Sufficient Optimality Conditions for Nonsmooth Optimal Control Problems

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von
Ricki Rosendahl
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auf Grund der Gutachten von Prof. Dr. Hans Joachim Oberle
und Prof. Dr. Matthias Gerdts.

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Prof Dr. Reiner Lauterbach
Leiter des Departments Mathematik

Für Frederik und Helene

Preface

The following work was mainly developed during my work at the department of mathematics at the univerity of Hamburg.

I would like to express my cordial thanks to my supervisor Prof. Dr. Hans Joachim Oberle for the excellent guidance and enriching discussions. He provided a very good atmosphere and working environment. I am grateful to my co-supervisor Prof. Dr. Matthias Gerdts for his support.

Special thanks go to my family for giving me constant strength and encouragement. I want to give very special thanks to my beloved husband for always being there for me and believing in me.

This work is dedicated to my children who show me a wonderful world.

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Introduction

In natural sciences and economics many processes can be described by mathematical models. The state of the model is determined by differential equations. In the case, these equations leave a degree of freedom, we are able to control the system in order to optimize a certain aspect. These processes therefore can be described by an *optimal control problem*.

Optimal control problems are infinite-dimensional optimization problems. The state variables are defined by ordinary or partial differential equations. The system can be influenced by control functions. There may occur boundary equality conditions for the state variable and mixed or pure state or control inequality constraints. We consider for example an Earth-Mars orbit transfer problem. The position and the velocity of the rocket are the state variables. The rocket can be controlled by its thrust and thrust angle. We are looking for control functions such that the transfer time or the fuel consumption is minimized.

In order to find a solution to an optimal control problem, necessary optimality conditions have been developed in the last 60 years. Two different approaches have asserted themselves. Following the way of indirect methods, necessary conditions in optimal control theory are applied to a problem [PBGM64], [Hes66],[BH75], [Neu76], [MZ79], [FH86], [MO98], [AS04], [Ger06]. The optimal control problem is transformed into a boundary value problem, which can be solved by hand or in most cases by means of a computer [Mau76], [Obe79], [Obe86], [OG89], [Obe90]. Using direct methods, the control function and/or the state variables are discretized. We then have to solve a finite dimensional optimization problem [Bue98], [BM00], [Ger03], [GK08]. The application of necessary conditions leads to stationary points of the optimal control problem. We have to check sufficient conditions to assure optimality.

In a first approach developing sufficient optimality conditions convex problems were treated [Man66], [Str95]. The verification of second order sufficient optimality conditions dealing with positive definiteness on certain sets can be applied to more general optimal control problems [MZ79], [Mau81], [Zei93], [Mal97], [MO98], [Mau99], [Zei01], [AS04], [Vos05], [BCT07]. A special approach yields in positive definiteness conditions on critical cones when treating problems with linearly appearing control functions [MO03], [MO04], [MBKK05]. In the case of nonlinear control functions, the Riccati approach has been investigated [Zei94], [MP95], [Str97], [Doe98], [MO02], [OL02] [MMP04].

We put the main focus on the development of new second order sufficient optimality conditions for special classes of optimal control problems following the Riccati approach. Here, the state variables are described by ordinary differential equations. The control functions usually enter into the problem nonlinearly. The verification of sufficient conditions using numerical methods is a second emphasis.

As a start, we introduce general optimal control problems in **chapter 1**. The state variables are described by ordinary differential equations. We recapitulate necessary and sufficient optimality conditions and apply these to an example in electrical engeneering. We take a closer look on how to reduce the sufficient conditions. These new results are useful for the application of sufficient optimality conditions to examples.

We go into detail describing optimal control problems with free final time in **chapter 2**. The theory implies a special structure of sufficient optimality conditions. We apply these conditions to the Re-entry optimal control problem. This chapter provides a basis for the succeeding chapters, in which new sufficient conditions are developed.

In **chapter 3**, we introduce nonsmooth optimal control problems. The ordinary differential equations describing the state variables may have discontinuities in the right hand side. Therefore the state variables are piecewise smooth. The points of discontinuity are determined implicitely by a switching function, which may change sign at finitely many times. The theory can be developed with the aid of a multiprocess technique. Thus, we obtain new sufficient conditions for this type of nonsmooth optimal control problems. By means of examples in electrical engeneering and navigation we explain how to apply sufficient conditions and point out difficulties in numerics.

Optimal control problems whose solutions may include *free control subarcs* are treated in **chapter 4**. Here, on subintervals, the appearing functions can be influenced only by some of the control components. This is the case for example in space-travel problems. If the thrust affecting to the space shuttle is zero on a certain time interval, the thrust angle cannot influence the system any more. Then a free control subarc occurs in the function describing the thrust angle. The verification of new sufficient optimality conditions will be discussed dealing with two examples. Numerical difficulties occur in the treatment of the Earth-Mars orbit transfer problem, that cannot be solved at the moment.

In **chapter 5** we introduce optimal control problems with *singular state subarcs*. Here nonsmooth optimal control problems that have already been investigated in chapter 3 are treated. The theory will be augmented to the case, where the switching function may vanish on whole subintervals. We deal with an augmentation of the example in electrical engeneering and apply the here developed sufficient conditions.

We conclude this work with a summary and an outlook pointing out further possibilities of generating sufficient conditions and combinations of the here developed theory.

Chapter 1

General Optimal Control Theory

In this chapter we summarize the current standards in optimal control theory. In section 1.1 we introduce a general optimal control problem followed by a list of conventions in section 1.2. Known first order necessary optimality conditions will be described in section 1.3. In section 1.4 we repeat second order sufficient optimality conditions.

1.1 The General Optimal Control Problem

We are concerned with optimal control problems, which are governed by ordinary differential equations. We treat a bounded time interval with a fixed starting time t_0. The final time t_f may be fixed or free. The trajectory of the *state variable*

$$x : [t_0, t_f] \to \mathbb{R}^n$$

is given by the differential equation

$$x'(t) = f(t, x(t), u(t)), \qquad t \in [t_0, t_f],$$

which we call *state equation*. The slope of the state variable can be influenced by the *control function*

$$u : [t_0, t_f] \to \mathbb{R}^m,$$

which describes the degree of freedom in the state equation. We look for a control function u, which minimizes the real-valued *cost functional*

$$J(x, u) = \phi(x(t_f)) + \int_{t_0}^{t_f} L(t, x(t), u(t))dt.$$

We deal with piecewise continuous control functions and continuous and piecewise continuous differentiable state variables:

Definition 1.1. *We call a function* $u : [t_0, t_f] \to \mathbb{R}^m$ piecewise continuous, *if there exists a subdivision* $t_0 < t_1 < \cdots < t_N = t_f, N \in \mathbb{N}$, *such that* u *is continuous on each open subinterval* $]t_{i-1}, t_i[, i = 1, \ldots, N$ *and the limits*

$$u(t_i^-) := \lim_{t \to t_i, t < t_i} u(t) \in \mathbb{R}^m \qquad \text{and} \qquad u(t_i^+) := \lim_{t \to t_i, t > t_i} u(t) \in \mathbb{R}^m,$$

$i = 1, \ldots, N-1$, *are real-valued numbers.*

We call a function $x : [t_0, t_f] \to \mathbb{R}^n$ piecewise continuous differentiable, *if there exists a subdivision* $t_0 < t_1 < \cdots < t_N = t_f, N \in \mathbb{N}$, *such that* x *is continuous differentiable on each open subinterval* $]t_{i-1}, t_i[, i = 1, \ldots, N$ *and the limits*

$$x'(t_i^-) := \lim_{t \to t_i, t < t_i} x'(t) \in \mathbb{R}^n \qquad \text{and} \qquad x'(t_i^+) := \lim_{t \to t_i, t > t_i} x'(t) \in \mathbb{R}^n,$$

$i = 1, \ldots, N-1$, *are real-valued numbers. The times* t_i^- *and* t_i^+ *are the times right before and right after the time* t_i *respectively.*

Furthermore, $|\cdot|$ *denotes the absolute value of a real number,* $||\cdot||$ *denotes the Euclidean norm of* $\mathbb{R}^n$, *and we use the* L^2*-norm*

$$||x||_2 := \left\{ \int_{t_0}^{t_f} \left(\sum_{i=1}^{n} |x_i(t)|^2 \right) dt \right\}^{\frac{1}{2}}$$

and the L^∞*-norm*

$$||x||_\infty := \max_{j=1,\ldots,n} \sup_{t_0 \le t \le t_f} |x_j(t)|$$

of functions $x : [t_0, t_f] \to \mathbb{R}^n$.

We say, that an equation holds almost everywhere *(a.e.), if the set of elements for which the equation does not hold is a set with measure zero.*

In practice there occur different types of restrictions. On the one hand, the state variable x may have to satisfy equality conditions at the initial and final time. These *general boundary conditions* are given in the form

$$r(x(t_0), x(t_f)) = 0.$$

If some components of the state variable are fixed at the starting or final time, we write

$$\begin{aligned} x_j(t_0) &= x_{sj}, \qquad j \in I_s \subset \{1, \ldots, n\}, \\ x_k(t_f) &= x_{fk}, \qquad k \in I_f \subset \{1, \ldots, n\}. \end{aligned}$$

On the other hand, the state variable x or control function u can be restricted on the entire time interval $[t_0, t_f]$ by *mixed state and control constraints*

$$C(t, x(t), u(t)) \le 0, \qquad t \in [t_0, t_f].$$

The corresponding state-control-set is defined by

$$\mathcal{U}(t) := \{(x(t), u(t)) \in \mathbb{R}^n \times \mathbb{R}^m \mid C(t, x(t), u(t)) \leq 0\}, \qquad t \in [t_0, t_f].$$

We often have to deal with affine linear constraints in form of *control bounds*

$$u_j(t) \in [u_{j\,\min}, u_{j\,\max}], \qquad t \in [t_0, t_f], j = 1, \ldots, m.$$

In summary, general optimal control problems are expressed in the following way.

OCP 1.1. *Determine a piecewise continuous control function $u : [t_0, t_f] \to \mathbb{R}^m$, such that the functional*

$$J(x, u) = \phi(x(t_f)) + \int_{t_0}^{t_f} L(t, x(t), u(t))dt \tag{1.1}$$

is minimized subject to

$$\begin{aligned} x'(t) &= f(t, x(t), u(t)), && t \in [t_0, t_f], a.e., && (1.2)\\ r(x(t_0), x(t_f)) &= 0, && && (1.3)\\ C(t, x(t), u(t)) &\leq 0, && t \in [t_0, t_f], a.e., && (1.4) \end{aligned}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$, $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^\ell$ and $C : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^d$ are twice continuous differentiable.

Definition 1.2. *A pair (x, u) is said to be* admissible *for OCP 1.1, if x is continuous and piecewise continuous differentiable, u is piecewise continuous, and the constraints (1.2)-(1.4) are satisfied.*

Definition 1.3. *An admissible pair (x_0, u_0) is called a* weak local minimum *for OCP 1.1 if for some $\epsilon > 0$, (x_0, u_0) minimizes $J(x, u)$ over all admissible pairs (x, u) satisfying*

$$||x - x_0||_\infty < \epsilon \quad \text{and} \quad ||u - u_0||_\infty < \epsilon.$$

The pair (x_0, u_0) is called a strong local minimum *if only the first inequality holds. We call (x_0, u_0)* global minimum *if the above condition is satisfied for $\epsilon = \infty$.*

A detailed examination of weak and strong minima can for example be found in [Fel01] or [Vin00].

The general optimal control problem OCP 1.1 may have a fixed or free final time t_f. A typical example of an optimal control problem with free final time is where the final time itself shall be minimized. Developing necessary and sufficient optimality conditions we distinguish between these cases.

We transform OCP 1.1 with free final time t_f into an auxiliary optimal control problem with fixed final time by introducing a new time variable $s \in [0,1]$ with $t = t_0 + s \cdot (t_f - t_0)$. We now define the new state variable

$$x_{n+1}(s) := \tau(s) := t_f - t_0, \qquad s \in [0,1].$$

The remaining functions are given by $x(s) := x(t_0 + s\tau(s))$ and $u(s) := u(t_0 + s\tau(s))$. If we differentiate the state variables x and x_{n+1} we get the corresponding state equations

$$\begin{aligned} x'(s) &= \tau(s) \cdot f(t_0 + s\tau(s), x(s), u(s)), \\ \tau'(s) &= 0. \end{aligned}$$

Thus, the transformed optimal control problem is given by

OCP 1.2. *Determine a piecewise continuous control function $u : [0,1] \to \mathbb{R}^m$, such that the functional*

$$J(x,u) = \phi(x(1)) + \int_0^1 \tau(s) \cdot L(t_0 + s\tau(s), x(s), u(s))ds$$

is minimized subject to

$$\begin{aligned} x'(s) &= \tau(s) \cdot f(t_0 + s\tau(s), x(s), u(s)), && s \in [0,1], a.e., \\ \tau'(s) &= 0, && s \in [0,1], a.e., \\ r(x(0), x(1)) &= 0, \\ C(t_0 + s\tau(s), x(s), u(s)) &\le 0, && s \in [0,1], a.e.. \end{aligned}$$

It is also possible, to transform OCP 1.1 with fixed final time in the same way as done above. We then have to add the boundary condition

$$\tau(0) = t_f - t_0.$$

to the transformed problem OCP 1.2.

We transform the general optimal control problem OCP 1.1 into an autonomous problem by introducing an additional state variable:

$$T(t) := t,\ t \in [t_0, t_f] \quad \Rightarrow \quad T'(t) = 1,\ T(t_0) = t_0.$$

The time appearing explicitely in L, f, and C must be replaced by $T(t)$.

OCP 1.3. *Determine a piecewise continuous control function $u : [t_0, t_f] \to \mathbb{R}^m$, such that the functional*

$$J(x,u) = \phi(x(t_f)) + \int_{t_0}^{t_f} L(T(t), x(t), u(t))dt \tag{1.5}$$

is minimized subject to

$$\begin{aligned}
x'(t) &= f(T(t), x(t), u(t)), && t \in [t_0, t_f], a.e., && (1.6)\\
T'(t) &= 1, && t \in [t_0, t_f], a.e., && (1.7)\\
r(x(t_0), x(t_f)) &= 0, && && (1.8)\\
T(t_0) &= t_0, && && (1.9)\\
C(T(t), x(t), u(t)) &\leq 0, && t \in [t_0, t_f], a.e.. && (1.10)
\end{aligned}$$

In practice there occur different kinds of formulations for general optimal control problems. The optimal control problem OCP 1.1 is written in Bolza-form. That is, the cost functional (1.1) is the sum of a function depending on the endpoint of the state variable and an integral. We call an optimal control problem to be in Mayer-form, if $L \equiv 0$. In the case $\phi \equiv 0$, we treat a so called optimal control problem in Lagrange-form. The optimal control problems in Mayer- or Lagrange-form can be seen as special types of optimal control problems in Bolza-form. It is easy to show the equivalence of the three types of optimal control problems.

For the transformation of OCP 1.1 into an optimal control problem in Mayer-form, we introduce a new state variable

$$x_{n+1}(t) := \int_{t_0}^{t} L(s, x(s), u(s)) ds, \qquad t \in [t_0, t_f].$$

We receive a new cost functional and an additional state equation and boundary condition. This results in the following problem.

OCP 1.4. *Determine a piecewise continuous control function $u : [t_0, t_f] \to \mathbb{R}^m$, such that the functional*

$$J(x, u) = \phi(x(t_f)) + x_{n+1}(t_f)$$

is minimized subject to

$$\begin{aligned}
x'(t) &= f(t, x(t), u(t)), && t \in [t_0, t_f], a.e.,\\
x'_{n+1}(t) &= L(t, x(t), u(t)), && t \in [t_0, t_f], a.e.,\\
r(x(t_0), x(t_f)) &= 0, &&\\
x_{n+1}(t_0) &= 0, &&\\
C(t, x(t), u(t)) &\leq 0, && t \in [t_0, t_f], a.e..
\end{aligned}$$

The transformation of an optimal control problem in Lagrange-form into Mayer-form can be treated in a similar way.

In the succeeding text, we will treat general Bolza-type optimal control problems OCP 1.1 with fixed or free final time.

1.2 Conventions

In this section we summarize all conventions which are used in the following work.

Column and row vectors. To avoid to transpose a lot of vectors, we are using the following arrangement.

The functions

$$\begin{aligned} x : [t_0, t_f] &\rightarrow \mathbb{R}^n \\ u : [t_0, t_f] &\rightarrow \mathbb{R}^m \end{aligned}$$

are considered as column vectors.
Whereas the functions

$$\begin{aligned} \lambda : [t_0, t_f] &\rightarrow \mathbb{R}^n \\ \mu : [t_0, t_f] &\rightarrow \mathbb{R}^d \end{aligned}$$

are treated as row vectors. Further, the multiplier $\nu \in \mathbb{R}^p$ and the element η of a kernel are row vectors.

Derivatives. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a C^1-function. Then

$$D_x f(x) := \begin{pmatrix} \frac{\partial}{\partial x_1} f_1(x) & \dots & \frac{\partial}{\partial x_n} f_1(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} f_m(x) & \dots & \frac{\partial}{\partial x_n} f_m(x) \end{pmatrix}$$

denotes the Jacobian of f.

Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^2-function. $(D_x g(x))^T$ is this gradient of the scalar function g. The Hessian of g is given by the symmetric matrix-function

$$D_x^2 g(x) := \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} g(x) & \dots & \frac{\partial^2}{\partial x_1 \partial x_n} g(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial x_n \partial x_1} g(x) & \dots & \frac{\partial^2}{\partial x_n^2} g(x) \end{pmatrix}.$$

For simplicity, we use the notation of subscripts for derivatives.

$$f_x(x) := D_x f(x), \qquad g_{xx} := D_x^2 g(x).$$

Abbreviations. Let the arguments of the functions ϕ and r be $x_s \in \mathbb{R}^n$ and $x_f \in \mathbb{R}^n$. The first and second order derivatives are denoted by D_{x_s}, D_{x_f} and $D_{x_s}^2, D_{x_s x_f}^2, D_{x_f}^2$. The subscript letters s and f are abbreviations for *start* and *final* respectively.

For simplicity, arguments of functions involving $x_0(t)$ and $u_0(t)$ will be abbreviated by $[t]$; for instance $\mathcal{H}[t] = \mathcal{H}(t, x_0(t), u_0(t), \lambda_0, \lambda(t))$.

If the meaning is clear from the context, we omit the argument t, for example $f(t, x(t), u(t))$ will be expressed simply by $f(t, x, u)$. Also the dependencies of ϕ and r are omitted. We write for example $D_{x_s}[\phi + \nu r]$ instead of $D_{x_s}[\phi(x_0(t_f)) + \nu r(x_0(t_0), x_0(t_f))]$.

If in a matrix not all elements are shown, these elements are zero.

Kernel. We will be dealing with Kernels of matrices. Let $M \in \mathbb{R}^{n\times m}$ be a matrix. Then

$$\ker\{M\} := \{\eta \in \mathbb{R}^n | M\eta^T = 0\}.$$

In this exposition, η will be treated as row vector.

Positive Definiteness. We use the reduced notation $M \geq 0$ or $M > 0$ for saying that the matrix M is positive semidefinite or positive definite respectively.

1.3 General Necessary Optimality Conditions

In this section we summarize the well known first order necessary optimality conditions for solutions of optimal control problems. There exists plenty of literature, which discribes necessary conditions [PBGM64], [Hes66], [BH75], [Neu76], [MZ79], [FH86], [MO98], [Ger06].

We follow the formulation of [Hes66] and define

Definition 1.4. *The* Hamiltonian *for the general optimal control problem OCP 1.1 is defined for $\lambda_0 \in \mathbb{R}$ and $\lambda \in \mathbb{R}^n$ by*

$$\begin{aligned} \mathcal{H} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n &\rightarrow \mathbb{R}, \\ \mathcal{H}(t, x, u, \lambda_0, \lambda) &:= \lambda_0 L(t, x, u) + \lambda f(t, x, u). \end{aligned}$$

The augmented Hamiltonian *for the general optimal control problem with mixed state and control constraints is defined for $\lambda_0 \in \mathbb{R}$, $\lambda \in \mathbb{R}^n$, and $\mu \in \mathbb{R}^d$ by*

$$\begin{aligned} \widetilde{\mathcal{H}} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^d &\rightarrow \mathbb{R}, \\ \widetilde{\mathcal{H}}(t, x, u, \lambda_0, \lambda, \mu) &:= \mathcal{H}(t, x, u, \lambda_0, \lambda) + \mu C(t, x, u). \end{aligned}$$

Let (x_0, u_0) be a weak local minimum. We emphasize extentions of second order sufficient optimality conditions. We do not regard pure state constraints in the treated examples. Therefore we introduce

Regularity Condition R1. *For each $t \in [t_0, t_f]$, the vectors*

$$D_u C_k(t, x_0(t), u_0(t)), \qquad k \in I^0(t) \tag{1.11}$$

are linear independent, with $I^0(t) := \{k \in \{1, \dots, d\} \mid C_k(t, x_0(t), u_0(t)) = 0\}$.

Regularity Condition R2.

$$rank\{D_{(x_s, x_f)} r(x_0(t_0), x_0(t_f))\} = \ell. \tag{1.12}$$

We can formulate first order necessary optimality conditions for OCP 1.1 with fixed final time t_f.

Theorem 1.5. *Let (x_0, u_0) be a weak local minimum of OCP 1.1. Let regularity conditions R1 and R2 be satisfied. Then there exist a continuous and piecewise continuous differentiable adjoint variable $\lambda : [t_0, t_f] \to \mathbb{R}^n$, a piecewise continuous multiplier function $\mu : [t_0, t_f] \to \mathbb{R}^d$, and multipliers $\nu \in \mathbb{R}^\ell$, and $\lambda_0 \in \mathbb{R}$ not all vanishing simultaneously on $[t_0, t_f]$, such that (x_0, u_0) satisfies*

(a) the adjoint equation

$$\lambda'(t) = -\widetilde{\mathcal{H}}_x(t, x_0(t), u_0(t), \lambda_0, \lambda(t), \mu(t)), \qquad t \in [t_0, t_f], a.e., \tag{1.13}$$

(b) the natural boundary conditions

$$\lambda(t_0) = -D_{x_s}\left[\nu r(x_0(t_0), x_0(t_f))\right], \tag{1.14}$$

$$\lambda(t_f) = D_{x_f}\left[\lambda_0 \phi(x_0(t_f)) + \nu r(x_0(t_0), x_0(t_f))\right], \tag{1.15}$$

(c) the complementary condition

$$\mu(t) \geq 0 \quad and \quad \mu(t) C(t, x_0(t), u_0(t)) = 0, \qquad t \in [t_0, t_f], a.e., \tag{1.16}$$

(d) the minimum principle

$$u_0(t) = argmin\{\widetilde{\mathcal{H}}(t, x_0(t), u, \lambda_0, \lambda(t), \mu(t)) | u \in \mathbb{R}^m\}, \; t \in [t_0, t_f], a.e., \tag{1.17}$$

(e) and the minimum condition

$$\widetilde{\mathcal{H}}_u(t, x_0(t), u_0(t), \lambda_0, \lambda(t), \mu(t)) = 0, \qquad t \in [t_0, t_f], a.e.. \tag{1.18}$$

Proof. [Hes66] chapters 6 and 7. □

More general formulations and extensions to the above are for example stated in [Neu76], [MO98], [Ger06].

Remark 1.6. *A nonzero Lagrange-multiplier* λ_0 *is called* normal. *There exist regularity conditions, which imply the normality of Lagrange-multipliers [Mal97], [MM01], [Mal03], [MMP04].*

In this exposition, we are concerned with sufficient conditions. We check these conditions for a solution candidate derived by the application of necessary optimality conditions. In the following, we shall make the hypothesis that first order conditions are satisfied in normal form. *We may choose this multiplier to satisfy* $\lambda_0 = 1$ *([FH86], corollar 6.1). The Hamiltonian and its augmentation are from now on given by* $\mathcal{H}(t, x, u, \lambda)$ *and* $\widetilde{\mathcal{H}}(t, x, u, \lambda, \mu)$.

In this work we search for solution candidates of problem OCP 1.1 with continuous control functions. Therefore we give the following definition.

Definition 1.7. *The Hamiltonian* $\mathcal{H}$ *of an optimal control problem is said to be* regular *with respect to the minimum principle, if the function* $u \mapsto \mathcal{H}(t, x_0(t), u, \lambda(t))$ *possesses a unique minimum* $(x_0(t), u_0(t)) \in \mathcal{U}(t)$ *for each* $t \in [t_0, t_f]$.

Remark 1.8. *If* $\mathcal{H}$ *is regular then the optimal control* u_0 *is continuous for all* $t \in [t_0, t_f]$ *([FH86], corollar 6.2). From now on we consider only regular optimal control problems.*

Applying the conditions stated in Theorem 1.5 to the general optimal control problem, we receive a two-point boundary value problem in $2n$ variables (x, λ). The control function is uniquely defined by (1.17) for regular optimal control problems. The boundary value problem has the following structure:

BVP 1.5. *Find continuous and piecewise continuous differentiable functions* $x_0 : [t_0, t_f] \to \mathbb{R}^n$ *and* $\lambda : [t_0, t_f] \to \mathbb{R}^n$ *satisfying the differential equations*

$$\begin{aligned} x_0'(t) &= f(t, x_0(t), u_0(t)), && t \in [t_0, t_f], \text{ a.e.}, \\ \lambda'(t) &= -\widetilde{\mathcal{H}}_x(t, x_0(t), u_0(t), \lambda(t), \mu(t)), && t \in [t_0, t_f], \text{ a.e.}, \end{aligned}$$

and boundary conditions

$$\begin{aligned} r(x_0(t_0), x_0(t_f)) &= 0, \\ \lambda(t_0) &= -D_{x_s}\left[\nu r(x_0(t_0), x_0(t_f))\right], \\ \lambda(t_f) &= D_{x_f}\left[\phi(x_0(t_f)) + \nu r(x_0(t_0), x_0(t_f))\right], \end{aligned}$$

where u_0 *is given by* $\widetilde{\mathcal{H}}_u[t] = 0$.

This boundary value problem can be solved by the multiple shooting code BNDSCO, [OG89]. The solution of the boundary value problem is a solution candidate for OCP 1.1. Since we have only applied necessary conditions, we still have to prove optimality. These sufficient conditions are presented in the next section.

1.4 General Second Order Sufficient Optimality Conditions

We summarize sufficient conditions for solutions of general optimal control problems. Exploiting the special structure of optimal control problems, different types of sufficient conditions were developed in the last 50 years, [Man66], [MZ79], [Mau81], [Zei93], [MP95], [MMP04], [Vos05].

In general, we want to check, whether a stationary point is really a minimum or not. In the following, we are calling this function a solution candidate.

Definition 1.9. *We call a pair* (x_0, u_0) solution candidate *for an optimal control problem, if* x_0 *is continuous and piecewise continuous differentiable, if* u_0 *is piecewise continuous, and if there exist multipliers* $\lambda : [t_0, t_f] \to \mathbb{R}^n$, $\mu : [t_0, t_f] \to \mathbb{R}^d$, *and* $\nu \in \mathbb{R}^\ell$ *such that first order necessary optimality conditions are fulfilled.*

A first result for convex optimal control problems is formulated by Mangasarian.

Theorem 1.10. *Let* (x_0, u_0) *be a solution candidate for OCP 1.1. If for every fixed* $t \in [t_0, t_f]$ *the Hamiltonian* $\mathcal{H}$ *with respect to* $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$, $\phi + \nu r$ *with respect to* (x_s, x_f), *and* $C(t, x, u)$ *with respect to* (x, u) *are convex functions, then* (x_0, u_0) *is a global minimum of OCP 1.1.*

Proof. [Man66]. □

Treating general optimal control problems, we adhere to the formulation of Maurer and Pickenhain [MP95]. Quite similar results are given in [Zei94]. Conditions of more theoretical use are no-gap sufficient conditions developed in [BH06]. These results are not practical for the examination of concrete examples.

Before we state the theorems, we need some definitions and boundary conditions. We consider a solution candidate (x_0, u_0) of OCP 1.1 with fixed final time. The Hamiltonian for OCP 1.1 and their extension are given by Definition 1.4.

Definition 1.11. *We say, that the* strict Legendre-Clebsch condition *is satisfied on* $[t_0, t_f]$, *if there exists* $\delta > 0$, *such that*

$$w^T \widetilde{\mathcal{H}}_{uu}[t] w \geq \delta w^T w$$

holds for all $t \in [t_0, t_f]$ *and* $w \in \mathbb{R}^m$.

Sufficient conditions involve the Riccati differential equation

$$Q' = -\widetilde{\mathcal{H}}_{xx} - Q f_x - f_x^T Q + (\widetilde{\mathcal{H}}_{xu} + Q f_u)(\widetilde{\mathcal{H}}_{uu})^{-1}(\widetilde{\mathcal{H}}_{xu} + Q f_u)^T \tag{1.19}$$

depending on a symmetric matrix-functin $Q : [t_0, t_f] \to \mathbb{R}^{n \times n}$, which has to satisfy the following boundary condition.

Boundary Condition B1. *The matrix*

$$M_R := \begin{pmatrix} Q(t_0) + D^2_{x_s}(\nu r) & D^2_{x_s x_f}(\nu r) \\ D^2_{x_f x_s}(\nu r) & -Q(t_f) + D^2_{x_f}(\phi + \nu r) \end{pmatrix}$$

is positive definite on the set

$$R_R := ker\{D_{(x_s,x_f)} r(x_0(t_0), x_0(t_f))\}.$$

With these conditions we can formulate second order sufficient optimality conditions. The following statements refer to an optimal control problem OCP 1.1 with fixed final time t_f.

Theorem 1.12. *Assume that the following conditions hold.*

1. (x_0, u_0) *is a solution candidate for OCP 1.1 with* $u_0 \in C([t_0, t_f], \mathbb{R}^m)$.
2. *The regularity conditions R1 and R2 hold.*
3. *The strict Legendre-Clebsch condition holds on* $[t_0, t_f]$.
4. *The Riccati differential equation (1.19) has a bounded solution Q on* $[t_0, t_f]$.
5. *This solution Q satisfies the boundary condition B1.*

Then, there exist $\epsilon > 0$ *and* $\delta > 0$, *such that*

$$J(x,u) \geq J(x_0,u_0) + \delta\{||(x,u) - (x_0,u_0)||_2^2 + ||(x(t_0), x(t_f)) - (x_0(t_0), x_0(t_f))||^2\}$$

holds for all admissible functions $(x, u) \in L^\infty([t_0, t_f], \mathbb{R}^{n+m})$ *with* $||(x,u) - (x_0,u_0)||_\infty \leq \epsilon$. *In particular,* (x_0, u_0) *is a weak local minimum.*

Proof. [MP95], pp. 657-661. □

A main result in the previous theorem is the usage of the two-norm discrepancy [Mau81], [AM95]. We are able to compare the solution candidate (x_0, u_0) with functions in an L^∞-neighborhood. This allows us to examine problems with discontinuous control functions.

If the strict Legendre-Clebsch condition is not satisfied, we can apply another theorem. By exploiting the special control structure, we loosen the strict Legendre-Clebsch condition and simplify the Riccati equation (1.19). That is, the right hand side of (1.19) will be only dependent on some of the control components.

We need a set of indices

$$I^+(t) := \{k \in \{1, \ldots, d\} \mid \mu_k(t) > 0\},$$

of active constraints to formulate the modified strict Legendre-Clebsch condition.

Regularity Condition R3. *There exists $\delta > 0$, such that*

$$\omega^T \widetilde{\mathcal{H}}_{uu}[t]\omega \geq \delta \omega^T \omega \tag{1.20}$$

for all $t \in [t_0, t_f]$ and ω belonging to the linear space

$$R_u^+(t) := \{\omega \in \mathbb{R}^m \mid D_u C_k(t, x_0(t), u_0(t))\omega = 0, k \in I^+(t)\}. \tag{1.21}$$

In order to establish the Riccati differential equation, we split the control function into active and inactive components for each $t \in [t_0, t_f]$

$$u_{\mathcal{A}}(t) = (u_k(t))_{k \in I^+(t)} \quad \text{and} \quad u_{\mathcal{I}}(t) = (u_k(t))_{k \notin I^+(t)} \, .$$

Accordingly, the function of active constraints is given by

$$C^+[t] := (C_k[t])_{k \in I^+(t)} \, .$$

With the regularity of the matrix $C^+_{u_{\mathcal{A}}}$ we obtain a new Riccati equation

$$E[t] - F[t]G[t]^{-1}F[t]^T = 0 \tag{1.22}$$

using

$$\begin{aligned}
E \;&:=\; Q' + \widetilde{\mathcal{H}}_{xx} + Qf_x + f_x^T Q \\
&\quad + \left((C^+_{\mathcal{A}})^{-1}C^+_x\right)^T \left(\tfrac{1}{2}\widetilde{\mathcal{H}}_{u_{\mathcal{A}}u_{\mathcal{A}}}(C^+_{u_{\mathcal{A}}})^{-1}C^+_x - \widetilde{\mathcal{H}}_{u_{\mathcal{A}}x} - f^T_{u_{\mathcal{A}}}Q\right) \\
&\quad + \left(\tfrac{1}{2}\widetilde{\mathcal{H}}_{u_{\mathcal{A}}u_{\mathcal{A}}}(C^+_{u_{\mathcal{A}}})^{-1}C^+_x - \widetilde{\mathcal{H}}_{u_{\mathcal{A}}x} - f^T_{u_{\mathcal{A}}}Q\right)^T \left((C^+_{u_{\mathcal{A}}})^{-1}C^+_x\right), \\
F \;&:=\; \widetilde{\mathcal{H}}_{xu_{\mathcal{I}}} + Qf_{u_{\mathcal{I}}} + \left((C^+_{u_{\mathcal{A}}})^{-1}C^+_x\right)^T \left(\tfrac{1}{2}\widetilde{\mathcal{H}}_{u_{\mathcal{A}}u_{\mathcal{A}}}(C^+_{u_{\mathcal{A}}})^{-1}C^+_{u_{\mathcal{I}}} - \widetilde{\mathcal{H}}_{u_{\mathcal{A}}u_{\mathcal{I}}}\right) \\
&\quad + \left(\tfrac{1}{2}\widetilde{\mathcal{H}}_{u_{\mathcal{A}}u_{\mathcal{A}}}(C^+_{u_{\mathcal{A}}})^{-1}C^+_x - \widetilde{\mathcal{H}}_{u_{\mathcal{A}}x} - f^T_{u_{\mathcal{A}}}Q\right)^T \left((C^+_{u_{\mathcal{A}}})^{-1}C^+_{u_{\mathcal{I}}}\right), \\
G \;&:=\; \widetilde{\mathcal{H}}_{u_{\mathcal{I}}u_{\mathcal{I}}} + \left((C^+_{u_{\mathcal{A}}})^{-1}C^+_{u_{\mathcal{I}}}\right)^T \left(\tfrac{1}{2}\widetilde{\mathcal{H}}_{u_{\mathcal{A}}u_{\mathcal{A}}}(C^+_{u_{\mathcal{A}}})^{-1}C^+_{u_{\mathcal{I}}} - \widetilde{\mathcal{H}}_{u_{\mathcal{A}}u_{\mathcal{I}}}\right) \\
&\quad + \left(\tfrac{1}{2}\widetilde{\mathcal{H}}_{u_{\mathcal{A}}u_{\mathcal{A}}}(C^+_{u_{\mathcal{A}}})^{-1}C^+_{u_{\mathcal{I}}} - \widetilde{\mathcal{H}}_{u_{\mathcal{A}}u_{\mathcal{I}}}\right)^T \left((C^+_{u_{\mathcal{A}}})^{-1}C^+_{u_{\mathcal{I}}}\right).
\end{aligned}$$

We apply the modified strict Legendre-Clebsch condition R3 and formulate weaker sufficient conditions as follows.

Theorem 1.13. *Assume that the following conditions hold.*

1. *(x_0, u_0) is a solution candidate for OCP 1.1 with $u_0 \in C([t_0, t_f], \mathbb{R}^m)$.*

2. *The regularity conditions R1 and R2 hold.*

3. *The regularity condition R3 holds on* $[t_0, t_f]$.

4. *The Riccati differential equation (1.22) has a bounded solution Q on* $[t_0, t_f]$.

5. *This solution Q satisfies the boundary condition B1.*

Then, there exist $\epsilon > 0$ *and* $\delta > 0$, *such that*

$$J(x,u) \geq J(x_0,u_0) + \delta\{||(x,u)-(x_0,u_0)||_2^2 + ||(x(t_0),x(t_f)) - (x_0(t_0),x_0(t_f))||^2\}$$

holds for all admissible functions $(x,u) \in L^\infty([t_0,t_f],\mathbb{R}^{n+m})$ *with* $||(x,u) - (x_0,u_0)||_\infty \leq \epsilon$. *In particular,* (x_0,u_0) *is a weak local minimum.*

Proof. [MP95], pp. 657-663. □

The above theorems correspond to general optimal control problems with fixed final time t_f. In chapter 2 we will augment the above statements for optimal control problems with free final time t_f.

Let us now develop helpful simplifications for special cases followed by a simple example for getting used to the formulations of sufficient optimality conditions.

1.5 Special Cases

In this section we develop statements for special cases of optimal control problems, which arise in this composition.

Lemma 1.14. *For optimal control problems with affine linear bounds, the regularity condition R1 is always satisfied.*

Proof. Let the constraints of an optimal control problem be of the form

$$u_j(t) \in [u_{j\,\min}, u_{j\,\max}], \qquad t \in [t_0, t_f]$$

with $u_{j\,\min} < u_{j\,\max}, j = 1,\ldots,m$. These contraints can be written in the general form

$$\begin{array}{lcrcl} C_{2j-1}(t,x,u) & = & u_{j\,\min} - u_j(t) & \leq & 0, \\ C_{2j}(t,x,u) & = & u_j(t) - u_{j\,\max} & \leq & 0, \end{array} \qquad j = 1,\ldots,m.$$

For each control component, the restrictions $u_{j\,\min} - u_j(t) \leq 0$ and $-u_{j\,\max} + u_j(t) \leq 0$ cannot be active at the same time. Thus, the vectors $D_u C_j(t,x,u)$ of active constraints C_j are linear independent for all $t \in [t_0,t_f]$. Regularity condition R1 is satisfied. □

The formulation of the matrices E, F and G in Theorem 1.13 can be reduced if the treated optimal control problem has only affine linear control constraints:

Lemma 1.15. *Suppose the optimal control problem OCP 1.1 has only control bounds of the form*

$$u_j(t) \in [u_{j\min}, u_{j\max}], \qquad t \in [t_0, t_f], j = 1, \dots, m. \tag{1.23}$$

Then the Riccati equation (1.22) in Theorem 1.13 reduces to

$$Q' = -\widetilde{\mathcal{H}}_{xx} - Qf_x - f_x^T Q + (\widetilde{\mathcal{H}}_{xu_\mathcal{I}} + Qf_{u_\mathcal{I}})(\widetilde{\mathcal{H}}_{u_\mathcal{I}u_\mathcal{I}})^{-1}(\widetilde{\mathcal{H}}_{xu_\mathcal{I}} + Qf_{u_\mathcal{I}})^T. \tag{1.24}$$

Proof. In the case of affine linear control constraints we have the derivatives

$$C^+_{u_\mathcal{A}} \text{ regular}, \qquad C^+_{u_\mathcal{I}} = 0, \qquad C^+_x = 0.$$

Therefore the Riccati equation (1.22) reduces to equation (1.24). □

We can reduce the boundary condition B1 of Theorems 1.12 and 1.13, if some components of the state variable x are fixed at the initial or final time. Assume that the first n_1 components of the state variable x are fixed in t_0

$$x_j(t_0) = x_{sj}, \qquad j = 1, \dots, n_1,\ n_1 \le n.$$

We define the reduced state variable containing those components, which are not fixed at the initial time

$$\bar{x}^T := (x_{n_1+1}, \dots, x_n).$$

The matrix-function Q, which is needed in the formulation of Theorems 1.12 and 1.13 can also be reduced in dependence of $\bar{x}$

$$\bar{Q} := \begin{pmatrix} Q_{n_1+1,n_1+1} & \cdots & Q_{n_1+1,n} \\ \vdots & & \vdots \\ Q_{n_1+1,n} & \cdots & Q_{n,n} \end{pmatrix}.$$

With these definitions we can formulate a helpful tool.

