Halfordered Chain Structures, Splittings by Chains and Convexity

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INTRODUCTION

In 1832 C.F. Gauß (cf. [7]) remarked that in Euclids system of axioms there is missing a description of the concept of "order". This gap was closed in 1882 by M. Pasch (cf. [32], § 1) by introducing the notion "betweenness", i.e. a ternary relation on the set of all collinear triples of points obeying certain axioms. D. Hilbert assumed the order axioms of Pasch in a modified way for his foundation of the Geometry in his "Grundlagen der Geometrie" (cf. [10]).

In 1949 E. Sperner tried to describe the orderstructure of a geometry by a function the so called orderfunction. This led to a generalization of the notion order, which Sperner called halforder. While an ordered affine geometry consists always of infinite many points there are halfordered affine geometries with a finite number of points. Already in a halfordered geometry one can introduce segments and halflines, and so the concept of convex subsets. Sperner called an orderfunction convex if all segments are convex subsets.

The development of the theory of orderfunctions in affine and projective geometries and their interplay with the corresponding halfordered and ordered algebraic structures initiated by Sperner [33],[34],[35] was continued by H. Karzel [16],[17],[18], J. Joussen [12],[13], F. Kalhoff [14],[15] and A. Kreuzer [24],[25],[26],[27]. H.-J. Kroll [29] studied in 1977 orderfunctions and so ordered and halfordered structures in circle geometries, in particular in hyperbola structures which contain the Minkowski planes. The hyperbola structures belong also to the class of chain structures (cf. §1.6).

The aim of this thesis is to introduce concepts of orderfunctions, orders and halforders in chain structures and to develop their theory, in particular with regard to convexity.

This thesis starts with the very general notion of an $I-$net (cf. 1.1). Examples of $I-$nets can be obtained from affine planes $(\mathcal{P}, \mathcal{L})$. If $\{\mathfrak{G}_i \mid i \in I\}$ is a set of parallelclasses of $(\mathcal{P}, \mathcal{L})$ then $(\mathcal{P}, \mathfrak{G}_i; i \in I)$ is an $I-$net.

In the following let $(\mathcal{P}, \mathfrak{G}_i; i \in I)$ be an $I-$net. Then for any $i, j \in I$, $i \neq j$ a binary operation $\square_{ij} : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ (cf. pag.17) can be defined in a natural way such that $(\mathcal{P}, \square_{ij})$ becomes a semigroup. A subset $M \subset \mathcal{P}$ will be called *joinable* (accordingly like in a Minkowski plane) if $\forall\ X \in \mathfrak{G} := \bigcup_{i\in I}\mathfrak{G}_i,\ |X \cap M| \leq 1$ and a *chain* if $\forall\ X \in \mathfrak{G},\ |X \cap M| = 1$. The set of all chains will be denoted by $\mathfrak{C}$. Each joinable subset is contained in a maximal joinable subset and each chain is a maximal joinable subset. For $|I| \geq 3$ there are examples of $I-$nets containing maximal joinable subsets which are not chains (cf. (1.2.5)). If $|I| \geq 3$ then for all $X, Y \in \mathfrak{G},\ |X| = |Y|$, if $|I| = 2$ hence $I = \{1,2\}$ then for $i \in I$ and $X, Y \in \mathfrak{G}_i,\ |X| = |Y|$ but if $A \in \mathfrak{G}_1$ and $B \in \mathfrak{G}_2$ then $|A| = |B|$ if and only if $\mathfrak{C} \neq \emptyset$ and then there is a one-to-one correspondence between $\mathfrak{C}$ and the symmetric group $Sym\mathfrak{G}_1$ (cf. (1.3.9)). For $|I| \geq 3$ it is not so easy to determine $\mathfrak{C}$. Clearly if $(\mathcal{P}, \mathfrak{G}_i; i \in I)$ is obtained from an affine plane $(\mathcal{P}, \mathcal{L})$ and $\mathfrak{G} \neq \mathcal{L}$ then $\mathcal{L} \setminus \mathfrak{G} \subset \mathfrak{C}$.

But in an affine plane $(\mathcal{P}_a, \mathcal{L}_a)$ one can define $3-$nets $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3)$ such that no chain is contained in a line of $\mathcal{L}_a$ by fixing a point $0 \in \mathcal{P}_a$ and two distinct lines $X_1, X_2 \in \mathcal{L}_a$ through 0 and defining:

$\mathcal{P} := \mathcal{P}_a \setminus (X_1 \cup X_2),\ \mathfrak{G}_i := \{L \cap \mathcal{P} \mid L \in \mathcal{L}_a \setminus \{X_i\} : L \parallel X_i\}$ for $i \in \{1,2\}$ and $\mathfrak{G}_3 := \{L \cap \mathcal{P} \mid L \in \mathcal{L}_a \setminus \{X_1, X_2\} : 0 \in L\}$.

If $(\mathcal{P}_a, \mathcal{L}_a)$ is an anti-Fano plane then one can associate to each point $p \in \mathcal{P}$ a chain $[p]_4$ consisting of collinear points but p (cf. (1.2.11)), if $(\mathcal{P}_a, \mathcal{L}_a)$ satisfies further conditions there is a set $\mathfrak{G}_4$ of conics which are chains of the $3-$net $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3)$ such that $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3, \mathfrak{G}_4)$ is even a $4-$net (cf. (1.2.10),(1.2.11)). The sets of chains $[\mathcal{P}]_4 := \{[p]_4 \mid p \in \mathcal{P}\}$ and $\mathfrak{G}_4$ coincide if and only if $(\mathcal{P}_a, \mathcal{L}_a)$ is of order 4; in this case $(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3 \cup \mathfrak{G}_4)$ is the affine plane of order 3.

These examples suggest to consider the following situation: Let $(\mathbb{K}, +, \cdot, \leq)$ be an ordered or halfordered field, let $\mathbb{K}_+ := \{\lambda \in \mathbb{K} \mid \lambda > 0\}$, $\mathcal{P} := \mathbb{K}_+ \times \mathbb{K}_+$, $\mathfrak{G}_1 := \{(x, \mathbb{K}_+) \mid x \in \mathbb{K}_+\}$, $\mathfrak{G}_2 := \{(\mathbb{K}_+, x) \mid x \in \mathbb{K}_+\}$, $\mathfrak{G}_3 := \{< m > := \{(x, mx) \mid x \in \mathbb{K}_+\} \mid m \in \mathbb{K}_+\}$. It turns out that $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3)$ is a $3-$net and for $\mathbb{K} = \mathbb{R}$ the set $M := \{(x, x^{-1}) \mid x \in \mathbb{K}_+\}$ is a chain, which can be altered to a maximal joinable set $M_{a,b}$ but which is not $i-$maximal for $i \in \{1,2,3\}$ (cf. (1.2.5)).

From chapter 1.3 on we consider only $2-$nets $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ for which the set $\mathfrak{C}$ of chains is not empty. Then there is only one binary operation

$\Box := \Box_{12} : \mathcal{P} \times \mathcal{P} \to \mathcal{P}; (x,y) \mapsto x\Box y := [x]_1 \cap [y]_2$ (cf. page 17)
and a subset $S \subset \mathcal{P}$ is joinable if for all $\{a,b\} \in \binom{S}{2} : a\Box b \notin S$, where $\binom{\mathcal{P}}{n} := \{\{x_1, x_2, ..., x_n\} \subset \mathcal{P} \mid |\{x_1, x_2, ..., x_n\}| = n\}$, with $n \in \mathbb{N}$. If $\mathfrak{K}$ is any subset of $\mathfrak{C}$ then $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is called *chain structure*. Classical chain structures are besides the webs (i.e. $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a 3−net) the 2−*structures* (cf. [22], p.87) characterized by
(I2) For all joinable $\{a,b\} \in \binom{\mathcal{P}}{2}$, $\exists_1\ K \in \mathfrak{K} : a, b \in K$,
the *affine plane* characterized by **(I2)** and
(P) For all $K \in \mathfrak{K}$, for all $p \in \mathcal{P} \setminus K$ there is exactly one $L \in \mathfrak{K} : p \in L$ and $L \cap K = \emptyset$,
the *hyperbola structures* (cf. [22], p.87) characterized by:
(I3) $\forall$ joinable $\{a,b,c\} \in \binom{\mathcal{P}}{3}$ $\exists_1\ K \in \mathfrak{K} : a, b, c \in K$
and the *Minkowski planes* characterized by **(I3)** and by
(B) For all $K \in \mathfrak{K}$, for all $k \in K$ and for all $p \in \mathcal{P} \setminus (K \cup [k]_1 \cup [k]_2)$ there is exactly one $L \in \mathfrak{K}$ such that $p \in L$ and $L \cap K = \{k\}$.

Each chain $E \in \mathfrak{C}$ can serve as a "frame of reference for a coordinate system" since the restriction
$\Box : E \times E \to \mathcal{P}; (x,y) \mapsto x\Box y$
is a bijection. The generators of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are represented by the sets $a\Box E$ and $E\Box a$ respectively, with $a \in E$.
On the other hand if E is a non empty set, E can be imbedded in the 2−net
$\nu(E) := (\mathcal{P} := E \times E,\ \mathfrak{G}_1 := \{(x,E) \mid x \in E\},\ \mathfrak{G}_2 := \{(E,x) \mid x \in E\})$
by identifying E with $E' := \{(x,x) \mid x \in E\}$. Then the set $\mathfrak{C}$ of chains of $\nu(E)$ is not empty since $E' \in \mathfrak{C}$.
Between the binary relations ρ of E and the subsets of $\mathcal{P}$ there is a one-to-one correspondence given by
$C(\rho) := \{x\Box y \mid x, y \in E : (x,y) \in \rho\}$.
In section 1.3 we consider the relations ρ of E for which the subset $C(\rho)$ is joinable, maximal joinable, i−maximal ($i = 1, 2$) or a chain respectively. $C(\rho)$ is a chain if and only if ρ defines a permutations of E.
In section 1.4 there are collected all known facts about chain operations. The set $\mathfrak{C}$ of all

chains can be provided with a ternary operation τ such that after fixing a chain $E \in \mathfrak{C}$, the binary operation

$A \cdot B := \tau(A, E, B)$

turns $\mathfrak{C}$ in a group $(\mathfrak{C}, \cdot)$, which proves to be isomorphic to $SymE$ (cf. (1.4.3)). With τ we can distinguish subsets $\mathfrak{S}$ of $\mathfrak{C}$, hence of chain structures. $\mathfrak{S}$ is called symmetric or double symmetric respectively if for all A, B of $\mathfrak{S}$, $\tau(A, B, A) \in \mathfrak{S}$ or for all A, B, C of $\mathfrak{S}$, $\tau(A, B, C) \in \mathfrak{S}$ (cf. 1.5) respectively. If $E \in \mathfrak{S}$ then $\mathfrak{S}$ is double symmetric if and only if $\mathfrak{S}$ is a subgroup of $(\mathfrak{C}, \cdot)$. Each $\mathfrak{A} \subset \mathfrak{C}$ possesses a symmetric closure $\mathfrak{A}^{\sim}$ and a double symmetric closure $\mathfrak{A}^{\sim\sim}$.

The interplay between subsets of the permutation group $SymE$ and the chain structures $\mathfrak{K}$ via the map (cf. (1.3.3))

$C : SymE \to \mathfrak{C}; \rho \mapsto C(\rho) := \{x \Box \rho(x) \mid x \in E\}$

is studied in 1.6.

Sperner started from the idea that in a geometry (in particular in a plane) each hyperplane (line) H splits the set of points in three classes, the points incident with H and the two "sides" of H. If $\mathcal{P}$ and $\mathfrak{H}$ denotes the set of points and hyperplanes respectively he described this fact by a function

$h : \mathfrak{H} \times \mathcal{P} \to \{0, 1, -1\}; (H, p) \mapsto H(p)$

which assumes exactly then the value 0, if p is incident with H. He called such a function *orderfunction*. To each orderfunction h Sperner associated *derived orderfunctions of degree* 1 and 2 by:

$h' : \mathfrak{H} \times \mathcal{P} \times \mathcal{P} \to \{0, 1, -1\}; (H, p, q) \mapsto (H|p, q) := H(p) \cdot H(q),$

$h'' : \mathfrak{H} \times \mathfrak{H} \times \mathcal{P} \times \mathcal{P} \to \{0, 1, -1\}; (H, J, p, q) \mapsto [H, J|p, q] := H(p) \cdot H(q) \cdot J(p) \cdot J(q).$

The derived orderfunction h' is characterized by

(O1) For all $H \in \mathfrak{H}$, for all $x, y, z \in \mathcal{P} \setminus H : (H|x, y) \cdot (H|y, z) = (H|x, z)$

and induces on each line E a betweenness function ξ if the following *hyperplane condition* is satisfied:

(O2) Let $a, b, c \in E$ and let $A, B \in \mathfrak{H}$ with $c \in A, B$ and $a, b \notin A, B$ then $(A|a, b) = (B|a, b)$.

In this case the function

$\xi : E^{3'} := \{(c,a,b) \in E^3 \mid a,b \neq c\} \to \{1,-1\}; (c,a,b) \mapsto (c|a,b) := (C|a,b)$

where C is a hyperplane with $c \in C$ and $a, b \notin C$ is welldefined and ξ obeys the law:

(Z) For all $a,b,c,d \in E,\ a \neq b,c,d : (a|b,c) \cdot (a|c,d) = (a|b,d)$.

The second part of this thesis starts with a halfordered set (E, ξ), i.e. the set E is provided with a betweenness function ξ, i.e. ξ satisfies **(Z)**. Following E.Sperner ([33], p.128) we associate to ξ three functions $k_\xi, \mathfrak{Z}_\xi$ (cf. p.55) and the separation function τ_ξ (cf. p.57). If E is a line of a halfordered affine plane $(\mathcal{P}_a, \mathcal{L}_a, h)$ and ξ induced by h then ξ satisfies **(R1)** (cf. p.57). But in general this condition is not valid and this leads to the concept "related halforders"; halforders ξ which satisfy **(R1)** are called *selfrelated* and these are characterized by the statement that k_ξ is a constant function (cf. (2.1.1)). Therefore according to Sperner the selfrelated halforders decompose in *harmonic* and *anharmonic* ones depending whether $k_\xi = -1$ or $k_\xi = 1$. ξ defines an order if $\mathfrak{Z}_\xi$ is constant assuming the value 1. This implies that ξ is harmonic. But there are harmonic halforders which are not orders. An harmonic halforder ξ is an order if and only if (E, ξ) is convex (cf. (2.2.1)).

Two betweenness functions ξ_1 and ξ_2 of E are called *equivalent* if $\tau_{\xi_1} = \tau_{\xi_2}$ and they are called *related* if for all distinct $a,b,c,d \in E,\ \tau_{\xi_1}(a,b,c,d) = \tau_{\xi_2}(c,d,a,b)$. All betweenness functions of E which are related to a fixed one form an equivalence class; in particular if ξ is selfrelated then the equivalence class of ξ consists of selfrelated betweenness functions. ξ_1 and ξ_2 are equivalent if and only if there exists a valuation $\beta : E \to \{1,-1\}; x \mapsto \overline{x}$ such that for $a,b,c \in E, a \neq b,c,\ \xi_2(a,b,c) = \xi_{1\beta}(a,b,c) := \xi_1(a,b,c) \cdot \overline{b} \cdot \overline{c}$ and then $k_{\xi_1} = k_{\xi_2}$ (cf. (2.4.2)).

The question whether an anharmonic halfordered set (E, ξ) can be convex is discussed in (2.2.2).

Sperner showed that between the halforders of a desarguesian affine geometry and the subgroups D of index 2 in the multiplicative group K^* of the corresponding field $(K,+,\cdot)$ there is a one-to-one correspondence. If moreover D is a semigroup with respect to "$+$" then D is a positive domain and by "$a < b \Leftrightarrow b - a \in D$", K becomes an ordered field $(K,+,\cdot,<)$.

If we set

$$sgn : K^* \to \{1,-1\}; \lambda \mapsto sgn\lambda := \begin{cases} 1 & \text{if } \lambda \in D \\ -1 & \text{if } \lambda \notin D \end{cases}$$

and $\xi : K^{3'} \to \{1,-1\}; (a,b,c) \mapsto sgn(b-a) \cdot sgn(c-a)$ then (K,ξ) is a halfordered set and moreover the translations $a^+ : K \to K; x \mapsto a+x$ for $a \in K$ are automorphisms of (K,ξ). In 2.3 we generalize this fact by replacing the translations by a "locally transitive permutation set" $(E,\Sigma;0)$ and the subgroup D by a "semidomain". Here it turns out that Σ is contained in the automorphism group $Aut(E,\xi)$ of (E,ξ) and that if $(E,\Sigma;0)$ is a locally transitive permutation set and $\Sigma \subset Aut(E,\xi)$ then $Aut(E,\xi)$ acts transitively on E (cf. (2.3.2)).

The separation function τ_ξ derived from a halforder ξ satisfies two laws $\mathbf{(Z)}_l$ and $\mathbf{(Z)}_r$ (cf. § 2.4), which are used to introduce the general notion "separation function".

In (2.4.3) there is answered the question when a separation function τ can be derived from a betweenness function ξ. This condition is satisfied if τ is selfrelated. According to the concept "ordered projective geometry", (E,τ) is called *ordered* if for all distinct $a,b,c,d \in E$ exactly one of the values $\tau(a,b,c,d), \tau(b,c,a,d), \tau(c,a,b,d)$ is equal -1. (E,τ) is ordered if and only if there are betweenness functions ξ which are orders with $\tau = \tau_\xi$ (cf. (2.4.5), (2.4.6)). From this fact follows: If (E,ξ) is a halfordered set then there is a valuation β such that ξ_β is an order if and only if (E,τ_ξ) is ordered.

In 2.4.1 we determine for a given order (E,ξ) the valuations β such that (E,ξ_β) is again an order.

In section 2.5 we consider the set $C' := \{(a,d;b,c) \in (\binom{E}{4}) \mid (a|b,c) = -(d|b,c) = 1\}$, where $(\binom{E}{4}) := \{(x,y,z,t) \in E^4 \mid \{x,y,z,t\} \in \binom{E}{4}\}$ and the subset $C := \{(a,d;b,c) \in C' \mid (a|b,d) = 1\}$ whose elements are called *convex quadrupels.* Then (E,ξ) is called *trivially convex* or *separation free* if $C' = \emptyset$ and *convex* if $C = C' \neq \emptyset$. If one looks at two open segments $]a,c[$ and $]b,d[$ with $b \in]a,c[$ and $c \in]b,d[$, then one has the impression that d is a point of the halfline $\overrightarrow{a,c}$. This picture leads to the definition of the sets $D' := \{(a,b : c,d) \in (\binom{E}{4}) \mid (c|d,b) = (b|c,a) = -1\}$ and $D := \{(a,b : c,d) \in D' \mid (a|b,d) = (d|a,c) = 1\}$ and we call (E,ξ) *D-convex*, if $D = D'$. The connections between these different convexity concepts, also in the case

of halfordered field, are examinated. Moreover a subsection is dedicated to the study of the concept of the $\tau-$convexity.
In the last section of chapter 2 the definition of a halfordered permutation set is given.

Chapter 3 is dedicated to the question
"What shall we understand under the notion "halfordered net" and "halfordered chain structure"?"
If we consider the attempts for an axiomatization of the classical absolute geometry we find with respect to order two basic starting points, the betweenness relation on the set of collinear point-triples (M. Pasch [32]) and the splittings of the set of points by lines or planes. This last idea we find already in D. Hilberts lecture notes of the years from 1891 to 1899 and then mainly in the mentioned publications of E. Sperner. From the so called axiom of Pasch it follows for the Euclidean geometry that the betweenness relations ξ is invariant against parallel perspectivities and if we extend the Euclidean Geometry to the projective one then the associated separation relation τ_ξ is invariant against central perspectivities. This led to the following notions of *halfordered affine* and *halfordered projective* spaces (cf. [19], pages 239-240 and pages 232-233). Let $(\mathcal{P},\mathcal{L})$ be an affine (projective) space, let $\mathcal{P}^{3'}$ $(\mathcal{P}^{4'})$ be the set of all collinear tripels (a,b,c) (collinear quadrupels (a,b,c,d)) of points with $a\neq b,c$ $(a,b\neq c,d)$ and let
$\xi:\mathcal{P}^{3'}\to\{1,-1\};(x,y,z)\mapsto(x|y,z)$
$(\tau:\mathcal{P}^{4'}\to\{1,-1\};(x,y,z,u)\mapsto[x,y|z,u])$
be a function such that for collinear point sets the condition **(Z)** (**(T)**) (cf. § 2.1) is satisfied and the function ξ (τ) is invariant against parallel (central) perspectivities then $(\mathcal{P},\mathcal{L},\xi)$ $((\mathcal{P},\mathcal{L},\tau))$ is called a *halfordered affine (projective) space.*
If we start from Sperners approach $(\mathcal{P},\mathfrak{H},h)$ where $(\mathcal{P},\mathfrak{H})$ is an affine (projective) space, if ξ is the induced betweenness function then $(\mathcal{P},\mathfrak{H},\xi)$ $((\mathcal{P},\mathfrak{H},\tau_\xi))$ is a halfordered affine (projective) space. Sperner showed vice versa that to a halfordered affine (projective) space there exists always an orderfunction h of $(\mathcal{P},\mathfrak{H})$ such that h induces ξ (τ).
Here in our case of a net $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2)$ or a chain structure $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{K})$ we follow these two approaches and we have to decide how to replace the notion "collinear".

A point set which is located on a common generator will still be called *collinear* and a point set which is located on a chain of the chain structure *cochainic*. Moreover we need compatibility conditions with respect to distinguished maps.
From the 2−net $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ there are comming two types of perspectivities, one along direction 1 the other along direction 2; they are given by: If $\{i,j\} = \{1,2\}$ and $A, B \in \mathfrak{G}_j \cup \mathfrak{K}$ then
$[A \stackrel{i}{\to} B] : A \to B; a \mapsto [a]_i \cap B.$

From the chains $K, L \in \mathfrak{K}$ there are comming the maps $\widetilde{K,L}$ and $\widetilde{K}$ (cf. p.41 and p.44).
In order to copy the definition of a halfordered affine plane we have to consider in our net $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ the set $\mathcal{P}^{3g}$ of all collinear tripels (a,b,c) with $a \neq b, c$ hence
$\mathcal{P}^{3g} = \{(a,b,c) \in \mathcal{P}^3 \mid a \neq b, c \wedge \exists\ i \in \{1,2\} : [a]_i = [b]_i = [c]_i\}$
and if even a chain structure is given also the set
$\mathcal{P}^{3\mathfrak{K}} := \{(a,b,c) \in \mathcal{P}^3 \mid a \neq b, c \wedge \exists\ K \in \mathfrak{K} : a,b,c \in K\}$
of cochainic point tripels.
Then we have to assume that there is given a map $\xi : \mathcal{P}^{3g} \to \{1,-1\}$ such that for each generator $G \in \mathfrak{G}_1 \cup \mathfrak{G}_2$ the restriction $\xi_{|G^{3'}}$ is a betweenness function of G (cf. $\mathbf{(Z1)}_g$, p.102) and that ξ is invariant with respect to the two types of perspectivities (cf. $\mathbf{(Z2)}_g$, p.102). This is already the definition of a halfordered 2−net $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2; \xi)$, a definition which is not yet very satisfactory. For if $G, H \in \mathfrak{G}_1 \cup \mathfrak{G}_2$ are two generators then the two halfordered sets $(G, \xi_{|G^{3'}})$ and $(H, \xi_{|H^{3'}})$ are isomorphic if G and H are of the same type but this must not be true if G and H are of different types.
For a halfordered 2−net $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2; \xi)$ we can state:
Let $G \in \mathfrak{G}_1$ and $H \in \mathfrak{G}_2$ then $(G, \xi_{|G^{3'}})$ and $(H, \xi_{|H^{3'}})$ are isomorphic if and only if $\mathfrak{C} \neq \emptyset$ and there is a $K \in \mathfrak{C}$ such that:
$\forall\ (a,b,c) \in \mathcal{P}^{3g} : (a|b,c) = (\widetilde{K}(a)|\widetilde{K}(b), \widetilde{K}(c)).$
This leads us to the definition of a halfordered chain structure $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \xi)$ (cf. § 3.1) and here the function ξ can be extended to a function $\overline{\xi}$ which is also defined on all point tripels which are cochainic with respect to $\mathfrak{K}$ (cf. § 3.2). Then for each $K \in \mathfrak{K}$,

$(K,\overline{\xi}_{|K^{3'}})$ is a halfordered set.
If $E\in\mathfrak{K}$ is fixed and $\mathfrak{C}_\xi := \{C\in\mathfrak{C}\ |\widetilde{C}\in Aut(\mathcal{P},\xi)\} \overset{(4.1.2)}{=} C(Aut(E,\overline{\xi}_{|E^{3'}})) \overset{(1.3.3)}{=} \{\{x\Box\alpha(x)\mid x\in E\}\mid \alpha\in Aut(E,\overline{\xi}_{|E^{3'}})\}$ and if $\mathfrak{K}^{\sim\sim}$ denotes the closure of $\mathfrak{K}$ with respect to the ternary operation $(\mathfrak{C},\tau)$ (cf. p.45) then $\mathfrak{K}\subset\mathfrak{K}^{\sim\sim}\subset\mathfrak{C}_\xi$ and with $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{K};\xi)$ also $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{K}^{\sim\sim};\xi)$ and $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{C}_\xi;\xi)$ are halfordered chain structures (cf. (3.1.7)). Moreover if $A\cdot B := \tau(A,E,B)$ then $\mathfrak{C}_\xi$ and $\mathfrak{K}^{\sim\sim}$ are subgroups of $(\mathfrak{C},\cdot)$ and $(\mathfrak{C}_\xi,\cdot)$ is isomorphic to $(Aut(E,\overline{\xi}_{|E^{3'}}),\circ)$ (cf. (4.1.3)).

In order to follow the second approach we start from a 2−net $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2)$ with $\mathfrak{C}\neq\emptyset$, fix a chain $E\in\mathfrak{K}$ and assume that E splits the points of $\mathcal{P}$ not incident with E in two parts, i.e. that there is given a function
$E_s : (\mathcal{P}\setminus E)\times(\mathcal{P}\setminus E)\to\{1,-1\}; (a,b)\mapsto E_s(a,b)$ with $E_s(a,b)\cdot E_s(b,c)=E_s(a,c)$ for $a,b,c\in\mathcal{P}\setminus E$.
A particular class of splittings are the *selfrelated* ones characterized by $E_s(a,b) = E_s(\widetilde{E}(a),\widetilde{E}(b))$ for $a,b\in\mathcal{P}\setminus E$ (cf. Def. 3.3.1).
To each splitting $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,E_s)$ of $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2)$ by a chain E there correspond two betweenness functions ξ_l,ξ_r of E defined by $\xi_l(a,b,c) := E_s(a\Box b,a\Box c)$ and $\xi_r(a,b,c) := E_s(b\Box a,c\Box a)$ (cf. (3.3.1)) which are related and which are equal (hence selfrelated) if and only if E_s is selfrelated.
On the other hand if ξ_1 and ξ_2 are related betweenness functions of E then ξ_1 and ξ_2 define exactly one splitting $E_s=E_s(\xi_1,\xi_2)$ of $\mathcal{P}$ by E such that $\xi_1=\xi_l(E_s)$ and $\xi_2=\xi_r(E_s)$ (cf. (3.3.2)). Moreover if E_s is selfrelated then $\xi:=\xi(E)$ is harmonic or anharmonic if for $x\in\mathcal{P}\setminus E$, $E_s(x,\widetilde{E}(x))=-1$ or $=1$ respectively.

Each halforder ξ of E can be induced by two binary relations " $<$ " on E characterized by
$(*)\ \forall\ a,b\in E, a\neq b$: either $a<b$ or $b<a$,
such that: If we set

$$(a|b)=\begin{cases} 1 & \text{if} \quad a<b\\ -1 & \text{if} \quad b<a\end{cases},$$

then $\xi(a,b,c) = (a|b,c) := (a|b)\cdot(a|c)$.

To such a relation " $<$ " there corresponds the subset

$D := C(<) := \{x\Box y \mid x,y \in E : x < y\}$ with the property:

(D2) $\exists\ E \in \mathfrak{C} : \mathcal{P} = D \mathbin{\dot\cup} E \mathbin{\dot\cup} \widetilde{E}(D)$ is a disjoint union.

A subset D of $\mathcal{P}$ satisfying **(D2)** will be called *E-domain*.

Also if E_s is a harmonic splitting of $\mathcal{P}$ by E and $d \in \mathcal{P}\setminus E$ then $D := \{x \in \mathcal{P}\setminus E \mid E_s(x,d) = 1\}$ is an E-domain.

Each E-domain D defines by "$a <_D b :\Leftrightarrow a\Box b \in D$" a binary relation on E satisfying $(*)$ (and we have $C(<_D) = D$) and by

$$E_s(D)(x,y) = \begin{cases} 1 & \text{if} \quad x,y \in D \quad \text{or} \quad x,y \in \widetilde{E}(D) \\ -1 & \text{if} \quad x \in D, y \in \widetilde{E}(D) \quad \text{or} \quad x \in \widetilde{E}(D), y \in D \end{cases},$$

a splitting of $\mathcal{P}$ by E.

If an E-domain D has the property

(1) $\forall\ a,b \in D$ with $a\Box b \notin D\cup E : b\Box a \in D \cup E$

or

(2) $\forall\ a,b \in D$ with $a\Box b \in E : b\Box a \in D \cup E$

or

(3) $\forall\ a,b \in D$ with $a\Box b \in E : b\Box a \in D$

respectively then the corresponding splitting is called *weakly convex* or *convex* or *strongly convex* respectively and D is called a *positive domain* if

(D1) $\forall\ a,b \in D$ with $a\Box b \notin D : b\Box a \in D$.

We show that for an E-domain D and the corresponding relation " $<$ " the statements "$\xi(E_s(D))$ is an order of E", " $<$ is a total order relation on E", "$D := C(<)$ is a positive domain" and "$E_s(D)$ is a strongly convex splitting" are equivalent (cf. (3.4.5), (3.4.6), (3.6.2), (3.6.3)). Moreover under the assumption that ξ is selfrelated, "ξ is $D-$convex" and "the splitting $E_s(\xi)$ is weakly convex" are equivalent (cf. (3.3.2), (3.4.3)).

In section 4.1 the close relations between halfordered chain structures and halfordered permutation sets are stated in two main theorems (cf. (4.1.1) and (4.1.2)).

In section 4.2 the connection between halfordered chain structures and splitting by chains is investigated : For any $p \in \mathcal{P}$ let $p_1 := [p]_1 \cap E$, $p_2 := [p]_2 \cap E$ hence $p = p_1 \Box p_2$ and let

$$\eta : \begin{cases} \mathcal{P}^{3g} & \to & \{1,-1\} \\ (a,b,c) & \mapsto & (a|b,c) = \begin{cases} E_s(b\Box a_1, c\Box a_1) & \text{if} \quad [a]_2 = [b]_2 = [c]_2 \\ E_s(a_2\Box b, a_2\Box c) & \text{if} \quad [a]_1 = [b]_1 = [c]_1 \end{cases} \end{cases}$$

then $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2; \eta)$ is a halfordered $2-$net which can be extended to a halfordered chain structure $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \eta)$ if and only if (E, ξ_l) and (E, ξ_r) are isomorphic. In this case, $E \in \mathfrak{C}_\eta$ (cf. p.13) if and only if E_s is selfrelated (cf. (4.2.2)).

If $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \xi)$ is a halfordered chain structure then to each $K \in \mathfrak{C}_\xi$ there corresponds a selfrelated splitting $K_s(\xi)$ (cf. (4.2.1)).

In the last section of chapter 4 the automorphism groups of splittings and positive domains of a $2-$net are studied, the main results are given in (4.3.3),(4.3.4) and (4.3.5).

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1 Nets, chains and permutations

1.0 NOTATIONS

If A, B are non empty sets let $M(A, B)$ be the set of all maps from A into B and let $Bij(A, B) := \{f \in M(A, B) \mid f \text{ is bijective }\}$. For $A = B$ we set $M(A) := M(A, A)$ and $SymA := Bij(A, A)$.

If A, B are subsets of a group $(G, \cdot)$ then we call A *normal* with respect to B if:

$$\forall\, b \in B : \quad b \cdot A \cdot b^{-1} = A.$$

If $\mathcal{P}$ is a non empty set let $J := \{\alpha \in Sym\mathcal{P} \mid \alpha^2 = id\}$, $J^* := J \setminus \{id\}$ (= set of all involutions) and $\binom{\mathcal{P}}{n} := \{\{x_1, x_2, ..., x_n\} \subset \mathcal{P} \mid |\{x_1, x_2, ..., x_n\}| = n\}$, with $n \in \mathbb{N}$.

1.1 I-NETS AND THEIR AUTOMORPHISM GROUPS

Let $\mathcal{P}$ be a set and let $\mathfrak{G}$ be a subset of the power set of $\mathcal{P}$; the elements of $\mathcal{P}$ and $\mathfrak{G}$ will be called *points* and *generators* respectively. The pair $(\mathcal{P}, \mathfrak{G})$ is called an $I-net$ if there is a partition $\mathfrak{G} = \bigcup_{i \in I} \mathfrak{G}_i$ of $\mathfrak{G}$ such that $\forall\ X \in \mathfrak{G} : |X| \geq 2$ and the following two conditions are valid:

(N1) For each point $x \in \mathcal{P}$ and each $i \in I$ there is exactly one generator $G \in \mathfrak{G}_i$ with $x \in G$; this generator will be denoted by $[x]_i$.

(N2) Any two generators of distinct classes $\mathfrak{G}_i \neq \mathfrak{G}_j$ intersect in exactly one point.

In the following let $(\mathcal{P}, \mathfrak{G})$ be an $I-$net, which we will also denote by $(\mathcal{P}, \mathfrak{G}_i; i \in I)$. If $|I| \geq 2$ we can form by **(N1)** and **(N2)** for each pair $(i, j) \in I^2$, with $i \neq j$, the following operation:

$$\square_{ij} : \begin{cases} \mathcal{P} \times \mathcal{P} & \rightarrow & \mathcal{P} \\ (x, y) & \mapsto & x \square_{ij} y := [x]_i \cap [y]_j \end{cases} .$$

For $|I| = 2$:

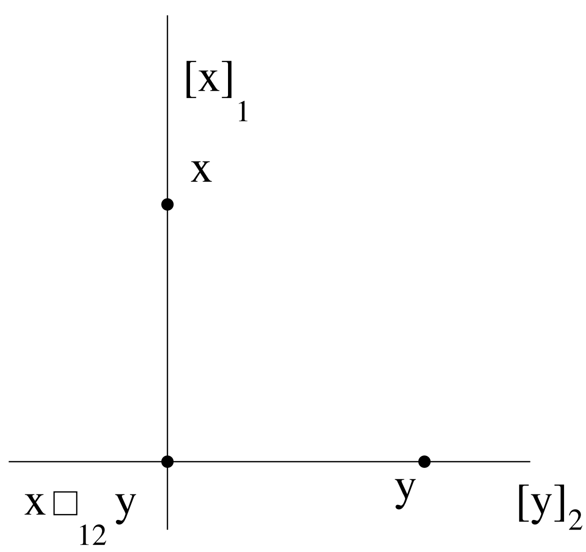

FIGURE 1.1.

(1.1.1) *For* $(\mathcal{P}, \mathfrak{G}_i; i \in I)$ *let* $|I| \geq 2$, $i, j \in I$ *with* $i \neq j$, $0 \in \mathcal{P}$ *be fixed and* $X := [0]_j, Y := [0]_i$. *Then:*

(1) *The map*:

$$\square_{ij_{|X \times Y}} : \begin{cases} X \times Y & \to \quad \mathcal{P} \\ (x, y) & \mapsto \quad x \square_{ij} y \end{cases},$$

is a bijection.

(2) $\forall\ i \in I,\ \forall\ G, H \in \mathfrak{G}_i :\ |G| = |H|$ *and* $|\mathcal{P}| = |X| \cdot |Y|$.

(3) *If* $|I| \geq 3$ *then* $\forall\ G, H \in \mathfrak{G} :\ |G| = |H|$ *and* $|\mathcal{P}| = |G|^2$.

Proof. (1) If we consider the map:

$$\varphi : \begin{cases} \mathcal{P} & \to \quad X \times Y \\ x & \mapsto \quad ([x]_i \cap X, [x]_j \cap Y) \end{cases},$$

we have $\square_{ij_{|X \times Y}} \circ \varphi = id_{\mathcal{P}}$ and $\varphi \circ \square_{ij_{|X \times Y}} = id_{X \times Y}$, hence $\square_{ij_{|X \times Y}}$ is a bijection.

(2) Let $i, j \in I$, with $i \neq j$, and $G, H \in \mathfrak{G}_i$; since the map:

$$[G \stackrel{j}{\to} H] : \begin{cases} G & \to \quad H \\ x & \mapsto \quad [x]_j \cap H \end{cases},$$

is a bijection, we obtain $|G| = |H|$. Moreover by (1) $|\mathcal{P}| = |X| \cdot |Y|$.

(3) Let $G, H \in \mathfrak{G}$, with $G \in \mathfrak{G}_i$ and $H \in \mathfrak{G}_j$, and let $k \in I \setminus \{i, j\}$; then the map:

$$[G \stackrel{k}{\to} H] : \begin{cases} G & \to \quad H \\ x & \mapsto \quad [x]_k \cap H \end{cases},$$

is a bijection, hence $|G| = |H|$. Moreover by (2) we have: $|\mathcal{P}| = |G|^2$. $\square$

Definition 1.1.1 Let $\mathfrak{D} \subset \mathfrak{P}(\mathcal{P})$. Then a permutation $\sigma \in Sym\mathcal{P}$ is called an *automorphism of the structure* $(\mathcal{P}, \mathfrak{D})$ if for all $X \in \mathfrak{P}(\mathcal{P})$ we have:

$$X \in \mathfrak{D} \Leftrightarrow \sigma(X) \in \mathfrak{D}.$$

If $Aut(\mathcal{P}, \mathfrak{D})$ denotes the set of all automorphisms of $(\mathcal{P}, \mathfrak{D})$ then $(Aut(\mathcal{P}, \mathfrak{D}), \circ)$ is a group.

Remark. In the case
(a) $(\mathcal{P}, \mathfrak{D}) := (\mathcal{P}, \mathcal{L})$ is an incidence space or
(b) $(\mathcal{P}, \mathfrak{D}) := (\mathcal{P}, \mathfrak{G}_i; i \in I)$ is an $I-$net, with $|I| \in \mathbb{N}$, we have:
if for $\sigma \in Sym\mathcal{P}$ and $\forall\ X \in \mathfrak{D}:\ \sigma(X) \in \mathfrak{D}$, then also $\forall\ X \in \mathfrak{D}:\ \sigma^{-1}(X) \in \mathfrak{D}$.

Proof. (a) Let $X \in \mathcal{L}$ and $x, y \in X$, with $x \neq y$. Then $\sigma^{-1}(x) \neq \sigma^{-1}(y)$ and there is exactly one $Y \in \mathcal{L}$ with $\sigma^{-1}(x), \sigma^{-1}(y) \in Y$. Then $x, y \in \sigma(Y) \in \mathcal{L}$ and so $\sigma(Y) = X$; this implies $\sigma^{-1}(X) = Y \in \mathcal{L}$.
(b) For $p \in \mathcal{P}$ let $\mathfrak{G}(p) := \{X \in \mathfrak{G} \mid p \in X\}$, then $|\mathfrak{G}(p)| = |I|$. If $X \in \mathfrak{G}, x \in X$ and $i \in I$ then by assumption there is exactly one $j \in I$ with $\sigma([\sigma^{-1}(x)]_i) = [x]_j \in \mathfrak{G}(x)$. Since $X \in \mathfrak{G}(x)$ and since $|I|$ is finite there is exactly one $k \in I$ with $\sigma([\sigma^{-1}(x)]_k) = X$. But then $\sigma^{-1}(X) = [\sigma^{-1}(x)]_k \in \mathfrak{G}$.□

We set $\overline{\Gamma} := Aut(\mathcal{P}, \mathfrak{G} := \bigcup_{i\in I} \mathfrak{G}_i)$, $\overline{\Gamma}^+ := Aut(\mathcal{P}, \mathfrak{G}_i; i \in I) := \{\sigma \in \overline{\Gamma} \mid \forall\ i \in I,\ \forall\ X \in \mathfrak{G}_i : \sigma(X) \in \mathfrak{G}_i\}$ and for each $i \in I$, $\overline{\Gamma}_i := \{\gamma \in \overline{\Gamma} \mid \forall\ x \in \mathcal{P} : [\gamma(x)]_i = [x]_i\}$. Then:

(1.1.2) *Let* $\sigma \in \overline{\Gamma}$, $X \in \mathfrak{G}_i$ *and* $\sigma(X) \in \mathfrak{G}_j$, *then* $\sigma(\mathfrak{G}_i) = \mathfrak{G}_j$ *and :*

$$\forall\ x \in \mathcal{P} : \sigma([x]_i) = [\sigma(x)]_j.$$

Proof. Let $X, Y \in \mathfrak{G}_i$, with $X \neq Y$, $x \in X$ and $y \in Y$; then $X = [x]_i$ and $Y = [y]_i$ and $X \cap Y = \emptyset$. Suppose $\sigma(X) = \sigma([x]_i) = [\sigma(x)]_j$ and $\sigma(Y) = \sigma([y]_i) = [\sigma(y)]_k$, with $k \neq j$; then $\emptyset = \sigma(\emptyset) = \sigma(X \cap Y) = \sigma(X) \cap \sigma(Y) = [\sigma(x)]_j \cap [\sigma(y)]_k \neq \emptyset$ which is a contradiction. Hence $k = j$. □

Remark. By (1.1.2) to each $\sigma \in \overline{\Gamma}$ there corresponds a permutation $\sigma' \in SymI$ defined by $\sigma([x]_i) = [\sigma(x)]_{\sigma'(i)}$ and this map

$$' : \begin{cases} \overline{\Gamma} & \to & SymI \\ \sigma & \mapsto & \sigma' \end{cases}$$

is a homomorphism with the kernel $\overline{\Gamma}^+$ since $[\tau\circ\sigma(x)]_{(\tau\circ\sigma)'(i)} = (\tau\circ\sigma)([x]_i) = \tau(\sigma([x]_i)) =$

$\tau([\sigma(x)]_{\sigma'(i)}) = [\tau \circ \sigma(x)]_{\tau'(\sigma'(i))}$. Consequently $(\tau \circ \sigma)' = \tau' \circ \sigma'$, i.e.:

(1.1.3) $\overline{\Gamma}^+ \trianglelefteq \overline{\Gamma}$.

(1.1.4) $\overline{\Gamma}_i \leq \overline{\Gamma}$.

Proof. Let $\gamma_1, \gamma_2, \gamma \in \overline{\Gamma}_i$; then $\forall\ x \in \mathcal{P}$ we have $[x]_i = [\gamma_1(x)]_i = [\gamma_2 \circ \gamma_1(x)]_i$, i.e. $\gamma_2 \circ \gamma_1 \in \overline{\Gamma}_i$ and $[x]_i = [\gamma(\gamma^{-1}(x))]_i = [\gamma^{-1}(x)]_i$, i.e. $\gamma^{-1} \in \overline{\Gamma}_i$.$\square$

(1.1.5) Let $I = \{1, 2\}$ *and* $\overline{\Gamma}^- := \overline{\Gamma} \setminus \overline{\Gamma}^+$. *Then:*
(1) $\forall\ i \in \{1, 2\}, \overline{\Gamma}_i \leq \overline{\Gamma}^+$.
(2) $\forall\ \chi \in \overline{\Gamma}^-, \forall\ x \in \mathcal{P} : \chi([x]_1) = [\chi(x)]_2$ *and* $\chi([x]_2) = [\chi(x)]_1$.
(3) *If there are* $X \in \mathfrak{G}_2, Y \in \mathfrak{G}_1$ *with* $|X| = |Y|$, *i.e. if there exists a bijection* $\pi : X \to Y$, *then the map:*

$$\gamma : \begin{cases} \mathcal{P} = X \square_{12} Y & \to \quad \mathcal{P} = X \square_{12} Y \\ (x \square_{12} y) & \mapsto \quad \pi^{-1}(y) \square_{12} \pi(x) \end{cases},$$

is an involution with $\gamma \in \overline{\Gamma}^-$ *and* $\overline{\Gamma} = \overline{\Gamma}^+ \rtimes \{id, \gamma\}$ *is a semidirect product.*

Proof. (1) If $\sigma \in \overline{\Gamma}_i$ then $\sigma'(i) = i$, so $\sigma' = id_I$ because $|I| = 2$ and hence $\sigma \in \overline{\Gamma}^+$.
(3) Let $x \in X, y \in Y$; then we have: $\gamma^2(x \square_{12} y) = \gamma(\pi^{-1}(y) \square_{12} \pi(x)) = \pi^{-1}(\pi(x)) \square_{12} \pi(\pi^{-1}(y)) = x \square_{12} y$, hence γ is an involution. Moreover, since $[x]_1 = \{x \square_{12} y \mid y \in Y\}$, we obtain: $\gamma([x]_1) = \{\gamma(x \square_{12} y) \mid y \in Y\} = \{\pi^{-1}(y) \square_{12} \pi(x) \mid y \in Y\} = X \square_{12} \pi(x) = [\pi(x)]_2$, hence $\gamma \in \overline{\Gamma} \setminus \overline{\Gamma}^+$.
Now let $\sigma \in \overline{\Gamma}$; if $\sigma \in \overline{\Gamma}^+$ we can write $\sigma = \sigma \circ id$. If $\sigma \notin \overline{\Gamma}^+$then $\sigma \circ \gamma := \tau \in \overline{\Gamma}^+$ and we can write $\sigma = \tau \circ \gamma^{-1} = \tau \circ \gamma$, hence $\overline{\Gamma} = \overline{\Gamma}^+ \circ \{id, \gamma\}$. Since $\overline{\Gamma}^+ \trianglelefteq \overline{\Gamma}, \{id, \gamma\} \leq \overline{\Gamma}$ and $\overline{\Gamma}^+ \cap \{id, \gamma\} = \{id\}$ we obtain: $\overline{\Gamma} = \overline{\Gamma}^+ \rtimes \{id, \gamma\}$.$\square$

Remark. If for each $i \in I$ we set $\overline{\Gamma}_i^+ := \{\gamma \in \overline{\Gamma}^+ \mid \forall\ x \in \mathcal{P} : [\gamma(x)]_i = [x]_i\}$ then we observe:

for $I = \{1,2\}$: $\overline{\Gamma}_i = \overline{\Gamma}_i^+$ (in fact let $j \in I \setminus \{i\}, \gamma \in \overline{\Gamma}_i$ and $x \in \mathcal{P}$; then $\gamma([x]_j) \neq [\gamma(x)]_i$ implies $\gamma([x]_j) = [\gamma(x)]_j$, i.e. $\gamma \in \overline{\Gamma}^+$);
for $|I| > 2$: $\overline{\Gamma}_i \neq \overline{\Gamma}_i^+$ (in fact if for example $I = \{1,2,3\}$ we can find elements of $\overline{\Gamma}_i$ which are not in $\overline{\Gamma}_i^+$: let $\mathcal{P} := \{a,b,c,d\}, \mathfrak{G}_1 := \{\{a,d\},\{b,c\}\}, \mathfrak{G}_2 := \{\{d,c\},\{a,b\}\}, \mathfrak{G}_3 := \{\{a,c\},\{d,b\}\}$ (cf. figure 1.2), let $\gamma \in \overline{\Gamma}$ with $\gamma(a) = a,\ \gamma(c) = c,\ \gamma(b) = d,\ \gamma(d) = b$. Then $\gamma \in \overline{\Gamma}_3$ but $\gamma \notin \overline{\Gamma}_3^+$ since $\gamma([d]_1) = [b]_2$.

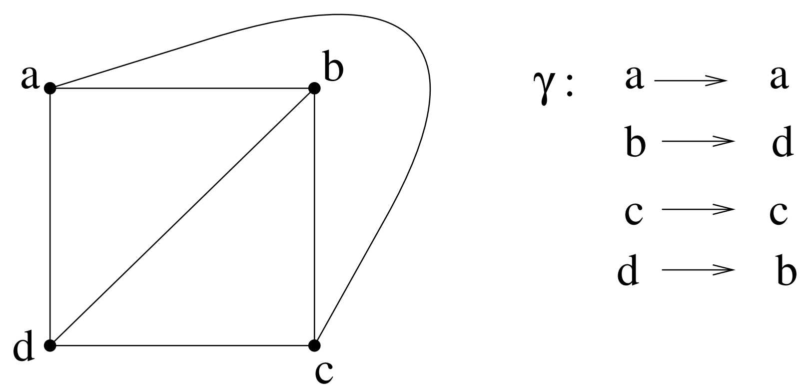

FIGURE 1.2.

1.2 JOINABLE SUBSETS OF AN I-NET AND CHAINS

A subset $M \subset \mathcal{P}$ is called *joinable* if

$$\forall\, X \in \mathfrak{G}: \ |M \cap X| \leq 1.$$

Let $\mathfrak{M}$ be the set of all joinable subsets M of our I-net $(\mathcal{P}, \mathfrak{G}_i; i \in I)$.

(1.2.1) (a) $\forall\, M \in \mathfrak{M}, \forall\, L \subset M : L \in \mathfrak{M}$.
(b) $\forall\, A \in \mathfrak{M}, \mathfrak{M}(A) := \{M \in \mathfrak{M} \mid A \subset M\}$ *is inductively ordered with respect to* " $\subset$ ".
(c) *Each* $M \in \mathfrak{M}(A)$ *is contained in a maximal joinable subset* M' *of* $\mathfrak{M}(A)$.

Proof.(b) Let $\mathfrak{K} \subset \mathfrak{M}(A)$ be totally ordered in $(\mathfrak{M}(A), \subset)$, let $L := \bigcup \mathfrak{K}$ and let $X \in \mathfrak{G}$. Suppose $a_1, a_2 \in L \cap X$. Then there are $K_1, K_2 \in \mathfrak{K}$ with $a_1 \in K_1, a_2 \in K_2$ and, since $\mathfrak{K}$ is totally ordered, let for istance $K_1 \subset K_2$, i.e. $a_1, a_2 \in K_2 \cap X$. Since K_2 is joinable, i.e. $|K_2 \cap X| \leq 1$ we obtain $a_1 = a_2$. This shows $L \in \mathfrak{M}(A)$ and $K \subset L$ for all $K \in \mathfrak{K}$.
(c) Since (b) is true, by Zorn's Lemma each $M \in \mathfrak{M}(A)$ is contained in a maximal joinable subset $M' \in \mathfrak{M}(A)$.□

Let $\mathfrak{M}_{max}$ be the set of all maximal joinable subsets of $\mathfrak{M}$.

(1.2.2) *Let* $A \in \mathfrak{M}, x \in \mathcal{P} \setminus A$; *then*:
(1) $A \cup \{x\} \in \mathfrak{M} \Leftrightarrow \ \forall\, i \in I : \ [x]_i \cap A = \emptyset$.
(2) $A \in \mathfrak{M}_{max} \Leftrightarrow \forall\, x \in \mathcal{P}, \exists\, i \in I : \ [x]_i \cap A \neq \emptyset$.
(3) *If* $|I| = 2$, $A \in \mathfrak{M}_{max}$ *and* $X \in \mathfrak{G}_1$ *with* $A \cap X = \emptyset$, *then* $\forall\, x \in X : [x]_2 \cap A \neq \emptyset$.

Proof. (2) " $\Leftarrow$ " Let $y \in \mathcal{P} \setminus A$ arbitrary; then $\exists\, i \in I : \emptyset \neq [y]_i \cap A = \overline{y}$ [1]. If we consider $A \cup \{y\}$ we have $(A \cup \{y\}) \cap [y]_i = \{y, \overline{y}\}$, hence $|(A \cup \{y\}) \cap [y]_i| = 2$ and this implies that $A \in \mathfrak{M}_{max}$.

[1] If the intersection of two sets A, B consists of a single element p, we will write instead of $A \cap B = \{p\}$ also $A \cap B = p$.

" $\Rightarrow$ " This is a consequence of (1).$\square$

A subset C of $\mathcal{P}$ is called a *chain* if the following condition **(N3)** is satisfied:

(N3) Every element X of $\mathfrak{G} := \bigcup_{i\in I} \mathfrak{G}_i$ intersects C in exactly one point.

Let $\mathfrak{C}$ be the set of all chains.

(1.2.3) (1) $\mathfrak{C} \subset \mathfrak{M}_{max}$.
(2) $\forall\ C, D \in \mathfrak{C}, \forall\ i, j \in I$, *with* $i \neq j$, *the map*

$$\varphi : \begin{cases} C \times D & \rightarrow & \mathcal{P} \\ (x, y) & \mapsto & x\square_{ij}y \end{cases},$$

is a bijection.
(3) $Aut(\mathcal{P}, \mathfrak{M}) = Aut(\mathcal{P}, \mathfrak{M}_{max}) = Aut(\mathcal{P}, \mathfrak{G}) \subset Aut(\mathcal{P}, \mathfrak{C})$.

Proof. (3) (a) " $Aut(\mathcal{P}, \mathfrak{M}) \subset Aut(\mathcal{P}, \mathfrak{M}_{max})$ ".
Let $\alpha \in Aut(\mathcal{P}, \mathfrak{M})$ and let $M \in \mathfrak{M}_{max}$. Then $\alpha(M) \in \mathfrak{M}$. Suppose $\alpha(M) \notin \mathfrak{M}_{max}$, i.e. there is a $M' \in \mathfrak{M}$ with $\alpha(M) \subset M'$ and $\alpha(M) \neq M'$. Since $Aut(\mathcal{P}, \mathfrak{M})$ is a group, we obtain $M \subset \alpha^{-1}(M') \in \mathfrak{M}$ and $M \neq \alpha^{-1}(M')$, i.e. M is not maximal which is a contradiction. Consequently $Aut(\mathcal{P}, \mathfrak{M}) \subset Aut(\mathcal{P}, \mathfrak{M}_{max})$.
(b) " $Aut(\mathcal{P}, \mathfrak{M}_{max}) \subset Aut(\mathcal{P}, \mathfrak{G})$ ".
Let $a, b \in \mathcal{P}$, with $a \neq b$, such that $[a]_i = [b]_i$, with $i \in I$, and let $\gamma \in Aut(\mathcal{P}, \mathfrak{M}_{max})$. Suppose $[\gamma(a)]_j \neq [\gamma(b)]_j$ for all $j \in I$ then $\{\gamma(a), \gamma(b)\} \in \mathfrak{M}$. By (1.2.1)$(c)$ $\exists\ C \in \mathfrak{M}_{max}$: $\gamma(a), \gamma(b) \in C$ and so, if we apply $\gamma^{-1} \in Aut(\mathcal{P}, \mathfrak{M}_{max})$, we obtain $a, b \in \gamma^{-1}(C)$, where $\gamma^{-1}(C) \in \mathfrak{M}_{max} \subset \mathfrak{M}$. Hence $\{a, b\} \in \mathfrak{M}$ which contradicts $a \neq b$ and $[a]_i = [b]_i$. Therefore there is an $j \in I$ such that $[\gamma(a)]_j = [\gamma(b)]_j$, i.e. $\gamma \in Aut(\mathcal{P}, \mathfrak{G})$.
(c) " $Aut(\mathcal{P}, \mathfrak{G}) \subset Aut(\mathcal{P}, \mathfrak{M})$ ".
Let $\gamma \in Aut(\mathcal{P}, \mathfrak{G})$, $M \in \mathfrak{M}$; then by definition we have: $\forall\ X \in \mathfrak{G} :\ |X \cap M| \leq 1$. Since $\gamma(X \cap M) = \gamma(X) \cap \gamma(M)$ and $\gamma(X) \in \mathfrak{G}$ we obtain $|\gamma(X) \cap \gamma(M)| \leq 1$, i.e. $\gamma(M) \in \mathfrak{M}$.
(d) " $Aut(\mathcal{P}, \mathfrak{G}) \subset Aut(\mathcal{P}, \mathfrak{C})$ ".

Let $\alpha \in Aut(\mathcal{P}, \mathfrak{G})$, $C \in \mathfrak{C}$ and $X \in \mathfrak{G}$. Then $\alpha^{-1}(X) \in \mathfrak{G}$, hence $C \cap \alpha^{-1}(X) =: \{p\}$, and so $\alpha(C) \cap X = \alpha(C \cap \alpha^{-1}(X)) = \{\alpha(p)\}$ shows that $\alpha(C) \in \mathfrak{C}$.□

A set $A \in \mathfrak{M}$ is called *$i-$maximal* if $i \in I$ and $[A]_i := \bigcup\{[a]_i \mid a \in A\} = \mathcal{P}$. We denote by $\mathfrak{M}_i$ the set of all $i-$maximal subsets of $(\mathcal{P}, \mathfrak{G}_i; i \in I)$, with $i \in I$.

(1.2.4) *Let $A \in \mathfrak{M}$, then:*

(1) A *is $i-$maximal* $\Rightarrow A \in \mathfrak{M}_{max}$.

