three dimensions the Poincaré group consists of the Lorentz group $SO(2,1)$ and translations in $\mathbb{R}^3$ therefore the particle spectrum and the connection between spin and statistics in three dimensions differs from that in four dimensions as already motioned above. In fact the projection group $SO(2) \subset SO(2,1)$ is isomorphic to the one sphere S^1 and the covering group is the real line $\mathbb{R}$. In four dimensions the irreducible unitary representations are labeled by integer or half integer numbers (spin) while the irreducible unitary representations of $\mathbb{R}$

are labeled by any real number s, which is also called spin.

To be more precise in the case of Fermi or Bose statistics in four dimensions the following hypothesis should hold true

$$e^{i2\pi s} = \mathrm{sign}\lambda, \tag{4.0.1}$$

with s being the spin of the elementary excitations of a quantum field and λ its exchange statistic parameter. In three spacetime dimensions there exists also representations of the braid group and signλ is replaced by a non-real complex number. The corresponding particles are called Plektons, or Anyons [Wil82] in the Abelian case, and have fractional spin. In [FM91] an extended version of the spin statistics theorem in three dimensions was found and in [Mun08] it is shown under more general assumptions that the hypothesis

$$e^{i2\pi s} = \frac{\lambda}{|\lambda|} \tag{4.0.2}$$

holds true for specific conditions on the mass spectrum in a local relativistic quantum theory. Contrary to the relativistic system in quantum mechanics models are known, which violate the spin statistics connection and additional assumptions have to be made, see [Kuc04] for the discussion for necessary and sufficient conditions concerning anyons, bosons and Fermions.

Having internalized that there is a fundamental difference between the physical impacts from four dimensions and three dimensions then we should have a more detailed look on the connection between spin and statistics in the case of composite Fermions with spin.

4.1 Quantum-Mechanical Projection onto $2+1$ Dimensions

The projection onto the two-plus-one dimensional system is derived by freezing out the component perpendicular to the inversion layer usually denoted as z-component. We are only interested in particles confined to some potential in this direction and by freezing out we mean evaluating quantum-mechanically the particles in the ground state of the assumed confinement potential. We introduce some formalism, which in a way combines relativistic quantum field theory (actually QED) and usual quantum mechanics, where we consider the coordinate $x^3 \equiv z$ and $p_3 = p_z$ as quantum-mechanical unbounded and selfadjoint operators on some infinite dimensional Hilbert space.

A fundamental postulate of the quantum theory is that observables are represented as Hermitian elements of a suitable C^*-algebra $\mathfrak{A}$ of operators on a Hilbert space and states are given by positive, linear and normalized functionals $\omega : \mathfrak{A} \to \mathbb{C}$ over this algebra. In our case we are interested in the linear functionals over the following Weyl system of the unbounded selfadjoint operators z and p_z

$$e^{i\alpha p_z} e^{i\beta z} = e^{\frac{1}{2} i\hbar\alpha\beta} e^{i(\alpha p_z + \beta z)},$$

where the Weyl (representation) operators $W(\alpha) := e^{i\alpha p_z}$, $W(\beta) := e^{i\beta z}$ might be considered as elements of a C^*-algebra[1] $\mathfrak{A}$ on a corresponding infinite dimensional separable Hilbert space

[1]The Weyl relation defines a product, an involution is given by $W(\cdot)^* = \bar{W}(\cdot)$ and a norm can be defined by $\|W\| := \sup W(\cdot)$, which satisfies the C^* condition $\|W^*W\| = \|W\|^2$.

($\mathscr{H}_z$). For the simplest model take for example a harmonic oscillator in z-direction. A state ω is defined as a linear functional from the algebra $\mathfrak{A}$ to the complex numbers $\omega : \mathfrak{A}(\mathscr{H}_z) \to \mathbb{C}$. A function of an operator may be defined by

$$f(z) := \int d\beta \ \check{f}(\beta) \ e^{i\beta z}$$

and

$$f(p_z) := \int d\alpha \ \check{f}(\alpha) \ e^{i\alpha p_z}$$

respectively and of course this is only possible iff $\check{f} \in L^1(\mathbb{R}) \cap \mathcal{F}L^1(\mathbb{R})$, where $\mathcal{F}$ denotes the Fourier transformation. Again the functions may be considered as elements of a suitable C^*-algebra, where the product can be defined via the Weyl system:

$$f(p_z) * g(z) = \int d\alpha d\beta \ \check{f}(\alpha)\check{f}(\beta) \ e^{\frac{1}{2}i\hbar\alpha\beta} e^{i(\alpha p_z + \beta z)},$$

the involution map can be defined in terms of the complex conjugation for instance:

$$f(p_z)^* := \int d\alpha \ \bar{\check{f}}(\alpha) \ e^{-i\alpha p_z} = \bar{f}(p_z)$$

and a norm is defined straigth forward for instance

$$\|f(p_z)\| := \int d\alpha |\check{f}(\alpha)|$$

and it is easyly checked that it fullfills the C^*-condition. We may denote the functions $f(\cdot)$ as Weyl symbols being an element af a C^*-algebra $\mathscr{A}$ and write for the evaluation of such a symbol in a state

$$\begin{aligned} \omega : \qquad & \mathscr{A} \to \mathbb{C} \\ & f \circ W(\cdot) \mapsto \omega(f \circ W(\cdot)) \equiv \langle \omega, f \circ W(\cdot) \rangle \end{aligned}$$

and define

$$\omega_a(f(z)) := \int d\beta \ \check{f}(\beta) \ \omega_a(e^{i\beta z}),$$

which is in the case of the ground state of a harmonic oscillator at z-position a

$$\omega_a(f(z)) = \int d\beta \ \check{f}(\beta) \ e^{-\frac{1}{2}\beta^2} \ e^{i\beta a}.$$

Since we have the corresponding creation and annihilation operators at hand we may prefer the representation

$$e^{i\beta z} = e^{i\frac{l_z\beta}{2}(\hat{a}^+ + \hat{a})} = e^{-\frac{l_z^2\beta^2}{8}} e^{i\frac{l_z\beta}{2}(\hat{a}^+)} e^{i\frac{l_z\beta}{2}(\hat{a})}. \tag{4.1.1}$$

$\hat{a} = (1/l_z)(z + i(l_z^2/2\hbar)p_z)$ annihilates the ground state $\hat{a}\,|0\rangle_z = 0$ with $l_z = \sqrt{2\hbar/m\omega_z}$ and the last equality in (4.1.1) holds true due to the Weyl condition. In the case of a more complicated potential, for example the inversion symmetry we have to know at least the ground state of the system. To be somehow more accurate to real systems we may then be able to introduce the band structure at this level.

The application of this formalism to quantum electrodynamics leads to an effective quasi two dimensional theory. Assume we have free fields $\psi(x)$, $x \in \mathbb{R}^{3+1}$ satisfying the Dirac field equation with classical background electromagnetic field A_μ

$$i(\not{\partial} - e\not{A} - m)\psi(x) = 0.$$

A_μ is constant in time and $\psi = (\varphi, \chi)$ is the usual four component spinor. Then we may have no problems in quantizing the free Dirac field later. The corresponding Hamilton operator is

$$H = \beta m + \alpha^i(p_i - eA_i) + eA_0,$$

where $\boldsymbol{\alpha} = \sigma^1 \otimes \boldsymbol{\sigma}$, $\beta = \sigma^3 \otimes I$ and in the example $eA^0 = 1/2\ m\omega^2(x_3)^2 = 2/(ml_z^2)z^2$, ($\hbar = c = 1$). Within this framework we easily can separate the z-contribution in the limit $eA^0 \ll 2m$ (low energy limit) and decouple the large φ from the small χ component for stationary states in (z, t) and derive the usual harmonic oscillator in z-direction with ground state $a\,|0\rangle_z = 0$ from above. However it is not possible to decouple the third component directly in the Dirac equation this way, but we will come to that point later on when we discuss relativistic Landau levels. It is clear that the theory then fails to transform covariantly under the action of the proper Lorentz group $\mathcal{L}_+^\uparrow(\mathbb{R}^4)$ and Poincaré group $\mathcal{P}_+^\uparrow(\mathbb{R}^4)$ respectively. But this should not lead to confusion since we are only interested in particles (fields) bounded to $2 + 1$ dimensions and we will introduce an approach which preserves covariance at least of a subgroup $\widetilde{\mathcal{P}}_+^\uparrow(\mathbb{R}^{2+1}) \subset \mathcal{P}(\mathbb{R}^4)$ of the Poincaré group. Again we stress that we want to have more control of the interaction of electrons (with spin) and electromagnetic fields so we wish to preserve Lorentz covariance as long as possible. However, the restriction of the Poincaré group to a subgroup needs some comment concerning the interpretation of the particles. We may consider here the definition of Wigner for relativistic particles defined as irreducible representations of the universal covering group of the Poincaré group. By reducing the Poincaré group we automatically change the particle itself. These new particles are labeled by the mass m and a real number s, the spin or helicity [FM88]. Since the statistics can be Bosonic or Fermionic or in-between Wilczek named these particles anyons [WZ83]. We may have no real anyons if we do not restrict the Poincaré group. However, then we have to discuss what effects high momenta in the third direction have. This should then be discussed in terms of a confinement potential vanishing at infinity. The particles interaction with this potential is then described by a scattering matrix where a free particle scatters with a potential and thus this is not the regime we are interested in. If we interprete $\psi(x)$ as a spinor wave function in Minkowski space $\mathbb{R}^{3+1}$ and $g(z) := \langle z|\,0_z\rangle$ the setup from above can be adopted:

$$\int_{\mathbb{R}^4} d^4k\ \check{\psi}(k) \int_{\mathbb{R}} dx_3\ g^*(x^3)g(x^3)e^{ik_3x^3} = \langle 0_z|\,\psi(x)\,|0_z\rangle = \langle \omega, \psi(x)\rangle$$

since ψ separates only if we do not take the separation ansatz in the low energy limit, but directly if possible.

Consider now some free quantum field operators ψ satisfy the canonical anti-commutation relations (CAR):

$$\{\psi(f), \psi^*(g)\} = (f, g), \text{ and } \{\psi(f), \psi(g)\} = 0$$

with some complex functions $f, g \in \mathcal{C}_0^\infty(\mathbb{R}^4)$ smearing the fields on some region in Minkowski space and

$$\psi(x) := \int_{\mathbb{R}^4} d^4k \ \check{\psi}(k) \otimes e^{ikx}.$$

A field (operator) ψ evaluated in a state ω of a confining potential is defined by

$$\psi_{[\omega]}(x^0, x^1, x^2) := \langle I \otimes \omega, \psi(x) \rangle.$$

Here and in the following we take the tensor product over the complex numbers $\otimes \equiv \otimes_{\mathbb{C}}$ if not explicitly stated from different and I is the identity map. Products of fields $\mathcal{W}(x_1, \ldots, x_n) := \psi(x_1) \otimes \psi(x_2) \otimes \ldots \otimes \psi(x_n)$ can be simultaneously projected via the map:

$$\begin{aligned}
\boldsymbol{P}_\omega^n(\mathcal{W}(x_1, \ldots, x_n)) &:= \langle \underbrace{(I \otimes \omega) \otimes \ldots \otimes (I \otimes \omega)}_{n\times}, \psi(x_1) \otimes \psi(x_2) \otimes \ldots \otimes \psi(x_n) \rangle \\
&= \psi_{[\omega]}(x_1^0, x_1^1, x_1^2) \otimes \psi_{[\omega]}(x_2^0, x_2^1, x_2^2) \otimes \ldots \otimes \psi_{[\omega]}(x_n^0, x_n^1, x_n^2).
\end{aligned}$$

Let ω_a be for example a state, usually the ground state, of a confinement potential in z-direction around a position z_0. It has not to be symmetric around that point only the wave functions may be chosen from Schwartz space $\mathcal{S}(\mathbb{R})$. In this sense we can view the operator ψ as a map $\omega \mapsto \psi(\omega)$ from the space of states of the confining system to the space of the in z-smeared field operators:

$$\begin{aligned}
\psi_{[\omega_{z_0}]}(x^0, x^1, x^2) &= \int_{\mathbb{R}^4} \frac{d^4a}{(2\pi)^4} \int_{\mathbb{R}^4} d^4k \ \langle I, \psi(a) e^{-ika} \rangle \ \omega_{z_0}(e^{ikx}) \\
&= \int_{\mathbb{R}^4} d^4a \ \psi(a) \ \prod_{i=0}^{2} \delta(x^i - a^i) \ \int_{\mathbb{R}} \frac{dk_3}{2\pi} \ \omega_{z_0}(e^{ik_3 x^3} \ ) e^{-ik_3 a^3} \\
&= \int_{\mathbb{R}} da_3 \ \psi(x^0, x^1, x^2, a^3) f(a^3 - z_0)
\end{aligned}$$

and $f \in \mathcal{S}(\mathbb{R})$. In order to consider field operators as operator valued distributions taking values in the compact operators of a suitable CAR algebra on an infinite dimensional Hilbert space we have to introduce some test function $g \in \mathcal{S}(\mathbb{R}^{2+1})$ and w.l.o.g $z_0 = 0$. The smeared field operator is then given by

$$\psi_{[\omega]}(g) = \int_{\mathbb{R}^4} d^4a \ \psi(a) \ g(a^0, a^1, a^2) \ f(a^3).$$