Lemma 1.16. *Let us consider a general optimal control problem OCP 1.1 with fixed final time. If some components of the state variable x are fixed at the initial time t_0*

$$x_j(t_0) = x_{sj}, \qquad j = 1, \dots, n_1,\ n_1 \le n,$$

then the boundary condition B1 reduces to:

Reduced Boundary Condition RB1. *The reduced matrix*

$$M_R^0 := \begin{pmatrix} \bar{Q}(t_0) + D^2_{\bar{x}_s}(\nu r) & D^2_{\bar{x}_s x_f}(\nu r) \\ D^2_{x_f \bar{x}_s}(\nu r) & -Q(t_f) + D^2_{x_f}(\phi + \nu r) \end{pmatrix}$$

is positive definite on the reduced set

$$R_R^0 := ker\{D_{(\bar{x}_s, x_f)} r(x_0(t_0), x_0(t_f))\}.$$

Proof. Suppose the general optimal control problem with fixed final time has $n_1 \leq n$ fixed components of the state variable at the initial time t_0. Then corresponding boundary conditions are given by the functions

$$r_j(x(t_0), x(t_f)) := x_j(t_0) - x_{sj} = 0, \qquad j = 1, \ldots, n_1. \tag{1.25}$$

For stating boundary conditions in Theorems 1.12 and 1.13 we need the kernel of the Jacobian of the entire function r

$$\begin{aligned} R_R &= \ker\{D_{(x_s,x_f)} r(x(t_0), x(t_f))\} \\ &= \{\eta = (\eta_{01}, \ldots, \eta_{0n}, \eta_{f1}, \ldots, \eta_{fn}) \in \mathbb{R}^{2n} \mid D_{(x_s,x_f)} r(x(t_0), x(t_f))\eta^T = 0\}. \end{aligned}$$

With the derivatives of the components in (1.25)

$$D_{(x_s,x_f)} r_j(x(t_0), x(t_f))^T = e_j \in \mathbb{R}^{2n}, \qquad j = 1, \ldots, n_1$$

the above set R_R reduces to

$$R_R^0 = R_R \cap \{(\eta_{01}, \ldots, \eta_{0n}, \eta_{f1}, \ldots, \eta_{fn}) \in \mathbb{R}^{2n} \mid \eta_{0j} = 0, j = 1, \ldots, n_1\}.$$

Because of this reduction we only have to show a sign condition for the lower submatrix $\bar{Q}$ in M_R. Condition B1 suffices as boundary condition in Theorems 1.12 and 1.13. □

Lemma 1.16 can be augmented to the case where arbitrary components of x are fixed in t_0. Obviously, the boundary conditions for $Q(t_0)$ vanish, if all state variables are fixed at the initial time.

Another way to weaken boundary condition B1 follows from the Embedding Theorem for differential equations (see for example [HG68]). A first approach in this direction is given in [MP95] and augmented in [MO02]. We generalize these statements and show the following simplification:

Lemma 1.17. *Let us consider a general optimal control problem OCP 1.1 with fixed final time. If some components of the state variable x are fixed at the initial time t_0*

$$x_j(t_0) = x_{sj}, \qquad j = 1, \ldots, n_1, \; n_1 \leq n,$$

then the boundary condition B1 reduces to

Reduced Boundary Condition RB2. *The reduced matrix M_R^0 is positive semidefinite on the set R_R^0 and the more reduced matrix*

$$M_R^{00} := \begin{pmatrix} \bar{Q}(t_0) + D^2_{\bar{x}_s}(\nu r) & D^2_{\bar{x}_s \bar{x}_f}(\nu r) \\ D^2_{\bar{x}_f \bar{x}_s}(\nu r) & -\bar{Q}(t_f) + D^2_{\bar{x}_f}(\phi + \nu r) \end{pmatrix}$$

is positive definite on the more reduced set

$$R_R^{00} := ker\{D_{(\bar{x}_s, \bar{x}_f)} r(x_0(t_0), x_0(t_f))\}.$$

For the proof of this lemma, we need an auxiliary statement:

Lemma 1.18. *Let A and B be two symmetric $n \times n$-matrices with $A - B$ being positive definite and B being positive semidefinite on a linear subspace $R \subset \mathbb{R}^n$. Then A is positive definite on R.*

Proof. Because of the positive definiteness of $A - B$ and the positive semidefiniteness of B, we get for all $x \in R \backslash \{0\}$:

$$0 < x^T(A - B)x = x^T Ax - x^T Bx \leq x^T Ax \quad \Rightarrow \quad 0 < x^T Ax.$$

Thus, matrix A is positive definite on R. □

Proof. (Lemma 1.17) Suppose the general optimal control problem with fixed final time has $n_1 \leq n$ fixed components of the state variable at the initial time t_0.

Then we know from Lemma 1.16, that the boundary conditions can be reduced to the positive definiteness of M_R^0 on the set R_R^0. The remaining components of the matrix $Q(t_0)$ can be chosen freely, which is necessary for the following argumentation.

If we find a bounded solution Q_0 of the Riccati differential equation (1.19)

$$Q_0' = -\widetilde{\mathcal{H}}_{xx} - Q_0 f_x - f_x^T Q_0 + (\widetilde{\mathcal{H}}_{xu} + Q_0 f_u)(\widetilde{\mathcal{H}}_{uu})^{-1}(\widetilde{\mathcal{H}}_{xu} + Q_0 f_u)^T$$

satisfying boundary condition RB2, then because of the Embedding Theorem, there also exists a solution Q_1 of the disturbed Riccati differential equation

$$Q_1' = -\widetilde{\mathcal{H}}_{xx} - Q_1 f_x - f_x^T Q_1 + (\widetilde{\mathcal{H}}_{xu} + Q_1 f_u)(\widetilde{\mathcal{H}}_{uu})^{-1}(\widetilde{\mathcal{H}}_{xu} + Q_1 f_u)^T + \epsilon I_n,$$

$\epsilon > 0$, which has a slope through a neighboring point satisfying the conditions that the matrix

$$\begin{aligned} & \begin{pmatrix} Q_1(t_0) + D^2_{x_s}(\nu r) & D^2_{x_s \bar{x}_f}(\nu r) \\ D^2_{\bar{x}_f x_s}(\nu r) & -\bar{Q}_1(t_f) + D^2_{\bar{x}_f}(\phi + \nu r) \end{pmatrix} \\ - & \begin{pmatrix} Q_0(t_0) + D^2_{x_s}(\nu r) & D^2_{x_s \bar{x}_f}(\nu r) \\ D^2_{\bar{x}_f x_s}(\nu r) & -\bar{Q}_0(t_f) + D^2_{\bar{x}_f}(\phi + \nu r) \end{pmatrix} \end{aligned}$$

is positive definite on R_R^0 and the equality

$$\begin{aligned} & \begin{pmatrix} \bar{Q}_1(t_0) + D^2_{\bar{x}_s}(\nu r) & D^2_{\bar{x}_s \bar{x}_f}(\nu r) \\ D^2_{\bar{x}_f \bar{x}_s}(\nu r) & -\bar{Q}_1(t_f) + D^2_{\bar{x}_f}(\phi + \nu r) \end{pmatrix} \\ = & \begin{pmatrix} \bar{Q}_0(t_0) + D^2_{\bar{x}_s}(\nu r) & D^2_{\bar{x}_s \bar{x}_f}(\nu r) \\ D^2_{\bar{x}_f \bar{x}_s}(\nu r) & -\bar{Q}_0(t_f) + D^2_{\bar{x}_f}(\phi + \nu r) \end{pmatrix} \end{aligned}$$

holds on the set R_R^{00}.

Applying Lemma 1.18, we have found a solution Q_1, which satisfies the boundary condition B1. This concludes our argumentation. □

We can treat optimal control problems, where some components of the state variable x are fixed at the final time t_f in an analogous way to the above. This yields in the following statements:

Lemma 1.19. *Let us consider a general optimal control problem OCP 1.1 with fixed final time. If some components of the state variable x are fixed at the final time t_f*

$$x_j(t_f) = x_{fj}, \qquad j = 1, \ldots, n_1,\ n_1 \leq n,$$

then the boundary condition B1 reduces to:

Reduced Boundary Condition RB3. *The reduced matrix*

$$M_R^f := \begin{pmatrix} Q(t_0) + D^2_{x_s}(\nu r) & D^2_{x_s \bar{x}_f}(\nu r) \\ D^2_{\bar{x}_f x_s}(\nu r) & -\bar{Q}(t_f) + D^2_{\bar{x}_f}(\phi + \nu r) \end{pmatrix}$$

is positive definite on the reduced set

$$R_R^f := ker\{D_{(x_s, \bar{x}_f)} r(x_0(t_0), x_0(t_f))\}.$$

Proof. Analogous to the proof of Lemma 1.16. □

Lemma 1.20. *Let us consider a general optimal control problem OCP 1.1 with fixed final time. If some components of the state variable x are fixed at the final time t_f*

$$x_j(t_f) = x_{fj}, \qquad j = 1, \ldots, n_1,\ n_1 \leq n,$$

then the boundary condition 1 reduces to

Reduced Boundary Condition RB4. *The reduced matrix M_R^f is positive semidefinite on the set R_R^f and the more reduced matrix*

$$M_R^{ff} := \begin{pmatrix} \bar{Q}(t_0) + D^2_{\bar{x}_s}(\nu r) & D^2_{\bar{x}_s \bar{x}_f}(\nu r) \\ D^2_{\bar{x}_f \bar{x}_s}(\nu r) & -\bar{Q}(t_f) + D^2_{\bar{x}_f}(\phi + \nu r) \end{pmatrix}$$

is positive definite on the more reduced set

$$R_R^{ff} := ker\{D_{(\bar{x}_s, \bar{x}_f)} r(x_0(t_0), x_0(t_f))\}.$$

Proof. Analogous to the proof of Lemma 1.17. □

It is possible to combine Lemmas 1.16, 1.17, 1.19, and 1.20. The resulting formulations are technical and the proofs work analogous to the above. Therefore we omit this augmentation. Note, that we do not have to show any boundary conditions for Q, if the state variable x is fixed at the initial and final time (example DIO1, section 1.6).

As an application of the above lemmas, we show how to treat a transformed Mayer-type optimal control problem.

Corollary 1.21. *Let us consider the optimal control problem OCP 1.4 with $n+1$ state variables $(x, x_{n+1}) \in \mathbb{R}^{n+1}$. Then it suffices to calculate the Riccati differential equation (1.19) or (1.22) with respect to the first $n \times n$ components of $Q : [t_0, t_f] \to \mathbb{R}^{(n+1)\times(n+1)}$.*

Proof. The additional state variable x_{n+1} enters into the optimal control problem OCP 1.4 in a special way. On the one hand, we get an additional boundary condition

$$r_{\ell+1}(x_{n+1}(t_0)) := x_{n+1}(t_0) = 0.$$

We can apply Lemmas 1.16 and 1.17. Thus, we do not have to show anything for the components of $Q(t_0)$ which refer to x_{n+1}. Furthermore, we only have to show positive semidefiniteness of the matrix where the corresponding components of $Q(t_f)$ enter. Note, that the second derivatives with respect to (x^T, x_{n+1}) of $r_{\ell+1}$ vanish.

On the other hand all first and second derivatives of the right hand side of the state equation

$$\begin{pmatrix} f(t, x(t), u(t)) \\ L(t, x(t), u(t)) \end{pmatrix}$$

are not explicitely dependent on the state variable x_{n+1}. Thus, the corresponding differential equation is homogeneous with respect to this component. The choice of the appropriate components of Q being zero is valid. We are able to reduce the system of Riccati differential equations and the boundary condition to an $n \times n$-matrix of Q. □

1.6 Example: Linear Regulator with Diode

This example is taken from the book of Clarke [Cla90], p. 213. It describes an electrical circuit consisting of a diode, a capacitor, and an impressed voltage. The circuit is shown in Figure 1.1. The state equation describing the linear regulator was modelled by McClamroch [McC80].

We treat the continuous case as a first example letting the reader get used to the formulations of sufficient optimality conditions. Later on we will augment this optimal control problem to the case of problems with discontinuous state equations (sections 3.5 and 5.4).

Problem DIO1. *Minimize the functional*

$$J(u) := \frac{1}{2}\int_0^2 u(t)^2 dt \tag{1.26}$$

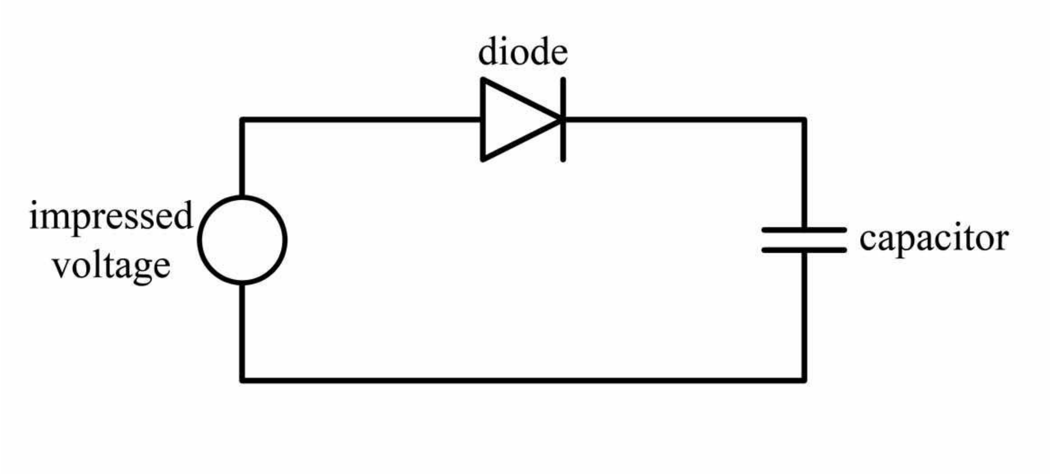

Figure 1.1: Problem DIO1, an electrical network

subject to the state equation

$$\dot{x} = 2(u - x) \tag{1.27}$$

and the boundary conditions

$$x(0) = 4, \qquad x(2) = 3, \tag{1.28}$$

This problem is an unconstrained optimal control problem with fixed final time.

With the Hamiltonian

$$\mathcal{H} = \frac{1}{2}u^2 + 2\lambda(u - x) \tag{1.29}$$

we apply first order necessary optimality conditions (Theorem 1.5) and receive the two-point boundary value problem.

Problem DIO1BVP. *Find continuous and piecewise continuous differentiable functions* $x_0 : [0, 2] \to \mathbb{R}$ *and* $\lambda : [0, 2] \to \mathbb{R}$ *satisfying the differential equations and boundary conditions*

$$\begin{aligned} \dot{x} &= 2(-2\lambda - x), \qquad & x(0) = 4, \\ \dot{\lambda} &= 2\lambda, & x(2) = 3. \end{aligned}$$

This boundary value problem can easily be solved. The solution is shown in Figure 1.2. In a second step we verify the optimality by applying sufficient conditions.

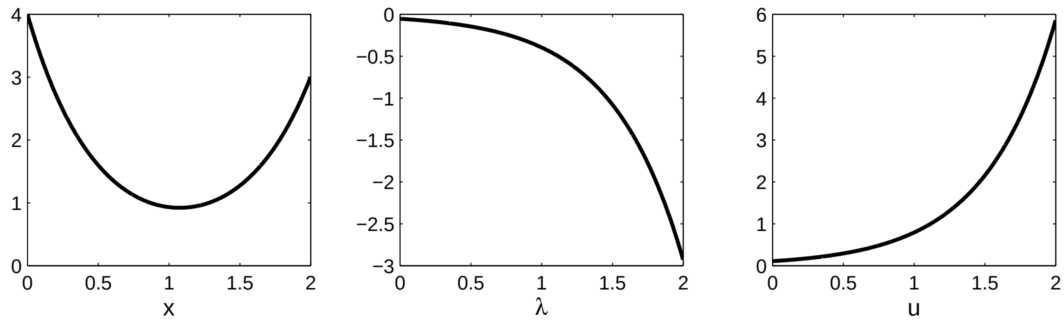

Figure 1.2: Problem DIO1, state and adjoint variables, control function

We show sufficient conditions by means of Theorem 1.10. Since the necessary conditions are satisfied for the solution candidate shown in Figure 1.2, we only have to show convexity of $\phi + \nu r$ as well as of the Hamiltonian (1.29).

$\phi + \nu r$ is convex, because the function is affine linear. The Hamiltonian (1.29) is convex because its Hessian

$$\mathcal{H}_{(x,u)(x,u)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.30}$$

is positive semidefinite for all $t \in [0, 2]$. The solution candidate received using necessary conditions is a global minimum.

For habituation, we show second order sufficient conditions in the way of Theorems 1.12 and 1.13. We verify all conditions stated in these theorems.

Since the solution candidate is calculated by using the first order necessary conditions, the solution candidate is admissible and the general necessary conditions are satisfied.

R1 is fulfilled because the treated example is an unconstrained optimal control problem.

The boundary conditions are given by

$$r(x(0), x(2)) = \begin{pmatrix} x(0) - 4 \\ x(2) - 3 \end{pmatrix} \in \mathbb{R}^2. \tag{1.31}$$

With $\operatorname{rank}\{D_{(x_s,x_f)} r(x(0), x(2))\} = 2$ regularity condition R2 is fulfilled.

For the investigation of the strict Legendre-Clebsch condition and regularity condition R3, we need the Hessian of the augmented Hamiltonian. Since this optimal control problem does not have any control constraints, the augmented Hamiltonian is the same as the original Hamiltonian (1.29). The strict Legendre-Clebsch condition and thus also regularity condition R3 is satisfied, since with (1.30) we get

$$\mathcal{H}_{uu} = 1 > 0.$$

We can apply both Theorems 1.12 and 1.13.

With the Hessian of the Hamiltonian (1.30) and the first partial derivatives of the right hand side of the state equation

$$f_x = -2, \qquad f_u = 2,$$

the differential equation (1.19) for a function $Q : [0,2] \to \mathbb{R}$ leads to the following differential equation:

$$\dot{Q} = 4Q + 4Q^2. \tag{1.32}$$

We deal with boundary condition B1 by applying Lemmas 1.16 and 1.19. The state variable is fixed at the initial time 0 and the final time 2. Therefore no boundary conditions regarding Q must be fulfilled.

It remains to find a solution of the following problem.

Problem DIO1SSC. *Find a continuous function $Q : [0,2] \to \mathbb{R}$ solving the ordinary differential equation (1.32).*

The function $Q(t) = 0, t \in [0,2]$, presents a solution of the above problem. We arrive at the following conclusion.

Result. *All second order sufficient optimality conditions stated in Theorems 1.12 and 1.13 are satisfied. The computed solution of the necessary conditions DIO1BVP is a weak local minimum of the corresponding optimal control problem DIO1.*

Chapter 2

Optimal Control Problems with Free Final Time

Now we focus on regular optimal control problems with free final time. We introduce this class of problems in section 2.1. The major difference to the theory introduced in chapter 1 lies in the second order sufficient optimality conditions. The Riccati differential equations have a special structure, which recurs in the proceeding chapters. Therefore we state necessary and sufficient optimality conditions in sections 2.2 and 2.3.

New results are established when treating the Re-entry problem in section 2.4. We are able to show sufficiency for the well-known solution candidate obtained by the application of necessary conditions.

2.1 Optimal Control Problems with Free Final Time

The optimal control problem has the same structure as the general problem OCP 1.1 in the last chapter.

OCP 2.1. *Determine a piecewise continuous control function $u : [t_0, t_f] \to \mathbb{R}^m$ and a final time t_f, such that the functional*

$$J(x, t_f, u) \quad = \quad \phi(x(t_f)) + \int_{t_0}^{t_f} L(t, x(t), u(t))dt \tag{2.1}$$

is minimized subject to

$$\begin{aligned} x'(t) &= f(t, x(t), u(t)), && t \in [t_0, t_f], a.e., && (2.2)\\ r(x(t_0), x(t_f)) &= 0, && && (2.3)\\ C(t, x(t), u(t)) &\leq 0, && t \in [t_0, t_f], a.e.. && (2.4) \end{aligned}$$

Our main interest lies in stating second order sufficient optimality conditions, which we apply to the Re-entry problem. Therefore we adhere to the formulation of Maurer and Oberle [MO02]. We define the new time variable $s \in [0,1]$ by $t = t_0 + s \cdot (t_f - t_0)$ and introduce the auxiliary problem with $\tau(s) := t_f - t_0$ analogous to the one in section 1.1:

OCP 2.2. *Determine a piecewise continuous control function $u : [0,1] \to \mathbb{R}^m$, such that the functional*

$$J(x,\tau,u) \quad = \quad \phi(x(1)) + \int_0^1 \tau(s) \cdot L(t_0 + s\tau, x(s), u(s))ds \tag{2.5}$$

is minimized subject to

$$\begin{aligned}
x'(s) &= \tau(s) \cdot f(t_0 + s\tau, x(s), u(s)), && s \in [0,1], a.e., && (2.6)\\
\tau'(s) &= 0, && s \in [0,1], a.e., && (2.7)\\
r(x(0), x(1)) &= 0, && && (2.8)\\
C(t_0 + s\tau, x(s), u(s)) &\le 0, && s \in [0,1], a.e.. && (2.9)
\end{aligned}$$

The transformed problem is equivalent to the primal problem OCP 2.1. The necessary and sufficient conditions stated in the following sections refer to OCP 2.2 which has the same properties as the general optimal control problem OCP 1.1, that is, the final time is fixed.

2.2 Necessary Optimality Conditions for Optimal Control Problems with Free Final Time

We follow the formulation of [MO02] for stating necessary optimality conditions. The variables related to the auxiliary problem OCP 2.2 are marked by an aterisk. With $\lambda^* = (\lambda, \lambda_{n+1})$ the augmented Hamiltonian is defined by

$$\begin{aligned}
&\widetilde{\mathcal{H}}^* : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^d \to \mathbb{R}\\
&\widetilde{\mathcal{H}}^*(s, x, \tau, u, \lambda_0, \lambda^*, \mu^*) :=\\
&\qquad \lambda_0 \tau L(t_0 + s\tau, x, u) + \lambda \tau f(t_0 + s\tau, x, u) + \mu^* C(t_0 + s\tau, x, u)
\end{aligned}$$

With the scaled multiplier $\mu := \mu^*/\tau$ and the Hamiltonian $\widetilde{\mathcal{H}}$ relating to OCP 2.1 we get

$$\widetilde{\mathcal{H}}^* = \tau(s)\left[\mathcal{H}(t_0 + s\tau, x, u) + \mu C(t_0 + s\tau, x, u)\right] = \tau \widetilde{\mathcal{H}}. \tag{2.10}$$

We are able to formulate first order necessary optimality conditions.

Theorem 2.1. *Let (x_0, τ_0, u_0) be a weak local minimum of OCP 2.2. Let regularity condition R1 and R2 be satisfied. Then there exist a continuous and piecewise continuous differentiable adjoint variable $\lambda^* : [0,1] \to \mathbb{R}^{n+1}$, $\lambda^* = (\lambda, \lambda_{n+1})$, a piecewise continuous multiplier function $\mu : [0,1] \to \mathbb{R}^d$, and multipliers $\nu \in \mathbb{R}^\ell$ and $\lambda_0 \in \mathbb{R}$ not all vanishing simultaneously on $[t_0, t_f]$, such that (x_0, τ_0, u_0) satisfies*

(a) the adjoint equations

$$\lambda'(s) = -\tau_0(s)\widetilde{\mathcal{H}}_x[s], \qquad s \in [0,1], a.e., \tag{2.11}$$

$$\lambda'_{n+1}(s) = -\widetilde{\mathcal{H}}^*_\tau[s] = -\widetilde{\mathcal{H}}[s] - \tau s \widetilde{\mathcal{H}}_t[s], \qquad s \in [0,1], a.e., \tag{2.12}$$

(b) the natural boundary conditions

$$\lambda(0) = -D_{x_s}\left[\nu r(x_0(0), x_0(1))\right], \tag{2.13}$$

$$\lambda(1) = \quad D_{x_f}\left[\lambda_0 \phi(x_0(1)) + \nu r(x_0(0), x_0(1))\right], \tag{2.14}$$

$$\lambda_{n+1}(0) = \lambda_{n+1}(1) = 0, \tag{2.15}$$

(c) the complementary condition

$$\mu(s) \geq 0 \quad \text{and} \quad \mu(s)C[s] = 0, \qquad s \in [0,1], a.e., \tag{2.16}$$

(d) the minimum principle

$$u_0(s) = argmin\{\tau_0(s)\widetilde{\mathcal{H}}(t_0 + s\tau_0, x_0(t), u, \lambda_0, \lambda(t), \mu(t)) | u \in \mathbb{R}^m\}, \quad s \in [0,1], a.e., \tag{2.17}$$

(e) and the minimum condition

$$\widetilde{\mathcal{H}}_u[s] = 0, \qquad s \in [0,1], a.e.. \tag{2.18}$$

Proof. [Hes66]. □

In the following, we assume that necessary conditions are satisfied in normal form and choose $\lambda_0 = 1$.

We are able to transform OCP 2.2 into a boundary value problem by applying Theorem 2.1. This boundary value problem consists of $2n + 2$ differential equations (2.6), (2.7), (2.11), (2.12) and corresponding boundary conditions (2.8), (2.13), (2.14), (2.15). The unique control function is given by (2.17). We calculate a solution candidate by hand or with the help of the software-rountine BNDSCO [OG89]. This solution candidate must be verified by sufficient conditions which are stated in the next section.

2.3 Sufficient Optimality Conditions for Optimal Control Problems with Free Final Time

Corresponding to the formulation of Mangasarian (Theorem 1.10), sufficient conditions for optimal control problems with free final time are given by the following statement.

Corollary 2.2. *Let (x_0, τ_0, u_0) be a solution candidate for OCP 2.1. If for every fixed $t \in [t_0, t_f]$ the matrix*

$$\begin{pmatrix} \tau(t)\mathcal{H}_{xx}[t] & \mathcal{H}_x^T[t] + s\tau(t)\mathcal{H}_{xt}[t] & \tau(t)\mathcal{H}_{xu}[t] \\ \mathcal{H}_x[t] + s\tau(t)\mathcal{H}_{tx}[t] & 2s\mathcal{H}_t[t] + s^2\tau(t)\mathcal{H}_{tt}[t] & \mathcal{H}_u[t] + s\tau(t)\mathcal{H}_{tu}[t] \\ \tau(t)\mathcal{H}_{ux}[t] & \mathcal{H}_u^T[t] + s\tau(t)\mathcal{H}_{ut}[t] & \tau(t)\mathcal{H}_{uu}[t] \end{pmatrix} \tag{2.19}$$

is positive semidefinite on $\mathbb{R}^{n+1+m}$ and the functions $\phi + \nu r$ with respect to (x_s, x_f) and $C(t, x, u)$ with respect to u are convex, then (x_0, τ_0, u_0) is a global minimum of OCP 2.1.

Proof. Consequene of sufficient conditions for nonsmooth optimal control problems. See page 44 for further explanations. □

Second order sufficient optimality conditions for general optimal control problems with free final time have been established in [MO02]. For the development of these conditions we correspond to the transformed problem OCP 2.2 with fixed final time. Since we use an augmented state variable $(x(s)^T, \tau(s))^T \in \mathbb{R}^{n+1}$, it is suggestive to divide the corresponding matrix-function $\widehat{Q} : [0,1] \to \mathbb{R}^{n+1 \times n+1}$ into

$$\widehat{Q}(s) := \begin{pmatrix} Q(s) & R(s) \\ R^T(s) & q(s) \end{pmatrix} \tag{2.20}$$

with $Q(s) \in \mathbb{R}^{n \times n}$, $R(s) \in \mathbb{R}^n$ and $q(s) \in \mathbb{R}$. These functions have to satisfy the system of Riccati differential equations

$$Q' = \tau \cdot \left[-\widetilde{\mathcal{H}}_{xx} - f_x^T Q - Q f_x + (\widetilde{\mathcal{H}}_{xu} + Q f_u)(\widetilde{\mathcal{H}}_{uu})^{-1}(\widetilde{\mathcal{H}}_{xu} + Q f_u)^T \right] \tag{2.21}$$

$$\begin{aligned} R' = & -\widetilde{\mathcal{H}}_x^T - s\tau\widetilde{\mathcal{H}}_{xt}^T - Q(f + s\tau f_t) - \tau f_x^T R \\ & + \left(\widetilde{\mathcal{H}}_{xu} + Q f_u\right)(\widetilde{\mathcal{H}}_{uu})^{-1}(\widetilde{\mathcal{H}}_u + s\tau\widetilde{\mathcal{H}}_{ut} + \tau R^T f_u)^T, \end{aligned} \tag{2.22}$$

$$\begin{aligned} q' = & -2s\widetilde{\mathcal{H}}_t - s^2\tau\widetilde{\mathcal{H}}_{tt} - 2R^T(f + s\tau f_t) \\ & + (\widetilde{\mathcal{H}}_u + s\tau\widetilde{\mathcal{H}}_{ut} + \tau R^T f_u)(\tau\widetilde{\mathcal{H}}_{uu})^{-1}(\widetilde{\mathcal{H}}_u + s\tau\widetilde{\mathcal{H}}_{ut} + \tau R^T f_u)^T \end{aligned} \tag{2.23}$$

Having stated the boundary condition

Boundary Condition B2. *The matrix*

$$M_R^* = \begin{pmatrix} Q(0) + D^2_{x_s}(\nu r) & R(0) & D^2_{x_s x_f}(\nu r) & 0 \\ R^T(0) & q(0) & 0 & 0 \\ D^2_{x_f x_s}(\nu r) & 0 & -Q(1) + D^2_{x_f}(\phi + \nu r) & -R(1) \\ 0 & 0 & -R^T(1) & -q(1) \end{pmatrix}$$

must be positive definite on the set

$$R_R^* = \left\{ (\eta_0, \eta_0^{(\tau)}, \eta_f, \eta_f^{(\tau)}) \in \mathbb{R}^{2(n+1)} | (\eta_0, \eta_f) \in ker \left\{ D_{(x_s, x_f)} r(x_0(0), x_0(1)) \right\} \right\}.$$

we are able to formulate second order sufficient optimality conditions.

Theorem 2.3. *Assume that the following conditions hold.*

1. (x_0, τ_0, u_0) *is a solution candidate for OCP 2.2 with* $\tau_0 > 0$ *and* $u_0 \in C([0,1], \mathbb{R}^m)$.
2. *The regularity conditions R1 and R2 hold.*
3. *The strict Legendre-Clebsch condition holds on* $[0, 1]$.
4. *The Riccati equations (2.21)-(2.23) have bounded solutions* Q, R *and* q.
5. *These solutions* Q, R *and* q *satisfy the boundary condition B2.*

Then, there exist $\epsilon > 0$ *and* $\delta > 0$, *such that*

$$\begin{aligned} J(x, \tau, u) \geq\ & J(x_0, \tau_0, u_0) \\ & + \delta \left\{ ||(x, \tau, u) - (x_0, \tau_0, u_0)||_2^2 + ||(x(0), x(1)) - (x_0(0), x_0(1))||^2 \right\} \end{aligned}$$

holds for all admissible functions $(x, \tau, u) \in L^\infty([0,1], \mathbb{R}^{n+1+m})$ *with* $||(x, \tau, u) - (x_0, \tau_0, u_0)||_\infty \leq \epsilon$. *In particular,* (x_0, τ_0, u_0) *is a weak local minimum.*

Proof. [MO02]. □

The theorem stated in [MO02] is more general and corresponds to an expansion of regularity condition R3. Our aim in this chapter is to show sufficient conditions for the Re-entry problem. For that purpose the stricter formulation above suffices.

Lemmas 1.16, 1.17, 1.19 and 1.20 can be applied to boundary condition B2. Note that the resulting reduction corresponds only to components of the functions Q and R. Now we show by means of an examble how to apply these Lemmas and reduce boundary condition B2.

2.4 Example: Re-Entry Problem

We consider the Re-entry optimal control problem stated in the book of Stoer and Bulirsch [SB02]. It describes the flight of an Apollo-type vehicle through the atmosphere of the earth. The aim is to minimize the heating-up of the vehicle while entering the atmosphere. For further explanations we refer to [SB02].
The problem is the following:

Problem REE. *Minimize the functional*

$$J(u, t_f) := \int_0^{t_f} 10v^3 \sqrt{\rho(\xi)}dt$$

subject to the state equations

$$\begin{aligned}
\dot{v} &:= V(v,\gamma,\xi,u) &&= -\frac{S}{2m}v^2\rho(\xi)C_D(u) - \frac{g\sin\gamma}{(1+\xi)^2},\\
\dot{\gamma} &:= \Gamma(v,\gamma,\xi,u) &&= \frac{S}{2m}v\rho(\xi)C_L(u) + \frac{v\cos\gamma}{\theta(1+\xi)} - \frac{g\cos\gamma}{v(1+\xi)^2},\\
\dot{\xi} &:= \Xi(v,\gamma,\xi,u) &&= \frac{v\sin\gamma}{\theta},
\end{aligned}$$

and the boundary conditions

$$\begin{aligned}
v(0) &= v_s, & \gamma(0) &= \gamma_s, & \xi(0) &= \xi_s,\\
v(t_f) &= v_f, & \gamma(t_f) &= \gamma_f, & \xi(t_f) &= \xi_f.
\end{aligned}$$

The functions of this problem are given by

$$\rho(\xi) = \rho_0 \exp(-\beta\theta\xi), \qquad C_D(u) = 1.174 - 0.9\cos u, \qquad C_L(u) = 0.6\sin u.$$

The parameters are shown in Table 2.1.

$v_s = 0.36,$	$v_f = 0.27,$	$\rho_0 = 2.704 \cdot 10^{-3},$	$\theta = 209,$
$\gamma_s = -8.1\frac{\pi}{360},$	$\gamma_f = 0,$	$g = 3.2172 \cdot 10^{-4},$	$\beta = 4.26,$
$\xi_s = \frac{4}{R},$	$\xi_f = \frac{2.5}{R},$	$S/m = 53200.$	

Table 2.1: Problem REE, parameters

This example is an unconstrained, continuous optimal control problem with free final time t_f. The state variables v, γ, and ξ are fixed at the initial and final time.

We resume necessary conditions stated in [SB02]. The conditions are adjusted to the transformed problem with fixed final time. Thus, the Hamiltonian is defined on $[0, 1]$ and given by

$$\mathcal{H}^* := \tau \left[\lambda_v V + \lambda_\gamma \Gamma + \lambda_\xi \Xi\right] = \tau \mathcal{H}.$$

The control function u is given by the minimum condition (2.18). This condition results in

$$\sin u = \frac{-0.6\lambda_\gamma}{\sqrt{(0.6\lambda_\gamma)^2 + (0.9v\lambda_v)^2}} \quad \text{and} \quad \cos u = \frac{-0.9v\lambda_v}{\sqrt{(0.6\lambda_\gamma)^2 + (0.9v\lambda_v)^2}}. \tag{2.24}$$

The corresponding boundary value problem is:

Problem REEBVP. *Find continuous and piecewise continuous differentiable functions* $x_0 = (v_0, \gamma_0, \xi_0, \tau_0)^T : [0, 1] \to \mathbb{R}^4$ *and* $\lambda = (\lambda_v, \lambda_\gamma, \lambda_\xi, \lambda_\tau) : [0, 1] \to \mathbb{R}^4$ *satisfying the differential equations and boundary conditions*

$$\begin{array}{llll}
\dot{v} = \tau V, & \dot{\lambda}_v = -\tau \mathcal{H}_v, & v(0) = v_s, & v(1) = v_f, \\
\dot{\gamma} = \tau \Gamma, & \dot{\lambda}_\gamma = -\tau \mathcal{H}_\gamma, & \gamma(0) = \gamma_s, & \gamma(1) = \gamma_f, \\
\dot{\xi} = \tau \Xi, & \dot{\lambda}_\xi = -\tau \mathcal{H}_\xi, & \xi(0) = \xi_s, & \xi(1) = \xi_f, \\
\dot{\tau} = 0, & \dot{\lambda}_\tau = -\mathcal{H}, & \lambda_\tau(0) = 0, & \lambda_\tau(1) = 0
\end{array}$$

with the control function u_0 *given by (2.24).*

The state and adjoint variables received via the multiple shooting code BNDSCO are shown in Figure 2.1.

A first attempt to show sufficient conditions was done by Strade [Str97]. Unfortunately, homogeneous differential equations were assumed, which turned out to be wrong.

We now investigate, if the solution candidate shown in Figure 2.1 is really optimal. Therefore we apply Theorem 2.3 to the problem and verify all conditions.

The solution candidate $(v_0, \gamma_0, \xi_0, \tau_0, u_0)$ was calculated by using the first order necessary conditions.

Since the optimal control problem has no constraints in state variables or control functions, regularity condition R1 is satisfied.

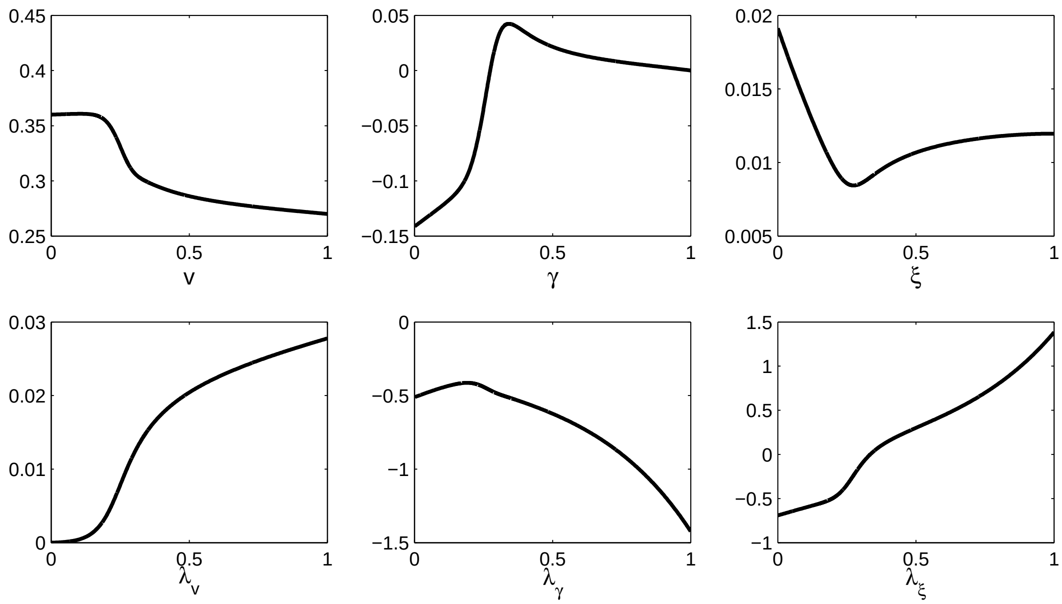

Figure 2.1: Problem REE, state and adjoint variables defined on $[0,1]$

The boundary conditions of problem REE are

$$r(x(0),x(t_f)) = \begin{pmatrix} v(0)-v_s \\ \gamma(0)-\gamma_s \\ \xi(0)-\xi_s \\ v(1)-v_f \\ \gamma(1)-\gamma_f \\ \xi(1)-\xi_f \end{pmatrix} \in \mathbb{R}^6. \tag{2.25}$$

With $\operatorname{rank}\{D_{(x_s,x_f)}r(x(0),x(1))\} = 6$ the regularity condition R2 is fulfilled.

Since we consider a one-dimensional control function, we can check the strict Legendre-Clebsch condition by investigating the time-development of the function $\mathcal{H}_{uu}[s], s \in [0,1]$. The second derivative of the Hamiltonian with respect to u can be calculated numerically while applying first order necessary conditions. This function is shown in Figure 2.2. The sign condition $\mathcal{H}_{uu}[s] > 0$ holds for all $s \in [0,1]$. Therefore the strict Legendre-Clebsch condition is satisfied.

We have to solve the system of Riccati differential equations for the symmetric matrix-function

$$\begin{pmatrix} Q(s) & R(s) \\ R^T(s) & q(s) \end{pmatrix} := \begin{pmatrix} Q_1(s) & Q_2(s) & Q_3(s) & R_1(s) \\ Q_2(s) & Q_4(s) & Q_5(s) & R_2(s) \\ Q_3(s) & Q_5(s) & Q_6(s) & R_3(s) \\ R_1(s) & R_2(s) & R_3(s) & q(s) \end{pmatrix}. \tag{2.26}$$

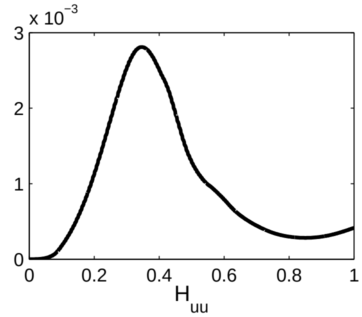

Figure 2.2: Problem REE, Hessian of the Hamiltonian defined on $[0,1]$

We calculate the right hand side of the Riccati differential equation numerically (chapter 6). We have to provide first and second partial derivatives of L, f and C. We quote these in the Appendix A.1.

We now treat boundary condition B2 with the symmetric matrix-function (2.26). We reduce the boundary condition by applying Lemmas 1.16, 1.17, 1.19, and 1.20 in the following way. Since the state variables v_0, γ_0, and ξ_0 are fixed at the starting time 0, no conditions must be shown for $Q(0)$. The conditions for $Q(1)$ can be neglected, because of the fixed state variables at the final time 1 (Lemmas 1.16 and 1.19). After this argumentation we only have to show positive definiteness of

$$\begin{pmatrix} q(0) & 0 \\ 0 & -q(1) \end{pmatrix}$$

on $\mathbb{R}^2$.

