ESSAYS IN THE THEORY OF VOTING POWER

by

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Contents

PREFACE

I entered the field of measures of voting power in early 1996 when Manfred Holler suggested that I write a Master's thesis (*Diplomarbeit* in German) on the application of measures of voting power to decision-making in firms. This culminated in a tome entitled, 'Power Indices and the Decision-making in Firms' which I submitted to the Faculty of Economics of the University of Hamburg in the summer of 1997. The present thesis is a continuation of this work and which I undertook while a research and teaching fellow at the Institut für Allokation und Wettbewerb (IAW), University of Hamburg.

In 1998, cooperation with Yener Altunbas and Shanti Chakravarty at the University of Wales, Bangor, started on questions of voting power and proportional representation in the national assemblies of Wales and Scotland. After two research visits to Bangor in 1999 and 2000 and various meetings at international conferences and workshops four papers were produced (Altunbas *et al*., 1999a, 1999b, 2000, 2002). These papers are partly published and have been presented at a number of international conferences and workshops. Altunbas *et al*. (1999a) is the basis of chapter 6 of this thesis. The cooperation with Yener Altunbas and Shanti Chakravarty is set to continue in the near future on a project entitled 'Economics of National Identity' financed by the DAAD and the British Council.

In 1999, having read a paper by Moshé Machover, Dan Felsenthal and William Zwicker (1998), I produced a paper on the question 'When is A Priori Voting Power Really A Priori?' (Steffen, 2000). This forms the basis of chapter 2. Following that I returned to the central idea of my Master's thesis and wrote a paper on 'Power and the Internal Organization of Firms' (Steffen, 1999) which is the basis of chapter 7. Both papers were the impetus for a very fruitful and close cooperation with Matthew Braham on the measurement of voting power during the last years. It has started with my request to Matthew, with whom I was working together on an experiment on distributional justice in the summer of 1999, to polish my English for both papers. Until now, the result of this cooperation are inter alia (i) five papers on the measurement of voting power dealing with questions of abstentions (Braham and Steffen [B&S], 2002a), voting power in hierarchical structures (B&S, 2001a, 2002b), partition effects in compound games (B&S, 2003), power and freedom (B&S, 2001b), and local monotonicity (B&S, 2002c). Chapters 3, 4, and 7 are based on B&S (2002a, 2002c, 2002b). Two of these papers are published in some form and all have been presented at a number of international conferences and workshops.

In 1999 Rie Ono from the University of Toyama (who has moved to Chiba University in 2002) spent a research semester at the IAW. Together with Rie and Manfred Holler I started to work on the question of monotonicity of measures of voting power – a problem with which I was already confronted in my master thesis as well as in Altunbas *et al*. (1999a, 2000), Steffen (1999) and in B&S (2001b, 2002a, 2002b, 2003). The first result of this research was a joint paper on 'Constrained Monotonicity and the Measurement of Power' (Holler *et al*., 2001) which is the basis of chapter 5 of this thesis. This paper has lead to two further strings of developments: one by Sören Schönfeld (2001) who has investigated the concept of constrained monotonicity in more detail, and one by Matthew Braham and myself, going deeper into the question of monotonicity per se (B&S, 2002c). As already mentioned, the research on the latter is the basis of chapter 4.

Chapter 1

INTRODUCTION*

1. Motivation

The measurement of voting power plays a useful role in the investigation of structural properties of collective decision-making rules which can be modelled as a simple (voting) game. Such rules can be found in legislative bodies, committees, and a variety of organizations. Measures of voting power have an established history in game and social choice theory, going back more than half a century. In the early 1980s, the field gained a reputation of having become a somewhat exhausted mine. However, since the early 1990s things have changed. The last decade has seen a resurgence of research into this field, with many new discoveries about the properties of classical power measures such as the Penrose (1946)/Banzhaf (1965) and Coleman (1971), the Public Good (Holler, 1982a; Holler and Packel, 1983; Holler and Li, 1995) and the Deegan-Packel (1978) measures and the Shapley-Shubik index (1954) as well as new developments in probabilistic techniques like Straffin's (1977) partial homogeneity approach based on Owen's (1972) multi-linear extension and new areas of applications. This thesis includes contributions to all these aspects. The theoretical contributions deal with the nature of a prioricty and monotonicity of measures of voting power and with the question of abstention. The applied contributions consist of applications of measures of voting power to a newly created institution (the National Assembly for Wales) and to hierarchical organizations.

The central aim of this introductory chapter is to discuss the meaning of the term 'voting power'. This is essential to understand to which purposes measures of voting power are applicable. The debates in the literature indicate that more attention on this issue is required especially in order to help those who are not deeply familiar to this area of research, but either seek to criticise it or just to apply its concepts and methods in an unreflective manner.

This chapter is organized as follows. In section 2 the term 'voting power' is introduced as a specific type of 'power'. Based on this the basic idea, a measure of

* The author is deeply indebted to Matthew Braham for helpful comments and intensive discussions.

voting power is introduced and a broad classification scheme between a priori and a posteriori measures is provided. Section 3 deals with the debate in the literature about the applicability of measures of voting power and tries to show that this debate is due largely to misunderstandings about the nature of voting power and the issues which a measure of voting power can deal with. Section 4 contains an outline and a summary of the main results of the thesis and the relationships between the different chapters.

2. Voting Power: Meaning and Measurement

A transparent discussion of the measurement of power requires at the very least a specification within the limits of ordinary language about what it is that should be measured. The term *power* is used in many situations in our daily life. However, if one asks people using this term to define it, one obtains a huge variety of answers. The same applies in the academic literature on power. A lot of work has been done in order to find a general characterization of the term power.[1] This thesis follows an outcome-orientated version of Barry (1976) and Morriss' (1987) understanding of power as an *ability* or *capacity to do* something or the possession of control in a social environment, which for the measurement of voting power following Brams (1975, p. 157) appears to be preferable to a player-orientated view.[2] Put in the words of Vannucci (2002): the measurement of voting power 'seems to be confined to analyzing the comparative "has more power than" relation among players ... as opposed to the more commonly used "has power over" relation'. That is, it is about 'influence rankings' of players in (collective) decision-making situations and not about so-called 'bossy' or bi-lateral relationships between players.[3]

In such situations power can be specified as the ability to determine the outcome of a (collective) decision-making situation based upon two components: the decision rule, commonly modelled by a mathematical structure known as a

[1] For a good collection of readings on the concept of power see Bell *et al*. (1969).

[2] A player-orientated view of power is what Barry (1976) calls *social power*, where a player, roughly speaking, has power if he or she has the ability to 'influence' the behavior of others.

[3] For clarifications of the distinction between both views of power assume we have a player i who has a set of actions or strategies $\{a_1, a_2\}$ which is mapped onto a set of outcomes $\{x_1, x_2\}$ such that if i chooses a_1, x_1 is the outcome; and if i chooses a_2, x_2 is the outcome. Under this set-up i is able to affect the outcome, i.e. i possesses 'power to' (do something) with respect to x_1 and x_2 (see Braham and Holler 2002 for a detailed discussion). Now assume that there is another player j who is able to determine i's choice concerning his or her action. Then j has 'power over' i, if i has an interest in the outcome. However, in general, this does not imply that i 'power to' with respect to x_1 and x_2.

simple game (or *simple voting game*),[4] and the decision-making structure. A player's power in such a decision-making situation – his or her *voting power* – then depends upon his or her resources given to him or her by both components.

Consider the case of a group of n players (or voters) who must collectively decide whether to accept or to reject a series of proposals. These may be members of a legislature who must decide on a series of bills or stockholders or managers in a corporation who must decide on management proposals. If such a collective decision-making body has a clearly defined decision rule as a means of specifying outcomes, it will do so by specifying which subsets of the group of players can ensure the acceptance of a proposal. Although the number of possible rules is often very large, one can distinguish three typical cases: the rule of consensus in which all n players must vote in favour; the rule of individual initiative – as Rae (1969) has called it – if at least one single player votes in favour; and the most common of all, majority rule with a specified quota between $n/2$ and n-1 of the members.

The information about which subset of players is *winning* defined by the decision rule also implies information about the distribution of power in the decision-making body in relation to the *bare* rule. Consider a group of three players a, b, and c with votes of 4, 2, and 1 respectively and a quota of 5 votes in a 'yes'-'no' voting situation. In measuring voting power we want to determine how probable it is that a player has the ability to force the outcome of the vote by either voting 'yes' or 'no' (or 'abstain' if possible). We do this by calculating the probability that each player is in a position to determine the outcome with his or her votes which in this case coincide with the players' resources (as no further structural information is known). At the first glance one may be attempted to take the players' votes and, thus, their resources as a predicator for their chances and, thus, as a measure of their voting power. Then a would be twice as powerful as b and b would have twice as much power as c. However, this turns out to be an

[4] Both terms are used in the literature on the theory of voting power. The term *simple game* goes back to Shapley (1953), who attributes the concept to von Neumann and Morgenstern (1944). But due to the fact that Shapley's class of simple games is wider than that by von Neumann and Morgenstern sometimes the term *simple voting game* is used to prevent confusion with von Neumann and Morgenstern's simple games (see, e.g., Felsenthal and Machover, 1998). Note, that essence of a *simple* (*voting*) *game* is just that it specifies which subsets of players can ensure the acceptance of a proposal. As, e.g., Moulin (1983) points out such a voting method is a constitution that can be viewed as a *game form*, a term introduced by Gibbard (1973) to describe games in which individual utilities are not yet attached to possible outcomes (in effect a game without payoff functions). That is, a game form is a system which allows each individual his or her choice among a set of *strategies*, and makes an *outcome* dependent, in a determinate way, on the strategy each individual chooses (see Miller, 1982, for a brief introduction to game forms). Obviously, for a voting situation as described above the strategies are $\langle$yes, no$\rangle$ and the outcome is an element of $\{0, 1\}$ (or $\langle$yes, abstain, no$\rangle$ and $\{1, 0, -1\}$ if abstention is possible).

inappropriate measure if one examines the situation more closely. If we look at the possible winning subsets which are $\{a, b\}$, $\{a, c\}$, and $\{a, b, c\}$ one can see that a is a member of three subsets while b and c are members of two subsets only. That is, without any further information an ordinal power ranking is possible: one can expect that the probability that a will be a member of a winning subset is higher than for b and c.

For a cardinal measure of power it is necessary to add *additional information* about the decision-making situation to the bare rule. Depending on this information one obtains measures of voting power such as the Penrose/Banzhaf or Coleman, the Public-Good, or the Deegan-Packel measures, or the Shapley-Shubik index. As discussed in chapter 3, the Penrose/Banzhaf or Coleman measures can be derived under the assumption that the players are individuals, while the Shapley-Shubik index can be derived under the assumption that the players behave as clones. The different measures of voting power which are subject of this thesis are discussed in detail in the different chapters, which are based on self contained papers. (This, unfortunately, leads to a certain amount of repetition.)

Note, that depending on the type of additional information, a further important distinction between the different measures of voting power is made: that between *a priori* and *a posteriori* measures.[5] While there seems to be a general agreement in the literature how these two classes of measures can be characterized, there is an ongoing discussion about the question which measures belongs to which class. An *a priori* measure is seen as a measure that evaluates the distribution of voting power behind a Rawlsian '*veil of ignorance*' while an *a posteriori* measure is seen as a measure that takes into account supplementary information. The core of the dispute in the literature is which information beyond the bare decision rule is *behind* or *before* the veil of ignorance.

The purist position is held by Felsenthal and Machover (1998, 2001a) and Felsenthal *et al*. (1998). They argue that only the Penrose/Banzhaf measure is a priori measure of voting power. From their point of view, the additional information that is necessary for a calculation of this measure and which is included in the measure as an implicit assumption, is the *nearest* to being behind the veil of ignorance relative to the bare decision rule because it represents the greatest ignorance of the decision-making structure. This position is based on the view that an a priori measure of voting power should in principle be based on, and

[5] The latter are also called *actual* or *real* measures, see Felsenthal and Machover (2000) or Stenlund *et al*. (1985), respectively.

only on, the bare decision rule which is regarded as an empty shell (and the distribution of resources given by that rule); i.e. the actual personalities of the players are ignored and must be ignored in order to provide a constitutional normative analysis. The justification for this position is that if designing a constitution, it would be mistaken tailor outcomes to a particular structure of preferences of the players and their affinities and disaffinities, because these are highly volatile and transient.

The position taken in thesis agrees with Felsenthal and Machover's argumentation in the sense that it would be wrong to include information about the players' preferences into the measurement of a priori voting power. However, on the other hand Felsenthal and Machover have neglected an important aspect that makes their point of view too restrictive. This is the existence of additional (a priori) information behind the veil of ignorance. If such type of information exists, one can show that their purist position can lead to the result that a measure of voting power is a priori but is not reasonable. Thus, the position taken in this thesis is that a measure should take into account all a priori information of the decision-making structure.

3. On the Applicability of Measures of Voting Power

In the literature some scholars criticize or even deny the usefulness of applying measures of voting power. Their criticism focuses mainly on two aspects: they maintain (i) that measures of voting power are incapable of taking into account complex decision-making procedures and (ii) that measures of voting power do not take into account the structure of preferences and affinities of the players (see, e.g., Garrett and Tsebelis, 1999a, 1999b, 2001; Tsebelis and Garrett, 1997; Steunenberg *et al*., 1999).

This criticism is more or less misguided. The critique (i) does not hold as one can apply the concept of a *composite* (or *compound*) voting game which allows the construction of extremely complex voting games from simpler ones, thus providing models for highly complex interactions among these simpler components (Felsenthal and Machover, 2001a).[6] Note, that this is also the case if abstentions are allowed.

[6] For an application to the EU decision-making procedure, see, e.g., Laruelle and Widgrén (2001). For the definition of *composite* (or *compound*) simple games, see Shapley (1962b) or Felsenthal and Machover (1998).

Critique (ii) can be countered as follows. Firstly, most applications of measures of voting power deal with *a priori* power in order to analyse institutions. As already mentioned above such an analysis only addresses the distribution of voting power under a given decision rule and, if applicable, additional a priori information concerning the decision-making structure, ignoring the actual personalities of the players. Thus, voting power is not meant to take into account preferences. This point of view was been argued by Felsenthal and Machover (2001a), Lane and Berg (1999) and Holler and Widgrén (1999).

Secondly, measures of voting power are in fact (at least from a technical perspective) capable of taking into account preferences and affinities via an appropriate enrichment of the structure of bare decision rules by a posteriori information of the decision-making structure.[7] Approaches to model preferences and affinities between players on the basis of bare rules have been proposed, for instance, in Owen (1971 and 1995), Stenlund *et al*. (1985), Straffin (1994), Holler and Widgrén (1999), Steunenberg *et al*. (1999) and Schmidtchen and Steunenberg (2002).

While Stenlund *et al*. consider empirical frequencies of coalitions players form from one voting occasion to another as an reflection of their preferences and affinities; Owen, Straffin, and Holler and Widgrén deal with preferences and affinities in *spatial voting models*. Owen and Straffin have suggested structures including additional information about players' *ideal points* assuming that a player would be happiest with a policy occupying his or her point; failing that he or she would like a position as close as possible to his or her point. Based on such a structure they have proposed measures of voting power for spatial voting games. That these and the 'classical' measures of voting power are far less exclusive than it is often argued (see, e.g., Garrett and Tsebelis, 1999a) is shown by Holler and Widgrén. They demonstrate how one can model (one-dimensional) ideal points under Straffin's (1977) partial homogeneity approach by assigning specific values to the elements of Straffin's acceptability vector, which originally includes the probabilities of each player to vote for a random proposal.

Another interesting instance of a unified approach is that proposed by Steunenberg *et al*. Their method is based on the average distance between players' ideal points and the equilibrium outcome in policy games where players have different abilities to affect the final outcome of the decision-making procedure

[7] However, whether this conceptually makes sense at all has been questioned recently in Braham and Holler (2002)

employing tools of non-cooperative game theory. Players' preferences, as well as the decision rule of the decision-making situation, are fully integrated into the analysis, in that it allows players to act strategically; this is the reason why they call their measure a *strategic power index* (or in the terminology of this thesis a *strategic measure of voting power*). Varying the preferences of the players, they consider the average distance between the equilibrium outcome and the ideal points of players as a proxy for their power. Even starting with a non-cooperative game theoretic setup 'before the veil of ignorance', Felsenthal and Machover (2001a) have shown that behind the veil of ignorance their strategic measure turns out to be the well known Penrose/Banzhaf measure.

4. Outline and Summary of Results

This thesis consists of two parts. Part I concerns theoretical aspects of the theory of voting power while part II deals with applications of the theory of voting power to political and organizational questions. The chapters of both parts are laced together by their common focus on questions of a prioricity and local monotonicity and by the analysis and application of Straffin's probabilistic partial homogeneity approach to the measurement of power.

Part I contains four theoretical chapters:

- Chapter 2: When is A Priori Voting Power Really A Priori ?
- Chapter 3: Voting Power in Games with Abstentions
- Chapter 4: Local Monotonicity and Straffin's Partial Homogeneity Approach to the Measurement of Voting Power
- Chapter 5: Constrained Monotonicity and the Measurement of Power

Chapter 2 concerns a discussion of a prioricity properties of measures of voting power, in particular, questioning the position taken by Felsenthal and Machover (see, e.g., Felsenthal and Machover, 1998; Felsenthal *et al*., 1998). The analysis in this paper is: (i) There is little ground to support Felsenthal and Machover's position that the Penrose/Banzhaf measure, derived from an assumption that each player behaves independently under Straffin's approach, is the only *pure* a priori measure or is 'more' a priori than the Shapley-Shubik index, which results from Straffin's approach if it is assumed that all players behave as clones according to the so-called 'homogeneity assumption'. (ii) That, in contrast to Straffin's (1977) statement that partial homogeneity assumptions are by their nature ad hoc, a partial homogeneity framework could also be a priori if the additional information which is used has an a priori 'character'. An example of the latter is given in chapter 7 where a priori voting power in hierarchical organizations is discussed.

In chapters 3 and 4 the a prioricity discussion is examined in more detail devoting a separate section of each chapter to this issue. While chapter 3 contains a more detailed discussion of the question whether one can distinguish between Straffin's independence and homogeneity assumption behind a Rawlsian 'veil of ignorance', chapter 4 elaborates when a measure based on a partial homogeneity structure fulfils the conditions to be aprioristic.

However, the main focus of chapters 3 and 4 are on different issues. Chapter 3 deals with the occurrence of abstentions in simple voting games. This is a very young and as yet under-developed part of the theory of voting power. Even in social choice, abstention is generally regarded as perfectly rational and normal.[8] It is rather surprising, therefore, that the literature on voting power has until quite recently ex- or implicitly ignored the phenomenon.[9]

A first approach to dealing with abstentions was made by Felsenthal and Machover (1997, 1998). They proposed a ternary voting game (TVG). Chapter 3 provides an alternative way to model abstention by using an abstention voting game (AVG). The basic difference is that a TVG treats 'yes, 'no' and 'abstain' as simultaneous choices, while under an AVG setup voting is conceptualised sequentially: a player first chooses whether to vote at all, and then, if he or she has decided to vote, between casting a 'yes' or 'no' vote.

Both approaches can be conceptually justified. As Machover (2002) has pointed out, we can distinguish between two different forms of abstention: *abstention by default* and *active abstention*. By the former is meant the act of not showing-up to vote; by the latter is meant the case of a player declaring 'I abstain'. While the TVG model can perhaps be regarded as assimilating all abstentions to those of the active kind, the AVG model, can be regarded doing the same for all abstentions that occur by default, i.e. abstentions that do not really figure as

[8] For a discussion see Green and Shapiro (1994, pp. 47-71) and the references they refer to.

[9] Note, that real-life decision rules (such as the UN Security Council) where abstention is in fact a tertium quid, are quite often misreported in the voting power literature as though they counted abstention as a 'no' vote. Apparently, scholars who assumed that abstention is irrational and undeserving of theoretical consideration fell into the trap of assuming that it, therefore, does not exist. Felsenthal and Machover (1997, 2001b, 2001c) cite many examples of such misreporting from the voting-power literature.

expressing an intermediate or even indeterminate degree of support between 'yes' and 'no', but as opting not to participate in a division.[10]

The focus of chapter 4 is on the ongoing and fundamental debate in the literature on voting power about what constitutes a 'reasonable' measure of *a priori* voting power. The reason is partly due to the fact that there is as yet no intuitively *compelling* and *complete* set of axioms that *uniquely* characterize a measure with the result that there are a variety of different measures that not only give different cardinal values but also different ordinal rankings of players.

A central topic in this debate is whether or not a reasonable measure of voting power should fulfil *local monotonicity* (LM) which is a specific version of a relation that goes back to Isbell (1958) and which is known as the *desirability* (Maschler and Peleg, 1966) or *dominance* (Felsenthal and Machover, 1995) *relation*. LM says that in weighted voting games (WVGs), i.e. simple voting games characterized by a vector of voting weights attached to each player and a quota, if a player i has at least as much weight as a player j, then player i should have as least as much power as player j. While the Shapley-Shubik index and the Penrose/Banzhaf or Coleman measures are locally monotonic, the Deegan-Packel and the Public-Good measures are not.

Some authors, notably Felsenthal and Machover (1998), have argued that the violation of LM is 'pathological' and thus measures of voting power that exhibit such behaviour are unreasonable. Other authors, noteably, Brams and Fishburn (1995) and Holler (1997, 1998), have argued that the violation of LM is a simple social fact of power and, therefore, LM cannot be used to determine the reasonableness of a measure of voting power.

So far the debate has ignored the violation of LM by another set of measures derived from Straffin's partial homogeneity approach. By examining violations of LM in this context it is shown that the different sides to this debate are in a sense 'both wrong'. It is argued that LM is a special case of a more general monotonicity condition that relates 'resources' to 'power'; in LM the resources are but the voting weights. However, given that it is not clear that a priori voting power is based on, and only on, the vector of voting weights and the decision rule, it turns

[10] Note, that abstention by default can be seen as a reversal of the 'new member story' from Brams (see Brams, 1975, pp. 178-180; Brams and Affuso, 1976; Rapoport and Cohen, 1984). This and also the relation of our approach to Saari and Siegberg's (2000) results for semivalue rankings after dropping players is subject for further research in progress together with Matthew Braham.

out that a violation of LM can be ‘reasonable’. This, however, does not imply that power is not monotonic in resources per se.

The issue of LM and its violation is also central to chapter 5. It deals with the violation of LM in voting weights by Public-Good measures which most prominent measure is the Public-Good Index (Holler, 1982a; Holler and Packel, 1983). The underpinning argument of the Public-Good measures is the existence of a decision-making structure that includes an incentive structure such that only those winning subset (coalitions) of players ought to form which contain no *excess-player*, i.e. the defection of each player makes the subset losing.

The chapter introduces two constrained versions of LM: (i) *player-constrained* LM by restricting the number of non-dummy players in a game and (ii) *partial* LM by applying specific constrains on voting weights. It is shown the Public-Good measures fulfil *partial* LM for every *proper* WVG, i.e. for WVGs in which two disjoint subsets are never winning at the same time, and *player-constrained* LM for every WVG with a simple majority rule and up to *four* non-dummy players. Later studies of *player-constrained* LM by Schönfeld (2001) have shown, that *player-constrained* LM is also fulfilled for *five* non-dummy players if the WVG is *self-dual* (or *constant sum*), i.e. *proper* and *strong*. A WVG is s*tron*g if there is no subset such that this and its complement are losing at the same time. Note, that a simple way to guarantee that a WVG is self-dual is to set the sum of voting weights to an odd-number. Due to the fact, that as the division voting weights becomes finer and finer, the probability that a non-self-dual game occurs converges to zero. Therefore, the probability that *player-constrained* LM will be violated for *five* non-dummy players converges to zero, while simulations for *six* and *seven* non-dummy-players have shown that the probability of a violation of LM converges to around 20 % and 35 %, respectively.

Chapter 5 concludes with a discussion that points out that whether a specific measure of voting power is appropriate depends on the properties of the model of collective decision-making which one wants to analyze, and not necessarily on some intuitive notions of monotonicity.

Part II of this thesis contains two applied chapters, which make use of parts of the results provided in the previous chapters:

- Chapter 6: Proportional Representation in the National Assembly for Wales
- Chapter 7: A Priori Voting Power in Hierarchical Organizations

While chapter 6 is an application of the theory of voting power to an actual decision-making situation, chapter 7 deals with what one may call a 'theoretical application', i.e. the application of voting power to answer questions in another theoretical area of research. In the case of chapter 7 this is the study decision-making situations and the nature of power in hierarchical organizations.

Chapter 6 contains an analysis of the voting rules for the National Assembly for Wales, which was established in 1999. The rule for electing members of the National Assembly for Wales is the *Additional Member System* (AMS), i.e. not the otherwise usual *first-past-the-post system* for Westminster Parliament. The AMS gives each voter two votes, to be cast at the Assembly Constituency level, and at the bigger Assembly Electoral Region level. One third of the members to the assembly are elected by a form of proportional representation, where party support is calculated by aggregating the two votes. The voters are allowed to cast the second vote in favour of a different party than the one they earlier voted for, at the Assembly Constituency level. It is shown that this additional degree of freedom can frustrate the objective of obtaining better correspondence between party support and the number of seats. Also, the effects of this additional degree of freedom on the voting power of the parties on the Assembly Electoral Region level are shown using Straffin's partial homogeneity approach.[11] Based on this analysis, a different system of proportional representation and a method of equating the distribution of voting power and seat distribution are proposed.

The main result of the study of the voting rules for the National Assembly for Wales is that the switch from the *first-past-the-post system* to the AMS for electing the assembly can frustrate voters and implies the possibility that some parties in the assembly will be rendered powerless, but may at least give some parties the chance of being involved in the business of government.

Chapter 7 deals with the nature of a priori voting power in hierarchical organizations. It is shown that every '*restricted*' *game with a permission structure*, which is a simple game where the winning subsets are additionally restricted by a *permission structure* (see Brink, 1994, 1997, 1999, 2001; Gilles *et al*., 1992; Gilles and Owen, 1994; Brink and Gilles, 1996), can be represented as a

[11] Note, that while writing this paper we were in the believe that the original positions of Labour and Liberal Democrats can be seen as historical based a priori information of the decision-making structure and thus could be used to analyse the a priori voting power of these. Unfortunately, when we finished this paper, Labour and Liberal Democrates have changed their positions. Thus our assumption has turned out to be inappropriate regarding our aim of an a priori analysis of the voting rules for the National Assembly for Wales. This has led to further research by the authors (see Altunbas *et al*., 2000, for first results).

compound game. Furthermore, it is pointed out that the existing research on voting power in hierarchical structures is necessary, but not sufficient to understand the nature of a priori voting power in hierarchical organizations, because it does not take into account: (i) that players who participate in a decision-making in hierarchical organizations in general have a *damatis personae*, which we model via Straffin's partial homogeneity approach applied as an aprioristic framework, and (ii) that the top of a hierarchical organization can have a board-structure.

Taking both aspects into account we not only come out with violation of LM which one would expect based on the results of chapter 4. Moreover, there are some further counterintuitive results, i.e. the violation of known monotonicity properties of power in hierarchical organizations such as (*weak*) *structural monotonicity* and *dis*- and *conjunctive fairness*. (*Weak*) *structural monotonicity* more or less says that a player in a hierarchy who dominates another player should have at least as much voting power as the dominated player; *dis*- and *conjunctive fairness* roughly stipulate that the deletion of a hierarchical relation between two players under *disjunctive fairness* should change their voting power and that of the superiors of the dominating player by the same amount and in the same direction, while under *conjunctive fairness* the voting power of the dominated player and his or her superiors should be changed by the same amount and in the same direction, i.e. the fairness conditions turn out to be very specific ones that appear to be more monotonicity than fairness conditions.

Moreover, it is illustrated that dropping a player belonging to an intermediate hierarchical level, does not necessarily imply that his or her voting power is transferred downwards to the lower hierarchical levels. This has important implications to two related management concepts which are known as *empowerment*[12] and *lean management*.[13] Both are based on the idea that: (i) by removing intermediate layers or parts of layers of a hierarchy power can be transferred downwards to employees on the lower levels and that (ii) such a change will lead to increased motivation due to employees having more of a say in the organization's destiny and thus, increased responsiveness and productivity gains for the organization. But as indicated above, (i) is not necessarily true if we remove layers or parts of layers. The practical implications of this perspective is that when we come to look at the performance of organizations, it is necessary to abstract from the particular personalities that are involved. The success or failure of an organization may not be so much a matter of its 'leadership' and

[12] See Gal-Or and Amit (1998) for a summary of empowerment.

