

# Galois and Hopf-Galois Theory for Associative $S$ -Algebras

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## Abstract

We define and investigate Galois and Hopf-Galois extensions of associative  $S$ -algebras, generalizing both the algebraic notions and the notions introduced by John Rognes for commutative  $S$ -algebras in [75]. We provide many examples such as matrix extensions, Thom spectra and extensions of Morava- $K$ -theory spectra induced from Lubin-Tate extensions.

We show three applications. First, we show the existence of associative  $S$ -algebras which have as homotopy groups a finite possibly associative Galois extension of the homotopy groups of a commutative  $S$ -algebra. Second, we show that  $B$  defines an element in the Picard group  $\text{Pic}(A[G])$  whenever  $A \rightarrow B$  is a Galois extension of associative  $S$ -algebras with finite abelian Galois group  $G$ . A third application concerns the calculation of the topological Hochschild homology of a Hopf-Galois extension of commutative  $S$ -algebras which we relate to the topological Hochschild homology of the Hopf-algebra involved.

The appendix contains a Galois correspondence for extensions of associative rings, generalizing at least two main theorems from literature.



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# Introduction and Outline

The fundamental development that makes this work possible stems from the mid-1990s. In technical terms, this is the construction of symmetric monoidal model categories of spectra as e.g. in [36]. Here, spectra are to be understood in the topologists' sense, i.e. as objects representing generalized cohomology theories like real or complex  $K$ -theory, cobordism, ordinary homology and stable homotopy. Generally speaking, the construction of symmetric monoidal model categories of spectra gives topologists a means to mimic algebraic theories in stable homotopy theory. It leads to what is sometimes called “brave new algebra”.

This thesis is about mimicking the algebraic theories of Galois and Hopf-Galois extensions in the world of spectra. Our framework is broad enough to include strictly associative, not necessarily commutative ring spectra. More precisely, we develop and investigate the notions of Galois and Hopf-Galois extensions for associative  $S$ -algebras in the sense of [36].

We are not the first and only ones to consider Galois and Hopf-Galois extensions of spectra. Galois extensions of strictly commutative ring spectra have been defined and thoroughly investigated by John Rognes [75]. For an example, we consider the map  $KO \rightarrow KU$  from the real to the complex cobordism spectrum given by complexification. Complex conjugation defines an action of the cyclic group  $\mathbb{Z}/2$  on  $KU$ . The homotopy fixed point spectrum  $KU^{h\mathbb{Z}/2}$  is equivalent to the spectrum  $KO$ . In addition, some unramification condition is satisfied making the map  $KO \rightarrow KU$  a  $\mathbb{Z}/2$ -Galois extension of commutative  $S$ -algebras, see [75, Prop. 5.3.1.].

Hopf-Galois extensions are a natural generalization of Galois extensions. Examples arise by replacing the action of a Galois group by the coaction of a Hopf algebra. The investigation of Hopf-Galois extensions of commutative  $S$ -algebras has also been started in [75] with an investigation of the complex cobordism spectrum  $MU$  as a Hopf-Galois extension of the sphere spectrum. To distinguish our work from that of John Rognes, note that this thesis deals with extensions of *associative* ring spectra dropping the commutativity assumption from [75].

A theory of Galois or Hopf-Galois extensions for *associative* ring spectra is desirable. One reason is that some of the fundamentally important objects of stable homotopy theory, the Morava- $K$ -theory spectra  $K(n)$ , are not strictly

commutative. Extensions involving these spectra, which play the role of fields in stable homotopy theory, can not be dealt with in a strictly commutative context. The associative context of this thesis however includes such examples. Another situation showing that associative objects in the new framework might be more important than in algebra is that in stable homotopy theory quotients of strictly commutative ring spectra need not be strictly commutative. There has been some interest in associative ring spectra often related with quotient constructions, see e.g. [4, 5, 39, 61, 81]. One result of this thesis is that many of these quotient constructions are Hopf-Galois extensions in a weak sense. This applies the quotient maps  $\widehat{E}(n) \rightarrow K(n)$  from a completed Johnson-Wilson spectrum  $\widehat{E}(n)$  to the corresponding Morava- $K$ -theory spectrum. Again, these and other examples can only be dealt with in the framework of an associative theory.

Developing Galois and Hopf-Galois theory with spectra includes

- defining the corresponding notions in a meaningful way,
- establishing some structural statements,
- providing examples and
- giving applications to show that the developed notions are useful.

We intend the outline below to summarize how we achieve these goals. For the moment, let us only comment on two properties which a meaningful definition should comprehend.

First, algebra is embedded in brave new algebra by passage to Eilenberg-MacLane spectra. The brave new notions should hence be compatible with the corresponding algebraic notions in the sense that Galois extensions of ordinary rings correspond to Galois extensions of Eilenberg-MacLane spectra. In other words, we want to develop a generalization of the algebraic theory.

Second, the category of  $S$ -algebras is a model category, and the notion of Galois and Hopf-Galois extensions for associative  $S$ -algebras should have homotopical meaning, i.e. the notions should be invariant under passage to weakly equivalent data. In consequence, isomorphisms in the algebraic theory have to be replaced by weak equivalences in the brave new theory, fixed points have to be replaced by homotopy fixed points etc. More generally, the brave new theory gets a model category theoretic flavour. In consequence it is sometimes pretty different from its algebraic counterpart.<sup>1</sup> Kathryn Hess started to investigate Hopf-Galois extensions in general model categories [44]. Our results fit into this framework.

Some hope in future research and achievements is connected with the study of (Hopf-)Galois extensions of structured ring spectra. Possible applications mentioned in [75] concern Galois descent and the understanding of the algebraic

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<sup>1</sup>“Brave new world” is of course most visible, horrifying and appealing, when it comes to techniques and technologies.



$K$ -theory of the sphere spectrum. Another possible application concerns the chromatic filtration on  $S$ -modules. John Rognes relates it to a chromatic filtration on  $MU$ -modules, which may be easier to understand. These filtrations are linked by maps with geometric content. More precisely, the linking maps are (pro-)Galois extensions or Henselian maps. We will contribute to this picture by showing that also weak Hopf-Galois extensions occur in this context, see the introduction to chapter 7.

Another direction concerns topological Hochschild homology spectra. We investigate the topological Hochschild homology of commutative Hopf-Galois extensions in chapter 10. We expect that the structure of a Hopf-Galois extension can help to understand topological Hochschild homology also in associative contexts. This would allow to investigate spectra like  $\mathrm{THH}^E(K)$  where  $E = \widehat{E}(n)$  or  $E_n$  and  $K = K(n)$  or  $K_n$ . Compare with [4, 5] for calculations and open questions in this area.

## Outline

In the following, we give an overview of the contents of this thesis. Furthermore, each chapter starts with a slightly more detailed introduction or summary. The thesis consists of three parts.

The aim of Part I is to define and to investigate the notion of Galois extensions for associative  $S$ -algebras. We start by briefly reviewing the algebraic notion of Galois extensions for associative rings in chapter 1. The theory to be developed should apply to maps of associative  $S$ -algebras  $A \rightarrow B$ . In this generality,  $B$  is not necessarily an associative  $A$ -algebra since in general no centrality condition holds. In other words,  $B$  is a unital not necessarily central  $A$ -algebra or an  $A$ -unca as we say for short. Unfortunately, this situation has not been investigated thoroughly enough for our purposes by other authors, and we take care to lay foundations in chapters 2 and 3. For instance, we investigate the model structure of the category of  $S$ -algebras under  $A$  and show that the smash products of cofibrant uncas and more generally all spectra belonging to the class of extended cell bimodules represent the derived smash product. This statement has a well known analog when  $A$  is a commutative  $S$ -algebra. Our statement is the key observation in order to prove that our definition of Galois extension is homotopically meaningful as explained above.

The main chapter of Part I is chapter 4. Here, we define and develop the Galois theory for associative  $S$ -algebras along the lines of [75] where the theory for commutative  $S$ -algebras has been established. We provide a number of examples such as trivial extensions, matrix extensions and extensions of Morava- $K$ -theory spectra induced from Lubin-Tate extensions. We prove an Eilenberg-MacLane embedding theorem for Galois extensions of associative rings with surjective trace. This theorem applies in particular whenever the ground ring is commutative. The topological theory hence generalizes the algebraic notion of Galois extensions at least when the ground ring is commutative.

We generalize the concept of dualizability to  $A$ -bimodules where  $A$  is an asso-

ciative  $S$ -algebra and show that if  $A \rightarrow B$  is a Galois extension of associative  $S$ -algebras, then  $B$  is dualizable over  $A$  and  $A^{\text{op}}$ . We characterize Galois extensions using dualizability and investigate conditions under which Galois extensions  $A \rightarrow B$  are preserved and detected when inducing up along a map of  $S$ -algebras  $A \rightarrow C$  so that we obtain the map  $C \rightarrow C \wedge_A B$ . We end the chapter by investigating the closure property of the spectrum  $K_n^{nr} = E_n^{nr} \wedge_{E_n} K_n$ , which occurs as a pro-Galois extension of  $K_n$  induced from the pro-Galois extension  $E_n \rightarrow E_n^{nr}$ .

Part II deals with Hopf-Galois extensions of associative  $S$ -algebras. We define and investigate Hopf-Galois extensions of associative  $S$ -algebras in chapter 5, generalizing the notion of Galois extension from Part I, the corresponding algebraic notion and the corresponding notion for commutative  $S$ -algebras introduced in [75]. We investigate under what conditions Hopf-Galois extensions  $A \rightarrow B$  are preserved or detected when inducing up along a map of  $S$ -algebras  $A \rightarrow C$  as we did for Galois extensions. We obtain necessary and sufficient conditions, thereby answering a question posed by John Rognes in the context of Galois extensions. Many of our results hold for a slightly more general notion of extension which we call coalgebra extension. We present basic examples for Hopf-Galois extensions.

In chapter 6, we investigate under which conditions Thom spectra give rise to Hopf-Galois extensions of associative  $S$ -algebras. The prototypical example of a Hopf-Galois extension of commutative  $S$ -algebras is the Thom spectrum  $MU$ , the complex cobordism spectrum [75]. More generally, let  $Mf$  be a Thom spectrum associated with a loop map  $f: X \rightarrow BGL_1 S = BF$ . The main result of chapter 6 is that  $S \rightarrow Mf$  is a Hopf-Galois extension of associative  $S$ -algebras if and only if  $Mf$  is orientable along  $H\mathbb{Z}$  or equivalently if and only if  $f$  lifts to  $BSF$ . This can also be formulated as a completion condition. It follows from this that most of the classical Thom spectra like  $MU$ ,  $MSU$  and  $MSO$  are Hopf-Galois extensions of the sphere spectrum. Though, note that by violation of the orientability condition,  $S \rightarrow MO$  is not a Hopf-Galois extension. Another example is the Thom spectrum  $M\xi$  from [8] which is associated with a map  $j: \Omega\Sigma CP^\infty \rightarrow BGL_1 S$ . The unit  $S \rightarrow M\xi$  is a Hopf-Galois extension of associative  $S$ -algebras and  $M\xi$  cannot be a commutative  $S$ -algebra because it is not even homotopy commutative.

As a second class of examples at least in a weak sense, we investigate regular quotients in chapter 7. Our main result is that regular quotient maps are weak Hopf-Galois extensions if and only if some completion condition is satisfied. This condition can be formulated in algebraic terms. Examples of weak Hopf-Galois extensions arising in this way include the maps  $MU \rightarrow H\mathbb{Z}$ ,  $\widehat{E(n)} \rightarrow K(n)$ ,  $E_n \rightarrow K_n$  and  $MU_{(p)} \rightarrow BP \rightarrow H\mathbb{Z}_{(p)}$ . Moreover, if completion conditions are satisfied, regular quotient constructions give rise to systems of weak Hopf-Galois extensions. For example, there is a system of weak Hopf-Galois extensions

$$\begin{aligned} \widehat{E(n)} &\longrightarrow \widehat{E(n)}/p &\longrightarrow \widehat{E(n)}/(p, v_0) &\longrightarrow \cdots \\ \cdots &\longrightarrow \widehat{E(n)}/(p, v_0, \dots, v_{n-1}) &= K(n) \end{aligned}$$

in which all maps and compositions are weak Hopf-Galois extensions.

Part III consists of three chapters, each including an application. Chapter 8 deals with a realization problem. Let  $A_* \rightarrow B_*$  be a finite Galois extension of graded associative rings, and assume that the ring  $A_*$  is given by the homotopy groups of a commutative  $S$ -algebra  $A$ . In this situation we show that there is an associative  $A$ -algebra  $B$  such that  $A \rightarrow B$  is a Galois extension that realizes the algebraic extension when passing to homotopy groups. In particular, the ring  $B_*$  can be realized as an associative  $S$ -algebra. We use this to show the existence of associative  $S$ -algebras which are cyclic and generalized quaternion extensions of suitable commutative  $S$ -algebras. This can be seen as an associative analog of the problem of adjoining roots of unity to strictly commutative ring spectra, compare with [77].

In chapter 9 we prove another structural statement on Galois extensions of associative  $S$ -algebras. The main result is that if  $A \rightarrow B$  is a faithful Galois extension with finite abelian Galois group  $G$ , then the  $A[G]$ -bimodule  $B$  defines an element in the Picard group  $\text{Pic}(A[G])$ . This generalizes results from [10, 75] to the associative context. More generally, we investigate the case of finite but not necessarily abelian Galois groups  $G$ . We show that  $B$  is an invertible  $(A[G], F_{A[G]}(B, B)^{\text{op}})$ -bimodule regardless whether  $G$  is abelian or not. If  $G$  is abelian, we also show that there is an equivalence  $F_A(B, B)^G \simeq F_A(B, B)^{hG}$ . In other words, the Sullivan conjecture holds for endomorphism ring spectra of faithful Galois extensions with finite abelian Galois group.

In chapter 10, we relate the topological Hochschild homology of a Hopf-Galois extension of commutative  $S$ -algebras to the topological Hochschild homology of the Hopf algebra. We recover the splitting  $\text{THH}^S(Mf) \simeq Mf \wedge BX_+$  from [13, 12] in case  $Mf$  is a Thom spectrum associated with an infinite loop map  $f: X \rightarrow BGL_1 S$ . We show that the splitting more generally holds for Hopf Galois extensions of commutative  $S$ -algebras  $A \rightarrow B$  with respect to a Hopf-algebra  $H$  with homotopy antipode. The splitting then takes the form

$$\text{THH}^A(B; M) \simeq M \wedge_A \mathcal{B}^A(A, H, A)$$

where  $\mathcal{B}^A(A, H, A)$  denotes the bar construction. Motivated by the algebraic results of [80], we construct a spectral sequence

$$E_{p,q}^2 = \text{Tor}_*^{\pi_*(K \wedge_R K)}(K_*, \pi_* \text{THH}^R(A; M)) \implies \pi_* \text{THH}^R(B; M)$$

where  $A \rightarrow B$  is a Hopf-Galois extension under some commutative  $S$ -algebra  $R$  with respect to a Hopf algebra  $H = A \wedge_R K$ .

The appendix contains a Galois correspondence for extensions of associative rings. Unlike the main parts of the thesis, the appendix is purely algebraic. The topologically interested reader may have a look at it to get an idea why a Galois correspondence in the associative context is much more difficult to obtain than in the commutative context. Optimistically speaking, the appendix might serve as a blueprint to obtain a Galois correspondence for associative extensions also in topology.



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To  
Vishnupriya  
and  
Anelbema





**Part I**

**Galois Extensions**



# Chapter 1

## Galois extensions of ordinary rings

The theory of Galois extensions of fields has a generalization to associative rings which we will briefly review. This is the natural starting point for the project of developing an analogous theory for associative  $S$ -algebras. The theory for fields was generalized to extensions of commutative rings by Auslander, Goldman, Chase, Harrison and Rosenberg [6, 24]. Several authors then studied Galois extensions of general rings, see e.g. [55, 56, 29, 51, 52, 53, 37, 34] and others. We review the definition and then present some basic structural statements and examples.

### 1.1 Galois extensions of associative rings

As a standing assumption, we assume that all rings have a unit and morphisms of rings preserve it. Rings may be graded but we will use this generality only in some examples. Unless explicitly said, rings are not assumed to be commutative.

**Definition 1.1** (Galois extensions of rings). Let  $\eta: R \rightarrow T$  be an extension of rings and let  $G$  be a finite subgroup of the group of automorphisms of rings of  $T$ . We call  $R \rightarrow T$  a  $G$ -Galois extension if

1. The map  $\eta$  induces an isomorphism  $R \cong T^G$  where  $T^G$  is the fixed ring of  $T$  under the action of the group  $G$ , and
2. the canonical map  $h: T \otimes_R T \rightarrow \text{Map}(G, T)$  sending  $t_1 \otimes_R t_2$  to the map that sends  $g$  to  $t_1 \cdot g(t_2)$  is surjective.

A proof that this generalizes the well-known notion of Galois extensions of fields can for example be found in [42]. For conditions equivalent to the conditions 1. and 2. above see [24, 37]. The ring of maps  $\text{Map}(G, T)$  is isomorphic to  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], T)$  as rings. Also there is an isomorphism of  $T$ -modules  $\text{Map}(G, T) \cong T[G]$  as  $G$  is supposed to be finite. Note that unless  $R$  and  $T$  are commutative, the morphism  $h$  need not be multiplicative. There is a left  $G$ -action on  $T \otimes_R T$  defined by the action of  $G$  on the second  $T$ -factor and a left  $G$ -action on  $\text{Map}(G, T)$  defined by the right action of  $G$  on itself. The

morphism  $h$  is always  $G$ -equivariant with respect to these actions and it is a morphism of left  $T$ -modules. Hence to check the surjectivity of  $h$ , it suffices to find a finite number of elements  $x_i, y_i \in T$  such that  $\sum_i x_i \sigma(y_i) = \delta_{e, \sigma}$  where  $\sigma \in G$  and  $\delta_{e, \sigma}$  is 0 or 1, the latter if and only if  $\sigma = e$  is the neutral element of  $G$ . We call such a collection of pairs  $(x_i, y_i)$  a *Galois system* for the extension  $R \rightarrow T$ .

We now collect some facts about Galois extensions of associative rings that we will use later. First recall the following definition:

**Definition 1.2** (Separable algebras). Let  $A \rightarrow B$  be a ring homomorphism. Then  $B$  is called  *$A$ -separable* if there are elements  $x_i, y_i \in B$  such that

- $\sum_i x_i y_i = 1$  and
- $\sum_i x x_i \otimes_A y_i = \sum_i x_i \otimes_A y_i x \in B \otimes_A B \quad \forall x \in B$ .

This is equivalent to saying that  $B$  is a projective  $B \otimes_A B^{op}$ -module [23, IX, 7.7].

**Lemma 1.3.** *Let  $\eta: R \rightarrow T$  be a Galois extension of rings with finite Galois group  $G$ . Then*

- (a) *The canonical map  $h: T \otimes_R T \rightarrow \text{Map}(G, T)$  is an isomorphism.*
- (b)  *$T$  is separable over  $R$ .*
- (c)  *$T$  is finitely generated and projective as a left  $R$ -module.*
- (d)  *$T$  is finitely generated and projective as a right  $R$ -module.*
- (e) *If  $R$  is commutative then  $T$  is faithfully flat as an  $R$ -module.*
- (f) *If  $R$  is commutative then the trace map  $\text{tr}: T \rightarrow R$  sending  $t$  to  $\sum_{\sigma \in G} \sigma(t)$  is surjective.*
- (g) *If the trace is surjective, then  $T$  is finitely generated and projective as a left  $R[G]$ -module.*

**Proof:** That  $h$  is an isomorphism can be proved as in [24], see also [37]. The statement on separability can be found in [47], also see the statement in [37]. We continue to prove (c) and (d). From the isomorphism  $h$  we obtain elements  $x_i, y_i \in T$  ( $1 \leq i \leq |G|$ ) such that  $\sum_i x_i \cdot \sigma(y_i) = \delta_{\sigma, e}$ . We apply  $\sigma^{-1}$  and observe that these elements satisfy  $\sum_i \sigma(x_i) \cdot y_i = \delta_{\sigma, e}$  as well. We use this equality in order to prove (c). Statement (d) can be shown using the former equality in an analogous way. We define left  $R$ -module maps  $\psi_i: T \rightarrow R$  by  $\psi_i(t) := \text{tr}(t \cdot x_i)$ . Hence for every  $t \in T$ , one can write

$$\begin{aligned}
 t &= \sum_{\sigma \in G} \sigma(t) \sum_i \sigma(x_i) y_i \\
 &= \sum_i \sum_{\sigma \in G} \sigma(t \cdot x_i) y_i \\
 &= \sum_i \psi_i(t) y_i.
 \end{aligned} \tag{1.1}$$

This reveals  $T$  as a left  $R$ -module direct summand of  $R^{|G|}$ , since by (1.1) the composition

$$T \xrightarrow{\bigoplus \psi_i} R^{|G|} \xrightarrow{\langle -, \vec{y} \rangle} T$$

is the identity.

We continue to prove (e). This is stated in [42, 0.1.9.] for Galois extensions of commutative rings and follows from [58, 1.10] (see also our proposition 5.2) when  $T$  is associative as every Galois extension can be looked at as a Hopf-Galois extension, see chapter 5.

For the proof of (f) note that  $I := \text{tr}(T)$  is an ideal in  $R$ . As  $\sum_i \psi_i(1)y_i = 1$  with the elements and notation from above, we know that  $I \otimes_R T \cong T$ . It then follows that  $\text{tr}(T) = I = R$  as  $T$  is faithfully flat as left  $R$ -module by (c).

We will now show the last point. It follows from (c) that  $T[G]$  is projective as a left  $R[G]$ -module. As in [24, p.28] we have an isomorphism of left  $T[G]$ -modules  $T \otimes_R T \cong T[G] \cong T \otimes_R R[G]$  where  $T$  acts on the first and  $G$  on the second factor. Then choose  $c \in T$  such that  $\text{tr}(c) = 1$  and consider the composite

$$R \otimes_R T \xrightarrow{\eta \otimes_R T} T \otimes_R T \xrightarrow{\text{tr}(- \cdot c) \otimes_R \text{id}} R \otimes_R T$$

which is the identity. It shows that  $T$  is an  $R[G]$ -direct summand of  $T \otimes_R T$  and hence is projective as a left  $R[G]$ -module as so is  $T \otimes_R T$ .  $\square$

## 1.2 Some examples

There are many examples of Galois extensions of rings. References explicitly including examples of Galois extensions of rings that are not commutative are not so easy to find, though. Some examples of Galois extensions of associative rings can e.g. be found in [37, 50, 83]. We give some general constructions of Galois extensions of associative rings in this section. Some of these constructions have analogues in the world of commutative rings (see references) but we did not find references that explicitly include the associative case.

**Example 1.4** (Trivial extensions). See [42] for trivial Galois extensions of commutative rings. Let  $R$  be any ring and  $G$  a finite group. Define  $T := \text{Map}(G; R)$  with the left  $G$ -action defined by the right  $G$ -action on  $G$ . Viewing  $R$  as the ring of constant maps from  $G$  to  $R$  exhibits  $R$  as a subring of  $T$  and the  $G$ -action on  $T$  is such that  $R \cong T^G$ . Moreover, the canonical morphism  $h$  is the obvious isomorphism  $T \otimes_R T \cong T \otimes_R \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]; R) \cong \text{Map}(G, T)$ . The morphism

$$R \longrightarrow \text{Map}(G; R)$$

is hence a  $G$ -Galois extension. Extensions of this form are called *trivial Galois extensions*.

**Example 1.5** (Matrix extensions). For any ring  $R$  let  $\mathcal{M}_n(R)$  be the ring of  $n \times n$ -matrices over  $R$ . Assume  $T$  is a ring with an action of a group  $G$  by homomorphisms of rings. Let  $R$  denote the fixed ring  $T^G$ . First note that the action of  $G$  on  $T$  gives an action of  $G$  on  $\mathcal{M}_n(T)$  defined componentwise. This

is an action by homomorphisms of rings as so is the action on  $T$ . It is clear that with this action  $\mathcal{M}_n(T)^G \cong \mathcal{M}_n(T^G) \cong \mathcal{M}_n(R)$ . Moreover it is true that

$$\mathcal{M}_n(R) \rightarrow \mathcal{M}_n(T)$$

is  $G$ -Galois if and only if so is  $R \rightarrow T$ . To see this, it remains to show that the canonical morphism

$$h_n: \mathcal{M}_n(T) \otimes_{\mathcal{M}_n(R)} \mathcal{M}_n(T) \longrightarrow \mathcal{M}_n(T)[G]$$

is an isomorphism if and only if so is  $h: T \otimes_R T \rightarrow T[G]$ . Let  $E_{i,j}$  be the elementary  $n \times n$ -matrix with 1 in position  $(i, j)$  and 0 everywhere else ( $1 \leq i, j \leq n$ ). Note that the source of  $h_n$  is generated as a  $T$ -bimodule by the elements  $E_{i,1} \otimes_{\mathcal{M}_n(R)} E_{1,j}$ . This follows as

$$E_{i,r} \otimes_{\mathcal{M}_n(R)} E_{s,j} = \begin{cases} E_{i,1} \otimes_{\mathcal{M}_n(R)} E_{1,j} & \text{if } r = s \\ 0 & \text{if } r \neq s. \end{cases}$$

Multiplication of matrices sends  $E_{i,1} \otimes_{\mathcal{M}_n(R)} E_{1,j}$  to  $E_{i,j}$ . Also note that  $\mathcal{M}_n(T)[G] \cong \mathcal{M}_n(T[G])$ . It follows that  $h_n$  decomposes in  $n^2$  copies of the morphism  $h$ , each copy defined on one of the direct summands  $T \cdot E_{i,1} \otimes_{\mathcal{M}_n(R)} E_{1,j} \cdot T$  with image  $E_{i,j} \cdot T[G]$ . Hence  $h_n$  is surjective if and only if so is  $h$ .

**Example 1.6** (Adjoining roots of unity). See [42] for adjoining roots of unity to commutative rings. Let  $\zeta_n$  be an  $n$ -th primitive root of unity. Then  $\mathbb{Z}[\frac{1}{n}] \rightarrow \mathbb{Z}[\frac{1}{n}, \zeta_n]$  is a  $(\mathbb{Z}/n)^\times$ -Galois extension. Let  $R$  be any ring, such that  $R \otimes \mathbb{Z}[\frac{1}{n}, \zeta_n]$  is non-zero. Then  $R \otimes \mathbb{Z}[\frac{1}{n}] \rightarrow R \otimes \mathbb{Z}[\frac{1}{n}, \zeta_n]$  is also  $(\mathbb{Z}/n)^\times$ -Galois. If elements  $(x_i, y_i)$  form a Galois system for the extension  $\mathbb{Z}[\frac{1}{n}] \rightarrow \mathbb{Z}[\frac{1}{n}, \zeta_n]$  then the elements  $(1 \otimes x_i, 1 \otimes y_i)$  form a Galois system for the extension  $R \otimes \mathbb{Z}[\frac{1}{n}] \rightarrow R \otimes \mathbb{Z}[\frac{1}{n}, \zeta_n]$ . In particular there is a  $(\mathbb{Z}/n)^\times$ -Galois extension

$$R \longrightarrow R[\zeta_n]$$

whenever  $n$  is invertible in  $R$ .

**Example 1.7** (Kummer extensions). See e.g. [42] for Kummer extensions of commutative rings. Let  $R$  be a ring and  $u \in R^\times$  a central unit in  $R$ , i.e. a unit that is also in the center of  $R$ . Define  $T := R[x]/(x^n - u)$ . We will write  $\bar{z}$  for the coset  $z + (x^n - u) \in T$  and assume that  $R$  contains an  $n$ -th primitive root of unity  $\zeta_n$  and that  $n$  is invertible in  $R$ . We can then define an automorphism of rings  $\sigma: T \rightarrow T$  of order  $n$  by setting  $\sigma(\bar{x}) := \overline{\zeta_n x}$ . This defines an action of the cyclic group  $C_n = \mathbb{Z}/n$  on  $T$  by homomorphisms of rings and  $T^{C_n} = R$ . Using that  $n$  is invertible in  $R$ , one can see that

$$R \rightarrow R[x]/(x^n - u)$$

is a  $C_n$ -Galois extension as  $\frac{1}{n \cdot u} \sum_{i=0}^{n-1} \bar{x}^{n-i} \tau(\bar{x}^i) = \delta_{1,\tau}$  for all  $\tau \in C_n$  (use  $\sum_{i=0}^{n-1} \zeta_n^{ki} = 0 \forall 1 \leq k \leq n-1$ ). One can allow that  $R = A_*$  is a graded ring. In this case the construction of Kummer extensions as above works equally well provided that the degree of the unit  $u$  is a multiple of  $2n$ , see [10].

**Example 1.8** (Skew-polynomial rings of homomorphism type, cyclic algebras). Let  $T$  be a ring and  $\sigma: T \rightarrow T$  an automorphism of rings of order  $n$ . Let  $G := \langle \sigma \rangle$  be the group generated by  $\sigma$ , denote the fixed ring  $T^G$  by  $R$  and let  $u$  be a central unit in  $R$ . Define the skew polynomial ring  $T[x; \sigma, u]$  to be the ring generated as an  $R$ -algebra by  $T$  and  $x$  modulo the relations  $x \cdot t = \sigma(t)x$  and  $x^n = u$ .

Now suppose that  $R \rightarrow T$  is  $G$ -Galois. The  $G$ -action on  $T$  can be extended to  $T[x; \sigma, u]$  by defining  $\sigma(f) := xfx^{-1}$  for  $f \in T[x; \sigma, u]$ . It is easy to check that  $T[x; \sigma, u]^G \cong R[x]/(x^n - u)$ . Moreover if  $R \rightarrow T$  is  $G$ -Galois then

$$R[x]/(x^n - u) \longrightarrow T[x; \sigma, u] \quad (1.2)$$

is also a  $G$ -Galois extension: Any Galois system for the extension  $R \rightarrow T$  is also a Galois system for the extension  $R[x]/(x^n - u) \rightarrow T[x; \sigma, u]$ . If  $R \rightarrow T$  is a  $\mathbb{Z}/n$ -Galois extension of fields, then  $T[x; \sigma, u]$  is called a *cyclic algebra* after Dickson, see [32, 66].

In particular this captures all finite dimensional central division algebras over the  $p$ -adic fields  $\mathbb{Q}_p$  and over algebraic number fields. It is a theorem of Hasse [43] that all finite dimensional skew-fields over the center  $\mathbb{Q}_p$  are cyclic algebras. This is also true for finite dimensional division algebras over an algebraic number field by a theorem of Albert, Brauer, Hasse and Noether [2, 19].

**Example 1.9** (Generalized quaternion algebras). A combination of the examples 1.7 and 1.8 produces generalized quaternion algebras as follows. We keep the setting from example 1.8 and assume that  $R \rightarrow R[x]/(x^n - u)$  is also a Galois extension. Denote the Galois group of this extension by  $H$ . Assume furthermore that the action of  $H$  can be extended to  $T[x; \sigma, u]$  and that this  $H$ -action commutes with the action of  $G$  so that there is an  $H \times G$ -action on  $T[x; \sigma, u]$ . E.g.  $R \rightarrow R[x]/(x^n - u)$  might be a Kummer extension as in example 1.7. Then the composite map

$$R \longrightarrow R[x]/(x^n - u) \rightarrow T[x; \sigma, u] \quad (1.3)$$

is in fact a  $G \times H$ -Galois extension: if  $(a_i, b_i)$  is a Galois system for the extension  $R \rightarrow T$  and  $(c_j, d_j)$  is a Galois system for the extension  $R \rightarrow R[x]/(x^n - u)$  then  $(c_j a_i, b_i d_j)$  is a Galois system for the extension  $R \rightarrow T[x; \sigma, u]$ . If  $T = R[y]/(y^n - v)$  is itself a Kummer extension then (1.3) takes the form

$$R \longrightarrow R[x, y; u, v]/(x^n - u, y^n - v, xy = \zeta_n yx). \quad (1.4)$$

If  $n = 2$  then  $R[x, y; u, v]/(x^2 - u, y^2 - v, xy = -yx)$  is called a *generalized quaternion algebra* [38].

As a concrete example we consider the Hamiltonian quaternions

$$\mathbb{H} := \mathbb{R}[i, j]/(i^2 = -1, j^2 = -1, ij = -ji).$$

We set  $k := ij$ . The extension  $\mathbb{R} \rightarrow \mathbb{C} \cong \mathbb{R}[i]/(i^2 = -1)$  is  $\mathbb{Z}/2$ -Galois with Galois group generated by complex conjugation. A Galois system is given by

$\frac{1}{2}(1 \otimes 1 - i \otimes i)$ . The situation for  $j$  instead of  $i$  is similar. We obtain a  $\mathbb{Z}/2 \times \mathbb{Z}/2$ -action on  $\mathbb{H}$  generated by

$$c_j : \mathbb{H} \rightarrow \mathbb{H} : r_0 + r_1i + r_2j + r_3k \mapsto r_0 - r_1i + r_2j - r_3k$$

$$c_i : \mathbb{H} \rightarrow \mathbb{H} : r_0 + r_1i + r_2j + r_3k \mapsto r_0 + r_1i - r_2j - r_3k$$

(conjugation with  $j$  resp.  $i$  where  $r_0, r_1, r_2, r_3 \in \mathbb{R}$ ). Obviously  $\mathbb{H}^{\mathbb{Z}/2 \times \mathbb{Z}/2} = \mathbb{R}$  and  $\frac{1}{4} \cdot (1 \otimes_{\mathbb{R}} 1 - i \otimes_{\mathbb{R}} i - j \otimes_{\mathbb{R}} j - k \otimes_{\mathbb{R}} k)$  is a Galois system for the extension  $\mathbb{R} \rightarrow \mathbb{H}$ .

**Example 1.10** (Skew-polynomial rings of derivation type, see [50, 55, 49]). Let  $R$  be a ring and  $\delta : R \rightarrow R$  a derivation. We define  $R[x; \delta]$  to be the associative ring with elements  $\sum_{i=0}^n x^i r_i$  and multiplication  $rx = xr + \delta(r)$  for all  $r \in R$ . We assume that there is some  $b \in R$  such that  $\delta^p(r) - \delta(r) = rb - br$  and  $\delta(b) = 0$ . Then  $f := x^p - x - b$  is central in the skew-polynomial ring  $R[x; \delta]$ . We also assume that  $T$  has characteristic  $p$ . Then there is a  $C_p = \mathbb{Z}/p$ -action on  $T := R[x; \delta]/f$  defined by  $\sigma(\bar{x}) := x + 1$  where  $\sigma$  is a generator of  $C_p$ . It is easy to see that  $T^{C_p} = R$ . The map

$$R \rightarrow R[x; \delta]/f$$

is in fact a  $C_p$ -Galois extension, see [50, 55, 49]. A Galois system can be obtained from the equations

$$\prod_{j=1}^{p-1} \frac{x+j}{j} - \sigma^k \left( \frac{x}{j} \right) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } 1 \leq k \leq p-1. \end{cases}$$

### 1.2.1 Pro-Galois extensions

We briefly mention a slight generalization of the notion of Galois extensions for pro-finite groups  $G = \lim_i G_i$  for a projective system of finite groups  $G_i$ . We say that  $R \rightarrow T$  is a *pro-Galois extension for the group  $G$*  if for each  $i$  there is a  $G_i$ -Galois extension  $R \rightarrow T_i$ , for every map  $G_k \rightarrow G_i$  there is a morphism of rings  $T_i \rightarrow T_k$  which is compatible with the group actions and if  $T = \text{colim}_i T_i$ . For any pro-finite group  $G$  the trivial Galois extensions  $R \rightarrow \text{Map}(G_i, R)$  assemble to a profinite Galois extension  $R \rightarrow \text{Map}(G, R)$  called a *trivial pro-Galois extension*.

The following example of a trivial pro-Galois extension arises from topology. We fix a prime  $p$ . Recall the Morava stabilizer group  $S_n$ . It is defined to be the pro-finite group of automorphisms of the height  $n$  Honda formal group law over the field  $\mathbb{F}_p$ . Let

$$\Sigma(n) := K(n)_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} K(n)_* \cong K(n)_*[t_1, t_2, \dots] / (v_n t_i^{p^n} - v_n^{p^i} t_i)$$

be the  $n$ -th Morava stabilizer algebra. Here  $K(n)$  is the  $n$ -th Morava  $K$ -theory spectrum defined at the prime  $p$  with homotopy groups  $K(n)_* = \mathbb{F}_p[v_n^{\pm}]$  and  $BP$  is the Brown-Peterson spectrum with homotopy groups  $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ . Let  $S(n)^*$  be the topological linear dual of  $S(n) := \Sigma(n) \otimes_{K(n)_*} \mathbb{F}_p$  [69, 70].



By [69, 70] there is an isomorphism of pro-finite Hopf-algebras  $S(n)^* \otimes \mathbb{F}_p^n \cong F_p^n[S_n]$  [70]. Analogously it follows that  $S(n) \otimes F_p^n \cong \text{Map}(S_n, F_p^n)$ . Hence the map

$$F_p^n \longrightarrow S(n) \otimes F_p^n$$

is the trivial pro-Galois extension of  $F_p^n$  for the Morava stabilizer group  $S_n$ .

This thesis is about generalizations of Galois extensions of rings to extensions of associative  $S$ -algebras, certain models for spectra as introduced by [36]. There is also a notion of pro-Galois extension for these objects. For example we will meet the Morava stabilizer group  $S_n$  again in section 4.6.1 where it occurs as the pro-finite Galois group of a (non-trivial) pro-Galois extension of the Morava  $K$ -theory spectrum  $K(n)$ . The term “generalization” is made precise in the Eilenberg-MacLane embedding theorem 4.5, see also section 5.3.1. There are possible consequences for the algebraic theory, e.g. equivalent formulations for the conditions 1. and 2. from definition 1.1 can be obtained from the topological characterizations from section 4.7.



## Chapter 2

# Algebras under associative algebras

Let  $R$  be a ring. An  $R$ -algebra  $T$  is a ring  $T$  together with a central morphism of rings  $R \rightarrow T$  [60]. As evident from chapter 1, we will need to consider homomorphisms of rings  $R \rightarrow T$  without this centrality condition. E.g., every Galois extension of a non-commutative ring  $R$  is an instance of this. We will need to consider the corresponding objects in a category of structured ring spectra and call them *unital not necessarily central algebras*, or *uncas* for short. We will start our investigation with an arbitrary symmetric monoidal category  $(\mathcal{M}_S, \wedge, S)$  with coequalizers and only later specify to the symmetric monoidal category of  $S$ -modules as introduced in [36]. In this case, the category of uncas inherits a model structure. Using results of this chapter, we will further investigate this model structure in chapter 3.

### 2.1 Unital not necessarily central algebras (uncas)

Let  $\mathcal{M}_S = (\mathcal{M}_S, \wedge, S)$  be a symmetric monoidal category in which coequalizers exist such as the category of  $S$ -modules introduced in [36]. An (associative)  $S$ -algebra is a monoid in  $\mathcal{M}_S$  and a commutative  $S$ -algebra is a commutative monoid in  $\mathcal{M}_S$ . The category of associative (commutative)  $S$ -algebras is denoted by  $\mathcal{A}_S$  ( $\mathcal{C}_S$ ). For an  $S$ -algebra  $A$  a (left)  $A$ -module is an algebra over the monad  $A \wedge (-)$  in  $\mathcal{M}_S$ . The category of  $A$ -modules is denoted by  $\mathcal{M}_A$ . If  $R$  is a commutative  $S$ -algebra,  $\mathcal{M}_R = (\mathcal{M}_R, \wedge_R, R)$  is again a symmetric monoidal category, the product  $N \wedge_R M$  of two  $R$ -modules  $N$  and  $M$  being defined as the coequalizer

$$N \wedge R \wedge M \rightrightarrows N \wedge M \longrightarrow N \wedge_R M$$

where the parallel arrows are given using the  $R$ -module structures of  $N$  and  $M$  respectively. If  $A$  is an associative  $S$ -algebra then so is  $A^e := A \wedge A^{\text{op}}$  and  $(\mathcal{M}_{A^e}, \wedge_A, A)$  is a (non-symmetric) monoidal category. This is of course the category of  $A$ -bimodules. Recall that if  $A$  is a commutative  $S$ -algebra then an  $A$ -algebra  $B$  is an  $S$ -algebra with a central map  $\eta: A \rightarrow B$  of  $S$ -algebras [36,

1.3.]. Here  $\eta$  is said to be central if the diagram

$$\begin{array}{ccccc} A \wedge B & \xrightarrow{\eta \wedge B} & B \wedge B & \xrightarrow{\mu} & B \\ \tau \downarrow & & & & \parallel \\ B \wedge A & \xrightarrow{B \wedge \eta} & B \wedge B & \xrightarrow{\mu} & B \end{array}$$

commutes where  $\tau$  is the isomorphism switching  $A$  and  $B$  and  $\mu$  is the multiplication. An  $A$ -algebra in particular is an  $S$ -algebra under  $A$ . The converse however is not true in general. For this reason we introduce the following terminology.

**Definition 2.1** ( $A$ -unca,  $A$ -algebra). Let  $A$  be an associative  $S$ -algebra and  $B$  an  $S$ -algebra under  $A$ , i.e. an object in the under-category  $A \downarrow \mathcal{A}_S$  of  $S$ -algebras under  $A$ . We call  $B$  a *unital not necessarily central algebra under  $A$*  or  $A$ -unca. An  $A$ -algebra is an  $S$ -algebra  $B$  with a central map of  $S$ -algebras  $A \rightarrow B$ . We denote the categories of  $A$ -uncas and  $A$ -algebras by  $A \downarrow \mathcal{A}_S$  and  $\mathcal{A}_A$  respectively.

The following observation will be important to us.

**Proposition 2.2.** *Let  $A$  be an associative  $S$ -algebra. The category of  $A$ -uncas is equal to the category of monoids in the monoidal category  $(\mathcal{M}_{A^e}, \wedge_A, A)$  of  $A$ -bimodules.*

Note that both categories are subcategories of the category  $\mathcal{M}_S$  and the statement says that these two subcategories coincide. We defer the straightforward but detailed and hence lengthy proof to section 2.2. We call an  $A$ -bimodule  $M$  central if

$$\begin{array}{ccc} A \wedge M & \xrightarrow{\tau} & M \wedge A \\ & \searrow \nu_\ell & \swarrow \nu_r \\ & M & \end{array}$$

commutes where  $\nu_\ell$  and  $\nu_r$  are the left and right action maps respectively. In other words a central bimodule is one where the left module structure determines the right one and vice versa. An  $A$ -algebra  $B$  is a central bimodule. So analogously to proposition 2.2 we have the following characterization of  $A$ -algebras.

**Corollary 2.3.** *Let  $A$  be an associative  $S$ -algebra. An  $A$ -algebra is a monoid in  $\mathcal{M}_{A^e}$  which is central as an  $A$ -bimodule.*

Note that if  $A$  is an associative but not a commutative  $S$ -algebra, then  $A$  is an  $A$ -unca but not an  $A$ -algebra. In particular the category of  $A$ -algebras is not monoidal in this case. We will give some examples after providing proofs of proposition 2.2 and corollary 2.3.

## 2.2 Characterizing $A$ -uncas and $A$ -algebras

The next two lemmata provide a proof of proposition 2.2. We first show that the two categories to be compared have the same objects.

**Lemma 2.4.** *Let  $B$  be an  $A$ -unca. Then  $B$  is a monoid in the category of  $A$ -bimodules  $\mathcal{M}_{A^e}$ . Conversely, a monoid in  $\mathcal{M}_{A^e}$  is an  $A$ -unca.*

**Proof:** Given a map  $\eta: A \rightarrow B$  of  $S$ -algebras, we obtain a left  $A$ -module on  $B$  structure via

$$A \wedge B \xrightarrow{\eta \wedge B} B \wedge B \rightarrow B \quad (2.1)$$

using that  $\eta: A \rightarrow B$  is a map of  $S$ -algebras. Similarly we obtain a right  $A$ -module structure and the structures define a bimodule-structure by associativity of the multiplication of  $B$ . We obtain a multiplication  $B \wedge_A B \rightarrow B$  from the multiplication  $B \wedge B \rightarrow B$  using associativity:

$$\begin{array}{ccccc} & & B \wedge B \wedge B & & \\ & \nearrow & & \searrow & \\ B \wedge A \wedge B & \xrightarrow{\quad} & B \wedge B & \xrightarrow{\quad} & B \wedge_A B \\ & & \downarrow & \swarrow & \\ & & B & & \end{array}$$

Here the middle row is the coequalizer defining  $B \wedge_A B$ . The product thus obtained is of course associative again. One can deduce unitality from the diagram

$$\begin{array}{ccccccc} B & \xrightarrow{\cong} & S \wedge B & \longrightarrow & B \wedge B & \xrightarrow{\mu} & B \\ \parallel & & \downarrow & & \downarrow & & \parallel \\ B & \xrightarrow{\cong} & A \wedge_A B & \xrightarrow{\eta \wedge_A B} & B \wedge_A B & \xrightarrow{\mu} & B \end{array} \quad (2.2)$$

where the upper composite is the identity by assumption. This proves the first part of the statement.

Conversely, if  $B$  is a monoid in  $\mathcal{M}_{A^e}$  it comes with a map  $\eta: A \rightarrow B$  and an associative multiplication  $\mu: B \wedge_A B \rightarrow B$ . We claim that  $B$  is an  $S$ -algebra with unit  $S \rightarrow A \xrightarrow{\eta} B$  where  $S \rightarrow A$  is the unit of  $A$ . Define the multiplication as the composite  $B \wedge B \rightarrow B \wedge_A B \xrightarrow{\mu} B$  which we denote again by  $\mu$ . It is clear, that this multiplication is associative. Left unitality follows again from diagram (2.2). Note that now the bottom map of (2.2) is the identity as  $B$  is assumed to be a monoid in  $\mathcal{M}_{A^e}$ . The middle square commutes since  $\eta$  is a morphism of  $A$ -bimodules. Right unitality can be proved in a similar way, showing that  $B$  is an  $S$ -algebra. It remains to show that  $\eta$  is a map of  $S$ -algebras. In the diagram

$$\begin{array}{ccccccc} A & \xleftarrow[\cong]{\mu} & A \wedge_A A & \xrightarrow{\eta \wedge_A \eta} & B \wedge_A B & \xrightarrow{\mu} & B \\ \downarrow \eta & & \downarrow A \wedge_A \eta & & \downarrow & & \parallel \\ B & \xleftarrow[\cong]{} & A \wedge_A B & \xrightarrow{\eta \wedge_A B} & B \wedge_A B & \xrightarrow{\mu} & B \end{array}$$

the left square commutes since  $\eta$  is a morphism of bimodules. The right square commutes as well and by unitality the lower composite map is the identity. Hence the diagram

$$\begin{array}{ccc}
A \wedge A & \xrightarrow{\eta \wedge \eta} & B \wedge B \\
\downarrow & & \downarrow \\
A \wedge_A A & \xrightarrow{\eta \wedge_A \eta} & B \wedge_A B \\
\downarrow \mu & & \downarrow \mu \\
A & \xrightarrow{\eta} & B
\end{array}$$

commutes, i.e.  $\eta$  commutes with the multiplications. Clearly it is a map under  $S$  and hence  $\eta$  is a map of  $S$ -algebras.  $\square$

Note that from unitality it follows that the unit  $S \rightarrow B$  is central even if  $A \rightarrow B$  is not. For morphisms we have the following statement.

**Lemma 2.5.** *Let  $B$  and  $B'$  be  $A$ -uncas and let us be given a morphism  $\alpha: B \rightarrow B'$  of  $S$ -modules. The following are equivalent:*

(a) *The morphism  $\alpha$  is a morphism of  $A$ -bimodules making the diagram*

$$\begin{array}{ccc}
B \wedge_A B & \xrightarrow{\alpha \wedge_A \alpha} & B' \wedge_A B' \\
\downarrow \mu & & \downarrow \mu \\
B & \xrightarrow{\alpha} & B'
\end{array}$$

*commute, i.e.  $\alpha$  is a morphism of monoids in the category of  $A$ -bimodules.*

(b) *The morphism  $\alpha$  is a morphism of  $S$ -algebras under  $A$ .*

(c) *The morphism  $\alpha$  is a morphism of  $S$ -algebras and one of  $A$ -bimodules.*

**Proof of lemma 2.5:** It is easy to show that (a) implies (b): We obtain a commuting diagram of  $S$ -algebras

$$\begin{array}{ccc}
B \wedge B & \xrightarrow{\alpha \wedge_A \alpha} & B' \wedge B' \\
\downarrow & & \downarrow \\
B \wedge_A B & \xrightarrow{\alpha \wedge_A \alpha} & B' \wedge_A B' \\
\downarrow \mu & & \downarrow \mu \\
B & \xrightarrow{\alpha} & B'
\end{array}$$

and note that the units  $S \rightarrow B$  and  $S \rightarrow B'$  factor through  $S \rightarrow A$ . In order to prove (c) using (b) consider the following diagram.

$$\begin{array}{ccc}
A \wedge B & \xrightarrow{\text{id} \wedge \alpha} & A \wedge B' \\
\downarrow & & \downarrow \\
B \wedge B & \xrightarrow{\alpha \wedge \alpha} & B' \wedge B' \\
\downarrow & & \downarrow \\
B & \xrightarrow{\alpha} & B'
\end{array}$$

It shows that  $\alpha$  is a morphism of left  $A$ -modules. One can show analogously that it is also a map of right  $A$ -modules and bimodules. We finally show that (c) implies (a): Since  $\alpha$  is a map of  $(A, A)$ -bimodules, we get a dotted arrow between coequalizers in the following diagram:

$$\begin{array}{ccccc}
 & & B \wedge A \wedge B & \xrightarrow{\quad} & B \wedge B & \xrightarrow{\quad} & B \wedge_A B \\
 & \swarrow & & \searrow^{\alpha \wedge \alpha} & \downarrow & & \downarrow \\
 B' \wedge A \wedge B' & \xrightarrow{\quad} & B' \wedge B' & \xrightarrow{\quad} & B' \wedge_A B' & & \\
 & \downarrow^{\mu} & \downarrow^{\mu} & & \downarrow & & \downarrow \\
 & & B & \xrightarrow{\quad} & B & \xrightarrow{\quad} & B \\
 & \swarrow^{\alpha} & & \searrow & \downarrow & & \downarrow \\
 & & B' & \xrightarrow{\quad} & B' & & \\
 & & & \swarrow & & & 
 \end{array}$$

The square on the left side of the cube commutes since  $\alpha$  is assumed to be a map of  $S$ -algebras. The right vertical maps exist by the universal property of coequalizers. We have to show that the right side of the cube commutes, i.e. that the two maps  $B \wedge_A B \rightarrow B'$  are the same. But since the left side of the cube commutes, both are induced by the same map  $B \wedge B \rightarrow B'$ . Hence, by the universal property of coequalizers, the right side of the cube commutes as well.  $\square$

**Proof of proposition 2.2:** Lemma 2.4 says that the two categories have the same objects and lemma 2.5 says that the morphism sets coincide as well.  $\square$

**Proof of corollary 2.3:** It is clear from the proof of lemma 2.4 that the morphism of algebras  $A \rightarrow B$  is central if and only if  $B$  is central as an  $A$ -bimodule. Corollary 2.3 then follows from proposition 2.2.  $\square$

Here is a characterization of central  $A$ -bimodules.

**Lemma 2.6.** 1. A central  $A$ -bimodule  $M$  is a left  $A$ -module, such that

$$\begin{array}{ccc}
 A \wedge A \wedge M & \xrightarrow{\tau \wedge M} & A \wedge A \wedge M \\
 \searrow^{\nu_1 \circ (A \wedge \nu_1)} & & \swarrow_{\nu_1 \circ (A \wedge \nu_1)} \\
 & M & 
 \end{array}$$

commutes.

2. (Bimodule-)morphisms between central  $A$ -modules are just morphisms of  $A$ -modules. Hence the category of central  $A$ -bimodules is a full subcategory of the category  $\mathcal{M}_A$  of  $A$ -modules.

**Proof:**

Ad 1: Given a central  $A$ -bimodule  $M$ , consider

$$\begin{array}{ccccccc}
 A_1 \wedge A_2 \wedge M & \xrightarrow{\tau} & A_1 \wedge M \wedge A_2 & \longrightarrow & M \wedge A_2 \wedge A_1 & \longrightarrow & A_2 \wedge A_1 \wedge M \\
 & \searrow & \downarrow & & \downarrow & & \swarrow \\
 & & A_1 \wedge M & \longrightarrow & M \wedge A_1 & & \\
 & & & \searrow & \downarrow & & \\
 & & & & M & & 
 \end{array}$$

The triangles commutes since  $M$  is a central  $A$ -bimodule. Conversely, a left  $A$ -module  $M$  is also a right  $A^{op}$ -module and if the given diagram commutes, it is also a right  $A$ -module. These structures combine to give a bimodule structure. The second point of the lemma is obvious.  $\square$

**Lemma 2.7.** *Let  $M$  and  $N$  be central  $A$ -bimodules. Then also  $M \wedge_A N$  is a central  $A$ -bimodule. Furthermore there is an isomorphism of  $A$ -bimodules*

$$M \wedge_A N \cong N \wedge_A M.$$

**Proof:**  $M \wedge_A N$  is obviously an  $A$ -bimodule, compare [36, prop.III.3.4]. It remains to check that  $M \wedge_A N$  is central. This follows as there are isomorphisms  $(A \wedge M) \wedge_A N \cong (M \wedge A) \wedge_A N \cong M \wedge_A (A \wedge N) \cong M \wedge_A (N \wedge A)$  over  $M \wedge_A N$ . To prove the isomorphism from the statement, note that there is always an isomorphism  $M \wedge_A N \simeq N \wedge_{A^{op}} M$  obtained from comparing the defining coequalizer diagrams, compare [36, lem. III.3.3]. Then it follows from lemma 2.6.1 that there is an isomorphism  $N \wedge_{A^{op}} M \cong N \wedge_A M$ .  $\square$

## 2.3 Some examples

There are some very basic examples for uncas and  $A$ -algebras.

- (a) Every associative  $S$ -algebra  $A$  is an  $A$ -unca. It is an  $A$ -algebra if and only if it is a commutative  $S$ -algebra.
- (b) If  $B$  is an  $A$ -unca and  $B'$  is an  $A'$ -unca, then  $B \wedge B'$  is an  $A$ - and an  $A'$ -unca. It is also an  $A \wedge A'$ -unca. This follows from proposition 2.2 using the isomorphism  $(B \wedge B') \wedge_{A \wedge A'} (B \wedge B') \cong (B \wedge_A B) \wedge (B' \wedge_{A'} B')$  (see [36, III.3.10.]) so that we can define the multiplications factorwise.