We have examined all conditions stated in Theorem 2.3. The following problem remains to be solved.

Problem REESSC. *Find continuous (symmetric) functions $Q : [0,1] \to \mathbb{R}^{3\times 3}$, $R : [0,1] \to \mathbb{R}^3$ and $q : [0,1] \to \mathbb{R}$, which satisfy the system of ordinary differential equations (2.21)-(2.23) and the boundary conditions $q(0) > 0$ and $q(1) < 0$.*

We get a solution of this problem by calculating initial value problems and changing the initial values by hand. See chapter 6 for further explanations. The above boundary conditions are satisfied using the initial conditions with $Q(0), R(0)$ and $q(0)$ stated in the Tables 2.2 and 2.3. The functions are shown in Figures 2.3-2.5.

Result. *We have shown all second order sufficient optimality conditions. Therefore the calculated solution candidate $(v_0, \gamma_0, \xi_0, \tau_0, u_0)$ is a weak local minimum of the corresponding problem REE.*

t	$Q_1(t)$	$Q_2(t)$	$Q_3(t)$	$Q_4(t)$	$Q_5(t)$	$Q_6(t)$
0	-1000.000	0.000	0.000	-1000.000	0.000	0.000
1	477.667	-74.227	-1060.310	-728.926	11077.045	-155178.638

Table 2.2: Problem REESSC, values at starting and final time for Q

t	$R_1(t)$	$R_2(t)$	$R_3(t)$	$q(t)$
0	0.000	0.000	0.000	$2.000 \cdot 10^{-6}$
1	-0.015	-0.055	0.895	$-3.151 \cdot 10^{-6}$

Table 2.3: Problem REESSC, values at starting and final time for R and q

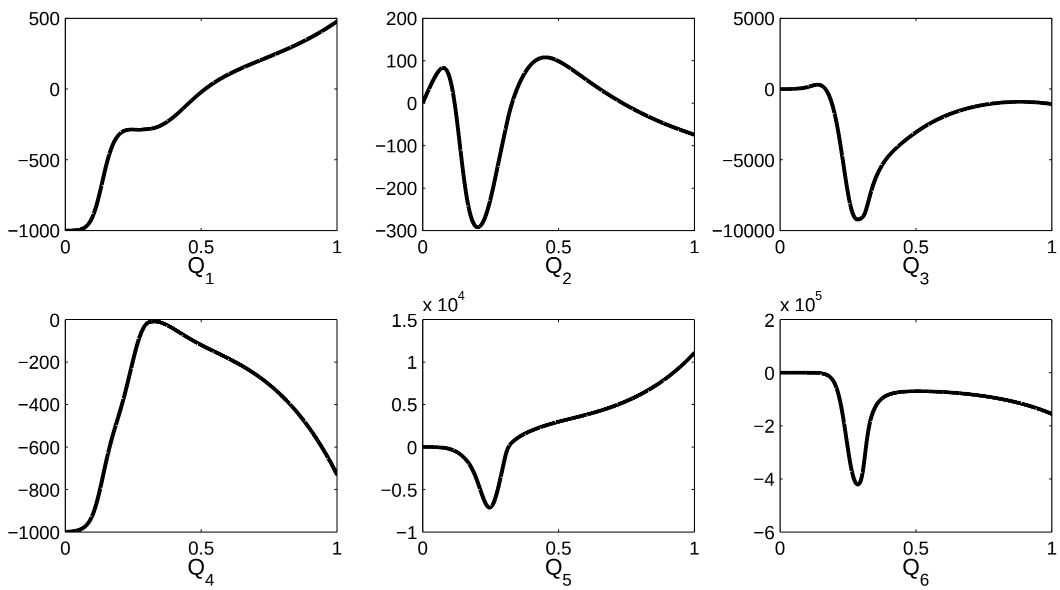

Figure 2.3: Problem REESSC, Q defined on $[0, 1]$

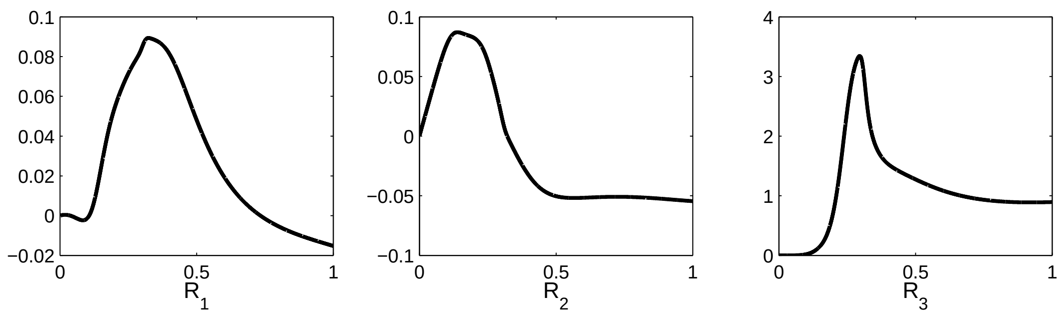

Figure 2.4: Problem REESSC, R defined on $[0, 1]$

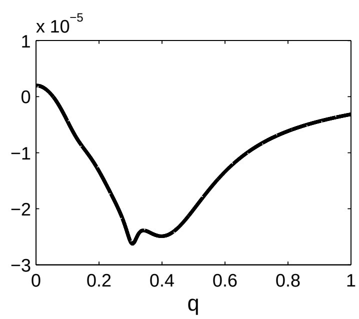

Figure 2.5: Problem REESSC, q defined on $[0, 1]$

Chapter 3

Optimal Control Problems with Discontinuous State Equations

In this chapter we take a closer look at nonsmooth optimal control problems. There occur discontinuities in the integrand of the cost functional and the right hand side of the state equation.

We start in section 3.1 with an introduction of the special class of optimal control problems. By means of a new piecewise metric developed in section 3.2 we are able to proof sufficient conditions. In section 3.3 we show results on first order necessary optimality conditions. We assume the switching function to satisfy a regularity condition which makes the theory more managable. New second order sufficient optimality conditions for this class of optimal control problems will be developed in section 3.4. In the last two sections of this chapter we examine the nonsmooth case of a linear regulator with diode (section 3.5) and a nonsmooth navigation problem (section 3.6).

In order to prove necessary and sufficient optimality conditions, a multiprocess technique is used, [CV89a], [CV89b]. A first approach developing sufficient conditions for optimal multiprocess control problems can be found in [AM00], [Vos05]. There, jump conditions are easier, since boundary conditions for the state variable are assumed to be affine linear. Note, that the jump conditions given explicitely in [AM00] for the cases that one or two components of the state variable are unspecified are not complete. Furthermore only problems without switching functions were treated. A study of a difficult nonsmooth optimal control problem with fixed switching times is given in [BP04].

The major achievement in this work is the development of more general sufficient conditions for nonsmooth optimal control problems including a switching function and general boundary conditions.

3.1 Nonsmooth Optimal Control Problems

We treat regular optimal control problems governed by ordinary differential equations. An extension to the previous chapters is, that the integrand of the cost functional and the right hand side of the state equation may contain discontinuities.

OCP 3.1. *Determine a piecewise continuous control function $u : [t_0, t_f] \to \mathbb{R}^m$, such that the functional*

$$J(x,u) \quad = \quad \phi(x(t_f)) + \int_{t_0}^{t_f} L(t, x(t), u(t))dt \tag{3.1}$$

is minimized subject to

$$\begin{aligned} x'(t) &= f(t, x(t), u(t)), && t \in [t_0, t_f], a.e., && (3.2)\\ r(x(t_0), x(t_f)) &= 0, && && (3.3)\\ C(t, x(t), u(t)) &\leq 0, && t \in [t_0, t_f], a.e., && (3.4) \end{aligned}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$, $r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^\ell$, and $C : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^d$ are twice continuous differentiable. The integrand of (3.1) and the right hand side of the state equation (3.2) are given in the following form:

$$L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \qquad L(t,x,u) = \begin{cases} L_1(t,x,u), & S(x,u) \leq 0 \\ L_2(t,x,u), & S(x,u) > 0 \end{cases},$$

$$f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \qquad f(t,x,u) = \begin{cases} f_1(t,x,u), & S(x,u) \leq 0 \\ f_2(t,x,u), & S(x,u) > 0 \end{cases}.$$

with a piecewise continuous differentiable switching function $S : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and twice continuous differentiable functions $L_k : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $f_k : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, k = 1, 2$.

Let (x_0, u_0) be a solution of OCP 3.1. In this chapter we assume the switching function to have only finitely many isolated roots along the optimal trajectory (x_0, u_0).

Assumption 3.1. *There exists a subdivision $t_0 < t_1 < \cdots < t_N = t_f, N \in \mathbb{N}$, with $N-1$ switching times, such that the switching function $S[t]$ is continuous on each open subinterval $]t_{i-1}, t_i[, i = 1, \ldots, N$ and*

$$\forall\ t \in [t_0, t_f] \backslash \{t_1, \ldots, t_{N-1}\} : \qquad S(x_0(t), u_0(t)) \neq 0.$$

Remark 3.2. *In this exposition we are concerned with finitely many different switching times t_i. The case, where switching times get close to each other, $(t_i - t_{i-1}) \to 0$, is not examined in this work.*

We omit the case $S \equiv const.$ in this chapter, since OCP 3.1 will be reduced to the general optimal control problem OCP 1.1.

Later on we are going to widen the problem. We will accept switching functions, which are allowed to vanish on whole subintervals (chapter 5). Let us now define a norm with which we can formulate a new neighborhood.

3.2 The Piecewise Metric

In chapters 3, 4, and 5, we deal with optimal control problems, whose solutions have a structure depending on a switching function. In order to compare the solution candidate with other functions of the same structure, we now introduce a new piecewise metric.

In the last chapter, the optimal control problem with free final time was transformed into an auxiliary problem with fixed final time. Sufficient conditions have been extended to a system of differential equations in functions Q, R and q which are defined on the normed interval $[0, 1]$. Having applied this transformations we were able to compare functions, which are defined on different intervals. The lengths of these intervals may vary in the amount of ϵ.

We use this idea for the optimal control problem with discontinuous state equations. We treat each subarc of a function, which is defined on a subinterval $[t_{i-1}, t_i]$, on its own. We transform these subarcs onto the normed interval. Corresponding lengths of subintervals are given by $|t_i - t_{i-1}|$.

Definition 3.3. *Let the functions $x : [t_0, t_f] \to \mathbb{R}^n$ and $\bar{x} : [\bar{t}_0, \bar{t}_f] \to \mathbb{R}^n$ be piecewise smooth with $N \in \mathbb{N}$ different switching times $t_0 < \cdots < t_{N-1} < t_f$ and $\bar{t}_0 < \cdots < \bar{t}_{N-1} < \bar{t}_f$. Then the distance of the functions is given by the* piecewise metric

$$d(x, \bar{x}) := \max_{i=1,\ldots,N} \{\max\{||x_i - \bar{x}_i||_\infty; |\tau_i - \bar{\tau}_i|\}\}$$

with $\tau_i := t_i - t_{i-1}$, $x_i : [0, 1] \to \mathbb{R}^n$, $x_i(s) := x(t_{i-1} + s\tau_i)$ and $\bar{\tau}_i := \bar{t}_i - \bar{t}_{i-1}$, $\bar{x}_i : [0, 1] \to \mathbb{R}^n$, $\bar{x}_i(s) := \bar{x}(\bar{t}_{i-1} + s\bar{\tau}_i)$, $i = 1, \ldots, N$.

This metric is defined on the space of piecewise continuous functions. Two functions are equal if and only if all function values and all switching times coincide. An example of an ϵ-neighborhood is shown in Figure 3.1.

The function x is given by the black line. The switching times t_1 and t_2 and the final time t_f may vary. Each function $\bar{x}$, which we compare x with, must lie within the grey area. The length of each subinterval may only differ in an amount of ϵ. We compare the function values by transforming each subarc of x and $\bar{x}$ onto the normed interval $[0, 1]$. The function values of these transformed functions $\bar{x}_1, \bar{x}_2$, and $\bar{x}_3$ may differ in an amount of ϵ from the function values of the transformed functions x_1, x_2, and x_3.

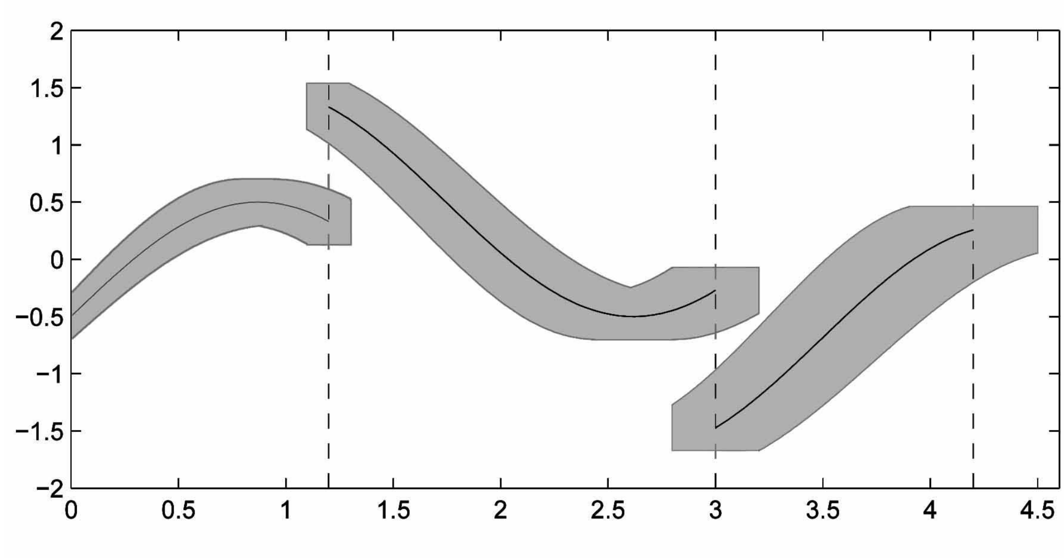

Figure 3.1: Piecewise ϵ-neighborhood of a piecewise continuous function

With this new metric, we can formulate sufficient optimality conditions for which it is possible to vary the switching times. Let us first formulate necessary conditions for OCP 3.1 with a switching function satisfying Assumption 3.1.

3.3 Necessary Optimality Conditions for Nonsmooth Optimal Control Problems

Before we can formulate necessary conditions for OCP 3.1, we take a closer look at the order of the switching function S:

Definition 3.4. *The switching function $S : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is said to be* of order *$p \in \mathbb{N}$, if p is the least degree of derivations for which S is explicitely dependent on the control function u, that is: $\exists\ p \in \mathbb{N}$:*

$$\begin{aligned} \forall k = 0, \dots, p-1: \qquad & D_u \frac{d^k}{dt^k} S(x(t), u(t)) = 0 \\ \textit{and} \qquad & D_u \frac{d^p}{dt^p} S(x(t), u(t)) \neq 0. \end{aligned}$$

The function $t \mapsto S(x_0(t), u_0(t))$ is continuous, if S is of order bigger than 0. In this case Assumption 3.1 yields in $S(x_0(t_i)) = 0, i = 1, \dots, N-1$. The examples treated in this work have switching functions of order 0 or 1. For these, necessary optimality conditions are stated in the following theorem. Note, that the Hamiltonian

$$\begin{aligned}&\widetilde{\mathcal{H}} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R},\\ &\widetilde{\mathcal{H}}(t, x, u, \lambda, \mu) := \lambda_0 L(t, x, u) + \lambda f(t, x, u) + \mu C(t, x, u),\end{aligned} \tag{3.5}$$

is defined piecewise in compliance with the integrand L and the right hand side of the state equation f.

Theorem 3.5. *Let (x_0, u_0) be a weak local minimum of OCP 3.1 with a switching function S of order 0 or 1 satisfying Assumption 3.1. Let regularity conditions R1 and R2 be satisfied. Then there exist a continuous and piecewise continuous differentiable adjoint variable $\lambda : [t_0, t_f] \to \mathbb{R}^n$, a piecewise continuous multiplier function $\mu : [t_0, t_f] \to \mathbb{R}^d$, and multipliers $\nu \in \mathbb{R}^\ell, \kappa \in \mathbb{R}^N$, and $\lambda_0 \in \mathbb{R}$ not all vanishing simultaneously on $[t_0, t_f]$, such that (x_0, u_0) satisfies*

(a) the adjoint equations

$$\lambda'(t) = -\widetilde{\mathcal{H}}_x(t, x_0(t), u_0(t), \lambda(t), \mu(t)), \qquad t \in [t_0, t_f], a.e., \tag{3.6}$$

(b) the natural boundary conditions

$$\lambda(t_0) = -D_{x_s}[\nu r(x_0(t_0), x_0(t_f))], \tag{3.7}$$

$$\lambda(t_f) = D_{x_f}[\lambda_0 \phi(x_0(t_f)) + \nu r(x_0(t_0), x_0(t_f))], \tag{3.8}$$

(c) the complementary condition

$$\mu(t) \geq 0 \quad \text{and} \quad \mu(t) C(t, x_0(t), u_0(t)) = 0, \qquad t \in [t_0, t_f], a.e., \tag{3.9}$$

(d) the minimum principle

$$u_0(t) = \text{argmin} \left\{ \widetilde{\mathcal{H}}(t, x_0(t), u, \lambda(t), \mu(t)) | u \in \mathbb{R}^m \right\}, \quad t \in [t_0, t_f], a.e., \tag{3.10}$$

(e) the minimum condition

$$\widetilde{\mathcal{H}}_u[t] = 0, \qquad t \in [t_0, t_f], a.e., \tag{3.11}$$

(f) the continuity condition

$$\widetilde{\mathcal{H}}[t_i^+] = \widetilde{\mathcal{H}}[t_i^-], \qquad i = 1, \ldots, N, \tag{3.12}$$

(g) and if the switching function is of order 0: the continuity condition

$$\lambda(t_i^+) = \lambda(t_i^-), \qquad i = 1, \ldots, N, \tag{3.13}$$

or if the switching function is of order 1: the jump condition

$$\lambda(t_i^+) = \lambda(t_i^-) + \kappa_i S_x(x_0(t_i)), \qquad i = 1, \ldots, N. \tag{3.14}$$

Proof. See Theorem 1.1 in [Obe07] and Theorem 2.1 in [OR06] and extend these to general constraints $C(t, x, u) \leq 0$. □

We assume that necessary optimality conditions are satisfied in normal form with $\lambda_0 = 1$.

Compared with the necessary conditions for general optimal control problems, the above conditions only differ in the continuity and jump conditions with respect to the adjoint variables and the Hamiltonian. We can transform OCP 3.1 into a boundary value problem in the same way as mentioned in section 1.3. In a further step, we solve this problem with the routine BNDSCO and derive a solution candidate for OCP 3.1.

3.4 Sufficient Optimality Conditions for Nonsmooth Optimal Control Problems

In this section we develop second order sufficient optimality conditions for optimal control problems with a nonsmooth cost functional and discontinuous state equations. The switching times which describe the change of the integrand and the right hand side are given by a switching function. The underlying problem OCP 3.1 is given in section 3.1.

For the formulation of sufficient conditions we need a function $\tau : [t_0, t_f] \to \mathbb{R}$ defined by

$$\begin{aligned} \tau(t) &:= t_i - t_{i-1}, \qquad \forall t \in [t_{i-1}, t_i[, i = 1, \dots, N \\ \text{and } \tau(t_f) &:= t_N - t_{N-1}. \end{aligned}$$

This function describes the lengths of subintervals which appear when Assumption 3.1 is fulfilled.

First we are concerned with sufficient optimality conditions for convex optimal control problems. The examples, which we will treat later, always have an affine linear switching function S. For this reason we formulate a rather strict result.

Theorem 3.6. *Let (x_0, u_0) be a solution candidate for OCP 3.1. Let $S : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a piecewise affine linear function which satisfies Assumption 3.1. If for every fixed $t \in [t_0, t_f]$ the matrix*

$$\begin{pmatrix} \tau(t)\mathcal{H}_{xx}[t] & \mathcal{H}_x^T[t] + s\tau(t)\mathcal{H}_{xt}[t] & \tau(t)\mathcal{H}_{xu}[t] \\ \mathcal{H}_x[t] + s\tau(t)\mathcal{H}_{tx}[t] & 2s\mathcal{H}_t[t] + s^2\tau(t)\mathcal{H}_{tt}[t] & \mathcal{H}_u[t] + s\tau(t)\mathcal{H}_{tu}[t] \\ \tau(t)\mathcal{H}_{ux}[t] & \mathcal{H}_u^T[t] + s\tau(t)\mathcal{H}_{ut}[t] & \tau(t)\mathcal{H}_{uu}[t] \end{pmatrix} \tag{3.15}$$

is positive semidefinite on $\mathbb{R}^{n+1+m}$ *and the functions* $\phi + \nu r$ *with respect to* (x_s, x_f) *and* $C(t, x, u)$ *with respect to* u *are convex, then* (x_0, u_0) *is a global minimum of OCP 3.1 compared with functions having the same switching structure as* (x_0, u_0).

The augmentation of the statement to convex switching functions of order 1 is not easy and is therefore omitted. This becomes apparent when regarding the proof of the theorem on page 50 (Remark 3.9).

We now fill a gap which occured in chapter 2 when treating optimal control problems with free final time. If we choose the switching function $S(x, u)$ to be constant, OCP 3.1 reduces to a general optimal control problem with free final time t_f. With $\tau := t_f - t_0$ Theorem 3.6 results in the statement on page 30 (Corollary 2.2).

We get results for optimal control problems which may be nonconvex, if we apply Theorems 1.12 or 1.13 to the auxiliary problem OCP 3.2.
The system of differential equations which must be solved for evaluating sufficient conditions is the following:

$$Q' = \tau \cdot \left[-\widetilde{\mathcal{H}}_{xx} - f_x^T Q - Q f_x + (\widetilde{\mathcal{H}}_{xu} + Q f_u)(\widetilde{\mathcal{H}}_{uu})^{-1}(\widetilde{\mathcal{H}}_{xu} + Q f_u)^T \right], \tag{3.16}$$

$$\begin{aligned} R' = {} & -\widetilde{\mathcal{H}}_x^T - s\tau \widetilde{\mathcal{H}}_{xt}^T - Q(f + s\tau f_t) - \tau f_x^T R \\ & + \left(\widetilde{\mathcal{H}}_{xu} + Q f_u\right)(\widetilde{\mathcal{H}}_{uu})^{-1}(\widetilde{\mathcal{H}}_u + s\tau \widetilde{\mathcal{H}}_{ut} + \tau R^T f_u)^T, \end{aligned} \tag{3.17}$$

$$\begin{aligned} q' = {} & -2s\widetilde{\mathcal{H}}_t - s^2 \tau \widetilde{\mathcal{H}}_{tt} - 2R^T (f + s\tau f_t) \\ & + (\widetilde{\mathcal{H}}_u + s\tau \widetilde{\mathcal{H}}_{ut} + \tau R^T f_u)(\tau \widetilde{\mathcal{H}}_{uu})^{-1}(\widetilde{\mathcal{H}}_u + s\tau \widetilde{\mathcal{H}}_{ut} + \tau R^T f_u)^T \end{aligned} \tag{3.18}$$

depending on the symmetric matrix-function $Q : [t_0, t_f] \to \mathbb{R}^{n \times n}$ and the functions $R : [t_0, t_f] \to \mathbb{R}^n$ and $q : [t_0, t_f] \to \mathbb{R}$. The functions $Q, R,$ and q must fulfill new boundary conditions and additional jump conditions.

Boundary Condition B3. *The matrix*

$$\begin{pmatrix} Q(t_0) + D^2_{x_s}(\nu r) & R(t_0) & D^2_{x_s x_f}(\nu r) & 0 \\ R^T(t_0) & q(t_0) & 0 & 0 \\ D^2_{x_f x_s}(\nu r) & 0 & -Q(t_f) + D^2_{x_f}(\phi + \nu r) & -R(t_f) \\ 0 & 0 & -R^T(t_f) & -q(t_f) \end{pmatrix} \tag{3.19}$$

must be positive definite on the set

$$\{(\eta_0, \eta_0^{(\tau)}, \eta_f, \eta_f^{(\tau)}) \in \mathbb{R}^{2(n+1)} | (\eta_0, \eta_f) \in ker\{D_{(x_s, x_f)} r(x_0(t_0), x_0(t_f))\}\}.$$

Jump Condition J3. *For all* $i = 1, \ldots, N-1$, *the matrices*

$$\begin{pmatrix} Q(t_i^+) - Q(t_i^-) & R(t_i^+) & -R(t_i^-) \\ R^T(t_i^+) & q(t_i^+) & 0 \\ -R^T(t_i^-) & 0 & -q(t_i^-) \end{pmatrix} \tag{3.20}$$

must be positive definite on the set

$$\begin{array}{ll} \{(\eta_1,\dots,\eta_n,\eta^+,\eta^-) \in \mathbb{R}^{n+2}\}, & \textit{if } S \textit{ is of order } 0, \\ \{(\eta_1,\dots,\eta_n,\eta^+,\eta^-) \in \mathbb{R}^{n+2} | \eta_j = 0, \textit{ if } D_{x_j}S[t] \neq 0\}, & \textit{if } S \textit{ is of order } 1. \end{array}$$

We use an argumentation as done in section 1.5 and reduce the jump condition J3. The matrix (3.20) can be reduced, if the switching function S depends explicitely on the state variable. If for example the switching function S depends explicitely on all components of the state variable, then J3 reduces to the sign condition

$$q(t_i^-) < 0 < q(t_i^+), \qquad i = 1,\dots,N-1.$$

Now we can formulate second order sufficient optimality conditions for general nonsmooth optimal control problems OCP 3.1 with a switching function S satisfying Assumption 3.1.

Theorem 3.7. *Assume that the following conditions hold.*

1. *(x_0, u_0) is a solution candidate for OCP 3.1 with a switching function S of order 0 or 1.*

2. *The regularity conditions R1 and R2 hold.*

3. *The strict Legendre-Clebsch condition is satisfied on each subinterval $[t_{i-1}, t_i]$, $i = 1,\dots,N$.*

4. *The system of differential equations (3.16)- (3.18) has bounded solutions Q, R, and q, which are piecewise continuous differentiable.*

5. *These solutions Q, R, and q satisfy boundary condition B3 and jump condition J3.*

Then, there exist $\epsilon > 0$ and $\delta > 0$, such that

$$\begin{aligned} J(x,\tau,u) \;\geq\; & J(x_0,\tau_0,u_0) \\ & +\delta\left\{||(x,\tau,u)-(x_0,\tau_0,u_0)||_2^2 + ||(x(t_0),x(t_f))-(x_0(t_0),x_0(t_f))||^2\right\} \end{aligned}$$

holds for all admissible functions $(x,\tau,u) \in L^\infty([t_0,t_f],\mathbb{R}^{n+1+m})$ with $d((x,\tau,u) - (x_0,\tau_0,u_0)) \leq \epsilon$. In particular, (x_0,u_0) is a weak local minimum compared with admissible pairs (x,u) having the same switching structure.

In the case, the strict Legendre-Clebsch condition is not satisfied, we may exploit the switching structure with respect to the general constraints $C(t,x,u)$.

Theorem 3.8. *Assume that the following conditions hold.*

1. (x_0, u_0) *is a solution candidate for OCP 3.1 with a switching function S of order* 0 *or* 1.
2. *The regularity conditions R1 and R2 hold.*
3. *The regularity condition R3 holds on each subinterval* $[t_{i-1}, t_i]$, $i = 1, \ldots, N$.
4. *The system of differential equations (3.16)- (3.18) with partial derivatives only in the direction of inactive control components* $u_{\mathcal{I}}$ *has bounded solutions* Q, R, *and* q, *which are piecewise continuous differentiable.*
5. *These solutions* Q, R, *and* q *satisfy boundary condition B3 and jump condition J3.*

Then, there exist $\epsilon > 0$ *and* $\delta > 0$, *such that*

$$\begin{aligned} J(x, \tau, u) \quad \geq \quad & J(x_0, \tau_0, u_0) \\ & + \delta \left\{ ||(x, \tau, u) - (x_0, \tau_0, u_0)||_2^2 + ||(x(t_0), x(t_f)) - (x_0(t_0), x_0(t_f))||^2 \right\} \end{aligned}$$

holds for all admissible functions $(x, \tau, u) \in L^\infty([t_0, t_f], \mathbb{R}^{n+1+m})$ *with* $d((x, \tau, u) - (x_0, \tau_0, u_0)) \leq \epsilon$. *In particular,* (x_0, u_0) *is a weak local minimum compared with admissible pairs* (x, u) *having the same switching structure.*

The jumps in the control function u and in the functions Q, R, and q may only occur at the switching times, which are defined in Assumption 3.1.

Lemmas 1.16, 1.17, 1.19, and 1.20 can be applied to boundary condition B3. One has to keep in mind, that the reduction must be undertaken with respect to the components of Q and R. In sections 3.5 and 3.6, we make this point clear by treating nonsmooth examples.

3.4.1 Proofs of Sufficient Optimality Conditions

The development of sufficient optimality conditions for nonsmooth optimal control problems is based on an auxiliary problem. We compare the solution candidate only with functions having the same switching structure. There may be a better solution for OCP 3.1 with different structure. The aim of this work is to give a first impression, whether the solution candidate is a local minimum or not. If the here developed sufficient conditions are not satisfied, it is likely not to have a local minimum at all.

Our way of argumentation looks as follows. We create an auxiliary problem for OCP 3.1 having the same properties as the general optimal control problem OCP 1.1. We then apply Theorems 1.10 or 1.12-1.13. By a backwards transformation we receive new sufficient optimality conditions for the nonsmooth problem OCP 3.1.

The Auxiliary Problem. First, we develop an auxiliary problem for OCP 3.1 which is based on a switching function $S(x, u)$ satisfying Assumption 3.1. We obtain the subdivision $t_0 < t_1 < \cdots < t_N = t_f$ of the interval $[t_0, t_f]$ into N subintervals

$$[t_{i-1}, t_i], \qquad i = 1, \ldots, N.$$

We define state variables and control functions on each subinterval through

$$x_i(s) := x(t_{i-1} + s\tau_i), \qquad u_i(s) := u(t_{i-1} + s\tau_i) \tag{3.21}$$

with the new time variable $s \in]0,1[$ and lengths of the subintervals $\tau_i(s) := t_i - t_{i-1}$, $i = 1, \ldots, N, s_i \in [0,1]$. We regard the continuous differentiable continuation of x_i and the continuous continuation of u_i on the time interval $[0,1]$. Obviously the condition

$$\sum_{i=1}^{N} \tau_i(s_i) = t_f - t_0 \tag{3.22}$$

must be satisfied for each $s_i \in [0,1], i = 1, \ldots, N$. We discover the signification of the variables s_i when treating example DIO2 (section 3.5). If we treat an optimal control problem with free final time t_f, the above condition (3.22) does not reduce the degree of freedom. The case of a fixed final time will be discussed later on (p. 55).

We differentiate the new state variables $(x_i, \tau_i), i = 1, \ldots, N$, and receive the auxiliary problem for problem OCP 3.1:

OCP 3.2. *Determine piecewise continuous control functions $u_i : [0,1] \to \mathbb{R}^m$, $i = 1, \ldots, N$, such that the functional*

$$J(x_1, \tau_1, \ldots, x_N, \tau_N, u_1, \ldots, u_N) = \phi(x_N(1)) + \sum_{i=1}^{N} \int_0^1 \tau_i(s) \cdot L(t_{i-1} + s\tau_i(s), x_i(s), u_i(s))ds \tag{3.23}$$

is minimized subject to

$$x_i'(s) = \tau_i(s) \cdot f(t_{i-1} + s\tau_i(s), x_i(s), u_i(s)), \qquad i = 1, \ldots, N, \tag{3.24}$$
$$\tau_i'(s) = 0, \qquad i = 1, \ldots, N, \tag{3.25}$$
$$r(x_1(0), x_N(1)) = 0, \tag{3.26}$$
$$x_i(0) - x_{i-1}(1) = 0, \qquad i = 2, \ldots, N, \tag{3.27}$$
$$C(t_{i-1} + s\tau_i(s), x_i(s), u_i(s)) \leq 0, \qquad i = 1, \ldots, N, \tag{3.28}$$

for almost every $s \in [0,1]$, and additionally, if the final time t_f is fixed

$$\sum_{i=1}^{N} \tau_i(s_i) - (t_f - t_0) = 0. \tag{3.29}$$

If we treat a switching function S of order 1, that is, S only depends on x, there occur additional constraints:

$$S(x_i(0)) = 0, \qquad i = 2, \ldots, N. \tag{3.30}$$

For our further work, we combine the vectors of state variables and control functions.

$$x^* := (x_1^T, \tau_1, \ldots, x_N^T, \tau_N)^T, \qquad u^* := (u_1^T, \ldots, u_N^T)^T. \tag{3.31}$$

The vector treating state variables as well as control functions is

$$y^* := (y_1^T, \ldots, y_N^T)^T, \quad \text{with} \quad y_i = (x_i^T, \tau_i, u_i^T)^T,\ i = 1, \ldots, N.$$

The combined right hand side of the state equation is given by

$$f^*(s, x^*, u^*) := \begin{pmatrix} \tau_1(s) \cdot f(t_0 + s\tau_1, x_1(s), u_1(s)) \\ 0 \\ \vdots \\ \tau_N(s) \cdot f(t_{N-1} + s\tau_N, x_N(s), u_N(s)) \\ 0 \end{pmatrix}. \tag{3.32}$$

Thus, the Hamiltonian of the auxiliary problem is defined for $\lambda^* := (\lambda_1, \lambda_1^{(\tau)}, \ldots, \lambda_N, \lambda_N^{(\tau)})$ by

$$\begin{aligned} \mathcal{H}^*(s, x^*, u^*, \lambda^*) &:= \textstyle\sum_{i=1}^N \tau_i \mathcal{H}_i(s, x_i, u_i, \lambda_i) \qquad \text{with} \\ \mathcal{H}_i(s, x_i, u_i, \lambda_i) &:= L(t_{i-1} + s\tau_i, x_i, u_i) + \lambda_i f(t_{i-1} + s\tau_i, x_i, u_i), \quad i = 1, \ldots, N. \end{aligned} \tag{3.33}$$

If we attach the control constraints with scaled multipliers $\tau_i \mu_i, i = 1, \ldots, N$, we derive the augmented Hamiltonian

$$\begin{aligned} \widetilde{\mathcal{H}}^*(s, x^*, u^*, \lambda^*, \mu^*) &:= \textstyle\sum_{i=1}^N \tau_i \widetilde{\mathcal{H}}_i(s, x_i, u_i, \lambda_i, \mu_i) \qquad \text{with} \\ \widetilde{\mathcal{H}}_i(s, x_i, u_i, \lambda_i, \mu_i) &:= \mathcal{H}_i(s, x_i, u_i, \lambda_i) + \mu_i C(t_{i-1} + s\tau_i, x_i, u_i), \quad i = 1, \ldots, N. \end{aligned} \tag{3.34}$$

Having defined the auxiliary optimal control problem and its state variables as well as control functions we are now able to proof the sufficient optimality conditions for OCP 3.1.

Proof of Nonsmooth Theorem of Mangasarian. The succeeding proof is based on the auxiliary problem OCP 3.2, which arises because of the special structure of the switching function S. Remember, that S is supposed to be piecewise affine linear.

Proof. (Theorem 3.6) Suppose (x_0, u_0) is a solution candidate for OCP 3.1. Let Assumption 3.1 be fulfilled. After the transformation of OCP 3.1 into the auxiliary problem OCP 3.2, the corresponding functions (x_0^*, u_0^*) are admissible and the general first order necessary optimality conditions (Theorem 1.5) are satisfied with corresponding multipliers λ^* and μ^*.

If for all $t \in [t_0, t_f]$ the matrix-function (3.15) is positive semidefinite, then, after transformation into problem OCP 3.2, all matrices

$$\mathcal{H}_{iy_iy_i} = \begin{pmatrix} \tau_i \mathcal{H}_{ix_ix_i} & \mathcal{H}_{ix_i}^T + s\tau_i \mathcal{H}_{ix_it_i} & \tau_i \mathcal{H}_{ix_iu_i} \\ \mathcal{H}_{ix_i} + s\tau_i \mathcal{H}_{it_ix_i} & s\mathcal{H}_{it_i} + s^2\tau_i \mathcal{H}_{it_it_i} & \mathcal{H}_{iu_i} + s\tau_i \mathcal{H}_{it_iu_i} \\ \tau_i \mathcal{H}_{iu_ix_i} & \mathcal{H}_{iu_i}^T + s\tau_i \mathcal{H}_{iu_it_i} & \tau_i \mathcal{H}_{iu_iu_i} \end{pmatrix}$$

are positive semidefinite. Therefore the Hessian of the Hamiltonian

$$\mathcal{H}^*_{y^*y^*} = \operatorname{diag}(\mathcal{H}_{1y_1y_1}, \ldots, \mathcal{H}_{Ny_Ny_N})$$

is positive semidefinite. The Hamiltonian corresponding to OCP 3.2 is convex.

Further, the function $\phi + \nu^* r^*$ of OCP 3.2 is given by

$$\begin{aligned} \phi + \nu^* r^* \ := \ & \phi(x_N(1)) + \nu r(x_1(0), x_N(1)) \\ & + \sum_{i=1}^{N} \tilde{\nu}_i (x_i(0) - x_{i-1}(1)) + \sum_{i=2}^{N} \kappa_i S(x_i(0)). \end{aligned}$$

In the case of a switching function S of order 0, all multipliers κ_i are zero. The function $\phi + \nu^* r^*$ is composed of the convex function $\phi + \nu r$ and affine linear functions. Therefore $\phi + \nu^* r^*$ itself is convex.

If $C(t, x, u)$ is convex for u with respect to OCP 3.1, then it is convex on each subinterval. Thus, the mixed control-state constraints of the auxiliary problem OCP 3.2 are convex.

All conditions for the application of Theorem 1.10 are fulfilled. The function (x_0^*, u_0^*) is a solution of OCP 3.2. If we only compare solution candidates having the same switching structure as (x_0, u_0), then (x_0, u_0) is a solution of OCP 3.1. □

Remark 3.9. *The above argumentation reveals why we cannot easily generalize Theorem 3.6 and use a convex switching function of order* 1. *For the additional equality constraint (3.30) we obtain multipliers κ_i, which have no sign constraints. Therefore the function $\phi + \nu^* r^*$ is not necessarily convex if S is convex.*

Remark 3.10. *We cannot extend the original proof [Man66] to optimal control problems with free final time or discontinuous state equations. An additional integral term appears, which neither vanishes nor can be estimated in a reasonable way.*

Now we are going to proof sufficient optimality conditions for more general nonsmooth optimal control problems then mentioned in Theorem 3.6.

Proof of Nonsmooth Second Order Sufficient Optimality Conditions. In the proofs of Theorems 3.7 and 3.8 we exploit the structure of the switching function S and use the auxiliary problem OCP 3.2. We treat an optimal control problem with free final time. The case of t_f being fixed will be treated later on (p. 55). Let the arguments of the function S be $x_p \in \mathbb{R}^n$ and $u_p \in \mathbb{R}^m$ (p stands for *partition*).

Proof. (Theorem 3.7, OCP 3.1 with free final time) We use the idea to verify all conditions of Theorem 1.12 regarding a solution candidate (x_0^*, u_0^*) of the auxiliary problem OCP 3.2. These are satisfied, if the conditions of Theorem 3.7 hold for a solution candidate (x_0, u_0) of OCP 3.1. This will lead to the optimality of (x_0, u_0) as shown in the last part of this proof.

Let conditions *1.-5.* of Theorem 3.7 be fulfilled. We now show that the corresponding conditions for the auxiliary problem are satisfied.

An admissable point (x_0, u_0) of problem OCP 3.1 which satisfies Assumption 3.1 is after transformation also admissible for the auxiliary problem OCP 3.2. By applying first order necessary conditions (Theorem 3.5) to OCP 3.1 we also satisfy the condition of Theorem 1.5; that is, general first order necessary conditions are satisfied for the auxiliary problem OCP 3.2. For a detailed explanation see [OR06].

Let R1 be satisfied for OCP 3.1. The vectors

$$D_u C_j(t, x_0(t), u_0(t)),$$

$j \in I^0(t)$, are linear independent for almost every $t \in [t_0, t_f]$. These vectors are linear independent especially on each subinterval $[t_{i-1}, t_i], i = 1, \ldots, N$. Treating the auxiliary problem OCP 3.2, this means, that all

$$D_{u_i} C_j(t_{i-1} + s\tau_i(s), x_i(s), u_i(s)),$$

$s \in [0, 1]$, are linear independent. Because of the block-diagonal structure of the Jacobian $D_{y^*} C(s, x_0^*(s), u_0^*(s))$, regularity condition R1 is satisfied for the auxiliary problem OCP 3.2.