[13] This concept goes back to Krafcik (1988).

'management style' – its 'corporate culture' – but of the interaction of *incentives* and *decision-making rules*.

PART I

THEORY

Chapter 2

WHEN IS A PRIORI VOTING POWER REALLY A PRIORI ?*

Abstract: This chapter concerns a discussion of a prioricity properties of measures of voting power, in particular, questioning the position taken by Felsenthal and Machover. The analysis in this paper is: (i) There is little ground to support Felsenthal and Machover's position that the Penrose/Banzhaf measure, derived from an assumption that each player behaves independently under Straffin's approach, is the only pure a priori measure or is 'more' a priori than the Shapley-Shubik index, which results from Straffin's so-called 'homogeneity assumption'. (ii) That, in contrast to Straffin's statement that partial homogeneity assumptions are by their nature ad hoc, a partial homogeneity framework could also be a priori if the additional information which is used has an a priori 'character'.

1. Introduction

Power is an important concept in economics and political science. We talk about market power, monopoly power, party power, and so forth. There is, however, little agreement on how power is to be defined and how to observe and measure it, although in collective decision-making situations where a decision is made by voting on a proposal that is pitted against the status quo, measures of voting power are used in order to measure the distribution of power. One family of measures of voting power are those based on Straffin's (1977) partial homogeneity approach. The best known measures within this family are its extreme cases: the Shapley-Shubik (1954) and the Penrose (1946)/Banzhaf (1965) measures. Most scholars classify these and all other measures belonging to this family as measuring *a priori* voting power, i.e. the voting power only results from logical conclusions which are independent from experience or observation. In this context, *a priori* is commonly taken to mean the abstraction from individual preferences and social and psychological influences of the members of the voting body.[1] Based on this, or at least on a very similar definition, Felsenthal *et al*. (1998) argue that only the

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* This chapter is based upon Steffen (2000). The author would like to thank Manfred Holler, Matthew Braham, Rie Ono, Christian Reuter, Mika Widgrén and two anonymous referees for helpful comments.

[1] An alternative way is that preferences are randomly distributed with respect to the decision-makers.

Penrose/Banzhaf measure is an *a priori* measure of voting power, while the Shapley-Shubik index and thus by implication all other measures of this family should be considered as *a posteriori* measures - because they take into account information obtained from experience or observation.

This chapter argues that this simple classification does not hold and that power is always measured a priori if it is not measured ex post (i.e. includes actual observations of voting behaviour), but that the degree of *a priority* varies with the extent to which additional information is used.

The chapter is organized as follows: section 2 contains a brief introduction to Straffin's partial homogeneity approach and some formal definitions. Section 3 illustrates Felsenthal *et al.*'s position and argues that they neglect some important aspects, especially concerning the application of measures of voting power to real problems. While this section only deals with the two aforementioned extreme cases of the partial homogeneity approach, Section 4 extends the considerations and introduces the idea of fuzzy standards. Section 5 offers a classification of three different levels of *a priori* information and delimits these from an *a posteriori* level. Concluding remarks are made in Section 6.

2. Measurement of Voting Power

Let $N = \{1, 2, ..., n\}$ be a set of *voters* (or *players*) of a *weighted voting game* $[q; w]$, where $q \in [0,1]$ is the *majority quota*, which is the voting weight needed to attain a certain end, i.e. to win or block a bill and $w = (w_1, w_2, ..., w_n)$ is the vector of voting weights of each voter $i \in N$. Furthermore, let $\mathcal{W}$ be a collection of subsets $S \subseteq N$ with $\sum_{i \in S} w_i \geq q$, which is called the set of all *winning coalitions*. Then, we define the set of all *crucial coalitions* $\mathcal{C}$ as a collection of subsets $S \in \mathcal{W}$ where for each S at least one voter $i \in S$ is a *crucial* voter. Voter i is called crucial for $S \in \mathcal{W}$, if S is a losing coalition without i: $S \backslash \{i\} \notin \mathcal{W}$. Finally, let $\mathcal{C}_i$ denote the class of crucial coalitions containing voter i as a crucial voter.

If we want to measure the distribution of voting power in a certain voting body following Straffin (1988) we need to ask the question 'What is the difference that voter i can make to the decision with its votes?' That is, voter i's vote makes a difference when i converts a losing coalition $S \backslash \{i\}$ to a winning one, - in other words, when i is crucial to a winning coalition S. Thus the power of voter i can be defined as the probability that a coalition S will be formed and that S belongs to the set $\mathcal{C}_i$.

For the calculation of this probability which is shown in Straffin (1977, 1988) we have to specify the probability model for the elements of the *acceptability vector* p, which includes the probabilities p_1, p_2, ..., p_n, with $p_i \in [0,1]$ with which each voter $i \in N$ will vote for a random bill. If we do not have any prior information of voters' attitude towards alternative bills per se, there are the following two standard assumptions:[2]

Independence assumption: Each p_i is chosen independently from the uniform distribution on [0,1], i.e. how one voter feels about a proposal has nothing to do with how any other voter feels.

Homogeneity assumption: Each $p_i = t$ and t is chosen from the uniform distribution on [0,1], i.e. all voters have the same probability of voting for a given proposal, but t varies from proposal to proposal. We could think of voters as judging bills by some common standard, and t as the bill's acceptability level by that standard.

It should be noted that the independence assumption is equivalent to assuming that each voter i will vote in favour for any bill with probability ½ (Straffin, 1977). Thus, the independence assumption, which implies so-called indifference, requires only that the mean value be specified, while the homogeneity assumption requires the specification of the whole distribution from which the elements of the probability vector are selected (Leech, 1990). In this regard Straffin (1977) demonstrates that the independence assumption leads to the Penrose/Banzhaf measure (which is also known as the absolute or non-normalized Banzhaf index), while the homogeneity assumption yields the Shapley-Shubik index.

In order to account for additional information concerning the relation of voters in a given voting body, we can combine these two assumptions leading to the *partial homogeneity assumption*. For example, we can assume that a group of voters $W \subset N$ has a certain standard which implies $p_{i \in W} = t$, whereas the standard of another group of voters $T \subset N$ with $W \cap T = \varnothing$ is exactly the opposite of the

[2] It should be pointed out that these assumptions do not include specific preferences as they are used in connection with spatial voting models (see Holler and Widgrén, 1999). While in the case of spatial voting models p includes real numbers as an expression for voters' preferences, here we only make assumptions concerning the probability distributions of the elements of p. Thus the partial homogeneity approach only employs relationships between standards of behavior of the voters which either can be a priori or a posteriori. For general considerations concerning other probability distributions of p_i, see Straffin (1978).

former one, i.e. $p_{i\in T} = 1 - t$, while all the other voters $i\in N \setminus \{W\cup T\}$ behave according to the independence assumption so that $p_{i\in N \setminus \{W \cup T\}} = ½$.[3]

3. Homogeneity vs. Independence Assumption?

Felsenthal *et al*. (1998) argue that in contrast to the independence assumption, the homogeneity assumption requires a very strong measure of uniformity among voters. They conclude that the choice between the homogeneity and independence assumption should not be made according to the degree of verisimilitude with regard to a real-life voting situation, but rather on the grounds of which is the more reasonable expression of *a priori* voting power. In their view, the independence assumption should be interpreted as embodying absence of a priori information about voters' motivations, intentions, and interests. They further held that even in situations where the homogeneity assumption may be preferable on the basis of some partial knowledge, the power measure is *a posteriori*. The position is supported by reference to Leech (1990), who points out that the distributional assumption underlying the homogeneity assumption is much stronger than behind the independence assumption. Consequently, the homogeneity assumption is appropriate only if there is good reason to assume a high degree of uniformity among voters.

There is no question that the formal arguments underlying Felsenthal *et al*.'s statement are correct. They do, however, seem to argue that we can only measure *a priori* voting power of a voting body behind a Rawlsian veil of ignorance (Rawls, 1971) if we know the majority quota q, the vector of voting weights w, and assume - due to the principle of insufficient reason - that the voters behave independently. At first glance, this looks like a real *a priori* position; although as it will be shown, this is mistaken.

Firstly, it is not clear that the independence assumption should be more appropriate than that of homogeneity if we have no information on voters' behaviour: the problem being that there is no justifiable reason to prefer one assumption or the other. The independence assumption implies that voters behave independently, while the homogeneity assumption implies a correlated behaviour and thus both assumptions are extreme cases.

[3] For the power value calculation for this case, see e.g. Kirman and Widgrén (1995).

Secondly, it appears reasonable to reject the independence assumption as inappropriate in cases when it is known that there is a common standard in the voting body. For instance this may be a historically determined pattern of behaviour and is thus valid behind the veil of ignorance. In such cases it is necessary to take into account such patterns by choosing the appropriate measure.

There is, moreover, a theoretical case behind the veil of ignorance where the homogeneity assumption is the appropriate one. This is when the relevant common standard for decision-making in a voting body is given by a Kantian categorical imperative (Kant, 1989), i.e. an *a priori norm* which is binding for all rational beings and is valid independent of all experience. There is, then, no reason to conclude that a measure of voting power that rests on a common standard is not an *a priori* one.

4. The Partial Homogeneity Assumption

Having considered the extreme cases of partial homogeneity, it is necessary to generalize the analysis. For this, we employ the arguments introduced above to further qualify why it seems inappropriate to conclude that only the independence assumption leads to a real *a priori* voting measure of voting power.

One point of departure is to translate Kant's a priori principles to the case of a political voting body. Here we can say that historically - at least in many countries - the voters of a left-wing party and the voters of a right-wing party, have a priori opposite standards. If, in addition, there is a centre party, its standard will lie between the other two former parties, i.e. the probability that voters of this party will vote in favour or against any given bill will be ½.[4]

According to Klingemann and Volkens (1992), Gallagher *et al*. (1995, pp. 157-162) and Roberts (1987), for instance, there is evidence to this effect in Germany since 1949: the left-wing Social Democrats (SPD) and the right-wing Christian Democrats (CDU) in general are diametrically opposed, while the Free Democrats (FDP) lie somewhere between the two. If we analyze the distribution of power in an assembly consisting of these three parties, and if we consider what we know about the position of the parties, we can model these positions as standards, i.e. $p_{\mathrm{SPD}} = t$, $p_{\mathrm{CDU}} = 1 - t$ and $p_{\mathrm{FDP}} = [t + (1 - t)]\ ½ = ½$, and we are still measuring *a priori* voting power.

[4] For the problems of choosing an one-dimensional political space, see e.g. Schmidt (1996).

Another example where diametrically opposed standards in voting games could occur is in the decision-making process of firms, the banking sector is such a case. Consider, for instance, the decision mechanism for granting a loan. This amounts more or less to a voting situation in which - in a simplified version - both the customer consultants and the credit department personnel have a vote. While a customer client is predominately interested in credit expansion, the credit department is on the whole interested in reducing the risk of the bank's credit-portfolio.[5] In this case we are again measuring *a priori* voting power despite the inclusion of supplementary information.

As the partial homogeneity approach modelled above uses diametrically opposed standards, it is possible to argue that it is inappropriate in cases where we do not know to what extent the standards differ. Here a possible solution is a 'fuzzy' version of the partial homogeneity approach. What this means is that we take intervals of standards rather than precisely opposite standards to analyze the 'stability' of the solution to the voting game that uses diametrically opposed standards. Defining intervals for a fuzzy measure requires, however, that the standards sufficiently differ. As an example, consider a politically divided voting body, divided into left-wing and right-wing groups, and that the left-wing voters, $i \in L$ with $L \subset N$, have a common standard $p_{i \in L} = \mathrm{p}_L = t$. Then we can analyze the effect of extending the common standard $p_{i \in R} = \mathrm{p}_R$ of the right-wing voters, $i \in R$ with $R = N \setminus L$, from $p_R = 1 - t$ to $p_R \in [1 - t - a; 1 - t + b]$ with $0 \leq b \leq t$, and $0 \leq a < 1 - 2t$ for $t < ½$.

If there exists a third group of voters $M \subset N$, with $M = N \setminus \{L \cup R\}$, whose members are provided with a common standard $p_{i \in M} = p_M$ and which lies between the standards of the right- and left-wing groups, we can define an additional standard interval for these voters. In this case we have to modify the interval for the right-wing voters $i \in R$ introduced above. Let $p_M \in [g - g; g + d]$ with $0 \leq g < g - t$ and $0 \leq d < 1 - t - a - g$ be the standard interval of the voters $i \in M$ in the central position then the corresponding interval for the right-wing voters is $p_R \in [1 - t - a; 1 - t + b]$ with $0 \leq b \leq t$ and $0 \leq a < 1 - t - g - d$.

In each case we can model the behaviour of additional voters without a common standard according to the independence assumption (see e.g. Altunbas *et al*., 1999a). This is shown for the case where we have a left-wing and a right-wing

[5] For a more detailed example and the application of measures of voting power to hierarchical organizations, see Steffen (1999) and Braham and Steffen (2001c, 2002b).

group of voters in the following example which illustrates the usage of probability intervals.

Example: Consider the weighted voting game [5; 2, 2, 1, 1, 1] and the acceptability vector $p = (p_L, p_L, ½, p_R, p_R)$. Applying a partial homogeneity structure with $p_L = t$ and $p_R = 1 - t$, we obtain Ef = (1.00, 1.00 ,0.50 ,0.50 ,0.50). After normalizing these values we obtain Ef^{nom} = (0.29, 0.29, 0.14 ,0.14 ,0.14). Now varying the common standard of the rightist voters $p_R \in [1 - t - a; 1 - t + b]$ in 5%-steps for $a|_{b=0} \in [0.00; 0.55]$ and $b|_{a=0} \in [0.00; 0.65]$, respectively, we obtain the result shown in Figure 1.

Figure 1 indicates the effects on relative (i.e. normalized) voting power resulting from such a variation of the standard p_R for all five voters in the game. Obviously it is unnecessary that the right-wing standard be exactly opposite to that of the left-wing voters $p_L = t$ to obtain an *approximate* relative power distribution close to the one by using diametrically opposed standards. For example, we can see from Figure 1 that this correspondence is true for $a = b \approx 0.15$. In this case there exists an interval spread of about 0.3.

But, one may also argue that there is a visible difference in the power distribution within the interval spread. However, we can encounter that these differences are negligible and that the ordinal power rankings remain the same within the spread. Note, that the ordinal rankings only change for $a > 0.3$, while they remain constant for $0.3 \geq a > 0$ and for all permissible values of b. For $a > 0.3$ which requires $t < 0.35$ and which is equal to the statement that the left-wing group of voters is a relative left-wing extremist group we obtain the result that these voters are no longer the most powerful voters. Figure 2 indicates the ordinal rankings resulting from variation in p_R.[6]

Additionally, we also see that total non-normalized power decreases with an increase of a, and increases with an increase of b, i.e. the more standards differ the more powerful are the voters and vice versa.

[6] For our analysis of figure 2, we have neglected ordinal ranking effects which result from relative power differences $\Delta Ef^{nom}_{ij} = Ef^{nom}_i - Ef^{nom}_j$ which are less than 0.003.

Figure 1: The Evolution of Normalized Voting Power Varying the Opposite Standard p_R

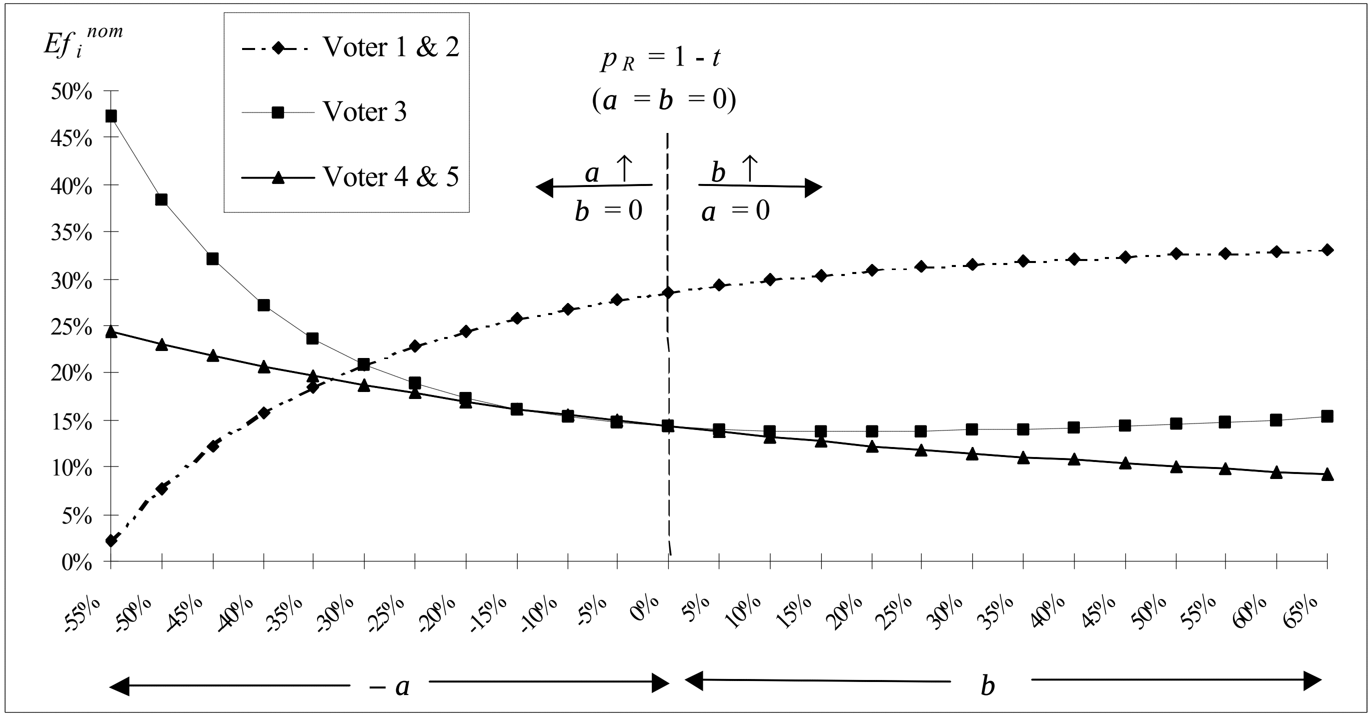


Figure 2: Ordinal Rankings of Voting Power

a	b	Ordinal Rankings of Voting Power
$a = 0.00$	$b = 0.00$	$Ef_1 \sim Ef_2 > Ef_3 \sim Ef_4 \sim Ef_5$
$a = 0.00$	$b > 0.00$	$Ef_1 \sim Ef_2 > Ef_3 > Ef_4 \sim Ef_5$
$0.20 \geq a > 0.00$	$b = 0.00$	$Ef_1 \sim Ef_2 > Ef_3 \sim Ef_4 \sim Ef_5$
$a = 0.25$	$b = 0.00$	$Ef_1 \sim Ef_2 > Ef_3 > Ef_4 \sim Ef_5$
$a = 0.30$	$b = 0.00$	$Ef_1 \sim Ef_2 \sim Ef_3 > Ef_4 \sim Ef_5$
$a \geq 0.35$	$b = 0.00$	$Ef_3 > Ef_4 \sim Ef_5 > Ef_1 \sim Ef_2$

5. A Classification of Three Different Levels of A Priori Information

In this section a classification consisting of three different levels of *a priori* voting power is put forward. Categories are distinguished by the nature of available information supplementary to the pure weighted voting game [q; w]. Each of the levels is demarcated from *a posteriori* voting power.

The first *a priori* level is described by cases where only information about the pure weighted voting game [q; w] is available. Here we cannot decide which assumption is more appropriate for modelling voter behaviour; the choice is, therefore, arbitrary.

The second level defines cases where general additional *a priori* information is available. This level includes the two cases of *a priori* additional information discussed above: that of a Kantian categorical imperative or as in the example of bank granting a loan; and that which refers to a historical precedent, such as the behaviour of voters when they are political parties.

The third level is a result of applying measures of voting power to analyze the distribution of power of a voting body when knowledge about individual voters is available. Here the formation of specific coalitions vis-à-vis particular issues can be taken into account - although in such cases we must exercise caution regarding what we mean by power. The problem here is that information derived from knowledge of individual voters is likely to be preference-based; and according to Miller (1982), we have to distinguish between preference and non-preference-based sources of success: a voter may be successful because his or her preferences are similar to those of most other voters, or he or she is in a median-voter position. This, however, says nothing about his or her power as he or she may no longer be successful if his or her preferences change. Instead, a powerful voter should be able (perhaps in a coalition with others) to bring out outcomes he or she likes or preclude outcomes he or she dislikes, regardless of the preferences or actions of other voters. In other words, power should be seen as a non-preference-based source of success. Hence, taking into account information on voting behaviour that may be blurred by preferences, makes it impossible to distinguish power from other components of success.

As a caveat, in contrast to *a priori* voting power which could be measured according to the classification above, it is possible to measure *a posteriori* voting power. In this case we would use quantitative information to model coalition formation, as for example, in Stenlund *et al*. (1985). In this paper the authors employed the relative frequencies of actual coalitions in the Swedish Riksdag from 1971 to 1979 as probability weights of the theoretically possible coalitions in order to measure the distribution of voting power. They call this a measure of *real voting power* in contrast to what they call *formal voting power*. Under formal voting power they understand voting power which is measured, for instance, by applying the measures based on the partial homogeneity approach.

6. Conclusion

Felsenthal *et al*. (1998) maintain that the only possibility to measure *a priori* voting power is to get away from all supplementary information and to stick with the pure voting game and then due to the principle of insufficient reason assume that all voters' behave according to the independence assumption. This chapter has argued that this appears inappropriate and that there is no unique measure of voting power which measures *a priori* voting power. In other words, power is always measured *a priori*, if it is not measured *a posteriori*, i.e. includes actual observations of voting behaviour. The point is to recognize that the degree of *a priority* varies with the amount of qualitative information.

Chapter 3

VOTING POWER IN GAMES WITH ABSTENTIONS*

Abstract: In general, analyses of voting power apply the notion of a simple voting game. This notion restricts each voter to two options: 'yes' or 'no'. Felsenthal and Machover introduced the concept of a ternary voting game in which 'abstain' is added as a third option. They derive analogues of the Shapley-Shubik and Banzhaf indices. We examine these analogues and show that the assumption upon which they are based, that yes, no, choices, may not be justified. We propose a sequential structure in which voters first choose between participation and abstention and then between yes and no. This results in a structure that we denote as an abstention voting game, which is a collection of possible games on possible assemblies.

1. Introduction

In a series of recent publications and papers, Felsenthal and Machover [F&M] (1997, 1998, 2001a, 2001b, 2001c) have noted that except in a few rare cases (e.g. Fishburn, 1973; Morriss, 1987), the published work on voting power has used, explicitly or implicitly, one and the same type of mathematical structure to model decision rules of voting bodies that make yes/no decisions. This is the structure known as a *simple voting game* (SVG).

The SVG set-up and that of an important sub-class known as *weighted voting games* (WVG) are, according to F&M, seriously wanting in one important respect: they are strictly *binary* in that they assume that in each division[1] a voter has just two options: voting 'yes' or 'no'. In their papers, F&M point out, and give examples of, the fact that many real-life decisions are *ternary* in the sense that they al-

* This chapter is based upon Braham and Steffen (2002a). The author wishes to thank the participants at the Institute of SocioEconomics' Research Seminar, University of Hamburg, for comments on a very early forerunner of this chapter as well as the participants at New Political Economy Meeting on Power and Fairness, 3–6 September 2000, Bad Segeberg, Germany. The author thanks Manfred Holler for helpful and insightful discussions at various stages in this research and Ines Lindner who has cooperated on various aspects of this chapter. The author also thanks William Zwicker for a valuable insight. Finally, the author is deeply indebted to Moshé Machover for helping him straighten out the crooked timber of his thinking. The usual disclaimer applies.

[1] Here we follow F&M's (1997, p. 335) use of the term as taken from English parliamentary parlance to denote the *collective* act of a voting body, whereby each individual member casts a vote.

low abstention as a *tertium quid*, the effect of which may be quite different from both a 'yes' and a 'no', and so cannot be assimilated to either. The implication is that one needs to take abstention seriously and account for it when we calculate the voting power of a voter in a given voting body.

In order to release decision rules from the SVG corset, F&M (1997, 1998) introduce a generalization of the SVG set-up called a *ternary voting game* (TVG) which recognizes abstention alongside 'yes' and 'no' votes. This extends some earlier work in this direction undertaken by Fishburn (1973, pp. 53–55). They then derive analogues for the most commonly used measures of voting-power, the Shapley-Shubik (1954) and Banzhaf (1965) indices.

In this chapter, we examine F&M's indices and show that they imply a particular decision-making structure, which, while defensible from a purely formal a priori position, is not necessarily the only one. We impute an alternative decision-making structure and hence derive an alternative family of measures of voting power for games with abstentions.

This chapter is organized as follows: section 2 reproduces the basic formal framework for a SVG, F&M's TVG, and the corresponding Shapley-Shubik and Banzhaf indices as the basis of our argumentation. Section 3 discusses and challenges the underlying and heuristic assumption in F&M's TVG set-up, that is, to abstain is symmetric to 'yes' and 'no'. It is proposed that the abstention decision may be of a different nature to 'yes' and 'no' and therefore should be characterized as a structurally separate category. Thus in contrast to F&M who assume a simultaneous choice structure, we assume a sequential one. Under this assumption we show in Section 4 that it is possible to derive a natural family of power indices for games with abstentions without departing from the traditional SVG structure, and characterize a SVG with abstentions as a set of possible games on possible assemblies. In section 5 some conceptual problems related to the choice of assumptions that have to be applied when using our framework are discussed. Section 6 concludes the chapter.

2. Basic Definitions and Terminology

For the basic definitions and terminology relating to the idea of a *simple voting game* (SVG) and *weighted voting game* (WVG), we refer the reader to Shapley (1962b) or to reproductions in F&M (1998) or Straffin (1983). However, in order to present F&M's derivation of analogues for the Shapley-Shubik and Banzhaf indices and to develop our own framework, we need to reiterate some of this here. In particular:

Definition 2.1 (i) A SVG is a pair $\boldsymbol{\Gamma} = (N, \mathcal{W})$, where N is a finite set and $\mathcal{W}$ is a collection of subsets of N, satisfying the following properties: $\varnothing \notin \mathcal{W}$; $N \in \mathcal{W}$; and if $S \in \mathcal{W}$ and $S \subseteq T$, then $T \in \mathcal{W}$.

(ii) By N is meant an *assembly* (or voting body) and is the largest in $\mathcal{W}$, its members are *voters* (or *players*), and its subsets are *coalitions*, and the members of $\mathcal{W}$ are *winning coalitions* of $\boldsymbol{\Gamma}$. If $S \subseteq N$ but $S \notin \mathcal{W}$, then it belongs to the set of losing coalitions $\mathcal{L}$. Further, if $i \in S$ and $S \in \mathcal{W}$ but $S - \{i\} \notin \mathcal{W}$, then voter i is called *critical* in the coalition S in the SVG $\boldsymbol{\Gamma}$.

(iii) Voters in a SVG $\boldsymbol{\Gamma}$ are identified by the integers 1, 2, …, n, where $n = |N|$.

There is an important sub-class of SVGs known as *weighted voting games* (WVG):

Definition 2.2 Let $w = (w_1, w_2, \ldots, w_n)$ be a nonnegative vector, and let q satisfy $0 \le q \le \sum\{w_i : i \in N\}$. Then in a WVG (q, w), $S \in \mathcal{W}$ iff $\sum\{w_i : i \in S\} > q$ and $S \in \mathcal{L}$ otherwise.

Remark 2.1 The numbers w_i are called (voting) weights, and q is called the (majority) quota.

Definition 2.3 A numerical measure of the power of voter i to determine a decision or series of decisions made in an assembly N is a vectorial function ξ associating to any SVG $\boldsymbol{\Gamma}$ an element of R_0^{+n}, i.e. ξ is any function assigning a nonnegative real number $\xi_i(\boldsymbol{\Gamma})$ to every voter i of every SVG $\boldsymbol{\Gamma}$ in which at least one voter i from each SVG $\boldsymbol{\Gamma}$ is assigned a value $\xi_i(\boldsymbol{\Gamma}) > 0$ and $\xi_i(\boldsymbol{\Gamma})$ is invariant under isomorphism.