The anti-commutator of these fields evaluated in states ω_1, ω_2 is written down straight forward

$$\{\psi_{\xi[\omega_1]}(g_1), \bar{\psi}_{\xi'[\omega_2]}(g_2)\} = i \int d^4a d^4b \ \Delta_{\xi,\xi'}(a - b) g_1(a^0, a^1, a^2) f_1(a^3) \ g_2(b^0, b^1, b^2) f_2(b^3)$$

$\xi\xi'$ are the spinor indices and the propagator is defined by

$$i\Delta_{\xi,\xi'}(a-b) = \{\psi_\xi(a), \bar{\psi}_{\xi'}(b)\}$$

with the commutator given by

$$\{\psi_\xi(a), \bar{\psi}_{\xi'}(b)\} = [i\not{\partial}_a - m]_{\xi\xi'} i\Delta(a-b)$$

and

$$i\Delta(a-b) = i\Im\Delta_+(a-b) = -\langle\Omega|\,\phi(a)\phi(b)\,|\Omega\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\, e^{ik(a-b)}.$$

At simultaneous time $a^0 = b^0 = t$ the commutator is as usual

$$\{\psi_\xi(t,\boldsymbol{a}), \bar{\psi}_{\xi'}(t,\boldsymbol{b})\} = [-\gamma^0_{\xi\xi'}\partial_0]\Delta(a^0-b^0, \boldsymbol{a}-\boldsymbol{b})|_{a^0=b^0} = \gamma^0_{\xi\xi'}\delta^3(\boldsymbol{a}-\boldsymbol{b}).$$

This should be clear if we follow for example the canonical quantization procedure for Fermi fields and can be verified by examining the energy momentum density, where we have to require that the energy is bounded from below. As an example we assume the harmonic potential in z-direction from above around $z_0 = 0$, thus $\omega \equiv \omega_0$. Then $\omega \circ W(z)$ is a Gaussian function around the origin and the projected Dirac field $\psi_{[\omega]}(x^0, x^1, x^2)$ may be defined by

$$\psi_{[\omega]}(x^0,x^1,x^2) = \int_{\mathbb{R}} da_3 \int_{\mathbb{R}^3} \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{m}{\omega_{\mathbf{k}}} \quad \textstyle\sum_\alpha \quad \big(b_\alpha(\mathbf{k})u^{(\alpha)}(\mathbf{k})e^{-ik_\mu x^\mu}\langle\omega, e^{ik_3(z-a_3)}\rangle \tag{4.1.2}$$

$$+ \quad d^+_\alpha(\mathbf{k})v^{(\alpha)}(\mathbf{k})e^{ik_\mu x^\mu}\langle\omega, e^{-ik_3(z-a_3)}\rangle\big) \tag{4.1.3}$$

where now $k_\mu x^\mu = \omega_{\mathbf{k}}x^0 - k_1x^1 - k_2x^2$ and

$$\langle\omega, e^{ik_3 z}\rangle = \langle\omega, e^{-ik_3 z}\rangle = e^{-\frac{1}{8}k_3^2}.$$

At least we would only require the ground state wave function to be in $\mathcal{S}(\mathbb{R})$. To drive this further we can completely eliminate the third component by introducing a delta distribution represented by the limit

$$\int_{\mathbb{R}} da_3\; h(a_3)\; f(a_3) \to \int_{\mathbb{R}} da_3\; h(a_3)\delta(a_3) \;=\; \int_{\mathbb{R}} da_3\; h(a_3)\; \lim_{l_z\to 0}\frac{1}{\sqrt{\pi}l_z}e^{-\frac{(a_3)^2}{l_z^2}} \tag{4.1.4}$$

$$= \int_{\mathbb{R}} da_3\; h(a_3)\; \lim_{l_z\to 0}\int_{\mathbb{R}}\frac{dk_3}{2\pi}\langle\omega^{l_z}_{a_3}, e^{ik_3 z}\rangle$$

for some test function $h \in \mathcal{S}(\mathbb{R})$. This limit has to be performed carefully since the operators $b_\alpha(\mathbf{k})\,|0\rangle = d_\alpha(\mathbf{k})\,|0\rangle$, $\mathbf{k} \in \mathbb{R}^3$ and the vacuum $|0\rangle$ respectively are defined in four rather than in three dimensions, which would lead to an undefined expression. The anti-commutator of $b_\alpha(\mathbf{k})$ with $b^+_\alpha(\mathbf{q})$ gives a further delta function $\delta(k_3 - q_3)$, which leads to the square of a delta function and that is not defined. Thus while performing this limit we also have to define the operators $b_\alpha(\mathbf{k})\,|0\rangle = d_\alpha(\mathbf{k})\,|0\rangle$ and the vacuum respectively with the momentum $\mathbf{k} \in \mathbb{R}^2$. This would be the case where the third component completely separates in the Dirac equation.

Of course a delta potential might be too simple to describe realistic systems. For example Rashba spin orbit coupling requires a gradient electric field in z-direction and that information would be completely lost. Therefore we have to choose a realistic confinement potential including the band structure of the material. But for a first analysis it may suffice.

The spinors u and v are derived from the Dirac equation and for the operators we require the commutation relation such that we have the translational symmetry

$$\psi(x+a) = e^{iP_\mu a^\mu}\Psi(x)e^{-iP_\mu a^\mu}$$

for $x = (x^0, x^1, x^2)$ and $a = (a^0, a^1, a^2)$ and the energy momentum P_μ:

$$P_\mu = \int d^2x \; {\Theta_{[\omega]}}^0{}_\mu, \quad \mu = 0, 1, 2.$$

more explicit

$$P_\mu \;=\; \int_{\mathbb{R}^2} da_3 db_3 \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{m}{\omega_\mathbf{k}} k_\mu \; \langle \omega, e^{-ik_3x_3}\rangle^2 \; \sum_\alpha [b^+_\alpha(\mathbf{k})b_\alpha(\mathbf{k}) - d_\alpha(\mathbf{k})d^+_\alpha(\mathbf{k})] \; e^{-ik_3(a_3-b_3)}$$

for $\mathbf{k} \in \mathbb{R}^3$ and in the limit (4.1.4)

$$P_\mu \;=\; \int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{m}{\omega_\mathbf{k}} k_\mu \;\; \sum_\alpha [b^+_\alpha(\mathbf{k})b_\alpha(\mathbf{k}) - d_\alpha(\mathbf{k})d^+_\alpha(\mathbf{k})]$$

with $\mathbf{k} \in \mathbb{R}^2$. The energy momentum tensor density ${\Theta_{[\omega]}}^{\mu\nu}$ is derived from a modified action for a (quasi) free Dirac particle in some classical background field

$$\boldsymbol{P}^2_{[\omega]}(S) = \int_{\mathbb{R}^3} d^3x \; \tfrac{i}{2}\bar{\psi}_{[\omega]} \overleftrightarrow{\not{D}} \psi_{[\omega]} - m\bar{\psi}_{[\omega]}\psi_{[\omega]}$$

where $\not{D} = \gamma^\mu D_\mu$ with $\mu = 0, 1, 2$ and the contribution of the third component with the confinement potential might be set to zero since once projected it gives only a neglectable contribution to the mass as we can verify within the analysis of the Dirac equation as stated before;

$$\boldsymbol{P}^3_{[\omega]}(\tfrac{i}{2}[\bar{\psi}\overleftrightarrow{D_3}\gamma^3\psi] + \bar{\psi}eA_0\gamma^0\psi) = \tfrac{i}{2}[\bar{\psi}_{[\omega]}\omega(\overleftrightarrow{A_3}\gamma^3)\psi_{[\omega]}(x^0, x^1, x^2)] + eA_0\gamma^0\bar{\psi}_{[\omega]}\psi_{[\omega]}.$$

The condition $eA \ll m$ from above might be assumed and we may include the term with A_3 later on. Then the standard derivation of ${\Theta_{[\omega]}}^{\mu\nu}$ is

$$\Theta^{\mu\nu}_{[\omega]} = D^\nu\bar{\psi}_{[\omega]}\frac{\delta\mathcal{L}}{\delta(D_\mu\bar{\psi}_{[\omega]})} + \frac{\delta\mathcal{L}}{\delta(D_\mu\psi_{[\omega]})}D^\nu\psi_{[\omega]} - g^{\mu\nu}\mathcal{L} = \frac{i}{2}\bar{\psi}_{[\omega]}\gamma^\mu\overleftrightarrow{D}^\nu\psi_{[\omega]}.$$

At first we discuss the case where $A_\mu = 0$. In order to make sure that the energy is bounded from below we have to require the (usual) anti-commutation relations

$$\{b_\alpha(\mathbf{k}), b^+_\beta(\mathbf{q})\} = \{d_\alpha(\mathbf{k}), d^+_\beta(\mathbf{q})\} = (2\pi)^3\tfrac{\omega_\mathbf{k}}{m}\delta^3(\mathbf{k}-\mathbf{q})\delta_{\alpha\beta}$$

where $\mathbf{k}, \mathbf{q} \in \mathbb{R}^3$ and in the case of (4.1.4) this can be reduced to $\mathbf{k}, \mathbf{q} \in \mathbb{R}^2$ and Paulis exclusion principle holds true since at equal times:

$$\{\psi_{[\omega]\xi}(t, \mathbf{x}), \psi^+_{[\omega]\xi'}(t, \mathbf{y})\} = \delta_{\xi\xi'}\delta(\mathbf{x} - \mathbf{y}).$$

Wick products can be defined as usual so we may use the definition of the total energy momentum as

$$P_\mu = \int_{\mathbb{R}} da_3 db_3 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{\omega_{\mathbf{k}}} k_\mu \ \langle \omega, e^{-ik_3 x_3} \rangle^2 \ \sum_\alpha [b^+_\alpha(\mathbf{k}) b_\alpha(\mathbf{k}) - d^+_\alpha(\mathbf{k}) d_\alpha(\mathbf{k})] \ e^{-ik_3(a_3 - b_3)}.$$

In the case of asymmetric states of the confinement

$$\langle \omega, e^{ik_3 z} \rangle - \langle \omega, e^{-ik_3 z} \rangle \neq 0$$

then there is only a marginal change in the smearing

$$P_\mu = \int_{\mathbb{R}} da_3 db_3 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{\omega_{\mathbf{k}}} k_\mu \ \langle \omega, e^{-ik_3(x_3 - b_3)} \rangle \langle \omega, e^{+ik_3(x_3 - a_3)} \rangle \ \sum_\alpha [b^+_\alpha(\mathbf{k}) b_\alpha(\mathbf{k}) - d^+_\alpha(\mathbf{k}) d_\alpha(\mathbf{k})]$$

and this does not affect the statistics. Through the anti-commutation relation the total energy is bounded from below as in the unprojected case. However the situation changes if we apply a constant background electromagnetic field.

4.1.1 Statistics of Quasi Two Dimensional Electrons and Anyons

In the last section we examined that we can derive the Fermi statistics for electrons, when quantum-mechanically confined to $2+1$ dimensions by the same argument of positivity of the energy as in four dimensions. We now show what happens when a second particle is present and we apply a uniform static (electro) magnetic background field with $A_0(x^3) \ll m$ and at first $A^3 \equiv 0$. The one particle action of such a system is given by

$$S_{[\omega]} = \int_{\mathbb{R}^3} d^3x \ \tfrac{i}{2}[\bar{\psi}_{[\omega]} \gamma^\mu \overleftrightarrow{\partial}_\mu \psi_{[\omega]} - e\gamma^\mu A_\mu \bar{\psi}_{[\omega]} \psi_{[\omega]} - m\bar{\psi}_{[\omega]} \psi_{[\omega]}], \quad \mu = 0, 1, 2 \tag{4.1.5}$$

and transforms covariantly under local $U(1)$ transformations

$$\psi_{[\omega]}(x) \ \rightarrow \ e^{i\varphi} \psi_{[\omega]}, \quad \varphi = \oint \mathcal{A} \tag{4.1.6}$$

$$eA_\mu \ \rightarrow \ eA_\mu + \mathcal{A}. \tag{4.1.7}$$

The generated field is a pure gauge field $\mathcal{A} = d\varphi$ as long as there are no topological defects like holes or in four dimensions lines or cylinders. In this case it then defines a cohomology in the sense that we look at differential forms, which are closed $d\mathcal{A} = 0$ but not exact $\mathcal{A} \neq d\varphi$. If we take a closer look at these transformations we observe that loops of electrons are also confined to $2 + 1$ dimensions being paths smeared in z-direction, see also figure 4.1. Furthermore particles themselves turn out to become topological defects when being projected, actually holes in the two dimensional space for other particles, much in the same way as in the low dimensional case. Due to the Fermi statistics the paths of two Fermions can then not pass each other. Thus if we interchange two electrons represented by the quantum fields $\psi_{[\omega]_i}$ we acquire a phase φ corresponding to the magnetic flux entering the surrounded area, since there are loops being no more contractible to a point and therefore $\mathcal{A}$ is closed but not exact:

$$\psi_{[\omega]_i}\psi_{[\omega]_j} = -e^{i\varphi}\psi_{[\omega]_j}\psi_{[\omega]_i}.$$

This leads to anyon statistics since the phase can have any value and the minus sign respects the fact that we still have electrons and the Fermi statistics has to be recovered when we turn off the background field. Later on we argue that this phase can be considered to take fractional values and thus lead to fractional statistics. This manifests itself in a cohomology class and we will see that this will define a nontrivial second Chern Class and thus a topological number, the so-called Chern number.