However, if  $B$  and  $B'$  are  $A$ -uncas the smash product  $B \wedge_A B'$  over  $A$  need not be an  $A$ -unca as  $B' \wedge_A B \cong B \wedge_{A^{op}} B'$  is different from  $B \wedge_A B'$  in general. On the other hand we have the following positive results:

**Lemma 2.8.** *Let  $A$  be an associative  $S$ -algebra and let  $B$  and  $C$  be  $A$ -algebras. Then  $C \wedge_A B$  is also an  $A$ -algebra, i.e. it admits an associative multiplication*

$$(C \wedge_A B) \wedge_A (C \wedge_A B) \rightarrow C \wedge_A B$$

and the map  $A \rightarrow C \wedge_A B$  is central. With the canonical maps  $C \rightarrow C \wedge_A B$  and  $B \rightarrow C \wedge_A B$  the smash product is also a  $C$ - and a  $B$ -unca.



**Proof:** We define the multiplication factorwise as  $(C \wedge_A B) \wedge_A (C \wedge_A B) \cong C \wedge_A C \wedge_A B \wedge_A B \rightarrow C \wedge_A B$ . Here the isomorphism in the middle is the one from lemma 2.7. It is clear that the maps from  $C$  and  $B$  into the smash product are maps of  $S$ -algebras.  $\square$

We will now specialize to structured ring and module spectra. More precisely let  $(\mathcal{M}_S, \wedge, S)$  now be the monoidal category of  $S$ -modules as introduced in [36]. In particular  $S$  denotes the sphere spectrum. The categories of  $S$ -modules and  $S$ -algebras are tensored and cotensored over the category of unbased topological spaces. Cotensors in the category of  $S$ -algebras are created in the category of  $S$ -modules. For an unbased space  $X$  and an  $S$ -module  $K$  the cotensor is denoted by  $F(X_+, K)$ . Here are some more examples of uncas.

- (c) If  $R \rightarrow T$  is a map of associative rings, then the induced map of Eilenberg-MacLane spectra  $HR \rightarrow HT$  gives  $HT$  the structure of an  $HR$ -unca. The spectrum  $HT$  is an  $HR$ -algebra if and only if  $R \rightarrow T$  is central.
- (d) Let  $K$  be an associative  $S$ -algebra and  $X \rightarrow Y$  a map of spaces. Then the function spectrum  $F(X_+, K)$  is an  $F(Y_+, K)$ -unca but in general not an  $F(Y_+, K)$ -algebra.
- (e) Let  $R$  be an even commutative  $S$ -algebra, i.e. a commutative  $S$ -algebra with homotopy groups concentrated in even degrees. Let  $I$  be a regular sequence in  $R_*$ . One can construct an associative  $S$ -algebra  $R/I$  that realizes the homotopy groups of  $R_*/I$ , see chapter 7 for details of this construction. Let  $I = J_1 + J_2$  be a decomposition of  $I$  into regular sequences  $J_1$  and  $J_2$ . Then  $R/I = R/J_1 \wedge_R R/J_2$  and the canonical maps  $R/J_i \rightarrow R/I$  make  $R/I$  into an  $R/J_i$ -unca.

## 2.4 Model structures

**Proposition 2.9.** *Let  $A$  be an associative  $S$ -algebra and assume that the category of associative  $S$ -algebras is a model category. Then so is the category of  $A$ -uncas where a map is a weak-equivalence, fibration or cofibration if it is so as a map in the model category  $\mathcal{A}_S$  of  $S$ -algebras. In particular this is the case when specifying to the category of  $S$ -modules from [36].*

**Proof:** This follows as the under-category  $A \downarrow \mathcal{A}_S$  inherits the model structure from  $\mathcal{A}_S$  with fibrations, cofibrations and weak equivalences as in the proposition, see [35, 3.10 p.15]  $\square$

We will again specialize to the category of  $S$ -modules from [36] for the rest of this chapter. In this case the category of associative  $S$ -algebras  $\mathcal{A}_S$  is a model category. Hence by the last proposition so is the category of  $A$ -uncas  $A \downarrow \mathcal{A}_S$ . Recall that for a commutative  $S$ -algebra  $A$  also the category  $\mathcal{C}_A$  of commutative  $A$ -algebras is a model category. Moreover the model structures on  $\mathcal{C}_A$ ,  $\mathcal{A}_A$  and  $A \downarrow \mathcal{A}_S$  are such that weak equivalences and fibrations are created in the category of spectra or likewise in the category of  $S$ -modules. This property is

special to the categories introduced in [36] and is in general not satisfied, see [79, Rem. 4.5.].

**Lemma 2.10** (Full-subcategory lemma). *Let  $A$  be an associative  $S$ -algebra in the sense of [36]. Then the inclusion*

$$\mathcal{A}_A \subset A \downarrow \mathcal{A}_S \tag{2.3}$$

*is an inclusion of a full subcategory. If  $A$  is a commutative  $S$ -algebra then*

$$\mathcal{C}_A \subset \mathcal{A}_A \tag{2.4}$$

*is an inclusion of a full subcategory.*

**Proof:** Maps in  $\mathcal{C}_A$  and  $\mathcal{A}_A$  are maps of  $S$ -algebras under  $A$ , compare also [36, 1.3.].  $\square$

The following is a formal consequence.

**Lemma 2.11.** *Let  $A$  be a commutative  $S$ -algebra. If  $\alpha$  is a map in  $\mathcal{A}_A$  that is a cofibration as a map in  $A \downarrow \mathcal{A}_S$ , then  $\alpha$  is a cofibration in  $\mathcal{A}_A$ . If  $\alpha$  is a map in  $\mathcal{C}_A$  that is a cofibration as a map in  $\mathcal{A}_A$ , then  $\alpha$  is a cofibration in  $\mathcal{C}_A$ .*

**Proof:** For the first case, note that every acyclic fibration in  $\mathcal{A}_A$  is also an acyclic fibration in  $A \downarrow \mathcal{A}_S$ . Hence  $\alpha$  has the left lifting property (LLP) with respect to every acyclic fibration in  $\mathcal{A}_A$  producing a lift in  $A \downarrow \mathcal{A}_S$ . By lemma 2.10 this lift is automatically in  $\mathcal{A}_A$ . The proof for the second statement is similar.  $\square$

## Chapter 3

# Model structures for uncas

The definition and examination of Galois extensions in stable homotopy theory was made possible by the construction of symmetric monoidal model categories of spectra. For several decades, such categories had been thought impossible to exist. When the goal was finally achieved in the mid-nineties, it was a real breakthrough. One of our main references, [36], gives such a construction and also investigates categories of module- and algebra-spectra. The categories of  $S$ -algebras,  $S$ -modules,  $R$ -algebras and  $R$ -modules constructed in [36] share a lot of good properties whence we decided to work with them as far as possible. However, our needs go beyond this volume as we will now explain. In short, we can say that [36] is written “under a *commutative*  $S$ -algebra”. Many results are only stated for the category of  $R$ -algebras, where  $R$  is a *commutative*  $S$ -algebra. The case of associative  $S$ -algebras under an *associative*  $S$ -algebra  $A$ , i.e. the category of objects we called  $A$ -uncas before, is not dealt with. However, as the previous chapter suggests and as will become evident in chapter 4, we need to consider this more general situation. In this chapter we generalize most of the statements of [36, chapter VII] to the category of  $S$ -algebras under  $A$ , where  $A$  is a not necessarily commutative  $S$ -algebra.

The aim of this chapter is to show that the smash product of cofibrant algebras is homotopically meaningful. This is made precise and proved at the end of this chapter in theorem 3.16. In order to achieve this, we have to gain better control over the cofibrations in the various categories under consideration. For this purpose, we establish the model structure on the category of  $A$ -uncas along the lines of [36], even though we already obtained the model structure almost for free in proposition 2.9. Philosophically the quintessence is that the categories of uncas share most of the good properties of the various categories introduced in [36]. This is not surprising, but we have to go through the necessary technicalities.

Again a comment on terminology seems appropriate: In this thesis, the term “cofibration” will *always* be used in the model category theoretic sense as is the usual usage in most publications. In [36] these maps were called “q-cofibrations”. Moreover, “cofibration” in [36] denotes a map which has the homotopy extension property (HEP), see definition 3.6. We will instead always

keep the term HEP whenever we are talking about maps with the homotopy extension property.

Let  $\mathcal{M}_S = (\mathcal{M}_S, \wedge, S)$  be the category of  $S$ -modules constructed in [36], so in particular  $S$  from now on denotes the sphere spectrum. We will also work with the categories of  $S$ -algebras  $\mathcal{A}_S$  and more generally with  $R$ -modules and  $R$ -algebras without further comments.

For the whole chapter, let  $A$  be a cofibrant associative  $S$ -algebra.

### 3.1 Basic properties of the category $A \downarrow \mathcal{A}_S$

As mentioned at the beginning of the chapter, we want to establish model structures and hence have to check several properties of the category of  $S$ -algebras under  $A$  in order to make the machinery of [36] work. We will do this in this and the next section. We collect formal properties and the first is the following.

The category  $A \downarrow \mathcal{A}_S$  is enriched over unbased spaces. (3.1)

This is rather trivial as the category of  $S$ -algebras  $\mathcal{A}_S$  is topologically enriched and the morphism sets in  $A \downarrow \mathcal{A}_S$  are subsets of the morphism sets in  $\mathcal{A}_S$ . For the other properties to be verified another description of  $A \downarrow \mathcal{A}_S$  is useful.

#### 3.1.1 An operadic description of uncas

A very helpful description of the category  $A \downarrow \mathcal{A}_S$  is via monads. In fact all the model structures in [36] are obtained as follows: One starts with a model category  $\mathcal{C}$  and wants to lift this structure to a category of algebras  $\mathcal{C}[\mathbb{T}]$  over a monad  $\mathbb{T}: \mathcal{C} \rightarrow \mathcal{C}$ . Assumptions under which this is possible are given in [36] and [79].

Recall from chapter 2 that an object in  $A \downarrow \mathcal{A}_S$  is just a monoid in  $\mathcal{M}_{A^e}$  (proposition 2.2). In other words it is an algebra over the monad given by the free algebra functor  $\mathbb{T}_A: \mathcal{M}_{A^e} \rightarrow \mathcal{M}_{A^e}$ . On objects this functor is given by  $\mathbb{T}_A(M) := \bigvee_{j \geq 0} M^{\wedge A^j}$ . Another convenient notation for the category of  $A$ -uncas hence is  $\mathcal{M}_{A^e}[\mathbb{T}_A]$ . Remember that  $\mathcal{M}_{A^e}$  itself has an analogous description as  $\mathcal{M}_{A^e} = \mathcal{M}_S[\mathbb{F}_{A^e}]$  with the monad  $\mathbb{F}_{A^e}: \mathcal{M}_S \rightarrow \mathcal{M}_{A^e}$  sending  $M$  to  $A \wedge M \wedge A$ . By [36, II.6.1] there is an equality  $\mathcal{M}_S[\mathbb{F}_{A^e}][\mathbb{T}_A] = \mathcal{M}_S[\mathbb{T}_A \circ \mathbb{F}_{A^e}]$  and this is our category of  $A$ -uncas as can be seen by the adjunction

$$A \downarrow \mathcal{A}_S(\mathbb{T}_A M, B) \cong \mathcal{M}_{A^e}(M, B).$$

It is clear that a bimodule map  $M \rightarrow B$  into an algebra  $B \in A \downarrow \mathcal{A}_S$  defines a map  $\mathbb{T}_A M \rightarrow B$  of uncas. Vice versa, given a map of uncas  $\mathbb{T}_A M \rightarrow B$  this provides a bimodule map  $M \rightarrow B$  by lemma 2.5. These mappings are inverse to each other. Hence  $\mathbb{T}_A$  is left adjoint to the forgetful functor

$$\mathbb{T}_A: A \downarrow \mathcal{A}_S \rightleftarrows \mathcal{M}_{A^e} : U.$$

We are therefore interested in the composite monad  $\mathbb{T} := \mathbb{T}_A \circ \mathbb{F}_A^e$  on the category  $\mathcal{M}_S$  of  $S$ -modules. Obviously

$$\mathbb{T}(M) = \bigvee_{j \geq 0} (A \wedge M \wedge A)^{\wedge_{A^j}} \cong A \wedge \bigvee_{j \geq 0} (M \wedge A)^{\wedge_j} \cong \mathbb{F}_A \circ \mathbb{T}_S \circ \mathbb{F}_A(M).$$

In order to avoid confusion we point out that this is to be read as a composition of functors not of monads. E.g.  $\mathbb{T}_S$  is not a monad in  $\mathcal{M}_S[\mathbb{F}_A]$ . This however will not affect the following.

Recall that a coequalizer

$$A \begin{array}{c} \xrightarrow{e} \\ \xrightarrow{f} \end{array} B \longrightarrow C$$

is called *reflexive*, if there is a map  $h: B \rightarrow A$  such that both  $e \circ h$  and  $f \circ h$  are the identity maps.

**Lemma 3.1.** *The monad  $\mathbb{T}$  for which  $A \downarrow \mathcal{A}_S = \mathcal{M}_S[\mathbb{T}]$  preserves reflexive coequalizers and is continuous, i.e. it also preserves all small limits as a functor  $\mathbb{T}: \mathcal{M}_S \rightarrow \mathcal{M}_S$ .*

**Proof:** This follows from  $\mathbb{T} \cong \mathbb{F}_A \circ \mathbb{T}_S \circ \mathbb{F}_A$  and the corresponding statements for the operads  $\mathbb{T}_S$  and  $\mathbb{F}_A$ . Note that  $\mathbb{F}_A$  preserves reflexive coequalizers as  $A \rightarrow A \rightarrow A$  (all identity maps) is a reflexive coequalizer and by [36, II.7.2] which says that smashing with this coequalizer preserves coequalizers. One can also mimic the proof of this statement directly, it is not necessary to work in a *symmetric* monoidal category as assumed in the reference.  $\square$

Proposition 2.10 in [36] then says that the forgetful functor  $A \downarrow \mathcal{A}_S \rightarrow \mathcal{A}_S$  creates all indexed limits and that  $A \downarrow \mathcal{A}_S$  has all indexed colimits. See [54] as a basic reference on indexed limits and colimits. Also cotensors and tensors are treated as a special case of these in [54, 3.7]. Our next observation hence is the following:

The category  $A \downarrow \mathcal{A}_S$  is topologically complete and cocomplete, i.e. it has all indexed limits and colimits. In particular it is tensored and cotensored. (3.2)

As usual cotensors are created in  $\mathcal{M}_S$  and also we have a map  $B \wedge X_+ \rightarrow B \otimes_{A \downarrow \mathcal{A}_S} X$  with properties analogous to those of [36, VII.2.11]. Proposition VII.2.10 also shows that the forgetful functor  $A \downarrow \mathcal{A}_S \rightarrow \mathcal{M}_S$  is continuous. Composition with the continuous forgetful functor  $\mathcal{M}_S \rightarrow \mathcal{S}$  gives:

The category  $A \downarrow \mathcal{A}_S$  has a continuous forgetful functor to the category of spectra  $\mathcal{S}$ . (3.3)

## 3.2 Realization of simplicial $S$ -algebras under $A$

Having set up the basic formal properties of the category  $A \downarrow \mathcal{A}_S$  we can now deal with geometric realization which is an important construction for the proof of the so called cofibration hypothesis.

First, as there are various forgetful functors from  $A \downarrow \mathcal{A}_S$  to less structured categories, there are a priori many ways to carry out geometric realization of a simplicial  $S$ -algebra under  $A$ .

**Lemma 3.2.** *The geometric realization of  $C_*$ , a simplicial  $S$ -algebra under  $A$ , may be carried out in any of the categories  $\mathcal{S}$ ,  $\mathcal{A}_S$ ,  $\mathcal{M}_S$ ,  $\mathcal{M}_A$  or  $\mathcal{M}_{A^e}$  without changing the result. In particular it can be calculated in the category of spectra  $\mathcal{S}$ , i.e.*

$$|C_*|_{\mathcal{S}} \cong |C_*|_{A \downarrow \mathcal{A}_S}.$$

Moreover for any simplicial space  $X_*$  and  $B \in A \downarrow \mathcal{A}_S$  there is a natural isomorphism

$$B \otimes_{A \downarrow \mathcal{A}_S} |X_*| \cong |B \otimes_{A \downarrow \mathcal{A}_S} X_*|_{A \downarrow \mathcal{A}_S}$$

Hence we may just write  $|C_*|$  without causing any ambiguity.

**Proof:** The results are known for  $S$ -algebras [36, prop.VII.3.3 and 3.2]. Note that by functoriality and the fact that the realization of the constant simplicial  $S$ -algebra  $\underline{A}$  is  $A$ , we know that realization of  $C_*$  carried out in  $\mathcal{A}_S$  provides in fact an  $S$ -algebra under  $A$ . So realization is structure preserving when done in any of the less structured categories mentioned in the lemma as this is the case for simplicial  $S$ -algebras and modules. Together with the results from the last section this is all we have to know in order to copy the proofs of [36, prop.VII.3.3 and 3.2].  $\square$

Next we want to study the tensors  $B \otimes_{A \downarrow \mathcal{A}_S} I$  where  $I$  is the unit interval. Therefore we introduce the following variant of the bar construction in the category  $A \downarrow \mathcal{A}_S$ .

**Definition 3.3.** Let  $B, B'$  and  $B''$  be  $S$ -algebras under  $A$  and also let us be given maps of  $S$ -algebras  $f: B \rightarrow B'$  and  $g: B \rightarrow B''$ . Together with the identities on  $B', B''$  and  $B$  these maps induce maps  $B' \sqcup B \rightarrow B', B \sqcup B'' \rightarrow B''$  and  $B \sqcup B \rightarrow B$ . We define a simplicial object  $\beta_*^{A \downarrow \mathcal{A}_S}(B', B, B'')$  by setting

$$\beta_n^{A \downarrow \mathcal{A}_S}(B', B, B'') := B' \sqcup \underbrace{B \sqcup \cdots \sqcup B}_{n} \sqcup B''$$

where all coproducts are taken in  $A \downarrow \mathcal{A}_S$ . The face and degeneracy maps are composed from identities, the unit  $A \rightarrow B$  and the maps  $B' \sqcup B \rightarrow B', B \sqcup B'' \rightarrow B''$  and  $B \sqcup B \rightarrow B$  already mentioned. Furthermore we define

$$\beta^{A \downarrow \mathcal{A}_S}(B', B, B'') := |\beta_*^{A \downarrow \mathcal{A}_S}(B', B, B'')|.$$

We want to compare this with the “double mapping cylinder” defined as

$$M^{A \downarrow \mathcal{A}_S}(B', B, B'') := B' \sqcup_B (B \otimes_{A \downarrow \mathcal{A}_S} I) \sqcup_B B''.$$

The collapse map  $I \rightarrow \{pt\}$  induces a map  $M^{A \downarrow \mathcal{A}_S}(B', B, B'') \rightarrow B' \sqcup_B B''$  and also  $\beta^{A \downarrow \mathcal{A}_S}(B', B, B'')$  has a map to  $B' \sqcup_B B''$  which can be seen as the constant simplicial  $S$ -algebra  $\beta_*^B(B', B, B'')$ .

**Proposition 3.4.** (compare [36, VII.3.7]) *There is a natural isomorphism*

$$\beta^A \downarrow \mathcal{A}_S(B', B, B'') \cong M^A \downarrow \mathcal{A}_S(B', B, B'').$$

*This isomorphism is under  $B' \sqcup B''$  and over  $B' \sqcup_B B''$ .*

**Proof:** It suffices to prove the result for  $B' = B = B''$ . In this case there are isomorphic simplicial objects  $\beta_*^A \downarrow \mathcal{A}_S(B, B, B)$  and  $B \otimes_A \downarrow \mathcal{A}_S I_*$  where  $I_*$  is the standard simplicial 1-simplex which has  $q+2$  simplicies in degree  $p$ . Passing to realizations gives the result.  $\square$

Here is finally the proposition we will refer to in the proof of the cofibration hypothesis:

**Proposition 3.5.** *For an  $S$ -module  $M$  and a map  $\mathbb{T}M \rightarrow B$  of  $S$ -algebras under  $A$ , there is a natural map of  $S$ -algebras under  $A$*

$$\psi: M^A \downarrow \mathcal{A}_S(\mathbb{T}CM, \mathbb{T}M, B) \rightarrow \mathbb{T}CM \sqcup_{\mathbb{T}M} B$$

*that is homotopic to an isomorphism.*

The proposition in particular implies that there is an isomorphism

$$M^A \downarrow \mathcal{A}_S(\mathbb{T}CM, \mathbb{T}M, B) \rightarrow \mathbb{T}CM \sqcup_{\mathbb{T}M} B.$$

**Proof:** Let  $CM$  denote the cone of  $M$  defined as the pushout (in  $\mathcal{M}_S$ ) of  $* \leftarrow M \rightarrow M \otimes_{\mathcal{M}_S} I$ . We apply  $\mathbb{T}$  to the map  $CM \sqcup_M (M \otimes_{\mathcal{M}_S} I) \rightarrow CM$  that corresponds to the map  $CX \cup_X (X \wedge I_+) \rightarrow CX$  for spaces which retracts the cylinder onto the base of the cone. These maps are homotopic to isomorphisms. As a left adjoint  $\mathbb{T}$  preserves colimits, hence coproducts. It also commutes with tensors as cotensors are created in  $\mathcal{M}_S$ . Hence we obtain a map

$$\mathbb{T}CM \sqcup_{\mathbb{T}M} (\mathbb{T}M \otimes_A \downarrow \mathcal{A}_S I) \rightarrow \mathbb{T}CM.$$

Applying  $-\sqcup_{\mathbb{T}M} B$  gives the map  $\psi$ .  $\square$

### 3.3 Cofibration hypothesis

We are almost ready to state and prove the cofibration hypothesis and recall the following definition from [36, I.1.].

**Definition 3.6.** A map  $L \rightarrow M$  in  $\mathcal{S}$  (or in some category of module spectra  $\mathcal{M}_A$ ) has the *homotopy extension property* (HEP) if for any solid diagram of the form

$$\begin{array}{ccc} L & \longrightarrow & L \wedge I_+ \\ \downarrow & & \downarrow \\ M & \longrightarrow & M \wedge I_+ \\ & \searrow & \dashrightarrow \\ & & K \end{array}$$

a dashed arrow exists in  $\mathcal{S}$  (respectively  $\mathcal{M}_A$ ). Here the horizontal maps are induced by the usual inclusion  $S^0 \rightarrow I_+$ . When a map  $L \rightarrow M$  has the HEP we will also say that  $L \rightarrow M$  is a HEP-map.

It is clear, that for any  $S$ -algebra  $A$  and any  $A$ -module  $M$ , the map  $* \rightarrow M$  has the HEP. Also inclusions of wedge summands have the HEP in any category of modules  $\mathcal{M}_A$  as  $(M_1 \vee M_2) \wedge I_+ \cong (M_1 \wedge I_+) \vee (M_2 \wedge I_+)$ . There are many operations that preserve HEP-maps: Compositions and pushforwards of HEP-maps are HEP-maps and more generally for a sequence of HEP-maps  $X_0 \rightarrow X_1 \rightarrow \dots$  also  $X_0 \rightarrow \operatorname{colim} X_i$  is a HEP-map. Also retracts of HEP-maps have the HEP. Finally, if  $L \rightarrow M$  is a map of left  $A$ -modules that has the HEP and if  $N$  is a right  $A$ -module then the adjunction  $\mathcal{M}_S(N \wedge_A L, K) \cong \mathcal{M}_A(L, F_S(N, K))$  shows that  $N \wedge_A L \rightarrow N \wedge_A M$  is a HEP-map of  $S$ -modules. A less obvious preservation statement is the following.

**Lemma 3.7.** *Let  $B$  be any  $S$ -algebra  $B$  under  $A$  and  $M$  any  $S$ -module. Then  $B \rightarrow \mathbb{T}M \sqcup B$  has the HEP in any of the categories  $\mathcal{M}_S, \mathcal{M}_A, \mathcal{M}_{A^{op}}$  or  $\mathcal{M}_{A^e}$  where the coproduct is taken in the category of  $S$ -algebras under  $A$ .*

**Proof:** Recall that  $\mathbb{T} = \mathbb{T}_A \circ \mathbb{F}_{A^e}$  and consider also its analogue  $\tilde{\mathbb{T}} := \mathbb{T}_B \circ \mathbb{F}_{B^e}$ . The adjunctions  $A \downarrow \mathcal{A}_S(\mathbb{T}(X), Y) \cong \mathcal{M}_S(X, Y) \cong B \downarrow \mathcal{A}_S(\tilde{\mathbb{T}}(X), Y)$  for any  $S$ -module  $X$  and  $S$ -algebra  $Y$  under  $B$  show that the solid diagram in  $A \downarrow \mathcal{A}_S$

$$\begin{array}{ccc}
 A & \longrightarrow & B \\
 \downarrow & & \downarrow \\
 \mathbb{T}(X) & \longrightarrow & \tilde{\mathbb{T}}(X) \\
 & \searrow & \dashrightarrow \\
 & & Y
 \end{array}$$

admits a unique extension by the dotted arrow. Hence  $\tilde{\mathbb{T}}(X)$  is isomorphic to the coproduct  $\mathbb{T}X \sqcup B$ . So in particular  $\mathbb{T}M \sqcup B \cong \bigvee_{n \geq 0} (B \wedge M \wedge B)^{\wedge_B(n)}$ . Hence  $B \rightarrow \mathbb{T}M \sqcup B$  is the inclusion of a wedge summand in any of the three categories of modules from the statement. It is hence a HEP-map as already mentioned after definition 3.6.  $\square$

Recall that a simplicial spectrum is called proper if the canonical maps  $sK_q \rightarrow K_q$  are HEP-maps. Here  $sK_p$  is defined to be the “union” of the subspectra  $s_j K_{p-1}$  where  $0 \leq j < p$ . This can be made precise by giving a definition of  $sK_p$  in terms of iterated pushout diagrams but we refer the reader to [36, X.2.] for this. The bar construction  $\beta_*^{A \downarrow \mathcal{A}_S}(\mathbb{T}CM, \mathbb{T}M, B)$  as in definition 3.3 is proper as all the degeneracies are of the form  $C \rightarrow C \sqcup \mathbb{T}M$  which are inclusions of wedge summands, hence HEP-maps by the last lemma. For later reference we recall that for a proper simplicial spectrum  $K_*$  also the map  $K_0 \rightarrow |K_*|$  from its zeroth space into its realization is a HEP-map (compare [36, VII.3.9, X.2.3]). This can be seen using the usual filtration  $F_q|K_*|$ . We have  $F_0|K_*| \cong K_0$  and



pushout diagrams

$$\begin{array}{ccc}
 (sK_q \wedge \Delta_{q+}) \cup (K_q \wedge \partial\Delta_{q+}) & \longrightarrow & F_{q-1}|K_*| \\
 \downarrow & & \downarrow \\
 K_q \wedge \Delta_{q+} & \longrightarrow & F_q|K_*|
 \end{array} \tag{3.4}$$

in which the left vertical maps are HEP-maps by [36, X.2.3]. So also the right vertical maps have the HEP and so does the map  $X_0 \rightarrow |K_*| = \operatorname{colim} F_q|K_*|$ .

Here is our version of the cofibration hypothesis adapted to the category  $A \downarrow \mathcal{A}_S$ . Recall that the cone  $CM$  of an  $S$ -module  $M$  is defined as the pushout of  $* \leftarrow M \cong M \wedge \{1\}_+ \rightarrow M \wedge I_+$ .

**Proposition 3.8.** *1. Let  $M$  be an  $S$ -module. For any pushout diagram*

$$\begin{array}{ccc}
 \mathbb{T}M & \longrightarrow & B \\
 \downarrow & & \downarrow \\
 \mathbb{T}CM & \longrightarrow & D
 \end{array}$$

*in  $A \downarrow \mathcal{A}_S$  the right vertical map has the HEP in either of the categories of  $S$ -modules, (left or right)  $A$ -modules and  $A^e$ -modules.*

*2. For a sequence of maps of  $S$ -algebras under  $A$  which have the HEP in the category  $A \downarrow \mathcal{A}_S$ , the underlying spectrum of their colimit formed in the category of  $S$ -algebras under  $A$  is their colimit as a sequence of maps of spectra.*

**Proof:** We start with proving the first point. Let  $\mathcal{C}$  be any of the categories of  $S$ -modules,  $A$ -modules and  $A^e$ -modules. By propositions 3.4 and 3.5 the pushout  $D$  is isomorphic to the realization of the bar construction  $\beta^A \downarrow \mathcal{A}_S(\mathbb{T}CM, \mathbb{T}M, B)$ . Its zeroth term is  $\mathbb{T}CM \sqcup B$ . The right vertical map from the statement factors as  $B \rightarrow \mathbb{T}CM \sqcup B \rightarrow \beta^A \downarrow \mathcal{A}_S(\mathbb{T}CM, \mathbb{T}M, B) \cong D$  and it suffices to show that the two maps from this factorization have the HEP in  $\mathcal{C}$ . For the first map this follows from the last lemma. The second map is a HEP-map as we just explained before stating the theorem as the simplicial spectrum  $\beta_*^A \downarrow \mathcal{A}_S = \beta_*^A \downarrow \mathcal{A}_S(\mathbb{T}CM, \mathbb{T}M, B)$  is proper in  $\mathcal{C}$ .

The second point follows immediately from the corresponding statement for  $S$ -algebras [36, VII.3.10].  $\square$

### 3.4 Setting up the model structure à la EKMM

In sections 3.1 and 3.3 we established all the necessary properties of the monad  $\mathbb{T}$  in order to apply [36, VII.4.7].

**Theorem 3.9.** *The category of  $S$ -modules  $\mathcal{M}_S$  creates a model structure on  $A \downarrow \mathcal{A}_S = \mathcal{M}_S[\mathbb{T}]$ . This model structure is the same as the one established in proposition 2.9.*

**Proof:** The first statement follows directly from [36, VII.4.7]. The terminology means that the weak equivalences and fibrations are created in  $\mathcal{M}_S$ . This determines the model structure completely. The same is the case for the model structure of proposition 2.9: Weak equivalences and fibrations are those maps that are such in the category  $\mathcal{A}_S$ . But  $\mathcal{A}_S$  as a model category is itself created by  $\mathcal{M}_S$ . Hence the second statement also holds.  $\square$

We now can say more precisely what the cofibrations are. Recall the pairs  $(CS^q, S^q)$  where  $S^q$  is a cell  $S$ -module of dimension  $q$  and  $C$  is the cone functor. In the following definition  $\mathbb{T}$  may be any monad in  $\mathcal{M}_S$ .

**Definition 3.10.** ([36, VII.4.11]) A *relative cell  $\mathbb{T}$ -algebra*  $Y$  under a  $\mathbb{T}$ -algebra  $Y_0$  is a  $\mathbb{T}$ -algebra  $Y = \operatorname{colim} Y_n$  where  $Y_{n+1}$  is obtained from  $Y_n$  as the pushout of a sum of attaching maps  $\mathbb{T}S^q \rightarrow Y_n$  along the coproduct of the natural maps  $\mathbb{T}S^q \rightarrow \mathbb{T}CS^q$ .

To be explicit, the diagrams to be considered are of the form

$$\begin{array}{ccc} \bigsqcup \mathbb{T}S^{q_i} & \longrightarrow & Y_n \\ \downarrow & & \downarrow \\ \bigsqcup \mathbb{T}CS^{q_i} & \longrightarrow & Y_{n+1}. \end{array} \quad (3.5)$$

Note that by our cofibration hypothesis the right vertical map is not only a cofibration in  $A \downarrow \mathcal{A}_S$  but also a HEP-map in  $\mathcal{M}_S, \mathcal{M}_A$  and  $\mathcal{M}_{A^e}$ .

Due to theorem 3.9 the model structure on  $\mathcal{M}_S[\mathbb{T}]$  can be established as in [36]. As a consequence we have the following description of cofibrations.

**Proposition 3.11.** (cp. [36, VII.4.14] and its proof) A map of  $\mathbb{T}$ -algebras is a cofibration in  $\mathcal{M}_S[\mathbb{T}]$  if and only if it is a retract of a relative cell  $\mathbb{T}$ -algebra. A cofibrant  $\mathbb{T}$ -algebra is a retract of a cell  $\mathbb{T}$ -algebra.  $\square$

### 3.5 Homotopical significance of the smash product in $A \downarrow \mathcal{A}_S$

In this section we finally prove that the smash product of two cofibrant  $S$ -algebras under  $A$  represents their derived smash product. Hence we generalize [36, ThmVII.6.7] which states the same for the case that  $A$  is a commutative  $S$ -algebra. Remember our standing assumption that  $A$  is a cofibrant  $S$ -algebra. We first show that all the possible ways to define derived smash products are equivalent. For this consider the following diagram. Note that in order to distinguish the different roles in which  $\wedge_A$  occurs we write  $\wedge_{A^e}$  for the product in the category of  $A^e$ -modules, though this is again smashing over  $A$ . As there will be no smash-products over  $A^e$  in this section, no confusion should arise.

$$\begin{array}{ccc}
 \mathcal{M}_{A^e} \times \mathcal{M}_{A^e} & \xrightarrow{V} & r\mathcal{M}_A \times l\mathcal{M}_A \xrightarrow{\wedge_A} \mathcal{M}_S \\
 \wedge_{A^e} \downarrow & & \nearrow U \\
 \mathcal{M}_{A^e} & & 
 \end{array} \tag{3.6}$$

Diagram (3.6) gives rise to several derived smash-products. The next lemma says that these are basically the same. As usual  $L(-)$  and  $R(-)$  denote total left and right derived functors. It is clear that all the possible derived smash products occurring in the following proposition exist.

**Proposition 3.12.** 1. *The derived smash product for the composite functor*

$$\mathcal{M}_{A^e} \times \mathcal{M}_{A^e} \xrightarrow{V} r\mathcal{M}_A \times l\mathcal{M}_A \xrightarrow{\wedge_A} \mathcal{M}_S$$

*is given by the composite  $L(\wedge_A) \circ R(V)$ , i.e. there is a natural isomorphism*

$$L(\wedge_A \circ V) \cong L(\wedge_A) \circ R(V).$$

2. *The derived smash product for the functor  $U \circ \wedge_{A^e}$  is given by the composite  $R(U) \circ L(\wedge_{A^e})$ , i.e. there is a natural isomorphism*

$$L(U \circ \wedge_{A^e}) \cong R(U) \circ L(\wedge_{A^e}).$$

*In particular the three possible definitions for the derived smash-product  $\mathcal{D}_{A^e} \times \mathcal{D}_{A^e} \rightarrow \mathcal{D}_S$  coincide.*

**Proof:** Using that  $V$  is the right adjoint of a Quillen equivalence one can check that  $L(\wedge_A) \circ R(V)$  has the universal property that makes it a left derived functor of the functor  $\wedge_A \circ V$  proving the first point. The second part may be proved analogously.  $\square$

**Corollary 3.13.** 1. *On two  $A$ -bimodules  $N$  and  $M$  the derived smash product  $\mathcal{D}_{A^e} \times \mathcal{D}_{A^e} \rightarrow \mathcal{D}_S$  is represented by  $N \wedge_A \Gamma_A M$  where  $\Gamma_A M$  is a cofibrant replacement of  $M$  in the category of left  $A$ -modules.*

2. *The derived smash product  $\mathcal{D}_{A^e} \times \mathcal{D}_{A^e} \rightarrow \mathcal{D}_S$  lifts to  $\mathcal{D}_{A^e}$  and is represented by  $N \wedge_A \Gamma_{A^e} M$  where  $\Gamma_{A^e} M$  is a cofibrant replacement of  $M$  in the category of  $A$ -bimodules.*

**Proof:** Part one follows from the last proposition and the fact that  $\wedge_A$  is represented by the point-set-level smash product with one argument cofibrantly replaced. The same is true for  $\wedge_{A^e}$  as we will now show. A priori both factors have to be replaced cofibrantly but we see that weak equivalences  $N \rightarrow \tilde{N}$  of  $A^e$ -modules are preserved by smashing with some cofibrant  $A^e$ -module  $M$  as follows. First if  $M = \mathbb{F}_{A^e} X$  for some CW-module  $X$ , we have  $M \wedge_A N \cong \mathbb{F}_A X \wedge N$  and the result follows from the corresponding statement for  $S$ -modules. Note that  $\mathbb{F}_A X$  is a cofibrant  $S$ -module as  $A$  is assumed to be a cofibrant  $S$ -algebra. The general statement then follows from the gluing lemma and passage to colimits. A similar argument is carried out in the proof of theorem 3.16  $\square$

We can see inductively, that there is a derived smash product  $\mathcal{D}_{A^e} \times \cdots \times \mathcal{D}_{A^e} \rightarrow \mathcal{D}_S$  that lifts to  $\mathcal{D}_{A^e}$  and is represented by

$$M_1 \wedge_A \Gamma[M_2 \wedge_A \Gamma[\cdots \Gamma[M_{n-1} \wedge_A \Gamma M_n] \cdots]]$$

where  $\Gamma$  is either  $\Gamma_A$  or  $\Gamma_{A^e}$ .

We now define a class of  $A$ -bimodules that will turn out to be well behaved with respect to the derived smash product. Recall that for  $A = S$  the class  $\overline{\mathcal{E}}_S$  defined in [36, VII.6.4] has this property by theorem [36, VII.6.7]. By [36, VII.6.5] the class  $\overline{\mathcal{E}}_S$  contains all cofibrant  $S$ -algebras.

**Definition 3.14** (Extended cell bimodules). Let  $A$  be an associative  $A$ -algebra. Define  $\mathcal{F}_A$  to be the collection of  $A^e$ -modules of the form

$$A \wedge_S X$$

where  $X$  is any spectrum in  $\overline{\mathcal{E}}_S$ . Define  $\overline{\mathcal{F}}_A$  to be the extension of  $\mathcal{F}_A$  being closed under all finite  $\wedge_A$ -products, wedges, pushouts along HEP-maps and colimits of countable sequences of HEP-maps and homotopy equivalences where all these operations are taken in  $A$ -bimodules. We call this class the class of *extended cell  $A$ -bimodules*.

**Proposition 3.15.** *Let  $A$  be an associative  $A$ -algebra and  $Y$  a cell  $A$ -unca, i.e. a cell algebra for the operad  $\mathbb{T}$ . Then  $Y$  is in  $\overline{\mathcal{F}}_A$ .*

**Proof:** First note that  $A$  is in  $\overline{\mathcal{F}}_A$  as  $S$  is in  $\overline{\mathcal{E}}_S$ . Then by definition  $Y$  is the colimit of  $Y_n$  as in definition 3.10. All the maps  $Y_n \rightarrow Y_{n+1}$  have the HEP by our cofibration hypothesis and  $Y_0 = A$ . So it suffices to show that with  $Y_n$  also  $Y_{n+1}$  is in  $\overline{\mathcal{F}}_A$ . In order to see this, let us write  $B \rightarrow D$  for the map  $Y_n \rightarrow Y_{n+1}$  and we factor this map as  $B \rightarrow D_0 = \mathbb{T}CM \sqcup B \rightarrow |D_*| \cong D$  where  $D_*$  is a proper spectrum as in the proof of proposition 3.8. By the proof of lemma 3.7 its zeroth term  $D_0$  is  $\bigvee_{n \geq 0} (B \wedge CM \wedge B)^{\wedge_B(n)} \cong B \vee (B \wedge_A \bigvee_{n \geq 1} (A \wedge CM \wedge B)^{\wedge_A(n)})$ . Hence  $D_0$  is in  $\overline{\mathcal{F}}_A$  as  $B$  is a cofibrant  $S$ -algebra by induction hypothesis and hence is in  $\overline{\mathcal{E}}_S$ . It remains to show that  $D$  is obtained from  $D_0$  by taking a colimit along HEP-maps. This is true as again we have a filtration of the realization  $|D_*|$  and pushouts analogous to those in diagram (3.4).  $\square$

**Theorem 3.16.** *Let  $A$  be a cofibrant  $S$ -algebra and for an  $A^e$ -bimodule  $M$  let  $\gamma_M: \Gamma M \rightarrow M$  be a cell approximation either in the category of (left)  $A$ -modules, e.g.  $\Gamma = \Gamma_A$  or likewise in  $A^e$ -modules, e.g.  $\Gamma = \Gamma_{A^e}$ . Then for  $M_1, \dots, M_n \in \overline{\mathcal{F}}_A$*

$$M_1 \wedge_A \Gamma[M_2 \wedge_A \Gamma[\cdots \Gamma[M_{n-1} \wedge_A \Gamma M_n] \cdots]] \rightarrow M_1 \wedge_A \cdots \wedge_A M_n \quad (3.7)$$

*is a weak equivalence. In other words: The smash product of modules in  $\overline{\mathcal{F}}_A$  represents the derived smash product in  $\mathcal{D}_{A^e}$ .*

**Corollary 3.17.** *The smash product of cofibrant  $S$ -algebras under  $A$  represents their derived smash product.*

**Proof:** For cell  $A$ -algebras this follows from theorem 3.16 and proposition 3.15. The general statement follows as every cofibrant  $S$ -algebra under  $A$  is a retract of a cell  $A$ -algebra and passing to retracts preserves weak equivalences.  $\square$

**Proof of theorem 3.16:** As for both  $\Gamma_A$  and  $\Gamma_{A^e}$  the left hand side of (3.7) represents the derived smash product, it suffices to prove the proposition for  $\Gamma = \Gamma_A$ . In order to prove (3.7) first note that the corresponding statement for commutative  $A$  and  $M_i \in \overline{\mathcal{E}}_A$  holds by [36, VII.6.7, III.3.8] and the 2 out of 3 property

$$\begin{array}{ccc} \Gamma M_1 \wedge_A \cdots \wedge_A \Gamma M_n & \xrightarrow{\cong} & M_1 \wedge_A M_2 \cdots \\ & \searrow \cong & \nearrow \\ & M_1 \wedge_A \Gamma[M_2 \wedge_A \cdots \wedge_A \Gamma M_n] & \end{array}$$

In other words, the theorem holds for  $A = S$  as  $\overline{\mathcal{E}}_S = \overline{\mathcal{F}}_S$ . From this we can continue as in the proof of [36, VII.6.7.]: First, equation (3.7) holds for  $M_i \in \mathcal{F}_A$  as in this case  $\Gamma_A M_i = A \wedge \Gamma_S N_i$  for certain  $N_i \in \overline{\mathcal{E}}_S$ . Hence  $M_1 \wedge_A \Gamma_A[M_2 \wedge_A \cdots \wedge_A \Gamma_A M_n] \cong A \wedge \Gamma_S N_1 \wedge \cdots \wedge \Gamma_S N_n$  and now we can apply the theorem for  $A = S$  as  $A \in \overline{\mathcal{E}}_S$ .

Then the statement for general  $M_i \in \overline{\mathcal{F}}_A$  may be proved by induction on the complexity of formulas of  $\overline{\mathcal{F}}_A$ . First note that it suffices to consider the case  $n = 2$ . By symmetry and as  $N \wedge_A \Gamma M \simeq \Gamma N \wedge_A M$  it suffices to prove that the statment remains true when replacing one of the factors corresponding to the operations building  $\overline{\mathcal{F}}_A$ . We will now assume  $N \wedge_A \Gamma M_i \simeq N \wedge_A M_i$  for  $N$  and  $M_i$  in  $\overline{\mathcal{F}}_A$ . It follows that for any finite set  $I$  also  $N \wedge_A \Gamma \bigwedge_I M_i \simeq N \wedge_A \bigwedge_I M_i$  as by induction hypothesis  $\bigwedge_I M_i$  represents the derived smash product and hence  $N \wedge_A \Gamma \bigwedge_I M_i \simeq N \wedge_A \Gamma[M_1 \wedge_A \Gamma[\cdots]] \simeq N \wedge_A \bigwedge_I M_i$ . For wedges this is immediate as  $\pi_*$  carries wedges to direct sums.

We come to the step replacing  $M$  by some  $\widetilde{M}$  obtained from  $M$  by a pushout along a HEP-map  $X \rightarrow Y$ . We want to conclude that  $\Gamma N \wedge_A M \simeq N \wedge_A M$  implies  $\Gamma N \wedge_A \widetilde{M} \simeq N \wedge_A \widetilde{M}$ . Note that we have a diagram

$$\begin{array}{ccc} \Gamma N \wedge_A Y & \xleftarrow{\text{HEP}} \Gamma N \wedge_A X & \longrightarrow \Gamma N \wedge_A M \\ \simeq \downarrow & & \downarrow \simeq \\ N \wedge_A Y & \xleftarrow{\text{HEP}} N \wedge_A X & \longrightarrow N \wedge_A M \end{array} \quad (3.8)$$

in which the left horizontal maps have the HEP and all vertical maps are weak equivalences. It is a standard argument also called the gluing lemma that the map  $\Gamma N \wedge_A \widetilde{M} \rightarrow N \wedge_A \widetilde{M}$  then also is a weak equivalence. We give a proof for the convenience of the reader: As HEP-maps induce long exact sequences in homotopy by [36, IV.1.2.(ii)] we see that the cofibers of the two left horizontal maps are weakly equivalent. Completing (3.8) to a cube by taking the pushouts of the top and bottom row yields two more HEP-maps and hence long exact sequences. Two out of three terms of these sequences are equivalent, one by the

assumption that  $\Gamma N \wedge_A M \simeq N \wedge_A M$  and the terms coming from the cofibers are those of the cofibers of the left horizontal maps in (3.8). It follows by the five lemma that the sequences are equivalent giving  $\Gamma N \wedge_A \widetilde{M} \simeq N \wedge_A \widetilde{M}$ . The statement remains true when passing to sequential colimits along HEP-maps: We have

$$\begin{aligned} \pi_*(\Gamma N \wedge_A \operatorname{colim} X_i) &\cong \pi_* \operatorname{colim}(\Gamma N \wedge_A X_i) \cong \operatorname{colim} \pi_*(\Gamma N \wedge_A X_i) \\ &\cong \operatorname{colim} \pi_*(N \wedge_A X_i) \cong \pi_* N \wedge_A \operatorname{colim} X_i \end{aligned}$$

as  $N \wedge_A (-)$  is a left adjoint and  $\pi_*$  commutes with sequential colimits. Lastly we have to treat with passing to homotopy equivalent objects. For this note that by [36, p.59] homotopy equivalences are preserved when smashing with another object.  $\square$

**Lemma 3.18.** *Let  $A$  be a cofibrant  $S$ -algebra and  $M \in \overline{\mathcal{F}}_A$ . Let  $\Gamma M \rightarrow M$  be a cell approximation in the category of left  $A$ -modules. Then for any left  $A$ -module  $X$  we have*

$$F_A(M, X) \simeq F_A(\Gamma M, X),$$

*i.e.  $F_A(M, X)$  represents the derived function spectrum.*

**Proof:** Again by induction on the complexity of formulas, one can show that every spectrum in  $\overline{\mathcal{F}}_A$  has the homotopy type of a  $CW$ -spectrum starting with the statement from [36, VII.6.6] that this is true for every spectrum in  $\overline{\mathcal{E}}_S$ . Hence the cell approximations  $\Gamma M \rightarrow M$  are homotopy equivalences and the statement follows.  $\square$

## Chapter 4

# Galois extensions of associative $S$ -algebras

John Rognes introduced the notion of Galois extensions for commutative  $S$ -algebras in [75]. In this chapter, we will generalize this notion to extensions of associative  $S$ -algebras. The quintessence of this chapter is that large parts of Rognes' theory have analogs for associative  $S$ -algebras. We show that there are lots of examples of Galois extensions of associative  $S$ -algebras. The first to be mentioned are the trivial and matrix extensions. We show that the Eilenberg-McLane embedding theorem holds whenever the trace of a Galois extension of associative rings is surjective, see theorem 4.5. In particular, this embeds all associative Galois extensions of a commutative ring in the world of Galois extensions of associative  $S$ -algebras.

We investigate conditions under which Galois extensions  $A \rightarrow B$  are preserved and detected when inducing up along a map of  $S$ -algebras  $A \rightarrow C$  so that we obtain a map  $C \rightarrow C \wedge_A B$ , see propositions 4.20 and 4.21. In this way, Galois extensions of the Morava  $K$ -theory spectra  $K(n)$  or  $K_n$  are obtained from the Lubin-Tate extensions of the  $K(n)$ -local sphere. In particular,  $K_n$  can be seen as a Galois extension of  $K(n)$  and there is a Galois extension  $K_n \rightarrow K_n^{nr}$  obtained from the maximal unramified extension  $E_n \rightarrow E_n^{nr}$ . Finally, we characterize Galois extensions using dualizability and conclude the chapter with an investigation of the closure property of the spectrum  $K_n^{nr}$ .

### 4.1 Definitions

Recall that for a topological space  $X$  the unreduced suspension spectrum  $\Sigma^\infty X_+$  is also denoted  $S[X]$  or  $S \wedge X_+$ . If  $G$  is a topological group and  $B$  a spectrum with left  $G$  action, the homotopy fixed point spectrum  $B^{hG}$  is defined as  $B^{hG} := F(EG_+, B)^G$  where  $EG$  is the free contractible left  $G$ -space. Also recall that a topological group  $G$  is stably dualizable if  $S[G]$  is dualizable over  $S$ , see [33, 64, 76]. The notion of dualizability is also recalled in section 4.5. Every finite discrete group is dualizable so one can keep this special test case in mind while reading the following. With these reminders we can approach the definition of a Galois extension of associative  $S$ -algebras.

Let  $A$  and  $B$  be associative  $S$ -algebras and  $\eta: A \rightarrow B$  a map of  $S$ -algebras making  $B$  an unca under  $A$  (“unital not necessarily central algebra under  $A$ ”, see chapters 2 and 3). Let  $G$  be a stably dualizable group acting on  $B$  from the left by  $S$ -algebra maps under  $A$ .

We will require some cofibrancy assumptions that make our definition homotopy invariant. First, we assume that the  $S$ -algebra  $A$  is cofibrant as an associative or commutative  $S$ -algebra. Second, we will require that smashing with  $B$  over  $A$  represents the derived smash product. If these two requirements are satisfied, we will say that our data satisfy *good cofibrancy assumptions*. Of course we have good cofibrancy assumptions if  $B$  is cofibrant in the category of  $A$ -uncas (i.e. the category of associative  $S$ -algebras under  $A$ ) by corollary 3.17. More generally, we have good cofibrancy assumptions by theorem 3.16 when  $B$  is in the class  $\overline{\mathcal{F}}_A$  of extended cell  $A$ -bimodules as introduced in definition 3.14. Note that we can always pass to a map  $A^c \rightarrow B^c$  that satisfies good cofibrancy assumptions as follows. First, replace  $A$  cofibrantly in the category of  $S$ -algebra providing  $A^c$ . In a second step, replace  $B$  cofibrantly in the category of  $A^c$ -uncas. As functorial cofibrant replacements exist in  $A \downarrow \mathcal{A}_S$ , the  $G$ -action transfers to the cofibrant replacement  $B^c$  of  $B$ . So the standard situation we will consider is that  $A$  is a cofibrant associative  $S$ -algebra and  $A \rightarrow B$  is a cofibration of  $S$ -algebras, i.e. of  $A$ -uncas.

*Remark 4.1.* By lemma 2.10 the group  $G$  always acts by maps in the appropriate category, i.e. if  $\eta$  is central, such that  $B$  is a central  $A$ -algebra, then  $G$  acts by maps of central  $A$ -algebras. If in addition  $A$  is commutative, such that  $B$  is an  $A$ -algebra in the sense of [36], then  $G$  acts by maps of  $A$ -algebras and if  $B$  is commutative then  $G$  acts by maps of commutative  $A$ -algebras. Moreover if  $A \rightarrow B$  is a map in  $\mathcal{A}_A$  or  $\mathcal{C}_A$  which is a cofibration in the category of  $A$ -uncas then the map  $A \rightarrow B$  is a cofibration also in  $\mathcal{A}_A$  or  $\mathcal{C}_A$  by lemma 2.11.

We define the maps

$$i: A \rightarrow B^{hG}$$

as the adjoint to the  $G$ -equivariant map  $EG_+ \wedge A \xrightarrow{\cong} A \rightarrow B$  first collapsing  $EG$  to a point, and

$$h: B \wedge_A B \rightarrow F(G_+, B)$$

as the right adjoint to  $B \wedge_A B \wedge G_+ \xrightarrow{B \wedge_A \tau} B \wedge_A G_+ \wedge B \xrightarrow{B \wedge_A \alpha} B \wedge_A B \xrightarrow{\mu} B$  where  $\tau$  is the twist map and  $\alpha: G_+ \wedge B \rightarrow B$  denotes the group action. The map  $h$  is a  $G$ -equivariant map of left  $B$ -modules and moreover, it is a left  $B[G]$ -module map where the  $G$ -action on the source is via the action on the second  $B$  factor and the left  $G$ -action on  $F(G_+, B)$  comes from the right action of  $G$  on itself. The map  $i$  is a map of  $S$ -algebras under  $A$ . The next definition generalizes [75, 4.3.1].

**Definition 4.2** (Galois extensions of associative  $S$ -algebras). Let  $\eta: A \rightarrow B$  be a map of  $S$ -algebras satisfying good cofibrancy assumptions, e.g.  $A$  could be a cofibrant associative  $S$ -algebra and  $A \rightarrow B$  a cofibration of  $S$ -algebras. Let  $G$  be a stably dualizable group acting on  $B$  by maps of  $S$ -algebras under



A. We call  $A \rightarrow B$  a  $G$ -Galois extension, if the maps  $i$  and  $h$  are both weak equivalences.

We will refer to the fact that  $i$  is a weak equivalence as the fixed point condition and to the fact that  $h$  is a weak equivalence as the unramification condition.