(2) $A \in \mathfrak{C} \Leftrightarrow \forall i \in I : A$ *is $i-$maximal.*

(3) $I = \{1, 2\}$ *and* $A \in \mathfrak{M}_{max} \Rightarrow \exists\ i \in \{1, 2\} :\ A$ *is $i-$maximal.*

Proof. (3) Let $[A]_2 \neq \mathcal{P}$, i.e. $\exists\ X \in \mathfrak{G}_2 :\ X \cap A = \emptyset$. Hence by (1.2.2.2) $\forall\ x \in X : [x]_1 \cap A \neq \emptyset$ thus $[A]_1 = \mathcal{P}$, i.e. A is $1-$maximal.□

If $|I| \geq 3$ there are examples of maximal joinable sets which are not $i-$maximal for all $i \in I$. We have:

(1.2.5) *Let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3)$ be a $3-$net such that the set $\mathfrak{C}$ of all chains is not empty, and let $C \in \mathfrak{C}$ be fixed. For $a, b \in C$ with $a \neq b$ let $\square := \square_{12}, c := [a\square b]_3 \cap C$ and $C_{a,b} := (C \cup \{a\square b\}) \setminus \{a, b, c\}$. Then $C_{a,b}$ is a maximal joinable set but not $1-$ nor $2-$ nor $3-$maximal.*

Proof. (1) $C_{a,b}$ is not $1-$maximal; in fact $\forall\ x \in \mathcal{P}$

$$[x]_1 \cap C_{a,b} = \begin{cases} \emptyset & if \quad x \in [b]_1, [c]_1 \\ a\square b & if \quad x \in [a]_1 \\ [x]_1 \cap\ C & if \quad x \notin [a]_1, [b]_1, [c]_1 \end{cases} ;$$

(2) $C_{a,b}$ is not 2−maximal; in fact $\forall\ x \in \mathcal{P}$

$$[x]_2 \cap C_{a,b} = \begin{cases} \emptyset & \text{if} \quad x \in [a]_2, [c]_2 \\ a\Box b & \text{if} \quad x \in [b]_2 \\ [x]_2 \cap\ C & \text{if} \quad x \notin [a]_2, [b]_2, [c]_2 \end{cases} ;$$

(3) $C_{a,b}$ is not 3−maximal; in fact $\forall\ x \in \mathcal{P}$

$$[x]_3 \cap C_{a,b} = \begin{cases} \emptyset & \text{if} \quad x \in [a]_3, [b]_3 \\ a\Box b & \text{if} \quad x \in [a\Box b]_3 \\ [x]_3 \cap\ C & \text{if} \quad x \notin [a]_3, [b]_3, [a\Box b]_3 \end{cases} .$$

Now let $x \in \mathcal{P} \setminus C_{a,b}$. We have to show : $\exists\ X \in \mathfrak{G} : |X \cap (C_{a,b} \cup \{x\})| \geq 2$.
If $x \notin [a]_1, [b]_1, [c]_1$ $(x \notin [a]_2, [b]_2, [c]_2)$ then $[x]_1 \cap (C_{a,b} \cup \{x\})$ $([x]_2 \cap (C_{a,b} \cup \{x\}))$ consists of the two distinct points x and $[x]_1 \cap C$ (x and $[x]_2 \cap C$). Since $a\Box b \in C_{a,b}$ it remains to consider the cases:

(1) if $x \in \{a\Box c, a\} : [x]_1 \cap (C_{a,b} \cup \{x\}) = \{x, a\Box b\}$;

(2) if $x \in \{b, c\Box b\} : [x]_2 \cap (C_{a,b} \cup \{x\}) = \{x, a\Box b\}$;

(3) if $x = c : [x]_3 \cap (C_{a,b} \cup \{x\}) = \{x, a\Box b\}$;

(4) if $x = b\Box a\ : [x]_3 \cap (C_{a,b} \cup \{x\}) = \begin{cases} \{x, a\Box b\} & \text{if} \quad [a\Box b]_3 = [b\Box a]_3 \\ \{x, [x]_3 \cap C\} & \text{if} \quad [a\Box b]_3 \neq [b\Box a]_3 \end{cases}$;

(5) if $x \in \{b\Box c, c\Box a\} : [x]_3 \cap (C_{a,b} \cup \{x\}) = \{x, [x]_3 \cap C\}$.□

One can obtain some examples of 3−nets $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3)$ with $\mathfrak{C} \neq \emptyset$ in the following way:

1) Projective embeddable examples: let $(\overline{\mathcal{P}}, \mathcal{L})$ be a projective plane, for $p \in \overline{\mathcal{P}}$ let $\mathcal{L}(p) := \{L \in \mathcal{L} \mid p \in L\}$ and for $x, y \in \overline{\mathcal{P}}, x \neq y$ let $\overline{x, y} := \mathcal{L}(x) \cap \mathcal{L}(y)$. Let $\{a_1, a_2, a_3\} \in \binom{\overline{\mathcal{P}}}{3}$ be fixed, $A_1 := \overline{a_2, a_3}$, $A_2 := \overline{a_3, a_1}$, $A_3 := \overline{a_1, a_2}$, let $\mathcal{P} := \overline{\mathcal{P}} \setminus (A_1 \cup A_2 \cup A_3)$, $\mathcal{L}_{\mathcal{P}} := \{L_{\mathcal{P}} := L \cap \mathcal{P} \mid L \in \mathcal{L} : L \cap \mathcal{P} \neq \emptyset\}$ and $\mathfrak{G}_i := \{L \cap \mathcal{P} \mid L \in \mathcal{L}(a_i) : L \cap \mathcal{P} \neq \emptyset\}$ for $i \in \{1, 2, 3\}$. Then $\text{Net}(a_1, a_2, a_3) := (\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3)$ is a 3−net; in fact:

(N1) let $x \in \mathcal{P}$ and $i \in \{1,2,3\}$ then $[x]_i = \overline{a_i, x} \setminus \{a_i, l_i\} \in \mathfrak{G}_i$, with $l_i := \overline{a_i, x} \cap A_i$, is the uniquely determined generator.

(N2) For $i \in \{1,2,3\}$ let $X_i = L_i \setminus \{a_i, l_i\} \in \mathfrak{G}_i$, with $L_i \in \mathcal{L}(a_i) \setminus \{A_1, A_2, A_3\}$ and $l_i := L_i \cap A_i$. If $\{i,j,k\} = \{1,2,3\}$ then L_i, L_j, A_k have no point in common and so $z := L_i \cap L_j$ is a single point with $z \neq a_1, a_2, a_3$ hence $z \in \mathcal{P}$ and so $X_i \cap X_j = \{z\} \neq \emptyset$.

We consider $\mathfrak{C} := \{C \in 2^{\mathcal{P}} \mid \forall\ X \in \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3 : |C \cap X| = 1\}$ the set of chains of the 3−net $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3)$. Clearly $\mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3 \subset \mathcal{L}_{\mathcal{P}} \setminus \mathfrak{C}$. Therefore let $L \in \mathcal{L} \setminus (\mathcal{L}(a_1) \cup \mathcal{L}(a_2) \cup \mathcal{L}(a_3))$ and $l_i := L \cap A_i$, then $L_{\mathcal{P}} := L \cap \mathcal{P} \in \mathcal{L}_{\mathcal{P}}$.
If $\{a_1, a_2, a_3\}$ are not collinear then $a_i \notin A_i$, hence $X_i := \overline{a_i, l_i} \setminus \{a_i, l_i\} \in \mathfrak{G}_i$ and $L_{\mathcal{P}} \cap X_i = \emptyset$, hence $L_{\mathcal{P}} \notin \mathfrak{C}$.
If $\{a_1, a_2, a_3\}$ are collinear then $(\mathcal{P}, \mathcal{L}_{\mathcal{P}})$ is an affine plane and L meets each generator $X \in \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3$ in exactly one point. This gives the result:

(1.2.6) *If* $\{a_1, a_2, a_3\}$ *are not collinear then* $\mathcal{L}_{\mathcal{P}} \cap \mathfrak{C} = \emptyset$, *if* $\{a_1, a_2, a_3\}$ *are collinear then* $\mathcal{L}_{\mathcal{P}} \setminus (\mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3) \subset \mathfrak{C}$.

In order to determine $\mathfrak{C}$ in the case that $\{a_1, a_2, a_3\}$ are not collinear we introduce the following definitions:

Definition 1.2.1 Let $C \subset \overline{\mathcal{P}}$. Then $p \in \overline{\mathcal{P}}$ is called a *node* or *n-point* of C if:
(n1) $p \notin C$,
(n2) $\forall\ X \in \mathcal{L}(p) : |X \cap C| = 1$,

and a *quadratic-point* or *q-point* if instead:
(q1) $p \in C$,
(q2) $\forall\ X \in \mathcal{L}(p) : |X \cap C| \leq 2$,
(q3) $|\{X \in \mathcal{L}(p) \mid |X \cap C| = 1\}| \leq 1$.

We denote with $N(C)$ (and we call $N(C)$ the *nucleus* of C) resp. $Q(C)$ the set of all n-points resp. q-points of C. If $(\overline{\mathcal{P}}, \mathcal{L})$ has order q and $N(C) \neq \emptyset$ then $|C| = q+1$. If $Q(C) \neq \emptyset$ then $|C| \leq q+2$. Moreover we have:

(1.2.7) *If* $|Q(C)| \geq 2$ *then* $|N(C)| \leq 1$.

Proof. Let $q_1, q_2 \in Q(C)$ with $q_1 \neq q_2$. If $n \in N(C)$ then $X_i := \overline{n, q_i}$ are lines such that $X_i \in \mathcal{L}(q_i)$, $|X_i \cap C| = 1$ and $X_1 \neq X_2$. By definition of q-point there is at most one line in $\mathcal{L}(q_i)$ with this property. This implies $N(C) = \{n\}$.□

(1.2.8) *Let* $C \subset \overline{\mathcal{P}}$, $\{q_1, q_2\} \in \binom{Q(C)}{2}$, $n \in N(C)$ *and let* $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3) :=$*Net*$(q_1, q_2, n)$. *If* $\mathfrak{C}$ *denotes the set of all chains of* $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3)$ *then* $C \setminus \{q_1, q_2\} \in \mathfrak{C}$.

Proof. For $\{i,j\} = \{1,2\}$ let $G \in \mathfrak{G}_i$. Then there is a $l_j \in \overline{n, q_j} \setminus \{n, q_j\}$ such that $G = \overline{q_i, l_j} \setminus \{q_i, l_j\}$. Since $n \in N(C)$ we have $|\overline{n, q_i} \cap C| = 1$ by **(n2)**, hence $|\overline{q_i, l_j} \cap C| = 2$ by **(q2)** and **(q3)**, i.e. $|G \cap (C \setminus \{q_1, q_2\})| = 1$. Now let $G \in \mathfrak{G}_3$, i.e. there is a $l_3 \in \overline{q_1, q_2} \setminus \{q_1, q_2\}$ with $G = \overline{n, l_3} \setminus \{n, l_3\}$ and $l_3 \notin C$ by **(q2)**. By **(n2)** there is exactly one point $x \in \overline{\mathcal{P}}$ with $\overline{n, l_3} \cap C = \{x\}$, where $x \neq q_1, q_2$. Since $l_3 \notin C$ and $n \notin C$ by **(n1)**, $G \cap (C \setminus \{q_1, q_2\}) = \{x\}$.□

(1.2.9) *Let* $\{a_1, a_2, a_3\} \in \binom{\overline{\mathcal{P}}}{3}$ *with* a_1, a_2, a_3 *not collinear and let* $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3) :=$ *Net*(a_1, a_2, a_3). *Moreover let* $\mathfrak{C}$ *the set of all chains of* $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3)$, $C^0 \in \mathfrak{C}$ *and* $C := C^0 \cup \{a_1, a_2\}$. *Then* $a_1, a_2 \in Q(C)$ *and* $a_3 \in N(C)$.

Proof. For $A_1 := \overline{a_2, a_3}$ and $A_2 := \overline{a_1, a_3}$ we have (for $i \in \{1,2\}$) $A_i \cap C^0 = \emptyset$ thus $A_1 \cap C = \{a_2\}$ and $A_2 \cap C = \{a_1\}$. For $X_i \in \mathcal{L}(a_i) \setminus \{A_1, A_2, A_3\}$, with $i \in \{1,2\}$ and $A_3 := \overline{a_1, a_2}$, we have $X_i \setminus \{a_i, l_i\} \in \mathfrak{G}_i$, with $l_i := X_i \cap A_i$, thus $|(X_i \setminus \{a_i, l_i\}) \cap C^0| = 1$ and therefore $|X_i \cap C| = 2$ and $A_3 \cap C = \{a_1, a_2\}$. So $a_1, a_2 \in Q(C)$. Let $Y \in \mathcal{L}(a_3) \setminus \{A_1, A_2\}$. Then $Y \setminus \{a_3, l_3\} \in \mathfrak{G}_3$, with $l_3 := A_3 \cap Y$, $a_1, a_2 \notin Y$ and $|(Y \setminus \{a_3, l_3\}) \cap C^0| = 1$. This implies $|Y \cap C| = 1$, so $a_3 \in N(C)$.□

By applying (1.2.8) we obtain the following examples:

(1.2.10) *Let* $(K, +, \cdot)$ *be a commutative field of char* 2 *with* $K^* := K \setminus \{0\} = K^{(2)} := \{x^2 \mid x \in K^*\}$ *and* $(\overline{\mathcal{P}}, \mathcal{L})$ *the projective plane over* $(K, +, \cdot)$ *hence* $\overline{\mathcal{P}} := (K^3)^*/K^* = \{K^*(x_0, x_1, x_2) \mid (x_0, x_1, x_2) \in K^3 \setminus \{(0,0,0)\}\}$. *Then the equation* $x_0^2 + x_1x_2 = 0$ *defines a conic* C *where* $e_0 := K^*(1,0,0)$ *is the nucleus, thus* $N(C) = \{e_0\}$ *(in fact:* $charK = 2$ *implies that* $\forall\ X \in \mathcal{L}(e_0) : |X \cap C| \leq 1$ *and* $K^* = K^{(2)}$ *implies that* $\forall\ X \in \mathcal{L}(e_0) : |X \cap C| = 1$*) and* $Q(C) = C$. *Therefore for any* $\{a_1, a_2\} \in \binom{C}{2}$ *(for instance* $a_1 = e_1 := K^*(0,1,0)$ *and* $a_2 = e_2 := K^*(0,0,1)$*) the set* $C^0 := C \setminus \{a_1, a_2\}$ *is a chain in the* $3-$*net* $Net(e_0, a_1, a_2)$.

Theorems (1.2.8) and (1.2.9) suggest to consider the affine plane $(\mathcal{P}_a, \mathcal{L}_a)$ where $\mathcal{P}_a := \overline{\mathcal{P}} \setminus A_3$ and $\mathcal{L}_a := \{L \cap \mathcal{P}_a \mid L \in \mathcal{L} \setminus \{A_3\}\}$. We set $0 := a_3$, $X_1 := A_2 \setminus \{a_1\}$ and $X_2 := A_1 \setminus \{a_2\}$. Then $X_1, X_2 \in \mathcal{L}_a$, $\mathcal{P} = \mathcal{P}_a \setminus (X_1 \cup X_2)$ and for $p \in \mathcal{P}$, $[p]_1 = (p \parallel X_1) \setminus X_2$, $[p]_2 = (p \parallel X_2) \setminus X_2$, $[p]_3 = \overline{0, p} \setminus \{0\}$.

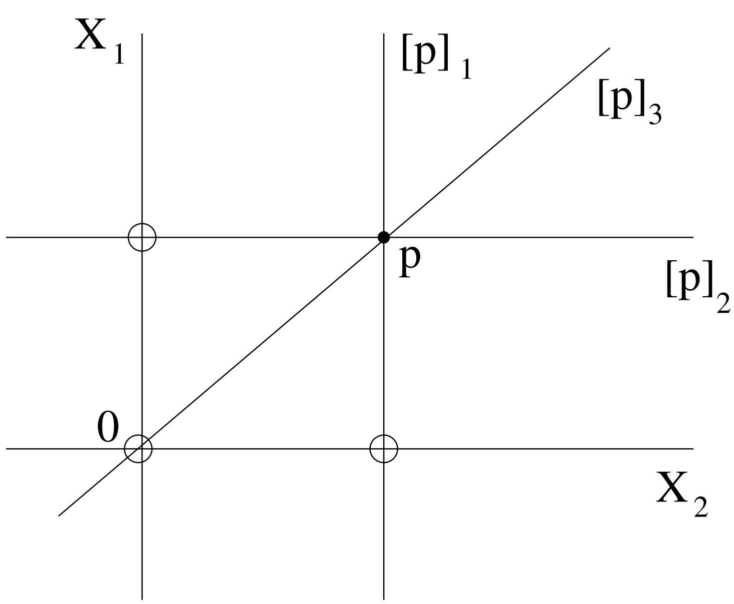

FIGURE 1.3.

By (1.2.6) if $A \in \mathcal{L}_a$ with A not parallel to X_1, X_2 and $0 \notin A$ then $A_{\mathcal{P}} := A \setminus (X_1 \cup X_2)$ is not in $\mathfrak{C}$ but $A_{\mathcal{P}}$ is a maximal joinable set if $(\mathcal{P}_a, \mathcal{L}_a)$ is a *Fano plane* [2]. In fact: let $x := A \cap X_1,\ y := A \cap X_2,\ z := [x]_2 \cap [y]_1$ and let $t \in \mathcal{P} \setminus A_{\mathcal{P}}$. If $t \in \mathcal{P} \setminus [x]_2$ then $(A_{\mathcal{P}} \cup \{t\}) \cap [t]_2 = \{t, A_{\mathcal{P}} \cap [t]_2\}$ consists of two distinct points, if $t \in [x]_2 \setminus \{z, x\}$ then $(A_{\mathcal{P}} \cup \{t\}) \cap [t]_1 = \{t, A_{\mathcal{P}} \cap [t]_1\}$ are two distinct points and if $t = z$ then $(A_{\mathcal{P}} \cup \{t\}) \cap [t]_3 = \{t, A_{\mathcal{P}} \cap [t]_3\}$ with $A_{\mathcal{P}} \cap [t]_3 \neq \emptyset$ and $t \neq A_{\mathcal{P}} \cap [t]_3$ if $(\mathcal{P}_a, \mathcal{L}_a)$ is a Fano plane. If $(\mathcal{P}_a, \mathcal{L}_a)$ is an *anti-Fano plane* then $A_{\mathcal{P}} \cap [z]_3 = \emptyset$ and $A_{\mathcal{P}} \cup \{z\} \in \mathfrak{C}$. Thus (1) of the following theorem is proved:

(1.2.11) *Let $(\mathcal{P}_a, \mathcal{L}_a)$ be an affine anti-Fano plane, let $\{0, e_1, e_2\} \in \binom{\mathcal{P}_a}{3}$ non collinear, $X_i := \overline{0, e_i},\ i \in \{1,2\}$ and let $Net_a(0; e_1, e_2) := (\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3)$ be the 3−net defined by $\mathcal{P} := \mathcal{P}_a \setminus (X_1 \cup X_2),\ \mathfrak{G}_i := \{G \cap \mathcal{P} \mid G \in \mathcal{L}_a \setminus \{X_i\} : G \parallel X_i\}$ for $i \in \{1,2\}$ and $\mathfrak{G}_3 := \{G \cap \mathcal{P} \mid G \in \mathcal{L}_a \setminus \{X_1, X_2\} : 0 \in G\}$. For any $p \in \mathcal{P},\ i \in \{1,2\}$ let $p_1 := X_1 \cap (p \parallel X_2),\ p_2 := X_2 \cap (p \parallel X_1)$ and $[p]_4 := \{p\} \cup \overline{p_1, p_2} \setminus \{p_1, p_2\}$ and let $[\mathcal{P}]_4 := \{[p]_4 \mid p \in \mathcal{P}\}$. Then:*

(1) *$[\mathcal{P}]_4$ is contained in the set $\mathfrak{C}$ of all chains of the $Net_a(0; e_1, e_2)$.*

(2) $\forall\ p, q \in \mathcal{P}, p \neq q : [p]_4 \cap [q]_4 = \emptyset \Leftrightarrow\ \exists\ i \in \{1,2,3\} : [p]_i = [q]_i$.

(3) *For $p, q \in \mathcal{P}$ with $[p]_i \neq [q]_i$ for all $i \in \{1,2,3\} : 1 \leq |[p]_4 \cap [q]_4| \leq 3$.*

(4) *If $(\mathcal{P}_a, \mathcal{L}_a)$ is a Pappus plane (cf. [23], pag.20) and if $(K, +, \cdot)$ is the commutative coordinate field of characteristic 2 such that $0 = (0,0), e_1 = (1,0)$ and $e_2 = (0,1)$ then each conic with nucleus 0 passing through the infinite points of X_1 and X_2 is given by an equation of the form $x_1 \cdot x_2 = a$ where $a \in K^* := K \setminus \{0\}$. For $a \in K^*$ let $C_a := \{(x_1, x_2) \in \mathcal{P}_a \mid x_1 \cdot x_2 = a\}$ and let $\mathfrak{G}_4 := \{C_a \mid a \in K^*\}$ then :*

(i) "$C_a \in \mathfrak{C} \Leftrightarrow\ K^{(2)} = K^*$" *and* "$C_a \cap C_b = \emptyset \Leftrightarrow\ a \neq b$".

(ii) *If $K^{(2)} = K^*$ then $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3, \mathfrak{G}_4)$ is a 4−net.*

(iii) $\mathfrak{G}_4 = [\mathcal{P}]_4\ \Leftrightarrow\ |K| = 4\ \Leftrightarrow\ \forall\ a, b \in K^* : [(a,b)]_4 = C_{a \cdot b}$.

(iv) *If $|K| = 4$ then $\mathfrak{C} = \mathfrak{G}_4 = [\mathcal{P}]_4$ and $(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3 \cup \mathfrak{G}_4)$ is the affine plane of*

[2] A quadrupel $(a,b,c,d) \in \binom{\mathcal{P}_a}{4}$ consisting of non collinear points are called a *parallelogram* if $\overline{a,b} \parallel \overline{c,d}$ and $\overline{b,c} \parallel \overline{d,a}$. Let $\square$ be the set of all parallelograms of $(\mathcal{P}_a, \mathcal{L}_a)$. Then the affine plane $(\mathcal{P}_a, \mathcal{L}_a)$ is called *Fano plane* if $\forall\ (a,b,c,d) \in \square : \overline{a,c} \cap \overline{b,d} \neq \emptyset$, *anti-Fano plane* if $\forall\ (a,b,c,d) \in \square : \overline{a,c} \cap \overline{b,d} = \emptyset$.

order 3.

Proof. (2) Since $(\mathcal{P}_a, \mathcal{L}_a)$ is an anti-Fano plane we have: $\forall\ p \in \mathcal{P} : \overline{0,p} \parallel \overline{p_1,p_2}$, with $\overline{0,p} \neq \overline{p_1,p_2}$. Now let $\{p,q\} \in \binom{\mathcal{P}}{2}$ then $\overline{p_1,p_2} \neq \overline{q_1,q_2}$. If $\overline{p_1,p_2} \parallel \overline{q_1,q_2}$ then $\overline{p_1,p_2} \cap \overline{q_1,q_2} = \emptyset$ and $\overline{0,p} \parallel \overline{0,q}$ hence $\overline{0,p} = \overline{0,q}$, i.e. $[p]_3 = [q]_3$ and $\overline{0,p} \cap \overline{p_1,p_2} = \emptyset$, $\overline{0,p} \cap \overline{q_1,q_2} = \emptyset$. This tell us $[p]_4 \cap [q]_4 = \emptyset$. If $\overline{p_1,p_2} \not\parallel \overline{q_1,q_2}$ let $\{r\} := \overline{p_1,p_2}\ \cap \overline{q_1,q_2}$. If $r \in X_i$ for one $i \in \{1,2\}$ then $r = p_i = q_i$ and so $\overline{p_1,p_2} \cap \overline{q_1,q_2} \cap \mathcal{P} = \emptyset$, $[p]_j = [q]_j$ where $\{i,j\} = \{1,2\}$, $p \notin \overline{q_1,q_2}$ and $q \notin \overline{p_1,p_2}$ implying $[p]_4 \cap [q]_4 = \emptyset$. If $r \notin X_1 \cup X_2$ hence $r \in \mathcal{P}$ then $r \in [p]_4 \cap [q]_4$, $[p]_1 \neq [q]_1$, $[p]_2 \neq [q]_2$ and $[p]_3 \neq [q]_3$ since $\overline{0,p} \parallel \overline{p_1,p_2} \not\parallel \overline{q_1,q_2} \parallel \overline{0,q}$.

(3) By the proof of (2), $[p]_4 \cap [q]_4 = ((\overline{p_1,p_2} \setminus \{p_1,p_2\}) \cup \{p\}) \cap ((\overline{q_1,q_2} \setminus \{q_1,q_2\}) \cup \{q\}) =$

$$\begin{cases} \{r\} & \text{if } \ p \notin \overline{q_1,q_2} \ \text{ and } \ q \notin \overline{p_1,p_2} \\ \{r,p\} & \text{if } \ p \in \overline{q_1,q_2} \ \text{ and } \ q \notin \overline{p_1,p_2} \\ \{r,q\} & \text{if } \ p \notin \overline{q_1,q_2} \ \text{ and } \ q \in \overline{p_1,p_2} \\ \{r,p,q\} & \text{if } \ p \in \overline{q_1,q_2} \ \text{ and } \ q \in \overline{p_1,p_2} \end{cases}.$$

(4) (i) The statement " $K^{(2)} = K^* \Rightarrow\ C_a \in \mathfrak{C}$ " is a consequence of (1.2.8) and (1.2.10). Let $b \in K^*$ be given. Then $(b,a) \in \mathcal{P}$ and by definition of $\mathfrak{G}_3$ the set $G_3 := \{(x_1,x_2) \in \mathcal{P} \mid b \cdot x_2 = a \cdot x_1\}$ is the generator of type 3 through the point (b,a). If $C_a \in \mathfrak{C}$ then by **(N3)** $|C_a \cap G_3| = 1$, i.e. $x_1^2 = a \cdot b \cdot a^{-1} = b$ has a solution and this implies $K^{(2)} = K^*$.

(ii) By (i), $\mathfrak{G}_4 \subset \mathfrak{C}$, i.e. each generator of $\mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3$ intersects each element of $\mathfrak{G}_4$ in exactly one point, and again by (i), any two different elements of $\mathfrak{G}_4$ have an empty intersection. If $p = (p_1,p_2) \in \mathcal{P}$ hence $p_1,p_2 \neq 0$ then $p \in C_{p_1 \cdot p_2} \in \mathfrak{G}_4$. Therefore $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3, \mathfrak{G}_4)$ is a 4−net.

(iii) Let $(a,b) \in \mathcal{P}$. Since $[(a,b)]_4 := \{(a,b)\} \cup \overline{(a,0),(0,b)} \setminus \{(a,0),(0,b)\}$, $\overline{(a,0),(0,b)} \setminus \{(a,0),(0,b)\} := \{(x, b \cdot a^{-1} \cdot (x+a)) \mid x \in K^* \setminus \{a\}\}$, $C_{a \cdot b} := \{(x_1,x_2) \in \mathcal{P} \mid x_1 \cdot x_2 = a \cdot b\}$, $(a,b) \in C_{a\cdot b}$ and for $x \in K^* \setminus \{a\}$,

$$(x, b \cdot a^{-1} \cdot (x+a)) \in C_{a\cdot b} \ \Leftrightarrow\ x \cdot [b \cdot a^{-1} \cdot (x+a)] = a \cdot b \ \Leftrightarrow\ (b \cdot x^2 + b \cdot a \cdot x) \cdot a^{-1} = a \cdot b \ \Leftrightarrow$$

$$x^2 + a \cdot x + a^2 = 0 \ \Leftrightarrow\ \left(\frac{x}{a}\right)^2 + \frac{x}{a} + 1 = 0 \ \overset{y := x \cdot a^{-1} \neq 0,1}{\Leftrightarrow}\ y^2 + y + 1 = 0.$$

Therefore we have: $\forall\ a,b \in K^* : [(a,b)]_4 \subset C_{a\cdot b} \ \Leftrightarrow\ \forall\ y \in K^* \setminus \{1\} : y^2+y+1=0 \Leftrightarrow$ $|K|=4$. Since $|C_{a\cdot b}| = |K^*| = |[(a,b)]_4|$ we obtain: $[(a,b)]_4 = C_{a\cdot b} \Leftrightarrow\ |K|=4$.
(iv) By (iii), $C_1, C_\alpha, C_{\alpha+1} \in \mathfrak{C}$. Since $|\mathfrak{C}(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{G}_3) \dot\cup \mathfrak{G}_3| \le |\mathfrak{C}(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2)|$ implies $|\mathfrak{C}(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{G}_3)| \le |\mathfrak{C}(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2)| - |\mathfrak{G}_3| \overset{(1.3.3)}{=} 3!-3=3$ we have $\mathfrak{C} = \mathfrak{G}_4$.
By (ii), $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{G}_3,\mathfrak{G}_4)$ is a 4−net. Let $\mathbb{L} := \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3 \cup \mathfrak{G}_4$; we prove that the two axioms for an affine plane:
(A1) $\forall\ \{p,q\} \in \binom{\mathcal{P}}{2},\ \exists_1\ L \in \mathbb{L} : p,q \in L,$
(A2) $\forall\ p \in \mathcal{P}, \forall\ L_1 \in \mathbb{L}$, with $p \notin L_1,\ \exists_1\ L_2 \in \mathbb{L} \setminus \{L_1\}$ with $p \in L_2$ and $L_1 \cap L_2 = \emptyset$,
are satisfied. Let $\{p,q\} \in \binom{\mathcal{P}}{2}$. Then we have to consider the following two cases:
case a): $\exists\ i \in \{1,2,3\}$ such that $[p]_i = [q]_i$. Then $[p]_i \in \mathbb{L}$ and $p,q \in [p]_i$. By **(N2)** $[p]_i$ is the only one.
case b): $\forall\ i \in \{1,2,3\} : [p]_i \neq [q]_i$. Then by (iii) $[p]_4 = [q]_4$ and so $[p]_4 \in \mathbb{L}$ with $p,q \in [p]_4$. By **(N2)** $[p]_4$ is the only one. Therefore **(A1)** is satisfied.
Now let $[x]_i \in \mathbb{L}$ with $i \in \{1,2,3,4\}$ and $p \in \mathcal{P} \setminus [x]_i$. Then by **(N1)** $\exists_1\ [p]_i \in \mathbb{L}$ and by **(N2)** $[p]_i \cap [x]_i = \emptyset$, i.e. **(A2)** is satisfied. Since $\forall\ L \in \mathbb{L} : |L| = 3$ we obtain the thesis.□

2) Let $(\mathbb{K},+,\cdot,\le)$ be an ordered field, let $\mathbb{K}^+ := \{x \in K \mid x > 0\}$ and let
$\mathcal{P} := \mathbb{K}^+ \times \mathbb{K}^+$,
$\mathfrak{G}_1 := \{(x,\mathbb{K}^+) \mid x \in \mathbb{K}^+\}$,
$\mathfrak{G}_2 := \{(\mathbb{K}^+,y) \mid y \in \mathbb{K}^+\}$,
$\mathfrak{G}_3 := \{< m > := \{(x,mx) \mid x \in \mathbb{K}^+\} \mid m \in \mathbb{K}^+\}$;
then $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{G}_3)$ is a 3−net. In fact: let $(x_0,y_0) \in \mathcal{P}$. Then $(x_0,\mathbb{K}^+), (\mathbb{K}^+,y_0)$ and $< y_0\, x_0^{-1} >$ are the three uniquely determined generators through the point (x_0,y_0), i.e. **(N1)** is satisfied. Moreover if $x,y,m > 0$ then $(x,\mathbb{K}^+) \cap (\mathbb{K}^+,y) = \{(x,y)\} \in \mathcal{P}$, $< m > \cap (x,\mathbb{K}^+) = \{(x,mx)\} \in \mathcal{P}$ and $< m > \cap (\mathbb{K}^+,y) = \{(m^{-1}y,y)\} \in \mathcal{P}$, i.e. **(N2)** is satisfied.

Now let $M := \{(x,x^{-1}) \mid x \in \mathbb{K}^+\}$ with $\mathbb{K} = \mathbb{R}$. Then M is a chain (in fact if $x,y,m > 0$ then $(x,\mathbb{R}^+) \cap M = \{(x,x^{-1})\} \in \mathcal{P}$, $(\mathbb{R}^+,y) \cap M = \{(y^{-1},y)\} \in \mathcal{P}$ and $< m > \cap M = \{(\sqrt{m}^{-1},\sqrt{m})\} \in \mathcal{P}$) and by (1.2.5) for $a,b \in M$ with $a \neq b$, $M_{a,b}$

is a maximal joinable set but not i-maximal for all $i \in \{1, 2, 3\}$. In the case where $a = (2, \frac{1}{2}), b = (\frac{1}{2}, 2)$ we obtain the following figure:

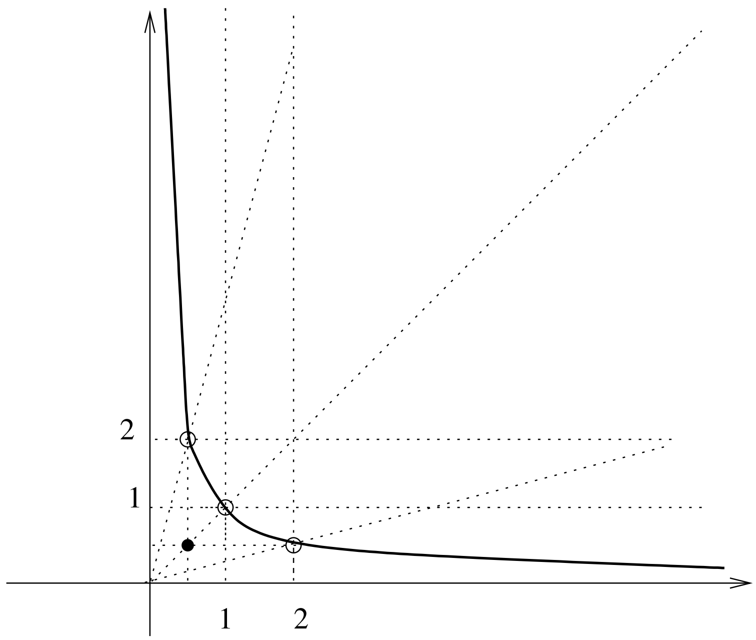

FIGURE 1.4.

We can obtain 3−nets with $\mathfrak{C} \neq \emptyset$ also if we consider the weaker structure $(\mathbb{K}, \mathbb{K}_+)$ of a *halfordered field*. Let $(\mathbb{K}, +, \cdot)$ be a field and let $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$. A subgroup $\mathbb{K}_+$ of index 2 of $\mathbb{K}^*$ is called a *halforder* (and $(\mathbb{K}, \mathbb{K}_+)$ a *halfordered field*).

Then if $(\mathbb{K}, \mathbb{K}_+)$ is a halfordered field, the structure $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3)$ defined by:

$\mathcal{P} := \mathbb{K}_+ \times \mathbb{K}_+$,

$\mathfrak{G}_1 := \{(x, \mathbb{K}_+) \mid x \in \mathbb{K}_+\}$,

$\mathfrak{G}_2 := \{(\mathbb{K}_+, y) \mid y \in \mathbb{K}_+\}$,

$\mathfrak{G}_3 := \{< m >:= \{(x, mx) \mid x \in \mathbb{K}_+\} \mid m \in \mathbb{K}_+\}$,

is a 3−net. We recall the following properties about halforders of a field:

(i) If $\mathbb{K}^{(2)} := \{\lambda^2 \mid \lambda \in \mathbb{K}^*\} \neq K^*$ then $(\mathbb{K}^{(2)}, \cdot) \leq (\mathbb{K}_+, \cdot)$.

(ii) If $[\mathbb{K}^* : \mathbb{K}^{(2)}] = 2$ then $\mathbb{K}^{(2)}$ is the only not trivial halforder of $(\mathbb{K}, +, \cdot)$.

(iii) If $|\mathbb{K}| \in \mathbb{N}$ then $\begin{cases} [\mathbb{K}^* : \mathbb{K}^{(2)}] = 2 & \Leftrightarrow \quad 2 \neq 0 \\ \mathbb{K}^* = \mathbb{K}^{(2)} & \Leftrightarrow \quad 2 = 0 \end{cases}$.

Therefore if we consider the 3−net constructed in the case of a halfordered field $(\mathbb{K}, \mathbb{K}_+)$

we obtain: $M := \{(x, x^{-1}) \mid x \in \mathbb{K}_+\}$ is a chain $\Leftrightarrow [\mathbb{K}^* : \mathbb{K}^{(2)}] = 2$ and $(-1 \notin \mathbb{K}^{(2)}$ or $-1 = 1)$. In fact if $x, m \in \mathbb{K}_+$ then always $(x, \mathbb{K}_+) \cap M = \{(x, x^{-1})\} \in \mathcal{P}$, $(\mathbb{K}_+, x) \cap M = \{(x^{-1}, x)\} \in \mathcal{P}$ since with $x \in \mathbb{K}_+$ also $x^{-1} \in \mathbb{K}_+$ but $| < m > \cap M| = 1$ if and only if the two conditions:

(c1) $< m > \cap M \neq \emptyset \Leftrightarrow$ the equation $m = x^{-2}$ has at least a solution $\Leftrightarrow \mathbb{K}_+ \subset \mathbb{K}^{(2)} \Leftrightarrow [\mathbb{K}^* : \mathbb{K}^{(2)}] \leq 2$,

(c2) $| < m > \cap M| \leq 1 \Leftrightarrow (x \in \mathbb{K}_+ \Rightarrow -x \notin \mathbb{K}_+)$ or $-1 = 1 \Leftrightarrow -1 \notin \mathbb{K}^{(2)}$ or $-1 = 1$,

are satisfied, i.e. $\forall\ G \in \mathfrak{G}_3 : |G \cap M| = 1 \Leftrightarrow [\mathbb{K}^* : \mathbb{K}^{(2)}] = 2$ and $(-1 \notin \mathbb{K}^{(2)}$ or $\mathrm{char}\mathbb{K} = 2)$.

Remarks. Let $(\mathbb{K}, +, \cdot)$ be a field. Then we can consider the following cases:

1) $\mathrm{char}\mathbb{K} = 0$. The field $\mathbb{C}$ does not satisfy **(c2)** since $-1 = i^2 \in \mathbb{C}^{(2)}$ and $\mathbb{Q}$ does not satisfy **(c1)** since $2 \notin \mathbb{Q}^{(2)} \,\dot{\cup}\, (-1)\mathbb{Q}^{(2)}$. But the real field $\mathbb{R}$ satisfies both conditions **(c1)** and **(c2)** since $\mathbb{R}^* = \mathbb{R}^{(2)} \,\dot{\cup}\, (-1)\mathbb{R}^{(2)}$.

2) $\mathrm{char}\mathbb{K} = 2$. Then $-1 = 1$ and so **(c2)** is satisfied. If $|\mathbb{K}| = q \in \mathbb{N}$ then $\mathbb{K}^* = \mathbb{K}^{(2)}$ and so for all finite fields $\mathbb{K}$ of char 2 the set M, above defined, is a chain. If $|\mathbb{K}| = \infty$ then we can consider $\mathbb{K} := Q(\mathbb{Z}_2[x])$ the quotient field of the polynomial ring $\mathbb{Z}_2[x]$ over $\mathbb{Z}_2$. For this field the condition **(c1)** is not satisfied since $x \notin \mathbb{K}^{(2)}$ and $x^3 + 1 \notin \mathbb{K}^{(2)} \,\dot{\cup}\, (x)\mathbb{K}^{(2)}$.

3) $\mathrm{char}\mathbb{K} = p$, with p an odd prime integer. Then $-1 \neq 1$. If $|\mathbb{K}| = p^n$, with $n \in \mathbb{N}$ then **(c1)** is always satisfied and **(c2)** is satisfied, i.e. $-1 \notin \mathbb{K}^{(2)}$, if and only if $p^n \equiv 3 \bmod 4$ (cf. [9],Th.82). If $|\mathbb{K}| = \infty$ then we can consider the quotient field $\mathbb{K} := Q(\mathbb{Z}_p[x])$ and as in the case of characteristic 2 the condition **(c1)** is not valid.

1.3 CHAINS AND BIJECTIONS

In this section let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ be a 2−net, let $\Box := \Box_{12}$ and as before let $\mathfrak{M}$, $\mathfrak{C}$ be the set of all joinable subsets, chains of $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ respectively.

(1.3.1) *For* $f \in M(\mathfrak{G}_1, \mathfrak{G}_2)$ *the subset* $C(f) := \{X \cap f(X) \mid X \in \mathfrak{G}_1\}$ of $\mathcal{P}$ *has the following properties*:
(1) $\forall\ G \in \mathfrak{G}_1 : \ |G \cap C(f)| = 1$.
(2) f *is injective* $\Leftrightarrow \forall\ H \in \mathfrak{G}_2 : \ |H \cap C(f)| \leq 1 \ \Leftrightarrow \ C(f) \in \mathfrak{M} \Leftrightarrow \ C(f) \in \mathfrak{M}_1$.
(3) f *is surjective* $\Leftrightarrow \forall\ H \in \mathfrak{G}_2 : H \Box C(f) = \mathcal{P}$.
(4) f *is bijective* $\Leftrightarrow C(f) \in \mathfrak{C}$.

Proof. (1) $G \cap C(f) = \{G \cap f(G)\} \Rightarrow |G \cap C(f)| = 1$.
(2) Since $H \cap C(f) = \{X \cap f(X) \mid f(X) = H\}$ we have the first equivalence; the second and the third equivalence follow by (1).
(3) Let $h \in H$ and $x := X \cap f(X) \in C(f)$. Then $[x]_2 = f(X)$, hence $h \Box x = [h]_1 \cap [x]_2 = [h]_1 \cap f(X)$. Consequently $H \Box C(f) = \{[h]_1 \cap f(X) \mid h \in H, X \in \mathfrak{G}_1\}$. Now let $p \in \mathcal{P}$. Then $h' := [p]_1 \cap H$ exists, $[h']_1 = [p]_1$ and so $p = [h']_1 \cap [p]_2$. Therefore $H \Box C(f) = \mathcal{P}$ if and only if for each $Y \in \mathfrak{G}_2$ there exists a $X \in \mathfrak{G}_1$ with $f(X) = Y$.
(4) Let $X \in \mathfrak{G}_1$, $Y \in \mathfrak{G}_2$. If $C(f) \in \mathfrak{M}_1$ then $|X \cap C(f)| = 1$ and $|Y \cap C(f)| \leq 1$. If $\mathcal{P} = H \Box C(f)$ then by (3) there is a $Z \in \mathfrak{G}_1$ with $f(Z) = Y$, hence $\emptyset \neq Y \cap Z = Z \cap f(Z) \in Y \cap C(f)$. Therefore $C(f) \in \mathfrak{M}_1$ and $\mathcal{P} = H \Box C(f)$ imply $C(f) \in \mathfrak{C}$, and so (4) is a consequence of (2) and (3).$\Box$

(1.3.2) *Let* M *be a subset of* $\mathcal{P}$ *with the property*:

$$\forall\ G \in \mathfrak{G}_1 : \ |G \cap M| = 1.$$

Then

$$\sigma(M) : \begin{cases} \mathfrak{G}_1 & \rightarrow & \mathfrak{G}_2 \\ X & \mapsto & [X \cap M]_2 \end{cases}$$

is a map with $M = C(\sigma(M))$.

Proof. By our assumption we have $M = \cup\{G \cap M \mid G \in \mathfrak{G}_1\}$ and by definition: $C(\sigma(M)) := \{X \cap (\sigma(M))(X) \mid X \in \mathfrak{G}_1\} = \{X \cap [X \cap M]_2 \mid X \in \mathfrak{G}_1\} = \{X \cap M \mid X \in \mathfrak{G}_1\} \overset{|X \cap M|=1}{=} M$.□

From now on we assume $\mathfrak{C} \neq \emptyset$. Then each chain $E \in \mathfrak{C}$ can serve as a "frame of reference for a coordinate system"; the map:

$$\square : \begin{cases} E \times E & \rightarrow & \mathcal{P} = E \square E \\ (x, y) & \mapsto & x \square y \end{cases}$$

is a bijection. If $p \in \mathcal{P}$, then $x := [p]_1 \cap E$ and $y := [p]_2 \cap E$ are the "coordinates" of p, $p = x \square y$, and $[p]_1 = x \square E = p \square E$ and $[p]_2 = E \square y = E \square p$ are the two generators through p. On the other hand:

each non empty set E determines a $2-$net $\nu(E)$ such that E can be considered as a chain of $\nu(E)$: let $\mathcal{P} := E \times E$, $\mathfrak{G}_1 := \{\{x\} \times E \mid x \in E\}, \mathfrak{G}_2 := \{E \times \{y\} \mid y \in E\}$; then $\nu(E) := (\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ is a $2-net$ and if we identify each $x \in E$ with (x, x) hence E with $\{(x, x) \mid x \in E\}$ then E is a chain. The point $p = (x, y) \in \mathcal{P}$ can be represented in the form $p = (x, x) \square (y, y) = x \square y$.

(1.3.3) *The set* $\mathfrak{C}$ *of all chains of* $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ *is in a one-to-one correspondence to the symmetric group SymE by the following maps:*

$$C : \begin{cases} SymE & \rightarrow & \mathfrak{C} \\ \gamma & \mapsto & C(\gamma) := \{(x, \gamma(x)) \mid x \in E\} = \{x \square \gamma(x) \mid x \in E\} \end{cases}$$

and

$$\sigma : \begin{cases} \mathfrak{C} & \rightarrow & SymE \\ K & \mapsto & \sigma_K : \begin{cases} E & \rightarrow & E \\ x & \mapsto & \pi_2((x, E) \cap K) \end{cases} \end{cases},$$

where $\pi_2 : \mathcal{P} \rightarrow E; x = (x_1, x_2) \mapsto x_2$.

Clearly if $\gamma \in SymE$ then for any $x \in E$ we have:
$(\sigma \circ C(\gamma))(x) = (\sigma \circ \{(t,\gamma(t)) \mid t \in E\})(x) = \pi_2((x,E) \cap \{(t,\gamma(t)) \mid t \in E\}) = \pi_2(x,\gamma(x)) = \gamma(x)$ hence $\sigma \circ C = id_{SymE}$ and if $A \in \mathfrak{C}$ then for any $x \in E$:
$C \circ \sigma_A(x) = C(\pi_2((x,E) \cap A)) = (x, (\pi_2(x,E) \cap A))$, hence $C(\sigma_A) = A$.

Now let $E \in \mathfrak{C}$ be fixed and let

$$\widetilde{E} : \begin{cases} \mathcal{P} = E\Box E & \to & \mathcal{P} \\ x\Box y & \mapsto & y\Box x \end{cases} .$$

Then $\widetilde{E} \in Sym\mathcal{P} \cap J^*$ and $\widetilde{E}_{|\mathfrak{G}_1} \in Bij(\mathfrak{G}_1, \mathfrak{G}_2)$, $\widetilde{E}_{|\mathfrak{G}_2} \in Bij(\mathfrak{G}_2, \mathfrak{G}_1)$.

(1.3.4) *Between the three sets of maps* $M(\mathfrak{G}_1, \mathfrak{G}_2), M(\mathfrak{G}_1)$ *and* $M(E)$ *there is a natural one-to-one correspondence depending on the choice of* $E \in \mathfrak{C}$. *If* $\mathrm{f} \in M(\mathfrak{G}_1, \mathfrak{G}_2)$ *then* $\widetilde{E}_{|\mathfrak{G}_2} \circ f \in M(\mathfrak{G}_1)$ *and* $f_E : E \to E; x \mapsto f([x]_1) \cap E \in M(E)$,
if $g \in M(\mathfrak{G}_1)$ *then* $\widetilde{E}_{|\mathfrak{G}_1} \circ g \in M(\mathfrak{G}_1, \mathfrak{G}_2)$ *and* $g_E : E \to E; x \mapsto g([x]_1) \cap E \in M(E)$,
and if $h \in M(E)$ *then* $h_1 : \mathfrak{G}_1 \to \mathfrak{G}_1; X \mapsto [h(X \cap E)]_1 \in M(\mathfrak{G}_1)$ *and* $h_{12} : \mathfrak{G}_1 \to \mathfrak{G}_2; X \mapsto [h(X \cap E)]_2 \in M(\mathfrak{G}_1, \mathfrak{G}_2)$.

Definition 1.3.1 If $f \in M(\mathfrak{G}_1, \mathfrak{G}_2)$ then $C(f) := \{X \cap f(X) \mid X \in \mathfrak{G}_1\}$,
if $g \in M(\mathfrak{G}_1)$ then $C_E(g) := C(\widetilde{E}_{|\mathfrak{G}_1} \circ g)$
and if $h \in M(E)$ then $C_E(h) := C(h_{12}) = \{x\Box h(x) \mid x \in E\}$ is called the *graph* of f, g and h respectively.

By (1.3.1) holds:
(1.3.5) *If* $f \in M(E)$ *and* $N := C_E(f)$ then:
(1) $\forall\ X \in \mathfrak{G}_1 :\ |X \cap N| = 1$.
(2) f *is injective* $\Leftrightarrow N \in \mathfrak{M} \ \Leftrightarrow\ N \in \mathfrak{M}_1$.
(3) f *is suriective* $\Leftrightarrow \forall\ Y \in \mathfrak{G}_2 :\ Y\Box N = \mathcal{P}$.
(4) $f \in SymE \ \Leftrightarrow\ N \in \mathfrak{C}$.

Proof. (1) Let $x := X \cap E$. Then $X \cap N = \{x \Box f(x)\}$.
(2) Let $x, y \in E$, with $x \neq y$. Then $x \Box f(x) \neq y \Box f(y)$ and $x \Box f(x), y \Box f(y) \in N \cap [x \Box f(x)]_2 \;\Leftrightarrow\; f(x) = f(y)$. Hence together with (1) we have the equivalences.
(3) Since $[N]_2 = E \Box f(E) = \{x \Box f(y) \mid x, y \in E\}$ we have: $[N]_2 = \mathcal{P} \;\Leftrightarrow\; f(E) = E$.
(4) By (2) and (3) follows: $f \in SymE \;\Leftrightarrow\; N$ is $1-$maximal and $[N]_2 = \mathcal{P} \;\Leftrightarrow\; N \in \mathfrak{C}$.□

Definition 1.3.2 For each $M \in \mathfrak{M} \setminus \{\emptyset\}$ let $M_1 := (M \Box E) \cap E, M_2 := (E \Box M) \cap E$ and

$$\sigma'_E(M) : \begin{cases} M_1 & \to & M_2 \\ x & \mapsto & [[x]_1 \cap M]_2 \cap E \end{cases};$$

if $M \in \mathfrak{M}_1$ (i.e. $M_1 = E$) let

$$\sigma(M) : \begin{cases} \mathfrak{G}_1 & \to & \mathfrak{G}_2 \\ X & \mapsto & [X \cap M]_2 \end{cases},$$

and $\sigma_E(M) = \widetilde{E}_{|\mathfrak{G}_2} \circ \sigma(M)$.

(1.3.6) *Let $M \in \mathfrak{M} \setminus \{\emptyset\}$ and $f := \sigma'_E(M)$. Then:*
(1) *f is a bijection and $M = \{x \Box f(x) \mid x \in M_1\}$.*
(2) *$M_1 = E \;\Leftrightarrow\; M$ is $1-$maximal.*
(3) *$M_2 = E \;\Leftrightarrow\; M$ is $2-$maximal.*
(4) *$f \in SymE \;\Leftrightarrow\; M \in \mathfrak{C}$.*
(5) *f can be extended to a permutation $\overline{f} \in SymE \Leftrightarrow$ there is a chain $C \in \mathfrak{C}$ with $M \subset C$.*
(6) *The two equivalent statements of* (5) *are valid if one of the following three conditions is satisfied:*
(i) *$M_1 = M_2$ (then $C := (E \setminus M_1) \cup M$ is such a chain);*
(ii) *$M_1 \cap M_2 = \emptyset$ (then $C := (E \setminus (M_1 \cup M_2)) \cup M \cup \widetilde{E}(M) \in \mathfrak{C}(M)$);*
(iii) *$|M| \in \mathbb{N}$ (if $C := M_2 \setminus M_1 = \{c_1, ..., c_q\}, D := M_1 \setminus M_2 = \{d_1, ..., d_q\}$ and $g : C \to D; c_i \mapsto d_i$ then $\overline{f} : E \to E$ with $\overline{f}(x) := x$ if $x \notin (M_1 \cup M_2), \overline{f}(x) = f(x)$ if $x \in M_1$ and $\overline{f}(x) = g(x)$ if $x \in C$ is such an extension with $\overline{f} \in SymE$).*

Proof. (1) For the map

$$f^{-1}:\begin{cases} M_2 & \to & E \\ b & \mapsto & [[b]_2\cap M]_1\cap E \end{cases},$$

we have: $f^{-1}(M_2)=M_1, f^{-1}\circ f=id_{M_1},\ f\circ f^{-1}=id_{M_2}$, hence f is a bijection.
Since for each $p\in\mathcal{P}$, $[p]_1=p\Box E$ and $[p]_2=E\Box p$ we have $M_i=\{m_i:=[m]_i\cap E\mid m\in M\}$ and so $m=m_1\Box m_2=m_1\Box f(m_1)$. This shows $M=\{x\Box f(x)\mid x\in M_1\}$ and moreover $[M]_i=[M_i]_i$, for $i\in\{1,2\}$.
(2), (3) By definition $M\in\mathfrak{M}$ is i-maximal if $\mathcal{P}=[M]_i=[M_i]_i$. But this is equivalent with $M_i=E$.
(4) By (2) and (3) the statement (4) is a consequence of (1.3.5).
(5) " $\Rightarrow$ " Let $\overline{f}\in SymE$ with $\overline{f}_{|M_1}=f$ and let $C:=\{x\Box\overline{f}(x)\mid x\in E\}$. By (4) $C\in\mathfrak{C}$ and $M=\{x\Box f(x)\mid x\in M_1\}\subset C$.
" $\Leftarrow$ " Now let $C\in\mathfrak{C}$ with $M\subset C$. Then the map

$$\overline{f}:\begin{cases} E & \to & E \\ x & \mapsto & [[x]_1\cap C]_2\cap E \end{cases},$$

is contained in $SymE$ and $\overline{f}_{|M_1}=f$.
(6) (i) If $C:=(E\backslash M_1)\cup M$ we have $C\in\mathfrak{C}$ and $M\subset C$. In fact let $X\in\mathfrak{G}_1$. If $X\cap M=\emptyset$ then $X\cap M_1=\emptyset$ and so $X\cap C=X\cap E$ consists of a single point. If $p:=X\cap M\neq\emptyset$ let $a:=[p]_1\cap E$ and $b:=[p]_2\cap E$. Then $a,b\in M_1=M_2$ and so $X\cap C=\{p\}$. Now let $Y\in\mathfrak{G}_2$. If $Y\cap M=\emptyset$ then $Y\cap M_2=\emptyset$ and so $|Y\cap C|=|Y\cap E|=1$. If $q:=Y\cap M\neq\emptyset$ and if $c:=[q]_1\cap E,\ d:=[q]_2\cap E$ then $c,d\in M_1=M_2$ and so $Y\cap C=\{q\}$.
(ii) Let $C:=(E\setminus(M_1\cup M_2))\cup M\cup\widetilde{E}(M)$, $X\in\mathfrak{G}_1$ and $Y\in\mathfrak{G}_2$. We have: $X\cap M\neq\emptyset \Leftrightarrow X\cap M_1\neq\emptyset$ and $X\cap\widetilde{E}(M)\neq\emptyset \Leftrightarrow X\cap M_2\neq\emptyset$. Since $M_1\cap M_2=\emptyset$ this shows $|X\cap C|=1$. In the same way $Y\cap M\neq\emptyset \Leftrightarrow Y\cap M_2\neq\emptyset$ and $Y\cap\widetilde{E}(M)\neq\emptyset \Leftrightarrow Y\cap M_1\neq\emptyset$, hence $|Y\cap C|=1$.
(iii) Let $E_0:=E\setminus(M_1\cup M_2)$. Then $E=E_0\dot\cup M_1\dot\cup C=E_0\dot\cup M_2\dot\cup D$ and by definition of $\overline{f}$, $\overline{f}_{|E_0}=id_{E_0}$, $\overline{f}_{|M_1}=f$ with $\overline{f}(M_1)=M_2$ and $\overline{f}_{|C}=g$ with $\overline{f}(C)=D$. Therefore these restrictions are bijections and so

$$h(x) = \begin{cases} x & if \quad x \in E_0 \\ f^{-1}(x) & if \quad x \in M_2 \\ g^{-1}(x) & if \quad x \in D \end{cases}$$

is also a bijection with $h \circ \overline{f} = id_E$. Consequently $\overline{f} \in SymE$ with $\overline{f}_{|M_1} = f.\Box$.

Remark. In the finite case we obtain $\mathfrak{M}_{max} = \mathfrak{C}$ by (1.3.6)(5) and (6) (iii). In the infinite case we have $\mathfrak{C} \subset \mathfrak{M}_{max}$ but $\mathfrak{C} \neq \mathfrak{M}_{max}$. In fact the graphs $gr(f) := \{x \Box f(x) \mid x \in E\}$ of all injective maps $f : E \to E$ are contained in $\mathfrak{M}_{max}$ but $gr(f)$ is a chain if and only if $f \in SymE$. For example if $E = \mathbb{N}$, $f : \mathbb{N} \to \mathbb{N}; x \mapsto 2x$ is injective but not surjective.