Remark 2.2 This very general definition, which is based upon Sagonti (1991), F&M (1995), and Felsenthal *et al*. (1998) encompasses three types of measures: (i) *scores* (counts or 'raw' indices), which we denote by $\kappa_i(\boldsymbol{\Gamma})$ for voter i; (ii) *indices*, obtained by normalization of κ, i.e. $\kappa_i/\sum\{\kappa_j(\boldsymbol{\Gamma}) : i \in N\}$; and (iii) *ratios* of $\kappa_i(\boldsymbol{\Gamma})$, obtained by dividing $\kappa_i(\boldsymbol{\Gamma})$ by an appropriate quotient.[2] We will make use of this definition later in the chapter in our derivation of a family of power indices that takes into account abstentions.

[2] The distinction between these three measures is made in Felsenthal *et al*. (1998).

The two most known and applied measures of voting power are the Shapley-Shubik and Banzhaf indices.

Definition 2.4 The *Shapley-Shubik index* (f) of voter i in a SVG $\boldsymbol{\Gamma}$ is defined as $f_i(\boldsymbol{\Gamma}) = \Sigma(|S| - 1)!(n - |S|)! / n!$, where the summation is over all coalitions S in which i is critical. As is easy to prove, the sum of $\Sigma(|S| - 1)!(n - |S|)!$ for all $i \in N$ is equal to $n!$, so that $\Sigma\{f_i(\boldsymbol{\Gamma}) : i \in N\} = 1$.

Definition 2.5 (i) The *Banzhaf score* or 'raw' Banzhaf index (h) of voter i in a SVG $\boldsymbol{\Gamma}$ is the number of coalitions in which i is critical.
(ii) The *Banzhaf index* (b) of voter i in a SVG $\boldsymbol{\Gamma}$ is obtained from $h_i(\boldsymbol{\Gamma})$ by normalization: $b_i(\boldsymbol{\Gamma}) = h_i(\boldsymbol{\Gamma})/\Sigma\{h_j(\boldsymbol{\Gamma}) : j \in N\}$, so that $\Sigma\{b_i(\boldsymbol{\Gamma}) : i \in N\} = 1$.
(iii) The *absolute Banzhaf measure* ($b^{¶}$) of voter i in a SVG $\boldsymbol{\Gamma}$ is defined by
$b_i{'}(\boldsymbol{\Gamma}) = h_i(\boldsymbol{\Gamma})/2^{n-1}$. (This measure is commonly known as the absolute Banzhaf index.)

As F&M (1998, p. 14) note, the basic SVG set-up and the corresponding measures of voting power are 'one-sided', for they only model the division of an assembly on a resolution by the coalition S of the members voting 'yes'. While admittedly this provides all the information needed when abstentions are not allowed, F&M take it to be too restrictive for modelling abstentions. Thus they introduce an alternative representation of a SVG that when extended can take abstentions into account. To acquaint the reader with this idea, we start with their concept of a SVG in which abstention is not a *tertium quid*, which they define in terms of a *bipartition* or *binary division*:

Definition 2.6 A *bipartition* or *binary division* of a set N is a map B from N to $\{-1, 1\}$. By B^- and B^+ are denoted the inverse images of $\{-1\}$ and $\{1\}$ respectively under B:

$$B^- = \{i \in N\colon Bi = -1\},\ B^+ = \{i \in N\colon Bi = 1\}$$

Thus for a SVG with an assembly N, the *bipartition rule* is the map $\mathcal{V}$ from the set ${}^N\{-1, 1\}$ of all bipartitions to $\{-1, 1\}$, such that for any $B \in {}^N\{-1, 1\}$,

$$\mathcal{V}(B) = \begin{cases} 1 & \text{if } B^+ \in \mathcal{W} \\ -1 & \text{otherwise} \end{cases}$$

Remark 2.3 In other words, a bipartition B represents a division of an assembly but one that does not allow abstentions: the sets B^- and B^+ represent the sets of 'no' and 'yes' voters respectively. Under this representation, a SVG is interpreted as a decision rule that assigns an outcome $\mathcal{V}(B)$ to each binary division: a negative outcome -1 or a positive outcome 1, according to whether the resolution is defeated or approved.

As is easy to see, the natural extension of this formal framework to the situation where abstentions are recognized alongside 'yes' and 'no' is a *ternary* or *tripartition*, i.e. classifying voters into three sets instead of two:

Definition 2.7 A *tripartition* or *ternary division* of a set N is a map T from N to $\{-1, 0, 1\}$. By T^-, T^0, and T^+ are denoted the inverse images of $\{-1\}$, $\{0\}$, and $\{1\}$ respectively under T:

$$T^- = \{i \in N\colon Ti = -1\}, \quad T^0 = \{i \in N\colon Ti = 0\}, \quad T^+ = \{i \in N\colon Ti = 1\}$$

That is, for a TVG with an assembly N, the *tripartition rule* is the map $\mathcal{U}$ from the set ${}^N\{-1, 0, 1\}$ of all tripartitions to $\{-1, 1\}$, such that for any $T \in {}^N\{-1, 0, 1\}$,

$$\mathcal{U}(T) = \begin{cases} 1 & \text{if } T^+ \in \mathcal{W} \\ -1 & \text{otherwise} \end{cases}$$

Remark 2.4 (i) Similar to Definition 2.6, a tripartition T represents a division of an assembly but one that now permits abstentions: the sets T^- and T^+ represent the sets of 'no' and 'yes' voters respectively and T^0 the voters who decide to abstain. Under this representation, a TVG is interpreted as a decision rule that assigns an outcome $\mathcal{U}(T)$ to each ternary division: a negative outcome -1 or a positive outcome 1, according to whether the resolution is defeated or approved.[3]

(ii) As TVGs are obtained from SVGs by allowing each voter three options instead of two, it is possible to generalize further and allow each voter k options, where $k > 1$. Each option can be interpreted as a 'degree of support' for a given resolution, ranging from complete opposition to total enthusiasm. Thus a TVG is a special case where $k = 3$ (Freixas and Zwicker, 2000).

The next step is to summarize F&M's derivation of the ternary analogues for the Shapley-Shubik and Banzhaf indices. For the Shapley-Shubik index, we need to note that they do so by way of an alternative representation that makes use of definitions 2.6 and 2.7 (F&M, 1996). That is, F&M define the Shapley-Shubik index in terms of a *roll-call model*, a model which disposes of the famed 'convenient device' of assuming that the all voters line up and all say 'yes' (or all say

[3] This structure is a generalization of Fishburn (1973, pp. 53–55).

'no') and replaces it with one in which each voter is assumed to vote 'yes' or 'no' with equal probability for bipartition games (Definition 2.6) or 'yes', 'no' or 'abstain' with equal probability for tripartition games (Definition 2.7).

The essence of a roll-call model is that a *pivotal* voter is the first voter whose vote – 'yes' or 'no' in the case of bipartitions or 'yes', 'no', or 'abstain' in the case of tripartitions – seals the outcome one way or the other, so that the votes of all subsequent voters no longer make any difference. Thus on the basis of the equiprobability assumption, the Shapley-Shubik index for voter i is the probability that i will be the unique voter (or pivot) in the roll-call space R_N, which is defined as the size of possible orders in N, $n!$, multiplied by the set of all possible partitions, 2^n or 3^n in the binary and ternary space respectively.

Calling the unique (or pivotal) voter i in each roll-call R of N for a binary partition (or SVG) $\mathcal{V}$ the *$\mathcal{V}$-pivot of* R, denoted as 'piv(R; $\mathcal{V}$); and calling the unique or pivotal voter i in each R of N for a ternary partition (or TVG) $\mathcal{U}$ the *$\mathcal{U}$-pivot of* R, denoted as 'piv(R; $\mathcal{U}$)', F&M obtain the following two characterizations of the Shapley-Shubik index:

Definition 2.8 (i) The *Shapley-Shubik index* (f) of voter i in a bipartition (or SVG) $\mathcal{V}$ is given by $f_i(\mathcal{V}) = |\{i = \text{piv}(R;\, \mathcal{V})\}| \,/\, 2^n\, n!$.
(ii) The *Shapley-Shubik index* (f) of voter i in a tripartition (or TVG) $\mathcal{U}$ is given by $f_i(\mathcal{U}) = |\{i = \text{piv}(R;\, \mathcal{U})\}| \,/\, 3^n\, n!$.

Our last step in this section is to show how F&M translate the ideas of a ternary space to defining the Banzhaf indices. Here the approach is more direct. First they demonstrate that it is possible to define the Banzhaf indices in terms of bipartitions, by showing that the Banzhaf score (see Definition 2.5) is equivalent to the number of bipartitions in which voter i is positively or negatively critical given that each partition is equally probable. Then they demonstrate that it is easy to extend this framework to the tripartition case.

To spare the formalities, F&M's intuition behind this idea is that what we are looking at is the agreement of voter i with the outcome of the bipartition B, which means that the decision goes i's way: in the case of positive agreement i votes 'yes' and the resolution is passed, and in the case of negative agreement i votes 'no' and the resolution fails. In other words, i's being critical (positively or negatively) for B means that i not only agrees with the outcome but also if i's vote were to be reversed, the outcome would likewise be reversed. The *absolute Banzhaf measure*

($b^{\P}$) is, therefore, the probability of obtaining a bipartition for which i is (positively or negatively) critical.

It is easy to see that this idea is extendible to the case of tripartitions, T, i.e. that is, instead of counting the number of times that i is critical in the binary space (i.e. the Banzhaf score, h), we count it in the ternary space. The *absolute Banzhaf measure* ($b^{\P}$) for a TVG is, then, the probability of obtaining a tripartition for which i is (positively or negatively) critical and so could change the outcome by reducing (switching from a yes to abstain or abstain to no) or increasing (switching from no to abstain or abstain to yes) his or her support for the resolution in question.

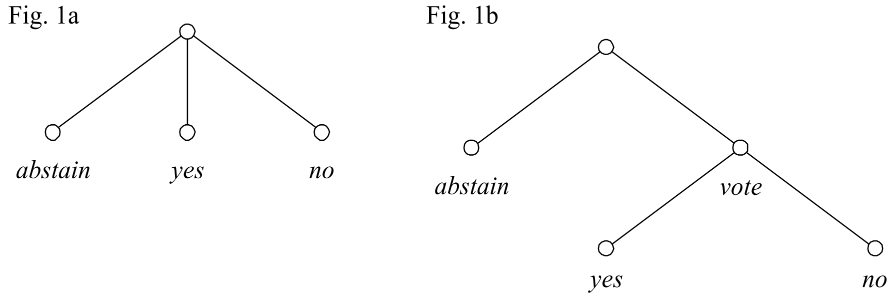
Definition 2.9 (i) The *Banzhaf score* or 'raw' Banzhaf index (h) of voter i in a TVG $\mathcal{U}$ is the number of coalitions in which i is critical.
(ii) The *Banzhaf index* (b) of voter i in a TVG $\mathcal{U}$ is obtained from h_i by normalization: $b_i(\mathcal{U}) = h_i(\mathcal{U})/\sum\{h_j(\mathcal{U}) : j \in N\}$, so that $\sum\{b_i(\mathcal{U}) : i \in N\} = 1$.
(iii) The *absolute Banzhaf measure* ($b^{\P}$) of voter i in a TVG $\mathcal{U}$ is defined by $b_i{}' = h_i(\mathcal{U}) / 3^{n-1}$.

3. Incomplete Assemblies and Possible Games

It should be immediate from Definition 2.8 parts (i) and (ii) and from comparing Definition 2.5 parts (i) – (iii) to Definition 2.9 parts (i) – (iii) that F&M have derived extremely natural analogues for the Shapley-Shubik and Banzhaf indices for voting games that allow abstentions as a *tertium quid*. In this section we offer a few reasons why, despite their naturalness, the method that F&M have followed is not the only one and that there is an equally natural and simple alternative, although the results are quite different.

In the first place, if we consider Definition 2.7 it should be obvious that this structure is not entirely innocent as it imposes a particular assumption on the nature of abstention: it is symmetric to 'yes' and 'no' (Figure 1a). To put the thought another way, when voter i is faced with a ternary decision rule, he or she evaluates 'yes', 'no' and 'abstain' in equal light. As F&M note (1998, p. 286), this is not unproblematic because it is not at all self-evident that abstention is symmetric to 'yes' and 'no'. While in their caution F&M are primarily concerned with the validity of assigning a priori probability 1/3 to each option, this may not be the central issue. For it may not be a matter of whether we should take the probability to abstain as a undetermined parameter in a general theory and leaving remaining probability to be shared equally between 'yes' and 'no' as F&M suggest, but whether the ternary structure is *per se* the appropriate – or only – way of modelling a decision rule that permits abstention.

Figure 1: Decision Structures



Consider, as F&M (1997, p. 336) do, the most commonly used rule in decision-making bodies: the *simple majority*, whereby a resolution is passed if, and only if, more members vote for it than against it. Unless specified otherwise, this rule does not treat abstentions as 'yes' or 'no'; for in the absence of a fixed voting quota, this rule *counts the votes only of those voting*, viz., abstention is treated as non-participation.[4] From the perspective, then, of the decision rule, to abstain is tantamount to making the assembly incomplete, which for weighted games, will re-scale the relative voting weights. The corollary is that a voting game that does not assign an abstention to either 'yes' or 'no' is a collection of possible games on possible assemblies. Essentially this amounts to assuming that in contrast to F&M's simultaneous choice structure, the choice is sequential: first voter i chooses to participate (or not as the case may be) in the division and then to chooses between 'yes' and 'no' (Figure 1b).[5] The upshot is that from an a priori standpoint voter i in an assembly of, for instance, $n = 3$ faces not 9 tripartitions where he or she could be positively (or negatively) critical but 8 sets of bipartitions (13 bi-

[4] A good example of this can be found in modern bankruptcy law in the UK and Germany where there are provisions for voting on so-called insolvency plans. In each jurisdiction, it is explicitly stated that majority quota is calculated only on the basis of participation in the division, which means casting a 'yes' or 'no' vote. See, Halsbury's Statutory Instruments (1991, pp. 257–8) and Balz and Landfermann (1999, pp. 497–8).

[5] The sequential structure is, in fact, more in line with the economic theory of voting than the simultaneous structure. See Downs (1957) and the extensions of the basic model in Riker and Ordeshook (1968). See also Feddersen and Pesendorfer (1999) and Rothenberg and Sanders (1999). An empirical test of the sequential structure can be found in Thurner and Eymann (2000). Note that the sequential structure is what is *observed*, not what actually happens. The decision to vote 'yes' or 'no' may have been made prior to the decision to 'abstain' or 'participate'. Reversing the sequence of decisions has no bearing upon our analysis.

partitions in all); or in terms of the roll-call model, they face not a ternary roll-call space of 162 possible roll calls where he or she may be pivotal, but 8 sets of binary roll-calls with 78 binary roll calls in all.

In the second place, the assumption of a simultaneous choice structure – irrespective of whether or not the a priori probabilities for each option are symmetric or asymmetric – does not come without a major philosophical difficulty, and that is the TVG analogue of the Shapley-Shubik index (or any other roll-call model for that matter) allows abstainers (non participants) to be pivotal. Whether this is defensible is clearly a moot point, and while this is not the place to examine the matter in depth, we would like to point out that Morriss (1987), in one of the more conceptually probing studies of power, is quite adamant that this is mistaken. In discussing his own and very loose attempt to derive an index of voting power that accounts for abstentions, he writes: '… a member who never takes his seat or votes can be ignored' (p. 171) and,

> … when you abstain, you are showing no interest in the outcome, and so what the eventual outcome is of no concern to us. We measure the power given by resources (here, votes) by seeing what you can obtain when you use these resources; what happens when you let them lie idle is not of any relevance (p. 173).

While it could be replied that the onus is equally on Morris or ourselves as to why power should not be assigned to abstainers given that they do affect outcomes, it should be noted that F&M's approach in this regard is not consistent: the TVG analogues for the Banzhaf indices do not assign a swing to abstainers. It could be argued, then, that on heuristic grounds alone, the sequential structure has the advantage of making the Shapley-Shubik and Banzhaf indices conceptually consistent in this respect.

4. Abstention Voting Games

In this section we formalize the rough notion introduced above that a decision rule that permits abstentions as a *tertium quid* is a collection of possible games played on possible assemblies. The gist of the idea is as follows: we have a fixed assembly N in which a certain number of voters abstain (do not participate in the division), this means we are in a situation where the remaining members are playing a binary voting game. Such a game we call a *possible game* and we denote the collection of possible games as an *abstention voting game* (AVG).

Definition 4.1 (i) Denote the power set of an assembly N, $2^{|N|}$, containing all possible subsets of members of N by $W(N) = \{W_g \mid (W_g \subseteq N)\}$, then a possible game is given as follows: if $\boldsymbol{\Gamma}$ is a SVG on N, then any game $\boldsymbol{\Gamma}_g$ on W_g is called a possible game on N.
(ii) Any possible game $\boldsymbol{\Gamma}_g$ is either itself a SVG or a degenerated simple voting game (DSVG) which is an 'extended' SVG where either all coalitions are winning or losing, i.e. $S \in \mathcal{W}\ \forall\ S \subseteq W_g$ or $S \in \mathcal{L}\ \forall\ S \subseteq W_g$, respectively.

Remark 4.1 (i) A DSVG is characterized by the absence of a critical voter in the assembly. This can occur in: (a) the extreme case where the assembly is the empty set which is either winning or losing;[6] or (b) where the assembly is not the empty set. An example of the latter is given in F&M (1997, p. 342): let $N = \{a, b, c\}$ and let the decision rule be that a resolution is carried if a supports it and at least one of the other two does not oppose it. If a abstains then we obtain a DSVG with assembly $\{b, c\}$ in which all coalitions lose (since it is given that a does not vote 'yes').
(ii) For the special case of a WVG, where voter i is assigned a fixed positive weight, each possible game $\boldsymbol{\Gamma}_g$ is characterized by different relative weights, i.e. the initial relative weights are re-scaled.

Applying Definition 4.1, we can define an AVG as:

Definition 4.2 An AVG $\mathcal{A}$ is a collection of possible games $\boldsymbol{\Gamma}_g$ such that $\mathcal{A} = \{\boldsymbol{\Gamma}_g \mid g = 1, 2, \ldots, 2^n\}$.

Remark 4.2 (i) To summarize this apparatus we can say that an AVG $\mathcal{A}$ is bundle of possible games $\boldsymbol{\Gamma}_g$ in the following sense: if the set of all voters is N, then from the moment the subset of non-abstainers W_g is specified we have a possible game $\boldsymbol{\Gamma}_g$ whose assembly is W_g. In a nutshell, an AVG $\mathcal{A}$ can be regarded as a family of possible games indexed by the subsets of N.[7]

[6] If the rule does not actually say what the outcome is under this circumstance, the game is, strictly speaking, not properly defined; or one could say that no decision is made, which is the same as if the assembly has not met at all.

[7] We are presupposing here that the decision rule *is* specified for the case of abstentions, which may not be the case. It is possible that the rule is given in an 'incomplete' or ambiguous form such as 'a simple majority', i.e. without defining the set of voters upon which the majority quota is calculated. In such cases one has to turn to supplementary material to find out how the decision rule has been interpreted with regard to abstentions. F&M (1997, pp. 348–350) discuss such cases.

(ii) An AVG can be considered as a generalization of a SVG, i.e. the 'canonical' assembly of size n is but a special case.[8]
(iii) Like a SVG, an AVG is a bare a priori structure, i.e. it rules out all considerations based on information not provided by the framework, still only classifying coalitions into winners and losers. Neither the propensities to abstain or vote 'yes' or 'no' are part of the definition.

By Definition 4.2 and using the very general definition of voting power for SVGs given by Definition 2.3, we define a measure of voting power for an AVG as:

Definition 4.3 For any AVG $\mathcal{A}$, a measure of voting power for voter i, $\xi_i(\mathcal{A})$, is the expected value of $\xi_i(\boldsymbol{\Gamma}_g)$, viz., $\mathrm{E}[\xi_i(\boldsymbol{\Gamma}_g)] = \sum\{a_g\, \xi_i(\boldsymbol{\Gamma}_g) : W_g \ni i\}$ with $a_g \geq 0$ and $\sum\{a_g : g = 1, 2, \ldots, 2^n\} = 1$ where a_g is a probability weighting on $\boldsymbol{\Gamma}_g$.

Remark 4.3 (i) As $\varnothing \in W(N)$ it follows that in general $\sum\{\xi_i(\mathcal{A}) : i \in N\} \neq 1$ if $a_g > 0$ for $\boldsymbol{\Gamma}_g(\varnothing, \mathcal{W})$. This violates Shapley's (1953) 'efficiency' axiom if $\xi_i(\mathcal{A})$ is an expected index, e.g. the expected Shapley-Shubik index.
(ii) The idea of an expected measure of voting power has already been mentioned by Dubey and Shapley (1979) and Holler (1985). And as Holler (1987, p. 430) has noted, if the rules of probability calculation apply to ξ, then $\xi_i(\mathcal{A})$ is characterized by the properties of the original index.

We specify the probability weighting a_g that a possible game $\boldsymbol{\Gamma}_g$ occurs by:

Definition 4.4 Let f be a mapping such that $f : \boldsymbol{\Gamma}_g \to a_g$ with $a_g \in [0,1]$ and $\sum\{a_g : g = 1, 2, \ldots, 2^n\} = 1$. The vector $\boldsymbol{a} = \{a_g \,|\, g = 1, 2, \ldots, 2^n\}$ is called the abstention vector.

Remark 4.4 (i) The function f maps to each possible game $\boldsymbol{\Gamma}_g$ on W_g in an AVG $\mathcal{A}$ the probability a_g that this particular game will occur. In other words, a_g is the device or indicator, that determines the probability with which each possible game of the bundle occurs: $a_g = \prod\{1 - a_i : i \in W_g\}\prod\{a_i : i \in N - W_g\}$, where $a_i \in [0,1]$ is the probability that voter i abstains in an AVG $\mathcal{A}$.
(ii) The probability weighting a_g is a priori in the sense that it does not in and of itself contain any empirical information or assumptions about voters intentions or motivations to abstain.

[8] Contrast with the sense in which F&M (1997, p. 338) consider a TVG as a generalization of a SVG. Under this structure a SVG can be considered as a somewhat degenerate TVG which conflates abstention with 'no' or 'yes'. Also contrast with a TVG being a special case of a further generalisation as in Remark 2.4 (ii).

It is straightforward to see that definitions 4.3 and 4.4 are insufficient for calculating voting power. To do so we must flesh out a_i, i.e. we need to make assumptions about the abstention behaviour of each voter in order to obtain the elements of the abstention vector a. From an a priori standpoint, there are two common assumptions that we can borrow and apply from the probabilistic interpretation of power indices (Straffin, 1977): the *independence* and the *homogeneity* assumptions.[9]

Assumption 4.1 The independence assumption implies that each a_i $\forall i$ is chosen *independently* from the uniform distribution on [0,1], i.e. the abstention decision of voter i has nothing to do with the abstention decision of voter j.

Remark 4.5 The independence assumption implies that we assign each a_g (that means each of the 2^n possible games) the same weight, i.e. a complete or an empty assembly occurs with the same probability as a partial assembly. Thus for calculating the voting power of an AVG $\mathcal{A}$ we assign each $\xi_i(\boldsymbol{\Gamma}_g)$ the same weight, thereby reducing $a_i = \prod\{1 - a_i : i \in W_g\}\prod\{a_i : i \in N - W_g\}$ to $1/2^n$. Hence, $\xi_i(\mathcal{A}) = 1/2^n \sum\{\xi_i(\boldsymbol{\Gamma}_g) : W_g \ni i\}$.

Assumption 4.2 The homogeneity assumption implies that each $a_i = g$ $\forall$ i and g is chosen from the uniform distribution on [0,1], i.e. all voters have the same probability to abstain for a random resolution, but g varies from resolution to resolution.

Remark 4.6 With $a_g = (g, g, \ldots, g)$ the homogeneity assumption assigns to each possible game $\boldsymbol{\Gamma}_g$ a weight of $(1 - g)^{w_g} \cdot g^{n-w_g}$. The expected power of voter i is, therefore, $\xi_i(\mathcal{A}) = \sum\{\int_0^1[(1 - g)^{w_g} \cdot g^{n-w_g}]\, dg\, \xi_i(\boldsymbol{\Gamma}_g) : W_g \ni i\}$, where the integral is the beta-function $\mathcal{B}(n - w_g + 1, w_g + 1)$. The expected power of voter i becomes: $\xi_i(\mathcal{A}) = \sum\{w_g!\,(n - w_g)! \,/\, (n + 1)! \cdot \xi_i(\boldsymbol{\Gamma}_g) : W_g \ni i\}$. Note that in contrast to the independence assumption, the homogeneity assumption leads to different weightings for each of the 2^n possible games (due to the different size of W_g), although the complete assembly and empty assembly have the same weight.

One general remark needs to be made that applies to both the independence and homogeneity assumptions. As noted in Remark 4.3, if $a_g > 0$ for $\boldsymbol{\Gamma}_g(\varnothing, \mathcal{W})$ then $\sum\{\xi_i(\mathcal{A}): i \in N\}$ will not in general sum to 1 if $\xi_i(\mathcal{A})$ is an *index*. It is easy to

[9] The two assumptions of independence and homogeneity can be considered as extreme cases of Straffin's (1977) more general framework of partial homogeneity. Formally, a partial homogeneity structure on N is a partition $\mathcal{P} = \{S_1, \ldots, S_m\}$ of N into disjoint subsets. If $\mathcal{P}$ is the discrete partition of N into one-voter subsets we have the independence assumption; if $\mathcal{P}$ is the indiscrete partition $\mathcal{P} = \{N\}$, we have the homogeneity assumption.

see that under both assumptions as n increases the probability of $\boldsymbol{\Gamma}_g(\varnothing, \mathcal{W})$ decreases so that at the limit $\sum\{\xi_i(\mathcal{A})\colon i \in N\}$ approaches 1, therefore fulfilling Shapley's (1953) second or 'efficiency' axiom. The assumptions differ, however, in the weight assigned to the empty and complete assemblies for $n > 1$: the independence assumption always assigning less weight than the homogeneity assumption.

5. A Prioricity

Astute readers will note that the AVG framework fails to provide a straightforward answer to the obvious question: which assumption on $\boldsymbol{a}_i$ should be applied? The problem is rather complicated as it touches upon some philosophical subtleties that require much more extensive analysis than that which we can provide here. What we can do, however, is sketch a line of reasoning that rejects the preference for one or the other assumptions on purely formal a priori grounds. Thus we remain, for the time being, agnostic on this matter, leaving a positive decision to future investigations.

A simple and attractive line of thought – and one that is repeatedly pointed out in the literature, or at least implicit in it – postulates a prioricity as *the* desideratum of a measure of voting power. The idea can be found in the original papers by Shapley and Shubik (1954), Banzhaf (1965) and Coleman (1971). As Roth (1988, p. 9) puts it:

> Analyzing voting rules that are modelled as [SVGs] abstracts from the particular personalities and political interests present in particular voting environments, but this abstraction is what makes the analysis focus on the rules themselves rather than on the other aspects of the political environment. This kind of analysis seems to be just what is needed to analyze the voting rules in a new constitution, for example, long before the specific issues to be voted on arise or the specific factions and personalities that will be involved can be identified.

Commenting on the passage quoted, F&M (1998, p. 20) point out that a SVG is an 'abstract shell, *uninhabited* by real agents, with real likes and dislikes, attractions, and repulsions'. Thus they insist that a truly a priori measure of voting power must not presuppose any specific information as to the interests of the voters or the affinities and disaffinities between them. By implication if $\xi_i(\mathcal{A})$ is to fulfil such a criterion, assumptions on $\boldsymbol{a}_i$ must too be a priori, something that is easy to accept if we believe that measures of voting power are to have any prescriptive application (e.g. designing a new constitution). In this regard a prioricity creates a 'level playing field', i.e. it is an impartiality criterion.