There is an  $E$ -local version of this definition if  $A$  and  $B$  are  $E$ -local for some  $S$ -module  $E$ . One can then ask that the maps  $i$  and  $h$  formed in the category  $\mathcal{M}_{S,E}$  of  $E$ -local  $S$ -modules are weak equivalences. The construction of the map  $h$  in  $\mathcal{M}_{S,E}$  includes an additional  $E$ -localization of  $B \wedge_A B$ . Hence the  $E$ -local version of a Galois extension amounts to asking that  $h$  is an  $E_*$ -equivalence. As for Galois extensions of commutative  $S$ -algebras, most of the statements below have analogs for  $E$ -local Galois extensions. However we restrict our presentation to the global case and feel that this restriction is not too big. To explain this, note that the main examples of local associative Galois extensions might occur as extensions from the local commutative Galois extensions presented in [75], see section 4.6.1. These extensions are  $K(n)$ -local for some Morava-  $K$ -theory spectrum  $K(n)$ . We will obtain associative Galois extensions inducing along a map into  $K(n)$  or some other  $K(n)$ -algebra in section 4.6.1. The extensions obtained in this way will automatically be global and so will be all the other non-commutative Galois extensions considered in this thesis. This makes a detailed consideration of local associative Galois extensions irrelevant for this document.

Due to our cofibrancy assumptions and as the smash-product of cofibrant uncas over  $A$  represents the derived smash-product in  $\mathcal{D}_{A^e}$  by theorem 3.16 we obtain the following invariance statement.

**Lemma 4.3** (Invariance up to changes by weak equivalences). *The property of being a Galois extension is homotopy invariant up to changes of  $A$ ,  $B$  and the stabilized group  $\Sigma^\infty G_+$ . In more detail: If we replace  $A$  by a weakly equivalent  $\tilde{A}$  or  $B$  by a  $G$ -equivariantly weakly equivalent  $\tilde{B}$  also satisfying the cofibrancy assumptions above, or  $G$  by a stably weakly equivalent group  $\tilde{G}$ , Galois extensions are preserved and detected.*

**Proof:** Let us first consider the fixed point condition and replacements by a weak equivalence  $B \rightarrow \tilde{B}$ . As this map is assumed to be  $G$ -equivariant there is a weak equivalence  $B^{hG} := F(EG_+, B)^G \rightarrow F(EG_+, \tilde{B})^G = \tilde{B}^{hG}$  by [36, III.6.8]. It is a map under  $A$ . So  $A \rightarrow B^{hG}$  is a weak equivalence if and only if  $A \rightarrow \tilde{B}^{hG}$  is. Of course an additional replacement by  $\tilde{A} \simeq A$  does not change this condition either.

For the unramification condition, consider two candidates for  $G$ -Galois extensions  $A \rightarrow B$  and  $\tilde{A} \rightarrow \tilde{B}$  with  $A \xrightarrow{\simeq} \tilde{A}$  and  $B \xrightarrow{\simeq} \tilde{B}$ . Note that the cofibrancy assumptions ensure that both  $B \wedge_A B$  and  $\tilde{B} \wedge_{\tilde{A}} \tilde{B}$  represent the derived smash products by 3.16, so the Künneth spectral sequence shows that

$\tilde{B} \wedge_{\tilde{A}} \tilde{B} \simeq B \wedge_A B$ . The commutative square

$$\begin{array}{ccc} B \wedge_A B & \xrightarrow{h} & F(G_+, B) \\ \simeq \downarrow & & \downarrow \simeq \\ \tilde{B} \wedge_{\tilde{A}} \tilde{B} & \xrightarrow{\tilde{h}} & F(G_+, \tilde{B}) \end{array}$$

then shows that  $h$  is a weak equivalence if and only if so is  $\tilde{h}$ . The right map in the diagram is a weak equivalence as  $G$  is assumed to be dualizable, hence topological. For the invariance with respect to changes in the stabilized group  $\Sigma^\infty G_+$  see [75, 4.1.4].  $\square$

## 4.2 First examples

### 4.2.1 Trivial extensions

For every cofibrant associative  $S$ -algebra  $A$  and stably dualizable group  $G$ , we have the trivial  $G$ -Galois-extension  $A \rightarrow B$  where  $B$  is a cofibrant replacement of  $F(G_+, A)$  in the category  $A \downarrow \mathcal{A}_S$  of associative  $S$ -algebras under  $A$ . The map  $A \rightarrow B$  is induced by the unique map  $G \rightarrow \{e\}$ . It is clear that  $B^{hG} \simeq F(EG_+, F(G_+, A))^G \simeq F(EG_+, A) \simeq A$ . Also  $h$  is a weak equivalence since it can be written as  $B \wedge_A B \simeq F(G_+, A) \wedge_A F(G_+, A) \rightarrow F(G_+, F(G_+, A)) \simeq F(G_+, B)$ . The map is a weak equivalence as  $G$  is stably dualizable. The first equivalence holds due to the fact that both smash products represent the derived smash product by theorem 3.16.

### 4.2.2 Matrix extensions of $S$ -algebras

For an associative  $S$ -algebra  $A$  and integers  $n, m$  define the matrix  $S$ -module  $\mathcal{M}_{n,m}(A)$  as

$$\mathcal{M}_{n,m}(A) := F_S(S \wedge m_+, A \wedge n_+).$$

where we look at an integer  $k$  as the set  $\{1, 2, \dots, k\}$ . As homotopy groups take wedges and products to direct sums there is an isomorphism  $\pi_*(\mathcal{M}_{n,m}(A)) \cong \mathcal{M}_{n,m}(\pi_*(A))$ . There is a map  $\mathcal{M}_{n,m}(A) \wedge \mathcal{M}_{m,k}(A) \rightarrow \mathcal{M}_{n,k}(A)$  defined to be adjoint to the iterated evaluation

$$\begin{aligned} & F(m_+, A \wedge n_+) \wedge F(k_+, A \wedge m_+) \wedge k_+ \xrightarrow{F(m_+, A \wedge n_+) \wedge \text{eval}} \\ & F(m_+, A \wedge n_+) \wedge m_+ \wedge A \xrightarrow{\text{eval} \wedge_A} A \wedge A \wedge n_+ \xrightarrow{\mu_A \wedge n_+} A \wedge n_+. \end{aligned}$$

If  $m = n$  this map makes  $\mathcal{M}_n(A) := \mathcal{M}_{n,n}(A)$  an associative  $S$ -algebra [36, III.6.12, VI.5]. Moreover if  $A \rightarrow B$  is a map of associative  $S$ -algebras then so is  $\mathcal{M}_n(A) \rightarrow \mathcal{M}_n(B)$ . In the next proposition we assume implicitly that  $\mathcal{M}_n(B)$  is cofibrant as an  $\mathcal{M}_n(A)$ -unca, making a cofibrant replacement if necessary.

**Proposition 4.4.** *Let  $A$  be an associative  $S$  algebra and  $B$  a cofibrant  $A$ -unca. Let  $G$  be a stably dualizable group that acts on  $B$  by maps of associative  $S$ -algebras under  $A$ . Then  $A \rightarrow B$  is a  $G$ -Galois extension if and only if so is the extension  $\mathcal{M}_n(A) \rightarrow \mathcal{M}_n(B)$  for some (hence every) positive integer  $n$ .*

**Proof:** Define the  $G$ -action on  $\mathcal{M}_n(B)$  by the action on  $B$ , i.e. as the adjoint to

$$G_+ \wedge \mathcal{M}_n(B) \wedge n_+ \xrightarrow{G_+ \wedge \text{eval}} G_+ \wedge B \wedge n_+ \xrightarrow{\alpha \wedge n_+} B \wedge n_+$$

where  $\alpha: G_+ \wedge B \rightarrow B$  denotes the group action. Then

$$\begin{aligned} \mathcal{M}_n(B)^{hG} &= F(EG_+, \mathcal{M}_n(B))^G \\ &\simeq F(n_+, F(EG_+, B \wedge n_+)^G) \\ &\simeq \mathcal{M}_n(B^{hG}) \end{aligned}$$

so  $\mathcal{M}_n(B)^{hG} \simeq \mathcal{M}_n(A)$  if and only if  $B^{hG} \simeq A$ . It remains to show that

$$h_n: \mathcal{M}_n(B) \wedge_{\mathcal{M}_n(A)} \mathcal{M}_n(B) \longrightarrow F(G_+, \mathcal{M}_n(B))$$

is a weak equivalence if and only if so is  $h: B \wedge_A B \rightarrow F(G_+, B)$ . This follows from the following commutative diagram.

$$\begin{array}{ccc} \mathcal{M}_n(B) \wedge_{\mathcal{M}_n(A)} \mathcal{M}_n(B) & \xrightarrow{h_n} & F(G_+, \mathcal{M}_n(B)) \\ \simeq \downarrow & & \downarrow \simeq \\ F(n_+, n_+) \wedge B \wedge_A B & \xrightarrow{h} & F(n_+, n_+) \wedge F(G_+, B) \end{array}$$

Here, the weak equivalence on the left uses the isomorphism  $(C \wedge B) \wedge_{C \wedge A} (C \wedge B) \cong C \wedge B \wedge_A B$  for  $C = F(n_+, S \wedge n_+)$  [36, Prop. 3.10.].  $\square$

### 4.2.3 The Eilenberg-MacLane embedding

**Theorem 4.5** (The Eilenberg-MacLane embedding). *Let  $R \rightarrow T$  be a homomorphism of rings and let  $G$  be a finite group acting on  $T$  by homomorphisms of rings under  $R$ . Assume that the trace  $\text{tr}: T \rightarrow R$  is surjective. Then  $R \rightarrow T$  is a Galois-extension of rings if and only if the induced map  $HR \rightarrow HT$  of Eilenberg-MacLane spectra is a Galois-extension of associative  $S$ -algebras.*

**Proof:** Assume first, that  $R \rightarrow T$  is  $G$ -Galois. Then  $T$  is projective as a left and right  $R$ -module by lemma 1.3 and hence  $\text{Tor}_s^R(T, T) = 0$  for  $s \neq 0$ . We will now show, that also  $\text{Ext}_{R[G]}^s(R, T)$  vanishes for  $s \neq 0$ : For its computation choose a free resolution of  $R$  as left  $R[G]$ -module. The resolution is also  $R$ -free. Now note that we have an isomorphism of left  $R[G]$ -modules  $R[G] \cong \prod_G R \cong \prod_G \text{Hom}_R(R, R) \cong \text{Hom}_R(\prod_G R, R) \cong \text{Hom}_R(R[G], R)$  and hence  $\text{Hom}_{R[G]}(R[G], R[G]) \cong R[G] \cong \text{Hom}_R(R[G], R)$ . It follows that

$$\text{Ext}_{R[G]}^s(R, R[G]) = \text{Ext}_R^s(R, R) = 0 \text{ for } s \neq 0. \quad (4.1)$$

We then use that  $T$  is projective as a left  $R[G]$  module by lemma 1.3 under the assumption that the trace  $\text{tr}: T \rightarrow R$  is surjective. From a split map  $R[G]^\ell \rightarrow T$  one obtains that

$$\text{Ext}_{R[G]}^s(R, T) = 0 \text{ for } s \neq 0 \quad (4.2)$$

since this is a retract of  $\text{Ext}_{R[G]}^s(R, R[G]^\ell)$  which also vanishes in this range. The algebraic data implies that the Künneth and the homotopy fixed point spectral sequence

$$E_{s,t}^2 = \text{Tor}_{s,t}^R(T, T) \implies \pi_{s+t}(HT \wedge_{HR} HT)$$

$$E_{s,t}^2 = H^{-s}(G; \pi_t(HT)) = \text{Ext}_{R[G]}^{-s, -t}(R, T) \implies \pi_{s+t}(HT^{hG})$$

collapse. A reference for this is [36, IV.4], the homotopy fixed point spectral sequence being the universal coefficient spectral sequence converging to  $\pi_* F_{R[G]}(R \wedge EG_+, HT)$ . Both spectral sequences are strongly convergent: For the Künneth spectral sequence this is always the case. The universal coefficient spectral sequence in general only converges conditionally, but the collapsing makes sure that we have strong convergence as well, see e.g. [14, 7]. It follows that  $HT \wedge_{HR} HT \simeq H(T \otimes_R T) \simeq H(\prod_G T) \simeq \prod_G HT$  and  $HR \simeq H(T^G) \simeq (HT)^{hG}$ .

Vice versa, assume that  $HR \rightarrow HT$  is a Galois extension as in definition 4.1. As the homotopy of  $HR$  and  $HT$  is concentrated in degree zero we see that the  $E^2$ -terms of the spectral sequences from above are concentrated in one line. Hence we see again that  $\prod_G T \cong \pi_* \prod_G HT \cong \pi_*(HT \wedge_{HR} HT) \cong \text{Tor}_0^R(T \otimes_R T) \cong T \otimes_R T$  and  $R \cong \pi_0 HR \cong \pi_0(HT^{hG}) \cong \pi_0 H(T^G) \cong T^G$ .  $\square$

The proposition says that the algebraic theory of associative Galois extensions with finite group  $G$  and surjective trace is embedded in the theory of Galois extensions of associative  $S$ -algebras as developed in this thesis. In particular, this embedding includes all Galois extensions with commutative base ring as in this case the trace is surjective by lemma 1.3. Moreover, the assumption on the trace is trivially satisfied if for example the order of  $G$  is invertible in  $R$ .

### 4.3 Faithful extensions

**Definition 4.6** (Faithful modules). Let  $A$  be an associative  $S$ -algebra and  $M$  a left  $A$ -module. If  $N \wedge_A M \simeq *$  implies  $N \simeq *$  for any right  $A$ -module  $N$  then we call  $M$  a (left) faithful  $A$ -module. A right module is *faithful* if it is so as a left module over the opposite ring.

All finite Galois extensions  $R \rightarrow T$  of commutative rings are faithfully flat, i.e. tensoring with  $T$  over  $R$  preserves and detects exact sequences. However, Galois extensions of  $S$ -algebras need not to be faithful as a counterexample by Ben Wieland shows (communicated by John Rognes). However for many examples faithfulness can be shown by hand. E.g. trivial extensions are always faithful as the retraction  $A \rightarrow F(G_+, A) \rightarrow A$  shows that the base ring is contained in the extension as a direct summand. We continue to examine the situation of the Eilenberg-MacLane embedding theorem.

**Lemma 4.7.** *Let  $R \rightarrow T$  be a Galois-extension of rings with finite Galois-group  $G$  and assume that the trace is surjective. Then  $T$  is faithful as a right (resp. left)  $R$ -module.*

**Proof:** Choose  $c \in T$  such that  $\text{tr}(c) = 1$ . The composition

$$R \longrightarrow T \xrightarrow{\text{tr}(c \cdot -)} R$$

equals the identity and hence shows that  $R$  is a right  $R$ -module direct summand in  $T$ . Using  $\text{tr}(- \cdot c)$  gives the same conclusion for left  $R$ -modules. Now assume that  $N$  is a left  $R$ -module such that  $T \otimes_R N$  vanishes. Tensoring the above sequence with  $N$  over  $R$  then shows, that  $N \cong *$ .  $\square$

**Lemma 4.8.** *If a map of  $S$ -algebras  $A \rightarrow B$  is such that  $B_*$  is faithfully flat as a right (left)  $A_*$ -module, then  $B$  is faithful as a right (left)  $A$ -module. In particular, if  $A_* \rightarrow B_*$  is a finite Galois extension of rings such that the trace is surjective, then  $A \rightarrow B$  is faithful.*

Proof: Assume  $B \wedge_A N \simeq *$ . Since  $B_*$  is flat as right  $A_*$ -module,  $\text{Tor}_{s,t}^{A_*}(B_*, N_*)$  vanishes for  $s \neq 0$ . The Künneth spectral sequence

$$E_{s,t}^2 = \text{Tor}_{s,t}^{A_*}(B_*, N_*) \implies \pi_{s+t}(B \wedge_A N)$$

is hence concentrated in one column and collapses. The assumption  $\pi_*(B \wedge_A N) = 0$  hence implies  $B_* \otimes_{A_*} N_* = \text{Tor}_0^{A_*}(A_*, N_*) = 0$ . Since  $B_*$  is assumed to be faithful as a right  $A_*$ -module, it follows that  $N_*$  vanishes.  $\square$

**Corollary 4.9.** *(EM-embedding and faithfulness). Assume that  $R \rightarrow T$  is a Galois-extension of rings such that the assumption from the embedding theorem hold. Then both the algebraic and the topological extension are faithful.*  $\square$

## 4.4 Extended equivalences

In this section we will define several maps that play an important role in the context of Galois extensions. Let  $A$  be an associative  $S$ -algebra and  $B$  an associative  $S$ -algebra under  $A$ . Also let  $G$  be a topological group acting on  $B$  from the left by maps of  $S$ -algebras under  $A$ .

In [75] the twisted group algebra  $B\langle G \rangle$  is defined to be  $B \wedge G_+$  with a twisted multiplication given by

$$\begin{aligned} B\langle G \rangle \wedge_A B\langle G \rangle &\xrightarrow{B \wedge \Delta \wedge B \wedge G_+} B \wedge_A G_+ \wedge G_+ \wedge B \wedge G_+ \xrightarrow{B \wedge_A G_+ \wedge \alpha \wedge G_+} \\ &B \wedge_A G_+ \wedge B \wedge G_+ \xrightarrow{B \wedge_A \tau \wedge G_+} B \wedge_A B \wedge G_+ \wedge G_+ \xrightarrow{\mu \wedge \mu} B\langle G \rangle. \end{aligned}$$

Here  $\alpha$  denotes the group action. In symbols, this product is given by

$$\begin{aligned} B\langle G \rangle \wedge_A B\langle G \rangle &\longrightarrow B\langle G \rangle \\ b\langle g \rangle \wedge_A \tilde{b}\langle \tilde{g} \rangle &\longmapsto bg\langle \tilde{b} \rangle \langle g\tilde{g} \rangle. \end{aligned}$$

$B\langle G \rangle$  comes with a map

$$\begin{aligned} j: B\langle G \rangle &\longrightarrow F_{A^{op}}(B, B) \\ j(b\langle g \rangle)(x) &:= bg(x). \end{aligned}$$

Formally  $j$  is adjoint to the map

$$B \wedge_{A^{op}} (B \wedge G_+) \xrightarrow{\tau} (B \wedge G_+) \wedge_A B \cong B \wedge_A (G_+ \wedge B) \xrightarrow{B \wedge_A \alpha} B \wedge_A B \xrightarrow{\mu} B.$$

Note that the target of  $j$  is the spectrum of right  $A$ -module maps. We chose this definition of  $j$  as the map thus defined is a map of associative  $A$ -algebras even if  $A$  and  $B$  are not commutative. We collect some structural observations in the next lemma.

**Lemma 4.10.** *Define the multiplication of  $B\langle G \rangle$  as above and left module structures on  $F_{A^{op}}(B, B)$  via left actions on the target. Then  $j$  is a map of left  $B$ - and of left  $A[G]$ -modules. It is also a map of  $S$ -algebras under  $A$  with the multiplication in  $F_{A^{op}}(B, B)$  given by composition of functions. Moreover,  $B\langle G \rangle$  is a central  $A$ -bimodule if and only if so is  $A$ .  $\square$*

There is another twisted algebra structure on  $B \wedge G_+$  and we denote the algebra thus obtained by  $B[G]$ . We define its product by

$$B[G] \wedge_A B[G] \xrightarrow{B \wedge \tau \wedge \Delta} (B \wedge_A B \wedge G_+ \wedge G_+) \wedge G_+ \xrightarrow{\tau} G_+ \wedge (B \wedge_A B \wedge G_+ \wedge G_+) \xrightarrow{\tilde{\alpha} \wedge B \wedge G_+ \wedge G_+} B \wedge_A B \wedge G_+ \wedge G_+ \xrightarrow{\mu \wedge \mu} B[G]$$

where  $\tilde{\alpha}$  is the group action precomposed with the involution  $g \mapsto g^{-1}$  on  $G$ , so the map above is

$$\begin{aligned} B[G] \wedge_A B[G] &\longrightarrow B[G] \\ b[g] \wedge_A \tilde{b}[\tilde{g}] &\longmapsto \tilde{g}^{-1}(b)\tilde{b}[g\tilde{g}] \end{aligned}$$

in symbols. The algebra  $B[G]$  also comes with an analogous map  $\tilde{j}$

$$\begin{aligned} \tilde{j}: B[G] &\longrightarrow F_A(B, B)^{op} \\ \tilde{j}(b[g])(x) &:= g^{-1}(x)b. \end{aligned}$$

Formally  $\tilde{j}$  is defined as being the adjoint to

$$(B \wedge_A B) \wedge G_+ \xrightarrow{\tau} G_+ \wedge (B \wedge_A B) \xrightarrow{\tilde{\alpha} \wedge_A B} B \wedge_A B \xrightarrow{\mu} B.$$

**Lemma 4.11.** *Define the multiplication on  $B[G]$  as above and right module structures on  $F_A(B, B)$  via the right actions on the target. Then  $\tilde{j}$  is a map of right  $B$ - and of right  $A[G]$ -modules. Let the multiplication in  $F_A(B, B)$  be given by composition of functions. In symbols the multiplication in  $F_A(B, B)^{op}$  is hence given by  $f \cdot \tilde{f} = \tilde{f} \circ f$ . Then  $\tilde{j}$  is a map of  $S$ -algebras under  $A$ . Moreover,  $B[G]$  is a central  $A$ -bimodule if and only if so is  $A$ .  $\square$*

**Lemma 4.12.** *Let  $A \rightarrow B$  be a map of  $S$ -algebras, let the stably dualizable group  $G$  act on  $B$  through  $A$ -algebra maps and assume that  $h: B \wedge_A B \rightarrow F(G_+, B)$  is a weak equivalence. E.g.  $A \rightarrow B$  could be a  $G$ -Galois extension.*

1. The right  $B$ -module map

$$\tilde{h}: B \wedge_A B \longrightarrow F(G_+, B)$$

defined to be adjoint to

$$G_+ \wedge B \wedge_A B \xrightarrow{\alpha \wedge_A B} B \wedge_A B \xrightarrow{\mu} B,$$

is a weak equivalence.

2. For any right  $B$ -module  $M$  there is a  $G$ -equivariant map

$$h_M: M \wedge_A B \rightarrow F(G_+, M)$$

which is a weak equivalence.

3. For any left  $B$ -module  $N$  there is a  $G$ -equivariant map

$$\tilde{h}_N: B \wedge_A N \rightarrow F(G_+, N)$$

which is a weak equivalence.

4. For each  $(A, B)$ -bimodule  $M$  there is a weak equivalence

$$j_M: M \wedge G_+ \rightarrow F_{A^{op}}(B, M).$$

It is a map of left  $A$ -modules and  $G$ -equivariant with respect to the right action of  $G$  on itself and the right action on  $F_{A^{op}}(B, M)$  defined via the left action of  $G$  on  $B$ .

5. The map

$$j: B\langle G \rangle \longrightarrow F_{A^{op}}(B, B)$$

is a weak equivalence.

6. For each  $(B, A)$ -bimodule  $N$  there is a weak equivalence

$$\tilde{j}_N: N \wedge G_+ \rightarrow F_A(B, N).$$

It is a map of right  $A$ -modules and  $G$ -equivariant with respect to the right  $G$ -actions defined as in 4.

7. The map

$$\tilde{j}: B[G] \longrightarrow F_A(B, B)^{op}$$

is a weak equivalence.

**Proof:** Ad 1: The map  $\tilde{h}$  is a weak equivalence if and only if  $h$  is since there is a commutative diagram

$$\begin{array}{ccc} B \wedge_A B & \xrightarrow{h} & F(G_+, B) \\ \parallel & & \downarrow \iota \\ B \wedge_A B & \xrightarrow{\tilde{h}} & F(G_+, B) \end{array}$$

where in symbols  $\iota$  is the involution on  $F(G_+, B)$  with  $\iota(f)(g) := g(f(g^{-1}))$ . It is strictly defined as being adjoint to

$$\begin{aligned} G_+ \wedge F(G_+, B) &\xrightarrow{\Delta \wedge \text{id}} G_+ \wedge G_+ \wedge F(G_+, B) \xrightarrow{\text{id} \wedge \text{inv} \wedge \text{id}} \\ &G_+ \wedge G_+ \wedge F(G_+, B) \xrightarrow{G_+ \wedge \text{eval}} G_+ \wedge B \xrightarrow{\alpha} B. \end{aligned}$$

Ad 2: Define  $h_M$  as

$$M \wedge_A B \cong M \wedge_B B \wedge_A B \xrightarrow{1 \wedge_B h} M \wedge_B F(G_+, B) \xrightarrow{\nu} F(G_+, M).$$

The last map  $\nu$  is a weak equivalence since  $G$  is stably dualizable. It then follows that  $h_M$  is a weak equivalence since also  $h$  is a weak equivalence. The map  $h_M$  is  $G$ -equivariant with respect to the left  $G$ -action on  $B$  in the source and the right action of  $G$  on itself on the target as  $h$  itself is  $G$ -equivariant.

Ad 3: This is proved analogously using the map  $\tilde{h}$ .

Ad 4: We define  $j_M$  to be adjoint to the composite

$$B \wedge_{A^{op}} (M \wedge G_+) \xrightarrow{\tau} (M \wedge G_+) \wedge_A B \cong M \wedge_A G_+ \wedge B \xrightarrow{M \wedge_A \alpha} M \wedge_A B \xrightarrow{\mu} M.$$

A look at section 4.5 and the adjunction formula (4.3) might be helpful. The map  $j_M$  factors in the stable homotopy category as the chain of weak equivalences

$$\begin{aligned} M \wedge G_+ &\xrightarrow{M \wedge \rho} M \wedge DDG_+ \xrightarrow{\nu} F(DG_+, M) \xrightarrow{\cong} \\ &F_{B^{op}}(DG_+ \wedge B, M) \xrightarrow{\tilde{h}^*} F_{B^{op}}(B \wedge_A B, M) \xrightarrow{\cong} F_{A^{op}}(B, M). \end{aligned}$$

The factorization is analogous to the one given in [75, 6.1.2]. As further hints to understand that it represents the map  $j_M$  we point out that restricted to  $M \wedge \{e\}_+$  both  $j_M$  and the factorization correspond to  $M \cong F_{B^{op}}(B, M) \rightarrow F_{A^{op}}(B, M)$ . Moreover both maps are  $G$ -equivariant where the action of  $G$  on  $F_A(B, M)$  is of course given by the action on  $B$ .

Ad 5: The map  $j$  is the special case  $j = j_B$  so it follows from 4.

Ad 6: We define  $\tilde{j}_N$  to be adjoint to the composite

$$(B \wedge_A N) \wedge G_+ \xrightarrow{\tau} G_+ \wedge B \wedge_A N \xrightarrow{\alpha \wedge_A N} B \wedge_A N \xrightarrow{\mu} N.$$

As above there is a factorization of  $\tilde{j}_N$  in the stable homotopy category as a chain of weak equivalences, this time given as

$$\begin{aligned} N \wedge G_+ &\xrightarrow{N \wedge \rho} N \wedge DDG_+ \xrightarrow{\nu} F(DG_+, N) \xrightarrow{\cong} \\ &F_B(B \wedge DG_+, N) \xrightarrow{h^*} F_B(B \wedge_A B, N) \xrightarrow{\cong} F_A(B, N). \end{aligned}$$

Ad 7: The map  $\tilde{j}$  is the special case  $\tilde{j} = \tilde{j}_B$  so it follows from 6.  $\square$

**Lemma 4.13.** *Let  $A \rightarrow B$  be a map of  $S$ -algebras, let the stably dualizable group  $G$  act on  $B$  through  $S$ -algebra maps under  $A$  and assume that  $h: B \wedge_A B \rightarrow F(G_+, B)$  is a weak equivalence. Then for any  $(A, B)$ -bimodule  $M$  the canonical map*

$$M \wedge_A B^{hG} \rightarrow (M \wedge_A B)^{hG}$$



is a weak equivalence. If  $B$  is a faithful central  $A$ -module then the map is an equivalence for any central  $A$ -bimodule  $M$ .

Analogously  $B^{hG} \wedge_A N \simeq (B \wedge_A N)^{hG}$  for any  $(B, A)$ -bimodule  $N$  and any central  $A$ -bimodule  $N$  if  $B$  is a faithful central  $A$ -module.

**Proof:** For right  $B$ -modules we look at the following composition.

$$M \simeq M \wedge_A B^{hG} \longrightarrow (M \wedge_A B)^{hG} \xrightarrow{h_M^{hG}} F(G_+, M)^{hG}$$

The composition above is  $M \simeq F(e_+, M) \xrightarrow{\simeq} F(G_+, M)^{hG}$  and so is a weak equivalence. The last map  $h_M^{hG}$  is a weak equivalence by lemma 4.12. So by 2 out of 3 also the map in question in the middle is a weak equivalence.

Now assume that  $B$  is a faithful central  $A$ -module. Then it suffices to check that  $B \wedge_A M \wedge_A B^{hG} \rightarrow B \wedge_A (M \wedge_A B)^{hG}$  is a weak equivalence. As  $M$  is now assumed to be central,  $B \wedge_A M$  is an  $(A, B)$ -bimodule and hence we already know that  $B \wedge_A M \wedge_A B^{hG} \simeq (B \wedge_A M \wedge_A B)^{hG}$ . The equivalence

$$(B \wedge_A M \wedge_A B)^{hG} \simeq (M \wedge_A F(G_+, B))^{hG} \simeq (F(G_+, M \wedge_A B))^{hG} \simeq B \wedge_A M.$$

finishes the statements for central  $M$ . The statements involving  $N$  are proved similarly using the weak equivalence  $\tilde{h}_N$ .  $\square$

## 4.5 Dualizability over associative algebras

In the theory of Galois extensions of commutative  $S$ -algebras, dualizability plays an important role as it replaces the algebraic notion of finite generation. We will need a canonical map

$$\nu: F_A(M, P) \wedge_A Z \longrightarrow F_A(M, P \wedge_A Z)$$

and define it shortly. We proceed in analogy to [33, 64] but their presentation assumes to work in a symmetric monoidal category, i.e. under a commutative algebra  $A$ . We will also allow  $A$  to be an associative  $S$ -algebra and hence have to go through the construction in order to see precisely at which statements we can arrive.

Duality theory is fundamentally based on the adjunction

$$\mathcal{M}_R(M \wedge_{R'} N, P) \cong \mathcal{M}_{R'}(N, F_R(M, P)) \quad (4.3)$$

defined for any  $(R, R')$ -bimodule  $M$ , left  $R'$ -module  $N$  and left  $R$ -module  $P$  ([36, III.6.2]). The function module  $F_R(M, P)$  is a left  $R'$ -module via the right action of  $R'$  on  $M$ . If  $P$  happens to be an  $(R, K)$ -bimodule, it induces the structure of an  $(R', K)$ -bimodule on  $F_R(M, P)$ . We will be most interested in the case when  $R = R' = P = A$ ,  $M = B$  is an  $A$ -bimodule and  $N = D_A B := F_A(B, A)$ . In this case, the function module  $F_A(B, A)$  is an  $A$ -bimodule itself. Though we keep the more general notation to better distinguish the different module structures.

We now review the construction of the map  $\nu$  from above in order to make visible that it extends to bimodules over associative algebras.

We start with the evaluation map

$$M \wedge_{R'} F_R(M, P) \longrightarrow P \quad (4.4)$$

which is adjoint to the identity on  $F_R(M, P)$ . It is a map of left  $R$ -modules and if  $P$  is an  $(R, K)$ -bimodule, the map is one of  $(R, K)$ -bimodules as can be seen by the adjunction

$$\mathcal{M}_{R \wedge K^{op}}(M \wedge_{R'} F_R(M, P), P) \cong \mathcal{M}_{R' \wedge K^{op}}(F_R(M, P), F_R(M, P)).$$

We assume that  $P$  is an  $(R, K)$ -bimodule and smash (4.4) with a left  $K$ -module  $Z$  and obtain a map

$$M \wedge_{R'} F_R(M, P) \wedge_K Z \longrightarrow P \wedge_K Z. \quad (4.5)$$

The map  $\nu$  is defined as the adjoint to (4.5) where  $R = R' = K = A$ . Dualizability is a condition in case  $R = R' = K = P = A$  and  $Z = M$ :

**Definition 4.14** (Dualizable bimodules). Let  $A$  be an associative  $S$ -algebra. We say that an  $A$ -bimodule  $M$  is *dualizable over  $A$*  if the canonical map  $\nu: F_A(M, A) \wedge_A M \longrightarrow F_A(M, M)$  is a weak equivalence.

The following proposition says that duality theory makes sense in any closed monoidal category. The monoidal structure needs not to be symmetric as assumed in [33] and [64]. If  $A$  is an associative  $S$ -algebra the statements from e.g. [64, III.1.3.] have the following generalizations.

**Proposition 4.15.** *Let  $A$  be an associative  $S$ -algebra and  $M$  an  $A$ -bimodule.*

1. *If  $M$  is dualizable over  $A$  the canonical map of  $A$ -bimodules*

$$\rho: M \rightarrow D_{A^{op}} D_A M$$

*is a weak equivalence.*

2. *If  $M$  is dualizable over  $A$  then the canonical map*

$$\nu: F_A(M, Y) \wedge_A Z \longrightarrow F_A(M, Y \wedge_A Z) \quad (4.6)$$

*is a weak equivalence for all  $A$ -bimodules  $Y$  and left  $A$ -modules  $Z$ .*

3. *If  $M$  is dualizable over  $A$  then  $D_A M := F_A(M, A)$  is dualizable over  $A^{op}$ .*
4. *If  $M, Y$  and  $Z$  are  $A$ -bimodules then (4.6) is also a weak equivalence if  $Z$  is dualizable over  $A$ .*

**Proof:** The canonical map  $\rho$  is defined to be the adjoint to the map  $D_A M \wedge_{A^{op}} M \cong M \wedge_A D_A M \rightarrow A$  where the last map is the evaluation, i.e. adjoint to the identity on  $D_A M$ . To prove the first statement, we give an inverse of  $\rho$  in the

stable homotopy category. Note that as  $M$  is assumed to be dualizable there is a weak equivalence  $F_A(M, M) \xleftarrow{\nu} D_A M \wedge_A M$ . This gives a map

$$\eta: A \rightarrow F_A(M, M) \rightarrow D_A M \wedge_A M \quad (4.7)$$

in the homotopy category which can be chosen to be  $A$ -bilinear. An inverse of  $\rho$  is then given as

$$D_{A^{op}} D_A M \cong A \wedge_{A^{op}} D_{A^{op}} D_A M \xrightarrow{\eta \wedge_{A^{op}} \text{id}} M \wedge_{A^{op}} D_A M \wedge_{A^{op}} D_{A^{op}} D_A M \xrightarrow{M \wedge_{A^{op}} \text{eval}} M.$$

The proof that this gives an inverse of  $\rho$  is left to the reader. One can also consult [33].

Now let us show that the map  $\nu$  from the second part of the statement is a weak equivalence if  $M$  is dualizable. Again we construct an inverse in the stable homotopy category. This inverse is given as

$$F_A(M, Y \wedge_A Z) \xrightarrow{\eta \wedge_A \text{id}} D_A M \wedge_A M \wedge_A F_A(M, Y \wedge_A Z) \xrightarrow{\text{eval}} D_A M \wedge_A Y \wedge_A Z \xrightarrow{\nu \wedge_A Z} F_A(M, Y) \wedge_A Z. \quad (4.8)$$

In order to prove the third part of the statement it suffices to prove the analogue of the second statement for the  $A^{op}$ -bimodule  $D_A M$ . Note that in the proof of the second part, we used the fact that the map  $\nu$  in the composition (4.8) was a weak equivalence only for the construction of the map  $\eta: A \rightarrow D_A M \wedge_A M$ . Hence for the proof of the third statement it suffices to find an analogue of this map, i.e. a map  $A \rightarrow D_{A^{op}} D_A M \wedge_{A^{op}} D_A M$  so that the proof of the second statement can be copied. Using (4.7) from above which exists as  $M$  is supposed to be dualizable over  $A$  this analogue is given as

$$A \xrightarrow{\eta} D_A M \wedge_A M \cong M \wedge_{A^{op}} D_A M \xrightarrow{\rho} D_{A^{op}} D_A M \wedge_{A^{op}} D_A M$$

also using the map  $\rho$  from part 1.

We come to the last part of the statement. The parts of the proposition we have already proved, provide the following chain of equivalences:

$$\begin{aligned} F_A(M, Y) \wedge_A Z &\simeq F_A(M, Y) \wedge_A D_{A^{op}} D_A Z \\ &\simeq D_{A^{op}} D_A Z \wedge_{A^{op}} F_A(M, Y) \\ &\simeq F_{A^{op}}(D_A Z, F_A(M, Y)) \\ &\simeq F_A(M, F_{A^{op}}(D_A Z, Y)) \\ &\simeq F_A(M, D_{A^{op}} D_A Z \wedge_{A^{op}} Y) \\ &\simeq F_A(M, Y \wedge_A Z) \end{aligned}$$

This proves the last point of the proposition.  $\square$

Following [76] we call a topological group (stably) dualizable when it is dualizable as an  $S$ -module. A first hint that dualizability might play a role in the topological theory of Galois extensions comes from the following observations:

First recall that for every algebraic Galois-extension  $R \rightarrow T$  the extension  $T$  is a finitely generated  $R$ -module, see lemma 1.3. This corresponds to the notion of dualizability defined above. Now, topologically we have the following characterization from [36, III.7.9].

**Lemma 4.16.** *Let  $A$  be a commutative  $S$ -algebra. Then  $M$  is a dualizable  $A$ -module if and only if it is weakly equivalent to a retract of a finite cell  $A$ -module, i.e. if and only if it is semi-finite.*

If  $A$  is connective this is the case if and only if  $M$  is a retract of a finite CW  $A$ -spectrum. In a local category  $\mathcal{M}_{S,E}$  of  $E$ -local  $S$ -modules where  $S \neq E$  however, there may be dualizable spectra that are not weakly equivalent to a retract of a finite cell module. The  $K(1)$ -local Galois extension  $L_{K(1)}S \rightarrow JU_2^\wedge$  into the complex image-of- $J$  spectrum  $JU_2^\wedge$  is an example for this. The spectrum  $JU_2^\wedge$  is a dualizable but not a semi-finite  $L_{K(1)}S$ -module, see [75, 6.2.2]. Galois extensions of commutative  $S$ -algebras are always dualizable by [75, 6.2.1.] and we will show in section 4.5.2 that this holds for Galois extensions of associative  $S$ -algebras as well.

#### 4.5.1 The norm map

For a spectrum  $X$  with left  $G$ -action recall the definition of the homotopy orbit spectrum

$$X_{hG} := EG_+ \wedge_G X$$

where  $EG = \mathcal{B}(*, G, G)$  is the standard contractible space with free right  $G$ -action given by a bar construction. Also recall the dualizing spectrum

$$S^{adG} := S[G]^{hG} = F(EG_+, S[G])^G$$

formed with respect to the right  $G$ -actions [76, 2.5.]. For any stably dualizable group and any  $S$ -module  $X$  with left  $G$ -action, there is a map

$$N: (X \wedge S^{adG})_{hG} \longrightarrow X^{hG}$$

called the *norm map* ([75, 3.6], [76, ch.5]). We refer the reader to these sources for details. We just recall for later use that the norm map is a weak equivalence whenever  $X$  is of the form  $W \wedge G_+$ , see [76, 5.2.5].

#### 4.5.2 Associative Galois extensions and dualizability

**Proposition 4.17.** *Let  $A$  be an  $S$ -algebra and  $A \rightarrow B$  a Galois extension of  $S$ -algebras with Galois group  $G$ . Then  $B$  is dualizable over  $A$  and over  $A^{op}$ .*

**Proof:** We prove that  $B$  is dualizable over  $A^{op}$  with the following diagram where  $M$  is a  $B$ -bimodule.

$$\begin{array}{ccc}
F_{A^{op}}(B, A) \wedge_{A^{op}} M & \xrightarrow{\nu} & F_{A^{op}}(B, A \wedge_{A^{op}} M) \\
\downarrow i_* & & \downarrow i_* \\
F_{A^{op}}(B, B^{hG}) \wedge_{A^{op}} M & \xrightarrow{\nu} & F_{A^{op}}(B, B^{hG} \wedge_{A^{op}} M) \\
\downarrow & & \downarrow \\
F_{A^{op}}(B, B^{hG}) \wedge_{A^{op}} M & & F_{A^{op}}(B, (B \wedge_{A^{op}} M)^{hG}) \\
\downarrow & & \downarrow \\
F_{A^{op}}(B, B)^{hG} \wedge_{A^{op}} M & \xrightarrow{\quad} & F_{A^{op}}(B, B \wedge_{A^{op}} M)^{hG} \\
\uparrow j^{hG} & & \uparrow (j_{M \wedge_A B})^{hG} \\
(B \wedge G_+)^{hG} \wedge_{A^{op}} M & \xrightarrow{\quad} & (B \wedge G_+ \wedge_{A^{op}} M)^{hG} \\
\cong \uparrow & & \cong \uparrow \\
M \wedge_A (B \wedge G_+)^{hG} & \xrightarrow{\quad} & (M \wedge_A B \wedge G_+)^{hG} \\
\uparrow M \wedge_A N & & \uparrow N \\
M \wedge_A (B \wedge G_+ \wedge S^{adG})_{hG} & \xrightarrow{\cong} & (M \wedge_A B \wedge G_+ \wedge S^{adG})_{hG}
\end{array}$$

It is clear that the diagram commutes and that the lower horizontal map is an isomorphism. We will prove that all the vertical maps are weak equivalences. The statement then follows from the case  $M = B$ . The top vertical maps are weak equivalences as  $i: A \rightarrow B^{hG}$  is an equivalence by assumption. The second top vertical map on the right is an equivalence by lemma 4.13. Then, one level further below, we have weak equivalences by a general property for homotopy fixed points that can be checked easily. The maps labeled  $j^{hG}$  and  $(j_{M \wedge_A B})^{hG}$  are weak equivalences by lemma 4.12. The isomorphisms are clear and the norm maps  $N$  are weak equivalences as the spectra in sight are of the form  $X \wedge G_+$ . The dualizability of  $B$  over  $A$  can be proved analogously using the maps  $\tilde{j}$ .  $\square$

**Lemma 4.18.** *Let  $B$  be an  $A$ -algebra with an action of a topological group  $G$  under  $A$ . Let  $M$  be dualizable over  $A^{op}$ . Then the natural map*

$$M \wedge_A B^{hK} \rightarrow (M \wedge_A B)^{hK}$$

*is a weak equivalence for any subgroup  $K$  of  $G$ . Analogously, if  $N$  is dualizable over  $A$  then the canonical maps*

$$B^{hK} \wedge_A N \rightarrow (B \wedge_A N)^{hK}$$

*are weak equivalences.*

**Proof:** As  $M$  is supposed to be dualizable over  $A^{op}$  there is a weak equivalence  $M \rightarrow D_A D_{A^{op}} M$  by proposition 4.15. We then get a commutative diagram

$$\begin{array}{ccccc}
 M \wedge_A B^{hK} & \xrightarrow{\rho \wedge \text{id}} & D_A D_{A^{op}} M \wedge_A B^{hK} & \xrightarrow{\nu} & F_A(D_{A^{op}} M, B^{hK}) \\
 \downarrow & & \downarrow & & \downarrow \simeq \\
 (M \wedge_A B)^{hK} & \xrightarrow{(\rho \wedge \text{id})^{hK}} & (D_A D_{A^{op}} M \wedge_A B)^{hK} & \xrightarrow{\nu^{hK}} & F_A(D_{A^{op}} M, B)^{hK}
 \end{array}$$

in which all the horizontal and hence also the vertical maps are weak equivalences. Again the statements involving  $N$  are proved analogously.  $\square$

**Corollary 4.19.** *For any Galois extension  $A \rightarrow B$  of associative  $S$ -algebras any subgroup  $K$  of the Galois group, the canonical maps*

$$B \wedge_A B^{hK} \rightarrow (B \wedge_A B)^{hK}$$

$$B^{hK} \wedge_A B \rightarrow (B \wedge_A B)^{hK}$$

are weak equivalences. Here the actions on  $B \wedge_A B$  are on the second factor in the first line and on the first factor in the second line.

**Proof:** This follows from lemma 4.18 as  $B$  is dualizable over  $A$  and over  $A^{op}$  by proposition 4.17.  $\square$

## 4.6 More examples

### 4.6.1 Induced extensions

Given a Galois extension  $A \rightarrow B$  and a map of  $S$ -algebras  $A \rightarrow C$  we want to know if also  $C \rightarrow C \wedge_A B$  is a Galois extension. In general, the smash product of two associative algebras over an associative algebra does not even have the structure of an associative algebra. Moreover, in general  $C \wedge_A B$  does not admit the structure of a right  $C$ -module. However, if  $B$  and  $C$  are not just uncas but (central)  $A$ -algebras, i.e. in  $\mathcal{A}_A$ , the product  $C \wedge_A B$  has more structure, see lemma 2.8. In particular,  $C \wedge_A B$  then is also an  $A$ -algebra and an unca under  $C$ .

For the rest of this section we assume that  $A$  is a cofibrant  $S$ -algebra and that  $B$  and  $C$  are  $A$ -algebras and cofibrant as associative  $S$ -algebras under  $A$ . We implicitly assume that  $C \wedge_A B$  is cofibrant as an associative  $S$ -algebra under  $A$ , making a cofibrant replacement if necessary. Note that under the above cofibrancy assumptions  $C \wedge_A B$  represents the derived smash product over  $C$  regardless whether a cofibrant replacement was made or not. So there will be no ambiguity in the sequel. We also assume that  $B$  has an action of a dualizable group  $G$  by maps of associative  $S$ -algebras under  $A$ . The smash product  $C \wedge_A B$  inherits this action of  $G$  by  $S$ -algebra maps and these maps are under  $C$ .

If  $A \rightarrow B$  is a Galois extension we can ask if  $C \rightarrow C \wedge_A B$  is a Galois extension as well. For this, we have to check the fixed point and the unramification

condition. The latter factors as follows and is always satisfied as the top map in the next diagram is a weak equivalence:

$$\begin{array}{ccc} C \wedge_A (B \wedge_A B) & \xrightarrow{C \wedge_A h} & C \wedge_A F(G_+, B) \\ \cong \downarrow & & \downarrow \simeq \\ (C \wedge_A B) \wedge_C (C \wedge_A B) & \longrightarrow & F(G_+, C \wedge_A B) \end{array} \quad (4.9)$$

The fixed point condition needs extra hypothesis and we have listed some sufficient conditions in the next proposition. For the question which hypothesis are really necessary, see section 5.4 and theorems 5.13 and 5.16.

**Proposition 4.20.** *Let  $A, B$  and  $C$  be as above and assume that  $A \rightarrow B$  is a Galois extension for the group  $G$ . Then  $C \rightarrow C \wedge_A B$  is a  $G$ -Galois extension under any of the following conditions:*

1.  $C$  is dualizable over  $A$  or
2.  $C$  is an  $(A, B)$ -bimodule or
3.  $B$  is faithful over  $A$ .

*in the last case, also  $C \rightarrow C \wedge_A B$  is faithful as a left  $C$ -module.*

The proposition has a converse under the condition that  $C$  is a faithful  $A$ -module:

**Proposition 4.21.** *Let  $A, B$  and  $C$  be as above and assume that  $C$  is faithful as an  $A$ -module. Let the group  $G$  act on  $B$  from the left through  $A$ -algebra morphisms such that  $C \rightarrow C \wedge_A B$  is a  $G$ -Galois extension. Then  $A \rightarrow B$  is a  $G$ -Galois extension if at least one of the following conditions holds:*

1.  $C$  is dualizable over  $A$  or
2.  $C$  is an  $(A, B)$ -bimodule or
3.  $B$  is faithful over  $A$ .

**Proof of propositions 4.20 and 4.21:**

The unramification condition follows from (4.9) so the only thing that remains to be checked is the fixed-point condition. For this we look at the following diagram.

$$\begin{array}{ccc} C \wedge_A A & \xrightarrow{\cong} & C \\ C \wedge_A i \downarrow & & \downarrow i \\ C \wedge_A B^{hG} & \longrightarrow & (C \wedge_A B)^{hG} \end{array} \quad (4.10)$$

Under the conditions of proposition 4.20 or 4.21 the bottom map is a weak equivalence by lemma 4.18 or lemma 4.13 respectively. So if  $A \simeq B^{hG}$  is a weak equivalence then so is the left map in the diagram and hence so is the map on the right. This proves proposition 4.20. Conversely, under the hypothesis of

proposition 4.21 the right vertical map is a weak equivalence. Hence so is the left and as  $C$  is supposed to be a faithful  $A$ -module also  $A \rightarrow B^{hG}$  is a weak equivalence.  $\square$

**Corollary 4.22.** *Any faithful Galois extension of a commutative  $S$ -algebra induces a faithful Galois extension along any map of  $S$ -algebras.*

*A Galois extension of a commutative  $S$ -algebra induces a Galois extension along any dualizable map of  $S$ -algebras.*  $\square$

**Example 4.23.** There is a  $C_{p-1}$ -action on  $p$ -completed  $K$ -theory  $KU_p$  with homotopy fixed point spectrum equivalent to the Johnson Wilson spectrum  $E(1)$ . In fact  $E(1)_p \rightarrow KU_p$  is a faithful  $C_{p-1}$ -Galois extension [10, 1.4.15]. The quotient map  $E(1)_p \rightarrow K(1) = E(1)_p/p$  to Morava- $K$ -theory induces an extension  $K(1) \rightarrow K(1) \wedge_{E(1)_p} KU_p \simeq KU/p$ . We investigate quotient maps more thoroughly in chapter 7. By corollary 4.22 the map  $K(1) \rightarrow KU/p$  is a faithful  $C_{p-1}$ -Galois extension of associative  $S$ -algebras.

Propositions 4.20 and 4.21 have analogs for local Galois extensions. We only point out that if the map  $h$  in diagram (4.9) is an  $E_*$ -equivalence then so is the bottom map. As a consequence,  $E$ -local Galois extensions are preserved when passing to an induced extension as in proposition 4.20 in the sense that we obtain again an  $E$ -local Galois extension. We will now look at some extensions induced from certain  $K(n)$ -local Galois extensions. We will induce up along maps into  $K(n)$  or some other  $K(n)$ -algebra. In particular, the extensions obtained are automatically maps between  $K(n)$ -local spectra. But as  $E_*$ -equivalences between  $E$ -local spectra are automatically weak equivalences in  $\mathcal{M}_S$ , the extensions obtained are automatically global Galois extensions.

With these remarks, we can obtain induced Galois extensions from the main examples of Galois extensions from [75, 5.4]. First fix a prime  $p$  and recall the Morava stabilizer group  $\mathbb{S}_n = \text{Aut}(\Gamma_n/\mathbb{F}_{p^n})$  of automorphisms of the height  $n$  Honda formal group law  $\Gamma_n$  over  $\mathbb{F}_{p^n}$ . The extended Morava stabilizer group  $\mathbb{G}_n$  is a semi-direct product  $\mathbb{G}_n = \mathbb{S}_n \rtimes \text{Gal}$  of  $\mathbb{S}_n$  with the finite Galois group  $\text{Gal} = \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n$ . The extended Morava stabilizer group acts on the coefficient ring  $\pi_*(E_n) = \mathbb{W}(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][[u^{\pm 1}]]$  of the  $n$ -th  $p$ -primary even periodic Lubin-Tate spectrum  $E_n$ . Here  $\mathbb{W}(-)$  denotes the ring of Witt vectors. By the Hopkins-Miller and Goerss-Hopkins theory this action lifts to an action of  $\mathbb{G}_n$  on  $E_n$  by maps of commutative  $S$ -algebras.

More generally, for the absolute Galois group  $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$ , the semidirect product  $\mathbb{G}_n^{nr} := \mathbb{S}_n \rtimes \hat{\mathbb{Z}}$  acts on the commutative  $S$ -algebra  $E_n^{nr}$  with coefficients  $\pi_*(E_n^{nr}) = \mathbb{W}(\overline{\mathbb{F}}_p)[[u_1, \dots, u_{n-1}]][[u^{\pm 1}]]$ . As expected in [75], the spectrum  $E_n^{nr}$  was recently shown to be the separable closure of  $E_n$  in the sense of definition 4.37, see [11]. Section 2 of [11] is also a good reference summarizing two different constructions of the spectrum  $E_n^{nr}$ .

The spectrum  $E_n$  is not a discrete  $\mathbb{G}_n$ -spectrum which makes it necessary to give an extended definition of homotopy fixed point spectra. Following Devinatz and Hopkins [31] we define for every closed subgroup  $K \subset \mathbb{G}_n$



$$E_n^{hK} := L_{K(n)}(\operatorname{colim}_i E_n^{hU_i K}) \quad (4.11)$$

where  $\{U_i\}$  is a descending sequence of open normal subgroups in  $\mathbb{G}_n$  with  $\bigcap_i U_i = \{e\}$ . This definition agrees with the usual definition of homotopy fixed points if  $K$  is finite. We refer to [31] and [75] for a discussion of this definition. In particular we have

$$E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S.$$

The following theorem taken from [75, 5.4.4] states that the definition of homotopy fixed point spectra (4.11) gives rise to a plethora of Galois extensions of commutative  $S$ -algebras. For the notion of pro-Galois extension see [75, 8.1.] or section 4.8.

**Theorem 4.24.** ( *Devinatz-Hopkins*)

1. For each pair of closed subgroups  $H \subset K \subset \mathbb{G}_n$  with  $H$  normal and of finite index in  $K$ , the map  $E_n^{hK} \rightarrow E_n^{hH}$  is a  $K(n)$ -local  $K/H$ -Galois extension.
2. In particular, for each finite subgroup  $K \subset \mathbb{G}_n$  the map  $E_n^{hK} \rightarrow E_n$  is a  $K(n)$ -local  $K$ -Galois extension.
3. Likewise, for each open normal subgroup  $U \subset \mathbb{G}_n$  (necessarily of finite index) the map

$$L_{K(n)}S \rightarrow E_n^{hU}$$

is a  $K(n)$ -local  $\mathbb{G}_n/U$ -Galois extension

4. A choice of a descending sequence  $\{U_i\}_i$  of open normal subgroups of  $\mathbb{G}_n$  with  $\bigcap_i U_i = \{e\}$  exhibits

$$L_{K(n)}S \rightarrow E_n$$

as a  $K(n)$ -local pro- $\mathbb{G}_n$ -Galois extension.