Concerning regularity condition R2 we combine the boundary conditions of problem OCP 3.2 with free final time t_f. If the switching function S is of order 1, we get:

$$r^*(x^*(0), x^*(1)) := \begin{pmatrix} r(x_1(0), x_N(1)) \\ x_2(0) - x_1(1) \\ S(x_2(0)) \\ \vdots \\ x_N(0) - x_{N-1}(1) \\ S(x_N(0)) \end{pmatrix} \in \mathbb{R}^{\ell + (n+1)\cdot(N-1)} . \tag{3.35}$$

The Jacobian of $r^*(x^*(0), x^*(1))$ is given by the matrix

$$\left(\begin{array}{cccccc|cccccc} D_{x_s}r & 0 & & & & & & & & & D_{x_f}r & 0 \\ & & I_n & 0 & & & -I_n & 0 & & & & \\ & & D_{x_p}S & 0 & & & 0 & 0 & & & & \\ & & & & \ddots & & & & \ddots & & & \\ & & & & I_n & 0 & & & -I_n & 0 & & \\ & & & & D_{x_p}S & 0 & & & 0 & 0 & & \end{array}\right).$$

It is of rank $\ell + (n+1)\cdot(N-1)$, if regularity condition R2 is satisfied for OCP 3.1. If the switching function S is of order 0, an analogous argumentation can be treated. S then does not enter into the boundary function r^*.

The strict Legendre-Clebsch condition must be satisfied for the auxiliary problem OCP 3.2. The Hessian of the corresponding augmented Hamiltonian (3.34) with respect to the control function u^* is given by

$$\widetilde{\mathcal{H}}^*_{u^*u^*} = \operatorname{diag}(\tau_1 \widetilde{\mathcal{H}}_{1u_1u_1}, \ldots, \tau_N \widetilde{\mathcal{H}}_{Nu_Nu_N}) \tag{3.36}$$

This matrix is positive definite on $\mathbb{R}^{m\cdot N}$, if every submatrix is positive definite. Since all $\tau_i, i = 1, \ldots, N$ are positive, this can be achieved if the strict Legendre-Clebsch condition is satisfied for OCP 3.1.

Condition *3.(c)* of Theorem 1.12 deals with the Riccati differential equation (1.19). This equation has to be solved for OCP 3.2. Therefore we develop first derivatives of f^* (Equation (3.32))

$$\begin{aligned} f^*_{x^*} &= \operatorname{diag}(FX_1, \ldots, FX_N) \qquad \text{with} \qquad FX_i := \begin{pmatrix} \tau_i f_{ix_i} & f_i + s\tau_i f_{it_i} \\ 0 & 0 \end{pmatrix}, \\ f^*_{u^*} &= \operatorname{diag}(FU_1, \ldots, FU_N) \qquad \text{with} \qquad FU_i := \begin{pmatrix} \tau_i f_{iu_i} \\ 0 \end{pmatrix} \end{aligned}$$

and second derivatives of $\widetilde{\mathcal{H}}^*$ (Equation (3.34))

$$\begin{aligned} \widetilde{\mathcal{H}}^*_{x^*x^*} &= \operatorname{diag}(HXX_1, \ldots, HXX_N),\ HXX_i := \begin{pmatrix} \tau_i \widetilde{\mathcal{H}}_{ix_ix_i} & \widetilde{\mathcal{H}}^T_{ix_i} + s\tau_i \widetilde{\mathcal{H}}_{ix_it_i} \\ \widetilde{\mathcal{H}}_{ix_i} + s\tau_i \widetilde{\mathcal{H}}_{it_ix_i} & s\widetilde{\mathcal{H}}_{it_i} + s^2\tau_i \widetilde{\mathcal{H}}_{it_it_i} \end{pmatrix}, \\ \widetilde{\mathcal{H}}^*_{x^*u^*} &= \operatorname{diag}(HXU_1, \ldots, HXU_N),\ HXU_i := \begin{pmatrix} \tau_i \widetilde{\mathcal{H}}_{ix_iu_i} \\ \widetilde{\mathcal{H}}_{iu_i} + s\tau_i \widetilde{\mathcal{H}}_{it_iu_i} \end{pmatrix}, \\ \widetilde{\mathcal{H}}^*_{u^*u^*} &= \operatorname{diag}(HUU_1, \ldots, HUU_N),\ HUU_i := \begin{pmatrix} \tau_i \widetilde{\mathcal{H}}_{iu_iu_i} \end{pmatrix}. \end{aligned}$$

The distinct block-diagonal structure of these matrices should not be neglected when calculating the right hand side of the Riccati differential equation. It is reasonable to choose a symmetric matrix-function $Q^* : [0,1] \to \mathbb{R}^{N(n+1)\times N(n+1)}$, which has the same structure

$$Q^* = \operatorname{diag}(Q_1^*, \ldots, Q_N^*) \quad \text{with} \quad Q_i^* := \begin{pmatrix} Q_i & R_i \\ R_i^T & q_i \end{pmatrix} \tag{3.37}$$

where $Q_i(s) \in \mathbb{R}^{n\times n}, R_i(s) \in \mathbb{R}^n, q_i(s) \in \mathbb{R}$. In the next step we show, that this choice is consistent with the boundary condition B1. If the system of differential equations (3.16)-(3.18) has bounded and piecewise discontinuous solutions Q, R, and q, then these solutions can be transformed into a valid solution for the Riccati differential equation referring to the auxiliary problem OCP 3.2.

On the one hand we need to set up the general boundary conditions for a solution Q^* of the Riccati equation. On the other hand we have to examine whether the choice of a block-diagonal form of Q^* is at all permissible. We use second derivatives for the Mayer-part $\phi(x_N(1))$ of the objective function and for the boundary conditions $r^*(x^*(0), x^*(1))$ (Equation (3.35)). Altogether we get

$$D^2_{(x_s^*, x_f^*)}(\phi + \nu^* r^*) = \left(\begin{array}{cccc|c} A_{11} & & & & A_{1N} \\ & B_2 & & & \\ & & \ddots & & \\ & & & B_N & \\ \hline & & & & \\ A_{N1} & & & & A_{NN} \end{array}\right)$$

with

$$A_{11} := \begin{pmatrix} D^2_{x_s}(\nu r) & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_{1N} := \begin{pmatrix} D^2_{x_s x_f}(\nu r) & 0 \\ 0 & 0 \end{pmatrix},$$

$$A_{N1} := \begin{pmatrix} D^2_{x_f x_s}(\nu r) & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_{NN} := \begin{pmatrix} D^2_{x_f}(\phi + \nu r) & 0 \\ 0 & 0 \end{pmatrix},$$

and if S is of order 1

$$B_i := \begin{pmatrix} D^2_{x_p}(\nu_i S) & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 2, \ldots, N,$$

or if S is of order 0

$$B_i := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 2, \ldots, N.$$

With the block-diagonal matrix Q^* (Equation (3.37)), we can formulate the following boundary conditions. The matrix

$$M_R = \begin{pmatrix} Q^*(0) & 0 \\ 0 & -Q^*(1) \end{pmatrix} + D^2_{(x_s^* x_f^*)}(\phi + \nu^* r^*) \tag{3.38}$$

must be positive definite on the set

$$\begin{aligned} R_R &= \ker\{D_{(x_s^*, x_f^*)} r^*(x^*(0), x^*(1))\} \\ &= \Big\{ (\eta_{s1}, \eta_{s1}^{(\tau)}, \ldots, \eta_{sN}, \eta_{sN}^{(\tau)}, \eta_{f1}, \eta_{f1}^{(\tau)}, \ldots, \eta_{fN}, \eta_{fN}^{(\tau)}) \in \mathbb{R}^{2N(n+1)} \; : \\ &\qquad (\eta_{s1}, \eta_{fN}) \in \; \ker\{D_{(x_s, x_f)} r(x_1(0), x_N(1))\}, \\ &\qquad \eta_{si} = \eta_{fi-1}, \; i = 2, \ldots, N, \\ &\qquad \eta_{si} \in \; \ker\{D_{x_p} S(x_i(0))\}, \; i = 2, \ldots, N \Big\}. \end{aligned}$$

Only in the case, the switching function S is of order 1, we get $D_{x_p} S \neq 0$. In particular, we treat all components x_j of the state variable, which occur explicitely in the switching function S. Then the following implications are fulfilled for every $i = 2, \ldots, N$.

$$\begin{aligned} D_{x_{p,j}} S \neq 0 \quad &\Rightarrow \quad \eta_{si,j} = 0 \\ \eta_{si,j} = \eta_{fi-1,j} \quad &\Rightarrow \quad \eta_{fi-1,j} = 0 \end{aligned}$$

We can reduce the set R_R. This is not possible for switching functions of order 0, because the switching function does not occur in the set R_R. It follows, that if the boundary condition B3 and the jump condition J3 are satisfied for OCP 3.1, then, after transformation, the matrix M_R is positive definite on R_R regarding the auxiliary problem OCP 3.2.

We have shown, that all conditions of Theorem 1.12 are satisfied for the auxiliary problem OCP 3.2. This means, that the solution candidate (x_0^*, u_0^*) is a local minimum.

If we only compare solution candidates for OCP 3.1 which have the same switching structure (Assumption 3.1), then (x_0, u_0) is a local minimum of the nonsmooth problem OCP 3.1. □

Now we are going to proof Theorem 3.8. As the statements of Theorems 3.7 and 3.8 diverge only in conditions *3.* and *4.*, also the proofs are alike.

Proof. (Theorem 3.8, OCP 3.1 with free final time) The proof is analogous to that of Theorem 3.7 except for conditions *3.* and *4.* stated in Theorem 1.13. We first replace the strict Legendre-Clebsch condition by regularity condition R3. The second difference lies in a different Riccati differential equation.

In order to apply Theorem 1.13, regularity condition R3 must be satisfied for the auxiliary problem OCP 3.2. The augmented Hamiltonian (3.34) has to be positive definite only on the set $R^+_{u^*}(s)$. Because of the block-diagonal structure of the Hessian (3.36) of $\widetilde{\mathcal{H}}^*$, this condition is satisfied, if regularity condition R3 holds on each subinterval $[t_{i-1}, t_i], i = 1, \ldots, N$.

We now treat the Riccati differential equation. Compared with equation (1.19), the difference to Theorem 1.13 lies in the active constraints C^+. If the reduced system of differential equations corresponding to Equation (1.22) has bounded, piecewise continuous solutions Q, R, and q, then these solutions can be transformed to a valid solution for the reduced differential equation referring to the auxiliary problem OCP 3.2.

The remaining conditions are ascribed to conditions included in both Theorems 1.12 and 1.13. We have shown the validity of these conditions in the proof of Theoerm 3.7. It follows, that the solution candidate (x_0^*, u_0^*) of the auxiliary problem OCP 3.2 is a local minimum.

If we only compare solution candidates for OCP 3.1 which have the same switching structure, (Assumption 3.1), then (x_0, u_0) is a local minimum of the nonsmooth problem OCP 3.1. □

The choice of Q^* having a block-diagonal structure is a restriction to all possible solution candidates for the differential equation (1.19) respectively (1.22). One should keep this restriction in mind and try the full matrix function Q^*, if the block-diagonal form does not give the desired solution.

Now we deal with an optimal control problem with fixed final time. The formulated sufficient conditions in Theorems 3.7 and 3.8 are also valid for this case.

Lemma 3.11. *Let us consider a nonmsooth optimal control problem OCP 3.1 satisfying Assumption 3.1 with fixed final time t_f. Then Theorems 3.7 and 3.8 are also valid for this problem.*

Proof. As mentioned above, there exists an additional condition (3.22). Since the variables $\tau_i(s)$ are constant, the formulation at one distinct time suffices. We use the initial time $t_0 = 0$.

$$\sum_{i=1}^{N} \tau_i(0) - (t_f - t_0) = 0. \tag{3.39}$$

This additional boundary condition enters into two conditions of Theorems 3.7 and 3.8:

First we have to treat regularity condition R2. The Jacobian of $r^*(x^*(0), x^*(1))$ is given by the augmented matrix

$$\left(\begin{array}{cccccccc|cccccccc}
D_{x_s}r & 0 & & & & & & & & & & & & & D_{x_f}r & 0 \\
& & I_n & 0 & & & & & -I_n & 0 & & & & & & \\
& & D_{x_p}S & 0 & & & & & 0 & 0 & & & & & & \\
& & & & \ddots & & & & & & \ddots & & & & & \\
& & & & & & I_n & 0 & & & & -I_n & 0 & & & \\
& & & & & & D_{x_p}S & 0 & & & & 0 & 0 & & & \\
0 & 1 & 0 & 1 & \dots & & 0 & 1 & 0 & 0 & \dots & 0 & 0 & & 0 & 0
\end{array}\right).$$

It is of rank $\ell + (n+1) \cdot (N-1) + 1$, if R2 is satisfied for OCP 3.1 and S is of order 1. In the case of an order 0 switching function, S does not enter as additional boundary condition.

Condition (3.39) influences boundary condition B3 and jump condition J3. The second derivatives of (3.39) with respect to τ_i vanish. But the set R_R will be reduced in the case of a fixed final time t_f. In the case of a switching function S of order 1, we get

$$\begin{aligned}
\widetilde{R}_R \; = \; \Big\{ & \big(\eta_{s1}, \eta_{s1}^{(\tau)}, \dots, \eta_{sN}, \eta_{sN}^{(\tau)}, \eta_{f1}, \eta_{f1}^{(\tau)}, \dots, \eta_{fN}, \eta_{fN}^{(\tau)}\big) \in \mathbb{R}^{2N(n+1)} \; : \\
& (\eta_{s1}, \eta_{fN}) \in \ker\{D_{(x_s,x_f)} r(x_1(0), x_N(1))\}, \\
& \eta_{si} = \eta_{fi-1}, \; i = 2, \dots, N, \\
& \eta_{si} \in \ker\{D_{x_p} S(x_i(0))\}, i = 2, \dots, N, \\
& \sum_{i=1}^{N} \eta_{si}^{(\tau)} = 0 \Big\} \quad \subset R_R.
\end{aligned}$$

If the matrix M_R (Equation (3.38)) is positive definite on the set R_R, then it is also positive definite on the reduced set $\widetilde{R}_R$. The case of S being of order 0 can be treated analogous.

We have shown, that the application of sufficient conditions in the form of Theorems 3.7 and 3.8 for OCP 3.1 is allowed independent of the final time t_f being free or fixed. □

Remark 3.12. *As a simplification, one should use the set $\widetilde{R}_R$ only in the case, when positive definiteness of the matrix M_R does not hold on the set R_R. Note, that the difference between the sets R_R and $\widetilde{R}_R$ gets smaller, the more subintervals we treat.*

Contrary to the proof of Lemma 3.11, we will see in Example DIO2 (section 3.5) that the choice of where to treat the additional boundary condition (3.22) really is important.

With these tools we are able to proof optimality of solution candidates for the following examples.

3.5 Example: Linear Regulator with Diode (nonsmooth)

We return to the example of a linear regulator with diode introduced in section 1.6. Clarke [Cla90] presents the following nonsmooth formulation of the problem.

Problem DIO2. *Minimize the functional*

$$J(u) := \frac{1}{2}\int_0^2 u^2(t)dt \tag{3.40}$$

subject to the state equation

$$\dot{x} = \begin{cases} a(u-x), & x-u \leq 0 \\ b(u-x), & x-u > 0 \end{cases}, \tag{3.41}$$

and the initial and end conditions

$$x(0) = 4, \qquad x(2) = 3. \tag{3.42}$$

The right hand side of the state equation is nonsmooth if we use the parameters $a = 4$ and $b = 2$. The switching function $S(x,u) := x - u$ is of order 0.

In [Obe07], Oberle applies first order necessary conditions (Theorem 3.5) to this problem and assumes a switching structure with one switching time $t_1 \in]0,2[$. Further in [OR08], this example is extended to a problem with a switching function of order 2.

For convenience, we introduce the following discontinuous variable:

$$c(t) := \begin{cases} b, & t \in [0, t_1[\\ a, & t \in [t_1, 2] \end{cases}.$$

Using the Hamiltonian

$$\mathcal{H}(x,u,\lambda) = \frac{1}{2}u^2 + \lambda c(u-x) \tag{3.43}$$

we receive the boundary value problem:

Problem DIO2BVP. *Find continuous and piecewise continuous differentiable functions $x_0 : [0,2] \to \mathbb{R}$ and $\lambda : [0,2] \to \mathbb{R}$ and a switching time $t_1 \in]0,2[$ satisfying the differential equations and boundary conditions*

$$\begin{aligned} \dot{x} &= -c(c\lambda + x), & x(0) &= 4, \\ \dot{\lambda} &= c\lambda, & x(2) &= 3, \end{aligned}$$
$$\frac{a+b}{2}\lambda(t_1) + x(t_1) = 0.$$

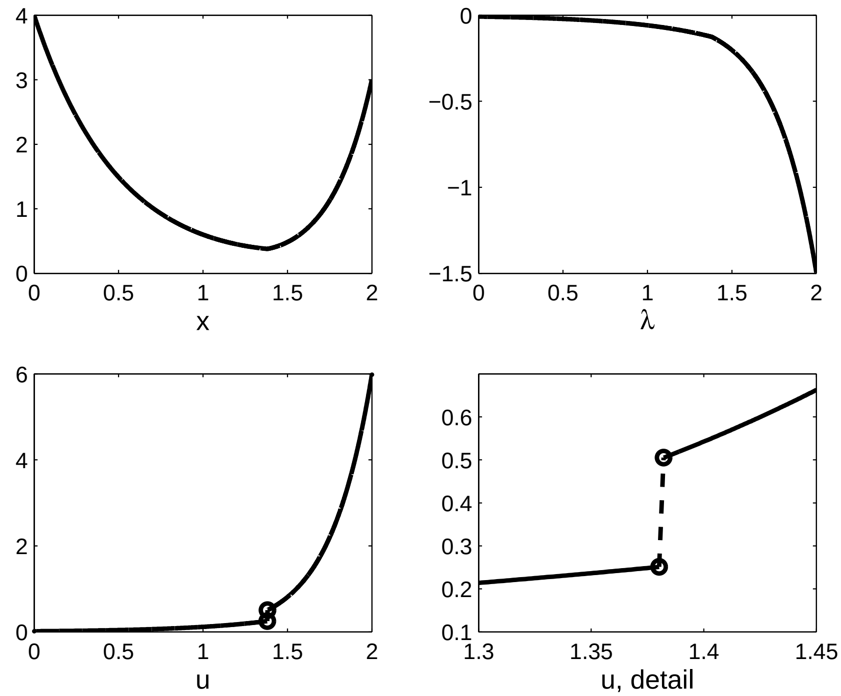

Figure 3.2: Problem DIO2, state and adjoint variables, control function

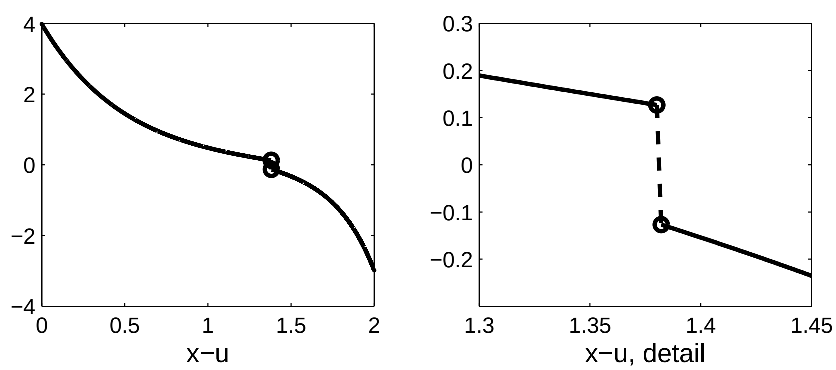

Figure 3.3: Problem DIO2, switching function

The solution shown in Figure 3.2 is calculated using the software-routine BNDSCO. In Figure 3.3 we can see the corresponding switching function S.

Let us now consider sufficient conditions for this solution candidate. First we apply the nonsmooth formulation of the Theorem of Mangasarian (Theorem 3.6). The solution candidate satisfies first order necessary conditions with admissible state variables and control function.

Matrix (3.15) in this example is given by

$$\begin{pmatrix} 0 & -\lambda c & 0 \\ -\lambda c & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $\lambda c \neq 0$. The offdiagonal entries occur because of the additional state variable τ. The matrix is not positive semidefinite for all $t \in [0, t_1]$ nor for all $t \in [t_1, 2]$. Thus, sufficient conditions by means of Theorem 3.6 are not satisfied.

We continue our investigation with general sufficient optimality conditions. Analogous to the continuous case in section 1.6 we verify all conditions listed in Theorems 3.7 and 3.8. For the sake of completeness we go through all conditions, whereby some resemble the conditions stated in section 1.6.

The solution candidate is calculated using first order necessary conditions.

R1 is fulfilled because the treated example is an unconstrained optimal control problem.

The boundary conditions are

$$r(x(0), x(2)) = \begin{pmatrix} x(0) - 4 \\ x(2) - 3 \end{pmatrix} \in \mathbb{R}^2. \tag{3.44}$$

With $\operatorname{rank}\{D_{(x_s,x_f)} r(x(0), x(2))\} = 2$ regularity condition R2 is fulfilled.

For the investigation of the strict Legendre-Clebsch condition and regularity condition R3, we need the Hessian of the augmented Hamiltonian. Since this optimal control problem does not have any control constraints, the augmented Hamiltonian is the same as the original Hamiltonian (3.43). Because of $\mathcal{H}_{uu} = 1 > 0$, the strict Legendre-Clebsch condition and thus also regularity condition R3 are satisfied. We can apply both Theorems 3.7 and 3.8.

The first derivatives of the right hand side of the state equation

$$f_x = -c, \qquad f_u = c,$$

and the first and second derivatives of the Hamiltonian

$$\mathcal{H}_x = -\lambda c, \qquad \mathcal{H}_u = u + \lambda c$$

$$\mathcal{H}_{xx} = 0, \quad \mathcal{H}_{xu} = 0, \quad \mathcal{H}_{uu} = 1$$

are needed to formulate the differential equations.
With functions $Q : [0, 2] \to \mathbb{R}$, $R : [0, 2] \to \mathbb{R}$, and $q : [0, 2] \to \mathbb{R}$ we obtain the system:

$$\dot{Q} = 2cQ + (cQ)^2, \tag{3.45}$$

$$\dot{R} = \frac{1}{\tau}[c\lambda - Qc(-\lambda c - x) + \tau cR + \tau(cQ)(cR)], \tag{3.46}$$

$$\dot{q} = \frac{1}{\tau}[-2Rc(-\lambda c - x) + \tau(Rc)^2]. \tag{3.47}$$

According to B3 and because all second derivatives of $\phi(x(2))$ and $r(x(0), x(2))$ vanish, we have to show positive definiteness of the matrix

$$\begin{pmatrix} Q(0) & R(0) & 0 & 0 \\ R(0) & q(0) & 0 & 0 \\ 0 & 0 & -Q(2) & -R(2) \\ 0 & 0 & -R(2) & -q(2) \end{pmatrix}$$

on the set $\{(\eta_s, \eta_s^{(\tau)}, \eta_f, \eta_f^{(\tau)}) \in \mathbb{R}^4 | (\eta_s, \eta_f) \in \ker\{D_{(x_s,x_f)} r(x_0(0), x_0(2)\}$. Because of the boundary conditions (3.44) in x, we can apply Lemmas 1.16 and 1.19. We just have to show positive definiteness of the matrix

$$\begin{pmatrix} q(0) & 0 \\ 0 & -q(2) \end{pmatrix}$$

on $\mathbb{R}^2$. In compliance with J3 an additional condition has to be satisfied at the switching time t_1. The following matrix must be positive definite on $\mathbb{R}^3$.

$$\begin{pmatrix} Q(t_1^+) - Q(t_1^-) & R(t_1^+) & -R(t_1^-) \\ R(t_1^+) & q(t_1^+) & 0 \\ -R(t_1^-) & 0 & -q(t_1^-) \end{pmatrix} \tag{3.48}$$

If we apply Remark 3.12 and add the condition for the fixed final time with $N = 2$ we are able to reduce the above jump condition. We reduce the jump conditions the most, if we choose the following additional boundary condition:

$$\tau_1(1) + \tau_2(0) - (t_f - t_0) = 0 \quad \Rightarrow \quad \eta_{f1}^{(\tau)} + \eta_{02}^{(\tau)} = 0.$$

Therefore we only have to show the positive definiteness of the matrix

$$\begin{pmatrix} Q(t_1^+) - Q(t_1^-) & R(t_1^+) + R(t_1^-) \\ R(t_1^+) + R(t_1^-) & q(t_1^+) - q(t_1^-) \end{pmatrix} \tag{3.49}$$

on $\mathbb{R}^2$. Compared with matrix (3.48), we now are able to choose other values for $R(t_1^-)$ and $R(t_1^+)$. It turns out to be important to use this reduction when searching for a solution of the left problem.

Problem DIO2SSC. *Find piecewise continuous functions $Q : [0, 2] \to \mathbb{R}$, $R : [0, 2] \to \mathbb{R}$ and $q : [0, 2] \to \mathbb{R}$ solving the system of ordinary differential equations (3.45)-(3.47) and satisfying the following boundary and jump conditions:*

1. *The boundary conditions $q(0) > 0$ and $q(2) < 0$ must hold.*

2. *The matrix (3.49) must be positive definite on $\mathbb{R}^2$.*

The functions shown in Figure 3.4 represent a solution of problem DIO2SSC. The functions are piecewise continuous. Their values at the starting and final times as well as at the switching time are shown in Table 3.1. Note that $Q(t) = -1, t \in [t_0, t_1^-[$, and $Q(t) = 0, t \in]t_1^+, t_f]$, holds on the entire subintervals. We receive this solution by using a backwards integration on the first subinterval and a forwards integration on the second subinterval. We vary the initial values $Q(t_1^-), R(t_1^-), q(t_1^-)$ and $Q(t_1^+), R(t_1^+), q(t_1^+)$ by hand (chapter 6).

This solution enables us to state the following

Result. *The solution candidate calculated by using necessary conditions DIO2BVP is a weak local minimum of the corresponding optimal control problem DIO2 compared with all functions having the same switching structure.*

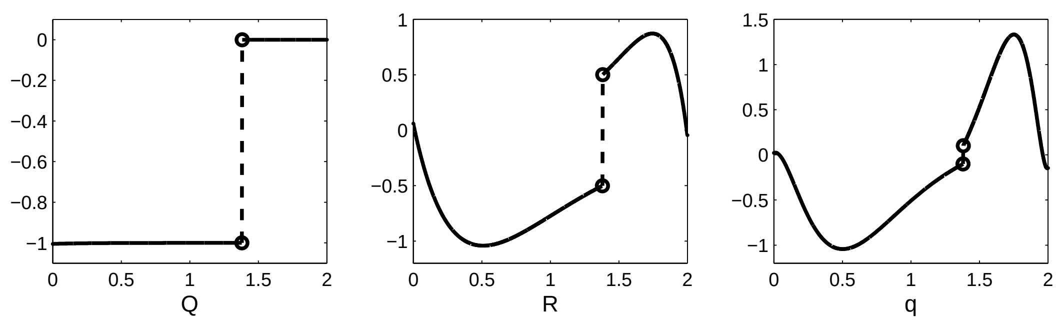

Figure 3.4: Problem DIO2SSC, functions Q, R, and q

t	$Q(t)$	$R(t)$	$q(t)$
t_0	-1.0000	0.0598	0.0214
t_1^-	-1.0000	-0.5000	-0.1000
t_1^+	0.0000	0.5000	0.1000
t_f	0.0000	-0.0448	-0.1460

Table 3.1: Problem DIO2SSC, values at starting, switching and final time

3.6 Example: Zermelo's Problem (nonsmooth)

Let us consider a nonsmooth navigation problem following the example treated in [BH75]. Necessary conditions for a nonsmooth and a singular navigation problem are treated in [OR08]. Compared to Example DIO2 we now investigate an example with an order 1 switching function and a free final time t_f.

The problem describes the guidance of a ship navigating over a river with nonsmooth current. The position of the ship is defined by the state variables $(x_1, x_2)^T$. The current is given by $V(x_1)$. We look for the control function u describing the steering of the ship in order to minimize the navigation time.

Problem ZER. *Minimize the functional*

$$J(u) := t_f = \int_0^{t_f} 1dt$$

subject to the state equation

$$\begin{aligned} \dot{x}_1 &= w \cos u, \\ \dot{x}_2 &= w \sin u + V(x_1), \end{aligned}$$

and the initial and end conditions

$$\begin{aligned} x_1(0) &= -1, \qquad & x_1(t_f) &= 1, \\ x_2(0) &= 0, \qquad & x_2(t_f) &= 0. \end{aligned}$$

The right hand side of the state equation is nonsmooth if we use the function $V(x_1)$ describing the current:

$$V(x_1) = \begin{cases} v(1 - x_1^2), & \alpha^2 - x_1^2 \leq 0 \\ v(1 - \alpha^2), & \alpha^2 - x_1^2 > 0 \end{cases} . \tag{3.50}$$

The switching function $S(x) := \alpha^2 - x_1^2$ is of order 1. Note, that S does not depend explicitely on the state variable x_2.
For calculations, we use the values $\alpha = 0.5, v = 1$, and $w = 2$.

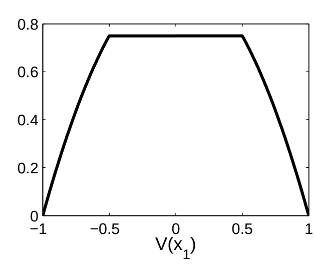

Figure 3.5: Problem ZER, current $V(x_1)$

We apply necessary conditions using the Hamiltonian

$$\mathcal{H}(x, u, \lambda) = 1 + \lambda_1(w \cos u) + \lambda_2(w \sin u + V(x_1)). \tag{3.51}$$

We receive the following normed boundary value problem:

Problem ZERBVP. *Find continuous and piecewise continuous differentiable functions $x_0 : [0,1] \to \mathbb{R}^2$ and $\lambda_2 : [0,1] \to \mathbb{R}$, switching times $t_1, t_2 \in\ ]0,1[, t_1 < t_2$, and a final time $t_f : [0,1] \to \mathbb{R}$ satisfying the differential equations and boundary conditions*

$$\begin{aligned}
\dot{x}_1 &= t_f(w\cos u), & x_1(0) &= -1,\\
\dot{x}_2 &= t_f(w\sin u + V(x_1)), & x_2(0) &= 0,\\
\dot{\lambda}_2 &= 0, & x_1(1) &= 1,\\
\dot{t}_f &= 0, & x_2(1) &= 0,\\
x_1(t_1) &= -\alpha, & x_1(t_2) &= \alpha
\end{aligned}$$

with the control variable $u : [0,1] \to \mathbb{R}$ given by the equations

$$\begin{aligned}
\sin u &= -\frac{w\lambda_2}{1+\lambda_2 V(x_1)},\\
\cos u &= \frac{\sqrt{(1+\lambda_2 V(x_1))^2 - (w\lambda_2)^2}}{1+\lambda_2 V(x_1)}.
\end{aligned}$$

The first adjoint variable $\lambda_1 : [0,1] \to \mathbb{R}$ can be computed by

$$\lambda_1 = -\frac{1}{w}\sqrt{(1+\lambda_2 V(x_1))^2 - (w\lambda_2)^2}. \tag{3.52}$$

The Hamiltonian is necessarily continuous and the jump conditions in the adjoint variables look as follows:

$$\begin{aligned}
\lambda_1(t_1^+) &= \lambda_1(t_1^-) + \kappa_1, & \lambda_1(t_2^+) &= \lambda_1(t_2^-) + \kappa_2,\\
\lambda_2(t_1^+) &= \lambda_2(t_1^-), & \lambda_2(t_2^+) &= \lambda_2(t_2^-).
\end{aligned}$$

We infer from equation (3.52), that the adjoint variable λ_1 is continuous in the switching times t_1 and t_2. In further computations we do not need the variable λ_1 explicitely.

The solution shown in Figure 3.6 is calculated using the software-routine BNDSCO. The optimal value of the objective function is $t_f = 1.0327$.

Let us now consider sufficient conditions for this solution candidate. The derivatives of f and $\mathcal{H}$ are given by

$$f_x = \begin{pmatrix} 0 & 0 \\ V'(x_1) & 0 \end{pmatrix}, \qquad f_u = \begin{pmatrix} -w\sin u \\ w\cos u \end{pmatrix},$$

$$\mathcal{H}_{xx} = \begin{pmatrix} \lambda_2 V''(x_1) & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{H}_{xu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathcal{H}_{uu} = -w\lambda_2/\sin u.$$

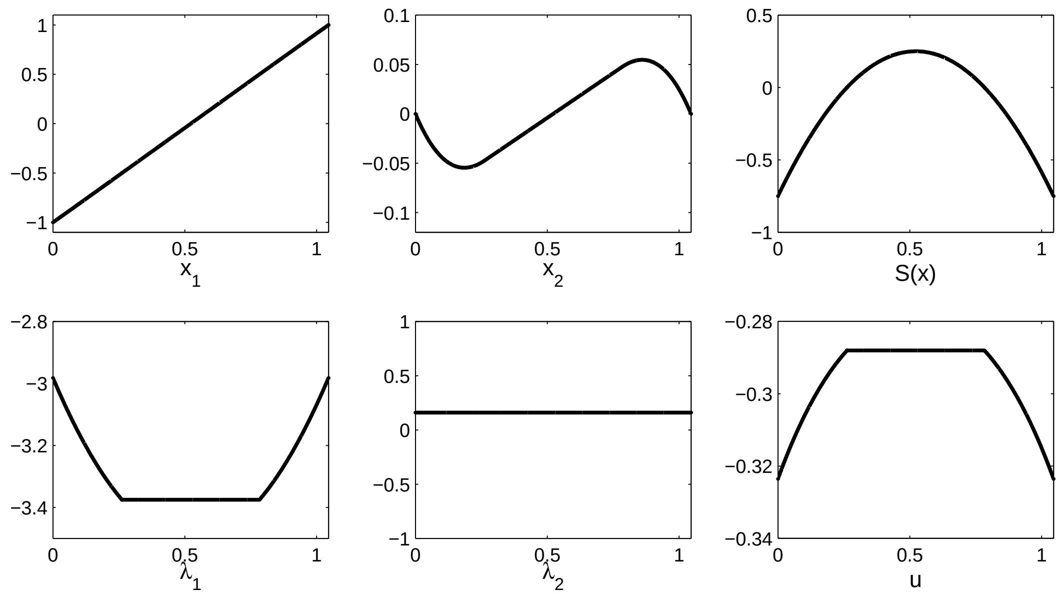

Figure 3.6: Problem ZER, state and adjoint variables, control function

Because of

$$\lambda_2 V''(x_1) = -2v\lambda_2 < 0, \qquad x_1 \in [-1, \alpha[\cup]\alpha, 1]$$

the nonsmooth sufficient conditions of Mangasarian (Theorem 3.6) cannot be applied.

We continue our investigation with general sufficient conditions. We verify all conditions listed in Theorems 3.7 and 3.8.

The solution candidate is calculated using first order necessary conditions.

R1 is fulfilled because the treated example is an unconstrained optimal control problem.

The boundary conditions are

$$r(x(0), x(t_f)) = \begin{pmatrix} x_1(0) + 1 \\ x_2(0) \\ x_1(t_f) - 1 \\ x_2(t_f) \end{pmatrix} \in \mathbb{R}^4. \tag{3.53}$$

With $\operatorname{rank}\{D_{(x_s, x_f)} r(x(0), x(t_f))\} = 4$ regularity condition R2 is fulfilled.

For the investigation of the strict Legendre-Clebsch condition and regularity condition R3, we need the Hessian of the augmented Hamiltonian. Since this optimal control problem does not have any control constraints, the augmented Hamiltonian is the same as the original Hamiltonian (3.51).

Because of $\mathcal{H}_{uu} = -w\lambda_2/\sin u > 0$ the strict Legendre-Clebsch condition and thus also regularity condition R3 are satisfied. We can apply both Theorems 3.7 and 3.8.

The first derivatives of the right hand side of the state equation and the first and second derivatives of the Hamiltonian are needed to formulate the Riccati differential equations. With functions $Q : [0, t_f] \to \mathbb{R}^{2\times 2}$, $R : [0, t_f] \to \mathbb{R}^2$, and $q : [0, t_f] \to \mathbb{R}$:

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_2 & Q_3 \end{pmatrix}, \qquad R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix},$$

we obtain the following system using relation (3.52):

$$\begin{aligned}
\dot{Q}_1 &= -\lambda_2 V''(x_1) - 2V'(x_1)Q_2 - \frac{w\sin u}{\lambda_2}(-\sin u\ Q_1 + \cos u\ Q_2)^2 && (3.54)\\
\dot{Q}_2 &= -V'(x_1)Q_3 - \frac{w\sin u}{\lambda_2}(-\sin u\ Q_1 + \cos u\ Q_2)(-\sin u\ Q_2 + \cos u\ Q_3) && (3.55)\\
\dot{Q}_3 &= -\frac{w\sin u}{\lambda_2}(-\sin u\ Q_2 + \cos u\ Q_3)^2 && (3.56)\\
\dot{R}_1 &= \frac{1}{\tau}\left[-\lambda_2 V'(x_1) - w\cos u\ Q_1 - (w\sin u\ + V(x_1))Q_2\right] - V'(x_1)R_2 && (3.57)\\
&\quad -\frac{w\sin u}{\lambda_2}(-\sin u\ Q_1 + \cos u\ Q_2)(-\sin u\ R_1 + \cos u\ R_2) &&\\
\dot{R}_2 &= \frac{1}{\tau}\left[-w\cos u\ Q_2 - (w\sin u\ + V(x_1))Q_3\right] && (3.58)\\
&\quad -\frac{w\sin u}{\lambda_2}(-\sin u\ Q_2 + \cos u\ Q_3)(-\sin u\ R_1 + \cos u\ R_2) &&\\
\dot{q} &= \frac{1}{\tau}\left[-2(w\cos u\ R_1 + (w\sin u\ + V(x_1))R_2)\right] && (3.59)\\
&\quad -\frac{w\sin u}{\lambda_2}(-\sin u\ R_1 + \cos u\ R_2)^2 &&
\end{aligned}$$

According to B3 and because all second derivatives of $r(x(0), x(t_f))$ vanish, we have to show positive definiteness of the matrix

$$\begin{pmatrix}
Q_1(0) & Q_2(0) & R_1(0) & 0 & 0 & 0\\
Q_2(0) & Q_3(0) & R_2(0) & 0 & 0 & 0\\
R_1(0) & R_2(0) & q(0) & 0 & 0 & 0\\
0 & 0 & 0 & -Q_1(t_f) & -Q_2(t_f) & -R_1(t_f)\\
0 & 0 & 0 & -Q_2(t_f) & -Q_3(t_f) & -R_2(t_f)\\
0 & 0 & 0 & -R_1(t_f) & -R_2(t_f) & -q(t_f)
\end{pmatrix}$$

on the set $\{(\eta_s, \eta_s^{(\tau)}, \eta_f, \eta_f^{(\tau)}) \in \mathbb{R}^6 | (\eta_s, \eta_f) \in \ker\{D_{(x_s,x_f)} r(x(0), x(t_f))\}$. Because of the fixed state variable at the starting and final time we can apply Lemmas

1.16 and 1.19. We just have to show

$$q(0) > 0 \quad \text{and} \quad q(t_f) < 0.$$

In compliance with J3 additional conditions have to be satisfied at the switching times t_1 and t_2. The conditions can be reduced because the switching function is of order 1 and it depends explicitely on x_1. Thus, the matrices

$$\begin{pmatrix} Q_3(t_i^+) - Q_3(t_i^-) & R_2(t_i^+) & -R_2(t_i^-) \\ R_2(t_i^+) & q(t_i^+) & 0 \\ -R_2(t_i^-) & 0 & -q(t_i^-) \end{pmatrix} \tag{3.60}$$

must be positive definite on $\mathbb{R}^3$ for $i = 1, 2$.

Finally, the following problem remains to be solved for showing general second order sufficient optimality conditions:

Problem ZERSSC. *Find piecewise continuous functions* $Q_j : [0, t_f] \to \mathbb{R}$, $j = 1, 2, 3$, $R_k : [0, t_f] \to \mathbb{R}, k = 1, 2$, *and* $q : [0, t_f] \to \mathbb{R}$ *solving the system of ordinary differential equations (3.54)-(3.59) and satisfying the boundary and jump conditions:*

1. *The boundary conditions* $q(0) > 0$ *and* $q(t_f) < 0$ *must hold.*

2. *The matrices (3.60),* $i = 1, 2$, *must be positive definite on* $\mathbb{R}^3$.