The problems arise, however, by the insistence – be it by F&M, Coleman (1971, p. 297), or others – that (in the context of voter propensities to choose 'yes' or 'no') only the independence assumption is truly a priori; the homogeneity assumption apparently being less so because it assumes affinity (or positive correlation) between voters. To put it another way, the independence assumption supposedly maximizes our ignorance regarding such affinities, although it does so at an equally high cost to that of the homogeneity assumption. Independence implies 'strict inaffinity' between voters, which is no less a strong assumption than that of affinity.[10]

While the defence of the independence assumption on the basis of a priori ignorance at first sight appears compelling, it has a snag: it is true if, and only if, we accept a particular metaphysical presupposition. That presupposition is, for want of better terminology, what Rae (1969, p. 42) has called *political individualism*, which is a model of a generic voter who is a discrete entity with a unique individuality and one that can make 'sincere' choices, in as much as a choice (whatever it is, be it selfish or altruistic) is made on the basis of what that individual thinks alone about the issue at stake and not what others think.[11] To give a literary flourish to this generic figure, voter *i* is a person who fulfils J. S. Mill's ideal of liberty: as one who, to paraphrase Isaiah Berlin's (1969, p. 160) description, is bold and non-conforming, who assets his own values in the face of the prevailing opinion; is a strong and self-reliant personality free from the leading strings of the instructors of society.[12]

This supposition, which is embedded in the insistence that a prioricity is the absence of relations between individuals, has its attractions and indeed is implicit in much of the analysis of what we call 'constitutional choice'. It is, however, essentially a normative principle and one that we do not necessarily have to hold. We could easily argue with the implicit script in Mill's ideal of liberty: that we are initially characterized by – or have tendency to – conformity from which we free ourselves. That is, political individualism is an ideal or end and not the starting point. But even if we wish to operate from ideals or the ends of life and character-

[10] The author owes this insight to Manfred Holler.

[11] This is what F&M (1998, pp. 18, 36) term as 'policy seeking' and use as the basis of their concept of I-power, or power as influence.

[12] As Rae (1969, p. 41) puts it, this generic voter 'wants to have his way by defeating proposals which his values lead him to dislike and by imposing those which they lead him to like'.

ize voter *i* as individualistic, we are not by necessity wedded to the independence assumption. For in place of political individualism we could have *Kantian individualism*: that voter *i* will behave in accord with what his 'rational will' ('real self') commands, which is to obey the categorical imperative. While the Kantian individualist may proclaim that he alone has given himself the order which he obeys, the fact is all rational individuals will choose likewise. Thus, starting with Mill's or with Kant's individualism we can still end up with homogeneity as the truly a priori assumption.

The point is we do not appear to be entitled to endorse one assumption or another as being a priori without making explicit our presuppositions. As a simple thought experiment, imagine you are told only that there is a group of voting entities behind the door. No other information is imparted. What grounds do you have for assuming that they will on average be as likely to behave in one way as the other (abstain or participate, or vote 'yes' or 'no') rather than assuming that on average that their behaviour will be highly correlated? The answer presents itself on its own: either you impute from your experience that voting entities behave one way or another, but then this is a posteriori information; or you resort to metaphysical presuppositions about the nature of voting entities. In short: voting games may, to borrow F&M's words, be uninhabited by real agents, but it is certainly inhabited by metaphysical ones.

To put the reasoning otherwise: the two assumptions differ not in the degree of a prioricity, but in their qualitative nature. For neither assumption specifies a particular value on a_i; each is only an assumption *about* behaviour – independence or homogeneity – and a *form of the distribution* of that behaviour (that it is uniform). As Dubey and Shapley (1979, p. 103) acutely observed, it is mistaken to believe that the independence assumption underpinning the Banzhaf indices is equivalent to being one of 'no assumption'. And nor can we argue as Leech (1990) has attempted to do, that the default assumption is independence (and therefore the true a priori one) because it can be expressed in a much weaker form than the homogeneity assumption. The argument is, in essence, that the condition that the probabilities are drawn from the uniform distribution is not necessary for the independence assumption: only that the distribution has a mean of 1/2 and that the probabilities are drawn independently. The homogeneity assumption, pace Leech, seemingly requires a specific form of distribution (that it is uniform), although Straffin (1978, p. 495) has argued otherwise. The seepage in Leech's argument is that it is not clear that there is any qualitative difference between defining the independence assumption on the basis of (i) independence and the form of the distribution; or (ii) independence and specifying the mean. Unless there is a difference, then there is no qualitative difference between homogeneity and independence in terms of the quantity of presupposed information, absence of which is the defining trait of a prioricity.[13]

Thus our AVG framework is quite independent from any suppositions about the generic voter. This ambivalence is certainly not without its problems, because it begs the question about which assumption should be applied on a_i for calculating voting power. While it is possible to argue that if we apply Straffin's (1977) probabilistic interpretation of voting power we should choose the assumption on a_i that is concordant with the assumptions on the propensity to vote 'yes' or 'no', i.e. homogeneity with homogeneity, independence with independence, this does not appear to us to be so straight-forward if abstention, as we have assumed, is structurally distinct from the 'yes/no' decision.

6. Conclusion

There is no need to summarise this chapter; the ideas are simple enough to stand on their own. In tying up this chapter we wish, however, to add two further remarks that concern further developments.

Firstly, our AVG structure is a fairly rough cut. In particular we have not imposed any internal consistency requirements that connect the possible assemblies in a non-arbitrary way and the weightings of the possible assemblies. What is also left to be explored is how the AVG structure can be further generalized possibly along the lines of the idea of multiple degrees of support (Freixas and Zwicker, 2000) as noted in Remark 2.4 (ii), for which F&M's TVG structure easily fits.

Secondly, there remains a much larger conceptual issue to be tackled: are the different frameworks comparable? That is, does it makes sense to pronounce that a measure of voting power for an AVG assigns to voter *i* this or that amount of power more or less than assigned by a TVG analogue?[14] One has to bear in mind

[13] In fact this conclusion is also reached in Felsenthal *et al*. (1998, p. 106) where they describe the independence and homogeneity assumptions in terms of entropy and show that both can achieve maximal entropy, and by analogy, maximal a priori ignorance. The homogeneity assumption implies, however, that voters are indistinguishable clones. That voters may not be is, by our reasoning, either a metaphysical presupposition or an empirical imputation.

[14] The results can differ by a fair amount. Consider the WVG [51; 50, 25, 25]. The absolute Banzhaf index under the TVG structure assigns 0.89, 0.11, and 0.11 respectively while under the AVG structure (assuming that each voter abstains with probability 0.5) the values are 0.94, 0.06, and 0.06 respectively.

that the different choice structures underlying the different approaches imply different games, although these games may derive from the same decision rule: one that permits abstention.[15] However, note that Lindner (2001) has shown that in those cases where $n > 15$ the results derived either from F&M's TVG or our AVG framework converge to the results of the 'usual' binary measures, put it in other words: abstention as a tertium quid has only an effect in small assemblies with less than 15 voters.

[15] See Lindner (2001) for an initial attempt to bring both the AVG and TVG structures into a single framework.

Chapter 4

LOCAL MONOTONICITY AND STRAFFIN'S PARTIAL HOMOGENEITY APPROACH TO THE MEASUREMENT OF VOTING POWER*

Abstract: There is a fundamental and on-going debate in the literature on voting power about what constitutes a 'reasonable' measure of a priori voting power. A central topic in this debate is whether or not a reasonable measure of voting power should fulfil local monotonicity (LM). While the Shapley-Shubik index and the Penrose/Banzhaf or Coleman measures are locally monotonic, the Deegan-Packel and the Public-Good measures are not. While some authors argue that the violation of LM is 'pathological' and thus measures of voting power that exhibit such behaviour are unreasonable, others say that the violation of LM is a simple social fact of power and, therefore, LM cannot be used to determine the reasonableness of a measure of voting power. However, so far the debate has ignored the violation of LM by another set of measures derived from Straffin's partial homogeneity approach. By examining violations of LM in this context it is shown that the different sides to this debate are in a sense 'both wrong'. It is argued that LM is a special case of a more general monotonicity condition that relates 'resources' to 'power'; in LM the resources are but the voting weights. However, given that it is not clear that a priori voting power is based on, and only on, the vector of voting weights and the decision rule, it turns out that a violation of LM can be 'reasonable'. This, however, does not imply that power is not monotonic in resources per se.

1. Introduction

The measurement of voting power is an important and established method for analysing the structural properties of collective decision making rules that can be modelled as a *simple game*, a mathematical structure that goes back to von

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* This chapter is based upon Braham and Steffen (2002c). The research on which this chapter is based has greatly benefited from intensive discussions with Manfred Holler and Moshé Machover. An early forerunner of the paper was presented at the Institute of SocioEconomics research seminar in June 2001.

Neumann and Morgenstern's tome, *Theory of Games and Economic Behavior* (1944). A simple game is one in which we can classify the players into sets of winning and losing coalitions and a measure of voting power essential counts the relative frequency that a player can change a winning coalition into a losing one.

There is, however, a fundamental and on-going debate in the literature – not unlike that found in the freedom of choice literature[1] – about what constitutes a 'reasonable' measure of *a priori* voting power, i.e. the power that each player has *ex ante*. The reason is in part due to the fact that there is as yet no intuitively *compelling* and *complete* set of axioms that *uniquely* characterize a measure with the result that there are a variety of different measures that not only give different cardinal values but also different ordinal rankings of players. And it is due in part to confusion about the nature and meaning of the term 'power' itself.

A central topic in this debate is whether or not a reasonable measure of voting power should fulfil *local monotonicity*. This is a postulates which says that in weighted voting games – simple games characterized by a vector of voting weights attached to each player and a quota – if a player i has at least as much weight as a player j, then player i should have at least as much power as player j. While the Shapley-Shubik (1954) index and the Penrose (1946)/Banzhaf (1965) or Coleman (1971) measures are locally monotonic, the Deegan-Packel (1978) index and the Public-Good Index (PGI) (Holler 1982a; Holler and Packel 1983) are not.

Freixas and Gambarelli (1997) and Felsenthal and Machover (1998) have taken the position that local monotonicity is such an intuitively compelling postulate that any measure that violates cannot be used as a reasonable yardstick of voting power.[2] This would mean that the Deegan-Packel index and PGI in a sense suffer 'pathological' defects.

On the other hand, Deegan-Packel (1978, 1983) and Holler (1997, 1998) as well as Brams and Fishburn (1995) take the position that if the rationale or 'story'

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[1] We are referring here to the debate about appropriate axioms for characterizing a measure of freedom. See, for example Jones and Sugden (1982), Pattanaik and Xu (1990, 1998), Sen (1991), (Sugden 1985), van Hees (2000).

[2] Actually the importance of local monotonicity as axiom or 'postulate' of power was noted already by Allingham (1975), although he did not take such a 'strong' position to that of Freixas and Gambarelli (1997) and Felsenthal and Machover (1998).

of a measure is reasonable and acceptable, then we are forced to accept that power is not locally monotone and that this is an inescapable fact of power being a social phenomenon (they cite empirical evidence to this effect). The underlying argument being that it is mistaken to take an axiomatic approach to the analysis of social interaction.

What this debate has ignored is a general violation of LM by another set of measures derived from Straffin (1977). This is a probabilistic interpretation of voting power based on Owen's (1972, 1975) *multilinear extension* (MLE) of a game. As is well known, the Shapley-Shubik and absolute Banzhaf indices can be derived as special cases of the MLE given a probability model of voter behaviour. Straffin's partial homogeneity approach allows us to mix these probability models so that we can derive an infinite set of families of power measures.

The reason for the violation of local monotonicity by the family of power measures derived by Straffin's approach is wholly different to that of the violation by the Deegan-Packel index and the PGI. In the latter case the reason is due to the fact that the measures are based only on *minimal winning coalitions* (coalitions in which no proper subset are winning), i.e. the domain of all conceivable winning coalitions is restricted to the set of all *minimal winning coalitions*. That is, these measures ignore certain coalitions in which i is critical (i.e. without i the coalition is losing) either on the grounds that these coalitions will not form or because they should be ignored (the rationale for this is given in section 2). While in the former case, the violation of local monotonicity is due to the fact that the partial homogeneity approach does not treat each coalition equally likely, with the result that coalitions are not equally weighted. Thus, there is no restriction of the domain of all conceivable winning coalitions in this case. Under Straffin's approach, the power of a player i depends not only upon the coalitions in i is critical but also upon the probability that such a coalition arises which is a function of voter propensities to vote 'yes' or 'no'. The greater the probability of a coalition arising in which i is critical, the larger is i's power.

Although at first sight it appears quite reasonable to measure a player's voting power as a function of being critical and of the probability such a critical coalition arising it is in fact inconsistent with the postulate of symmetry: that the *a priori* voting power of a player i should depend on, and only on, the position of that player in a game. In other words, such a measure should rule out by default all information not provided by the framework of winning and losing coalitions. A measure that goes beyond this sparse informational framework, such as including data on player preferences, is usually called *a posteriori* voting power. The notion of *a priori* power as a sparse framework can be found in the original papers by Shapley and Shubik (1954), Banzhaf (1965) and Coleman (1971). As Roth (1988, p. 9) puts it:

> Analyzing voting rules that are modelled as [SVGs] abstracts from the particular personalities and political interests present in particular voting environments, but this abstraction is what makes the analysis focus on the rules themselves rather than on the other aspects of the political environment. This kind of analysis seems to be just what is needed to analyze the voting rules in a new constitution, for example, long before the specific issues to be voted on arise or the specific factions and personalities that will be involved can be identified.

Commenting on the passage quoted, Felsenthal and Machover (1998, p. 20) point out that a simple game is an 'abstract shell, *uninhabited* by real agents, with real likes and dislikes, attractions, and repulsions'. It is for this reason that they insist that a truly *a priori* measure of voting power must not presuppose any specific information as to the interests of the players or the affinities and disaffinities between them. According to this position, Straffin's partial homogeneity approach cannot be considered as truly *a priori* and as a corollary its violation of local monotonicity is irrelevant to the nature of *a priori* voting power. (This may explain why it has so far been ignored in the monotonicity discussion.)

Such a position – although widely held – is mistaken because it is not in fact possible to calculate voting power merely on the collection of winning coalitions. As we will argue, the classical measures of voting power *do* presuppose more information than simply the collection of winning and losing coalitions. This additional information can be interpreted as a combination of a presupposed probability model of voting behaviour (this is not a new insight) *and* a particular type of decision-making structure (this is a new insight). And it is only a particular combination of the probability model and decision-making structure that will guarantee fulfilment of the postulates of symmetry and local monotonicity (if one applies the probability model).[3] Once this is recognized, it is easy to see that Straffin's partial homogeneity approach is not necessarily any less *a priori* than say the Shapley-Shubik index or the Banzhaf measure. Thus the violation of local monotonicity by Straffin's approach *is* relevant to the nature of *a priori* voting power. Its relevance can be stated as follows: neither symmetry – or its more general form, *iso-invariance* – nor local monotonicity are compelling postulates or axioms of *a priori* voting power.

However, having done away with local monotonicity, we argue – some what paradoxically it may seem – that the position taken by Deegan and Packel (1978, 1983), Holler (1997, 1998) and Brams and Fishburn (1995) that power must be accepted to be not locally monotonic is not entirely correct either. Their position is essentially one of argument by analogy. Drawing on experimental and political and social evidence they say that the fact that a player *j* who has less weight than a player *i* in a weighted voting game can have more power than player *i* is simply an instance of the violation of local monotonicity in resources that we observe all the time. Put differently, these authors believe that power is not necessarily increasing in the 'resources', or to borrow Dahl's (Dahl, 1957) terminology, in the 'base of power'. Voting weights are just a particular kind of resource or 'base of power'.

The problem here is that once we take the position that calculating voting power actually presupposes a probability model and a decision-making structure, the resources or 'base of power' is no longer restricted to only the voting weights. These weights may be augmented by the assumptions about how players behave (whether or not their behaviour is correlated), which is contingent on the *a priori* incentive structures given by the decision-making structure. Hence a player *i* may have more weight than a player *j* but due to the incentive structures that govern coalition formation, *i* may be in a weaker position because certain coalitions where *i* is critical may have a smaller probability of occurring than the coalitions in which *j* is critical. In a sense a player *i*'s weight is modified by the number of other players with whom *i* is correlated. Thus a violation of local monotonicity as defined by voting weights only does not imply a violation of local monotonicity

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[3] Note that one can also take the position that the classical measures of voting power are just functions (recipes) which satisfy specific axioms. Following this line of thought, which is not done in this chapter, voting power is not (necessarily) a probabilistic concept.

when defined in terms of resources or the 'base of power' more generally. It is our belief that power *is* locally monotonic in this latter sense. As one can imagine, this throws up all sorts of theoretical problems because it requires a much more complicated definition of resources and a method for calculating the value of these resources in a voting game.

The primary contention of this chapter is that an examination of the general violation of local monotonicity by Straffin's approach throws light on the debate and on the nature of *a priori* voting power; and in particular on the problems of the 'axiomatic' or 'postulate' approach to the measure of voting power.

This chapter is organized as follows. section 2 reproduces the basic formal framework for simple games and the measurement of voting power (readers familiar with this material may wish to skip this section). Section 3 outlines the five basic postulates – local monotonicity is one of them – that are generally taken to be necessary for defining a reasonable measure of *a priori* voting power. Section 4 considers the derivation of local monotonicity from the more general desirability (also called dominance) relation and its connection to *a prioroicity*. In this section it is shown that despite the intuitive appeal of the desirability relation and, therefore, local monotonicity the arguments in its favour break down once we recognize that voting is embedded in particular a decision-making structures. In section 5 the more general issue of 'power and resources' is examined. It is here that we show that local monotonicity in voting weights is a special case of a more general local monotonicity based upon resources or the 'base of power' and that a violation of the former does not entail a violation of the latter. Section 6 concludes the chapter.

2. Simple Games and the Measurement of Voting Power

In order to develop our argument we need to restate the basic definitions of the theory of simple games and voting power. We refer the reader to Shapley (1962b), Felsenthal and Machover (1998), and Taylor and Zwicker (1999) for additional background and results.

The most important definition that we require is that of a *decision rule* which we will first formulate informally as follows. Let a n-member decision-making body be denoted by a set N. A decision rule specifies which subsets of N can ensure the acceptance of a proposal. Formally:

Let $N = \{1, 2, \dots, n\}$ be the set of players. $\wp(N) = \{0, 1\}^n$ is the set of feasible coalitions. The *simple game* v is characterized by the set $\mathcal{W}(v) \subseteq \wp(N)$ of *winning coalitions*. $\mathcal{W}(v)$ satisfies $\varnothing \notin \mathcal{W}(v)$; $N \in \mathcal{W}(v)$; and if $S \in \mathcal{W}(v)$ and $S \subseteq T$ then $T \in \mathcal{W}(v)$. Further, v can also be described by a characteristic function, $v: \wp(N) \to \{0, 1\}$ with $v(S) = 1$ iff $S \in \mathcal{W}(v)$ and 0 otherwise.

By $\mathcal{G}^N$ we denote the set of all such n-person simple games. *Weighted voting games* are a special sub-class of simple games characterized by a non-negative real vector $(w_1, w_2, \dots, w_n)$ where w_i represents player i's voting weight and a quota q which is the quota of votes necessary to establish a winning coalition, such that *quota* $0 < q \leq \sum_{i \in N} w_i$. A weighted voting game is represented by $[q; w_1, w_2, \dots, w_n]$.

Power, in the generic sense of an ability or capacity to determine an outcome, is represented in a simple game as the ability of a player i to change the outcome of a play of the game. We say that a player i who by leaving a winning coalition $S \in \mathcal{W}(v)$ turns it into a losing coalition $S \setminus \{i\} \notin \mathcal{W}(v)$ has a *swing* in S and is called a *critical member* of S. Coalitions where i has a swing are called *critical coalitions with respect to i*. Let us denote the set of critical coalitions w.r.t i as $\mathcal{C}_i$. A concise description of v can be given by a set $\mathcal{M}(v)$, where $S \in \mathcal{W}(v)$ but no subset of S is in $\mathcal{W}(v)$, i.e. all members of S are critical. We call such a coalition a *minimal winning coalition* (MWC). Further, we denote by $\eta_i(v)$ the number of swings of player i in v. Thus, $\eta_i(v) =_{def} |\mathcal{C}_i(v)|$. A player i for which $\eta_i(v) = 0$ is called a *dummy* in v, i.e. it is never the case that i can turn a winning coalition into a losing coalition (it is easy to see that i is a dummy iff it is never a member of a MWC; and i is a *dictator* if $\{i\}$ is the sole MWC).

A *measure of voting power* is a mapping $\xi: \mathcal{G}^N \to \mathbb{R}^n_+$ that assigns to each player $i \in N$ a number $\xi_i(v)$ that indicates i's power in the game v. As we have already mentioned in the introduction, there are a number of well known measures, namely, the Shapley-Shubik index, the Banzhaf index, the Deegan-Packel index, and the Holler-Packel or Public Good Index.

The Shapley-Shubik (1954) index (S-S) is a special case of the Shapley (1953) value for cooperative games. In this measure power equals the relative number of pivotal ('swing') positions of a player *i* in a simple game *v* assuming all player permutations are equally probable. The idea (or 'story') is that the players line up to vote yes and the player that turns a losing coalition into a winning coalition is the pivot ('swing'). The S-S is given by:

$$f_i(v) =_{def} \sum_{\substack{S \in \mathcal{W} \\ i \in S \\ S \setminus i \notin \mathcal{W}}} \frac{(|S|-1)!(n-|S|)!}{n!}$$

Whereas the S-S is concerned with the order in which a winning coalition may form, the Banzhaf (1965) index (Bz) examines any winning coalition, irrespective of the order in which it may be formed and considers any player to have power from having a swing in it.[4] The Penrose/Banzhaf measure (which is also known as the 'absolute' or non-normalized Bz) is given by:

$$b_i'(v) =_{def} \frac{h_i(v)}{2^{n-1}}$$

The Bz is obtained by normalization:[5]

$$b_i'(v) =_{def} \frac{h_i(v)}{\sum_{j=1}^{n} n_j(v)}$$

The Deegan-Packel (1978) index (D-P) is based on three assumptions: that only MWCs will form; all MWCs are equally likely; and the MWC that is formed will divide the payoff equally among its members. Subject to these assumptions, the D-P index assigns to each player power proportional to the player's expected

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[4] The Banzhaf measure is in fact a rediscovery of Penrose (1946) and was later independently rediscovered by Rae (1969) and Coleman (1971). A history of the measure of voting power is contained in Felsenthal and Machover (1998).

[5] Here we follow Felsenthal and Machover (1998) and reserve the term 'index' for measures in which $\sum_{i \in N} \xi_i(v) = 1$. See section 3.

payoff. Denote by $\mathcal{M}_i(v)$ the set of MWCs to which player *i* belongs. The D-P index is given by:

$$r_i(v) =_{def} \frac{1}{|\mathcal{M}(v)|} \frac{1}{\sum_{S \in \mathcal{M}_i} |S|}$$

The Public Good Index (PGI) (which is also known as the Holler-Packel index) (Holler, 1982a; Holler and Packel, 1983) is also based on MWCs, although the story is different. Whereas the D-P index is based sharing the spoils of victory, the PGI is based upon the essential characteristic of a public good: non-rivalry in consumption and non-excludability in access. Thus if the outcome of a game *v* is the provision of a public good, each member of the winning coalition will receive the undivided value of the coalition. Only MWCs are taken into account not because winning coalitions with excess players will not form, but when it comes to the provision of a public good they will only form by sheer 'luck' because of the potential for free-riding.[6] Assuming that all MWCs are equally likely, the PGI is given by:

$$h_i(v) =_{def} \frac{|\mathcal{M}_i(v)|}{\sum_{j=1}^{n} |\mathcal{M}_j(v)|}$$

The non-normalized or 'absolute' PGI is given by:

$$h_i'(v) =_{def} \frac{|\mathcal{M}_i(v)|}{|\mathcal{M}(v)|}$$

Power in simple games can also be modelled in probabalistic setting. As we have already said in section 1, this is what Straffin's (1977, 1988) *partial homogeneity approach* is all about. It is a particular interpretation and extension of Owen's (1972, 1975) *multilinear extension* (MLE) of a game *v*.[7]

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[6] This rationale is based on Barry (1980a, 1980b).

[7] See also Laruelle and Valenciano (2001a) for an attempted synthesis of the probabilistic models.

Instead of deterministic coalitions $S \subseteq N$ that correspond to corner points $s \in \{0, 1\}^n$ of the n-dimensional unit cube, one considers random coalitions $\mathcal{S}$ represented by the points $p_i \in [0, 1]^n$ anywhere in the cube. Each p_i is interpreted as the probability of a player i deciding in favour of a random proposal or participating in a random coalition; p_i is also known as a player's *acceptance rate*.

Assuming that acceptance decisions are independent, the probability **P** of a given coalition $S \subseteq N$ is $\mathbf{P}(\mathcal{S}=S)=\prod_{i\in S} p_i \prod_{j\notin S}(1-p_j)$. If we extend the characteristic function v of a simple game by weighting each $v(S)$ with the respective probability of formation, we obtain the MLE $f:[0, 1]^n \rightarrow [0, 1]$ of a game v:

$$
\begin{aligned}
f(p_1,\ldots,p_n) &= \sum_{S\subseteq N} \prod_{i\in S} p_i \prod_{j\notin S}(1-p_j)v(S) \\
&= \sum_{S\in \mathcal{W}(v)} \prod_{i\in S} p_i \prod_{j\notin S}(1-p_j)
\end{aligned}
$$

For fixed acceptance rates, the MLE gives the probability that a winning coalition S will form in v, and thus the expected value of v. The partial derivative $\partial f/\partial p_i$ of v's MLE w.r.t to p_i is called by Straffin (1977, 1988) a player's *power polynomial*, which we denote by f_i.

$f_i(p_1, \ldots p_n)$ is, then, the probability of i having a swing (i.e. having power in the generic sense) in a random coalition in a game v. If a player's acceptance rates are themselves random variables with a joint distribution P, the expectation $Ef_i = \int f_i(p_1, \ldots p_n)dP$ is i's power in a game v. The probabilistic *measure of power* $Ef_i(v)$ coincides with some of the classical measures under different probability models.

Independence $p_i \sim U(0, 1)\ \forall\ i \in N$ (A1)

i.e. the decision of i has nothing to do with decision of j.[8]

Homogeneity $t \sim U(0, 1), p_i = t\ \forall\ i \in N$ (A2)

i.e. each i approves or rejects a proposal with the same probability t but t varies from proposal to proposal.

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[8] Actually one does not necessarily need the uniform distribution. Leech (1990) has shown that distribution must only have a mean of 0.5.

It is a well-known result from Straffin that applying (A1) we obtain the Penrose/Banzhaf measure; applying (A2) we obtain the S-S; and as Brueckner (2001) has shown (A1) in combination with counting only MWCs (i.e. $\mathcal{M}_i(v)$) gives the non-normalized or absolute PGI.

It is easy to see that this probability model is extremely flexible and allows us to create families of power measures that lie between the extremes of (A1) and (A2) by mixing these assumptions. This is what Straffin meant by *partial homogeneity structure* on N which is a partition $\mathcal{P} = \{G_1, \ldots, G_m\}$ of N into disjoint subsets. If $\mathcal{P}$ is the discrete partition of N into one-player subsets we have (A1); if $\mathcal{P}$ is the indiscrete partition $\mathcal{P} = \{N\}$, we have (A2). Formally,

Partial homogeneity $\mathcal{P} = \{G_1, \ldots, G_m\}$ (A3)

$$\mathrm{G_k} \cap \mathrm{G_l} = \varnothing \textit{ if } k \neq l, \bigcup G_k = N$$

$$t_k \sim U\,(0, 1), p_i = t_k \; \forall \; i \in G_k, \; k = 1, \ldots m$$

(Note that we will use $Ef_i(v)$ when referring to a measure derived from Straffin's approach and $\xi_i(v)$ for measures in general).

3. Postulates of Power

Given this variety of power measures and the fact there is as yet no intuitively *complete* and *compelling* set of axioms that uniquely characterize a measure but only *individual* axioms – some of which are compelling and others opaque unconvincing[9] – there has been a number of attempts to reduce the set of measures by eliminating those that violate certain properties that are considered intuitively reasonable for a measure of *a priori* power. Here the literature very much concurs on three basic postulates or properties.[10] An *a priori* measure of voting power $\xi_i(v)$ should at the very least satisfy:

Iso-invariance (P1)

if there is an isomorphism of v to v' that maps a player i to i', then $\xi_i(v) = \xi_{i'}(v')$.

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[9] See, for example, Straffin's (1983, pp. 292–297) discussion of the axiomation of the S-S and Bz.

[10] Postulates (P1)–(P3) appear in all axiomatisations of measures of voting power as well as in comparisons of the different measures, e.g. Allingham (1975), Felsenthal and Machover (1998), Freixas and Gambarelli (1997), Laruelle (1999), Straffin (1983).

Ignoring dummies (P2)

if v and v' have exactly the same MWCs, i.e. $\mathcal{M}(v) = \mathcal{M}(v')$, then $\xi_i(v) = \xi_i(v')$ for any player i common to both.

Vanishing for dummies (P3)

$\xi_i(v) = 0$ if i is a dummy in v.