Statistical Transmutation and Dimension

In the last section we saw that the statistical behavior of a particle depends intrinsically on its physical environment. We are interested in the situation where charged particles are confined to a $2+1$ dimensional sub-manifold $\mathbb{R}\times\Sigma$, Σ being a two dimensional Cauchy hyper-surface embedded in the usual Minkowski space $\mathbb{R}^{3+1}$. In this section we consider a simple approach just $\Sigma = \mathbb{R}^2$. We apply also an external magnetic field having at least a contribution perpendicular to Σ.

The standard textbook example from section 2.3.1 is also valid here. Let us have a look at the projected Lagrangian from the action (4.1.5). As in section 2.3.1 we want to add a total derivative $\bar{\psi}_{[\omega]}\psi_{[\omega]}d\theta$ of the polar angular function $\theta(\mathbf{x}_1 - \mathbf{x}_2)$ but now with the projected fields $\psi_{[\omega]}$ This is again only possible on $\mathbb{R}^2 - \mathbb{R}_+$. So we cannot describe closed loops within this approach. The path-integral point of view is here the same as in the usual case which means that the angular function gives a phase.

4.1.2 Equations of Motion and Lagrangian for Chern Simons Fields

There are of course again the inhomogeneous $d*\mathcal{F} = -*j$ and the homogeneous (structure equation) $d\mathcal{F} = 0$ Maxwell equations. As in the usual case electric transport properties are described by Ohm's law $\mathbf{j} = \sigma\mathbf{E}$ leading to the diffusion equation for the fields $\mathcal{A}$. Therefore we have to consider a quasi static system. However, Ohm's law intrinsically violates causality and is not useful for high energies. A relativistic generalization $\mathbf{j} = \gamma\sigma(\mathbf{E}+\boldsymbol{\alpha}\times\mathbf{B}-\boldsymbol{\alpha}(\boldsymbol{\alpha}\times\mathbf{E}))+\rho\boldsymbol{\alpha}$. may be considered elsewhere. It is so far not clear how to build up a relativistic generalization. For discussions on that topic see [Mei04] and references therein. We assume here the simplest case $\mathbf{j} = \sigma\mathbf{E}$, where the conductivity tensor is given by $\sigma = \sigma_H i\sigma_2 + \sigma_L\mathbb{1}_{2\times2}$. The following steps are analog to the usual procedure. For $\mathcal{A}_0 \equiv 0$ being a pure gauge, set to zero we derive from Ampére's law the diffusion equation $\nabla^2\mathcal{A}_i = \sigma_l\partial_t\mathcal{A}_i$. This means the gauge fields $\mathcal{A}_i$ have a imaginary contribution to the 'mass' σ_l. Since the diffusion equation is a non-relativistic equation we may consider instead the Dirac like equation $(i\gamma^\mu\partial_\mu - i\sigma_l)\Phi_i = 0$, $\Phi_i = (\mathcal{A}_i, \mathcal{C}_i)$ being a four component spinor. In the limit $\sigma_l \to 0$ we reobtain then the usual wave equation $\partial_\mu \partial^\mu \mathcal{A}_i = 0$ for massless (photon) fields (here we may neglect the spin properties), which we may obtain when σ_H also vanishes. However, in a fractional Hall state σ_H is nonzero and thus the Chern Simons gauge fields are massive. This appears naturally since they describe a massive current. In a fractional Hall state the longitudinal conductivity σ_l vanishes and thus we consider in the following only the case where $\sigma_l \equiv 0$. The fields $\mathcal{A}_i$ get again the contribution to the mass, if we move away from a Hall state. We have the same additional Chern Simons term in the Lagrangian as in the usual case (2.3.6):

$$\mathcal{L}_{\text{CS}} = \sigma_H \ \mathcal{A} \wedge \mathcal{F} = \sigma_H \mathcal{A} \wedge d\mathcal{A} \ = \sigma_H \ \varepsilon^{\mu\nu\rho} \mathcal{A}_\mu \partial_\nu \mathcal{A}_\rho \ d^3x.$$

Let us consider the non-projected Lagrangian in $3+1$ dimensions. Here we can also perform a local gauge transformation of the form (4.1.6). However the situation concerning the statistics is completely different. Closed loops are always contractible to a point as long as there are no one dimensional topological defects like lines or circles or we may also have cylinders or tori. In order to recover the situation of the $2+1$ dimensional system we have to choose at first a somehow artificial geometry. For example lets focus on the space $\mathcal{O} := \mathbb{R}^4 - \{x, y \in \mathbb{R} | x^2 + y^2 \geq l_B^2, l_B > 0\}$ where we removed the cylinder around the $z = x^3$ axis. The situation is now nearly the same as from above, there are now loops which can not be continuously transformed to a point and thus we have a nontrivial de Rahm cohomology. We may choose the one form

$$w := \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy + f(x) dz$$

with $f(x)$ being at least a smooth function on $\mathcal{O}$, but w.l.o.g. we can set $f \equiv 0$. Or mathematically more precise we can introduce an equivalence class $[w]$, with $w_1 \sim w_2$ iff $w_2 = w_1 + f(x)dz$. Since this is again pure classical electrodynamics we can proceed as usual and derive a field strength $f = da$, $a = eA - \mathcal{A}$ ($\mathcal{A}|_C = w|_C$ with $d\mathcal{A} \neq 0$) via the Yang-Mills construction and introduce a coupling to a current. The particles get a phase. We can consider a Maxwell like term in the Lagrangian to derive the equations of motions as mentioned above. But for Ohms Law we would require a Chern Simons term. Here we may be tempted to use the antisymmetric term of $\sigma_H f \wedge f = \sigma_H dW$, with $W = a \wedge f$. However, since this is a total derivative of the Chern Simons Lagrangian it is defined only on some boundary of a spacetime region and if we take the whole Minkowski space this term vanishes due to the vanishing of all fields at infinity. We may consider the three sphere S^3 as boundary of our system and obtain Ohm's law on the S^3 (respectively $\mathbb{R} \times S^2$). But the Chern Simons term onto that manifold can not be mapped onto a Hall sample without doing some artificial assumptions, following the discussion in [Kos09]. Furthermore the quantum-mechanical projection of this sometimes called Pontryagin term results not in the Ohm's equations of motion.

The statistics of the particles is also not changed as long as we do not consider the electron moving with this cylinder together and we require that the cylinders can not pass each other. Two such objects of course would lead to anyon statistics since their motion is limited. This sounds somehow artificially, however this is now exactly the point where the projection enters successfully. Via the quantum-mechanically projection the system turns

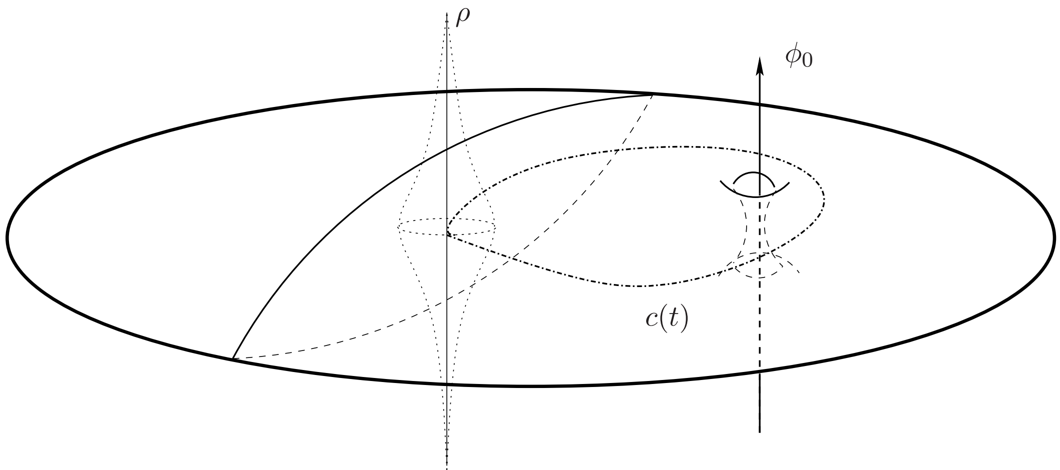


Figure 4.1: The flux ϕ_0 can be attached to the projected electron by a Chern Simons transformation. The integral along the closed path $c(t)$ over the closed form ω depeding on the charge density ρ generates a nontrivial phase.

over in the case where the cylinder can not pass each other due to the repulsive interaction between the electrons compare Figure 4.1. So we find $\mathcal{A} \to \mathcal{A}_{[\omega]}$. When we require Ohm's law we find the usual Chern Simons Lagrangian and equations of motion

$$j^{\mu}_{[\omega]} = \sigma_H \varepsilon^{\mu\nu\rho} f_{[\omega]\nu\rho}.$$

This justifies also the naive approach from the previous section which can now be replaced by the projection method.

Equations of Motion for Nonabelian Chern Simons Gauge Fields

The considerations from above can easily be generalized to nonabelian gauge fields. If we choose a system with more than one Hall layers or at least some domain with arbitrary angular momentum, the electron couples in the most general case to a nonabelian field, a $Lie(U(n))$ algebra valued one form $\mathcal{A} = \sum_i T^i \alpha_\mu(x) dx^\mu$. T^i being the generators of the Lie algebra denoted in the previous chapter. As in the last section we can derive the field strength tensor (curvature) via the covariant derivative $D_\mu = i\partial_\mu - eA_\mu + \mathcal{A}_\mu$, $\mathcal{A}_\mu(x) = \sum_i T^i \alpha_\mu(x)$ and use the Yang-Mills construction

$$f_{\mu\nu} = [D_\mu, D_\nu] \;\Leftrightarrow\; f = d^a a = da + a \wedge a,$$

with $a := -eA + \mathcal{A}$ and the (exterior) covariant derivative $d^a = d + 1/2[a, \cdot]_\wedge$. If we now ask for the equations of motion then we have instead of the Maxwell equations the homogeneous (structure equation) $d^a * f = 0$ and the inhomogeneous (second Bianchi identity) $d^a f = 0$ Yang-Mills equations in the absence of matter fields. Again we require Ohm's law and derive for $\sigma_l = 0$

$$j^\mu := \sum_A j^\mu_A = \sigma_H \ \mathrm{tr}[\varepsilon^{\mu\nu\rho} \hat{e} \ f_{\nu\rho}], \tag{4.1.8}$$

with $\hat{e} = \sum_A T^A$ leading to an nonabelian Lagrangian

$$\mathcal{L} = -a \wedge *j + \sigma_H \ \mathrm{tr} a \wedge f.$$

So far these equations are similar to the one in the abelian case at first glance. However, there is a significant difference concerning the cubic self-interacting term in the Lagrangian resulting in a quadratic contribution to the current. If we compare this Lagrangian to the usual Chern Simons Lagrangian we find that

$$\mathcal{L}_{\mathrm{CS}} = \mathcal{L} - \tfrac{1}{3}\mathrm{tr}[a \wedge a \wedge a].$$

The gauge field a considered as being represented by a Hermitian $n \times n$ matrix-valued one form leads to the expression of the cubic term:

$$\mathrm{tr}[a \wedge a \wedge a] = a^i{}_j \wedge a^j{}_k \wedge a^k{}_i = (a_\mu)^i{}_j \ (a_\nu)^j{}_k \ (a_\rho)^k{}_i \ dx^\mu \wedge dx^\nu \wedge dx^\rho.$$

We may therefore require the current

$$j^\mu = \sigma_H \ \mathrm{tr}[\varepsilon^{\mu\nu\rho} \ \hat{e}(\partial_\nu a_\nu + \frac{1}{3} a_\nu a_\rho)]$$

as Chern Simons current, however this is not a gauge invariant expression contrary to (4.1.8). We may require gauge invariance only for the equation of motion and not for the action, since neither Lagrangians are gauge invariant in general, see the discussion in the previous chapter.