The functions shown in Figure 3.7 represent one solution of problem ZERSSC. The functions are piecewise continuous. Their values at the starting and final times as well as at the switching times are shown in Table 3.2. Note that $Q_2(t) = Q_3(t) = R_2(t) = 0$ holds on the subinterval $t \in [0, t_1]$. We receive this solution by using a forwards integration on all subintervals. We vary the initial values $Q_1(0), Q_3(0), q(0)$, $Q_1(t_1^-), Q_3(t_1^-)$, $q(t_1^-), Q_1(t_2^-), Q_3(t_2^-)$, and $q(t_2^-)$ by hand (chapter 6).

Result. *We have shown second order sufficient optimality conditions for problem ZER. The solution candidate computed by applying necessary conditions ZERBVP is a weak local minimum compared with all functions having the same switching structure.*

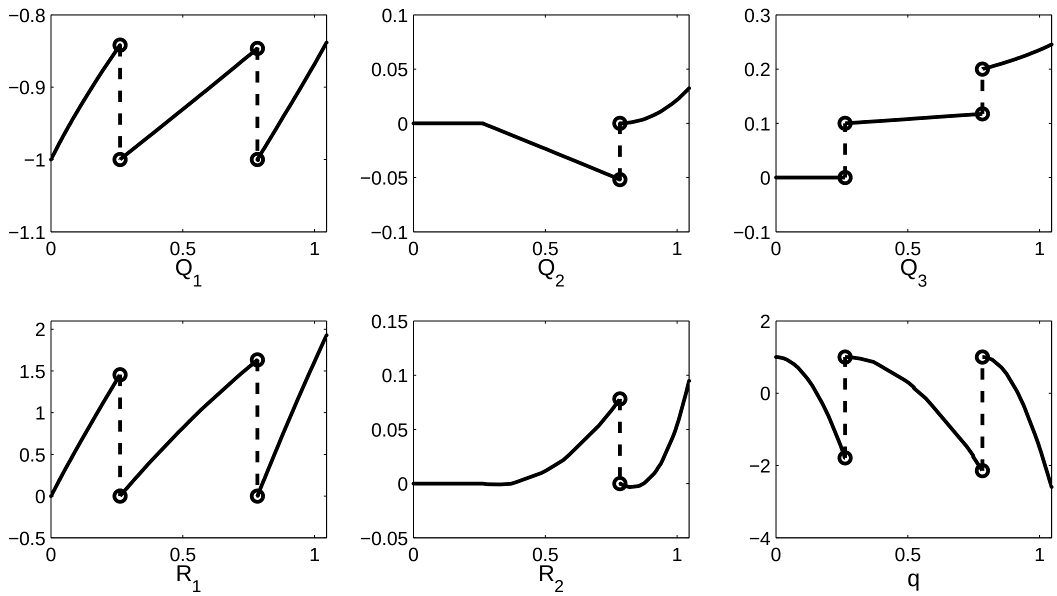

Figure 3.7: Problem ZERSSC, functions Q, R, and q

t	$Q_1(t)$	$Q_2(t)$	$Q_3(t)$	$R_1(t)$	$R_2(t)$	$q(t)$
t_0	-1.0000	0.0000	0.0000	0.0000	0.0000	1.0000
t_1^-	-0.8419	0.0000	0.0000	1.4555	0.0000	-1.7911
t_1^+	-1.0000	0.0000	0.1000	0.0000	0.0000	1.0000
t_2^-	-0.8464	-0.0518	0.1175	1.6325	0.0780	-2.1447
t_2^+	-1.0000	0.0000	0.2000	0.0000	0.0000	1.0000
t_f	-0.8384	0.0323	0.2453	1.9282	0.0947	-2.5960

Table 3.2: Problem ZERSSC, values at starting, switching and final time

Chapter 4

Optimal Control Problems with Free Control Subarcs

In practice there occur dependent and independent control functions. This means, control components enter into the objective function, the state equation, and the mixed constraints by multiplication. If the factor vanishes on a subinterval, the dependent controls can be chosen freely. We call a dependent control on this subinterval *free control subarc*. This class of optimal control problems is introduced in section 4.1.

The solution of the optimal control problem is no longer unique. Most users ignore this fact when applying first order necessary optimality conditions. We will see in section 4.2 that this approach is allowed. Because general necessary conditions may be too strict we develop new necessary conditions in this section.

In section 4.3 we show, that the known second order sufficient optimality conditions [MP95] cannot be applied to optimal control problems with free control subarcs. We develop new sufficient conditions in this section and verify these conditions treating an example in section 4.4. In the second example (section 4.5) we are concerned with an orbit transfer problem.

4.1 Optimal Control Problems with Free Control Subarcs

We consider the optimal control problem stated in chapter 1. The difference lies in the structure of the Lagrange-part of the objective function, the right hand side of the state equation, and the mixed control-state constraints. For the sake of clearness, we treat an automomous problem. The following theory can be applied to non-autonomous problems by a transformation to an autonomous auxiliary problem.

OCP 4.1. *Determine a piecewise continuous control function* $u : [t_0, t_f] \to \mathbb{R}^m$*, such that the functional*

$$J(x,u) = \phi(x(t_f)) + \int_{t_0}^{t_f} L(x(t),u(t))dt \tag{4.1}$$

is minimized subject to

$$\begin{aligned} x'(t) &= f(x(t),u(t)), \qquad t \in [t_0,t_f], a.e., &(4.2)\\ r(x(t_0),x(t_f)) &= 0, &(4.3)\\ C(x(t),u(t)) &\le 0, \qquad t \in [t_0,t_f], a.e., &(4.4) \end{aligned}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ *and* $r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^\ell$ *are twice differentiable. The control function can be split into two parts* $u(t) = (v(t), w(t)) \in \mathbb{R}^{m_v+m_w}$, $t \in [t_0,t_f]$, $m_w > 0$, *such that the integrand of (4.1), the right hand side of the state equation (4.2), and the constraints (4.4) have the following structure*

$$\begin{aligned} L(x,u) &= L^1(x,v) + S(x,v) \cdot L^2(x,v,w),\\ f(x,u) &= f^1(x,v) + S(x,v) \cdot f^2(x,v,w),\\ C(x,u) &= C^1(x,v) + S(x,v) \cdot C^2(x,v,w) \end{aligned}$$

with a piecewise continuous differentiable function $S : \mathbb{R}^n \times \mathbb{R}^{m_v} \to \mathbb{R}$ *and twice continuous differentialbe functions* $L^1 : \mathbb{R}^n \times \mathbb{R}^{m_v} \to \mathbb{R}$, $L^2 : \mathbb{R}^n \times \mathbb{R}^{m_v+m_w} \to \mathbb{R}$, $f^1 : \mathbb{R}^n \times \mathbb{R}^{m_v} \to \mathbb{R}^n$, $f^2 : \mathbb{R}^n \times \mathbb{R}^{m_v+m_w} \to \mathbb{R}^n$, $C^1 : \mathbb{R}^n \times \mathbb{R}^{m_v} \to \mathbb{R}^d$, *and* $C^2 : \mathbb{R}^n \times \mathbb{R}^{m_v+m_w} \to \mathbb{R}^d$.

The function S can be regarded as a switching function. We use similar ideas when developing the theory as used in chapter 3. Let us first introduce some notations.

Definition 4.1. *The control components w are called* dependent controls*, since they enter into the problem by multiplication with the switching function S. The control components v are called* independent controls*.*

Independent controls do not have to exist if we use a switching function $S(x)$ of order 1. If the control is a scalar function, it may not appear on subintervals, where S vanishes.

We now treat the special case, where S vanishes on subintervals. Let (x_0, u_0) with $u_0 = (v_0, w_0)$ be a solution of OCP 4.1. Further let the following assumption be fulfilled.

Assumption 4.2. *There exists a subdivision* $t_0 < t_1 < \cdots < t_{N-1} < t_N := t_f, N \in \mathbb{N}$*, of the interval* $[t_0,t_f]$ *into N subintervals, such that for every* $i = 1, \ldots, N$ *either one of the following conditions is satisfied*

$$\begin{aligned} & S(x_0(t),v_0(t)) = 0 \quad \text{for every } t \in [t_{i-1},t_i],\\ \text{or} \quad & S(x_0(t),v_0(t)) \neq 0 \quad \text{for almost every } t \in]t_{i-1},t_i[. \end{aligned}$$

The case of S having isolated roots is included in the second expression above. Let us define two sets of indices which distinguish between intervals, where S vanishes or not:

$$\begin{aligned} J^0 &:= \{i \in \{1,\dots,N\} \mid \forall t \in [t_{i-1}, t_i] : S(x_0(t), v_0(t)) = 0\}, \\ J^1 &:= \{1,\dots,N\} \backslash J^0. \end{aligned}$$

On the subintervals $[t_{i-1}, t_i], i \in J^0$, the dependent control w_0 has no influence on the integrand of the objective function (4.1), on the state equation (4.2), and on the mixed control-state constraints (4.4) any more. Therefore we define:

Definition 4.3. *If Assumption 4.2 is fulfilled, we call the subarcs $w_0(t), t \in [t_{i-1}, t_i], i \in J^0$, free control subarcs.*

The functions (x_0, u_0) then describe a solution of OCP 4.1 with free control subarcs.

After this introduction we are able to develop new necessary and sufficient optimality conditions.

4.2 Necessary Optimality Conditions for Optimal Control Problems with Free Control Subarcs

In this section, we develop necessary optimality conditions for the optimal control problem with free control subarcs. These are less strict compared with general first order necessary conditions.

Let the switching function S which occurs in the functions L, f, and C additionally satisfy the following condition.

Assumption 4.4. *The switching function S satisfies:*

$$\exists j \in \{1,\dots,m_v\} : \qquad D_{v_j} S(x, v) \neq 0.$$

This means that there exists a control component v_j, for which we can apply the Implicit Function Theorem. A continuous function V exists, which solves the equation $S(x, v) = 0$ uniquely. Without loss of generality, we assume $j = m_v$ and devide the control components into $v = (\tilde{v}^T, V)^T$. The function defined by $S(x, v) = 0$ is labelled by $v_{m_v} = V(t; x, \tilde{v})$.

If Assumption 4.4 is satisfied, the switching function is of order 0. In general it suffices to demand $D_v S(x, v) \neq 0$, that is, the switching function S shall be of order 0. This condition is more general, since the control components which occur explicitely in the switching function may change. One then has to apply the multiprocess-technique with a subdivision regarding these changes. For simplicity, we restrict ourselves to a switching function, which satisfies Assumption 4.4.

First order necessary optimality conditions can be stated using the Hamiltonian

$$\begin{aligned}
&\widetilde{\mathcal{H}} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R},\\
&\widetilde{\mathcal{H}}(x,u,\lambda,\mu) := \lambda_0 L(x,u) + \lambda f(x,u) + \mu C(x,u)
\end{aligned}$$

Theorem 4.5. *Let (x_0, u_0) be a weak local minimum of OCP 4.1 with a switching function S satisfying Assumptions 4.2 and 4.4. Let regularity conditions R1 and R2 be satisfied. Then there exist a continuous and continuous differentiable adjoint variable $\lambda : [t_0, t_f] \to \mathbb{R}^n$, a piecewise continuous multiplier function $\mu : [t_0, t_f] \to \mathbb{R}^d$, and multipliers $\nu \in \mathbb{R}^\ell$ and $\lambda_0 \in \mathbb{R}$ not all vanishing simultaneously on $[t_0, t_f]$, such that (x_0, u_0) satisfies*

(a) the adjoint equations

$$\lambda'(t) = -\widetilde{\mathcal{H}}_x(x_0(t), u_0(t), \lambda_0, \lambda(t), \mu(t)), \qquad t \in [t_0, t_f], \textit{a.e.}, \tag{4.5}$$

(b) the natural boundary conditions

$$\lambda(t_0) = -D_{x_s}[\nu r(x_0(t_0), x_0(t_f))], \tag{4.6}$$
$$\lambda(t_f) = D_{x_f}[\lambda_0 \phi(x_0(t_f)) + \nu r(x_0(t_0), x_0(t_f))], \tag{4.7}$$

(c) the complementary condition

$$\mu(t) \geq 0 \quad \textit{and} \quad \mu(t) C(x_0(t), u_0(t)) = 0, \qquad t \in [t_0, t_f], \textit{a.e.}, \tag{4.8}$$

(d) the minimum principle

$$\begin{aligned}
\forall i \in J^1 : \quad & u_0(t) = \textit{argmin}\left\{\widetilde{\mathcal{H}}(x_0(t), u, \lambda_0, \lambda(t), \mu(t)) | u \in \mathbb{R}^m\right\},\\
\forall i \in J^0 : \quad & \tilde{v}_0(t) = \textit{argmin}\left\{\widetilde{\mathcal{H}}(x_0(t), \tilde{v}, \lambda_0, \lambda(t), \mu(t)) | \tilde{v} \in \mathbb{R}^{m_v - 1}\right\},\\
\textit{and} \quad & v_{m_v}(t) = V(t; x, \tilde{v}),
\end{aligned} \tag{4.9}$$

for almost every $t \in [t_0, t_f]$,

(e) the minimum condition

$$\begin{aligned}
\forall i \in J^1 : \quad & \widetilde{\mathcal{H}}_u[t] = 0,\\
\forall i \in J^0 : \quad & \widetilde{\mathcal{H}}_{\tilde{v}}[t] = 0 \textit{ and } S(x,v) = 0,
\end{aligned} \tag{4.10}$$

for almost every $t \in [t_0, t_f]$,

(f) and the continuity conditions

$$\widetilde{\mathcal{H}}[t_i^+] = \widetilde{\mathcal{H}}[t_i^-], \qquad i = 1, \dots, N, \tag{4.11}$$
$$\lambda(t_i^+) = \lambda(t_i^-), \qquad i = 1, \dots, N. \tag{4.12}$$

We assume that these conditions are satisfied in normal form and set $\lambda_0 = 1$.

We can generalize the problem by treating a higher dimensional switching function S or different control functions which are obtained by applying the Implicit Function Theorem. However since these generalizations are quite cumbersome, we restrict the discussion to the special case.

4.2.1 Proof of Necessary Optimality Conditions

For the proof of Theorem 4.5, we use the same technique as used in chapter 3. We develop first order necessary optimality conditions for optimal control problems with free control subarcs by transforming OCP 4.1 into an auxiliary optimal control problem for which the necessary conditions stated in section 1.3 hold. After applying Theorem 1.5 to the auxiliary problem, we will get new necessary conditions by transforming the results back to the initial problem.

The Auxiliary Problem. Let (x_0, u_0) be a solution of OCP 4.1 with free control subarcs and a switching function satisfying Assumption 4.4. We compare the optimal solution only with admissible solutions (x, u) which have the same switching structure (Assumption 4.2).

For all $i = 1, \dots, N$ we define new state variables

$$x_i(s) := x(t_{i-1} + s(t_i - t_{i-1})), \tag{4.13}$$

continous differentiable continued, depending on the new time variable $s \in [0, 1]$. Since the switching times $t_i, i = 1, \dots, N-1$ are not known, these times have to be calculated in the optimal control problem. We introduce additional state variables with corresponding state equations

$$\tau_i(s) := t_{i-1} - t_i \quad \Rightarrow \quad \tau_i'(s) = 0,$$

$i = 1, \dots, N, s \in [0, 1]$, defining the lengths of the subintervals. With this formulation the equation

$$\sum_{i=1}^{N} \tau_i(s_i) = t_f - t_0, \qquad s_i \in [0, 1], i = 1, \dots, N \tag{4.14}$$

must be valid. If t_f is free, the above condition does not reduce the degree of freedom. If t_f is fixed, (4.14) can be treated as an additional boundary condition, because all functions τ_i are constant. For instance, it suffices to demand equation (4.14) to be valid in $s_i = 0, i = 1, \dots, N$. The choice of different values for s_i is useful for the application of sufficient conditions (example FCS, pp. 85).

We separate the control function as mentioned above into $u = (v, w)$. Since we treat subintervals corresponding to Assumption 4.2, only for $i \in J^1$ all components of the control function influence the optimal control problem. We define these components as the continuous continuation of

$$u_i(s) := u(t_{i-1} + s\tau_i(s)), \qquad s \in]0,1[, i \in J^1.$$

on $[0,1]$. The dependent controls w are free control subarcs on the subintervals $[t_{i-1}, t_i]$, $i \in J^0$ and thus have no influence to the problem. Furthermore because of Assumption 4.4 we can apply the Implicit Function Theorem and split the independent control components into two parts $v(t) = (\tilde{v}(t), V(t)) \in \mathbb{R}^{m_v - 1} \times \mathbb{R}$. Further control components which enter into the auxiliary problem are

$$\tilde{v}_i(s) := \tilde{v}(t_{i-1} + s\tau_i(s)), \qquad s \in [0,1], i \in J^0.$$

The differentiation of the state variables x_i yields in

$$x_i'(s) = \tau_i(s) f(x_i(s), u_i(s)),$$

For all $i \in J^0$ the switching function S vanishes. The right hand side of the state equation reduces to:

$$S(x_i(s), u_i(s)) = 0 \quad \Rightarrow \quad x_i'(s) = \tau_i(s) f^1(x_i(s), v_i(s)).$$

An analogous argumentation yields in

$$\begin{aligned}\int_{t_0}^{t_f} L(x(t), u(t))dt &= \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} L(x(t), u(t))dt \\ &= \sum_{i \in J^1} \int_0^1 \tau_i(s) L(x_i(s), u_i(s))ds \\ &+ \sum_{i \in J^0} \int_0^1 \tau_i(s) L^1(x_i(s), v_i(s))ds.\end{aligned}$$

We formulate the auxiliary problem:

OCP 4.2. *Determine piecewise continuous control functions $u_i : [0,1] \to \mathbb{R}^m$, $i \in J^1$ and $\tilde{v}_i : [0,1] \to \mathbb{R}^{m_{\tilde{v}}}$, $i \in J^0$, such that the functional*

$$\begin{aligned}&J(x_1, \tau_1, \dots, x_N, \tau_N, u_1, \dots, u_N) = \\ &\quad \phi(x_N(1)) + \sum_{i \in J^1} \int_0^1 \tau_i(s) L(x_i(s), u_i(s))ds \\ &\qquad + \sum_{i \in J^0} \int_0^1 \tau_i(s) L^1(x_i(s), \tilde{v}_i(s), V_i(s; x_i(s), \tilde{v}_i(s)))ds\end{aligned}$$

is minimized subject to:

$$\begin{aligned}
x_i'(s) =& \ \tau_i(s) f(x_i(s), u_i(s)), && i \in J^1, \\
\tau_i'(s) =& \ 0, && i \in J^1, \\
x_i'(s) =& \ \tau_i(s) f^1(x_i(s), \tilde{v}_i(s), V_i(s; x_i(s), \tilde{v}_i(s))), && i \in J^0, \\
\tau_i'(s) =& \ 0, && i \in J^0, \\
r(x_1(0), x_N(1)) =& \ 0, && \\
x_i(0) - x_{i-1}(1) =& \ 0, && i = 2, \dots, N, \\
C(x_i(s), u_i(s)) \leq& \ 0, && i \in J^1, \\
C^1(x_i(s), \tilde{v}_i(s), V_i(s; x_i(s), \tilde{v}_i(s))) \leq& \ 0, && i \in J^0,
\end{aligned}$$

for almost every $s \in [0,1]$, and additionally if t_f is fixed

$$\sum_{i=1}^{N} \tau_i(s_i) - (t_f - t_0) = 0, \qquad s_i \in [0,1],\ i = 1, \dots, N. \tag{4.15}$$

Problem OCP 4.2 is a regular, constrained optimal control problem with continuous right hand side and general boundary conditions on the fixed interval $[0,1]$. Before we apply the general statements of chapter 1 to this problem, we introduce some notational conventions. All state variables and control functions of the auxiliary problem OCP 4.2 can be summarized by the vectors

$$x^* := (x_1^T, \tau_1, \dots, x_N^T, \tau_N)^T \qquad \text{and} \qquad u^* := (u_1^T, \dots, u_N^T)^T. \tag{4.16}$$

We distinguish between subintervals with usual control subarcs or free control subarcs.

$$u_i = \begin{cases} (\tilde{v}_i^T, V_i, w_i^T)^T, & i \in J^1, \\ (\tilde{v}_i), & i \in J^0. \end{cases} \tag{4.17}$$

The dimension m_i of the control function $u_i(t)$ is m respectively $m_{\tilde{v}}$. The following index set $I_S(s)$ contains all indices of active control constraints as well as control components which are free control subarcs or which can be computed by applying the Implicit Function Theorem to $S[s] = 0$.

$$I_S(s) := \begin{cases} I^+(s) \cup \{m_v, m_v + 1, \dots, m\}, & S[s] = 0, \\ I^+(s), & S[s] \neq 0. \end{cases}$$

Regarding all state variables and control functions together, we introduce the vector

$$y^* := (y_1^T, \dots, y_N^T)^T, \quad \text{with} \quad y_i(t) := (x_i^T, \tau_i, u_i^T)^T,\ i = 1, \dots, N. \tag{4.18}$$

The combined right hand side of the state equation is given by

$$f^*(x^*, u^*) := \begin{pmatrix} \tau_1(s) \cdot f(x_1(s), u_1(s)) \\ 0 \\ \vdots \\ \tau_N(s) \cdot f(x_N(s), u_N(s)) \\ 0 \end{pmatrix}. \tag{4.19}$$

Thus, the Hamiltonian of the auxiliary problem is defined with the multiplier $\lambda^* := (\lambda_1, \lambda_1^{(\tau)}, \ldots, \lambda_N, \lambda_N^{(\tau)})$ by

$$\begin{aligned} \mathcal{H}^*(x^*, u^*, \lambda_0, \lambda^*) &:= \sum_{i=1}^{N} \tau_i \mathcal{H}_i(x_i, u_i, \lambda_0, \lambda_i) \qquad (4.20) \\ \text{with} \quad \mathcal{H}_i(x_i, u_i, \lambda_0, \lambda_i) &:= \lambda_0 L(x_i, u_i) + \lambda_i f(x_i, u_i), \quad i = 1, \ldots, N. \end{aligned}$$

If we attach the control constraints with scaled multipliers $\tau_i \mu_i, i = 1, \ldots, N$, we derive the augmented Hamiltonian

$$\begin{aligned} \widetilde{\mathcal{H}}^*(x^*, u^*, \lambda_0, \lambda^*, \mu^*) &:= \sum_{i=1}^{N} \tau_i \widetilde{\mathcal{H}}_i(x_i, u_i, \lambda_0, \lambda_i, \mu_i) \qquad (4.21) \\ \text{with} \quad \widetilde{\mathcal{H}}_i(x_i, u_i, \lambda_0, \lambda_i, \mu_i) &:= \mathcal{H}_i(x_i, u_i, \lambda_0, \lambda_i) + \mu_i C(x_i, u_i), \quad i = 1, \ldots, N. \end{aligned}$$

The augmented performance index is given by

$$\Phi := \lambda_0 \phi(x_N(1)) + \nu r(x_1(0), x_N(1)) + \sum_{i=2}^{N} \tilde{\nu}_i (x_i(0) - x_{i-1}(1))$$

After these definitions we are able to proof the necessary conditions for OCP 4.1.

Proof. (Theorem 4.5) Let (x_0, u_0) be a solution of OCP 4.1 with free control subarcs. Let the switching function S fulfill Assumptions 4.2 and 4.4. Then (x_0^*, u_0^*) is a solution of OCP 4.2.

The application of Theorem 1.5 to the auxiliary problem OCP 4.2 yields in the existence of multipliers $\lambda^*(s) \in \mathbb{R}^{N(n+2)}$, $\mu_i(s) \in \mathbb{R}^d, i = 1, \ldots, N$, $\lambda_0 \in \mathbb{R}, \nu \in \mathbb{R}^\ell$, and $(\tilde{\nu}_1, \ldots, \tilde{\nu}_{N-1}) \in \mathbb{R}^{(N-1)n}$, which satisfy the following equations for almost every $s \in [0, 1]$

$$\lambda_i'(s) = -\tau_i(s)\widetilde{\mathcal{H}}_{ix_i}[s], \qquad i = 1,\dots,N, \tag{4.22}$$

$$\lambda_i^{(\tau)\prime}(s) = -\widetilde{\mathcal{H}}_i[s], \qquad i = 1,\dots,N, \tag{4.23}$$

$$\lambda_1(0) = -D_{x_s}\left[\nu r(x_{01}(0), x_{0N}(1))\right], \tag{4.24}$$

$$\lambda_N(1) = D_{x_f}\left[\lambda_0\phi(x_{0N}(1)) + \nu r(x_{01}(0), x_{0N}(1))\right], \tag{4.25}$$

$$\lambda_{i+1}(0) = -\tilde{\nu}_i, \quad \lambda_i(1) = -\tilde{\nu}_i, \qquad i = 1,\dots,N-1, \tag{4.26}$$

$$\mu_i(s) \geq 0, \quad \mu_i C(x_{0i}(s), u_{0i}(s)) = 0, \qquad i = 1,\dots,N, \tag{4.27}$$

$$u_{0i}(s) = \operatorname{argmin}\{\widetilde{\mathcal{H}}_i(x_{0i}(s), u, \lambda_0, \lambda_i) | u_i \in \mathbb{R}^{m_i}\}, \qquad i = 1,\dots,N, \tag{4.28}$$

$$\widetilde{\mathcal{H}}_{iu_i}[s] = 0, \qquad i = 1,\dots,N. \tag{4.29}$$

If the final time t_f is free, we get

$$\lambda_i^{(\tau)}(0) = 0, \quad \lambda_i^{(\tau)}(1) = 0, \qquad i = 1,\dots,N. \tag{4.30}$$

Otherwise we have to regard an additional boundary condition (4.15) in the state variables τ_i. Without loss of generality, we choose $s_i = 0, i = 1,\dots,N$ in this equation. This condition results in the existence of an additional multiplier $\nu^{(\tau)} \in \mathbb{R}$ with

$$\lambda_i^{(\tau)}(0) = -\nu^{(\tau)}, \quad \lambda_i^{(\tau)}(1) = 0, \qquad i = 1,\dots,N. \tag{4.31}$$

Because the Hamiltonian $\widetilde{\mathcal{H}}_i[t]$ is constant, equations (4.23) and (4.30) respectively (4.31) yield in multipliers $\lambda_i^{(\tau)}$ which all have the same slope. Thus the Hamiltonian is continuous; Equation (4.11).

If we recombine the multipliers

$$\lambda(t) := \lambda_i\Big(\frac{t - t_{i-1}}{\tau_i}\Big), \quad \mu(t) := \mu_i\Big(\frac{t - t_{i-1}}{\tau_i}\Big), \qquad \text{for } t \in [t_{i-1}, t_i], i = 1,\dots,N,$$

we obtain the adjoint equation (4.5), the continuity of the adjoint variables (4.12) and the transversality condition (4.8). The boundary conditions (4.24) and (4.25) result in conditions (4.6) and (4.7).

The optimal control u_0 can be recombined as well. Thus, (4.28) and (4.29) yield in the minimum principle (4.9) and the minimum condition (4.10). $\square$

4.3 Sufficient Optimality Conditions for Optimal Control Problems with Free Control Subarcs

In this section we show, that general second order sufficient optimality conditions cannot always be applied to problem OCP 4.1. Therefore we develop new sufficient conditions.

Lemma 4.6. *The strict Legendre-Clebsch condition is not satisfied for problem OCP 4.1 with free control subarcs.*

Proof. Let (x_0, u_0) be admissible for OCP 4.1 with free control subarcs. Let Assumption 4.2 be satisfied and $J^0 \neq \emptyset$. With $S(x,v) = 0$ on $[t_{i-1}, t_i], i \in J^0$, also the partial derivative $S_v(x,v)$ vanishes. Thus, we get for all $t \in [t_{i-1}, t_i], i \in J^0$

$$\widetilde{\mathcal{H}}_{uu} = \begin{pmatrix} \widetilde{\mathcal{H}}_{vv} & 0 \\ 0 & 0 \end{pmatrix} \tag{4.32}$$

with $\widetilde{\mathcal{H}}_{vv} = L^1_{vv}(x,v) + \lambda f^1_{vv}(x,v) + \mu C^1_{vv}(x,v)$. The matrix $\widetilde{\mathcal{H}}_{uu}$ is not positive definite. Thus, the strict Legendre-Clebsch condition is not satisfied. □

Because of the above lemma we need to develop new sufficient conditions for problem OCP 4.1 with free control subarcs.

Let (x_0, u_0) be a solution candidate for problem OCP 4.1 with free control subarcs. That is, let Assumptions 4.2 and 4.4 be satisfied. Let the function $\tau : [t_0, t_f] \to \mathbb{R}$ defined by

$$\begin{aligned} \tau(t) &:= t_i - t_{i-1}, \qquad \forall t \in [t_{i-1}, t_i[, i = 1, \dots, N \\ \text{and } \tau(t_f) &:= t_N - t_{N-1} \end{aligned}$$

describe the lengths of subintervals which arise when Assumption 4.2 is fulfilled.

Regarding the essential control functions

$$u_{\mathcal{J}}(t) := \begin{cases} (\tilde{v}(t)^T, V(t), w(t)^T)^T, & t \in [t_{i-1}, t_i], i \in J^1, \\ \tilde{v}(t), & t \in]t_{i-1}, t_i[, i \in J^0, \end{cases}$$

we formulate the system of Riccati differential equations as

$$Q' = -\widetilde{\mathcal{H}}_{xx} - f_x^T Q - Q f_x + (\widetilde{\mathcal{H}}_{xu_{\mathcal{J}}} + Q f_{u_{\mathcal{J}}})(\widetilde{\mathcal{H}}_{u_{\mathcal{J}} u_{\mathcal{J}}})^{-1}(\widetilde{\mathcal{H}}_{xu_{\mathcal{J}}} + Q f_{u_{\mathcal{J}}})^T, \tag{4.33}$$

$$R' = \frac{1}{\tau}\left[-\widetilde{\mathcal{H}}_x^T - Qf\right] - f_x^T R + \left(\widetilde{\mathcal{H}}_{xu_{\mathcal{J}}} + Q f_{u_{\mathcal{J}}}\right)(\widetilde{\mathcal{H}}_{u_{\mathcal{J}} u_{\mathcal{J}}})^{-1} f_{u_{\mathcal{J}}}^T R, \tag{4.34}$$

$$q' = -\frac{2}{\tau} R^T f + R^T f_{u_{\mathcal{J}}} (\widetilde{\mathcal{H}}_{u_{\mathcal{J}} u_{\mathcal{J}}})^{-1} f_{u_{\mathcal{J}}}^T R. \tag{4.35}$$

depending on the symmetric matrix-function $Q : [t_0, t_f] \to \mathbb{R}^{n\times n}$ and the functions $R : [t_0, t_f] \to \mathbb{R}^n$ and $q : [t_0, t_f] \to \mathbb{R}$. The boundary and jump conditions are the same as in chapter 3 (page 45). Note, that we only deal with switching functions of order 0.

With these formulations the theorem corresponding to Theorem 1.12 is the following.

Theorem 4.7. *Assume that the following conditions hold.*

1. (x_0, u_0) *is a solution candidate for problem OCP 4.1.*
2. *The regularity conditions R1 and R2 hold.*
3. *The strict Legendre-Clebsch condition is satisfied with respect to* $u_{\mathcal{J}}$.
4. *The system of differential equations (4.33)- (4.35) has bounded solutions* Q, R, *and* q, *which are piecewise continuous differentiable.*
5. *These solutions* Q, R, *and* q *satisfy boundary condition B3 and jump condition J3.*

Then, there exist $\epsilon > 0$ *and* $\delta > 0$, *such that*

$$\begin{aligned} J(x,\tau,v) \;\geq\; & J(x_0,\tau_0,v_0) \\ & +\delta\{||(x,\tau,v)-(x_0,\tau_0,v_0)||_2^2 + ||(x(t_0),x(t_f))-(x_0(t_0),x_0(t_f))||^2\} \end{aligned}$$

holds for all admissible functions $(x,\tau,u) \in L^\infty([t_0,t_f],\mathbb{R}^{n+1+m})$ *with* $d((x,\tau,u) - (x_0,\tau_0,u_0)) \leq \epsilon$. *In particular,* (x_0,u_0) *is a weak local minimum compared with admissible pairs* (x,u) *having the same switching structure.*

Corresponding to Theorem 1.13 we get the following statement.

Theorem 4.8. *Assume that the following conditions hold.*

1. (x_0, u_0) *is a solution candidate for problem OCP 4.1.*
2. *The regularity conditions R1 and R2 hold.*
3. *The regularity condition R3 holds with respect to* $u_{\mathcal{J}}$.
4. *The system of differential equations (4.33)-(4.35) with partial derivatives only in the direction of inactive control components of* $u_{\mathcal{J}}$ *has bounded solutions* Q, R *and* q, *which are piecewise continuous differentiable.*
5. *These solutions* Q, R, *and* q *satisfy boundary condition B3 and jump condition J3.*

Then, there exist $\epsilon > 0$ and $\delta > 0$, such that

$$\begin{aligned} J(x,\tau,v) \;\geq\; & J(x_0,\tau_0,v_0) \\ & +\delta\{||(x,\tau,v)-(x_0,\tau_0,v_0)||_2^2 + ||(x(t_0),x(t_f))-(x_0(t_0),x_0(t_f))||^2\} \end{aligned}$$

holds for all admissible functions $(x,\tau,u) \in L^\infty([t_0,t_f],\mathbb{R}^{n+1+m})$ with $d((x,\tau,u) - (x_0,\tau_0,u_0)) \leq \epsilon$. In particular, (x_0,u_0) is a weak local minimum compared with admissible pairs (x,u) having the same switching structure.

The jumps in the control function u and the functions Q, R, and q may only occur at the switching times, which are defined in Assumption 4.2.

In the case of a higher dimensional function S the results can be extended. We abstain from developing sufficient conditions for the general case since this will lead to an unnecessary complexity.

Second order sufficient optimality conditions for similar problems have already been investigated by Maurer and Osmolovskii [OM07]. They treat the case, where v enters into the problem linearly while w enters nonlinearly. The optimal control obtained by applying necessary conditions is a bang-bang control v and a continuous control w. The switching functions σ is related to the linearly appearing control function v. This switching function can be different to S, which we treat in this chapter.

Second order sufficient optimality conditions are given for the special case of bang-bang and continuous control functions. Maurer and Osmolovskii combine the theory of positive definiteness on a critical cone and the Riccati approach. Comparing Theorem 4.1 in [OM07] with Theorems 4.7 and 4.8, the boundary condition (d) is the same as B3. The jump conditions (c) differ from J3.

In the discussed paper, the solution candidate must be strictly Legendrian (Definition 3.2). This condition is not satisfied for problems with free control subarcs (Lemma 4.6). The here developed theory is an extension to [OM07] under the assumption that we compare functions of the same structure.

Let us now proof Theorems 4.7 and 4.8.

4.3.1 Proof of Second Order Sufficient Optimality Conditions

The new sufficent conditions stated in Theorems 4.7 and 4.8 can be obtained by applying Theorems 1.12 and 1.13 to the auxiliary problem OCP 4.2. We compare solution candidates with functions of the same structure.

We first treat an optimal control problem OCP 4.1 with free final time. The case of t_f being fixed will be treated later on (p. 84).

Proof. (Theorem 4.7, OCP 4.1 with free final time) In a fist step, we verify all conditions of Theorem 1.12 regarding a solution candidate (x_0^*, u_0^*) of the auxiliary problem OCP 4.2. These are satisfied, if the conditions of Theorem 4.7 hold for a solution candidate (x_0, u_0) of OCP 4.1. In a second step, we show that this leads to the optimality of (x_0, u_0).

Let conditions *1.-3.* of Theorem 4.7 be fulfilled. Then the following argumentation shows that the corresponding conditions for the auxiliary problem are satisfied.

An admissable point (x_0, u_0) of problem OCP 4.1 which satisfies Assumptions 4.2 and 4.4 is after transformation also admissible for the auxiliary problem OCP 4.2. By applying first order necessary conditions (Theorem 4.5) to problem OCP 4.1 the condition of Theorem 1.12 holds; that is, first order necessary conditions shall be satisfied for the auxiliary problem OCP 4.2.

Let R1 be satisfied for OCP 4.1. The vectors

$$D_u C_j(x_0(t), u_0(t)),$$

$j \in I^0(t)$, are linear independent for almost every $t \in [t_0, t_f]$. These vectors are especially linear independent on each subinterval $[t_{i-1}, t_i], i = 1, \ldots, N$. Treating the auxiliary problem OCP 4.2, this means, that all

$$D_{u_i} C_j(x_i(s), u_i(s))$$

are linear independent. And because of the block-diagonal structure of the Jacobian $D_{y^*} C(x_0^*(s), u_0^*(s))$, regularity condition R1 is satisfied for the auxiliary problem OCP 4.2.

Concerning regularity condition R2 we need the boundary conditions of problem OCP 4.2:

$$r^*(x^*(0), x^*(1)) := \begin{pmatrix} r(x_1(0), x_N(1)) \\ x_2(0) - x_1(1) \\ \vdots \\ x_N(0) - x_{N-1}(1) \end{pmatrix} \in \mathbb{R}^{\ell + n \cdot (N-1)}. \tag{4.36}$$

The Jacobian of $r^*(x^*(0), x^*(1))$ is given by the matrix

$$\left(\begin{array}{cccccccc|cccccccc} D_{x_s} r & 0 & & & & & & & & & & & & & D_{x_f} r & 0 \\ & & I_n & 0 & & & & & -I_n & 0 & & & & & & \\ & & & & \ddots & & & & & & \ddots & & & & & \\ & & & & & & I_n & 0 & & & & & -I_n & 0 & & \end{array}\right). \tag{4.37}$$

It is of rank $\ell + n \cdot (N-1)$, if and only if regularity condition R2 is satisfied for problem OCP 4.1.

Now we treat the strict Legendre-Clebsch condition. The augmented Hamiltonian is given by equation (4.21). Regarding the control components (4.16) and (4.17) the Hessian of $\widetilde{\mathcal{H}}^*$

$$\widetilde{\mathcal{H}}^*_{u^*u^*} = \operatorname{diag}(\tau_1 \widetilde{\mathcal{H}}_{1u_1u_1}, \ldots, \tau_N \widetilde{\mathcal{H}}_{Nu_Nu_N}) \tag{4.38}$$

is positive definite for all $s \in [0,1]$, if all submatrices are positive definite. This is the case, because the strict Legendre-Clebsch condition shall be satisfied on each subinterval $[t_{i-1}, t_i]$ (Theorem 4.7) and $\tau_i > 0$ holds for all $i = 1, \ldots, N$.

We need the derivatives of f^* and $\widetilde{\mathcal{H}}^*$ to establish the Riccati equation for the auxiliary problem OCP 4.2. In the following we use formulations (4.16)-(4.21).

$$f^*_{x^*} = \operatorname{diag}(FX_1, \ldots, FX_N) \qquad \text{with} \qquad FX_i := \begin{pmatrix} \tau_i f_{ix_i} & f_i \\ 0 & 0 \end{pmatrix},$$

$$f^*_{u^*} = \operatorname{diag}(FU_1, \ldots, FU_N) \qquad \text{with} \qquad FU_i := \begin{pmatrix} \tau_i f_{iu_i} \\ 0 \end{pmatrix}.$$

Second derivatives of the Hamiltonian are

$$\widetilde{\mathcal{H}}^*_{x^*x^*} = \operatorname{diag}(HXX_1, \ldots, HXX_N) \qquad \text{with} \qquad HXX_i := \begin{pmatrix} \tau_i \widetilde{\mathcal{H}}_{ix_ix_i} & \widetilde{\mathcal{H}}^T_{ix_i} \\ \widetilde{\mathcal{H}}_{ix_i} & 0 \end{pmatrix},$$

$$\widetilde{\mathcal{H}}^*_{x^*u^*} = \operatorname{diag}(HXU_1, \ldots, HXU_N) \qquad \text{with} \qquad HXU_i := \begin{pmatrix} \tau_i \widetilde{\mathcal{H}}_{ix_iu_i} \\ \widetilde{\mathcal{H}}_{iu_i} \end{pmatrix},$$

$$\widetilde{\mathcal{H}}^*_{u^*u^*} = \operatorname{diag}(HUU_1, \ldots, HUU_N) \qquad \text{with} \qquad HUU_i := \begin{pmatrix} \tau_i \widetilde{\mathcal{H}}_{iu_iu_i} \end{pmatrix}.$$

Note, that on subintervals with free control subarcs L, f, and C reduce to L^1, f^1, and C^1, respectively. We exploit the block-diagonal structure of the matrices above. Therefore we introduce the symmetric matrix-function $Q^* : [0,1] \to \mathbb{R}^{N(n+1)\times N(n+1)}$, which has the same block-diagonal form

$$Q^* = \operatorname{diag}(Q^*_1, \ldots, Q^*_N) \quad \text{with} \quad Q^*_i := \begin{pmatrix} Q_i & R_i \\ R_i^T & q_i \end{pmatrix} \tag{4.39}$$

where $Q_i(s) \in \mathbb{R}^{n\times n}, R_i(s) \in \mathbb{R}^n, q_i(s) \in \mathbb{R}$. We have to show the consistency of this choice with boundary condition B1. If the functions Q, R, and q solve the system of Riccati differential equations (4.33)-(4.35) as demanded in Theorem 4.7, we can create a solution Q^* of the Riccati differential equation (1.19) correspoding to the auxiliary problem OCP 4.2.