(P1) generally trades under the name of *symmetry*, which is a special case of iso-invariance in which we have an automorphism of v (i.e. an isomorphism of v to itself). This postulate requires that $\xi_i(v)$ be symmetric (i.e. invariant under any automorphism): if players i and j have symmetric positions w.r.t to v they have equal power. Note that (P1) implies that $\xi_i(v)$ should depend only on the collection $\mathcal{W}(v)$ of winning coalitions and nothing more. Felsenthal and Machover (1995, p. 204) claim that to deny (P1) would be tantamount to denying that simple games provide an adequate framework for theorizing about *a priori* voting power. Felsenthal and Machover buttress their position by saying that all authors dealing with voting power within the framework of simple games implicitly if not explicitly accept (P1). However, as we will discuss later, (P1) is a very restrictive way of defining *a prioricity* – a restriction that is a cause for much confusion about the nature of *a priori* voting power per se. That is, *a priori* voting power can be shown to presuppose (P1) only under very specific conditions; and these are shown up under Straffin's partial homogeneity approach. This leads us to the conclusion that the deterministic simple game framework may not in fact be always adequate for theorizing about *a priori* voting power. This does not mean that we reject it outright; rather it means that it is too restrictive.

(P2) means that the value of ξ_i for any player i in the simple game v is unchanged if v is extended to v' by adding new dummy players (or equivalently, removing a dummy player from v will not alter the value of ξ_i. (P3) is obvious: dummy players have no power.

A forth postulate, that of normalization has also frequently been put forward:

Normalization (P4)

$\sum_{i \in N} x_i(v) = 1$.

The meaning of (P4) is straightforward: it is a way of answering questions like 'What fraction of the power in this game do I hold?' However, as a postulate of power it is not without technical difficulties because for the Bz it is not necessarily meaningful (Dubey and Shapley, 1979; Shapley, 1977) and in particular distorts the probabalistic interpretation of the Bz measure. Furthermore, in contrast to (P1)–(P3), (P4) is not without conceptual problems, in that there is no intuitive justification for saying that a measure of voting power ought – either naturally or artifically – to sum to unity (Laruelle and Valenciano, 1999, 2001b) and thus should not be used to eliminate a measure as being unreasonable.[11]

Finally, we come to the fifth and central postulate of this chapter:

Local monotonicity (P5)

If in a weighted voting game v, $w_i \geq w_j$ then $\xi_i(v) \geq \xi_j(v)$.

(P5) can be expressed in two ways: that a $\xi_i(v)$ should preserve the order of weights; or more rhetorically, 'having extra votes cannot hurt you, although will not necessarily help you.'

All the classical measures that we have listed above satisfy the first three postulates; normalization is naturally satisfied by the S-S and D-P indices (the Bz and PGI are 'normalized'); and the D-P index and the PGI violate (P5) as will in general the family of measures $Ef_i(v)$ that can be derived from Straffin's partial homogeneity structure as represented by (A3). For illustration, consider the following three examples:

Example 3.1 Assume the weighted voting game [51; 30, 26, 16, 12, 9, 7].

(i) The D-P index values are $\rho_1 = 0.23$, $\rho_2 = 0.18$, $\rho_3 = 0.21$, $\rho_4 = \rho_5 = 0.16$, and $\rho_6 = 0.07$.

(ii) The PGI values are $h_1 = 0.21$, $h_2 = 0.17$, $h_3 = 0.21$, $h_4 = h_5 = 0.17$, $h_6 = 0.08$.

(iii) Assume (A3) as follows: player 1 behaves independently, while players 2, 3,

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[11] Actually the problem is not restricted to voting power. There was a fair amount of controversy among political scientists and sociologists from the 1950s to the 1970s about whether or not power had a constant sum property. See Nagel (1973). The question has in fact been reincarnated in Felsenthal and Machovers's (1998) distinction between what they call 'power as influence' (I-power) and 'power as prize' (P-power), the latter is considered to be a zero-sum game while the former not.

and 4 form a standard t and players 5 and 6 form a standard (1-t). Then we have $Ef_1(v) = 0.40$, $Ef_2(v) = 0.47$ $Ef_3(v) = 0.38$, $Ef_4(v) = 0.30$, $Ef_5(v) = 0.12$, and $Ef_6(v) = 0.03$.

Example 3.2 Assume the weighted voting game [51; 30, 30, 18, 10, 9, 3].
(ii) The D-P index values are $\rho_1 = \rho_2 = 0.19$, $\rho_3 = 0.22$, and $\rho_4 = \rho_5 = \rho_6 = 0.13$.
(ii) The PGI values are $h_1 = h_2 = h_3 = h_4 = h_5 = 0.18$, and $h_6 = 0.09$.
(iii) Assume (A3) as follows: player 1 behaves independently, while players 2, 3, and 4 form a standard t and players 5 and 6 form a standard (1-t). Then we have $Ef_i(v)$ are $Ef_1(v) = 0.37$, $Ef_2(v) = 0.50$ $Ef_3(v) = 0.42$, $Ef_4(v) = 0.25$, and $Ef_5(v) = Ef_6(v) = 0.08$.

It is easy to see that the violation of (P5) – which we will henceforth denote as LM – by the Deegan-Packel and PGI is for an entirely different conceptual reason than the violation as a result of applying (A3) to Straffin's probabilistic approach. In the first case the reason lies with the fact that both measures are based only on MWCs, i.e. the domain of the coalitions is restricted. According to the Deegan-Packel 'story' only MWCs will form; according to the PGI 'story' only MWCs form intentionally (excess sized coalitions are a matter of 'luck') and express power so that only they should be counted in the calculation of power. This means that a certain number of a player's swings are not counted in the final measure of voting power, i.e. those in $\mathcal{C}_i \backslash \mathcal{M}_i$. It can be the case that a 'large' player is 'crowded out' by many smaller players, who may have far more opportunities to form MWCs. The violation of LM is a result of all conceivable coalitions no longer being equally probable; only MWCs are considered with equal probability while other winning coalitions obtain a probability of zero. In the second case, the domain of all conceivable coalitions is not restricted. However, also in this case we do not assign the same probability to all conceivable coalitions. E.g., in Example 3.1 the violation of LM is a result of the coalitions no longer being equally probable. Player 2 gets a 'boost' in power over and above player 1 because it is critical in a winning coalition that occurs with a probability of 0.0833, which is significantly larger than the probability of any of winning coalitions in which player 1 is critical. This compensates for the fact that player 2 has four less swings than player 1. Table 1 gives the probabilities of each of the winning coalitions.

Table 1: Winning Coalitions for Example 3.1

				$\mathcal{W}$ Player i						
S	1	2	3	4	5	6	$\sum_{i \in S} wi$	*MWC*	$\mathbf{P}(S)_{(A1)}$	$\mathbf{P}(S)_{(A3)}$
1	$\underline{30}$	$\underline{26}$					56	yes	0.0156	0.0083
2		$\underline{26}$	$\underline{16}$		$\underline{9}$		51	yes	0.0156	0.0083
3		$\underline{26}$	$\underline{16}$	$\underline{12}$			54	yes	0.0156	0.0833
4	$\underline{30}$			$\underline{12}$	$\underline{9}$		51	yes	0.0156	0.0083
5	$\underline{30}$		$\underline{16}$			$\underline{7}$	53	yes	0.0156	0.0083
6	$\underline{30}$		$\underline{16}$		$\underline{9}$		55	yes	0.0156	0.0083
7	$\underline{30}$		$\underline{16}$	$\underline{12}$			58	yes	0.0156	0.0167
8	$\underline{30}$	$\underline{26}$				7	63	no	0.0156	0.0083
9	$\underline{30}$	$\underline{26}$			9		65	no	0.0156	0.0083
10	$\underline{30}$	$\underline{26}$		12			68	no	0.0156	0.0167
11	$\underline{30}$	$\underline{26}$	16				72	no	0.0156	0.0167
12		$\underline{26}$		$\underline{12}$	$\underline{9}$	$\underline{7}$	54	yes	0.0156	0.0083
13		$\underline{26}$	$\underline{16}$		$\underline{9}$	7	58	no	0.0156	0.0083
14		$\underline{26}$	$\underline{16}$	$\underline{12}$		7	61	no	0.0156	0.0167
15		$\underline{26}$	$\underline{16}$	12	9		63	no	0.0156	0.0167
16	$\underline{30}$			$\underline{12}$	$\underline{9}$	7	58	no	0.0156	0.0167
17	$\underline{30}$		$\underline{16}$		9	7	62	no	0.0156	0.0167
18	$\underline{30}$		$\underline{16}$	12		7	65	no	0.0156	0.0083
19	$\underline{30}$		16	12	9		67	no	0.0156	0.0083
20	$\underline{30}$	$\underline{26}$			9	7	72	no	0.0156	0.0167

…/ Table 1 cont.

				𝒲 Player *i*						
S	1	2	3	4	5	6	$\sum_{i\in S} wi$	*MWC*	$\mathbf{P}(S)_{(A1)}$	$\mathbf{P}(S)_{(A3)}$
21	<u>30</u>	<u>26</u>		12		7	75	no	0.0156	0.0083
22	<u>30</u>	26		12	9		77	no	0.0156	0.0083
23	<u>30</u>	26	16			7	79	no	0.0156	0.0083
24	30	26	16		9		81	no	0.0156	0.0083
25	30	26	16	12			84	no	0.0156	0.0833
26		<u>26</u>	16	12	9	7	70	no	0.0156	0.0083
27	<u>30</u>		16	12	9	7	74	no	0.0156	0.0083
28	30	26		12	9	7	84	no	0.0156	0.0083
29	30	26	16		9	7	88	no	0.0156	0.0083
30	30	26	16	12		7	91	no	0.0156	0.0167
31	30	26	16	12	9		93	no	0.0156	0.0167
32	30	26	16	12	9	7	100	no	0.0156	0.0083
$\lvert\mathscr{C}_i\rvert$	18	14	10	6	6	2			0.5000	0.5000
$\lvert\mathscr{M}_i\rvert$	5	4	5	4	4	2				

Note: Critical player is underlined.

4. The Desirability Relation and A Prioricity

Given the conviction that LM is taken to be such an intuitively compelling postulate of *a priori* voting power it is necessary to recap its justification in some detail. Once this is done it is possible to show that LM is compelling only under a very narrow definition of *a prioricity*. That is, once we widen the notion of *a prioricity*, the argument in favour of LM as it is defined above is seriously weakened.

As a number of authors have pointed out (Felsenthal and Machover, 1995, 1998, pp. 241–246; Freixas and Gambarelli 1997), LM is a special case of the *desirability* (also called *dominance*) relation, $\succeq$, which is a pre-ordering (i.e. it is transitive and reflexive) of the players in a simple game v.[12] The idea is that we can order the players in terms of their contribution to a coalition. Formally,

$i \succeq j$ iff $S \cup \{j\} \in \mathcal{W}(v)$ implies $S \cup \{i\} \in \mathcal{W}(v)$.

In words, player i is at least as desirable as j in coalition S in a game v if interchanging i and j does not change S from winning to losing. If we have $i \succeq j$ but not $j \succeq i$, then $i \succ j$, i.e. player i is strictly more desirable than player j, which says that whatever j can contribute to the passing of a bill i can do as well (is at least as desirable) and in some cases more (is more desirable). Thus,

$i \succ j$ then $\xi_i(v) > \xi_j(v)$.

It is also easy to see that if players i and j in v are interchangeable, then by symmetry $\xi_i(v) = \xi_j(v)$, and,

$i \succeq j$ then $\xi_i(v) \geq \xi_j(v)$.

For a weighted voting game it clearly follows that if $w_i \geq w_j$ then $i \succeq j$, i.e. anything that w_j can do, w_i can also do because a winning coalition cannot become a losing coalition if it gains more weight (but it does not necessarily follow that if

[12] The desirability relation was first introduced by Isbell (1958) and later generalized by Maschler and Peleg (1966). See also Taylor and Zwicker (1999, pp. 86–92).

$w_i > w_j$ then $i \succ j$). It is therefore straightforward that if $w_i \geq w_j$ then $\xi_i(v) \geq \xi_j(v)$, viz. precisely LM as in (P5).

It is clearly difficult to quarrel with this argument. Felsenthal and Machover (1998, p. 245) have expressed it forcefully: 'In our view, any reasonable measure of *a priori* power … must respect dominance [desirability]. The case for this postulate is so strong that it hardly needs spelling out.' That is, if desirability is respected – and it must be respected – then it logically follows that *a priori* power is locally monotone. And this certainly backs up the very common sense intuition that power is monotonically increasing in resources, and voting weights can clearly be taken to be a resource. This would seem to justify Felsenthal and Machover's (1998, pp. 221–223) strong position that any *a priori* measure of voting power that violates LM is 'pathological' and should be disqualified as serving as a valid yardstick.

This position, we contend is unwarranted as it is easy to show that it hinges on the particular notion of *a prioricity* that has traditionally been used in voting power. This is the view that an *a priori* measure of power is one that is dependent on, and only on, the collection of subsets $\mathcal{W}(v)$ (or the characteristic function). That is, no other information is used other than the decision rule itself. Clearly the argument in favour of accepting the logic of the desirability relation – and hence LM – follows from this perspective.

Under this notion of *a prioricity* it is clear that the application of (A3) means that any resulting measure Ef_i will not only violate iso-invariance (P1), it also follows that it is not *a priori*. Does this mean that LM does not necessarily apply and thus its violation has no relevance to understanding if *a priori* voting power and the choice of a reasonable measure? The answer is 'no'.

Firstly, it is not quite accurate to say that an *a priori* measure of power is based only on $\mathcal{W}(v)$. It is in fact not possible to calculate $\xi_i(v)$ in absence of an assumption of how the players behave.[13] If we assume that all possible configurations of players are equally probable so that each subset of N (coalitions) is equally likely we are in effect assuming that for a random bill put before the assembly each player votes 'yes' or 'no' with equal probability.[14] This is precisely the idea underpinning the Bz. It also underpins the D-P index and PGI, albeit

[13] This point was already recognized by Dubey and Shapley (1979, p. 103).

[14] We are ignoring the case of abstentions. See Felsenthal and Machover (1997) and Braham and Steffen (2002a).

under a modified form. Here the set of winning coalitions is restricted to $\mathcal{M}$. For the S-S index, it is not the equi-probability of all combinations of players but equi-probability of all permutations, but this too results directly from the assumption that each player votes 'yes' or 'no' with a probability of 0.5.[15] Thus the *a prioricity* of a measure of voting power is contingent upon a probability model of voting behaviour. The general belief is that a probability model that treats all players symmetrically (i.e. iso-invarance) is *a priori* while one that does not is *a posteriori*. Further, it is believed that the only really *a prioristic* model is one derived from the Bernoullian principle of insufficient reason which assigns equi-probabilities to each strategy that a player faces, i.e. each player votes 'yes' or 'no' with equi-probability.

Secondly, in the same manner that $\xi_i(v)$ requires specification of behavioural assumptions, it also requires a specification of the decision-making situation or structure. That is, a probability model that treats all players symmetrically implies far more information than is generally thought to be the case. It presumes that the voting body has a homogeneous structure; there is no structural differentiation in the types of players: it is *flat*. In many instances this is true: voting in a parliament is flat in the sense that there are no structural difference between its members, although there obviously will be temporal differences as a matter of political and personal predilection resulting in a correlation of voting behaviour – but this is not 'structure'.[16] We can characterise a flat decision structure by saying that each member has complete freedom of choice.[17] For such a structure a symmetric probability model that assigns equal likelihood to each of the options for each player is obviously the most appropriate; and possibly the Bz will turn out to be the measure to use.[18] But the fact that such a measure obeys iso-invariance, dominance, and thus LM is a happy coincidence. There is no way we can

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[15] That is, the S-S index does not necessarily depend upon all voters lining up to vote 'yes' or all lining up to vote 'no'. See Felsenthal and Machover (1996).

[16] Our position is in no way to be confused with that of Brams (1975, p. 202) who takes the environmental constraints or decision-making structure that we are dealing with to be preference based: 'One such constraint is the organitzational ties of players, which may limit their freedom to select other players as coalition partners. In many legislatures, for example, the structure of the party system is all-important in determining what coalitions form. When strict party discipline prevails, a legislator always votes with his party and has no opportunity to seek out potential coalition partners among nonparty members.'

[17] This concurs with Dubey and Shapley's (1979, p. 103) discussion of the Bz.

[18] Whether this is the case, i.e. that the independence assumption is the most appropriate *a priori* assumption of a flat structure, or whether it is *a priori* impossible to differentiate between the independence and the homogeneity assumption as the two extreme cases of partial homogeneity is discussed in Steffen (1999) and Braham and Steffen (2002a).

conclude from this that voting power either *is*, or *ought* to be, locally monotone as defined by (P5).

But there are also cases, perhaps more common than realized in the voting power literature, in which the decision-making structure is *not* flat, but differentiated and hierarchized in some manner or another. This is obviously the case of a bureaucracy or firm. In such a setting the players occupy positions and have to make choices that pertain to the aims of the department or that part of the organization to which they belong. In contrast to a flat structure the player's freedom of choice is constrained by the system of incentives rewards used to make sure that each player makes choices that are concordant with their department or section and that of the organization as a whole. This will mean that players belonging to the same department or section of the organization will have highly correlated voting behaviour. That is, an organization is a series of arrangements between individuals with possibly differing goals.[19] For instance, a bank will have staff that are responsible for expanding credit and staff responsible for managing risk. The granting of a large loan will usually require consent of both sections. It is reasonable to assume that the staff responsible for expanding credit will all have one standard of behaviour, while those responsible for managing risk will have an opposing standard.[20] Example 3.3 above captures this structure in the definition of two opposing standards of t and $(1-t)$. Note, also, that in this example we have not actually defined p_i at all; we have only assumed certain patterns of correlated voting behaviour.

Hence any reasonable model of voting power associated with committee voting in such structures requires that we take into account these different behavioural standards, i.e. apply the partial homogeneity structure of (A3). Furthermore, it is essential to recognize that in no way is this a violation of *a prioricity* because there are no 'flesh and blood' individuals in the model: all the sociological, psychological, and political – and dare say even the psychiatric –

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[19] Shubik (1962) discussed this issue some forty years ago. See also the much earlier attempt to formalize this issue by Morgenstern (1951).

[20] See Steffen (1999) for a detailed example and Braham and Steffen (2001c) for a more general investigation of this case which also includes another example, that of a United Nations field office responsible for development projects that are a part of a refugee repatriation programme. In many instances, such projects have to be approved by the finance section of the agency headquarters which may have interests completely at odds with those of the field office. The field office is concerned with the welfare of particular refugees; the goal of the finance office is maximising donor contributions, which often leads to a tendency to support 'high visibility' projects that are popular with donors but have little value to the refugees. Alternatively put, the finance office has a tendency to turn down useful 'low visibility' projects proposed by the field office.

aspects of the players are ignored. It is not the names of individuals who are on the ballot papers but the positions in an organization.[21] The structure is still, to use Felsenthal and Machover's (1998, p. 20) own words, an 'abstract shell, *uninhabited* by real agents, with real likes and dislikes, attractions, and repulsions' and is therefore totally in accord with the position taken by Roth (1988) that we cited earlier in this essay. In a nutshell, there is nothing necessarily *a posteriori* about (A3) – although it can of course be so.

Essentially what our argument is boiling down to is that the belief that (A3) necessarily contains more information from outside of $\mathcal{W}(v)$ than either (A1) or (A2) is mistaken; (A3) *contains more structure*. And here is where we do agree with the position that (A3) contains more information but only insofar as differentiation implies information and not in terms of the *a prioricity* of this information, i.e. a three dimensional space can contain more information than a two dimensional space. In the light of this reasoning we can say that a flat structure in which we treat players symmetrically is a limiting case of (A3). Thus, it is neither here nor there that a measure of voting power for these situations will generally violate iso-invariance, dominance, and by implication LM.

To concur, then, with the proponents of iso-invariance, dominance, and LM postulates for selecting out an *a priori* measure of voting power can – will – lead to methodological absurdity. In many a situation we would end up employing a measure which is *a priori* in the very restrictive sense of being a limiting case of partial homogeneity but it would be entirely inappropriate (although 'reasonable' because it respects LM); or employing a measure that would be appropriate but neither *a priori* in the limiting sense nor reasonable because of its violation of LM. In the first case the analysis may turn out to be misplaced; while in the second case the analysis may be unwittingly discarded for normative reasons because of the attractive ethical appeal of *a prioricity*: it corresponds to a 'veil of ignorance' argument á la Harsanyi (1955) and Rawls (1971).[22] The important methodological implication of this fine grained and even somewhat pedantic

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[21] Straffin was in fact lead astray here: 'Partial homogeneity assumptions are by their nature ad hoc; they would be out of place in theoretical analysis of abstract political structures where the level of abstraction requires symmetrical treatment of the players' (Straffin 1978, p. 493). Straffin is of course correct if he is referring only to parliamentary decision-making structures.

[22] The importance of the veil of ignorance character of *a prioricity* to the analysis of voting power is stressed in Holler and Widgrén (1999) and Felsenthal and Machover (2001a) in their reply to the critical attack on voting power measures by Garrett and Tsebelis (1999a) who argue for a preference-based approach to the measure of voting power.

analysis is that the usual notion of *a prioricity* can lead one astray.[23] To belabour the point and even be a little rhetorical, is it reasonable to take the position that from behind a veil of ignorance the world should be treated as flat?

5. Power and Resources

Although we have shown via a discussion of Straffin's partial homogeneity approach that in its present form LM is untenable as a postulate of *a priori* power, this does not imply that we can take the position of Deegan and Packel (1983), Brams and Fishburn (1995), and especially Holler (1997, 1998) that the violation of LM simply reflects a social and political fact that there is an inverse relationship between power (in whatever form) and resources. This may seem paradoxical given our conclusion in the previous section, but this is not too difficult to resolve. As it turns out, the underlying intuition of LM is not necessarily wrong; only its definition is too restrictive.

If we abstract from the particular definition of LM to that of monotonicity *simpliciter* we find a very general principle which states that as the underlying data of a problem changes, so does its solution. LM merely takes as its underlying data the vector of voting weights (w_1, w_2, $\ldots, w_n$). There lies the problem. As we have argued in the previous section, the underlying data of voting game is actually more than this: it is made up of (i) the voting weights and (ii) the players positions within the decision-making structure. The interaction of both these components are what we can call the resources or, to use Dahl's (1957) terminology, the 'base' of (voting) power. Under what we have called a flat structure, the position of each 'vote' of a player's voting weight (which is merely the sum of a players 'votes') is by definition symmetric and therefore each 'vote' has the same ability to make a difference to the outcome irrespective of who possesses these votes. In a flat structure, resources (or base of power) and weight happen to coincide; in a differentiated structure they do not. For the sake of illustration, consider a committee of five players and a simple majority rule, which can be represented as the weighted voting game [3; 1, 1, 1, 1, 1]. We can construct the following seven scenarios.

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[23] Although in their attempt at a probabilistic refoundation of power measures Laruelle and Valenciano (2001a) recognise that the decision rule $\mathcal{W}(v)$ is not a 'game' and requires a specification of a probability model (actually this insight can be found explicitly in Straffin (1983, 1988, 1994)) they do not push their analysis far enough. The result is that they err in their conclusion that '… from a normative point of view the Banzhaf index is no doubt the best candidate as a reference for the design of voting procedures, where any information about the voters should be ignored even if available' (p. 26).

Example 5.1 Assume (A1) for all players, we have $Ef_1 = Ef_2 = Ef_3 = Ef_4 = Ef_5 =$ 0.38. (This is the Penrose/Banzhaf measure b'.)

Example 5.2 Assume (A2) for all players, we have $Ef_1 = Ef_2 = Ef_3 = Ef_4 = Ef_5 =$ 0.20 (This is the S-S index f.)

Example 5.3 Assume (A3) as follows: players 1, 2, 3 form a standard t and players 4 and 5 a standard $(1–t)$. We have $Ef_1 = Ef_2 = Ef_3 = 0.53$ and $Ef_4 = Ef_5 = 0.30$.

Example 5.4 Assume (A3) as follows: players 1, 2 form a standard t and players 3 and 4 a standard $(1–t)$ and player 5 behaves independently. We have $Ef_1 = Ef_2 =$ $= Ef_3 = Ef_4 = 0.42$ and $Ef_5 = 0.53$.

Example 5.5 Assume (A3) as follows: players 1, 2, 3, 4 form a standard t and player 5 behaves independently. We have $Ef_1 = Ef_2 = Ef_3 = Ef_4 = 0.25$ and $Ef_5 =$ 0.20.

Example 5.6 Assume (A3) as follows: players 1, 2, 3 form a standard t and players 4 and 5 behave independently. We have $Ef_1 = Ef_2 = Ef_3 = 0.33$ and $Ef_4 = Ef_5 =$ 0.25.

Example 5.7 Assume (A3) as follows: players 1, and 2 form a standard t and players 3, 4, 5 behave independently. We have $Ef_1 = Ef_2 = 0.38$ and $Ef_3 = Ef_4 =$ $Ef_5 = 0.33$.

Observe that except for the extreme cases of applying (A1) and (A2), Ef_i is always less for the independent players (for this committee) except in Example 5.4, where it is greater than for the players belonging to either t or $(1–t)$. This makes intuitive sense because the two ‘groups’ (or more accurately, the collection of players conforming to a given standard) are of equal size and ‘antagonistic’ which leaves the neutral party in a more powerful position.[24] The reason is simple, each of the players in t and $(1–t)$ is more likely to form a coalition with the independent than with players from the antagonistic standard. In a sense we could say that the antagonism ‘depletes’ the resources (i.e. weights) of the members of these groups, and as a consequence neutrality increases the value of the independent player.

[24] One could say that it is a form of a quarrel, although our examples in no way display the so-called paradox of quarrelling members. See Brams (1975, p. 314).

The outcome for Example 5.3 where Ef_i for the players in t is greater than the players in $(1–t)$ also makes sense. Here we again have two opposing 'groups' and the members of the largest 'group' have a greater probability to be decisive than the smaller 'group'. Clearly – and obviously – Ef_i depends on the size of the group. This certainly makes sense; it confirms the idea that under certain circumstances there is power in numbers. We see this again in Examples 4–6.

Thus the underlying data of a voting game needs to be clearly specified before we can calculate the resources or the 'base of power'. This is the reason why we say that it is mistaken to believe that a violation of LM defined only by voting weights implies that power is not necessarily locally monotone in resources. It is beyond the scope of this chapter, but we posit that a reasonable method for calculating a quantitative value of a player's resources put altogether in a voting game would probably produce a resulting measure of voting power that is locally monotonic in this quantity.

Where those who deny this monotonic relationship err and equally where the proponents of LM also err is in the calculation of the quantity of the resource. Both sides of the debate focus *only* on the vector of voting weights as given by $(w_1, w_2, \ldots, w_n)$. As we have argued above this is mistaken in the same way it is mistaken to say that a superbly outfitted army that is defeated by a band of poorly equipped guerrillas is evidence that power is not locally monotonic in military resources. True, military power is not necessarily monotone in guns; but guns do not fully describe the underlying data of the situation, which includes military intelligence, knowledge of local geography, and even physical acclimatisation to the theatre of operations. We would argue that such a ill-equipped band of guerrillas probably does have more resources than its well-equipped enemy.

6. Conclusion

The main result of this chapter is in one sense depressing. By showing that neither iso-invariance (P1) nor local monotonicity (P5) are necessarily compelling axioms or postulates of *a priori* voting power we have whittled down the remaining set to the somewhat trivial and related axioms or postulates of *ignoring dummies* (P2) and *vanishing for dummies* (P3).

For those familiar with the axiomatic approach to social choice problems, this result should come as no surprise. The related field of the measurement of freedom suffers from similar problems. Pattanaik and Xu (1990, 1998), for instance, have shown that an apparently harmless and compelling set of axioms uniquely characterize a 'naive' counting rule that measures an agent's total freedom by simply counting the number of options open to that agent. This is not the place to go into this debate, but suffice to say Pattanaik and Xu's axiomatic structure has been dissected and severely criticized and in one important contribution to the debate (Carter 2001) the axioms have been shown to be inconsistent with the very basic definition of freedom itself.

Returning to the question of power, we would like to close by pointing out that we are not saying that LM in its present form (i.e. based only on the vector of voting weights) is irrelevant to the study *a priori* voting power. Far from it. It has an important place to play as a *normative* criterion in institutional design. That is, *if* we desire to preserve the ranking of influence over social outcomes with that of the ranking of voting weights, *then* (i) we are compelled to create a voting structure in which there are no incentives such that players will *a priori* correlate their behaviour in one way or the other; and (ii) create conditions such that *a priori* not only MWCs will form. This perspective points to the possibility that the debate about LM has been confused by the ethical appeal of this postulate – it seems to reflect a requirement of fairness that goes back to Aristotle. We have not examined it, but it seems straightforward to assume that behind a veil of ignorance rational players would choose a voting system that respects LM in those situations where the players have a *personal* interest in the outcome of a vote. It would seem inappropriate however to characterize (describe) voting power by axioms or postulates that capture, directly, our moral intuitions.[25]

[25] There is an interesting parallel of this problem related to the monotonicity axiom in bargaining theory. See Roemer (1986).