From a systematic point of view the latter definition of the current might appear to be more convenient since one can derive it consistently via the Chern Simons Lagrangian for any $Lie(U(n))$ valued gauge fields. The former definition is a naive extrapolation of the abelian constrained. However, if we require gauge invariance we have to add the cubic term in the Lagrangian. So far the cubic term is not discussed in the context of $SU(2)$ composite Fermions since the coupling enters cubically rather than quadratically. However it may play a rôle in the quantization procedure of the Chern Simons gauge fields.

4.2 Covariant Effective Model

So far we have established a rather abstract formalism, but it is interesting to see how the common theories change if we use the projection method to derive a quasi covariant (only $SO(2,1)$ invariant) composite Fermion model. The Lagrangian (2.3.8) changes to the covariant Lagrangian where we assume the z-confinement to be delta like. We start with the ansatz $x \in \mathbb{R}^3$:

$$\mathcal{L}^P(x) \quad = \quad \mathcal{L}_0^P(x) + \mathcal{L}_{\mathrm{CS}} P(x) + \mathcal{L}_C^P(x), \tag{4.2.1}$$

the first part is the projected free fermion part

$$\mathcal{L}_0^P(x) \quad = \quad \bar{\psi}(x)^P \left[\gamma^\mu \big(p - e(A^P(x) - \mathcal{A}^P(x))\big)_\mu - m - \gamma^0 \mu\right] \psi^P(x) \tag{4.2.2}$$

with $\mu = 0, 1, 2$, and for the Chern Simons action we use the abelian $U(1)$ version

$$\mathcal{L}^P_{\mathrm{CS}}(x) \quad = \quad \frac{e}{\tilde{\varphi}\phi_0}\varepsilon^{\mu\nu\rho}\mathcal{A}^P_\mu(x)\partial_\nu\mathcal{A}^P_\rho(x). \tag{4.2.3}$$

In the quantization procedure we may proceed in the Coulomb gauge to be able to directly compare the theory with the common theories. The Coulombian interaction is here a problem since it violates intrinsically causality. In the usual quantum electrodynamics this Coulombian term is removed by counter-terms to repair covariance and to get correct (!) results. In this spirit one may introduce the Coulombian interaction and discuss what the impacts are and if it can be removed in an S-matrix formulation. We take for the potential $V(\mathbf{x}-\mathbf{y}) = e^2/\epsilon|\mathbf{x}-\mathbf{y}|$:

$$\mathcal{L}^P_{\mathrm{C}}(x) \quad = \quad -\frac{1}{2}\int d\mathbf{y}\ \rho^P(x)V(\mathbf{x}-\mathbf{y})\rho^P(y). \tag{4.2.4}$$

The equations of motion for the Chern Simons fields are again obtained by varying the action with respect to the fields $\mathcal{A}^P_\mu$

$$\frac{\delta S^P}{\delta\mathcal{A}^P_\mu} = 0$$

the zero component leads to the relation discussed above

$$\varepsilon^{ij}\partial_i\mathcal{A}^P_j(\mathbf{x},t) = \tilde{\varphi}\phi_0\ \rho^P(\mathbf{x},t) \tag{4.2.5}$$

and the charge density $\rho^P(\boldsymbol{\eta},t)$ can in this way also be replaced in the Coulomb part of the action by $(\tilde{\varphi}\phi_0)^{-1}\varepsilon^{ij}\partial_i\mathcal{A}^P_j(\mathbf{x},t)$.

4.2.1 Mean Field and Random Phase Approximation in the Covariant Model

We can divide the magnetic field as in the common theory in a mean field with a dynamical field respecting the fluctuations

$$b(x) = B - \mathcal{B} = B - \tilde{\varphi}\phi_0\rho^P(x). \tag{4.2.6}$$

The Chern Simons field is then transformed also in a mean field and a dynamical field:

$$a^P_\mu(\boldsymbol{\eta},t) = \mathcal{A}^P_\mu(\mathbf{x},t) - \langle\mathcal{A}^P_\mu\rangle. \tag{4.2.7}$$

Since the average $\langle\mathcal{A}^P_\mu\rangle$ is per definition not dynamical it gives no contribution to the equations of motion. Then the effect that it reduces the external field is the same as in the common theory. Explicitly $\partial_i\langle\mathcal{A}^P_j\rangle = 0$ and only the dynamical part a^P_μ contributes to the equation of motions and this leads to the relation:

$$\rho^P(x) = \frac{1}{\tilde{\varphi}\phi_0}\ \varepsilon^{ij}\partial_i a^P_j(x). \tag{4.2.8}$$

Single Particle Treatment

We proceed with the single particle Lagrangian with the chemical potential $\mu = 0$. The projected Fermionic fields $\psi^P = \psi_{[\omega]}$ is given by (4.1.2):

$$\psi_{[\omega]}(x^0, x^1, x^2) = \int_{\mathbb{R}} da_3 \int_{\mathbb{R}^3} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{\omega_{\mathbf{k}}} \quad \sum_\alpha \quad \big(b_\alpha(\mathbf{k}) u^{(\alpha)}(\mathbf{k}) e^{-ik_\mu x^\mu} \langle \omega, e^{ik_3(z-a_3)} \rangle + d_\alpha^+(\mathbf{k}) v^{(\alpha)}(\mathbf{k}) e^{ik_\mu x^\mu} \langle \omega, e^{-ik_3(z-a_3)} \rangle \big).$$

We choose a delta like confinement potential which leads to the fields with $\mu = 0, 1, 2$:

$$\psi(x^0, x^1, x^2) = \int_{\mathbb{R}^3} \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{m}{\omega_{\mathbf{k}}} \quad \sum_\alpha \quad \big(b_\alpha(\mathbf{k}) u^{(\alpha)}(\mathbf{k}) e^{-ik_\mu x^\mu} \tag{4.2.9}$$

$$+ \quad d_\alpha^+(\mathbf{k}) v^{(\alpha)}(\mathbf{k}) e^{ik_\mu x^\mu} \big). \tag{4.2.10}$$

The free propagator of the covariant free massive charge carriers can be derived in the usual way through the two point function

$$\begin{aligned} \langle \psi_a(x) \bar{\psi}_b(y) \rangle &= \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{m^2}{\omega_{\mathbf{k}}^2} \sum_\alpha u^{(\alpha)}(\mathbf{k})_a v^{(\alpha)}(\mathbf{k})_b e^{-ik_\mu(x^\mu - y^\mu)} \\ &= (i\partial\!\!\!/_x + m)_{ab} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{m^2}{\omega_{\mathbf{k}}^2} e^{-ik_\mu(x^\mu - y^\mu)} \end{aligned} \tag{4.2.11}$$

and

$$\begin{aligned} \langle \bar{\psi}_b(y) \psi_a(x) \rangle &= \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{m^2}{\omega_{\mathbf{k}}^2} \sum_\alpha v^{(\alpha)}(\mathbf{k})_b u^{(\alpha)}(\mathbf{k})_a e^{ik_\mu(x^\mu - y^\mu)} \\ &= -(i\partial\!\!\!/_x + m)_{ab} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{m^2}{\omega_{\mathbf{k}}^2} e^{-ik_\mu(x^\mu - y^\mu)} \end{aligned} \tag{4.2.12}$$

such that the retarded Green's function is

$$S_R^{ab}(x - y) = \theta(x^0 - y^0) \langle \{\psi_a(x), \psi_b(y)\} \rangle. \tag{4.2.13}$$

Being a Green's function of the projected Dirac operator

$$(i\partial\!\!\!/_x - m) S_R(x - y) = i\delta^{(3)}(x - y) \mathbb{1}_{3\times 3} \tag{4.2.14}$$

the Fourier transformed Green's function $\check{S}_r(k)$ is determined by the equation

$$\int \frac{d^3k}{(2\pi)^3} (p\!\!\!/ - m) \; e^{-ik_\mu(x^\mu - y^\mu)} \; \check{S}_r(k) = i\delta^{(3)}(x - y). \tag{4.2.15}$$

So the Fourier transformed propagator

$$\check{S}_R = \frac{i}{p\!\!\!/ - m} = \frac{i(p\!\!\!/ + m)}{p^2 - m^2} \tag{4.2.16}$$

can be integrated in the complex plane in k^0 with $\epsilon > 0$.

$$S_F(x-y) = \int d^3k(2\pi)^3 \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} e^{-ik_\mu(x^\mu - y^\mu)} \tag{4.2.17}$$

$$= \begin{cases} + & \langle \psi(x)\bar{\psi}(y)\rangle \text{ for } x^0 > y^0 \\ - & \langle \bar{\psi}(y)\psi(x)\rangle \text{ for } y^0 > x^0 \end{cases} \tag{4.2.18}$$

and the time-ordered product can be defined through the propagator by

$$\langle T^{[x^0,y^0]}\psi(x)\psi(y)\rangle := S_F(x-y) \tag{4.2.19}$$

The Chern Simons Lagrangian is the same as in the low energy limit and if we introduce a coulomb interaction term then the polarization function of the gauge field remains the same. Then the contribution in the random phase approximation is also the same. The only problem we have is that a Coulomb interaction term violates causality.

4.3 Projection onto the Lowest Landau Level

4.3.1 Relativistic Landau Levels

In this section we follow some discussion on stationary states solutions of the Dirac equation in view of Landau levels. A similar example can be found in some textbooks for example [CI06]. Nevertheless it should be mentioned here since it has been paid not so much attention so far in context of electrons in quasi two dimensions.

We apply a homogeneous constant magnetic field in "z-direction" to the Dirac equation (in four dimensions) and choose a fixed gauge, the symmetric gauge $A_1 = 1/2Bx^2$ and $A_2 = -1/2Bx^1$, $A_0 = A_3 = 0$. Then a charged particle performs a cyclotron motion. For a stationary solution of the energy $\mathcal{E}$ take a separation ansatz $\psi = exp\{-i\mathcal{E}x^0\}(\varphi, \chi)$. The Dirac equation gives two coupled equations:

$$\begin{aligned} (\mathcal{E}-m)\varphi &= \boldsymbol{\sigma}\cdot(\mathbf{p}-e\mathbf{A})\chi, \\ (\mathcal{E}+m)\chi &= \boldsymbol{\sigma}\cdot(\mathbf{p}-e\mathbf{A})\phi. \end{aligned}$$

These equations can be decoupled and we receive a Hamiltonian of a harmonic oscillator and a free part in the third component.

$$(\mathcal{E}^2 - m^2)\varphi = \left[\mathbf{p}^2 + \frac{e^2B^2}{4}(x_1^2 + x_2^2) - \frac{eB}{2}(\sigma_3 - 2x_1p_2 + 2x_2p_1)\right]\varphi. \tag{4.3.1}$$

We also see that we could have introduced an electrostatic (harmonic) confinement potential in the third component and would have a quasi two dimensional system. Now we search for solutions of the form $\varphi(\mathbf{x}) = \exp\{i(k_3x_3)\}f(x_1, x_2)$ and introduce new coordinates. We split the coordinates (x_1, x_2) in relative coordinates:

$$\zeta_1 = \tfrac{1}{eB}\left(p_2 + \tfrac{eB}{2}x_1\right), \quad \zeta_2 = -\tfrac{1}{eB}\left(p_1 - \tfrac{eB}{2}x_2\right)$$

and guiding center

$$\eta_1 = x_1 - \tfrac{1}{eB}\big(p_2 + \tfrac{eB}{2}x_1\big), \quad \eta_2 = x_2 + \tfrac{1}{eB}\big(p_1 - \tfrac{eB}{2}x_2\big).$$

They fulfill the commutation relations $[\eta_1, \eta_2] = il_B^2$ and $[\zeta_1, \zeta_2] = -il_B^2$ with $l_B = \sqrt{\hbar/eB}$. ζ_1 is therefore the canonical conjugate operator to ζ_2 and so is η_1 to η_2, and we may write them in the Schroedinger representation $\zeta_2 = -i\partial_{\zeta_1}$ or vice versa. The equation (4.3.1) then defines a harmonic oscillator and is degenerated due to absence of the guiding center coordinates. We may introduce the corresponding creation and annihilation operators which fulfill the commutation relations $[a, a^+] = [b, b^+] = 1$:

$$a = \tfrac{1}{\sqrt{2}l_B}(\zeta_2 - i\zeta_1), \quad b = \tfrac{1}{\sqrt{2}l_B}(\eta_2 - i\eta_1)$$

and the corresponding ground states $a\,|0_\zeta\rangle = b\,|0_\eta\rangle = 0$. Finally we end up with the equation:

$$\mathcal{E}^2 f = [2eB(a^+a + \tfrac{1}{2}) + m^2 + k_3^3 + \tfrac{eB}{2}\sigma_3]f.$$

As usual we introduce $n = \langle a^+a\rangle$ to be the eigenvalues of the number operator and we may also define the cyclotron frequency by $\omega_c = eB/m$. Furthermore if f is an eigenfunction of $\hat{s} = \sigma_3/2$ with eigenvalues $s = \pm 1/2$ we have the energy levels:

$$\mathcal{E}^2 = m\omega_c[(n + \tfrac{1}{2}) - s] + m^2 + k_3^2. \tag{4.3.2}$$

The energy is degenerated in $(n, s = -1/2) \hat{=} (n+1, s = 1/2)$ and in b^+b which refers to the arbitrariness of the angular momentum. As mentioned before we could have turned on a confining potential and instead of k_3^2 we would have some descrete energy levels $\mathcal{E}_3^i$. The solutions of the system (4.3.2) are easily written down:

$$\varphi_{n,l,s}(x_1, x_2) = \frac{1}{l_B\sqrt{(2\pi)n!(l+n)!}}(a^+)^n(b^+)^{n+l}e^{-\frac{x_1^2+x_2^2}{4l_B^2}}\ v_s\ e^{-ik_3x_3}$$

and $v = (1, 0)$ for $s = 1/2$ and $v = (0, -1)$ for $s = -1/2$. The energy spectrum can take continuous values as long as we do not confine the system in the third component, but once we turn on the confinement potential the spectrum is discrete and the discrete levels we may call relativistic Landau levels (RLL). From a physical point of view we could prefer only the low energy limit of the spectrum (4.3.2): $E = \omega(n + 1/2 + s) + m$. However, from a technical point of view there is so far no reason for that, furthermore we take automatically care of the spin.