In particular this theorem gives rise to the following diagram taken from [75, 5.4.6]. Here  $M$  is a maximal finite subgroup of  $\mathbb{G}_n$  which is unique up to conjugacy if  $p$  is odd and  $n = (p-1)k$  with  $k$  prime to  $p$  or for  $p = 2$  and  $n = 2k$  with  $k$  odd by [46]. The fixed point spectra  $E_n^{hM}$  are then known to be the  $n$ -th higher real  $K$ -theory spectra  $EO_n$  of Hopkins and Miller. In the diagram groups label faithful Galois or pro-Galois extensions of commutative  $S$ -algebras.

$$\begin{array}{ccccc}
& & \hat{\mathbb{Z}} & & \\
& & \curvearrowright & & \\
E_n^{h\text{Gal}} & \xrightarrow{\text{Gal}} & E_n & \xrightarrow{n\hat{\mathbb{Z}}} & E_n^{nr} \\
\uparrow & \nearrow M & \uparrow & \nearrow & \uparrow \\
EO_n & & \mathbb{G}_n & & \mathbb{S}_n \\
\uparrow & \nearrow & \uparrow & \nearrow & \uparrow \\
L_{K(n)}S & \xrightarrow{\text{Gal}} & E_n^{h\mathbb{S}_n} & \xrightarrow{n\hat{\mathbb{Z}}} & (E_n^{nr})^{h\mathbb{S}_n} \\
& & \uparrow \mathbb{G}_n^{nr} & & \uparrow \mathbb{S}_n \\
& & \hat{\mathbb{Z}} & & 
\end{array} \tag{4.12}$$

We also obtain the completed Johnson-Wilson spectra  $\widehat{E(n)}$  as the homotopy fixed point spectra  $E_n^{hK}$  for the group  $K = \mathbb{F}_p^* \rtimes \text{Gal}$ . We will come back to the completed Johnson-Wilson spectra in chapter 7. In the same way as above we hence obtain the following diagram.

$$\begin{array}{ccccc}
& & \hat{\mathbb{Z}} & & \\
& & \curvearrowright & & \\
E_n^{h\text{Gal}} & \xrightarrow{\text{Gal}} & E_n & \xrightarrow{n\hat{\mathbb{Z}}} & E_n^{nr} \\
\uparrow & \nearrow K & & & \\
\widehat{E(n)} & & & & 
\end{array} \tag{4.13}$$

We can now obtain Galois extensions of associative  $S$ -algebras from these diagrams. First recall the Morava  $K$ -theory spectra  $K_n$  and  $K(n)$  defined for every prime  $p$ . Recall from [75, 5.6.4] that  $K(n)$  does not admit the structure of a commutative  $S$ -algebra. Though  $K(n)$  respectively  $K_n$  admits the structure of an associative  $\widehat{E(n)}$ - respectively  $E_n$ -algebra, e.g. by [4]. So there are central maps  $E_n \rightarrow K_n$  and  $\widehat{E(n)} \rightarrow K(n)$  of  $S$ -algebras. Inducing up along the map  $L_{K(n)}S \rightarrow K_n$  turns every Galois extension from diagram (4.12) into a Galois extension under  $K_n$  yielding the following diagram.

$$\begin{array}{ccccc}
& & \hat{\mathbb{Z}} & & \\
& & \curvearrowright & & \\
E_n^{h\text{Gal}} \wedge_{L_{K(n)}S} K_n & \xrightarrow{\text{Gal}} & E_n \wedge_{L_{K(n)}S} K_n & \xrightarrow{n\hat{\mathbb{Z}}} & E_n^{nr} \wedge_{L_{K(n)}S} K_n \\
\uparrow & \nearrow M & \uparrow & \nearrow & \uparrow \\
EO_n \wedge_{L_{K(n)}S} K_n & & \mathbb{G}_n & & \mathbb{S}_n \\
\uparrow & \nearrow & \uparrow & \nearrow & \uparrow \\
K_n & \xrightarrow{\text{Gal}} & E_n^{h\mathbb{S}_n} \wedge_{L_{K(n)}S} K_n & \xrightarrow{n\hat{\mathbb{Z}}} & (E_n^{nr})^{h\mathbb{S}_n} \wedge_{L_{K(n)}S} K_n \\
& & \uparrow \mathbb{G}_n^{nr} & & \uparrow \mathbb{S}_n \\
& & \hat{\mathbb{Z}} & & 
\end{array} \tag{4.14}$$

An analogous diagram makes sense with  $K_n$  replaced by  $K(n)$ . Inducing up along the map  $\widehat{E(n)} \rightarrow K(n)$  gives a diagram corresponding to (4.13). We observe that  $E_n \wedge_{\widehat{E(n)}} K(n) \simeq K_n$ . This equivalence follows easily from the theory of regular quotients as  $K(n) = \widehat{E(n)}/I$  for the regular ideal  $I = (p, v_1, \dots, v_{n-1})$  in  $\pi_*(\widehat{E(n)})$ , see [36] and chapter 7. It makes sense to set  $K_n^{nr} := E_n^{nr} \wedge_{E_n} K_n \simeq E_n^{nr} \wedge_{\widehat{E(n)}} K(n)$ . This spectrum has homotopy groups  $\pi_*(E_n^{nr})/I$  and we give a further justification for the notation  $K_n^{nr}$  in section 4.9. We end up with the following diagram.

$$\begin{array}{ccc}
 E_n^{h \text{ Gal}} \wedge_{\widehat{E(n)}} K(n) & \xrightarrow{\text{Gal}} & K_n \xrightarrow{n\hat{\mathbb{Z}}} K_n^{nr} \\
 \uparrow & \nearrow K & \\
 K(n) & & 
 \end{array} \tag{4.15}$$

In particular this exhibits the map  $K(n) \rightarrow K_n$  as a  $K(n)$ -local Galois extension for the group  $K$  as above. Since every  $K(n)$ -module spectrum is  $K(n)$ -local, the  $K(n)$  local Galois extensions of  $K(n)$  respectively  $K_n$  are automatically global Galois extensions. The next proposition then follows immediately from corollary 4.22.

**Proposition 4.25.** *The labeled maps in diagrams (4.14) and (4.15) are (pro-) Galois extensions with respect to the indicated groups.  $\square$*

### 4.6.2 Intermediate algebras: Half a Galois correspondence

In the commutative context, inducing extensions along a map of commutative  $S$ -algebras leads to a proof of one part of a Galois correspondence [75, 7.2.]. The analogous statement holds for a possibly associative Galois extension  $B$  of a commutative  $S$ -algebra  $A$ . Recall the following definition from [75, 7.2.1].

**Definition 4.26** (Allowable subgroup). Let  $G$  be a stably dualizable group and  $K < G$  a subgroup.  $K$  is said to be an *allowable subgroup* if

- $K$  is dualizable,
- as a continuous map of spaces the projection  $G \rightarrow G/K$  onto the orbit space under the action of  $K$  admits a section up to homotopy and
- the collapse map  $G \times_K EK \rightarrow G/K$  induces a stable equivalence  $S[G \times_K EK] \rightarrow S[G/K]$ .

In particular every subgroup of a discrete group  $G$  is allowable.

**Theorem 4.27.** *Let  $A$  be a commutative  $S$ -algebra and  $B$  a possibly associative  $S$ -algebra such that  $A \rightarrow B$  is a faithful  $G$ -Galois extension. Let  $K < G$  be an allowable subgroup. Then  $B^{hK} \rightarrow B$  is a faithful  $K$ -Galois extension. If moreover  $K \triangleleft G$  is an allowable normal subgroup then  $A \rightarrow B^{hK}$  is a faithful  $G/K$ -Galois extension.*

**Proof:** We just sketch the proof as it is analogous to the proof of [75, 7.2.3.]. For the first part, it suffices to check that the induced extension  $B \wedge_A B^{hK} \rightarrow B \wedge_A B$  is a faithful  $K$ -Galois extension. Note that  $B \wedge_A B^{hK} \simeq F(G/K_+, B)$  and  $B \wedge_A B \simeq F(G_+, B) \simeq F(K_+, F(G/K_+, B))$  as  $K$  is an allowable subgroup of  $G$ . In fact the extension  $B \wedge_A B^{hK} \rightarrow B \wedge_A B$  corresponds to the trivial  $K$ -Galois extension of  $F(G/K_+, B)$  and so the claim follows.

For the second part, it suffices to show that  $B \rightarrow B \wedge_A B^{hK}$  is a faithful  $G/K$ -Galois extension. Again this extension corresponds to the trivial extension  $B \rightarrow F(G/K_+, B)$  from which the claim follows.  $\square$

For more examples obtained by realizing algebraic Galois extensions over a given commutative  $S$ -algebra, see chapter 8.

## 4.7 Characterizing associative Galois extensions

In the algebraic setting, the fact that a map  $R \rightarrow T$  is a Galois extension can be expressed in several ways, see e.g. [24, 37]. Some of these characterizations have been generalized to the commutative topological setting in [75, 6.3]. We give equivalent characterizations for the associative case.

**Proposition 4.28** (Characterization). *Let  $A \rightarrow B$  be a map of associative  $S$ -algebras and let the stably dualizable group  $G$  act on  $B$  from the left by  $S$ -algebra maps under  $A$ . The following are equivalent:*

1.  $A \rightarrow B$  is a  $G$ -Galois extension, i.e. the canonical maps  $i$  and  $h$  are weak equivalences.
2. The maps  $i$  and  $\tilde{j}$  are equivalences and  $B$  is dualizable over  $A$ .
3. The maps  $i$  and  $j$  are equivalences and  $B$  is dualizable over  $A^{op}$ .

If in addition  $B$  is faithful as a right  $A$ -module then these statements are also equivalent to the following:

4. The canonical map  $h$  is a weak equivalence and  $B$  is faithful as a right  $A$ -module and dualizable over  $A^{op}$ .

If in addition  $B$  is faithful as a left  $A$ -module then the statements 1.-3. are also equivalent to the following:

5. The canonical map  $h$  is a weak equivalence and  $B$  is faithful as a left  $A$ -module and dualizable over  $A$ .

**Proof:** (1)  $\Rightarrow$  (2): This follows from lemma 4.12 and proposition 4.17.

(2)  $\Rightarrow$  (1): We have to show that  $h_M$  is a weak equivalence in the case  $M = B$ . This is true as in the stable homotopy category  $h_M$  factors as

$$\begin{aligned} M \wedge_A B &\cong B \wedge_{A^{op}} M \xrightarrow{\rho \wedge_{A^{op}} M} D_{A^{op}} D_A B \wedge_{A^{op}} M \xrightarrow{\nu} F_{A^{op}}(D_A B, M) \\ &\cong F_{B^{op}}(D_A B \wedge_A B, M) \xleftarrow{\nu^*} F_{B^{op}}(F_A(B, B)^{op}, M) \\ &\xleftarrow{\tilde{j}^*} F_{B^{op}}(B[G], M) \cong F(G_+, M). \end{aligned}$$

Here, we are first using the canonical map  $\rho$ , then that  $D_A B$  is dualizable over  $A^{op}$  and that  $B$  is dualizable over  $A$ . Finally the last map is the one induced by  $\tilde{j}$ .

(1)  $\Rightarrow$  (3): Again this follows from lemma 4.12 and proposition 4.17.

(3)  $\Rightarrow$  (1): Analogously to the proof of (2)  $\Rightarrow$  (1) one can show that  $\tilde{h}$  is a weak equivalence. Then  $h$  is a weak equivalence by the proof of lemma 4.12.1.

(1)  $\Rightarrow$  (4): Again, this follows from lemma 4.17.

(4)  $\Rightarrow$  (1): We have to show that  $i$  is a weak equivalence and by the faithfulness assumption it suffices to show that  $B \wedge_A i$  is a weak equivalence. This map factors in the homotopy category as

$$B \cong B \wedge_A A \simeq B \wedge_A B^{hG} \longrightarrow (B \wedge_A B)^{hG} \xrightarrow{h^{hG}} F_A(G_+, B)^{hG} \simeq B.$$

Here the last map is a weak equivalence as  $h$  is assumed to be one and the map before is a weak equivalence by lemma 4.18.

(1)  $\Leftrightarrow$  (5) is proved similarly. Recall that  $h$  is a weak equivalence if and only if  $\tilde{h}$  is.  $\square$

#### 4.7.1 Preservation and detection statements for induced extensions

The characterizations above may be used to obtain preservation and detection statements for induced extensions, compare [75]. We first consider the dualizability property.

**Lemma 4.29.** *Let  $A$  be an associative  $S$ -algebra and  $C$  a central  $(A, A)$ -bimodule.*

1. *If  $B$  is dualizable over  $A$  then  $C \wedge_A B$  is dualizable over  $C$ .*
2. *If  $B$  and  $C$  are central and  $C$  is faithful and dualizable over  $A$  and  $C \wedge_A B$  is dualizable over  $C$  then  $B$  is dualizable over  $A$ .*

**Proof:** There is a commutative square

$$\begin{array}{ccc} F_A(B, A) \wedge_A C \wedge_A B & \xrightarrow{\nu} & F_A(B, C \wedge_A B) \\ \downarrow \nu & & \downarrow \cong \\ F_A(B, C) \wedge_A B & & \\ \cong \downarrow & & \\ F_C(C \wedge_A B, C) \wedge_C C \wedge_A B & \xrightarrow{\nu} & F_C(C \wedge_A B, C \wedge_A B). \end{array}$$

If  $B$  is dualizable over  $A$  then the upper horizontal and the left top vertical maps are weak equivalences by proposition 4.15.2 and hence so is the lower map which shows part 1. For part 2 note that the dualizability of  $C \wedge_A B$  over

$C$  provides a diagram

$$\begin{array}{ccc}
F_A(B, A) \wedge_A B \wedge_A C & \xrightarrow{\nu \wedge_A C} & F_A(B, B) \wedge_A C \\
\cong \downarrow & & \downarrow \nu \simeq \\
F_A(B, A) \wedge_A C \wedge_A B & & F_A(B, B \wedge_A C) \\
\simeq \downarrow \nu & & \downarrow \cong \\
F_A(B, C) \wedge_A B & & F_A(B, B \wedge_A C) \\
\cong \downarrow & & \downarrow \cong \\
F_C(C \wedge_A B, C) \wedge_C C \wedge_A B & \xrightarrow{\nu} & F_C(C \wedge_A B, C \wedge_A B).
\end{array}$$

in which the lower map is a weak equivalence as  $C \wedge_A B$  is dualizable over  $C$ . The vertical maps are weak equivalences by proposition 4.15.4 as  $C$  is assumed to be dualizable over  $A$ . Hence the upper map is also a weak equivalence and by faithfulness of  $C$  we see that  $B$  is dualizable over  $A$ .  $\square$

**Corollary 4.30.** *If  $G$  is a stably dualizable group, then  $A[G]$  is dualizable over  $A$  for any associative  $S$ -algebra  $A$ .*  $\square$

Next, we investigate faithfulness in the context of induced extensions.

**Lemma 4.31** (Faithfulness preservation lemma). *If  $B$  is a right  $A$ -module and  $M$  is a left faithful  $A$  module, then  $B \wedge_A M$  is a faithful left  $B$ -module.*

Proof:  $N \wedge_B B \wedge_A M \simeq *$  implies  $N \wedge_A M \simeq *$  and hence  $N \simeq *$  since  $M$  is faithful as a left  $A$ -module.  $\square$

**Lemma 4.32** (Faithfulness detection lemma). *Let  $C$  and  $B$  be central  $A$ -bimodules,  $C \wedge_A B$  faithful as left  $C$ -module and  $C$  faithful as left  $A$ -module. Then  $B$  is a faithful left  $A$ -module.*

**Proof:** Assume  $N \wedge_A B \simeq *$ . Then  $N \wedge_A C \wedge_A B \cong N \wedge_A B \wedge_A C \simeq *$  as  $B$  and  $C$  are central and hence  $(N \wedge_A C) \wedge_C (C \wedge_A B) \simeq *$ . As  $(C \wedge_A B)$  is a faithful left  $C$ -module it follows that  $N \wedge_A C$  is contractibel. Hence also  $N \simeq *$  as  $C$  is a faithful left  $A$ -module.  $\square$

The next observation summarizes the contents of lemma 4.32 and 4.31.

**Corollary 4.33.** *An extension induced along a central map is faithful if and only if the original extension is faithful.*  $\square$

Also recall the statements from lemma 4.29 that dualizability is preserved by induced extensions and for central extensions is detected by induced extensions along faithful and dualizable maps. Together with the characterizations of Galois extensions from proposition 4.28 this allows to derive the preservation and detections statements for Galois extensions (propositions 4.20 and 4.21) again.

## 4.8 Pro-Galois extensions

The slightly more general notion of a pro-Galois extension refers to certain colimits of finite Galois extensions as we now review. Fix a cofibrant associative  $S$ -algebra  $A$  and consider a directed system of finite Galois extensions  $A \rightarrow B_\alpha$  of associative  $S$ -algebras  $B_\alpha$  with Galois groups  $G_\alpha$ . We assume several coherence properties: For each  $\alpha \leq \beta$  we assume that  $B_\alpha \rightarrow B_\beta$  is a cofibration of associative  $S$ -algebras and  $B_\alpha \simeq (B_\beta)^{h_{G_{\alpha\beta}}}$  for the kernel  $G_{\alpha\beta}$  of a preferred surjection  $G_\beta \rightarrow G_\alpha$ . Hence  $A \rightarrow B_\alpha$  is a sub Galois extension of  $A \rightarrow B_\beta$ . Define  $B$  to be the colimit  $B := \operatorname{colim}_\alpha B_\alpha$  formed in the category of  $S$ -algebras and let  $G$  be the limit  $G := \lim_\alpha G_\alpha$ .

**Definition 4.34** (Pro-Galois extensions of associative  $S$ -algebras). In this situation we define the map  $A \rightarrow B$  to be a *pro-Galois extension* with Galois group  $G$ .

The fundamental weak equivalences  $h, \tilde{h}, j$  and  $\tilde{j}$  from lemma 4.12 defined for every Galois extension extend to weak equivalences for pro-Galois extensions in the following way: For each  $\alpha$  the weak equivalence  $h_\alpha: B_\alpha \wedge_A B_\alpha \rightarrow F_A(G_{\alpha+}, B_\alpha)$  provides a weak equivalence  $h_{\alpha,B}: B \wedge_A B_\alpha \rightarrow F_A(G_{\alpha+}, B)$  by lemma 4.12.2. The colimit over these gives a weak equivalence

$$h: B \wedge_A B \longrightarrow F_A((G_+, B))$$

where we use the notation  $F_A((G_+, B)) := \operatorname{colim}_\alpha F_A(G_{\alpha+}, B)$ . We have

$$\operatorname{colim}_\alpha B \wedge_A B_\alpha \cong B \wedge_A \operatorname{colim}_\alpha B_\alpha \cong B \wedge_A B$$

as smash product commute with sequential colimits in the category of associative  $S$ -algebras. The morphisms  $j_\alpha$  extend to a morphism

$$B\langle\langle G \rangle\rangle := \lim_\alpha B\langle G_\alpha \rangle \longrightarrow F_A(B, B).$$

Let  $A \rightarrow B$  be a pro-Galois extension of associative  $S$ -algebras  $A \rightarrow B$  and  $A \rightarrow C$  a map of associative  $S$ -algebras such that for each  $\alpha$ , the map  $C \rightarrow C \wedge_A B_\alpha$  is again a  $G_\alpha$ -Galois extension. The system  $C \rightarrow C \wedge_A B_\alpha$  again assembles to a pro-Galois extension  $C \rightarrow \operatorname{colim}_\alpha C \wedge_A B_\alpha \simeq C \wedge_A B$ . Again there is an  $E$ -local version of pro-Galois extensions introduced in [75, 8.1.] for extensions of commutative  $S$ -algebras. As examples we recall the systems of  $K(n)$ -local (pro-) Galois extensions over  $L_{K(n)}S$  and other intermediate  $S$ -algebras into  $E_n^{nr}$  introduced in section 4.6.1. Starting with these extensions and inducing extensions along maps as in section 4.6.1 produces associative pro-Galois extensions in the sense of definition 4.34.

## 4.9 Separability, connectedness and separable closure

For every commutative ring  $R$  there is a separable extension  $\overline{R}$  that contains every finite projective connected separable commutative extension of  $R$ . The

ring  $\overline{R}$  is called the separable closure of  $R$  and  $R \rightarrow \overline{R}$  is a pro-Galois extension [30].

The picture has been partially generalized to the topological context and we recall the definitions, partly allowing associative ring spectra as well.

Recall that a ring is *connected* if it does not split as the product of two non-trivial rings. Connected rings are also called indecomposable. Recall that a possibly associative ring is connected if and only if it does not have any non-trivial *central* idempotents [59, §21].

**Definition 4.35** (Connected associative  $S$ -algebras). Let  $B$  be an associative  $S$ -algebra. We say that it is *connected* if it is not equivalent to the product  $B_1 \times B_2$  of two non-trivial  $S$ -algebras  $B_1$  and  $B_2$  under  $B$ .

Note that the product can likewise be seen as being formed in algebras or modules as the forgetful functor as a right adjoint commutes with limits.

**Lemma 4.36.** *An associative  $S$ -algebra  $B$  is connected if and only if  $\pi_0(B)$  is connected as an associative ring, i.e. it does not contain any central idempotents besides 0 and 1.*

**Proof:** If  $B$  is not connected, choose  $B_1$  and  $B_2$  with  $B_i \not\cong *$  such that  $B \simeq B_1 \times B_2$ . Then the canonical map  $B_1 \times B_2 \rightarrow B_1$  and the trivial map  $B_1 \times B_2 \rightarrow B_2$  induce a self map  $B \rightarrow B$ . This map is idempotent and is neither trivial nor it represents 1 in  $\pi_0(B)$ . We call the idempotent that it represents  $e$  and have to show that it represents a central element in  $\pi_0(B)$ . For this note that  $\pi_*(B_1) \oplus \pi_*(B_2) \cong \pi_*(B) \cong [B, B]^B \cong [B_1, B_1]^B \oplus [B_2, B_2]^B \oplus [B_1, B_2]^B \oplus [B_2, B_1]^B$ . We have  $[B_1, B_1]^B \cong [B_1, B_1]^{B_1} \cong \pi_*(B_1)$  and similarly for  $B_2$ . The map  $\pi_*(B_1) \oplus \pi_*(B_2) \cong \pi_*(B_1) \oplus \pi_*(B_2) \oplus [B_1, B_2]^B \oplus [B_2, B_1]^B$  allows to conclude that  $[B_1, B_2]^B \cong 0$ . Dealing with non-commutative rings, such a conclusion may be wrong in general but here it follows as the isomorphism is given by the canonical inclusion. As  $\pi_*(B_i)$  is a direct summand in  $\pi_*(B)$  we see that the Künneth spectral sequence

$$\mathrm{Ext}_{\pi_*(B)}(\pi_*(B_1), \pi_*(B_2)) \implies \pi_*(F_B(B_1, B_2))$$

collapses and hence gives  $\mathrm{Hom}_{\pi_*(B)}(\pi_*(B_1), \pi_*(B_2)) \cong [B_1, B_2]^B \cong *$ . By [59, 21.6] this means that  $(1 - e)\pi_*(B)e \cong *$  and by [59, 21.5] it follows that  $e$  is central.

Vice versa if  $\pi_0(B)$  is not connected choose a non-trivial central idempotent  $e \in \pi_0(B) \setminus \{0, 1\}$ . Choose  $B$ -module maps  $f_0$  and  $f_1$  representing  $e$  and  $1 - e$  in  $\pi_0(B) = [B, B]_0^B$ . Form the mapping telescopes  $B[f_i^{-1}]$  and define  $B_i := L_{B[f_i^{-1}]}B$ . Localization preserves algebra structures and under the assumption that  $e$  is central we can show that  $B_1 \times B_2$  is equivalent to  $B$ . For this, recall the decomposition  $B_* = eB_*e \oplus eB_*(1 - e) \oplus (1 - e)B_*e \oplus (1 - e)B_*(1 - e)$  for any idempotent  $e$  of  $B_*$ . We see that  $(B_1)_* \cong eB_*e$  and  $(B_2)_* \cong (1 - e)B_*(1 - e)$ . As  $eB_*(1 - e)$  and  $(1 - e)B_*e$  are trivial if and only if  $e$  is central, we see that under the centrality assumption of  $e$  we have  $\pi_*(B_1 \times B_2) \cong B_*$ .  $\square$



Of course in a commutative ring any idempotent is central and hence our approach generalizes the corresponding definition and statement for commutative  $S$ -algebras from [75, 10.2]. Recall the following definition from [75, 10.3].

**Definition 4.37** (Separably closed commutative  $S$ -algebras). A commutative  $S$ -algebra  $A$  is *separably closed* if for every finite commutative  $G$ -Galois extension  $A \rightarrow B$  either  $B$  is not connected or  $G = \{e\}$ . A separable closure of a commutative ring spectrum  $A$  is a commutative pro-Galois extension  $A \rightarrow \overline{A}$  such that  $\overline{A}$  is separably closed.

In the topological setting only few examples of separably closed commutative ring spectra are known. John Rognes has shown that the sphere spectrum is separably closed [75] and Andrew Baker and Birgit Richter have shown that  $E_n^{nr}$  is the separable closure of  $E_n$  in the commutative sense [11]. Both proofs rely on the algebraic property that the coefficients  $\mathbb{Z}$  respectively  $(E_n^{nr})_*$  do not have non-trivial connected Galois extensions of commutative rings, i.e. they are separably closed as commutative rings.

We hesitate to make an analogous definition for associative ring spectra as this would not generalize the commutative definition but gave something new. Note that the algebraic closure in the commutative sense above might very well admit non-trivial *associative* Galois extensions. Instead of persuing to find a meaningful definition we investigate the theories  $K_n$  directly. Recall that  $(K_n)_* \cong \mathbb{F}_p[u^\pm]$  and the spectrum  $K_n^{nr}$  has coefficients  $(K_n^{nr})_* \cong \overline{\mathbb{F}}_p[u^\pm]$ . Also recall from proposition 4.25 and section 4.8 that the pro-Galois extension  $E_n \rightarrow E_n^{nr}$  induces a pro-Galois extension  $K_n \rightarrow K_n^{nr}$ .

$$\begin{array}{ccc} E_n^{nr} & \longrightarrow & K_n^{nr} \\ \hat{\mathbb{Z}} \uparrow & & \uparrow \hat{\mathbb{Z}} \\ E_n & \longrightarrow & K_n \end{array}$$

**Proposition 4.38.** *Let  $p$  be an odd prime and let  $B$  be a Galois extension of  $K_n^{nr}$  with non-trivial Galois group and  $B_*$  commutative. Then  $B$  is not connected. In other words  $K_n^{nr}$  is a maximal connected Galois extension of  $K_n$  with commutative coefficients.*

The following lemma gives the link to algebra that will allow to prove the last proposition.

**Lemma 4.39.** *Let  $K$  be a ring spectrum such that  $K_*$  is a graded field. Then every finite topological Galois extension induces an algebraic Galois extension of associative rings on  $\pi_*$ .*

**Proof:** Let  $K \rightarrow B$  be a Galois extension. As  $K_*$  is a graded field the Künneth spectral sequence

$$\mathrm{Tor}^{K_*}(B_*, B_*) \implies \pi_*(B \wedge_K B)$$

collapses and gives  $\pi_*(B \wedge_K B) \cong B_* \otimes_{K_*} B_*$ . The topological unramification condition hence gives

$$B_* \otimes_{K_*} B_* \cong \pi_*(B \wedge_K B) \cong \prod_G B_* \tag{4.16}$$

For the fixed point condition note that  $B_*$  as a  $K_*$ -module, i.e. as a  $K_*$ -vector space, is finitely generated by the last formula as  $G$  is supposed to be finite. The fixed points  $B_* \otimes_{K_*} B_*^G$  are isomorphically mapped onto a subspace of  $(\prod_G B_*)^G \cong B_*$ . Counting dimensions as in [11] then shows that  $B_*^G$  must be isomorphic to  $K_*$ .  $\square$

**Proof of proposition 4.38:** Suppose  $B$  is a finite Galois extension of  $K_n^{nr}$  such that  $B_*$  is a commutative ring. The coefficients  $\pi_*(K_n^{nr}) \cong \overline{\mathbb{F}}_p[u^{\pm 1}]$  are a graded field so the last lemma applies and says that  $B_*$  is a finite Galois extension of  $(K_n^{nr})_*$ . As  $p$  is assumed to be odd, it follows as in the proof of [11, 5.1.] that  $\pi_1(B) = *$ . It then follows that  $\pi_0(B)$  is a separable extension of the field  $\overline{\mathbb{F}}_p$  which is separably closed. So by [30, II.2.4]

$$\pi_0(B) \cong \prod \overline{\mathbb{F}}_p.$$

Hence either  $G$  is trivial or  $B_*$  is not connected. In case  $B_*$  is not connected lemma 4.36 says that  $B$  is not connected as an associative  $S$ -algebra.  $\square$

**Part II**

**Hopf-Galois Extensions**



## Chapter 5

# Hopf-Galois extensions of associative $S$ -algebras

In this chapter we define the notion of a Hopf-Galois extension for associative  $S$ -algebras generalizing the notion for commutative  $S$ -algebras introduced in [75]. Galois extensions  $A \rightarrow B$  in the sense of definition 4.2 with finite Galois group give rise to Hopf-Galois extensions as defined below. We investigate under which conditions Hopf-Galois extensions  $A \rightarrow B$  are preserved or detected when inducing up along a map of  $S$ -algebras  $A \rightarrow C$ . We obtain necessary and sufficient conditions, thereby answering a question posed by Rognes. Many of our results hold for a slightly more general notion of extensions to be defined which we call coalgebra extension. We give basic examples for Hopf-Galois extensions. Thom spectra and regular quotients are dealt with in chapters 6 and 7.

### 5.1 Hopf-Galois extensions of ordinary rings

Let  $R \rightarrow T$  be an extension of rings. The notion of Galois extensions has been generalized to situations where one does not necessarily have a group acting on the extension object  $T$ . Instead one requires a Hopf algebra  $K$  to coact on  $T$ . This leads to Hopf-Galois extensions. The concept is due to Chase and Sweedler [25] for extensions of commutative rings and was generalized to extensions of associative rings by Kreimer and Takeuchi [58]. We work with a Hopf algebra  $K$  over  $R$ . Some authors instead work with Hopf algebras  $\tilde{K}$  over some field  $k$  but this is included in our framework by setting  $K := R \otimes_k \tilde{K}$ . We require the Hopf algebra  $K$  to coact on  $T$  under  $R$ , meaning that there is a coassociative and counital morphism  $\beta: T \rightarrow T \otimes_R K$  of algebras under  $R$ . In particular we require  $T \otimes_R K$  to be an  $R$ -algebra which is not automatically the case when  $R$  is not commutative. However this is always the case when  $K = R \otimes_k \tilde{K}$  or when  $T$  and  $K$  are central  $R$ -algebras regardless whether  $R$  is commutative or not. Recall the definition of the  $R$ -module of cofixed points  $T^{\text{co}K}$  which is defined as

$$T^{\text{co}K} := \{t \in T \mid \beta(t) = t \otimes_R 1\}.$$

This is the same as the equalizer of the maps  $\beta$  and  $T \otimes_R \eta$  where  $\eta: R \rightarrow K$  is the unit map:

$$T^{\text{co}}K \longrightarrow T \underset{T \otimes_R \eta}{\overset{\beta}{\rightrightarrows}} T \otimes_R K. \quad (5.1)$$

**Definition 5.1** (Hopf-Galois extension of rings). Let  $R, T$  and  $K$  be as above. The ring map  $R \rightarrow T$  then is a *Hopf-Galois extension* if

1. The unit map  $R \rightarrow T^{\text{co}}K$  is an isomorphism and
2. the canonical map  $h: T \otimes_R T \xrightarrow{T \otimes_R \beta} T \otimes_R T \otimes_R K \xrightarrow{\mu \otimes_R K} T \otimes_R K$  is an isomorphism.

There is a lot of recent literature on this subject, see e.g. [26, 21] and the references therein. We are very brief here and just say the minimum we need for motivating our topological work.

We just mention that every Galois extension with finite Galois group  $G$  provides a Hopf-Galois extension. As Hopf algebra one simply can take the functional dual of the group ring  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], R)$ . This and many more examples can be found in the literature cited above.

We will develop a theory of Hopf-Galois extensions for associative ring spectra. When dealing with (possibly associative) extensions of some commutative ring or ring-spectrum it will turn out that this generalizes the algebraic notion via an Eilenberg-MacLane embedding theorem as it was the case for honest Galois extensions. For this we will need the following result which we state from [58, 1.10].

**Proposition 5.2.** ([58, 1.10]) *Let  $R$  be a commutative ring and  $R \rightarrow T$  a Hopf-Galois extension with respect to a  $K$  which is finitely generated projective as an  $R$ -module. Then  $T$  is faithfully flat as an  $R$ -module.*  $\square$

This allows to state the cofixed point condition yet in another way. For this we recall the definition of the Amitsur complex for a ring map  $R \rightarrow T$ . Recall that a cosimplicial object  $C^\bullet(T/R)$  is given by

$$C^q(T/R) := T^{\otimes_R(q+1)}$$

where the structure maps are defined using the unit  $R \rightarrow T$  and the multiplication  $T \otimes_R T \rightarrow T$ . The Amitsur complex then is the associated cochain complex.

**Proposition 5.3.** *If  $R \rightarrow T$  is faithfully flat, then the Amitsur complex associated with the ring map  $R \rightarrow T$  is acyclic.*

**Proof:** An argument similar to the following is given in [21, B.0.4] for commutative rings. We check that this argument also applies when  $R$  is associative. As  $T$  is faithfully flat over  $R$  it suffices to show that  $T \otimes_R C^q(T/R) \otimes_R T$  is

acyclic. Let us show that this complex is exact in  $T^{\otimes_R(k+2)}$  and choose a cycle  $x = \sum_j t_{0,j} \otimes_R t_{1,j} \otimes_R \cdots \otimes_R t_{k+1,j} \in T^{\otimes_R(k+2)}$ . Then

$$\begin{aligned} & \sum_j t_{0,j} \otimes_R 1 \otimes_R t_{1,j} \otimes_R t_{2,j} \otimes_R \cdots \otimes_R t_{k+1,j} \\ &= \sum_j t_{0,j} \otimes_R t_{1,j} \otimes_R 1 \otimes_R t_{2,j} \otimes_R \cdots \otimes_R t_{k+1,j} \\ & \quad - \sum_j t_{0,j} \otimes_R t_{1,j} \otimes_R t_{2,j} \otimes_R 1 \otimes_R \cdots \otimes_R t_{k+1,j} \\ & \quad \pm \cdots \\ & \quad (-1)^{k+1} \sum_j t_{0,j} \otimes_R t_{1,j} \otimes_R t_{2,j} \otimes_R \cdots \otimes_R 1 \otimes_R t_{k+1,j} \end{aligned}$$

Multiplying the first two factors we obtain  $x = (\text{id}_T \otimes_R \partial \otimes_R \text{id}_T)(\sum_j t_{0,j} t_{1,j} \otimes_R \cdots \otimes_R t_{k+1,j})$  where  $\partial: T^{\otimes_R(k-1)} \rightarrow T^{\otimes_R(k)}$  is the boundary map in the cochain complex associated with  $C^\bullet(T/R)$ . So the cycle  $x$  is a boundary which proves the proposition.  $\square$

Given a Hopf-Galois extension  $R \rightarrow T$ , the two maps  $\delta^0, \delta^1: T \rightarrow T \otimes_R T$  from the Amitsur complex correspond to the maps  $\beta$  and  $T \otimes_R \eta$  from the equalizer diagram (5.1). So the cofixed points occur as the zeroth cohomology group of the Amitsur complex and when the extension is faithfully flat, this is the only non-vanishing Amitsur cohomology group by proposition 5.2. We now turn to our topological definitions.

## 5.2 Definitions

### 5.2.1 Hopf-Algebras in stable homotopy theory

**Definition 5.4** (Bialgebras). Let  $A$  be an associative  $S$ -algebra. We define a *bialgebra  $H$  under  $A$*  to be an  $A$ -unca that is also a coalgebra, hence equipped with a unit  $\eta: A \rightarrow H$  and an associative multiplication  $\mu: H \wedge_A H \rightarrow H$ , together with a counit  $\epsilon: H \rightarrow A$  and a coproduct  $\Delta: H \rightarrow H \wedge_A H$  in the category of  $S$ -algebras under  $A$  satisfying the usual coassociativity and counitality assumptions.

For  $A = S$  a bialgebra  $H$  with structure maps in the category of commutative  $S$ -algebras was called a *Hopf algebra* in [75]. In algebra, a Hopf algebra by definition is a bialgebra with antipode where an antipode is a map  $\lambda: H \rightarrow H$  such that  $\mu(H \wedge_A \lambda) \Delta = \eta \epsilon = \mu(\lambda \wedge_A H) \Delta$ . The antipode condition was dropped in [75] as this would restrict the number of examples. Although in the examples we will usually have maps that satisfy the antipode condition up to homotopy we cannot always strictify these maps to obtain an antipode in the category of  $S$ -algebras under  $A$ . So we do not include this condition in our point-set-level definition. We might call the  $A$ -bialgebras defined above  *$A$ -Hopf algebras* when we agree that in stable homotopy theory the existence of an antipode is usually

not included in the definition. We will do so and instead always emphasize when a Hopf algebra has a (homotopy) antipode. In this sense, for  $A = S$  the definition above is the definition of [75] with the difference, that we work with associative, not commutative  $S$ -algebras. Note that in general it is no longer automatic that  $H \wedge_A H$  is an  $S$ -algebra. It is of course an  $S$ -algebra if  $A$  is commutative and  $H$  is an associative  $A$ -algebra or if the map  $A \rightarrow H$  is central or more generally if  $H$  is of the form  $H = A \wedge \tilde{H}$  for some  $S$ -bialgebra  $\tilde{H}$ . In any of our examples we will be in one of these situations. Moreover the multiplicative structure of the Hopf algebra is not needed for many of the statements we will make. We can often work with objects of the following type.

**Definition 5.5** (Unital coalgebra). Let  $A$  be an associative  $S$ -algebra. We define a *unital coalgebra*  $H$  (or *coalgebra under  $A$* ) to be an  $A$ -bimodule under  $A$  with a counit  $\epsilon: H \rightarrow A$  and a coassociative, counital comultiplication  $\Delta: H \rightarrow H \wedge_A H$  in the category of  $A$ -bimodules under  $A$ .

Here are some examples:

- (a) First note that for any (pointed) space  $X$  the group ring  $S[X] := S \wedge X_+$  has the structure of a unital coalgebra. This structure is induced by the inclusion of the basepoint  $* \rightarrow X$ , the diagonal  $X \rightarrow X \times X$  and the projection  $X \rightarrow *$ . Moreover if  $X$  is an  $n$ -fold loop space,  $S[X]$  inherits the  $E_n$ -operad action from  $X$  and for  $n \geq 1$  we obtain a Hopf algebra in the sense above, see proposition 6.3.
- (b) For a finite group  $G$  the functional dual  $DG_+ := F(G_+, S) \simeq \prod_G S$  can be rigidified to a commutative Hopf algebra under  $S$ . Again smashing with any unca  $A$  provides a Hopf-algebra  $H \simeq F(G_+, S) \wedge A$  under  $A$  which is equivalent to  $F(S[G], A)$  by the dualizability of  $G$ .
- (c) We can obtain new Hopf algebras from old ones: The smash product of Hopf algebras under some commutative  $S$ -algebra is again a Hopf algebra under the same commutative  $S$ -algebra. And for any Hopf algebra  $H$  under  $S$  also  $A \wedge H$  is a Hopf algebra under  $A$  for any  $S$ -algebra  $A$ .

### 5.2.2 Coaction of a Hopf-algebra and Hopf-Galois extensions

**Definition 5.6.** (Coaction) Let  $A$  be an associative  $S$ -algebra,  $B$  an  $A$ -unca and  $H$  a Hopf-algebra under  $A$ . We say that  $H$  *coacts on  $B$  under  $A$*  if there is a map

$$\beta: B \rightarrow B \wedge_A H$$

of  $S$ -algebras under  $A$ , again satisfying coassociativity and counitality. More generally,  $H$  can just be a coalgebra in the category of  $A$ -bimodules under  $A$  in which case  $\beta$  is just required to be a coassociative and counital map of  $A$ -bimodules under  $A$ .

We will now approach the definition of Hopf-Galois extension and see that this definition also makes sense in case a unital coalgebra  $H$  coacts on some unca  $B$



under  $A$ . In this case, note that we can immediately define the morphism  $h$  as

$$h: B \wedge_A B \xrightarrow{B \wedge_A \beta} B \wedge_A B \wedge_A H \xrightarrow{\mu \wedge_A H} B \wedge_A H \quad (5.2)$$

which is a map of  $(B, A)$ -bimodules under  $A$ .

Formulating a cofixed point condition in analogy to the algebraic situation needs a little more work. First note that there is a cosimplicial object, the *Hopf-cobar complex*  $C_A^\bullet(H; B)$  which under  $A$  is defined by

$$C_A^q(H; B) := B \wedge_A \underbrace{H \wedge_A \cdots \wedge_A H}_q$$

with coface maps  $\delta_0 := \beta \wedge 1^q$ ,  $\delta_i := 1^i \wedge_A \Delta \wedge_A 1^{q-i}$  for  $0 < i < q$  and  $\delta_q := 1^q \wedge \eta$ . The codegeneracies are given by  $\sigma_i := 1^i \wedge \epsilon \wedge 1^{q-i}$ . We denote the totalization of this cosimplicial object as  $C_A(H; B)$  and have a coaugmentation

$$i: A \rightarrow C_A(H; B). \quad (5.3)$$

We will require some cofibrancy assumptions that make the theory homotopy invariant. Having a map of  $S$ -algebras  $A \rightarrow B$  and a unital coalgebra  $H$  coacting on  $B$  under  $A$  we will first require that  $A$  is cofibrant as an  $S$ -algebra. Second we will require that smashing with  $B$  respectively  $H$  over  $A$  represents the derived smash product. If these two requirements are satisfied, we will say that our data satisfy *good cofibrancy assumptions*. Of course we have good cofibrancy assumptions if  $B$  and  $H$  are cofibrant objects in the category of  $A$ -uncas by corollary 3.17 but we have to allow more flexibility here to include the coalgebra case. Moreover, as we do not have model structures on the category of comodules or coalgebras, at least we do not know that such structures exist, it might not be possible to keep the comodule and coalgebra structure when cofibrantly replacing  $B$  respectively  $H$  in the category of  $A$ -uncas. So even in the Hopf algebra situation we might not be able to force cofibrancy in this sense if it is not given anyway. It is therefore worth noting that by theorem 3.16 we have good cofibrancy assumptions whenever  $B$  and  $H$  are in the class  $\overline{\mathcal{F}}_A$  of extended cell  $A$ -bimodules as introduced in definition 3.14.

Under good cofibrancy assumptions,  $C(H; B)$  is a good replacement for the homotopy cofixed points as we will now explain. An investigation of the derived functor of the cofixed point functor in general model categories based on cobar constructions and the investigation of Hopf-Galois extensions in this context has been announced by Kathryn Hess, see [44, 45]. We mentioned that in the algebraic context the cofixed points are exactly given by the equalizer of the unit and coaction that correspond to the maps  $\delta^0, \delta^1: C^0 \rightarrow C^1$  in the Hopf-cobar complex. This equalizer is just  $\text{Hom}_\Delta(*, C^\bullet)$  where  $*$  is the constant cosimplicial object with a point in each degree and it is clear that a homotopy invariant version should be  $\text{Hom}_\Delta$  evaluated at a cofibrant and fibrant replacement, and this is  $\text{Hom}_\Delta(\Delta, C^{\bullet f}) = \text{Tot}(C^{\bullet f})$ . We will show below that in the situation

of the next definition  $C^\bullet$  itself is good enough i.e.  $\text{Tot}(C^\bullet) \simeq \text{Tot}(C^{\bullet f})$  has the correct homotopy type. Here is the basic definition of this chapter:

**Definition 5.7** (Hopf-Galois extensions and coalgebra extensions of associative  $S$ -algebras). Let  $A$  be a cofibrant  $S$ -algebra and  $H$  a Hopf algebra that coacts on  $B$  over  $A$  as above. Assume that smashing with  $B$  respectively  $H$  represents the derived smash product over  $A$ , e.g. this is the case when  $B$  and  $H$  are cofibrant  $S$ -algebras under  $A$ . We call  $B$  an  *$H$ -Hopf-Galois extension* of  $A$ , if the maps  $h$  and  $i$  from equations (5.2) and (5.3) are both weak equivalences. More generally, if  $H$  is just a coalgebra coacting on  $B$  under  $A$  such that the morphisms  $h$  and  $i$  are weak equivalences we call  $A \rightarrow B$  a *coalgebra extension*.

There is a nice criterion in [75, prop. 12.1.8] for when a map  $S \rightarrow B$  of commutative  $S$ -algebras is a Hopf-Galois extension. For a map of  $S$ -algebras  $A \rightarrow B$  define the Amitsur complex to be the cosimplicial object  $C^\bullet(B/A)$ , analogously defined to the object  $C^\bullet(T/R)$  from section 5.1, see [75, 8.2.1]. Then define  $A_B^\wedge := \text{Tot } C^\bullet(B/A)$  to be the completion of  $A$  along  $B$  formed in  $A$ -modules, [75, 8.2.1]. This is one example of a  $\mathcal{G}$ -completion as defined in [17]. More precisely  $A_B^\wedge$  is the  $\mathcal{G}$ -completion of  $A$  defined as the totalization of a cosimplicial resolution of  $A$  in the  $\mathcal{G}$ -model structure on the category of cosimplicial  $A$ -modules  $(\mathcal{M}_A)^\Delta$  for the class of injective models  $\mathcal{G} = \{G | G \text{ a left } B\text{-module}\}$ . At least for commutative  $A$  this completion  $A_B^\wedge$  is equivalent to Bousfield's  $B$ -nilpotent completion  $\widehat{L}_B^A A$  from [15] understood to be defined in the category of  $A$ -modules. This follows by comparison of [17] with [15, 5.6.5.8], a proof if  $A \rightarrow B$  is a map of commutative  $S$ -algebras is given in [75, 8.2.3]. The proof applies equally well if  $B$  is an associative  $S$ -algebra. If  $A$  is just in  $\mathcal{A}_S$  we have to require that  $A \rightarrow B$  is central in order that the nilpotent resolution given in [75, 8.2.2] is one of  $A$ -modules. We will need to compare these completions only in the case where  $A$  is a commutative  $S$ -algebra.

**Criterion 5.8.** *Let  $H$  coact on  $B$  over  $A$  as above with appropriate cofibrancy assumptions. Then  $B$  is an  $H$ -Hopf-Galois-extension of  $A$  if and only if*

1. *the canonical map  $h: B \wedge_A B \rightarrow B \wedge_A H$  is a weak equivalence and*
2. *the canonical map  $A \rightarrow A_B^\wedge$  is a weak equivalence, i.e.  $A$  is complete along  $B$ .*

**Proof of criterion 5.8:** We have to show that  $\text{Tot}(C^\bullet(B, H))$  defines the  $\mathcal{G}$ -completion of  $A$  for the class of injective models  $\mathcal{G}$  as above. In the terminology of [17] we can prove the criterion by showing that  $C^\bullet(B, H)$  defines a weak  $\mathcal{G}$ -resolution of  $A$  as we will now recall.

First look at the Amitsur complex  $C^\bullet(B/A)$  defined as  $C^k(B/A) := B^{\wedge_{A^{k+1}}}$  with the obvious coface and codegeneracy maps, i.e. given by unit and multiplication maps as in the algebraic case. The Amitsur complex is exactly the triple resolution for the triple  $(B \wedge_A -, \eta, \nu)$ . This is a weak  $\mathcal{G}$ -resolution of  $A$  by [17, 7.4] and by [17, 6.5] weak resolutions can be used to calculate completions.

Now let us show that also  $C^\bullet(B, H)$  defines a weak  $\mathcal{G}$ -resolution of  $A$ . Note that there is a cosimplicial weak equivalence  $C^\bullet(B/A) \rightarrow C^\bullet(B, H)$  defined

using the map  $h$  iteratedly which is a weak equivalence by assumption. The claim then follows as the resolution  $C^\bullet(B, H)$  is clearly termwise  $\mathcal{G}$ -injective (it suffices that all the terms are  $B$ -modules). For commutative  $S$ -algebras this result is stated in [75, 12.1.7, 12.1.8].  $\square$

The criterion and the arguments in its proof particularly show that  $\text{Tot } C^\bullet(B, H)$  has the correct homotopy type as claimed above. Moreover, due to the cofibrancy assumptions it is now clear that the definition of Hopf-Galois extension or coalgebra extension is homotopy invariant. For the unramification condition this follows analogously to the Galois case, see lemma 4.3. For the completion condition this follows as the  $\mathcal{G}$ -completion up to natural equivalence is independent of the choice of the  $\mathcal{G}$ -resolution [17, 5.7.].

The prototypical example of a Hopf-Galois extension of  $S$ -algebras is the unit map  $S \rightarrow MU$  from the sphere spectrum into the complex cobordism spectrum  $MU$  [75, 12]. The Hopf algebra for this extension is the spherical group ring  $S[BU]$ . The map  $h: MU \wedge MU \rightarrow MU \wedge S[BU]$  is the Thom-equivalence. The target of the map  $i: S \rightarrow S_{MU}^\wedge := \text{Tot } C^\bullet(MU/S)$  is the abutment of the Bousfield-Kan spectral sequence associated to the cosimplicial commutative  $S$ -algebra  $C^\bullet(MU/S)$ . But this spectral sequence is the Adams-Novikov-spectral sequence, hence converges to  $\pi_* S$  and  $i$  is a weak equivalence as well. We will generalize this example in chapter 6. It is maybe interesting to note that the extension  $S \rightarrow MU$  is not thh-étale and hence cannot be a Galois extension for any group  $G$ . Though the Hopf-Galois extension  $S \rightarrow MU$  contains a lot of information on the infinitely many pro-Galois extensions  $L_{K(n)} S \rightarrow E_n$  for each prime  $p$  and  $n \geq 0$ . We refer the reader to [75, chapter 12] for more details on this.

## 5.3 Some examples

### 5.3.1 Hopf-Galois extensions from Galois extensions

Let  $A \rightarrow B$  be a map of associative  $S$ -algebras and  $G$  a finite group acting on  $B$  by maps of associative  $S$ -algebras under  $A$ . For the case  $A = S$  and  $B$  a commutative  $S$ -algebra it is stated in [75] that  $A \rightarrow B$  is a Hopf-Galois extension if and only if it is a Galois extension. The argument from [75, 12.1.6] carries over to our more general situation as we now recall. First,  $H = A \wedge F(G_+, S)$  is a Hopf-algebra under  $A$  in the sense of definition 5.4. The group action  $\alpha: G_+ \wedge B \rightarrow B$  gives a coaction  $\beta: B \rightarrow F(G_+, B) \cong B \wedge_A H$  of  $H$  on  $B$  under  $A$  by adjunction. It is immediate from this definition that the canonical map  $B \wedge_A B \rightarrow F(G_+, B)$  for the Galois extension (i.e. defined using the group action) is a weak equivalence if and only if so is the canonical map  $B \wedge_A B \rightarrow B \wedge_A H$  for the Hopf-Galois extension (defined using the coaction of  $H$ ). The Hopf cobar complex  $C^\bullet(H; B)$  is the group cobar complex given in codegree  $q$  by  $F(G_+^q, B)$  with totalization  $B^{hG}$ . So we see that  $A$  is complete along  $B$ , i.e.  $A \simeq \text{Tot } C^\bullet(H; B)$  if and only if  $A \simeq B^{hG}$ . It follows that  $A \rightarrow B$  is a Galois extension if and only if it is a Hopf-Galois extension with Hopf algebra and

coaction defined as above.

### 5.3.2 Trivial extensions

We call Hopf-Galois extensions  $A \rightarrow H$  given by the unit map into a Hopf algebra  $H$  under  $A$  *trivial*. The Galois, hence Hopf-Galois extensions of  $S$ -algebras  $A \rightarrow F(G_+, A)$  for a finite group  $G$  are examples for this. Not every unit map  $A \rightarrow H$  needs to be a Hopf-Galois extension and the following definition is important.

**Definition 5.9** (Homotopy antipode). Let  $H$  be a Hopf algebra under  $A$ . We call a map  $\lambda: H \rightarrow H$  of  $A$ -bimodules such that

$$\mu(H \wedge_A \lambda)\Delta \simeq \eta\epsilon \simeq \mu(\lambda \wedge_A H)\Delta$$

a *homotopy antipode*.

If  $H$  is a Hopf algebra under  $A$  as in definition 5.4 such that  $A$  is complete along  $H$  and if  $H$  admits a homotopy antipode  $\lambda$ , then  $A \rightarrow H$  is a Hopf-Galois extension. In this case, an inverse of  $h$  in the homotopy category is given by

$$H \wedge_A H \xrightarrow{H \wedge_A \Delta} H \wedge_A H \wedge_A H \xrightarrow{H \wedge_A \lambda \wedge_A H} H \wedge_A H \wedge_A H \xrightarrow{\mu \wedge_A H} H \wedge_A H.$$

A diagram proving that this is a left inverse for  $h$  is the following:

$$\begin{array}{ccccccc} H \wedge H & \xrightarrow{H \wedge \Delta} & H \wedge H \wedge H & \xrightarrow{H \wedge \lambda \wedge H} & H \wedge H \wedge H & \xrightarrow{\mu \wedge H} & H \wedge H \\ \downarrow H \wedge \Delta & & \downarrow H \wedge H \wedge \Delta & & \downarrow H \wedge H \wedge \Delta & & \downarrow H \wedge \Delta \\ H \wedge H \wedge H & \xrightarrow{H \wedge \Delta \wedge H} & H^{(4)} & \xrightarrow{H \wedge \lambda \wedge H \wedge H} & H^{(4)} & \xrightarrow{\mu \wedge H \wedge H} & H \wedge H \wedge H \\ \downarrow H \wedge \epsilon \wedge H & & & & \downarrow H \wedge \mu \wedge H & & \downarrow \mu \wedge H \\ H \wedge A \wedge H & \xrightarrow{H \wedge \eta \wedge H} & & & H \wedge H \wedge H & \xrightarrow{\mu \wedge H} & H \wedge H \end{array}$$

The left vertical and bottom horizontal compositions are both the identity map. The diagram proving that the map in question is also a right inverse for  $h$  is left to the reader.

Besides trivial Galois extensions, one class of trivial Hopf-Galois extensions is given by the Hopf algebras  $A[X]$  for any loop space  $X$ . Taking loop inverses defines a homotopy antipode. The completion condition  $A \simeq A_{A[X]}^\wedge$  can be checked directly by an Adams spectral sequence argument, see section 6.2. The point of view taken in section 6.2 is that  $S[X]$  is the Thom spectrum associated with the trivial map  $X \rightarrow BF$  and this map clearly lifts to  $BSF$ , i.e. the Thom spectrum  $S[X]$  is orientable along  $H\mathbb{Z}$ . We hence conclude that  $S \simeq S_{S[X]}^\wedge$  (proposition 6.9). The claim  $A_{A[X]}^\wedge \simeq A$  follows by our preservation theorem for induced extensions (theorem 5.13). The next lemma shows that the completion condition is always satisfied for the unit map of a Hopf algebra.