Chapter 5

CONSTRAINED MONOTONICITY AND THE MEASUREMENT OF POWER*

Abstract: This chapter introduces two constrained versions of local monotonicity: (i) player-constrained local monotonicity by restricting the number of non-dummy players in a game and (ii) partial local monotonicity by applying specific constrains on voting weights. It is shown the Public-Good measures fulfil partial local monotonicity for every proper weighted voting game and player-constrained local monotonicity for every weighted voting game with a simple majority rule and up to four non-dummy players. The discussion points out that whether a specific measure of voting power is appropriate depends on the properties of the model of collective decision-making which one wants to analyze, and not necessarily on some intuitive notions of monotonicity.

1. Introduction

When it comes to monotonicity of power with respect to voting weights, it is important to note that none of the existing measures guarantee that the power value of an individual player i will not decrease if his or her voting weight increases. Fischer and Schotter (1978) demonstrate this result, i.e. the so-called paradox of redistribution, for the Shapley-Shubik (1954) and (normalized) Banzhaf (1965) indices.

The paradox of redistribution stresses upon the fact that power is a social concept: if we discuss the power of an individual member of a group in isolation from his or her social context, i.e. related only to his or her individual resources, we may experience all sorts of 'paradoxical results'. It seems that sociologists are quite aware of this problem (see, e.g., Caplow, 1968). Political scientists, however, often see the non-monotonicity of power as a threat to the principle of democracy. To them, it is hard to accept that by increasing the number of votes, a group could decrease its power. However, it seems that there is ample empirical evidence for this phenomenon.[1] On the other hand, an increase of votes is more likely to imply

* This chapter is a based upon Holler, Ono and Steffen (2001). The author would like to thank Matthew Braham and two anonymous referees for helpful comments.

[1] See Brams and Fishburn (1995) for references.

increasing than decreasing power given a random distribution of voting weights of the other players and corresponding re-distributions (see Fischer and Schotter, 1978).

In general, economists also assume that controlling more resources is more likely to mean more power than less. However, they also deal with concepts like monopoly power, bargaining, and exploitation which stress the social context of power and the social value of resources (assets, money, property, etc.). Note that in the discussion of power indices, voting weights are a proxy for resources.

The discussion of measures of voting power follows the strategy to constrain the redistribution of weights in order to derive monotonicity properties for the various measures and to discriminate the measures with respect to these properties. For example, a well-known result is that the Shapley-Shubik index satisfies *global monotonicity*,[2] while the normalized Banzhaf index does not and thus suffers from the *donation paradox*:[3] a player may experience a loss in his or her index value, although his or her voting weight increases, *given that the voting weight of no other player increases*. Note that global monotonicity can only be applied if this constraint holds: therefore, it is a *constrained monotonicity concept*.

Both, the Shapley-Shubik and the (normalized) Banzhaf index satisfy another monotonicity property: *local monotonicity* (LM). This property says that a player with a greater voting weight cannot have less power than a player with a smaller voting weight. The violation of LM is considered by many scholars as counter-intuitive[4] and implies, in terms of Machover (2000), a contradiction to the pre-formal notion of what voting power is. LM is a compelling property and it seems straightforward to argue that indices which do not satisfy this property, such as the Public-Good Index (Holler, 1982b; Holler and Packel, 1983) or the Deegan-Packel index (Deegan and Packel, 1978; Packel and Deegan, 1982), should not be applied for the measurement of power. It is, however, interesting to note that the very same scholars who are strongly propagating this position emphasize that the power of players has to be discussed in the context of the distribution of votes between *all players* in the game when their favourite measure faces the paradox of re-distribution. We do not know of any study that discusses the properties of the distribution of votes so that LM is satisfied. For example, it is obvious that LM

[2] See Turnovec (1998) and the generalization in Levinský and Silársky (2002).

[3] See Felsenthal and Machover (1998, p. 252).

[4] See Freixas and Gambarelli (1997) and the discussion which accompanies the publication of this article in the same volume.

will not be violated by any of the known power measures, including the Public-Good Index and the Deegan-Packel index, for n players and if n-2 players are dummies. It is, however, less obvious that LM is also satisfied for the Public-Good Index (and the Deegan-Packel index) if the games have n-4 dummies.

In the following, we will elaborate some results of constraining the distribution of votes, such that LM is satisfied for the Public-Good measures of voting power. Section 2 outlines the theory of simple voting games and introduces the Public-Good measures of voting power and its motivation. Sections 3 and 4 present our results for a constrained LM of Public-Good measures of voting power. Section 5 concludes with a discussion that shows that whether a specific measure of voting power is appropriate depends on the properties of the model of collective decision-making which one wants to analyze, and not necessarily on some intuitive notions of monotonicity.

2. Simple Voting Games and the Public-Good Measures of Voting Power

Before we proceed to discuss our results, let us briefly review the Public-Good measures of voting power. This family contains the Public-Good Index, the Public Value (Holler and Li, 1995) and the Public-Good Measure.[5] These measures have been introduced as a solution concepts for *simple voting games* (SVG). For the basic definitions and terminology relating to the idea of a SVG and the subclass of *weighted voting games* (WVG), we refer the reader to Shapley (1962b) or to reproductions in Felsenthal and Machover (1998) or Taylor and Zwicker (1999). However, in order to develop our argument, we need to reiterate some of this here.

Definition 2.1 (i) A simple voting game (SVG) is a pair (N, $\mathcal{W}$) where $\mathcal{W}$ is a collection of subsets of a finite set N, satisfying the following three conditions:
$\emptyset \notin \mathcal{W}$; $N \in \mathcal{W}$; and (monotonicity) if $S \in \mathcal{W}$ and $S \subseteq T$, then $T \in \mathcal{W}$.
(ii) By N is meant an assembly (or voting body) and is the largest in $\mathcal{W}$, its members are *players*, and its subsets are *coalitions*. A coalition S is said to be *winning* or *losing* according to whether $S \in \mathcal{W}$ or $S \notin \mathcal{W}$.
(iii) Players in a SVG $\mathcal{W}$ are identified by the integers 1, 2, ..., n, where $n = |N|$.
(iv) A coalition S is called a *minimal winning coalition* (MWC) if $S \in \mathcal{W}$, but no subset of S is in $\mathcal{W}$. The set of MWCs is denoted by $\mathcal{W}^m$.

[5] The Public Good Index is axiomatized in Holler and Packel (1983). Recently, Napel (2001) has completed the axiomatization by proving that Holler and Packel's axiomatization contains no redundant axioms.

Remark 2.1 (i) A SVG can be represented by $\mathcal{W}$ because N is uniquely determined by $\mathcal{W}$ (its largest member). We will therefore follow this notation through this chapter.
(ii) Monotonicity and the finiteness of N imply that $\mathcal{W}^m$ completely determines $\mathcal{W}$.

Definition 2.2 In a SVG $\mathcal{W}$ let S be a coalition and i a player. We say that: (i) i is *critical* in S if $S \in \mathcal{W}$ but $S - \{i\} \notin \mathcal{W}$; (ii) i is a *dummy* if it never happens that $S \notin \mathcal{W}$ but $S \cup \{i\} \in \mathcal{W}$.

Remark 2.2 (i) It is easy to see that (i) a coalition is a MWC iff each of its members is critical; and (ii) a player is a dummy iff it is never a member of a MWC.

Alternatively, a SVG can be represented by a characteristic function:

Definition 2.3 Let $\mathcal{W}$ be a SVG on an assembly N. The characteristic function of $\mathcal{W}$ is a mapping $v : 2^N \rightarrow \{0, 1\}$ such that for any coalition S, $v(S) = 1$ iff $S \in \mathcal{W}(v)$ and 0 otherwise.

Remark 2.3 (i) Using the characteristic function a SVG $\mathcal{W}$ can be represented as a pair (N, v) that satisfies the conditions of Def. 2.1. In particular the monotonicity condition translates into $v(S) \leq v(T)$ whenever $S \subseteq T$.

Furthermore, we need the definition of that important sub-class of SVGs known as *weighted voting games* (WVG):

Definition 2.4 (i) A SVG $\mathcal{W}$ is a WVG if there are nonnegative weights $w_1,\ldots,w_n$ allocated to the players and a *quota* $0 < q \leq \sum\{w_i : i \in N\}$. such that $S \in \mathcal{W}$ iff $\sum\{w_i : i \in S\} > q$.
(ii) A WVG can be represented by $[q; w_1,\ldots,w_n]$

Definition 2.5 A category is a set of WVGs which yield the same $\mathcal{W}^m$.

Rermark 2.4 Since a category is described by $\mathcal{W}^m$, we use the set $\mathcal{W}^m$ to characterize the different categories.

Example 2.1 Consider (½; w_1, w_2, w_3). For this type of game, we can identify two different categories of MWC's, which are $\mathcal{W}^{m1} = \{\{1,2\}, \{1,3\}\}$ and $\mathcal{W}^{m2} = \{\{1,2\}, \{1,3\}, \{2,3\}\}$, i.e., each possible distribution of voting weights w leads to a set of MWC's which is $\mathcal{W}^{m1}$ or $\mathcal{W}^{m2}$, respectively, if the players are ordered with decreasing voting weights, $w_1 \geq w_2 \geq w_3$ and $w_1 \leq$ ½. For example, a game with the distribution $w = (0.50, 0.25, 0.25)$ belongs to $\mathcal{W}^{m1}$, while a game with $w = (0.33, 0.33, 0.33)$ or $w = (0.40, 0.40, 0.20)$ is in $\mathcal{W}^{m2}$.

Before we introduce the Public-Good measures of voting power we define what we understand under a measure of voting power and how we are going to characterise such a measure.

Definition 2.6 A numerical measure of the power of player i to influence a decision or series of decisions made in an assembly N is a vectorial function x associating to any SVG $\mathcal{W}$ an element of R_0^{+n}, i.e. x is any function assigning a non-negative real number $x_i(\mathcal{W})$ to every player i of every SVG $\mathcal{W}$ in which at least one player i from each SVG $\mathcal{W}$ is assigned a value $x_i(\mathcal{W}) > 0$ and $x_i(\mathcal{W})$ is invariant under isomorphism.

Remark 2.5 This very general definition, which is based upon Sagonti (1991), F&M (1995), and Felsenthal *et al*. (1998) encompasses three types of measures: (i) *scores* (counts or 'raw' indices), which we denote by $\kappa_i(\mathcal{W})$ for player i; (ii) *indices*, obtained by normalization of κ, i.e. $\kappa_i / \sum\{\kappa_j(\mathcal{W}) : i \in N\}$; and (iii) *ratios* of $\kappa_i(\mathcal{W})$, obtained by dividing $\kappa_i(\mathcal{W})$ by an appropriate quotient.[6]

The Public-Good measures of voting power belong to the group of measures which are based on counting MWCs.

Definition 2.7 (i) The *Public Value* (h'') for player i in a SVG $\mathcal{W}$, which is a score, is derived from counting the MWCs which have i as a member: $h''_i(\mathcal{W}) = |\{S \in \mathcal{W}_i^m\}|$.
(ii) The *Public-Good Measure* (h') for player i in a SVG $\mathcal{W}$, which is a ratio, is obtained from its $h''(\mathcal{W})$ by division of the cardinality of the set of MWCs: $h'_i(\mathcal{W}) = h''_i(\mathcal{W}) \,/\, |\mathcal{W}^m|$.
(iii) The *Public-Good Index* (h) for player i in a SVG $\mathcal{W}$ is derived by the normalization of $h''(\mathcal{W})$ or $h'(\mathcal{W})$: $h_i(\mathcal{W}) = h''_i(\mathcal{W}) \,/\, \sum\{h''_i(\mathcal{W}): i \in N\} = h'_i(\mathcal{W}) \,/\, \sum\{h'_i(\mathcal{W}): i \in N\}$ so that $\sum\{h_i(\mathcal{W}): i \in N\} = 1$.

Remark 2.6 (i) The Public-Good measures are based on the concept of MWC's. This specification is supported by the fact that the value of a coalition is considered to be a Public-Good. If the players in $\mathcal{W}$ consider the value $v(S) = 1$ as a Public-Good there should be no rivalry in consumption and each member of S will enjoy this value if coalition S is formed. Furthermore, if there are no entry cost or transaction cost of coalition formation, S will be formed - given that S is a

[6] The distinction between these three measures is made in Felsenthal *et al*. (1998).

coalition in $\mathcal{W}^m$ so that no possibility of free-riding exists. Thus, we can state that non-rivalry in consumption and the exclusion of free-riding are the core elements from which a Public-Good measure of voting power is derived.

(ii) If $S \in \mathcal{W}$ but $S \notin \mathcal{W}^m$ then it is assumed that because of the potential of free-riding, it will only be formed by 'luck'[7] - and not because of the power of its members - or that the outcome of S is identical with an MWC $K \subset S$. In the latter interpretation, it leads to double counting, if S is taken into consideration.

(iii) Although we take only MWC's into account for the calculation of a Public-Good measure of voting power we do not claim that no other coalitions will be formed. It is only assumed that these coalitions do not matter and thus should not be taken into consideration when it comes to measuring power.[8]

3. Player-Constrained Local Monotonicity

In sections 3 and 4, we will consider the violation of LM for voting bodies when power is measured by a Public-Good measure of voting power. Our results so far are rather general in that we do not restrict our analysis to specific voting weights. However, as the most common voting rule used in voting bodies is the simple majority, we only consider this rule for our results in this section. In particular, we show that there exists a class of WVGs which is 'violation proof', that is, for which we cannot find a distribution of votes which violates LM.

Proposition 3.1 The Public-Good measures satisfy LM for every WVG with n-g dummy players and simple majority rule (i.e., $q = ½$), if $g \leq 4$ (*player-constrained* LM).

Proof For games with n-g dummy players, we have g players which are decisive at least in one case and thus will have some power. Thus, we have to prove that LM is guaranteed for games with $q = ½$ and $g = 1, ..., 4$. We can do this by listing all possible distributions of voting power which can occur using the Public-Good Index for these games and then showing that there is no violation of monotonicity in any case. For this we refer to Brams and Fishburn (1995) and Fishbrun and Brams (1996), who have already derived all possible categories of MWC's for these games. By calculating h'', h', and h for these categories, we derive all possible values of the measures of voting power of the regarded classes of games. As h'' and h' are linear transformations of h it is enough to prove that h

[7] For an explicit distinction of power and luck, see Barry (1980a, 1980b).

[8] See, e.g. Holler (1998) for this argument.

fulfils LM. This is shown by the results in table 1 and 2 which indicate all values of h for $g \leq 4$ are locally monotonic. ð

Table 1: The Public-Good Index values for g = 3

Categories of MWC's for g =3	$c_i(v)$			$h_i(v)$		
	1	2	3	1	2	3
{1,2}, {1,3}	2	1	1	0.50	0.25	0.25
{1,2}, {1,3}, {2,3}	2	2	2	0.33	0.33	0.33

Table 2: The Public-Good Index values for g = 4

Categories of MWC's for g = 4	$c_i(v)$				$h_i(v)$			
	1	2	3	4	1	2	3	4
{1,2}, {1,3,4}, {2,3,4}	2	2	2	2	0.25	0.25	0.25	0.25
{1,2}, {1,3}, {2,3,4}	2	2	2	1	0.29	0.29	0.29	0.14
{1,2}, {1,3}, {1,4}, {2,3,4}	3	2	2	2	0.33	0.22	0.22	0.22
{1,2}, {1,3}, {2,3}	2	2	2	0	0.33	0.33	0.33	0.00
{1,2}, {1,3}, {1,4}	3	1	1	1	0.50	0.17	0.17	0.17
{1,2,3}, {1,2,4}, {1,3,4}, {2,3,4}	2	2	2	2	0.25	0.25	0.25	0.25

Furthermore, for g = 5, the violation occurs between the two players with the largest voting weights in only one category (5%), while in the rest of the cases (4 categories = 20%) the violation occurs between the players with the second and third largest voting weight, while the two players with the lowest weights are never affected. Moreover, in only two cases (10%) the violation leads to the result that a player with less weight has more power than the player with the largest weight.

For g = 6 in ten categories (9%), the violation occurs between the players with the largest weights and in four categories (3%), the violation occurs between the players with the third and fourth largest voting weight, while all other twenty-seven violations (23%) occur between the players with the second and third largest voting weight. Moreover, in eighteen cases (16%), the violation leads to the result that a player with less weight has more power than the player with the largest weight.

Further, we have found out that in both cases (g = 5 and g =6), the player with the largest voting weight seems to be affected only by the violation of LM, if his or her relative voting weight lies in the interval [0.24, 0.36]. For the players with the second and third largest voting weight, the corresponding intervals where they are not affected by non-monotonicity seem to be [0.20, 0.35] and [0.15, 0.21], respectively.

A possible conclusion from these results on monotonicity presented in this section is the following. Assume that the players, for e.g., are parties in an election game under proportional representation from which they only know that they have to decide on collective goods under a simple majority rule wherein they do not know their voting weights or they suppose that these will lie under 15% or over 36%. Then the best strategy to maximize their power is to maximize their votes, i.e. their voting weight, especially if they could expect that there will not be more than four, five or six parties which will have a significant party support to win seats such that they are not dummies.

4. Partial Local Monotonicity

In Section 3, we have shown that the Public-Good measures of voting power are locally monotonic for all WVGs with a simple majority rule and with (n-g) - dummy players for $g \leq 4$. In this section, we prove that LM can be achieved for the these measures if we apply specific constraints on voting weights (different from the dummy condition of the previous section).

Definition 4.1 A WVG is proper if two disjoint coalitions are never winning at the same time; formally, $q \geq \sum \{w_i\colon i \in N\} / 2 = ½$.

Proposition 4.1 The Public-Good measures satisfy LM for the players i and j for every proper WVG so that $h''_i(\mathcal{W}) \geq h''_j(\mathcal{W})$, $h'_i(\mathcal{W}) \geq h'_j(\mathcal{W})$, and $h_i(\mathcal{W}) \geq h_j(\mathcal{W})$, if $w_i > w_j$ and $w_k = w'$ for all other players $k \neq i, j$ (*partial* LM).

Proof Without a loss of generality, we assume $i = 1$ and $j = 2$. Further, we classify the MWC's into the following four subclasses:

$\mathcal{W}^{m12} \quad = \{S \in \mathcal{W}^m \mid 1,2 \in S\}$

$\mathcal{W}^{m1} \quad = \{S \in \mathcal{W}^m \mid 1 \in S, 2 \notin S\}$

$\mathcal{W}^{m2} \quad = \{S \in \mathcal{W}^m \mid 1 \notin S, 2 \in S\}$

$\mathcal{W}^{m\varnothing} \quad = \{S \in \mathcal{W}^m \mid 1, 2 \notin S\}$

Thus the Public-Good Index for the players 1 and 2 can be written as

(1) $h_1(\mathcal{W}) = (\,|\mathcal{W}^{m12}| + |\mathcal{W}^{m1}|\,) / \sum\{h''_i(\mathcal{W}) : i \in N\}$ and

$h_2(\mathcal{W}) = (\,|\mathcal{W}^{m12}| + |\mathcal{W}^{m2}|\,) / \sum\{h''_i(\mathcal{W}) : i \in N\}$,

where $\sum\{h''_i(\mathcal{W}): i \in N\}$ is only used for normalization and $|\mathcal{W}^m|$ represents the cardinality of $\mathcal{W}^m$. Thus, $\sum\{h''_i(\mathcal{W}): i \in N\}$ and $|\mathcal{W}^{m12}|$ have no influence on the relative distribution of power among the players 1 and 2 and we can focus on $\mathcal{W}^{m1}$ and $\mathcal{W}^{m2}$. As it is easy to see this implies that this proof also hold for *h''* and *h'*. By Proposition 4.1 it is assumed that each player $k \in N - \{i, j\}$ has the same weight w'. As a consequence any coalition in $\mathcal{W}^{mf}$ ($f = 1, 2$) has the same size, say, $r_f + 1$ (*r* '*k*-members' plus player *f*). Formally, $r_f := \min\{r |\, q - w_f < r\, w'\}$; it is clear that

(2) $r_1 \leq r_2$.

Furthermore, $|\mathcal{W}^{mf}| = \binom{n-2}{r_f}$. Since the WVG is proper,

(3) $(r_1 + 1) + (r_2 + 1) > n$

holds (otherwise, there exist two MWC's). The inequalities (2) and (3) imply:

(4) $|\mathcal{W}^{m1}| \geq |\mathcal{W}^{m2}| \Leftrightarrow \binom{n-2}{r_1} \geq \binom{n-2}{r_2}$,

i.e., a constrained LM *holds* with respect to player 1 and 2.ð

Remark 4.1 (i) Note that in Proposition 4.1, nothing is said about the power relationship of *i* and *k* and *j* and *k*.
(ii) Note, that the fact of monotonicity being fulfilled for the players 1 and 2 according to Proposition 4.1 does not mean that all index values will be locally monotonic in these cases. For instance, for the game (½; 0.4, 0.3, 0.1, 0.1, 0.1) the Public-Good Index is (0.27, 0.13, 0.20, 0.20, 0.20). This example illustrates partial LM of players 1 and 2, however, (unconstrained) LM is not satisfied.

5. Discussion

In this chapter, we have discussed constraints on the number of (non-dummy) players and on the distribution of votes such that LM is satisfied for the Public-Good measures of voting power. We have, furthermore, identified the player-constrained and partial LM concepts along with it.[9]

An alternative approach to discuss monotonicity properties would be to choose constrained sets of coalitions (or permutations) which have *i* and *j* as members. For instance, the price monotonicity, introduced in Felsenthal *et al.* (1998), says that a power measure x should satisfy the condition $x_i(v) \geq x_j(v)$ if *i* is in more coalitions a swinger[10] than *j*. Since the Banzhaf index is based on the swing concept, it trivially satisfies price monotonicity while it is easy to show that the Shapley-Shubik index which refers to pivotal players and permutations fails to do so. Needless to say that the Public-Good measures of voting power or the Deegan-Packel index also lack price monotonicity defined by swing coalitions. However, if we re-define price monotonicity by stating that the power measures have to satisfy $x_i(v) \geq x_j(v)$ if *i* is in more MWC's than *j*, then measures which satisfies this monotonicity are Public-Good measures.

If the collective decision-making can be described by a model of weighted voting which consists of a decision rule (e.g. simple majority) and a distribution of votes, then a measure of voting power should answer the question: what is the probability that a player *i* is decisive for the collective outcome, given that we have no other information on the voting (or the players) and the forming of coalitions. On this level of abstraction, the measure represents an a priori evaluation of the WVG to each player. It has been argued that 'power index speaks about the properties of a model, not about the properties of the power as such' (Turnovec, 1997, p. 613). Whether a specific measure of voting power is appropriate depends on the properties of the model of collective decision-making which one wants to analyze, and not necessarily on some intuitive notions of monotonicity.

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[9] For further studies of player-constrained local monotonicity see Schönfeld (2001).

[10] Player *i* is a swinger of coalition *S* if he or she is critical in *S*.

Appendix

Table 3: The Public-Good Index values for g = 5

Categories of MWC's for g = 5	c_i(v)					$h_i(v)$				
	1	2	3	4	5	1	2	3	4	5
{1,2}, {1,3,4}, {1,3,5}, {1,4,5}, {2,3,4,5}	4	2	3	3	3	0.27	0.13	0.20	0.20	0.20
{1,2}, {1,3,4}, {1,3,5}, {2,3,4}, {2,3,5}	3	3	4	2	2	0.21	0.21	0.29	0.14	0.14
{1,2}, {1,3,4}, {1,3,5}, {1,4,5}, {2,3,4}	4	2	3	3	2	0.29	0.14	0.21	0.21	0.14
{1,2}, {1,3,4}, {1,3,5}, {1,4,5}, {2,3,4}, {2,3,5}	4	3	4	3	3	0.24	0.18	0.24	0.18	0.18
{1,2}, {1,3,4}, {1,3,5}, {1,4,5}, {2,3,4}, {2,3,5}, {2,4,5}	4	4	4	4	4	0.20	0.20	0.20	0.20	0.20
{1,2}, {1,3}, {2,3,4}, {2,3,5}	2	3	3	1	1	0.20	0.30	0.30	0.10	0.10
{1,2}, {1,3}, {1,4,5}, {2,3,4,5}	3	2	2	2	2	0.27	0.18	0.18	0.18	0.18
{1,2}, {1,3}, {1,4,5}, {2,3,4}	3	2	2	2	1	0.30	0.20	0.20	0.20	0.10
{1,2}, {1,3}, {1,4,5}, {2,3,4}, {2,3,5}	3	3	3	2	2	0.23	0.23	0.23	0.15	0.15
{1,2}, {1,3}, {1,4}, {2,3,4,5}	3	2	2	2	1	0.30	0.20	0.20	0.20	0.10
{1,2}, {1,3}, {1,4}, {2,3,4}	3	2	2	2	0	0.33	0.22	0.22	0.22	0.00
{1,2}, {1,3}, {1,4}, {1,5}, {2,3,4,5}	4	2	2	2	2	0.33	0.17	0.17	0.17	0.17
{1,2}, {1,3}, {1,4}, {1,5}	4	1	1	1	1	0.50	0.13	0.13	0.13	0.13
{1,2}, {1,3}, {2,3}	2	2	2	0	0	0.33	0.33	0.33	0.00	0.00
{1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,3,5}, {1,4,5}, {2,3,4,5}	6	4	4	4	4	0.27	0.18	0.18	0.18	0.18
{1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,3,5}, {1,4,5}, {2,3,4}	6	4	4	4	3	0.29	0.19	0.19	0.19	0.14
{1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,3,5}, {2,3,4}, {2,3,5}	5	5	5	3	3	0.24	0.24	0.24	0.14	0.14
{1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,3,5}, {1,4,5}, {2,3,4}, {2,3,5}	6	5	5	4	4	0.25	0.21	0.21	0.17	0.17
{1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,3,5}, {1,4,5}, {2,3,4}, {2,3,5}, {2,4,5}	6	6	5	5	5	0.22	0.22	0.19	0.19	0.19
{1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,3,5}, {1,4,5}, {2,3,4}, {2,3,5}, {2,4,5}, {3,4,5}	6	6	6	6	6	0.20	0.20	0.20	0.20	0.20

PART II

APPLICATIONS

PROPORTIONAL REPRESENTATION IN THE NATIONAL ASSEMBLY FOR WALES*

Abstract: The rule for electing members to the National Assembly for Wales gives each voter two votes, to be cast at the Assembly Constituency level, and at the bigger Assembly Electoral Region level. One third of the members to the assembly are elected by a form of proportional representation, where party support is calculated by aggregating the two votes. The voters are allowed to cast the second vote in favour of a different party than the one they earlier voted for, at the Assembly Constituency level. It is shown that this additional degree of freedom can frustrate the objective of obtaining better correspondence between party support and the number of seats. Also, the effects of this additional degree of freedom on the voting power of the parties on the Assembly Electoral Region level are shown using Straffin's partial homogeneity approach. Based on this analysis, a different system of proportional representation and a method of equating the distribution of voting power and the seat distribution are proposed.

1. Introduction

Few people would object to the idea that a system of election which allows a legislature to be captured by organised minorities is flawed. Even fewer people would argue that a party should secure greater representation or power in the parliament because the electoral support in favour of another party has increased. It is in keeping with the first of the ideas above that there is a move towards embracing some form of proportional representation in the United Kingdom. A tentative step towards experimenting with proportional representation was taken by central government in framing the rule for electing the National Assembly for Wales, which was elected the first time in May 1999 to take over the responsibilities that the Secretary of State exercises in Wales. Yet, this particular system of proportional representation chosen by government, described first in the White Paper entitled 'A Voice for Wales' (HMSO, 1997) and now part of the

* This chapter is a based upon Altunbas, Chrakravarty and Steffen (1999a). The author is grateful for helpful comments to Matthew Braham and Manfred Holler. An earlier version of this paper by Altunbas and Chakravarty (2000) entitled 'Proportional Representation in the Welsh Assembly' is published in *Public Choice* 103, 85-94.