The discussion from above may also be applied to some situation in graphene, where the charge carriers at the so-called Dirac point are described by massless Dirac Fermions. Just send the mass in the upper equation to zero. The cyclotron frequency for massless particles then appears to be defined by the applied magnetic field only $\omega_c^0 := eB$. However the situation in graphene is a little bit more complicated which we will not discuss here.

Projection onto the Lowest Landau Level

We recognized in the last section that a charged particle propagates with cyclotron motion in the presence of an external magnetic field. We assume the external magnetic field again to be parallel to the $x_3 = z$-axis (otherwise perform a rotation) and decompose similarly the coordinates in the x-y-plane into the guiding center

$$\eta_1 = x - \frac{1}{eB_z}P_y,\ \eta_2 = x + \frac{1}{eB_z}P_x$$

and relative coordinates

$$\zeta_1 = \frac{1}{eB_z}P_y,\ \zeta_2 = -\frac{1}{eB_z}P_x.$$

Where $P_i = -i\hbar\partial_i - eA_i$ and in the case of composite particles $P_i = -i\hbar\partial_i - a_i$ and b_z instead of eB_z. We may pull out the factor $eB_z/2$ and respectively $b_z/2$ from the gauge fields and define the magnetic length as before $l_B = \sqrt{\hbar/eB_z}$ and the composite Fermion magnetic length as $l_b = \sqrt{\hbar/b_z}$ and require the following Weyl conditions:

$$e^{i\alpha_1\eta_1}e^{i\alpha_2\eta_2} = e^{i(l_B^2/2)\alpha_1\alpha_2}e^{i(\alpha_1\eta_1+\alpha_2\eta_2)}$$

and

$$e^{i\beta_1\zeta_1}e^{i\beta_2\zeta_2} = e^{-i(l_B^2/2)\beta_1\beta_2}e^{i(\beta_1\zeta_1+\beta_2\zeta_2)}$$

A similar construction to the localization in the z-direction is applied to this case here. The Fourier factor can be expressed in terms of the guiding center and relative coordinates and the latter we transform into the usual creation and annihilation operators:

$$e^{ik_1x^1+ik_2x^2} = e^{ik_1\eta^1+ik_2\eta^2+ik_1\zeta^1+ik_2\zeta^2} = e^{-\frac{l_B^2}{4}(k_1^2+k_2^2)}e^{ik_1\eta^1+ik_2\eta^2}e^{ka^+}e^{i\bar{k}a}$$

with $k = k_1 + ik_2$ and $a = 1/(\sqrt{2}l_B^2)(\zeta_2 - i\zeta_1)$, such that $a\,|0\rangle_a = 0$. The projection on the lowest Landau level now means that we evaluate the Weyl operator in the ground state with respect to a.

$$\omega_{\mathrm{LLL}}(e^{ik_1x^1+ik_2x^2}) = e^{-\frac{l_B^2}{4}(k_1^2+k_2^2)}e^{ik_1\eta^1+ik_2\eta^2}.$$

So the x-y-plane is transformed into a noncommutative plane with coordinates (η_1, η_2) due to the noncommutativity of the η_i. These operators are actually unbounded, selfadjoint operators (per definition of the domains $\mathscr{D}(\eta_i)$) on an infinite dimensional Hilbert space $\mathscr{H}_{\mathrm{LLL}}$. On the noncommutative plane the translation form a nonabelian group by

$$e^{i\mathbf{k}\boldsymbol{\eta}}e^{i\mathbf{q}\boldsymbol{\eta}} = e^{i(\mathbf{k}+\mathbf{q})\boldsymbol{\eta}}\ e^{il_B^2\det(\mathbf{k},\mathbf{q})}$$

or

$$[e^{i\mathbf{k}\boldsymbol{\eta}}, e^{i\mathbf{q}\boldsymbol{\eta}}] = 2i\ e^{i(\mathbf{k}+\mathbf{q})\boldsymbol{\eta}}\ \sin(l_B^2\det(\mathbf{k},\mathbf{q})).$$

Thus the translation becomes a magnetic translation, which means that we get in addition a multiplicative phase factor or twist factor.

In an homogeneous external field the translation invariance, Poincaré invariance respectively, is broken by fixing a gauge, thus a translation must be accompanied by a compensating gauge transformation. In this sense the fields transform covariantly under translation.

4.3.2 Combined Projection of Fields

The upper projections can be performed by a single projection map $(P_\eta)^2 = P_\eta$, which can be defined by

$$
\begin{aligned}
P_\eta(f) &= \langle I \otimes \omega_{z_0} \otimes \omega_{\mathrm{LLL}}, f \rangle \\
&= \int \frac{d^3\mathbf{k}}{\sqrt{2\pi}^3} \langle I, \check{f}(k) \rangle \langle \omega_a, e^{ik_3 x^3} \rangle \langle \omega_{\mathrm{LLL}}, e^{ik_1 x^1 + ik_2 x^2} \rangle \\
&= \int \frac{d^3\mathbf{k}}{\sqrt{2\pi}^3} \check{f}(k) \, e^{-\frac{1}{4}(l_i k^i)^2} e^{i\mathbf{k}\boldsymbol{\eta}}
\end{aligned}
$$

with $\mathbf{l} = (l_1, l_2, l_3) = (l_B, l_B, l_z/\sqrt{2})$ and $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$, $\eta_3 = z_0$ from above. Therefore we may define the new fields by

$$\psi(\eta) := P_\eta(\psi(x)).$$

However, at this point we should be more precise on the objects we have so far derived. A smeared field operator we can define straight forward by evaluating the projected field again in not necessarily coherent states ω_0, more precisely we evaluate the guiding center coordinates in a state. Choosing a pure state corresponds to the best possible localization in space. We also have to introduce an adiabatic cut-off in time $g \in \mathcal{S}(\mathbb{R})$:

$$
\begin{aligned}
\psi(\eta) \;\mapsto\; \langle I \otimes \omega_0, \psi(\eta) \rangle(g) &= \int_{\mathbb{R}^4} d^4k \int_{\mathbb{R}^4} \frac{d^4a}{(2\pi)^4} \langle I, \psi(a) e^{-ik_\mu a^\nu} \rangle \, e^{-\frac{(l_i k^i)^2}{4}} \langle \omega_0, e^{ik_\mu \eta^\mu} \rangle \\
&= \int_{\mathbb{R}^4} d^4a \; \psi(a) g(a^0) h(\boldsymbol{a}).
\end{aligned}
$$

From the LLL and $2+1d$ projection it follows that $h \in \mathcal{S}(\mathbb{R}^3)$ if the corresponding states belong to the Schwarz space:

$$h(\boldsymbol{a}) := \int \frac{d^3\mathbf{k}}{(2\pi)^3} \, e^{-\frac{(l_i k^i)^2}{4}} \langle \omega_0, e^{ik_\mu \eta^\mu} \rangle \, e^{-ik_\mu a^\nu}.$$

As an example we can apply the projection map P_η to the Dirac field operator:

$$P_\eta(\psi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} d^3\mathbf{k} \, \frac{m}{\omega_{\mathbf{k}}} \, e^{-\frac{1}{4}(l_i k^i)^2} \sum_\alpha \left(b_\alpha(\mathbf{k}) u^{(\alpha)}(\mathbf{k}) e^{-ik_\mu \eta^\mu} + d_\alpha^+(\mathbf{k}) v^{(\alpha)}(\mathbf{k}) e^{ik_\mu \eta^\mu} \right)$$

where $(\eta_\mu) = (\eta_0, \eta_1, \eta_2, \eta_3)$ with $\eta_0 = x_0$. W.l.o.g we can set $\eta_3 \equiv 0$. The Dirac field operator in the lowest Landau level on the plane is then defined by $\psi(\eta) := P_\eta(\psi)$. However, now we have fields depending on coordinates which are themselves non-commuting operators. In this sense we have a special realization of a noncommutative field theory in 2+1 dimensions, where the time is central.

4.4 Products and Wick Products of Projected Fields

Defining the Wick products in the projected case needs some effort and there is up to date no unique ansatz. A suitable definition of the Wick products, which also works in curved spacetimes, is given by i.e. (n=2):

$$:\psi(x_1)\psi(x_2): := \lim_{x\to y} \psi(x_1)\psi(x_2) - \langle\Omega|\,\psi(x_1)\psi(x_2)\,|\Omega\rangle\,.$$

Being well defined for common field operators on the four dimensional Minkowski space this definition looses its meaning in the case of the projected fields $\psi(\eta)$, since the limit of coinciding points $\eta_{i1} \to \eta_{i2}$ of the noncommutative coordinates can not be performed. At this point the best we can do is evaluating the relative coordinates $\eta_{i12} := \frac{1}{2}(\eta_{i1} - \eta_{i2})$ in best localized states, the coherent states, which may be represented by the ground state of an harmonic oscillator. This is much in the spirit of the Quantum Diagonal Map introduced by Bahns, Doplicher, Fredenhagen and Piacitelli in order to get well defined Wick-ordered products of fields on the noncommutative Minkowski space [BDFP03]. Indeed the situation with the projected field operators is somehow a special realization of a field theory on a two-plus-one dimensional noncommutative Minkowski space where the time commutes with the spacial coordinates. Furthermore our case seems to be rather harmless since we already have smeared field operators, they are smeared intrinsically with a Gaussian function. There are some more or other choices of the Wick product [DFR95] and we will comment on that in this section.

But first we have to say what products of projected fields are. The projected fields describe a noncommutative quantum field theory, a field theory on a two dimensional noncommutative space with coordinates (η^1, η^2), which are considered as unbounded, selfadjoint operators on an infinite dimensional Hilbert space and $\eta^0 \equiv x^0$ is central. They fulfill the commutation relation

$$[\eta^1, \eta^2] = il_B^2 \text{ and } [\eta^0, \eta^i] = 0.$$

This means that the commutator and the zero component are trivial elements of the center, which is itself trivial in this case, of the algebra. Through the introduction of the Weyl system the problem with unbounded operators and their domains is circumvented and we can define a noncommutative algebra with the product ($\mu = 0, 1, 2$ and $i = 1, 2$, $k \in \mathbb{R}^3$ and $\mathbf{k} \in \mathbb{R}^2$)

$$(f \star g)(\eta) = \frac{1}{(2\pi)^3} \int d^3k d^3q\ \check{f}(k)\check{g}(q)\ e^{-\frac{l_B^2 \mathbf{k}^2}{4}} e^{-\frac{l_B^2 \mathbf{q}^2}{4}}\ e^{-\frac{il_B^2}{2}\det(\mathbf{k},\mathbf{q})}\ e^{-i(k_\mu+q_\mu)\eta^\mu}.$$

This product is related to the (Wigner-) Moyal star product or twisted star product and so the noncommutative plane is related to the Moyal plane. The difference is the appearance of the Gaussian functions in the spacial like components. An involution is given by

$$f(\eta)^* = \frac{1}{\sqrt{2\pi}^3} \int d^3k\ \overline{\check{f}}(k)\ e^{-\frac{l_B^2 \mathbf{k}^2}{4}}\ e^{ik_\mu \eta^\mu}.$$

A C^*-norm can be defined by

$$\|f\| := \frac{1}{(\sqrt{2\pi})^3} \int d^3k |\check{f}(k)|\ e^{-\frac{l_B^2 \mathbf{k}^2}{4}}$$

which fulfills the C^*-condition:

$$\begin{aligned} \|f^* f\| &= \frac{1}{(2\pi)^3}\int d^3k|\check{f}(k)|^2\; e^{-\frac{l_B^2\mathbf{k}^2}{2}} \\ &= \|f\|^2. \end{aligned}$$