**Lemma 5.10.** *Let  $H$  be a Hopf algebra under  $A$ . Then  $A$  is complete along  $H$ , i.e.  $A \simeq A_H^\wedge$ .*

**Proof:** The coaugmented Amitsur complex  $A \rightarrow C^\bullet(H/A)$ ,

$$A \longrightarrow H \rightrightarrows H \wedge_A H \cdots$$

admits a contraction defined in degree  $q$  to be the map  $\sigma_0: H^{\wedge_A q+1} \xrightarrow{\epsilon \wedge_A H^q} H^q$ . So for the cohomotopy terms  $\pi^s \pi_t(C^\bullet(H/A))$  we have

$$\pi^s \pi_t(C^\bullet(H/A)) = \begin{cases} \pi_t(A) & \text{if } s = 0 \\ 0 & \text{if } s > 0. \end{cases}$$

We hence see that the homotopy spectral sequence

$$E_2^{s,t} = \pi^s \pi_t(C^\bullet(H/A)) \implies \text{Tot}(C^\bullet(H/A)) \simeq A_H^\wedge$$

[17, 2.9,6,7] collapses which yields  $A \simeq A_H^\wedge$ .  $\square$

**Proposition 5.11.** *Assume that  $H$  is a Hopf-algebra under  $A$  which has a homotopy antipode. Then  $A \rightarrow H$  is a Hopf-Galois extension with Hopf algebra  $H$ .*

**Proof:** We have explained above that the existence of a homotopy antipode implies that the canonical map  $h$  is a weak equivalence. The completion condition holds by lemma 5.10. Hence  $A \rightarrow H$  is a Hopf-Galois extension by criterion 5.8.  $\square$

## 5.4 Induced extensions

We saw in lemma 2.8 that for an associative  $S$ -algebra  $A$  and  $A$ -algebras  $B$  and  $C$  also  $B \wedge_A C$  is an  $A$ -algebra and an unca under  $B$ . Moreover, a Hopf algebra  $\tilde{H}$  under  $A$  gives rise to a Hopf algebra  $H := C \wedge_A \tilde{H}$  under  $C$  if  $\tilde{H}$  and  $C$  are  $A$ -algebras and any unital  $A$ -coalgebra  $D$  gives rise to a unital  $C$ -coalgebra  $C \wedge_A D$  without further assumptions. If  $A \rightarrow B$  is as above and a Hopf-Galois extension for some Hopf algebra  $\tilde{H}$  it makes hence sense to ask whether  $C \rightarrow C \wedge_A B$  is a Hopf-Galois extension for  $H$ . We also treat a detection statement. We formulate our results for Hopf-Galois extensions but the statements have analogs for coalgebra extensions with the same proofs.

For the next statements recall that the completion  $C_{C \wedge_A B}^\wedge$  of  $C$  along  $C \wedge_A B$  formed in the category of  $C$ -modules by definition is  $\text{Tot} C^\bullet(C \wedge_A B/C)$  and the completion  $C_B^\wedge$  of  $C$  along  $B$  formed in  $A$ -modules is  $\text{Tot}(C \wedge_A C^\bullet(B/A))$ .

**Lemma 5.12.** *Let  $A$  be an associative  $S$ -algebra and  $B$  and  $C$  be  $A$ -algebras. Then the completion  $C_{C \wedge_A B}^\wedge$  formed in the category of  $C$ -modules is equivalent to the completion  $C_B^\wedge$  formed in  $A$ -modules.*

**Proof:** The triple resolution  $\Gamma_{\mathcal{M}_C}^\bullet(C \wedge_A B) = C \wedge_A \Gamma_{\mathcal{M}_A}^\bullet(B)$  by [17, 7.4] is a weak resolution of  $C$  along  $C \wedge_A B$  in  $C$ -modules. We show that it is also a weak resolution of  $C$  along  $B$  in  $\mathcal{M}_A$ . As all the terms of the resolution are  $B$ -modules by our centrality assumptions, it is clear that the resolution is

termwise  $\mathcal{G}$ -injective for the class of  $B$ -module spectra  $\mathcal{G}$ . A sufficient condition for that we even get a weak resolution as wanted is that the coaugmentation  $C \rightarrow C \wedge_A \Gamma_{\mathcal{M}_A}^\bullet(B)$  induces a weak equivalence  $\mathrm{Hom}_{\mathcal{M}_A}(C \wedge_A \Gamma_{\mathcal{M}_A}^\bullet(B); I) \rightarrow \mathrm{Hom}_{\mathcal{M}_A}(C; I)$  for every  $\mathcal{G}$ -injective  $I$ , see [17, 7.2,7.4]. This is the case by [17, 7.3] as a contraction  $s_{-1}$  of this augmented simplicial set is given by  $s_{-1}f := \mu f \Gamma_{\mathcal{M}_A}$ .  $\square$

**Theorem 5.13.** (*Preservation theorem*) *Let  $A \rightarrow B$  be a Hopf-Galois extension with respect to a Hopf algebra  $H$  and let  $B$  and  $C$  be cofibrant  $A$ -algebras under some associative  $S$ -algebra  $A$ . Then  $C \rightarrow C \wedge_A B$  is a Hopf-Galois extension with Hopf algebra  $C \wedge_A H$  if and only if  $C$  is complete along  $B$  in the category of  $A$ -modules.*

**Proof:** The canonical maps  $h$  for the two extensions in question are related by  $h_{C \wedge_A B/C} = C \wedge_A h_{B/A}$ . So if  $h_{B/A}$  is a weak equivalence then so is  $h_{C \wedge_A B/C}$ . The rest follows from the last lemma.  $\square$

We will apply the following corollary in the next chapter. For a detection theorem the following property turns out to be crucial.

**Corollary 5.14.** *Let  $A \rightarrow B$  be a cofibration of cofibrant associative  $S$ -algebras and let a Hopf algebra  $H$  coact on  $B$  over  $A$  such that the canonical map  $h$  is a weak equivalence. Assume that  $A \rightarrow B$  is central and let  $C$  be an  $A$ -algebra which is complete along  $B$  in the category of  $A$ -modules. Then  $C \rightarrow C \wedge_A B$  is a Hopf-Galois extension with the Hopf algebra  $C \wedge_A H$ .*  $\square$

**Definition 5.15.** Let  $M$  be a left  $A$ -module. We say that the completion of  $M$  along  $B$  in  $\mathcal{M}_A$  is *smashing* if  $M_B^\wedge \cong A_B^\wedge \wedge_A M$ .

For example this is the case for all  $A$ -modules  $M$  if the completion is a smashing Bousfield localization in the category of  $A$ -modules. We can now formulate our detection theorem.

**Theorem 5.16.** (*Detection theorem*) *Let  $B$  and  $C$  be  $A$ -algebras under some associative  $S$ -algebra  $A$ , let  $C \rightarrow C \wedge_A B$  be a Hopf-Galois extension with respect to a Hopf algebra  $C \wedge_A H$  and assume that  $A \rightarrow C$  is faithful. Then  $A \rightarrow B$  is Hopf-Galois with Hopf algebra  $H$  if and only if the completion of  $C$  along  $B$  formed in  $A$ -modules is smashing.*

**Proof:** As  $h_{C \wedge_A B/C} = C \wedge_A h_{B/A}$  and  $A \rightarrow C$  is faithful it follows that  $h_{B/A}$  is a weak equivalence. We have to check whether the completion  $A_B^\wedge$  of  $A$  along  $B$  formed in  $\mathcal{M}_A$  is equivalent to  $A$ . By faithfulness this condition exactly asks whether  $C \simeq A_B^\wedge \wedge_A C$ . By assumption and lemma 5.12 we have  $C \simeq C_{C \wedge_A B}^\wedge \simeq C_B^\wedge$ . So  $C \simeq A_B^\wedge \wedge_A C$  if and only if  $C_B^\wedge \simeq A_B^\wedge \wedge_A C$ , i.e. if and only if the completion is smashing.  $\square$

When working with commutative  $S$ -algebras, theorems 5.13 and 5.16 give precise statements under which conditions Hopf-Galois extensions are preserved or detected by a basechange along a (faithful) map  $A \rightarrow C$ . By the embedding of Galois theory into Hopf-Galois theory the theorems apply to Galois

extensions and so answer the question posed in [75, 7.1] which conditions are actually necessary assumption for preservation and detection statements for Galois extensions. The following lemma is a nice corroboration.

**Lemma 5.17.** *If  $M$  is a dualizable  $A$ -module, then the completion of  $M$  along  $B$  in  $\mathcal{M}_A$  is smashing.*

**Proof:** We have

$$\begin{aligned} M \wedge_A A_B^\wedge &\simeq F_{A^{\text{op}}}(D_A M, \text{Tot } C^\bullet(B/A)) \\ &\cong \text{Tot } F_{A^{\text{op}}}(D_A M, C^\bullet(B/A)) \simeq \text{Tot } M \wedge_A C^\bullet(B/A) \simeq M_B^\wedge, \end{aligned}$$

compare [75, 8.2.4]. □

Moreover comparison with the results about induced extensions from section 4.6.1 provide the following proposition.

**Proposition 5.18.** *Let  $A \rightarrow B$  be a central Galois extension of  $S$ -algebras.*

- (a) *If  $C$  is dualizable over  $A$  or if  $C$  is an  $(A, B)$ -bimodule then  $C$  is complete along  $B$  in the category of  $A$ -modules.*
- (b) *If  $A \rightarrow B$  is faithful, then any associative  $A$ -algebra is complete along  $B$  in the category of  $A$ -modules.*

*Conversely if  $C \rightarrow C \wedge_A B$  is a Galois extension where  $G$  acts on  $B$  and  $C$  is central and faithful over  $A$  we have the following conclusions:*

*If  $C$  is dualizable over  $A$  or  $C$  is a right  $B$ -module or  $A \rightarrow B$  is faithful then the completion of  $C$  along  $B$  formed in  $A$ -modules is smashing.*

**Proof:** Follows from comparing theorem 5.13 with proposition 4.20 and 5.16 with proposition 4.21. □





## Chapter 6

# Thom spectra

The prototype of a Hopf-Galois extension of commutative  $S$ -algebras presented in [75] is the unit map  $S \rightarrow MU$  from the sphere spectrum into the complex cobordism spectrum. This spectrum arises as the Thom spectrum  $Mf$  associated to the canonical map  $f: BU \rightarrow BF = BGL_1S$ , the classifying space of stable spherical fibrations or the space of units of  $S$ . The Hopf algebra for this extension is the spherical group ring  $S[BU]$  and the coaction is given by the Thom diagonal. The canonical map  $h: MU \wedge MU \rightarrow MU \wedge S[BU]$  is the Thom equivalence. The completion condition  $S \simeq S_{MU}^\wedge$  follows from the convergence of the Adams-Novikov spectral sequence.

Thom spectra associated with maps  $X \rightarrow BGL_1S$  have been constructed in [64] and among experts it has been known for some time that the construction of Thom spectra works more generally starting with a map  $X \rightarrow BGL_1R$  into the space of units of any  $E_\infty$ -ring spectrum  $R$ . The construction of Thom spectra has recently been generalized and updated in [3]. In particular, Thom spectra constructions that apply even when  $R$  is just an  $A_\infty$  ring spectrum are given and orientations are investigated.

In section 6.1 we briefly sketch the construction of Thom spectra using the category of  $*$ -monoids [13, 3], a rigidified model of  $A_\infty$  spaces based on the linear isometries operad in the style of [36]. If  $Mf$  is an  $R$ -algebra Thom spectrum associated with a loop map  $f: X \rightarrow BGL_1R$ , the spectrum  $R[X]$  is a Hopf-algebra that coacts on  $Mf$ . It also follows from the theory of orientations that the canonical map  $h: Mf \wedge_R Mf \rightarrow Mf \wedge_R R[X]$  is a weak equivalence in this case. The map  $R \rightarrow Mf$  is hence a Hopf-Galois extension if and only if  $R$  is complete along  $Mf$ . In general this is a non-empty condition and needs to be checked. The following theorem answers the question whether  $R$  is complete along  $Mf$  in case  $R = S$ .

**Theorem 6.1.** *Let  $X$  be path connected,  $f: X \rightarrow BGL_1S = BF$  a loop map and let  $Mf$  be the associated Thom spectrum. Then  $S$  is complete along  $Mf$  if and only if  $f$  lifts to  $BSF$ , i.e. if and only if  $Mf$  is orientable along  $H\mathbb{Z}$ . Likewise the map  $S \rightarrow Mf$  is an associative Hopf-Galois extension with Hopf algebra  $S[X]$  and coaction defined by the Thom diagonal if and only if  $f$  lifts to  $BSF$ .*

In other words,  $S \rightarrow Mf$  is a Hopf-Galois extension if and only if the  $Mf$ -

based Adams spectral sequence converges to  $\pi_*S$ . It follows that most of the classical Thom spectra, e.g.  $MU$ ,  $MSU$  and  $MSO$  are Hopf-Galois extensions of the sphere spectrum, though  $MO$  is not. There is also a Hopf-Galois extension  $S \rightarrow H\mathbb{Z}$  associated with a map  $\Omega^2 S^3 \langle 3 \rangle \rightarrow BGL_1 S$ . Next recall the Thom spectrum denoted  $M\xi$  which is associated with a map  $j: \Omega\Sigma CP^\infty \rightarrow BGL_1 S$  and was investigated in [8]. The unit  $S \rightarrow M\xi$  is a Hopf-Galois extension of associative  $S$ -algebras and  $M\xi$  cannot be a commutative  $S$ -algebra as it is not even homotopy commutative.

If  $f$  does not lift to  $BSF$  we can at least produce Thom spectra after inducing up along the map  $S \rightarrow S_{S\mathbb{Z}/2}^\wedge$ , where  $S\mathbb{Z}/2$  denotes the Moore spectrum modulo 2. In this case, completion along  $S\mathbb{Z}/2$  is completion along  $Mf$  or likewise localization at  $S\mathbb{Z}/2$ .

**Proposition 6.2.** *If  $X$  is path connected and  $f: X \rightarrow BGL_1 S = BF$  a loop map that does not lift to  $BSF$ , the induced extension  $S_{S\mathbb{Z}/2}^\wedge \rightarrow S_{S\mathbb{Z}/2}^\wedge \wedge Mf$  is a Hopf-Galois extension with the induced Hopf algebra  $S_{S\mathbb{Z}/2}^\wedge \wedge S[X]$ .*

Examples for maps  $S \rightarrow Mf$  where  $Mf$  is a Thom spectrum which is not complete along  $S$  are given by the Thom spectra associated with systems of groups like e.g. Braid groups. It is known that these Thom spectra are wedges of Eilenberg McLane spectra  $H\mathbb{Z}/2$  and so  $Mf$  is not orientable along  $H\mathbb{Z}$  in this case.

For experts theorem 6.1 may not be surprising and some result in this direction has already been indicated in [75]. However, we think that the role of the orientability condition has not been clear so far. Theorem 6.1 follows mainly by combination of various results from the literature. We review these results and think that it is worth giving an account based on [3].

## 6.1 Units, Thom spectra and orientations

### 6.1.1 Units of ring spectra and Thom spectra

Recall that for an  $S$ -algebra  $R$  the space of units is defined to consist of those components of  $\Omega^\infty R$  that correspond to units in the ring  $\pi_0 R$ . In other words, there is a pullback

$$\begin{array}{ccc} GL_1 R & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ \pi_0(R)^\times & \longrightarrow & \pi_0(R) \end{array}$$

in the category of unpointed spaces. Now let us suppose that  $R$  is a commutative  $S$ -algebra. Then  $GL_1 R$  is an infinite loop space. We can hence form the principal fibration  $GL_1 R \rightarrow EGL_1 R \rightarrow BGL_1 R$  and for every map  $f: X \rightarrow BGL_1 R$  the (homotopy) pullback

$$\begin{array}{ccc} Pf & \longrightarrow & EGL_1 R \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BGL_1 R \end{array} \quad (6.1)$$

in the category of unpointed spaces. The Thom spectrum associated to the map  $f: X \rightarrow BGL_1R$  is then defined as the derived smash product

$$Mf := \Sigma_+^\infty Pf \wedge_{\Sigma_+^\infty GL_1R}^L R. \quad (6.2)$$

The classical Thom spectra arise in this way from maps  $f: X \rightarrow BGL_1S = BF$ .

If  $R$  is just an associative  $S$ -algebra, then  $GL_1R$  is just a group-like  $A_\infty$  space and  $GL_1R \rightarrow EGL_1R \rightarrow BGL_1R$  is not a principal fibration. In order to define Thom spectra one hence needs a replacement of diagram (6.1). One solution can be given by working in the more rigid category of  $*$ -modules  $\mathcal{M}_*$ . We refer to [13, 3] for the construction and a more thorough treatment of the properties of the category  $\mathcal{M}_*$  but give some indications here. The category of  $*$ -modules is constructed in analogy to the category of  $S$ -modules  $\mathcal{M}_S$ , just for spaces instead of spectra. For the construction we have to fix a countably infinite-dimensional real inner product space  $U$  and let  $\mathcal{L}(n)$  be the  $n$ -th space of the linear isometries operad  $\mathcal{L}$ , i.e.  $\mathcal{L}(n) = \mathcal{L}(U^n, U)$  is the space of linear isometries  $U^n \rightarrow U$ . In particular  $\mathcal{L}(1) = \mathcal{L}(U, U)$  is a topological monoid. Then one first defines the category of  $\mathcal{L}(1)$ -spaces which has objects the unpointed spaces  $X$  with an associative and unital action  $\mathcal{L}(1) \times X \rightarrow X$ . The category of  $\mathcal{L}(1)$ -spaces can be equipped with the structure of a model category [13, 4.16] and has an associative and commutative product given by

$$X \times_{\mathcal{L}} Y := \mathcal{L}(2) \times_{\mathcal{L}(1) \times \mathcal{L}(1)} X \times Y.$$

The category of  $*$ -modules  $\mathcal{M}_*$  is the full subcategory of objects for which this product is unital. It can be equipped with the structure of a monoidal model category [13, 4.22]. Monoids with respect to the product  $\times_{\mathcal{L}}$  correspond to  $A_\infty$  spaces [13, 4.8] and  $*$ -monoids are just  $\times_{\mathcal{L}}$ -monoids which are  $*$ -modules. In particular, for any associative  $S$ -algebra  $R$ , the space of units  $GL_1(R)$  gives rise to a  $*$ -monoid in the following way. Note that every space  $X$  is a trivial  $\mathcal{L}$ -space and  $* \times_{\mathcal{L}} X$  is a  $*$ -module. Now by a mild abuse of notation and as in [3] we set

$$GL_1R = * \times_{\mathcal{L}} (GL_1R)^c$$

where  $(GL_1R)^c$  is a cofibrant replacement of the  $A_\infty$  space  $GL_1R$  in the category of  $\times_{\mathcal{L}}$ -monoids.  $GL_1R$  thus redefined is a monoid in  $\mathcal{M}_*$ . The monoidal structure of the category of  $*$ -modules then allows to define a refined version of diagram (6.1). Note that for a monoid  $G$  in  $\mathcal{M}_*$  we can form the usual classifying space construction. We define

$$E_{\mathcal{L}}G := |B_\bullet(*, G, G)|_{\mathcal{M}_*}$$

$$B_{\mathcal{L}}G := |B_\bullet(*, G, *)|_{\mathcal{M}_*}$$

as realizations of the bar constructions. The generalization of (6.1) is then given as follows. For a map of  $*$ -modules  $f: X \rightarrow B_{\mathcal{L}}GL_1R$  we can take the pullback

$$\begin{array}{ccc} P_{\mathcal{L}}f & \longrightarrow & E_{\mathcal{L}}GL_1R \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & B_{\mathcal{L}}GL_1R \end{array}$$

in the category of right  $GL_1R$ -modules and define the Thom spectrum as

$$Mf := \Sigma_+^\infty P_{\mathcal{L}}f \wedge_{\Sigma_+^\infty GL_1R}^L R \quad (6.3)$$

[3, section 5]. This generalizes (6.2). Henceforth we will apply the same notation regardless whether we work in the category of  $*$ -modules or not, e.g. we will write  $Pf$  also for  $P_{\mathcal{L}}f$ . Using functorial cofibrant replacements we can make the Thom spectrum construction into a functor on the category of  $*$ -modules over  $BGL_1R$  to the category of  $R$ -modules. Directly from the construction we see that

$$\begin{aligned} P* &\cong GL_1R, \\ f \simeq g &\Rightarrow Pf \simeq Pg, \\ P(X \xrightarrow{*} BGL_1R) &\cong X \times GL_1R, \end{aligned}$$

so that

$$\begin{aligned} M* &\cong R, \\ f \simeq g &\Rightarrow Mf \simeq Mg, \\ M(X \xrightarrow{*} BGL_1R) &\cong R \wedge_{\Sigma_+^\infty X} = R[X]. \end{aligned}$$

**Proposition 6.3.** *Let  $X$  be a  $*$ -monoid or a loop space. Then  $M(X \xrightarrow{*} BGL_1R) = R[X]$  is an associative Hopf algebra under  $R$ . If  $X$  is a commutative  $*$ -monoid or an infinite loop space then  $R[X]$  is a commutative Hopf algebra under  $R$ . More generally for any  $R$ -algebra Thom spectrum  $Mf$  associated with a map  $f: X \rightarrow BGL_1R$  the Hopf algebra  $R[X]$  coacts on the associated Thom spectrum  $Mf$  under  $R$ .*

**Proof:** By [13, A.1] loop maps over a grouplike  $*$ -monoid can be rigidified to maps of  $*$ -monoids and the analogous statement for  $n$ -fold loop maps holds as well [13, A.2.]. So it suffices to prove the statement for maps of (commutative)  $*$ -monoids. The structure maps are induced by the corresponding maps on the level of spaces which are

$$\begin{array}{ccccccc} * & \longrightarrow & X & \xrightarrow{\Delta} & X \times X & \xrightarrow{\mu} & X & \longrightarrow & * \\ & & \searrow & & \downarrow & & \swarrow & & \\ & & & & BGL_1R & & & & \end{array}$$

and

$$\begin{array}{ccccc} X & \xrightarrow{\Delta} & X \times X & \xrightarrow{pr_1} & X \\ & \searrow f & \downarrow f \times * & \swarrow f & \\ & & BGL_1R & & \end{array}$$

This proves the proposition.  $\square$

If  $R$  is a commutative  $S$ -algebra (infinite) loop maps  $f: X \rightarrow BGL_1R$  give rise to associative (commutative)  $R$ -algebra Thom spectra  $Mf$ , see [3]. However if

$R$  is only associative,  $Mf$  is defined as a smash product over the (associative)  $S$ -algebra  $\Sigma_+^\infty GL_1 R$ . Smash products over associative  $S$ -algebras need not have the structure of an  $S$ -algebra unless centrality conditions hold, compare chapter 2. In particular there is no reason for  $Mf$  having the structure of an  $S$ -algebra in this case.

### 6.1.2 Orientations

Given a map of  $S$ -algebras  $R \rightarrow T$  and a map  $f: X \rightarrow BGL_1 R$ , we can compare the Thom spectrum  $Mf$  with the one associated with the composite map

$$g: X \xrightarrow{f} BGL_1 R \rightarrow BGL_1 T. \quad (6.4)$$

In this case, there is an equivalence

$$Mg \simeq Mf \wedge_R T \quad (6.5)$$

[3]. Moreover, if the composite map  $g$  is trivial we get a lift in the following diagram where  $B(R, T)$  is the pullback in the category of  $GL_1 R$ -modules.

$$\begin{array}{ccc} B(R, T) & \longrightarrow & EGL_1 T \\ & \searrow \tilde{a} & \downarrow \\ X & \xrightarrow{f} & BGL_1 R \longrightarrow BGL_1 T \end{array}$$

By [3, 5.30] the space of such lifts is equivalent to the space of  $T$ -orientations of  $Mf$ . As an example  $B(S, H\mathbb{Z}) = BSF$ . Given a lift  $\tilde{a}$  as in the diagram, we can pass to Thom spectra and get a map  $a: Mg \rightarrow T$  which is called the  $T$ -orientation associated with the lift  $\tilde{a}$ .

**Theorem 6.4.** [3, 5.43] *Let  $Mg$  be a Thom spectrum associated to a map  $g: X \rightarrow BGL_1 T$  and let us be given a  $T$ -orientation  $a: Mg \rightarrow T$ . Then the map*

$$Mg \xrightarrow{\Delta} Mg \wedge_R R[X] \xrightarrow{a \wedge_R R[X]} T \wedge_R R[X]$$

*is a weak equivalence.*

## 6.2 Thom spectra and Hopf-Galois extensions

The next lemma follows immediately.

**Proposition 6.5.** *Let  $X$  be path connected and let  $Mf$  an  $R$ -algebra Thom spectrum associated to a map  $f: X \rightarrow BGL_1 R$ . Then the canonical map*

$$h: Mf \wedge_R Mf \rightarrow Mf \wedge_R R[X]$$

*is a weak equivalence.*

**Proof:** Set  $T = Mf$  and let  $g$  be the composition as in (6.4). Then  $Mg \simeq Mf \wedge_R Mf$  by (6.5). The multiplication  $\mu: Mf \wedge_R Mf \rightarrow Mf$  is an  $Mf$ -orientation of  $Mf \wedge_R Mf$ , e.g. due to the unit condition by [3, 5.40]. The canonical map  $h$  then is the map from proposition 6.4 for the orientation  $a = \mu$ .  $\square$

In order to investigate whether a loop map  $f: X \rightarrow BGL_1R$  actually gives rise to a Hopf-Galois extension  $R \rightarrow Mf$  we have to check whether  $R$  is complete along  $Mf$ . The answer on this depends on the situation. We will investigate the case  $R = S$ . The following proposition is well known.

**Proposition 6.6.** *Let  $Mf$  be a Thom spectrum associated to a loop map  $f: X \rightarrow BGL_1S$  for some path connected space  $X$ . Then  $\pi_0(Mf)$  is either  $\mathbb{Z}$  or  $\mathbb{Z}/2$ , the former if and only if  $Mf$  is orientable along  $H\mathbb{Z}$ , i.e. if and only if  $f$  lifts to  $BSF = B(S, H\mathbb{Z})$ .*

**Proof:** As  $X$  is path connected,  $Mf$  is  $-1$ -connected. So if  $Mf$  is orientable along  $H\mathbb{Z}$  then the Thom equivalence  $H\mathbb{Z} \wedge Mf \simeq H\mathbb{Z} \wedge S[X]$  shows that  $\pi_0 Mf \cong \mathbb{Z}$  using the Künneth spectral sequence. Vice versa, if  $\pi_0 Mf \cong H\mathbb{Z}$  then an orientation  $Mf \rightarrow H\mathbb{Z}$  is given by the map from  $Mf$  into the first stage of its Postnikov tower. As  $* \simeq BGL_1\mathbb{Z}/2 = EGL_1\mathbb{Z}/2$ , every Thom spectrum associated with a map  $X \rightarrow BGL_1S$  is orientable along  $H\mathbb{Z}/2$ . Again the Thom isomorphism shows that the 2-torsion part of  $\pi_0 Mf$  is  $\mathbb{Z}/2$ . Then note that the inclusion of the fibre  $GL_1S \rightarrow P^*$  is surjective on  $\pi_0$  and hence  $S \rightarrow Mf$  is surjective on  $\pi_0$ . So  $\pi_0 Mf$  is cyclic which completes the argument.  $\square$

We will now investigate the completeness condition, i.e. whether  $S_{Mf}^\wedge$  is equivalent to  $S$  by use of Bousfield's analysis of the Adams spectral sequence. First, note that for the construction of the Adams Spectral sequence based on a ring-spectrum  $E$  no commutativity assumptions are needed [1, 15]. So we can work with the Adams spectral sequence based on the Thom spectrum  $Mf$  for any loop map  $f$ . For the convergence result, we state Bousfield's theorems 6.5 and 6.6 from [15]. Here  $cR := \{r \in R : r \otimes 1 = 1 \otimes r \text{ in } R \otimes_{\mathbb{Z}} R\}$  is the core of the ring  $R$ . By  $SG$  we denote the Moore spectrum associated with the abelian group  $G$ . By  $Y_E$  we denote the Bousfield localization of  $Y$  at  $E$  and  $J$  is a set of primes.

**Theorem 6.7.** ([15, 6.5]) *Let  $E$  be a connective ring spectrum with  $c\pi_0 E = \mathbb{Z}[J^{-1}]$ , and let  $Y$  be a connective spectrum. Then  $Y_E \simeq \widehat{L}_E^S Y$  and  $\widehat{L}_E^S Y \simeq Y_{S\mathbb{Z}[J^{-1}]}$ . If  $X$  is a finite CW-spectrum, then the  $E_*$ -Adams spectral sequence for  $X$  and  $Y$  is strongly Mittag-Leffler and converges completely to  $[X, \widehat{L}_E^S Y]_*$ .*

**Theorem 6.8.** ([15, 6.6]) *Let  $E$  be a connective ring spectrum with  $c\pi_0 E = \mathbb{Z}/n$  for  $n \geq 2$ , and let  $Y$  be a connective spectrum. Then  $Y_E \simeq \widehat{L}_E^S Y$  and  $\widehat{L}_E^S Y \simeq Y_{S\mathbb{Z}/n}$ . If  $X$  is a finite CW-spectrum and if each  $[X, Y]_m$  has  $n$ -torsion of bounded order, then the  $E_*$ -Adams spectral sequence for  $X$  and  $Y$  is Mittag-Leffler and converges completely to  $[X, \widehat{L}_E^S Y]_*$ .*

In our case, i.e. for  $X = Y = S$  and  $E = Mf$  either theorem 6.7 or 6.8 applies in view of proposition 6.6. The abutment of the  $Mf$ -based Adams spectral sequence hence is  $S_{Mf} = S_{H\mathbb{Z}} = S$  or  $S_{Mf} = S_{S\mathbb{Z}/2}$  according to the question whether  $f$  lifts over  $BSF$  or not. Note again that the  $\mathcal{G}$ -completion coincides with Bousfield's nilpotent completion as explained after criterion 5.8. We obtain the following proposition.

**Proposition 6.9.** *Let  $X$  be a path connected loop-space and  $f: X \rightarrow BF$  a loop-map. Then  $S$  is complete along the Thom spectrum  $Mf$  if and only if  $Mf$  is orientable along  $H\mathbb{Z}$ .*

**Proof:** We have to show, that the unit map  $S \rightarrow S_{Mf}^\wedge$  is a weak equivalence. This follows from theorem 6.7 together with proposition 6.6 in case  $f$  lifts to  $BSF$ . If a loop map  $f: X \rightarrow BF$  does not lift to  $BSF$ , then  $S \rightarrow Mf$  is definitely not an  $S[X]$ -Hopf-Galois extension as then by proposition 6.6 and theorem 6.8 the completion condition is violated as  $\pi_* S_{H\mathbb{Z}/2}^\wedge \simeq \mathbb{Z}_2^\wedge \otimes \pi_* S$  by [15, 2.5].  $\square$

**Proof of theorem 6.1:** We verify the conditions of the criterion 5.8: The first condition of criterion 5.8 always holds as was already shown in proposition 6.5. The second condition of criterion 5.8 by proposition 6.9 holds if and only if  $Mf$  is orientable along  $H\mathbb{Z}$ .  $\square$

In the situation of theorem 6.8 the completion is localization and hence idempotent. We hence obtain the following proposition which corresponds to proposition 6.2.

**Proposition 6.10.** *Let  $X$  be a path connected loop-space and  $f: X \rightarrow BF$  be a loop-map, that does not lift to  $BSF$ . Then  $S_{S\mathbb{Z}/2}^\wedge \rightarrow S_{S\mathbb{Z}/2}^\wedge \wedge Mf$  is an  $S_{S\mathbb{Z}/2}^\wedge[X]$ -Hopf-Galois extension of associative  $S$ -algebras in the sense of definition 5.7.*

**Proof:** Due to proposition 6.5 and corollary 5.14 we only have to check that  $S_{S\mathbb{Z}/2}^\wedge$  is complete along  $S\mathbb{Z}/2$  which is obvious since by theorem 6.8 this completion is localization and hence is idempotent.  $\square$

**Examples 6.11.** (a) Let  $G$  be one of the infinite classical groups  $U$ ,  $SU$ ,  $O$ ,  $SO$ ,  $Sp$  or  $Spin$ . The Thom spectra  $MG$  ([64, 13]) are associated with the canonical infinite loop maps  $BG \rightarrow BGL_1 S$  and  $S \rightarrow MG$  is a Hopf-Galois extension of commutative  $S$ -algebras whenever  $\pi_0(MG) \cong \mathbb{Z}$ . For  $G = O$  this is not the case as  $\pi_* MO$  is a  $\mathbb{Z}/2$ -algebra. For all the other  $G$  from the above list however, we have  $\pi_0(MG) \cong \mathbb{Z}$ . It is well known that the cobordism ring  $\pi_*(MU)$  is an integral polynomial ring and  $\pi_0(MSO) \cong \mathbb{Z}$  was calculated in [84]. From this it follows that also  $MSpin$  and  $MSU$  are orientable using the morphisms of groups  $Spin(n) \rightarrow SO(n)$  and  $SU(n) \rightarrow U(n)$ . It follows from theorem 6.1 that  $MU, MSU, MSO$  and  $MSpin$  are Hopf-Galois extensions of the sphere spectrum, but  $MO$  is not.

- (b) The Eilenberg-McLane spectrum  $H\mathbb{Z}$  is the Thom spectrum associated to a loop map  $\Omega^2 S^3 \langle 3 \rangle \rightarrow BSF$  [65, 28, 12]. So  $S \rightarrow H\mathbb{Z}$  is a Hopf-Galois extension by theorem 6.1.
- (c) The canonical inclusion  $\mathbb{C}P^\infty = BU(1) \rightarrow BU$  can be extended to a loop map  $j: \Omega\Sigma CP^\infty \rightarrow BU$ . The associated bundle is denoted  $\xi$  and the Thom spectrum  $M\xi$  is the Thom spectrum associated with the map  $j: \Omega\Sigma CP^\infty \rightarrow BU$ , see [8]. The spectrum  $M\xi$  is hence an associative  $S$ -algebra. So  $S \rightarrow M\xi$  is a Hopf-Galois extension of associative  $S$ -algebras by theorem 6.1. It was shown in [8] that  $H_*(\Omega\Sigma CP^\infty)$  is isomorphic to the (non-commutative) algebra of non-symmetric functions. Hence also  $H_*(M\xi) \cong H_*(\Omega\Sigma CP^\infty)$  is not commutative and it follows that  $M\xi$  can not be a commutative  $S$ -algebra.
- (d) Several Thom spectra arise from systems of groups  $G_n \rightarrow O(n)$ . For instance let  $G_n$  be the braid group  $Br_n$ , the symmetric group  $\Sigma_n$ , the general linear group  $GL_n(\mathbb{Z})$  or the  $n$ -th group  $Br(C_n)$  from the Coxeter group  $C$ -series. There is also a chain of morphisms of groups

$$Br_n \longrightarrow \Sigma_n \longrightarrow GL_n(\mathbb{Z}) \longrightarrow O(n).$$

Each of the families comes with pairings  $G_n \times G_m \rightarrow G_{n+m}$  providing a loop map  $BG_\infty \rightarrow BO \rightarrow BGL_1 S$  [20, 27, 85, 13]. We denote the Thom spectrum associated with the system  $\{G_n\}$  by  $MG$ . It is an associative  $S$ -algebra. By [85]  $MBr(C)$  is a wedge of Eilenberg-McLane spectra  $H\mathbb{Z}/2$  and it was shown in [27] that  $MBr$  is a model for  $H\mathbb{Z}/2$ . So none of the Thom spectra  $MG$  for  $G$  as above is orientable along  $H\mathbb{Z}$ . By proposition 6.2 we obtain Hopf-Galois extensions  $S_{S\mathbb{Z}/2}^\wedge \rightarrow S_{S\mathbb{Z}/2}^\wedge \wedge MG$ .



# Chapter 7

## Regular Quotients

### 7.1 Statement of results

In this chapter we will show that under mild hypotheses regular quotients give rise to Hopf-Galois extensions in a weak sense. Our definition from chapter 5 is not met exactly as we do not know whether there is a Hopf algebra in the category of associative algebras as required by definition 5.7. Though the canonical morphisms from criterion 5.8 can be defined and are weak equivalences. This applies to some very famous maps of structured ring spectra including the maps from the completed Johnson-Wilson theories  $\widehat{E}(n)$  to the Morava  $K(n)$ -theories, the maps from the Lubin-Tate spectra  $E_n$  to  $K_n$ , and many others. We state the main theorem before explaining the details.

**Theorem 7.1.** *Let  $R$  be an even commutative  $S$ -algebra and  $I = \vec{x}$  a finite or infinite regular sequence in  $R_*$ . For every  $x_i \in I$  we arbitrarily fix a structure of an associative  $R$ -algebra on  $R/x_i$  thereby fixing such a structure on  $R/I$ . Then the quotient map  $R \rightarrow R/I$  is a weak Hopf-Galois extension if and only if  $R_*$  is  $I$ -adically complete.*

When the completion condition from the last theorem is not satisfied one can hope that an adapted statement holds. Let  $\widehat{R}_* = (R_*)_I^\wedge$  be the completion of  $R_*$  along  $I$  and let  $\widehat{L}_{R/I}^R R$  be Bousfield's nilpotent completion of  $R$  along  $R/I$  formed in  $R$ -modules, see section 7.4.

**Theorem 7.2.** *Under the same assumptions, define  $\widehat{R} := \widehat{L}_{R/I}^R R$  and assume that  $\widehat{R}_*/I \cong R_*/I$ . Then the induced morphism of  $S$ -algebras  $\widehat{R} \rightarrow R/I$  is a weak Hopf-Galois extension. In particular this is the case whenever  $I$  is generated by a finite regular sequence.*

Assuming that the algebraic condition holds, the statement says that  $R \rightarrow R/I$  respectively  $\widehat{R} \rightarrow R/I$  is a weak Hopf-Galois extension for every  $A_\infty$  structure on  $R/I$  when  $I = x$  has length one. However if  $I$  is not generated by a single element, there may be more multiplicative structures than those coming from multiplications on each  $R/x_i$  separately. These were called “mixed” multiplications in [4, 9.1]. The statements then refer to all  $A_\infty$  structures on  $R/I$  which are not mixed.

If  $I = x$  is a regular sequence of length one and  $x$  is of degree  $d$ , the Hopf algebra in a weak sense for the extension  $R \rightarrow R/x$  is given by the square zero extension  $R \vee \Sigma^{d+1}R$ . We look at this square zero extension as the suspension spectrum induced from the  $A_\infty$  space  $\mathbb{S}^0 \vee \mathbb{S}^{d+1} \cong \mathbb{S}_+^{d+1}$ . The coaction is defined using the Bockstein map  $\overline{\beta}_x: R/x \rightarrow \Sigma^{d+1}R/x$ , see (7.7). This map is a homotopy derivation and we will show that it lifts to a strict derivation. Any choice of such a strict derivation then defines a coaction of  $R \vee \Sigma^{d+1}R$  on  $R/x$  under  $R$  and the quotient map  $R \rightarrow R/x$  is weakly Hopf-Galois with respect to any such coaction. For general finite or infinite sequences  $I = (x_1, x_2, \dots)$  the weak Hopf-algebra is just the smash product of the weak Hopf algebras defined for each  $x_i$  and similarly for the coactions. The weak Hopf algebras and coactions for the extensions  $\widehat{R} \rightarrow R/I$  arise in the same way.

This way of constructing the coaction implies that we work with a multiplicative structure  $\Phi$  on  $R/I$  which is obtained from multiplications  $\Phi_i$  on  $R/x_i$ . It is hence clear that our theorem cannot refer to any of the “mixed products” on  $R/I$  that are not of this form.

Our theorems have the following consequences: Let  $p$  be an odd prime and recall the Johnson-Wilson spectra  $E(n)$  with  $\pi_*(E(n)) = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_{n-1}, v_n^\pm]$ . Let  $v_0 = p$  and  $I(n) = (v_0, \dots, v_{n-1})$ . The completed Johnson-Wilson spectrum  $R := \widehat{E(n)}$  is a commutative  $S$ -algebra with homotopy groups

$$\pi_* \widehat{E(n)} = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]_{I(n)}^\wedge.$$

We obtain the  $n$ -th Morava  $K$ -theory at  $p$  as the regular quotient  $K(n) = \widehat{E(n)}/I(n) = E(n)/I(n)$  (see e.g. [48, p.7]). Similarly for  $R := E_n$ , the even periodic Lubin-Tate-spectrum with

$$\pi_*(E_n) = \mathbb{W}(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$$

and  $I_n = (p, u_1, \dots, u_{n-1})$  we have  $K_n = E_n/I_n$ . Our theorems imply the following.

**Proposition 7.3.** *The quotient maps*

$$\begin{array}{ccc} \widehat{E(n)} & \longrightarrow & K(n) \\ \text{and} & & \\ E_n & \longrightarrow & K_n \end{array}$$

*are weak Hopf-Galois extensions.*

There are also interesting examples when the ideal  $I$  is infinite:

**Proposition 7.4.** *The quotient map*

$$MU \longrightarrow H\mathbb{Z}$$

*is a weak Hopf-Galois extension.*

Also recall the spectrum  $\widehat{L}_{K(n)}^{MU} MU$  from [75, 9.6]. Defining  $J_n$  to be the kernel of the map  $\pi_* MU_{(p)} \rightarrow \pi_* E(n)$ , the homotopy groups of  $\widehat{L}_{K(n)}^{MU} MU$  are given by  $\pi_* MU_{(p)}[v_n^{-1}]_{I_n+J_n}^\wedge$ . We have the following result:

**Proposition 7.5.** *The quotient map*

$$\widehat{L}_{K(n)}^{MU} MU \longrightarrow \widehat{L}_{K(n)}^{MU} MU/J_n \quad (7.1)$$

*is a weak Hopf-Galois extension.*

It was assumed in [75] that  $\widehat{L}_{K(n)}^{MU} MU/J_n$  equals  $\widehat{E(n)}$  but this is actually not true, see lemma 7.16. The composition  $\widehat{L}_{K(n)}^{MU} MU \rightarrow \widehat{L}_{K(n)}^{MU} MU/J_n \rightarrow \widehat{E(n)}$  plays an important role in John Rognes' work on Galois extensions as it is part of a splitting  $L_{K(n)} S \rightarrow \widehat{L}_{K(n)}^{MU} MU \rightarrow \widehat{E(n)} \rightarrow E_n$  that relates the chromatic filtration on  $S$ -modules to a chromatic filtration on  $MU$ -modules by the maps  $\widehat{L}_{K(n)}^{MU} MU \rightarrow \widehat{E(n)} \rightarrow E_n$  and  $L_{K(n)} S \rightarrow E_n$  with geometric content: The map  $\widehat{E(n)} \rightarrow E_n$  is a  $K(n)$ -local Galois extension,  $L_{K(n)} S \rightarrow E_n$  is a  $K(n)$ -local pro-Galois extension and  $\widehat{L}_{K(n)}^{MU} MU \rightarrow \widehat{E(n)}$  is a Henselian map. We think that this interpretation remains true despite of lemma 7.16 though at the moment we are not sure whether  $\widehat{L}_{K(n)}^{MU} MU$  is really a commutative  $S$ -algebra (this would follow from  $\widehat{L}_{K(n)}^{MU} MU/J_n = \widehat{E(n)}$ ). In this context the interpretation of (7.1) as a weak Hopf-Galois extension might be of interest.

## 7.2 Regular quotients revisited

Let  $R$  be a commutative  $S$ -algebra and  $x \in \pi_d R$ . We obtain a map

$$\Sigma^d R \cong \Sigma^d S \wedge R \xrightarrow{x \wedge R} R \wedge R \xrightarrow{\mu} R$$

of  $R$ -modules which we also denote by  $x$ . The cofiber of this map in the category of  $R$ -modules is denoted by  $R/x = R/(x)$ . This gives a diagram

$$\Sigma^d R \xrightarrow{x} R \xrightarrow{\rho_x} R/x \xrightarrow{\beta_x} \Sigma^{d+1} R. \quad (7.2)$$

If  $x \in \pi_d(R)$  is not a zero-divisor we have  $\pi_*(R/x) \cong R_*/(x)$ , so  $R/x$  realizes the algebraic quotient  $R_*/(x)$ . More generally, for a sequence  $I = (x_1, x_2, \dots)$  of elements  $x_i \in \pi_* R$  define  $R/I$  as the smash product  $\bigwedge_R R/x_i$ . Now suppose that  $R$  is even, i.e. that all odd-dimensional homotopy groups vanish, and suppose that  $I$  is a regular sequence in  $R_*$ . By this we mean that multiplication with  $x_i$  is injective on  $R_*/(x_1, \dots, x_{i-1})$ . In particular all the  $x_i$  are not zero-divisors and  $R/I$  realizes the algebraic quotient  $R_*/I$ . Recall from [81] that  $R/x$  and hence also  $R/I$  admits a homotopy associative multiplication. There may be several different multiplications on  $R/x$  and on  $R/I$  and moreover, there may also be ‘‘mixed’’ multiplications on  $R/I$  that are not obtained by smashing multiplications on the  $R/x_i$  together. However, independent of the

fact whether the multiplication is mixed or not, it can be rigidified to a strictly associative one making  $R/I$  an associative  $R$ -algebra ([81, 4]). There are a lot of examples for this construction and regular quotients occur at several places in the literature, see e.g. [81, 61, 62, 39, 36, 7, 4, 86, 87]. The following is a list of some spectra and their homotopy groups arising in this context.

$$\begin{aligned}
\pi_*(MU) &= \mathbb{Z}[x_1, x_2, \dots]; & |x_i| &= 2i; \\
\pi_*(BP) &= \mathbb{Z}_{(p)}[v_1, v_2, \dots]; & |v_i| &= 2(p^i - 1); \\
\pi_*(BP\langle n \rangle) &= \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n]; \\
\pi_*(k(n)) &= \mathbb{F}_p[v_n]; \\
\pi_*(E(n)) &= \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_{n-1}, v_n^\pm]; & (7.3) \\
\pi_*(K(n)) &= \mathbb{F}_p[v_n^\pm]; \\
\pi_*(P(n)) &= \mathbb{F}_p[v_n, v_{n+1}, v_{n+2}, \dots]; \\
\pi_*(E_n) &= \mathbb{W}(\mathbb{F}_{p^n})[[u_1, u_2, \dots, u_{n-1}]] [u^\pm]; & |u_i| &= 0; \quad |u| = -2; \\
\pi_*(K_n) &= \mathbb{F}_{p^n}[u^\pm]; \\
\pi_*(E_n^{nr}) &= \mathbb{W}(\overline{\mathbb{F}}_p)[[u_1, u_2, \dots, u_{n-1}]] [u^\pm]; \\
\pi_*(K_n^{nr}) &= \overline{\mathbb{F}}_p[u^\pm];
\end{aligned}$$

The following calculation was a motivating observation which inspired the search for theorem 7.1 as we will explain after the statement.

**Lemma 7.6.** *Let  $R$  be an even commutative  $S$ -algebra,  $I$  a regular sequence in  $R_*$  and  $R/I$  a quotient  $S$ -algebra as defined above. Then there is an isomorphism of groups*

$$\pi_*(R/I \wedge_R R/I) \cong \Lambda_{R_*/I}(e_i : i \geq 1)$$

where the  $e_i$  are elements associated to the  $x_i$  of degree  $|e_i| = d_i + 1 := |x_i| + 1$ . Here  $\Lambda_{R_*/I}$  denotes the exterior algebra on  $R_*/I$ .

This calculation can be found at different places in the literature, e.g. see [7, 5.1]. For us, the lemma serves only as a motivation and the statement can actually also be derived from the results of this chapter.

A necessary condition for  $R \rightarrow R/I$  being a (weak) Hopf-Galois extension hence is that  $\Lambda_{R_*/I}(e_i : i \geq 1)$  equals  $\pi_*(R/I \wedge_R H)$  for some (weak) Hopf algebra  $H$  under  $R$ . This would immediately follow from the existence of a canonical weak equivalence  $h: R/I \wedge_R R/I \rightarrow R/I \wedge_R H$ . If  $I = x$  and  $|x| = d$ , the square zero extension  $R/I \vee \Sigma^{d+1} R/I \cong R/I \wedge_R (R \vee R^{d+1})$  has the same homotopy groups as  $R/I \wedge_R R/I$ . We do not know whether  $R \vee R^{d+1}$  is a Hopf algebra in the category of associative  $R$ -algebras but it is a coalgebra or a Hopf algebra in a weak sense to be made precise. As an algebraic analogue note that for every commutative ring  $T$  the square zero extension  $T \oplus \Sigma^{d+1} T$  is the same as the exterior algebra  $\Lambda_T(e) = T[e]/e^2$  generated by an element  $e$  of degree  $d + 1$ . This exterior algebra is a Hopf algebra with structure maps  $\Delta(e) := 1 \otimes e + e \otimes 1$ ,  $\epsilon(e) := 0$  and  $\lambda(e) := -e$ . Similarly, for an exterior algebra on a set of generators we can define a Hopf algebra structure by similar structure maps.

### 7.3 Defining the Hopf algebra and the coaction

We make two preparatory statements on smash products of Hopf algebras and coactions that are rather clear in case we have a smash product with only finitely many factors. Briefly said, the next two lemmata imply that the case of a quotient  $R/x$  by a single element is a building block for the more general cases where  $I$  has length greater than one.

**Lemma 7.7.** *The smash product over a commutative  $S$ -algebra  $R$  of two Hopf-algebras under  $R$  is again a Hopf-algebra under  $R$  with factorwise defined maps. More generally, the statement is also true for an infinite smash product  $\bigwedge_i H_i$  of Hopf-algebras defined as the colimit of all finite smash-products (all smash products taken over  $R$ ).*

**Proof:** It suffices to prove the case  $R = S$ . It is easy to check the various identities, compare [82, III] for the algebraic analog in the finite case. For the infinite case, set  $A_j := \bigwedge_{i=1}^j H_i$  and use that  $(\operatorname{colim} A_j) \wedge (\operatorname{colim} A_j) \cong \operatorname{colim}(A_j \wedge_R A_j)$  (e.g. compare [36, p.138]).  $\square$

It will be convenient to have a name for  $S \vee S^k$  that reminds one of a Hopf algebra and so we write  $H(S, k) := S \vee S^k$ . More generally and according to the last lemma, for every  $S$ -algebra  $B$  and every finite or infinite sequence  $\vec{k} = (k_1, k_2, \dots)$  of non-negative integers, we have Hopf algebras

$$H(B, \vec{k}) := B \wedge \bigwedge_i H(S, k_i) = B \wedge (S \vee S^{k_1}) \wedge \dots \wedge (S \vee S^{k_n}) \wedge \dots \quad (7.4)$$

**Lemma 7.8.** *Let  $R$  be a commutative  $S$ -algebra and let  $H_i$  be Hopf algebras under  $R$  that coact on algebras  $R_i$  by maps  $\beta_i: R_i \rightarrow R_i \wedge_R H_i$ . Then the Hopf algebra  $\bigwedge_i H_i$  coacts on  $\bigwedge_i R_i$ .*

**Proof:** Again it suffices to prove the case  $R = S$ . For two factors define the coaction as  $\beta: R_1 \wedge R_2 \rightarrow (R_1 \wedge H_1) \wedge (R_2 \wedge H_2) \cong R_1 \wedge R_2 \wedge H_1 \wedge H_2$ . This is clearly a coassociative and counital map under  $S$ . By induction, the statement follows for all finite smash products. For infinite products the statement follows as above from the isomorphism

$$\operatorname{colim}_j \left( \bigwedge_{i=1}^j R_i \wedge \bigwedge_{i=1}^j H_i \right) \cong \operatorname{colim}_j \left( \bigwedge_{i=1}^j R_i \right) \wedge \operatorname{colim}_j \left( \bigwedge_{i=1}^j H_i \right).$$

$\square$

With these preparatory remarks we can come back to regular quotients. The calculation of  $\pi_*(R/x \wedge_R R/x)$  from lemma 7.6 suggests that for a quotient  $R/x$  generated by one element, we work with the weak Hopf algebra  $H(R, |x| + 1)$ . Now recall that for every sequence  $I = \vec{x}$  the quotient  $R/I$  is defined to be the smash product  $\bigwedge_i R/x_i$ . So by lemma 7.6 and 7.7 the corresponding weak Hopf algebra can just be the smash product of the weak Hopf algebras defined by the  $x_i$  separately. This is the algebra  $H(R, \vec{k})$  as defined in (7.4) where  $k_i = |x_i| + 1$ .

Our next aim is to define a coaction of  $H(R, \vec{k})$  on  $R/I$  under  $R$ , i.e. in particular a map  $R/I \rightarrow R/I \wedge_R H(R, \vec{k})$  of  $S$ -algebras under  $R$ . Such maps should be looked at as derivations, and we review some definitions and basic results to continue our investigation with this point of view.

### 7.3.1 Derivations and coactions

Let  $R$  be a commutative  $S$ -algebra which serves as our ground ring throughout this section and let  $A$  be an  $R$ -algebra. Following Lazarev [61] we define  $\Omega_A$  to be a cofibrant replacement in  $\mathcal{M}_{A \wedge_R A^{op}}$  of the homotopy fiber of the multiplication on  $A$  providing a homotopy fiber sequence of  $R$ -modules

$$\Omega_A \rightarrow A \wedge_R A \rightarrow A.$$

We then define the  $R$ -module of derivations from  $A$  into some  $A$ -bimodule  $M$  to be

$$\mathrm{Der}_R(A, M) := F_{A \wedge_R A^{op}}(\Omega_A, M).$$

Moreover, we call the elements of

$$\mathrm{Der}_R^{-k}(A, M) := \pi_k \mathrm{Der}_R(A, M)$$

*strict derivations from  $A$  to  $M$  of degree  $k$ .* The following theorem is very important for our purposes:

**Theorem 7.9.** ([61, 2.2]) *There is an isomorphism*

$$\mathrm{Der}_R^*(A, M) \cong \mathrm{Ho}\mathcal{A}_{R/A}^*(A, A \vee M).$$

There is a weaker notion of derivations which we define after Strickland [81]. We call these maps *homotopy derivations* to distinguish them from the derivations defined above. We define a homotopy derivation from  $A$  to  $M$  under  $R$  to be a map  $Q: A \rightarrow M$  of  $R$ -(bi)modules that satisfies

$$Q \circ \mu \simeq \mu \circ (1 \wedge Q + Q \wedge 1) \tag{7.5}$$

as maps  $A \wedge A \rightarrow M$  and where  $\mu$  is the multiplication on  $A$ . Define

$$\mathcal{D}\mathrm{er}_R^k := \{Q \in [A, \Sigma^k M]_{R\text{-bimod}} \mid Q \text{ is a homotopy derivation}\}.$$

Due to theorem 7.9 every strict derivation defines a homotopy derivation by neglect of structure. So there is a map

$$\mathrm{Der}_R^*(A, M) \longrightarrow \mathcal{D}\mathrm{er}_R^*(A, M), \tag{7.6}$$

see also [87, 3.3.]. With these preliminary remarks we can now explain how we define the coaction of  $H(R, \vec{k})$  on  $R/I$  under  $R$ . By view of lemma 7.8 it suffices again to define a coaction of  $H(R, k_i)$  on  $R/x_i$  under  $R$  for each  $i$  separately. We formulate a corollary to theorem 7.9 for this case.