Now we obtain the following supplement of (1.2.3.3):

(1.3.7) *Let $(\mathcal{P}, \mathfrak{G}_i; i \in I)$ be an I-net with $|I| \geq 2$ and $\mathfrak{C} \neq \emptyset$; then $Aut(\mathcal{P}, \mathfrak{C}) = Aut(\mathcal{P}, \mathfrak{G})$.*

Proof. Let $\sigma \in Aut(\mathcal{P}, \mathfrak{C})$. Suppose there are $x, y \in \mathcal{P}$, with $x \neq y$, $i \in I$ with $[x]_i = [y]_i$ and $[\sigma(x)]_j \neq [\sigma(y)]_j$ for all $j \in I$. Then $M := \{\sigma(x), \sigma(y)\} \in \mathfrak{M}$ and by (1.3.6)(5) and (6) (iii) there is a $C \in \mathfrak{C} : M \subset C$. Since $\sigma \in Aut(\mathcal{P}, \mathfrak{C})$, hence $\sigma^{-1} \in Aut(\mathcal{P}, \mathfrak{C})$; so we have $\{x, y\} \subset \sigma^{-1}(C) \in \mathfrak{C} \subset \mathfrak{M}$ which contradicts $[x]_i = [y]_i$. $\Box$

From (1.3.1), (1.3.5) and (1.3.6) follows:

(1.3.8) *Let $f \in M(\mathfrak{G}_1, \mathfrak{G}_2), g \in M(\mathfrak{G}_1)$ and $h \in M(E)$ and $N \in \mathfrak{M}_1$. Then:*

(1) $C(f) \in \mathfrak{M}_1$ *or* $C_E(g) \in \mathfrak{M}_1$ *or* $C_E(h) \in \mathfrak{M}_1 \;\Leftrightarrow\; f$ *or* g *or* h is injective respectively.

(2) $C(f) \in \mathfrak{C}$ *or* $C_E(g) \in \mathfrak{C}$ *or* $C_E(h) \in \mathfrak{C} \;\Leftrightarrow\; f$ *or* g *or* h is bijective respectively.

(3) *$\sigma(N)$, $\sigma_E(N)$ and $\sigma'_E(N)$ are injective maps with $\sigma(N) \in M(\mathfrak{G}_1, \mathfrak{G}_2)$, $\sigma_E(N) \in M(\mathfrak{G}_1)$ and $\sigma'_E(N) \in M(E)$.*

(1.3.9) *Let $E \in \mathfrak{C}$. Then the maps*

$$C_E : \begin{cases} Sym\mathfrak{G}_1 & \to & \mathfrak{C} \\ \gamma & \mapsto & C_E(\gamma) = C(\widetilde{E}_{|\mathfrak{G}_1} \circ \gamma) = \{X \cap \widetilde{E} \circ \gamma(X) \mid X \in \mathfrak{G}_1\} \end{cases}$$

and

$$\sigma_E : \begin{cases} \mathfrak{C} & \to & Sym\mathfrak{G}_1 \\ A & \mapsto & \sigma_E(A) = \widetilde{E}_{|\mathfrak{G}_2} \circ \sigma(A) : \begin{cases} \mathfrak{G}_1 & \to & \mathfrak{G}_1 \\ X & \mapsto & \widetilde{E}([X \cap A]_2) \end{cases} \end{cases}$$

are bijections with $\sigma_E \circ C_E = id_{Sym\mathfrak{G}_1}$ *and* $C_E \circ \sigma_E = id_{\mathfrak{C}}$.

1.4 CHAIN OPERATIONS

In a 2−net $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ with $\mathfrak{C} \neq \emptyset$ we associate to any pair (A, B) of chains the following maps:
the two *perspectivities* for $i \in \{1, 2\}$,
$[A \overset{i}{\to} B] : A \to B; a \mapsto [a]_i \cap B$
and the permutations

$$\widetilde{A, B} : \begin{cases} \mathcal{P} & \to \quad \mathcal{P} \\ x & \mapsto \quad [[x]_1 \cap A]_2 \cap [[x]_2 \cap B]_1 \end{cases},$$

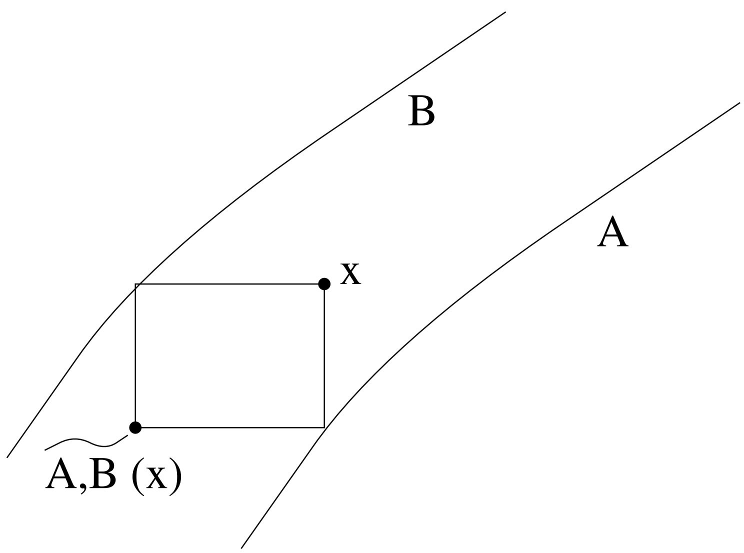

FIGURE 1.5.

$$(\overrightarrow{A, B})_1 : \begin{cases} \mathcal{P} & \to \quad \mathcal{P} \\ x & \mapsto \quad [x]_1 \cap [[[x]_2 \cap A]_1 \cap B]_2 \end{cases}$$

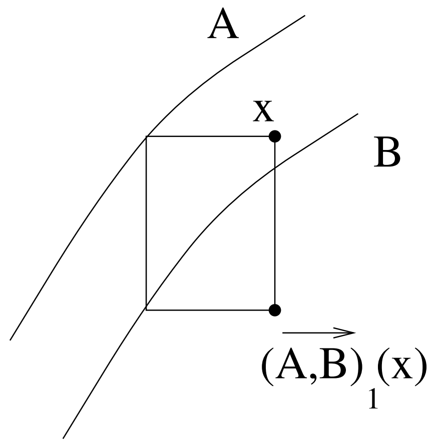

FIGURE 1.6.

and

$$\overrightarrow{(A,B)}_2 : \begin{cases} \mathcal{P} \to \mathcal{P} \\ x \mapsto [x]_2 \cap [[[x]_1 \cap A]_2 \cap B]_1 \end{cases}.$$

We recall that $\mathcal{P}$ has the representation $\mathcal{P} = A\square B = \{a\square b \mid a \in A, b \in B\}$.

(1.4.1) *If* $A, B, C, D \in \mathfrak{C}$ *then:*

(1) $\widetilde{A,B} : \mathcal{P} = A\square B \to \mathcal{P} = B\square A; a\square b \mapsto b\square a.$

(2) $\forall\ a \in A,\ \forall\ b \in B:$

$$\widetilde{A,B}([a]_1) = [a]_2,\ \widetilde{A,B}([b]_2) = [b]_1.$$

(3) $(\widetilde{A,B})^{-1} = \widetilde{B,A},\ Fix\widetilde{A,B} = A \cap B$ *and* $\widetilde{A,B} \in \overline{\Gamma}^-$.

(4) $\widetilde{A,B}_{|A} = [A \xrightarrow{2} B],\ \widetilde{A,B}_{|B} = [B \xrightarrow{1} A].$

(5) $\widetilde{A,B} = \widetilde{C,D} \iff A = C$ *and* $B = D$.

(6) *The map:*

$$\psi : \begin{cases} \mathfrak{C} \times \mathfrak{C} \to \overline{\Gamma}^- \\ (A,B) \mapsto \widetilde{A,B} \end{cases}$$

is a bijection.
(7) $\forall\, \gamma \in \overline{\Gamma} : \gamma \circ \widetilde{A,B} \circ \gamma^{-1} = \widetilde{\gamma(A),\gamma(B)}$.
(8) $\widetilde{A,B}(C) \in \mathfrak{C}$.

Proof. (1) Let $x = a\Box b$, with $a \in A, b \in B$, then $[x]_1 = [a]_1$ and $[x]_2 = [b]_2$ and so $\widetilde{A,B}(x) = [[a]_1 \cap A]_2 \cap [[b]_2 \cap B]_1 = [a]_2 \cap [b]_1 = b\Box a$.
(2) Since $[a]_1 = a\Box B, [a]_2 = B\Box a, [b]_1 = b\Box A$ and $[b]_2 = A\Box b$ we have by (1):

$$\widetilde{A,B}([a]_1) = B\Box a = [a]_2$$

and

$$\widetilde{A,B}([b]_2) = b\Box A = [b]_1.$$

(3) By (1) follows $(\widetilde{A,B})^{-1} = \widetilde{B,A}$ and so $\widetilde{A,B} \in Sym\mathcal{P}$. If $x = a\Box b \in Fix\widetilde{A,B}$ with $a \in A, b \in B$ then by (1) $a\Box b = \widetilde{A,B}(a\Box b) = b\Box a$ hence $a = b$ and so $Fix\widetilde{A,B} = A \cap B$. Now let $X \in \mathfrak{G}_1, Y \in \mathfrak{G}_2, a := X \cap A$ and $b := Y \cap B$. Then $X = [a]_1, Y = [b]_2$ and by (2), $\widetilde{A,B}(X) = \widetilde{A,B}([a]_1) = [a]_2 \in \mathfrak{G}_2$ and $\widetilde{A,B}(Y) = \widetilde{A,B}([b]_2) = [b]_1 \in \mathfrak{G}_1$. This shows $\widetilde{A,B} \in \overline{\Gamma}^-$.
(4) Let $a \in A$ and $b := [a]_2 \cap B = [A \xrightarrow{2} B](a)$ then $a = a\Box b$ and by (1), $\widetilde{A,B}(a) = b$.
(5) Let $x \in \mathcal{P}, a = [x]_1 \cap A, c := [x]_1 \cap C, b := [x]_2 \cap B, d := [x]_2 \cap D$. Then $x = a\Box b = c\Box d$ hence $\widetilde{A,B}(x) = b\Box a$ and $\widetilde{C,D}(x) = d\Box c$. Consequently:
$\widetilde{A,B}(x) = \widetilde{C,D}(x) \;\Leftrightarrow\; b\Box a = d\Box c \;\Leftrightarrow\; [b]_1 = [d]_1$ and $[a]_2 = [c]_2$, i.e. $b = d$ and $a = c$ since $[b]_2 = [x]_2 = [d]_2$ and $[a]_1 = [x]_1 = [c]_1$. This shows (5).
(6) By (3) and (5) ψ is an injection. Now let $\alpha \in \overline{\Gamma}^-$ be given. We set $A := \{X \cap \alpha(X) \mid X \in \mathfrak{G}_1\}$ and $B := \{Y \cap \alpha(Y) \mid Y \in \mathfrak{G}_2\}$. We claim that $A, B \in \mathfrak{C}$. For $G \in \mathfrak{G}_1$ we have by definition $G \cap A = G \cap \alpha(G)$ and since $\alpha(G) \in \mathfrak{G}_2$ (by $\alpha \in \overline{\Gamma}^-$) the set $G \cap \alpha(G)$ is a single point, hence $|G \cap A| = 1$. If $H \in \mathfrak{G}_2$ then $H \cap A = \alpha^{-1}(H) \cap H$ is also a single point. Thus $A \in \mathfrak{C}$ and in the same way $B \in \mathfrak{C}$. By definition of A (B) and by (2) we have for $a \in A$ ($b \in B$): $\alpha([a]_1) = [a]_2 = \widetilde{A,B}([a]_1)$ and $\alpha([b]_2) = [b]_1 = \widetilde{A,B}([b]_2)$, hence, if $x = a\Box b = [a]_1 \cap [b]_2 \in \mathcal{P}$ with $a \in A, b \in B$ then $\widetilde{A,B}(x) = \widetilde{A,B}([a]_1) \cap \widetilde{A,B}([b]_2) = \alpha([a]_1) \cap \alpha([b]_2) = \alpha(a\Box b) = \alpha(x)$.
(7) For all $x \in \mathcal{P}$ we have:

$\gamma \circ \widetilde{A,B} \circ \gamma^{-1}(x) = \gamma \circ \widetilde{A,B}(\gamma^{-1}(x)) = \gamma([[\gamma^{-1}(x)]_1 \cap A]_2 \cap [[\gamma^{-1}(x)]_2 \cap B]_1) = [[x]_1 \cap \gamma(A)]_2 \cap [[x]_2 \cap \gamma(B)]_1 = \widetilde{\gamma(A),\gamma(B)}(x)$, where $\gamma(A), \gamma(B) \in \mathfrak{C}$ by (1.2.3.3).

(8) If $X \in \mathfrak{G}_1 \cup \mathfrak{G}_2$ then $\widetilde{B,A}(X) \in \mathfrak{G}_1 \cup \mathfrak{G}_2$ by (2) hence $1 = |\widetilde{B,A}(X) \cap C| \overset{(3)}{=} |X \cap \widetilde{A,B}(C)|$ and so $\widetilde{A,B}(C) \in \mathfrak{C}$. □

For each $A \in \mathfrak{C}$ let $\widetilde{A} := \widetilde{A,A}$ and $\widetilde{\mathfrak{C}} := \{\widetilde{C} \mid C \in \mathfrak{C}\}$. From (1.4.1) we obtain:

(1.4.2) *The map:*

$$\psi_0 : \begin{cases} \mathfrak{C} & \rightarrow \quad \overline{\Gamma}^- \\ C & \mapsto \quad \psi(C,C) = \widetilde{C} \end{cases}$$

is an injection which maps $\mathfrak{C}$ *onto the set of all involutions of* $\overline{\Gamma}^-$ *and moreover we have for all* $A, B, C \in \mathfrak{C}$ *:*

(1) $Fix\widetilde{A} = A$.

(2) $\widetilde{\widetilde{A}(B)} = \widetilde{A} \circ \widetilde{B} \circ \widetilde{A}$.

(3) $\widetilde{A} \circ \widetilde{B} \circ \widetilde{C} \ \in J^* \Leftrightarrow \ \widetilde{A} \circ \widetilde{B} \circ \widetilde{C} \ \in \ \widetilde{\mathfrak{C}}$.

(4) $\widetilde{A}_{|\mathfrak{G}_1} = \widetilde{B}_{|\mathfrak{G}_1} \Leftrightarrow \ A = B \ \Leftrightarrow \ \widetilde{A}_{|\mathfrak{G}_2} = \widetilde{B}_{|\mathfrak{G}_2}$.

(5) $\forall\ X \in \mathfrak{G}_1 : \widetilde{A,B}(X) = \widetilde{A}(X)$ *and* $\forall\ Y \in \mathfrak{G}_2 : \widetilde{A,B}(Y) = \widetilde{B}(Y)$.

Proof. By (1.4.1.6) "ψ" is a bijection from $\mathfrak{C} \times \mathfrak{C}$ onto $\overline{\Gamma}^-$ and by (1.4.1.3), since $(\widetilde{A,B})^{-1} = \widetilde{A,B} \overset{(1.4.1.5)}{\Leftrightarrow} A = B$, "$\psi_0$" is a bijection from $\mathfrak{C}$ onto $\overline{\Gamma}^- \cap J^*$.

(1) $Fix\widetilde{A} \overset{\textbf{def.}}{=} Fix\widetilde{A,A} \overset{(1.4.1.3)}{=} A \cap A = A$.

(2) Since by (1.4.1.3) $\widetilde{A} \in \overline{\Gamma}^-$ we have the thesis by (1.4.1.7) .

(3) Since with $\widetilde{A}, \widetilde{B}, \widetilde{C} \in \overline{\Gamma}^-$ also $\widetilde{A} \circ \widetilde{B} \circ \widetilde{C} \in \overline{\Gamma}^-$ and since "ψ_0" is a bijection from $\mathfrak{C}$ onto $\overline{\Gamma}^- \cap J^*$ we have the thesis.

(4) Let $a \in A$ and $b := B \cap [a]_1$. Then $[a]_1 = [b]_1$ and by (1.4.1.2) we have: $\widetilde{A}([a]_1) = [a]_2 = \widetilde{B}([b]_1) = [b]_2$ and so $a = b$, i.e. $A = B$. In the same way we obtain:

$$\widetilde{A}_{|\mathfrak{G}_2} = \widetilde{B}_{|\mathfrak{G}_2} \ \Leftrightarrow \ A = B.$$

(5) Let $X \in \mathfrak{G}_1,\ Y \in \mathfrak{G}_2,\ a := A \cap X$ and $b := B \cap Y$; then by (1.4.1.2) we have:

$$\widetilde{A,B}(X) = \widetilde{A,B}([a]_1) = [a]_2 = \widetilde{A,A}([a]_1) = \widetilde{A}(X)$$

and

$$\widetilde{A,B}(Y) = \widetilde{A,B}([b]_2) = [b]_1 = \widetilde{B,B}([b]_2) = \widetilde{B}(Y). \ \square$$

By (1.4.1.8) $\tau : \mathfrak{C}^3 \to \mathfrak{C};\ (A,B,C) \mapsto \widetilde{A,C}(B)$ is a *ternary operation* with $\tau(A,A,B) = \tau(B,A,A) = B$ (by (1.4.1.4)) and moreover:

(1.4.3) *Let $E \in \mathfrak{C}$ be fixed. Then:*
(1) $(\mathfrak{C},\cdot)$ *with* $A \cdot B := \tau(A,E,B) = \widetilde{A,B}(E)$ *is a group.*
(2) *The groups* $(SymE, \circ)$ *and* $(\mathfrak{C},\cdot)$ *are isomorphic and graph* $: SymE \to \mathfrak{C}; \gamma \mapsto \widehat{\gamma} := \{x\square\gamma(x) \mid x \in E\}$ *is an isomorphism with graph*$^{-1} : \mathfrak{C} \to SymE; A \mapsto \overline{A} : E \to E; x \mapsto [[x]_1 \cap A]_2 \cap E$.
(3) *For the map* $\sigma_E : \mathfrak{C} \to Sym\mathfrak{G}_1$ *(defined in* (1.3.9)*) we have the representation:*

$$\sigma_E : \begin{cases} \mathfrak{C} & \to & Sym\mathfrak{G}_1 \\ A & \mapsto & \widetilde{E} \circ \widetilde{A}_{|\mathfrak{G}_1} \end{cases} .$$

Proof. (1), (2) It is enough to show that the function graph is a homomorphism. Let $\alpha, \beta \in SymE$. Then: $\widehat{\alpha} \cdot \widehat{\beta} = \{[[x]_1 \cap \widehat{\alpha}]_2 \cap [[x]_2 \cap \widehat{\beta}]_1 \mid x \in E\} = \{\beta^{-1}(x)\square\alpha(x) \mid x \in E\} = \{x\square(\alpha \circ \beta)(x) \mid x \in E\} = \widehat{\alpha \circ \beta}$.
(3) Since $\forall\ X \in \mathfrak{G}_1 : \widetilde{A}(X) = [X \cap A]_2$, we obtain the thesis by (1.3.9).$\square$

(1.4.4) *Let* $A, B, C \in \mathfrak{C},\ D := \widetilde{A,B}(C),\ X \in \mathfrak{G}_1$ *and* $Y \in \mathfrak{G}_2$. *Then:*
(1) $(\overrightarrow{C,B})_1 = \widetilde{B,A} \circ \widetilde{A,C},\ (\overrightarrow{C,B})_2 = \widetilde{A,B} \circ \widetilde{C,A},\ (\overrightarrow{C,B})_1(C) = (\overrightarrow{C,B})_2(C) = B$.
(2) $(\overrightarrow{A,B})_1, (\overrightarrow{A,B})_2 \in \overline{\Gamma}^+$ *and* $(\overrightarrow{A,B})_1(X) = X, (\overrightarrow{A,B})_2(Y) = Y$.
(3) $\widetilde{C,D}(A) = B,\ \widetilde{D,C}(B) = A$.
(4) $(\overrightarrow{C,B})_2 = (\overrightarrow{A,D})_2,\ (\overrightarrow{D,A})_2 = (\overrightarrow{B,C})_2$.

(5) $(\overrightarrow{C,A})_1 = (\overrightarrow{B,D})_1, \ (\overrightarrow{D,B})_1 = (\overrightarrow{A,C})_1$.

Proof. (1) Let $x \in \mathcal{P}$ and $a \in A, \ c \in C$ such that $x := a \square c$ and let $b := [c]_1 \cap B$. Then: $(\overrightarrow{C,B})_1(x) = (\overrightarrow{C,B})_1(a \square c) \overset{\textbf{def.}}{=} [a]_1 \cap [[[c]_2 \cap C]_1 \cap B]_2 = [a]_1 \cap [[c]_1 \cap B]_2 = [a]_1 \cap [b]_2 = a \square b \overset{(1.4.1.1)}{=} \widetilde{B,A}(b \square a) = \widetilde{B,A}(c \square a) \overset{(1.4.1.1)}{=} \widetilde{B,A} \circ \widetilde{A,C}(a \square c) = \widetilde{B,A} \circ \widetilde{A,C}(x)$.

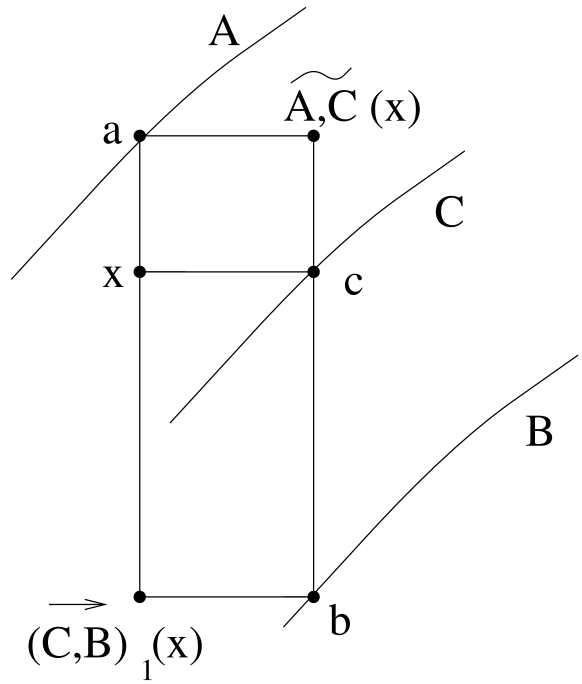

FIGURE 1.7.

In the same way $(\overrightarrow{C,B})_2 = \widetilde{A,B} \circ \widetilde{C,A}$ and by (1.4.1.4) $(\overrightarrow{C,B})_1(C) = \widetilde{B,A} \circ \widetilde{A,C}(C) = \widetilde{B,A}(A) = B$ and $(\overrightarrow{C,B})_2(C) = \widetilde{A,B} \circ \widetilde{C,A}(C) = \widetilde{A,B}(A) = B$.
(2) By (1) and (1.4.1.3) $(\overrightarrow{A,B})_1, (\overrightarrow{A,B})_2 \in \overline{\Gamma}^+$ and for all $x \in X$ we have $(\overrightarrow{A,B})_1 \ (x) \in X$ by definition of $(\overrightarrow{A,B})_1$. Now let $y \in X$ be arbitrary and let $\overline{x} = [y]_1 \cap [[[y]_2 \cap B]_1 \cap A]_2$. Then we have $(\overrightarrow{A,B})_1 \ (\overline{x}) = y$, hence $(\overrightarrow{A,B})_1 \ (X) = X$. In the same way $(\overrightarrow{A,B})_2 \ (Y) = Y$.
(3) Let $a \in A$ and $c := [a]_1 \cap C, \ d := [a]_2 \cap D, \ b := [c]_2 \cap B$. Then $c = a \square b$ and so $\widetilde{A,B}(c) = b \square a \in D$. From this we obtain $d = b \square a$, hence $a = c \square d, \ b = d \square c$ and so $\widetilde{C,D}(a) = b$, i.e. $\widetilde{C,D}(A) = B$.

(4) Let $x \in \mathcal{P}$, $c := [x]_1 \cap C$, $a := [x]_1 \cap A$, $b := [c]_2 \cap B$, $d := [a]_2 \cap D$. Then we have $\overrightarrow{(C,B)}_2\,(x) = b \Box x$ and $\overrightarrow{(A,D)}_2\,(x) = d \Box x$. Since $D = \widetilde{A,B}(C)$ we have $[b]_1 = [d]_1$ and so $\overrightarrow{(C,B)}_2\,(x) = b \Box x = d \Box x = \overrightarrow{(A,D)}_2\,(x)$.

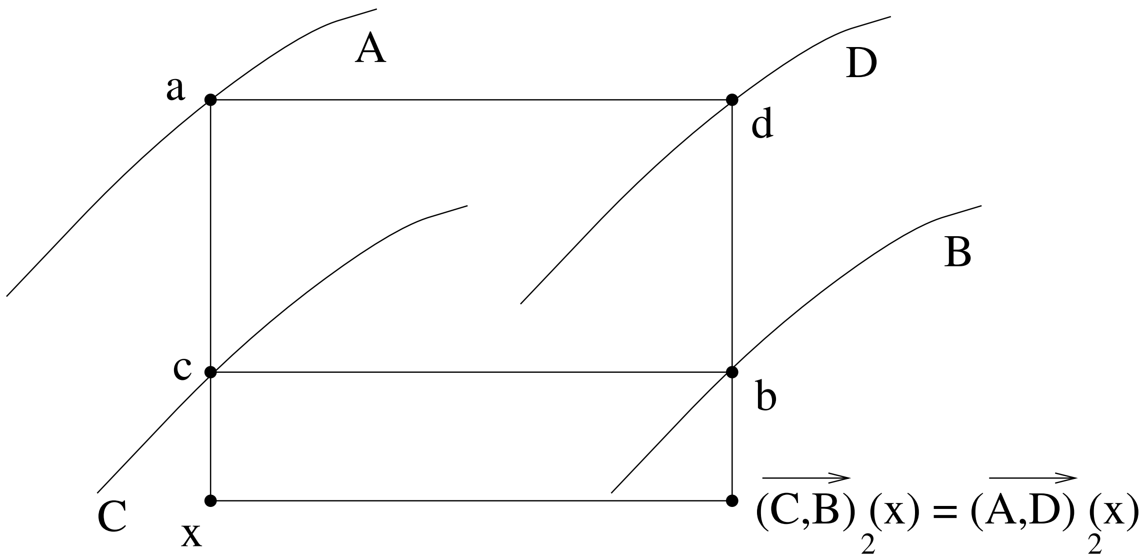

FIGURE 1.8.

(5) Let $x \in \mathcal{P}$, $b := [x]_2 \cap B$, $c := [x]_2 \cap C$, $a := [c]_1 \cap A$, $d := [b]_1 \cap D$. Then we have $\overrightarrow{(C,A)}_1\,(x) = x \Box a$ and $\overrightarrow{(B,D)}_1\,(x) = x \Box d$. Since $D = \widetilde{A,B}(C)$ we have $[a]_2 = [d]_2$ and so $\overrightarrow{(C,A)}_1\,(x) = x \Box a = x \Box d = \overrightarrow{(B,D)}_1\,(x)$.

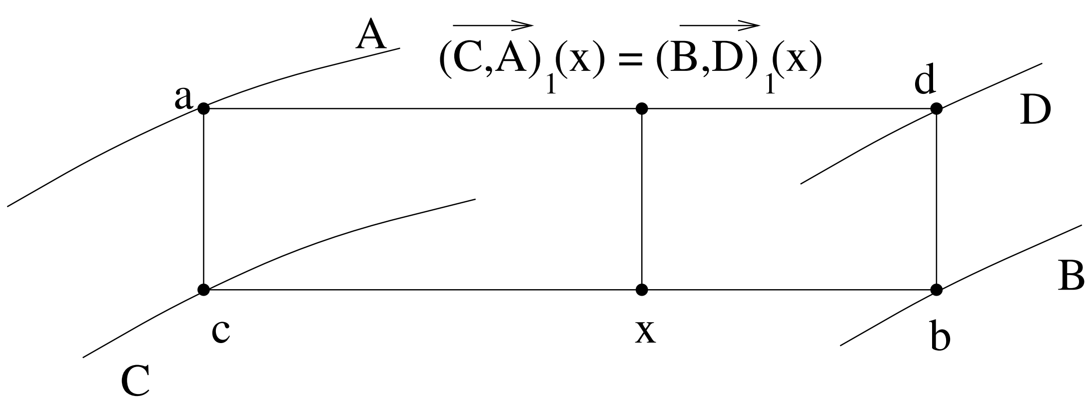

FIGURE 1.9.

$\Box$

CASE $|I| = 3$

In the case $I = \{1, 2, 3\}$ we have a 3−net $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3)$ and if $\mathfrak{C} \neq \emptyset$ we can associate to any chain $C \in \mathfrak{C}$ the following maps:

$$\widetilde{C}_{ij} : \begin{cases} \mathcal{P} & \rightarrow & \mathcal{P} \\ x & \mapsto & [[x]_i \cap C]_j \cap [[x]_j \cap C]_i \end{cases}$$

where $i, j \in I$, with $i \neq j$.

We observe that if $|I| = 2$ then $\widetilde{C}_{ij} \equiv \widetilde{C}$.

For example if $i = 2, j = 3$ we have

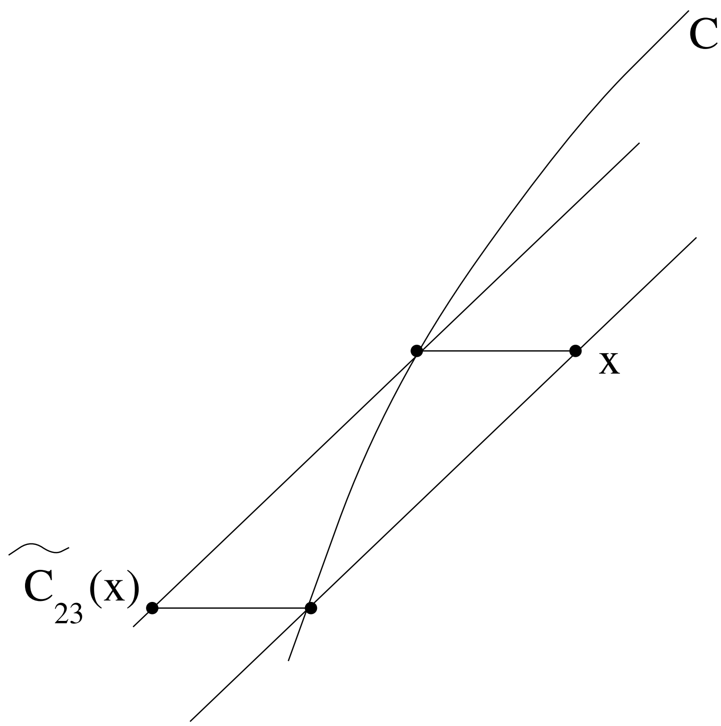

FIGURE 1.10.

We define also $\mathfrak{C}_{ij} := \{C \in 2^{\mathcal{P}} \mid \forall\ X \in \mathfrak{G}_i \cup \mathfrak{G}_j : |C \cap X| = 1\}$.

Let $\gamma \in \overline{\Gamma}_2 \setminus \overline{\Gamma}^+$ (cf. sec. 1.1) then:

1. $\gamma_{|\mathfrak{G}_2} = id_{\mathfrak{G}_2}$;

2. $\forall\ X \in \mathfrak{G}_1 \ \Rightarrow\ \gamma(X) \in \mathfrak{G}_3$;

3. $\forall\ Y \in \mathfrak{G}_3 \ \Rightarrow\ \gamma(Y) \in \mathfrak{G}_1$.

Moreover if $A \in \mathfrak{G}_1$ and $B := \gamma(A) \in \mathfrak{G}_3$ then $\forall\ X \in \mathfrak{G}_1 : \gamma(X) = [[X \cap B]_2 \cap A]_3$ and $Fix(\gamma) \in \mathfrak{C}_{13}$. In fact let $X \in \mathfrak{G}_1$; then $\gamma(X) \in \mathfrak{G}_3$ and $\emptyset \neq X \cap \gamma(X) \in Fix(\gamma)$ implies $|X \cap Fix(\gamma)| = 1$. In the same way we have $|Y \cap Fix(\gamma)| = 1,\ \forall\ Y \in \mathfrak{G}_3$.

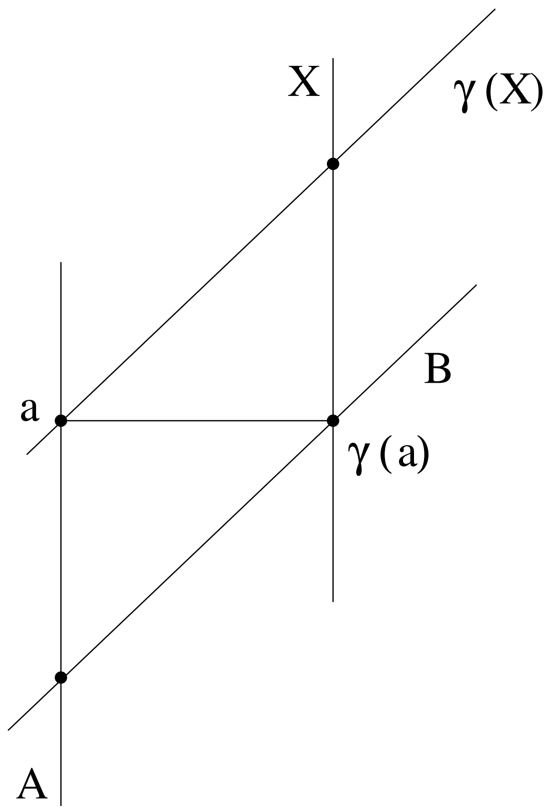

FIGURE 1.11.

Remark. If $I = \{1,2,3\}$ and $\{i,j,h\} \in \binom{I}{3}$ then $\forall\ x \in \mathcal{P}, \forall\ C \in \mathfrak{C}$:

$$\widetilde{C}_{ji} \circ \widetilde{C}_{hi}([x]_h) = \widetilde{C}_{ij} \circ \widetilde{C}_{ih}([x]_h) = [[x]_h \cap C]_j.$$

1.5 CLOSURE OPERATIONS

In this section let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ be a 2−net with $\mathfrak{C} \neq \emptyset$. By (1.4.1) if $A, B \in \mathfrak{C}$ then $\widetilde{A,B}$ and $\widetilde{A}$ are automorphisms of $(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2)$ contained in the coset $\overline{\Gamma}^-$ of $\overline{\Gamma}^+$.

Definition 1.5.1 A subset $\mathfrak{S} \subset \mathfrak{C}$ is called *symmetric* if $\forall A, B \in \mathfrak{S} : \tau(A, B, A) = \widetilde{A}(B) \in \mathfrak{S}$; *double symmetric* if $\forall A, B, C \in \mathfrak{S} : \tau(A, B, C) = \widetilde{A,C}(B) \in \mathfrak{S}$.

Clearly each double symmetric subset $\mathfrak{S}$ is also symmetric, and the set of all symmetric and double symmetric subsets is closed with respect to the intersection. This allows us to define the two closure operations: if $\mathfrak{A}$ is an arbitrary subset of $\mathfrak{C}$ and if $\mathfrak{C}_S$ ($\mathfrak{C}_{SS}$) denotes the set of all symmetric (double symmetric) subsets of $\mathfrak{C}$ let $\mathfrak{A}^\sim := \bigcap\{\mathfrak{S} \subset \mathfrak{C}_S \mid \mathfrak{A} \subset \mathfrak{S}\}$ $(\mathfrak{A}^{\sim\sim} := \bigcap\{\mathfrak{S} \subset \mathfrak{C}_{SS} \mid \mathfrak{A} \subset \mathfrak{S}\})$ be the smallest symmetric (double symmetric) subset of $\mathfrak{C}$ containing $\mathfrak{A}$.

(1.5.1) *For* $\mathfrak{A} \subset \mathfrak{C}$ *and* $n \in \mathbb{N}$ *let* $\mathfrak{A}_1 = \mathfrak{A}'_1 := \mathfrak{A}$, $\mathfrak{A}_{n+1} := \{\widetilde{X}(Y) \mid X, Y \in \mathfrak{A}_n\}$ *and* $\mathfrak{A}'_{n+1} := \{\widetilde{X,Y}(Z) \mid X, Y, Z \in \mathfrak{A}'_n\}$. *Then* $\mathfrak{A}^\sim = \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$ *and* $\mathfrak{A}^{\sim\sim} = \bigcup_{n\in\mathbb{N}} \mathfrak{A}'_n$ *and*

$\subset \mathfrak{A}^\sim \subset \mathfrak{A}^{\sim\sim}$.

$\widetilde{X}(X) = X$, with $X \in \mathfrak{C}$ and definition 1.5.1 we have:

$$(*)\ \forall\, n \in \mathbb{N} :\ \mathfrak{A} \subset \mathfrak{A}_n \subset \mathfrak{A}_{n+1} \subset \mathfrak{A}^\sim,$$

$\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n \subset \mathfrak{A}^\sim$.

$B \in \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$, thus there are $n, m \in \mathbb{N}$ with $A \in \mathfrak{A}_n,\ B \in \mathfrak{A}_m$. By $(*)$ we $m \leq n$, i.e. $A, B \in \mathfrak{A}_n$. Then $\widetilde{A}(B),\ \widetilde{B}(A) \in \mathfrak{A}_{n+1} \subset \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$ and this $\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$ is symmetric, hence $\mathfrak{A}^\sim \subset \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$. In the same way we obtain $\mathfrak{A}'_n$.

$\widetilde{X,X}$ we have:

$$\mathfrak{A}_2 = \{\widetilde{X}(Y) \mid X, Y \in \mathfrak{A}\} \subset \{\widetilde{X,Y}(Z) \mid X, Y, Z \in \mathfrak{A}\} = \mathfrak{A}'_2$$

and by induction we obtain $\mathfrak{A}_n \subset \mathfrak{A}'_n$ for all $n \in \mathbb{N}$. This shows, with the first part, $\mathfrak{A} \subset \mathfrak{A}^{\sim} \subset \mathfrak{A}^{\sim\sim}$.□

Let $\widetilde{\mathfrak{A}} := \{\widetilde{A} \mid A \in \mathfrak{A}\}$ and $\widetilde{\widetilde{\mathfrak{A}}} := \{\widetilde{A,B} \mid A, B \in \mathfrak{A}\}$.
From (1.4.1) we obtain:

(1.5.2) *Let* $\mathfrak{A} \subset \mathfrak{C}$. *Then:*
(1) $\mathfrak{A} = \mathfrak{A}^{\sim} \;\Leftrightarrow\; \widetilde{\mathfrak{A}}$ *is normal in* $\widetilde{\mathfrak{A}}$, *i.e.* $\forall\, \alpha, \beta \in \widetilde{\mathfrak{A}} : \alpha \circ \beta \circ \alpha \in \widetilde{\mathfrak{A}}$.
(2) $\mathfrak{A} = \mathfrak{A}^{\sim\sim} \;\Leftrightarrow\; \widetilde{\mathfrak{A}}$ *is normal in* $\widetilde{\widetilde{\mathfrak{A}}}$, *i.e.* $\forall\, \alpha \in \widetilde{\mathfrak{A}},\ \forall\, \beta \in \widetilde{\widetilde{\mathfrak{A}}} : \beta \circ \alpha \circ \beta^{-1} \in \widetilde{\mathfrak{A}}$.

Proof. (1) Let $A, B \in \mathfrak{A}$; then $\widetilde{\widetilde{A}(B)} \overset{(1.4.2.2)}{=} \widetilde{A} \circ \widetilde{B} \circ \widetilde{A} \in \widetilde{\mathfrak{A}} \overset{(1.4.2)}{\Leftrightarrow} \widetilde{A}(B) \in \mathfrak{A}$.
(2) Let $A, B, C \in \mathfrak{A}$; then $\widetilde{\widetilde{B,C}(A)} \overset{(1.4.1.7)}{=} \widetilde{B,C} \circ \widetilde{A} \circ (\widetilde{B,C})^{-1} \in \widetilde{\mathfrak{A}} \overset{(1.4.2)}{\Leftrightarrow} \widetilde{B,C}(A) \in \mathfrak{A}$.□

(1.5.3) *Let* $\Delta \subset \overline{\Gamma}^{-} \cap J^*$ *and let* $\psi_0 : \mathfrak{C} \to \overline{\Gamma}^{-} \cap J^*; C \mapsto \widetilde{C}$. *Then:*
(1) Δ *is normal in* $\Delta \;\Leftrightarrow\; \psi_0^{-1}(\Delta)$ *is symmetric.*
(2) Δ *is normal in* $\widetilde{\widetilde{(\psi_0^{-1}(\Delta))}} \;\Leftrightarrow\; \psi_0^{-1}(\Delta)$ *is double symmetric.*

Proof. (1) By (1.4.2) the map ψ_0 is a bijection, then by (1.5.2.1) we have the thesis.
(2) By (1.4.2) the map ψ_0 is a bijection, then by (1.5.2.2) we have the thesis.□

Now we consider the relations between $\mathfrak{C}$ and $Sym\mathfrak{G}_1$ via the maps σ_E und C_E (cf. (1.3.9)).

(1.5.4) *Let* $E \in \mathfrak{A} \subset \mathfrak{C}$. *Then:*
$\mathfrak{A} = \mathfrak{A}^{\sim} \;\Leftrightarrow\; \sigma_E(\mathfrak{A}) =: \mathbb{A} \subset Sym\mathfrak{G}_1$ *satisfies:* $\forall\, \alpha, \beta \in \mathbb{A} : \alpha \circ \beta^{-1} \circ \alpha \in \mathbb{A}$.

Proof. Let $A, B, C \in \mathfrak{A}$. Then by (1.4.3.3) $\alpha := \sigma_E(A) = \widetilde{E} \circ \widetilde{A}_{|\mathfrak{G}_1}$, $\beta := \sigma_E(B) = \widetilde{E} \circ \widetilde{B}_{|\mathfrak{G}_1}$, $\gamma := \sigma_E(C) = \widetilde{E} \circ \widetilde{C}_{|\mathfrak{G}_1}$ and $\sigma_E(\widetilde{A}(B)) = \widetilde{E} \circ \widetilde{\widetilde{A}(B)}_{|\mathfrak{G}_1} \overset{(1.4.2.2)}{=} \widetilde{E} \circ \widetilde{A} \circ \widetilde{B} \circ \widetilde{A}_{|\mathfrak{G}_1} = \alpha \circ \widetilde{B} \circ \widetilde{E} \circ \widetilde{E} \circ \widetilde{A}_{|\mathfrak{G}_1} = \alpha \circ \beta^{-1} \circ \alpha$. Hence: $\widetilde{A}(B) \in \mathfrak{A} \;\Leftrightarrow\; \alpha \circ \beta^{-1} \circ \alpha \in \mathbb{A}$ and this proves the thesis.□

1.6 CHAIN STRUCTURES

A quadruple $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is called a *chain net* or a *chain structure* if $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ is a 2−net and $\mathfrak{K}$ is a non empty subset of $\mathfrak{C}$.

Definition 1.6.1 A chain structure $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ such that $\mathfrak{K}$ is symmetric is called a *symmetric chain structure.*

In the following let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a chain net and let $E \in \mathfrak{K}$ be fixed. Then:

(1.6.1) *The following statements are equivalent*:
(1) $\forall\ A, B \in \mathfrak{K}:\ \widetilde{A, B}(E) \in \mathfrak{K}$, *i.e.* $\widetilde{\widetilde{\mathfrak{K}}}\ (E) \subset \mathfrak{K}$.
(2) $\sigma_E(\mathfrak{K})$ *is a semigroup.*
(3) $\forall\ \alpha \in \widetilde{\widetilde{\mathfrak{K}}}: \alpha \circ \widetilde{E} \circ \alpha^{-1} \in \widetilde{\mathfrak{K}}$.

Proof. (1) $\Leftrightarrow$ (2). By (1.4.3.3) we have $\sigma_E(\mathfrak{K}) = \{\widetilde{E} \circ \widetilde{K}_{|\mathfrak{G}_1} \mid K \in \mathfrak{K}\}$. Now let $A, B \in \mathfrak{K}$ then $\sigma_E(A) \circ \sigma_E(B) = \widetilde{E} \circ \widetilde{A} \circ \widetilde{E} \circ \widetilde{B}_{|\mathfrak{G}_1} \overset{(1.4.2.5)}{=} \widetilde{E} \circ \widetilde{A, B} \circ \widetilde{E} \circ \widetilde{B, A}_{|\mathfrak{G}_1} \overset{(1.4.1.7)}{=}$
$\widetilde{E} \circ \widetilde{\widetilde{A, B}(E)}_{|\mathfrak{G}_1} \in \sigma_E(\mathfrak{K}) \ \Leftrightarrow\ \widetilde{A, B}(E) \in \mathfrak{K}$.
(1) $\Leftrightarrow$ (3). Let $\alpha := \widetilde{A, B} \in \widetilde{\widetilde{\mathfrak{K}}}$. Then $\alpha \circ \widetilde{E} \circ \alpha^{-1} \overset{(1.4.1.7)}{=} \widetilde{\alpha(E)} \in \widetilde{\mathfrak{K}} \overset{(1.4.2)}{\Leftrightarrow} \alpha(E) \in \mathfrak{K}$.□

(1.6.2) *The following statements are equivalent:*
(1) $\forall\ A \in \mathfrak{K}:\ \widetilde{E}(A) \in \mathfrak{K}$, *i.e.* $\widetilde{E}(\mathfrak{K}) \subset \mathfrak{K}$.
(2) $\sigma_E(\mathfrak{K}) = (\sigma_E(\mathfrak{K}))^{-1}$.
(3) $\widetilde{E} \circ \widetilde{\mathfrak{K}} \circ \widetilde{E} = \widetilde{\mathfrak{K}}$.

Proof. (1) $\Leftrightarrow$ (2). Let $A \in \mathfrak{K}$. Then:
$(\sigma_E(A))^{-1} \overset{(1.4.3.3)}{=} (\widetilde{E} \circ \widetilde{A})^{-1}_{|\mathfrak{G}_1} = \widetilde{A} \circ \widetilde{E}_{|\mathfrak{G}_1} = \widetilde{E} \circ \widetilde{E} \circ \widetilde{A} \circ \widetilde{E}_{|\mathfrak{G}_1} \overset{(1.4.2.2)}{=} \widetilde{E} \circ \widetilde{\widetilde{E}(A)}_{|\mathfrak{G}_1} \in \sigma_E(\mathfrak{K}) \ \Leftrightarrow$
$\widetilde{E}(A) \in \mathfrak{K}$.
(1) $\Leftrightarrow$ (3). $\widetilde{E}(\mathfrak{K}) \subset \mathfrak{K} \overset{(1.4.2)}{\Leftrightarrow} \widetilde{\widetilde{E}(\mathfrak{K})} \overset{(1.4.2.2)}{=} \widetilde{E} \circ \widetilde{\mathfrak{K}} \circ \widetilde{E} \subset \widetilde{\mathfrak{K}}$ implies $\widetilde{\mathfrak{K}} \subset \widetilde{E} \circ \widetilde{\mathfrak{K}} \circ \widetilde{E}$ since $\widetilde{E} \circ \widetilde{E} = id$.□

(1.6.3) *The following statements are equivalent:*

(1) $\widetilde{\widetilde{\mathfrak{K}}}\,(E) \subset \mathfrak{K}$ *and* $\widetilde{E}(\mathfrak{K}) \subset \mathfrak{K}$.

(2) $\mathfrak{K}$ *is double symmetric.*

(3) $\sigma_E(\mathfrak{K}) \leq Sym\mathfrak{G}_1$.

(4) $\widetilde{\mathfrak{K}}$ *is normal in* $\widetilde{\widetilde{\mathfrak{K}}}$.

Proof. (1) $\Leftrightarrow$ (3). By $(1.6.1.1), (1.6.1.2)$ and $(1.6.2.1), (1.6.2.2)$ follows the equivalence of (1) and (3).

(2) $\Leftrightarrow$ (4). Now let $A, B, C \in \mathfrak{K}$. By $(1.4.1.3), (1.4.1.6)$ and $(1.4.1.7)$ we have $\widetilde{\widetilde{A,B}(C)} = \widetilde{A,B} \circ \widetilde{C} \circ (\widetilde{A,B})^{-1}$ and so by (1.4.2), $\widetilde{A,B}(C) \in \mathfrak{K} \Leftrightarrow \widetilde{A,B} \circ \widetilde{C} \circ (\widetilde{A,B})^{-1} \in \widetilde{\mathfrak{K}}$. This shows the equivalence of (2) and (4).

(4) $\Rightarrow$ (1). Let $A \in \mathfrak{K}$ and $\gamma \in \widetilde{\widetilde{\mathfrak{K}}}$. Since also $E \in \mathfrak{K}$ we have $\widetilde{E} \in \widetilde{\widetilde{\mathfrak{K}}}$, $\widetilde{\gamma(E)} \overset{(1.4.1.7)}{=} \gamma \circ \widetilde{E} \circ \gamma^{-1} \in \widetilde{\mathfrak{K}}$ by (4) hence $\gamma(E) \in \mathfrak{K}$ by (1.4.2), i.e. $\widetilde{\widetilde{\mathfrak{K}}}\,(E) \subset \mathfrak{K}$ and $\widetilde{\widetilde{E}(A)} \overset{(1.4.2.2)}{=} \widetilde{E} \circ \widetilde{A} \circ \widetilde{E} \in \widetilde{\mathfrak{K}}$ by (4) hence $\widetilde{E}(A) \in \mathfrak{K}$ by (1.4.2), i.e. $\widetilde{E}(\mathfrak{K}) \subset \mathfrak{K}$. Hence (4) implies (1).

(3) $\Rightarrow$ (2). Let $A, B, C \in \mathfrak{K}$. Then $\widetilde{\widetilde{A,B}(C)}_{|\mathfrak{G}_1} \overset{(1.4.1.7)}{=} \widetilde{A,B} \circ \widetilde{C} \circ (\widetilde{A,B})^{-1}_{|\mathfrak{G}_1} \overset{(1.4.1.3)}{=} \widetilde{A,B} \circ \widetilde{C} \circ \widetilde{B,A}_{|\mathfrak{G}_1} \overset{(1.4.2.5)}{=} \widetilde{A} \circ \widetilde{C} \circ \widetilde{B}_{|\mathfrak{G}_1}$. Since $\sigma_E(\mathfrak{K})$ is a group and $\widetilde{E} \circ \widetilde{E} = id$ we obtain $\widetilde{E} \circ \widetilde{A} \circ \widetilde{C} \circ \widetilde{B}_{|\mathfrak{G}_1} = \sigma_E(A) \circ \widetilde{C} \circ \widetilde{E} \circ \widetilde{E} \circ \widetilde{B}_{|\mathfrak{G}_1} = \sigma_E(A) \circ (\sigma_E(C))^{-1} \circ \sigma_E(B) \in \sigma_E(\mathfrak{K})$. Consequently there is a $D \in \mathfrak{K}$ with $\widetilde{E} \circ \widetilde{A} \circ \widetilde{C} \circ \widetilde{B}_{|\mathfrak{G}_1} = \sigma_E(D) = \widetilde{E} \circ \widetilde{D}_{|\mathfrak{G}_1}$ and so $\widetilde{\widetilde{A,B}(C)}_{|\mathfrak{G}_1} = \widetilde{A} \circ \widetilde{C} \circ \widetilde{B}_{|\mathfrak{G}_1} = \widetilde{D}_{|\mathfrak{G}_1}$. By (1.4.2.4) this gives us $\widetilde{A,B}(C) = D \in \mathfrak{K}$, i.e. $\mathfrak{K}$ is double symmetric.□

(1.6.4) *The following statements are equivalent:*

(1) $\mathfrak{K}$ *is symmetric.*

(2) $\sigma_E(\mathfrak{K})$ *has the property:* $\forall\ \alpha, \beta \in \sigma_E(\mathfrak{K}) : \alpha \circ \beta^{-1} \circ \alpha \in \sigma_E(\mathfrak{K})$.

(3) $\widetilde{\mathfrak{K}}$ *is normal in* $\widetilde{\mathfrak{K}}$.

Proof. (1) $\Leftrightarrow$ (2). Let $\alpha := \widetilde{E} \circ \widetilde{A}_{|\mathfrak{G}_1}$ and $\beta := \widetilde{E} \circ \widetilde{B}_{|\mathfrak{G}_1}$; then $\alpha \circ \beta^{-1} \circ \alpha = \widetilde{E} \circ \widetilde{A} \circ \widetilde{B} \circ \widetilde{E} \circ \widetilde{E} \circ \widetilde{A}_{|\mathfrak{G}_1} = \widetilde{E} \circ \widetilde{A} \circ \widetilde{B} \circ \widetilde{A}_{|\mathfrak{G}_1} \overset{(1.4.2.2)}{=} \widetilde{E} \circ \widetilde{\widetilde{A}(B)}_{|\mathfrak{G}_1}$ and so: $\alpha \circ \beta^{-1} \circ \alpha \in \sigma_E(\mathfrak{K}) \Leftrightarrow \widetilde{A}(B) \in \mathfrak{K}$.

Hence (1) and (2) are equivalent.

(1) $\Leftrightarrow$ (3). Let $A, B \in \mathfrak{K}$. Then the statement "$\widetilde{A}(B) \in \mathfrak{K} \overset{(1.4.2)}{\Leftrightarrow} \widetilde{\widetilde{A}(B)} \overset{(1.4.2.2)}{=} \widetilde{A} \circ \widetilde{B} \circ \widetilde{A} \in \widetilde{\mathfrak{K}}$" shows the equivalence of (1) and (3).□

2 Halfordered sets, semidomains and convexity

2.0 NOTATIONS

Let E be a set of points, with $|E| \geq 4$, $E^{3'} := \{(a,b,c) \in E^3 \mid a \neq b, c\}$ and let $SymE$ be the set of all permutations of E. If $E = \{1,2,...,n\}$ we write $S_n := Sym\{1,2,...,n\}$. Let

$$\xi : \begin{cases} E^{3'} & \rightarrow \quad \{1,-1\} \\ (a,b,c) & \mapsto \quad (a|b,c) \end{cases}$$

be a map such that the axiom

(Z) $\forall\ a,b,c,d \in E,\ a \neq b,c,d : (a|b,c) \cdot (a|c,d) = (a|b,d)$

is satisfied. Then the function ξ is called a *betweenness function* or a *halforder* of E and the pair (E,ξ) a *halfordered set*. A halforder ξ of E is called *trivial* if $\forall\ (a,b,c) \in E^{3'} : (a|b,c) = 1$. We observe that by **(Z)** for each halforder the following property is satisfied:

$$\forall\ (a,b,c) \in E^{3'} : (a|b,c) = (a|c,b).$$

To the halforder ξ we associate the functions

$$k_\xi : \begin{cases} \binom{E}{3} & \rightarrow \quad \{1,-1\} \\ \{a,b,c\} & \mapsto \quad (a|b,c)\cdot(b|c,a)\cdot(c|a,b) \end{cases}$$

and

$$\mathfrak{z}_\xi : \begin{cases} \binom{E}{3} & \rightarrow \quad \mathbb{N} \cup \{0\} \\ \{a_1,a_2,a_3\} & \mapsto \quad \frac{1}{2}\cdot|\{\sigma \in S_3 \mid (a_{\sigma(1)}|a_{\sigma(2)},a_{\sigma(3)}) = -1\}| \end{cases}.$$

If moreover

(Z0) the function $\mathfrak{z}_\xi$ is constant $= 1$

then the halforder ξ is called an *order* and the pair (E,ξ) an *ordered set.*

For all $a, b \in E$, with $a \neq b$, the sets $\overrightarrow{a,b} := \{x \in E \mid (a|b,x) = 1\}$, $\overrightarrow{a,b}^- := \{x \in E \mid (a|b,x) = -1\}$ are called *halflines* of E and the set $]a,b[:= \{x \in E \mid (x|a,b) = -1\}$ an *open segment* of E.
We observe if $b' \in \overrightarrow{a,b}$ then $\overrightarrow{a,b} = \overrightarrow{a,b'}$; in fact we have: $(a|b,b') = 1$ and $x \in \overrightarrow{a,b}$ (i.e. $(a|b,x) = 1$) implies $1 = (a|b,b') \cdot (a|b,x) = (a|b',x)$, and so we obtain $x \in \overrightarrow{a,b'}$. In the same way if $c \in \overrightarrow{a,b}^-$ then $\overrightarrow{a,b}^- = \overrightarrow{a,c}$; in fact we have: $(a|b,c) = -1$ and $x \in \overrightarrow{a,b}^-$ (i.e. $(a|b,x) = -1$) implies $1 = (-1) \cdot (-1) = (a|b,c) \cdot (a|b,x) = (a|c,x)$ and so $x \in \overrightarrow{a,c}$.

Definition 2.0.1 Let $K \subset E$; then a function

$$K_s : \begin{cases} (E \setminus K) \times (E \setminus K) & \rightarrow & \{1, -1\} \\ (a,b) & \mapsto & K_s(a,b) \end{cases}$$

is called a *splitting of* E *by* K if the following condition
(S1) $\forall\ a,b,c \in E \setminus K : K_s(a,b) \cdot K_s(b,c) = K_s(a,c)$,
is satisfied. A splitting K_s of E by K is called *trivial* if $\forall\ (a,b) \in (E \setminus K) \times (E \setminus K)$: $K_s(a,b) = 1$.

Definition 2.0.2 A subset $A \subset E$ is called *convex* if $\forall\ a,b \in A :]a,b[\subset A$.
A halfordered set (E,ξ) is called *convex* if $\forall\ a,b \in E, a \neq b : \overrightarrow{a,b}$ is convex, i.e.
(C0) $(a|b,c) = 1$ and $(x|b,c) = -1 \ \Rightarrow\ (a|b,x) = 1$
and *trivially convex* if $\forall\ \{a,b,c,d\} \in \binom{E}{4} : (a|c,d) = (b|c,d)$.
If the condition **(C0)** is satisfied for a fixed $a \in E$, then (E,ξ) is called *convex in* a.