Now let $\psi_1,\dots\psi_n$ be the projected fields, thus the non-commutative field operators. A product of fields is now defined via the star product:

$$(\psi_1\star\dots\star\psi_n)(\eta) := \frac{1}{(2\pi)^{3n}}\int d^3k_1\dots d^3k_n\; r(k_1,\dots,k_n;\eta)\otimes\check{\psi}_1(k_1)\dots\check{\psi}_n(k_n)$$

where $r(k_1,\dots,k_n;\eta)$ is a non-local kernel and the Gaussian factors. We may formulate this in terms of symbols [DFR95] so let $f(\eta)=\int d^3k\;\check{f}(k)e^{ik\eta}$. The symbol of $f(\eta)$ is defined as $f(x)=\int d^3k\;\check{f}(k)e^{ikx}$. The multiplication of these symbols is given by

$$(f_1\dots f_n)(x) = \int d^3x_1\dots d^3x_n\check{f}(x_1)\dots\check{f}(x_n)C_n(x_1-x,\dots,x_n-x) \tag{4.4.1}$$

with the non local kernel

$$C_n(x_1-x,\dots,x_n-x) = \frac{1}{(2\pi)^3}\int d^3k_1\dots d^3k_n\; e^{i(\sum_j k_jx_j+\frac{l_B^2}{2}\sum_{j<l}\varepsilon^{lm}k_{lj}k_{mi})} \tag{4.4.2}$$

with $j,i=1,\dots,n$ and $l,m=1,2$. So we may write for the kernel

$$r(k_1,\dots,k_n;\eta) = e^{i(\sum_j k_j\eta_j+\frac{l_B^2}{2}\sum_{j<l}\varepsilon^{lm}k_{lj}k_{mi})}. \tag{4.4.3}$$

A simple choice of the Wick product is then the following:

$$\begin{aligned} (\psi_1\star\dots\star\psi_n)(\eta) := \frac{1}{(2\pi)^{3n}}\int d^3k_1\dots d^3k_n \quad & \mathbf{:}\check{\psi}_1(k_1)\dots\check{\psi}_n(k_n)\mathbf{:}\otimes \\ & \otimes e^{-\frac{l_B^2\mathbf{k}_1^2}{4}}e^{ik_1\eta_1}\dots e^{-\frac{l_B^2\mathbf{k}_n^2}{4}}e^{ik_n\eta_n}. \end{aligned} \tag{4.4.4}$$

Due to the non-local kernel these Wick products are not local in terms of the concept of q-locality proposed in [BDFP05], which enters here since $\boldsymbol{\eta}=\pi(q)$ is a special realization of the quantum coordinates q^μ. In addition some finite terms are also subtracted. The concept of quasi-planar Wick products [BDFP05] addresses these problems, only the infinite and local counterterms are subtracted.

We may comment that in a noncommutative field theory the commutative C^*-algebra of the usual commutative field theory on the commutative Minkowski space is replaced by noncommutative C^*-algebras of the symbols, roughly speaking. The requirement of a C^*-algebra in general is useful to define a representation on a suitable Hilbert space. It can be proved that every C^*-algebra is isomorphic to a norm closed algebra of bounded operators on a Hilbert space [BR02, Th:2.1.11] or [Dix82, RS75]. Though the represetaion of a given C^*-algebra is not unique in general. The construction of such a Hilbert space is known under the GNS-construction after Gelfand Neimark and Segal [BR02, Dix82, RS75]. Here we just point out that the projected fields allow for such a treatment but we will not follow the details further.

The projected fields are fields on the noncommutative plane and we also need to define derivatives on that spacetime since we want to define field equations. In the usual commutative Minkowski space we define derivatives with the generators of the infinitesimal spacetime translations. We consider in a general case the action of the Poincaré group in three dimensions $\tau_{a,\Lambda} \in \mathcal{P}^{\uparrow}$, $a \in \mathbb{R}^3$, $\Lambda \in \mathcal{L}(\mathbb{R}^3)$ and $\det \Lambda = \pm 1$ on the symbols $f \in \mathcal{C}_{\prime} \times \mathbb{R}^3$:

$$(\tau_{a,\Lambda} f)(\eta) := f(\lambda^{-1}(\eta - a)) \det \Lambda.$$

The derivation on the symbols $f(\eta)$ can be defined with the automorphism $\tau_{a,\mathbb{1}}$ of the spacetime translations in the following sense:

$$\begin{aligned} \partial_\mu f(\eta) &:= -\frac{d}{da^\mu}\Big|_{a=0} (\tau_{a,\mathbb{1}} f)(\eta) &\quad (4.4.5)\\ &= \frac{d}{da^\mu}\Big|_{a=0} f(\eta + a). &\quad (4.4.6) \end{aligned}$$

In this sense the usual derivative in the unprojected field equation is replaced by the non-commutative analog when we do the projection:

$$\frac{\partial}{\partial x^\mu}\ f(x) \quad \mapsto \quad \partial_\mu f(\eta) = \frac{d}{da^\mu}\Big|_{a=0} f(\eta + a).$$

4.4.1 Simultaneous Projection of Products of Quantum Fields

On a formal level we derive these products of noncommutative field operators if we project the usual product of commutative fields. So let P_η be the projection of a single field as defined in the previous section, let $W(\psi_1, \ldots, \psi_n)$ be a product of field operators on a suitable (Fermionic) Fock space. The map $\boldsymbol{P}_\eta^n$ defined by

$$\boldsymbol{P}_\eta^n(W) := \langle P_\eta \otimes \ldots \otimes P_\eta, \psi_1 \otimes \ldots \otimes \psi_n \rangle$$

maps simultaneously all n-fields in the lowest Landau level on a quasi two-plus-one dimensional system.

$$\boldsymbol{P}_\eta^n(W) = \psi_1^P * \psi_2^P * \cdots * \psi_n^P$$

with $\psi_i^P \equiv \psi(\eta)$, defined in the previous section and the star denotes the product on the field algebra of the projected fields, which is related to the Wigner Moyal star product ($\star$) on the noncommutative plane. For example we may apply this map to a Lagrangian term:

$$\boldsymbol{P}_\eta^3(\bar{\psi} \otimes \gamma^\mu a_\mu \otimes \psi) = (\bar{\psi}^P \star (\gamma^\mu a_\mu^P) \star \psi^P)(\eta).$$

instead of the map $\langle I \otimes I \otimes I, \bar{\psi} \otimes \gamma^\mu a_\mu \otimes \psi \rangle$. The meaning of $a_\mu^P \equiv a_\mu(\eta)$ we will discuss later on.

Projection of a Coulomb Potential

The problem which arises in context with the Coulomb potential is, that it is not apriori Fourier transformable. However this problem can be circumvented by the use of a suitable limit of a Yukawa like potential together with the theorem of B. Levi on the monotonic convergence, the theorem on the majorized convergence by Lebesgue also holds. The projection is applied to a special representation of the Coulomb potential in terms of a Yukawa potential, which is Fourier transformable. So let $V(\mathbf{x}-\mathbf{y})$ be the usual Coulomb potential in three dimensions and let $Y_\alpha(\mathbf{x}-\mathbf{y})$ be a Yukawa potential with dumping parameter $\alpha > 0$. The Coulomb potential expressed in terms of the Yukawa potential is given by

$$V(\mathbf{x}-\mathbf{y}) = \lim_{\alpha\to 0} Y_\alpha(\mathbf{x}-\mathbf{y}) = \lim_{\alpha\to 0} \frac{-g^2\; e^{-\alpha|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}.$$

Where $g^2 = e^2/(4\pi\epsilon)$, $\mathbf{x},\mathbf{y} \in \mathbb{R}^3$ and let $\check{Y}_\alpha(\mathbf{k})$, $\mathbf{k} \in \mathbb{R}^3$ be the Fourier-transformed Yukawa potential:

$$\check{Y}_\alpha(\mathbf{k}) = -4\pi g^2\; \frac{1}{k^2+\alpha^2}.$$

The projected Coulomb potential is given by

$$V^P(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2) = \langle P_{\eta_1}\otimes P_{\eta_2}, V(\mathbf{x}-\mathbf{y})\rangle$$

and is expressed through

$$V^P(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2) = \lim_{\alpha\to 0} \frac{1}{(2\pi)^3} \int d^3\mathbf{k}\; \check{Y}_\alpha(\mathbf{k})\; e^{-\frac{1}{2}(l_i k^i)^2}\; e^{-i\mathbf{k}\boldsymbol{\eta}_1 + i\mathbf{k}\boldsymbol{\eta}_2} \tag{4.4.7}$$

since $[\eta_1^i, \eta_2^j] = 0$ and w.l.o.g. $\eta_3 = 0$. At this point it should be mentioned that the classical Limit $x \to y$ does not make sense for the unbounded selfadjoint operators η_i^j. A suitable (or pragmatically) treatment is to evaluate the difference $\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2$ in appropriate states. It means that we can minimize the distance but it will never go to zero. Therefore the Pauli exclusion principle is always fulfilled. There exists automatically a minimal distance and this is where the difference is evaluated in coherent states, for example in the ground state of a harmonic oscillator. In this case it is clear that these states are those where we should obtain an incompressible (Fermi) liquid.

If we evaluate the differences of the non-commuting coordinates in a (coherent) state ω_0, then we obtain the minimal distance by evaluating

$$\begin{aligned} \omega_0(V^P(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2)) &= \lim_{\alpha\to 0} \frac{1}{(2\pi)^3} \int d^3\mathbf{k}\; \check{Y}_\alpha(\mathbf{k})\; e^{-\frac{1}{2}(l_i k^i)^2}\; \omega_0\Big(e^{-i\mathbf{k}\boldsymbol{\eta}_1} e^{i\mathbf{k}\boldsymbol{\eta}_2}\Big) \\ &= \lim_{\alpha\to 0} \frac{1}{(2\pi)^3} \int d^3\mathbf{k}\; \check{Y}_\alpha(\mathbf{k})\; e^{-l_B^2(k_1^2+k_2^2)+\frac{1}{4}l_z^2 k_3^2}. \end{aligned} \tag{4.4.8}$$

The minimal distance $r_{\min}$ can be defined by the integral and depends on the evaluation in coherent states:

$$\frac{1}{r_{\min}} := \lim_{\alpha\to 0} \frac{4\pi}{(2\pi)^3} \int d^3\mathbf{k}\; \frac{e^{-\frac{1}{2}(l_i k^i)^2}}{k^2+\alpha^2}\; \omega_0\Big(e^{-i\mathbf{k}\boldsymbol{\eta}_1} e^{i\mathbf{k}\boldsymbol{\eta}_2}\Big). \tag{4.4.9}$$

This integral is finite at least if we chose states from Schwartz space the coherent states respectively. The Schwartz function regularizes the integral for large momenta and since $\alpha > 0$ it is well defined on all on $\mathbb{R}^3$.

The projection of the Coulomb potential leads to an intrinsic quantised distance between the particles for fixed magnetic fields since it depends on the magnetic length. The simplest example for a minimal distance we calculate for $l_z = 4l_B$ and obtain

$$\omega_0(V^P(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2)) \quad = \quad -2g\frac{\sqrt{\pi}}{l_B} \tag{4.4.10}$$

thus $r_{\min} = l_B/(2\sqrt{\pi})$. The third component l_z may be chosen different depending on the confinement potential. It may be reduced such that it gives just a neglectable contribution.

We mapped the Coulomb potential in three dimensions by freezing out the third component and the radial contribution in coherent states to a screened Coulomb potential in two non-commutative dimensions, which leads in the minimal distance regime to a constant factor and thus to an incompressible liquid. A related discussion on the evaluation of the Coulomb potential can be found as a remark in [BvS94], however they consider the potential only in two dimensions. We stress that the Coulomb potential and the Yukawa potential respectively differs in two dimensions from that in tree dimensions and the evaluation in states leads then to different results which can be explained for instance with the integration measure.

Projection of the Charge Density

The projection of the charge density can easily be performed in our formalism. It is just the projection of the product of the spinors $\psi^+(x)\psi(x)$ so we obtain

$$\begin{aligned} \rho^P(x^0, \boldsymbol{\eta}) \quad &:= \quad P^2_\eta(\psi^+(x)\otimes\psi(x)) \\ &= \quad ((\psi^+)^P \star \psi^P)(x^0, \boldsymbol{\eta}) \end{aligned}$$

and the Coulomb interaction term of two particles would result in

$$\rho(x)V(\mathbf{x}-\mathbf{y})\rho(y) \mapsto \rho^P(x^0_1, \boldsymbol{\eta}_1) \star V^P(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2) \star \rho^P(x^0_2, \boldsymbol{\eta}_2)\ , \quad x^0_1 \equiv x^0_2.$$

Projection of an Action

Some more questions arise in this context, first how can we formulate normal-ordering and then how can we formulate a propagator of the projected fields or an action principle. It is the problem which also arises in a field theory on the noncommutative Minkowski space but with a slight modification due to the different commutation relation between the (quantum) coordinates. However since we have a special realization we may use the same concepts.