On the one hand we need to set up the general boundary conditions for a solution Q^* of the Riccati equation. On the other hand it still must be examined whether the choice of a block-diagonal form of Q^* is at all permissible. We use second derivatives of the Mayer-part in the objective function $\phi(x_N(1))$ and of the boundary conditions $r^*(x^*(0), x^*(1))$ from equation (4.36). Altogether we get

$$D^2_{(x_s^*, x_f^*)}(\phi + (\nu^*)^T r^*) = \left(\begin{array}{cccc|c} A_{11} & & & & A_{1N} \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ \hline & & & & \\ A_{N1} & & & & A_{NN} \end{array}\right)$$

with

$$A_{11} := \begin{pmatrix} D^2_{x_s}(\nu r) & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_{1N} := \begin{pmatrix} D^2_{x_s x_f}(\nu r) & 0 \\ 0 & 0 \end{pmatrix},$$

$$A_{N1} := \begin{pmatrix} D^2_{x_f x_s}(\nu r) & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_{NN} := \begin{pmatrix} D^2_{x_f}(\phi + \nu r) & 0 \\ 0 & 0 \end{pmatrix}.$$

With the block-diagonal matrix Q^* from equation (4.39), we can formulate boundary conditions. The matrix

$$M_R = \begin{pmatrix} Q^*(0) & 0 \\ 0 & -Q^*(1) \end{pmatrix} + D^2_{(x_s^*, x_f^*)}(\phi + \nu^* r^*) \tag{4.40}$$

must be positive definite on the set

$$\begin{aligned} R_R &= \ker\{D_{(x_s^*, x_f^*)} r^*(x^*(0), x^*(1))\} \\ &= \Big\{(\eta_{01}, \eta_{01}^{(\tau)}, \ldots, \eta_{0N}, \eta_{0N}^{(\tau)}, \eta_{f1}, \eta_{f1}^{(\tau)}, \ldots, \eta_{fN}, \eta_{fN}^{(\tau)}) \in \mathbb{R}^{2N(n+1)} \ : \\ &\qquad (\eta_{01}, \eta_{fN}) \in \ \ker\{D_{(x_s, x_f)} r(x_1(0), x_N(1))\}, \\ &\qquad \eta_{0i} = \eta_{fi-1},\ i = 2, \ldots, N\Big\}. \end{aligned}$$

We exploit the condition $\eta_{0i} = \eta_{fi-1}, i = 2, \ldots, N$. After some calculations we find, that if boundary condition B3 and jump condition J3 are satisfied for OCP 4.1, then the matrix M_R is positive definite on R_R regarding the auxiliary problem OCP 4.2.

We have shown, that under the conditions of Theorem 4.7 for the optimal control problem OCP 4.1 with free control subarcs all conditions of Theorem 1.12 are satisfied for the auxiliary problem OCP 4.2. This means, that the solution candidate (x_0^*, u_0^*) is a local minimum.

If we only compare solution candidates for OCP 4.1 which have the same switching structure (Assumption 4.2), then (x_0, u_0) is a local minimum of the problem OCP 4.1. □

We proof Theorem 4.8 in a similar way to the above. The two differences concern regularity condition R3 and the development of the Riccati differential equation. Therefore we can shorten the proof to the following.

Proof. (Theorem 4.8, OCP 4.1 with free final time) The proof is analogous to that of Theorem 4.7 except for conditions *3.* and *4.* stated in Theorem 1.13. We replace the strict Legendre-Clebsch condition by regularity condition R3. The second difference lies in a different Riccati differential equation.

The weaker condition R3 can be treated analogous to the strict Legendre-Clebsch condition. The difference lies in the treated set $R^+_{u^*}(s)$ on which the Hessian (4.38) of the augmented Hamiltonian must be positive definite. If R3 is satisfied on each subinterval $[t_{i-1}, t_i], i = 1, \dots, N$, this condition is also satisfied for the transformed problem OCP 4.2.

Analogous to Theorem 1.13, we exploit active constraints and formulate Riccati differential equations with reduced right hand sides. Compared with equation (1.19), the difference to Theorem 1.13 lies in active constraints C^+. We follow an analogous argumentation to that in the proof of Theorem 4.7. If the reduced Riccati equation corresponding to equation (1.22) has bounded, piecewise continuous solutions Q, R, and q, then these solutions can be transformed to a valid solution for the reduced Riccati differential equation referring to the auxiliary problem OCP 4.2.

Having treated both conditions, we only have to refer to the remaining conditions in the proof of Theorem 4.7 which also appear in Theorem 1.12 and have already been proofed. Thus, we have shown the validity of all conditions of Theorem 1.13. The solution candidate (x^*_0, u^*_0) of the auxiliary problem OCP 4.2 is a local minimum. If we only compare solution candidates for OCP 4.1 which have the same switching structure, (Assumption 4.2), then (x_0, u_0) is a local minimum of OCP 4.1. □

We can use the full matrix Q^*, if we cannot find a solution in block-diagonal form.

We now treat optimal control problems with fixed final time. We can formulate sufficient conditions for problem OCP 4.1 independent of the final time t_f being free or fixed.

Lemma 4.9. *Let us consider an optimal control problem OCP 4.1 with free control subarcs; that is, Assumption 4.2 is satisfied. Then Theorems 4.7 and 4.8 are also valid for this problem if the final time t_f is fixed.*

Proof. Analogous to the proof of Lemma 3.11. □

One should reduce the set $\widetilde{R}_R$, which results by applying Lemma 4.9, only in the case, the matrices stated in B3 and J3 are not positive definite on the whole set R_R.

4.4 Example: OCP with Free Control Subarc

As a start, we consider the following optimal control problem with free control subarc. The state variable $(x_1, x_2)^T$ and the control function $(v, w)^T$ are two-dimensional functions. Following the notation of problem OCP 4.1, v is the independent and w the dependent control component.

Problem FCS. *Minimize the functional*

$$J(v, w) := -x_2(1)$$

subject to the state equations

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 v \sin w, && (4.41)\\ \dot{x}_2 &= -1 + x_1 v \cos w, && (4.42) \end{aligned}$$

the boundary conditions

$$x_1(0) = -0.4, \qquad x_2(0) = 1, \tag{4.43}$$

and the control constraint

$$v_{\min} \le v \le v_{\max}$$

with $v_{\min} = -1$ *and* $v_{\max} = 0$.

The independent control v enters linearly into the problem. Whereas the dependent component w enters nonlinearly. If v vanishes on a subinterval, problem FCS does not depend on the control w any more. The control variable w then describes a free control subarc.

Necessary conditions are developed using the Hamiltonian

$$\mathcal{H}(x, v, w, \lambda) = \lambda_1 x_2 + \lambda_1 x_1 v \sin w - \lambda_2 + \lambda_2 x_1 v \cos w.$$

Adjoint equations and natural boundary conditions are given by

$$\begin{aligned} \dot{\lambda}_1 &= -\lambda_1 v \sin w - \lambda_2 v \cos w, &\qquad \lambda_1(1) &= 0, && (4.44)\\ \dot{\lambda}_2 &= -\lambda_1, & \lambda_2(1) &= -1. && (4.45) \end{aligned}$$

Applying the minimum condition and second order necessary conditions, we can describe the continuous control function w by the equations

$$\sin w = \frac{-\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}}, \qquad \cos w = \frac{-\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}. \tag{4.46}$$

Since the component v enters into the problem linearly, we consider v to be a bang-bang control with the following structure

$$v(t) = \begin{cases} -1, & S[t] > 0 \\ 0, & S[t] < 0 \end{cases}, \qquad \text{with } S(x, v, w, \lambda) = -x_1\sqrt{\lambda_1^2 + \lambda_2^2}$$

In order to get a solution candidate, we have to solve the following boundary value problem.

Problem FCSBVP. *Find continuous and piecewise continuous differentiable functions $x_0 : [0,1] \to \mathbb{R}^2$ and $\lambda : [0,1] \to \mathbb{R}^2$ and a switching time $t_1 \in]0,1[$ such that the system of ordinary differential equations (4.41)-(4.42) and (4.44)-(4.45) holds with boundary conditions (4.43) and (4.44)-(4.45). The continuous control function w is given by (4.46). The control function v has the following structure*

$$v(t) = \begin{cases} -1, & t \in [0, t_1[, \\ 0, & t \in [t_1, 1], \end{cases}$$

where the switching time $t_1 \in]0,1[$ is given by the condition

$$-x_1(t_1)\sqrt{\lambda_1^2(t_1) + \lambda_2^2(t_1)} = 0.$$

Obviously, on $[t_1, 1]$ there exists a free control subarc. The state and adjoint variables are shown in Figure 4.1.

The switching time which occurs regarding the control bounds $v_{\min}$ and $v_{\max}$ is the same switching time as the one when regarding the structure of the problem with respcet to the free control subarc. Therefore the jump condition of $(v, w)^T$ is consistent with the continuity condition for the control function stated in Theorems 4.7 and 4.8.

We check all conditions of Theorems 4.7 or 4.8 to verify whether the solution candidate shown in Figure 4.1 is really a weak local minimum compared with functions of the same structure.

Since the solution candidate is calculated by using general first order necessary conditions, the solution candidate is admissible and the necessary conditions stated in Theorem 4.5 are satisfied.

Regularity condition R1 is satisfied, because the mixed control-state constraints are affine linear (Lemma 1.14).

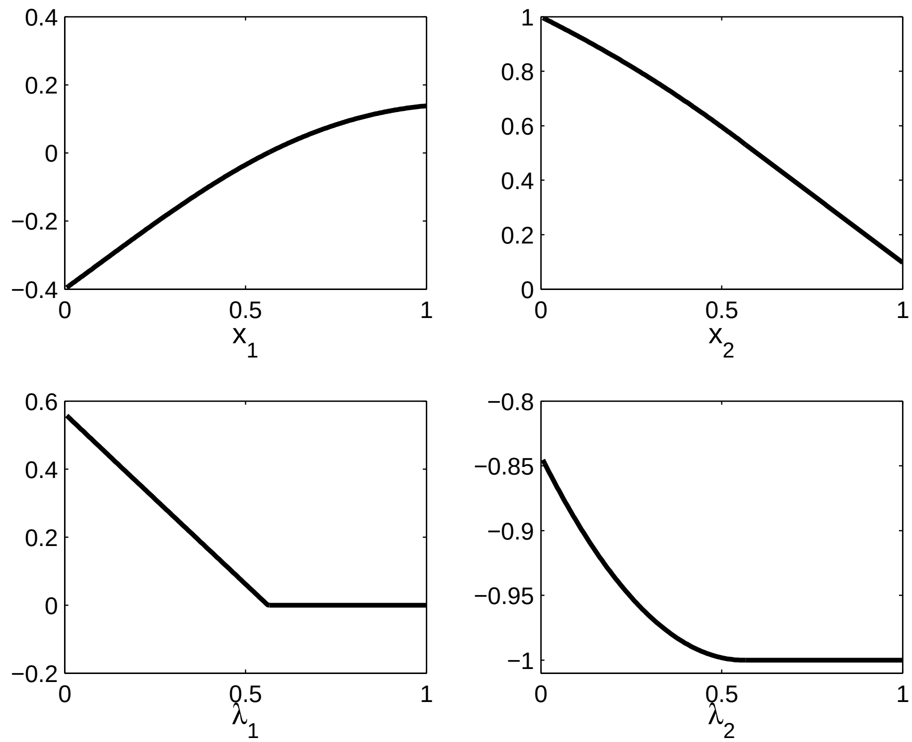

Figure 4.1: Problem FCS, state and adjoint variables

The boundary conditions are

$$r(x(0), x(1)) = \begin{pmatrix} x_1(0) + 0.4 \\ x_2(0) - 1 \end{pmatrix} \in \mathbb{R}^2. \tag{4.47}$$

With $\operatorname{rank}\{D_{(x_s,x_f)} r(x(0), x(1))\} = 2$ regularity condition R2 is fulfilled.

For the investigation of the strict Legendre-Clebsch condition we need the Hessian of the augmented Hamiltonian. With the control constraint

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} v_{\min} - v(t) \\ v(t) - v_{\max} \end{pmatrix} \leq 0$$

the derivatives of the augmented Hamiltonian

$$\begin{aligned} \widetilde{\mathcal{H}}(x, v, w, \lambda, \mu) &= \lambda_1 x_2 + \lambda_1 x_1 v \sin w - \lambda_2 + \lambda_2 x_1 v \cos w \\ &\quad + \mu_1 (v_{\min} - v) + \mu_2 (v - v_{\max}), \end{aligned}$$

with respect to the control function $u = (v, w)^T$ are

$$\begin{aligned}
\widetilde{\mathcal{H}}_u &= \begin{pmatrix} \lambda_1 x_1 \sin w + \lambda_2 x_1 \cos w \\ \lambda_1 x_1 v \cos w - \lambda_2 x_1 v \sin w \end{pmatrix}^T, \\
\widetilde{\mathcal{H}}_{uu} &= \begin{pmatrix} 0 & \lambda_1 x_1 \cos w - \lambda_2 x_1 \sin w \\ \lambda_1 x_1 \cos w - \lambda_2 x_1 \sin w & -\lambda_1 x_1 v \sin w - \lambda_2 x_1 v \cos w \end{pmatrix}.
\end{aligned}$$

The Hessian $\widetilde{\mathcal{H}}_{uu}$ is not positive definite. Therefore the strict Legendre-Clebsch condition is not satisfied. We cannot apply Theorem 4.7.

Regarding regularity condition R3, we require the set $R_u^+(t), t \in [0, 1]$. We develop this set by calculating the multipliers μ_1 and μ_2. By exploiting the mimimum condition it follows

$$\mu_1 = x_1 \sqrt{\lambda_1^2 + \lambda_2^2}, \qquad \mu_2 = -x_1 \sqrt{\lambda_1^2 + \lambda_2^2}.$$

For all $t \in [0, t_1[$ the following implication holds.

$$\mu_2(t) > 0 \quad \Rightarrow \quad I^+(t) = \{2\} \quad \Rightarrow \quad R_u^+(t) = \{\omega \in \mathbb{R}^2 | \omega_1 = 0\}.$$

Thus we have to check the following sign condition

$$\widetilde{\mathcal{H}}_{ww} = x_1 v \sqrt{\lambda_1^2 + \lambda_2^2} > 0.$$

This condition holds on $[0, t_1[$, because there $x_1 < 0, \lambda_2 < 0$ and $v = -1 < 0$ are satisfied. On the subinterval $[t_1, 1]$ a free control subarc in the control component w occurs because of $v = 0$. We do not have to show any sign condition for this subinterval. Therefore $\widetilde{\mathcal{H}}_{uu}$ is positive definite on the set $R_u^+(t)$ and regularity condition R3 is satisfied. The application of Theorem 4.8 is still allowed.

Since we try to apply Theorem 4.8, we have to distinguish between active and inactive control functions as well as subintervals, where free control subarcs occur. The control function v has always active constraints whereas w is unbounded. We have to calculate first and second partial derivatives with respect to the state variable x and the control component w. On the subinterval $[t_1, 1]$ the differential equations reduce to:

$$\begin{aligned}
\dot{Q} &= -\widetilde{\mathcal{H}}_{xx} - f_x^T Q - Q f_x, \\
\dot{R} &= \frac{1}{\tau}[-\widetilde{\mathcal{H}}_x^T - Qf] - f_x^T R, \\
\dot{q} &= -\frac{2}{\tau} R^T f.
\end{aligned}$$

We abstain from stating the derivatives here. It is useful, not to establish the Riccati differential equations analytically but to use the matrix operations in for example the computer program MATLAB. First and second partial derivatives are shown in the Appendix A.2.

We receive boundary condition B3 and jump condition J3 for

$$Q(t) = \begin{pmatrix} Q_1(t) & Q_2(t) \\ Q_2(t) & Q_3(t) \end{pmatrix}, \quad R(t) = \begin{pmatrix} R_1(t) \\ R_2(t) \end{pmatrix}, \quad \text{and } q(t), \qquad t \in [0,1],$$

by calculating the derivatives of $\phi + \nu r$. The boundary condition r for problem FCS is stated in equation (4.47). It is helpful that r as well as the Mayer-part of the performance index ϕ are decoupled and affine linear with respect to $x(0)$ and $x(1)$. The state variable x is fixed at the initial time 0. We can apply Lemmas 1.16 and 1.17. At the switching time, there has to occur a jump of the form, that the matrix (3.20) is positive definite. We exploit the fact, that the final time is fixed. We apply the additional condition (4.15) and use

$$\tau_1(1) + \tau_2(0) = t_f - t_0$$

in order to reduce the jump condition J3.

Altogether, we have to find a solution of the following problem.

Problem FCSSSC. *Find piecewise continuous (symmetric) functions $Q : [0,1] \to \mathbb{R}^{2\times 2}$, $R : [0,1] \to \mathbb{R}^2$, and $q : [0,1] \to \mathbb{R}$ which solve the system of Riccati differential equations (4.33)-(4.35) and satisfy the following boundary and jump conditions.*

1. *The boundary condition $q(0) > 0$ must hold.*

2. *The matrix*
$$\begin{pmatrix} -Q_1(1) & -Q_2(1) & -R_1(1) \\ -Q_2(1) & -Q_3(1) & -R_2(1) \\ -R_1(1) & -R_2(1) & -q(1) \end{pmatrix}$$
must be positive semidefinite on $\mathbb{R}^3$ and $q(1) < 0$ must hold.

3. *The matrix*
$$\begin{pmatrix} Q_1(t_1^+) - Q_1(t_1^-) & Q_2(t_1^+) - Q_2(t_1^-) & R_1(t_1^+) + R_1(t_1^-) \\ Q_2(t_1^+) - Q_2(t_1^-) & Q_3(t_1^+) - Q_3(t_1^-) & R_2(t_1^+) + R_2(t_1^-) \\ R_1(t_1^+) + R_1(t_1^-) & R_2(t_1^+) + R_2(t_1^-) & q(t_1^+) - q(t_1^-) \end{pmatrix}$$
must be positive definite on $\mathbb{R}^3$.

In order to get a solution of problem FCSSSC, we use two backwards integrations on the subintervals $[0, t_1]$ and $[t_1, 1]$. We start with initial values $Q(1), R(1)$, and $q(1)$, which satisfy the boundary conditions *2.* stated in problem FCSSSC. We then adjust $Q(t_1^-), R(t_1^-)$, and $q(t_1^-)$ in such a way that conditions *1.* and *3.* are satisfied.

One solution of problem FCSSSC is shown in Figure 4.2. Starting, jump and final values are shown in Table 4.1.

Result. *Second order sufficient optimality conditions stated in Theorem 4.8 are satisfied. The solution candidate is a weak local minimum of the problem FCS compared with candidates of the same structure.*

t	$Q_1(t)$	$Q_2(t)$	$Q_3(t)$	$R_1(t)$	$R_2(t)$	$q(t)$
0	-3.5112e-03	-5.1969e-04	-3.4904e-03	9.9988e-01	5.5359e-01	4.4370e-01
t_1^-	-3.0000e-03	-4.4000e-04	-3.0000e-03	2.0864e-09	-1.8307e-09	-3.0000e-03
t_1^+	-1.0000e-03	-4.3400e-04	-1.1884e-03	-9.9051e-05	9.4837e-04	-1.9808e-03
1	-1.0000e-03	-6.8858e-12	-1.0000e-03	-3.1018e-06	1.3688e-06	-9.8822e-04

Table 4.1: Problem FCSSSC, values at starting, switching and final time for Q, R, and q

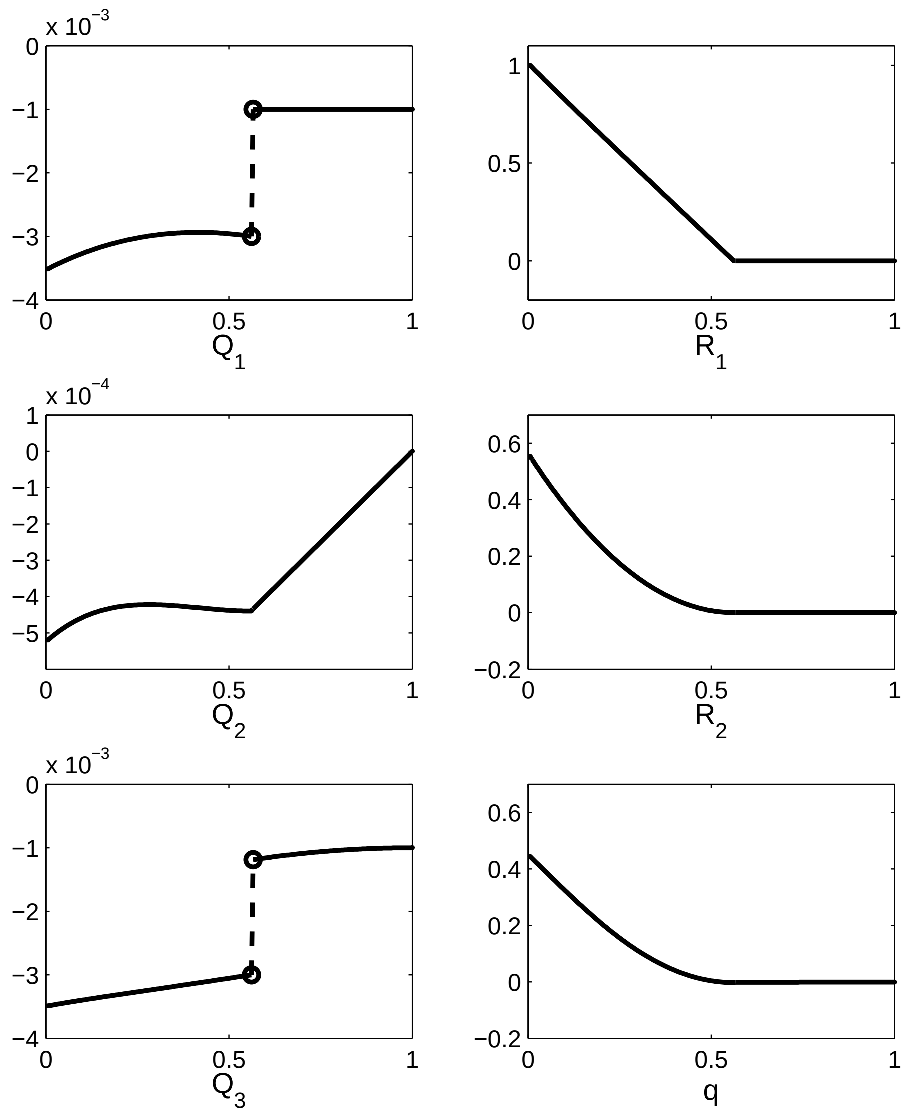

Figure 4.2: Problem FCSSSC, functions Q, R, and q

4.5 Example: Orbit Transfer Problem

Typical examples for optimal control problems with free control subarcs are space-travel problems. The thrust and the thrust angel of a rocket arise as controls where the thrust angle appears as free control subarc, if the thrust itself vanishes on a subinterval.

Rocket flights have been studiet extensively in the literature [Kel62],[MP64], [KM66], [Obe90], [OT97], [MO02], ,[MW05], [MWW05], [BC07]. Sufficient conditions have only been shown for problems with minimal flight time [Str97], [MO02]. In this case free control subarcs occur. A fist approach for the fuel-minimal case was investigated by Dölker [Doe98]. There, each phase of thrust was examined on its own.

We follow the example of a fuel-optimal Earth-Mars orbit transfer problem treated in [OT97]. This example describes an optimal control problem with free final time t_f. The state variables are the *position of the rocket* in polar coordinates (r, ϕ), the *velocity* given in Cartesian coordinates (w, v), and the *mass* m, whereas the rocket is modeled by a mass point. The control functions are given by the *thrust* β and the *thrust angle* ψ. We are able to leave out the calculation of the polar angle ϕ because it does not influence the cost functional or another state variable. The example is, after scaling, given by the following problem.

Problem ORB. *Minimize the functional*

$$J(\beta, \psi, t_f) := -m(t_f)$$

subject to the state equations

$$\dot{r} = w, \tag{4.48}$$

$$\dot{w} = \frac{v^2}{r} - \frac{1}{r^2} + \frac{c\beta}{m}\sin\psi, \tag{4.49}$$

$$\dot{v} = -\frac{wv}{r} + \frac{c\beta}{m}\cos\psi, \tag{4.50}$$

$$\dot{m} = -\beta, \tag{4.51}$$

the boundary conditions

$$r(0) = r_s, \qquad w(0) = w_s, \qquad v(0) = v_s, \qquad m(0) = m_s, \tag{4.52}$$

$$r(t_f) = r_f, \qquad w(t_f) = w_f, \qquad v(t_f) = v_f, \tag{4.53}$$

and the control constraint

$$\beta_{\min} \le \beta(t) \le \beta_{\max}.$$

The parameters are shown in Table 4.2.

$r_s = 1,$	$w_s = 0,$	$v_s = 1,$	$m_s = 1,$
$r_f = 1.525,$	$w_f = 0,$	$v_f = 1/\sqrt{(r_f)},$	
$c = 1.872,$	$\beta_{\min} = 0,$	$\beta_{\max} = 0.075.$	

Table 4.2: Problem ORB, parameters

This optimal control problem has got the structure described in section 4.1. The independent control function is the thrust β, whereas the thrust angle ψ is a dependent control function. If β vanishes on a subinterval, problem ORB does not depend on ψ any more. The control component ψ then describes a free control subarc. Using the notation of section 4.1, we define the switching function

$$S(x, \beta) := \beta.$$

Necessary conditions were developed in [OT97]. The resulting boundary value problem is the following.

Problem ORBBVP. *Find continuous and piecewise continuous differentiable functions $x_0 = (r_0, w_0, v_0, m_0)^T : [0, t_f] \to \mathbb{R}^4$ and $\lambda = (\lambda_r, \lambda_w, \lambda_v, \lambda_m) : [0, t_f] \to \mathbb{R}^4$ and a final time t_f, such that the system of ordinary differential equations (4.48)-(4.51) together with*

$$\begin{aligned}
\dot{\lambda}_r &= \lambda_w\Big(\frac{v^2}{r^2} - \frac{2}{r^3}\Big) - \lambda_v\frac{wv}{r^2} \\
\dot{\lambda}_w &= -\lambda_r + \lambda_v\frac{v}{r} \\
\dot{\lambda}_v &= -\lambda_w\frac{2v}{r} + \lambda_v\frac{w}{r} \\
\dot{\lambda}_m &= \frac{c\beta}{m^2}(\lambda_w \sin\psi + \lambda_v \cos\psi)
\end{aligned}$$

holds with boundary conditions (4.52)-(4.53) as well as $\lambda_m(t_f) = -1$. Since the final time t_f is free, the additional condition $\mathcal{H}[t_f] = 0$ must hold.

The independent control function β enters into the problem linearly. The optimal control β_0 will be of bang-bang or bang-singular structure. Here we check sufficient conditions for solution candidates with pure bang-bang arcs with respect to β. The control function ψ is continuous. The switching times which occur regarding the control bounds $\beta_{\min}$ and $\beta_{\max}$ are the same switching times as those when regarding the switching function S. Therefore the jump condition of the solution candidate $(\beta_0, \psi_0)^T$ is consistent with the continuity condition for the control function stated in Theorems 4.7 and 4.8.

We treat the case of β having the switching structure

$$\beta(t) = \begin{cases} \beta_{\max}, & t \in [0, t_1[\\ \beta_{\min}, & t \in [t_1, t_2] \\ \beta_{\max}, & t \in]t_2, t_f] \end{cases} . \tag{4.54}$$

Obviously, on $[t_1, t_2]$ there exists a free control subarc. The state and adjoint variables are shown in Figures 4.3 and 4.4. For a detailed derivation of the necessary conditions and the solution candidates we refer to [OT97].

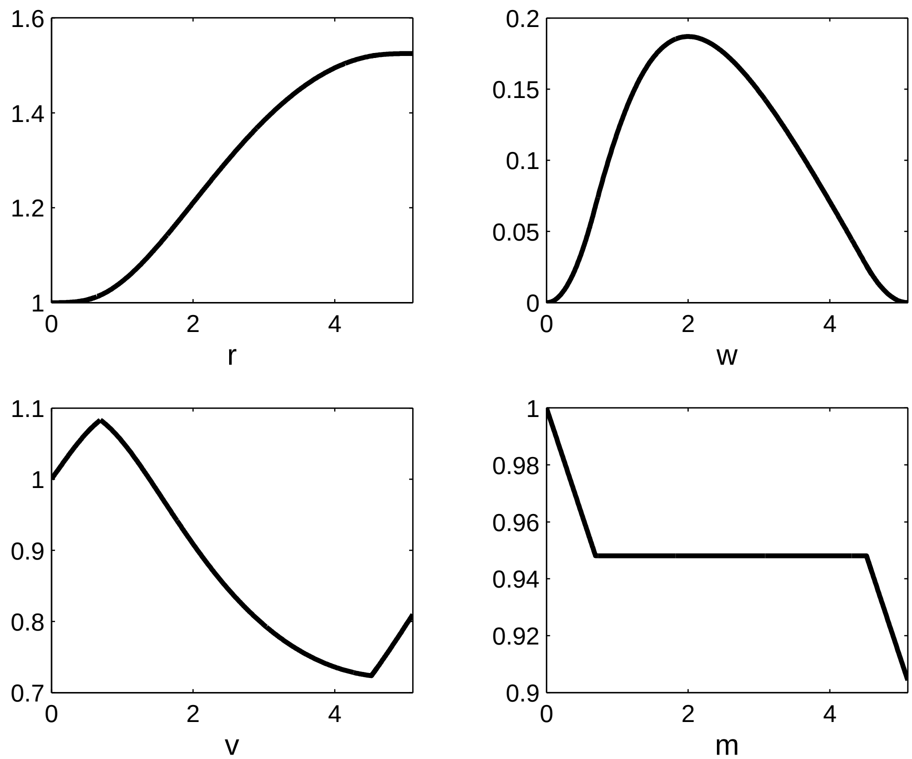

Figure 4.3: Problem ORB, state variables

We check all conditions of Theorems 4.7 or 4.8 to verify whether a weak local minimum really exists compared with functions of the same structure.

Since the solution candidate is calculated by using general first order necessary conditions, the solution candidate is admissible and the necessary conditions stated in Theorem 4.5 are satisfied.

Regularity condition R1 is satisfied, because the control constraints are affine linear (Lemma 1.14).

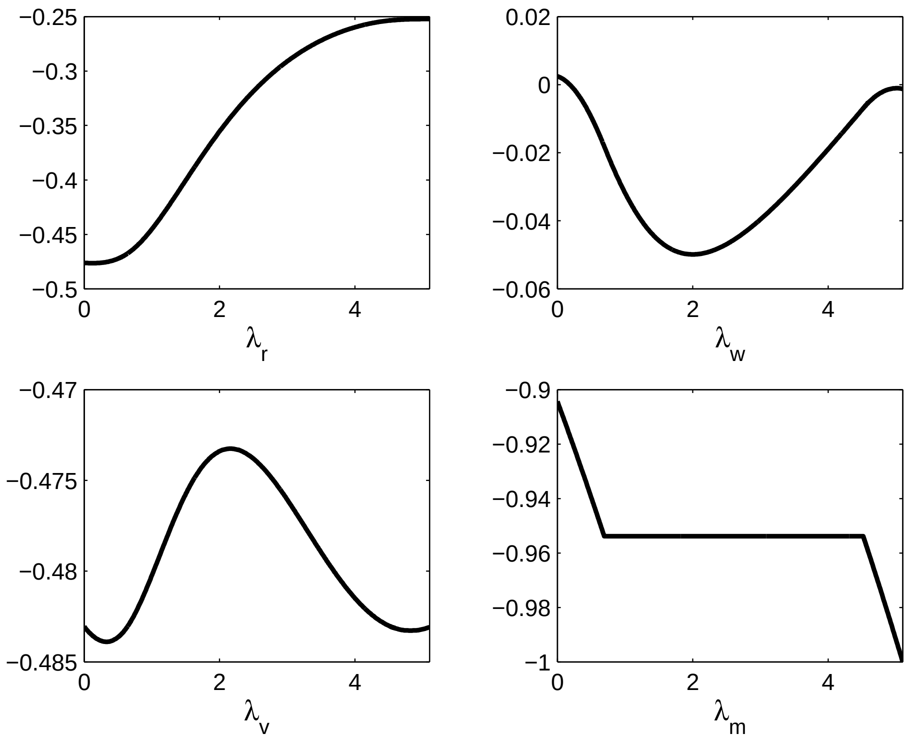

Figure 4.4: Problem ORB, adjoint variables

The boundary conditions are

$$\tilde{r}(x(0), x(t_f)) = \begin{pmatrix} r(0) - r_s \\ w(0) - w_s \\ v(0) - v_s \\ m(0) - m_s \\ r(t_f) - r_f \\ w(t_f) - w_f \\ v(t_f) - v_f \end{pmatrix} \in \mathbb{R}^7. \tag{4.55}$$

With $\operatorname{rank}\{D_{(x_s,x_f)}\tilde{r}(x(0), x(t_f))\} = 7$ regularity condition R2 is fulfilled.

For the investigation of the strict Legendre-Clebsch condition we need the Hessian of the augmented Hamiltonian. With the control constraint

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \beta_{\min} - \beta(t) \\ \beta(t) - \beta_{\max} \end{pmatrix} \leq 0$$

the derivatives of the augmented Hamiltonian

$$\begin{aligned} \widetilde{\mathcal{H}} \;&=\; \lambda_r w + \lambda_w \left(v^2/r - 1/r^2 + \frac{c\beta}{m} \sin\psi \right) + \lambda_v \left(-wv/r + \frac{c\beta}{m} \cos\psi \right) \\ &\quad + \lambda_m(-\beta) + \mu_1 \left(\beta_{\min} - \beta \right) + \mu_2 \left(\beta - \beta_{\max} \right), \end{aligned}$$

with respect to the control function $u = (\beta, \psi)$ are

$$\begin{aligned}
\widetilde{\mathcal{H}}_u &= \begin{pmatrix} \lambda_w \frac{c}{m} \sin\psi + \lambda_v \frac{c}{m} \cos\psi - \lambda_m - \mu_1 + \mu_2 \\ \lambda_w \frac{c\beta}{m} \cos\psi - \lambda_v \frac{c\beta}{m} \sin\psi \end{pmatrix}^T, \\
\widetilde{\mathcal{H}}_{uu} &= \begin{pmatrix} 0 & \lambda_w \frac{c}{m} \cos\psi - \lambda_v \frac{c}{m} \sin\psi \\ \lambda_w \frac{c}{m} \cos\psi - \lambda_v \frac{c}{m} \sin\psi & -\lambda_w \frac{c\beta}{m} \sin\psi - \lambda_v \frac{c\beta}{m} \cos\psi \end{pmatrix}.
\end{aligned}$$

We have to examine each subinterval $[0, t_1[$, $[t_1, t_2]$, and $]t_2, t_f]$ separately. In the case of $t \in [t_1, t_2]$ the control function $\beta(t) = \beta_{\min} = 0$ vanishes on this subinterval. The component $\psi(t)$ is a free control subarc and can be omitted. Because of $\widetilde{\mathcal{H}}_{\beta\beta} = 0$, the strict Legendre-Clebsch condition is not satisfied. We cannot apply Theorem 4.7.

Regarding regularity condition R3, we require the set $R_u^+(t), t \in [0, t_f]$. We develop this set by calculating the multipliers μ_1 and μ_2.

By exploiting the mimimum condition it follows

$$\begin{aligned}
\mu_1 &= \lambda_w \frac{c}{m} \sin\psi + \lambda_v \frac{c}{m} \cos\psi - \lambda_m, \\
\mu_2 &= -\lambda_w \frac{c}{m} \sin\psi - \lambda_v \frac{c}{m} \cos\psi + \lambda_m.
\end{aligned}$$

The control function β and the multiplier μ_2 are shown in Figure 4.5.

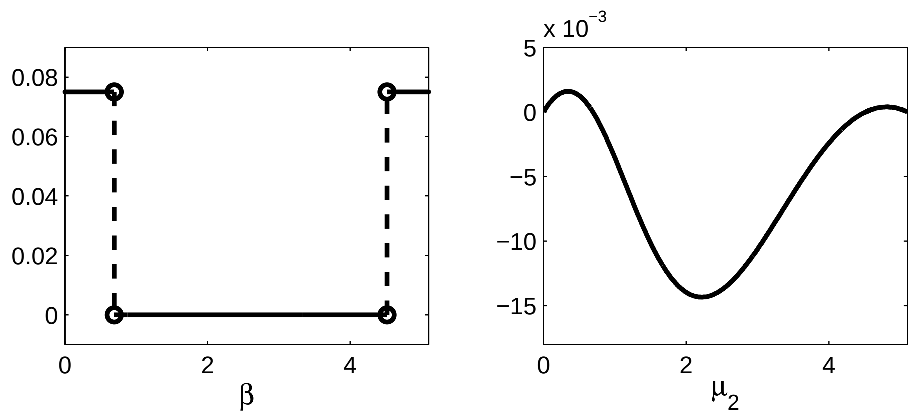

Figure 4.5: Problem ORB, control function β with corresponding multiplier μ_2

The sign condition $\mu_2 \leq 0$ is satisfied on the subinterval $[t_1, t_2]$, where $\beta = \beta_{\min}$. The multiplier $\mu_1 = -\mu_2$ is less or equal to zero on $[0, t_1] \cup [t_2, t_f]$. For all $t \in [t_1, t_2]$ we get

$$\mu_2 \leq 0 \quad \Rightarrow \quad I^+(t) \subset \{1\} \quad \Rightarrow \quad R_\beta^+(t) \subset \{0\}.$$

Thus, the Hessian $\widetilde{\mathcal{H}}_{\beta\beta}$ is positive definite on the set $R^+_\beta(t)$. On the subintervals $[0, t_1[\cup]t_2, t_f]$ the inequality $\beta(t) = \beta_{\max} > 0$ implies that we have to treat both control components $u = (\beta, \psi)^T$ for the development of the set $R^+_u(t)$. Here

$$\mu_1 \leq 0 \quad \Rightarrow \quad I^+(t) \subset \{2\} \quad \Rightarrow \quad R^+_u(t) \subset \{(\eta_1, \eta_2) \in \mathbb{R}^2 | \eta_1 = 0\}$$

implies the positive definiteness of $\widetilde{\mathcal{H}}_{uu}$ on $R^+_u(t)$, if $\widetilde{\mathcal{H}}_{\psi\psi} > 0$ holds for all $t \in [0, t_1[\cup]t_2, t_f]$. In [OT97] the relations

$$\cos\psi = -\lambda_v / \sqrt{\lambda_w^2 + \lambda_v^2} \qquad \text{and} \qquad \sin\psi = -\lambda_w / \sqrt{\lambda_w^2 + \lambda_v^2}$$

are stated, which are valid for $\beta > 0$. With these equations

$$\widetilde{\mathcal{H}}_{\psi\psi} = \frac{c\beta}{m}\sqrt{\lambda_w^2 + \lambda_v^2} > 0$$

holds. Therefore $\widetilde{\mathcal{H}}_{uu}$ is positive definite on the set $R^+_u(t)$ and regularity condition R3 is satisfied. The application of Theorem 4.8 is still allowed.

Since we try to apply Theorem 4.8, we have to distinguish between active and inactive control functions as well as subintervals, where free control subarcs occur. The control function β has always active constraints whereas ψ is unbounded. We have to calculate first and second partial derivatives with respect to the state variable $x = (r, w, v, m)^T$ and the control component ψ. On the subinterval $[t_1, t_2]$ the differential equation reduces to

$$\dot{\widehat{Q}} = -\widetilde{\mathcal{H}}_{xx} - f_x^T \widehat{Q} - \widehat{Q} f_x. \tag{4.56}$$

We abstain from stating the derivatives here. It is useful, not to establish the Riccati differential equations analytically but to use matrix operations in for example the computer program MATLAB. First and second partial derivatives are shown in the Appendix A.3.

We receive boundary condition B3 and jump condition J3 for

$$\widehat{Q}(t) = \begin{pmatrix} Q_1 & Q_2 & Q_3 & Q_4 & R_1 \\ Q_2 & Q_5 & Q_6 & Q_7 & R_2 \\ Q_3 & Q_6 & Q_8 & Q_9 & R_3 \\ Q_4 & Q_7 & Q_9 & Q_{10} & R_4 \\ R_1 & R_2 & R_3 & R_4 & q \end{pmatrix}(t), \qquad t \in [0, t_f],$$

by calculating the derivatives of $\phi + \nu\tilde{r}$. The boundary condition $\tilde{r}$ for problem ORB is stated in Equation (4.55). It is helpful that $\tilde{r}$ as well as the Mayer-part of the performance index ϕ are decoupled and affine linear with respect to $x(0)$ and $x(t_f)$. The state variable $x = (r, w, v, m)^T$ is fixed at the starting time 0. We can apply Lemmas 1.16 and 1.17. At the final time the components r, w, and v are fixed, which reduces the boundary condition even more (Lemma 1.19). At the switching times, there have to occur jumps of the form, that the matrix (3.20) is positive definite.