If (E,ξ) is a halfordered set, we set:

$$Aut(E,\xi) := \{\gamma \in SymE \mid \forall\ (a,b,c) \in E^{3'} :\ (a|b,c) = (\gamma(a)|\gamma(b),\gamma(c))\}.$$

We call the elements of the group $Aut(E,\xi)$ the *betweenness preserving permutations* of the halfordered set (E,ξ).

If $\Sigma \subset SymE$ is fixed we define $\forall\ a,b \in E$:

$$[a \rightarrow b] := \{\sigma \in \Sigma\ : \sigma(a) = b\}.$$

2.1 RELATED HALFORDERS

To each halfordered set (E,ξ) we associate a "separation function" :

$$\tau_\xi : \begin{cases} E^{4'} := \{(a,b,c,d) \in E^4 \mid a,b \neq c,d\} & \to & \{1,-1\} \\ (a,b,c,d) & \mapsto & [a,b|c,d] := (a|c,d)\cdot(b|c,d) \end{cases},$$

which has by **(Z)** the following property:

(T) $\forall\ (a,b,c,d),(a,b,d,e) \in E^{4'}$:

$$[a,b|c,d]\cdot[a,b|d,e] = [a,b|c,e].$$

Each function $\tau : E^{4'} \to \{1,-1\}$ which satisfies **(T)** is called a *separation function.* The set $T(E)$ of all separation functions forms a commutative group of exponent 2: let τ_1,τ_2 be separation functions of E, $(a,b,c,d) \in E^{4'}$ then

$$(\tau_1\cdot\tau_2)(a,b,c,d) = [a,b|c,d]_1 \cdot [a,b|c,d]_2.$$

Two halforders ξ_1 and ξ_2 of E are called *related*, denoted by $\xi_1\ rel\ \xi_2$, if:
(R) $\forall\ (a,b,c,d) \in E^{4'} : [a,b|c,d]_1 = [c,d|a,b]_2$;
a halforder ξ of E is called *selfrelated* if $\xi\ rel\ \xi$, i.e. if
(R1) For all $(a,b,c,d) \in E^{4'} : [a,b|c,d] = [c,d|a,b]$
is satisfied.

Remark. The relation "*rel*" is symmetric, i.e. if $\xi_1\ rel\ \xi_2$ then also $\xi_2\ rel\ \xi_1$.

(2.1.1) *For a halfordered set* (E,ξ), *with* $|E| \geq 4$, *the following statements are equivalent*:
(1) *The function* k_ξ *is constant.*
(2) ξ *is selfrelated.*

Proof. $\forall\ \{a,b,c,d\} \in \binom{E}{4}$ we have:
$k_\xi(\{a,c,d\}) = k_\xi(\{b,c,d\}) \Leftrightarrow\ 1 = (a|c,d)\cdot(c|d,a)\cdot(d|a,c)\cdot(b|c,d)\cdot(c|d,b)\cdot(d|b,c) \overset{(\mathbf{Z})}{=}$ $(a|c,d)\cdot(c|a,b)\cdot(d|a,b)\cdot(b|c,d) \Leftrightarrow [a,b|c,d] = [c,d|a,b]$. □

Remark. If (E,ξ) is a halfordered set then each point $0 \in E$ defines a splitting

$$0_s : \begin{cases} (E\setminus\{0\})\times(E\setminus\{0\}) & \rightarrow & \{1,-1\} \\ (a,b) & \mapsto & (0|a,b) \end{cases}$$

of E by 0.

Now let $\{0,1,\infty\} \in \binom{E}{3}$ be fixed and let $(a,b,c) \in E^{3'}$. If ξ' *rel* ξ we obtain:
(1) $(a|b,c)' = [b,c|0,a]\cdot(0|b,c)'$, if $b,c \neq 0$.
(2) $(a|b,0)' = [0,b|1,a]\cdot(1|b,0)'$, if $b \neq 1, a \neq 0$.
(3) $(1|b,0)' \overset{(\mathbf{Z})}{=} (1|0,\infty)'\cdot(1|\infty,b)' \overset{(\mathbf{1})}{=} (1|0,\infty)'\cdot[\infty,b|0,1]\cdot(0|b,\infty)'$, if $b \neq 1, 0$.

Hence from **(2)** and **(3)**:
(4) $(a|b,0)' = [0,b|1,a]\cdot[\infty,b|0,1]\cdot(0|b,\infty)'\cdot(1|0,\infty)' \overset{(\mathbf{Z})}{=} (0|1,a)\cdot(b|0,a)\cdot(\infty|0,1)\cdot$ $(0|b,\infty)'\cdot(1|0,\infty)'$, if $a,b \neq 0$.

The equations **(1)** and **(4)** show that ξ' is completely determined by ξ, the splitting in 0 derived from ξ' and the value $(1|0,\infty)'$. Moreover we have:

(2.1.2) *If (E,ξ) is a halfordered set, $\{0,1,\infty\} \in \binom{E}{3}$ fixed, 0_s a splitting of E by 0 and $\epsilon \in \{1,-1\}$ then the function $\xi'(\xi,0_s,\epsilon) = \xi'$ defined by*

$$\xi' : \begin{cases} E^{3'} \quad \rightarrow \quad \{1,-1\} \\ (a,b,c) \ \mapsto \ (a|b,c)' := \begin{cases} [b,c|0,a]\cdot 0_s(b,c), & if \quad b,c \neq 0 \\ (0|1,a)\cdot(b|0,a)\cdot(\infty|0,1)\cdot 0_s(b,\infty)\cdot\epsilon, & if \quad b \neq 0, c = 0 \\ (0|1,a)\cdot(c|0,a)\cdot(\infty|0,1)\cdot 0_s(c,\infty)\cdot\epsilon, & if \quad b = 0, c \neq 0 \\ 1 & if \quad b = c = 0 \end{cases} \end{cases}$$

is a halforder related to ξ.

Each halforder related to ξ *is completely determined by the data* ξ, *a splitting* 0_s *of* E *by* 0 *and a value* $\epsilon \in \{1, -1\}$, *and this is a one-to-one correspondence.*

Proof. ξ' is a halforder; in fact $(a|b,c)' = (a|c,b)'$ and

a) if $a = 0$ then $\xi' = 0_s$ and so **(Z)** is satisfied by **(S1)**;

b) if $a \neq 0$ then:

b1) if $b = c = 0$ or $c = d = 0$,

$(a|0,0)' \cdot (a|0,d)' = 1 \cdot (a|0,d)' = (a|0,d)'$ or

$(a|b,0)' \cdot (a|0,0)' = (a|b,0)' \cdot 1 = (a|b,0)'$.

b2) if $c = 0$ and $b, d \neq 0$,

$$(a|b,0)' \cdot (a|0,d)' :=$$

$$(0|1,a) \cdot (b|0,a) \cdot (\infty|0,1) \cdot 0_s(b,\infty) \cdot \epsilon \cdot (0|1,a) \cdot (d|0,a) \cdot (\infty|0,1) \cdot 0_s(d,\infty) \cdot \epsilon \overset{(\mathbf{S1})}{=}$$

$$(b|0,a) \cdot (d|0,a) \cdot 0_s(b,d) = [b,d|0,a] \cdot 0_s(b,d) =: (a|b,d)'.$$

b3) if $c \neq 0$ and $b = d = 0$,

$(a|0,c)' \cdot (a|c,0)' = 1 = (a|0,0)'$.

b4) if $b, c \neq 0$ and $d = 0$,

$$(a|b,c)' \cdot (a|c,0)' :=$$

$$(b|0,a) \cdot (c|0,a) \cdot 0_s(b,c) \cdot (0|1,a) \cdot (c|0,a) \cdot (\infty|0,1) \cdot 0_s(c,\infty) \cdot \epsilon \overset{(\mathbf{S1})}{=}$$

$$(b|0,a) \cdot (0|1,a) \cdot (\infty|0,1) \cdot 0_s(b,\infty) \cdot \epsilon =: (a|b,0)'.$$

b5) if $b, c, d \neq 0$,

$$(a|b,c)' \cdot (a|c,d)' :=$$

$$(b|0,a) \cdot (c|0,a) \cdot 0_s(b,c) \cdot (c|0,a) \cdot (d|0,a) \cdot 0_s(c,d) \overset{(\mathbf{S1})}{=}$$

$$(b|0,a) \cdot (d|0,a) \cdot 0_s(b,d) =: (a|b,d)'.$$

Now we want to prove: $\xi'\ rel\ \xi$, i.e. **(R)** is satisfied. If $a = b$ or $c = d$, the value of $[a, b|c, d]'$ and $[a, b|c, d]$ is $= 1$. Therefore let $a \neq b$ and $c \neq d$

<u>case 1.</u> $a = 0$.

$$(0|c, d)' \cdot (b|c, d)' :=$$

$$0_s(c, d) \cdot (c|0, b) \cdot (d|0, b) \cdot 0_s(c, d) = (c|0, b) \cdot (d|0, b).$$

<u>case 2.</u> $c = 0$.

$$(a|0, d)' \cdot (b|0, d)' := (d|0, a) \cdot (0|1, a) \cdot (\infty|0, 1) \cdot 0_s(d, \infty) \cdot \epsilon \cdot$$

$$(d|0, b) \cdot (0|1, b) \cdot (\infty|0, 1) \cdot 0_s(d, \infty) \cdot \epsilon \overset{\mathbf{(Z)}}{=} (d|a, b) \cdot (0|a, b).$$

<u>case 3.</u> $a, b, c, d \neq 0$.

$$(a|c, d)' \cdot (b|c, d)' :=$$

$$(c|0, a) \cdot (d|0, a) \cdot 0_s(c, d) \cdot (c|0, b) \cdot (d|0, b) \cdot 0_s(c, d) \overset{\mathbf{(Z)}}{=} (c|a, b) \cdot (d|a, b). \ \square$$

Remark. If ξ_1, ξ_2 are two halforders of a non empty set E such that $\xi_1\ rel\ \xi_2$ and ξ_1 is selfrelated, then also ξ_2 is selfrelated.

(2.1.3) *Every order ξ is selfrelated.*

Proof. By **(Z0)**, $k_\xi = -1$ hence by (2.1.1) $\xi\ rel\ \xi$. $\square$

(2.1.4) *Let E be a non empty set and let ξ_1, ξ_2 be two halforders of E and k_{ξ_1}, k_{ξ_2} the corresponding functions. Then:*

(1) *If $\xi_1\ rel\ \xi_2$ then $k_{\xi_1} = k_{\xi_2}$.*

(2) *If $k_{\xi_1} = k_{\xi_2}$ then $\forall\ \{a, b, c, d\} \in \binom{E}{4}$:*

$$[a, b|c, d]_1 \cdot [c, d|a, b]_1 = [a, b|c, d]_2 \cdot [c, d|a, b]_2.$$

Proof. (1) Let $\{a, b, c, d\} \in \binom{E}{4}$ then :

$$k_{\xi_1}(\{a, b, c\}) := (a|b, c)_1 \cdot (b|a, c)_1 \cdot (c|a, b)_1 \overset{\mathbf{(Z)}}{=}$$

$$(a|b,d)_1 \cdot (a|d,c)_1 \cdot (b|c,d)_1 \cdot (b|d,a)_1 \cdot (c|a,d)_1 \cdot (c|d,b)_1 \overset{\textbf{def.}}{=}$$

$$[a,c|b,d]_1 \cdot [a,b|c,d]_1 \cdot [b,c|a,d]_1 \overset{\xi_1 \ rel \ \xi_2}{=} [d,b|a,c]_2 \cdot [c,d|a,b]_2 \cdot [a,d|b,c]_2 \overset{\textbf{def.}}{=}$$

$$(d|a,c)_2 \cdot (b|a,c)_2 \cdot (c|a,b)_2 \cdot (d|a,b)_2 \cdot (a|b,c)_2 \cdot (d|b,c)_2 \overset{(\mathbf{Z})}{=}$$

$$(b|a,c)_2 \cdot (c|a,b)_2 \cdot (a|b,c)_2 =: k_{\xi_2}(\{a,b,c\}).$$

(2)

$$[a,b|c,d]_1 := (a|c,d)_1 \cdot (b|c,d)_1 =$$

$$k_{\xi_1}(\{a,c,d\}) \cdot (c|d,a)_1 \cdot (d|c,a)_1 \cdot k_{\xi_1}(\{b,c,d\}) \cdot (c|b,d)_1 \cdot (d|b,c)_1 \overset{k_{\xi_1}=k_{\xi_2} \wedge (\mathbf{Z})}{=}$$

$$k_{\xi_2}(\{a,c,d\}) \cdot k_{\xi_2}(\{b,c,d\}) \cdot (c|a,b)_1 \cdot (d|a,b)_1 =$$

$$(a|c,d)_2 \cdot (c|a,d)_2 \cdot (d|a,c)_2 \cdot (b|c,d)_2 \cdot (c|b,d)_2 \cdot (d|b,c)_2 \cdot [c,d|a,b]_1 \overset{(\mathbf{Z})}{=}$$

$$[a,b|c,d]_2 \cdot (c|a,b)_2 \cdot (d|a,b)_2 \cdot [c,d|a,b]_1 = [a,b|c,d]_2 \cdot [c,d|a,b]_2 \cdot [c,d|a,b]_1. \ \square$$

Remark. There are halforders ξ_1 and ξ_2 of E such that $k_{\xi_1} = k_{\xi_2}$ but ξ_1 is not related to ξ_2. In fact already in the smallest possible set $E := \{a,b,c,d\}$ we can find two such halforders ξ_1, ξ_2 if we take for instance $\xi_1 = 1$ and for ξ_2 :

$$(d|a,c)_2 = (b|c,d)_2 = (a|b,c)_2 = (b|a,c)_2 = 1,$$

$$(a|c,d)_2 = (c|a,d)_2 = (c|b,d)_2 = (d|b,c)_2 = -1.$$

Then $\forall \ \{x,y,z\} \in \binom{E}{3} : k_{\xi_1}(\{x,y,z\}) = k_{\xi_2}(\{x,y,z\}) = 1$ but $1 = [a,b|c,d]_1 \neq [c,d|a,b]_2 = -1$.

2.2 SELFRELATED HALFORDERS

Let $E \neq \emptyset$ and let ξ be a selfrelated halforder. By (2.1.1) k_ξ is a constant function. Following E. Sperner [33] we call the halforder ξ *harmonic* if $k_\xi = -1$ and *anharmonic* if $k_\xi = 1$. Clearly in the harmonic case, $\mathfrak{Z}_\xi(\binom{E}{3}) \subset \{1,3\}$, in the anharmonic case, $\mathfrak{Z}_\xi(\binom{E}{3}) \subset \{0,2\}$ and ξ is an order iff $\mathfrak{Z}_\xi(\binom{E}{3}) = \{1\}$.

(2.2.1) *For* $|E| \geq 4$ *the statements are equivalent*:
(1) (E,ξ) *is an ordered set.*
(2) (E,ξ) *is harmonic and convex.*

Proof. (1) $\Rightarrow$ (2). The function ξ is an order iff $\mathfrak{Z}_\xi(\binom{E}{3}) = \{1\}$ and this implies $k_\xi = -1$, i.e. ξ is harmonic. Now we prove that (E,ξ) is convex:
let $\{a,b\} \in \binom{E}{2}$, $c \in \overrightarrow{a,b}$ and $d \in]c,b[$; we have to show: $(a|b,d) = 1$. Since $c \in \overrightarrow{a,b}$, by definition $(a|b,c) = 1$ and since $k_\xi = -1$, $(b|a,c) \cdot (c|a,b) = -1$, i.e. $(*)$ $(b|a,c) = -(c|a,b)$. Since $d \in]c,b[$, i.e. $(d|b,c) = -1$ and since $k_\xi = -1$, $(**)$ $(b|c,d) = (c|d,b) = 1$. $(*)$, $(**)$ and **(Z)** imply $(b|a,d) = -(c|a,d)$. If $(b|a,d) = -1$ then $(a|b,d) = 1$, if $(c|a,d) = -1$, then $(a|c,d) = 1$ and so $(a|b,d) = (a|b,c) \cdot (a|c,d) = 1 \cdot 1 = 1$.
(2) $\Rightarrow$ (1). Since (E,ξ) is harmonic, we have $\mathfrak{Z}_\xi(\{x,y,z\}) \in \{1,3\}$ for all $\{x,y,z\} \in \binom{E}{3}$. Let us assume there are $\{a,b,c\} \in \binom{E}{3}$ with $\mathfrak{Z}_\xi(\{a,b,c\}) = 3$ and let $d \in E \setminus \{a,b,c\}$. Then $-1 = (a|b,c) = (a|b,d) \cdot (a|d,c)$ and so for instance $(a|b,d) = 1$ and $(a|d,c) = -1$. If $(c|b,d) = -1$ then the convexity implies $(a|b,c) = 1$ contradicting $\mathfrak{Z}_\xi(\{a,b,c\}) = 3$. Hence $(c|b,d) = 1$, and since ξ is harmonic we have : $(b|c,d) = -(d|b,c)$.
<u>case 1.</u> $(b|c,d) = -(d|c,b) = 1$.
Then $(a|c,d) = -1$ and the convexity imply $(b|a,c) = 1$ contradicting $\mathfrak{Z}_\xi(\{a,b,c\}) = 3$.
<u>case 2.</u> $(b|c,d) = -1$ and $(d|c,b) = 1$.
Here $(a|b,c) = -1$ and $(d|b,c) = 1$ imply $(d|a,c) = 1$. Since $(a|c,d) = -1$ we have $(c|a,d) = 1$ (ξ is harmonic !) and so $(c|b,d) = (c|a,d) \cdot (c|a,b) = 1 \cdot (-1) = -1$ which contradicts $(c|b,d) = 1$. $\square$

Now we consider the case (E, ξ) in which ξ is anharmonic hence $k_\xi = 1$ but not trivial. Then there are three distinct elements $a, b, c \in E$ with $(a|b, c) = (b|a, c) = -1$ and $(c|a, b) = 1$. If $E = \{a, b, c\}$ then $\overrightarrow{c, a}$ is the only halfline which contains two points namely a, b. Since $]a, b[= \emptyset$, (E, ξ) is convex. Therefore we may assume $|E| \geq 4$. Here we restrict ourself on the case $|E| = 4$, hence $E = \{a, b, c, d\}$.

Possible cases

1. $(d|a, b, c) = 1$ [3] . Then:

a) If (E, ξ) is convex then:

(1) $(a|b, d) = (b|a, d) = -1$ and $(a|c, d) = (b|c, d) = 1$.

(2) $(c|a, b, d) = 1$.

Proof. (1) If $(a|b, d) = 1$ then $(a|c, d) = (a|b, d) \cdot (a|b, c) = 1 \cdot (-1) = -1$ and, since $(d|a, c) = 1$ and $k_\xi = 1$, $(c|a, d) = -1$. Then $(c|b, d) = (c|a, b) \cdot (c|a, d) = 1 \cdot (-1) = -1$. $(a|b, d) = 1$ and $(c|b, d) = -1$ implies $(a|b, c) = 1$ by **(C0)** contradicting $(a|b, c) = -1$. Therefore $(a|b, d) = -1$ and $(a|c, d) = (a|b, c) \cdot (a|b, d) = (-1) \cdot (-1) = 1$ and since our assumptions are symmetric in a, b also $(b|a, d) = -1$ and $(b|c, d) = 1$.

(2) $(d|a, c) = (a|c, d) = 1$ (cf. (1)) and $k_\xi = 1$ imply $(c|a, d) = 1$ and since $(c|a, b) = 1$ we have $(c|a, b, d) = 1$. □

b) If $(a|b, d) = 1$ then: $(a|c, d) = (a|b, c) \cdot (a|b, d) = (-1) \cdot 1 = -1$, $(b|a, d) = k_\xi \cdot (a|b, d) \cdot (d|a, b) = 1 \cdot 1 \cdot 1 = 1$, $(b|c, d) = (b|a, c) \cdot (b|a, d) = (-1) \cdot 1 = -1$, $(c|a, d) = k_\xi \cdot (a|c, d) \cdot (d|a, c) = 1 \cdot (-1) \cdot 1 = -1$ and $(c|b, d) = (c|a, b) \cdot (c|a, d) = 1 \cdot (-1) = -1$.

2. $(d|a, b) = 1, (d|a, c) = (d|b, c) = -1$

a) $(c|a, d) = 1$, hence $(c|a, b, d) = 1$. Then:

(1) $(a|b, d) = (b|a, d) = 1$ and $(a|c, d) = (b|c, d) = -1$.

Proof. $(d|a, c) = -1, (c|a, d) = 1$ and $k_\xi = 1$ imply $(a|c, d) = -1$ and $(a|b, d) = (a|b, c) \cdot (a|c, d) = (-1) \cdot (-1) = 1$. Still the assumptions are symmetric in a, b and so all is proved.□

b) $(c|a, d) = -1$ hence $(c|b, d) = (c|a, b) \cdot (c|a, d) = 1 \cdot (-1) = -1$. Then:

[3] $(d|a, b, c) = 1 :\Leftrightarrow (d|a, b) = (d|a, c) = 1$.

(1) $(a|b,d) = (b|a,d) = -1$ and $(a|c,d) = (b|c,d) = 1$.

Proof. $(d|a,c) = -1, (c|a,d) = -1$ and $k_\xi = 1$ imply $(a|c,d) = 1$ and $(a|b,d) = (a|b,c) \cdot (a|c,d) = (-1) \cdot 1 = -1$. Again the assumptions are symmetric in a and b. □

3.$(d|a,c) = -(d|a,b) = 1$ (and so $(d|b,c) = -1$ and $(a|b,d) = -(b|a,d)$ by $k_\xi = 1$). [4] Then:

(1) (E,ξ) is not convex.

Proof. $(d|a,c) = 1$, $(b|a,c) = -1$ and **(C0)** imply $(d|a,b) = 1$, a contradiction.□

a) If $(a|b,d) = -(b|a,d) = 1$ then:

(2) $(a|c,d) = -1, (b|c,d) = 1, (c|a,d) = (c|b,d) = -1$.

Proof. $(a|b,d) = 1, (b|a,d) = -1 \;\Rightarrow\; (a|c,d) = (a|b,d) \cdot (a|b,c) = 1 \cdot (-1) = -1$ and $(b|c,d) = (b|a,c) \cdot (b|a,d) = (-1) \cdot (-1) = 1$; $(d|a,c) = 1, (a|c,d) = -1$ and $k_\xi = 1$ imply $(c|a,d) = -1$. □

b) If $(b|a,d) = -(a|b,d) = 1$ then:

(3) $(a|c,d) = 1, (b|c,d) = -1, (c|a,d) = (c|b,d) = 1$.

Proof. $(a|b,d) = -1, (b|a,d) = 1 \;\Rightarrow\; (a|c,d) = (a|b,d) \cdot (a|b,c) = (-1) \cdot (-1) = 1$ and $(b|c,d) = (b|a,c) \cdot (b|a,d) = (-1) \cdot 1 = -1$; $(d|a,c) = 1, (a|c,d) = 1$ and $k_\xi = 1$ imply $(c|a,d) = 1$ and $(c|b,d) = (c|a,b) \cdot (c|a,d) = 1 \cdot 1 = 1$. □

We have proved:

(2.2.2) *Let $(\{a,b,c\},\xi)$ be a halfordered set with $(a|b,c) = (b|a,c) = -1$ and $(c|a,b) = 1$ and let $E = \{a,b,c,d\}$. Then there are the following possibilities of anharmonic extensions of ξ* onto E:

1. $(d|a,b,c) = 1$

a) $(c|a,b,d) = 1, (a|b,d) = (b|a,d) = -1, (a|c,d) = (b|c,d) = 1$

(*a convex case*)

b) $(c|a,d) = (c|b,d) = -1, (a|b,d) = (b|a,d) = 1, (a|c,d) = (b|c,d) = -1$

(*a non convex case*)

[4] The assumption $(d|b,c) = 1,\ (d|a,b) = (d|a,c) = -1$ yields isomorphic examples.

2. $(d|a,b)=1, (d|a,c)=(d|b,c)=-1$

a) $(c|a,b,d)=1, (a|b,d)=(b|a,d)=1, (a|c,d)=(b|c,d)=-1$

(*a trivially convex case*)

b) $(c|a,d)=(c|b,d)=-1, (a|b,d)=(b|a,d)=-1, (a|c,d)=(b|c,d)=1$

(*a trivially convex case*)

3. $(d|a,c)=1, (d|a,b)=(d|b,c)=-1$

a) $(a|b,d)=1, (b|a,d)=-1, (a|c,d)=-1, (b|c,d)=1, (c|a,d)=(c|b,d)=-1$

(*a non convex case*)

b) $(a|b,d)=-1, (b|a,d)=1, (a|c,d)=1, (b|c,d)=-1, (c|a,d)=(c|b,d)=1$

(*a non convex case*)

4. $(d|b,c)=1, (d|a,b)=(d|a,c)=-1$

a) $(a|b,d)=1, (b|a,d)=-1, (a|c,d)=-1, (b|c,d)=1, (c|a,d)=(c|b,d)=1$

(*a non convex case*)

b) $(a|b,d)=-1, (b|a,d)=1, (a|c,d)=1, (b|c,d)=-1, (c|a,d)=(c|b,d)=-1$

(*a non convex case*)

2.3 SEMIDOMAINS AND HALFORDERED SETS

Let $(E,\Sigma;0)$ be a locally transitive permutation set (i.e. $E\neq\emptyset$, $\Sigma\subset SymE$, $0\in E$ and $\Sigma(0)=E$), for $a\in E$ let $[0\to a]:=\{\sigma\in\Sigma\mid\sigma(0)=a\}$ and let $\Delta_0:=<\{\gamma^{-1}\circ\alpha\circ\beta\mid\alpha,\ \beta\in\Sigma,\ \gamma\in[0\to\alpha\circ\beta(0)]\}>$ be the *structure group* of $(E,\Sigma;0)$. Then a subset $P\subset E$ is called a *semidomain* of $(E,\Sigma;0)$ if the two conditions:

(P1) $0\notin P$

(P2) $\forall\ \delta\in\Delta_0\ :\delta(P)=P$ or $\delta(P)=P_-:=E\setminus(P\cup\{0\})$

are satisfied.

(2.3.1) *Let (E,ξ) be a halfordered set such that $Aut(E,\xi)$ acts transitively on E and let $0,1\in E$ be two distinct points. Then $(E,\Sigma;0)$, with $\Sigma:=Aut(E,\xi)$, is a locally transitive permutation set and $P_\xi:=\overrightarrow{0,1}$ is a semidomain of $(E,\Sigma;0)$.*

Proof. We have $0\notin\overrightarrow{0,1}$ by definition and since in our case $\Delta_0\leq Aut(E,\xi)_0:=\{\gamma\in Aut(E,\xi)\mid\gamma(0)=0\}$ we have to prove: for all $\delta\in\Delta_0\ :\delta(\overrightarrow{0,1})=\overrightarrow{\delta(0),\delta(1)}=\overrightarrow{0,\delta(1)}=\overrightarrow{0,1}$ or $\delta(\overrightarrow{0,1})=\overrightarrow{0,1}^-$. If $x\in\overrightarrow{0,1}$ we obtain:
$1=(0|1,x)=(0|\delta(1),\delta(x))\overset{(\mathbf{Z})}{=}(0|1,\delta(1))\cdot(0|\delta(x),1)$. Hence if $(0|1,\delta(1))=1$ then $(0|1,\delta(x))=1$, i.e. $\delta(\overrightarrow{0,1})=\overrightarrow{0,1}$ and if $(0|1,\delta(1))=-1$ then $(0|1,\delta(x))=-1$, i.e. $\delta(\overrightarrow{0,1})=\overrightarrow{0,1}^-$. Thus $\overrightarrow{0,1}$ is a semidomain of $(E,\Sigma;0)$. □

(2.3.2) *Let $(E,\Sigma;0)$ be a locally transitive permutation set and let $P\subset E$ be a semidomain of $(E,\Sigma;0)$.* We set:

$$\varepsilon:=\varepsilon_P:\begin{cases}E\setminus\{0\}&\to\quad\{1,-1\}\\ x&\mapsto\quad\varepsilon(x):=\begin{cases}1&if\quad x\in P\\-1&if\quad x\notin P\end{cases}\end{cases}$$

and

$$\xi_P:\begin{cases}E^{3'}&\to\quad\{1,-1\}\\(a,b,c)&\mapsto\quad(a|b,c):=\varepsilon(\alpha^{-1}(b))\cdot\varepsilon(\alpha^{-1}(c))\end{cases},$$

where $\alpha \in [0 \to a]$.

Then (E, ξ_P) *is a halfordered set, with* $\Sigma \subset Aut(E, \xi_P)$.

Proof. From the definition of a semidomain follows (cf. **(P2)**) :

$$(*)\ \forall\ x, y \in E \setminus \{0\}, \forall \delta \in \Delta_0 : \varepsilon(x) \cdot \varepsilon(y) = \varepsilon(\delta(x)) \cdot \varepsilon(\delta(y)).$$

With $(*)$ we show that ξ_P is welldefined. Let $\alpha, \alpha_1 \in [0 \to a]$ and $\omega \in [0 \to 0]$, then $\delta := \alpha_1^{-1} \circ \alpha \circ \omega \in \Delta_0$, $\omega = \omega^{-1} \circ \omega \circ \omega \in \Delta_0$ and so $\alpha_1^{-1} \circ \alpha = \delta \circ \omega^{-1} =: \delta_1 \in \Delta_0$, i.e. $\alpha_1^{-1} = \delta_1 \circ \alpha^{-1}$. Therefore $\varepsilon(\alpha^{-1}(b)) \cdot \varepsilon(\alpha^{-1}(c)) \overset{(*)}{=} \varepsilon(\delta_1 \circ \alpha^{-1}(b)) \cdot \varepsilon(\delta_1 \circ \alpha^{-1}(c)) = \varepsilon(\alpha_1^{-1}(b)) \cdot \varepsilon(\alpha_1^{-1}(c))$.

Now we have to prove **(Z)**: let $a, b, c, d \in E, a \neq b, c, d$ then $(a|b,c) \cdot (a|c,d) := \varepsilon(\alpha^{-1}(b)) \cdot \varepsilon(\alpha^{-1}(c)) \cdot \varepsilon(\alpha^{-1}(c)) \cdot \varepsilon(\alpha^{-1}(d)) := (a|b,d)$, where $\alpha \in [0 \to a]$. Hence (E, ξ_P) is a halfordered set.

Finally let $\beta \in \Sigma$ and $(a, b, c) \in E^{3'}$ be given. Let $\alpha \in [0 \to a]$ and $\gamma \in [0 \to \beta \circ \alpha(0)]$. Then $\delta := \gamma^{-1} \circ \beta \circ \alpha \in \Delta_0$ and so :

$$(\beta(a)|\beta(b), \beta(c)) = (\beta \circ \alpha(0)|\beta \circ \alpha(\alpha^{-1}(b)), \beta \circ \alpha(\alpha^{-1}(c)) :=$$

$$\varepsilon(\gamma^{-1}(\beta \circ \alpha \circ \alpha^{-1}(b))) \cdot \varepsilon(\gamma^{-1}(\beta \circ \alpha \circ \alpha^{-1}(c))) \overset{def.}{=} \varepsilon(\delta \circ \alpha^{-1}(b)) \cdot \varepsilon(\delta \circ \alpha^{-1}(c)) \overset{(*)}{=}$$

$$\varepsilon(\alpha^{-1}(b)) \cdot \varepsilon(\alpha^{-1}(c)) = (a|b,c).\ \square$$

(2.3.3) (1) *Let* (E, ξ) *be a halfordered set such that* $Aut(E, \xi)$ *acts transitively on* E, *let* $0, 1 \in E$, *with* $0 \neq 1$ *and let* $P_\xi = \overrightarrow{0,1}$ *be the corresponding semidomain (cf.* (2.3.1)) *of* $(E, Aut(E, \xi); 0)$. *If* ξ_{P_ξ} *is the halforder of* E *corresponding to* P_ξ *according to* (2.3.2) *then* $\xi = \xi_{P_\xi}$.

(2) *Conversely let* $(E, \Sigma; 0)$ *be a locally transitive permutation set,* P *a semidomain of* $(E, \Sigma; 0), 1 \in P$, ξ_P *the associated halforder of* E *and* P_{ξ_P} *the semidomain of* $(E, Aut(E, \xi_P); 0)$. *Then* $P_{\xi_P} = P$.

Proof. (1) We have $P_\xi = \overrightarrow{0,1} := \{x \in E \setminus \{0\} \mid (0|1,x) = 1\}$ and

$$\varepsilon := \varepsilon_{P_\xi} : \begin{cases} E \setminus \{0\} & \to & \{1,-1\} \\ x & \mapsto & \varepsilon(x) := (0|1,x)_\xi \end{cases},$$

hence for $(a,b,c) \in E^{3'}$, $\alpha \in [0 \to a]$:

$$\xi_{P_\xi}(a,b,c) = (a|b,c)_{\xi_{P_\xi}} = \varepsilon(\alpha^{-1}(b)) \cdot \varepsilon(\alpha^{-1}(c)) := (0|1,\alpha^{-1}(b))_\xi \cdot (0|1,\alpha^{-1}(c))_\xi \stackrel{(\mathbf{Z})}{=}$$

$$(0|\alpha^{-1}(b), \alpha^{-1}(c))_\xi = (a|b,c)_\xi,$$

since $\alpha \in Aut(E,\xi)$. Hence $\xi = \xi_{P_\xi}$.

(2) Let $x \in E \backslash \{0\}$ and let $\omega \in [0 \to 0]$. Then $\omega = \omega^{-1} \circ \omega \circ \omega \in \Delta_0$, hence $\omega(P) \in \{P, P_-\}$ and $x \in P_{\xi_P} \Leftrightarrow 1 = (0|1,x)_{\xi_P} = \varepsilon_P(\omega^{-1}(1)) \cdot \varepsilon_P(\omega^{-1}(x)) \Leftrightarrow (\star)\ \varepsilon_P(\omega^{-1}(1)) = \varepsilon_P(\omega^{-1}(x))$.

Firstly we assume $\omega(P) = P$. Then: $1 \in \omega(P) \Leftrightarrow \omega^{-1}(1) \in P \Leftrightarrow 1 = \varepsilon_P(\omega^{-1}(1)) \stackrel{(\star)}{=} \varepsilon_P(\omega^{-1}(x)) \Leftrightarrow \omega^{-1}(x) \in P \Leftrightarrow x \in \omega(P) = P$.

Now let $\omega(P) = P_-$. Then:

$1 \notin \omega(P) \Leftrightarrow \omega^{-1}(1) \notin P \Leftrightarrow -1 = \varepsilon_P(\omega^{-1}(1)) \stackrel{(\star)}{=} \varepsilon_P(\omega^{-1}(x)) \Leftrightarrow \omega^{-1}(x) \notin P \Leftrightarrow x \notin \omega(P) = P_- \Leftrightarrow x \in P$. $\square$

(2.3.4) *Let P be a semidomain of the locally transitive permutation set $(E,\Sigma;0)$, let $SymE_{\{P,P_-\}} := \{\sigma \in SymE \ : \sigma(P) = P$ and $\sigma(P_-) = P_-$ or $\sigma(P) = P_-$ and $\sigma(P_-) = P\}$ and let $\Gamma := \{\sigma \in SymE_{\{P,P_-\}} \mid \forall\ \alpha \in \Sigma,\ \forall\ \alpha_1 \in [0 \to \sigma \circ \alpha(0)] : \alpha_1^{-1} \circ \sigma \circ \alpha \in SymE_{\{P,P_-\}}\}$. Then $Aut(E,\xi_P) = <\Sigma, \Gamma>$, where ξ_P is the halforder associated to P (cf. (2.3.2)).*

Proof. By (2.3.2) we have : $\Sigma \subset Aut(E,\xi_P)$. Let $(a,b,c) \in E^{3'}$ and let $\sigma \in \Gamma, \alpha \in [0 \to a]$ and $\alpha_1 \in [0 \to \sigma(a)]$; then $(a|b,c)_{\xi_P} := \varepsilon(\alpha^{-1}(b)) \cdot \varepsilon(\alpha^{-1}(c)) \stackrel{b' := \alpha^{-1}(b), c' := \alpha^{-1}(c)}{=} \varepsilon(b') \cdot \varepsilon(c') \stackrel{def.\ \Gamma}{=} \varepsilon(\alpha_1^{-1} \circ \sigma \circ \alpha(b')) \cdot \varepsilon(\alpha_1^{-1} \circ \sigma \circ \alpha(c')) = \varepsilon(\alpha_1^{-1} \circ \sigma \circ \alpha \circ \alpha^{-1}(b)) \cdot \varepsilon(\alpha_1^{-1} \circ \sigma \circ \alpha \circ \alpha^{-1}(c)) = (\sigma(a)|\sigma(b),\sigma(c))_{\xi_P}$, hence $\Gamma \subset Aut(E,\xi_P)$. This implies $<\Sigma,\Gamma> \leq Aut(E,\xi_P)$.

Now let $\gamma \in Aut(E,\xi_P)$ with $\gamma(0) = 0$; then for all $(a,b,c) \in E^{3'}$: $(a|b,c)_{\xi_P} = (\gamma(a)|\gamma(b),\gamma(c))_{\xi_P}$ and if $a = 0$, we obtain $(0|b,c)_{\xi_P} = \varepsilon_P(b) \cdot \varepsilon_P(c) = (0|\gamma(b),\gamma(c))_{\xi_P} =$

$\varepsilon_P(\gamma(b)) \cdot \varepsilon_P(\gamma(c))$ and this implies $\gamma \in SymE_{\{P,P_-\}}$. If $\alpha \in \Sigma$ and $\alpha_1 \in [0 \to \gamma \circ \alpha(0)]$ then $\alpha_1^{-1} \circ \gamma \circ \alpha \in Aut(E, \xi_P)$ and $\alpha_1^{-1} \circ \gamma \circ \alpha(0) = 0$. So we have, as before, $\alpha_1^{-1} \circ \gamma \circ \alpha \in SymE_{\{P,P_-\}}$, i.e. $\gamma \in \Gamma$. Moreover if $\sigma \in Aut(E, \xi_P)$, $a := \sigma(0), \alpha \in [0 \to a]$, then $\gamma := \alpha^{-1} \circ \sigma \in Aut(E, \xi_P)$ and $\gamma(0) = 0$, hence $\gamma \in \Gamma$ (as we have proved above) and so $\sigma = \alpha \circ \gamma \in \Sigma \circ \Gamma$, i.e. $Aut(E, \xi_P) \leq < \Sigma, \Gamma >$. $\square$

Remark. We observe that we can find functions $\sigma \in SymE_{\{P,P_-\}}$ such that $\sigma(0) = 0$ but $\sigma \notin Aut(E, \xi_P)$. For example if we consider:
$E := \mathbb{Z}_5$,
$\Sigma := \{1^+, -1^+, 2^+, 3^+, 0^+\}$,
then $(E, \Sigma; 0)$ is a locally transitive permutation set;
$P := \{1, -1\},\ P_- := \{3, 2\}$ and $\Delta_0 = \{id\}$ since $\mathbb{Z}_5$ is a group. If we set:
$\sigma : 0 \mapsto 0,\ 2 \mapsto 2,\ 3 \mapsto 3,\ 1 \mapsto -1,\ -1 \mapsto 1,$
then we have:
$(1| -1, 2) \overset{def.}{=} (0|3, 1) = -1$ and $(\sigma(1)|\sigma(-1), \sigma(2)) = (-1|1, 2) \overset{def.}{=} (0|2, 3) = 1$ and so $\sigma \notin Aut(E, \xi_P)$.

2.4 RELATED HALFORDERS AND VALUATIONS

Let (E,ξ) be a halfordered set with $|E|\geq 4$. Like in sec. 2.1 we associate to ξ the function:

$$\tau_\xi : E^{4'} \to \{1,-1\}; (a,b,c,d) \mapsto [a,b|c,d] := (a|c,d)\cdot(b|c,d).$$

Then by **(Z)** τ_ξ satisfies the conditions:
$\mathbf{(Z)}_l$ $\forall\ (a,b,c,d),(a,b,d,e)\in E^{4'} : [a,b|c,d]\cdot[a,b|d,e] = [a,b|c,e]$;
$\mathbf{(Z)}_r$ $\forall\ (a,b,d,e),(b,c,d,e)\in E^{4'} : [a,b|d,e]\cdot[b,c|d,e] = [a,c|d,e]$.
Each function $\tau : E^{4'} \to \{1,-1\}$ which satisfies $\mathbf{(Z)}_l$ and $\mathbf{(Z)}_r$ is called a *separation function* of E.[5]

Now let τ be a separation function, $(\binom{E}{4}) := \{(a,b,c,d)\in E^4 \mid \{a,b,c,d\}\in\binom{E}{4}\}$ and let

$$k_{\tau_r} : (\tbinom{E}{4}) \to \{1,-1\}; (a,b,c,d) \mapsto [a,b|c,d]\cdot[b,c|a,d]\cdot[c,a|b,d],$$

$$k_{\tau_l} : (\tbinom{E}{4}) \to \{1,-1\}; (a,b,c,d) \mapsto [a,b|c,d]\cdot[a,c|b,d]\cdot[a,d|b,c].$$

Remark. For all $x\in E$ with $(a,b,c,x)\in(\binom{E}{4})$ and $(x,b,c,d)\in(\binom{E}{4})$ the values $k_{\tau_r}(a,b,c,x)$ and $k_{\tau_l}(x,b,c,d)$ are constant and these values do not change if the elements a,b,c and b,c,d are permuted respectively.

(2.4.1) *If ξ is a halforder of E and $\tau := \tau_\xi$ the corresponding separation function then τ satisfies (besides* $\mathbf{(Z)}_l$ and $\mathbf{(Z)}_r$*) the condition*:
$\mathbf{(T)}^*$ $\forall\ (a,b,c,d)\in(\binom{E}{4}) : k_{\tau_r}(a,b,c,d) = k_{\tau_l}(d,a,b,c) = k_\xi(\{a,b,c\})$.

Proof. Let $(a,b,c,d)\in(\binom{E}{4})$. Then $k_{\tau_r}(a,b,c,d) := [a,b|c,d]\cdot[b,c|a,d]\cdot[c,a|b,d] \overset{\tau=\tau_\xi}{=} (a|c,d)\cdot(b|c,d)\cdot(b|a,d)\cdot(c|a,d)\cdot(c|b,d)\cdot(a|b,d) \overset{\mathbf{(Z)}}{=} (a|b,c)\cdot(b|a,c)\cdot(c|a,b) \overset{\mathbf{(Z)}}{=} (d|b,c)\cdot(a|b,c)\cdot(d|a,c)\cdot(b|a,c)\cdot(d|a,b)\cdot(c|a,b) \overset{\tau=\tau_\xi}{=} [d,a|b,c]\cdot[d,b|a,c]\cdot[d,c|a,b] =: k_{\tau_l}(d,a,b,c)$.□

[5]This definition of a separation function is stronger as in sec. 2.1 where only $\mathbf{(Z)}_l$ was claimed.

We denote the halforder (and the separation) function which is constant $= 1$ by 1. Two halforders ξ_1 and ξ_2 of E are called *equivalent*, denoted by $\xi_1 \approx \xi_2$, if the corresponding separation functions τ_{ξ_1} and τ_{ξ_2} are equal.
A function $\beta : E \to \{1, -1\}; x \mapsto \overline{x}$ which maps each $x \in E$ on one of the values 1 or -1 is called a *valuation*. If β is a valuation and ξ a halforder of E then also

$$\xi_\beta : E^{3'} \to \{1, -1\}; (a, b, c) \mapsto (a|b, c)_\xi \cdot \overline{b} \cdot \overline{c}$$

is a halforder of E. In fact $\forall\ (a, b, c), (a, c, d) \in E^{3'}$:

$$\xi_\beta(a, b, c) \cdot \xi_\beta(a, c, d) := (a|b, c)_\xi \cdot \overline{b} \cdot \overline{c} \cdot (a|c, d)_\xi \cdot \overline{c} \cdot \overline{d} \overset{(\mathbf{Z})}{=} (a|b, d)_\xi \cdot \overline{b} \cdot \overline{d} =: \xi_\beta(a, b, d),$$

i.e. ξ_β satisfies **(Z)**.

In sec. 2.1 two halforders ξ_1 and ξ_2 of E were called *related*, denoted by $\xi_1\ rel\ \xi_2$, if $\forall\ (a, b, c, d) \in E^{4'} : [a, b|c, d]_1 = [c, d|a, b]_2$ and a halforder ξ *selfrelated* if $\xi\ rel\ \xi$. Here we call a separation function τ *selfrelated* if $\forall\ (a, b, c, d) \in E^{4'} : [a, b|c, d] = [c, d|a, b]$.
We have the following

(2.4.2) *Let ξ_1, ξ_2 be two halforders of E. Then:*
(1) *If $\xi_1 \approx \xi_2$ then $k_{\xi_1} = k_{\xi_2}$ and : ξ_1 is selfrelated $\Leftrightarrow$ ξ_2 is selfrelated.*
(2) *$\xi_1 \approx \xi_2$ $\Leftrightarrow$ there is a valuation β with $\xi_1 = \xi_{2\beta}$.*

Proof. (1) Let $(a, b, c, d) \in (\binom{E}{4})$ then by (2.4.1):

$$k_{\xi_1}(\{a, b, c\}) = k_{\tau_{1_r}}(a, b, c, d) \overset{\xi_1 \approx \xi_2}{=} k_{\tau_{2_r}}(a, b, c, d) = k_{\xi_2}(\{a, b, c\}).$$

(2) " $\Rightarrow$ " Let $\{0, 1, \infty\} \in \binom{E}{3}$ be fixed; then $\forall\ x \in E \setminus \{1\}$ we set $\beta(x) = \overline{x} := (1|0, x)_{\xi_1} \cdot (1|0, x)_{\xi_2}$, and $\beta(1) = \overline{1} := (\infty|1, 0)_{\xi_1} \cdot (\infty|1, 0)_{\xi_2}$. We prove: $\forall\ (x, y, z) \in E^{3'}$:

$$\xi_1(x, y, z) = (x|y, z)_{\xi_2} \cdot \overline{y} \cdot \overline{z}.$$

1. case: $y, z \neq 1$:

$$(x|y, z)_{\xi_2} \cdot \overline{y} \cdot \overline{z} := (x|y, z)_{\xi_2} \cdot (1|0, y)_{\xi_1} \cdot (1|0, y)_{\xi_2} \cdot (1|0, z)_{\xi_1} \cdot (1|0, z)_{\xi_2} \overset{(\mathbf{Z})}{=}$$

$$(x|y,z)_{\xi_2}\cdot(1|y,z)_{\xi_1}\cdot(1|y,z)_{\xi_2}\overset{\xi_1\approx\xi_2}{=}(x|y,z)_{\xi_1}\cdot(1|y,z)_{\xi_1}\cdot(1|y,z)_{\xi_1}=(x|y,z)_{\xi_1}.$$

2. case: $y=1, z\neq 1,\infty$ (or $z=1, y\neq 1,\infty$) : $(x|1,z)_{\xi_2}\cdot\overline{1}\cdot\overline{z}:=$

$$(x|1,z)_{\xi_2}\cdot(\infty|1,0)_{\xi_1}\cdot(\infty|1,0)_{\xi_2}\cdot(1|0,z)_{\xi_1}\cdot(1|0,z)_{\xi_2}\overset{\xi_1\approx\xi_2}{=}$$

$$(x|1,z)_{\xi_2}\cdot(\infty|1,0)_{\xi_1}\cdot(\infty|1,0)_{\xi_2}\cdot(\infty|0,z)_{\xi_1}\cdot(\infty|0,z)_{\xi_2}\overset{(\mathbf{Z})}{=}$$

$$(x|1,z)_{\xi_2}\cdot(\infty|1,z)_{\xi_1}\cdot(\infty|1,z)_{\xi_2}\overset{\xi_1\approx\xi_2}{=}(x|1,z)_{\xi_1}\cdot(\infty|1,z)_{\xi_1}\cdot(\infty|1,z)_{\xi_1}=(x|1,z)_{\xi_1}.$$

3. case: $y=1, z=\infty$ (or $z=1, y=\infty$) and $x\neq 0$: $(x|1,\infty)_{\xi_2}\cdot\overline{1}\cdot\overline{\infty}\overset{(\mathbf{Z})}{=}$

$$(x|0,1)_{\xi_2}\cdot(x|0,\infty)_{\xi_2}\cdot(\infty|1,0)_{\xi_1}\cdot(\infty|1,0)_{\xi_2}\cdot(1|0,\infty)_{\xi_1}\cdot$$

$$(1|0,\infty)_{\xi_2}\overset{\xi_1\approx\xi_2}{=}(x|0,1)_{\xi_1}\cdot(\infty|0,1)_{\xi_1}\cdot(x|0,\infty)_{\xi_1}\cdot(1|0,\infty)_{\xi_1}\cdot(\infty|1,0)_{\xi_1}\cdot$$

$$(1|0,\infty)_{\xi_1}\overset{(\mathbf{Z})}{=}(x|1,\infty)_{\xi_1}.$$

4. case: $y=1, z=\infty, x=0$ (or $y=\infty, z=1, x=0$):

$$(0|1,\infty)_{\xi_2}\cdot\overline{1}\cdot\overline{\infty}=(0|1,\infty)_{\xi_2}\cdot(\infty|1,0)_{\xi_1}\cdot(\infty|1,0)_{\xi_2}\cdot(1|0,\infty)_{\xi_1}\cdot(1|0,\infty)_{\xi_2}=$$

$$k_{\xi_1}(\{0,1,\infty\})\cdot(0|1,\infty)_{\xi_1}\cdot k_{\xi_2}(\{0,1,\infty\})\overset{(1)\wedge\xi_1\approx\xi_2}{=}(0|1,\infty)_{\xi_1}.$$

5. case: $y=z=1$:

$$(x|1,1)_{\xi_2}\cdot\overline{1}\cdot\overline{1}=1=(x|1,1)_{\xi_1}.$$

" $\Leftarrow$ " $\forall\ (a,b,c,d)\in E^{4'}$:

$$[a,b|c,d]_{\xi_2}:=(a|c,d)_{\xi_2}\cdot(b|c,d)_{\xi_2}=$$

$$(a|c,d)_{\xi_2}\cdot\overline{c}\cdot\overline{d}\cdot(b|c,d)_{\xi_2}\cdot\overline{c}\cdot\overline{d}=:(a|c,d)_{\xi_1}\cdot(b|c,d)_{\xi_1}=:[a,b|c,d]_{\xi_1},$$

i.e. $\xi_2\approx\xi_1$. □

In particular if $\xi\approx 1$ then there is a valuation β with $\xi=1_\beta$.

A separation function $\tau: E^{4'}\to\{1,-1\}$ is called an *order* if $\forall\ \{a,b,c,d\}\in\binom{E}{4}$ exactly one of the values

$$[a,b|c,d],[a,c|d,b],[a,d|b,c]$$

is equal -1. Then the pair (E,τ) is called an *ordered set.*

(2.4.3) *Let τ be a separation function of a set E with $|E|\geq 4$ and let $\{0,1,\infty\}\in\binom{E}{3}$ be fixed. Then:*
(1) Each of the functions $f_i : E\setminus\{0,1,\infty\}\rightarrow\{1,-1\}$, with $i\in\{1,2\}$, $f_1(x):=k_{\tau_r}(\infty,1,0,x)$ and $f_2(x):=k_{\tau_l}(x,\infty,1,0)$ is constant $=:\gamma_i$.
(2) If $\gamma_1=\gamma_2=:\gamma$ then there is exactly one halforder $\xi=\xi(\tau;0,1,\infty)$ of E with
1. $\tau=\tau_\xi$
2. $\forall\ x,y\in E\setminus\{\infty\}:(\infty|x,y)=1$
3. $(0|1,\infty)=1$
and ξ has the form
(i) $(1|\infty,0)=\gamma$
(ii) $(a|b,c):=[\infty,a|b,c]$, *if* $b,c\neq\infty$
(iii) $(a|\infty,c):=\gamma\cdot[a,1|0,\infty]\cdot[\infty,a|0,c]$, *if* $a\neq 0,c\neq\infty$
(iv) $(0|\infty,c):=[\infty,0|1,c]$, *if* $c\neq\infty$
(v) $(a|\infty,\infty):=1.$

Proof. (1) By definition, $f_1(x)\cdot f_1(y)=[\infty,1|0,x]\cdot[1,0|\infty,x]\cdot[0,\infty|1,x]\cdot[\infty,1|0,y]\cdot[1,0|\infty,y]\cdot[0,\infty|1,y]\overset{(\mathbf{Z})_l}{=}[\infty,1|x,y]\cdot[1,0|x,y]\cdot[0,\infty|x,y]\overset{(\mathbf{Z})_r}{=}[\infty,0|x,y]\cdot[0,\infty|x,y]=1.$
In the same way $f_2(x)\cdot f_2(y)=1$.
(2) Let ξ be a halforder satisfying the conditions **1.**,**2.** and **3.**. Since $|E|\geq 4$ there is an $u\in E\setminus\{0,1,\infty\}$ and so
(i) $(1|\infty,0)\overset{\mathbf{1.}}{=}(u|\infty,0)\cdot[u,1|\infty,0]\overset{(2.4.3.1)}{=}(u|\infty,0)\cdot\gamma\cdot[u,0|1,\infty]\cdot[u,\infty|0,1]\overset{\mathbf{1.}}{=}$

$$\gamma\cdot(u|\infty,0)\cdot(u|1,\infty)\cdot(0|1,\infty)\cdot(u|0,1)\cdot(\infty|0,1)\overset{(\mathbf{Z})\wedge\ \mathbf{3.}}{=}\gamma\cdot(\infty|0,1)\overset{\mathbf{2.}}{=}\gamma.$$

Now let $(a,b,c)\in E^{3'}$. Then **1.**,**2.** and **3.** imply:
(ii) for $b,c\neq\infty$:

$$(a|b,c)\overset{\mathbf{2.}}{=}(a|b,c)\cdot(\infty|b,c)\overset{\mathbf{1.}}{=}[\infty,a|b,c];$$

(iii) for $a\neq 0,c\neq\infty,b=\infty$:

$$(a|\infty,c)\overset{(\mathbf{Z})}{=}(a|0,c)\cdot(a|\infty,0)\overset{(ii)\wedge\ \mathbf{1.}}{=}[\infty,a|0,c]\cdot[a,1|\infty,0]\cdot(1|\infty,0)\overset{(i)}{=}\gamma\cdot[a,1|\infty,0]\cdot[\infty,a|0,c];$$

(iv) for $c \neq \infty$ $(a = 0, b = \infty)$

$$(0|\infty, c) \overset{(\mathbf{Z})}{=} (0|1,\infty) \cdot (0|1,c) \overset{\mathbf{3.}\ \wedge\ (ii)}{=} [\infty, 0|1, c].$$

Consequently ξ is uniquely determined by these data.

It remains to show that if ξ is given by these data, ξ is a halforder satisfying **1.**,**2.** and **3.**. Let $(a,b,c),(a,c,d) \in E^{3'}$:

<u>case 1</u>: $b, c, d \neq \infty$: by (ii)

$$(a|b,c) \cdot (a|c,d) = [\infty, a|b,c] \cdot [\infty, a|c,d] \overset{(\mathbf{Z})_l}{=} [\infty, a|b,d] =: (a|b,d);$$

<u>case 2</u>: $a \neq 0; b, c \neq \infty; d = \infty$: by (ii) and (iii)

$$(a|b,c) \cdot (a|c,\infty) = [\infty, a|b,c] \cdot \gamma \cdot [a,1|\infty,0] \cdot [\infty, a|0,c] \overset{(\mathbf{Z})_l}{=}$$

$$\gamma \cdot [a,1|\infty,0] \cdot [\infty, a|b,0] =: (a|b,\infty);$$

<u>case 3</u>: $a \neq 0; b, d \neq \infty; c = \infty$: by (iii)

$$(a|b,\infty) \cdot (a|\infty,d) = \gamma \cdot [a,1|\infty,0] \cdot [\infty,a|0,b] \cdot \gamma \cdot [a,1|\infty,0] \cdot [\infty,a|0,d] \overset{(\mathbf{Z})_l}{=}$$

$$[\infty, a|b,d] =: (a|b,d);$$

<u>case 4</u>: $a = 0; b, d \neq \infty; c = \infty$: by (iv)

$$(0|b,\infty) \cdot (0|\infty,d) = [\infty,0|1,b] \cdot [\infty,0|1,d] \overset{(\mathbf{Z})_l}{=} [\infty,0|b,d] =: (0|b,d);$$

<u>case 5</u>: $a = 0; b = \infty; c, d \neq \infty$: by (ii) and (iv)

$$(0|\infty,c) \cdot (0|c,d) = [\infty,0|1,c] \cdot [\infty,0|c,d] \overset{(\mathbf{Z})_l}{=} [\infty,0|1,d] =: (0|\infty,d).$$

Hence ξ is a halforder.

By (iv) and $(\mathbf{Z})_l$, $(0|1,\infty) = 1$ and by (ii) $(\infty|x,y) = [\infty,\infty|x,y]$. But $(\mathbf{Z})_r$ implies $[a,a|d,e] = 1$ for all $(a,d,e) \in E^{3'}$. Therefore ξ has the properties **2.** and **3.**.