**Corollary 7.10.** *Every strict  $R$ -module derivation from  $R/x$  to  $\Sigma^k R/x$  defines a coaction of  $H(R, k)$  on  $R/x$  under  $R$ .*

**Proof:** This follows immediately from theorem 7.9: Given a strict derivation as in the statement, we choose a map  $\beta \in \mathcal{A}_{R/(R/x)}(R/x, R/x \wedge_R H(R, k))$  that represents it. As  $\beta$  is a map over  $R/x$ , it is counital. The counit of the weak Hopf algebra  $H(R, k) = R \vee \Sigma^k R$  is defined to be the projection on the first summand. Moreover, its comultiplication comes from the space level diagonal on  $\mathbb{S}_+^k$ . So the fact that  $\beta$  is a map over  $R/x$  also implies that  $\beta$  is coassociative.  $\square$

We emphasize that the coaction  $\beta: R/x \rightarrow R/x \vee \Sigma^k R/x$  provides  $R/x \vee \Sigma^k R/x$  with the structure of an  $R/x$ -algebra. The coaction  $\beta$  is the unit map for this structure and is in general *not* the inclusion in the first factor. However the  $R$ -algebra structure on  $R/x \vee \Sigma^k R/x$  obtained by precomposing with the quotient map  $R \rightarrow R/x \rightarrow R/x \vee \Sigma^k R/x$  is the quotient map followed by the inclusion in the first factor since  $\beta$  is a map under  $R$ .

By corollary 7.10 we have hence reduced the definition of a coaction to finding strict derivations  $R/x_i \rightarrow \Sigma^{k_i} R/x_i$  for each  $i$ . We achieve this in two steps, first defining a homotopy derivation and then showing that it lifts to a strict derivation via the map (7.6). For this, we use the maps from diagram (7.2) and define the Bockstein maps

$$\overline{\beta}_x: R/x \xrightarrow{\beta_x} \Sigma^k R \xrightarrow{\Sigma^k \rho_x} \Sigma^k R/x, \quad (7.7)$$

again setting  $k = d + 1 = |x| + 1$ . It is known from [81, prop. 3.14] that these maps are homotopy derivations with respect to any multiplication on  $R/x$ .

**Proposition 7.11.** *The homotopy derivation  $\overline{\beta}_x$  can be lifted to a strict derivation.*

**Proof:** Recall that  $I = x$ , i.e.  $I$  is generated by a single element. In [81, 4.16] Strickland constructs a map  $\mathcal{D}er_R^*(R/I, R/I) \rightarrow \text{Hom}_{R_*}^{*-1}(I, R_*/I)$  that sends the derivation  $\overline{\beta}_x \in \mathcal{D}er_R^*(R/I, R/I)$  to the projection  $\Sigma I \simeq \Sigma^{d+1} R \rightarrow \Sigma^{d+1} R/x \simeq \Sigma I/I^2$ . The equivalences result from comparisons of cofiber sequences. Strickland's map passes to an isomorphism

$$\mathcal{D}er_R^*(R/I, R/I) \xrightarrow{\cong} \text{Hom}_{R_*}^{*-1}(I/I^2, I/I^2)$$

and it is clear that the map in  $\text{Hom}_{R_*}(I/I^2, I/I^2)$  associated to  $\overline{\beta}_x$  is the identity. It is shown in [87, 4.3] that the homotopy derivation  $\overline{\beta}_x$  (i.e.  $\delta_0$  in Wüthrich's notation) lifts to a strict derivation.  $\square$

We are now ready to define the weak Hopf algebras and coactions needed for theorem 7.1. We will do this now, prove theorem 7.1 and present some examples.

## 7.4 Quotients $R/I$ as weak Hopf-Galois extensions of $R$

As before let us be given a regular sequence  $I \subset R_*$ . For every  $x_i \in I$  we fix an  $A_\infty$  structure on  $R/x_i$  thereby also fixing such a structure on  $R/I$ .

**Definition 7.12.** For  $I = x$  let  $\beta(x): R/x \rightarrow R/x \wedge_R H(R, k)$  be a coaction of  $\underline{H}(R, k)$  on  $R/x$  under  $R$  defined by (choosing) a lift of the homotopy derivation  $\overline{\beta}_x$  to a strict derivation as stated in corollary 7.10. If  $I = \vec{x}$  let  $\beta(I)$  be the smash products of the maps  $\beta(x_i)$  so that  $\beta(I)$  defines a coaction of  $H(R, \vec{k})$  on  $R/I$  under  $R$  by lemma 7.8.

*Remark 7.13.* By construction the coactions are maps of  $S$ -algebras under  $R$  with respect to the  $A_\infty$  structures fixed at the beginning of this section. Although we allowed any of the multiplicative structures on the  $R/x_i$  there may be still multiplicative structures on  $R/I$  that do not come from multiplications on the factors  $R/x_i$  (as long as there is some homotopy in  $\pi_{k_i+k_j+2}R/I$ , see [4, p.70]). We have not constructed coactions with respect to these mixed multiplications. For this it would be desirable to apply theorem 7.9 directly to  $A = R/I$  equipped with the mixed multiplication. With our method this however is not possible for sequences  $I$  of length at least 2, as the weak Hopf algebra  $H(R, \vec{k})$  is only a square zero extension of  $R$  if  $I$  is of length one.

Now, as we have constructed weak Hopf algebras and coactions, we can ask whether the map  $R \rightarrow R/I$  is a weak Hopf-Galois extension in the sense that the two canonical morphisms from criterion 5.8 are weak equivalences. In other words we have to show that the canonical map  $h$  is a weak equivalence and that  $R$  is complete along  $R/I$ .

**Lemma 7.14.** *The canonical map*

$$h(I): R/I \wedge_R R/I \xrightarrow{1 \wedge_R \beta(I)} R/I \wedge_R R/I \wedge_R H(R, \vec{k}) \xrightarrow{\mu \wedge_R 1} R/I \wedge_R H(R, \vec{k})$$

*is a weak equivalence.*

**Proof:** We first prove the lemma for a sequence of length one. So let  $I = (x)$  and consider the following diagram where the vertical maps are cofiber sequences. The upper left square commutes, as  $\beta(x)$  is a map under  $R$ .

$$\begin{array}{ccccc} R/x \wedge_R R & \xrightarrow{\cong} & R/x \wedge S & \xrightarrow{\cong} & R/x \\ \downarrow R/x \wedge_R \rho_x & & \downarrow & & \downarrow \iota_1 \\ R/x \wedge_R R/x & \xrightarrow{h} & R/x \wedge H(S, k) & \xrightarrow{\cong} & R/x \vee \Sigma^k R/x \\ \downarrow R/x \wedge_R \beta_x & & \downarrow & & \downarrow pr_2 \\ R/x \wedge_R \Sigma^k R & \xrightarrow{\quad} & \Sigma^k R/x & \xlongequal{\quad} & \Sigma^k R/x \end{array}$$

We have to show that the map  $h = h(x) := (\mu \wedge H(S, k)) \circ (R/x \wedge_R \beta(x))$  is a weak equivalence. We therefore investigate the bottom half of the diagram. Note that  $\beta(x) \simeq \text{id} \vee \overline{\beta}_x$  as  $\beta(x)$  realizes the homotopy derivation  $\overline{\beta}_x$ . The bottom square hence extends to the following homotopy commutative diagram.

$$\begin{array}{ccc} R/x \wedge_R R/x & \xrightarrow{id \wedge_R (id \vee \overline{\beta}_x)} & R/x \wedge_R (R/x \vee \Sigma^k R/x) \xrightarrow{\mu} R/x \vee \Sigma^k R/x \\ \downarrow R/x \wedge_R \beta_x & & \downarrow pr_2 \\ R/x \wedge_R \Sigma^k R & \xrightarrow{\quad} & \Sigma^k R/x \end{array}$$



In order to prove that the upper map is a weak equivalence, we will investigate the lower map. Note that the right vertical map is the projection on the second summand as can be seen since the right cofiber sequence in the first diagram splits. Remember that  $\overline{\beta}_x$  is  $\rho_x \circ \beta_x$ . Hence it follows from the commutativity of the diagrams, that the lower map is  $\mu \circ (R/x \wedge_R \Sigma^k \rho_x)$ . But  $\rho_x$  is the  $R$ -algebra unit for  $R/x$ , hence the lower horizontal map and thus also  $h(x)$  is a weak equivalence.

In order to prove the lemma for general  $I$ , note that the multiplication on  $R/I$  is defined factorwise,  $H(S, \vec{k})$  is the product of the  $H(S, k_i)$  and also  $\beta(I)$  is the product of the  $\beta(x_i)$  up to switch maps  $\tau$ . Now checking that  $h(I)$  up to switch maps  $\tau$  is as well the product of the  $h(x_i)$  completes the proof.  $\square$

One might want to use the weak equivalence  $h$  in order to get information about the homotopy of  $R/I \wedge_R R/I$ . From lemma 7.14 one can easily read off the additive structure of  $\pi_*(R/I \wedge_R R/I)$ . We hence can obtain lemma 7.6 as a corollary. But note that  $h$  is not an algebra map unless the multiplication on  $R/I$  is commutative. Under the assumption that  $R/I$  is homotopy commutative, one could moreover read off the algebra structure of  $\pi_*(R/I \wedge_R (R/I)^{\text{op}})$  which then coincides with the one from  $\pi_*H(R; \vec{k}) \cong \Lambda_{R_*/I}(e_i : i \geq 1)$ , compare lemma 7.6.

We are now going to attack the criterion's second point. Recall again that the completion  $R_{R/I}^\wedge$  of  $R$  along  $R/I$  is equivalent to Bousfield's  $R/I$ -nilpotent completion  $\widehat{L}_{R/I}^R R$  of  $R$  formed in  $R$ -modules. Its homotopy groups are the abutment of the Adams-spectral sequence for  $R$  based on  $R/I$  constructed in  $R$ -modules and was investigated by Andrew Baker and Andrey Lazarev in [7], see [7, Thm 2.3] for the convergence statement. For an even commutative  $S$ -algebra  $R$  and a regular ideal  $I$ , Baker and Lazarev construct an internal  $I$ -adic tower

$$R/I \leftarrow R/I^2 \leftarrow R/I^3 \leftarrow \dots \tag{7.8}$$

that realizes the short exact sequences

$$0 \leftarrow R_*/(I_*)^s \leftarrow R_*/(I_*)^{s+1} \leftarrow (I_*)^s/(I_*)^{s+1} \leftarrow 0$$

in homotopy [7, thm. 5.9]. It is shown that the tower is an  $R/I$ -nilpotent resolution of  $R$  and hence has homotopy inverse limit weakly equivalent to  $\widehat{L}_{R/I}^R R$  [7, thm. 6.1, Rem. 2.2]. Furthermore one obtains

$$\pi_* R_{R/I}^\wedge \cong \widehat{L}_{R/I}^R R \cong \lim_s R_*/(I_*)^s$$

since the maps in the tower are surjective in homotopy and hence the  $\lim^1$ -term in the Milnor sequence vanishes. The right hand side is usually defined to be the completion of the ring  $R_*$  along the ideal  $I_*$ .

$$(R_*)_{I_*}^\wedge := \lim_s R_*/(I_*)^s \tag{7.9}$$

One says that  $R_*$  is  $I_*$ -adically complete if the canonical map  $R_* \rightarrow (R_*)_{I_*}^\wedge$  is an isomorphism. In particular we obtain the following algebraic criterion (compare [7, 6.3]).

**Criterion 7.15.** *There is an equivalence  $R \simeq R_{R/I}^\wedge$  if and only if  $R_* \cong (R_*)_{I_*}^\wedge$ . In particular, if  $R_*$  is  $I_*$ -adically complete, the map  $R \rightarrow R_{R/I}^\wedge$  is a weak equivalence and  $R$  is complete along  $R/I$ .*

**Proof:** This follows as  $\pi_* R_{R/I}^\wedge$  is the  $I_*$ -adical completion  $(R_*)_{I_*}^\wedge$ .  $\square$

**Proof of theorem 7.1:** Theorem 7.1 now follows from criterion 5.8 together with lemma 7.14 and criterion 7.15.  $\square$

## 7.5 Examples I

Let  $R$  be a commutative  $S$ -algebra and  $R/I$  a regular quotient. Due to the algebraic condition from theorem 7.1 we have to investigate the algebraic completion (7.9) in order to figure out whether  $R \rightarrow R/I$  is a weak Hopf-Galois extension. We restrict to situations that are of interest in the topological context. To begin with, assume that  $R_* = \mathbb{Z}[x_1, x_2, \dots]$  is a polynomial algebra with finitely or infinitely many generators and  $I = (x_1, x_2, \dots)$ . If all the  $x_i$  have degree 0, it is easy to see that

$$\widehat{R}_* := (R_*)_{I_*}^\wedge = \mathbb{Z}[[x_1, x_2, \dots]].$$

Note that if  $I$  is infinitely generated, in general  $\widehat{R}_*/I \neq R_*/I$ . If  $|x_i| = 0$  for all  $i$  an easy example for this is the element  $\sum_{i=0}^{\infty} x_i \in \mathbb{Z}[[x_1, x_2, \dots]]$  which is nonzero in  $\widehat{R}_*/I$  as the infinite sum is not in  $I \cdot \widehat{R}_*$ . Note that for this example it is crucial that all  $x_i$  are in degree 0. We will use gradings on the  $x_i$  to exclude these kinds of examples below. In general, if one wants to obtain  $R_*/I$  as a quotient of  $\widehat{R}_*$ , one has to quotient out the completed ideal  $\widehat{I}$  that is defined to be

$$\widehat{I} := \{(r_1, r_2, \dots) \in \widehat{R}_* \subset \prod R_*/I^s : r_1 = 0\}$$

and for which

$$\widehat{R}_*/\widehat{I} \cong R_*/I$$

always holds. Clearly, the condition  $\widehat{R}_*/I \cong R_*/I$  hence holds if  $I \cdot \widehat{R}_* = \widehat{I}$ . It is also easy to see that this is always true whenever  $I$  is finite.

If  $R_*$  is a graded ring, it may very well happen, that  $\widehat{R}_*/I = R_*/I$  even if  $I$  is infinitely generated. For this look at the example

$$R_* = MU_* = \mathbb{Z}[x_1, x_2, \dots]$$

with  $|x_i| = 2i$ . We set  $I = (x_1, x_2, \dots)$  as before. Then it is easy to see that  $\pi_n(MU/I^s) = \pi_n(MU)$  whenever  $2s > n$  as each element in  $I^s$  has at least degree  $2s$ . It follows that

$$MU_I^\wedge \simeq MU$$

and hence also  $(MU_*)_I^\wedge/I \cong MU_*/I$ . It follows that  $MU \rightarrow H\mathbb{Z}$  is a weak Hopf-Galois extension.

**Proof of proposition 7.4:** Clearly  $H\mathbb{Z} = MU/I$  for the regular ideal  $I = (x_1, x_2, \dots)$ . We have just shown that  $MU_*$  is complete along this ideal  $I$ . The lemma then follows from theorem 7.1.  $\square$

Now recall the spectrum  $\widehat{L}_{K(n)}^{MU}MU$  with homotopy groups  $\pi_*MU_{(p)}[v_n^{-1}]_{I_n+J_n}^\wedge$  where  $J_n$  is the kernel of the map  $\pi_*MU_{(p)} \rightarrow \pi_*E(n)$ . We can now show that the map  $\widehat{L}_{K(n)}^{MU}MU \rightarrow \widehat{L}_{K(n)}^{MU}MU/J_n$  is a weak Hopf-Galois extension.

**Proof of proposition 7.5:** Define  $R = \widehat{L}_{K(n)}^{MU}MU$ . The spectrum  $R$  is also given by the completion  $T_{J_n}^\wedge$  where  $T = MU_{(p)}[v_n^{-1}]_{I_n}^\wedge$  is a commutative  $S$ -algebra. Moreover,  $R$  is at least an associative  $T$ -algebra as the tower (7.8) can be realized as a tower of associative  $T$ -algebras [87]. The map  $R \rightarrow R/J_n$  in question is then obtained from the quotient  $T \rightarrow T/J_n$  by inducing up along the map  $T \rightarrow R$ . It follows that the canonical map  $h$  is a weak equivalence. One can check by hand that  $R_*$  is complete along  $R_*/J_n$ , see also [75, 9.6]. Hence  $R$  is complete along  $R/J_n$ . It then follows from criterion 5.8 that  $R \rightarrow R/I$  is a Hopf-Galois extension.  $\square$

At this place we should explain why  $R/J_n = \widehat{L}_{K(n)}^{MU}MU/J_n$  is not the same as  $\widehat{E(n)}$ .

**Lemma 7.16.** *The quotient  $\widehat{L}_{K(n)}^{MU}MU/J_n$  is different from  $\widehat{E(n)}$ .*

**Proof:** Set  $R := \pi_*\widehat{L}_{K(n)}^{MU}MU = \pi_*MU_{(p)}[v_n^{-1}]_{I_n+J_n}^\wedge$ . The regular ideal  $J_n$  contains the infinitely many elements  $v_k$  for  $k > n$ . The set of these  $v_k$  ( $k > n$ ) decomposes in classes depending on the congruence class of  $|v_k^k|$  modulo  $2(p^n - 1)$ . At least one of these classes contains infinitely many  $v_k$  and we let  $K$  be an infinite subset of the integers that indexes one of these classes. For  $k > n$  let  $\alpha_k$  be the unique integer such that  $0 \leq |v_n^{-\alpha_k}v_k^k| < 2(p^n - 1)$ . Then

$$x := \sum_{k \in K} v_n^{-\alpha_k}v_k^k$$

is an infinite sum of elements of the same degree and  $x$  is an element of  $R$ . The element  $x$  is not in  $J_n$  as the elements in  $J_n = J_n \cdot R$  are the *finite* sums over  $\sum j_i r_i$  where  $j_i \in J_n$ ,  $r_i \in R$ . The element  $x$  can not be written as such a finite sum as  $J_n$  and hence the infinite sequence of the  $v_k$  involved is regular. So  $x$  is nonzero in  $R/J_n$ . But its image in  $\widehat{E(n)}_*$  is zero.  $\square$

A similar construction shows that  $\pi_*\widehat{E(n)} = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]_{I(n)}^\wedge$  is different from  $\mathbb{Z}_p^\wedge[[v_1, \dots, v_{n-1}]][[v_n^{\pm 1}]]$ . The ring  $\mathbb{W}(\mathbb{F}_{p^n})[[u_1, u_2, \dots, u_{n-1}]][[u^\pm]] = \pi_*E_n$  however is complete along  $I_n$ . Comparing these two calculations once again shows that the different gradings of the  $v_k$  and the  $u_k$  are responsible and cause these different behaviours.

## 7.6 Quotients $R/I$ as weak Hopf-Galois extensions of $\widehat{R}$

Let  $\widehat{R} := \widehat{L}_{R/I}^R R$  be Bousfield's nilpotent completion of  $R$  as above. If  $R$  is not complete along  $R/I$  we still may be lucky and exhibit the canonical map  $\widehat{R} \rightarrow R/I$  as a Hopf-Galois extension. Note that Samuel Wüthrich has shown that the tower (7.8) can be realized as a tower of associative  $R$ -algebras [87]. Hence the map  $\widehat{R} \rightarrow R/I$  can be realized as a map of associative  $R$ -algebras as well. We do not know whether  $\widehat{R}$  is a commutative  $S$ -algebra in general. However we have the following result.

**Proposition 7.17.** *Assume that  $\widehat{R}_*/I \cong R_*/I$ . Then  $\widehat{R} \simeq L_{R/I}^R R$ , i.e. Bousfield's nilpotent completion equals localization. In particular,  $\widehat{R}$  can then be realized as a commutative  $S$ -algebra.*

**Proof:** For finite  $I$  the proposition is exactly [7, Thm 6.3]. Its proof carries over to our proposition which is just slightly more general. The statement that  $\widehat{R}$  can be realized as a commutative  $S$ -algebra follows as localization preserves commutative  $S$ -algebra structures ([36]).  $\square$

Under the assumption that  $\widehat{R}_*/I \cong R_*/I$  we can hence go through the quotient construction starting with  $\widehat{R}$  instead of  $R$  again, obtaining a map  $\widehat{R} \rightarrow \widehat{R}/I \simeq R/I$  which corresponds to the canonical map  $\widehat{R} \rightarrow R/I$  from the definition of  $\widehat{R}$  as a limit over the  $R/I^s$ . Moreover, in this situation there are isomorphisms  $\widehat{R}_*/I^s \cong R_*/I^s$  as  $\widehat{R}_*/I \cong R_*/I$  implies  $I \cdot \widehat{R}_* \cong \widehat{I}$  and hence  $I^s \cong \widehat{I}^s$ . Theorem 7.2 can hence be derived from theorem 7.1.

**Proof of theorem 7.2:** By Proposition 7.17,  $\widehat{R}$  can then be realized as a commutative  $S$ -algebra. As  $I$  is a regular sequence also in  $\widehat{R}$  we can apply theorem 7.1 to the quotient map  $\widehat{R} \rightarrow \widehat{R}/I$ . The only thing we have to check is that  $\widehat{R}_*$  is  $I$ -adically complete. But by assumption  $\widehat{R}_*/I \cong R_*/I$  and hence  $(\widehat{R}_*)_I^\wedge \cong \lim_s \widehat{R}_*/I^s \cong \lim_s R_*/I^s \cong \widehat{R}_*$  as we just explained. For finite sequences  $I$  we always have  $\widehat{R}_*/I \cong R_*/I$  as already mentioned in section 7.5 or likewise see [7]. This proves the last sentence of the theorem.  $\square$

In view of proposition 7.17 and theorem 7.2 it is interesting to give conditions under which  $\widehat{R}_*/I \cong R_*/I$ . As we mentioned above, this isomorphism always holds when  $I$  is finite. When  $I$  is infinite the next proposition gives a sufficient condition.

**Proposition 7.18.** *Let  $R_*$  be a graded ring concentrated in non negative degrees ( $R$  is connective) and let  $I = (x_1, x_2, \dots) \subset R_*$  be a (finite or infinite) regular sequence of elements of positive degree. Assume that for every  $N > 0$  there are only finitely many  $x_i$  such that  $|x_i| \leq N$ . Then  $I \cdot \widehat{R}_* = \widehat{I}$  and hence  $\widehat{R}_*/I = R_*/I$ .*

**Proof:** We have to show that  $\widehat{I} \subset I$ . For this let  $z = (0, r_2, r_3, \dots) \in \widehat{I}$ . We can write  $r_s = r_{s-1} + \tilde{r}_s$  with  $\tilde{r}_s \in I^{s-1}$ . Set  $N := |z|$  and let  $x_1, \dots, x_{k_N}$  be

the finitely many elements of  $I$  with degree  $\leq N$ . Each  $\tilde{r}_s$  can be written as  $\tilde{r}_s = \sum_{i=1}^{k_N} x_i \tilde{r}_{s,i}$  with  $\tilde{r}_{s,i} \in I^{s-2}$ . The elements  $z_i := x_i(0, \tilde{r}_{2,i}, \tilde{r}_{2,i} + \tilde{r}_{3,i}, \dots)$  are in  $I \cdot \widehat{R}_*$  and  $z = \sum_{i=1}^{k_N} z_i$ . Hence  $z \in I$ . This shows  $I \cdot \widehat{R}_* = \widehat{I}$  and hence  $\widehat{R}_*/I = R_*/I$ .  $\square$

**Proposition 7.19.** *Let  $R_*$  be a graded ring concentrated in non negative degrees and let  $I = (x_1, x_2, \dots) \subset R_*$  be a regular sequence of elements of positive degree. Assume that for every  $N > 0$  there are only finitely many  $x_i$  such that  $|x_i| \leq N$ . Then  $\widehat{R}_* = (\widehat{R}_*)_{I=\widehat{R}_*}^\wedge$ , i.e. completion of  $R_*$  along  $I$  is idempotent.*

**Proof:** We only have to show that  $\widehat{R}_*/I^s = R_*/I^s$  for all  $s \geq 1$ . Proposition 7.18 states that  $I \cdot \widehat{R}_* = \widehat{I}$  and hence  $\widehat{R}_*/I^s = \widehat{R}_*/\widehat{I}^s = R_*/I^s$ . The conclusion follows.  $\square$

## 7.7 Examples II

We can now prove that  $E_n \rightarrow K_n$  and  $\widehat{E(n)} \rightarrow K(n)$  are weak Hopf-Galois extensions.

**Proof of proposition 7.3:** First, recall that  $E_n$  admits the structure of a commutative  $S$ -algebra by Goerss-Hopkins theory [41]. One can check that  $\pi_*(E_n) = \mathbb{W}(\mathbb{F}_{p^n})[[u_1, u_2, \dots, u_{n-1}]][[u^\pm]]$  is  $I_n$ -adically complete where  $I_n = (p, u_1, \dots, u_{n-1})$ . Hence  $E_n \rightarrow K_n$  is a weak Hopf-Galois extension by theorem 7.1.

The spectrum  $\widehat{E(n)}$  admits the structure of a commutative  $S$ -algebra as follows e.g. from [75, 5.4.9]. It is complete along the ideal  $I(n) = (v_0, \dots, v_{n-1})$  as  $\pi_* \widehat{E(n)} = (E(n)_*)_{I(n)}^\wedge$  and  $I(n)$  is finite. Hence  $\widehat{E(n)} \rightarrow K(n)$  is a weak Hopf-Galois extension by theorem 7.1.  $\square$

Examples of regular quotients usually arise as regular quotients of the complex cobordism spectrum  $MU$  or of some localized or periodic version of it. It is easy to build some examples arising from the spectra listed in (7.3). E.g.  $MU_I^\wedge \rightarrow H\mathbb{Z}/p$  with  $I = (p, x_1, x_2, \dots)$  is a weak Hopf-Galois extension as can e.g. be seen using proposition 7.19. We have  $\pi_*(MU_I^\wedge) = \mathbb{Z}_p^\wedge[x_1, x_2, \dots]$ . However,  $MU_{(p)} \rightarrow H\mathbb{Z}/p$  is not Hopf-Galois even not in a weak sense as the completion condition is violated. The map  $MU_{(p)} \rightarrow H\mathbb{Z}_{(p)}$  is a weak Hopf-Galois extension where we look at  $H\mathbb{Z}_{(p)}$  as the regular quotient of  $MU_{(p)}$  by the ideal  $(x_1, x_2, \dots)$ . Moreover, this extension factors as

$$MU_{(p)} \rightarrow BP \rightarrow H\mathbb{Z}_{(p)} \quad (7.10)$$

and in fact all three maps are weak Hopf-Galois extensions. In general, if completeness conditions are satisfied, a regular quotient construction gives rise to a whole system of weak Hopf-Galois extensions as is stated in the following theorem.

**Theorem 7.20.** *Let  $R$  be an even commutative  $S$ -algebra,  $I + J$  a regular ideal in  $R_*$  and let  $R/I$ ,  $R/J$  and  $R/(I + J) = R/I \wedge_R R/J$  be regular quotients.*

Assume that  $R_*/I$  is complete with respect to the ideal  $J = (j_1, j_2, \dots)$ . Then  $R/I \rightarrow R/(I+J)$  is a weak Hopf-Galois extension with respect to the weak Hopf algebra  $H(R/I, \vec{k})$  where  $k_i = |j_i| + 1$ .

**Proof:** We look at the map  $R/I \rightarrow R/(I+J) = R/I \wedge_R R/J$  as an induced extension. Then the statement follows from corollary 5.14 together with criterion 7.15.  $\square$

The theorem applies to the tower (7.10) and so in particular  $BP \rightarrow H\mathbb{Z}_{(p)}$  is a weak Hopf-Galois extension. As another example we can split the weak Hopf-Galois extension  $\widehat{E}(n) \rightarrow K(n)$  as the sequence

$$\begin{aligned} \widehat{E}(n) &\longrightarrow \widehat{E}(n)/p &\longrightarrow \widehat{E}(n)/(p, v_0) &\longrightarrow \dots \\ & &\dots &\longrightarrow \widehat{E}(n)/(p, v_0, \dots, v_{n-1}) = K(n) \end{aligned}$$

where all maps and composites of maps in the tower are weak Hopf-Galois extensions.

Combining results from this and the last chapter, we also have the sequence

$$\underbrace{\overbrace{S \longrightarrow MU}^{S[BU]} \longrightarrow \overbrace{H\mathbb{Z}}^{H(MU, \vec{k})}}_{S[\Omega^2 S^3 \langle 3 \rangle]}$$

which exhibits  $S \rightarrow MU \rightarrow H\mathbb{Z}$  as part of a possible Hopf-Galois tower in which both (weak) Hopf-Galois extensions arising from Thom spectra and from regular quotients are involved. We can not say how or if the three Hopf algebras are related but note that  $H(MU, \vec{k}) \simeq MU[SU]$ , at least as modules, as follows from the cell structure of the space  $SU$ . Note that there are algebraic examples of Hopf-Galois extensions where the Hopf-algebra is not uniquely determined. So maybe in general one should not expect a too strong relation between the Hopf-algebras in a tower of extensions.

**Part III**  
**Applications**





## Chapter 8

# Realizing algebraic extensions

Let  $A_* \rightarrow B_*$  be a  $G$ -Galois extension of (graded) rings and  $A$  an  $S$ -algebra with  $\pi_* A \cong A_*$ . One can ask whether there is an  $S$ -algebra  $B$  with an action of  $G$  by maps of  $S$ -algebras under  $A$  such that  $A \rightarrow B$  is a Galois extension of associative  $S$ -algebras realizing the extension  $A_* \rightarrow B_*$  on homotopy groups. This question was originally raised by John Rognes. If  $A$  is a commutative  $S$ -algebra and  $B_*$  a commutative ring, Baker and Richter gave a positive answer to this question in [10]. We will show that there is still a positive answer in case  $B_*$  is a possibly associative ring. We keep the assumption that the ground ring  $A_*$  is commutative and moreover that  $A$  is a commutative  $S$ -algebra, see theorem 8.7. We closely follow [10] and prove theorem 8.7 in several steps. First, we show in section 8.1 that realizations of  $B_*$  and of the group action exist as a ring object and as maps in the homotopy category of  $A$ -ring spectra. We will then rigidify these in section 8.2 to objects and actions in the category of associative  $A$ -algebras using obstruction theories that go back to Robinson [73, 74] and Goerss-Hopkins [41, 40]. In our case, we have to rigidify an  $A$ -ring spectrum  $B$  which is not homotopy commutative but homotopy commutativity is assumed by both Robinson and Goerss-Hopkins. We show that in our situation the obstruction theories nevertheless work, respectively we refer to Angeltveit [5] for this. We also have to vary the original arguments to identify the obstruction groups using algebraic properties of the Galois extension  $A_* \rightarrow B_*$ . As an application, we realize certain cyclic and generalized quaternion algebras as associative  $S$ -algebras.

### 8.1 Realizing algebraic extensions up to homotopy

**Lemma 8.1.** *Let  $A$  be a commutative  $S$ -algebra and  $B_*$  a possibly associative Galois extension of  $A_*$  with finite Galois group  $G$ . Then  $B_*$  admits a realization as an  $A$ -bimodule.*

**Proof:** By lemma 1.3,  $B_*$  is a finitely generated projective left  $A_*$ -module. Hence  $B_*$  is a direct summand in a finitely generated free  $A_*$ -module. Hence it

is the image of an idempotent self-map

$$e_*: \bigoplus_{i=1}^n \Sigma^{m_i} A_* \longrightarrow \bigoplus_{i=1}^n \Sigma^{m_i} A_*$$

which is a left  $A_*$ -module map. Since  $\mathrm{Hom}_{A_*}(A_*, A_*) = A_* = [S, A] = [A, A]_A$ , we can model the map  $e$  by an  $A$ -module self-map on a suspension of copies of  $A$ . Iterating this map gives a sequence of left  $A$ -modules

$$\bigvee_{i=1}^n \Sigma^{m_i} A \longrightarrow \bigvee_{i=1}^n \Sigma^{m_i} A \longrightarrow \bigvee_{i=1}^n \Sigma^{m_i} A \cdots$$

We call the colimit of this sequence  $B$  as its homotopy is the colimit of the sequence induced in homotopy which is the image of  $e_*$ , i.e.  $B_*$ . The system is one of  $A$ -modules, so  $B$  is a left  $A$ -module and hence an  $A$ -bimodule as  $A$  is commutative.  $\square$

**Proposition 8.2** (weak realization theorem). *Let  $A$  be a commutative  $S$ -algebra and  $B_*$  a Galois extension of  $A_*$  with finite Galois group  $G$ . Then  $B$  admits a homotopy associative and unital multiplication  $B \wedge_A B \rightarrow B$ . In other words  $B_*$  admits a realization as an associative  $A$ -ring spectrum. Furthermore, there is an action of  $G$  on  $B$  by morphisms of  $A$ -ring spectra that induces the action of  $G$  on  $B_*$ .*

**Proof:** By lemma 8.1  $B_*$  admits a realization  $B$  as an  $A$ -bimodule. So we can form the product  $B \wedge_A B$  and calculate its homotopy as

$$\pi_*(B \wedge_A B) \cong B_* \otimes_{A_*} B_*. \quad (8.1)$$

This follows from the projectivity of  $B_*$  over  $A_*$  and the Künneth spectral sequence

$$\mathrm{Tor}^{A_*}(B_*, B_*) \implies \pi_*(B \wedge_A B).$$

More generally, we find inductively that  $B_*^{\otimes_{A_*} n}$  is also projective as (left and right)  $A_*$ -module (and also as  $B_*$ -module) using the splitting as  $A$ -bimodules from the proof of lemma 8.1. So by induction, the Künneth spectral sequence tells us that

$$\pi_* B^{\wedge_A n} \cong B_*^{\otimes_{A_*} n}. \quad (8.2)$$

We now use the universal coefficient spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_{A_*}^{p,q}(\pi_* B^{\wedge_A n}, B_*) \implies B_A^*(B^{\wedge_A n})$$

to see that  $B$  admits a multiplicative structure. The universal coefficient spectral sequence also collapses and

$$B_A^{-t}(B^{\wedge_A n}) \cong E_2^{0,t} \cong \mathrm{Hom}_{A_*}^t(B_*^{\wedge_A n}, B_*) \cong \mathrm{Hom}_{A_*}^t(\pi_* B^{\otimes_{A_*} n}, B_*). \quad (8.3)$$

In particular, there is a unit map  $A \rightarrow B$  and the homotopy classes of  $A$ -module maps  $B \wedge_A B \rightarrow B$  are in bijection with the elements in  $\mathrm{Hom}_{A_*}(B_* \otimes_{A_*} B_*, B_*)$ .

The multiplication on  $B_*$  is such a map and induces hence a multiplication on  $B$  over  $A$ . The bijection between algebraic and derived Hom-sets also proves that this multiplication is unital and associative up to homotopy. The  $G$ -action can be lifted analogously as  $[B, B]_A \cong \text{Hom}_{A_*}(B_*, B_*)$ .  $\square$

**Proposition 8.3** (uniqueness). *Let  $A$  be a commutative  $S$ -algebra and  $G$  a finite group acting on the  $A$ -ring spectra  $B$  and  $C$  such that both  $A_* \rightarrow B_*$  and  $A_* \rightarrow C_*$  are  $G$ -Galois extensions of associative rings. Assume furthermore that there is a  $G$ -equivariant ring isomorphism  $\Phi: B_* \rightarrow C_*$  under  $A_*$ . Then there is a map of  $A$ -ring spectra  $B \rightarrow C$  inducing  $\Phi$  which is  $G$ -equivariant up to homotopy.*

**Proof:** From the proof of the last proposition we know that the universal coefficient spectral sequence

$$\text{Ext}_{A_*}(B_*^{\otimes A_* n}, C_*^{\otimes A_* n}) \implies \pi_* F_A(B^{\wedge A n}, C^{\wedge A n})$$

collapses and gives  $[B^{\wedge A n}, C^{\wedge A n}]_A \cong \text{Hom}_{A_*}(B_*^{\otimes A_* n}, C_*^{\otimes A_* n})$ . In particular the isomorphism  $B_* \rightarrow C_*$  provides an  $A$ -module map  $B \rightarrow C$ . To see that it even provides a map of  $A$ -ring spectra, recall that the set of  $A_*$ -algebra maps is given by the equalizer

$$\text{Hom}_{A_*\text{-alg}}(B_*, C_*) \longrightarrow \text{Hom}_{A_*}(B_*, C_*) \rightrightarrows \text{Hom}_{A_*}(B_* \otimes_{A_*} B_*, C_*).$$

A similar description exists for the derived Hom-sets which shows that homotopy classes of maps of  $A$ -ring spectra correspond to  $A_*$ -algebra maps from  $B_*$  to  $C_*$ . Analogously, as  $G$  is finite, the  $G$ -equivariant algebra maps may be written as an equalizer showing that  $[B, C]_{A\text{-ring}}^G \cong \text{Hom}_{A_*\text{-alg}}(B_*, C_*)^G$ . As  $\Phi$  is an element of the latter, it has a realization as stated in the proposition.  $\square$

## 8.2 Strong realization

We saw in the last section that for a finite Galois extension of rings  $A_* \rightarrow B_*$  with  $A_*$  commutative and any commutative  $S$ -algebra  $A$  with  $\pi_* A \cong A_*$  there is a realization  $B$  of  $B_*$  as an  $A$ -ring spectrum and also the  $G$ -action can be realized by self- $A$ -ring-maps of  $B$ . We want to refine these realizations to realizations in the category of associative  $A$ -algebras.

For the  $A$ -algebra structure on  $B$  this can be done using Robinson's obstruction theory in the form presented by Angeltveit [5]. In order to apply Robinson directly one has to assume that  $B$  is a homotopy commutative ring spectrum, an assumption which is not satisfied if  $A_* \rightarrow B_*$  is a non-commutative Galois extension of a graded commutative ring  $A_*$ . To include these cases in our result we refer to Angeltveit's extended version of Robinson's obstruction theory where homotopy commutativity is not assumed.

For the realization of the  $G$ -action we have a similar situation. Knowing that  $B$  is an associative  $A$ -algebra we have to examine the space  $\mathcal{A}_A(B, B)$  of  $A$ -algebra self-maps of  $B$ . If  $B$  was a homotopy commutative  $A$ -ring spectrum, this could

be done by the Goerss-Hopkins-Miller spectral sequence [41, 40, 72]. But again we want to consider cases where  $B$  is not homotopy commutative. We will show that the approach of Goerss-Hopkins-Miller nevertheless works in case  $A_* \rightarrow B_*$  is a possibly non-commutative Galois extension of a commutative ring  $A_*$ . For instance the Goerss-Hopkins spectral sequence exists and collapses, showing that

$$\pi_i \mathcal{A}_A(B, B) \cong \begin{cases} \text{Hom}_{A_*\text{-alg}}(B_*, B_*) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases} \quad (8.4)$$

for cofibrant  $A$ -algebras  $B$ .

### 8.2.1 Rigidifying the ring-structure of the Galois extension

**Proposition 8.4.** *Let  $A$  be a commutative  $S$ -algebra and  $B$  an  $A$ -ring spectrum. Assume that  $B_A^{-t}(B^{\wedge A^s}) \cong \text{Hom}_{A_*}^t(B_*^{\otimes A_*^s}, B_*)$ . Then the obstructions for refining the ring spectrum structure on  $B$  to an  $A_\infty$ -structure under  $A$  lie in  $HH_{A_*}^{n, n-3}(B_*)$ . In particular, if  $A_* \rightarrow B_*$  is a finite Galois extension, then there is an associative  $A$ -algebra  $B$  realizing  $B_*$ .*

It was shown in [73, 74] that the obstructions for realizing an  $A$ -ring spectrum  $B$  as an associative  $S$ -algebra, lie in the Hochschild cohomology  $HH^*(\pi_*(B \wedge_A B)|_{B_*, B_*})$  provided that  $B$  is homotopy commutative and certain algebraic properties are satisfied (these properties are collected under the term “perfect universal coefficient formula”; the generalization from  $S$ - to  $A$ -ring spectra where  $A$  is any commutative  $S$ -algebra is just a formality). If one can identify  $\pi_*(B \wedge_A B)$  with  $B_* \otimes_{A_*} B_*$  as it is the case for a Galois extension  $A_* \rightarrow B_*$  by (8.1), there is an isomorphism  $HH^*(\pi_*(B \wedge_A B)|_{B_*, B_*}) \cong HH_{A_*}^*(B_*)$ . This isomorphism is due to an isomorphism of the defining cochain complexes and identifies the obstruction groups from proposition 8.4 with those from [73, 74] in case Robinson’s hypotheses are satisfied as well.

**Proof of proposition 8.4:** Note that as  $B$  is an  $A$ -ring spectrum, it comes with an  $A_3$ -structure. Analogously to [5] we define  $E_1^{s,t} := B^{-t}(B^{\wedge A^s})$ . This can be seen as a graded cosimplicial group [5, 3] and we let  $E_2^{*,*}$  be the homology of the associated graded cochain complex. For  $n \geq 4$  the obstruction to extending an  $A_{n-1}$ -structure on  $B$  to an  $A_n$ -structure lies in  $E_2^{n, n-3}$  by [5, thm 3.5]. Here, one allows the  $A_{n-1}$ -structure to vary while fixing the  $A_{n-2}$ -structure. By assumption  $E_1^{s,t} \cong \text{Hom}_{A_*}^t(B_*^{\otimes A_*^s}, B_*)$ . The cochain complex associated with  $E_1^{*,*}$  defines Hochschild cohomology so  $E_2^{n, n-3} \cong HH_{A_*}^{n, n-3}(B_*)$ . In case  $A_* \rightarrow B_*$  is a Galois extension this also follows from the remark after the statement of [5, thm 3.5].

It only remains to prove the last assertion, i.e. the one involving the Galois extension. We already know by proposition 8.2 that there is a realization of  $B_*$  as an  $A$ -ring spectrum  $B$  and by (8.3) the identification of  $B^*(B^{\wedge A^n})$  with the given Hom-set holds. It remains to check that the obstructions for refining this structure vanish. For this it suffices to show that the groups  $HH_{A_*}^{n, 3-n}(B_*)$  are zero for every  $n \geq 3$ . We outsource this statement to the next lemma.  $\square$

**Lemma 8.5.** *If  $A_* \rightarrow B_*$  is a Galois extension with  $A_*$  commutative then the Hochschild cohomology groups  $HH_{A_*}^n(B_*)$  vanish for every  $n \geq 1$ .*

**Proof:** As  $B_*$  is  $A_*$ -projective (lemma 1.3) it follows that  $HH_{A_*}^n(B_*) \cong \text{Ext}_{B_* \otimes_{A_*} B_*^{\text{op}}}^n(B_*, B_*)$ . The vanishing of  $\text{Ext}_{B_* \otimes_{A_*} B_*^{\text{op}}}^n(B_*, B_*)$  then follows from the separability of  $B_*$  over  $A_*$ , i.e. from the fact that  $B_*$  is  $B_* \otimes_{A_*} B_*^{\text{op}}$ -projective (lemma 1.3).  $\square$

### 8.2.2 Realizing the $G$ -action by maps of associative $A$ -algebras

In order to strictly realize the  $G$ -action we will now investigate  $\pi_* \mathcal{A}_A(B; B)$  under cofibrancy assumptions. This will turn out to be the abutment of a Goerss-Hopkins spectral sequence [41, 40]. For this we fix a cofibrant commutative  $S$ -algebra  $A$  which we consider as our ground ring throughout this section. Let  $C$  be a cofibrant  $A_\infty$ -operad and  $B$  a cofibrant  $C$ -algebra. Then  $\text{Alg}_C(B; B)$  has the homotopy type of the derived mapping space  $\text{map}_{A_\infty}(B, B)$  of associative  $A$ -algebra self maps of  $B$ . There is a simplicial associative  $A$ -algebra  $C^{\bullet+1}B$  and a cosimplicial space  $Y^\bullet$  given in codegree  $n$  by

$$Y^n := \text{Alg}_C(C^{n+1}B, B).$$

The cofibrancy assumptions ensure that  $|C^{\bullet+1}B| \rightarrow B$  and

$$\text{Alg}_C(B, B) \xrightarrow{\simeq} \text{Alg}_C(|C^{\bullet+1}B|, B) \simeq \text{Tot } Y^\bullet \quad (8.5)$$

are weak equivalences [72, §13,14]. We give some hints how (8.5) can also be derived from [40], using simplicial algebras over simplicial operads. For this write  $C_\bullet$  for  $C$  seen as a constant simplicial operad and note that  $C^{\bullet+1}B$  is the resolution of  $B$  associated with the triple defined by  $C$ . There is hence a weak equivalence  $\text{Alg}_C(B, B) \simeq \text{Alg}_{|C_\bullet|}(|C^{\bullet+1}B|, B)$ . Then there is an adjunction  $\text{Alg}_{|C_\bullet|}(|C^{\bullet+1}B|, B) \simeq \text{Alg}_{C_\bullet}(C^{\bullet+1}B, B^\Delta)$  [40, Thm 1.3.2] and again by adjunction this is  $\text{Tot}(\text{Alg}_C(C^{\bullet+1}B, B)) = \text{Tot } Y^\bullet$  establishing (8.5).

So in order to get our hands on the space  $\text{map}_{A_\infty}(B, B)$  we can investigate the homotopy spectral sequence associated with the cosimplicial space  $Y^\bullet$ . Suppose we have chosen a  $C$ -algebra map  $f: B \rightarrow B$ . This gives us a base-point in  $\text{Tot } Y^\bullet$  and also base-points  $C^{n+1}B \rightarrow B \xrightarrow{f} B$  in all the  $Y^n$  which we also denote by  $f$ . Vice versa we will see below that in our case every basepoint in  $Y^0$  that lifts to  $\text{Tot}^1 Y^\bullet$  also lifts to a basepoint in  $\text{Tot } Y^\bullet$ . So we can also start with a “weak basepoint”  $f \in \text{Tot}^1 Y^\bullet$ . The homotopy spectral sequence mentioned then has the form

$$E_2^{s,t} = \pi^s \pi_t(Y^\bullet, f) \implies \pi_{t-s}(\text{Tot } Y^\bullet, f). \quad (8.6)$$

As usual this spectral sequence is concentrated in the range  $t \geq s \geq 0$  and has differentials

$$d_r: E_r^{s,t} \rightarrow E_r^{s+r, t+r-1}.$$

Note that  $\pi_t(Y^n)$  is abelian for all  $t$  as the  $Y^n$  are loop spaces. So the cohomotopy terms  $E_2^{s,t} = \pi^s \pi_t(Y^\bullet, f)$  are all given as the cohomology of the

cochain complex associated to the cosimplicial object  $\pi_t(Y^\bullet)$ , see [18, X §7]. In particular  $E_2^{0,t}$  is an equalizer

$$E_2^{0,t} \longrightarrow \pi_t Y^0 \rightrightarrows \pi_t Y^1$$

and this holds for all  $t \geq 0$ .

**Proposition 8.6.** *Let  $A$  be a commutative  $S$ -algebra as above and let  $A \rightarrow B$  be a map of  $S$ -algebras such that  $B$  is cofibrant as an  $A$ -algebra and assume that  $A_* \rightarrow B_*$  is a Galois extension. Then the  $E_2$ -term of the homotopy spectral sequence (8.6) takes the form*

$$E_2^{s,t} \cong \begin{cases} \text{Hom}_{A_*\text{-alg}}(B_*, B_*) & \text{if } s = t = 0 \\ 0 & \text{else.} \end{cases} \quad (8.7)$$

Hence the spectral sequence converges and

$$\pi_i \text{map}_{A_\infty}(B, B) \cong \begin{cases} \text{Hom}_{A_*\text{-alg}}(B_*, B_*) & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases} \quad (8.8)$$

**Proof:** We closely follow the argument of [72, ch. 18] but some changes are necessary as in our case  $B$  is not homotopy commutative. First consider the case  $t = s = 0$ . We have to identify the left term in the equalizer diagram

$$E_2^{0,0} \longrightarrow \pi_0 Y^0 \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \end{array} \pi_0 Y^1$$

with  $\text{Hom}_{A_*\text{-alg}}(B_*, B_*)$ . Let  $T = T_{A_*}$  be the tensor algebra functor over  $A_*$  given by  $T(X_*) := \bigoplus_{n \geq 0} X_*^{\otimes_{A_*} n}$ . Note that there is a commutative diagram

$$\begin{array}{ccc} \pi_0 \text{Alg}_C(CX, B) & \longrightarrow & \text{Hom}_{A_*\text{-alg}}(TX_*, B_*) \\ \cong \downarrow & & \downarrow \cong \\ [X, B]_A & \longrightarrow & \text{Hom}_{A_*}(X_*, B_*) \end{array} \quad (8.9)$$

If  $X_*$  is a projective  $A_*$ -module, the lower horizontal map is an isomorphism as once again follows from the universal coefficient spectral sequence. As  $A_* \rightarrow B_*$  is a Galois extension this is the case for  $X = B$  or  $X = CB$  (and more generally for  $X = C^n B$ ) and we can hence identify  $\pi_0 Y^n = \pi_0 \text{Alg}_C(C^{n+1}B, B)$  with  $\text{Hom}_{A_*}(\pi_*(C^n B), B_*)$ . As  $A_* \rightarrow B_*$  is a Galois extension there is an isomorphism  $\pi_*(C^n B) \cong T^n(B_*)$  by equation (8.2). We can then identify  $E_2^{0,0}$  with the equalizer of

$$\text{Hom}_{A_*}(B_*, B_*) \rightrightarrows \text{Hom}_{A_*}(TB_*, B_*)$$

where the maps send a function  $f$  to  $T(f)$  followed by the operad action and to the operad action followed by  $f$ . So the equalizer clearly is

$$E_2^{0,0} \cong \text{Hom}_{A_*\text{-alg}}(B_*, B_*).$$

Now consider the case  $t > 0$ . We will first show that

$$\pi_t(Y^n, f) \cong \text{Hom}_{A_*\text{-alg}\downarrow B_*}(T^{\bullet+1}B_*, B_* \oplus B_{*+t}) \quad (8.10)$$

where  $f \in \text{Alg}_C(C^{n+1}B, B)$  is a basepoint,  $B_* \oplus B_{*+t}$  is a square-zero extension and the map  $T^{\bullet+1}B_* \rightarrow B_*$  is given by  $\pi_*f$ . First note that by adjunction we have

$$[\mathbb{S}^t, Y^n] = [\mathbb{S}^t, \text{Alg}_C(C^{n+1}B, B)] \quad (8.11)$$

$$\begin{aligned} &\cong [\mathbb{S}^t, A\text{-mod}(C^n B, B)] \\ &\cong [*, A\text{-mod}(C^n B, B^{\mathbb{S}^t})] \\ &\cong [*, A\text{-mod}(C^n B, B \vee \Sigma^{-t}B)] \end{aligned} \quad (8.12)$$

The last isomorphism is induced by an equivalence of  $A$ -algebras  $B^{\mathbb{S}^t} \simeq B \vee \Sigma^{-t}B$  where  $B \vee \Sigma^{-t}B$  is considered as a square-zero extension [63, 8.2.]. Restricting to those maps in 8.11 that fix the basepoint of  $\mathbb{S}^t$  corresponds to restricting to those maps in 8.12 that are over  $B$ , i.e. to  $A\text{-mod}\downarrow B(C^n B, B \vee \Sigma^{-t}B)$ . We conclude that

$$\begin{aligned} \pi_t(Y^n, f) &\cong [*, A\text{-mod}\downarrow B(C^n B, B \vee \Sigma^{-t}B)] \\ &\cong \pi_0 C\text{-alg}\downarrow B(C^{n+1}B, B \vee \Sigma^{-t}B). \end{aligned} \quad (8.13)$$

Analogously to (8.9) there is a commutative diagram

$$\begin{array}{ccc} \pi_0 \text{Alg}_C(CX, B \vee \Sigma^{-t}B) & \longrightarrow & \text{Hom}_{A_*\text{-alg}}(TX_*, B_* \oplus B_{*+t}) \\ \cong \downarrow & & \downarrow \cong \\ [X, B \vee \Sigma^{-t}B] & \longrightarrow & \text{Hom}_{A_*}(X_*, B_* \oplus B_{*+t}) \end{array}$$

and again the lower map is an isomorphism for  $X = C^n B$ . We can hence identify  $\pi_0 \text{Alg}_C(C^{n+1}B, B^{\mathbb{S}^t})$  with  $\text{Hom}_{A_*\text{-alg}}(T^{n+1}B_*, B_* \oplus B_{*+t})$ . Restricting to those maps that are over  $B$  or  $B_*$  respectively, equation (8.10) then follows from (8.13). It then follows that  $E_2^{s,t} = \pi^s \pi_t Y^\bullet \cong H^s \text{Hom}_{A_*\text{-alg}\downarrow B_*}(T^{\bullet+1}B_*, B_* \oplus B_{*+t})$  and

$$\begin{aligned} E_2^{s,t} &\cong H^s \text{Hom}_{A_*\text{-alg}\downarrow B_*}(T^{\bullet+1}B_*, B_* \oplus B_{*+t}) \\ &\cong H^s \text{Der}_{A_*}(T^{\bullet+1}B_*, B_{*+t}) \\ &\cong \text{Der}_{A_*}^s(B_*, B_{*+t}). \end{aligned} \quad (8.14)$$

The last isomorphism holds as  $\text{Der}_{A_*}(T^{\bullet+1}B_*, B_{*+t})$  is the only non-vanishing column of the  $E_1$ -term of a spectral sequence computing  $\text{Der}_{A_*}^s(B_*, B_{*+t})$  as explained in [72, 18.4, 18.5]. This spectral sequence is constructed starting with the double complex which is  $T^{p+q+1}B_*$  in dimension  $(p, q)$  and has  $E_1^{p,q} = \text{Der}_{A_*}^p(T^{q+1}B_*, B_{*+t})$ . All except the first column of this spectral sequence vanish as  $\text{Der}_{A_*}^p(TB_*, M) \cong \text{Ext}_{A_*}^p(B_*, M)$  which is zero for  $q \geq 1$  by projectivity of  $B_*$ . This shows the last isomorphism in (8.14). By the same projectivity argument the terms  $E_2^{s,t} = \text{Der}_{A_*}^s(B_*, B_{*+t})$  of our initial spectral sequence

vanish for  $s > 0$ . In particular, the vanishing of the terms  $E_s^{s,s-1}$  shows that there are no obstructions to lifting a base-point from  $\text{Tot}^{s-1}(Y^\bullet)$  to  $\text{Tot}^s(Y^\bullet)$  by [16, 5.2].