Altogether, we have to find a solution of the following problem.

Problem ORBSSC. *Find piecewise continuous (symmetric) functions $Q : [0, t_f] \to \mathbb{R}^{4\times 4}$, $R : [0, t_f] \to \mathbb{R}^4$, and $q : [0, t_f] \to \mathbb{R}$ which solve the system of Riccati differential equations (4.33)-(4.35) and satisfy the following boundary and jump conditions.*

1. *The matrix*
$$\begin{pmatrix} -Q_{10}(t_f) & -R_4(t_f) \\ -R_4(t_f) & -q(t_f) \end{pmatrix}$$
must be positive semidefinite on $\mathbb{R}^2$.

2. *The boundary conditions $q(0) > 0$ and $q(t_f) < 0$ must hold.*

3. *The matrices*
$$\begin{pmatrix} Q(t_i^+) - Q(t_i^-) & R(t_i^+) & -R(t_i^-) \\ R^T(t_i^+) & q(t_i^+) & 0 \\ -R^T(t_i^-) & 0 & -q(t_i^-) \end{pmatrix}$$
must be positive definite on $\mathbb{R}^{4+2}, i = 1, 2$.

Current situation. We use different techniques described in chapter 6 for the search of a solution of the above problem. Unfortunaltely, we are not able to give a solution to problem ORBSSC at the moment.

The search over a grid including different optimization techniques failes because of too high dimensional problems and a small local search area. Searching by hand does not lead to a solution because the different functions influence each other in a negative way. While improving one value to satisfy the boundary and jump conditions, other values change for the worse.

The next idea is to follow the homotopy path from the time minimal case to the fuel minimal case as done when applying necessary conditions [Obe77]. We suggest the solution of sufficient conditions for the time minimal case in [MO02] as a starting point. Hopefully one will find the right way of homotopy to the fuel minimal case.

This example shows the difficulties of checking sufficient conditions for more complex optimal control problems. At this point further theoretical investigations would be helpful to loosen the here developed sufficient conditions. One probably should try to use different conditions to those developed in [MP95].

Chapter 5

Optimal Control Problems with Singular State Subarcs

In this chapter we continue the investigation of nonsmooth optimal control problems. We allow the switching function to vanish on whole subintervals. On these subintervals we call the state variable to form a *singular state subarc*.

We follow the theory of [OR06], [Obe07], and [OR08] and introduce the optimal control problem in section 5.1. In section 5.2 we summarize necessary conditions for switching functions of order 0 and 1. New necessary optimality conditions are shown for a special class of optimal control problems.

We develop new second order sufficient optimality conditions in section 5.3. We conclude this chapter in section 5.4 with the treatment of a nonsmooth model of an electric circuit. The new necessary conditions can be applied for this type of problem.

5.1 Optimal Control Problems with Singular State Subarcs

We treat the nonsmooth optimal control problem introduced in chapter 3 following the idea introduced in [OR06], [Obe07] and [OR08]. As an extension, the switching function may have a different structure.

Again, we develop the theory for autonomous problems. We can widen the result to non-autonomous problems by transforming the problem into an autonomous auxiliary problem.

OCP 5.1. *Determine a piecewise continuous control function $u : [t_0, t_f] \to \mathbb{R}^m$, such that the functional*

$$J(x,u) \quad = \quad \phi(x(t_f)) + \int_{t_0}^{t_f} L(x(t),u(t))dt \tag{5.1}$$

is minimized subject to

$$\begin{aligned} x'(t) &= f(x(t),u(t)), \qquad t \in [t_0,t_f], a.e., && (5.2)\\ r(x(t_0),x(t_f)) &= 0, && (5.3)\\ C(x(t),u(t)) &\leq 0, \qquad t \in [t_0,t_f], a.e., && (5.4)\end{aligned}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ and $r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^\ell$ are sufficiently smooth. The integrand of the objective function (5.1), the right hand side of the state equation (5.2), and the mixed constraints (5.4) are given in the following way

$$L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \qquad L(x,u) = \begin{cases} L_1(x,u), & S(x,u) < 0 \\ L_2(x,u), & S(x,u) = 0 \\ L_3(x,u), & S(x,u) > 0 \end{cases},$$

$$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \qquad f(x,u) = \begin{cases} f_1(x,u), & S(x,u) < 0 \\ f_2(x,u), & S(x,u) = 0 \\ f_3(x,u), & S(x,u) > 0 \end{cases},$$

$$C : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^d, \qquad C(x,u) = \begin{cases} C_1(x,u), & S(x,u) < 0 \\ C_2(x,u), & S(x,u) = 0 \\ C_3(x,u), & S(x,u) > 0 \end{cases}$$

with a piecewise continuous differentiable switching function $S : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and twice continuous differentiable functions $L_k : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $f_k : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $C_k : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^d, k = 1,2,3$.

Let (x_0, u_0) be a solution of OCP 5.1. Treating the switching function along the optimal trajectory (x_0, u_0), we now allow it to vanish on whole subintervals.

Assumption 5.1. *There exists a subdivision $t_0 < t_1 < \cdots < t_N = t_f, N \in \mathbb{N}$, with $N-1$ switching times, such that for all $i = 1, \ldots, N$ the switching function $S[t]$ is continuous on each open subinterval $]t_{i-1}, t_i[$ and*

$$\begin{aligned} \forall\, t \in\,]t_{i-1}, t_i[: &\qquad S(x_0(t), u_0(t)) \neq 0, \\ \text{or } \forall\, t \in [t_{i-1}, t_i] : &\qquad S(x_0(t), u_0(t)) = 0. \end{aligned}$$

The set of indices denoting all subintervals, where the switching function S vanishes, is called J^0:

$$\begin{aligned} J^0 &:= \{i \in \{1,\ldots,N\} | \forall t \in [t_{i-1},t_i] : S(x_i(t),u_i(t)) = 0\}, \\ J^1 &:= \{1,\ldots,N\} \backslash J^0. \end{aligned}$$

Assumption 5.1 allows the switching function to jump in the way treated in chapter 3. Additionally the switching function may vanish on whole subintervals. We call t_{i-1} the *entry time* and t_i the *exit time* of the subinterval $[t_{i-1}, t_i], i \in J^0$. On subintervals $[t_{i-1}, t_i], i \in J^0$, we obtain so called *singular state subarcs* and *singular control subarcs*.

5.2 Necessary Optimality Conditions for Optimal Control Problems with Singular State Subarc

As done in chapter 3, we consider switching functions of order 0 or 1. We connect the switching function to the Hamiltonian. This is done in a similar way to the theory of pure state constrained optimal control problems.

If the switching function is of order 0, we treat the Hamiltonian

$$\widetilde{\mathcal{H}} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R},$$
$$\widetilde{\mathcal{H}}(x, u, \lambda, \mu, \vartheta) := \lambda_0 L(x, u) + \lambda f(x, u) + \mu C(x, u) + \vartheta S(x, u).$$

If the switching function is of order 1, the first total time derivative is explicitely depending on the control function.

$$S^{(1)}(x(t), u(t)) := \frac{d}{dt} S(x(t)) = S_x(x(t)) f(x(t), u(t)).$$

Therefore we treat the following Hamiltonian.

$$\widetilde{\mathcal{H}} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R},$$
$$\widetilde{\mathcal{H}}(x, u, \lambda, \mu, \vartheta) := \lambda_0 L(x, u) + \lambda f(x, u) + \mu C(x, u) + \vartheta S^{(1)}(x, u).$$

Theorem 5.2. *Let (x_0, u_0) be a weak local minimum of OCP 5.1 with a switching function S of order 0 or 1 satisfying Assumption 5.1. Let regularity conditions R1 and R2 be satisfied. Then there exist a continuous and piecewise continuous differentiable adjoint variable $\lambda : [t_0, t_f] \to \mathbb{R}^n$, piecewise continuous multiplier functions $\mu : [t_0, t_f] \to \mathbb{R}^d$, $\vartheta : [t_0, t_f] \to \mathbb{R}$, multipliers $\nu \in \mathbb{R}^\ell, \kappa \in \mathbb{R}^N$, and $\lambda_0 \in \mathbb{R}$ not all vanishing simultaneously on $[t_0, t_f]$, and a function $\widetilde{\mathcal{H}}$, such that (x_0, u_0) satisfies*

(a) the adjoint equations

$$\lambda'(t) = -\widetilde{\mathcal{H}}_x(x_0(t), u_0(t), \lambda(t), \mu(t), \vartheta(t)), \qquad t \in [t_0, t_f], a.e., \tag{5.5}$$

(b) the natural boundary conditions

$$\lambda(t_0) = -D_{x_s}[\nu r(x_0(t_0), x_0(t_f))], \tag{5.6}$$
$$\lambda(t_f) = D_{x_f}[\lambda_0 \phi(x_0(t_f)) + \nu r(x_0(t_0), x_0(t_f))], \tag{5.7}$$

(c) *the complementary conditions*

$$\mu(t) \geq 0 \quad \text{and} \quad \mu(t)C(x_0(t), u_0(t)) = 0, \qquad t \in [t_0, t_f], a.e., \tag{5.8}$$
$$\text{and} \quad \vartheta(t)S[t] = 0, \qquad t \in [t_0, t_f], a.e.,$$

(d) *the minimum principle*

$$u_0(t) = \text{argmin}\left\{\widetilde{\mathcal{H}}(x_0(t), u, \lambda(t), \mu(t), \vartheta(t)) | u \in \mathbb{R}^m\right\}, \ t \in [t_0, t_1], a.e., \tag{5.9}$$

(e) *the minimum condition*

$$\widetilde{\mathcal{H}}_u[t] = 0, \qquad t \in [t_0, t_f], a.e., \tag{5.10}$$

(f) *the continuity condition*

$$\widetilde{\mathcal{H}}[t_i^+] = \widetilde{\mathcal{H}}[t_i^-], \qquad i = 1, \dots, N, \tag{5.11}$$

(g) *and if the switching function is of order* 0*: the continuity condition*

$$\lambda(t_i^+) = \lambda(t_i^-), \qquad i = 1, \dots, N, \tag{5.12}$$

or if the switching function is of order 1*: the jump condition*

$$\lambda(t_i^+) = \lambda(t_i^-) + \kappa_i S_x(x_0(t_i^-)), \qquad i = 1, \dots, N. \tag{5.13}$$

We assume, that necessary conditions are satisfied in normal form and choose $\lambda_0 = 1$. Necessary conditions for OCP 5.1 have already been developed in [Obe07] and [OR06]. In favor of a uniform theory we give a different proof for the case of an order 0 switching function.

5.2.1 Proof of Necessary Optimality Conditions

We exploit the fact, that the switching function S fulfills Assumption 5.1. As done in chapter 4, we transform problem OCP 5.1 into an auxiliary problem.

OCP 5.2. *Determine piecewise continuous control functions $u_i : [0, 1] \to \mathbb{R}^m$, $i = 1, \dots, N$, such that the functional*

$$J(x_1, \tau_1, \dots, x_N, \tau_N, u_1, \dots, u_N) = \phi(x_N(1)) + \sum_{i=1}^{N} \int_0^1 \tau_i(s) \cdot L(x_i(s), u_i(s)) ds \tag{5.14}$$

is minimized subject to

$$
\begin{aligned}
&x_i'(s) = \tau_i(s) \cdot f(x_i(s), u_i(s)), && i = 1, \ldots, N, && (5.15)\\
&\tau_i'(s) = 0, && i = 1, \ldots, N, && (5.16)\\
&r(x_1(0), x_N(1)) = 0, && && (5.17)\\
&x_i(0) - x_{i-1}(1) = 0, && i = 2, \ldots, N, && (5.18)\\
&S(x_i(s), u_i(s)) = 0, && i \in J^0, && (5.19)\\
&C(x_i(s), u_i(s)) \leq 0, && i = 1, \ldots, N, && (5.20)
\end{aligned}
$$

for almost every $s \in [0,1]$, and additionally, if the final time t_f is fixed

$$\sum_{i=1}^{N} \tau_i(s_i) - (t_f - t_0) = 0. \tag{5.21}$$

Applying necessary and sufficient optimality conditions to the auxiliary problem, we use the same notations as done in chapter 4 (Equations (4.16)-(4.21)). Note, that the above problem has differential algebraic equations as constraints, namely Equation (5.19), if S is of order bigger than 0.

Proof. (Theorem 5.2) In the case, the switching function is of order 1, we refer to Theorem 4.1 in [OR06]. We extend this statements to general constraints $C(x,u) \leq 0$.

The second case, where S is of order 0, is treated in a similar way. We connect the switching function as a new equality constraint to the Hamiltonian. Necessary conditions for the above auxiliary problem are for example given in [BH75], section 3.3. The Hamiltonian of the auxiliary problem is given by

$$
\begin{aligned}
\widetilde{\mathcal{H}}^*(x^*, u^*, \lambda^*, \mu^*, \vartheta^*) &:= \sum_{i=1}^{N} \tau_i \widetilde{\mathcal{H}}_i(x_i, u_i, \lambda_i, \mu_i, \vartheta_i) && (5.22)\\
\text{with} \quad \widetilde{\mathcal{H}}_i(x_i, u_i, \lambda_i, \mu_i, \vartheta_i) &:= \lambda_0 L(x_i, u_i) + \lambda_i f(x_i, u_i)\\
&\quad + \mu_i C(x_i, u_i) + \vartheta_i S(x_i, u_i), \quad i = 1, \ldots, N.
\end{aligned}
$$

with $\vartheta_i = 0$ for all $i \in J^1$. The augmented performance index is given by

$$\Phi := \lambda_0 \phi(x_N(1)) + \nu r(x_1(0), x_N(1)) + \sum_{i=2}^{N} \tilde{\nu}_i (x_i(0) - x_{i-1}(0)) \ .$$

The application of general first order necessary conditions yields in

(a) $\lambda_i'(s) = -\tau_i(s)\widetilde{\mathcal{H}}_{ix_i}[s],$
$\lambda_i^{(\tau)\prime}(s) = -\widetilde{\mathcal{H}}_i[s],$

(b) $\lambda_1(0) = -D_{x_s}\Phi = -D_{x_s}[\nu r],$
$\lambda_N(1) = \ D_{x_f}\Phi = D_{x_f}[\lambda_0\phi + \nu r],$
$\lambda_i(0) = -\tilde{\nu}_i, \quad \lambda_{i-1}(1) = -\tilde{\nu}_i, \qquad i = 2, \dots, N,$
$\lambda_i^{(\tau)}(0) = 0, \quad \lambda_i^{(\tau)}(1) = 0, \qquad i = 1, \dots, N,$

(c) $\mu_i(s) = 0$ and $\mu_i(s)C(x_i(s), u_i(s)) = 0,$
$\vartheta_i(s)S[s] = 0,$

(d) $u_{0i}(s) = \operatorname{argmin}\{\widetilde{\mathcal{H}}_i(x_i, u, \lambda_i, \mu_i, \vartheta_i) | u \in \mathbb{R}^m\},$

(e) $\widetilde{\mathcal{H}}_{iu_i}[s] = 0.$

We recombine the adjoint variable

$$\lambda(t) := \begin{cases} \lambda_i\left(\frac{t-t_{i-1}}{t_i-t_{i-1}}\right), & t \in [t_{i-1}, t_i[\ , \quad i = 1, \dots, N \\ \lambda_N(1), & t = t_f \end{cases}$$

and the state variable x, control function u and multiplier functions μ and ϑ accordingly. This yields in conditions (a)-(e) and (g) in Theorem 5.2. We receive the continuity of the Hamiltonian (condition (f)) because of $\widetilde{\mathcal{H}}$ being constant.

All conditions of Theorem 5.2 are shown. □

This statement differs from the necessary conditions developed by Oberle [Obe07]. Here, the results for order 0 and 1 look similar. The statement in [Obe07] has got a similar structure as the necessary conditions for optimal control problems with free control subarcs.

5.2.2 Continuous Control

In special cases, the continuity of the control function is a consequence of the necessary conditions. Since we treat an example of this kind, we focus in this section on linear-quadratic optimal control problems.

Let the control function be one-dimensional.

OCP 5.3. *Determine a piecewise continuous control function $u : [t_0, t_f] \to \mathbb{R}$, such that the functional*

$$J(x,u) \;=\; \phi(x(t_f)) + \int_{t_0}^{t_f} \frac{1}{2}u(t)^2 + b(x(t))u(t) + c(x(t))dt \tag{5.23}$$

is minimized subject to

$$\begin{aligned} x'(t) &= g(x(t))(u(t)+e(x(t))) \qquad t \in [t_0,t_f], a.e., &(5.24)\\ r(x(t_0),x(t_f)) &= 0, &(5.25)\\ u(t) &\in [u_{\min}, u_{\max}], \qquad t \in [t_0,t_f], a.e., &(5.26) \end{aligned}$$

where $b : \mathbb{R}^n \to \mathbb{R}$, $c : \mathbb{R}^n \to \mathbb{R}$, $e : \mathbb{R}^n \to \mathbb{R}$, and $r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^\ell$, are twice continuous differentiable.

The state equation (5.24) may be discontinuous in the following way:

$$g(x(t)) = \begin{cases} g_1(x(t)), & S(x(t),u(t)) < 0 \\ g_2(x(t)), & S(x(t),u(t)) = 0 \\ g_3(x(t)), & S(x(t),u(t)) > 0 \end{cases}$$

with a switching function

$$S(x(t),u(t)) := h(x(t))(u(t)+e(x(t))),$$

where $h : \mathbb{R}^n \to \mathbb{R}$ and $g_k : \mathbb{R}^n \to \mathbb{R}, k = 1,2,3$, are twice continuous differentiable.

For this problem, the following statement can be prooven.

Lemma 5.3. *Let (x_0, u_0) be a weak local minimum of problem OCP 5.3. Let the switching function S satisfy Assumption 5.1. Then u_0 is continuous at all switching times:*

$$\forall i = 1, \ldots, N : \qquad u_0(t_i^+) = u_0(t_i^-).$$

Proof. Let (x_0, u_0) be a weak local minimum of problem OCP 5.3. For clearness, we omit writing the subscribt zero at the solution and we omit writing the time variable t in this proof. We want to investigate the continuity of the optimal control function u at a switching time. We treat a switching function with the following structure and corresponding multiplier

$$S[t] \begin{cases} < 0, & t \in [t_0, t_1^-[, \\ = 0, & t \in [t_1^+, t_f], \end{cases} \qquad \Rightarrow \qquad \vartheta(t) \begin{cases} = 0, & t \in [t_0, t_1^-[, \\ = \vartheta, & t \in [t_1^+, t_f], \end{cases}$$

Other switching structures can be treated in an analogous way. The switching function is of order 0. Therefore the Hamiltonian looks as follows.

$$\begin{aligned} \mathcal{H} &= \lambda_0 \frac{1}{2}u^2 + b(x)u + c(x) + \lambda g(x)\big(u+e(x)\big) + \vartheta h(x)\big(u+e(x)\big) \\ &= \lambda_0 \frac{1}{2}u^2 + u\big(b(x) + \lambda g(x) + \vartheta h(x)\big) + \big(c(x) + \lambda g(x)e(x) + \vartheta h(x)e(x)\big) \end{aligned}$$

On the subinterval $[t_0, t_1^-[$ the control function is calculated using the minimum condition.

$$0 = \mathcal{H}_u[t] \quad \Rightarrow \quad \lambda_0 u = -b(x) - \lambda g_1(x) - \vartheta h(x) = -b(x) - \lambda g_1(x).$$

On the subinterval $[t_1^+, t_f]$ the control function is calculated using the Implicit Function Theorem.

$$0 = S[t] \quad \Rightarrow \quad u = -e(x)$$

The multiplier ϑ can be determined by applying the minimum condition.

Inserting the optimal control into the Hamiltonian at the switching time t_1, we get

$$\begin{aligned}
\mathcal{H}[t_1^-] &= -\frac{1}{2}(b(x) + \lambda g_1(x))^2 + c(x) + \lambda g_1(x) e(x) \\
\mathcal{H}[t_1^+] &= \frac{1}{2} e(x)^2 - b(x)e(x) + c(x)
\end{aligned}$$

One necessary condition is the continuity of the Hamiltonian at each switching time. Thus with $0 = -\mathcal{H}[t_1^-] + \mathcal{H}[t_1^+]$ it follows:

$$\begin{aligned}
0 &= \frac{1}{2}(b(x) + \lambda g_1(x))^2 - \lambda g_1(x)e(x) + \frac{1}{2}e(x)^2 - b(x)e(x) \\
&= \frac{1}{2}\big(b(x) + \lambda g_1(x) - e(x)\big)^2 \\
&\Rightarrow -b(x) - \lambda g_1(x) = -e(x)
\end{aligned}$$

Thererfore, u is continuous at the switching time t_1. □

Lemma 5.3 plays an importent role in the development of sufficient conditions for OCP 5.3. We need this statement for the varification of less strict sufficient conditions for problem DIO3 (pp. 110).

5.3 Sufficient Optimality Conditions for Optimal Control Problems with Singular State Subarcs

In this section we develop second order sufficient optimality conditions for optimal control problems with singular state subarcs.

Let the switching function S fulfill Assumption 5.1. The auxiliary problem OCP 5.2 is stated on page 102. In the case of a higher order switching function, we have to treat a differential algebraic constrained optimal control problem. At this time, there do not exist general sufficient conditions for problems with differential algebraic equations. Therefore we restrict our investigations to switching functions of order 0.

For simplicity, we make the use of Assumption 4.4.

Assumption 4.4. *The switching function S satisfies:*

$$\exists j \in \{1,\dots,m\}: \qquad D_{u_j}S(x_0(t),u_0(t)) \neq 0$$

We can apply the Implicit Function Theorem to $S(x,u) = 0$, and calculate one component of the control function: $u_j = V(t;x)$. Without loss of generality, we assume $j = m$. Note, that V has to satisfy the mixed control-state constraints (5.4). We separate the control function in the following way:

$$u = (\tilde{u}^T, V)^T.$$

The optimal control problem either depends on all components u or we insert $V(t;x)$ into the problem. Regarding the essential control functions

$$u_{\mathcal{J}}(t) := \begin{cases} u(t), & t \in [t_{i-1}, t_i], i \in J^1, \\ \tilde{u}(t), & t \in]t_{i-1}, t_i[, i \in J^0, \end{cases}$$

we formulate the system of Riccati differential equations as

$$Q' = -\widetilde{\mathcal{H}}_{xx} - f_x^T Q - Qf_x + (\widetilde{\mathcal{H}}_{xu_{\mathcal{J}}} + Qf_{u_{\mathcal{J}}})(\widetilde{\mathcal{H}}_{u_{\mathcal{J}}u_{\mathcal{J}}})^{-1}(\widetilde{\mathcal{H}}_{xu_{\mathcal{J}}} + Qf_{u_{\mathcal{J}}})^T, \tag{5.27}$$

$$R' = \frac{1}{\tau}\left[-\widetilde{\mathcal{H}}_x^T - Qf\right] - f_x^T R + \left(\widetilde{\mathcal{H}}_{xu_{\mathcal{J}}} + Qf_{u_{\mathcal{J}}}\right)(\widetilde{\mathcal{H}}_{u_{\mathcal{J}}u_{\mathcal{J}}})^{-1} f_{u_{\mathcal{J}}}^T R, \tag{5.28}$$

$$q' = -\frac{2}{\tau} R^T f + R^T f_{u_{\mathcal{J}}}(\widetilde{\mathcal{H}}_{u_{\mathcal{J}}u_{\mathcal{J}}})^{-1} f_{u_{\mathcal{J}}}^T R. \tag{5.29}$$

depending on the symmetric matrix-function $Q : [t_0, t_f] \to \mathbb{R}^{n\times n}$ and the functions $R : [t_0, t_f] \to \mathbb{R}^n$ and $q : [t_0, t_f] \to \mathbb{R}$. The boundary and jump conditions introduced in chapter 3, page 45, can also be used in this chapter. We only treat switching functions of order 0.

Theorem 5.4. *Assume that the following conditions hold.*

1. *(x_0, u_0) is a solution candidate for OCP 5.1 with an order-zero switching function.*
2. *The regularity conditions R1 and R2 hold.*
3. *The strict Legendre-Clebsch condition is satisfied with respect to $u_{\mathcal{J}}$.*
4. *The system of differential equations (5.27)- (5.29) has bounded solutions Q, R, and q, which are piecewise continuous differentiable.*
5. *These solutions Q, R, and q satisfy boundary condition B3 and jump condition J3.*

Then, there exist $\epsilon > 0$ and $\delta > 0$, such that

$$\begin{aligned} J(x,\tau,u) \;\geq\;& J(x_0,\tau_0,u_0) \\ &+\delta\{||(x,\tau,\tilde{u}) - (x_0,\tau_0,\tilde{u}_0)||_2^2 + ||(x(t_0),x(t_f)) - (x_0(t_0),x_0(t_f))||^2\} \end{aligned}$$

holds for all admissible functions $(x, \tau, u) \in L^\infty([t_0, t_f], \mathbb{R}^{n+1+m})$ *with* $d((x, \tau, u) - (x_0, \tau_0, u_0)) \leq \epsilon$. *In particular,* (x_0, u_0) *is a weak local minimum compared with admissible pairs* (x, u) *having the same switching structure.*

Again, we formulate sufficient optimality conditions for problems, where the strict Legendre-Clebsch condition is not satisfied:

Theorem 5.5. *Assume that the following conditions hold.*

1. (x_0, u_0) *is a solution candidate for OCP 5.1 with an order-zero switching function.*
2. *The regularity conditions R1 and R2 hold.*
3. *The regularity condition R3 holds with respect to* $u_{\mathcal{J}}$.
4. *The system of differential equations (5.27)- (5.29) with partial derivatives only in the direction of inactive control components of* $u_{\mathcal{J}}$ *has bounded solutions* Q, R, *and* q, *which are piecewise continuous differentiable.*
5. *These solutions* Q, R, *and* q *satisfy boundary condition B3 and jump condition J3.*

Then, there exist $\epsilon > 0$ *and* $\delta > 0$, *such that*

$$\begin{aligned} J(x, \tau, u) \quad \geq \quad & J(x_0, \tau_0, u_0) \\ & + \delta\{||(x, \tau, \tilde{u}) - (x_0, \tau_0, \tilde{u}_0)||_2^2 + ||(x(t_0), x(t_f)) - (x_0(t_0), x_0(t_f))||^2\} \end{aligned}$$

holds for all admissible functions $(x, \tau, u) \in L^\infty([t_0, t_f], \mathbb{R}^{n+1+m})$ *with* $d((x, \tau, u) - (x_0, \tau_0, u_0)) \leq \epsilon$. *In particular,* (x_0, u_0) *is a weak local minimum compared with admissible pairs* (x, u) *having the same switching structure.*

5.3.1 Proof of Sufficient Optimality Conditions

We use the same idea for prooving the last two theorems as done in the previous chapter. In this section we only treat switching functions of order 0.

Proof. (Theorems 5.4 and 5.5) The proof goes in an analogous way as done in chapter 4. We have to treat auxiliary problem OCP 5.2 which does not contain the equality constraints (5.19), since the switching function is of order 0.

Regarding the auxiliary problem OCP 5.2, we introduce

$$u_i(s) = \begin{cases} u(t_{i-1} + s\tau_i), & i \in J^1, \\ \tilde{u}(t_{i-1} + s\tau_i), & i \in J^0, \end{cases}$$

The switching function S may jump, since it is depending on the control function. There do not occur point constraints in S. □

Let us now take a closer look to problems with continuous controls. If we treat an optimal control problem like OCP 5.3, we are able to reduce the jump conditions.

Remark 5.6. *Treating the special problem OCP 5.3, the matrix (3.20) defining the jumps of functions Q, R, and q must only be positive definite on the set*

$$\{(\eta_1, \ldots, \eta_n, \eta^+, \eta^-) \in \mathbb{R}^{n+2} | \eta_i = 0, \text{ if } D_{x_i}S[t] \neq 0\}.$$

The matrix itself may contain additional second derivatives of the switching function with respect to the state variable.

We have to make the following considerations justifying the above remark. Treating the special problem OCP 5.3, additional constraints in the entry and exit points of the form

$$\begin{array}{rcll} S(x_{i-1}(0), u_{i-1}(0)) & = & 0, & i \in J^0, \\ S(x_i(1), u_i(1)) & = & 0, & i \in J^0 \end{array}$$

only occur in the auxiliary problem, if

$$D_x S[t] \neq 0. \tag{5.30}$$

We treat the augmented boundary conditions with respect to the state variable

$$r^*(x^*(0), x^*(1)) := \begin{pmatrix} r(x_1(0), x_N(1)) \\ x_2(0) - x_1(1) \\ \vdots \\ x_N(0) - x_{N-1}(1) \\ S(x_{i-1}(0), u_{i-1}(0)), i \in J^0 \\ S(x_i(1), u_i(1)), i \in J^0 \end{pmatrix}.$$

Regularity condition R2 is fulfilled, if condition (5.30) is fulfilled.

The kernel $\ker\{D_{(x_s,x_f)} r(x_0(t_0), x_0(t_f))\}$ on which the matrix (3.20) must be positive definite reduces to the set

$$\{(\eta_1, \ldots, \eta_n, \eta^+, \eta^-) \in \mathbb{R}^{n+2} | \eta_i = 0 \text{ if } D_{x_i}S[t] \neq 0\}.$$

Second derivatives of the switching function enter into the matrix (3.20).

In the next section we treat a singular case of the linear regulator with diode. It will become apparent, why we need Lemma 5.3 and Remark 5.6.

5.4 Example: Linear Regulator with Diode (singular)

We return to the example of a linear regulator with diode treated in sections 1.6 and 3.5. This time, the switching function vanishes on a subinterval.

Problem DIO3. *Minimize the functional*

$$J(u) := \frac{1}{2}\int_0^2 u^2(t)dt \tag{5.31}$$

subject to the state equation

$$\dot{x} = \begin{cases} a(u-x), & x-u \leq 0 \\ b(u-x), & x-u > 0 \end{cases}, \tag{5.32}$$

and the initial and end conditions

$$x(0) = 4, \qquad x(2) = 3. \tag{5.33}$$

The right hand side of the state equation is nonsmooth if we use the parameters $a = 2$ and $b = 4$. The switching function $S(x,u) := x - u$ is of order 0.

According to [Obe07] the switching function has the following structure:

$$S(x,u) = \begin{cases} > 0, & t \in [0, t_1[\\ = 0, & t \in [t_1, t_2] \\ < 0, & t \in]t_2, 2] \end{cases}.$$

We use the Hamiltonian

$$\mathcal{H}(x,u,\lambda) = \begin{cases} \frac{1}{2}u^2 + \lambda b(u-x), & t \in [0, t_1[\\ \frac{1}{2}x^2, & t \in [t_1, t_2] \\ \frac{1}{2}u^2 + \lambda a(u-x), & t \in]t_2, 2] \end{cases} \tag{5.34}$$

and receive the boundary value problem:

Problem DIO3BVP. *Find continuous and piecewise continuous differentiable functions $x_0 : [0,2] \to \mathbb{R}$ and $\lambda : [0,2] \to \mathbb{R}$ and switching times $t_1, t_2 \in]0,2[, t_1 \neq t_2$ satisfying the differential equations and boundary conditions*

$$\dot{x} = \begin{cases} -b(b\lambda + x), & t \in [0, t_1[\\ 0, & t \in [t_1, t_2] \\ -a(a\lambda + x), & t \in]t_2, 2] \end{cases}, \qquad \dot{\lambda} = \begin{cases} b\lambda, & t \in [0, t_1[\\ -x, & t \in [t_1, t_2] \\ a\lambda, & t \in]t_2, 2] \end{cases}$$

$$x(0) = 4, \qquad x(2) = 3,$$

$$x(t_1) + b\lambda(t_1) = 0 \qquad x(t_2) + a\lambda(t_2) = 0.$$

The solution shown in Figure 5.1 is calculated using the software-routine BNDSCO. In Figure 5.2 we can see the corresponding switching function S.

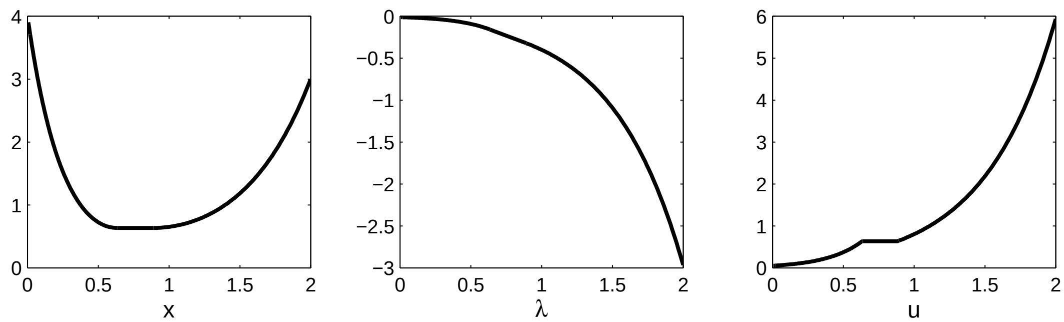

Figure 5.1: Problem DIO3, state and adjoint variables, control function

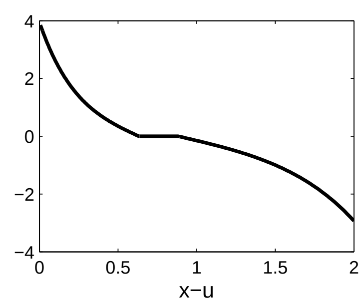

Figure 5.2: Problem DIO3, switching function

Let us now consider sufficient conditions for this solution candidate. Analogous to the cases in sections 1.6 and 3.5 we verify all conditions listed in Theorems 5.4 and 5.5.

The solution candidate is calculated using first order necessary conditions.

R1 is fulfilled because the treated example is an unconstrained optimal control problem.

The boundary conditions are

$$r(x(0), x(2)) = \begin{pmatrix} x(0) - 4 \\ x(2) - 3 \end{pmatrix} \in \mathbb{R}^2. \tag{5.35}$$

With $\operatorname{rank}\{D_{(x_s,x_f)} r(x(0), x(2))\} = 2$ the Regularity condition R2 is fulfilled.

For the investigation of the strict Legendre-Clebsch condition, we need the Hessian of $\mathcal{H}$ with respect to the control function:

$$\mathcal{H}_{uu}[t] = \begin{cases} 1, & t \in [t_0, t_1[\\ 0, & t \in [t_1, t_2] \\ 1, & t \in]t_2, t_f] \end{cases} .$$

The strict Legendre-Clebsch condition is satisfied for all subintervals $]t_{i-1}, t_i[, i \in J^1$, i. e. for $[t_0, t_1[\cup]t_2, t_f]$. Thus also regularity condition R3 is satisfied. We can apply both Theorems 5.4 and 5.5.

The first derivatives of the right hand side of the state equation

$$f_x(t) = \begin{cases} -b, & t \in [t_0, t_1[\\ -a, & t \in [t_1, t_2] \\ -a, & t \in]t_2, t_f] \end{cases}, \qquad f_u(t) = \begin{cases} b, & t \in [t_0, t_1[\\ a, & t \in [t_1, t_2] \\ a, & t \in]t_2, t_f] \end{cases}$$

and first and second derivatives of the Hamiltonian

$$\mathcal{H}_x[t] = \begin{cases} -\lambda b, & t \in [t_0, t_1[\\ x, & t \in [t_1, t_2] \\ -\lambda a, & t \in]t_2, t_f] \end{cases}, \qquad \mathcal{H}_u[t] = \begin{cases} u + \lambda b, & t \in [t_0, t_1[\\ x, & t \in [t_1, t_2] \\ u + \lambda a, & t \in]t_2, t_f] \end{cases},$$

$$\mathcal{H}_{xx}[t] = \begin{cases} 0, & t \in [t_0, t_1[\\ 1, & t \in [t_1, t_2] \\ 0, & t \in]t_2, t_f] \end{cases}, \qquad \mathcal{H}_{xu}[t] = \begin{cases} 0, & t \in [t_0, t_1[\\ 0, & t \in [t_1, t_2] \\ 0, & t \in]t_2, t_f] \end{cases}$$

are needed to formulate the system of differential equations. With functions $Q : [0,2] \to \mathbb{R}$, $R : [0,2] \to \mathbb{R}$, and $q : [0,2] \to \mathbb{R}$ we obtain differential equations depending on the structure of the switching function.

$$\dot{Q} = \begin{cases} 2bQ + (bQ)^2, & t \in [t_0, t_1[\\ 2aQ, & t \in [t_1, t_2] \\ 2aQ + (aQ)^2, & t \in]t_2, t_f] \end{cases}, \tag{5.36}$$

$$\dot{R} = \begin{cases} \frac{1}{\tau}[b\lambda - Qb(-\lambda b - x) + \tau bR + \tau(bQ)(bR)], & t \in [t_0, t_1[\\ \frac{1}{\tau}[a\lambda + \tau aR], & t \in [t_1, t_2] \\ \frac{1}{\tau}[a\lambda - Qa(-\lambda a - x) + \tau aR + \tau(aQ)(aR)], & t \in]t_2, t_f] \end{cases}, \tag{5.37}$$

$$\dot{q} = \begin{cases} \frac{1}{\tau}[-2Rb(-\lambda b - x) + \tau(Rb)^2], & t \in [t_0, t_1[\\ 0, & t \in [t_1, t_2] \\ \frac{1}{\tau}[-2Ra(-\lambda a - x) + \tau(Ra)^2], & t \in]t_2, t_f] \end{cases}. \tag{5.38}$$

Note, that $q(t) = const$ for all $t \in [t_1, t_2]$.

The boundary condition B3 can be reduced. Because all second derivatives of $\phi(x(2))$ and $r(x(0), x(2))$ vanish and because of the boundary conditions (5.35) in x, we just have to show positive definiteness of the matrix

$$\begin{pmatrix} q(0) & 0 \\ 0 & -q(2) \end{pmatrix}$$

on $\mathbb{R}^2$. In compliance with J3 additional conditions have to be satisfied at the switching times t_1 and t_2. Regarding Remark 5.6, since $D_x S[t] \neq 0$, the following sign conditions must hold:

$$q(t_i^-) < 0 < q(t_i^+), \qquad i = 1, 2.$$

These sign conditions are contradictory to the condition $q(t) = const$ for all $t \in [t_1, t_2]$. We can help ourselves by using an additional condition following from the fixed end time $t_f = 2$.

$$\tau_1(s) + \tau_2(s) + \tau_3(s) - (t_f - t_0) = 0 \qquad \Rightarrow \qquad \eta_1 + \eta_2 + \eta_3 = 0,$$

with different $\eta_1, \eta_2, \eta_3 \in \{\eta_{s1}, \eta_{s2}, \eta_{s3}, \eta_{f1}, \eta_{f2}, \eta_{f3}\}$. Using $\eta_1 = -\eta_2 - \eta_3$ the boundary conditions reduce according to

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}^T \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ \eta_3 \end{pmatrix}^T \begin{pmatrix} q_2 + q_1 & -q_1 \\ -q_1 & q_3 + q_1 \end{pmatrix} \begin{pmatrix} \eta_2 \\ \eta_3 \end{pmatrix} > 0$$

If q_2 and q_3 are big enough, the choice of $q_1 < 0$ is admissible. In this problem, $q_1 := q(t_1^+) = q(t_2^-) = const$ can be chosen such that the above 2×2-matrix is positive definite.

Finally, the following problem remains to be solved for showing general second order sufficient optimality conditions:

Problem DIO3SSC. *Find piecewise continuous functions $Q : [0, 2] \to \mathbb{R}$, $R : [0, 2] \to \mathbb{R}$ and $q : [0, 2] \to \mathbb{R}$ solving the system of ordinary differential equations (5.36)-(5.38) and satisfying the boundary and jump conditions:*

1. *The boundary conditions $q(0) > 0$ and $q(2) < 0$ must hold.*
2. *The sign conditions $q(t_1^-) < 0$ and $q(t_2^+) > 0$ must hold.*

We use forwards integration and solve initial value problems. On the first and third subinterval, we check, where the Riccati differential equation in Q has a solution at all:

$$\begin{aligned} Q(t_0) &\in \;]-\infty, 0.0032029], \\ Q(t_2^+) &\in \;]-\infty, 0.01156]. \end{aligned}$$

Holding $q(t_0) = q(t_2^+) = 0$, we choose $Q(t_0), R(t_0)$ and $Q(t_2^+), R(t_2^+)$ such that $q(t_1^-)$ and $q(t_f)$ are as small as possible. We then select $q(t_1^+) = q(t_2^-)$ such that the jump conditions are satisfied. The initial values $Q(t_1^+)$ and $R(t_1^+)$ can be choosen freely.