In order to prove: $\tau = \tau_\xi$, let $(a,b,c,d) \in E^{4'}$; we have to consider the following cases:

a) $c, d \neq \infty$:

$$\tau_\xi(a,b,c,d) := (a|c,d) \cdot (b|c,d) := [\infty,a|c,d] \cdot [\infty,b|c,d] \overset{(\mathbf{Z})_r}{=} [a,b|c,d];$$

b) $a, b \neq 0, c = \infty, d \neq \infty$:

$$\tau_\xi(a,b,\infty,d) := (a|\infty,d)\cdot(b|\infty,d) := \gamma\cdot[a,1|0,\infty]\cdot[\infty,a|0,d]\cdot\gamma\cdot[b,1|0,\infty]\cdot[\infty,b|0,d] \overset{(\mathbf{Z})_r}{=}$$

$$[a,b|0,\infty]\cdot[a,b|0,d] \overset{(\mathbf{Z})_l}{=} [a,b|d,\infty];$$

c) $c, d = \infty$:

$$\tau_\xi(a,b,\infty,\infty) := (a|\infty,\infty)\cdot(b|\infty,\infty) := 1\cdot 1 = 1 = [a,b|\infty,\infty];$$

d) $a = b = 0, c = \infty, d \neq \infty$:

$$\tau_\xi(0,0,\infty,d) := (0|\infty,d)\cdot(0|\infty,d) = 1 = [0,0|\infty,d];$$

e) $a \neq 0, b = 0, c = \infty, d \neq \infty, 1$ (we substitute γ):

$$\tau_\xi(a,0,\infty,d) := (a|\infty,d)\cdot(0|\infty,d) := \gamma\cdot[a,1|0,\infty]\cdot[\infty,a|0,d]\cdot[\infty,0|1,d] \overset{d\neq 1}{=}$$

$$[\infty,1|0,d]\cdot[1,0|\infty,d]\cdot[0,\infty|1,d]\cdot[a,1|0,\infty]\cdot[\infty,a|0,d]\cdot[\infty,0|1,d] \overset{(\mathbf{Z})_r}{=}$$

$$[1,a|0,d]\cdot[1,a|0,\infty]\cdot[1,0|\infty,d] \overset{(\mathbf{Z})_l}{=} [1,a|\infty,d]\cdot[1,0|\infty,d] \overset{(\mathbf{Z})_r}{=} [a,0|\infty,d];$$

f) $a \neq 0, b = 0, c = \infty, d = 1$:

$$\tau_\xi(a,0,\infty,1) := (a|\infty,1)\cdot(0|\infty,1) := \gamma\cdot[a,1|0,\infty]\cdot[\infty,a|0,1]\cdot 1 \overset{(2.4.3.1)}{=} [a,0|\infty,1]. \ \square$$

Remark. If the hypotheses of (2.4.3) are satisfied then

1. If τ is selfrelated then by (2.1.1) $\gamma_1 = \gamma_2 = \gamma = k_{\tau_l} = k_{\tau_r}$.

2. The following statements are equivalent:

(1) The functions k_{τ_l} and k_{τ_r} are constant.

(2) τ is selfrelated.

In fact $\forall\ (a,b,c,d) \in (\binom{E}{4})$:
$k_{\tau_l}(a,b,c,d) = k_{\tau_l}(c,d,a,b) \ \Leftrightarrow\ 1 = [a,b|c,d]\cdot[a,c|b,d]\cdot[a,d|b,c]\cdot[c,d|a,b]\cdot[c,a|d,b]\cdot[c,b|a,d] \overset{(2.4.3)}{=} (a|c,d)\cdot(b|c,d)\cdot(a|b,d)\cdot(c|b,d)\cdot(a|b,c)\cdot(d|b,c)\cdot(c|a,b)\cdot(d|a,b)\cdot(c|d,b)\cdot(a|d,b)\cdot(c|a,d)\cdot(b|a,d) = (a|b,d)\cdot(b|a,c)\cdot(d|a,c)\cdot(c|b,d) \ \Leftrightarrow\ [a,c|b,d] = [b,d|a,c]$.

(2.4.4) *Let τ be a separation function on a set E with $|E| \geq 4$. If τ is an order then τ is selfrelated.*

Proof. Let $\{a, b, 0, \infty\} \in \binom{E}{4}$. Since τ is an order exactly one of the values $[\infty, 0|a, b], [\infty, a|b, 0]$, $[\infty, b|0, a]$ and $[a, b|0, \infty], [a, 0|\infty, b], [a, \infty|b, 0]$ is equal -1 respectively.
If $[\infty, a|b, 0] = -1$ then $[\infty, 0|a, b] = [\infty, b|0, a] = [a, b|0, \infty] = [a, 0|\infty, b] = 1$ and so we have the thesis.
If $[\infty, a|b, 0] = 1$ we may assume $[\infty, 0|a, b] = -1$. Then $[\infty, b|0, a] = 1$. Suppose $[a, b|0, \infty] = 1$, then, since τ is an order, $[b, 0|a, \infty] = -1$. This contradicts $[\infty, 0|a, b] = -1$. Thus τ is selfrelated.□

(2.4.5) (1) *If a separation function τ is an order of E then each corresponding betweenness function $\xi := \xi(\tau; 0, 1, \infty)$ (cf. (2.4.3)) is an order of E.*
(2) *If (E, ξ) is an ordered set, then the separation function $\tau := \tau_\xi$ is an order.*

Proof. (1) By (2.4.4) τ is selfrelated and so the constants γ_1, γ_2 of (2.4.3.1) are equal. Therefore by (2.4.3.2) $\xi := \xi(\tau; 0, 1, \infty)$ is a halforder of E. Let $\{a, b, c\} \in \binom{E}{3}$. If $a, b, c \neq \infty$ then
$(a|b, c) = [\infty, a|b, c], \ (b|a, c) = [\infty, b|a, c], \ (c|a, b) = [\infty, c|a, b]$
and since τ is an order exactly one of these three values equals -1 and therefore $\gamma = -1$.
If $a, b \neq 0$ and $c = \infty$ then $(\infty|a, b) = [\infty, \infty|a, b] = 1$ by $\mathbf{(Z)}_r$, and since $\gamma = -1$, $(a|\infty, b) = -[a, 1|0, \infty] \cdot [\infty, a|0, b]$ and $(b|\infty, a) = -[b, 1|0, \infty] \cdot [\infty, b|0, a]$.
Now $(a|\infty, b) \cdot (b|\infty, a) = [a, b|0, \infty] \cdot [\infty, a|0, b] \cdot [\infty, b|0, a] \overset{(2.4.4)}{=} [\infty, 0|a, b] \cdot [\infty, a|0, b] \cdot [\infty, b|0, a] = k_{\tau_l} = \gamma = -1$, i.e. exactly one of the values $(\infty|a, b), (a|\infty, b), (b|\infty, a)$ equals -1.
If $a = 0$ then again
$(\infty|0, b) = [\infty, \infty|0, b] = 1, \ (0|\infty, b) = [0, \infty|1, b]$ and
$(b|\infty, 0) = \gamma \cdot [b, 1|0, \infty] \cdot [\infty, b|0, 0] = -[b, 1|0, \infty] \overset{(2.4.4)}{=} -[0, \infty|1, b]$.
Hence ξ is an order of E.
(2) Now let (E, ξ) be an ordered set and let $\{a, b, c, d\} \in \binom{E}{4}$ then

$[a, b|c, d] = (a|c, d) \cdot (b|c, d)$, $[a, c|b, d] = (a|b, d) \cdot (c|b, d)$, $[a, d|b, c] = (a|b, c) \cdot (d|b, c)$ and exactly one of the values $(b|c, d), (c|b, d), (d|b, c)$ is -1, for instance $(b|c, d) = -1$. Then $[a, b|c, d] = -(a|c, d)$, $[a, c|b, d] = (a|b, d)$ and $[a, d|b, c] = (a|b, c) = (a|c, d) \cdot (a|b, d)$. If $(a|c, d) = -1$ then $[a, b|c, d] = 1$ and $[a, d|b, c] = (a|b, c) = -(a|b, d) = -[a, c|b, d]$. If $(a|c, d) = 1$ then $[a, b|c, d] = -1$ and $[a, c|b, d] = (a|b, d) = (a|b, c) = [a, d|b, c]$. Suppose $(a|b, c) = (a|b, d) = -1$ then $(b|a, c) = (b|a, d) = 1$; hence $(b|c, d) = (b|a, c) \cdot (b|a, d) = 1 \cdot 1 = 1$, contradicting $(b|c, d) = -1$.□

By (2.4.2), (2.4.3) and (2.4.5) follows

(2.4.6) *Let (E, ξ) be a halfordered set. There is a valuation $\beta : E \to \{1, -1\}; x \mapsto \overline{x}$ such that the function $\xi_\beta : E^{3'} \to \{1, -1\}; (a, b, c) \mapsto (a|b, c)_\xi \cdot \overline{b} \cdot \overline{c}$ is an order if and only if (E, τ_ξ) is an ordered set.*

Remarks. There are pairs (ξ_1, ξ_2) of equivalent halforders (i.e. $\xi_1 \approx \xi_2$) of a set E such that:

1. ξ_1 is an order and ξ_2 is not an order.

2. ξ_1 is selfrelated, convex but not an order and ξ_2 is not convex.

3. ξ_1 is convex but not selfrelated and ξ_2 is not convex.

4. There are no particular relations between $\mathfrak{Z}_{\xi_1}$ and $\mathfrak{Z}_{\xi_2}$.

This shows that for halforders ξ and the corresponding separation functions τ_ξ we have the following diagram:

$$\xi \textit{ order} \;\Rightarrow\; \tau_\xi \textit{ order} \not\Rightarrow\; \xi \textit{ order},$$

(i.e. there are halforders ξ which are not orders but where τ_ξ is an order).

Proof. Let $E = \{a, b, c, d\}$ and let β be the valuation given by $\overline{a} := \overline{c} := \overline{d} := -\overline{b} := 1$.

1. An order ξ_1 on E is completely determined by $(b|a, c)_{\xi_1} := (c|b, d)_{\xi_1} := -1$ (implying $(b|a, d)_{\xi_1} = (c|a, d)_{\xi_1} = -1$) and $\xi_2 := \xi_{1\beta}$ is not an order.

2. If ξ_1 is defined by $(d|a, b, c)_{\xi_1} := 1$, $(c|a, b, d)_{\xi_1} := 1$, $(a|b, d)_{\xi_1} := (b|a, d)_{\xi_1} := -1$, $(a|c, d)_{\xi_1} := (b|c, d)_{\xi_1} := 1$, then (cf. (2.2.2) 1.$a$)) ξ_1 is selfrelated, convex but not an

order and $\xi_2 := \xi_{1\beta}$ is not convex since $(c|a,d)_{\xi_2} = -(b|a,d)_{\xi_2} = 1$ but $(c|b,d)_{\xi_2} = -1$.
3. Let ξ_1 be defined by $(a|b,c,d)_{\xi_1} = (c|a,b,d)_{\xi_1} = (d|a,b,c)_{\xi_1} = (b|c,d)_{\xi_1} = 1$ and $(b|a,c)_{\xi_1} = (b|a,d)_{\xi_1} = -1$, then ξ_1 is not selfrelated ($[a,c|b,d]_{\xi_1} = 1 = -[b,d|a,c]_{\xi_1}$) but convex and $\xi_2 := \xi_{1\beta}$ is not convex ($1 = (d|a,c)_{\xi_2} = -(b|a,c)_{\xi_2}$ and $(d|b,c)_{\xi_2} = -1$).$\square$

2.4.1 ORDERS AND VALUATIONS

Let (E,ξ) be an ordered set and let V be the set of all valuations of E; then V is a group with respect to the following operation: $\forall\, \beta_1, \beta_2 \in V : (\beta_1 \cdot \beta_2)(x) := \beta_1(x) \cdot \beta_2(x)$. Now we consider the set V_1 of all valuations $\beta : E \to \{1,-1\}; x \mapsto \overline{x}$ such that if ξ is an order of E then the functions $\xi' : E^{3'} \to \{1,-1\}; (a,b,c) \mapsto (a|b,c)_\xi \cdot \overline{b} \cdot \overline{c}$ are orders of E. We start from the case $E := \{a,b,c\}$. Then up to isomorphisms there is only one order which let be given by $(b|a,c) = -1, (a|b,c) = (c|a,b) = 1$ and so we can construct 8 valuations:

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8
$\beta(a)$	1	-1	1	1	-1	-1	-1	1
$\beta(b)$	1	-1	1	-1	1	-1	1	-1
$\beta(c)$	1	-1	-1	1	-1	1	1	-1

We observe that $\beta_i \in V \setminus V_1 \Leftrightarrow\ \beta_i(a) = \beta_i(c) = -\beta_i(b)$, therefore $V \setminus V_1 = \{\beta_4, \beta_5\}$; we remark moreover that V_1 is not a group because if we consider the product of β_3 and β_7 we obtain $\beta_5 \notin V_1$. If instead we compose $\beta_4 \cdot \beta_5$ we obtain $\beta_2 \in V_1$.

Definition 2.4.1 Two halforders ξ_1, ξ_2 of E, with $|E| = n$, are *isomorphic* if $\exists\, \sigma \in S_n$: $\xi_1 = \xi_2 \circ \sigma$.

A linearly ordered set $(E,<)$ can be described in the form (E,α) where α is the function

$$\alpha : \begin{cases} E^{2'} := \{(a,b) \in E^2 \mid a \neq b\} & \rightarrow & \{1,-1\} \\ (a,b) & \mapsto & (a|b) := \begin{cases} 1 & \text{if } a<b \\ -1 & \text{if } b<a \end{cases} \end{cases}.$$

If $|E| = n \in \mathbb{N}$ then E has $n!$ total orders α. To each such order α there corresponds an order betweenness function:

$$\xi_\alpha : \begin{cases} E^{3'} & \rightarrow & \{1,-1\} \\ (a,b,c) & \mapsto & (a|b,c) := (a|b)\cdot(a|c) \end{cases}.$$

On the other hand if $\xi : E^{3'} \mapsto \{1,-1\}$ is an order betweenness function then there are exactly two total orders α_1, α_2 of E such that $\xi = \xi_{\alpha_1} = \xi_{\alpha_2}$ and moreover for $(a,b) \in E^{2'}$, $\alpha_1(a,b) = -\alpha_2(a,b)$ (cf. [23],(13.3)).
Consequently a finite set E with $n := |E|$ can be provided with exactly $\frac{n!}{2}$ order betweenness functions and all these orders are isomorphic. Hence for $E = \{a,b,c,d\}$ there are 12 orders ξ of E which are all isomorphic and one ξ of them is defined by $(a|b,c) = (a|b,d) = (c|a,d) = (c|b,d) = -1$.

There are 2^4 possible valuations (we denote with V the set of all valuations) :

	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8
$\beta(a)$	1	-1	1	1	1	-1	-1	-1
$\beta(b)$	1	1	-1	1	1	-1	1	1
$\beta(c)$	1	1	1	-1	1	1	-1	1
$\beta(d)$	1	1	1	1	-1	1	1	-1

	β_9	β_{10}	β_{11}	β_{12}	β_{13}	β_{14}	β_{15}	β_{16}
$\beta(a)$	1	1	1	-1	-1	-1	1	-1
$\beta(b)$	-1	-1	1	-1	1	-1	-1	-1
$\beta(c)$	-1	1	-1	-1	-1	1	-1	-1
$\beta(d)$	1	-1	-1	1	-1	-1	-1	-1

As before we denote by V_1 the set of all valuations β_i of E such that the functions $\xi' : E^{3'} \to \{1,-1\}; (a,b,c) \mapsto (a|b,c)_\xi \cdot \beta_i(b) \cdot \beta_i(c)$ are orders of E. Then we have $V_1 = \{\beta_1, \beta_3, \beta_5, \beta_6, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{16}\}$. Since $\beta_5 \cdot \beta_6 = \beta_{14} \notin V_1$, the set V_1 is not a group. For the elements of $V \setminus V_1$ we obtain $\beta_7 \cdot \beta_8 = \beta_{11} \in V_1,\ \beta_{14} \cdot \beta_{15} = \beta_7 \in V \setminus V_1$. So also $V \setminus V_1$ is not a group. If we look for the set V_2 of all valuations β_i of E such that for each order ξ of E the functions $\xi' : E^{3'} \to \{1,-1\}; (a,b,c) \mapsto (a|b,c)_\xi \cdot \beta_i(b) \cdot \beta_i(c)$ is an order of E we obtain $V_2 = \{\beta_1, \beta_{16}\}$ and this is a group.

For the general case we introduce two new definitions: let $\xi : E^{3'} \to \{1,-1\}$ be an order and let $\beta : E \to \{1,-1\}$ be a valuation. Then ξ and β are called *not compatible* if $\exists\ \{a,b,c\} \in \binom{E}{3}$ with $(b|a,c) = -1$ and $\beta(a) = \beta(c) = -\beta(b)$; otherwise ξ and β are called *compatible*. Moreover let $\xi_\beta : E^{3'} \to \{1,-1\}$ with $\xi_\beta(a,b,c) := (a|b,c) \cdot \beta(b) \cdot \beta(c)$ (cf. p.71). Then:

(2.4.7) *If ξ is an order of E and β is a valuation of E then:*
the function ξ_β is an order $\Leftrightarrow$ ξ and β are compatible.

Proof. Let $\{a,b,c\} \in \binom{E}{3}$ with $(b|a,c) = -1$, hence $(a|b,c) = (c|a,b) = 1$. We restrict our considerations onto $\{a,b,c\}$. If $\beta = 1$ or $\beta = -1$ then $\xi_\beta = \xi$ and β is compatible with ξ. Let $\beta \neq 1,-1$ and let $\beta(b) = 1$ then we may assume $\beta(a) = -1$ and obtain $(b|a,c)_{\xi_\beta} = \beta(c), (a|b,c)_{\xi_\beta} = \beta(c), (c|a,b)_{\xi_\beta} = -1$. Hence: ξ_β is an order $\Leftrightarrow$ $\beta(c) = 1 \Leftrightarrow$ β is compatible with ξ.
If $\beta(b) = -1$ we may assume $\beta(a) = 1$ and obtain $(b|a,c)_{\xi_\beta} = -\beta(c), (a|b,c)_{\xi_\beta} = -\beta(c), (c|a,b)_{\xi_\beta} = -1$. Thus: ξ_β is an order $\Leftrightarrow$ $\beta(c) = -1 \Leftrightarrow$ β is compatible with ξ.$\square$

From (2.4.7) follows:
(2.4.8) *If ξ is an order of a finite set E then there are $2 \cdot |E|$ valuations β of E such that ξ_β is an order.*
Proof. Since all orders of a finite set are isomorphic we may assume $E = \mathbb{Z}_n = \{1,2,\ldots,n\}$ and if $a,b,c \in E$ then : $(b|a,c) = -1 \ \Leftrightarrow\ a < b < c$ or $c < b < a$. Let B_n

be the set of all compatible valuations of $\mathbb{Z}_n$ and for $k \leq n$ and $i_1, \ldots, i_k \in \{1, -1\}$ let $B_{i_1 \ldots i_k}$ be the set of all compatible valuations β with $\beta(l) = i_l$ for $l \in \{1, \ldots, k\}$. Then $B_{i_1 \ldots i_k} = \emptyset$ if there are $a, b, c \in \{1, \ldots, k\}$ with $a < b < c$ and $i_a = i_c = -i_b$ and so $B_n = B_{111} \cup B_{-1-1-1} \cup B_{11-1} \cup B_{-111} \cup B_{-1-11} \cup B_{1-1-1}$ if $n \geq 3$. Each of the four last sets consists of exactly one valuation β : for $m \in \mathbb{Z}_n \setminus \{1, 2, 3\}$ we have $\beta(m) = \beta(3)$. For the two first sets we have if $n > 3$: $B_{111} = B_{1111} \cup B_{111-1}$ with $B_{111-1} = \{\beta_{111-1}\}$ where $\beta_{111-1}(m) = -1$ for all $m \in \mathbb{Z}_n \setminus \{1, 2, 3\}$, $B_{-1-1-1} = B_{-1-1-1-1} \cup B_{-1-1-11}$ with $\beta_{-1-1-11}(m) = 1$ for all $m \in \mathbb{Z}_n \setminus \{1, 2, 3, 4\}$. This gives us the result $|B_3| = 6 = 2 \cdot 3$, $|B_4| = 4 + 2 \cdot 2 = 8 = 2 \cdot 4$ and
(∗) if $n > 4$: $|B_n| = 4 + 2 + |B_{1111}| + |B_{-1-1-1-1}|$.
For $m \in \mathbb{N}$ let $B_{m \cdot 1} := B_{\underbrace{1 \ldots 1}_{m}}$, $B_{m \cdot (-1)} := B_{\underbrace{-1 \ldots -1}_{m}}$, $B_{m \cdot 1, -1} := B_{\underbrace{1 \ldots 1}_{m} -1}$, $B_{m \cdot (-1), 1} := B_{\underbrace{-1 \ldots -1}_{m} 1}$; then for each m with $3 \leq m < n$, $B_{m \cdot 1} = B_{(m+1) \cdot 1} \dot{\cup} B_{m \cdot 1, -1}$, $B_{m \cdot (-1)} = B_{(m+1) \cdot (-1)} \dot{\cup} B_{m \cdot (-1), 1}$ with $|B_{m \cdot 1, -1}| = |B_{m \cdot (-1), 1}| = 1$. Hence:
(∗∗) $\forall\, m \in \mathbb{N}$ with $4 \leq m < n : |B_{m \cdot 1}| = |B_{(m+1) \cdot 1}| + 1$ and $|B_{m \cdot (-1)}| = |B_{(m+1) \cdot (-1)}| + 1$.
Since $|B_{n \cdot 1}| = |B_{n \cdot (-1)}| = 1$ one obtains from (∗∗) $|B_{4 \cdot 1}| = (n - 4) + 1 = n - 3 = |B_{4 \cdot (-1)}|$.
Thus by (∗) $|B_n| = 6 + 2 \cdot (n - 3) = 2 \cdot n$.□

2.5 CONVEX QUADRUPLES

Let (E,ξ) be a halfordered set and let τ_ξ be the corresponding separation function; $(a,b;c,d) \in (\binom{E}{4})$ is called a *convex quadruple* if $(a|b,c,d) = 1$ and $(b|c,d) = -1$. Let C be the set of all convex quadruples and $C' := \{(a,b;c,d) \in (\binom{E}{4}) \mid (a|c,d) = -(b|c,d) = 1\}$. If $M \subset E$ let $C(M) := M^4 \cap C$ and $C'(M) := M^4 \cap C'$.

Remarks. $C \subset C'$. If $(a,b;c,d) \in C'$ then $[a,b|c,d] = -1$ and if $(a,b;c,d) \in C$ then $(a,b;d,c) \in C$.

A subset $A \subset E$ is called *halfbounded* if there is an element $s \in E \setminus A$ with $(s|A) = 1$, i.e. $\forall\ a,b \in A : (s|a,b) = 1$.
$a \in E$ is called *end(point)* if $\exists\ b \in E :\ [a,b] = E$, *exceptional(point)* if $\forall\ x,y,z \in E \setminus \{a\}$ with $x \neq y,z : [a,x|y,z] = 1$. $\{a,b\} \in \binom{E}{2}$ are called *neighbour(point)s* if $]a,b[= \emptyset$.
A halforder ξ is called *separationfree* if one of the equivalent conditions is satisfied:
(a) τ_ξ is constant $= 1$;
(b) $C' = \emptyset$, i.e. (E,ξ) is *trivially convex*;
(c) $E^{2''} := \{\{a,b\} \in \binom{E}{2} \mid\ ]a,b[\ \neq \emptyset \wedge [a,b] \neq E\} = \emptyset$;
(d) $\forall\ a \in E,\ a$ is exceptional.

Proof. (a) $\Leftrightarrow$ (b). Let $(a,b;c,d) \in C'$; then by defintion $(a|c,d)\cdot(b|c,d) = [a,b|c,d] = -1$ and so τ_ξ is not constant $= 1$.
(a) $\Leftrightarrow$ (c). Suppose $E^{2''} \neq \emptyset$. Then $\exists\ \{a,b\} \in \binom{E}{2}\ :\ ]a,b[\ \neq \emptyset$ and $[a,b] \neq E$, i.e. $\exists\ c,d \in \binom{E\setminus\{a,b\}}{2} : (c|a,b) = -(d|a,b) = 1$. This implies : $[c,d|a,b] = -1$ and so τ_ξ is not constant $=1$. Vice versa let $\{x,y,a,b\} \in \binom{E}{4} : \tau_\xi(x,y,a,b) = (x|a,b)\cdot(y|a,b) = -1$. If $(x|a,b) = -(y|a,b) = 1$ (in the same way if $(y|a,b) = -(x|a,b) = 1$) then $\{a,b\} \in E^{2''}$.
(a) $\Leftrightarrow$ (d). Let $a \in E$ such that $\exists\ x,y,z \in E \setminus \{a\}$ with $x \neq y,z$ and $[a,x|y,z] = -1$. Then τ_ξ is not constant $= 1$. Vice versa let $(a,x,y,z) \in E^{4'} : \tau_\xi(a,x,y,z) = -1$. Then $a \in E$ is not exceptional.$\square$

From now on we assume that τ_ξ is not constant. Then by Definition 2.0.2 (E,ξ) is convex if $C' = C$. We recall (cf. (2.2.1)) that the statements "(E,ξ) is an order set" and "(E,ξ) is convex and harmonic" are equivalent.[6]

Let $C_r := \{(a,b;c,d) \in C \mid (a,c;b,d) \notin C$ and $(a,d;b,c) \notin C\}$ be the set of all *regular convex quadruples* and $C_i := \{(a,b;c,d) \in C \mid (a,c;b,d) \in C$ or $(a,d;b,c) \in C\}$ the set of all *irregular convex quadruples.*

Now we assume moreover that (E,ξ) is selfrelated, i.e. $\forall\ \{a,b,c,d\} \in \binom{E}{4} : [a,b|c,d] = [c,d|a,b]$. Then $(a,b;c,d) \in C'$ implies $-1 = [a,b|c,d] = (c|a,b)\cdot(d|a,b)$. Therefore exactly one of the values $(c|a,b),(d|a,b)$ is $=1$ and we can order the elements of C' such that $(d|a,b) = 1$. Let $\mathbf{C}' := \{(a,b;c,d) \in C' \mid (d|a,b) = 1\}$ and if $(a,b;c,d) \in \mathbf{C}'$ then $(d|a,c) = (d|b,c)$ by $(d|a,b) = 1$ and therefore there is the decomposition of $\mathbf{C}'$ by

$$\mathbf{C}'^{+} := \{(a,b;c,d) \in C' \mid (d|a,b,c) = 1\}$$

and

$$\mathbf{C}'^{-} := \{(a,b;c,d) \in C' \mid (d|a,c) = (d|b,c) = -1\}.$$

We set $\mathbf{C} := \mathbf{C}' \cap C$, $\mathbf{C}^+ := \mathbf{C}'^+ \cap C$, $\mathbf{C}^- := \mathbf{C}'^- \cap C$ (then $\mathbf{C} = \mathbf{C}^+ \dot{\cup} \mathbf{C}^-$), $\mathbf{C}_r := C_r \cap \mathbf{C}'$ and $\mathbf{C}_i := C_i \cap \mathbf{C}'$.

Now let $F := \{a,b,c,d\} \in \binom{E}{4}$. If $(a,b;c,d) \in \mathbf{C}_r$ then $(a|b,c,d) = 1, (b|c,d) = -1, (c|b,d) = (d|b,c) = 1$ and $(d|a,b) = 1$ hence $(d|a,b,c) = 1$ and $\mathfrak{Z}_\xi(\{b,c,d\}) = 1$ implying $k_\xi = -1$ and $\mathbf{C}_r(F) \subset \mathbf{C}^+(F)$. $(a|b,d) = (d|a,b) = 1$ and $k_\xi = -1$ gives $(b|a,d) = -1$ hence $\mathfrak{Z}_\xi(\{a,b,d\}) = 1$. $(a|d,c) = (d|a,c) = 1 = -k_\xi$ gives $\mathfrak{Z}_\xi(\{a,d,c\}) = 1$. Finally $(b|a,c) = (b|a,d)\cdot(b|c,d) = (-1)\cdot(-1) = 1$ and $(a|b,c) = 1 = -k_\xi$ implies $(c|a,b) = -1$ and $\mathfrak{Z}_\xi(\{a,b,c\}) = 1$. We have proved the first part of:

(2.5.1) *Let (E,ξ) be a selfrelated halfordered set, $(a,b;c,d) \in \mathbf{C}_r$ and $F := \{a,b,c,d\}$ then:*

1. *(E,ξ) is harmonic, the pair $(F,\xi_{|F})$ is an ordered set and* $\mathbf{C}(F) = \{(a,b;c,d),(d,c;b,a)\}$.

[6] A selfrelated halforder ξ is called *harmonic* if $k_\xi = -1$ and *anharmonic* if $k_\xi = 1$.

2. *If* $\mathbf{C}' = \mathbf{C}_r$ *then* (E,ξ) *is an ordered set.*

3. *If* $C = C_r \neq \emptyset$ *then* (E,ξ) *is an ordered set or there are three points* $\{u,v,w\}$ with $\mathfrak{Z}_\xi(\{u,v,w\}) = 3$ *which are not halfbounded (and then all* $3-$*sets* $\{x,y,z\}$ *with* $\mathfrak{Z}_\xi(\{x,y,z\}) = 3$ *are not halfbounded).*

Proof. 2. By **1.**, (E,ξ) is harmonic, and $\mathbf{C}' = \mathbf{C}_r$ implies that (E,ξ) is convex. Consequently by (2.2.1) (E,ξ) is an ordered set.

3. By **1.**, (E,ξ) is harmonic. If (E,ξ) is not ordered then there are $\{u,v,w\}$ with $(u|v,w) = (v|u,w) = (w|u,v) = -1$. If $\{u,v,w\}$ is halfbounded by t, hence $(t|u,v,w) = 1$ then $(t,u;v,w),(t,v;u,w),(t,w;u,v) \in C$, i.e. $(t,u;v,w) \notin C_r$ contradicting $C = C_r$.□

Now let $(a,b;c,d) \in \mathbf{C}_i^+ := \mathbf{C}_i \cap \mathbf{C}'^+$ hence $(a|b,c,d) = 1$, $(b|c,d) = -1$, $(d|c,b,a) = 1$ and so $(c|b,d) = -1$. Therefore $\mathfrak{Z}_\xi(\{b,c,d\}) = 2$ implying $k_\xi = 1$ (i.e. (E,ξ) is anharmonic) and so $(a,c;b,d) \in \mathbf{C}_i^+$ and $\mathfrak{Z}_\xi(\{a,b,d\}) = \mathfrak{Z}_\xi(\{a,c,d\}) = 0$. Now $(c|a,b) = (c|b,d)\cdot(c|d,a) = (-1)\cdot 1 = -1$ and $(a|b,c) = k_\xi = 1$ implies $(b|a,c) = -1$, i.e. $(d,b;c,a) \in \mathbf{C}^+$. Then $(a|b,c) = 1 = k_\xi$ implies $(c|a,b) = -1$ hence $(d,c;b,a) \in \mathbf{C}^+$ and so $(d,b;c,a),(d,c;b,a) \in \mathbf{C}_i^+$. Together one has:
$\mathbf{C}'(F) = \mathbf{C}(F) = \mathbf{C}_i^+(F) = \{(a,b;c,d),(a,c;b,d),(d,c;b,a),(d,b;c,a)\}$ and $(F,\xi_{|F})$ is the convex anharmonic halfordered set of four elements with $\mathfrak{Z}_\xi(\{a,b,c\}) = \mathfrak{Z}_\xi(\{b,c,d\}) = 2$ and $\mathfrak{Z}_\xi(\{a,c,d\}) = \mathfrak{Z}_\xi(\{a,b,d\}) = 0$ (cf. (2.2.2) 1.a)).

Finally let $(a,b;c,d) \in \mathbf{C}_i^- := \mathbf{C}_i \cap \mathbf{C}'^-$ thus $(a|b,c,d) = 1$, $(b|c,d) = (d|a,c) = (d|b,c) = -1$. This implies $k_\xi = (b|c,d)\cdot(c|b,d)\cdot(d|b,c) = (-1)\cdot(c|b,d)\cdot(-1) = (c|b,d) = (c|a,d)\cdot(a|c,d)\cdot(d|a,c) = (c|a,d)\cdot 1\cdot(-1) = -(c|a,d) \Rightarrow (c|a,b) = (c|b,d)\cdot(c|a,d) = k_\xi\cdot(-k_\xi) = -1$. Now $k_\xi = (a|b,c)\cdot(b|c,a)\cdot(c|a,b) = 1\cdot(b|c,a)\cdot(-1)$ hence $(b|c,a) = -k_\xi$ and $(b|a,d) = (-k_\xi)\cdot(b|c,d) = (-k_\xi)\cdot(-1) = k_\xi$. Together we have $k_\xi = (b|a,d) = -(b|a,c) = (c|b,d) = -(c|a,d)$.

Case: $k_\xi = -1 \Rightarrow \mathfrak{Z}_\xi(\{a,b,c\}) = \mathfrak{Z}_\xi(\{a,c,d\}) = \mathfrak{Z}_\xi(\{a,b,d\}) = 1$, $\mathfrak{Z}_\xi(\{b,c,d\}) = 3$, $\mathbf{C}(F) = \{(a,b;c,d),(a,d;b,c),(a,c;d,b)\} = \mathbf{C}_i^-(F)$ and $\mathbf{C}'(F)\setminus \mathbf{C}(F) = \{(d,c;b,a),(c,b;d,a),(b,d;c,a)\}$.

Case: $k_\xi = 1 \Rightarrow \mathfrak{Z}_\xi(\{a,b,c\}) = \mathfrak{Z}_\xi(\{a,c,d\}) = \mathfrak{Z}_\xi(\{b,c,d\}) = 2,\ \mathfrak{Z}_\xi(\{a,b,d\}) = 0$, $\mathbf{C}(F) = \{(a,b;c,d),(a,d;c,b)\}$ and $\mathbf{C}'(F)\setminus \mathbf{C}(F) = \{(b,c;d,a),(d,c;b,a)\}$ (cf. (2.2.2) 1. b)).

We have proved the following

(2.5.2) *Let (E,ξ) be a selfrelated halfordered set, $F := \{a,b,c,d\}$ and $(a,b;c,d) \in \mathbf{C}_i$.*
1. *If $k_\xi = -1$ and $(d|a,c) = (d|b,c) = -1$, i.e. $(a,b;c,d) \in \mathbf{C}'^{-}$, then $\mathfrak{Z}_\xi(\{a,b,c\}) = \mathfrak{Z}_\xi(\{a,c,d\}) = \mathfrak{Z}_\xi(\{a,b,d\}) = 1,\ \mathfrak{Z}_\xi(\{b,c,d\}) = 3$, $\mathbf{C}(F) = \{(a,b;c,d),(a,d;b,c),(a,c;d,b)\} = \mathbf{C}_i^-(F)$ and $\mathbf{C}'(F)\setminus \mathbf{C}(F) = \{(d,c;b,a),(c,b;d,a),(b,d;c,a)\}$.*
2. *If $k_\xi = 1$ and $(d|a,b,c) = 1$, i.e. $(a,b;c,d) \in \mathbf{C}'^{+}$, then $\mathbf{C}'(F) = \mathbf{C}(F) = \mathbf{C}_i^+(F) = \{(a,b;c,d),(a,c;b,d),(d,c;b,a),(d,b;c,a)\}$ and $(F,\xi_{|F})$ is the convex anharmonic halfordered set of four elements with $\mathfrak{Z}_\xi(\{a,b,c\}) = \mathfrak{Z}_\xi(\{b,c,d\}) = 2$ and $\mathfrak{Z}_\xi(\{a,c,d\}) = \mathfrak{Z}_\xi(\{a,b,d\}) = 0$.*
3. *If $k_\xi = 1$ and $(d|a,c) = (d|b,c) = -1$, i.e. $(a,b;c,d) \in \mathbf{C}'^{-}$, then $\mathfrak{Z}_\xi(\{a,b,c\}) = \mathfrak{Z}_\xi(\{a,c,d\}) = \mathfrak{Z}_\xi(\{b,c,d\}) = 2,\ \mathfrak{Z}_\xi(\{a,b,d\}) = 0$, $\mathbf{C}(F) = \{(a,b;c,d),(a,d;c,b)\}$ and $\mathbf{C}'(F)\setminus \mathbf{C}(F) = \{(b,c;d,a),(d,c;b,a)\}$.*

2.5.1 WEAK CONVEXITY

Let $D' := \{(c,a:b,d) \in (\binom{E}{4}) \mid (a|b,c) = (b|a,d) = -1\}$ and $D := \{(c,a:b,d) \in D' \mid (c|a,d) = (d|b,c) = 1\}$. We observe that if $(c,a:b,d) \in D'$ then $(d,b:a,c) \in D'$.

(E,ξ) is called *D-convex* if $D' = D$.

(2.5.3) *A set $E := \{a,b,c,d\}$ of four elements has up to isomorphisms two selfrelated halforders with $D' = D$ and $C' \neq C$, one harmonic with $|C'| = 12$ and $|C| = 6$ and one anharmonic with $|C'| = 8$ and $|C| = 4$.*

Proof. $C \subset C'$, with $C \neq C'$, implies $C' \neq \emptyset$. Since ξ has to be selfrelated we have to look for a quadruple $(d,b;a,c) \in \mathbf{C}' \setminus C$, hence a quadruple with $(d|a,c) = 1,\ (b|a,c) =$

-1, $(c|b,d) = 1$ and $(d|a,b) = (d|b,c) = -1$.

Then $(a|b,d) = (c|b,d) \cdot (b|a,c) \cdot (d|a,c) = 1 \cdot (-1) \cdot 1 = -1$ and so:

(i) $(a|b,c) = -(a|c,d)$

(ii) $(b|a,d) = -(b|c,d)$

(iii) $(c|a,b) = (c|a,d)$

$k_\xi = -1$ (harmonic case): $(d|a,b) = (a|b,d) = -1$ implies $(b|a,d) = -1$ and by (ii) $(b|c,d) = 1$.

If $(a|b,c) = -1$ then $(d,b:a,c) \in D' \setminus D$, a contradiction to $D' = D$. Hence $(a|b,c) = 1$ and $(a|c,d) = -1$. $(a|b,c) = 1$ and $(b|a,c) = -1$ imply (by $k_\xi = -1$), $1 = (c|a,b) \overset{(iii)}{=} (c|a,d)$. By this data a harmonic halforder ξ with $D = D'$ is completely determined and one obtains: $D = D' = \{(c,b:a,d),(c,a:d,b),(c,d:b,a),(d,a:b,c),(b,d:a,c),(a,b:d,c)\}$, $C' = \{(a,d;b,c),(c,d;a,b),(c,b;a,d),(b,a;c,d),(c,a;b,d),(d,b;a,c),(a,d;c,b),$ $(c,d;b,a),(c,b;d,a),(b,a;d,c),(c,a;d,b),(d,b;c,a)\}$, $C = \{(c,d;a,b),(c,b;a,d),(c,a;b,d),$ $(c,d;b,a),(c,b;d,a),(c,a;d,b)\}$ and moreover $\tau_{\xi_{|\binom{E}{4}}} = -1$, i.e. $\forall\ \{x,y,z,u\} \in \binom{E}{4}$: $[x,y|z,u] = -1$.

$k_\xi = 1$ (anharmonic case): $(d|a,b) = (a|b,d) = -1$ implies here $(b|a,d) = 1$ and so by (ii), $(b|c,d) = -1$.

If $(c|a,b) = -1$ then $(a,c:b,d) \in D' \setminus D$ (since $(d|a,b) = -1$), a contradiction to $D' = D$. Hence $(c|a,b) = 1$ and by (iii), $(c|a,d) = 1$. $(d|a,c) = (c|a,d) = k_\xi = 1$ imply $(a|c,d) = 1$ and by (i), $(a|b,c) = -1$. Here (E,ξ) is an anharmonic halfordered set with $D = D' = \{(c,b:d,a),(c,b:a,d),(a,d:b,c),(d,a:b,c)\}$, $C' = \{(c,a;b,d),(c,a;d,b),(c,d;a,b),(c,d;b,a),(a,b;c,d),(a,b;d,c),(d,b;a,c),(d,b;c,a)\}$ and $C = \{(c,a;b,d),(c,a;d,b),(c,d;a,b),(c,d;b,a)\}$. □

Now let (E,ξ) be selfrelated with $k_\xi = 1$ and convex, i.e. $C = C'$ and let $\{a,b,c,d\} \in \binom{E}{4}$, with $(a,b:c,d) \in D'$ (hence $(b|a,c) = (c|b,d) = -1$). Then $(d|a,c) = -(b|a,c) \cdot (d|a,c) = -[b,d|a,c] = -[a,c|b,d] = -(a|b,d) \cdot (c|b,d) = (a|b,d)$. Hence if $(d|a,c) = (a|b,d) = 1$ then $(a,b:c,d) \in D$.

If $(d|a,c) = (a|b,d) = -1$ then besides

$$(i)\ (b|a,d) = -(b|c,d),$$
$$(ii)\ (c|a,b) = -(c|a,d), \quad \text{we have}$$
$$(iii)\ (a|b,c) = -(a|c,d),$$
$$(iv)\ (d|a,b) = -(d|b,c).$$

We discuss the two cases:
a) $(c|a,b) = 1$. Then $(c|a,d) = -1$ by (ii) and since $k_\xi = 1$, $(a|c,d) \overset{(iii)}{=} -(a|b,c) = 1$. Suppose $(d|a,b) = -1$, i.e. $(c,d;a,b) \in C'$; then $C' = C$ implies $(c|a,d) = 1$. Thus $(d|a,b) = 1$ and $(d|b,c) = -1$ by (iv). $(a|b,d) = -1$, $(d|a,b) = 1$ and $k_\xi = 1$ gives $(b|c,d) = -(b|a,d) = 1$. Then $\xi_{|\{a,b,c,d\}} \approx 1_{|\{a,b,c,d\}}$.
b) $(c|a,b) = -1$. Then $(c|a,d) = 1$ by (ii) and since $k_\xi = 1$ we obtain $(a|b,c) = -(a|c,d) = 1$. If $(b|a,d) = -1$ then $(c,b;a,d) \in C'$ hence $(c|a,b) = 1$ by $C = C'$ a contradiction to $(c|a,b) = -1$. Hence $1 = (b|a,d) = -(b|c,d)$ by (i). This together with $k_\xi = 1$ and $(c|b,d) = -1$ gives $(d|b,c) = -(d|a,b) = 1$. Then also in this case $\xi_{|\{a,b,c,d\}} \approx 1_{|\{a,b,c,d\}}$.
We have proved the result:

(2.5.4) *Let (E,ξ) be selfrelated with $k_\xi = 1$, convex and let $(a,b:c,d) \in D'$. If $(a,b:c,d) \notin D$ then $(\{a,b,c,d\}, \xi_{|\{a,b,c,d\}})$ is trivially convex.*

(2.5.5) *Let (E,ξ) be selfrelated with $k_\xi = -1$ and convex. Then (E,ξ) is ordered and $D = D'$.*

Proof. By (2.2.1) ξ is an order. Let $(c,a:b,d) \in D'$; then by definition $(a|b,c) = (b|a,d) = -1$. Since ξ is an order $(c|a,b) = (b|a,c) = 1$ and by **(Z)** we obtain $(b|c,d) = -1$ Since ξ is an order $(d|b,c) = (c|b,d) = 1$ and by **(Z)** $(c|a,d) = 1$ This implies $(c,a:b,d) \in D$. □

(2.5.6) *Let (E,ξ) be selfrelated, with $C = C'$* and *$D = D' \neq \emptyset$. For each $(a,b:c,d) \in D$ we have:*

(1) $(a|b,c,d) = (d|a,b,c) = 1, (b|a,c) = (c|b,d) = -1, (b|a,d) = (c|a,d)$ *and* $(b|c,d) = (c|a,b)$.

(2) *If* $(b|a,d) = (c|a,d) = -1$ *then* $(b|c,d) = (c|a,b) = 1$ *and* ξ *induces an order on* $\{a,b,c,d\}$.

(3) *If* $(b|a,d) = (c|a,d) = 1$ *then* $(b|c,d) = (c|a,b) = -1, k_\xi = 1$ *and* $\forall\ \{x,y,z\} \in \binom{E}{3}$: $\mathfrak{Z}_\xi(\{x,y,z\}) \subset \{0,2\}$.

Proof. (1) If $(a,b:c,d) \in D$ then by definition of D $(a|b,d) = (d|a,c) = 1 = -(b|a,c) = -(c|b,d)$ and this implies $(d,b;a,c),(a,c;b,d) \in C' = C$. Then by definition of C $(a|b,c,d) = 1 = (d|a,b,c)$. Moreover $(b|a,d) \cdot (c|a,d) = [b,c|a,d] \overset{\xi\ self.}{=} [a,d|b,c] = (a|b,c) \cdot (d|b,c) = 1$ implies $(b|a,d) = (c|a,d)$ and since $(b|a,c) = (c|b,d)$,we obtain by **(Z)** $(b|c,d) = (c|a,b)$. □

(2.5.7) *A set $E = \{a,b,c,d,e\}$ of five elements has two selfrelated halforders with $C = C', D = D', k_\xi = 1$ and $(a,b:c,d) \in D$, namely $(a|b,c,d,e) = (d|a,b,c,e) = (e|a,b,c,d) = (b|a,d,e) = (c|a,d,e) = -(b|a,c) = -(b|c,e) = -(c|a,b) = -(c|b,e) = 1$ and $(a|b,c,d,e) = (d|a,b,c,e) = (b|a,d) = (b|c,e) = (c|a,d) = (c|b,e) = (e|a,d) = (e|b,c) = -(b|a,c) = -(b|d,e) = -(c|b,d) = -(c|a,e) = -(e|a,b) = -(e|a,c) = 1$.*

Proof. By (2.5.6.1) we have $1 = (a|b,c,d) = (d|a,b,c) = -(b|a,c) = -(c|b,d)$. The assumption $k_\xi = 1$ implies by (2.5.6.2) and (2.5.6.3), $(b|a,d) = (c|a,d) = 1$ and $(b|c,d) = (c|a,b) = -1$. Moreover $(a|e,d) = (d|e,a) = 1$; for $(a|e,d) = -1$ implies $(a|e,c) = -(a|c,d) = -1$, i.e. $(e,a:c,b) \in D' = D$. Thus by (2.5.6.1), $(e|a,b,c) = (b|a,c,e) = (a|e,b) = (c|e,b) = 1$, $(a|c,b) = (c|e,a) = -1$ and $(a|e,c) = (c|a,b) = -1$, a contradiction to $(a|c,b) = 1$. Together $(a|b,c,d,e) = (d|a,b,c,e) = 1$. By $(a|e,b) = 1$ and $k_\xi = 1$, $(b|a,e) = (e|a,b)$; therefore we have to consider the following two cases:

1. case $(b|a,e) = (e|a,b) = 1$. By **(Z)** we obtain $(b|c,e) = (b|a,e) \cdot (b|a,c) = 1 \cdot (-1) = -1$ and so $(d,c:b,e) \in D'$ implying $(e|b,c,d) = 1$ and $(c|d,e) = (b|d,e) = 1$ by (2.5.6.3).

Hence $(e|a,b,c,d)=1$ and $(c|e,b)=(c|b,d)\cdot(c|e,d)=(-1)\cdot 1=-1$.

2. case $(b|a,e)=(e|a,b)=-1$. Then $(b|c,e)=(b|a,e)\cdot(b|a,c)=(-1)\cdot(-1)=1$ and since $(c|b,d)=-1$, $-(e|b,d)=(c|b,d)\cdot(e|b,d)\overset{\xi\ self.}{=}(b|e,c)\cdot(d|e,c)=1\cdot 1=1$, i.e. $(e|b,d)=-1$. Now $(d|b,e)=1$ and $k_\xi=1$ implies $(b|e,d)=-1$. Then $(e|a,d)=(e|b,d)\cdot(e|a,b)=(-1)\cdot(-1)=1$. Since $(a|c,e)=1=(b|e,c)$ we have by $k_\xi=1$ $(e|a,c)=(c|a,e),(e|b,c)=(c|b,e)$ and from $(e|a,b)=-1=(c|a,b)$ we obtain $(e|a,c)=-(e|b,c),(c|a,e)=-(c|b,e)$. If $(e|a,c)=-(e|b,c)=1$ then $(a,b:e,c)\in D'=D$ and so $(c|a,b,e)=1$ contradicting $(c|a,b)=-1$. Therefore $(e|a,c)=-1$. $\square$

2.5.2 CONVEX QUADRUPLES IN A HALFORDERED FIELD

We begin with a supplement of sec. 2.3. Let P be a semidomain of the locally transitive permutation set $(E,\Sigma;0)$ and ξ_P the associated halforder (cf. (2.3.2)). For $n\in\mathbb{N}$ and $T\subset Aut(E,\xi_P)$ let $(T^n)':=\{(\alpha_1,...,\alpha_n)\in T^n \mid |\{\alpha_1(0),...,\alpha_n(0),0\}|=n+1\}$ and let

$$k:\begin{cases} ((Aut(E,\xi_P))^3)' & \to \quad \{1,-1\} \\ (\alpha,\beta,\gamma) & \mapsto \quad k(\alpha,\beta,\gamma):=\varepsilon(\alpha^{-1}\circ\beta(0))\cdot\varepsilon(\beta^{-1}\circ\alpha(0))\cdot\varepsilon(\beta^{-1}\circ\gamma(0))\cdot \\ & \qquad \varepsilon(\gamma^{-1}\circ\beta(0))\cdot\varepsilon(\gamma^{-1}\circ\alpha(0))\cdot\varepsilon(\alpha^{-1}\circ\gamma(0)) \end{cases}.$$

P is called *T-selfrelated* if the restriction $k_{|(T^3)'}$ is constant, and *selfrelated* if P is $\Sigma-$selfrelated.

(2.5.8) (Theorem) (1) *If there is a $\Phi\subset Aut(E,\xi_P)$ with $\Phi(0)=E$ such that P is $\Phi-$selfrelated then for each $T\subset Aut(E,\xi_P)$, P is $T-$selfrelated.*

(2) *ξ_P is selfrelated $\Leftrightarrow$ P is selfrelated.*

(3) *If there is a $\Phi\subset Aut(E,\xi_P)$ with $\Phi(0)=E$ and a map $\omega:(\Phi^2)'\to SymE_{\{P,P_-\}};(\alpha,\beta)\mapsto\omega_{\alpha,\beta}$ with $\omega_{\alpha,\beta}\circ\beta^{-1}\circ\alpha(0)=\alpha^{-1}\circ\beta(0)$ and $\varepsilon(\omega_{\alpha,\beta}\circ\beta^{-1}\circ\gamma(0))\cdot\varepsilon(\omega_{\alpha,\gamma}\circ\gamma^{-1}\circ\beta(0))$ is constant for all $(\alpha,\beta,\gamma)\in(\Phi^3)'$ then P is selfrelated.*

(4) *If* $Aut(E,\xi_P) =< \Sigma,\Gamma >$ *(cf.* (2.3.4)) *acts* 2−*transitively* [7] *on* E *and if there is a* $\Phi \subset Aut(E,\xi_P)$ *with* $\Phi(0) = E$*, a* $\tau \in \Phi'$ *and a map* $\omega : \Phi' \to SymE_{\{P,P_-\}}; \alpha \mapsto \omega_\alpha$ *with* $\omega_\alpha \circ \alpha^{-1}(0) = \alpha(0)$ *and* $\varepsilon(\omega_\tau \circ \tau^{-1} \circ \alpha(0)) \cdot \varepsilon(\omega_\alpha \circ \alpha^{-1} \circ \tau(0))$ *is constant for all* $\alpha \in \Phi'$ *then* P *is selfrelated.*

(5) *If there is a* $\Phi \subset Aut(E,\xi_P)$ *with* $\Phi(0) = E$ *and* $\varepsilon(\alpha^{-1} \circ \beta(0)) \cdot \varepsilon(\beta^{-1} \circ \alpha(0))$ *is constant* $= \varepsilon_0$ *for all* $(\alpha,\beta) \in (\Phi^2)'$ *then* P *is selfrelated (* $k(\alpha,\beta,\gamma) = \varepsilon_0^3 = \varepsilon_0$*).*

Proof. (1) Clearly for each $T \subset \Phi$, P is T−selfrelated. Therefore it is enough to discuss the case $T = Aut(E,\xi_P)$. Let $(\alpha,\beta,\gamma) \in (Aut(E,\xi_P)^3)'$ and $\alpha',\beta',\gamma' \in \Phi$ with $\alpha'(0) = \alpha(0)$, $\beta'(0) = \beta(0)$, $\gamma'(0) = \gamma(0)$ (possible since $\Phi(0) = E$). Then $k(\alpha,\beta,\gamma) \overset{(2.3.2)}{=} (\alpha(0)|\beta(0),\gamma(0)) \cdot (\beta(0)|\gamma(0),\alpha(0)) \cdot (\gamma(0)|\alpha(0),\beta(0)) = k(\alpha',\beta',\gamma')$.

(2) Let $\{a,b,c\} \in \binom{E}{3}$ and let $\alpha,\beta,\gamma \in \Sigma$ with $\alpha(0) = a$, $\beta(0) = b$, $\gamma(0) = c$. Then $k_\xi(\{a,b,c\}) = (a|b,c) \cdot (b|c,a) \cdot (c|a,b) \overset{(2.3.2)}{=} k(\alpha,\beta,\gamma)$.

(3) Let $(\alpha,\beta,\gamma) \in (\Phi^3)'$. Then by assumption $\varepsilon(\beta^{-1}\circ\gamma(0))\cdot\varepsilon(\beta^{-1}\circ\alpha(0)) = \varepsilon(\omega_{\alpha,\beta}\circ\beta^{-1}\circ\gamma(0))\cdot\varepsilon(\alpha^{-1}\circ\beta(0))$ and $\varepsilon(\gamma^{-1}\circ\beta(0))\cdot\varepsilon(\gamma^{-1}\circ\alpha(0)) = \varepsilon(\omega_{\alpha,\gamma}\circ\gamma^{-1}\circ\beta(0))\cdot\varepsilon(\alpha^{-1}\circ\gamma(0))$ and therefore $k(\alpha,\beta,\gamma) = \varepsilon(\omega_{\alpha,\beta}\circ\beta^{-1}\circ\gamma(0))\cdot\varepsilon(\omega_{\alpha,\gamma}\circ\gamma^{-1}\circ\beta(0))$.

(4) Let $\{x,y,z\} \in \binom{E}{3}$. Since $Aut(E,\xi_P)$ is 2−transitive there is a $\psi \in Aut(E,\xi_P)$ with $\psi(x) = 0$, $\psi(y) = \tau(0)$ and $k_\xi(\{x,y,z\}) = k_\xi(\{0,\tau(0),\psi(z)\})$. Let $\alpha \in \Phi$ with $\alpha(0) = \psi(z)$. Then $k_\xi(\{x,y,z\}) = (0|\tau(0),\alpha(0))\cdot(0|\tau^{-1}\circ\alpha(0),\tau^{-1}(0))\cdot(0|\alpha^{-1}(0),\alpha^{-1}\circ\tau(0)) = (0|\tau(0),\alpha(0))\cdot(0|\omega_\tau\circ\tau^{-1}\circ\alpha(0),\tau(0))\cdot(0|\alpha(0),\omega_\alpha\circ\alpha^{-1}\circ\tau(0)) = (0|\omega_\tau\circ\tau^{-1}\circ\alpha(0),\omega_\alpha\circ\alpha^{-1}\circ\tau(0)) = \varepsilon(\omega_\tau\circ\tau^{-1}\circ\alpha(0))\cdot\varepsilon(\omega_\alpha\circ\alpha^{-1}\circ\tau(0))$.□

In the following let $(K,+,\cdot)$ be a field. For $a \in K$ let $a^+ : K \to K; x \mapsto a + x$ and $a^\bullet : K \to K; x \mapsto a \cdot x$. Then $Aff(K,+,\cdot) := \{a^+ \circ b^\bullet \mid a \in K, b \in K^* := K \setminus \{0\}\}$ is a subgroup of $SymK$, called the *affine group* of the field K and $Aff(K,+,\cdot)$ acts sharply 2−transitively on K. In fact if a_1,a_2,b_1,b_2 are elements of K, with $a_1 \neq a_2$, $b_1 \neq b_2$,

[7] A subgroup A of the symmetric group $SymK$ of a set K acts (sharply) 2−transitively on K if for any two pairs $(a_1,a_2),(b_1,b_2)$ of K^2, with $a_1 \neq a_2$ and $b_1 \neq b_2$, there exists (exactly) one $g \in A$ such that $g(a_i) = b_i$, with $i \in \{1,2\}$.

then (since K is a field) there exists exactly the element

$$g := (\frac{a_1 \cdot b_2 - b_1 \cdot a_2}{a_1 - a_2})^+ \circ (\frac{b_1 - b_2}{a_1 - a_2})^\bullet \in Aff(K, +, \cdot)$$

such that $g(a_i) = b_i$, with $i \in \{1, 2\}$. From this follows that $(K, \Sigma := Aff(K, +, \cdot); 0)$ is a *locally transitive permutation set* (cf. sec. 2.3) and since Σ is a group the *structure group* Δ_0 consists of the elements of $K^{*\,\bullet} := \{a^\bullet \mid a \in K \setminus \{0\}\}$ and so $(\Delta_0, \circ) \cong (K^*, \cdot)$. If $\sigma : (K^*, \cdot) \to (\{1, -1\}, \cdot)$ is a homomorphism, then $(K, +, \cdot; \sigma)$ is called a *halfordered field.*

Now we assume that σ is an epimorphism. Then $P := \sigma^{-1}(1)$ is a subgroup of $(K^*, \cdot)$ of index 2 and $0 \notin P$. Therefore for each $a^\bullet \in \Delta_0$ we obtain either $a^\bullet(P) = a \cdot P = P$ or $a^\bullet(P) = a \cdot P = K^* \setminus P = P_-$. This shows that P is a semidomain of $(K, \Sigma; 0)$ and with (2.3.2) and (2.5.8) we have proved Sperner's theorem:

(2.5.9) (Sperner) *Let $(K, +, \cdot; \sigma)$ be a halfordered field and let*

$$\xi_\sigma : \begin{cases} K^{3'} & \to \quad \{1, -1\} \\ (a, b, c) & \mapsto \quad (a|b, c)_{\xi_\sigma} := \sigma((b - a) \cdot (c - a)) = \sigma(b - a) \cdot \sigma(c - a) \end{cases} .$$

Then (K, ξ_σ) is a selfrelated halfordered set with $k_\xi = \sigma(-1)$ and $Aff(K, +, \cdot) \le Aut(K, \xi_\sigma)$.