The QED action given in the previous section has to be projected to the Hall system. The problem which arises here is how to interpret the integration measure. Here the methods of measure theory should clarify what we have to do. The usual measure d^4x is turned into an operator valued measure, the spectral measure, which we will denote by $dE(\psi_1, \dots, \psi_n; \boldsymbol{P}^n_\eta)$ and the integration should be considered as a positive trace in a similar way to [DFR95].

$$S = \int_{\mathcal{O}} d^4x \; \mathcal{L}(\psi_1, \dots, \psi_n) \to S^P = \int dE(\psi_1, \dots, \psi_n; \boldsymbol{P}^n_\eta) \; \mathcal{L}^P(\psi_1^P, \dots, \psi_n^P).$$

For example the term where the Fermions ψ couples to a field a_μ is given by

$$\begin{aligned} S_I^P &= \int dx^0 \int dE(\bar{\psi}, a_\mu^P, \psi; \boldsymbol{P}^3_\eta) \; \mathcal{L}_I(\psi_1^P, a_\mu^P, \psi_n^P) \\ &= \int dx^0 d\eta_1 d\eta_2 \; (\psi_1^P \star \gamma^\mu a_\mu^P \star \psi_n^P)(x^0, \boldsymbol{\eta}) \\ &= \int dx^0 \; \mathrm{tr}_{\boldsymbol{\eta}}[(\psi_1^P \star \gamma^\mu a_\mu^P \star \psi_n^P)(x^0, \boldsymbol{\eta})]. \end{aligned}$$

The trace $\mathrm{tr}_{\boldsymbol{\eta}}$ has to be well defined or in other words it has to be a positive linear functional.

Projection of the Yang Mills and Chern Simons Term

If we take the definition of the fluctuating $U(1)$ Chern Simons gauge field a_μ from section 4.1 in four or in three dimensions respectively and if we assume that there exists the Fourier transformed

$$a_\mu(x) = \frac{1}{(2\pi)^4} \int d^4k \; a_\mu(k) \; e^{ikx}$$

and their derivative fields

$$a_{\mu,\nu}(x) = \partial_\nu a_\mu = \frac{1}{(2\pi)^4} \int d^4k \; a_\mu(k) \; k_\nu \; e^{ikx},$$

then we may define a projection of the field strength

$$\begin{aligned} f_{\mu\nu}^P(\eta) := \boldsymbol{P}^n_\eta(f(x)) &= \boldsymbol{P}^n_\eta(f_{\mu\nu}) \\ &= f_{\mu\nu}^P(\eta) \\ &= (a_{\mu,\nu}^P - a_{\nu,\mu}^P + a_\mu^P \star a_\nu^P - a_\nu^P \star a_\mu^P)(\eta) \end{aligned}$$

with the noncommutative analog of the derivative defined via (4.4.5):

$$a_{\mu,\nu}^P(\eta) = \frac{1}{(2\pi)^4} \int d^4k \; a_\mu(k) \; k_\nu \; e^{-\frac{(l_i k^i)^2}{4}} \; e^{ik\eta}.$$

In the following we drop the index "P", the Yang Mills term is then given in non-local coordinates:

$$YM(\eta) = \tfrac{1}{4} \; (f_{\mu\nu} \star f^{\mu\nu})(\eta) \; d\eta_0 d\eta_1 d\eta_2 d\eta_3.$$

In the same manner the Pontryagin term in non-local coordinates is:

$$PJ(\eta) = \tfrac{1}{4} \; \varepsilon^{\mu\nu\rho\sigma} f_{\mu\nu} \star f_{\rho\sigma} \; d\eta_0 d\eta_1 d\eta_2 d\eta_3.$$

The Chern Simons term on a suitable surface is also projected straight forward and expressed in non-local coordinates we have:

$$W(\eta) = (\varepsilon^{\mu\nu\rho}(a_\mu \star a_{\rho,\mu} + \tfrac{2}{3} a_\mu \star a_\nu \star a_\rho))(\eta) d\eta_0 d\eta_1 d\eta_2.$$

Therefore the Chern Simons theory in a Landau level is a non-commutative field theory the noncommutative Chern Simons theory. Susskind derived a similar form of the Chern Simons term via fluid dynamics, where it is considered that the particles occupy a finite, hard disc to satisfy a granular microscopic picture of a Hall fluid [Sus01].

What we immediately recognize is that we fail if we want to obtain the Chern Simons term via projection from the Pontryagin Lagrangian. So far there is a further difficulty since we have to say what Stokes theorem, Poincaré's lemma and (cyclic)-cohomology is in a noncommutative space. Nevertheless we observe

$$\begin{tikzcd}
\int d^4x\ \varepsilon^{\mu\nu\rho\sigma} f_{\mu\nu}(x) f_{\rho\sigma}(x) \arrow[r, "(P{,}4d\to 3d \text{ and LLL})"] \arrow[d, "f\wedge f = dW"] & \mathrm{tr}\varepsilon^{\mu\nu\rho\sigma} \hat{f}_{\mu\nu}(\eta) \star \hat{f}_{\rho\sigma}(\eta) \arrow[d, "?"] \\
\int d^3 \varepsilon^{\mu\nu\rho\sigma}(a_\mu \partial_\nu a_\sigma + \tfrac{2}{3} a_\mu a_\nu a_\sigma) \arrow[r, "(P{,}\text{LLL})"] & \mathrm{tr}\varepsilon^{\mu\nu\sigma}(\hat{a}_\mu \star \partial_\nu \hat{a}_\sigma + \tfrac{2}{3} \hat{a}_\mu \star \hat{a}_\nu \star \hat{a}_\sigma)
\end{tikzcd}.$$

The Chern Simons Lagrangian describes a gauge invariant theory modulo $\mathbb{Z}$ on a manifold without boundary especially in $\mathbb{R}^3$ where all fields vanish at infinity. On manifolds with boundary we may require that all gauge transformations on the boundary vanish. This restricts the underlying gauge group as already mentioned in the previous chapter.

After we project the gauge fields we have a noncommutative version of the Chern Simons Lagrangian. However we may clarify whether we still have at least a noncommutative version of a gauge invariant theory. There arises for instance some problems if we want to describe noncommutative gauge fields. It turns out that there exist no $SU(N)$ noncommutative fields due to a lack of a proper definition of a determinant. From that point of view we would prefer the $U(N)$ gauge fields only. We may quantize the theory in terms of (noncommutative) BRST quantization. However the description of noncommutative gauge theories is even in the classical case still an open question. There are some different approaches to define gauge theories on noncommutative spacetime. To discuss them all is beyond of the scope of this theses, the interested reader may follow the discussion in [Zah06].

One approach to handle noncommutative gauge theories is the use of the Seiberg Witten map, where the noncommutative $U(N)$ theory is mapped to a commutative $U(N)$ theory [SW99]. Roughly speaking it gives an expression of the noncommutative field in terms of the commutative field.

4.5 Low Energy Effective Noncommutative Model

It is interesting to see how the common theories change if we project them into the lowest Landau level. We project the low energy Lagrangian (2.3.8)

$$\mathcal{L} \mapsto \mathcal{L}^P = P^n_\eta(\mathcal{L})$$

and obtain a low energy effective theory in the lowest Landau level the potential in z-direction is of course assumed to be a delta potential:

$$\mathcal{L}^P(\boldsymbol{\eta},t) = \mathcal{L}_0^P(\boldsymbol{\eta},t)+\mathcal{L}_{\rm CS}^P(\boldsymbol{\eta},t)+\mathcal{L}_C^P(\boldsymbol{\eta},t), \tag{4.5.1}$$

wherein the noncommutative version of the free part is given by

$$\mathcal{L}_0^P(\boldsymbol{\eta},t) = \psi(\boldsymbol{\eta},t)^{P+}\star\big[-\frac{1}{2m}\big(\mathbf{p}-e(\mathbf{A}^P(\boldsymbol{\eta})-\boldsymbol{\mathcal{A}}^P(\boldsymbol{\eta},t))\big)^2+ \tag{4.5.2}$$
$$+i\partial_t+\mu+e(A_0^P(\boldsymbol{\eta})-\mathcal{A}_0^P(\boldsymbol{\eta},t))\big]\star\psi^P(\boldsymbol{\eta},t) \tag{4.5.3}$$

and for the noncommutative Chern Simons action we skip in a first approach the cubic term

$$\mathcal{L}_{\rm CS}^P(\boldsymbol{\eta},t) = \frac{e}{\tilde{\varphi}\phi_0}\varepsilon^{\mu\nu\rho}\mathcal{A}_\mu^P(\boldsymbol{\eta},t)\star\partial_\nu\mathcal{A}_\rho^P(\boldsymbol{\eta},t). \tag{4.5.4}$$

and we should as in the common case restrict the discussion to

$$\mathcal{L}_{\rm CS}^P(\boldsymbol{\eta},t) = \frac{e}{\tilde{\varphi}\phi_0}\varepsilon^{ij}\mathcal{A}_0^P(\boldsymbol{\eta},t)\star\partial_i\mathcal{A}_j^P(\boldsymbol{\eta},t). \tag{4.5.5}$$

to be able to compare both theories. The quantization procedure needs a bit care but we will discuss it in more detail below. The noncommutative version of the Coulombian interaction $V(\boldsymbol{\eta}_1-\boldsymbol{\eta}_2)$ is to be understood in terms of (4.4.7), a suitable limit of the Fourier transformed Yukawa potential:

$$\mathcal{L}_{\rm C}^P(\boldsymbol{\eta}_1,t) = -\frac{1}{2}\int d\boldsymbol{\eta}_2\ \rho^P(\boldsymbol{\eta}_1,t)\star V(\boldsymbol{\eta}_1-\boldsymbol{\eta}_2)\star\rho^P(\boldsymbol{\eta}_2,t). \tag{4.5.6}$$

The equations of motions for the noncommutative Chern Simons fields are obtained by varying the action with respect to the fields $\mathcal{A}_\mu^P$

$$\frac{\delta S^P}{\delta\mathcal{A}_\mu^P}=0$$

and the zero component leads also to the relation (we skipped the cubic part)

$$\varepsilon^{ij}\partial_i\mathcal{A}_j^P(\boldsymbol{\eta},t)=\tilde{\varphi}\phi_0\ \rho^P(\boldsymbol{\eta},t). \tag{4.5.7}$$

The charge density $\rho^P(\boldsymbol{\eta},t)$ can in this way be replaced in the Coulomb part of the action by $(\tilde{\varphi}\phi_0)^{-1}\varepsilon^{ij}\partial_i\mathcal{A}_j^P(\boldsymbol{\eta},t)$.

4.5.1 Mean Field and Random Phase Approximation in the Non-relativistic, Noncommutative Model

Since we completely replaced the coordinate $\mathbf{x}$ by the noncommutative coordinate $\boldsymbol{\eta}$ through projection, the total magnetic field is given by

$$b(\boldsymbol{\eta},t)=B-\mathcal{B}=B-\tilde{\varphi}\phi_0\rho^P(\boldsymbol{\eta},t). \tag{4.5.8}$$

Then we can also divide the noncommutative Chern Simons field in a mean field and a dynamical field respecting the fluctuations:

$$a^P_\mu(\boldsymbol{\eta},t) = \mathcal{A}^P_\mu(\boldsymbol{\eta},t) - \langle \mathcal{A}^P_\mu \rangle. \tag{4.5.9}$$

The average $\langle \mathcal{A}^P_\mu \rangle$ is per definition not dynamical and it gives no contribution to the equation of motions. Then the effect that it reduces the external field is the same as in the commutative analog. Only the dynamical part a^P_μ is the relevant contribution to the equation of motions and this leads to the relation:

$$\rho^P(\boldsymbol{\eta},t) = \frac{1}{\tilde{\varphi}\phi_0}\ \varepsilon^{ij}\partial_i a^P_j(\boldsymbol{\eta},t). \tag{4.5.10}$$

To calculate the free propagator of the low energy free massive charge carriers we can go straight forward, only the free propagator of the noncommutative Chern Simons field and the interacting part requires more theory, which we will discuss later. The free part of the noncommutative action is

$$S_{\mathrm{F}} = \int d\boldsymbol{\eta}dt\ \psi^{+P}(\boldsymbol{\eta},t) \star \big[-\frac{\nabla^2_{\boldsymbol{\eta}}}{2m} + i\partial_t + \mu\big] \star \psi^P(\boldsymbol{\eta},t) \tag{4.5.11}$$

with the projected fields. We write from now on $\psi(\boldsymbol{\eta},t)$ for the spin-less projected fields and the magnetic length is set to one $l_B \equiv c \equiv 1$:

$$\psi(\boldsymbol{\eta},t) \quad = \quad \int \frac{d^2\mathbf{k}d\omega}{(2\pi)^3}\ e^{-\frac{\mathbf{k}^2}{4}}(a^+(\mathbf{k})\otimes e^{i(\mathbf{k}\boldsymbol{\eta}-\omega t)} + a(\mathbf{k})\otimes e^{-i(\mathbf{k}\boldsymbol{\eta}-\omega t)}) \tag{4.5.12}$$

$$\check{\psi}(\mathbf{k},\omega) \quad = \quad \int d^2\boldsymbol{\eta}dt\ \psi(\boldsymbol{\eta},t)\otimes e^{-i(\mathbf{k}\cdot\boldsymbol{\eta}-\omega t)}. \tag{4.5.13}$$