The functions shown in Figure 5.3 represent a solution of problem DIO3SSC. The functions are piecewise continuous. Their values at the starting and final times as well as at the switching times are shown in Table 5.1.

Having calculated this solution we are able to say:

Result. *The solution candidate derived by using necessary conditions is a weak local minimum of the corresponding optimal control problem DIO3 compared with all functions having the same switching structure.*

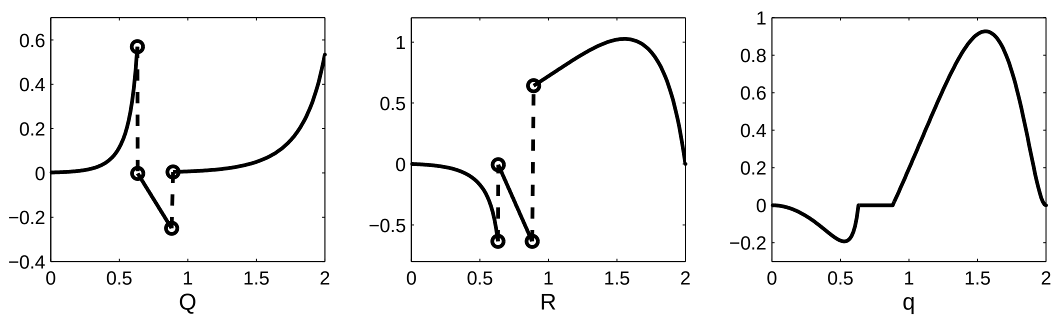

Figure 5.3: Problem DIO3SSC, functions Q, R, and q

t	$Q(t)$	$R(t)$	$q(t)$
t_0	0.0017	0.0000	$2.50e-7$
t_1^-	0.5692	−0.6342	$-3.15e-7$
t_1^+	0.0000	0.0000	$-1.00e-7$
t_2^-	−0.2500	−0.6342	$-1.00e-7$
t_2^+	0.0040	0.6342	$5.00e-6$
t_f	0.5339	−0.0013	$-5.14e-6$

Table 5.1: Problem DIO3SSC, values at starting, switching and final time

Chapter 6

Numerics

Showing second order sufficient optimality conditions for general optimal control problems, the main difficulty lies in finding a solution of a system of Riccati differential equations, which satisfies also boundary and jump conditions. In this chapter, we describe how this system of ordinary differential equations can be solved.

As we have seen in the previous chapters, the task is to find a solution of the following general problem.

Problem SSC. *Find a piecewise continuous symmetric matrix-function $Q : [t_0, t_f] \to \mathbb{R}^{n\times n}$, and piecewise continuous functions $R : [t_0, t_f] \to \mathbb{R}^n$ and $q : [t_0, t_f] \to \mathbb{R}$ of the system of Riccati differential equations*

$$\begin{aligned}
Q' &= -\widetilde{\mathcal{H}}_{xx} - f_x^T Q - Q f_x + (\widetilde{\mathcal{H}}_{xu} + Q f_u)(\widetilde{\mathcal{H}}_u)^{-1}(\widetilde{\mathcal{H}}_{xu} + Q f_u)^T, \\
R' &= \frac{1}{\tau}\left[-\widetilde{\mathcal{H}}_x^T - Qf\right] - s\widetilde{\mathcal{H}}_{xt}^T - sQf_t - f_x^T R \\
&\quad + \left(\widetilde{\mathcal{H}}_{xu} + Qf_u\right)(\tau\widetilde{\mathcal{H}}_{uu})^{-1}(\widetilde{\mathcal{H}}_u + s\tau\widetilde{\mathcal{H}}_{ut} + \tau R^T f_u)^T, \\
q' &= -\frac{2}{\tau}(-\widetilde{\mathcal{H}}_t - R^T f) - s^2\widetilde{\mathcal{H}}_{tt} - 2sR^T f_t \\
&\quad + (\widetilde{\mathcal{H}}_u + s\tau\widetilde{\mathcal{H}}_{ut} + \tau R^T f_u)(\tau^2\widetilde{\mathcal{H}}_{uu})^{-1}(\widetilde{\mathcal{H}}_u + s\tau\widetilde{\mathcal{H}}_{ut} + \tau R^T f_u)^T
\end{aligned}$$

that satisfy the following boundary and jump conditions

Boundary Condition B3. *The matrix*

$$\begin{pmatrix}
Q(t_0) + D^2_{x_s}(\nu r) & R(t_0) & D^2_{x_s x_f}(\nu r) & 0 \\
R^T(t_0) & q(t_0) & 0 & 0 \\
D^2_{x_f x_s}(\nu r) & 0 & -Q(t_f) + D^2_{x_f}(\phi + \nu r) & -R(t_f) \\
0 & 0 & -R^T(t_f) & -q(t_f)
\end{pmatrix} \tag{6.1}$$

must be positive definite on the set

$$\{(\eta_0, \eta_0^{(\tau)}, \eta_f, \eta_f^{(\tau)}) \in \mathbb{R}^{2(n+1)} | (\eta_0, \eta_f) \in ker\{D_{(x_s,x_f)} r(x_0(t_0), x_0(t_f))\}\}.$$

Jump Condition J3. *For all $i = 1, \ldots, N-1$, the matrices*

$$\begin{pmatrix} Q(t_i^+) - Q(t_i^-) & R(t_i^+) & -R(t_i^-) \\ R^T(t_i^+) & q(t_i^+) & 0 \\ -R^T(t_i^-) & 0 & -q(t_i^-) \end{pmatrix} \tag{6.2}$$

must be positive definite on the set

$$\{(\eta_1, \ldots, \eta_n, \eta^+, \eta^-) \in \mathbb{R}^{n+2}\}, \qquad \textit{if } S \textit{ is of order } 0,$$
$$\{(\eta_1, \ldots, \eta_n, \eta^+, \eta^-) \in \mathbb{R}^{n+2} | \eta_j = 0, \textit{ if } D_{x_j} S[t] \neq 0\}, \qquad \textit{if } S \textit{ is of order } 1.$$

Only in few cases, problem SSC is so simple, that we can actually read off a solution or receive a solution analytically (Example DIO1). The interesting case occurs, when the system of differential equations and the boundary and jump conditions are more complicated.

Searching for a solution of Problem SSC, we reduce the problem to one with easier differential equations and fewer boundary and jump conditions. Then we point out how an expedient implementation should look like. At last we suggest different approaches for finding a solution of the reduces Problem SSC.

6.1 Reduction of Problem SSC

Problem SSC can be reduced, depending on the special strucutre of the underlying optimal control problem. In the case of a continuous problem with fixed final time, the differential equations for R and q vanish. Furthermore no jump conditions have to be satisfied. In the case of a continuous problem with free final time, again, we do not have to satisfy any jump conditions.

On closer inspection we find a Riccati differential equation only in the function Q. The differential equation with respect to R is linear, whereas we only have to solve an integral equation in q.

It is useful and often necessary to reduce the boundary and jump conditions as much as possible to get a solution of Problem SSC.

On the one hand, the boundary conditions B3 can be reduced, if components of the state variable are fixed at the starting or final time. We can apply Lemmas 1.16, 1.17, 1.19, and 1.20. If for example all components of the state variable are fixed at the starting and final time, we only have to show the boundary conditions $q(t_0) > 0$ and $q(t_f) < 0$. A similar reduction can be achieved for the jump condition J3, if the switching function S is of order 1, or is necessarily continuous (Remark 5.6).

On the other hand, we can reduce the jump conditions, if we treat an optimal control problem with fixed final time. Because of the additional boundary condition in the state variables τ_i

$$\sum_{i=1}^{N} \tau_i(s_i) - (t_f - t_0) = 0$$

the set is smaller, on which the matrices given in J3 must be positive definite. This reduction makes sense, especially in the case treating an optimal control problem with only one switching time ($N = 2$). The jump condition J3 reduces to:

$$\begin{pmatrix} Q(t_1^+) - Q(t_1^-) & R(t_1^+) + R(t_1^-) \\ R^T(t_1^+) + R^T(t_1^-) & q(t_1^+) - q(t_1^-) \end{pmatrix} \tag{6.3}$$

must be positive definite on the set

$$\begin{aligned} &\{(\eta_1, \dots, \eta_n, \eta^+, \eta^-) \in \mathbb{R}^{n+2}\}, && \text{if } S \text{ is of order } 0, \\ &\{(\eta_1, \dots, \eta_n, \eta^+, \eta^-) \in \mathbb{R}^{n+2} | \eta_j = 0, \text{ if } D_{x_j}S[t] \neq 0\}, && \text{if } S \text{ is of order } 1. \end{aligned}$$

In Example DIO3 we can see, that this reduction also is useful for two switching times ($N = 3$).

Having done this preparatory work, we now have to solve the reduced problem SSC numerically. In the next section we give hints how to implement this problem.

6.2 Implementation of Problem SSC

The calculation of the right hand side of the Riccati differential equation analytically (by hand or with the help of software like MAPLE or MATHEMATICA) is rather cumbersome. This will always be the case, when treating more realistic or higher-dimensional optimal control problems. It is reasonable to compute the right hand side automatically. Therefore we use the following idea.

In the here developed method, we use matrix-vector-calculations. The right hand side of the Riccati differential equation can be calculated, if all components are known in matrix-formulation. This involves the Jacobian of the right hand side f of the state equation, the Hessian of the Hamiltonian $\widetilde{\mathcal{H}}$ and the functions $Q, R,$ and q. We obtain the Jacobian of f and the Hessian of $\widetilde{\mathcal{H}}$ by using all first and second order partial derivatives of L, f and C with respect to the state variable x and the control function u. These partial derivatives must be provided by the user of our method. We suggest the user to compute these derivatives by using a software like MAPLE or MATHEMATICA. Our method then provides the needed vectors and matrices.

The solution of problem SSC may have discontinuities. These are only allowed at switching times which occur because of the structure of the underlying optimal control problem.

- Regarding general optimal control problems, Q must be continuous (chapter 1).

- Regarding optimal control problems with free final time, Q, R, and q must be continuous (chapter 2).

- Regarding optimal control problems with discontinuities, Q, R, and q must satisfy jump conditions at times of discontinuity which appear in the right hand side of the state equation (chapter 3).

- Regarding optimal control problems with free control subarcs, Q, R, and q must satisfy jump conditions at junction times between general and free control subarcs (chapter 4).

- Regarding optimal control problems with singular state subarcs, Q, R, and q must satisfy jump conditions at times of discontinuity and junction times between general and singular state subarcs (chapter 5).

For numerical calculations it is useful to separate the interval $[t_0, t_f]$ into N subintervals

$$[t_{i-1}, t_i], \quad i = 1, \dots, N,$$

on which the functions Q, R, and q are continuous. Note that all functions x, τ, u, f and $\mathcal{H}$ are defined on each subinterval. Boundary and jump conditions now occur at the starting and final times of $Q_i : [t_{i-1}, t_i] \to \mathbb{R}^{n \times n}$, $R_i : [t_{i-1}, t_i] \to \mathbb{R}^n$, and $q_i : [t_{i-1}, t_i] \to \mathbb{R}$, $i = 1, \dots, N$.

The problem is to find continuous functions Q_i, R_i, and q_i, $i = 1, \dots, N$, which solve the differential equations on each subinterval and satisfy corresponding boundary conditions.

6.3 Approaches for Finding a Solution of Problem SSC

We solve the reduced problem SSC on the subintervals $[t_{i-1}, t_i]$ by calculating initial value problems. Varying the initial values, we try to find initial and final values in Q_i, R_i, and q_i, that satisfy all boundary conditions. Depending on the boundary conditions, it may be useful to run the integration backwards. This should be done, if there are many boundary conditions for Q_i, R_i, and q_i at the final time and only few boundary conditions at the starting time (Example DIO2).

We use the software routines `ode45` or `ode113` in MATLAB as well as the FORTRAN-codes DOPRI5, DOP853 and RADAU5 ([HNW93], [HW96]) for integrating the system of Riccati differential equations.

It is useful to get an overview, on which subset of initial values the routines converge at all. For that purpose, we first calculate the system of differential equations depending only on the varialbe Q. Thus the allowed set of initial values can be reduced and we do not have to search for initial values on the entire $\mathbb{R}^{n(n+1)/2+n+1}$.

In order to varify the boundary and jump conditions, different evaluations are possible. Positive definiteness of the matrices in B3 and J3 is fulfilled, if for example all eigenvalues are positive, if all of the leading principal minors are positive, or if the criterion of diagonally dominance is fulfilled.

If we use existing routines to calculate the eigenvalues like DSYEV in FORTRAN or `eig` in MATLAB, this calculation is not continuous since the eigenvalues are sorted by their size. Using diagonally dominance, there occurs a nondifferential absolute value in the calculations. Furthermore, this criterion is sufficient but not necessary. This condition may be too strong. Depending on the way of evaluating the Sylvester criterion, the problem may be ill posed.

Let us now summarize different possibilities of finding a solution of problem SSC.

The first approach. We try to solve problem SSC by calculating initial value problems. In a fist step, we use typical initial values $Q_i(t_{i-1}), R_i(t_{i-1})$, and $q_i(t_{i-1}), i = 1, \ldots, N$. We set all values to zero, use unit matrices or try initial values which satisfy parts of the boundary conditions. In a second step, we change these initial values by hand until we find a solution of the system of Riccati differential equations, which satisfies all boundary conditions at the final time.

This procedure can be tedious. Since it is not guaranteed that this way leads to a solution of problem SSC, one may choose another technique with less human effort.

Search on a grid. If the first approach does not lead to consistent values, it may be helpful to search over a whole grid of initial values. In general, the dimension of the problem yields in difficulties of solving problem SSC.

We consider N subintervals. On each subinterval, the symmetric matrix $Q(t) \in \mathbb{R}^{n\times n}$ has $n(n+1)/2$ initial values. Additionally, n initial values for the vector function $R(t) \in \mathbb{R}^n$ and one value for $q(t) \in \mathbb{R}$ occur. Thus, we can choose

$$N_0 := N \cdot \left(\frac{n(n+1)}{2} + n + 1\right)$$

different initial values. If we choose initial values for each of theses variables over a grid with m values, we get m^{N_0} different initial value problems. If we choose only the diagonal values of Q to be different form zero, the amount of variables reduces to $N \cdot (n + n + 1)$. We can also try to vary only the initial values of $Q_1(t_0), R_1(t_0)$ and $q_1(t_0)$ and adapt the initial values on the remaining subintervals in a way, that the jump conditions are satisfied. This choice yields in only $n(n+1)/2 + n + 1$ different values. But there is no reason, why these choices suffice in finding a solution of problem SSC.

On the whole, we should use the search on a grid only for optimal control problems with a small number of switching times and a small dimension of the state variable. A combination of searching on a grid and optimization techniques might be a better way of solving problem SSC.

Optimization techniques. The next idea is to use existing optimization routines to ease the calculations of problem SSC. With a first guess of initial values $Q_i(t_{i-1}), R_i(t_{i-1})$, and $q_i(t_{i-1}), i = 1, \ldots, N$, we start an optimization algorithm having these inital values as optimization variables. We formulate the following algorithm.

1. Guess initial values for Q_i, R_i, and $q_i, i = 1, \ldots, N$.
2. These initial values are the optimization variables for an unconstrained finite optimization problem:
   (a) Minimize a performance index, which depends on the boundary and jump conditions of problem SSC.
   (b) These boundary and jump conditions depend on the initial and final values of Q_i, R_i, and $q_i, i = 1, \ldots, N$. Therefore we have to use

an integration routine, which solves the system of Riccati differential equations.

Different formulations of the performance index are reasonable. Our aim is to get positive definite matrices defined in B3 and J3. First we can sum up all negative eigenvalues of these matrices and maximize this sum. A second performance index results from exploiting the criterium of diagonally dominance. This choice has the advantage, that we avoid the calculation of eigenvalues, which might be expensive. A drawback in both formulations is the nondifferentiability of the perfomance index due to either the calculation of eigenvalues or the absolute value when using the criterion of diagonally dominance. A differentiable alternative is using the Sylvester criterion.

In every function evaluation, the optimization algorithm has to solve an initial value problem on each subinterval. On the one hand, the usage of existing optimization techniques supports us in solving problem SSC. On the other hand this calculation may take a long run time. The underlying optimization algorithm may search only very close to the inital guess. This forces us to start this algorithm with different initial guesses. Thus, problems arise similar to those when searching over a grid.

We formulate a constrained optimization problem by exploiting the necessary condition, that every diagonal element of a positive definite matrix must be positive. Constrained optimization algorithms reduce the search directions. The calculation time decreases.

As an alternative calculation we use a constrained optimization problem, where the performance index is constant. The aim is to satisfy the constraints, which imply the boundary and jump conditions B3 and J3.

In this composition, we use the MATLAB routines `fminunc` and `fmincon` as well as the FORTRAN-code DONLP2 [Spe98a], [Spe98b].

Keeping the here developed examples in mind, the search by hand for a solution of problem SSC turnes out to be the best method.

If we deal with more realistic problems like Example ORB (section 4.5), the search over a grid combined with optimization routines may give an overview on where the integration routine converges at all. But we get a better feeling for the problem when searching initial values by hand. Thus an idea, in which direction the search should continue, can be achieved.

Conclusions and Outlook

Many different techniques and software routines are available to solve optimal control problems. There exist solvers based on direct and indirect methods and solve either problems with ordinary or partial differential equations. The derived solution candidate often fulfills necessary conditions.

The idea of this work is to develop a routine which checks sufficient conditions for solution candidates of optimal control problems governed by ordinary differential equations. One goal is to develop a code that can easily be attached to typical solvers and checks whether the calculated solution candidate is really a weak local minimum or not.

As a result of this work, we give new sufficient optimality conditions for nonsmooth problems. We use the multiprocess technique to develop sufficient conditions for problems with discontinuous state equations. Unlike previous works on nonsmooth optimal control problems we develop a theory including problems with a switching function. This switching function depending on state variables and control functions implicitely determines the times of discontinuity.

Depending on the structure of the problem, we either treat a problem with free final time, with points of discontinuity, with free control subarcs, or with singular state subarcs. We develop sufficient conditions for switching functions of order zero and one. The sufficient conditions for problems with singular state subarcs are not given for switching functions of order one because of the lack of general sufficient conditions for optimal control problems with differential algebraic equations.

Different examples are discussed to show difficulties when applying sufficient conditions. We are able to show sufficient conditions for the Re-Entry problem with free final time as well as for nonsmooth problems in electrical engineering and navigation. The fuel-optimal case of an Earth-Mars orbit transfer problem turns out to be more complicated. At this moment we are not able to show sufficient optimality conditions.

Difficulties arise when we solve the Riccati differential equations with additional boundary and jump conditions. We develop tools to weaken these conditions in general depending on the structure of the problem. In special cases we use particular utilities for the reduction of the boundary and jump conditions. Further we give hints on how to search for a solution of these differential equations.

There are drawbacks of developing a routine which checks sufficient conditions. On the one hand we are forced to weaken boundary and jump conditions. Thus we have to examine the examples in each single case. On the other hand the reduced problem still is hard to solve. This gets obvious when treation more realistic examples. Furthermore we cannot exclude the possibility of having a weak local minimum in the case we do not find a solution of the Riccati differential equation.

In the example of the Earth-Mars orbit transfer problem, one might have a chance of showing sufficient optimality conditions using the path of homotopy from the time-minimal problem.

In general we suggest to use an existing theory with other sufficient optimality conditions. One can use this work as a sample for the development of new conditions. Thus weaker sufficient optimality conditions can be developed as well as further conditions for problems with singular state subarcs and a switching function of order one. The here developed sufficient conditions for different cases of optimal control problems can be combined. The nonsmooth economic optimal control problem treated in [OR06] contains a dependent control subarc and a singular state subarc. Further, the development of a software code using automatic differentiation to establish the Riccati differential equations might also be of interest.

For nonsmooth optimal control problems with small dimensions the here developed sufficient optimality conditions lead to good results.

Appendix A

Partial derivatives

A.1 Example REE of section 2.4

We provide first and second partial derivatives of the integrand L of the cost functional and the right hand side $f = (V, \Gamma, \Xi)$ of the state equation regarding problem REE. We have

$$
\begin{aligned}
L &:= 10v^3\sqrt{\rho(\xi))}, \\
V &:= -\frac{Sv^2}{2m}\rho(\xi)C_D(u) - \frac{g\sin\gamma}{(1+\xi)^2}, \\
\Gamma &:= \frac{Sv}{2m}\rho(\xi)C_L(u) + \frac{v\cos\gamma}{\theta(1+\xi)} - \frac{g\cos\gamma}{(1+\xi)^2}, \\
\Xi &:= \frac{v\sin\gamma}{\theta}.
\end{aligned}
$$

First partial derivatives with respect to $x = (v, \gamma, \xi)$ and u are

$$
\begin{aligned}
&L_v = 30v^2\sqrt{\rho(\xi)}, && \Gamma_v = \tfrac{S}{2m}\rho(\xi)C_L(u) + \tfrac{\cos\gamma}{\theta(1+\xi)}, \\
&L_\gamma = 0, && \Gamma_\gamma = -\tfrac{v\sin\gamma}{\theta(1+\xi)} - \tfrac{g\sin\gamma}{(1+\xi)^2}, \\
&L_\xi = 5v^3\tfrac{\rho'(\xi)}{\sqrt{\rho(\xi)}}, && \Gamma_\xi = \tfrac{Sv}{2m}\rho'(\xi)C_L(u) - \tfrac{v\cos\gamma}{\theta(1+\xi)^2} + \tfrac{2g\cos\gamma}{(1+\xi)^3}, \\
&V_v = -\tfrac{Sv}{m}\rho(\xi)C_D(u), && \Xi_v = \tfrac{\sin\gamma}{\theta}, \\
&V_\gamma = -\tfrac{g\cos\gamma}{(1+\xi)^2}, && \Xi_\gamma = \tfrac{v\cos\gamma}{\theta}, \\
&V_\xi = -\tfrac{Sv^2}{2m}\rho'(\xi)C_D(u) + \tfrac{2g\sin\gamma}{(1+\xi)^3}, && \Xi_\xi = 0, \\
&L_u = 0, && \Gamma_u = \tfrac{Sv}{2m}\rho(\xi)C_L'(u), \\
&V_u = -\tfrac{Sv^2}{2m}\rho(\xi)C_D'(u), && \Xi_u = 0.
\end{aligned}
$$

Second partial derivatives with respect to x and u are

$$
\begin{aligned}
L_{vv} &= 60v\sqrt{\rho(\xi)}, & \Gamma_{vv} &= 0,\\
L_{v\gamma} &= 0, & \Gamma_{v\gamma} &= -\tfrac{\sin\gamma}{\theta(1+\xi)},\\
L_{v\xi} &= 15v^2\tfrac{\rho'(\xi)}{\sqrt{\rho(\xi)}}, & \Gamma_{v\xi} &= \tfrac{S}{2m}\rho'(\xi)C_L(u) - \tfrac{\cos\gamma}{\theta(1+\xi)},\\
L_{\gamma\gamma} &= 0, & \Gamma_{\gamma\gamma} &= -\tfrac{v\cos\gamma}{\theta(1+\xi)} - \tfrac{g\cos\gamma}{(1+\xi)^2},\\
L_{\gamma\xi} &= 0, & \Gamma_{\gamma\xi} &= \tfrac{v\sin\gamma}{\theta(1+\xi)^2} - \tfrac{2g\sin\gamma}{(1+\xi)^3},\\
L_{\xi\xi} &= \tfrac{5v^3}{2}\tfrac{2\rho(\xi)\rho''(\xi)-\rho'(\xi)^2}{\sqrt{\rho(\xi)}\rho(\xi)}, & \Gamma_{\xi\xi} &= \tfrac{Sv}{2m}\rho''(\xi)C_L(u) + \tfrac{2v\cos\gamma}{\theta(1+\xi)^3} - \tfrac{6g\cos\gamma}{(1+\xi)^4},\\
V_{vv} &= -\tfrac{S}{m}\rho(\xi)C_D(u), & \Xi_{vv} &= 0,\\
V_{v\gamma} &= 0, & \Xi_{v\gamma} &= \tfrac{\cos\gamma}{\theta},\\
V_{v\xi} &= -\tfrac{Sv}{m}\rho'(\xi)C_D(u), & \Xi_{v\xi} &= 0,\\
V_{\gamma\gamma} &= \tfrac{g\sin\gamma}{(1+\xi)^2}, & \Xi_{\gamma\gamma} &= -\tfrac{v\sin\gamma}{\theta},\\
V_{\gamma\xi} &= \tfrac{2g\cos\gamma}{(1+\xi)^3}, & \Xi_{\gamma\xi} &= 0,\\
V_{\xi\xi} &= -\tfrac{Sv^2}{2m}\rho''(\xi)C_D(u) - \tfrac{6g\sin\gamma}{(1+\xi)^4}, & \Xi_{\xi\xi} &= 0,\\
L_{vu} &= 0, & \Gamma_{vu} &= \tfrac{S}{2m}\rho(\xi)C_L'(u),\\
L_{\gamma u} &= 0, & \Gamma_{\gamma u} &= 0,\\
L_{\xi u} &= 0, & \Gamma_{\xi u} &= \tfrac{Sv}{2m}\rho'(\xi)C_L'(u),\\
V_{vu} &= -\tfrac{Sv}{m}\rho(\xi)C_D'(u), & \Xi_{vu} &= 0,\\
V_{\gamma u} &= 0, & \Xi_{\gamma u} &= 0,\\
V_{\xi u} &= -\tfrac{Sv^2}{2m}\rho'(\xi)C_D''(u), & \Xi_{\xi u} &= 0,\\
L_{uu} &= 0, & \Gamma_{uu} &= \tfrac{Sv}{2m}\rho(\xi)C_L''(u),\\
V_{uu} &= -\tfrac{Sv^2}{2m}\rho(\xi)C_D''(u), & \Xi_{uu} &= 0
\end{aligned}
$$

with

$$
\begin{aligned}
\rho(\xi) &:= \rho_0\exp(-\beta\theta\xi), & \rho'(\xi) &= -\rho_0\beta\theta\exp(-\beta\theta\xi), & \rho''(\xi) &= \rho_0\beta^2\theta^2\exp(-\beta\theta\xi),\\
C_D(u) &:= 1.174 - 0.9\cos u, & C_D'(u) &= 0.9\sin u, & C_D''(u) &= 0.9\cos u,\\
C_L(u) &:= 0.6\sin u, & C_L'(u) &= 0.6\cos u, & C_L''(u) &= -0.6\sin u.
\end{aligned}
$$

A.2 Example FCS of section 4.4

We provide first and second partial derivatives of the right hand side f of the state equation regarding problem ORB. This example is given in Mayer-form. Thus, we have

$$\begin{aligned} f_1 &:= x_2 + x_1 v \sin w, \\ f_2 &:= -1 + x_1 v \cos w, \end{aligned}$$

First partial derivatives with respect to $x = (x_1, x_2)^T$ and w are

$$\begin{array}{lll} f_{1x_1} = v \sin w \,, & f_{1x_2} = 1 \,, & f_{1w} = x_1 v \cos w, \\ f_{1x_1x_1} = 0 \,, & f_{1x_1x_2} = 0 \,, & f_{1x_2x_2} = 0, \\ f_{1x_1w} = v \cos w \,, & f_{1x_2w} = 0 \,, & f_{1ww} = -x_1 v \sin w, \\ f_{2x_1} = v \cos w \,, & f_{2x_2} = 0 \,, & f_{2w} = -x_1 v \sin w, \\ f_{2x_1x_1} = 0 \,, & f_{2x_1x_2} = 0 \,, & f_{2x_2x_2} = 0, \\ f_{2x_1w} = -v \sin w \,, & f_{2x_2w} = 0 \,, & f_{2ww} = -x_1 v \cos w. \end{array}$$

A.3 Example ORB of section 4.5

We provide first and second partial derivatives of the right hand side f of the state equation regarding problem ORB. This example is given in Mayer-form. Thus, we have

$$\begin{aligned} f_1 &:= w, \\ f_2 &:= \frac{v^2}{r} - \frac{1}{r^2} + \frac{c\beta}{m} \sin\psi, \\ f_3 &:= -\frac{wv}{r} + \frac{c\beta}{m} \cos\psi, \\ f_4 &:= -\beta. \end{aligned}$$

First partial derivatives with respect to $x = (r, w, v, m)$ and $u = (\beta, \psi)$ are

$$\begin{array}{llll} f_{1r} = 0, & f_{2r} = -\frac{v^2}{r^2} + \frac{2}{r^3}, & f_{3r} = \frac{wv}{r^2}, & f_{4r} = 0, \\ f_{1w} = 1, & f_{2w} = 0, & f_{3w} = -\frac{v}{r}, & f_{4w} = 0, \\ f_{1v} = 0, & f_{2v} = \frac{2v}{r}, & f_{3v} = -\frac{w}{r}, & f_{4v} = 0, \\ f_{1m} = 0, & f_{2m} = -\frac{c\beta}{m^2} \sin\psi, & f_{3m} = -\frac{c\beta}{m^2} \cos\psi, & f_{4m} = 0, \\ & & & \\ f_{1\beta} = 0, & f_{2\beta} = \frac{c}{m} \sin\psi, & f_{3\beta} = \frac{c}{m} \cos\psi, & f_{4\beta} = -1, \\ f_{1\psi} = 0, & f_{2\psi} = \frac{c\beta}{m} \cos\psi, & f_{3\psi} = -\frac{c\beta}{m} \sin\psi, & f_{4\psi} = 0. \end{array}$$

Second partial derivatives with respect to x and u are

$$
\begin{array}{llll}
f_{1rr} = 0, & f_{2rr} = \frac{2v^2}{r^3} - \frac{6}{r^4}, & f_{3rr} = -\frac{2wv}{r^3}, & f_{4rr} = 0, \\
f_{1rw} = 0, & f_{2rw} = 0, & f_{3rw} = \frac{v}{r^2}, & f_{4rw} = 0, \\
f_{1rv} = 0, & f_{2rv} = -\frac{2v}{r^2}, & f_{3rv} = \frac{w}{r^2}, & f_{4rv} = 0, \\
f_{1rm} = 0, & f_{2rm} = 0, & f_{3rm} = 0, & f_{4rm} = 0, \\
f_{1ww} = 0, & f_{2ww} = 0, & f_{3ww} = 0, & f_{4ww} = 0, \\
f_{1wv} = 0, & f_{2wv} = 0, & f_{3wv} = -\frac{1}{r}, & f_{4wv} = 0, \\
f_{1wm} = 0, & f_{2wm} = 0, & f_{3wm} = 0, & f_{4wm} = 0, \\
f_{1vv} = 0, & f_{2vv} = \frac{2}{r}, & f_{3vv} = 0, & f_{4vv} = 0, \\
f_{1vm} = 0, & f_{2vm} = 0, & f_{3vm} = 0, & f_{4vm} = 0, \\
f_{1mm} = 0, & f_{2mm} = \frac{2c\beta}{m^3} \sin\psi, & f_{3mm} = \frac{2c\beta}{m^3} \cos\psi, & f_{4mm} = 0, \\
f_{1r\beta} = 0, & f_{2r\beta} = 0, & f_{3r\beta} = 0, & f_{4r\beta} = 0, \\
f_{1w\beta} = 0, & f_{2w\beta} = 0, & f_{3w\beta} = 0, & f_{4w\beta} = 0, \\
f_{1v\beta} = 0, & f_{2v\beta} = 0, & f_{3v\beta} = 0, & f_{4v\beta} = 0, \\
f_{1m\beta} = 0, & f_{2m\beta} = -\frac{c}{m^2} \sin\psi, & f_{3m\beta} = -\frac{c}{m^2} \cos\psi, & f_{4m\beta} = 0, \\
f_{1r\psi} = 0, & f_{2r\psi} = 0, & f_{3r\psi} = 0, & f_{4r\psi} = 0, \\
f_{1w\psi} = 0, & f_{2w\psi} = 0, & f_{3w\psi} = 0, & f_{4w\psi} = 0, \\
f_{1v\psi} = 0, & f_{2v\psi} = 0, & f_{3v\psi} = 0, & f_{4v\psi} = 0, \\
f_{1m\psi} = 0, & f_{2m\psi} = -\frac{c\beta}{m^2} \cos\psi, & f_{3m\psi} = \frac{c\beta}{m^2} \sin\psi, & f_{4m\psi} = 0, \\
f_{1\beta\beta} = 0, & f_{2\beta\beta} = 0, & f_{3\beta\beta} = 0, & f_{4\beta\beta} = 0, \\
f_{1\beta\psi} = 0, & f_{2\beta\psi} = \frac{c}{m} \cos\psi, & f_{3\beta\psi} = -\frac{c}{m} \sin\psi, & f_{4\beta\psi} = 0, \\
f_{1\psi\psi} = 0, & f_{2\psi\psi} = -\frac{c\beta}{m} \sin\psi, & f_{3\psi\psi} = -\frac{c\beta}{m} \cos\psi, & f_{4\psi\psi} = 0.
\end{array}
$$

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b, c	functions describing special optimal control problem	105
C, C^+	mixed state and control constraints, active constraints ...	6, 16
C_1, C_2, C_3	different mixed state and control constraints	100
d	dimension of mixed control state constraints	7
e	function describing special optimal control problem	105
e_i	ith unit vector	19
E, F, G	matrices describing modified Riccati differential equation	16
f	right hand side of the state equation	5
f_1, f_2, f_3	different right hand sides	40, 100
f^*	combined right hand sides	49
g, g_1, g_2, g_3	functions describing special optimal control problem	105
h	function describing special optimal control problem	105
$\mathcal{H}, \widetilde{\mathcal{H}}$	Hamiltonian, augmented Hamiltonian	11
$\mathcal{H}^*, \widetilde{\mathcal{H}}^*$	Hamiltonian, augmented Hamiltonian, related to auxiliary problem	28, 49
i	index of switching times	6
I_n	$n \times n$ identity matrix	52
I_s, I_f	index set of fixed state variables at starting res. final time	6
$I^0(t)$	index set of active mixed constraints	12
$I^+(t)$	index set of positive multiplier corresp. to mixed constraints	15
j	index of state variable and control function components	6
J	cost functional	5
J^0, J^1	index sets describing where S vanishes or not	71
ℓ	dimension of boundary condition	7
L	Lagrange-part of the cost functional	5
L_1, L_2, L_3	different Lagrange-parts of cost functional	40, 100
m	dimension of control function	5
m_v	dimension of independent control function	70
m_w	dimension of dependent control function	70

M_R	matrix describing sufficient boundary conditions	15
M_R^0, M_R^{00}	reduced matrices describing sufficient boundary conditions	18, 19
M_R^f, M_R^{ff}	reduced matrices describing sufficient boundary conditions	21, 21
M_R^*	matrix describing sufficient boundary conditions, ocp free final time	31
n	dimension of the state variable	5
N	number of switching times	6
p	order of switching function	42
Q, R, q	variables of system of Riccati differential equations	14, 30
Q_i, R_i, q_i, Q^*	variables on subinterval, combined	53
Q_1	solution of disturbed Riccati differential equation	20
$\bar{Q}, \widehat{Q}$	reduced res. augmented symmetric matrix function	19, 30
r, r^*	general boundary condition, combined	6, 51
R_R	set corresp. to sufficient boundary conditions	15
R_R^0, R_R^{00}	reduced sets corresp. to sufficient boundary conditions	18, 19
R_R^f, R_R^{ff}	reduced sets corresp. to sufficient boundary conditions	21, 21
R_R^*	set corresp. to sufficient boundary conditins, ocp free final time	31
R_u^+	set corresp. to Regularity Condition R3	16
s, s_i	time variables on normed interval	8, 48
S	switching function	40
t	time variable	5
t_0, t_f, t_i	starting time, final time, switching time	5, 6
t_i^-, t_i^+	time right before res. right after t_i	6
T	state variable describing the time	8
u, u_0	control function, solution of ocp	5, 7
u_i, u^*, u_0^*	control variable on subinterval, combined, solution	48, 49, 51
u_p	argument of function S	51
$u_{i\,\min}, u_{i\,\max}$	lower and upper control boundary	7
$u_{\mathcal{A}}, u_{\mathcal{I}}$	active and inactive components of control function	16
$u_{\mathcal{J}}$	essential control components	78
$\mathcal{U}$	state-control-set	7
v, v_0	independent control function, solution	70
$\tilde{v}, V$	independent control components, solution of $S = 0$	71
w, w_0	dependent control function, solution	70

Variables of Problems DIO1, DIO2, DIO3 (sections 1.6, 3.5, 5.4)

Variables of Problem REE (section 2.4)

Variables of Problem ZER (section 3.6)

c	describes nonsmooth current ......
v, w	velocity ......
V	current of river ......
θ	control function, steering of the ship ......

Variables of Problem ORB (section 4.5)

c	constant describing amongst others gravity, mass of the sun .....
m	state variable, mass of the rocket ......
r, ϕ	state variables, position of the rocket ......
v, w	state variables, velocity of the rocket ......
β, ψ	control variables, thrust and thrust angle ......

This list of symbols is not complete. There occur symbols in the text, which are defined and used in small neighborhoods. Few symbols have two meanings. In each section, these symbols are used in just one way. For example, treating the theory of optimal control problems, the dimension of the control function is given by the symbol m. Only in examples REE and ORB, m is used for the mass of a rocket.

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Zusammenfassung

In dieser Arbeit werden hinreichende Optimalitätsbedingungen für nichtglatte Optimalsteuerungsaufgaben betrachtet. Die hier behandelten Optimalsteuerungsaufgaben sind unendlichdimensionale Optimierungsaufgaben mit gewöhnlichen Differentialgleichungsnebenbedingungen. Nichglatte Aufgaben treten auf, wenn der Integrand der Zielfunktion oder die rechte Seite der Differentialgleichung unstetig ist. Abhängig von dem Vorzeichen einer Schaltfunktion wird zwischen unterschiedlichem dynamischen Verhalten geschaltet.

Einführend werden glatte Optimalsteuerungsaufgaben mit freier Endzeit betrachtet. Dann werden Aufgaben mit endlich vielen Schaltpunkten im regulären und singulären Fall behandelt. Eine weitere Aufgabenklasse sind Optimalsteuerungsaufgaben mit freier Steuerung. Hier verschwindet auf Teilintervallen der Einfluss mancher Steuerkomponenten.

Durch die Anwendung der Multiprozesstechnik werden neue hinreichende Optimalitätsbedingungen entwickelt. Dabei wird die nichtglatte Aufgabe durch Telescoping in eine glatte Optimalsteuerungsaufgabe transformiert. Auf diese Aufgabe können bekannte hinreichende Optimalitätsbedingungen angewendet werden. Eine Rücktransformierung führt zu den gewünschten Aussagen. Zum Einen werden die hinreichenden Optimalitätsbedigungen von Mangasarian auf die transformierte Aufgabe angewendet. Zum Anderen werden allgemeinere Bedingungen von Maurer und Pickenhain betrachtet. So erhält man eine Matrix-Riccati-Differentialgleichung mit zusätzlichen Vorzeichen- und Sprungbedingungen.

Anhand von Beispielen wird die Anwendung der hinreichenden Bedingungen erläutert und Schwierigkeiten bei der numerischen Berechnung angegeben. Hinreichende Bedingungen für das zeitminimale ReEntry-Problem können gezeigt werden. An einem Diodenproblem aus der Elektrotechnik und einem Navigationsproblem werden hinreichende Bedingungen für nichtglatte Optimalsteuerungsaufgaben im regulären und singulären Fall untersucht. Zwei Beispiele mit freier Steuerung werden angegeben. Dabei konnten bei dem Erde-Mars Orbit Transfer Problem die hinreichenden Bedingungen noch nicht gezeigt werden. Für einfache nichtglatte Optimalsteuerungsaufgaben mit wenigen Schaltpunkten führen die hier entwickelten Bedingungen zu guten Resultaten.

Lebenslauf

Ricki Rosendahl geboren am 19.08.1975 in Göttingen.

Wissenschaftlicher Werdegang:

2004-2009	Promotionsstudent im Department Mathematik, Universität Hamburg
SS 2002	Auslandssemester an der Technical University of Denmark in Lyngby
1999-2004	Studium: Mathematik Diplom, Universität Hamburg
1998-1999	Studium: Lehramt für Schulen der Oberstufe an der Universität Hamburg, Nebenfächer: Mathematik und Physik
1997-1998	Studium: Medientechnik-Ingenieur, Fachhochschule Hamburg
1992-1993	Schulbesuch in den USA mit Abschluss: Graduation
1982-1996	Schulausbildung in Göttingen, Abitur am 21.05.1996

Anstellungen:

2004-2008	wissenschafliche Mitarbeiterin an dem Department Mathematik, Universität Hamburg
2001-2004	Studentische Hilfskraft am Fachbereich Mathematik, Universität Hamburg
2001	Praktikum in der mathematischen Abteilung der Hamburg-Mannheimer Versicherungs-AG