Now we like to determine the sets $\mathbf{C}'$,C',$\mathbf{C}$,C of (K, ξ_σ). Clearly if $\gamma \in Aut(K, \xi_\sigma)$ and $(u, x; y, z) \in \mathbf{C}'$ then $(\gamma(u), \gamma(x); \gamma(y), \gamma(z)) \in \mathbf{C}'$ and since $\beta := ((y - u)^{-1})^\bullet \circ (-u)^+ \in Aff(K, +, \cdot) \le Aut(K, \xi_\sigma)$ (cf. (2.5.9)) we have for $\{u, x, y, z\} \in \binom{K}{4}$:

$$(u, x; y, z) \in \mathbf{C}' \Leftrightarrow \; (0, (y - u)^{-1} \cdot (x - u); 1, (y - u)^{-1} \cdot (z - u)) \in \mathbf{C}'.$$

Therefore it is enough to determine the elements $a, b \in K \setminus \{0, 1\}$, with $(0, a; 1, b) \in \mathbf{C}'$. We prove:

(2.5.10) (Theorem) *Let $(K, +, \cdot; \sigma)$ be a non trivially halfordered field, i.e. $P = \sigma^{-1}(1) \ne K^*$, let $(K, \xi) := (K, \xi_\sigma)$ be the corresponding selfrelated halfordered set and let $a, b \in$*

$K \setminus \{0,1\}$, with $a \neq b$. Then:

(1) $(0,a;1,b) \in \mathbf{C}' \Leftrightarrow \ \sigma(b) = \sigma(b-a) = 1 \wedge \ \sigma(1-a) = -1$.

(2) $(0,a;1,b) \in \mathbf{C} \Leftrightarrow \ \sigma(a) = \sigma(b) = \sigma(b-a) = 1 \wedge \ \sigma(1-a) = -1$.

(3) $\mathbf{C} = \mathbf{C}' \Leftrightarrow \ \forall\, a,b \in K\setminus\{0,1\},\ a \neq b$ *with* $\sigma(b) = \sigma(b-a) = 1 = -\sigma(1-a) : \sigma(a) = 1$.

(4) $(1,0;a,b) \in C' \Leftrightarrow \ \sigma(a) = -\sigma(b) \ \wedge \ \sigma(a-1) = \sigma(b-1)$.

(5) $C = C' \ \Leftrightarrow \ \forall\, a,b \in K \setminus \{0,1\},\ a \neq b$ *with* $\sigma(a) = -\sigma(b)$ *and* $\sigma(a-1) = \sigma(b-1) :$ $\sigma(a-1) = \sigma(-1)$.

(6) *If* $\sigma(-1) = 1$ *then* : *a)* $C \neq C'$.

b) $D \neq D'$.

(7) *If* $\sigma(-1) = -1$ *and* $C = C'$ *then* $(K,+,\cdot;\sigma)$ *is an ordered field.*

(8) *If* $\sigma(-1) = -1$ *and* $D = D'$ *then* $(K,+,\cdot;\sigma)$ *is either an ordered field or the prime field* $\mathbb{Z}_3$ *with* $\sigma(1) = 1$ *and* $\sigma(-1) = -1$.

Proof. (1) $(0,a;1,b) \in \mathbf{C}' :\Leftrightarrow \ (0|1,b)_{\xi_\sigma} = (b|0,a)_{\xi_\sigma} = -(a|1,b)_{\xi_\sigma} = 1, \ \Leftrightarrow \ \sigma(b) = (0|1,b)_{\xi_\sigma} = 1,\ \sigma(b)\cdot\sigma(b-a) = (b|0,a)_{\xi_\sigma} = 1$ and $\sigma(1-a)\cdot\sigma(b-a) = (0|1-a,b-a)_{\xi_\sigma} = (a|1,b)_{\xi_\sigma} = -1 \ \Leftrightarrow \ \sigma(b) = \sigma(b-a) = 1$ and $\sigma(1-a) = -1$.

(2) $(0,a;1,b) \in \mathbf{C} \Leftrightarrow \ \sigma(b) = \sigma(b-a) = 1 = -\sigma(1-a) \wedge \ 1 = (0|1,a)_{\xi_\sigma} = \sigma(a)$.

(3) This statement is a consequence of (1) and (2).

(4) $(1,0;a,b) \in C' \Leftrightarrow \ (1|a,b)_{\xi_\sigma} = -(0|a,b)_{\xi_\sigma} = 1 \Leftrightarrow \ \sigma(a-1) = \sigma(b-1)$ and $\sigma(a) = -\sigma(b)$.

(5) $(1,0;a,b) \in C \Leftrightarrow \ \sigma(a-1) = \sigma(b-1),\ \sigma(a) = -\sigma(b)$ (by (4)) and $1 = (1|0,a,b)_{\xi_\sigma} = (0|-1,a-1,b-1)_{\xi_\sigma}$, i.e. $\sigma(-1) = \sigma(a-1) = \sigma(b-1)$.

(6) a) Let us assume $C = C'$. Since $\sigma(-1) = 1$ we have by (5):

(5′) $\forall\ a,b \in K \setminus \{0,1\}$ with $a \neq b$, $\sigma(a-1) = \sigma(b-1)$ and $\sigma(a) = -\sigma(b) : \sigma(a-1) = 1$.

If we set $c := a-1,\ d := b-1$ then $c,d \in K \setminus \{0,1\}, c \neq d, \sigma(c) = \sigma(d), \sigma(c+1) = -\sigma(d+1)$ and (5′) can be expressed by

(5″) $\forall\ a,b \in K \setminus \{0,1\}$ with $a \neq b$, $\sigma(a) = \sigma(b)$ and $\sigma(a+1) = -\sigma(b+1) : \sigma(a) = 1$.

From (5″) we obtain:

(5‴) $\forall a,b \in K \setminus \{0,1\}$ with $a \neq b$, $\sigma(a) = \sigma(b) = -1 \ \Rightarrow \ \sigma(a+1) = \sigma(b+1)$.

Now let $P := \{x \in K^* \mid \sigma(x) = 1\}$ and $N := \{x \in K^* \mid \sigma(x) = -1\}$. If there is a $c \in N \cap (P-1)$, (i.e. $\sigma(c) = -1$ and $\sigma(c+1) = 1$) then by $(5''')$, $N+1 \subset P$. Since σ is a homomorphism this gives us $N \cdot P + P = N + P \subset P \cdot P = P$ and $c \cdot N + c \cdot P = P + N \subset c \cdot P = N$, and so $\emptyset \neq N + P \subset N \cap P = \emptyset$, a contradiction. Consequently $N \cap (P-1) = \emptyset$, i.e. $(N+1) \cap P = \emptyset$, i.e. $N+1 \subset N$. This implies $N + P = (N+1) \cdot P \subset N \cdot P = N$ and for $n \in N$, $P = n \cdot N \supset n \cdot (N+P) = P + N$. Hence again $\emptyset \neq N + P \subset N \cap P = \emptyset$ the same contradiction. Therefore $C \neq C'$.

(7) Let $a, b \in P$ hence $(0|a,b) = 1$. We have only to show $a + b \in P$. By (2.5.9) and (2.2.1) ξ_σ is an order. Since $\sigma(-b) = \sigma(-1) \cdot \sigma(b) = (-1) \cdot 1 = -1$, $-b \in N$ and so $(0|b,-b) = -1$. Suppose $(a|0,-b) = -1$ then (since ξ_σ is an order and $(0|a,b) = 1$) $1 = (0|a,-b) = (0|a,-b) \cdot (0|a,b) = (0|b,-b) = -1$, a contradiction. Therefore $(a|0,-b) = 1$ and since $(-1)^\bullet, a^+ \in Aut(K, \xi_\sigma)$ (cf. (2.5.9)), $1 = (-a|0,b) = (0|a,a+b)$, i.e. $\sigma(a+b) = \sigma(a)$, i.e. $a + b \in P$.

(8) and (6) b) Now we study the case $D = D'$:

$(a,0:1,b) \in D' \;\Leftrightarrow\; (0|a,1) = (1|0,b) = -1 \;\Leftrightarrow\; \sigma(a) = \sigma(1-b) = -1$;

$(a,0:1,b) \in D \;\Leftrightarrow\; \sigma(a) = \sigma(1-b) = -1 \;\wedge\; 1 = (a|0,b) = \sigma(-a) \cdot \sigma(b-a) = (b|1,a) = \sigma(1-b) \cdot \sigma(a-b) \;\Leftrightarrow\; \sigma(a) = \sigma(1-b) = -1 = \sigma(a-b)$.

Therefore: $D = D' \;\Leftrightarrow\; \forall\, a,b \in K \setminus \{0,1\},\ a \neq b : \sigma(a) = \sigma(1-b) = -1 \;\Rightarrow\; \sigma(a-b) = -1$.

We put $c := 1 - b$ (so $c \neq 0, 1$) and obtain:

$D = D' \Leftrightarrow$ "$\forall\, a, c \in K \setminus \{0,1\}$, with $a \neq 1 - c, \sigma(a) = \sigma(c) = -1 \;: \sigma(a + c - 1) = -1$" $\Leftrightarrow$ "$\forall\, a, c \in N$, with $a + c - 1 \neq 0 : a + c - 1 \in N$" $\Leftrightarrow N + N - 1 \subset N \cup \{0\} \Leftrightarrow N + N \subset (N+1) \cup \{1\}$.

If $N \neq \emptyset$ and $n_0 \in N$ then:

$D = D' \Leftrightarrow\; P + P \subset (P + n_0) \cup \{n_0\} \Leftrightarrow\; P + P - N \subset P \cup \{0\}$.

We discuss the cases:

1. $-1 \in N$. Then $2 := 1 + 1 \neq 0$ and (we put $n_0 = -1$) :

$D = D' \;\Leftrightarrow P + P + 1 \subset P \cup \{0\} \;\Leftrightarrow\; P + P + P \subset P \cup \{0\}$. Firstly we assume $P + P + P \subset P \cup \{0\}$ and $\mathrm{char}(K) \neq 3$. Since $K^{(2)} := \{\lambda^2 \;|\lambda \in K^*\} \subset P$ we have $1, 2^2 = 4 \in P$ and so $4 + 1 + 1 = 6 \in P$ and $1 + 1 + 1 = 3 \in P$ hence $2 = \frac{6}{3} \in P$. If $a, b \in P$ then $\frac{b}{2} \in P$ and so $a + b = a + \frac{b}{2} + \frac{b}{2} \in P + P + P \subset P \cup \{0\}$. $a + b = 0$

implies $b = -a$ hence $-1 = a \cdot (-a)^{-1} = a \cdot b^{-1} \in P$ contradicting $-1 \in N$. Therefore $P + P \subset P$, i.e. $(K, +, \cdot; \sigma)$ is an ordered field.
Now let $P + P + P \subset P \cup \{0\}$ and $\mathrm{char} K = 3$. Then $1 \in P$ and $-1 = 1 + 1 \in N$ imply $-1 + P \subset P \cup \{0\}$ hence $P \cdot (-1 + P) = -P + P = N + P \subset P \cup \{0\}$ and $(-1) \cdot (N + P) = N + P \subset -P \cup \{0\} = N \cup \{0\}$. Consequently $N + P \subset (P \cup \{0\}) \cap (N \cup \{0\}) = \{0\}$ implying $P = \{1\}, N = \{-1\}$, i.e. $K = \mathbb{Z}_3$.

2. $-1 \in P$. Then $0 = 1 - 1 \in (P + n_0) \cup \{n_0\}$ hence $0 \in P + n_0$, i.e. $-n_0 \in P$ and so $n_0 = (-1) \cdot (-n_0) \in P$, a contradiction.□

2.5.3 $\tau-$CONVEXITY

A separation function $\tau : E^{4'} \to \{1, -1\}$ is called *$\tau-$convex* if $\forall\ (a, b, c, d), (a, e, c, d) \in E^{4'}$:

$$[a, b|c, d] = 1 \wedge\ [a, e|c, d] = -1\ \Rightarrow\ [a, b|e, c] = 1,$$

trivially $\tau-$convex if $\tau = 1$.

Remarks. 1. If τ is not selfrelated we can introduce "by dualisation" a second concept of $\tau-$convexity by: $\forall\ (a, b, c, d), (a, b, c, e) \in E^{4'}$:

$$[a, b|c, d] = 1 \wedge\ [a, b|c, e] = -1\ \Rightarrow\ [a, e|c, d] = 1.$$

But these concepts are equivalent, for suppose that (E, τ) is $\tau-$convex in the first sense but not in the second, then there are $(a, b, c, d), (a, b, c, e) \in E^{4'}$ such that (i) $[a, b|c, d] = 1$, (ii) $[a, b|c, e] = -1$ and (iii) $[a, e|c, d] = -1$. But by the $\tau-$convexity in the first sense, (i) and (iii) we have $[a, b|c, e] = 1$ contradicting (ii).
2. If $|E| = 4$ then a separation function τ is always $\tau-$convex.

(2.5.11) *If (E, τ) is an order then τ is $\tau-$convex and selfrelated.*

Proof. By (2.4.4) τ is selfrelated. Let $(a,b,c,d),(a,e,c,d) \in E^{4'}$ with $[a,b|c,d] = 1, [a,e|c,d] = -1$. Then, since τ is an order, $[a,d|e,c] = 1$ and so $[b,e|c,d] \overset{\mathbf{(Z)_r}}{=} [a,b|c,d] \cdot [a,e|c,d] = 1 \cdot (-1) = -1$ hence $[b,d|e,c] = 1$ and $[a,b|e,c] \overset{\mathbf{(Z)_r}}{=} [a,d|e,c] \cdot [b,d|e,c] = 1 \cdot 1 = 1$. □

Vice versa:

(2.5.12) *Let (E,τ) be selfrelated (then by (2.1.1) and (2.4.3) $k_\tau := k_{\tau_l} = k_{\tau_r}$ is constant equal either 1 or -1) and $\tau-$convex and let $|E| \geq 5$. Then:*
a) $k_\tau = 1 \;\Rightarrow\; \tau$ is not necessarily trivially $\tau-$convex.
b) $k_\tau = -1 \;\Rightarrow\; \tau$ is an order.

Proof. a) Let $E = \{a,b,c,d,e\}$ and let ξ be the betweenness function given by $(b|a,c,d,e) = (c|a,b,d,e) = (e|a,b,c,d) = (a|b,c,e) = (d|b,c,e) = -(a|b,d) = -(d|a,b) = 1$. Then $Sym\{a,d\}, Sym\{b,c,e\} \leq Aut(E,\xi)$. For the corresponding separation function $\tau := \tau_\xi$ we have by (2.4.1) $k_\tau = k_\xi$ and the $\tau-$convexity can be translated in the condition:
(τ) $\forall\ (x_1,x_2,x_3,x_4),(x_1,x_5,x_3,x_4) \in E^{4'}$:

$$(x_1|x_3,x_4) = (x_2|x_3,x_4) = -(x_5|x_3,x_4) \;\Rightarrow\; (x_1|x_3,x_5) = (x_2|x_3,x_5).$$

By definition follows for $\{x,y,z\} \in \binom{E}{3}$:

$$(x|y,z) = -1 \;\Leftrightarrow\; \{x,y\} = \{a,d\} \text{ or } \{x,z\} = \{a,d\}.$$

This shows $k_\tau = k_\xi = 1$ and that the assumptions of (τ) are only verified if $\{x_5,x_3\} = \{a,d\}$ or $\{x_5,x_4\} = \{a,d\}$. By $Sym\{a,d\}, Sym\{b,c,e\} \leq Aut(E,\xi)$ we may assume $x_5 = a, x_3 = d, x_4 = b, x_1 = c, x_2 = e$. But then $(x_1|x_3,x_5) = (c|d,a) = 1$ and $(x_2|x_3,x_5) = (e|d,a) = 1$.
b) $k_\tau = -1$ implies that $\forall\ (a,b,c,d) \in (\binom{E}{4}) : [a,b|c,d] = [a,c|b,d] = [a,d|b,c] = -1$ or exactly one of the values $[a,b|c,d],[a,c|b,d],[a,d|b,c]$ is equal -1. Let us assume there are $(x,y,z,t) \in (\binom{E}{4})$:

$$(*) \qquad\qquad [x,y|z,t] = [x,z|y,t] = [x,t|y,z] = -1$$

and let $w \in E \setminus \{x, y, z, t\}$. Then $-1 = [x, y|z, t] = [x, y|z, w] \cdot [x, y|t, w]$ and so for instance $[x, y|z, w] = 1$ and $[x, y|t, w] = -1$. If $[x, t|z, w] = -1$ then the $\tau-$convexity implies $[x, y|z, t] = 1$, a contradiction. Hence $[x, t|z, w] = 1$ and since $k_\tau = -1$ we have : $[x, z|t, w] = -[x, w|t, z]$.

case 1: $[x, z|t, w] = -[x, w|t, z] = 1$.
Then $[x, y|t, w] = -1$ and the $\tau-$convexity imply $[x, z|y, t] = 1$, a contradiction to $(*)$.
case 2: $[x, z|t, w] = -1$ and $[x, w|t, z] = 1$.
Here $[x, y|z, t] = -1$ and $[x, w|z, t] = 1$ imply $[x, w|y, t] = 1$. Since $[x, y|t, w] = -1$ we have $[x, t|y, w] = 1$ ($k_\tau = -1!$) and so $[x, t|w, z] = [x, t|y, w] \cdot [x, t|y, z] = 1 \cdot (-1) = -1$ which contradicts $[x, t|w, z] = 1$.□

Remark. One can find examples of not selfrelated $\tau-$convex sets (E, τ_ξ): let $E := \{a, b, c, d, e\}$ and let $(a|b, c, d, e)$[8]$= (c|a, b, d, e) = (d|a, b, c, e) = (e|a, b, c, d) = (b|a, c) = (b|d, e) = -(b|a, d) = 1$. Here τ_ξ is $\tau-$convex because $\forall\ (x, y, z, t) \in E^{4'}$, if $x, y \neq b$ then $[x, y|z, t] = 1$. Moreover τ_ξ is not selfrelated: in fact $[a, b|c, d] = -[c, d|a, b]$.

There are halfordered convex sets (E, ξ) (not selfrelated) such that (E, τ_ξ) is not $\tau-$convex: let $E := \{a, b, c, d, e\}$ and let $(a|b, c, e) = (b|a, d, e) = (c|a, b, d, e) = (d|a, b, c, e) = (e|a, b, c, d) = -(a|b, d) = -(b|a, c) = 1$. Then (E, ξ) is convex. In fact: since $(c|a, b, d, e) = (d|a, b, c, e) = (e|a, b, c, d) = 1$ we have only to check the cases $(b|d, e) = -(a|d, e) = 1$ and $(a|c, e) = -(b|c, e) = 1$. Since $(b|a, d) = (a|b, c) = 1$ (E, ξ) is convex but not $\tau-$convex, because $[a, b|c, d] = -[a, e|c, d] = 1$ and $[a, b|e, c] = -1$, and not selfrelated, because $[a, e|b, d] = -[b, d|a, e]$.

On the other hand there are examples of not selfrelated halfordered sets (E, ξ) such that (E, τ_ξ) is $\tau-$convex but (E, ξ) is not convex: let $E = \{a, b, c, d, e\}$ and $(a|b, e) = (a|c, d) = (b|a, d, e) = (c|a, d, e) = (d|a, e) = (d|b, c) = (e|a, c, d) = -(a|b, c) = -(b|a, c) = -(c|a, b) = -(d|b, a) = -(e|a, b) = 1$. Then (E, ξ) is not convex because $(a|c, d) = -(b|c, d) = 1$ and $(a|b, c) = -1$ and not selfrelated because $[a, b|c, d] =$

[8] $(a|b, c, d, e) := (a|b, c) = (a|c, d) = (a|d, e) = 1$.

$-[c,d|a,b]$. (E,τ_ξ) is $\tau-$convex: since $]a,b[=\{c,d,e\}$, $]a,c[=\{b,d\}$, $]a,d[=\emptyset$, $]a,e[=\emptyset$, $]b,c[=\{a,e\}$, $]b,d[=\{a,c,e\}$, $]b,e[=\{c,d\}$, $]c,d[=\{b\}$, $]c,e[=\{a,b,d\}$, $]d,e[=\{a\}$ and since the assumptions of (τ) are only verified for $|]x_3,x_4[|=1$ or $=2$ and the hypothesis is satisfied for $|]x_3,x_5[|$, $|]x_4,x_5[|=0$ or $=3$ we have only to check the 5 cases where $\{x_3,x_4\},\{x_3,x_5\},\{x_4,x_5\}\notin M:=\{\{a,b\},\{a,d\},\{a,e\},\{b,d\},\{c,e\}\}$:

1. $\{x_3,x_4\}=\{a,c\}:\Rightarrow\ \{x_1,x_2\}=\{b,d\}, x_5=e$ and so $\{x_3,x_5\}\in M$ or $\{x_4,x_5\}\in M$.
2. $\{x_3,x_4\}=\{b,c\}:\Rightarrow\ \{x_1,x_2\}=\{a,e\}, x_5=d$ and so $\{x_3,x_5\}\in M$ or $\{x_4,x_5\}\in M$.
3. $\{x_3,x_4\}=\{b,e\}:\Rightarrow\ \{x_1,x_2\}=\{c,d\}, x_5=a$ and so $\{x_3,x_5\}\in M$ or $\{x_4,x_5\}\in M$.
4. $\{x_3,x_4\}=\{c,d\}:\Rightarrow\ \{x_1,x_2\}=\{e,a\}, x_5=b$ and so $\{x_3,x_5\}\in M$ or $\{x_4,x_5\}\in M$.
5. $\{x_3,x_4\}=\{d,e\}:\Rightarrow\ \{x_1,x_2\}=\{b,c\}, x_5=a$ and so $\{x_3,x_5\}\in M$ or $\{x_4,x_5\}\in M$.

Definition 2.5.1 Let (E,ξ) be a halfordered set and let $M\subset E$. An element $a\in E$ is called *bound* of M if $\forall\ x,y\in M:(a|x,y)=1$.

(2.5.13) *Let (E,ξ) a halfordered set and let (E,τ_ξ) $\tau-$ convex. If for every four points there is a bound then also (E,ξ) is convex.*

Proof. Let $\{a,b,c,d\}\in\binom{E}{4}:(a|c,d)=1$ and $(b|c,d)=-1$. Then there is a bound $\infty\in E\setminus\{a,b,c,d\}:(\infty|a,b)=(\infty|a,c)=(\infty|a,d)=1$. Since $[\infty,a|c,d]=1$ and $[\infty,b|c,d]=-1$ and τ_ξ is $\tau-$convex we have $[\infty,a|b,c]=1$ and this implies $(a|b,c)=1$, i.e. (E,ξ) is convex.□

Now let (E,ξ) be a selfrelated convex halfordered set. Then there is the harmonic and the anharmonic case.

<u>harmonic case:</u> by definition the function k_ξ is constant $=-1$. We have the following diagram: $k_\xi=-1$ and ξ convex $\overset{(2.2.1)}{\Leftrightarrow}$ ξ order $\overset{(2.4.5.2)}{\Rightarrow}$ τ_ξ order $\overset{(2.5.11)\ \wedge\ (2.5.12)}{\Longleftrightarrow}$ τ_ξ is $\tau-$convex.

<u>anharmonic case:</u> for a set $E=\{a,b,c,d,e\}$ with five elements there are halforders such that:

a) (E,ξ) is convex, anharmonic and τ_ξ is $\tau-$convex:

let $(a|b,c,d,e) = (d|a,b,c,e) = (e|a,b,c,d) = (b|a,d,e) = (c|a,d,e) = -(b|a,c) = -(c|a,b) = 1$. Since a,d,e are bounds (resp. of $\{b,c,d,e\},\{a,b,c,e\}$ and $\{a,b,c,d\}$) and $(bc) \in Aut(E,\xi)$ it is enough to check the convexity in the point b: but since $\overrightarrow{b,c} = \{c\}$, $\overrightarrow{b,a} = \{a,d,e\}$ and a,d,e are bounds, the assumptions for the convexity in b are not satisfied. The function k_ξ is clearly $= 1$. Moreover $< Sym\{a,d,e\} \cup Sym\{b,c\} >\leq Aut(E,\tau)$ and $]a,d[=]a,e[=]d,e[=]b,c[= \emptyset$, $]a,b[=]b,d[=]b,e[= \{c\}$, $]a,c[=]c,d[=]c,e[= \{b\}$. Therefore it is enough to consider the case $\{x_3,x_4\} = \{a,b\}$ implying $x_5 = c$ and $\{x_1,x_2\} = \{d,e\}$. But then $(x_1|x_3,x_5) = (d|a,c) = 1 = (x_2|x_3,x_5) = (e|a,c)$.

b) (E,ξ) is convex, anharmonic and not $\tau_\xi-$convex:

let $(b|a,c,d,e) = (a|b,c) = (a|e,c) = (c|b,a) = (c|b,d) = (d|b,c) = (d|e,c) = (e|b,d) = -(a|b,d) = -(c|e,a) = -(d|b,a) = -(e|b,c) = -(e|a,c) = 1$. The function $k_\xi = 1$ and τ is not $\tau-$convex: in fact $[e,a|c,d] = 1$ and $[e,b|c,d] = -1$ but $[e,a|b,c] = -1$. (E,ξ) is convex: since $(ad),(ce) \in Aut(E,\xi)$ we have only to consider the cases $(c|b,d) = -(a|b,d) = 1, (b|c,d) = -(a|c,d) = 1, (b|a,e) = -(c|a,e) = 1$ and $(a|b,e) = -(c|b,e) = 1$. Since $(c|a,b) = (b|a,c) = (a|c,b) = 1$ we have the convexity.

c) (E,ξ) is not convex, anharmonic and $\tau_\xi-$convex:

let $(a|b,c,e) = (c|b,d,e) = (b|a,d) = (b|e,c) = (d|a,c) = (d|e,b) = (e|a,d) = (e|b,c) = -(a|e,d) = -(c|e,a) = -(b|a,c) = -(d|a,b) = -(e|a,b) = 1$. The function $k_\xi = 1$ and since $]a,b[= \{c,d,e\}$, $]a,c[= \{b,e\}$, $]a,d[= \{c\}$, $]a,e[= \{c,b,d\}$, $]b,c[= \{d\}$, $]b,d[= \{a,e\}$, $]b,e[= \emptyset$, $]c,d[= \{a,b,e\}$, $]c,e[= \{d\}$, $]d,e[= \{a,b\}$ as before (cf. example before the Definition 2.5.1) for the $\tau-$convexity we have only to check the 6 cases where $\{x_3,x_4\},\{x_3,x_5\},\{x_4,x_5\} \notin N := \{\{a,b\},\{a,e\},\{b,e\},\{c,d\}\}$:

1. $\{x_3,x_4\} = \{a,c\} :\Rightarrow \{x_1,x_2\} = \{b,e\}, x_5 = d$ and so $\{x_3,x_5\} \in N$ or $\{x_4,x_5\} \in N$.
2. $\{x_3,x_4\} = \{a,d\} :\Rightarrow \{x_1,x_2\} = \{b,e\}, x_5 = c$ and so $\{x_3,x_5\} \in N$ or $\{x_4,x_5\} \in N$.
3. $\{x_3,x_4\} = \{b,c\} :\Rightarrow \{x_1,x_2\} = \{a,e\}, x_5 = d$ and so $\{x_3,x_5\} \in N$ or $\{x_4,x_5\} \in N$.
4. $\{x_3,x_4\} = \{b,d\} :\Rightarrow \{x_1,x_2\} = \{a,e\}, x_5 = c$ and so $\{x_3,x_5\} \in N$ or $\{x_4,x_5\} \in N$.
5. $\{x_3,x_4\} = \{c,e\} :\Rightarrow \{x_1,x_2\} = \{a,b\}, x_5 = d$ and so $\{x_3,x_5\} \in N$ or $\{x_4,x_5\} \in N$.
6. $\{x_3,x_4\} = \{d,e\} :\Rightarrow \{x_1,x_2\} = \{a,b\}, x_5 = c$ and so $\{x_3,x_5\} \in N$ or $\{x_4,x_5\} \in N$.

Moreover (E,ξ) is not convex because $(c|b,d) = -(a|b,d) = 1$ and $(c|a,b) = -1$.

d) (E,ξ) is not convex, anharmonic and not $\tau_\xi-$convex:

let $(d|a,b,c,e) = (a|b,e) = (a|c,d) = (b|a,c,e) = (c|a,d,e) = (e|a,b) = (e|c,d) = -(a|b,c) = -(b|a,d) = -(c|a,b) = -(e|a,c) = 1$. (E,ξ) is not convex because $(a|c,d) = -(b|c,d) = 1$ and $(a|b,c) = -1$ and not $\tau-$convex because $[b,e|a,d] = -[b,c|a,d] = 1$ and $[b,e|a,c] = -1$. Also here the function k_ξ is $= 1$.

Remark. One can find examples of not convex, anharmonic and $\tau_\xi-$convex sets (E,ξ) with $|E| = 6$. For this purpose we start from example c) which we enlarge by one element f and extend ξ as follows: $(b|a,f) = (c|a,f) = (d|a,f) = (e|a,f) = (f|a,d) = (f|b,c,e) = 1$ and $(a|b,f) = (f|a,b) = -1$. Then k_ξ has still the value 1 and since $]a,b[= \{c,d,e,f\}$, $]a,c[= \{b,e,f\}$, $]a,d[= \{c\}$, $]a,e[= \{b,c,d,f\}$, $]a,f[= \emptyset$, $]b,c[= \{d\}$, $]b,d[= \{a,e,f\}$, $]b,e[= \emptyset$, $]b,f[= \{a,c,d,e\}$, $]c,d[= \{a,b,e,f\}$, $]c,e[= \{d\}$, $]c,f[= \{a,b,e\}$, $]d,e[= \{a,b,f\}$, $]d,f[= \{c\}$, $]e,f[= \{a,b,c,d\}$ for the $\tau-$convexity we have only to check the 8 cases where $\{x_3,x_4\},\{x_3,x_5\},\{x_4,x_5\} \notin R := \{\{a,b\},\{a,e\},\{a,f\}, \{b,e\},\{b,f\},\{c,d\},\{e,f\}\}$:

1. $\{x_3,x_4\} = \{a,c\} :\Rightarrow \{x_1,x_2\} \subset \{b,e,f\}, x_5 = d$ and so $\{x_3,x_5\} \in R$ or $\{x_4,x_5\} \in R$.
2. $\{x_3,x_4\} = \{a,d\} :\Rightarrow \{x_1,x_2\} \subset \{b,e,f\}, x_5 = c$ and so $\{x_3,x_5\} \in R$ or $\{x_4,x_5\} \in R$.
3. $\{x_3,x_4\} = \{b,c\} :\Rightarrow \{x_1,x_2\} \subset \{a,e,f\}, x_5 = d$ and so $\{x_3,x_5\} \in R$ or $\{x_4,x_5\} \in R$.
4. $\{x_3,x_4\} = \{b,d\} :\Rightarrow \{x_1,x_2\} \subset \{a,e,f\}, x_5 = c$ and so $\{x_3,x_5\} \in R$ or $\{x_4,x_5\} \in R$.
5. $\{x_3,x_4\} = \{c,e\} :\Rightarrow \{x_1,x_2\} \subset \{a,b,f\}, x_5 = d$ and so $\{x_3,x_5\} \in R$ or $\{x_4,x_5\} \in R$.
6. $\{x_3,x_4\} = \{c,f\} :\Rightarrow \{x_1,x_2\} \subset \{a,b,e\}, x_5 = d$ and so $\{x_3,x_5\} \in R$ or $\{x_4,x_5\} \in R$.
7. $\{x_3,x_4\} = \{d,e\} :\Rightarrow \{x_1,x_2\} \subset \{a,b,f\}, x_5 = c$ and so $\{x_3,x_5\} \in R$ or $\{x_4,x_5\} \in R$.
8. $\{x_3,x_4\} = \{d,f\} :\Rightarrow \{x_1,x_2\} \subset \{a,b,e\}, x_5 = c$ and so $\{x_3,x_5\} \in R$ or $\{x_4,x_5\} \in R$.

2.6 HALFORDERED PERMUTATION SETS

In the following let $E \neq \emptyset$ be a fixed set and let $\Sigma(E)$ be the set of all halforders of E. Then (cf. [34]):

(2.6.1) *$\Sigma(E)$ is a commutative group of exponent 2 with respect to the operation: for $\xi_i \in \Sigma(E)$, $i \in \{1,2\}$ and $(a,b,c) \in E^{3'}$ let $\xi_i(a,b,c) = (a|b,c)_i$; then let*

$$\xi_1 \cdot \xi_2(a,b,c) := (a|b,c)_1 \cdot (a|b,c)_2.$$

Proof. Let $(a,b,c),(a,b,d) \in E^{3'}$, then $\xi_1 \cdot \xi_2(a,b,c) \cdot \xi_1 \cdot \xi_2(a,b,d) = (a|b,c)_1 \cdot (a|b,c)_2 \cdot (a|b,d)_1 \cdot (a|b,d)_2 \overset{(\mathbf{Z})}{=} (a|c,d)_1 \cdot (a|c,d)_2 = \xi_1 \cdot \xi_2(a,c,d)$, i.e. **(Z)** is satisfied for $\xi_1 \cdot \xi_2$. Moreover by definition of $\xi \in \Sigma(E)$, every element ξ is an involution, hence $\xi^{-1} = \xi \in \Sigma(E)$.□

For $\sigma \in SymE$ and $n \in \mathbb{N}$ let $\sigma_n : E^n \to E^n; (x_1,...,x_n) \mapsto (\sigma(x_1),...,\sigma(x_n))$. Then we have:

(2.6.2) *Let $\sigma \in SymE$, $\Gamma \subset SymE$, $\xi \in \Sigma(E)$ and $A \subset \Sigma(E)$ and let*

$$\xi \circ \sigma := \xi \circ \sigma_3 : E^{3'} \to \{1,-1\}; (a,b,c) \mapsto (\sigma(a)|\sigma(b),\sigma(c)).$$

Then:

(1) *$\xi \circ \sigma \in \Sigma(E)$ and the map*

$$\zeta_\sigma : \Sigma(E) \to \Sigma(E); \xi \mapsto \xi \circ \sigma,$$

is an automorphism of the group $(\Sigma(E),\cdot)$.

(2) *$Fix_3(\Gamma) := \{\xi \in \Sigma(E) \mid \forall\, \gamma \in \Gamma : \xi \circ \gamma = \xi\}$ is a subgroup of $(\Sigma(E),\cdot)$.*

(3) *The stabilizer $(SymE)_A := \{\sigma \in SymE \mid \forall\, \xi \in A : \xi \circ \sigma = \xi\}$ is a subgroup of $SymE$.*

(4) *$\Gamma \subset (SymE)_{Fix_3(\Gamma)}$ and $A \subset Fix_3((SymE)_A)$.*

Proof. (1) $\forall\, (a,b,c),(a,c,d) \in E^{3'}$ we have $\xi \circ \sigma(a,b,c) \cdot \xi \circ \sigma(a,c,d) = (\sigma(a)|\sigma(b),\sigma(c)) \cdot (\sigma(a)|\sigma(c),\sigma(d)) \overset{(\mathbf{Z})}{=} (\sigma(a)|\sigma(b),\sigma(d)) = \xi \circ \sigma(a,b,d)$, i.e. $\xi \circ \sigma$ satisfies **(Z)**. Moreover

$\forall\ \xi_1,\xi_2 \in \Sigma(E),\ \forall\ (a,b,c) \in E^{3'}$, $(\xi_1 \cdot \xi_2) \circ \sigma(a,b,c) = (\xi_1 \cdot \xi_2)(\sigma(a),\sigma(b),\sigma(c)) = (\sigma(a)|\sigma(b),\sigma(c))_1 \cdot (\sigma(a)|\sigma(b),\sigma(c))_2 = (\xi_1 \circ \sigma) \cdot (\xi_2 \circ \sigma)(a,b,c)$, hence ζ_σ is an automorphism of the group $(\Sigma(E),\cdot)$.

(2) $\forall\ \xi_1,\xi_2 \in Fix_3(\Gamma)$ and $\forall\ \gamma \in \Gamma$ we have $(\xi_1 \cdot \xi_2) \circ \gamma \overset{(1)}{=} (\xi_1 \circ \gamma) \cdot (\xi_2 \circ \gamma) \overset{\xi_1,\xi_2 \in Fix_3(\Gamma)}{=} \xi_1 \cdot \xi_2$, hence $Fix_3(\Gamma) \leq (\Sigma(E),\cdot)$.

(3) $\forall\ \sigma_1,\sigma_2 \in (SymE)_A,\ \forall\ \xi \in A$ we have $\xi \circ (\sigma_1 \circ \sigma_2) = \xi \circ \sigma_2 = \xi$, hence $(SymE)_A$ is a subgroup of $SymE$.$\square$

Definition 2.6.1 If (E,ξ) is a halfordered set and $\Gamma \subset Aut(E,\xi)$, then (E,Γ,ξ) is called a *halfordered permutation set.*

3 Halfordered chain structures, splittings and convexity

3.1 HALFORDERED NETS AND HALFORDERED CHAIN STRUCTURES

Definition 3.1.1 Let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a chain structure,

$$\mathcal{P}^{3g} := \{(a,b,c) \in \mathcal{P}^{3'} \mid \exists\ i \in \{1,2\} : [a]_i = [b]_i = [c]_i\}$$

and

$$\mathcal{P}^{3\mathfrak{K}} := \{(a,b,c) \in \mathcal{P}^{3'} \mid \exists\ K \in \mathfrak{K} : a,b,c \in K\}.$$

Let

$$\xi : \mathcal{P}^{3g} \to \{1,-1\}; (a,b,c) \mapsto (a|b,c)$$

be a map such that:

$(\mathbf{Z1})_g$ $\forall\ G \in \mathfrak{G}_1 \cup \mathfrak{G}_2$, $(G, \xi_{|G^{3'}})$ is a halfordered set;

$(\mathbf{Z2})_g$ $\forall\ j \in \{1,2\}, \forall\ G, L \in \mathfrak{G}_j, \forall\ (a,b,c) \in G^{3'}$ if $i \in \{1,2\} \setminus \{j\}$, then
$(a|b,c) = ([a]_i \cap L \mid [b]_i \cap L, [c]_i \cap L)$;

$(\mathbf{ZC})_g$ $\forall\ K \in \mathfrak{K}, \forall\ (a,b,c) \in \mathcal{P}^{3g} : (a|b,c) = (\widetilde{K}(a)|\widetilde{K}(b), \widetilde{K}(c))$.

Then $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \xi)$ is called a *halfordered chain structure.*

Definition 3.1.2 Let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ a 2−net and let $\xi : \mathcal{P}^{3g} \to \{1,-1\}; (a,b,c) \mapsto (a|b,c)$ be a map such that the two conditions $(\mathbf{Z1})_g$ and $(\mathbf{Z2})_g$ are satisfied. Then $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2; \xi)$ is called a *halfordered* 2−*net.*

Now let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \xi)$ be a halfordered chain structure. Let $Aut(\mathcal{P}, \xi) := \{\gamma \in Sym\mathcal{P} \mid \forall (a,b,c) \in \mathcal{P}^{3g} : (\gamma(a), \gamma(b), \gamma(c)) \in \mathcal{P}^{3g}$ and $(a|b,c) = (\gamma(a)|\gamma(b), \gamma(c))\}$ be the group of all *betweenness preserving permutations.*

Remark. We recall that $\widetilde{\mathfrak{C}} := \{\widetilde{C} \mid C \in \mathfrak{C}\} \subset \overline{\Gamma}^-$ (cf. (1.4.2)). Therefore if $(a,b,c) \in \mathcal{P}^{3g}$ then $(\widetilde{C}(a), \widetilde{C}(b), \widetilde{C}\ (c)) \in \mathcal{P}^{3g}$ for all $C \in \mathfrak{C}$.

From this definition follows:

(3.1.1) (1) $Aut(\mathcal{P},\xi) \leq \overline{\Gamma}$.
(2) *If* $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2;\xi)$ *is a halfordered* $2-$*net and* $\emptyset \neq \mathfrak{K} \subset \mathfrak{C}$ *then* $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{K};\xi)$ *is a halfordered chain structure if and only if* $\widetilde{\mathfrak{K}} \subset Aut(\mathcal{P},\xi)$.

Proof. (2) Let $(a,b,c) \in \mathcal{P}^{3g}$ and $K \in \mathfrak{K}$. Then $(\widetilde{K}(a),\widetilde{K}(b),\widetilde{K}(c)) \in \mathcal{P}^{3g}$ by $\widetilde{K} \in \overline{\Gamma}^{-}$ (cf. (1.4.1.3)) and so $(a|b,c) = (\widetilde{K}(a)|\widetilde{K}(b),\widetilde{K}(c))$ is equivalent with $\widetilde{K} \in Aut(\mathcal{P},\xi)$.□

ξ induces on $\mathfrak{G}_1$ and on $\mathfrak{G}_2$ resp. a halforder ξ_1 and ξ_2 respectively: let $A \in \mathfrak{G}_2$ and $B \in \mathfrak{G}_1$ be fixed; then let

$$\xi_1 : \begin{cases} \mathfrak{G}_1^{3'} & \rightarrow & \{1,-1\} \\ (X,Y,Z) & \mapsto & (X|Y,Z)_{\xi_1} := (X\cap A \mid Y\cap A, Z\cap A)_{\xi} \end{cases}$$

and

$$\xi_2 : \begin{cases} \mathfrak{G}_2^{3'} & \rightarrow & \{1,-1\} \\ (X,Y,Z) & \mapsto & (X|Y,Z)_{\xi_2} := (X\cap B \mid Y\cap B, Z\cap B)_{\xi} \end{cases} .$$

A consequence of (3.1.1) is the following:

(3.1.2) *Let* $\mathfrak{C}_\xi := \{C \in \mathfrak{C} \mid \widetilde{C} \in Aut(\mathcal{P},\xi)\}$, *then* $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{C}_\xi;\xi)$ *is a (the greatest) halfordered chain structure with* $\mathfrak{K} \subset \mathfrak{C}_\xi$.

By (3.1.1.1) we can define:

$$Aut(\mathcal{P},\xi)^+ := Aut(\mathcal{P},\xi) \cap \overline{\Gamma}^{+},$$

$$Aut(\mathcal{P},\xi)^- := Aut(\mathcal{P},\xi) \cap \overline{\Gamma}^{-}.$$

By $\mathbf{(Z2)}_g$ we have:

(3.1.3) *The definition of the halfordered sets* $(\mathfrak{G}_1, \xi_1)$ *and* $(\mathfrak{G}_2, \xi_2)$ *does not depend on the choice of* $A \in \mathfrak{G}_2$ *and* $B \in \mathfrak{G}_1$.

(3.1.4) $Aut(\mathcal{P}, \xi)^+_{|\mathfrak{G}_1} \leq Aut(\mathfrak{G}_1, \xi_1)$.

Proof. Let $\alpha \in Aut(\mathcal{P}, \xi)^+, (X_1, X_2, X_3) \in \mathfrak{G}_1^{3'}, Y \in \mathfrak{G}_2$ and $y_i := X_i \cap Y$. Then $X_i = [y_i]_1,\ \alpha(X_i) = [\alpha(y_i)]_1,\ \alpha(y_i) \in \alpha(Y) \in \mathfrak{G}_2$ and so $(X_1|X_2, X_3)_{\xi_1} \stackrel{def.}{=} (y_1|y_2, y_3)_\xi = (\alpha(y_1)|\alpha(y_2), \alpha(y_3))_\xi \stackrel{def.}{=} ([\alpha(y_1)]_1|[\alpha(y_2)]_1, [\alpha(y_3)]_1)_{\xi_1} = (\alpha(X_1)|\alpha(X_2), \alpha(X_3))_{\xi_1}$. Hence $\alpha_{|\mathfrak{G}_1} \in Aut(\mathfrak{G}_1, \xi_1)$.$\square$

(3.1.5) *Let* $\mathfrak{C}_\xi \neq \emptyset$, *then:*
(1) $\forall\ K \in \mathfrak{C}_\xi$,

$$\widetilde{K}_{|\mathfrak{G}_1} : \mathfrak{G}_1 \to \mathfrak{G}_2;\ X \mapsto \widetilde{K}(X)$$

is an isomorphism between $(\mathfrak{G}_1, \xi_1)$ *and* $(\mathfrak{G}_2, \xi_2)$, *hence* $(\mathfrak{G}_1, \xi_1)$ *and* $(\mathfrak{G}_2, \xi_2)$ *are isomorphic.*
(2) $\{\widetilde{A, B} \mid A, B \in \mathfrak{C}_\xi\} = Aut(\mathcal{P}, \xi)^-$ *and* $\forall\ \alpha \in Aut(\mathcal{P}, \xi)^-$,

$$\alpha_{|\mathfrak{G}_1} : \mathfrak{G}_1 \to \mathfrak{G}_2;\ X \mapsto \alpha(X)$$

is an isomorphism between $(\mathfrak{G}_1, \xi_1)$ *and* $(\mathfrak{G}_2, \xi_2)$.
(3) *Let* $\beta_1 \in Aut(\mathfrak{G}_1, \xi_1),\ K \in \mathfrak{C}_\xi,\ \beta_2 := \widetilde{K}_{|\mathfrak{G}_1} \circ \beta_1 \circ \widetilde{K}_{|\mathfrak{G}_2}$ *and*

$$\beta : \begin{cases} \mathcal{P} & \to & \mathcal{P} \\ x & \mapsto & \beta_1([x]_1) \cap \beta_2([x]_2) \end{cases},$$

then $\beta \in Aut(\mathcal{P}, \xi)^+$ *and* $\beta_{|\mathfrak{G}_1} = \beta_1,\ \beta_{|\mathfrak{G}_2} = \beta_2$.
(4) $Aut(\mathcal{P}, \xi)^+_{|\mathfrak{G}_1} = Aut(\mathfrak{G}_1, \xi_1)$.

Proof. (1) Let $K \in \mathfrak{C}_\xi,\ (X_1, X_2, X_3) \in \mathfrak{G}_1^{3'}$, and $k \in K$. If we set $k_i := X_i \cap K,\ x_i := X_i \cap [k]_2$ and $y_i = \widetilde{K}(X_i) \cap [k]_1$ then $\widetilde{K}([k]_2) = [k]_1$ and so $\widetilde{K}(x_i) = y_i$ and $\widetilde{K}(X_i) = [y_i]_2$. By definition of ξ_i and (3.1.2) we have $(X_1|X_2, X_3)_{\xi_1} = (x_1|x_2, x_3)_\xi \stackrel{(\mathbf{ZC})_g}{=} (y_1|y_2, y_3)_\xi = ([y_1]_2|[y_2]_2, [y_3]_2)_{\xi_2} = (\widetilde{K}(X_1)|\widetilde{K}(X_2), \widetilde{K}\ (X_3))_{\xi_2}$. Hence, since $(\widetilde{K}_{|\mathfrak{G}_1})^{-1} = \widetilde{K}_{|\mathfrak{G}_2},\ \widetilde{K}_{|\mathfrak{G}_1}$ is an isomorphism between $(\mathfrak{G}_1, \xi_1)$ and $(\mathfrak{G}_2, \xi_2)$.

(2) Let $\alpha \in Aut(\mathcal{P}, \mathfrak{G})^-$, $(x_1, x_2, x_3) \in \mathcal{P}^{3g}$ and $x_i' = \alpha(x_i)$. By (1.4.1.6) $\exists_1\ (A, B) \in \mathfrak{C}^2$: $\alpha = \widetilde{A, B}$. If $[x_1]_1 = [x_2]_1 = [x_3]_1$ then $[x_1']_2 = [x_2']_2 = [x_3']_2$, $[x_i']_1 = [\widetilde{B}(x_i)]_1$ and so by **(Z2)**$_g$, $(x_1'|x_2', x_3') = (\widetilde{B}(x_1)|\widetilde{B}(x_2), \widetilde{B}(x_3))$. Hence $(x_1|x_2, x_3) = (x_1'|x_2', x_3') \ \Leftrightarrow\ B \in \mathfrak{C}_\xi$. If $[x_1]_2 = [x_2]_2 = [x_3]_2$ then $[x_i']_2 = [\widetilde{A}(x_i)]_2$ and so $(x_1|x_2, x_3) = (x_1'|x_2', x_3') \ \Leftrightarrow\ A \in \mathfrak{C}_\xi$. Let $K \in \mathfrak{C}_\xi$, hence $\widetilde{K} \in Aut(\mathcal{P}, \xi)^-$ (cf. (1.4.1.3) and (3.1.2)). Then $\widetilde{K} \circ \alpha \in Aut(\mathcal{P}, \xi)^+$, hence by (3.1.4) $\beta := \widetilde{K} \circ \alpha_{|\mathfrak{G}_1} \in Aut(\mathfrak{G}_1, \xi_1)$ and so by (1), $\alpha_{|\mathfrak{G}_1} = \widetilde{K}_{|\mathfrak{G}_1} \circ \beta$ is an isomorphism from $(\mathfrak{G}_1, \xi_1)$ onto $(\mathfrak{G}_2, \xi_2)$.

(3) By (1), $\beta_2 \in Aut(\mathfrak{G}_2, \xi_2)$ and by the definition of β, $\beta \in \overline{\Gamma}^+$. Now let $X \in \mathfrak{G}_i$ and $x \in X$, hence $X = [x]_i$ and so $\beta(X) = \beta_i(X)$, i.e. $\beta_{|\mathfrak{G}_i} = \beta_i$.

Now let $(a, b, c) \in \mathcal{P}^{3g}$ and for instance $Y := [a]_2 = [b]_2 = [c]_2$. Then $(a|b, c)_\xi = ([a]_1|[b]_1, [c]_1)_{\xi_1}$ and

$$(\beta(a)|\beta(b), \beta(c))_\xi \overset{\textbf{def.}}{=} (\beta_1([a]_1)\ \cap\ \beta_2(Y)|\beta_1([b]_1)\ \cap\ \beta_2(Y), \beta_1([c]_1)\ \cap\ \beta_2(Y))_\xi \overset{\textbf{def.}}{=}$$

$$= (\beta_1([a]_1)|\beta_1([b]_1), \beta_1([c]_1))_{\xi_1} \overset{\beta_1 \in Aut(\mathfrak{G}_1, \xi_1)}{=} ([a]_1|[b]_1, [c]_1)_{\xi_1} = (a|b, c)_\xi.$$

This shows $\beta \in Aut(\mathcal{P}, \xi)^+$.

(4) This statement is a consequence of (3.1.4) and (3).□

(3.1.6) *If $\mathfrak{K} \neq \emptyset$ then $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \xi)$ is completely determined by the halfordered set $(\mathfrak{G}_1, \xi_1)$.*

Proof. Let $G \in \mathfrak{G}_1 \cup \mathfrak{G}_2$ and $(a, b, c) \in G^{3'}$. If $G \in \mathfrak{G}_2$ then $(a|b, c)_\xi = ([a]_1|[b]_1, [c]_1)_{\xi_1}$, if $G \in \mathfrak{G}_1$ we choose a $K \in \mathfrak{K}$ and then $(a|b, c)_\xi = (\widetilde{K}([a]_2)|\widetilde{K}([b]_2), \widetilde{K}([c]_2))_{\xi_1}$.□

(3.1.7) *Let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \xi)$ be a halfordered chain structure, let $\mathfrak{K}^\sim$ be the symmetric and $\mathfrak{K}^{\sim\sim}$ the double symmetric closure of $\mathfrak{K}$ (cf. section* 1.5). *Then:*

(1) $\mathfrak{K} \subset \mathfrak{K}^\sim \subset \mathfrak{K}^{\sim\sim} \subset \mathfrak{C}_\xi$.

(2) $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}^\sim; \xi)$, $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}^{\sim\sim}; \xi)$ (*besides* $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C}_\xi; \xi)$ *cf.* (3.1.2)), *are halfordered chain structures.*

Proof. (1) Since $\mathfrak{K}^{\sim} \subset \mathfrak{K}^{\sim\sim}$ (cf. (1.5.1)) and $\mathfrak{K}'_1 = \mathfrak{K} \subset \mathfrak{C}_\xi$ let $K \in \mathfrak{K}^{\sim\sim} \setminus \mathfrak{K}$. By (1.5.1) there is an $n \in \mathbb{N}$ with $K \in \mathfrak{K}'_{n+1}$ We assume $\mathfrak{K}'_n \subset \mathfrak{C}_\xi$. Then there are $A, B, C \in \mathfrak{K}'_n$

$K = \widetilde{A,B}(C)$ and by (1.4.1.3) and (1.4.1.7), $\widetilde{K} = \widetilde{A,B} \circ \widetilde{C} \circ \widetilde{B,A}$. By (3.1.5.2),

$\widetilde{C}, \widetilde{B,A} \in Aut(\mathcal{P}, \xi)^-$ hence $\widetilde{K} \in Aut(\mathcal{P}, \xi)^-$, i.e. $K \in \mathfrak{C}_\xi$.

y (1) and by (3.1.1.2), each of the quintuples $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}^{\sim}; \xi)$ and $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}^{\sim\sim};$

a halfordered chain structure. □

3.2 EXTENSION OF ξ ONTO $\mathcal{P}^{3\mathfrak{K}}$

Let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \xi)$ be a halfordered chain structure. We will extend the halforder ξ onto $\mathcal{P}^{3\mathfrak{K}}$. By $\mathbf{(Z2)}_g$ the perspectivities in the two directions given by $\mathfrak{G}_1$ and $\mathfrak{G}_2$ preserve the betweenness relation. Since we want this property kept we define the extension $\overline{\xi}$ by:

$$\overline{\xi} : \begin{cases} \mathcal{P}^{3\mathfrak{K}} & \to & \{1, -1\} \\ (a, b, c) & \mapsto & (a|b, c)_{\overline{\xi}} := ([a]_1|[b]_1, [c]_1)_{\xi_1} \end{cases} .$$

By axiom $\mathbf{(ZC)}_g$ we have $(a|b, c)_{\overline{\xi}} = ([a]_2|[b]_2, [c]_2)_{\xi_2}$.

Then the following axioms are satisfied:

(Z1) $\forall\ X \in \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{K},\ (X, \overline{\xi}_{|X^{3'}})$ is a halfordered set.

(Z2) $\forall\ G, L \in \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{K},\ \forall\ (a, b, c) \in G^{3'},\ \forall i, j \in \{1, 2\}$ with $i \neq j$:

(a) if $G, L \in \mathfrak{G}_i$:

$$(a|b, c)_{\overline{\xi}} = ([a]_j \cap L|[b]_j \cap L, [c]_j \cap L)_{\overline{\xi}} \ ;$$

(b) if $G, L \in \mathfrak{K}$:

$$(a|b, c)_{\overline{\xi}} = ([a]_i \cap L|[b]_i \cap L, [c]_i \cap L)_{\overline{\xi}} \ ;$$

(c) if $G \in \mathfrak{K},\ L \in \mathfrak{G}_i$ (or $G \in \mathfrak{G}_i,\ L \in \mathfrak{K}$):

$$(a|b, c)_{\overline{\xi}} = ([a]_j \cap L|[b]_j \cap L, [c]_j \cap L)_{\overline{\xi}}.$$

The axiom:

(ZC) $\forall\ K \in \mathfrak{K},\ \forall\ (a, b, c) \in \mathcal{P}^{3\mathfrak{K}} \cup \mathcal{P}^{3g}$:

$$(a|b, c)_{\overline{\xi}} = (\widetilde{K}(a)|\widetilde{K}(b), \widetilde{K}(c))_{\overline{\xi}},$$

makes sense only if $\mathfrak{K}$ is symmetric. In this case **(ZC)** is satisfied; in fact:

$(a|b, c)_{\overline{\xi}} = ([a]_1|[b]_1, [c]_1)_{\xi_1} \overset{(3.1.5.1)}{=} (\widetilde{K}([a]_1)|\widetilde{K}([b]_1), \widetilde{K}([c]_1))_{\xi_2} = (\widetilde{K}(a)|\widetilde{K}(b), \widetilde{K}(c))_{\overline{\xi}}$.

Definition 3.2.1 Let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a symmetric chain structure, let

$$\mathcal{P}^{3g\cup\mathfrak{K}} := \{(a,b,c) \in \mathcal{P}^{3'} \mid \exists\, L \in \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{K} : \ a,b,c \in L\}$$

and let

$$\xi : \mathcal{P}^{3g\cup\mathfrak{K}} \to \{1,-1\}; (x,y,z) \mapsto (x|y,z)$$

be a function such that the axioms **(Z1)**,**(Z2)** and **(ZC)** are satisfied. Then $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \xi)$ is called a *halfordered symmetric chain structure.*

By (3.1.7) and this extension process we have the result:

(3.2.1) *Let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \xi)$ be a halfordered chain structure, let ξ', ξ'' and ξ''' be the extension of ξ onto the symmetric set of chains $\mathfrak{K}^\sim$, $\mathfrak{K}^{\sim\sim}$ and $\mathfrak{C}_\xi$ respectively; then $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}^\sim; \xi')$, $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}^{\sim\sim}; \xi'')$ and $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C}_\xi; \xi''')$ are halfordered symmetric chain structures.*

3.3 HALFORDERS AND SPLITTING BY A CHAIN

In this section let E be a non empty set and let $\nu(E) := (\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ be the corresponding greatest chain structure (where $\mathcal{P} := E \times E, \mathfrak{G}_1 := \{\{x\} \times E \mid x \in E\}, \mathfrak{G}_2 := \{E \times \{y\} \mid y \in E\}$ and $\mathfrak{C} := \{C \in 2^{\mathcal{P}} \mid \forall\, X \in \mathfrak{G}_1 \cup \mathfrak{G}_2 : |C \cap X| = 1\} \neq \emptyset$). The elements of $\mathcal{P}$, $\mathfrak{G}_1 \cup \mathfrak{G}_2$ and $\mathfrak{C}$ will be called *points, generators* and *chains* respectively. The triple $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$, called $2-$*net*, has the two properties **(N1)** and **(N2)** (cf. sec. 1.1) :
We will use the operation $\Box := \Box_{12}$ (cf. sec. 1.1).