The action in the Fourier transformed space is then straight forward. The derivative $\nabla_{\boldsymbol{\eta}}$ on the field $\psi(\boldsymbol{\eta},t)$ is defined in the sense of equation (4.4.5) and from now on we skip the tensor sign, since there should be no confusion.

$$\begin{aligned} S_{\mathrm{F}} = \int \frac{d^2\mathbf{k}d\omega}{(2\pi)^3} \int d\boldsymbol{\eta}dt \qquad & (a(\mathbf{k})e^{-i(\mathbf{k}\boldsymbol{\eta}-\omega t)} + a(\mathbf{k})e^{i(\mathbf{k}\boldsymbol{\eta}-\omega t)})\ e^{-\frac{\mathbf{k}^2}{2}}\ \big[\omega - \frac{k^2}{2m} - \mu\big] \\ & (a^+(\mathbf{k})\ e^{i(\mathbf{k}\boldsymbol{\eta}-\omega t)} + a(\mathbf{k})\ e^{-i(\mathbf{k}\boldsymbol{\eta}-\omega t)}). \end{aligned} \tag{4.5.14}$$

The propagator is nearly the same as in the common case, but we should keep in mind, that the fields are smeared with a Gaussian function. We may shift this Gaussian factor to the propagator and obtain free unsmeared fields and a smeared propagator. However, the smearing enters the vertices and thus we keep the Gaussian factor in the fields:

$$S_{\mathrm{F}} \quad = \quad \int \frac{d\mathbf{k}d\omega}{(2\pi)^3}\ \check{\psi}(k,\omega)\underbrace{\big[\omega - \frac{k^2}{2m} - \mu\big]}_{[G^0(\mathbf{k},\omega)]^{-1}}\check{\psi}(\mathbf{k},t). \tag{4.5.15}$$

The low energy Fermionic Green's function for the projected Lagrangian is then ($\epsilon > 0$):

$$G^0(\mathbf{k},\omega) = [\omega - \frac{k^2}{2m} - \mu + i\epsilon \mathrm{sign}(\omega)]^{-1}. \tag{4.5.16}$$

The gauge field propagator of the projected Chern Simons gauge fields needs more care since we deal with a noncommutative gauge field theory. The BRST quantization method introduced in the previous chapter for usual gauge fields with compact gauge group may provide the correct treatment here too. However this would require some more research since the methods for example described in [Zah06] have to be adapted and combined with the Chern Simons model. In general we expect within this noncommutative setup a smoother behavior, or at best a vanishing, of UV or IR divergences since this is what happens in some models of quantum field theory on noncommutative spacetimes [BDFP03] and this circumstance is also supported by the intrinsic Gaussian factors appearing naturally in our setup. Therefore the dispersion relation (2.3.46) introduced in chapter 2 for the usual theory should be strongly modified due to the modified structure at the scale of the magnetic length and this has then consequences for the effective composite Fermion masses (2.3.50). We also expect that the 'classical' results are reobtained in the large scale limit where the noncommutativity vanishes which is also true in some scalar field theories on noncommutative spacetimes [Kos08]. In general this setup may provide a useful proper treatment of low energy composite Fermions in a Landau, or composite Fermion Landau level, which is needed also in the case of second generation composite Fermions. For instance in the fractional Hall state with $\nu = 4/11$, the filling factor ν can be mapped to a composite Fermion filling factor of $\nu_{CF} = 4/3 = 1 + 1/3$ thus the second composite Fermion Landau level is filled 1/3 and we may project our field theory to exactly this situation.

4.6 Quasi Covariant Effective Noncommutative Model

We can also combine the covariant model with the noncommutative model to obtain a quasi covariant composite Fermion model in the relativistic lowest Landau level. Since the projection onto the lowest Landau level is a quantum mechanical projection the resulting Lagrangian is no more covariant at the scale of the magnetic length. However, covariance can be restored in the limit $l_B \to 0$, which classifies the large scale limit. This happens also if we require scale invariance for example in some conformal field theory approach. Then the noncommutativity vanishes and we obtain the usual commutative theory. Since relativistic covariance breaks down at the magnetic length, the resulting spacetime structure leads to a modified dispersion relation. We start with the projected effective covariant Lagrangian

$$\mathcal{L}^P(\eta) \;=\; \mathcal{L}^P_0(\eta) + \mathcal{L}^P_{\mathrm{CS}}(\eta) + \mathcal{L}^P_C(\eta), \tag{4.6.1}$$

with the free part

$$\mathcal{L}^P_0(\eta) \;=\; \bar{\psi}(\eta) \star \left[\gamma^\mu \big(p - e(a(\eta))\big)_\mu - m - \gamma^0\mu\right] \star \psi(\eta) + h.c.. \tag{4.6.2}$$

For simplicity we start with the noncommutative $U(1)$ Chern Simon term where we may skip in a first treatment the cubic part. However, for introducing spin couplings as in the chapter before we may prefer a noncommutative $U(N)$ Chern Simons theory instead of an $SU(N)$ nonabelian commutative Chern Simons theory since there exists no noncommutative analog of the the determinant.

$$\mathcal{L}_{\mathrm{CS}}^{P}(\eta) = \frac{e}{\tilde{\varphi}\phi_0}\varepsilon^{\mu\nu\rho}a_\mu(\eta)\star\partial_\nu a_\rho(\eta) \tag{4.6.3}$$

and as in the examples before we take the zero part only:

$$\mathcal{L}_{\mathrm{CS}}^{P}(\eta) = \frac{e}{\tilde{\varphi}\phi_0}\varepsilon^{ij}a_0(\eta)\star\partial_i a_j(\eta). \tag{4.6.4}$$

The noncommutative version of the Coulombian interaction $V(\boldsymbol{\eta}_1-\boldsymbol{\eta}_2)$ we understand in the sense of the Fourier transformed Yukawa potential:

$$\mathcal{L}_{\mathrm{C}}^{P}(\eta_1) = -\frac{1}{2}\int d\boldsymbol{\eta}_2\ \rho(\eta_1)\star V(\boldsymbol{\eta}_1-\boldsymbol{\eta}_2)\star\rho(\eta_2). \tag{4.6.5}$$

The variation with respect to a_0 gives the relation

$$\varepsilon^{ij}\partial_i a_j(\eta) = \tilde{\varphi}\phi_0\ \rho(\eta). \tag{4.6.6}$$

and we observe again that the charge density can be replaced by

$$\rho(\eta) = \frac{1}{\tilde{\varphi}\phi_0}\ \varepsilon^{ij}\partial_i a_j(\eta). \tag{4.6.7}$$

The task is now to calculate the propagators of the relativistic noncommutative composite Fermions and of the quantized gauge fields. What we expect from this approach is also a modified dispersion relation and a modified effective composite Fermion mass. The common field theory of composite Fermions we should obtain in the limit where the velocity is low and the magnetic length vanishes. By performing these limits we should be able to control correction terms in the sense of spin effects coming from the Dirac theory and smearing or smoothing effects coming from the nonlocal noncommutative setup.

Chapter 5

Summary and Outlook

The incorporation of the spin dynamics in the fractional quantum Hall effect is a challenging mission. On the one hand we can introduce phenomenological models where the Hamiltonian or Lagrangian is modified and tuned in such a way that experiments can be fitted at best with suitable accuracy. Led by the knowledge of the behavior of electrons in solids we can extrapolate these systems to composite Fermions in fractional Hall states and if we fit experimental data we are finished. We reviewed some possibilities of such models in the introduction. However, we choose here a different point of view and move one step deeper in the analysis: the spin of the electron is a relativistic effect and therefore the spin of composite Fermions should also be a relativistic effect and we ask for the possibility to derive the spin effects of composite Fermions from a relativistic equation. Since the relativistic equations for electrons require the four dimensional Minkowski space and a global $SU(2)$ symmetry, we are encouraged to start at this point for composite Fermions, too. The naive one particle approach in chapter three immediately lead to correction terms in the low energy limit. The corrections like Rashba, Dresselhaus and Zeemann etc. are in principle the same as for the electrons but with the charge and particles now corresponding to the composite Fermions. We can therefore support the phenomenological approaches which propose a Rashba spin orbit coupling term for composite Fermions and we motivate moreover additional correction terms such as Dresselhaus or zitterbewegung. Our model however is based on some heuristic consideration we made at the beginning. We modified the spacetime topology in the way that we removed one dimensional objects, lines or two dimensional cylinders representing the flux quanta. In chapter four we then looked deeper in the analysis and developed a mechanism to quantum-mechanically project four dimensional quantum fields to three dimensions. The charged particles are then smeared in the projected component depending on the underlying confinement potential. This mechanism allows to project the relativistic Dirac particles to three dimensions and followed by a Chern Simons transformation the heuristic arguments from chapter three can be replaced by a systematic mechanism which leads to relativistic composite Fermions in three dimensions. Relativistic now means that the particles transform covariantly only under a subgroup of the Poincaré group. We showed however that the Fermi statistics is preserved. The composite Fermions are quasi particles consisting of an electron with Fermi statistics and an even number of flux quanta which can be treated as particles obeying fractional statistics and having therefore fractional spin. The combined particle, the composite Fermion also obey fractional statistics and the spin statistic theorem holds.

The next problem is to describe composite Fermions in Landau levels or in composite Fermion Landau levels. We introduced therefore the projection of quantum fields onto Landau levels. Since there exists relativistic Landau levels it is possible to project also the relativistic particles. The two projections can be combined and we have a mechanism, a map, to project Dirac particles in four dimensions to composite Fermion in three dimensions in a required Landau level.

In chapter three we discussed also the possibility to derive a Chern Simons action from an antisymmetric combination of the field strength, the Pontryagin term in the Lagrangian. By comparing this approach with the projection method we observed that it is not clear how both methods can be incorporated.

The projection method generates a theory of composite Fermions in a Landau level. This is a realization of a quantum field theory on the noncommutative plane. The corresponding Chern Simons theory is then a noncommutative gauge theory on a noncommutative space. Gauge theories on noncommutative spacetimes are subjected to actual research and not fully understood. However, since the fractional Hall effect is experimentally accessible it may be interesting to study these field theories and their limitations. Especially the quantization of such theories are rather interesting. This is just one motivation for the introduction of the BRST quantization of Chern Simons theories on classical manifolds. The incorporation of the spin is a further motivation for this quantization procedure. In some phenomenological models there appear nonabelian Chern Simons fields. The quantization in a perturbative treatment for a composite Fermion model then requires ghost fields and the BRST cohomology. Therefore we explained how a $SU(2)$ Chern Simons model will look like. The consequences on experimental data, for example the effective composite Fermion mass, is an interesting subject and requires more research. A further interesting aspect of the Chern-Simons theories and BRST cohomology is the relation to Wess Zumino Witten theories on manifolds with boundary. The connection to composite Fermions allows to relate the composite Fermion filling factor with the central charge of the corresponding affine Lie algebra (Kac-Moody-algebra) of the WZW-Model. The BRST cohomology may play a crucial rôle in this context and this strongly motivates more research.

We investigated composite Fermion models within the projection methods. The influence for instance on the dispersion relation can be successively and systematically explored. In particular we expect that in the low energy limit the relativistic composite Fermion model should give corrections in terms of effects coming from the spin. The noncommutative model we expect to correspond to the common model in the limit where the magnetic length vanishes. This might be the case when the correlation length diverges, however, to make precise statements more work and research is needed.

To conclude this discussion: The quantum field theory of composite Fermions in fractional Hall states is on the constructive mathematical side far from being completely understood. However the phenomenological success in the prediction of experimental data, even on mean field level of spinless composite Fermions, strongly encourages more research. We have given an answer to one aspect to the incorporation of the spin of composite Fermions and the corresponding quantization of the Chern Simons theories derived from relativistic quantum electrodynamics. We have set up different models in this framework, which require more research but may provide also more insight into the physical structure of the fractional Hall effect and the theory of composite Fermions.

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Erklärung gemäß Prüfungsordnung (Promotion)

Hiermit versichere ich, dass ich die vorliegende Dissertation selbstständig und nur mit Hilfe der angegebenen Quellen verfasst habe. Desweiteren gestatte ich die Veröffentlichung dieser Arbeit.

Marcel Kossow, Hamburg Februar 2009.