For  $s = 0$  the term  $E_2^{0,t}$  by definition is the equalizer

$$E_2^{0,t} \longrightarrow \pi_t Y^0 \rightrightarrows \pi_t Y^1$$

and by the argument just above we identified  $\pi_t(Y^n, f)$  with the set of homomorphisms  $\text{Hom}_{A_*\text{-alg}|_{B_*}}(T^{n+1}B_*, B_* \oplus B_{*+t})$ . In the same way as we already showed for  $E_2^{0,0}$ , the equalizer  $E_2^{0,t}$  is hence given by the equalizer of the two canonical maps

$$\text{Hom}_{A_*}(B_*, B_{*+t}) \rightrightarrows \text{Hom}_{A_*}(TB_*, B_{*+t}).$$

This equalizer only contains the zero map since the multiplication in  $B_{*+t}$  is trivial. All in all, the only non-vanishing term in the spectral sequence is  $E_2^{0,0}$  and we find that  $\pi_0 \text{Alg}_C(B, B) \cong \text{Hom}_{A_*\text{-alg}}(B_*, B_*)$  and all the components are contractible as there are no higher homotopy groups. This proves (8.7) and the spectral sequence collapses and hence converges to  $\pi_* \mathcal{A}_A(B, B)$ . This proves (8.8).  $\square$

**Theorem 8.7.** *Let  $A_* \rightarrow B_*$  be a  $G$ -Galois extension of rings with finite group  $G$  and  $A_*$  commutative. Let  $A$  be a cofibrant commutative  $S$ -algebra with  $\pi_* A = A_*$ . Then there exists a cofibrant associative  $A$ -algebra  $B$  acted on by  $G$  by maps of associative  $A$ -algebras. Moreover,  $A \rightarrow B$  is a  $G$ -Galois extension of associative  $S$ -algebras realizing the Galois extension  $A_* \rightarrow B_*$  on homotopy groups.  $\square$*

**Proof:** We know by proposition 8.4 that  $B_*$  has a realization as an associative  $A$ -algebra and we choose  $B$  to be a cofibrant  $A$ -algebra with this property. Proposition 8.6 applies and the isomorphism

$$\pi_0(\text{map}_{A_\infty}(B, B)) \cong \text{Hom}_{A_*\text{-alg}}(B_*, B_*)$$

shows that each group element  $g \in G$  seen as an element in the homomorphism set  $\text{Hom}_{A_*\text{-alg}}(B_*, B_*)$  can be realized by a self map of  $B$  in the category of associative  $A$ -algebras. Moreover, again by proposition 8.6 all the components of  $\text{map}_{A_\infty}(B, B)$  are contractible. This observation is important in order to make sure that the lifts of the group elements  $g$  to  $A$ -algebra self-maps of  $B$  can be chosen such that they assemble to an action of the group  $G$  at least on some cofibrant  $A$ -algebra weakly equivalent to  $B$ . In the language of [31, Def. 3.1] we have an  $h_\infty$ -diagram in the category of associative  $A$ -algebras indexed by our group  $G$  seen as a one-element category. In this case the homotopy  $G$ -action can be rigidified to an action by maps of associative  $A$ -algebras. We refer to the proof of the analogous statement for commutative  $S$ -algebras which can be found in [31, 3]. Hence we can choose  $B$  to be a cofibrant associative  $A$ -algebra with an action of  $G$  by maps of associative  $A$ -algebras realizing  $B_*$  with the given  $G$ -action.



It remains to show that  $A \rightarrow B$  is a Galois extension. For the fixed point condition, the homotopy fixed point spectral sequence shows that

$$\pi_* B^{hG} \cong \mathrm{Hom}_{A_*[G]}(A_*; B_*) \cong B_*^G = A_*.$$

Here we use the vanishing of the higher Ext-terms as in (4.2). Using the projectivity of  $B_*$  over  $A_*$  then shows that the Künneth spectral sequence converging to  $\pi_*(B \wedge_A B)$  collapses and

$$\pi_*(B \wedge_A B) \cong B_* \otimes_{A_*} B_* \cong B_*[G] \cong \pi_*(B[G]).$$

This completes the proof.  $\square$

**Example 8.8** (cyclic algebras). We can use the results from this chapter to prove the existence of certain  $S$ -algebras that realize cyclic algebras. Assume that  $A$  is a commutative  $S$ -algebra with homotopy groups  $A_* \cong R_*[x]/(x^n - u)$  where  $u$  is a unit in  $R_*$ . Such commutative  $S$ -algebras  $A$  exist, e.g. by the realization theorem for commutative Galois extensions, see [10]. Let  $T[x, \sigma, u]$  be a cyclic algebra over  $A_*$  which is a Galois extension of  $A_*$ , see example 1.8. Then by theorem 8.7 there is an associative  $S$ -algebra  $B$  with homotopy groups  $B_* \cong T[x, \sigma, u]$  and such that  $A \rightarrow B$  is a Galois extension of associative  $S$ -algebras.

**Example 8.9** (generalized quaternion algebras). Also recall the generalization of quaternion algebras

$$R_* \longrightarrow R_*[x, y; u, v]/(x^n - u, y^n - v, xy = \zeta_n yx) \quad (8.15)$$

which is a Galois extension and where  $\zeta_n \in R_*$  is an  $n$ -th primitive root of unity and  $u, v$  are central units in  $R_*$  of degree a multiple of  $2n$ . Assume that there is a commutative  $S$ -algebra  $A$  realizing  $R_*$ . Then by theorem 8.7 there is an associative  $S$ -algebra  $B$  realizing  $R_*[x, y; u, v]/(x^n - u, y^n - v, xy = \zeta_n yx)$  such that  $A \rightarrow B$  is a Galois extension.

To give a concrete example recall the completed Johnson-Wilson spectra  $\widehat{E}(n)$  defined at a prime  $p$  and choose some integer  $m$  such that  $m$  and  $p$  are coprime. Then there is a commutative  $S$ -algebra  $A$  with  $A_* \cong \widehat{E}(n)_*[\zeta_m]$ , e.g. by [10]. We write  $\widehat{E}(n)[\zeta_m]$  for  $A$ . Then there is a Galois extension of associative  $S$ -algebras  $\widehat{E}(n)[\zeta_m] \rightarrow B$  with

$$B_* \cong \widehat{E}(n)_*[\zeta_m, x, y]/(x^m - 1, y^m - 1, xy = \zeta_m yx).$$

We write  $\widehat{E}(n)[\zeta_m, x, y]/(x^m - 1, y^m - 1, xy = \zeta_m yx)$  for  $B$ . If  $m = p^i - 1$  the unit  $v_n \in \widehat{E}(n)_*$  has degree  $2m$ . Then  $\widehat{E}(n)_*[\zeta_m] \rightarrow \widehat{E}(n)_*[\zeta_m, x, y]/(x^m - v_n, y^m - v_n, xy = \zeta_m yx)$  is a Galois extension and there is an associative  $S$ -algebra  $\widehat{E}(n)[\zeta_m, x, y]/(x^m - v_n, y^m - v_n, xy = \zeta_m yx)$  and a Galois extension

$$\widehat{E}(n)[\zeta_m] \longrightarrow \widehat{E}(n)[\zeta_m, x, y]/(x^m - v_n, y^m - v_n, xy = \zeta_m yx)$$

of associative  $S$ -algebras where the homotopy groups are what the notation suggests.



## Chapter 9

# Invertibility, Picard groups and Morita context

If  $A \rightarrow B$  is a Galois-extension of commutative rings with finite abelian Galois-group  $G$  then  $B$  defines an element in the Picard group  $\text{Pic}(A[G])$  of  $A[G]$ , see e.g. [42]. The corresponding statement for Galois extensions of commutative  $S$ -algebras has been proved in [10, 75]. We extend these result to associative Galois extensions with finite abelian Galois group. The first part of this chapter is purely algebraic and serves as a blueprint for the second part which deals with the corresponding topological statements. Without assuming that  $G$  is abelian we still can prove that  $B$  is an invertible  $(A[G], \text{Hom}_{A[G]}(B, B)^{\text{op}})$ -bimodule and so defines a Morita equivalence. The corresponding topological result holds as well.

### 9.1 Invertibility, Picard groups and Morita context in algebra

#### 9.1.1 Invertible bimodules

Let  $\Lambda$  and  $\Delta$  be rings and recall that a  $(\Lambda, \Delta)$ -bimodule  $M$  is invertible if there is a  $(\Delta, \Lambda)$ -bimodule  $N$  with isomorphisms of bimodules

$$M \otimes_{\Delta} N \cong \Lambda; \quad N \otimes_{\Lambda} M \cong \Delta. \quad (9.1)$$

In this case,  $\Delta$  is necessarily isomorphic to  $\text{Hom}_{\Lambda}(M, M)^{\text{op}}$  as a ring, compare [71, Thm. 16.14] for this and the following statements. Note that the multiplication in  $\text{Hom}_{\Lambda}(M, M)$  seen as set of *left*  $\Lambda$ -module homomorphisms is given by  $f \cdot \tilde{f} := \tilde{f} \circ f$ . As a ring,  $\Delta$  is isomorphic to this Hom-set with the opposite multiplication and  $M$  is a right  $\text{Hom}_{\Lambda}(M, M)^{\text{op}}$ -module via the evaluation map. There is also an isomorphism  $N \cong \text{Hom}_{\Lambda}(M, \Lambda)$  of  $(\Delta, \Lambda)$ -bimodules where the bimodule structure of  $\text{Hom}_{\Lambda}(M, \Lambda)$  is given via the structures of its first and second argument, i.e.  $N$  is isomorphic to the  $\Lambda$ -dual of  $M$ .

Moreover, the isomorphisms in (9.1) are given by evaluation and the canonical

map  $\nu$  respectively,

$$\text{eval} : M \otimes_{\text{Hom}_\Lambda(M, M)^{\text{op}}} \text{Hom}_\Lambda(M, \Lambda) \longrightarrow \Lambda \quad (9.2)$$

$$\nu : \text{Hom}_\Lambda(M, \Lambda) \otimes_\Lambda M \longrightarrow \text{Hom}_\Lambda(M, M), \quad (9.3)$$

where  $[\nu(f \otimes_\Lambda m)](\tilde{m}) = f(\tilde{m})m$ . This is a map of  $\text{Hom}_\Lambda(M, M)^{\text{op}}$ -bimodules where the left and right  $\text{Hom}_\Lambda(M, M)^{\text{op}}$ -module structures on  $\text{Hom}_\Lambda(M, M)$  are given via the action on the first and second argument. Clearly (9.2) is a map of  $\Lambda$ -bimodules. There are criteria, when these maps are isomorphisms as we will recall now. The next lemma is analogous to [71, Thm. 16.7] and follows together with [71, Thm. 16.14]. Note that by  $\text{Hom}_R$  we always denotes morphisms of *left*  $R$ -modules whereas in [71] the same denotes *right* module morphisms. We choose this presentation as it is better adapted to the topological parts of this thesis.

**Lemma 9.1.** *The evaluation map (9.2) is an isomorphism if and only if  $M$  is finitely generated projective as a left  $\Lambda$ -module. The map  $\nu$  from (9.3) is an isomorphism if and only if  $\Sigma_{f \in \text{Hom}_\Lambda(M, \Lambda)} f(M) = \Lambda$ .*

The term  $\Sigma_{f \in \text{Hom}_\Lambda(M, \Lambda)} f(M)$  is called the *trace ideal* of  $M$ . The trace ideal is isomorphic to  $\Lambda$  if and only if  $M$  is a generator of the category of left  $\Lambda$ -modules by [71, Cor. 15.5]. An  $M$  satisfying both conditions from the last lemma is called a *progenerator* for the category of left  $\Lambda$ -modules.

### 9.1.2 Invertible bimodules and Galois extensions

We want to verify the conditions from lemma 9.1 for certain  $G$ -Galois extensions  $A \rightarrow B$  of rings. Hence we set  $M = B$ ,  $\Lambda = A[G]$ ,  $N := \text{Hom}_{A[G]}(B, A[G])$  and  $\Delta := \text{Hom}_{A[G]}(B, B)$ . Recall that if the trace is surjective, the projectivity condition in the last lemma is satisfied by lemma 1.3. Here is a criterion implying that  $B$  is a generator for the category of left  $A[G]$ -modules:

**Proposition 9.2.** *Let  $A \rightarrow B$  be a Galois-extension of rings with finite Galois group  $G$ . Assume that  $\text{tr}: B \rightarrow A$  is surjective. Then*

$$\Sigma_{f \in \text{Hom}_{A[G]}(B, A[G])} f(B) = A[G].$$

Note that by lemma 4.7 the surjectivity of the trace implies the faithfulness of the extension  $A \rightarrow B$ . Moreover, Galois extensions  $A \rightarrow B$  with  $A$  commutative are always faithful and hence have surjective trace, see lemma 1.3.

**Proof of proposition 9.2:** As the canonical map  $h: B \otimes_A B \rightarrow B[G]$  is an isomorphism also the map  $\tilde{h}: B \otimes_A B \rightarrow B[G]$  sending  $b \otimes_A \tilde{b}$  to  $(g \mapsto g(b) \cdot \tilde{b})$  is an isomorphism. This is analogous to lemma 4.12.1 in the topological situation. As  $\text{tr}(B) = A$ , there is a surjection

$$\text{tr} \circ \tilde{h}: B \otimes_A B \rightarrow B[G] \rightarrow A[G] \quad (9.4)$$

of left  $A[G]$ -modules using the  $G$  action on the first  $B$  factor in the source. Hence for every  $\tilde{b} \in B$  we obtain a map in  $\text{Hom}_{A[G]}(B, A[G])$  by precomposing

$tr \circ \tilde{h}$  with  $(-) \otimes_A \tilde{b}$ . If  $b \otimes_A \tilde{b}$  maps to  $x \in A[G]$ , we hence constructed a morphism in  $\text{Hom}_{A[G]}(B, A[G])$  with  $x$  in its image. The surjectivity of (9.4) hence proves the claim.  $\square$

We obtain the following theorem.

**Theorem 9.3.** *Let  $A \rightarrow B$  be a Galois-extension of rings with finite Galois group  $G$ . Assume that the trace  $tr: B \rightarrow A$  is surjective. Then  $B$  is an invertible  $(A[G], \text{Hom}_{A[G]}(B, B)^{\text{op}})$ -bimodule.*

**Proof:** By lemma 9.1 we have to check that  $B$  is finitely generated projective as a left  $A[G]$ -module and that the trace ideal of  $B$  equals  $A[G]$ . The former is stated in lemma 1.3 and the latter is proposition 9.2.  $\square$

*Remark 9.4* (Morita equivalence). Proposition 9.3 can be reformulated as follows. The  $(A[G], \text{Hom}_{A[G]}(B, B)^{\text{op}})$ -bimodule  $B$  defines a Morita context such that  $A[G]$  and  $\text{Hom}_{A[G]}(B, B)^{\text{op}}$  are Morita-equivalent whenever  $A \rightarrow B$  is a finite Galois extension with surjective trace, compare [71, Cor. 16.9].

### 9.1.3 Picard groups

If  $\Delta = \Lambda$ , the invertible  $(\Lambda, \Delta)$ -bimodules define a group  $\text{Pic}(\Lambda)$ , the Picard group of  $\Lambda$ . Of course, in the situation of theorem 9.3, the interesting question is, whether there is an isomorphism of rings  $\text{Hom}_{A[G]}(B, B)^{\text{op}} \cong A[G]$ . We have the following result:

**Proposition 9.5.** *Let  $A \rightarrow B$  be a Galois-extension of rings with abelian Galois group  $G$ . Then  $A[G] \cong \text{Hom}_{A[G]}(B, B)^{\text{op}}$ .*

**Proof:** As in the topological situation (section 4.4) we define a ring  $B[G]$  that is  $B[G]$  as an abelian group but with multiplication  $b[g] \cdot \tilde{b}[\tilde{g}] := \tilde{g}^{-1}(b)\tilde{b}[g\tilde{g}]$ . We can then define a morphism

$$\tilde{j}: B[G] \rightarrow \text{Hom}_A(B, B)^{\text{op}} \tag{9.5}$$

mapping  $b[g]$  to  $x \mapsto g^{-1}(x) \cdot b$ . This is a morphism of rings with the multiplication in  $\text{Hom}_A(B, B)$  given by  $f \cdot \tilde{f} := \tilde{f} \circ f$ . We show in the next lemma that  $\tilde{j}$  is an isomorphism. Furthermore the map  $\tilde{j}$  is  $G$ -equivariant: On the source, consider the  $G$ -action on  $B$  and on the target the  $G$ -action defined by  $[h, f](\tilde{b}) := h(f(h^{-1}(\tilde{b})))$ . Then  $[h, \tilde{j}(b, g)]$  maps  $\tilde{b}$  to  $h((g^{-1}h^{-1}(\tilde{b}) \cdot b)) = hg^{-1}h^{-1}(\tilde{b}) \cdot h(b)$ . This shows that  $\tilde{j}$  is  $G$ -equivariant if and only if  $hgh^{-1} = g$  for all  $g, h \in G$ , i.e. if and only if  $G$  is abelian.

If  $G$  is abelian, the isomorphism  $\tilde{j}$  restricts to an isomorphism on  $G$ -fixed points and this is a morphism  $A[G] = B[G]^G \rightarrow \text{Hom}_A(B, B)^G = \text{Hom}_{A[G]}(B, B)$ .  $\square$

**Lemma 9.6.** *If  $A \rightarrow B$  is  $G$ -Galois then the map  $\tilde{j}$  from equation (9.5) is an isomorphism.*

**Proof:** As in the proof of lemma 1.3 we choose elements  $x_i, y_i \in B$  such that  $\sum_i \sigma(x_i) \cdot y_i = \delta_{\sigma, e}$  and also we define left  $A$ -module maps  $\psi_i: B \rightarrow A$  by

$\psi_i(t) := \text{tr}(t \cdot x_i)$ . As in the proof of lemma 1.3 we see that

$$\sum_i \psi_i(b) y_i = b \quad (9.6)$$

for all  $b \in B$ . We first show that  $\tilde{j}$  is onto. For  $u \in \text{Hom}_A(B, B)$  define  $z := \sum_{g,i} g(x_i) \cdot u(y_i) [g^{-1}] \in B[G]$ . Due to equation (9.6) It suffices to show that  $\tilde{j}(z)(y_k) = u(y_k)$  for all  $k$ . We calculate:

$$\begin{aligned} \tilde{j}(z)(y_k) &= \tilde{j} \left( \sum_{g,i} g(x_i) \cdot u(y_i) [g^{-1}] \right) (y_k) \\ &= \sum_{g,i} g(y_k) g(x_i) \cdot u(y_i) \\ &= \sum_i \psi_i(y_k) u(y_i) \\ &= u \left( \sum_i \psi_i(y_k) y_i \right) \\ &= u(y_k) \end{aligned}$$

For the injectivity of  $\tilde{j}$  note that with  $v := \sum b_g [g] \in B[G]$  we have

$$\sum_{\sigma,i} \sigma(x_i) [\tilde{j}(v)(y_i)] [\sigma] = \sum_{\sigma,g,i} \sigma(x_i) g(y_i) b_g [\sigma] = \sum_g b_g [g] = v$$

as  $\sum_i \sigma(x_i) g(y_i) = \delta_{\sigma,g}$ . □

If the Galois group  $G$  is abelian we can hence reinterpret theorem 9.3:

**Theorem 9.7.** *Let  $A \rightarrow B$  be a Galois-extension of rings with finite abelian Galois group  $G$  and assume that the trace  $\text{tr}: B \rightarrow A$  is surjective. Then  $B$  is an element in the Picard group  $\text{Pic}(A[G])$ .*

**Proof:** This follows from theorem 9.3 and proposition 9.5. □

## 9.2 Invertibility, Picard groups and Morita context in topology

We come back to topology and present the notions of invertibility and Picard group in this context. They are defined in the derived category. We will obtain analogs of theorem 9.3 and 9.7.

### 9.2.1 Invertible bimodules

Let  $R, \hat{R}$  be associative  $S$ -algebras. We say that an  $(R, \hat{R})$ -bimodule  $M$  is invertible if there is an  $(\hat{R}, R)$ -bimodule  $N$  such that

$$M \wedge_{\hat{R}}^L N \simeq R \quad \text{and} \quad N \wedge_R^L M \simeq \hat{R} \quad (9.7)$$

as  $R$ - respectively  $\widehat{R}$ -bimodules. In this case there are retractions  $\mathcal{D}_{\widehat{R}}(X, Y) \rightarrow \mathcal{D}_R(M \wedge_{\widehat{R}}^L X, M \wedge_{\widehat{R}}^L Y) \rightarrow \mathcal{D}_{\widehat{R}}(N \wedge_R^L M \wedge_{\widehat{R}}^L X, N \wedge_R^L M \wedge_{\widehat{R}}^L Y) \cong \mathcal{D}_{\widehat{R}}(X, Y)$  and  $\mathcal{D}_R(M \wedge_{\widehat{R}}^L X, M \wedge_{\widehat{R}}^L Y) \rightarrow \mathcal{D}_{\widehat{R}}(X, Y) \rightarrow \mathcal{D}_R(M \wedge_{\widehat{R}}^L X, M \wedge_{\widehat{R}}^L Y)$  so both  $M \wedge_{\widehat{R}}^L -$  and  $N \wedge_R^L -$  define equivalences of derived categories. We see that  $N$  is equivalent to the derived  $R$ -dual  $\widetilde{F}_R(M, R) := F_R^R(M, R)$  of  $M$ ,

$$N \simeq \widetilde{F}_R(M, R) \quad (9.8)$$

as  $\mathcal{D}_{\widehat{R}}(X, N) \cong \mathcal{D}_R(M \wedge_{\widehat{R}}^L X, M \wedge_{\widehat{R}}^L N) \cong \mathcal{D}_R(M \wedge_{\widehat{R}}^L X, R) \cong \mathcal{D}_{\widehat{R}}(X, \widetilde{F}_R(M, R))$  for any left  $\widehat{R}$ -module  $X$ . We think of  $\widetilde{F}_R(M; R)$  or more generally  $\widetilde{F}_R(M, Y)$  as being represented by  $F_R(\widetilde{M}, Y)$  where  $\widetilde{M}$  is a cofibrant replacement of  $M$  in the category of left  $R$ -modules. Also there is an equivalence  $\widetilde{F}_R(M, Y) \simeq \widetilde{F}_R(M, R) \wedge_R^L Y$  for every left  $R$ -module  $Y$  as

$$\begin{aligned} \mathcal{D}_{\widehat{R}}(X, \widetilde{F}_R(M, Y)) &\cong \mathcal{D}_R(M \wedge_{\widehat{R}}^L X, Y) \\ &\cong \mathcal{D}_{\widehat{R}}(N \wedge_R^L M \wedge_{\widehat{R}}^L X, N \wedge_R^L Y) \\ &\cong \mathcal{D}_{\widehat{R}}(X, \widetilde{F}_R(M, R) \wedge_R^L Y). \end{aligned}$$

Hence we conclude that  $\widehat{R} \simeq N \wedge_R^L M \simeq \widetilde{F}_R(M, R) \wedge_R^L M \simeq \widetilde{F}_R^L(M, M)$ , i.e.

$$\widehat{R} \simeq \widetilde{F}_R(M, M). \quad (9.9)$$

Analogously to the algebraic situation, the equivalences from equation (9.7) lead us to look at the morphisms

$$\begin{aligned} \text{eval: } \quad &\widetilde{M} \wedge_{F_R(\widetilde{M}, \widetilde{M})^{\text{op}}}^L F_R(\widetilde{M}, R) \longrightarrow R \\ \text{and } \quad &\nu: \quad \widetilde{F}_R(M, R) \wedge_R^L M \longrightarrow \widetilde{F}_R(M, M)^{\text{op}}. \end{aligned}$$

where  $\widetilde{M}$  is a cofibrant replacement of  $M$  in the category of left  $R$ -modules. Analogously to the algebraic situation, the right  $F_R(\widetilde{M}, \widetilde{M})^{\text{op}}$ -module structure on  $\widetilde{M}$  is given by evaluation making  $\widetilde{M}$  and  $(R, F_R(\widetilde{M}, \widetilde{M})^{\text{op}})$ -bimodule. Then  $F_R(\widetilde{M}, R)$  is an  $(F_R(\widetilde{M}, \widetilde{M})^{\text{op}}, R)$ -bimodule via the right  $F_R(\widetilde{M}, \widetilde{M})^{\text{op}}$ -module structure on  $\widetilde{M}$  and the right  $R$ -module structure on  $R$ .

## 9.2.2 Invertible bimodules and Galois extensions

In analogy to proposition 9.2 we have the following.

**Proposition 9.8.** *Let  $A \rightarrow B$  be a Galois-extension of associative  $S$ -algebras with finite Galois group  $G$ . Assume that  $A \rightarrow B$  is faithful. Then the canonical map*

$$\nu: F_{A[G]}(B, A[G]) \wedge_{A[G]}^L B \longrightarrow F_{A[G]}(B, B)^{\text{op}}$$

*is a weak equivalence.*

Before proving this proposition we want to note that the function spectra involved actually represent the derived function spectra.

**Lemma 9.9.** *Let  $A \rightarrow B$  be a faithful Galois extension of associative  $S$ -algebras with finite Galois group  $G$ . Let  $\tilde{B} \rightarrow B$  be a cofibrant replacement of  $B$  as a left  $A[G]$ -module. Then there are equivalences*

$$\begin{aligned} F_{A[G]}(B, B) &\longrightarrow F_{A[G]}(\tilde{B}, B) \\ F_{A[G]}(B, A[G]) &\longrightarrow F_{A[G]}(\tilde{B}, A[G]). \end{aligned}$$

**Proof:** As  $B$  is faithful as a left  $A$ -module it suffices to check that we obtain equivalences after smashing with  $B$  over  $A$  from the right. There is an equivalence  $F_{A[G]}(\tilde{B}, B) \wedge_A B \simeq B[G]$  given by  $F_{A[G]}(\tilde{B}, B) \wedge_A B \simeq F_{A[G]}(\tilde{B}, B \wedge_A B) \simeq \tilde{F}_{A[G]}(\tilde{B}, B[G]) \simeq \tilde{F}_{B[G]}(B \wedge_A B, B[G]) \simeq \tilde{F}_{B[G]}(B[G], B[G]) \simeq B[G]$ . Here we are using that  $B$  is dualizable as an  $A$ -module, the weak equivalence  $\tilde{h}: B \wedge_A B \rightarrow F(G_+, B) \cong B[G]$  from lemma 4.12 which is a map of left  $A[G]$ -modules with respect to the action of  $G$  on the first factor of  $B \wedge_A B$  and that  $B \wedge_A B$  represents the derived smash product. Similar equivalences hold for the other function spectra from the statement. Assembling these equivalences to commutative diagrams proves the claim.  $\square$

**Proof of proposition 9.8:** By faithfulness of  $B$  as an  $A$ -module it suffices to show that  $\nu \wedge_A B$  is a weak equivalence. We have the following diagram.

$$\begin{array}{ccc} F_{A[G]}(B, A[G]) \wedge_{A[G]}^L B \wedge_A B & \xrightarrow{\nu \wedge_A B} & F_{A[G]}(B, B) \wedge_A B \\ \downarrow F_{A[G]}(B; A[G]) \wedge_{A[G]}^L \tilde{h} & & \downarrow \nu \\ F_{A[G]}(B, A[G]) \wedge_{A[G]}^L B[G] & & \\ \simeq \downarrow & & \\ F_{A[G]}(B, A[G]) \wedge_A B & & \\ \downarrow \nu & & \downarrow \nu \\ F_{A[G]}(B, B[G]) & \xleftarrow[\simeq]{\tilde{h}_*} & F_{A[G]}(B, B \wedge_A B) \end{array}$$

The vertical maps labeled  $\nu$  are weak equivalences as  $B$  is dualizable as an  $A$ -module. It follows that also the top horizontal map is a weak equivalence which proves the proposition.  $\square$

**Proposition 9.10.** *Let  $A \rightarrow B$  be a Galois-extension of associative  $S$ -algebras with finite Galois group  $G$ . Assume that  $A \rightarrow B$  is faithful. Then*

$$\text{eval}: B \wedge_{F_{A[G]}(B, B)^{\text{op}}}^L F_{A[G]}(B, A[G]) \longrightarrow A[G]$$

*is a weak equivalence.*



**Proof:** By assumption,  $A \rightarrow B$  is faithful and  $B \wedge_A B \simeq B[G]$ . So  $\text{eval}$  is a weak equivalence if and only if  $\text{eval} \wedge_A^L B$ ,  $\text{eval} \wedge_{A[G]}^L B[G]$ ,  $\text{eval} \wedge_{A[G]}^L B \wedge_A B$  or  $\text{eval} \wedge_{A[G]}^L B$  is a weak equivalence. We will check this for the last map which is

$$\text{eval} \wedge_{A[G]}^L B: B \wedge_{F_{A[G]}(B, B)^{\text{op}}} F_{A[G]}(B, A[G]) \wedge_{A[G]}^L B \longrightarrow B. \quad (9.10)$$

The map  $\text{eval} \wedge_{A[G]}^L B$  factors as

$$\begin{aligned} B \wedge_{F_{A[G]}(B, B)^{\text{op}}} F_{A[G]}(B, A[G]) \wedge_{A[G]}^L B &\xrightarrow{B \wedge \nu} \\ B \wedge_{F_{A[G]}(B, B)^{\text{op}}} F_{A[G]}(B, B) &\xrightarrow{\text{eval}} B \end{aligned} \quad (9.11)$$

where  $\nu$  is the weak equivalence from proposition 9.8. The evaluation in (9.11) is clearly an isomorphism and so (9.11) is a weak equivalence. So also  $\text{eval} \wedge_{A[G]}^L B$  and hence  $\text{eval}$  are weak equivalences.  $\square$

**Theorem 9.11.** *Let  $A \rightarrow B$  be a Galois-extension of associative  $S$ -algebras with finite Galois group  $G$ . Assume that  $A \rightarrow B$  is faithful. Then  $B$  is an invertible  $(A[G], F_{A[G]}(B, B)^{\text{op}})$ -bimodule.*

**Proof:** This follows directly from propositions 9.8 and 9.10.  $\square$

*Remark 9.12* (Morita equivalence). As a consequence, the derived smash product functor  $B \wedge_{A[G]}^L -$  defines an equivalence of the derived categories of left  $A[G]$ - and left  $F_{A[G]}(B, B)$ -modules. This can be interpreted as a Morita equivalence as in [78].

The main question now is whether we can replace the right  $F_{A[G]}(B, B)^{\text{op}}$ -module structure on  $B$  by a right  $A[G]$ -module structure.

### 9.2.3 Picard groups

For an associative  $S$ -algebra  $R$  we define  $\text{Pic}(R)$  to be the collection of weak equivalence classes of invertible  $R$ -bimodules. It follows from [9, Prop. 16] that  $\text{Pic}(R)$  is a set and hence an abelian group, the *Picard group of  $R$* . We will see that if  $A \rightarrow B$  is a faithful Galois extension with finite abelian Galois group  $G$  then  $B$  gives rise to an element in the Picard group  $\text{Pic}(A[G])$ . Henceforth we assume that  $G$  is abelian. This is necessary in order that an  $A[G]$ -bimodule structure on  $B$  is defined.

**Lemma 9.13.** *Let  $A \rightarrow B$  be a faithful Galois extension with finite abelian Galois group  $G$ . Then there is a map*

$$\tilde{j}: A[G] \rightarrow F_{A[G]}(B, B)^{\text{op}} \quad (9.12)$$

*which is a weak equivalence of  $S$ -algebras.*

**Proof:** We define the map  $\tilde{j}$  analogously to the algebraic situation, i.e. being right adjoint to the module action  $B \wedge A[G] \rightarrow B$ . It is clear that  $\tilde{j}$  gives a map of  $S$ -algebras, compare lemma 4.11. Since  $B$  is faithful as left  $A$ -module, it suffices to check that the map  $B[G] \rightarrow F_{A[G]}(B, B) \wedge_{A[G]} B[G]$  is a weak equivalence. This follows as in the proof of lemma 9.9.  $\square$

**Theorem 9.14.** *Let  $A \rightarrow B$  be a Galois-extension of rings with finite Galois group  $G$ . Assume that  $A \rightarrow B$  is faithful and that  $G$  is abelian. Then  $B$  is an invertible  $A[G]$ -bimodule.*

**Proof:** This follows from propositions 9.8 and 9.10 together with lemma 9.13.  $\square$

We conclude this chapter with some remarks concerning the comparison of fixed points with homotopy fixed points. Given a space or spectrum  $X$  with  $G$ -action, the contraction of  $EG$  onto a point provides a comparison map

$$X^G \cong F(S, X)^G \longrightarrow F(EG_+, X)^G = X^{hG}.$$

For finite  $G$ -complexes  $X$  it was originally conjectured by Sullivan that this map is a weak equivalence after  $p$ -adic completion. Proofs of the Sullivan conjecture were given in [67, 68, 22].

Our results allow to compare fixed points and homotopy fixed points of the spectrum  $F_A(B, B)$  where  $A \rightarrow B$  is a faithful Galois extension with finite abelian Galois group.

**Proposition 9.15.** *Let  $A \rightarrow B$  be a faithful Galois extension of  $S$ -algebras with finite abelian Galois group  $G$ . Then*

$$F_A(B, B)^G \simeq F_A(B, B)^{hG}.$$

*In other words: The Sullivan conjecture holds for endomorphism ring spectra of finite abelian faithful Galois extensions.*

**Proof:** By (9.12) there is a weak equivalence  $\tilde{j}: A[G] \rightarrow F_{A[G]}(B, B) \cong F_A(B, B)^G$ . Recall that in section 4.4 we introduced a weak equivalence  $\tilde{j}: B[G] \rightarrow F_A(B, B)^{op}$ . It is  $G$ -equivariant with  $G$ -actions defined analogously to the algebraic situation, see the proof of proposition 9.5. In particular we consider the  $G$ -action in the source to be given by the  $G$ -action on  $B$ . Passing to homotopy fixed points we obtain a weak equivalence  $(\tilde{j})^{hG}: A[G] \simeq B[G]^{hG} \rightarrow F_A(B, B)^{hG}$ . The weak equivalences  $\tilde{j}$  and  $(\tilde{j})^{hG}$  fit into a commutative diagram

$$\begin{array}{ccc} & A[G] & \\ \simeq \swarrow & & \searrow \simeq \\ F_A(B, B)^G & \longrightarrow & F_A(B, B)^{hG} \end{array}$$

proving the proposition.  $\square$

## Chapter 10

# Topological Hochschild homology and Hopf-Galois extensions

Given a Hopf-Galois extension of commutative  $S$ -algebras  $A \rightarrow B$  with Hopf algebra  $H$  and a  $B$ -bimodule  $M$  we construct an  $H$ -bimodule action on  $M$  such that

$$\mathrm{THH}^A(B; M) \simeq \mathrm{THH}^A(H; M). \quad (10.1)$$

We present two applications resulting from this equivalence. First, for a commutative Hopf algebra  $H$  under  $A$  with homotopy antipode we use this bimodule structure to show that  $\mathrm{THH}^A(H; M) \simeq M \wedge_A \mathcal{B}^A(A, H, A)$  where  $\mathcal{B}^A(A, H, A)$  denotes the bar construction. More generally

$$\mathrm{THH}^A(B; M) \simeq M \wedge_A \mathcal{B}^A(A, H, A) \quad (10.2)$$

for any Hopf-Galois extension of commutative  $S$ -algebras  $A \rightarrow B$  with respect to the Hopf algebra  $H$ . If  $B = Mf$  is a Thom spectrum associated with an infinite loop map  $f: X \rightarrow BGL_1 A$  this shows that

$$\mathrm{THH}^A(Mf) \simeq Mf \wedge BX_+. \quad (10.3)$$

For  $A = S$  the equivalence (10.3) is a theorem due to Blumberg, Cohen and Schlichtkrull [13, 12]. Note that (10.2) is neither restricted to Thom spectra nor to the case  $A = S$ .

As a second application we establish an equivalence

$$\mathrm{THH}^R(B; M) \simeq \mathrm{THH}^A(H; M \wedge_{A \wedge_R A} A)$$

where  $R$  is a commutative  $S$ -algebra mapping to  $A$ . Often, the Hopf algebra  $H$  of a Hopf-Galois extension  $A \rightarrow B$  has the form  $H = A \wedge_R K$  for some Hopf algebra  $K$  under  $R$ . In this case, the last equivalence takes the form

$$\mathrm{THH}^R(B; M) \simeq \mathrm{THH}^R(K; M \wedge_{A \wedge_R A}^L A).$$

We obtain a spectral sequence

$$E_{p,q}^2 = \mathrm{Tor}_{\pi_*}^{\pi_*(K \wedge_R K)}(K_*, \pi_* \mathrm{THH}^R(A; M)) \implies \pi_* \mathrm{THH}^R(B; M)$$

which under flatness assumptions has  $E_{p,q}^2 \cong HH_*(K_*; \pi_* \mathrm{THH}^R(A; M))$ . This spectral sequence generalizes an algebraic analog constructed by Stefan [80] and was our initial motivation for pursuing the approach of this chapter.

The completion condition on the extension  $A \rightarrow B$  is actually not used in this chapter and the statements in fact hold whenever the canonical map  $h$  associated with a coaction of a Hopf algebra is a weak equivalence of commutative  $S$ -algebras. In particular this includes all Thom spectra associated with infinite loop maps, not just oriented ones.

## 10.1 Connection between the Hochschild homology of the extension and its Hopf algebra

**Lemma 10.1.** *Let  $A \rightarrow B$  be a map of commutative  $S$ -algebras and let  $H$  be a commutative Hopf algebra under  $A$  that coacts on  $B$  under  $A$ . Assume that the canonical morphism  $h: B \wedge_A B \rightarrow B \wedge_A H$  is a weak equivalence. Assume further that  $B$  and  $H$  are cofibrant as commutative  $A$ -algebras. Then there is a morphism*

$$\Phi: B \wedge_A H \longrightarrow B \wedge_A B$$

*of commutative  $B$ -algebras that is an inverse to  $h$  up to homotopy, in particular  $\Phi$  is a weak equivalence. Here the  $B$ -algebra structures of  $B \wedge_A B$  and  $B \wedge_A H$  are given by inclusion of  $B$  in the first factor.*

**Proof:** First, note that since we deal with a Hopf-Galois extension of commutative  $S$ -algebras, the canonical map  $h$  is a map of commutative  $B$ -algebras. Then note that there is a pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ H & \longrightarrow & B \wedge_A H \end{array}$$

in the category of commutative  $A$ -algebras. Hence  $B \rightarrow B \wedge_A H$  is a cofibration of commutative  $A$ -algebras as cofibrations are preserved by base change. This means that  $B \wedge_A H$  is a cofibrant commutative  $B$ -algebra as the category of commutative  $B$ -algebras  $\mathcal{C}_B$  is the category of commutative  $S$ -algebras under  $B$ . It follows analogously that  $B \wedge_A B$  is a cofibrant commutative  $B$ -algebra. Also note that all objects in  $\mathcal{C}_B$  are fibrant and recall that a map between fibrant cofibrant objects is a weak equivalence if and only if it has a homotopy inverse [35, 4.24]. So we can take  $\Phi: B \wedge_A H \rightarrow B \wedge_A B$  to be a homotopy inverse of  $h$  in the category of commutative  $B$ -algebras. As  $h$  is a weak equivalence so is the map  $\Phi$ .  $\square$

As a map of  $B$ -algebras,  $\Phi$  is of course a map of  $B$ -bimodules. In order to avoid confusion we point out that the left and right  $B$ -module structures coming from this  $B$ -algebra structure are both given via the first  $B$ -factor. On  $B \wedge_A B$  there is a different right  $B$ -module structure being induced from the action on the

second  $B$ -factor. However, there is no ambiguity about the right (and left)  $A$ -module structures on  $B \wedge_A B$  as we smash over  $A$ .

In order to motivate the next step we recall the definition of topological Hochschild homology spectra. For a commutative  $S$ -algebra  $A$ , an associative  $A$ -algebra  $B$  and a  $B$ -bimodule  $M$  relative  $A$ , i.e. a  $B \wedge_A B^{\text{op}}$ -module, there is a simplicial  $A$ -module  $\text{thh}^A(B; M)_*$  given in degree  $p$  by  $M \wedge_A B^{\wedge p}$  with face and degeneracy maps given by

$$d_i = \begin{cases} \nu_r \wedge_A \text{id}^{p-1} & \text{if } i = 0 \\ \text{id} \wedge_A \text{id}^{i-1} \wedge_A \mu \wedge_A \text{id}^{p-i-1} & \text{if } 1 \leq i < p \\ (\nu_l \wedge_A \text{id}^{p-1}) \circ \tau & \text{if } i = p \end{cases}$$

and  $s_i = \text{id} \wedge_A \text{id}^i \wedge_A \eta \wedge_A \text{id}^{p-i}$ . Here  $\nu_l$  and  $\nu_r$  denote the left and right module actions,  $\tau: (M \wedge_A B^{p-1}) \wedge_A B \rightarrow B \wedge_A (M \wedge_A B^{p-1})$  denotes the switch map and  $\eta: A \rightarrow B$  is the unit. The topological Hochschild homology spectrum  $\text{thh}^A(B; M)$  is then defined to be the geometric realization of the simplicial  $A$ -module  $\text{thh}^A(B; M)_*$

$$\text{thh}^A(B; M) := |\text{thh}^A(B; M)_*|$$

[36, IX.2]. This definition is sometimes referred to as the algebraic definition of topological Hochschild homology. In the derived category, topological Hochschild homology of  $B$  with coefficients in  $M$  is defined to be

$$\text{THH}^A(B; M) := M \wedge_{B \wedge_A B^{\text{op}}}^L B.$$

If  $B$  is cofibrant as an  $A$ -algebra and  $M$  is a cell  $B \wedge_A B^{\text{op}}$ -module then there is an equivalence

$$\text{thh}^A(B; M) \simeq M \wedge_{B \wedge_A B^{\text{op}}} B \simeq \text{THH}^A(B; M)$$

[36, IX.2.5]. Assuming that  $B$  is cofibrant as a commutative  $A$ -algebra, a cell  $B \wedge_A B^{\text{op}}$ -module  $M$  represents the derived smash product over  $A$ . Now let  $N$  be a  $B \wedge_A B^{\text{op}}$ -module which is equivalent to  $M$  and which is also an extended cell  $A$ -module. Then the proper simplicial spectra  $\text{thh}_*^A(B; N)$  and  $\text{thh}_*^A(B; M)$  are degreewise equivalent and it follows that  $\text{thh}^A(B; N) \simeq \text{thh}^A(B; M)$ . Hence  $\text{thh}^A(B; N) \simeq \text{THH}^A(B; N)$  also holds when  $N$  is an extended cell  $A$ -module, e.g. if  $N = B$ .

Coming back to our initial situation we would like to compare  $\text{thh}^A(B; M)$  with  $\text{thh}^A(H; M)$ . For this, given a  $B$ -bimodule  $M$  we need to define an  $H$ -bimodule structure on  $M$ . More generally than in lemma 10.1, for every right  $B$ -module  $M$  we can define a map

$$\Phi_M: M \wedge_A H \rightarrow M \wedge_A B$$

as the composite

$$M \wedge_A H \cong M \wedge_B B \wedge_A H \xrightarrow{M \wedge_B \Phi} M \wedge_B B \wedge_A B \cong M \wedge_A B.$$

Our notation is such that  $\Phi_B = \Phi$ .

**Definition 10.2.** Let  $A$ ,  $B$  and  $H$  be as in lemma 10.1 and let  $M$  be a  $B$ -bimodule relative  $A$ , i.e. a  $B \wedge_A B$ -module. We define an  $H$ -bimodule structure on  $M$  by

$$\begin{aligned} \nu_r: M \wedge_A H &\xrightarrow{\Phi_M} M \wedge_A B \rightarrow M && \text{and} \\ \nu_l: H \wedge_A M &\xrightarrow{\tau} M \wedge_A H \xrightarrow{\Phi_M} M \wedge_A B \xrightarrow{\tau} B \wedge_A M \rightarrow M \end{aligned}$$

where the last maps in each row are given by the right respectively left  $B$ -module structure on  $M$ .

Since  $\Phi$  is a map of  $B$ -algebras it follows that  $\nu_r$  and  $\nu_l$  define right and left  $H$ -module structures on  $M$  and they combine to an  $H$ -bimodule structure because  $M$  is a  $B$ -bimodule relative  $A$ .

**Proposition 10.3.** *Let  $A$ ,  $B$  and  $H$  be as in lemma 10.1 and let  $M$  be a  $B$ -bimodule relative  $A$  which is an extended cell  $B$ -module (e.g.  $M = B$ ). With the  $H$ -bimodule structure on  $M$  from definition 10.2 there is a weak equivalence*

$$\Phi_*: \mathrm{thh}^A(H; M) \xrightarrow{\cong} \mathrm{thh}^A(B; M).$$

*The statement also holds when  $M$  is a cell  $B \wedge_A B$ -module.*

There is also a map

$$\alpha: H \xrightarrow{\eta} B \wedge_A H \xrightarrow{\Phi} B \wedge_A B \xrightarrow{\mu} B \quad (10.4)$$

which is a map of  $A$ -algebras. One can also use this map to define an  $H$ -(bi)module structure on a given  $B$ -(bi)module  $M$ . The right action  $\nu_r$  from the last definition is the one induced by  $\alpha$ . However  $\alpha$  induces the left module structure  $\nu_l$  only if  $M$  is a central  $B$ -bimodule. If  $M$  is not central as a  $B$ -bimodule, we need the bimodule structure from definition 10.2 to prove proposition 10.3 as will be evident from the proof.

**Proof of proposition 10.3:**

We will establish a map between the defining simplicial objects, given in degree  $n$  by a weak equivalence  $M \wedge_A H^{\wedge n} \rightarrow M \wedge_A B^{\wedge n}$ . Both source and target are proper simplicial spectra by [36, IX.2.8] and so it follows by [36, X.2.4] that degreewise weak equivalences give an equivalence of geometric realizations.

The maps  $M \wedge_A H^{\wedge n} \rightarrow M \wedge_A B^{\wedge n}$  can be defined as the composites

$$\begin{aligned} M \wedge_A H^{\wedge n} &\cong M \wedge_B (B \wedge_A H)^{\wedge n} \\ &\xrightarrow{M \wedge_B \Phi^{\wedge n}} M \wedge_B (B \wedge_A B)^{\wedge n} \cong M \wedge_A B^{\wedge n}. \end{aligned}$$

The map in the middle is a weak equivalence since  $\Phi$  is a weak equivalence and all smash factors represent the derived smash product. We have to check that these maps assemble to a map of simplicial objects, i.e. that these maps commute with degeneracy and face maps. This is clear for the degeneracies

since in our case they are all given by unit maps  $A \rightarrow H$  or  $A \rightarrow B$  and  $\Phi$  is a map under  $A$ . The face maps  $d_0$  are given by the right module actions of  $H$  respectively  $B$  on  $M$  and commutativity follows since the action  $\nu_r$  of  $H$  on  $M$  is precisely given by the action of  $B$  precomposed with the map  $\Phi_M$ . For the  $d_i$  with maximal index  $i$  commutativity follows from the definition of the left module action  $\nu_l$  on  $M$  in definition 10.2 and actually this is what motivated this definition. For the intermediate  $d_i$  commutativity follows by multiplicativity of  $\Phi$ . The following diagram gives an example for this.

$$\begin{array}{ccc}
 M \wedge_B (B \wedge_A H) \wedge_B (B \wedge_A H) & \xrightarrow{M \wedge_B \Phi \wedge_B \Phi} & M \wedge_B (B \wedge_A B) \wedge_B (B \wedge_A B) \\
 \downarrow M \wedge_B \mu & & \downarrow M \wedge_B \mu \\
 M \wedge_B (B \wedge_A H) & \xrightarrow{M \wedge_B \Phi} & M \wedge_B B \wedge_A B
 \end{array}$$

This completes the proof.  $\square$

By the cofibrancy assumptions in proposition 10.3 the terms  $M \wedge_A H^{\wedge k}$  and  $M \wedge_A B^{\wedge k}$  represent the derived smash products and so we get weak equivalences  $\mathrm{THH}^A(H; M) \simeq \mathrm{thh}^A(H; M)$  and  $\mathrm{THH}^A(B; M) \simeq \mathrm{thh}^A(B; M)$  as in [36, IX.2.]. So there is at least a zigzag of weak equivalences  $\mathrm{THH}^A(H; M) \simeq \mathrm{THH}^A(B; M)$ .

The next lemma shows that if  $M$  is central as a  $B$ -bimodule, this equivalence is induced by the map  $\alpha$ .

**Lemma 10.4.** *Assume that  $A$ ,  $B$  and  $H$  are as in lemma 10.1 and  $M$  is a  $B$ -bimodule relative  $A$ . Assume moreover that  $M$  is central as a  $B$ -bimodule. Then the map  $\alpha$  induces a map*

$$\alpha_*: \mathrm{THH}^A(H; M) \xrightarrow{\simeq} \mathrm{THH}^A(B; M)$$

which is also a weak equivalence.

**Proof:** For the proof we can assume that  $M$  is a cell  $B$ -module. We will use the equivalences  $\mathrm{THH}^A(H; M) \simeq \mathrm{thh}^A(H; M)$  and  $\mathrm{THH}^A(B; M) \simeq \mathrm{thh}^A(B; M)$  and that  $\alpha_*$  corresponds to the weak equivalence  $\Phi_*$  from proposition 10.3. Since the  $B$ -bimodule  $M$  is central, the  $H$ -bimodule structure on  $H$  is induced by the map  $\alpha$ . We first want to mention that the maps  $\alpha_*$  and  $\Phi_*$  however do not coincide on the simplicial level. The following diagram shows why this is not the case.

$$\begin{array}{ccccccc}
 \Phi_M: M \wedge_A H & \xrightarrow{\cong} & M \wedge_B B \wedge_A H & \xrightarrow{M \wedge_B \Phi} & M \wedge_B B \wedge_A B & \xrightarrow{\cong} & M \wedge_A B \\
 \parallel & & \uparrow & & \uparrow & & \parallel \\
 M \wedge_A H & \xrightarrow{\eta} & M \wedge_A B \wedge_A H & \xrightarrow{M \wedge_A \Phi} & M \wedge_A B \wedge_A B & \xrightarrow{\mu \wedge_A B} & M \wedge_A B \\
 & & & \searrow M \wedge_A \alpha & \downarrow M \wedge_A \mu & & \\
 & & & & M \wedge_A B & & 
 \end{array}$$

Expressing  $\Phi_M$  as in the second line, we see that the difference to  $M \wedge_A \alpha$  is the multiplication in the end which is  $\mu \wedge_A B$  or  $M \wedge_A \mu$  respectively. We will now

show that this difference does not matter on the level of THH. Note that the map  $M \wedge_A B \rightarrow M \wedge_{B \wedge_A B} B$  equalizes the two maps  $\mu \wedge_A B$  and  $M \wedge_A \mu$ . There is hence the following commutative diagram where the vertical compositions are coequalizers.

$$\begin{array}{ccc}
M \wedge_A (H \wedge_A H) \wedge_A H & \xrightarrow[\cong]{\Phi_{(M \wedge_A B \wedge_A B)} \circ \Phi_{(M \wedge_A B)} \circ \Phi_M} & M \wedge_A B \wedge_A B \wedge_A B \\
\downarrow & & \downarrow \\
M \wedge_A H & \xrightarrow{\quad} & M \wedge_A B \wedge_A B \xrightarrow[\cong]{\mu \wedge_A B} M \wedge_A B \\
\downarrow & & \downarrow \\
M \wedge_{H \wedge_A H} H & \xrightarrow{(\Phi_M)_* = \alpha_*} & M \wedge_{B \wedge_A B} B
\end{array}$$

The upper horizontal maps in the diagram induce a map  $(\Phi_M)_*$  on coequalizers and the lower horizontal maps in the diagram induce a map  $\alpha_*$ . As the lower right vertical map equalizes  $\mu \wedge_A B$  and  $M \wedge_A \mu$ , the maps  $\Phi_*$  and  $\alpha_*$  coincide. We obtain the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{thh}^A(H; M) & \xrightarrow[\simeq]{\Phi_*} & \mathrm{thh}^A(B; M) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathrm{THH}^A(H; M) & & \mathrm{THH}^A(B; M) \\
\parallel & & \parallel \\
M \wedge_{H \wedge_A H}^L H & \xrightarrow{\alpha_*} & M \wedge_{B \wedge_A B}^L B
\end{array}$$

Using that  $M$  is a cell  $B$ -module and  $B$  is an extended cell  $A$ -module it follows that the vertical maps in the diagram are weak equivalences, the crucial property being that  $M \wedge_A X$  represents the derived smash product for every extended cell  $A$ -module  $X$ . It follows from the last diagram that  $\alpha_*: \mathrm{THH}^A(H; M) \rightarrow \mathrm{THH}^A(B; M)$  is a weak equivalence as well.  $\square$

## 10.2 Hochschild homology of Hopf algebras with homotopy antipode

Recall from proposition 5.11 that if  $H$  is a Hopf algebra under some commutative  $S$ -algebra  $A$  which has a homotopy antipode  $\lambda$ , then the unit map  $A \rightarrow H$  is a (trivial) Hopf-Galois extension. Moreover, a homotopy inverse to the canonical map  $h$  in this case is given by the map

$$\Psi: H \wedge_A H \xrightarrow{H \wedge_A \Delta} H \wedge_A H \wedge_A H \xrightarrow{H \wedge_A \lambda \wedge_A H} H \wedge_A H \wedge_A H \xrightarrow{\mu \wedge_A H} H \wedge_A H.$$

Now let  $M$  be a central  $H$ -bimodule. Then setting  $B = H$  in equation (10.4) we get a morphism of commutative  $A$ -algebras  $\alpha: H \rightarrow H$ . The self map  $\alpha: H \rightarrow H$  induces a new  $H$ -bimodule structure on  $M$ . Let us write  $M_\alpha$  for



$M$  regarded as an  $H$ -bimodule with this structure. Also note that there is a homotopy commutative diagram

$$\begin{array}{ccccc}
 \alpha: H & \xrightarrow{\eta \wedge_A H} & H \wedge_A H & \xrightarrow{\Phi} & H \wedge_A H & \xrightarrow{\mu} & H \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 H & \xrightarrow{\eta \wedge_A H} & H \wedge_A H & \xrightarrow{\Psi} & H \wedge_A H & \xrightarrow{\mu} & H \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 H & \xrightarrow{\epsilon} & A & \xrightarrow{\eta} & H & & H
 \end{array} \tag{10.5}$$

The square in the middle of the upper part of the diagram commutes up to homotopy as both  $\Phi$  and  $\Psi$  are homotopy inverses to the canonical map  $h$ . The lower part of the diagram commutes up to homotopy due to the antipode property. Note that the bottom map  $\eta \circ \epsilon$  is also a map of commutative  $A$ -algebras and so induces yet another  $H$ -bimodule structure on  $M$ . We will write  $M_\epsilon$  for  $M$  with the  $H$ -bimodule structure induced by  $\eta \circ \epsilon$ . As the diagram above commutes up to homotopy there is a weak equivalence  $\mathrm{THH}^A(H; M_\alpha) \simeq \mathrm{THH}^A(H; M_\epsilon)$  as follows by a Künneth spectral sequence argument.