(3.3.1) *Let* K_s *be a splitting of* $\mathcal{P}$ by $K \in \mathfrak{C}$ *and let* $\xi_l = \xi_l(K_s)$ and $\xi_r = \xi_r(K_s)$ *be the two maps defined by* :

$$\xi_l : \begin{cases} K^{3'} & \to & \{1, -1\} \\ (a, b, c) & \mapsto & (a|b, c)_l := K_s(a\Box b, a\Box c) \end{cases}$$

and

$$\xi_r : \begin{cases} K^{3'} & \to & \{1, -1\} \\ (a, b, c) & \mapsto & (a|b, c)_r := K_s(b\Box a, c\Box a) \end{cases} .$$

Then ξ_l and ξ_r *are halforders of* K *and* ξ_l *rel* ξ_r.

Proof. We have to show that the two functions ξ_l and ξ_r satisfy the condition **(Z)**.
$\xi_l)$ $\forall\ (a, b, c), (a, c, d) \in K^{3'}$:

$$(a|b, c)_l \cdot (a|c, d)_l := K_s(a\Box b, a\Box c) \cdot K_s(a\Box c, a\Box d) \overset{\textbf{(S1)}}{=}$$

$$K_s(a\Box b, a\Box d) =: (a|b, d)_l;$$

$\xi_r)$ $\forall\ (a, b, c), (a, c, d) \in K^{3'}$:

$$(a|b, c)_r \cdot (a|c, d)_r := K_s(b\Box a, c\Box a) \cdot K_s(c\Box a, d\Box a) \overset{\textbf{(S1)}}{=}$$

$$K_s(b\Box a, d\Box a) =: (a|b, d)_r.$$

Now we prove that the condition **(R)** is valid, i.e. ξ_l *rel* ξ_r :
$\forall\ (a,b,c,d) \in K^{4'}$ we have :

$$[a,b|c,d]_l := (a|c,d)_l \cdot (b|c,d)_l := K_s(a\Box c, a\Box d) \cdot K_s(b\Box c, b\Box d) \overset{\textbf{(S1)}}{=}$$
$$K_s(a\Box c, b\Box c) \cdot K_s(b\Box c, a\Box d) \cdot K_s(b\Box c, a\Box d) \cdot K_s(a\Box d, b\Box d) =$$
$$K_s(a\Box c, b\Box c) \cdot K_s(a\Box d, b\Box d) =: (c|a,b)_r \cdot (d|a,b)_r =: [c,d|a,b]_r. \ \Box$$

Remark. There are halforders ξ_l and ξ_r (derived by a splitting K_s by a chain K) such that ξ_l *rel* ξ_r, ξ_l *rel* ξ_l, ξ_r *rel* ξ_r and $\xi_l \neq \xi_r$:
$(b|a,c)_l = (b|a,d)_l = (c|a,d)_l = (d|a,c)_l = -(a|b,c)_l = -(a|b,d)_l = -(c|a,b)_l = -(d|a,b)_l = 1$ and $\xi_r = 1$. In fact $[a,b|c,d]_l = [a,c|b,d]_l = [a,d|c,b]_l = [d,b|a,c]_l = [b,c|a,d]_l = [c,d|a,b]_l = 1$, i.e. ξ_l *rel* ξ_r. Since $\tau_{\xi_r} = 1$ we have ξ_r *rel* ξ_r and this implies ξ_l *rel* ξ_l.

(3.3.2) *Let* $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ *be a chain structure with* $|\mathfrak{G}_1| \geq 4$, *let* $K \in \mathfrak{C}$ *and for* $p \in \mathcal{P}$ *let* $p_i := K \cap [p]_i$ *(hence* $p = p_1 \Box p_2$*). Moreover let* ξ_1, ξ_2, ξ *be halforders of* K, *where* ξ_1, ξ_2 *are related. Then:*
(1) *For* $x, y \in \mathcal{P} \setminus K$ *and* $0 \in K \setminus \{x_1, y_1\}$ *let*
$K(x,y)_0 := (x_1|0,x_2)_1 \cdot (y_1|0,y_2)_1 \cdot (0|x_1,y_1)_2$
then $K(x,y)_0 = K(x,y)_{0'}$ *for* $0, 0' \in K \setminus \{x_1, y_1\}$ *(we set* $K(x,y) := K(x,y)_0$ *where* $0 \in K \setminus \{x_1, y_1\}$*) and the function*
$K_s := K_s(\xi_1, \xi_2) : (\mathcal{P} \setminus K) \times (\mathcal{P} \setminus K) \to \{1, -1\}; (x,y) \mapsto K(x,y)$
is a splitting of $\mathcal{P}$ *by* K *with* $\xi_l(K_s) = \xi_1$ *and* $\xi_r(K_s) = \xi_2$.
(2) *Let* $\{0, 1, \infty\} \in \binom{K}{3}$ *be fixed and for* $x \in \mathcal{P} \setminus K$ *let*

$$K(x) := \begin{cases} (x_1|0,x_2) \cdot (0|1,x_1) & \text{for} \quad x_1 \neq 0 \\ (0|\infty,x_2) \cdot (\infty|1,0) \cdot (1|0,\infty) = (0|1,x_2) \cdot k_\xi(\{0,1,\infty\}) & \text{for} \quad x_1 = 0 \end{cases} .$$

Then the function

$K_s := K_s(\xi) : (\mathcal{P} \setminus K) \times (\mathcal{P} \setminus K) \to \{1, -1\}; (x, y) \mapsto K(x) \cdot K(y)$
is a splitting of $\mathcal{P}$ by K *with* $\xi_l(K_s(\xi)) = \xi$ *and* $\xi_r(K_s(\xi)) = \xi'(\xi, 0_s, \epsilon)$ *where* $\forall\ a, b \in K \setminus \{0\} : 0_s(a, b) := (0|a, b)$ *and* $\epsilon = (1|0, \infty)$ (*cf.* (2.1.2)).
(3) *The splitting* K_s *defined in* (2) *is independent of the choice of the elements* $\{0, 1, \infty\} \in \binom{K}{3}$ *if and only if* ξ *is selfrelated.*

Proof. (1) $K(x, y)_0 \cdot K(x, y)_{0'} = (x_1|0, 0')_1 \cdot (y_1|0, 0')_1 \cdot [0, 0'|x_1, y_1]_{\xi_2} = [x_1, y_1|0, 0']_{\xi_1} \cdot [0, 0'|x_1, y_1]_{\xi_2}$ hence $K(x, y)_0 = K(x, y)_{0'}$.
We observe that by definition, $K(x, y) = K(y, x)$.
Now let $x, y, z \in \mathcal{P} \setminus K$. Since $|K| = |\mathfrak{G}_1| \geq 4$ there is an element $0 \in \mathcal{P} \setminus \{x_1, y_1, z_1\}$ and hence:
$K_s(x, y) \cdot K_s(y, z) = (x_1|0, x_2)_1 \cdot (y_1|0, y_2)_1 \cdot (0|x_1, y_1)_2 \cdot (y_1|0, y_2)_1 \cdot (z_1|0, z_2)_1 \cdot (0|y_1, z_1)_2 \overset{(\mathbf{Z})}{=} (x_1|0, x_2)_1 \cdot (0|x_1, z_1)_2 \cdot (z_1|0, z_2)_1 =: K_s(x, z)$.
Now let $(a, b, c) \in K^{3'}$; then $(a|b, c)_l := K_s(a\Box b, a\Box c) := (a|0, b)_1 \cdot (a|0, c)_1 \cdot (0|a, a)_2 \overset{(\mathbf{Z})}{=} (a|b, c)_1$ and $(a|b, c)_r := K_s(b\Box a, c\Box a) := (b|0, a)_1 \cdot (c|0, a)_1 \cdot (0|b, c)_2 \overset{\xi_1\ rel\ \xi_2}{=} (0|b, c)_2 \cdot (a|b, c)_2 \cdot (0|b, c)_2 = (a|b, c)_2$.
(2) For $x, y, z \in \mathcal{P} \setminus K : K_s(x, y) \cdot K_s(y, z) = K(x) \cdot K(y) \cdot K(y) \cdot K(z) = K_s(x, z)$, i.e. $K_s(\xi)$ is a splitting of $\mathcal{P}$ by K.
For $(a, b, c) \in K^{3'}$ with $b \neq c$:

$$(a|b, c)_l = K_s(a\Box b, a\Box c) = K(a\Box b) \cdot K(a\Box c) =$$

$$\begin{cases} (a|0, b) \cdot (0|a, 1) \cdot (a|0, c) \cdot (0|a, 1) \overset{(\mathbf{Z})}{=} (a|b, c) & if \quad a \neq 0 \\ (0|\infty, b) \cdot (\infty|0, 1) \cdot (1|0, \infty) \cdot (0|\infty, c) \cdot (\infty|0, 1) \cdot (1|0, \infty) \overset{(\mathbf{Z})}{=} (0|b, c) & if \quad a = 0 \end{cases}$$

hence $\xi_l(K_s(\xi)) = \xi$ and by (2.1.2),

$$(a|b, c)_r = K_s(b\Box a, c\Box a) = K(b\Box a) \cdot K(c\Box a) =$$

$$\begin{cases} (b|0, a) \cdot (0|b, 1) \cdot (c|0, a) \cdot (0|c, 1) \overset{(\mathbf{Z})}{=} [b, c|0, a]_\xi \cdot (0|b, c) = (a|b, c)' & if \quad b, c \neq 0 \\ (b|0, a) \cdot (0|b, 1) \cdot (0|\infty, a) \cdot (\infty|0, 1) \cdot (1|0, \infty) & if \quad c = 0 \neq b \end{cases}.$$

Since by **(Z)**, $(0|1, b) \cdot (0|\infty, a) = (0|1, a) \cdot (0|\infty, b)$ we have also $(a|b, 0)_r = (a|0, b)_r =$

$(a|b,0)'$ thus $\xi_r(K_s(\xi)) = \xi'(\xi, 0_s, (1|0,\infty))$.

(3) Let $\{0,1,\infty\}, \{0',1',\infty'\} \in \binom{K}{3}$ and let $x = x_1 \Box x_2, y = y_1 \Box y_2 \in \mathcal{P} \setminus K$; then:

a) $0 = 0'$: by **(Z)** we obtain:

$$K(x) \cdot K'(x) = \begin{cases} (0|1,1') & \text{if} \quad x_1 \neq 0 \\ (0|1,1') \cdot k_\xi(\{0,1,\infty\}) \cdot k_\xi(\{0,1',\infty'\}) & \text{if} \quad x_1 = 0 \end{cases},$$

therefore

$$K(x) \cdot K'(x) \cdot K(y) \cdot K'(y) =$$

$$\begin{cases} 1 & \text{if} \quad x_1, y_1 \neq 0 \quad \text{or} \quad x_1 = y_1 = 0 \\ k_\xi(\{0,1,\infty\}) \cdot k_\xi(\{0,1',\infty'\}) & \text{if} \quad x_1 \neq 0, y_1 = 0 \quad \text{or} \quad x_1 = 0, y_1 \neq 0 \end{cases}.$$

b) $0 \neq 0'$: by **(Z)** we obtain:

$$K(x) \cdot K'(x) = \begin{cases} (x_1|0,0') \cdot (0|1,x_1) \cdot (0'|1',x_1) & \text{if} \quad x_1 \neq 0, 0' \\ k_\xi(\{0,1,\infty\}) \cdot (0|0',1) \cdot (0'|0,1') & \text{if} \quad x_1 = 0 \\ k_\xi(\{0',1',\infty'\}) \cdot (0|0',1) \cdot (0'|0,1') & \text{if} \quad x_1 = 0' \end{cases}.$$

Therefore again by **(Z)**:

$$K(x) \cdot K'(x) \cdot K(y) \cdot K'(y) =$$

$$\begin{cases} [x_1, y_1|0,0'] \cdot [0,0'|x_1,y_1] & \text{if} \quad x_1, y_1 \neq 0, 0' \\ k_\xi(\{0,1,\infty\}) \cdot k_\xi(\{0,0',x_1\}) & \text{if} \quad x_1 \neq 0, 0', y_1 = 0 \\ k_\xi(\{0',1',\infty'\}) \cdot k_\xi(\{0,0',x_1\}) & \text{if} \quad x_1 \neq 0, 0', y_1 = 0' \\ k_\xi(\{0,1,\infty\}) \cdot k_\xi(\{0',1',\infty'\}) & \text{if} \quad x_1, y_1 \in \{0,0'\} \end{cases}.$$

This shows: $K(x) \cdot K(y) = K'(x) \cdot K'(y) \;\Leftrightarrow\; \xi$ is selfrelated.□

In the chain structure $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ we associate to any chain $A \in \mathfrak{C}$ the following map (cf. section 1.4):

$$\widetilde{A} : \begin{cases} \mathcal{P} & \to \quad \mathcal{P} \\ x & \mapsto \quad [[x]_1 \cap A]_2 \cap [[x]_2 \cap A]_1 \end{cases}.$$

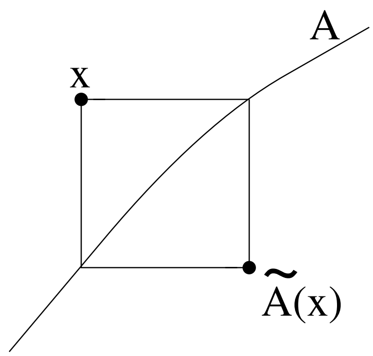

FIGURE 3.1.

Remark. If $a, b \in A \in \mathfrak{C}$ then : $\widetilde{A}(a\Box b) = b\Box a$. $(\diamond\diamond)$

(3.3.3) *Let K_s be a splitting of $\mathcal{P}$ by $K \in \mathfrak{C}$, $\overline{a} \in \mathcal{P} \setminus K$, $\mathcal{P}^+ := \overrightarrow{K, \overline{a}} := \{\overline{x} \in \mathcal{P} \setminus K \mid K_s(\overline{a}, \overline{x}) = 1\}$, $\mathcal{P}^- := (\overrightarrow{K, \overline{a}})^- := \{\overline{x} \in \mathcal{P} \setminus K \mid K_s(\overline{a}, \overline{x}) = -1\}$ and let ξ_l and ξ_r be the two associated halforders of K (cf. (3.3.1)). Then the following statements are equivalent:*

(1) $\xi_l = \xi_r$ *(then $\xi := \xi_l = \xi_r$ is selfrelated and (by (2.1.1)) k_ξ is a constant function).*

(2) $\widetilde{K} \in Aut(\mathcal{P}, \{\mathcal{P}^+, \mathcal{P}^-\})$.

(3) $\forall\ \overline{x} = x_1\Box x_2 \in \mathcal{P} \setminus K$, *with* $x_1, x_2 \in K$ *and* $0 \in K \setminus \{x_1, x_2\}$,

$$K_s(\overline{x}, \widetilde{K}(\overline{x})) = (x_1|0, x_2) \cdot (x_2|0, x_1) \cdot (0|x_1, x_2) = k_\xi(\{0, x_1, x_2\}) = constant.$$

(4) $\forall\ \{a, b, c\} \in \binom{K}{3} : k_\xi(\{a, b, c\}) = constant.$

Proof. (2) $\Rightarrow$ (1). Let $(a, b, c) \in K^{3'}$. Then

$$(a|b, c)_l := K_s(a\Box b, a\Box c) \overset{(\diamond\diamond)}{=}$$

$$K_s(\widetilde{K}(b\Box a), \widetilde{K}(c\Box a)) \overset{(2)}{=} K_s(b\Box a, c\Box a) =: (a|b, c)_r.$$

(1) $\Rightarrow$ (2). Let $\overline{x}, \overline{y} \in \mathcal{P} \setminus K$ and $x_i := K \cap [\overline{x}]_i, y_i := K \cap [\overline{y}]_i$, then $\overline{x} = x_1\Box x_2, \overline{y} = y_1\Box y_2$

and if $x_1 \neq y_2$,

$$K_s(\overline{x},\overline{y}) = K_s(x_1\Box x_2, y_1\Box y_2) \overset{(\mathbf{S1})}{=}$$

$$K_s(x_1\Box x_2, x_1\Box y_2)\cdot K_s(x_1\Box y_2, y_1\Box y_2) = (x_1|x_2,y_2)_l\cdot(y_2|x_1,y_1)_r$$

and

$$K_s(\widetilde{K}(\overline{x}),\widetilde{K}(\overline{y})) \overset{(\diamond\diamond)}{=} K_s(x_2\Box x_1, y_2\Box y_1) \overset{(\mathbf{S1})}{=}$$

$$K_s(x_2\Box x_1, y_2\Box x_1)\cdot K_s(y_2\Box x_1, y_2\Box y_1) = (x_1|x_2,y_2)_r\cdot(y_2|x_1,y_1)_l.$$

If $x_2 \neq y_1$ then $K_s(\overline{x},\overline{y}) = (x_2|x_1,y_1)_r\cdot(y_1|x_2,y_2)_l$ and $K_s(\widetilde{K}(\overline{x}),\widetilde{K}(\overline{y})) = (x_2|x_1,y_1)_l\cdot(y_1|x_2,y_2)_r$.

Since (1) is valid, $K_s(\overline{x},\overline{y}) = K_s(\widetilde{K}(\overline{x}),\widetilde{K}(\overline{y}))$.

If $x_1 = y_2$ and $y_1 = x_2$ then $K_s(\overline{x},\overline{y}) = K_s(x_1\Box x_2, x_2\Box x_1)$ and $K_s(\widetilde{K}(\overline{x}),\widetilde{K}(\overline{y})) \overset{(\diamond\diamond)}{=} K_s(x_2\Box x_1, x_1\Box x_2)$, hence $K_s(\overline{x},\overline{y}) = K_s(\widetilde{K}(\overline{x}),\widetilde{K}(\overline{y}))$.

(3) $\Leftrightarrow$ (2). By **(S1)** we have $\forall\ \overline{a},\overline{b}\in\mathcal{P}\setminus K$:

$$K_s(\overline{a},\overline{b})\cdot K_s(\widetilde{K}(\overline{a}),\widetilde{K}(\overline{b}))\cdot K_s(\overline{a},\widetilde{K}(\overline{a}))\cdot K_s(\overline{b},\widetilde{K}(\overline{b})) = 1,$$

and so

$$K_s(\overline{a},\overline{b})\cdot K_s(\widetilde{K}(\overline{a}),\widetilde{K}(\overline{b})) = K_s(\overline{a},\widetilde{K}(\overline{a}))\cdot K_s(\overline{b},\widetilde{K}(\overline{b})).$$

By this formula we obtain that the two statements (3) and (2) are equivalent.

(3) $\Rightarrow$ (4). Let $a,b,c\neq 0$ then $k_\xi(\{a,b,c\}) = k_\xi(\{a,b,0\})\cdot k_\xi(\{b,c,0\})\cdot k_\xi(\{c,a,0\})$. $\Box$

Definition 3.3.1 A splitting K_s of $\mathcal{P}$ by $K\in\mathfrak{C}$ is called *selfrelated* if one of the three equivalent statements of (3.3.3) is valid.

Now we can formulate the following main result:

(3.3.4) *Let $K\in\mathfrak{C}$ be a chain of a chain structure $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{C})$, let $\Sigma_{self}(K)$ be the set of all selfrelated halforders of K and let $S_{self}(K)$ be the set of all selfrelated splittings of $\mathcal{P}$ by K. Then there is exactly one bijection*

$\varphi : \Sigma_{self}(K) \rightarrow\ S_{self}(K); \xi \mapsto K_s := \varphi(\xi)$
such that $\forall\ a,b,c,d,0 \in K, a \neq b, c \neq d, 0 \neq a,b :$

$$K_s(a\Box b, c\Box d) = \begin{cases} (a|b,d)\cdot(d|a,c) & \textit{if} \quad a \neq d \\ (c|d,b)\cdot(b|a,c) & \textit{if} \quad c \neq b \\ k_\xi(\{a,b,0\}) & \textit{if} \quad a=d, b=c \end{cases}$$

and $\forall\ (a,b,c) \in K^{3'}:\ (a|b,c) = K_s(a\Box b, a\Box c)$.

In the section 3.1 we introduced the notion of a halfordered chain structure $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \xi)$, that means that $\mathfrak{K}$ is a non empty subset of the set $\mathfrak{C}$ of all chains and $\xi : \mathcal{P}^{3g} \rightarrow \{1,-1\}$ a function defined on the set $\mathcal{P}^{3g} := \{(a,b,c) \in \mathcal{P}^{3'} \mid \exists\ i \in \{1,2\} : [a]_i = [b]_i = [c]_i\}$ such that the axioms $\mathbf{(Z1)}_g$, $\mathbf{(Z2)}_g$ and $\mathbf{(ZC)}_g$ are satisfied. Then

(3.3.5) *Let* $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \xi)$ *be a halfordered chain structure,* $a \in \mathcal{P}$ *and* $i \in \{1,2\}$ *fixed and let* $j \in \{1,2\} \setminus \{i\}$*. For each* $K \in \mathfrak{K}$ *let*

$$\xi_K : \begin{cases} K^{3'} & \rightarrow & \{1,-1\} \\ (x,y,z) & \mapsto & ([x]_j \cap [a]_i \mid [y]_j \cap [a]_i, [z]_j \cap [a]_i)_\xi = (a\Box x|a\Box y, a\Box z)_\xi = \\ & & (x\Box a|y\Box a, z\Box a)_\xi. \end{cases}$$

Then (K, ξ_K) *is a halfordered set isomorphic to* $([a]_i, \xi_{|[a]_i^{3'}})$ *(cf.* $\mathbf{(Z1)}_g$*) which is independent of the choice of* $a \in \mathcal{P}$ *(cf.* $\mathbf{(Z2)}_g$*) and of* i *(cf.* $\mathbf{(ZC)}_g$*).*

Definition 3.3.2 The halforder ξ_K defined in (3.3.5) on K is called the *induced halforder*. $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \xi)$ is called *selfrelated* if one (and then all) halfordered set $(G, \xi_{|G^{3'}})$, with $G \in \mathfrak{G}_1 \cup \mathfrak{G}_2$, is selfrelated.

3.4 CONVEXITY FOR SPLITTING

In this section let E be a non empty set and let $\nu(E) := (\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ be the corresponding greatest *chain structure* (cf. sec. 3.3).

A splitting K_s by a chain $K \in \mathfrak{C}$ is called *harmonic* or *anharmonic* resp. if $\forall\ a, b \in K, a \neq b : K_s(a\Box b, b\Box a) = -1$ or $= 1$ respectively. By (3.3.3) each selfrelated splitting is either harmonic or anharmonic and each harmonic or anharmonic splitting is selfrelated.

(3.4.1) *A splitting K_s by a chain is harmonic if and only if $\forall\ a, b \in \mathcal{P} \setminus K$ with $a\Box b \in K$ and $K_s(a, b) = 1 : b\Box a \notin K$.*

Proof. " $\Rightarrow$ " Let $a, b \in \mathcal{P} \setminus K$ such that $c := a\Box b \in K$ and $K_s(a, b) = 1$. If $d := b\Box a \in K$, then $a = c\Box d$, $b = d\Box c$, $1 = K_s(a, b) = K_s(c\Box d, d\Box c)$ and so K_s is not harmonic.
" $\Leftarrow$ " Let $a, b \in K$, with $a \neq b$ and let $c := a\Box b, d := b\Box a$. Then $c, d \in P\setminus K$, $c\Box d = a \in K$ and $d\Box c = b \in K$. Thus our condition implies $K_s(c, d) = -1$, i.e. K_s is harmonic.□

In the following let K_s be a splitting by a chain K. Then K_s is called
weakly convex if $\forall\ a, b \in \mathcal{P} \setminus K$ with $a\Box b, b\Box a \notin K, K_s(a, b) = 1$ and $K_s(a, a\Box b) = -1$: $K_s(a, b\Box a) = 1$ (and so $K_s(b, b\Box a) = 1$).

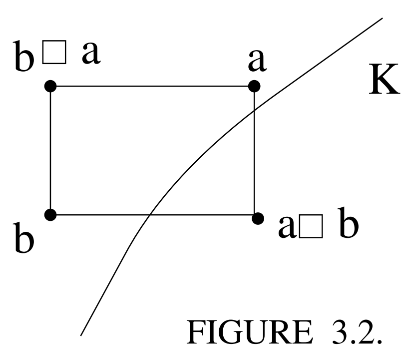

FIGURE 3.2.

convex if $\forall\ a, b \in \mathcal{P} \setminus K$ with $a\Box b \in K$, $b\Box a \notin K$ and $K_s(a, b) = 1 : K_s(a, b\Box a) = 1$.

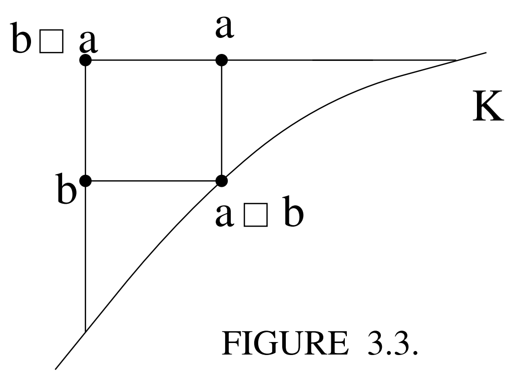

FIGURE 3.3.

strongly convex if $\forall\ a,b \in \mathcal{P} \setminus K$ with $a\Box b \in K$ and $K_s(a,b) = 1 : b\Box a \notin K$ and $K_s(a, b\Box a) = 1$.

(3.4.2) (1) *If K_s is strongly convex then K_s is convex.*
(2) *If K_s is convex then K_s is weakly convex.*

Proof. (1) follows directly from the definitions.
(2) Let $a,b \in \mathcal{P} \setminus K$ with $a\Box b, b\Box a \notin K$ and $K_s(a,b) = -K_s(a,a\Box b) = 1$ and let $c := [a]_1 \cap K$. Then $a\Box(b\Box c) = c \in K$ and $b\Box c,\ (b\Box c)\Box a = b\Box a \notin K$. If $K_s(a,b\Box c) = 1$ then by the convexity $K_s(a,b\Box a) = 1$. Therefore let $K_s(a,b\Box c) = -1$ and let $a' :=$ $a\Box b,\ b' := b\Box c$. Then $K_s(a',b') = K_s(a,a') \cdot K_s(a,b') = (-1)\cdot(-1) = 1,\ a'\Box b' =$ $(a\Box b)\Box(b\Box c) = a\Box c = c \in K$ and $b'\Box a' = (b\Box c)\Box(a\Box b) = b \notin K$, hence by the convexity $1 = K_s(a', b'\Box a') = K_s(a\Box b, b)$. But $-1 = (-1)\cdot 1 = K_s(a,a\Box b)\cdot K_s(a\Box b,b) = K_s(a,b)$ contradicts $K_s(a,b) = 1$. Thus $K_s(a,b\Box c)$ is equal 1 and so $K_s(a,b\Box a) = 1$.□

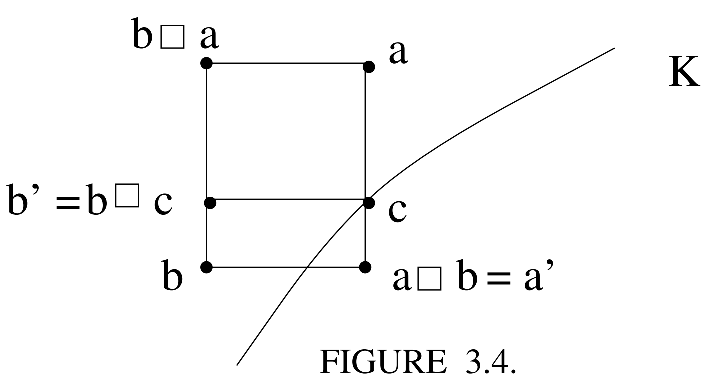

FIGURE 3.4.

(3.4.3) *Let ξ_l and ξ_r be the two halforders of K corresponding with K_s (cf.* (3.3.1)). *Then:*

(1) *The following statements are equivalent*:

(i) K_s *is convex*

(ii) $\forall\ \{a,b,c\} \in \binom{K}{3}, (a|b,c)_l = (c|a,b)_r \ \Rightarrow\ (a|b,c)_l = 1$

(iii) $\forall\ \{a,b,c\} \in \binom{K}{3}, (a|b,c)_l = -1 \Rightarrow\ (c|a,b)_r = (b|a,c)_r = 1$

(2) K_s *is weakly convex* $\Leftrightarrow\ \forall\ \{a,b,c,d\} \in \binom{K}{4}$ with $(a|b,d)_l = (d|a,c)_r = -1 : (b|a,c)_r = 1$ *and also* $(c|b,d)_l = 1$.

(3) *If* $\xi_l = \xi_r$ *then* : K_s *is weakly convex* $\Leftrightarrow$ (K,ξ_l) *is D-convex.*

Proof. (1) Let $\{a,b,c\} \in \binom{K}{3}$ such that $1 = K_s(a\Box b, b\Box c) \overset{(\mathbf{S1})}{=} K_s(a\Box b, a\Box c)\cdot K_s(a\Box c, b\Box c) \overset{(3.3.1)}{=} (a|b,c)_l \cdot (c|a,b)_r$, i.e. $(a|b,c)_l = (c|a,b)_r$. Since $K_s(b\Box c, a\Box c) = (c|a,b)_r$ we have: K_s is convex $\Leftrightarrow$ $(a|b,c)_l = (c|a,b)_r$ implies $(a|b,c)_l = 1$.

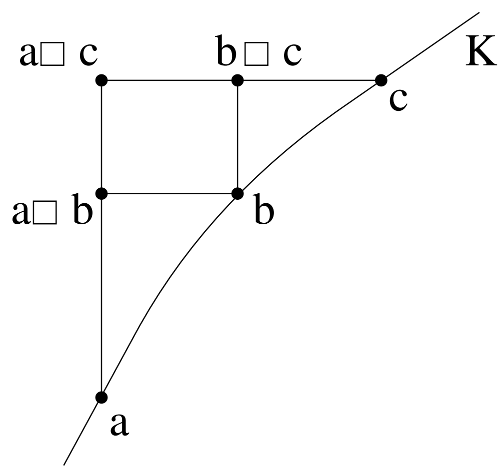

FIGURE 3.5.

(2) Let $a,b,c,d \in K$ such that $a' := a\Box b,\ b' := c\Box d,\ a'\Box b' = a\Box d,\ b'\Box a' = c\Box b \notin K$. Then $a \neq b \neq c \neq d \neq a$, $K_s(a',b') = K_s(a\Box b, a\Box d)\cdot K_s(a\Box d, c\Box d) = (a|b,d)_l \cdot (d|a,c)_r, K_s(a', a'\Box b') = K_s(a\Box b, a\Box d) = (a|b,d)_l$ and $K_s(a', b'\Box a') = K_s(a\Box b, c\Box b) = (b|a,c)_r$. Therefore:

$K_s(a',b') = -K_s(a', a'\Box b') = 1 \ \Leftrightarrow\ (a|b,d)_l = (d|a,c)_r = -1 \ \Rightarrow\ \{a,b,c,d\} \in \binom{K}{4}$

and this implies the equivalence.

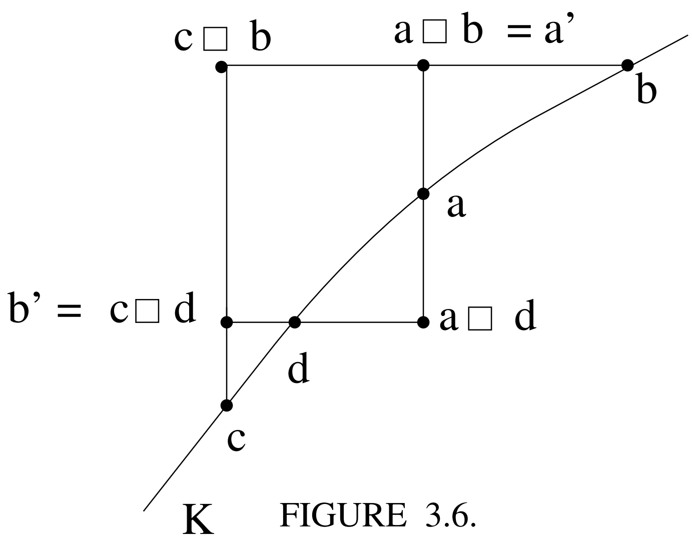

FIGURE 3.6.

(3) " $\Rightarrow$ " Let $(c, a : b, d) \in D'$. Then by definition $(a|b,c)_l = (b|a,d)_l = -1$ and by (2) we obtain $(c|a,d)_l = (d|b,c)_l = 1$. So $(c, a : b, d) \in D$ and this implies : (K, ξ_l) is $D-$convex.

" $\Leftarrow$ " By $D = D'$ and (2) we have the thesis.□

(3.4.4) *For a halfordered set* (E, ξ) *the following conditions are equivalent*:

(1) $\forall \ \{a,b,c\} \in \binom{E}{3} : (a|b,c) = (b|a,c) \ \Rightarrow \ (a|b,c) = 1$

(2) $\forall \ \{a,b,c\} \in \binom{E}{3} : (a|b,c) = -1 \ \Rightarrow \ (b|a,c) = (c|a,b) = 1$

(3) $\forall \ \{a,b,c\} \in \binom{E}{3} : \mathfrak{z}_\xi(\{a,b,c\}) \subset \{0,1\}$

If (E, ξ) *is selfrelated and one of the equivalent conditions* (1), (2), (3) *is satisfied then* (E, ξ) *is ordered or trivial.*

Proof. Let (E, ξ) be selfrelated (i.e. k_ξ is constant (by (2.1.1))) and the equivalent conditions satisfied. Then k_ξ constant and (3) imply either $\mathfrak{z}_\xi \equiv 0$, i.e. $\xi = 1$, or $\mathfrak{z}_\xi \equiv 1$, i.e. ξ is an order. □

As before in the chain structure $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ we associate to any chain $A \in \mathfrak{C}$ the map $\widetilde{A}$ (cf. sec. 3.3).

(3.4.5) *Let K_s be a convex splitting of P by $K \in \mathfrak{C}$, with $|K| \geq 4$, $\overline{a} \in \mathcal{P}\backslash K$, $\mathcal{P}^+ := \overrightarrow{K, \overline{a}} := \{\overline{x} \in \mathcal{P} \setminus K \mid K_s(\overline{a}, \overline{x}) = 1\}$, $\mathcal{P}^- := (\overrightarrow{K, \overline{a}})^- := \{\overline{x} \in \mathcal{P} \setminus K \mid K_s(\overline{a}, \overline{x}) = -1\}$ and ξ_l, ξ_r the corresponding halforders of K. If $\widetilde{K} \in Aut(\mathcal{P}, \{\mathcal{P}^+, \mathcal{P}^-\})$ then $\xi := \xi_l = \xi_r$ and ξ is trivial or an order of K.*

Proof. By (3.3.3) we have $\xi := \xi_l = \xi_r$ and so by (3.3.1) ξ is selfrelated. By (3.4.3.1) and (3.4.4) : K_s is convex $\Leftrightarrow$ $\forall \, \{a, b, c\} \in \binom{K}{3}$: $\mathfrak{z}_\xi(\{a, b, c\}) \subset \{0, 1\}$. Hence by (3.4.4) we have the result. □

(3.4.6) *A splitting K_s* of $\mathcal{P}$ by $K \in \mathfrak{C}$ *is strongly convex if and only if K_s is convex, not trivial and $\widetilde{K} \in Aut(\mathcal{P}, \{\mathcal{P}^+, \mathcal{P}^-\})$.*

Proof. " $\Rightarrow$ " Let K_s be strongly convex. Then K_s is convex by (3.4.2.1) and harmonic by (3.4.1). Therefore K_s is selfrelated not trivial and so $\widetilde{K} \in Aut(\mathcal{P}, \{\mathcal{P}^+, \mathcal{P}^-\})$ by (3.3.3).
" $\Leftarrow$ " By (3.4.5) $\xi := \xi_l = \xi_r$ and ξ is an order of K and so by (3.3.3.3) we have $K_s(x, \widetilde{K}(x)) = -1$ for all $x \in \mathcal{P} \setminus K$.
Now let $a, b \in \mathcal{P} \setminus K$ with $c := a \Box b \in K$ and $K_s(a, b) = 1$. Assume $d := b \Box a \in K$, then $a = c \Box d, b = d \Box c$, hence $b = \widetilde{K}(a)$, and so $K_s(a, b) = K_s(a, \widetilde{K}(a)) = -1$ a contradiction. Consequently $d = b \Box a \notin K$ and by the convexity $K_s(a, b \Box a) = 1$. □

Remark. There are examples of convex but not selfrelated splittings K_s with $\xi_l \neq \xi_r$ where even ξ_l and ξ_r are selfrelated.
Case: $K = \{a, b, c\}$.
If we set $(a|b, c)_l := (b|a, c)_l := -1$ then by convexity (cf. (3.4.3.1)) we have to put $(c|a, b)_r := (b|a, c)_r := (a|b, c)_r := 1$, hence $\xi_{r|\{a,b,c\}} = 1$ and this implies $k_{\xi_r} = 1$. By (3.3.1) and (2.1.4.1) we have to reach $k_{\xi_r} = k_{\xi_l}$ thus $(c|a, b)_l := 1$. Then $\xi_l \neq \xi_r$.

Case: $K = \{a, b, c, d\}$.
Since we are looking for an example with $\xi_l \neq \xi_r$ we may assume that ξ_l is not tri-

vial, i.e. $\exists\ (a,b,c) \in K^{3'}$: $(a|b,c)_l = -1$ and by (3.4.3.1) we have to set $(b|a,c)_r := (c|a,b)_r := 1$. Since $-1 = (a|b,c)_l = (a|b,d)_l \cdot (a|d,c)_l$ we may put $(a|b,d)_l := -1$ and $(a|d,c)_l = 1$ and by (3.4.3.1) we have to define $(b|a,d)_r := (d|a,b)_r := 1$. Then $(b|a,c,d)_r = 1$. Moreover since ξ_l has to be related with ξ_r we have to set $(b|c,d)_l := (a|c,d)_l \cdot (c|a,b)_r \cdot (d|a,b)_r = 1 \cdot 1 \cdot 1 = 1$. If we define $(b|a,c)_l := -1$ then by (3.4.3.1) we have to set $(a|b,c)_r := 1$ and so $\{a,b,c\} \in T_r := \{\{a,b,c\} \in \binom{K}{3} \mid (a|b,c)_r = (b|a,c)_r = (c|a,b)_r = 1\}$. By (2.1.4) the equation $1 = k_{\xi_r}(\{a,b,c\}) = k_{\xi_l}(\{a,b,c\}) = (a|b,c)_l \cdot (b|c,a)_l \cdot (c|a,b)_l = (-1) \cdot (-1) \cdot (c|a,b)_l$ has to be satisfied, hence $(c|a,b)_l := 1$. Since $(b|c,d)_l = -(b|a,c)_l = 1$ we have to put $(b|a,d)_l := -1$, so by (3.4.3.1) $(a|b,d)_r := 1$. Then $\{a,b,d\} \in T_r$ and as before we have to set $(d|a,b)_l := 1$. Till now ξ_l and ξ_r are determined except for $(c,a,d),(c,b,d),(d,a,c),(d,b,c)$. Since $(c|a,b)_l = (c|a,b)_r = (d|a,b)_l = (d|a,b)_r = 1$ we have to observe $(c|a,d)_l = (c|b,d)_l, (c|a,d)_r = (c|b,d)_r, (d|a,c)_l = (d|b,c)_l, (d|a,c)_r = (d|b,c)_r$ and $-(c|a,d)_l = (b|a,d)_l \cdot (c|a,d)_l = (a|b,c)_r \cdot (d|b,c)_r = (d|b,c)_r$ hence $(d|a,c)_r = (d|b,c)_r = -(c|a,d)_l = -(c|b,d)_l$ and $(c|a,d)_r = (b|a,d)_r \cdot (c|a,d)_r = (a|b,c)_l \cdot (d|b,c)_l = -(d|b,c)_l$ hence $(c|b,d)_r = (c|a,d)_r = -(d|b,c)_l = -(d|a,c)_l$. Therefore if we set $(c|b,d)_l := -1, (c|b,d)_r := -1$ then $(d|a,c)_l = (d|a,c)_r = 1$ and so $(d|a,b,c)_l = (d|a,b,c)_r = 1$.

3.5 SPLITTINGS AND VALUATIONS

In this section let E be a non empty set.

There are the following relations between splittings and valuations:

a) Let $\beta : E \to \{1,-1\}; a \mapsto \overline{a}$ be a valuation. For any subset $K \subset E$ the following function:

$$K_s(\beta) : \begin{cases} E \setminus K \times E \setminus K & \to \quad \{1,-1\} \\ (a,b) & \mapsto \quad K_s(a,b) := \overline{a} \cdot \overline{b} \end{cases}$$

is a splitting of E by K called the *corresponding splitting.* In fact:

$$\forall\ a,b,c \in E \setminus K : K_s(a,b) \cdot K_s(b,c) := \overline{a} \cdot \overline{b} \cdot \overline{b} \cdot \overline{c} = \overline{a} \cdot \overline{c} =: K_s(a,c),$$

i.e. the condition **(S1)** is satisfied.

Vice versa: Let K_s be a splitting of E by K where $K \subset E$ and let $e \in E \setminus K$ be fixed. Then the following function:

$$\beta := \beta(K_s,e) : \begin{cases} E \quad \to & \{1,-1\} \\ a \quad \mapsto & \beta(K_s)(a) = \begin{cases} K_s(e,a) & \text{if} \quad a \notin K \\ -1 & \text{if} \quad a \in K \end{cases} \end{cases} .$$

is a valuation with $K_s(\beta(K_s,e)) = K_s$: for if $x,y \in E \setminus K$ then $K_s(\beta(K_s,e))(x,y) = \beta(K_s,e)(x) \cdot \beta(K_s,e)(y) = K_s(e,x) \cdot K_s(e,y) \overset{\textbf{(S1)}}{=} K_s(x,y)$.

Remark. If $e' \in E \setminus (K \cup \{e\})$ and $\beta' := \beta(K_s,e')$ then $\forall\ a \notin K : \beta(a) \cdot \beta'(a) = K(e,e')$, i.e. $\beta = \beta'$ or $\beta = -\beta'$.

b) Let $\beta_1, \beta_2 : E \to \{1,-1\}$ be two valuations and for $i \in \{1,2\}$ let

$$K_s(\beta_i) : \begin{cases} E \setminus K \times E \setminus K & \to \quad \{1,-1\} \\ (a,b) & \mapsto \quad \beta_i(a) \cdot \beta_i(b) \end{cases}$$

be the corresponding splittings of E by K. When are the two splittings $K_s(\beta_1)$ and $K_s(\beta_2)$ equal?

$$K_s(\beta_1) = K_s(\beta_2) :\Leftrightarrow$$
$$\forall\ (a,b) \in E\setminus K \times E\setminus K : \beta_1(a)\cdot\beta_1(b) = \beta_2(a)\cdot\beta_2(b)\ \Leftrightarrow$$
$$\beta_1(a)\cdot\beta_2(a) = \beta_1(b)\cdot\beta_2(b)\ \Leftrightarrow (\beta_1\cdot\beta_2)(a) = (\beta_1\cdot\beta_2)(b)\ :\Leftrightarrow$$
$$\beta_1\cdot\beta_{2|_{E\setminus K}} = \text{const.}\ \Leftrightarrow\ \beta_{1|_{E\setminus K}} = \beta_{2|_{E\setminus K}} \text{ or } \beta_{1|_{E\setminus K}} = -\beta_{2|_{E\setminus K}}.$$

Now let K_s and L_s be splittings of E by the subsets $K, L \subset E$ and let $e, e' \in E\setminus(K\cup L)$. Then:

case 1. If $e = e'$ then $\forall\ x \in E\setminus(K\cup L) : \beta(K_s,e)(x) = \beta(L_s,e)(x) \Rightarrow\ \forall\ x,y \in E\setminus(K\cup L) : K_s(x,y) = L_s(x,y)$.
case 2. If $e \neq e'$ and $e'' \in E\setminus(K\cup L)$ then $\forall\ x \in E\setminus(K\cup L)$:

$$\beta(K_s,e)(x) = \beta(L_s,e')(x) \Leftrightarrow\ K_s(e'',x)\cdot K_s(e,e'') = K_s(e,x) = L_s(e',x) = L_s(e'',x)\cdot L_s(e',e'').$$

a) If $K_s(e,e'') = L_s(e',e'')$ then $\forall\ x,y \in E\setminus(K\cup L),\ K_s(e'',x) = L_s(e'',x)$ and $K_s(e'',y) = L_s(e'',y)$ and by **(S1)** we obtain $K_s(x,y) = L_s(x,y)$.
b) If $K_s(e,e'') = -L_s(e',e'')$ then in the same way we have $K_s(x,y) = L_s(x,y)$, a contradiction.

So in both cases we obtain $\forall\ x,y \in E\setminus(K\cup L) : K_s(x,y) = L_s(x,y)$.

c) Let $\nu(E) := (\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ be a chain structure (cf. sec. 3.3), let $K \in \mathfrak{C}$ be a chain and let $\beta : \mathcal{P} \to \{1,-1\}; a \mapsto \overline{a}$ be a valuation. Then $K_s(\beta)$ is a splitting of $\mathcal{P}$ by K (cf. **a)**).
Which properties has the splitting $K_s(\beta)$?
By (3.3.1) $K_s(\beta)$ induces the two halforders

$$\xi_l : \begin{cases} K^{3'} & \to & \{1,-1\} \\ (a,b,c) & \mapsto & K_s(\beta)(a\Box b, a\Box c) = \beta(a\Box b)\cdot\beta(a\Box c) \end{cases}$$

and

$$\xi_r : \begin{cases} K^{3'} & \rightarrow & \{1, -1\} \\ (a, b, c) & \mapsto & K_s(\beta)(b\Box a, c\Box a) = \beta(b\Box a) \cdot \beta(c\Box a) \end{cases}$$

on K. By (3.3.3) and definition 3.3.1 $K_s(\beta)$ is selfrelated iff $\xi_r(K_s(\beta)) = \xi_l(K_s(\beta))$ hence if $\forall\ (a, b, c) \in K^{3'}$:

$$\beta(a\Box b) \cdot \beta(a\Box c) = \beta(b\Box a) \cdot \beta(c\Box a).$$

Since $b\Box a = \widetilde{K}(a\Box b), c\Box a = \widetilde{K}(a\Box c)$ we obtain : $K_s(\beta)$ selfrelated $\Leftrightarrow$ $\forall x \in \mathcal{P} \setminus K$: $\beta(x) = \beta(\widetilde{K}(x))$ or $\forall x \in \mathcal{P} \setminus K : \beta(x) = -\beta(\widetilde{K}(x))$.

Moreover if the valuation β satisfies the following condition: $\forall\ a, b \in \mathcal{P} \backslash K$ with $a\Box b \in K$ and $\beta(a) \cdot \beta(b) = 1 : b\Box a \notin K$ and $\beta(a) \cdot \beta(b\Box a) = 1$ then by sec. 3.4 (p.117) $K_s(\beta)$ is strongly convex. So by (3.4.5) $K_s(\beta)$ induces an order $\xi = \xi_l = \xi_r$ on K.

3.6 POSITIVE DOMAINS IN CHAIN STRUCTURES

Let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ be a $2-$ net such that $\mathfrak{C} \neq \emptyset$. A subset $\mathcal{P}^+ \subset \mathcal{P}$ is called *positive domain* of $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ if the following two conditions are satisfied:

(D1) $\forall\ a, b \in \mathcal{P}^+$ with $a \Box b \notin \mathcal{P}^+ : b \Box a \in \mathcal{P}^+$

(D2) $\exists\ K \in \mathfrak{C} : \mathcal{P} = \mathcal{P}^+ \mathbin{\dot\cup} K \mathbin{\dot\cup} \widetilde{K}(\mathcal{P}^+)$ is a disjoint union (K is called the corresponding chain of $\mathcal{P}^+$).

(3.6.1) *If $\mathcal{P}^+$ is a positive domain of $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ and K is the corresponding chain of $\mathcal{P}^+$ then also $\widetilde{K}(\mathcal{P}^+)$ is a positive domain of $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ and K the corresponding chain.*

Proof. Since $\widetilde{K}$ is an involutory antiautomorphism of $(\mathcal{P}, \Box)$, $\widetilde{K}(\mathcal{P}^+)$ satisfies **(D2)**. If $x, y \in \widetilde{K}(\mathcal{P}^+)$ with $x \Box y \notin \widetilde{K}(\mathcal{P}^+)$ then $\widetilde{K}(x), \widetilde{K}(y) \in \mathcal{P}^+$ and $\widetilde{K}(x \Box y) = \widetilde{K}(y) \Box \widetilde{K}(x) \notin \mathcal{P}^+$ hence by **(D1)** (for $\mathcal{P}^+$), $\widetilde{K}(x) \Box \widetilde{K}(y) = \widetilde{K}(y \Box x) \in \mathcal{P}^+$ implying $y \Box x \in \widetilde{K}(\mathcal{P}^+)$. $\Box$

(3.6.2) *If $\mathcal{P}^+$ is a positive domain of $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ and K the corresponding chain let: for $a, b \in K : a < b \Leftrightarrow\ a \Box b \in \mathcal{P}^+$,*

$$K_{\mathcal{P}^+} : \begin{cases} \mathcal{P} \setminus K & \rightarrow \quad \{1, -1\} \\ \quad x & \mapsto \quad K_{\mathcal{P}^+}(x) = \begin{cases} 1 & \text{if} \quad x \in \mathcal{P}^+ \\ -1 & \text{if} \quad x \in \widetilde{K}(\mathcal{P}^+) \end{cases} \end{cases}$$

and $K_s(\mathcal{P}^+) : (\mathcal{P} \setminus K) \times (\mathcal{P} \setminus K) \rightarrow \{1, -1\}; (x, y) \mapsto K_s(\mathcal{P}^+)(x, y) := K_{\mathcal{P}^+}(x) \cdot K_{\mathcal{P}^+}(y)$. Then:

(1) *$(K, <)$ is a totally ordered set.*

(2) *$K_s(\mathcal{P}^+)$ is a strongly convex splitting.*

(3) *$\xi_l(K_s(\mathcal{P}^+))$ is an order of K and moreover $\forall\ \{a, b, c\} \in \binom{K}{3} : (b|a, c)_l = -1 \Leftrightarrow a < b < c$ or $c < b < a$.*

Proof. (1) Let $\{a, b, c\} \in \binom{K}{3}$. Then $a \Box b \notin K$ and either $a \Box b \in \mathcal{P}^+$ (and so $b \Box a = \widetilde{K}(a \Box b) \notin \mathcal{P}^+$) or $a \Box b \notin \mathcal{P}^+$ (and so $b \Box a = \widetilde{K}(a \Box b) \in \mathcal{P}^+$). Hence either $a < b$ or

$b < a$. Assume $a < b$ and $b < c$, i.e. $a\Box b, b\Box c \in \mathcal{P}^+$. Then $(b\Box c)\Box(a\Box b) = b \in K$ hence $b \notin \mathcal{P}^+$ and so by **(D1)**, $(a\Box b)\Box(b\Box c) = a\Box c \in \mathcal{P}^+$, i.e. $a < c$.
(2) Let $a, b \in \mathcal{P} \setminus K$ with $a\Box b \in K$ and $1 = K_s(\mathcal{P}^+)(a,b) = K_{\mathcal{P}^+}(a) \cdot K_{\mathcal{P}^+}(b)$ hence either $a, b \in \mathcal{P}^+$ or $a, b \in \widetilde{K}(\mathcal{P}^+)$. By (3.6.1) we may assume $a, b \in \mathcal{P}^+$. Since $a\Box b \in K$, hence $a\Box b \notin \mathcal{P}^+$, by **(D1)**, $b\Box a \in \mathcal{P}^+$ and therefore $b\Box a \notin K$ (by **(D2)**) and $K_s(\mathcal{P}^+)(a, b\Box a) = K_{\mathcal{P}^+}(a) \cdot K_{\mathcal{P}^+}(b\Box a) = 1 \cdot 1 = 1$.
(3) From (2), (3.4.5) and (3.4.6) follows that $\xi_l(K_s(\mathcal{P}^+))$ is an order. By (3.3.1), $(b|a,c)_l = K_s(\mathcal{P}^+)(b\Box a, b\Box c) = K_{\mathcal{P}^+}(b\Box a) \cdot K_{\mathcal{P}^+}(b\Box c) = -1 \;\Leftrightarrow\; (b\Box a \in \mathcal{P}^+$ and $b\Box c \in \widetilde{K}(\mathcal{P}^+))$ or $(b\Box a \in \widetilde{K}(\mathcal{P}^+)$ and $b\Box c \in \mathcal{P}^+) \;\Leftrightarrow\; (b < a$ and $c < b$ (since $c\Box b = \widetilde{K}(b\Box c) \in \mathcal{P}^+$)) or $(a < b$ and $b < c)$. □

(3.6.3) *Let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ be a $2-$net, with $\mathfrak{C} \neq \emptyset$, let $K \in \mathfrak{C}$ be a chain furnished with a total order " $<$ ". Then $\mathcal{P}^+ := \{a\Box b \mid a, b \in K : a < b\}$ is a positive domain of $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ with the corresponding chain K.*

Proof. Let $a\Box b, c\Box d \in \mathcal{P}^+$ and $(a\Box b)\Box(c\Box d) = a\Box d \notin \mathcal{P}^+$ hence $a < b, c < d$ and $a \not< d$ i.e. $a = d$ or $d < a$. Then $c < d \leq a < b$ thus $c < b$ and so $(c\Box d)\Box(a\Box b) = (c\Box b) \in \mathcal{P}^+$. Moreover if $p \in \mathcal{P} \setminus K$ and $p_i := [p]_i \cap K$, then $p = p_1\Box p_2$ and $p_1 \neq p_2$. Thus either $p_1 < p_2$ (and so $p \in \mathcal{P}^+$) or $p_2 < p_1$ and so $\widetilde{K}(p) = \widetilde{K}(p_1\Box p_2) = p_2\Box p_1 \in \mathcal{P}^+$. □

Definition 3.6.1 Let $\mathcal{P}^+$ be a positive domain of a $2-$net $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ and $(K, <)$ the corresponding ordered chain (cf. (3.6.2)). For $i \in \{1, 2\}$ a point $p \in \mathcal{P}^+$ is called *i-rand point* if $[p]_i \subset \mathcal{P}^+ \cup K$.

Then we have the following:

(3.6.4) *Let $\mathcal{P}^+$ be a positive domain of a $2-$net $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ and $(K, <)$ the correspon-ding ordered chain (cf. (3.6.2)). Then:*
(1) *$\mathcal{P}^+$ contains a $1-$rand point $\Leftrightarrow$ $(K, <)$ has a smallest element.*
(2) *$\mathcal{P}^+$ contains a $2-$rand point $\Leftrightarrow$ $(K, <)$ has a greatest element.*

(3) *If p is a $1-$ (or a $2-$)rand point then $[p]_1 \cap K$ ($[p]_2 \cap K$) is the smallest (greatest) element of $(K,<)$.*

(4) *If a (b) is the smallest (greatest) element of $(K,<)$ then $[a]_1 \setminus \{a\}$ ($[b]_2 \setminus \{b\}$) is the set of all $1-$ ($2-$)rand points of $\mathcal{P}^+$ and, if both a and b exist, then $a\Box b$ is the only one $1-$ and $2-$rand point of $\mathcal{P}^+$.*

Proof. (1) " $\Rightarrow$ " and (3). Let $p \in \mathcal{P}^+$ such that $[p]_1 \subset K \cup \mathcal{P}^+$. Then $a := [p]_1 \cap K \in K$ and $\forall\ x \in K \setminus \{a\}$ we have $a\Box x \notin K$ and $a\Box x \in a\Box(K \setminus \{a\}) = [p]_1 \setminus \{a\} \subset \mathcal{P}^+$ hence $a < x$. In the dual way one proves (2) " $\Rightarrow$ ".

(1) " $\Leftarrow$ " and (4). Let $a \in K$ be the smallest element of $(K,<)$. Then $\forall\ x \in K \setminus \{a\}$: $a < x \overset{(3.6.2)}{:\Leftrightarrow} a\Box x \in \mathcal{P}^+$ and this implies $a\Box(K \setminus \{a\}) = [a]_1 \setminus \{a\} \subset \mathcal{P}^+$ (In the dual way one proves (2) " $\Leftarrow$ "). Therefore $\forall\ p \in [a]_1 \setminus \{a\} : [p]_1 \subset \mathcal{P}^+ \cup K$ and so $p \in \mathcal{P}^+$, i.e. $[a]_1 \setminus \{a\}$ is the set of $1-$randpoints of $(K,<)$. Moreover if a resp. b is the smallest resp. greatest element of $(K,<)$ then $([a]_1 \setminus \{a\}) \cap ([b]_2 \cap \{b\}) = a\Box b$ is the only one $1-$ and $2-$randpoint of $\mathcal{P}^+$.$\Box$

4 Relations between halfordered chain structures, halfordered permutation sets and splittings by chains

4.1 HALFORDERED CHAIN STRUCTURES AND HALFORDERED PERMUTATION SETS

(4.1.1) *Let* $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \xi)$ *be a halfordered chain structure and let* $E \in \mathfrak{K}$ *be fixed, then the following triples are halfordered permutation sets*:

(1) $(\mathfrak{G}_1, \sigma_E(\mathfrak{K}), \xi_1)$ (*cf.* (1.4.3) *and section* 3.1).