‘Government of Wales Act 1998 (chapter 38)’ (HMSO, 1998), suffers from the flaw that Party *a* might find that its representation (in seats) in the National Assembly for Wales has decreased in favour of Party *b* because Party *b* has increased support (in votes) at the expense of Party *c*.

This state of affairs comes about because voters are given two votes, one to cast at the Assembly Constituency level and another to cast at the Assembly Electoral Region level. These two votes are aggregated for arriving at the percentage share of party support, the figure which enters into the calculation of the additional member seats in the National Assembly for Wales. A problem with an electoral system which suffers from the anomaly described above is that the legitimacy of the assembly elections might come into question when the voters discover that the system of election is capable of producing unintended consequences of the votes cast. This is especially important because one of the objectives of introducing the system of proportional representation is to make representation in the assembly more reflective of the pattern of votes cast (see, e.g., Michael, 1999).[1]

The procedure for allocating the additional member seats per Assembly Electoral Region entails four sets of calculations. Party support is measured using results of these calculations. The party gaining the greatest amount of support after the first set of calculations is allocated the first additional member seat, and so on.

The chapter is organised as follows: section 2 describes the rule for electing the assembly while section 3 outlines Straffin’s partial homogeneity approach to the measurement of voting power, explaining how voting power is measured in this chapter and why this should be done so. Section 4 illustrates above mentioned anomalies and their effects on voting power starting with party support based on the general election results in May 1992. Section 5 illustrates these anomalies and effects starting with party support on the 1997 general election data.[2] In continuation of the results obtained in sections 4 and 5, section 6 offers a different system of proportional representation and section 7 suggests a (theoretical) method to equate distribution of voting power and seat distribution. The conclusions are contained in section 8.

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[1] Another anomaly of the election rules for the National Assembly for Wales concerning the election the First Minister due to a fall in support for his own party is discussed in Altunbas *et al*. (2002).

[2] The data on parliamentary votes in the above elections are taken from a database supplied by UK-Elect.

2. The Electoral Rule

The electoral rules, the Additional Member System (AMS), are described in Sections 6 and 7 of the ‘Government of Wales Act 1998 (chapter 38)’ (HMSO, 1998).[3] Accordingly, each voter is to have two votes, and each voter will choose more than one member of the assembly. The first vote is to be cast to elect an assembly member by the *first-past-the-post system* (FPTPS) from the voter’s Assembly Constituency. There are 40 constituencies and 40 members of the assembly will be chosen by the FPTPS. These Assembly Constituencies which are the same as the existing Parliamentary Constituencies are then grouped into bigger units, comprising of the current boundaries for selecting members for the European Parliament. There are five such European Parliamentary Constituencies which here are called Assembly Electoral Regions. Each Assembly Electoral Region is allowed to contribute four additional members, a total of 20 for Wales, by a form of proportional representation. The composition of the five bigger constituencies is described in table 1.

The voters will be able to cast their second vote for the purpose of electing the additional members, and they can cast it differently from their first one. The sum of the votes - the votes cast on the Parliamentary and on the Assembly Electoral Region level - will then be recorded for each party in each of the larger Assembly Electoral Regions. An arithmetic formula is proposed to translate these total votes to allocate seats for the additional members from each Assembly Electoral Region using a modification of d’Hont’s allocation method as is described by De Meur (1987).

The four additional seats from each Assembly Electoral Region will be determined by (HMSO, 1997, p. 36):

1. counting the number of votes cast for each party list in the Assembly Electoral Region;
2. calculating the number of constituency seats won by each party throughout the Assembly Electoral Region;
3. dividing the number of each party’s party list votes by the number of constituency seats won by the party, *plus one*. The party with the highest number of votes after that calculation gains the first additional member;

[3] For a more easily readable formulation concerning the most important rules, see Annex C, entitled ‘Electoral Arrangements’, of the devolution White Paper (HMSO 1997).

Table 1: Composition of the Assembly Electoral Regions

Assembly Electoral Region	Corresponding Assembly Constituency	Number of Constituency Seats	Number of Assembly Seats
North Wales Add’l seats: 4	1. Ynys Mon 2. Caernarfon 3. Conwy 4. Clwyd West 5. Vale of Clwyd 6. Clwyd South 7. Delyn 8. Ayln and Deeside 9. Wrexham	9	13
Mid and West Wales Add’l seats: 4	10.Meirionnydd Nant Conwy 11.Ceredigion 12.Preseli Pembrokeshire 13.Carmarthen West and South Pembrokeshire 14.Carmarthen East and Dinefwr 15.Llanelli 16.Montgomeryshire 17.Brecon and Radnorshire	8	12
South Wales West Add’l seats: 4	18.Gower 19.Swansea East 20.Swansea West 21.Neath 22.Aberavon 23.Bridgend 24.Ogmore	7	11
South Wales Central Add’l seats: 4	25.Vale of Glamorgan 26.Pontypridd 27.Rhondda 28.Cynon Valley 29.Cardiff North 30.Cardiff West 31.Cardiff Central 32.Cardiff South and Penarth	8	12
South Wales East Add’l seats: 4	33.Newport East 34.Newport West 35.Monmouth 36.Torfaen 37.Islwyn 38.Caerphilly 39.Blaenau Gwent 40.Merthyr Tydfiland and Rhymney	8	12

4. repeating the calculation for the second to the fourth additional members, but in each case dividing the party list votes by the number of constituency seats won, *plus one*, *plus any additional seats allocated in previous rounds*.[4]

The system should ensure that all parties which command a significant level of support across Wales win some seats in the assembly.

3. Voting Power: Concept and Measurement

3.1 The Concept of Voting Power

Power is a central concept in political science. However, there is little agreement on how power is to be defined, how to observe and measure it. In this chapter, we follow Barry (1976) and Morriss' (1987) understanding of power as an *ability* or *capacity to do* something or the possession of control in a social environment. If a decision-making situation can be represented as a voting situation the literature on the theory and measurement of voting power provides a way to 'measure power' defined in that manner by determining how probable it is that a decision-maker has such an ability. This is done by calculating the probability that a decision-maker has the ability to do something, i.e. to force the outcome of a vote.

In our case the voting situation is that in the National Assembly for Wales. It is assumed that decision-makers are political parties. They are considered as being unions formed in order to carry out the common will of their members through parliamentary decision making, i.e. by voting on proposed motions. To simplify, we will assume each party's representatives vote as one bloc. So the number of votes of each party in the assembly - the voting weight of a party - is given by their number of seats. If the decision rule implicates some majority rule, only one of possible opposing motions on a certain issue will pass. The higher the likelihood that a party wins a poll, the higher will be the probability for the party to enforce its will and hence the 'more powerful' this party will be. Thus such probability - the probability with which each party will participate in a majority coalition - is used to measure the voting power of a party.

Given the distribution of parliamentary seats, we can calculate the probabilities of different parties using a measure of voting power. In this chapter, we use a measure based on Straffin's (1977) partial homogeneity approach.

[4] For the procedure as what to should happen in the event of two parties registering a tie at one of these stages of calculations - the White Paper (HMSO, 1997) is silent - see The „Government of Wales Act 1998 (chapter 38)" (HMSO, 1998, Section 7 (7) - (9)).

A special feature of these and some familiar measures is that preparatory work, i.e. the work which is needed for the creation of proposals, the bargaining process of amending the draft proposals before they are passed or rejected, and the bargaining for coalitions, is not modelled. It is simply assumed that the voters' resources in the final voting define a limiting condition for the bargaining and so the decision-making is modelled as a simple voting game (using the subclass of weighted voting games).

3.2 Straffin's Approach to the Measurement of Voting Power

Let $N = \{1, 2, ..., n\}$ be a set of parties (players) of a weighted voting game $[q; w]$, where $q \in [0,1]$ is the majority quota, which is the sum voting weights (number of seats) needed to attain a certain end (i.e. to win or block a bill), and $w = (w_1, w_2, ..., w_n)$ is the vector of voting weights (or seats) of each party $i \in N$. Let $\mathcal{W}$ be a collection of subsets $S \subseteq N$ with $\sum_{i \in S} w_i > q$, which is called the class of all winning coalitions. Then, define the subset of all crucial coalitions $\mathcal{C}$ as a collection of subsets $S \in \mathcal{W}$ where for each S at least one party $i \in S$ is a crucial party. Party i is called crucial for $S \in \mathcal{W}$, if S is a losing coalition without i: $S \backslash \{i\} \notin \mathcal{W}$. Finally, let $\mathcal{W}_i$ and $\mathcal{C}_i$ denote the classes of winning coalitions containing party i in the first case and additionally requiring player i to be crucial in the latter.

We have already defined above the voting power of a party as the probability with which a party will participate in a majority coalition. We now refine this definition by saying that the power of a party i is the probability that party i is a crucial member of a winning coalition, assuming that the parties acceptance decisions are independent: $\mathbf{P}_i(\mathcal{S}=S,\ S \in \mathcal{C}_i) = \mathbf{P}_i(\mathcal{S}=S,\ S \in \mathcal{W}_i) - \mathbf{P}_i(\mathcal{S}=S\backslash\{i\},\ S\backslash\{i\} \notin \mathcal{W})$, where $\mathcal{S}$ is a randomly chosen coalition. Calculating this probability, we obtain

$$(1) \qquad \mathbf{P}_i(\mathcal{S}=S,\ S \in \mathcal{C}_i) = \sum_{\substack{S \in \mathcal{C}_i \\ j \in S}} \prod_{j \in S \backslash \{i\}} p_j \prod_{\substack{j \notin S \backslash \{i\} \\ j \neq i}} (1 - p_j) = f'_i(p)$$

where p_j is the probability of party $j \in N$ to vote for a random bill to and p is the vector of such probabilities for all parties: $p = (p_1, p_2, ..., p_n)$.

We now have to make an assumption concerning relations of the elements of the 'acceptability vector' p. If we do not have any prior information of parties' attitude towards alternative bills per se, there are the following two standard assumptions:[5]

[5] Straffin (1977) shows that if we use the independence assumption party i's power polynomial leads to the Penrose (1946) / Banzhaf (1965) measure, which is known as the absolute or non-

- Independence assumption: $p_i \sim U[0,1]$
- Homogeneity assumption : $p_i = t$ with $t \sim U[0,1]$

The main difference between these is that under the homogeneity assumption, there is a common standard by which the parties evaluate a bill and thus the probabilities of parties' decisions are strictly correlated, while under the independence assumption, parties have the same distribution of voting for a proposal but act independently of each other. As $E[p_i] = ½ \ \forall \ i \in N$ under the independence assumption, we can say that this assumption is equivalent to assuming that any party will vote for any bill with probability ½ (Straffin, 1977, 1988).

In order to account for additional information concerning the relation of parties in a given voting body, we can combine these two assumptions leading to a partial homogeneity structure. For instance, we can assume that a party j has a certain standard ($p_j = t$), whereas the standard of another party h is exactly the opposite of the first one ($p_h = 1\text{-}t$), while all the other parties $l \in N \backslash \{j, h\}$ behave independently ($p_l = ½$). For this case, the mean value of party i's power polynomial can be written as (Kirman and Widgrén, 1995):

$$(2) \qquad Ef_i = \int_0^1 \ldots \int_0^1 f'_i(t, p_{\# j+1}, \ldots, p_{n-\# j-\# h}, 1-t)\, dt \; dp_{\# j+1} \ldots \, dp_{n-\# j-\# h}$$

where Ef_i is party i's voting power value, which we will use in our following analysis.[6]

3.3 Why Measuring Voting Power?

With the calculation of voting power values using measures of voting power, we can not only gain some information about the decisiveness of parties in parliamentary decision-making, but also the relation between party support, seat distribution and different decisions rules. This can be helpful for designing 'efficient' collective decision units and decision rules. Thus, concerning the design, there are two important problems to solve: The creation of the aggregation mechanism for seat distribution based on party support and the choice of the

normalized Banzhaf index, while the homogeneity assumption yields to the Shapley-Shubik (1954) index. For general considerations concerning other probability distributions of p_i, see Straffin (1978).

[6] For further applications, see, e.g., Kirman and Widgrén (1995), Straffin (1977) and (1988), Straffin, Davis and Brams (1982) and Widgrén (1993).

decision rule. We suggest a possible solution concerning these problems in section 6 and 7. It should be mentioned in this context that in many representational bodies, seats are distributed among the parties roughly in proportion of their relative support. This practice seems to be based on the idea that voting power is reflected in the relative distribution of seats. But there is a number of studies suggesting that the voting power of parties in general does not coincide with their share of the seats[7], even if the discrepancy of seat and power distribution varies with the choice of the measure of voting power. In sections 4 and 5 we will see that this is even true for the National Assembly for Wales.

4. The 1992 Election

We start our analyses with an investigation using the data of the general election results in May 1992 as a basis to compute the composition of the National Assembly for Wales.[8]

We first calculate the distribution of assembly seats, assuming that voters cast both votes identically. Thus, the total votes received by one party in an Assembly Electoral Region are twice the sum of votes gained in the Assembly Constituencies belonging to that region. After that, we calculate the voting power of each party for the cases of an absolute (q = 1/2) and a qualified majority quota (q = 2/3) on the basis of the original votes and seats determined in the step before. For computing the power values, we make the following assumptions concerning the acceptability vector $p = (p_{\text{Labour}}, p_{\text{Conservaties}}, p_{\text{Liberal Democrats}}, p_{\text{Plaid Cymru}})$, which contains the p_i's of the four most important parties in Wales:[9] It is well known that it is possible to position Labour, Conservative and Liberal Democrats in an one dimensional ideological space, putting Labour on the left, Conservative on the right and Liberal Democrats between them, and that in general, each of these parties has a common standard.[10] Keeping this in mind, we can model the p_i's of

[7] See, e.g., Banzhaf (1968), Bomsdorf (1980; 1982), Brams (1975), Holler (1982a), Shapley and Shubik (1954), Turnovec (1994), Weiersmüller (1971), Widgrén (1994).

[8] Note, that this study was undertaken before the first ever elections of the National Assembly for Wales in May 1999. For an initial study of the actual 1999 election data see Altunbas *et al.* (2000).

[9] For purposes of calculation, we have also considered the Greens and for the 1997 election, the Referendum Party as a fifth homogenous party, with $p_{\text{Green}} = t'$ and $p_{\text{Referendum Party}} = t'$, respectively, which have voting power in some cases even if they will never get a seat under the current rules for the seat distribution, while the other small parties can be ignored because their influence is negligible, i.e. they additionally have no voting power in the decision rules under consideration, and they also do not have an impact on the voting power of the other parties.

[10] See, e.g., Gallagher, Laver and Mair (1995, pp. 152-157). Note, that this was in fact the case when this chapter was written. Unfortunately, when the research was finished, Labour and

these parties using a partial homogeneity structure with $p_{\text{Labour}} = t$, $p_{\text{Conservaties}} = 1-t$, and $p_{\text{Liberal Democrats}} = (t + (1-t)) / 2 = ½$. In contrast to these parties, Plaid Cymru is assumed to behave independently, i.e. $p_{\text{Plaid Cymru}} = ½$, because it contains three different political groups (cultural nationalists, a right-wing extremist fraction and social democrats), which hold together only by their common language (Welsh), that is spoken by about 20% of the Welsh population and by their opinion that there is a separate Welsh identity.

For reasons of simplicity, we restrict our analysis to some of the Assembly Electoral Regions instead of analysing the effects for the whole National Assembly for Wales, i.e. we make a partial analysis, because effects and structure of results will be the same in case of a total analysis.

We start our analysis with the Assembly Electoral Region of South West Wales. Table 4.1. shows that the relation between party support in South West Wales and seat distribution is relatively close though the chosen rule for proportional representation seems to be adequate.

Concerning the distribution of voting power, the case of absolute majority seems obvious enough to need no further attention. This is quite different for the rule of qualified majority. A detailed look at the party support and distribution of voting power shows us that these values are in an inverse proportion for Liberal Democrats and Conservatives. We can observe the same in the case of seat distribution and voting power as well. It is what is in general called the violation of *local monotonicity*. That is that property, that a party *i* with a larger party support (in general terms ′with more resources′) than a party *j*, should have at least as much voting power than party *j*.

Liberal Democrates have changed their positions. Thus the assumption that the above standards of the parties can be seen as historical based *a priori information* (see Steffen, 2000) has turned out to be inappropriate regarding the aim of an a priori analysis of the voting rules for the National Assembly for Wales. This has led to further research on this issue. First results have been represented at the 2000 Annual Meeting of the European Public Choice Society, Sienna, 26-29 April 2000 (see Altunbas *et al*., 2000).

Table 4.1: Votes go from Labour to Plaid Cymru but the Conservatives lose one Seat

Assembly Electoral Region: South West Wales								
Party	Total votes (in %)		Voting power (in %)		Seats		Voting power (in %)	
	1st vote	2nd vote	absolute majority	qualified majority	absolute	in %	absolute majority	qualified majority
Both votes identical								
Labour	61.26	61.26	100.00	81.25	7	63.64	100.00	75.00
Conservative	23.72	23.72	0.00	18.75	3	27.27	0.00	25.00
Liberal Democrats	9.25	9.25	0.00	25.00	1	9.09	0.00	33.33
Plaid Cymru	5.15	5.15	0.00	8.33	0	0.00	0.00	0.00
Second vote cast differently								
Labour	61.26	**55.00**	100.00	81.25	7	63.64	100.00	37.50
Conservative	23.72	23.72	0.00	18.75	**2**	18.18	0.00	25.00
Liberal Democrats	9.25	9.25	0.00	25.00	1	9.09	0.00	16.67
Plaid Cymru	5.15	**11.41**	0.00	8.33	**1**	9.09	0.00	16.67

As argued in Felsenthal and Machover (1995, 1998) local monotonicity should be fulfilled by any reasonable measure of (a priori) voting power. We agree on this but on the other hand, we have to take into account that, as discussed in Braham and Steffen (2002c), the common notion of local monotonicity is too restrictive. Under Straffin's approach it is only applicable to the independence and the homogeneity assumption as in both cases 'party support' and 'resources' coincide. But this does no longer hold if we apply a partial homogeneity assumption as we have done here. In this case, where we have allowed for additional information, we have to use a more general definition of local monotonicity which also takes into account parties' 'standards of behavior' (i.e. the different assumptions about the elements of the acceptability vector) and their interaction as a further component of parties' resources resulting from the additional information. Then, following Braham and Steffen's line of thought, the violation of local monotonicity (in weights) in our case turns out to be an expected and intuitively reasonable result.

Another interesting aspect follows from the comparison of the voting power for party support and seat distribution. Here, we see that the chosen rule for proportional representation leads to a decrease in power for Labour from 81.25 % to 75.00 % (and Plaid Cymru from 8.33 % to 0.00%) while the power of

Conservatives and Liberal democrats increases from 18.75 % to 25.00 % and from 25.00 % to 33.33 % respectively.

In a next step, we simulate what would happen if the second vote was cast differently, leaving the votes cast at the Assembly Constituency level untouched. That is, the simulations affect only the election of additional members.

First, we assume that many of those who vote Labour in the constituencies transfer their allegiance to Plaid Cymru in exercising their second ballot. But the seat distribution reveals that Plaid Cymru gains a seat from the Conservatives instead from Labour. Concerning the distribution of voting power, nothing has changed in the case of party support. But looking at the distribution of voting power in relation to the new seat distribution, we can see that Labour and Liberal Democrats lose voting power without a change in their number of seats, while voting power of Plaid Cymru and the Conservatives, which lose a seat, remains constant or increases. This effect is known as the paradox of redistribution in literature (Fischer and Schotter, 1978; Schotter, 1982) and is based on possibilities of forming a winning coalition.[11]

Table 4.2: Votes go from Labour to Liberal Democrates but the Conservatives lose one Seat

Assembly Electoral Region: South West Wales								
Party	Total votes (in %)		Voting power (in %)		Seats		Voting power (in %)	
	1st vote	2nd vote	absolute majority	qualified majority	absolute	in %	absolute majority	qualified majority
Both votes identical								
Labour	61.26	61.26	100.00	81.25	7	63.64	100.00	75.00
Conservative	23.72	23.72	0.00	18.75	3	27.27	0.00	25.00
Liberal Democrats	9.25	9.25	0.00	25.00	1	9.09	0.00	33.33
Plaid Cymru	5.15	5.15	0.00	8.33	0	0.00	0.00	0.00
Second vote cast differently								
Labour	61.26	**47.76**	100.00	75.00	7	63.64	100.00	75.00
Conservative	23.72	23.72	0.00	25.00	**2**	18.18	0.00	25.00
Liberal Democrats	9.25	**22.75**	0.00	33.33	**2**	18.18	0.00	33.33
Plaid Cymru	5.15	5.15	0.00	0.00	0	0.00	0.00	0.00

[11] For an interpretation and some comments on this paradox, see Straffin (1982).

We obtain similar results if we assume that the Liberal Party gains votes from Labour, as has been shown in table 4.2. In this case, Liberal Democrats gain an assembly seat at the expense of the Conservatives, but regarding the distribution of voting power nothing changes. In contrast to the simulation before, the distribution of power in case of party support has changed now: while the direction of the variation in party support and voting power is concurring for Labour and Liberal Democrats, Conservatives gain some voting power and Plaid Cymru loses its voting power without any change in its party support. Furthermore, it is interesting to note that in this case, voting power regarding party support and seat distribution are identical.

In table 4.3, we illustrate a case where an increase in the Liberal Democrats' second votes at the expense of the Conservatives does not benefit the Liberals; instead Labour gains an extra seat and all voting power without obtaining any extra votes.

Table 4.3: Votes go from Conservatives to Liberal Democrats but Labour gains one Seat

Assembly Electoral Region: South West Wales								
Party	Total votes (in %)		Voting power (in %)		Seats		Voting power (in %)	
	1st vote	2nd vote	absolute majority	qualified majority	absolute	in %	absolute majority	qualified majority
Both votes identical								
Labour	61.26	61.26	100.00	81.25	7	63.64	100.00	75.00
Conservative	23.72	23.72	0.00	18.75	3	27.27	0.00	25.00
Liberal Democrats	9.25	9.25	0.00	25.00	1	9.09	0.00	33.33
Plaid Cymru	5.15	5.15	0.00	8.33	0	0.00	0.00	0.00
Second vote cast differently								
Labour	61.26	61.26	100.00	81.25	**8**	72.73	100.00	100.00
Conservative	23.72	**17.72**	0.00	18.75	**2**	18.18	0.00	0.00
Liberal Democrats	9.25	**15.25**	0.00	25.00	1	9.09	0.00	0.00
Plaid Cymru	5.15	5.15	0.00	8.33	0	0.00	0.00	0.00

We obtain a similar example in table 4.4 for the Assembly Electoral Region Mid and West Wales, where an increase in Plaid Cymru votes in the exercise of the second set of ballot papers at the expense of the Conservatives could give an extra seat to the Liberal Democrats. Additionally, we have some interesting effects on

Table 4.4: Votes go from Conservatives to Plaid Cymru but Liberal Democrats gain one Seat

Assembly Electoral Region: Mid and West Wales								
Party	Total votes (in %)		Voting power (in %)		Seats		Voting power (in %)	
	1st vote	2nd vote	absolute majority	qualified majority	absolute	in %	absolute majority	qualified majority
Both votes identical								
Labour	32.58	32.58	75.00	37.50	4	33.33	25.00	25.00
Conservative	28.88	28.88	25.00	37.50	4	33.33	25.00	25.00
Liberal Democrats	20.25	20.25	33.33	41.67	2	16.67	33.33	8.33
Plaid Cymru	17.67	17.67	33.33	41.67	2	16.67	33.33	8.33
Second vote cast differently								
Labour	32.58	32.58	75.00	37.50	4	33.33	50.00	37.50
Conservative	28.88	**24.88**	25.00	37.50	**3**	25.00	25.00	25.00
Liberal Democrats	20.25	20.25	33.33	41.67	**3**	25.00	16.67	25.00
Plaid Cymru	17.67	**21.67**	33.33	41.67	2	16.67	16.67	25.00

voting power here. Where the voting power concerning party support remains unchanged as in the simulation before, the effects on voting power related to seat distribution are quite different. Whilst the direction of the change in power for Conservatives and Liberal Democrats corresponds with variation in seat distribution for the qualified majority quota, we have an opposite effect for the absolute majority quota and on the other hand a reverse change in power for both quotas for Labour and Plaid Cymru without any change in their number of seats in the assembly.

Another example regarding the effects on the seat distribution but with some different effects on voting power is given by table 4.5 for the Assembly Electoral Region of North Wales, where a decrease in Labour votes in favour of Plaid Cymru could give an extra assembly seat instead to Liberal Democrats.

In all examples above, a voter can only predict that the party from which the allegiance is taken away in the exercise of casting the second vote cannot gain seats as a result, but the voter cannot predict that this is the party which will necessarily gain fewer seats as a consequence. A voter cannot predict who will be losing a seat. Another feature of the examples above is that the party which is more successful in increasing the share of second votes may find that the reward of an

Table 4.5: Votes go from Labour to Plaid Cymru but Liberal Democrats gain one Seat

Assembly Electoral Region: North Wales								
Party	Total votes (in %)		Voting power (in %)		Seats		Voting power (in %)	
	1st vote	2nd vote	absolute majority	qualified majority	absolute	in %	absolute majority	qualified majority
Both votes identical								
Labour	37.32	37.32	75.00	50.00	5	38.46	50.00	50.00
Conservative	35.72	35.72	25.00	50.00	5	38.46	50.00	50.00
Liberal Democrats	12.98	12.98	33.33	0.00	1	7.69	0.00	0.00
Plaid Cymru	13.35	13.35	33.33	0.00	2	15.38	66.67	0.00
Second vote cast differently								
Labour	37.32	**26.32**	50.00	50.00	**4**	30.77	25.00	37.50
Conservative	35.72	35.72	50.00	50.00	5	38.46	75.00	62.50
Liberal Democrats	12.98	12.98	0.00	0.00	**2**	15.38	33.33	16.67
Plaid Cymru	13.35	**24.35**	66.67	0.00	2	15.38	33.33	16.67

extra assembly seat goes to another party. Thus, a party could gain or lose seats without any change in the proportion of votes cast in its favour. Concerning the effects on voting power, we have seen that voting power regarding party support and seat distribution does not correspond in general. Looking upon the power effects resulting from our simulations and comparing the initial distribution of voting power with the distribution where the second ballot is cast differently, we cannot observe any systematic variation of the distribution of voting power.

5. The 1997 Election

The pattern of simulation results reported in section 4 can be repeated for the 1997 General Election. An illustration is given in table 5.1 and table 5.2. One interesting feature of these results is that Plaid Cymru will have to cause a collapse of Labour votes to gain an extra seat in North Wales, but Liberal Democrats can capture an extra seat with more modest transfer of votes from Labour. In both cases, Conservatives lose one seat in consequence of the decline in Labour support in the second vote.

Regarding the relation between party support and voting power, we can observe that if both votes are cast identically, we have a contrary relation between party support and voting power for Conservatives, Liberal Democrats and Plaid Cymru for the case of the absolute majority quota and that voting power for all parties, except Liberal Democrats, remains constant for the resulting seat distribution in the assembly, while the latter lose all their power even if they receive one seat in the assembly.

Table 5.1: Votes go from Labour to Plaid Cymru but Conservatives lose one Seat

Assembly Electoral Region: North Wales								
Party	Total votes (in %)		Voting power (in %)		Seats		Voting power (in %)	
	1st vote	2nd vote	absolute majority	qualified majority	absolute	in %	absolute majority	qualified majority
Both votes identical								
Labour	46.72	46.72	75.00	62.50	7	53.85	100.00	75.00
Conservative	24.28	24.28	25.00	37.50	**3**	23.08	0.00	25.00
Liberal Democrats	11.78	11.78	33.33	16.67	1	7.69	0.00	0.00
Plaid Cymru	14.47	14.47	33.33	16.67	**2**	15.38	0.00	33.33
Second vote cast differently								
Labour	46.72	25.72	75.00	56.25	7	53.85	100.00	75.00
Conservative	24.28	24.28	25.00	31.25	**2**	15.38	0.00	25.00
Liberal Democrats	11.78	11.78	33.33	20.83	1	7.69	0.00	0.00
Plaid Cymru	14.47	35.47	33.33	20.83	**3**	23.08	0.00	33.33

Looking upon the simulation when the second vote is cast differently, in table 5.1, we see that the hypothetical transfer of votes has no effect on voting power with the exception of absolute majority quota in case of party support, where the effect on the power of Labour and Plaid Cymru coincides with the change in party support, whereas Conservatives lose and Liberal Democrats gain voting power without a change in their votes. A similar effect occurs in table 5.2 in case of qualified majority quota for the seat distribution. While the variation of power corresponds to the change in seat distribution for Conservatives and Liberal Democrats, Labour gains and Plaid Cymru loses some power without any change in their number of seats.