In order to proceed, we have to recall the bar construction, see [36, IV.7.]. Let  $A$  be a commutative  $S$ -algebra,  $B$  an associative  $A$ -algebra,  $M$  a right and  $N$  a left  $B$ -module. We define a simplicial  $S$ -module  $\mathcal{B}_*^A(M, B, N)$  by

$$\mathcal{B}_p^A(M, B, N) := M \wedge_A B^{\wedge p} \wedge_A N,$$

with face and degeneracy operators

$$d_i = \begin{cases} \mu \wedge_A \mathrm{id}^{p-1} \wedge_A \mathrm{id}_N & \text{if } i = 0 \\ \mathrm{id}_M \wedge_A \mathrm{id}^{i-1} \wedge_A \mu \wedge_A \mathrm{id}^{p-i-1} \wedge_A \mathrm{id}_N & \text{if } 1 \leq i < p \\ \mathrm{id}_M \wedge_A \mathrm{id}^{p-1} \circ \mu & \text{if } i = p \end{cases}$$

and  $s_i = \mathrm{id}_M \wedge_A \mathrm{id}^i \wedge_A \eta \wedge_A \mathrm{id}^{p-i} \wedge_A \mathrm{id}_N$ .

**Theorem 10.5.** *Let  $A$  be a commutative  $S$ -algebra and  $H$  a commutative Hopf algebra under  $A$  with homotopy antipode. Assume that  $A$  is cofibrant as a commutative  $S$ -algebra and  $H$  is cofibrant as a commutative  $A$ -algebra. Let  $M = M_\epsilon$  be an extended cell  $A$ -module which we see as a central  $H$ -bimodule via the map  $\epsilon: H \rightarrow A$ . Then*

$$\mathrm{thh}^A(H, M) \simeq M \wedge_A \mathcal{B}^A(A, H, A). \tag{10.6}$$

**Proof:** Comparing the definitions of the simplicial spectra  $\mathrm{thh}_*$  and  $\mathcal{B}_*$  we see that  $\mathrm{thh}_*^A(H; M_\epsilon) \cong \mathcal{B}_*^A(M_\epsilon, H, A)$  where  $A$  is seen as a left  $H$ -module via  $\epsilon: H \rightarrow A$ . Here we are using that also the  $H$ -module structures on  $M_\epsilon$  are defined via the map  $\epsilon$ . Analogously it follows that  $\mathcal{B}_*^A(M_\epsilon, H, A) \cong M_\epsilon \wedge_A \mathcal{B}_*^A(A, H, A)$ . Passing to realizations then gives the stated result.  $\square$

**Proposition 10.6.** *Let  $A$  be a commutative  $S$ -algebra and  $H$  a commutative Hopf algebra under  $A$  with homotopy antipode. Assume that  $A$  is cofibrant as a commutative  $S$ -algebra and  $H$  is cofibrant as a commutative  $A$ -algebra. Let  $M$  be an  $H$ -bimodule relative  $A$  which is central and an extended cell  $A$ -module. Then*

$$\mathrm{THH}^A(H; M) \simeq M \wedge_A \mathcal{B}^A(A, H, A).$$

*In particular there is an equivalence*

$$\mathrm{THH}^A(H) \simeq H \wedge_A \mathcal{B}^A(A, H, A).$$

**Proof:** We can see  $A \rightarrow H$  as a Hopf-Galois extension and have the following chain of weak equivalences.

$$\begin{aligned} \mathrm{THH}^A(H, M) &\simeq \mathrm{THH}^A(H; M_\alpha) && \text{lemma 10.4} \\ &\simeq \mathrm{THH}^A(H; M_\epsilon) && \text{diagram (10.5)} \\ &\simeq \mathrm{thh}^A(H; M_\epsilon) && M = M_\epsilon \text{ ext. cell } A\text{-module} \\ &\simeq M \wedge_A \mathcal{B}^A(A, H, A) && \text{theorem 10.5} \end{aligned}$$

This proves the first equivalence. The second equivalence is just the case  $M = H$ .  $\square$

**Theorem 10.7.** *Let  $A \rightarrow B$  be a Hopf-Galois extension of commutative  $S$ -algebras with Hopf algebra  $H$  which has a homotopy antipode. Assume that  $A$  is a cofibrant commutative  $S$ -algebra and  $B$  and  $H$  are cofibrant commutative  $A$ -algebras. Let  $M$  be a central  $B$ -bimodule relative  $A$  which is an extended cell  $A$ -module. Then*

$$\mathrm{THH}^A(B; M) \simeq M \wedge_A \mathcal{B}^A(A, H, A).$$

*If  $H$  is of the form  $H = A \wedge K$  then*

$$\mathrm{THH}^A(B; M) \simeq M \wedge \mathcal{B}^S(S, K, S).$$

**Proof:** There is the following chain of weak equivalences.

$$\begin{aligned} \mathrm{THH}^A(B; M) &\simeq \mathrm{THH}^A(H; M_\alpha) && \text{lemma 10.4} \\ &\simeq M \wedge_A \mathcal{B}^A(A, H, A) && \text{as in the proof above} \end{aligned}$$

This proves the first equivalence and the second in case  $H = A \wedge K$  is a direct consequence.  $\square$

In the following corollary we implicitly assume that the usual cofibrancy assumptions are satisfied.

**Corollary 10.8.** *Let  $Mf$  be a Thom spectrum associated with an infinite loop map  $f: X \rightarrow BGL_1(A)$ . Then*

$$\mathrm{THH}^A(Mf) \simeq Mf \wedge BX_+ \tag{10.7}$$

**Proof:** Recall from chapter 6 that the group ring  $H := A[X]$  is a Hopf algebra with homotopy antipode coacting on  $Mf$  such that the canonical map  $h$  is a weak equivalence of commutative  $A$ -algebras. It follows as in the last theorem that there is a weak equivalence  $\mathrm{THH}^A(Mf) \simeq M \wedge \mathcal{B}^S(S, S[X], S)$ . Then as the space level and the spectrum level bar constructions commute, e.g. by [36, X.1.3.], we see that  $\mathcal{B}^S(S, \Sigma^\infty X, S) \cong \Sigma^\infty \mathcal{B}(*, X, *) = S[BX]$ .  $\square$

In case  $A = S$  the statement was formulated and proven in [13, 12] where [13] also includes Thom spectra arising from at least 3-fold loop maps  $X \rightarrow BGL_1(S)$ . We can not reproduce the result for finite loop maps by our approach since we used that the canonical morphism  $h$  is a morphism of commutative  $S$ -algebras which is only the case when starting with an infinite loop map. However, none of the references deals with  $A$  different from  $S$ .

### 10.3 A spectral sequence converging to the Hochschild homology of a Hopf-Galois extension

So far we have only dealt with topological Hochschild homology relative to the base spectrum  $A$  of a Hopf-Galois extension  $A \rightarrow B$ . We also want to work relative  $S$  and express  $\mathrm{THH}^S(B; M)$  in terms of the topological Hochschild homology of  $A$  and the Hopf algebra  $H$ . More generally, let  $R \rightarrow A$  be a map of commutative  $S$ -algebras mapping a commutative  $S$ -algebra  $R$  to a Hopf-Galois extension  $A \rightarrow B$ . The next statement describes the topological Hochschild homology of  $B$  relative  $R$  in a different way. We assume that  $R$  is a cofibrant commutative  $S$ -algebra,  $A$  is a cofibrant commutative  $R$ -algebra and  $B$  and  $H$  are cofibrant commutative  $A$ -algebras.

**Theorem 10.9.** *Let  $R \rightarrow A$  be a map of commutative  $S$ -algebras and let  $A \rightarrow B$  be a Hopf-Galois extension with respect to a Hopf algebra  $H$ . Assume that the above cofibrancy hypotheses hold. Let  $M$  be a  $B$ -bimodule relative  $R$  and let  $\widetilde{M} \rightarrow M$  be a cell approximation in the category of  $B \wedge_R B$ -modules. Then*

$$\mathrm{THH}^R(B; M) \simeq \mathrm{THH}^A(H; \widetilde{M} \wedge_{A \wedge_R A} A). \quad (10.8)$$

*If moreover  $H$  is of the form  $H = A \wedge_R K$  for a Hopf algebra  $K$  under  $R$  then this equivalence takes the form*

$$\mathrm{THH}^R(B; M) \simeq \mathrm{THH}^R(K; \widetilde{M} \wedge_{A \wedge_R A} A). \quad (10.9)$$

**Proof:** There is the following chain of weak equivalences.

$$\begin{aligned} \mathrm{THH}^R(B; M) &\simeq \widetilde{M} \wedge_{B \wedge_R B} B && \widetilde{M} \text{ cell } B \wedge_R B\text{-module} \\ &\cong \widetilde{M} \wedge_{B \wedge_R B} (B \wedge_A B) \wedge_{B \wedge_A B} B \\ &\simeq \mathrm{THH}^A(B; \widetilde{M} \wedge_{B \wedge_R B} (B \wedge_A B)) && N := \widetilde{M} \wedge_{B \wedge_R B} (B \wedge_A B) \\ &\simeq \mathrm{thh}^A(B; \widetilde{M} \wedge_{B \wedge_R B} (B \wedge_A B)) && \text{cell } B \wedge_A B\text{-module} \\ &\simeq \mathrm{thh}^A(H; \widetilde{M} \wedge_{B \wedge_R B} (B \wedge_A B)) && \text{proposition 10.3} \\ &\simeq \mathrm{THH}^A(H; \widetilde{M} \wedge_{B \wedge_R B} (B \wedge_A B)) && N \text{ extended cell } A\text{-module} \\ &\simeq \mathrm{THH}^A(H; \widetilde{M} \wedge_{A \wedge_R A} A) \end{aligned}$$

The last equivalence is due to isomorphisms

$$\begin{aligned}
\widetilde{M} \wedge_{B \wedge_R B} B \wedge_A B &\cong (B \wedge_B \widetilde{M}) \wedge_{B \wedge_R B} (B \wedge_A B) \\
&\cong (B \wedge_B B) \wedge_{B \wedge_R A} (\widetilde{M} \wedge_B B) \\
&\cong (B \wedge_A A) \wedge_{B \wedge_R A} (\widetilde{M} \wedge_A A) \\
&\cong (B \wedge_B \widetilde{M}) \wedge_{A \wedge_R A} (A \wedge_A A) \\
&\cong \widetilde{M} \wedge_{A \wedge_R A} A
\end{aligned}$$

where the second and fourth isomorphism is due to a comparison of coequalizer diagrams, compare with [36, III.3.10.]. This proves (10.8). Equation (10.9) is a direct consequence.  $\square$

The last theorem allows the construction of spectral sequences.

**Theorem 10.10.** *Let  $R, A, B, K, M$  and  $\widetilde{M}$  be as in theorem 10.9. There is a spectral sequence*

$$\mathrm{Tor}_*^{\pi_*(K \wedge_R K)}(\pi_* \mathrm{THH}^R(A; M), K_*) \implies \pi_* \mathrm{THH}^R(B; M).$$

**Proof:** This is the usual spectral sequence converging to  $\pi_* \mathrm{THH}^R(B; M)$  from [36, IX.1.6.] and nothing but a Künneth spectral sequence. Here we are using theorem 10.9 which says that  $\pi_* \mathrm{THH}^R(K; \widetilde{M} \wedge_{A \wedge_R A} A) \cong \pi_* \mathrm{THH}^R(B; M)$ . We are also using that  $\widetilde{M} \wedge_{A \wedge_R A} A \simeq \mathrm{THH}^R(A; M)$ . To prove this equivalence, we have to show that  $\widetilde{M} \wedge_{A \wedge_R A} A$  represents the derived smash product. This follows from our cofibrancy assumptions as the following chain of equivalences shows. We write  $\widetilde{A}$  for a cell approximation of  $A$  in the category of  $A \wedge_R A$ -modules.

$$\begin{aligned}
\widetilde{M} \wedge_{A \wedge_R A} A &\cong \widetilde{M} \wedge_{B \wedge_R B} (B \wedge_R B) \wedge_{A \wedge_R A} A \\
&\cong \widetilde{M} \wedge_{B \wedge_R B} (B \wedge_A A \wedge_A B) \\
&\simeq \widetilde{M} \wedge_{B \wedge_R B} (B \wedge_A \widetilde{A} \wedge_A B) \\
&\cong \widetilde{M} \wedge_{B \wedge_R B} (B \wedge_R B) \wedge_{A \wedge_R A} \widetilde{A} \\
&\cong \widetilde{M} \wedge_{A \wedge_R A} \widetilde{A}
\end{aligned}$$

This completes the proof.  $\square$

Under flatness assumptions the spectral sequence simplifies, e.g. to a spectral sequence

$$HH_*(K_*; \pi_* \mathrm{THH}^R(A; M)) \implies \pi_* \mathrm{THH}^R(B; M).$$

Moreover,  $\pi_* \mathrm{THH}^R(A; M)$  may be given by ordinary Hochschild homology. For instance, assume that  $R, A$  and  $K$  are Eilenberg-McLane spectra and  $A_*$  and  $K_*$  are flat as  $R_*$ -modules. Then the spectral sequences specializes to a spectral sequence of the form

$$HH_*^{R_*}(K_*; HH_*^{R_*}(A_*; M_*)) \implies \pi_* \mathrm{THH}^R(B; M)$$

and generalizes the spectral sequence for Hopf-Galois extensions of ordinary rings from [80].

# Appendix



# Appendix A

## A Galois correspondence for associative ring extensions

The classical Galois correspondence for field extensions has a generalization to commutative ring theory. In this more general situation, the subgroups of the Galois group  $G$  correspond to the intermediate rings that have the additional properties of being separable and  $G$ -strong [24], [42], see definitions 1.2 and A.4. In non-commutative ring theory it is not clear how to characterize those intermediate rings that correspond to subgroups of the Galois group in general and Galois correspondences have only been obtained under additional assumptions. Ferrero obtains a Galois correspondence in [37]. He supposes some so-called property (H) which in the commutative case is fulfilled if and only if there are no nontrivial idempotents, see definition A.2.

The aim of this appendix is to set up a Galois correspondence for Galois extensions  $R = S^G \rightarrow S$  of associative rings under the assumptions that the trace  $tr : S \rightarrow R$  is surjective and that all idempotents of  $S$  lie in the center of  $S$ . The intermediate rings corresponding to subgroups of  $G$  then are exactly those which are  $G$ -strong and fulfill a certain separability condition, see definition A.7. This condition equals the usual condition of separability if  $R$  and  $S$  are commutative or under the assumptions of the main theorem of [37]. Our main theorem is hence a generalization of those of [24],[42] and [37].

### A.1 A short review

The easy part of a Galois correspondence for ring extensions reads as follows, see e.g. [37] for a reference including the associative case.

**Proposition A.1** ([37] prop. 3.1 p.83).

*Let  $R \rightarrow S$  be  $G$ -Galois. Then for any subgroup  $H < G$  the inclusion  $T := S^H \rightarrow S$  is an  $H$ -Galois extension and  $H$  is the subset of  $G$  leaving  $T$  pointwise fixed. If  $tr(S) = R$ , then  $T$  is separable and if  $H$  is normal in  $G$ , then  $R \rightarrow T$  is a  $G/H$ -Galois extension.*

To obtain the converse implication of a main-theorem, Ferrero supposes some property called (H) in [37].

**Definition A.2** (Property (H)). Let  $\mu: S \otimes_{\mathbb{Z}} S^{\text{op}} \rightarrow S$  be the multiplication. We say that  $S$  verifies (H) if:

$$z \in S \otimes_{\mathbb{Z}} S^{\text{op}}, \quad \mu(z) = \mu(z^2) \implies \mu(z) = 0 \quad \text{or} \quad \mu(z) = 1.$$

Obviously, if  $S$  verifies (H), it has no idempotents other than 0 or 1. Also for the converse statement of a Galois theorem, recall the following definitions:

**Definition A.3** (strongly distinct morphisms). Let  $f, g: A \rightarrow B$  be two homomorphisms into a ring  $B$ . Then  $f$  and  $g$  are called *strongly distinct*, if for every non-zero idempotent  $e \in B$  there is an  $a \in A$  such that  $f(a)e \neq g(a)e$ .

If  $B$  only has the trivial idempotents 0 and 1, then obviously  $f$  and  $g$  as above are strongly distinct if and only if they are distinct.

**Definition A.4** ( $G$ -strong intermediate rings). Let  $S$  be a  $G$ -Galois extension of  $R$  and  $T$  an intermediate ring. Then  $T$  is called  $G$ -strong if the restrictions to  $T$  of any two elements of  $G$  are either equal or strongly distinct as maps from  $T$  to  $S$ .

The parts from the main theorems from [24],[42] and [37] which are interesting for our purposes can be stated as follows:

**Theorem A.5** ([24] theorem 2.3).

*Let  $R \rightarrow S$  be a  $G$ -Galois extension of commutative rings. Then there is a one to one correspondence between subgroups of  $G$  and  $R$ -separable intermediate rings that are  $G$ -strong.*

In the non-commutative case of [37] it takes the following form:

**Theorem A.6** ([37] theorem 3.3).

*Let  $S$  be a  $G$ -Galois extension of  $R$ . If  $S$  verifies (H) and  $\text{tr}(S) = R$ , there is a one to one correspondence between subgroups of  $G$  and  $R$ -separable intermediate rings.*

Note that  $\text{tr}(S) = R$  is always satisfied if  $R$  and  $S$  are commutative ([24, Lemma 1.6]) and that (H) implies that all intermediate rings are  $G$ -strong.

## A.2 Galois-separable rings and a Galois correspondence

I want to prove a main theorem under different assumptions. The following property is important.

**Definition A.7** (Galois-separable intermediate rings). Let  $R \rightarrow S$  be a  $G$ -Galois extension  $T$  an intermediate ring  $R \subset T \subset S$ . We call  $T$  *Galois-separable for the  $G$ -Galois extension  $R \rightarrow S$* , if  $h(S \otimes_R T) \subset E := F(G, S) \cong \prod_G S$  satisfies the following separability condition:

There exist elements  $x_i, y_i \in h(S \otimes_R T)$  such that

- $\sum_i x_i y_i = 1$



$$\bullet \sum_i x x_i \otimes_S y_i = \sum_i x_i \otimes_S y_i x \quad \in E \otimes_S E \quad \forall x \in h(S \otimes_R T).$$

In this case, most of the time we will just say that  $T$  is Galois-separable, since the group  $G$  and the extension  $R \rightarrow S$  will be clear from the context.

If  $h(S \otimes_R T)$  is a subring of  $E$ , then  $T$  is Galois-separable if and only if  $h(S \otimes_R T)$  is  $S$ -separable. If furthermore  $h$  is a morphism of rings this is equivalent to saying that  $S \otimes_R T$  is  $S$ -separable. If  $R$  and  $S$  are commutative, we obtain even more:

**Lemma A.8.**

*If  $R \rightarrow S$  is a  $G$ -Galois extension of commutative rings, then an intermediate ring  $T$  is Galois-separable if and only if it is  $R$ -separable.*

**Proof:** By the remark preceding the Lemma, we have to show that  $T$  is  $R$ -separable if and only if  $S \otimes_R T$  is  $S$ -separable. It is a standard argument, that if  $T$  is  $R$ -separable, then  $S \otimes_R T$  is  $S$ -separable; one just can construct elements witnessing the conclusion.

For the other direction assume that  $S \otimes_R T$  is  $S$ -separable. We then use that in our situation  $S$  is an invertible  $R[G]$ -module. Hence by tensoring with an inverse for  $S$  one can obtain as above that  $R[G] \otimes_R T$  is  $R[G]$ -separable. This amounts to saying that the multiplication map  $R[G] \otimes_R T \otimes_R T \rightarrow R[G] \otimes_R T$  has a section. Regard  $R \rightarrow R[G]$  as the trivial  $G$ -Galois extension and note that this section is a map of left  $R[G]$ -modules. Hence by descent theory (e.g. [57, II.5.2]) it is induced by a (unique)  $R$ -module map  $s: T \rightarrow T \otimes_R T$ . The same is true for the multiplication and the uniqueness property tells us that  $s$  is a section for  $\mu: T \otimes_R T \rightarrow T$ . Hence  $T$  is  $R$ -separable.  $\square$

Note that for  $G$ -strong rings  $T$  the lemma is a corollary of theorem A.5. One just has to use that  $S \rightarrow S \otimes_R S$  is also  $G$ -Galois with action on the second factor. The Lemma is also a corollary if one compares theorem A.11 with theorem A.5 and their proofs.

The last definition allows us to formulate the following lemma which will be crucial in the proof of our main theorem. The lemma is an adaption of [24, lemma 1.2] to the situation for non-commutative rings. Note that in the situation of the next lemma,  $h(S \otimes_R T)$  is contained in the fixed subring  $E^H$  of  $E = F(G, S)$ .

**Lemma A.9.**

*Let  $R \rightarrow S$  be a  $G$ -Galois extension. Let  $T$  be a Galois-separable intermediate ring as above and  $H < G$  the subgroup of elements of  $G$  leaving  $T$  pointwise fixed. Suppose  $f: E^H \rightarrow S$  is an  $S$ -algebra-morphism and that all idempotents of  $S$  are central. Then there exists an idempotent  $e \in h(S \otimes_R T) \subset E^H$  such that*

1.  $f(e) = 1$
2.  $f(x)e = xe \quad \forall x \in h(S \otimes_R T)$
3. If  $f_1, \dots, f_n$  are homomorphisms from  $E^H$  to  $S$  as above and their restrictions to  $h(S \otimes_R T)$  are pairwise strongly distinct, then the corresponding idempotents  $e_1, \dots, e_n$  are pairwise orthogonal and  $f_i(e_j) = \delta_{ij}$ .

**Proof:** (compare the proof of [24, 1.2.]

Define  $e := \sum_i f(x_i)y_i \in h(S \otimes_R T)$  for  $x_i, y_i$  as in definition A.7. Then we have  $f(e) = f(\sum_i f(x_i)y_i) = \sum_i f(x_i)f(y_i) = f(\sum_i x_i y_i) = f(1) = 1$ . By applying  $f \otimes_S \text{id}$  to the second point in definition A.7 we obtain  $1 \otimes_S f(x)e = 1 \otimes_S f(x) \sum_i f(x_i)y_i = \sum_i f(x)f(x_i) \otimes_S y_i = (f \otimes_S \text{id})(\sum_i x x_i \otimes_S y_i) = (f \otimes_S \text{id})(\sum_i x_i \otimes_S y_i x) = 1 \otimes_S \sum_i f(x_i)y_i x = 1 \otimes_S e x \quad \forall x \in h(S \otimes_R T)$ . Using that  $E$  is  $S$ -free, we obtain  $f(x)e = e x$ . Hence  $e$  is an idempotent in  $E$ . As all the idempotents of  $E = \prod_G S$  are in the center of  $E$  we obtain  $f(x)e = e x = x e$ . To prove the last statement of the lemma use that  $f_i(x)f_i(e_j) = f_i(xe_j) = f_i(f_j(x)e_j) = f_j(x)f_i(e_j)$ . Hence since the restrictions of the  $f_k$  to  $h(S \otimes_R T)$  are strongly distinct, we obtain  $f_i(e_j) = \delta_{ij}$  and  $e_i e_j = f_j(e_i)e_j = \delta_{ij}e_j$ .  $\square$

We can now come to the converse part of the Galois correspondence:

**Proposition A.10.**

*Let  $R \rightarrow S$  be  $G$ -Galois, assume that all idempotents of  $S$  are central and that the trace  $\text{tr}: S \rightarrow R$  is surjective. Let  $T$  be a  $G$ -strong, Galois-separable intermediate ring  $R \subset T \subset S$ . Then there exists a subgroup  $H < G$  such that  $T = S^H$ .*

**Proof:** (compare [24, theorem 2.3])

Define  $H := \{g \in G \mid g(t) = t \quad \forall t \in T\}$ . We have to show that  $S^H \subset T$ . Therefore, let  $G = \bigcup_{i=1}^r \sigma_i H$  (disjoint union). Then  $E^H$  is the set of functions from  $G$  to  $S$  which are constant on each coset  $\sigma_i H$ . Define the morphisms of algebras

$$f_i : E^H \rightarrow S : v \mapsto v(\sigma_i).$$

Claim: The  $f_i$  are strongly distinct as morphisms of left  $S$ -modules  $f_i : h(S \otimes_R T) \rightarrow S$ .

Proof of the claim: The  $\sigma_i$  are strongly distinct as homomorphisms  $T \rightarrow S$ , hence for all nonzero idempotents  $e \in S, i \neq j$ , there exists  $t \in T$  such that  $\sigma_i(t)e \neq \sigma_j(t)e$ . Therefore  $f_i(h(1 \otimes_R t))e = \sigma_i(t)e \neq \sigma_j(t)e = f_j(h(1 \otimes_R t))e$ . Now, since  $T$  is supposed to be Galois-separable, we may apply lemma A.9. We obtain pairwise orthogonal idempotents  $e_1, \dots, e_r \in h(S \otimes_R T)$  such that

1.  $e_j(\sigma_i) = f_i(e_j) = \delta_{i,j}$
2.  $f_i(x)e_i = x e_i \quad \forall x \in E^H$  (sic!)

It follows that the  $e_i$  form an  $S$ -basis of  $E^H$ , hence  $h(S \otimes_R S^H) \subset E^H \subset h(S \otimes_R T)$ . We obtain  $S \otimes_R S^H \subset S \otimes_R T$  by applying  $h^{-1}$ . Then by applying  $\text{tr} \otimes_R 1$ , since the trace is surjective and  $S$  is  $R$ -projective (lemma 1.3),  $R$  is an  $R$ -module direct summand in  $S$  and we obtain  $S^H \subset T$ .  $\square$

Now we can come to the following main theorem and its proof.

**Theorem A.11.**

*Let  $R \rightarrow S$  be  $G$ -Galois and assume that all idempotents of  $S$  are central and*

that the trace  $\text{tr}: S \rightarrow R$  is surjective. Then there is a correspondence between subgroups of  $G$  and intermediate rings  $T$  that are  $G$ -strong and Galois-separable.

**Proof:** Given a ring  $T$  as in the theorem, a subgroup  $H < G$  is given by proposition A.10 and its proof.

Let us now be given any subgroup  $H < G$ . We have to show, that  $T := S^H$  is  $G$ -strong and Galois-separable.

First note, that also  $\text{tr}^H := \sum_{\tau \in H}$  is surjective, since if  $\text{tr}(d) = 1$  then we have  $\text{tr}^H(\sum_{i=1}^r \sigma_i(d)) = \text{tr}(d) = 1$  with the notation from the last proof. Let  $c \in S$  be such that  $\text{tr}^H(c) = 1$ , e.g.  $c := \sum_{i=1}^r \sigma_i(d)$ . Second, since  $R \rightarrow S$  is a  $G$ -Galois extension,  $h: S \otimes_R S \rightarrow E$  is surjective. Hence there are elements  $x_i, y_i \in S$  such that  $\sum_i x_i \sigma(y_i) = \delta_{\sigma,1}$ . We will now argue as in [24, p.23] that  $T$  is  $G$ -strong: With  $a_i := \sum_{\tau \in H} \tau(cx_i)$  and  $b_i := \sum_{g \in H} g(y_i) \in T$  we have

$$\begin{aligned}
\sum_i a_i \sigma(b_i) &= \sum_i \left( \sum_{\tau \in H} \tau(cx_i) \right) \sigma \left( \sum_{g \in H} g(y_i) \right) \\
&= \sum_i \sum_{\tau \in H} \sum_{g \in H} \tau(cx_i) \sigma(g(y_i)) \\
&= \sum_{\tau \in H} \tau \left( c \sum_{g \in H} \sum_i x_i \tau^{-1} \sigma g(y_i) \right) \tag{A.1} \\
&= \sum_{\tau \in H} \tau \left( c \sum_{g \in H} \delta_{\tau^{-1} \sigma g, 1} \right) \\
&= \begin{cases} \sum_{\tau \in H} \tau(c \cdot 1) = \text{tr}^H(c) = 1 & \text{if } \sigma \in H \\ \sum_{\tau \in H} \tau(c \cdot 0) = \text{tr}^H(0) = 0 & \text{if } \sigma \notin H. \end{cases}
\end{aligned}$$

Now for  $\sigma, \rho \in G$  which do not coincide on all of  $T$ , we have  $\rho^{-1} \sigma \notin H$ . Hence if  $e \in S$  is an idempotent, such that  $\sigma(t)e = \rho(t)e$  for all  $t \in T$ , we have  $\rho^{-1} \sigma(t) \rho^{-1}(e) = t \rho^{-1}(e)$ , hence

$$\rho^{-1}(e) = \sum_i a_i b_i \rho^{-1}(e) = \sum_i a_i \rho^{-1} \sigma(b_i) \rho^{-1}(e) = 0$$

by equation (A.1) and therefore also  $e = 0$  which shows that  $T$  is  $G$ -strong.

For the proof that  $T := S^H$  is Galois-separable, note that equation (A.1) also shows, that  $h(S \otimes_R T)$  is all of  $E^H = h((S \otimes_R S)^H)$ . Hence  $S \otimes_R S^H = (S \otimes_R S)^H$ . Now recall the trivial  $G$ -Galois extension  $S \rightarrow E$ . As the trace of the trivial  $G$ -Galois extension of  $S$  is surjective, we can apply proposition A.1 to obtain that  $E^H = h((S \otimes_R S)^H) = h(S \otimes_R S^H)$  is separable. Hence  $T = S^H$  is Galois-separable.  $\square$

Here is an example, which is neither covered by the main-theorem of [24], [42] nor by that of [37]. It also shows, that not every separable intermediate ring is also Galois-separable.

**Example A.12** (Quaternions). Let  $\mathbb{H}$  be the Hamiltonian quaternions and recall from example 1.9 that  $\mathbb{R} \rightarrow \mathbb{H}$  is a  $G$ -Galois extension with  $G \cong \mathbb{Z}/2 \times \mathbb{Z}/2$  generated by

$$c_j : \mathbb{H} \rightarrow \mathbb{H} : r_0 + r_1 i + r_2 j + r_3 k \mapsto r_0 - r_1 i + r_2 j - r_3 k$$

$$c_i : \mathbb{H} \rightarrow \mathbb{H} : r_0 + r_1i + r_2j + r_3k \mapsto r_0 + r_1i - r_2j - r_3k.$$

As  $\mathbb{H}$  is not commutative, the main-theorem of [24] does not apply and also that of [37] does not. To see that  $\mathbb{H}$  does not satisfy (H) consider  $z := \frac{1}{2}(1 \otimes_{\mathbb{Z}} 1 + i \otimes_{\mathbb{Z}} j)$ . Then  $z^2 = z$  and  $\mu(z) = \mu(z^2) = \frac{1}{2}(1 + k) \notin \{0, 1\}$ . However the example fits in the situation of our main-theorem: As a skew-field,  $\mathbb{H}$  does not have a non-trivial idempotent and since the order of  $G$  is invertible in  $\mathbb{R}$ , the trace is surjective.

We can now identify the Galois-separable intermediate rings  $\mathbb{R}[i]/(i^2 + 1) \cong \mathbb{H}^{<c_i>}$ ,  $\mathbb{R}[j]/(j^2 + 1) \cong \mathbb{H}^{<c_j>}$  and  $\mathbb{R}[k]/(k^2 + 1) \cong \mathbb{H}^{<c_i c_j>}$  corresponding to the subgroups  $\langle c_i \rangle$ ,  $\langle c_j \rangle$  and  $\langle c_i c_j \rangle$  of  $G$  of index 2. But there are other intermediate rings, for example  $T := \{r_1 + r_2(i + j + k) \in \mathbb{H} | r_1, r_2 \in \mathbb{R}\}$ . The ring  $T$  is isomorphic to  $\mathbb{C}$  hence  $\mathbb{R}$ -separable, but cannot be Galois-separable for the given extension by theorem A.11.

More examples to which theorem A.11 applies are given by cyclic algebras and generalized quaternion algebras that are subalgebras of a skew-field extension. Starting the constructions of these algebras with a Kummer extensions  $R \rightarrow T$  we know that the trace is surjective, as the the order of the Galois group is invertible in  $R$ . Moreover, subrings of skew-fileds have only trivial idempotents so that the hypotheses of theorem A.11 are satisfied.

### A.3 Relation to other main theorems

In the commutative case, the trace of a Galois extension  $R \rightarrow S$  is always surjective and also the assumption that all idempotents of  $S$  are central holds trivially. Lemma A.8 then shows that theorem A.11 specializes to the Galois correspondence of [24] and [42] in the commutative case. It is a real generalization of these theorems, since there are non-commutative rings all of whose idempotents are central, e.g. skew-fields or any other connected ring.

We want to shed more light on the relationship between “separable” and “Galois-separable”. In view of propositions A.10 and A.1 a Galois extension  $R \rightarrow S$  with  $\text{tr}(S) = R$  and  $\{\text{idempotents of } S\} \subset \text{Center}(S)$  leads to the following chain of inclusions which maps a ring  $T$  to itself:

$$\begin{array}{ccc} \{\text{Galois-separable, } G\text{-strong rings } T\} & \hookrightarrow & \{\text{subgroups of } G\} \\ & \searrow & \\ & & \{\text{separable intermediate rings } T\} \end{array} \quad (\text{A.2})$$

Hence in this situation every Galois-separable  $G$ -strong ring is separable. It would be interesting to know when exactly “Galois-separable” is the same as “separable”. This is a difficult question: An answer under settings leading to diagram (A.2) immediately implies a Galois correspondence, since the answer tells us also, when the inclusions in diagram (A.2) are in fact bijections. Example A.12 shows, that there are separable intermediate rings which are not Galois-separable.

Vice versa, having a diagram (A.2) where the inclusions are bijections, we know of course that “Galois-separable” equals “separable” for  $G$ -strong rings. For instance, assume that  $\text{tr}(S) = R$  and  $S$  verifies (H) i.e. assume the situation of Ferrero’s main theorem (theorem A.6). In this case, since  $S$  then only has trivial idempotents, every intermediate ring is automatically  $G$ -strong and  $\{\text{idempotents of } S\} \subset \text{Center}(S)$  holds obviously. Hence the assumptions of our main theorem A.11 are also satisfied and the inclusions in diagram A.2 are bijections. This shows that “Galois-separable” is also the same as “separable” for intermediate rings  $T$  of a Galois extension  $R \rightarrow S$  if the trace is surjective and  $S$  verifies (H). Hence our main-theorem can also be seen as a generalization of that of [37].



# Bibliography

- [1] J. F. Adams. *Stable homotopy and generalised homology*. University of Chicago Press, Chicago, Ill., 1974. Chicago Lectures in Mathematics.
- [2] A. A. Albert and H. Hasse. A determination of all normal division algebras over an algebraic number field. *Trans. Amer. Math. Soc.*, 34(3):722–726, 1932.
- [3] M. Ando, A. Blumberg, D. Gepner, M. Hopkins, and C. Rezk. Units of ring spectra and Thom spectra. Preprint, arXiv:0810.4535v2.
- [4] V. Angeltveit. *Noncommutative ring spectra*. PhD thesis, Massachusetts Institute of Technology, 2006. Available from <http://www.math.uchicago.edu/~vigleik/>.
- [5] V. Angeltveit. Topological Hochschild homology and cohomology of  $A_\infty$  ring spectra. *Geom. Topol.*, 12(2):987–1032, 2008.
- [6] M. Auslander and O. Goldman. The Brauer group of a commutative ring. *Trans. Amer. Math. Soc.*, 97:367–409, 1960.
- [7] A. Baker and A. Lazarev. On the Adams spectral sequence for  $R$ -modules. *Algebr. Geom. Topol.*, 1:173–199 (electronic), 2001.
- [8] A. Baker and B. Richter. Quasisymmetric functions from a topological point of view. *Mathematica Scandinavica*, 103, issue 2, (2008) 208–242.
- [9] A. Baker and B. Richter. Invertible modules for commutative  $\mathbb{S}$ -algebras with residue fields. *Manuscripta Math.*, 118(1):99–119, 2005.
- [10] A. Baker and B. Richter. Realizability of algebraic Galois extensions by strictly commutative ring spectra. *Trans. Amer. Math. Soc.*, 359(2):827–857 (electronic), 2007.
- [11] A. Baker and B. Richter. Galois extensions of Lubin-Tate spectra. *Homology, Homotopy and Applications*, 10(3):27–43, 2008. Proceedings of a Conference in honor of Douglas C. Ravenel and W. Stephen Wilson.
- [12] A. Blumberg. THH of Thom spectra that are  $E_\infty$  ring spectra. Preprint, arXiv:0811.0803v1.

- 
- [13] A. Blumberg, R. Cohen, and C. Schlichtkrull. Topological Hochschild homology of Thom spectra and the free loop space. Preprint, arXiv:0811.0553v1.
- [14] J. M. Boardman. Conditionally convergent spectral sequences. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 49–84. Amer. Math. Soc., Providence, RI, 1999.
- [15] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.
- [16] A. K. Bousfield. Homotopy spectral sequences and obstructions. *Israel J. Math.*, 66(1-3):54–104, 1989.
- [17] A. K. Bousfield. Cosimplicial resolutions and homotopy spectral sequences in model categories. *Geom. Topol.*, 7:1001–1053 (electronic), 2003.
- [18] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin, 1972.
- [19] R. Brauer, H. Hasse, and E. Noether. Beweis eines Hauptsatzes in der Theorie der Algebren. *J. Reine Angew. Math.*, 167:399–404, 1932.
- [20] S. Bullett. Permutations and braids in cobordism theory. *Proc. London Math. Soc. (3)*, 38(3):517–531, 1979.
- [21] S. Caenepeel. *Brauer groups, Hopf algebras and Galois theory*, volume 4 of *K-Monographs in Mathematics*. Kluwer Academic Publishers, Dordrecht, 1998.
- [22] G. Carlsson. Equivariant stable homotopy and Sullivan’s conjecture. *Invent. Math.*, 103(3):497–525, 1991.
- [23] H. Cartan and S. Eilenberg. *Homological algebra*. Princeton University Press, Princeton, N. J., 1956.
- [24] S. U. Chase, D. K. Harrison, and A. Rosenberg. Galois theory and Galois cohomology of commutative rings. *Mem. Amer. Math. Soc. No.*, 52:15–33, 1965.
- [25] S. U. Chase and M. E. Sweedler. *Hopf algebras and Galois theory*. Lecture Notes in Mathematics, Vol. 97. Springer-Verlag, Berlin, 1969.
- [26] L. N. Childs. *Taming wild extensions: Hopf algebras and local Galois module theory*, volume 80 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.
- [27] F. R. Cohen. Braid orientations and bundles with flat connections. *Invent. Math.*, 46(2):99–110, 1978.
- [28] F. R. Cohen, J. P. May, and L. R. Taylor.  $K(\mathbf{Z}, 0)$  and  $K(\mathbf{Z}_2, 0)$  as Thom spectra. *Illinois J. Math.*, 25(1):99–106, 1981.



- 
- [29] F. R. DeMeyer. Some notes on the general Galois theory of rings. *Osaka J. Math.*, 2:117–127, 1965.
- [30] F. R. DeMeyer and E. Ingraham. *Separable algebras over commutative rings*. Lecture Notes in Mathematics, Vol. 181. Springer-Verlag, Berlin, 1971.
- [31] E. S. Devinatz and M. J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topology*, 43(1):1–47, 2004.
- [32] L. E. Dickson. Linear algebras in which division is always uniquely possible. *Bull. Amer. Math. Soc.*, 12:441–442, 1905-6.
- [33] A. Dold and D. Puppe. Duality, trace, and transfer. In *Proceedings of the International Conference on Geometric Topology (Warsaw, 1978)*, pages 81–102, Warsaw, 1980. PWN.
- [34] A. W. M. Dress. Basic noncommutative Galois theory. *Adv. Math.*, 110(1):141–173, 1995.
- [35] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.
- [36] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.
- [37] M. Ferrero. On the Galois theory of non-commutative rings. *Osaka J. Math.*, 7:81–88, 1970.
- [38] P. Gille and T. Szamuely. *Central simple algebras and Galois cohomology*, volume 101 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.
- [39] P. Goerss. Associative  $MU$ -algebras. Preprint available from <http://www.math.northwestern.edu/~pgoerss/>.
- [40] P. Goerss and M. Hopkins. Moduli problems for structured ring spectra. Preprint available from <http://www.math.northwestern.edu/~pgoerss/>.
- [41] P. Goerss and M. Hopkins. Moduli spaces of commutative ring spectra. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 151–200. Cambridge Univ. Press, Cambridge, 2004.
- [42] C. Greither. *Cyclic Galois extensions of commutative rings*, volume 1534 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1992.
- [43] H. Hasse. Über  $\wp$ -adische Schiefkörper und ihre Bedeutung für die Arithmetik hyperkomplexer Zahlssysteme. *Math. Ann.*, 104(1):495–534, 1931.

- 
- [44] K. Hess. The cobar construction: A modern perspective. Lecture notes from a minicourse at Louvain-la-Neuve, available from <http://sma.epfl.ch/~hessbell/>.
- [45] K. Hess. Homotopic Hopf-Galois extensions: Foundations and examples. Preprint, arXiv:0902.3393v1.
- [46] T. Hewett. Finite subgroups of division algebras over local fields. *J. Algebra*, 173(3):518–548, 1995.
- [47] K. Hirata and K. Sugano. On semisimple extensions and separable extensions over non commutative rings. *J. Math. Soc. Japan*, 18:360–373, 1966.
- [48] M. Hovey and N. P. Strickland. Morava  $K$ -theories and localisation. *Mem. Amer. Math. Soc.*, 139(666), 1999.
- [49] S. Ikehata. A note on separable polynomials in skew polynomial rings of derivation type. *Math. J. Okayama Univ.*, 22(1):59–60, 1980.
- [50] S. Ikehata. On  $H$ -separable and Galois polynomials of degree  $p$  in skew polynomial rings. *Int. Math. Forum*, 3(29-32):1581–1586, 2008.
- [51] T. Kanzaki. On commutator rings and Galois theory of separable algebras. *Osaka J. Math.* 1, 103-115; correction, *ibid.*, 1:253, 1964.
- [52] T. Kanzaki. On Galois algebra over a commutative ring. *Osaka J. Math.*, 2:309–317, 1965.
- [53] T. Kanzaki. On Galois extension of rings. *Nagoya Math. J.*, 27:43–49, 1966.
- [54] G. M. Kelly. *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1982.
- [55] K. Kishimoto. On abelian extensions of rings. I. *Math. J. Okayama Univ.*, 14:159–174, 1969/1970.
- [56] K. Kishimoto. On abelian extensions of rings. II. *Math. J. Okayama Univ.*, 15:57–70, 1971/72.
- [57] M.-A. Knus and M. Ojanguren. *Théorie de la descente et algèbres d’Azumaya*, volume 389 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1974.
- [58] H. F. Kreimer and M. Takeuchi. Hopf algebras and Galois extensions of an algebra. *Indiana Univ. Math. J.*, 30(5):675–692, 1981.
- [59] T. Y. Lam. *A first course in noncommutative rings*, volume 131 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001.

- [60] S. Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.
- [61] A. Lazarev. Homotopy theory of  $A_\infty$  ring spectra and applications to  $MU$ -modules. *K-Theory*, 24(3):243–281, 2001.
- [62] A. Lazarev. Towers of  $MU$ -algebras and the generalized Hopkins-Miller theorem. *Proc. London Math. Soc. (3)*, 87(2):498–522, 2003.
- [63] A. Lazarev. Cohomology theories for highly structured ring spectra. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 201–231. Cambridge Univ. Press, Cambridge, 2004.
- [64] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.
- [65] M. Mahowald. Ring spectra which are Thom complexes. *Duke Math. J.*, 46(3):549–559, 1979.
- [66] J. C. McConnell. Division algebras—Beyond the quaternions. *Amer. Math. Monthly*, 105(2):154–162, 1998.
- [67] H. Miller. The Sullivan conjecture on maps from classifying spaces. *Ann. of Math. (2)*, 120(1):39–87, 1984.
- [68] H. Miller. Correction to: “The Sullivan conjecture on maps from classifying spaces” [Ann. of Math. (2) **120** (1984), no. 1, 39–87; MR0750716 (85i:55012)]. *Ann. of Math. (2)*, 121(3):605–609, 1985.
- [69] D. C. Ravenel. The structure of Morava stabilizer algebras. *Invent. Math.*, 37(2):109–120, 1976.
- [70] D. C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986.
- [71] I. Reiner. *Maximal orders*, volume 28 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, Oxford, 2003. Corrected reprint of the 1975 original, With a foreword by M. J. Taylor.
- [72] C. Rezk. Notes on the Hopkins-Miller theorem. In *Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997)*, volume 220 of *Contemp. Math.*, pages 313–366. Amer. Math. Soc., Providence, RI, 1998.
- [73] A. Robinson. Obstruction theory and the strict associativity of Morava  $K$ -theories. In *Advances in homotopy theory (Cortona, 1988)*, volume 139 of *London Math. Soc. Lecture Note Ser.*, pages 143–152. Cambridge Univ. Press, Cambridge, 1989.

- [74] A. Robinson. Classical obstructions and  $S$ -algebras. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 133–149. Cambridge Univ. Press, Cambridge, 2004.
- [75] J. Rognes. Galois extensions of structured ring spectra. *Mem. Amer. Math. Soc.*, 192(898):1–98, 2008.
- [76] J. Rognes. Stably dualizable groups. *Mem. Amer. Math. Soc.*, 192(898):99–132, 2008.
- [77] R. Schwänzl, R. M. Vogt, and F. Waldhausen. Adjoining roots of unity to  $E_\infty$  ring spectra in good cases—a remark. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 245–249. Amer. Math. Soc., Providence, RI, 1999.
- [78] S. Schwede. Morita theory in abelian, derived and stable model categories. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 33–86. Cambridge Univ. Press, Cambridge, 2004.
- [79] S. Schwede and B. E. Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000.
- [80] D. Ştefan. Hochschild cohomology on Hopf Galois extensions. *J. Pure Appl. Algebra*, 103(2):221–233, 1995.
- [81] N. P. Strickland. Products on  $MU$ -modules. *Trans. Amer. Math. Soc.*, 351(7):2569–2606, 1999.
- [82] M. E. Sweedler. *Hopf algebras*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.
- [83] G. Szeto and L. Xue. On splitting rings for Azumaya skew polynomial rings. *Algebra Colloq.*, 8(1):25–32, 2001.
- [84] R. Thom. Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.*, 28:17–86, 1954.
- [85] V. V. Vershinin. Thom spectra of generalized braid groups. *J. London Math. Soc. (2)*, 61(1):245–258, 2000.
- [86] S. Wüthrich.  $I$ -adic towers in topology. *Algebr. Geom. Topol.*, 5:1589–1635 (electronic), 2005.
- [87] S. Wüthrich. Infinitesimal thickenings of Morava  $K$ -theories. *J. Pure Appl. Algebra*, 212(1):99–121, 2008.

# Zusammenfassung

Ausgehend von der Definition von Galois Erweiterungen für assoziative Ringe und der entsprechenden Definition für kommutative  $S$ -Algebren nach John Rognes [75], haben wir Galois Erweiterungen von assoziativen  $S$ -Algebren definiert. Hierzu haben wir zunächst die Modellstruktur auf der Kategorie der  $S$ -Algebren unter einer gegebenen kofibranten  $S$ -Algebra  $A$  analog zum Vorgehen in [36] untersucht. Das Hauptergebnis ist, dass das Smashprodukt von kofibranten  $S$ -Algebren unter  $A$  das derivierte Smashprodukt repräsentiert. Dies ist wesentlich, um die Homotopieinvarianz unserer Definition von Galois Erweiterungen sicher zu stellen.

Wir haben dann Galois Erweiterungen von assoziativen  $S$ -Algebren weiter untersucht. Wir zeigen, dass Dualisierbarkeit auch im assoziativen Kontext Sinn macht, und dass jede Galois Erweiterung  $B$  einer assoziativen  $S$ -Algebra  $A$  dualisierbar über  $A$  und  $A^{\text{op}}$  ist. Das konnte benutzt werden, um Galois Erweiterungen zu charakterisieren. Wir behandeln viele Beispiele, beginnend mit den sogenannten trivialen Erweiterungen und Erweiterungen von Matrix  $S$ -Algebren. Wir beweisen ein Eilenberg-MacLane Einbettungstheorem für Galois Erweiterungen von assoziativen Ringen mit surjektiver Spur, das insbesondere auf alle assoziativen Galois Erweiterungen von kommutativen Grundringen anwendbar ist. Somit ist die algebraische Theorie in diesen Fällen in der topologischen Theorie enthalten.

Desweiteren haben wir induzierte Erweiterungen untersucht. Hierfür beweisen wir Entdeckungs- und Erhaltungssätze. Insbesondere liefern die Lubin-Tate Erweiterungen aus [75] Galois Erweiterungen von assoziativen  $S$ -Algebren durch Induzieren entlang von Abbildungen in entsprechende Morava- $K$ -Theorie Spektren. Wir erhalten so auch eine pro-Galois Erweiterung  $K_n \rightarrow K_n^{nr} = E_n^{nr} \wedge_{E_n} K_n$  und zeigen, dass das Spektrum  $K_n^{nr}$  zumindest an ungeraden Primzahlen ein separabler Abschluss bezüglich Erweiterungen mit kommutativen Homotopiegruppen ist.

In Teil II haben wir den allgemeineren Begriff von Hopf-Galois Erweiterungen für assoziative  $S$ -Algebren definiert. Erneut untersuchen wir induzierte Erweiterungen. Die allgemeinere Sichtweise erlaubt es, nicht nur hinreichende sondern auch notwendige Bedingungen für die Erhaltung und Erkennung von Hopf-Galois Erweiterungen zu erhalten, was eine Frage von Rognes beantwortet. Wir haben uns dann mit Thomspektren und regulären Quotienten beschäftigt. Analog zur Hopf-Galois Erweiterung  $S \rightarrow MU$  war allgemein erwartet worden, dass Thomspektren zu unendlichen Schleifenabbildungen Hopf-Galois Erweiterungen sind. Wir konnten diesen Ansatz verallgemeinern und präzisieren. Insbesondere

liefern Hopf-Galois Erweiterungen zu (endlichen oder unendlichen) Schleifenabbildungen in den klassifizierenden Raum  $BF = BGL_1S$  für stabile sphärische Faserungen genau dann Hopf-Galois Erweiterungen, wenn das Thomspektrum orientierbar entlang  $H\mathbb{Z}$  ist.

In Teil III haben wir drei Anwendungen behandelt. Wir beschäftigten uns zunächst mit einem Realisierungsproblem und konnten so insbesondere die Existenz von Ringspektren zeigen, die zyklische und verallgemeinerte Quaternionenerweiterungen von bestimmten kommutativen  $S$ -Algebren sind.

In einem weiteren Kapitel haben wir gezeigt, dass für treue Galois Erweiterungen  $A \rightarrow B$  von assoziativen  $S$ -Algebren das Erweiterungsspektrum  $B$  stets ein invertierbarer  $(A[G], F_{A[G]}(B, B)^{\text{op}})$ -Bimodul ist. Ist die Galoisgruppe zudem abelsch, so gibt es eine schwache Äquivalenz  $F_{A[G]}(B, B)^{\text{op}} \simeq A[G]$ , sodass  $B$  ein Element in der Picardgruppe  $\text{Pic}(A[G])$  definiert.

Im letzten Kapitel haben wir die topologische Hochschildhomologie von Hopf-Galois Erweiterungen kommutativer  $S$ -Algebren untersucht. Für Thomspektren  $Mf$  zu einer unendlichen Schleifenabbildung  $f: X \rightarrow BGL_1S$  konnten wir die Äquivalenz  $\text{THH}^S(Mf) \simeq Mf \wedge BX_+$  [12, 13] erneut beweisen. Ist allgemeiner  $A \rightarrow B$  eine Hopf-Galois Erweiterung in Bezug auf eine Hopfalgebra  $H$  mit Homotopieantipode, so gibt es eine Äquivalenz

$$\text{THH}^A(B; M) \simeq M \wedge_A \mathcal{B}^A(A, H, A),$$

wobei  $\mathcal{B}^A(A, H, A)$  die Barkonstruktion bezeichnet. Zum Schluss haben wir gezeigt, dass es für kommutative Hopf-Galois Erweiterungen unter einer kommutativen  $S$ -Algebra  $R$  mit Hopfalgebra  $H = A \wedge_R K$  eine Spektralsequenz der Form

$$E_{p,q}^2 = \text{Tor}_*^{\pi_*(K \wedge_R K)}(K_*, \pi_* \text{THH}^R(A; M)) \implies \pi_* \text{THH}^R(B; M)$$

gibt.

Der Anhang enthält eine Galoiskorrespondenz für assoziative Ringerweiterungen.

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