(2) $(\mathfrak{G}_1, \sigma_E(\mathfrak{K}^{\sim}), \xi_1)$ *and* $\forall\ \alpha,\ \beta \in \sigma_E(\mathfrak{K}^{\sim}):\ \alpha \circ \beta^{-1} \circ \alpha \in \sigma_E(\mathfrak{K}^{\sim})$ (*cf.* (1.5.4)).

(3) $(\mathfrak{G}_1, \sigma_E(\mathfrak{K}^{\sim\sim}), \xi_1)$ *and* $\sigma_E(\mathfrak{K}^{\sim\sim})$ *is a group* (*cf.* (1.6.3.3)).

(4) $(\mathfrak{G}_1, \sigma_E(\mathfrak{C}_\xi), \xi_1)$ *and* $\sigma_E(\mathfrak{C}_\xi) = Aut(\mathfrak{G}_1, \xi_1)$.

Proof. Since $\mathfrak{K} \subset \mathfrak{K}^{\sim} \subset \mathfrak{K}^{\sim\sim} \subset \mathfrak{C}_\xi$ (cf. (3.1.7.1)), hence $\sigma_E(\mathfrak{K}) \subset \sigma_E(\mathfrak{K}^{\sim}) \subset \sigma_E(\mathfrak{K}^{\sim\sim}) \subset \sigma_E(\mathfrak{C}_\xi) := \{\widetilde{E} \circ \widetilde{C}_{|\mathfrak{G}_1} \mid C \in \mathfrak{C}_\xi\}$ (cf. (1.4.3)), $\widetilde{E},\ \widetilde{C} \in Aut(\mathcal{P}, \xi)$ by (3.1.1.2) and so $\widetilde{E} \circ \widetilde{C} \in Aut(\mathcal{P}, \xi)^+$, we have $\widetilde{E} \circ \widetilde{C}_{|\mathfrak{G}_1} \in Aut(\mathfrak{G}_1, \xi_1)$ by (3.1.4). Therefore $\sigma_E(\mathfrak{C}_\xi) \subset Aut(\mathfrak{G}_1, \xi_1)$ proves that the four triples are halfordered permutation sets. The statements (2),(3) and (4) are consequences of (1.5.4), (1.6.3.3) and (3.1.5.4) respectively.□

By (4.1.1) we see that to each halfordered chain structure $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \xi)$ there corresponds a halfordered permutation set $(\mathfrak{G}_1, \Gamma, \xi_1)$ and that the set of permutations $\Gamma \subset Aut(\mathfrak{G}_1, \xi_1)$ has certain additional properties if $\mathfrak{K}$ is symmetric, double symmetric or $\mathfrak{K} = \mathfrak{C}_\xi$.

Now we start with a halfordered permutation set (E, Γ, η) (with $\Gamma \subset Aut(E, \eta)$). Let $\nu(E) := (\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ be the corresponding 2−net, let $\mathfrak{K} := C(\Gamma) := \{C(\gamma) \mid \gamma \in \Gamma\}$, where $C(\gamma) := \{(x, \gamma(x)) \mid x \in E\}$ (cf. (1.3.3)), and for $a = (a_1, a_2),\ b = (b_1, b_2),\ c = (c_1, c_2)$, with $(a, b, c) \in \mathcal{P}^{3g}$, we set:

$$(a|b,c)_\xi := \begin{cases} (a_1|b_1,c_1)_\eta & if \quad a_2 = b_2 = c_2 \\ (a_2|b_2,c_2)_\eta & if \quad a_1 = b_1 = c_1 \end{cases} .$$

Then $\nu(E,\eta) := (\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2; \xi)$ satisfies the axioms $\mathbf{(Z1)}_g$ and $\mathbf{(Z2)}_g$, and $(\mathfrak{G}_1, \xi_1)$ and $(\mathfrak{G}_2, \xi_2)$ are isomorphic to (E,η).
For $\sigma \in SymE,\ K := C(\sigma)$ and $x = (x_1, x_2) \in \mathcal{P}$ we obtain $\widetilde{K}(x) = (\sigma^{-1}(x_2), \sigma(x_1))$

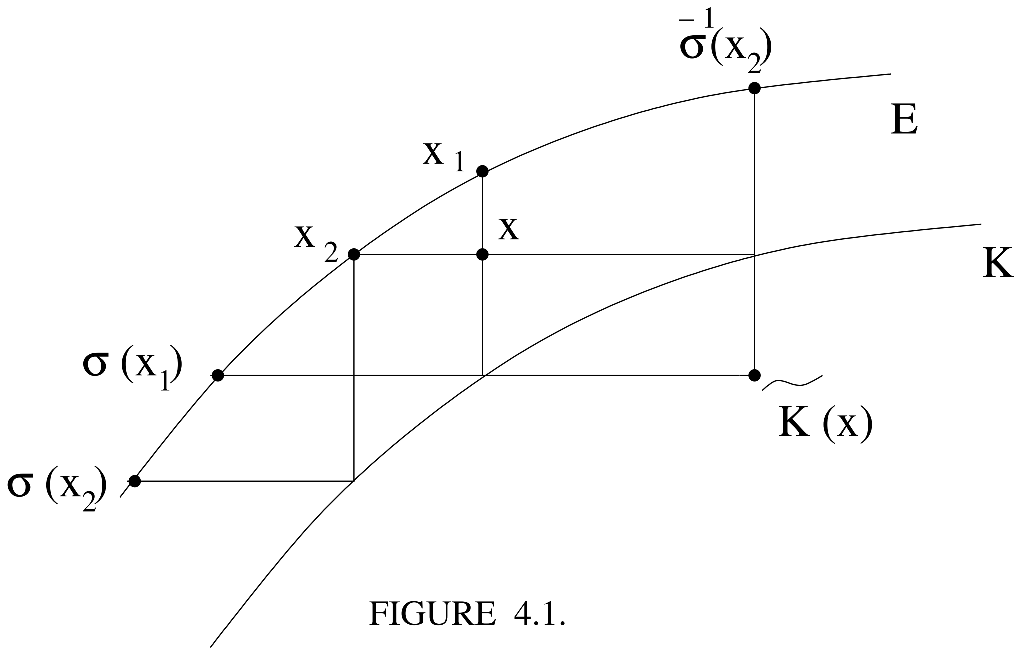

FIGURE 4.1.

hence

$$(\widetilde{K}(a)|\widetilde{K}(b), \widetilde{K}(c))_\xi :=$$
$$\begin{cases} (\sigma(a_1)|\sigma(b_1), \sigma(c_1))_\eta & if \quad \sigma^{-1}(a_2) = \sigma^{-1}(b_2) = \sigma^{-1}(c_2) \quad \Leftrightarrow \quad a_2 = b_2 = c_2 \\ (\sigma^{-1}(a_2)|\sigma^{-1}(b_2), \sigma^{-1}(c_2))_\eta & if \quad \sigma(a_1) = \sigma(b_1) = \sigma(c_1) \quad \Leftrightarrow \quad a_1 = b_1 = c_1 \end{cases}.$$

Hence $\widetilde{K} \in Aut(\mathcal{P}, \xi)$ with $K = C(\sigma)$ if and only if $\sigma \in Aut(E,\eta)$. Moreover for $\alpha, \beta \in SymE, A := C(\alpha), B := C(\beta)$ and $y = (y_1, y_2) \in \mathcal{P}$ we obtain $\widetilde{A,B}(y) = (\beta^{-1}(y_2), \alpha(y_1))$, hence

$$(\widetilde{A,B}(a)|\widetilde{A,B}(b), \widetilde{A,B}(c))_\xi :=$$
$$\begin{cases} (\alpha(a_1)|\alpha(b_1), \alpha(c_1))_\eta & if \quad \beta^{-1}(a_2) = \beta^{-1}(b_2) = \beta^{-1}(c_2) \quad \Leftrightarrow \quad a_2 = b_2 = c_2 \\ (\beta^{-1}(a_2)|\beta^{-1}(b_2), \beta^{-1}(c_2))_\eta & if \quad \alpha(a_1) = \alpha(b_1) = \alpha(c_1) \quad \Leftrightarrow \quad a_1 = b_1 = c_1 \end{cases}.$$

Hence $\widetilde{A,B} \in Aut(\mathcal{P}, \xi)$ with $A = C(\alpha), B = C(\beta)$ if and only if $\alpha, \beta \in Aut(E,\eta)$. We have proved the main theorem:

(4.1.2) *Let (E,η) be a halfordered set, let $\nu(E,\eta) := (\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2;\xi)$ be the corresponding halfordered 2−net and $\mathfrak{K} := C(Aut(E,\eta))$. Then $\mathfrak{K} = \mathfrak{C}_\xi$ and $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{K};\xi)$ is a maximal halfordered chain structure (which is symmetric) with $(\mathfrak{G}_1,\xi_1) \cong (E,\eta)$.*

(4.1.3) *Let $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{K};\xi)$ be a maximal halfordered chain structure (obtained as in* (4.1.2)*) and for all $A,B \in \mathfrak{C}$ let $A\cdot B := \tau(A,E,B)$* (cf. p.45). *Then $\mathfrak{C}_\xi$ and $\mathfrak{K}^{\sim\sim}$ are subgroups of $(\mathfrak{C},\cdot)$ and $(\mathfrak{C}_\xi,\cdot)$ is isomorphic to $(Aut(E,\overline{\xi}_{|E^{3\prime}}),\circ)$* (cf. §3.2).

Proof. By (4.1.2) $\mathfrak{C}_\xi := \{C \in \mathfrak{C} \mid \widetilde{C} \in Aut(\mathcal{P},\xi)\} = C(Aut(E,\overline{\xi}_{|E^{3\prime}}))$ and by (1.4.3.2) the function $C : SymE \to \mathfrak{C}; \alpha \mapsto C(\alpha) := \{x\square\alpha(x) \mid x \in E\}$ is an isomorphism between the groups $(SymE,\circ)$ and $(\mathfrak{C},\cdot)$. Since $Aut(E,\overline{\xi}_{|E^{3\prime}})$ is a subgroup of $(SymE,\circ)$, $\mathfrak{C}_\xi = C(Aut(E,\overline{\xi}_{|E^{3\prime}})$ is a subgroup of $(\mathfrak{C},\cdot)$ and $(Aut(E,\overline{\xi}_{|E^{3\prime}}),\circ)$ is isomorphic to $(\mathfrak{C}_\xi,\cdot)$. By definition of "double symmetric", $\mathfrak{K}^{\sim\sim}$ is closed with respect to the binary operation "$\cdot$". Now let $A \in \mathfrak{K}^{\sim\sim}$. Then by definition of τ, $A^{-1} \overset{(1.4.3)}{=} \widetilde{E}(A)$ and since $\mathfrak{K}^{\sim\sim}$ is double symmetric we have $\widetilde{E}(A) = \widetilde{E,E}(A) \in \mathfrak{K}^{\sim\sim}$, i.e. $\mathfrak{K}^{\sim\sim}$ is a subgroup of $(\mathfrak{C},\cdot)$.$\square$

4.2 HALFORDERED CHAIN STRUCTURES AND SPLITTINGS BY CHAINS

Let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ be a 2−net and let K_s be a splitting of $\mathcal{P}$ by a chain $K \in \mathfrak{C}$. Then for $i \in \{1,2\}$ the function

$$\xi_i(K_s) : \begin{cases} \mathfrak{G}_i^{3'} & \to & \{1,-1\} \\ (A,B,C) & \mapsto & (A|B,C)_{K_s\ i} := K_s([a]_j \cap B, [a]_j \cap C) \end{cases},$$

where $a := K \cap A$ and $j \in \{1,2\} \setminus \{i\}$ is a halforder of the set $\mathfrak{G}_i$. Thus if $b := K \cap B$, $c := K \cap C$ and $(A,B,C) \in \mathfrak{G}_1^{3'}$ then $(A|B,C)_{K_s\ 1} = K_s(b \Box a, c \Box a)$ and if $(A,B,C) \in \mathfrak{G}_2^{3'}$ then $(A|B,C)_{K_s\ 2} = K_s(a \Box b, a \Box c)$.

Definition 4.2.1 Two splittings K_s, L_s of $\mathcal{P}$ by chains are called *$i-$compatible*, $i \in \{1,2\}$, if $\xi_i(K_s) = \xi_i(L_s)$.

There is the following connection between halfordered chain structures and splittings:

(4.2.1) *Let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \xi)$ be a halfordered chain structure, let $K \in \mathfrak{K}$ and $\{0,1,\infty\} \in \binom{K}{3}$ be fixed, let ξ_K be the induced halforder of K according to* (3.3.5) *(hence $\xi_K : K^{3'} \to \{1,-1\}; (a,b,c) \mapsto (a|b,c)_{\xi_K} := (a|a\Box b, a \Box c)_\xi$) and let $K_s := K_s(\xi_K; \{0,1,\infty\})$ be the splitting of $\mathcal{P}$ by K depending on $\{0,1,\infty\}$ according to* (3.3.2.2) (*hence $K_s(x,y) := K(x) \cdot K(y)$ where $K(x) := (x_1|0,x_2)_{\xi_K} \cdot (0|1,x_1)_{\xi_K}$ for $x_1 \neq 0, K(x) := (0|1,x_2)_{\xi_K} \cdot k_{\xi_K}(\{0,1,\infty\})$ for $x_1 = 0$ and $x_i := [x]_i \cap K$). Then:*
(1) $\forall\, (a,b,c) \in K^{3'} : (a|b,c)_l := K_s(a \Box b, a \Box c) = (a|b,c)_{\xi_K} = (a|a \Box b, a \Box c)_\xi = (a|b \Box a, c \Box a)_\xi$ (*cf.* (3.3.1)).
(2) $\forall\ \{a,b,c\} \in \binom{K}{3} : (a|b,c)_r := K_s(b \Box a, c \Box a) = K(b \Box a) \cdot K(c \Box a) = (a|b,c)_{\xi_K} \cdot k_{\xi_K}(\{a,b,c\}) \cdot k_{\xi_K}(\{0,b,c\}) = (a|b \Box a, c \Box a)_\xi \cdot k_{\xi_K}(\{a,b,c\}) \cdot k_{\xi_K}(\{0,b,c\})$ *if* $b,c \neq 0$,
$(a|0,c)_r = (a|0,c)_{\xi_K} \cdot k_{\xi_K}(\{a,0,c\}) \cdot k_{\xi_K}(\{0,1,\infty\}) = (a|0 \Box a, c \Box a)_\xi \cdot k_{\xi_K}(\{a,0,c\}) \cdot k_{\xi_K}(\{0,1,\infty\})$.
(3) *The following statements are equivalent:*
(*i*) *ξ is selfrelated.*

(ii) ξ_K *is selfrelated.*

(iii) K_s *is selfrelated.*

(4) *If* ξ *is selfrelated then:*

(i) The splitting $K_s := K_s(\xi_K; \{0,1,\infty\})$ *is independent of the choice of* $\{0,1,\infty\} \in \binom{K}{3}$ *(by* (3.3.2.3)).

(ii) $\forall\ (a,b,c) \in K^{3'} : (a|b,c)_l = (a|b,c)_r = (a|b,c)_{\xi_K} = (a|a\Box b, a\Box c)_\xi = (a|b\Box a, c\Box a)_\xi$ *(by* (3.3.3) *and Def.* 3.3.1).

(iii) $\forall\ K, L \in \mathfrak{K}$ *the splittings* $K_s := K_s(\xi_K)$ *and* $L_s := L_s(\xi_L)$ *are* $1-$ *and* $2-$*compatible.*

Proof. (1) and the first part of (2) are a consequence of the definitions, so let $\{a,b,c\} \in \binom{K}{3}$. If $b, c \neq 0$ then $(a|b,c)_{\xi_K} \cdot k_{\xi_K}(\{a,b,c\}) \cdot k_{\xi_K}(\{0,b,c\}) = (b|a,c)_{\xi_K} \cdot (c|a,b)_{\xi_K} \cdot (0|b,c)_{\xi_K} \cdot (b|0,c)_{\xi_K} \cdot (c|0,b)_{\xi_K} \overset{(\mathbf{Z})}{=} (b|0,a)_{\xi_K} \cdot (c|0,a)_{\xi_K} \cdot (0|1,b)_{\xi_K} \cdot (0|1,c)_{\xi_K} \cdot \overset{(3.3.2.2)}{=} K(b\Box a) \cdot K(c\Box a)$, if $b = 0$ then $(a|0,c)_{\xi_K} \cdot k_{\xi_K}(\{a,0,c\}) \cdot k_{\xi_K}(\{0,1,\infty\}) = (0|a,c)_{\xi_K} \cdot (c|0,a)_{\xi_K} \cdot k_{\xi_K}(\{0,1,\infty\}) = (0|1,c)_{\xi_K} \cdot (0|1,a)_{\xi_K} \cdot (c|0,a)_{\xi_K} \cdot k_{\xi_K}(\{0,1,\infty\}) \overset{(3.3.2.2)}{=} K(0\Box a) \cdot K(c\Box a)$.

(3) By (3.3.5) and Def. 3.3.2, (i) and (ii) are equivalent, and by (3.3.3) and Def. 3.3.1, (ii) and (iii) are equivalent.

(4) (iii) Let $(A,B,C) \in \mathfrak{G}_i^{3'}$ and $a := A \cap K, a' := A \cap L, b := B \cap K, b' := B \cap L, c := C \cap K, c' := C \cap L$. If $i = 1$ then

$$(A|B,C)_{K_s\ 1} = K_s(b\Box a, c\Box a) \overset{(2)}{=} (a|b,c)_r \overset{(4)(ii)}{=}$$

$$(a|b\Box a, c\Box a)_\xi \overset{(\mathbf{Z2})_g}{=} (a'|b'\Box a', c'\Box a')_\xi = (A|B,C)_{L_s\ 1}.$$

If $i = 2$ then

$$(A|B,C)_{K_s\ 2} = K_s(a\Box b, a\Box c) \overset{(1)}{=} (a|a\Box b, a\Box c)_\xi \overset{(\mathbf{Z2})_g}{=}$$

$$(a'|a'\Box b', a'\Box c')_\xi \overset{(1)}{=} L_s(a'\Box b', a'\Box c') = (A|B,C)_{L_s\ 2}.$$

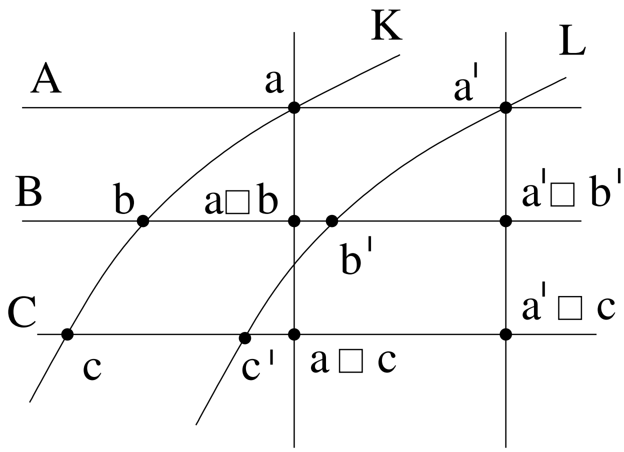

FIGURE 4.2.

□

(4.2.2) *Let* $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ *be a* $2-$*net such that* $\mathfrak{C} \neq \emptyset$, *let* K_s *be a splitting of* $\mathcal{P}$ *by a chain* $K \in \mathfrak{C}$, *let* $\xi_l = \xi_l(K_s)$ *and* $\xi_r = \xi_r(K_s)$ *be the corresponding halforders of* K *according to* (3.3.1). *For each* $p \in \mathcal{P}$ *let* $p_i := [p]_i \cap K$ *and let*

$$\xi : \begin{cases} \mathcal{P}^{3g} & \rightarrow & \{1,-1\} \\ (a,b,c) & \mapsto & (a|b,c) := \begin{cases} (a_1|b_1,c_1)_r & if \quad [a]_2 = [b]_2 = [c]_2 \\ (a_2|b_2,c_2)_l & if \quad [a]_1 = [b]_1 = [c]_1 \end{cases} \end{cases} .$$

Then:

(1) $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \{K\}; \xi)$ *satisfies the axioms* $\mathbf{(Z1)}_g$ *and* $\mathbf{(Z2)}_g$.

(2) $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \{K\}; \xi)$ *is a halfordered chain structure (i.e.* $\mathbf{(ZC)}_g$ *is satisfied) if and only if* K_s *is selfrelated.*

Proof. (2) Let $(a,b,c) \in \mathcal{P}^{3g}$, with $[a]_2 = [b]_2 = [c]_2$. Then $(a|b,c) := (a_1|b_1,c_1)_r$ and $(\widetilde{K}(a)|\widetilde{K}(b), \widetilde{K}(c)) := (a_1|b_1,c_1)_l$. By definition 3.3.1 this shows the equivalence in (2). □

Remark. The halfordered chain structure $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \{K\}; \xi(K_s))$ obtained from a selfrelated splitting via (4.2.2) is symmetric and also double symmetric (cf. sec. 3.1) since $\mathfrak{K} = \{K\}$ consists of a single chain and this implies $\{K\} = \{K\}^{\sim} = \{K\}^{\sim\sim} \subset \mathfrak{C}_\xi := \{C \in \mathfrak{C} \mid \widetilde{C} \in Aut(\mathcal{P}, \xi)\}$ (cf. (3.1.2)). Hence $\mu(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, K_s) := (\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C}_\xi; \xi)$ is the greatest halfordered chain structure with $K \in \mathfrak{C}_\xi$ and for each $L \in \mathfrak{C}_\xi$,

$$L_s : \begin{cases} (\mathcal{P} \setminus L) \times (\mathcal{P} \setminus L) & \to \quad \{1, -1\} \\ (a, b) & \mapsto \quad L_s(a, b) := ([a]_2 |\ \widetilde{L}([a]_1), \widetilde{L}([b]_1)) \cdot (\widetilde{L}([b]_1) |\ [a]_2, [b]_2) \end{cases}$$

is a selfrelated splitting of $\mathcal{P}$ by L and $\mu(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, L_s) = \mu(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, K_s)$ (cf. sec. 3.1).

4.3 AUTOMORPHISMS OF SPLITTINGS AND POSITIVE DOMAINS

Definition 4.3.1 Let $\mathcal{P}, \mathcal{P}' \neq \emptyset$, $E \subset \mathcal{P}$ and $E' \subset \mathcal{P}'$. If E_s resp. E'_s is a splitting of $\mathcal{P}$ resp. $\mathcal{P}'$ by E resp. E' then a bijection $\varphi : \mathcal{P} \to \mathcal{P}'$ is called an *isomorphism between the splittings* E_s *and* E'_s if $\varphi(E) = E'$ and if $\forall\ a, b \in \mathcal{P} \setminus E$:

$$E_s(a,b) = E'_s(\varphi(a), \varphi(b)).$$

If $\mathcal{P} = \mathcal{P}'$ and $E_s = E'_s$ then φ is called an *automorphism*. Let $Aut(\mathcal{P}, E_s)$ be the group of all these automorphisms.
Two splittings E_s of $\mathcal{P}$ and E'_s of $\mathcal{P}'$ are called *isomorph* if such an isomorphism exists.

Now we consider the case $\mathcal{P} = \mathcal{P}'$ and we assume that $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ is a chain structure. Let E_s be a splitting of $\mathcal{P}$ by a chain $E \in \mathfrak{C}$. By $Aut(\mathcal{P}, \mathfrak{C}) = Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2)$ (cf. (1.3.7)) if $\varphi \in Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2)$ then $\varphi(E) =: K_\varphi \in \mathfrak{C}$ and if we set $\forall\ a', b' \in \mathcal{P} \setminus K_\varphi$:

$$(K_\varphi)_s(a', b') := E_s(\varphi^{-1}(a'), \varphi^{-1}(b'))$$

then $(K_\varphi)_s$ is a splitting of $\mathcal{P}$ by K_φ and φ is an isomorphism between these two splittings E_s and $(K_\varphi)_s$.

We set $Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, E_s) := Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2) \cap Aut(\mathcal{P}, E_s)$.

We have:
(4.3.1) *Let* $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ *be a 2-net and* $\overline{\Gamma}^+, \overline{\Gamma}^-$ *defined as in section* 1.1. *Then:*
a) $\alpha \in \overline{\Gamma}^+ \Leftrightarrow\ \alpha \in Aut(\mathcal{P}, \square)$
b) $\alpha \in \overline{\Gamma}^- \Leftrightarrow\ \alpha \in Antiaut(\mathcal{P}, \square)$.

Proof. a) " $\Rightarrow$ " Let $a, b \in \mathcal{P}$. Then:
$\alpha(a \square b) = \alpha([a]_1 \cap [b]_2) \overset{\alpha \in \overline{\Gamma}^+}{=} [\alpha(a)]_1 \cap [\alpha(b)]_2 = \alpha(a) \square \alpha(b)$.
" $\Leftarrow$ " Let $a \in \mathcal{P}$. Then, since $[a]_1 = a \square \mathcal{P}$ and $[a]_2 = \mathcal{P} \square a$, we have:
$\alpha([a]_1) \overset{\alpha \in Aut(\mathcal{P}, \square)}{=} \alpha(a) \square \alpha(\mathcal{P}) = [\alpha(a)]_1$ and

$\alpha([a]_2) \overset{\alpha\in Aut(\mathcal{P},\square)}{=} \alpha(\mathcal{P})\square\alpha(a) = [\alpha(a)]_2$.

b) " $\Rightarrow$ " Let $a, b \in \mathcal{P}$. Then:

$\alpha(a\square b) = \alpha([a]_1 \cap [b]_2) \overset{\alpha\in\overline{\Gamma}^-}{=} [\alpha(a)]_2 \cap [\alpha(b)]_1 = \alpha(b)\square\alpha(a)$.

" $\Leftarrow$ " Let $a \in \mathcal{P}$. Then, since $[a]_1 = a\square\mathcal{P}$ and $[a]_2 = \mathcal{P}\square a$, we have:

$\alpha([a]_1) \overset{\alpha\in Antiaut(\mathcal{P},\square)}{=} \alpha(\mathcal{P})\square\alpha(a) = [\alpha(a)]_2$ and

$\alpha([a]_2) \overset{\alpha\in Antiaut(\mathcal{P},\square)}{=} \alpha(a)\square\alpha(\mathcal{P}) = [\alpha(a)]_1$. □

By (4.3.1) we obtain:

(4.3.2) *Let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ be a chain structure, $A, B \in \mathfrak{C}$ and $\gamma \in Bij(A, B)$.*

(1) *The following extensions*

$$\gamma^+ : \begin{cases} \mathcal{P} = A\square A & \to & \mathcal{P} = B\square B \\ x\square y & \mapsto & \gamma(x)\square\gamma(y) \end{cases}$$

and

$$\gamma^- : \begin{cases} \mathcal{P} = A\square A & \to & \mathcal{P} = B\square B \\ x\square y & \mapsto & \gamma(y)\square\gamma(x) \end{cases}$$

are elements of $\overline{\Gamma}^+$ and $\overline{\Gamma}^-$ such that $\gamma^+(A) = B, \gamma^-(A) = B$.

(2) *If $\sigma \in Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2)$ and $A \in \mathfrak{C}$. Then:*

a) If $\sigma \in \overline{\Gamma}^+$ then $\sigma = (\sigma_{|A})^+$

b) If $\sigma \in \overline{\Gamma}^-$ then $\sigma = (\sigma_{|A})^-$.

We define $(Bij(A, B))^+ := \{\gamma^+ \mid \gamma \in Bij(A, B)\}$, $(Bij(A, B))^- := \{\gamma^- \mid \gamma \in Bij(A, B)\}$. Then if $E \in \mathfrak{C}$, $Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, E) = (SymE)^+ \cup (SymE)^-$ (by (4.3.2.2)).

(4.3.3) *Let $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ be a chain structure, E_s a selfrelated splitting of $\mathcal{P}$ by a chain $E \in \mathfrak{C}$ and $\xi := \xi(E_s)$ the associated selfrelated halforder of E according to* (3.3.1). *Then: $Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, E_s) = Aut(E, \xi)^+ \cup Aut(E, \xi)^-$.*

Proof. Let $\sigma \in Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, E_s)$ and $(a,b,c) \in E^{3'}$. Then $(a|b,c) = E_s(a\Box b, a\Box c) = E_s(\sigma(a\Box b), \sigma(a\Box c)) =$

$$\left\{ \begin{array}{ll} E_s(\sigma(a)\Box\sigma(b), \sigma(a)\Box\sigma(c)) & \text{if} \quad \sigma \in \overline{\Gamma}^+ \\ E_s(\sigma(b)\Box\sigma(a), \sigma(c)\Box\sigma(a)) & \text{if} \quad \sigma \in \overline{\Gamma}^- \end{array} \right\} \overset{E_s\ self.}{=} (\sigma(a)|\sigma(b), \sigma(c)),$$

hence $\sigma_{|E} \in Aut(E,\xi)$ and so $(\sigma_{|E})^+ = \sigma$ if $\sigma \in \overline{\Gamma}^+$ and $(\sigma_{|E})^- = \sigma$ if $\sigma \in \overline{\Gamma}^-$ by (4.3.2.2). Consequently $Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, E_s) \subset Aut(E,\xi)^+ \cup Aut(E,\xi)^-$.

Now let $\sigma \in Aut(E,\xi)$, $a, b \in \mathcal{P} \setminus E$ and for $i \in \{1,2\}, a_i := [a]_i \cap E, b_i := [b]_i \cap E$. If $a\Box b, b\Box a \in E$ then $\widetilde{E}(a) = b, \widetilde{E}(\sigma^+(a)) = \sigma^+(b)$ and $\widetilde{E}(\sigma^-(a)) = \sigma^-(b)$ hence by (3.3.3) $k_\xi = E_s(a,b) = E_s(\sigma^+(a), \sigma^+(b)) = E_s(\sigma^-(a), \sigma^-(b))$. If for instance $b\Box a \notin E$ then $a = a_1\Box a_2, b\Box a = b_1\Box a_2, b = b_1\Box b_2$

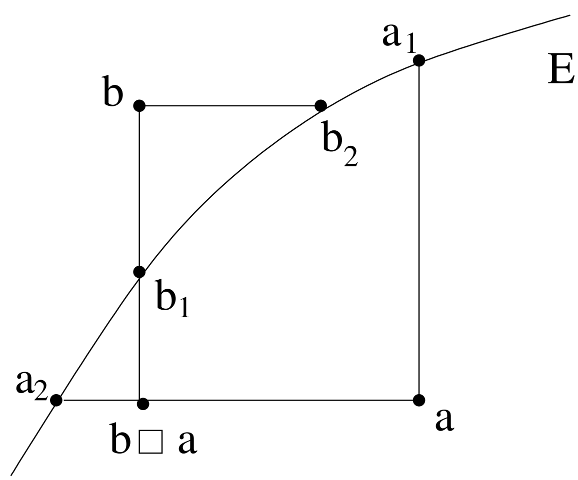

FIGURE 4.3.

and so $E_s(a,b) = E_s(a, b\Box a) \cdot E_s(b\Box a, b) = (a_2|b_1, a_1) \cdot (b_1|a_2, b_2) =$

$$(\sigma(a_2)|\sigma(b_1), \sigma(a_1)) \cdot (\sigma(b_1)|\sigma(a_2), \sigma(b_2)) =$$

$$E_s(\sigma(a_1)\Box\sigma(a_2), \sigma(b_1)\Box\sigma(a_2)) \cdot E_s(\sigma(b_1)\Box\sigma(a_2), \sigma(b_1)\Box\sigma(b_2)) =$$

$$E_s(\sigma(a_1)\Box\sigma(a_2), \sigma(b_1)\Box\sigma(b_2)) = E_s(\sigma^+(a), \sigma^+(b)) \overset{E_s self.}{=}$$

$$E_s(\sigma(a_2)\Box\sigma(a_1), \sigma(a_2)\Box\sigma(b_1)) \cdot E_s(\sigma(a_2)\Box\sigma(b_1), \sigma(b_2)\Box\sigma(b_1)) =$$

$$E_s(\sigma(a_2)\Box\sigma(a_1), \sigma(b_2)\Box\sigma(b_1)) = E_s(\sigma^-(a), \sigma^-(b)),$$

i.e. $\sigma^+, \sigma^- \in Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, E_s)$. □

(4.3.4) *Let* $E \neq \emptyset$, *let* $\nu(E) := (\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ *be the corresponding* $2-$*net, let* E_s *be a selfrelated splitting of* $\mathcal{P}$ *by* E *and let* $\xi := \xi(E_s)$ *be the corresponding selfrelated halforder of* E *according to* (3.3.1). *Moreover for each* $p \in \mathcal{P}$ *let* $p_i := [p]_i \cap E$. *Then* $Aut(E_s) := Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, E_s)_{|E} = Aut(E, \xi(E_s))$ *and if*

$$\overline{\xi} : \begin{cases} \mathcal{P}^{3g} & \to & \{1, -1\} \\ (a,b,c) & \mapsto & (a|b,c) := \begin{cases} (a_1|b_1, c_1)_{\xi(E_s)} & if \quad [a]_2 = [b]_2 = [c]_2 \\ (a_2|b_2, c_2)_{\xi(E_s)} & if \quad [a]_1 = [b]_1 = [c]_1 \end{cases} \end{cases}$$

and $\mathfrak{K} := C(Aut(E_s))$ *then* $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \overline{\xi})$ *is a selfrelated halfordered chain structure such that* $\forall\ A, B \in \mathfrak{K}$ *the corresponding splittings* $A_s := \varphi(\xi_A)$ *and* $B_s := \varphi(\xi_B)$ *(cf.* (3.3.4)*) are selfrelated and* $1-$ *and* $2-$*compatible (cf. Definition* 4.2.1*).*

Proof. By (4.3.3) $Aut(E_s) = Aut(E, \xi(E_s))$. By (4.1.2) $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \overline{\xi})$ is a maximal halfordered chain structure which is (by definition of $\overline{\xi}$) selfrelated if and only if $\xi(E_s)$ is selfrelated. The selfrelated halforder $\overline{\xi}$ induces on each chain $K \in \mathfrak{K}$ a selfrelated halforder ξ_K (cf. (3.3.5) and Def. 3.3.2) and so by (3.3.4) we obtain a selfrelated splitting $K_s := \varphi(\xi_K)$ and by (4.2.1)(iv) $\forall\ A, B \in \mathfrak{K}$ the splittings $A_s := \varphi(\xi_A)$ and $B_s := \varphi(\xi_B)$ are $1-$ and $2-$compatible. □

Definition 4.3.2 A quadruple $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}_s)$ is called *splitting chain structure* if $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ is a $2-$net and $\mathfrak{K}_s$ is a set of selfrelated splittings by chains $K \in \mathfrak{K} \subset \mathfrak{C} \neq \emptyset$ which are all $1-$ and $2-$ compatible.

By (4.3.4) we obtain the following

(4.3.5) *If* K_s *is a selfrelated splitting of* $\mathcal{P}$ *by a chain* K *then there exists a greatest splitting chain structure* $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}_s)$ *with* $K_s \in \mathfrak{K}$.

Finally:

(4.3.6) *Let $\mathcal{P}^+$ be the positive domain of a 2−net $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$, K the corresponding chain, $Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, \mathcal{P}^+) := \{\sigma \in Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2) \mid \sigma(\mathcal{P}^+) = \mathcal{P}^+\}$ the automorphism group of the positive domain $\mathcal{P}^+$ and let " $<$ " the associated order of K according to* (3.6.2). *Then:*

(1) $\forall\, \sigma \in Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, \mathcal{P}^+) : \sigma(K) = K$ *and* $\sigma_{|K}$ *is an isoton or antiton permutation of* $(K, <)$ *resp. if* $\sigma \in \overline{\Gamma}^+$ *or* $\sigma \in \overline{\Gamma}^-$ *respectively.*

(2) *If* τ *is an isoton resp. antiton permutation of* $(K, <)$ *then* $\tau^+, \widetilde{K} \circ \tau^- \in Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, \mathcal{P}^+) \cap \overline{\Gamma}^+$ *resp.* $\tau^-, \widetilde{K} \circ \tau^+ \in Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, \mathcal{P}^+) \cap \overline{\Gamma}^-$.

(3) *Let* $\sigma \in Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, \mathcal{P}^+)$, *if* $\sigma \in \overline{\Gamma}^+$ *then* $\sigma_{|K}$ *is an isoton permutation of* $(K, <)$ *and* $(\sigma_{|K})^+ = \sigma$, *if* $\sigma \in \overline{\Gamma}^-$ *then* $\sigma_{|K}$ *is an antiton permutation of* $(K, <)$ *and* $(\sigma_{|K})^- = \sigma$.

(4) *If* $|\mathcal{P}|$ *is finite then* $|Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, \mathcal{P}^+)| = 2$ *and* $Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, \mathcal{P}^+) \cap \overline{\Gamma}^+ = \{id\}$.

Proof. (1) Let $\sigma \in Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, \mathcal{P}^+)$, $a \in K$ and assume $\sigma(a) \notin K$. Then $\sigma(a) \in \widetilde{K}(\mathcal{P}^+)$ and if $a_i := [\sigma(a)]_i \cap K$ for $i \in \{1, 2\}$ then $a_1 \neq a_2$ and $\sigma(a) = a_1 \Box a_2$. This implies $a_2 \Box a_1 = \widetilde{K}(a_1 \Box a_2) = \widetilde{K}(\sigma(a)) \in \mathcal{P}^+$. Since also $\sigma^{-1} \in Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, \mathcal{P}^+)$ we have $\sigma^{-1}(a_2 \Box a_1) \in \mathcal{P}^+$, $\sigma^{-1}(a_i) \notin \mathcal{P}^+$ and $\sigma^{-1}(a_i) \in \sigma^{-1}([\sigma(a)]_i) = [\sigma^{-1}(\sigma(a))]_i = [a]_i$ with $\sigma^{-1}(a_i) \neq a$ implying $\sigma^{-1}(a_i) \in \widetilde{K}(\mathcal{P}^+)$ and $a = \sigma^{-1}(a_1) \Box \sigma^{-1}(a_2)$. Since $a \notin \widetilde{K}(\mathcal{P}^+)$ we obtain by (3.6.1) $\sigma^{-1}(a_2) \Box \sigma^{-1}(a_1) \in \widetilde{K}(\mathcal{P}^+)$. If $\sigma \in \overline{\Gamma}^+$ then $\sigma^{-1}(a_2) \Box \sigma^{-1}(a_1) = \sigma^{-1}(a_2 \Box a_1) \in \widetilde{K}(\mathcal{P}^+)$ contradicting $\sigma^{-1}(a_2 \Box a_1) \in \mathcal{P}^+$. If $\sigma \in \overline{\Gamma}^-$ then $\sigma^{-1}(a_2) \Box \sigma^{-1}(a_1) = \sigma^{-1}(a_1 \Box a_2) = \sigma^{-1}(\sigma(a)) = a \in K \cap \widetilde{K}(\mathcal{P}^+) = \emptyset$ also a contradiction. Therefore $\sigma(K) = K$ and $\sigma_{|K}$ is a permutation of K. Now let $a, b \in K$ with $a < b$ hence $a \Box b \in \mathcal{P}^+$. Since $\sigma(\mathcal{P}^+) = \mathcal{P}^+$ we obtain :

$$\sigma(a \Box b) = \begin{cases} \sigma_{|K}(a) \Box \sigma_{|K}(b) \in \mathcal{P}^+ & \text{if } \sigma \in \overline{\Gamma}^+ \quad \text{by (4.3.1}a) \\ \sigma_{|K}(b) \Box \sigma_{|K}(a) \in \mathcal{P}^+ & \text{if } \sigma \in \overline{\Gamma}^- \quad \text{by (4.3.1}b) \end{cases}$$

and so by definition of " $<$ " : $\sigma_{|K}(a) < \sigma_{|K}(b)$ if $\sigma \in \overline{\Gamma}^+$ and $\sigma_{|K}(b) < \sigma_{|K}(a)$ if $\sigma \in \overline{\Gamma}^-$. Therefore $\sigma_{|K}$ is an isoton or antiton permutation of $(K, <)$ resp. if $\sigma \in \overline{\Gamma}^+$ or $\sigma \in \overline{\Gamma}^-$ respectively.

(2) By (4.3.2.1) $\tau^+ \in \overline{\Gamma}^+$, $\tau^- \in \overline{\Gamma}^-$ and by (1.4.2) $\widetilde{K} \in \overline{\Gamma}^-$. We prove: $\tau^+(\mathcal{P}^+) = \mathcal{P}^+$ and $\tau^-(\mathcal{P}^+) = \mathcal{P}^+$. Let $x := a \Box b \in \mathcal{P}^+$ with $a, b \in K$ then $a < b$, $\tau^+(a) = \tau^-(a) =$

$\tau(a), \tau^+(b) = \tau^-(b) = \tau(b) \in K$ and $\tau(a) < \tau(b)$ or $\tau(b) < \tau(a)$ resp. hence $\tau^+(x) = \tau(a)\Box\tau(b) \in \mathcal{P}^+$ and $\tau^-(x) = \tau(b)\Box\tau(a) \in \widetilde{K}(\mathcal{P}^+)$ or $\tau^+(x) = \tau(a)\Box\tau(b) \in \widetilde{K}(\mathcal{P}^+)$ and $\tau^-(x) = \tau(b)\Box\tau(a) \in \mathcal{P}^+$ respectively.

(3) By (1) and (2).

(4) By (3) we have $Aut(\mathcal{P}, \mathfrak{G}_1 \cup \mathfrak{G}_2, \mathcal{P}^+) = \{\sigma \in \overline{\Gamma} \mid \sigma_{|K}$ is an isoton permutation of $(K,<)\} \dot{\cup} \{\sigma \in \overline{\Gamma} \mid \sigma_{|K}$ is an antiton permutation of $(K,<)\}$ and since $(K,<)$ is a finite ordered set we may assume $K = \{a_1, a_2, \ldots, a_n\}$ with $a_i < a_{i+1},\ \forall\ i \in \{1, \ldots, n-1\}$.

<u>Case 1</u>: let $\sigma_{|K}$ be isoton. We show:

$(*)$ If ρ is an isoton permutation of K then $\rho(a_1) = a_1$.

Assume $\rho(a_1) \neq a_1$. Then $a_1 < \rho(a_1)$ and $\exists\ a_i \in \{a_2, \ldots, a_n\} : \rho(a_i) = a_1$. This implies $\rho(a_i) = a_1 < \rho(a_1)$ and since ρ is an isoton permutation of $(K,<)$ we obtain $a_i < a_1$, a contradiction. Thus $\rho(a_1) = a_1$ and so $\rho(\{a_2, \ldots, a_n\}) = \{a_2, \ldots, a_n\}$, i.e. $\rho_{|K\setminus\{a_1\}}$ is an isoton permutation of $K \setminus \{a_1\}$. From $(*)$ follows $\rho(K \setminus \{a_1\}) = K \setminus \{a_1\}$, i.e. $\rho_{|K\setminus\{a_1\}}$ is an isoton permutation of $K \setminus \{a_1\}$ and so by $(*)$, $\rho(a_2) = a_2$ and since K is a finite set we obtain $\rho = id$, i.e. $\sigma_{|K} = id$.

<u>Case 2</u>: let $\sigma_{|K}$ be antiton. The function

$$\omega := \begin{cases} \{(a_1 a_n)(a_2 a_{n-1}) \cdots (a_{\frac{n}{2}} a_{\frac{n}{2}+1})\} & \text{if } n \text{ is even} \\ \{(a_1 a_n)(a_2 a_{n-1}) \cdots (a_{\frac{n-1}{2}} a_{\frac{n+3}{2}})\} & \text{if } n \text{ is odd} \end{cases}$$

is an antiton permutation of $(K,<)$, so $\sigma_{|K} \circ \omega$ is an isoton permutation of $(K,<)$ and by the case 1 we have $\sigma_{|K} \circ \omega = id$, i.e. $\sigma_{|K} = \omega$.$\Box$

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AXIOMS

(A1) $\forall\ \{p,q\} \in \binom{\mathcal{P}}{2},\ \exists_1\ L \in \mathbb{L} : p,q \in L.$

(A2) $\forall\ p \in \mathcal{P}, \forall\ L_1 \in \mathbb{L}$, with $p \notin L_1,\ \exists_1\ L_2 \in \mathbb{L} \setminus \{L_1\}$ with $p \in L_2$ and $L_1 \cap L_2 = \emptyset$.

(B) For all $K \in \mathfrak{K}$, for all $k \in K$ and for all $p \in \mathcal{P} \setminus (K \cup [k]_1 \cup [k]_2)$ there is exactly one $L \in \mathfrak{K}$ such that $p \in L$ and $L \cap K = \{k\}$.

(C0) $(a|b,c) = 1$ and $(x|b,c) = -1\ \Rightarrow\ (a|b,x) = 1.$

(D1) $\forall\ a,b \in \mathcal{P}^+$ with $a \Box b \notin \mathcal{P}^+ : b \Box a \in \mathcal{P}^+$.

(D2) $\exists\ K \in \mathfrak{C} : \mathcal{P} = \mathcal{P}^+ \mathbin{\dot{\cup}} K \mathbin{\dot{\cup}} \widetilde{K}(\mathcal{P}^+)$ is a disjoint union.

(I2) For all joinable $\{a,b\} \in \binom{\mathcal{P}}{2},\ \exists_1\ K \in \mathfrak{K} : a,b \in K.$

(n1) $p \notin C.$

(n2) $\forall\ X \in \mathcal{L}(p) : |X \cap C| = 1.$

(N1) For each point $x \in \mathcal{P}$ and each $i \in I$ there is exactly one generator $G \in \mathfrak{G}_i$ with $x \in G$.

(N2) Any two generators of distinct classes $\mathfrak{G}_i \neq \mathfrak{G}_j$ intersect in exactly one point.

(N3) Every element X of $\mathfrak{G} := \bigcup_{i \in I} \mathfrak{G}_i$ intersects C in exactly one point.

(O1) For all $H \in \mathfrak{H}$, for all $x,y,z \in \mathcal{P} \setminus H : (H|x,y) \cdot (H|y,z) = (H|x,z).$

(O2) Let $a,b,c \in E,\ a,b \neq c$ and let $A,B \in \mathfrak{H}$ with $c \in A,B$ and $a,b \notin A,B$ then $(A|a,b) = (B|a,b).$

(P) For all $K \in \mathfrak{K}$, for all $p \in \mathcal{P} \setminus K$ there is exactly one $L \in \mathfrak{K} : p \in L$ and $L \cap K = \emptyset$.

(P1) $0 \notin P.$

(P2) $\forall\ \delta \in \Delta_0\ : \delta(P) = P$ or $\delta(P) = P_- := E \setminus (P \cup \{0\}).$

(q1) $p \in C.$

(q2) $\forall\ X \in \mathcal{L}(p) : |X \cap C| \leq 2.$

(q3) $|\{X \in \mathcal{L}(p) \mid |X \cap C| = 1\}| \leq 1.$

(R) $\forall\ (a,b,c,d) \in E^{4'} : [a,b|c,d]_1 = [c,d|a,b]_2.$

(R1) For all $(a,b,c,d) \in E^{4'} : [a,b|c,d] = [c,d|a,b].$

(S1) $\forall\ a,b,c \in E \setminus K : (K_s|a,b) \cdot (K_s|b,c) = (K_s|a,c).$

(T) $\forall\ (a,b,c,d),(a,b,d,e) \in E^{4'} :\ [a,b|c,d] \cdot [a,b|d,e] = [a,b|c,e].$

(T)* $\forall\ (a,b,c,d) \in (\binom{E}{4}) : k_{\tau_r}(a,b,c,d) = k_{\tau_l}(d,a,b,c) = k_\xi(\{a,b,c\}).$

(Z) $\forall\ (a,b,c),(a,c,d)\in E^{3'} : (a|b,c)\cdot(a|c,d)=(a|b,d)$.

(Z0) $\forall\ \{a,b,c\}\in\binom{E}{3}$ exactly one of the values $(a|b,c),(b|c,a),(c|a,b)$ is equal -1.

(Z1)$_g$ $\forall\ G\in\mathfrak{G}_1\cup\mathfrak{G}_2,\ (G,\xi_{|G^{3'}})$ is a halfordered set.

(Z2)$_g$ $\forall\ j\in\{1,2\},\forall\ G,L\in\mathfrak{G}_j,\forall\ (a,b,c)\in G^{3'}$ if $i\in\{1,2\}\setminus\{j\}$, then $(a|b,c)=([a]_i\cap L\mid[b]_i\cap L,[c]_i\cap L)$.

(ZC)$_g$ $\forall\ K\in\mathfrak{K},\forall\ (a,b,c)\in\mathcal{P}^{3g} : (a|b,c)=(\widetilde{K}(a)|\widetilde{K}(b),\widetilde{K}(c))$.

(Z1) $\forall\ X\in\mathfrak{G}_1\cup\mathfrak{G}_2\cup\mathfrak{K},\ (X,\overline{\xi}_{|X^{3'}})$ is a halfordered set.

(Z2) $\forall\ G,L\in\mathfrak{G}_1\cup\mathfrak{G}_2\cup\mathfrak{K},\ \forall\ (a,b,c)\in G^{3'}$:

(a) if $G,L\in\mathfrak{G}_i$: $(a|b,c)_{\overline{\xi}}=([a]_j\cap L|[b]_j\cap L,[c]_j\cap L)_{\overline{\xi}}$, where $i,j\in\{1,2\}$ and $i\neq j$;

(b) if $G,L\in\mathfrak{K}$: $(a|b,c)_{\overline{\xi}}=([a]_i\cap L|[b]_i\cap L,[c]_i\cap L)_{\overline{\xi}}$, where $i\in\{1,2\}$;

(c) if $G\in\mathfrak{K},\ L\in\mathfrak{G}_i$ (or $G\in\mathfrak{G}_i,\ L\in\mathfrak{K}$): $(a|b,c)_{\overline{\xi}}=([a]_j\cap L|[b]_j\cap L,[c]_j\cap L)_{\overline{\xi}}$, where $i,j\in\{1,2\}$ and $i\neq j$.

(ZC) $\forall\ K\in\mathfrak{K},\ \forall\ (a,b,c)\in\mathcal{P}^{3\mathfrak{K}}\cup\mathcal{P}^{3g}$: $(a|b,c)_{\overline{\xi}}=(\widetilde{K}(a)|\widetilde{K}(b),\widetilde{K}(c))_{\overline{\xi}}$.

(Z)$_l$ $\forall\ (a,b,c,d),(a,b,d,e)\in E^{4'} : [a,b|c,d]\cdot[a,b|d,e]=[a,b|c,e]$.

(Z)$_r$ $\forall\ (a,b,d,e),(b,c,d,e)\in E^{4'} : [a,b|d,e]\cdot[b,c|d,e]=[a,c|d,e]$.

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ABSTRACT

In 1949 E. Sperner [33],[34] generalized by his "orderfunction in a geometry" the concept "ordered" to "halfordered affine or projective geometry". If (K,t) is the ternary ring coordinatizing an affine plane then there is a one-to-one correspondence between the halforders η of the plane and the betweenness functions ξ of K which turn (K,t) in a halfordered ternary ring (K,t,ξ).

In 1981 H. Karzel and H.-J. Kroll [21] introduced the notions "$I-$net" and "$I-$chain net" in particular $2-$net $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2)$ and $2-$chain net $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{K})$, also called chain structure (affine planes can be considered as particular chain structures). If $\mathfrak{C}$ denotes the set of all chains of a $2-$net $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2)$ then there is a ternary operation $\tau : \mathfrak{C}^3 \to \mathfrak{C}$ such that for a fixed $E \in \mathfrak{C}$, $(\mathfrak{C},\cdot)$ with $A \cdot B := \tau(A,E,B)$ is a group which is isomorphic to the symmetric group $SymE$.

The aim of this thesis is to introduce concepts of orderfunction, order and halforder in chain structures, to develop their theory and to relate them with "one dimensional" orderstructures which are given for instance on the points of a chain E.

It will be shown: If $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{K};\eta)$ is a halfordered chain structure then the halforder η induces on each chain $E \in \mathfrak{K}$ a betweenness function ξ and with E a splitting E_s of $\mathcal{P}$ (i.e. a partition of $\mathcal{P} \setminus E$ into two parts). Between the splittings E_s of $\mathcal{P}$ by E and the betweenness functions ξ on E there is a bijective connection. If α denotes the isomorphism from $SymE$ onto $(\mathfrak{C},\cdot)$ and $Aut(E,\xi)$ the permutations of E preserving ξ, then $\mathfrak{K} \subset \mathfrak{C}_\xi := \alpha(Aut(E,\xi)) \leq (\mathfrak{C},\cdot)$ and η can be extended to a halforder $\overline{\eta}$ of $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{C}_\xi)$ such that $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{K};\eta)$ appears as a substructure of the *envelopping* (maximal) halfordered chain structure $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{C}_\xi;\overline{\eta})$.

In this frame the interplay between properties (like related, selfrelated, harmonic, anharmonic, convex or order) of halfordered sets (E,ξ), halfordered chain structures $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{K};\eta)$ and splittings by a chain $(\mathcal{P},\mathfrak{G}_1,\mathfrak{G}_2;E_s)$ is studied.

In this way halfordered algebraic structures like halfordered or ordered loops, groups or fields can be lifted to halfordered or ordered envelopping chain structures.

ZUSAMMENFASSUNG

1949 verallgemeinerte E. Sperner [33],[34] durch seine „Ordnungsfunktion einer Geometrie“ den Begriff „angeordnete Geometrie“ zu „halbgeordnete affine oder projektive Geometrie“. Ist (K, t) der ternäre Ring, der eine affine Ebene koordinatisiert, so gibt es eine eineindeutige Beziehung zwischen den Halbordnungen η der Ebene und den Zwischenfunktionen ξ von K. Dadurch wird (K, t) ein halbgeordneter ternärer Ring (K, t, ξ).
H. Karzel und H.-J. Kroll [21] führten 1981 die Begriffe „$I-$net“ und „$I-$chain net“ ein. Besondere Beachtung erfuhren dabei die $2-$Netze $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$ und die $2-$chain nets $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$, welche auch Kettenstrukturen genannt werden. Affine Ebenen können als besondere Kettenstrukturen angesehen werden. Bezeichnet $\mathfrak{C}$ die Menge aller Ketten eines $2-$Netzes $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2)$, dann gibt es eine ternäre Operation $\tau : \mathfrak{C}^3 \to \mathfrak{C}$, so dass für ein ausgezeichnetes $E \in \mathfrak{C}$ durch $A \cdot B := \tau(A, E, B)$ eine Gruppenstruktur definiert ist, welche zur Symmetrischen Gruppe $SymE$ isomorph ist.
Ziel dieser Dissertation ist die Begriffe Ordnungsfunktion, Anordnung und Halbordnung in Kettenstrukturen einzuführen, ihre Theorie zu entwickeln und sie mit eindimensionalen Ordnungsstrukturen, zum Beispiel gegeben durch die Punkte einer Kette, zu verbinden.
Es wird gezeigt werden, dass wenn $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \eta)$ eine halbgeordnete Kettenstruktur ist, die Halbordnung η auf jeder Kette $E \in \mathfrak{K}$ eine Zwischenfunktion ξ induziert und durch E eine Seiteneinteilung E_s von $\mathcal{P}$ gegeben ist (also eine Zerlegung von $\mathcal{P} \setminus E$ in zwei Teile). Es gibt einen bijektiven Zusammenhang zwischen den Zwischenfunktionen ξ auf E und den Seiteneinteilungen E_s, die durch E auf $\mathcal{P}$ induziert werden. Bezeichnet α den Isomorphismus von $SymE$ nach $(\mathfrak{C}, \cdot)$ und $Aut(E, \xi)$ die Permutationen von E, die ξ erhalten, dann ist $\mathfrak{K} \subset \mathfrak{C}_\xi := \alpha(Aut(E, \xi)) \leq (\mathfrak{C}, \cdot)$ und η kann auf eine Halbordnung $\overline{\eta}$ von $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C}_\xi)$ ausgedehnt werden, wodurch $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}, \eta)$ zur Unterstruktur einer maximalen halbgeordneten Kettenstruktur $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C}_\xi, \overline{\eta})$ wird.
In diesem Zusammenhang werden die Beziehungen zwischen Eigenschaften von halbgeordneten Mengen (E, ξ), halbgeordneten Kettenstrukturen $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}; \eta)$ und Seiteneinteilungen durch eine Kette $(\mathcal{P}, \mathfrak{G}_1, \mathfrak{G}_2; E_s)$ untersucht.

Auf diese Weise können halbgeordnete algebraische Strukturen wie halbgeordnete oder angeordnete Loops, Gruppen oder Körper zu halbgeordneten oder angeordneten Kettenstrukturen angehoben werden.

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1979-1984	Grundschule in Bergamo, Italien
1984-1987	Mittelschule in Bergamo
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August '99	4th International Conference on Geometry and Applications, Varna,Bulgarien
Februar '00	Arbeitstagung über Geometrie, München
Mai '00	Combinatorics 2000, Gaeta, Italien
Februar '02	Arbeitstagung über Geometrie, Berlin