Table 5.2: Votes go from Labour to Liberals but Conservatives lose one Seat

Assembly Electoral Region: North Wales								
Party	Total votes (in %)		Voting power (in %)		Seats		Voting power (in %)	
	1st vote	2nd vote	absolute majority	qualified majority	absolute	in %	absolute majority	qualified majority
Both votes identical								
Labour	46.72	46.72	75.00	62.50	7	58.33	100.00	75.00
Conservative	24.28	24.28	25.00	37.50	3	25.00	0.00	25.00
Liberal Democrats	11.78	11.78	33.33	16.67	1	8.33	0.00	0.00
Plaid Cymru	14.47	14.47	33.33	16.67	2	16.67	0.00	33.33
Second vote cast differently								
Labour	46.72	36.72	75.00	62.50	7	58.33	100.00	87.50
Conservative	24.28	24.28	25.00	37.50	2	16.67	0.00	12.50
Liberal Democrats	11.78	21.78	33.33	16.67	2	16.67	0.00	16.67
Plaid Cymru	14.47	14.47	33.33	16.67	2	16.67	0.00	16.67

6. A Model for Designing a Proportional Representation

As we have seen from the last two sections, the election rule proposed in the ‘Government of Wales Act 1998 (chapter 38)’ poses a dilemma and some counter-intuitive results. In this section, inspired by Gambarelli (1999) and Potthoff and Brams (1998) and Te Riele (1978), we propose a system of proportional representation dealing with the assignment of seats to parties according to their support using the method of integer quadratic programming (IQP).[12] But before presenting this model, we have to make some general considerations concerning proportional representation.

The purpose of proportional representation is the delegation of ‘authority’ or ‘power’ from a larger body of persons to a smaller one. While designing the rule for this procedure, it has to be ensured that the relative frequency of occurrence of some characteristics in the larger body should be roughly identical to those in the smaller one (Nurmi, 1984). Thus, our first step will be to lay down what should be the larger body and who are the persons in this body. Following Laruelle and Widgrén (1998), we can distinguish between three different cases:

[12] The assignment of parliament seats to electoral districts is another issue of proportional representation.

(i) The case of an association of Assembly Constituencies where each Assembly Constituency is treated equally.

(ii) The case of a single state where each individual citizen is treated equally.

(iii) The case of a federal state where each Assembly Constituency is treated in accordance to a weighted average between the extreme cases (i) and (ii).

Considering the proposed system for the proportional representation in the National Assembly for Wales, we notice that this belongs to the case of a federal state.

While designing a model of proportional representation, we also have to ensure that the elected body is able to make decisions, i.e. that there are not too many parties represented in the assembly (Laakso and Taagepera, 1982). This criteria of stability which is a trade-off relation to proportional representation seems to be satisfied by the above mentioned rule for the National Assembly for Wales, where no more than four parties will obtain a seat in the assembly. But indeed, this bias in the electoral system should not be as extreme as to permit a single party with considerably less than 50% of the party support to form a majority government.

The number of parties in the assembly can be controlled by denying representation to small parties whose vote falls below a certain threshold like it is used in Germany or by using a simple majority rule with single-seat districts as in the case of the National Assembly for Wales.

We now propose a model of proportional representation with reference to the above mentioned case of a federal state maintaining the Assembly Constituencies and the Assembly Electoral Regions introduced in section 2.[13] Mathematically, the problem consists of transforming an ordered set of non-negative real numbers (voting weights) into integers (seats) with respect to certain constraints (Gambarelli, 1999). We suggest minimising 'misrepresentation'[14] in the assembly, by minimising the sum of the quadratic differences between share of votes w_{ij} and seats s_{ij}/S_j for each party i and each Assembly Electoral Region j, where s_i denotes the number of seats for a party, and hence is to be modelled as a general integer (*gin*), while S denotes the total number of seats in the assembly.[15] For reason of

[13] But with only a few simple modifications the proposed model can also be used to model the other two cases mentioned above, e.g., dropping out the Assembly Electoral Region index j for the case of a single state in the following model.

[14] Gambarelli (1999) calls this the percentage error.

[15] For some alternative distance minimising procedures, see, e.g., Te Riele (1978).

stability, we furthermore suggest introducing a threshold $D > 0$. Therefore, we have to minimise the following objective function (3) subject to the constraints (4) to (11):

(3) $\min_{s_{ij}} \sum_{i,j} \left(w_{ij}^{ad} - s_{ij} / S_j \right)^2$ s.t.

(4) $\sum_i s_{ij} = S_j \qquad \forall j$

(5) $\sum_j w_{ij} / \#j = w_i \qquad \forall i$

(6) $w_i \ (1\text{-}m_i) < D \qquad \forall i$

(7) $w_i \geq D m_i \qquad \forall i$

(8) $s_{ij} \leq \mu_i \ M \qquad \forall i, j$ M: sufficiently large number

(9) $\mu_i = 0 \vee 1 \qquad \forall i$

(10) $w_{ij}^{ad} = m_i w_{ij} \Big/ \sum_i w_{ij} m_i \qquad \forall i, j$

(11) $gin \ s_{ij} \qquad \forall i, j$

Constraint (4) ensures that all seats for each Assembly Electoral Region will be distributed, while (5) to (9) are needed to implement the threshold D concerning party support as a necessary condition to gain a seat in the assembly. In this context, using the binary variable μ_i (6) to (8) ensure that only a party whose party support is equal or larger than D can gain a seat. Constraint (10) is an adjustment constraint which normalises voting weights of the parties with $w_i > D$. The integer restriction (11) concludes the model.

A pleasant property of the suggested model which has been implicitly proved by Te Riele (1978), is that this model fulfils the Hare constraints and an intuitive monotonicity condition together Gambarelli (1999) has called fundamental criteria for proportional representation.[16]

The Hare constraints ensure, that no party would obtain fewer seats than rounding down or up its Hare quota[17], which is defined as the product of the voting

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[16] Te Riele (1978) has proved that the proposed model is equivalent to the method of the Greatest Remainders, also known as the method of Roget (Lisman, 1973), which was proposed already in 1792 by Hamilton (Syrett, 1966) and which obviously fulfils the mentioned criteria. The method of Greatest Remainders is used, e.g., in Greece (Gallagher, Laver and Mair ,1995, p. 282).

[17] Lisman (1973) calls this the 'exact distribution'.

weight of party i in constituency j and the total number of seats in constituency j taking into account the threshold D via the binary variable μ_i:

(12) $h_{ij} = w_{ij}^{ad} S_j m_i$ $\quad\forall\ i, j$ („Hare quota")

(13) $h_{ij}^{\min} = \text{round down}\ (h_{ij})$ $\quad\forall\ i, j$ („Hare minimum")

(14) $h_{ij}^{\max} = \text{round up}\ (h_{ij})$ $\quad\forall\ i, j$ („Hare maximum")

(15) $h_{ij}^{\min} \leq s_{ij}$ $\quad\forall\ i, j$

(16) $h_{ij}^{\max} \geq s_{ij}$ $\quad\forall\ i, j$

while monotonicity is defined here as follows: A party i which has a smaller party support than party k should not have more seats than party k:

(17) $\left(w_{kj}^{ad} - w_{ij}^{ad}\right)\left(s_{kj} - s_{ij}\right) \geq 0 \quad \forall\ k, i, j$ and $k \neq i$

Applying the above model to the case of the National Assembly for Wales for the case of a federal state (j = 5) and for the case of a single state dropping out the Assembly Electoral Region index j, we obtain the results shown in table 6.

Table 6: Proportional Representation: A Comparison of Different Systems (60 Seats)

The 1992 Election								
Party	Party Support		Seats AMS		Seats IQP (j = 1)		Seats IQP (j = 5)	
	original	adjusted	Total	in %	Total	in %	Total	in %
Labour	49.68	50.02	31	51.67	30	50.00	30	50.00
Conservatives	28.39	28.58	19	31.67	17	28.33	18	30.00
Liberal Democrats	12.43	12.51	6	10.00	8	13.33	7	11.67
Plaid Cymru	8.83	8.89	4	6.67	5	8.33	5	8.33

The 1997 Election								
Party	Party Support		Seats AMS		Seats IQP (j = 1)		Seats IQP (j = 5)	
	original	adjusted	Total	in %	Total	in %	Total	in %
Labour	54.74	56.63	37	61.67	34	56.67	33	55.00
Conservatives	19.60	20.28	13	21.67	12	20.00	13	21.67
Liberal Democrats	12.36	12.79	6	10.00	8	13.33	7	11.67
Plaid Cymru	9.96	10.30	4	6.67	6	10.00	7	11.67

We can see that the seat distributions using the IQP model lead to results which are closer to (adjusted) party support than the results which are obtained by using the proposed method of the ‘Government of Wales Act 1998 (chapter 38)’, which leads to a distortion in favour of stronger parties and to the anomalies mentioned above. The anomalies will be prevented by using the IQP model.

Moreover, the effect that we may call ‘cost of federalism’ is obvious, i.e. the distortion with respect to (adjusted) party support is larger for the case of a federal state than for a single state.

7. Equating Voting Power to Seat Distribution

After solving the problem of proportional seat distribution, the next step is to address how to trigger the change from a given electoral system to one in which the distribution of party support is identical with the distribution of voting power. This seems to be worthwhile, because it is evident that the idea of representational democracy rests on this identity.

An obvious answer to the problem just posed is to derive the seat distribution from the distribution of party support so that the latter is first transformed into power values of each party and then, each party is given the share of seats that corresponds to its share of power. This procedure would technically work if just one decision rule is applied in the voting body. In most parliaments, this is not the case; some issues require simple majority decisions, other 2/3 or even larger majorities. Thus the problem becomes now one of weighing different values of the chosen measure of voting power for different decision rules. In the case of the National Assembly for Wales, where the assumed decision rules are ½ and 2/3, we probably can use an unweighted average of the chosen power values as the measure of the overall power of parties. Indeed, this may be a justifiable solution. Intuitively, one could suggest that the issues requiring larger majorities are more important than others and that consequently, the power values for those decision rules should be given more weight than others. However, the question of how much more weight remains a problem. On the other hand, one can propose weighing the values using the relative frequencies by which issues belonging to the domain of a given decision rule have in the past been put into the agenda of the voting body. This would, in most cases, mean putting most weight on the simple majority rule (Nurmi, 1982).

A possible solution to this problem is given by Berg and Holler (1986) and Holler (1985, 1987) where the concept of strict proportional power (SPP) is proposed, by using a randomised decision rule and interpreting the power values as expectation values. When a random sample over some discrete decision rules is allowed, we can obtain a power distribution equating the seat distribution, if the following equation system is met with $\sum_{j=1}^{m} g_j = 1$ and $0 > g_j > 1 \; \forall \, j$:

(18) $$\left(g_1, g_2, \ldots, g_m\right)\left\{Ef\left(q_1\right), Ef\left(q_2\right), \ldots, Ef\left(q_m\right)\right\} = \left(w_1, w_2, \ldots, w_n\right)$$

$$\Leftrightarrow \; \sum_{j=1}^{m} g_j p_i\left(q_j\right) = w_i \qquad \forall \; i = 1, 2, \ldots, n$$

$$\Leftrightarrow \; g_1 Ef_i(q_1) + g_2 Ef_i(q_2) + \ldots + g_m Ef_i(q_m) = w_i \; \forall \; i = 1, 2, \ldots, n$$

where the $g_j \in g = (g_1, g_2, \ldots, g_m)$ are the probabilities or relative frequencies to derive with which the different decision rules $q_j \in q = (q_1, q_2, \ldots, q_m)$ should occur and the $Ef_i(q_j)$'s are the different power values for each decision rule q_j for a party i. The solution of this system is possible, if we find m g_j-levels, with corresponding power vectors $Ef(q_j)$ which are linearly independent, such that the vector of voting weights w is interior to the (convex) space generated by the m power vectors $Ef(q_j)$ (Berg and Holler, 1986).

If we interpret the g_j's as relative frequencies, this implies that each decision rule $q_j \in q$ will be realised g_j times, i.e. each decision rule will be applied in a legislative period. Using the alternative interpretation and considering the g_j's as probabilities, each decision rule $q_j \in q$ has the probability g_j to be put into reality, i.e. in the extreme case only one $g_j \in g$ will be singled out by some random mechanism.

The first possibility will be preferred by risk averse voters. But this procedure might affect stability and continuity of the decision making process because decisions made by a voting body often form only a part of an integrated whole. Instead, the latter possibility does not pose these problems as it is theoretically sufficient to make a random choice of a decision rule at the beginning of a election period, when the seat distribution is determined. For purposes of application, however, we think that the decision rule should be changed at shorter intervals, e.g., every year within the legislative period of four years for the National Assembly for Wales. Over a period of time, the actual voting power obtained by the parties is then likely to be close to the voting weights or seat distribution. Furthermore, Berg and Holler (1986) show that it is normally possible to prevent including decision rules that might endanger the functioning of a political system, such as a veto rule. Moreover, the proposed concept avoids majority (of votes) dominating the minority all the time, such that every party in the assembly will have a chance to influence decision making.

Table 7: The Probability vector g for the 1992 Election

The 1992 Election					
Party i	Party Support	Seats IQP Total	in %	Voting power $Ef_{ij}(q_j)$ (in %)	Probability g_j
Labour	49.68	30	50.00	0.50: 87.50	19.28 %
				0.60: 75.00	10.80 %
				0.65: 62.50	4.57 %
				0.80: 37.50	58.27 %
Conservatives	28.39	18	30.00	0.50: 12.50	19.28 %
				0.60: 25.00	10.80 %
				0.65: 31.25	4.57 %
				0.80: 37.50	58.27 %
Liberal Democrats	12.43	7	11.67	0.50: 16.67	19.28 %
				0.60: 33.33	10.80 %
				0.65: 16.67	4.57 %
				0.80: 8.33	58.27 %
Plaid Cymru	8.83	5	8.33	0.50: 16.67	19.28 %
				0.60: 0.00	10.80 %
				0.65: 16.67	4.57 %
				0.80: 8.33	58.27 %

Applying the concept of SPP to the creation of the National Assembly for Wales on the basis of the 1992 Election, we have chosen vector q = (0.50, 0.60, 0.65, 0.80) as the decision rule vector. Solving the equation system (18), we obtain g = (0.1928, 0.1080, 0.0457, 0.5827) as the corresponding vector of relative frequencies of the decision rules. This is shown in table 7 which additionally includes the power values for the different parties resulting from the different decision rules.

Even though we have obtained a situation in which the distribution of party support is identical to the distribution of voting power, this result leads to a problem concerning the ability of the voting body to make decisions as we have a high probability for q = 0.80, and as has been shown by Duncan Black, the greater the size of a majority quota, the lower will be the chance to displace a status quo by a new decision (McLean *et al*., 1998, pp. 103-118).[18]

[18] This reponsiveness to the overall nature of a decision rule could be analyzed applying Coleman's (1971) *power of the collectivitiy to act* or Felsenthal and Machover's (1998, p. 62) *coefficient of resistance* $R(\mathcal{W})$ which is a linear transformation of Coleman's measure. One can think of Coleman's measure in the following sense. In an assembly of n members, there are 2^n distinct partitions into the set of 'yes' and 'no' voters, then the *power of a collectivity to act* is

8. Conclusion

There is growing uneasiness in Great Britain about the lack of correspondence between party support at the polls and party seats. While there is little chance of constitutional reform to embrace proportional representation at Westminster Parliament, there is willingness within the government to experiment with proportional representation elsewhere. The 'Government of Wales Act 1998 (chapter 38)' (HMSO, 1998) contains the implicit admission that the application of the FPTPS might keep out Conservative party from the assembly, even with 20 % of the votes. Table 8.1 outlines what would happen if there were 40 assembly members, elected from the current parliamentary boundaries using the FPTPS. The results based on the voting patterns in 1992 and 1997 are shown in this table.

Table 8.1: First-Past-the-post (40 Seats)

Party	Votes (in %)	Voting power (in %)		Seats		Voting power (in %)	
		absolute majority	qualified majority	absolute	in %	absolute majority	qualified majority
1992							
Labour	49.68	87.50	62.50	27	67.50	100.00	100.00
Conservative	28.39	12.50	37.50	8	20.00	0.00	0.00
Liberal Democrats	12.43	16.67	16..67	1	2.50	0.00	0.00
Plaid Cymru	8.83	16.67	16..67	4	10.00	0.00	0.00
1997							
Labour	54.74	100.00	81.25	34	85.00	100.00	100.00
Conservative	19.60	0.00	18.75	0	0.00	0.00	0.00
Liberal Democrats	12.36	0.00	25.00	2	5.00	0.00	0.00
Plaid Cymru	9.96	0.00	8.33	4	10.00	0.00	0.00

Proportional representation is needed not just to allow representation by small parties in the assembly, but to allow for minimal representation by the Conservative Party. The voting system proposed for the assembly will augment the

defined as the a priori probability that a proposal that is put before the assembly will be approved, i.e. it is a measure of success or 'complaisance' of $\mathcal{W}$. Formally, Coleman defines his measure as: $A(\mathcal{W}) =_{\text{def}} |\mathcal{W}|/2^n$. However, $A(\mathcal{W})$ implies independence as an underlying assumption. For an analysis of a structured decision-making situation we have to generalize $A(\mathcal{W})$ to be capable of a partial homogeneity structure. It is easy to see that we can derive such an generalization, $\tilde{A}(\mathcal{W})$, which contains $A(\mathcal{W})$ under independence as a special case as $\tilde{A}(\mathcal{W}) =_{\text{def}} \sum_{S\in\mathcal{W}} \int \dots \int f(p_1,\dots,p_n)\, dp_1,\dots,dp_n$.

number of seats by 20, and these seats will be allocated by a formula which departs from the FPTPS. Including these additional seats, the composition of the assembly is given in table 8.2.

Table 8.2: AMS (60 Seats)

Party	Votes (in %)	Voting power (in %)		Seats		Voting power (in %)	
		absolute majority	qualified majority	absolute	in %	absolute majority	qualified majority
1992							
Labour	49.68	87.50	62.50	31	51.67	100.00	62.50
Conservative	28.39	12.50	37.50	19	31.67	0.00	37.50
Liberal Democrats	12.43	16.67	16.67	6	10.00	0.00	16.67
Plaid Cymru	8.83	16.67	16.67	4	6.67	0.00	16.67
1997							
Labour	54.74	100.00	81.25	37	61.67	100.00	87,50
Conservative	19.60	0.00	18.75	13	21.67	0.00	12.50
Liberal Democrats	12.36	0.00	25.00	6	10.00	0.00	16.67
Plaid Cymru	9.96	0.00	8.33	4	6.67	0.00	16.67

Table 8.2 assumes that the voters cast both the ballots identically. The second ballot is superfluous. However, the ‘Government of Wales Act 1998 (chapter 38)’ contains the possibility that the second ballot is cast differently from the first one. We have argued in the above sections that the purpose of this extra degree of freedom is unclear in the context of an attempt to achieve some measure of proportionality in assembly representation. We have given examples illustrating that this extra degree of freedom can cause departure from the concept of proportionality.

Furthermore, we have also shown that if we want to distribute power in the assembly according to party support, we have to change our methods of allocating the seat distribution and the decision rules as far as the proposed method for measuring voting power is accepted. We have suggested a method which leads to a seat distribution that is closer to party support than the AMS. Additionally, it rules out the possibility of the occurrence of the above mentioned anomalies resulting from the AMS, and have shown that there exists a (theoretical) possibility to equate seat distribution and voting power in the assembly using a randomised decision rule.

From all this, one might ask whether it is worth having an electoral reform as has been introduced by the ‘Government of Wales Act 1998 (chapter 38)’. The answer given by many proponents of the electoral reform in Great Britain is the following (Johnston, 1998): The actual reform is a first step to resolve one of the major problems that they identify with the British constitution and form of Government - its unrepresentative ‘elected dictatorship’. As we have demonstrated above, the switch from the FPTPS to the AMS for electing the assembly may at least give some parties the chance of being involved in the business of government. Moreover, it is argued that the issue of the mentioned electoral reform should not be considered separately from the much wider and deeper one of the constitutional reform (Plant, 1991).

If one accepts this point of view, the consequences of using the AMS should be borne in mind: It can frustrate voters and it implies the possibility that some parties in the assembly will be rendered powerless.

Chapter 7

A PRIORI VOTING POWER IN HIERACHRICAL ORGANIZATIONS*

Abstract: Power in hierarchical organizations can be investigated in different ways This chapter focuses on the study of a priori voting power in such organizations. It is shown that every 'restricted' game with a permission structure can be represented as a compound game. Furthermore, it is pointed out that the existing research in voting power in hierarchical structures is necessary, but not sufficient to understand the nature of a priori voting power in hierarchical organizations. This is because it does not take into account: (i) that players who participate in a decision-making in hierarchical organizations in general have a damatis personae, and (ii) that the top of a hierarchical organizations can have a board-structure. Once we account for these two factors we arrive at some counterintuitive characteristics of hierarchies, i.e. the violation of known monotonicity properties of power in hierarchical organizations such as structural monotonicity and dis- and conjunctive fairness.

1. Introduction

Power in hierarchical organizations can be investigated in different ways. One possibility is to study power arising only from the position of a player within the hierarchical structure of an organization, i.e. the analysis of 'has power over' relations. Brink and Gilles (1994) and Brink (1994) call this type of power *relational power*.

This chapter focuses on a alternative method and type of power in hierarchical organizations: that of voting power. This will allow us to simultaneously take into account the relational power of the players and the existence of a decision-making process which takes place within the hierarchical structured organization. This type of power refers to a player's *ability* to approve

* This chapter is a based upon Steffen (1999) and Braham and Steffen (2001a, 2002b). The author is indebted to Matthew Braham and Moshé Machover for their assistance in clarifying the fundamental concepts presented in this chapter as well as to Manfred Holler for the same reason and giving the initial impetus to this research. The author is also grateful for helpful comments to Yener Altunbas and Stefan Napel. Furthermore, this chapter has benefited from consultations with Deutsche Bank AG, Siemens Medical Solutions AG, and the Office of the United Nations High Commissioner for Refugees (UNHCR), Geneva. Moreover, the author thanks Deutsche Bank AG in Hamburg for financial support.

projects and measures the ‘has more power than’ relations among players. We will refer to this type of power as *voting power in hierarchical organizations*.[1]

Although the analysis of hierarchical organizations using the tools of cooperative game theory was foreseen many years ago by Morgenstern (1951), Shapley (1962b, p. 66), and Shubik (1962), it is a subject that economists have more or less left by the wayside. Notable exceptions are Brink (1994, 1997, 1999; 2001), Gilles *et al*. (1992), Gilles and Owen (1994), and Brink and Gilles (1996). In order to take into account the aspects of voting power in hierarchical organizations they represent the decision-making situation as a combination of a simple game which represents the decision rule and a so-called *permission structure* which takes into account the hierarchical properties of the organization. Based on this combination they come out with what they call a *permission value* which can be interpreted as a measure of *a priori* voting power in a hierarchical structured organization (see Gilles *et al*., 1992), i.e. it assigns values of voting power to each player in terms of his or her probability of being able to approve projects *in a hierarchical organization*.[2]

Although Brink, Gilles, and Owen’s method is able to capture essential properties of a priori voting power in hierarchical organizations, it is not sufficient, because it does not take into account all social asymmetries that play an important role. The missing element is that they do not consider the fact that players who participate in a decision-making in hierarchical organizations in general have a *damatis personae*: they are the bearers of predetermined attributes and modes of behaviour.[3] That is, players in (hierarchical) organizations often play predetermined *roles,* such as salesman, financial officer, head of external affairs, etc., which are equipped with a bundle of incentive structures and, thus, leads to specific standards of behavior. Thus determining the power of a player in such a structure must take such information into account because it is likely that players

[1] Gilles *et al*. (1992) call this *hierarchical power*, a term we do not use in order to avoid confusion with the concept of *relational power.*

[2] Note, that the central aim of their analysis is the creation of allocation mechanisms for the distribution of value within hierarchical organizations, e.g. for payment schemes for employees of a firm.

[3] Other types of social asymmetries discussed in the literature are, for instance, *limited communication possibilities* between players; see, e.g., Myerson (1977, 1980), Owen (1986), and Borm, Owen and Tijs (1992). Such *communication structures* can be described by using undirected graphs where the vertices represent the players and the edges connecting the vertices indicate the communications possibilities between the players. Following Brink and Gilles (1994) one can increase this type of social asymmetry by imposing dominance relations on such a communication structure. However, how the results of this chapter are related to limited communication possibilities (e.g. as indicated by Fn. 11 in section 3.3) is an issue of future research by the author.

with same incentive structure will act in the same way. One way to include this information is to partition the set of players into a priori subsets.[4]

This idea was already noted by Shubik (1962, p. 330) when he wrote that 'an organization is a series of arrangements between individuals with possibly differing goals'. For example, a bank will have staff that are responsible for expanding credit and staff responsible for managing risk. The granting of a risky loan will usually require consent of both sections. It is reasonable to assume that the staff responsible for expanding credit will all have one standard of behaviour, while those responsible for managing risk will have an opposing standard.[5]

One way of integrating such behavioural standards into the analysis of a priori voting power in hierarchical organizations is to apply Straffin's (1977, 1988) partial homogeneity approach to the calculation of voting power. To do this we will model decision-making in a hierarchy by a *composite* (or *compound*) simple game which we will show to be an equivalent to the method of Brink, Gilles and Owen. Although this approach has the drawback that it loses the explicit description of the hierarchical properties of the decision-making situation, it does have the advantage that we can immediately apply known results from the theory and measurement of voting power.

An important fact is that if a priori standards are considered we are likely to encounter violations of *local monotonicity*, which is that property of a measure of voting power that ranks a player's voting weight (number of votes) in the same order as his or her voting power. The reason is that under the partial homogeneity structure the winning coalitions do not occur with equal probability. This will also lead to violations of other monotonicity properties, namely, *weak structural monotonicity* (Brink, 2001) and its stronger version *structural monotonicity* (Brink and Gilles, 1996) – both properties more or less say that a player in a hierarchy who dominates another player should have at least as much voting power as the dominated player, and *disjunctive fairness* (Brink, 1997) and *conjunctive fairness* (Brink, 2001) – these properties roughly stipulate that the deletion of a hierarchical relation between two players under *disjunctive fairness* should change their voting power and that of the superiors of the dominating player by the same amount and in the same direction, while under *conjunctive fairness* the voting

[4] Note, that we use the term 'act in the same way' in order to point out that we are not talking about 'cooperation' of players in the sense of Aumann and Drèze (1974).

[5] See Braham and Steffen (2001a, 2002b) for further examples.

power of the dominated player and his or her superiors should be changed by the same amount and the same direction.

Moreover, an advantage of the (partial homogeneity) approach taken here is that it is more flexible than that of Brink, Owen, and Gilles because it can be used to represent 'real-life' hierarchical decision-making situations which cannot be represented by a combination of a simple game and a permission structure (see Example 6 in section 3.3).

This chapter is organised as follows: section 2 describes what we understand under a hierarchical organization and how this is related to a hierarchical structure and a permission structure. Section 3 reproduces the basic formal framework for simple games including compound games and discusses how a decision-making situation in a hierarchical structure can be modelled as such. In section 4 Straffin's partial homogeneity approach is introduced. On the basis of the properties of this approach, i.e. the likely violation of local monotonicity, section 5 contains some general properties of voting power in hierarchical structures referring to the violation of (weak) structural monotonicity and dis- and conjunctive fairness. Section 6 concludes.

2. Hierarchical Organizations, Hierarchical Structures, Permission Structures, and Their Relationship

Following Brink (1994) we define a *hierarchical structure* to be a system consisting of a finite set of positions (which can be players) and binary relations between these positions, called *superior to* or *dominates*. The relations are such that except for the position(s) at the top of the structure, each position has at least one predecessor, i.e. a hierarchical structure is a kind of tree structure.[6]

Building up on this definition of a hierarchical structure we can define a hierarchical organization to be a system that can be described by a finite set of players affiliated to the positions, dominance relations between these players, and a rule according to which (collective) decision-making takes place within the organization. By a dominance relation is meant a relation that indicates that one player has a direct influence on the set of actions that are available to the other player. In this context we refer to predecessors of a player as *principals* and to successors as *agents*. In common parlance the predecessors are what we call 'bosses' or 'managers'.

[6] See Radner (1992) for a general description of hierarchies.

The distinction between a hierarchical organization and a hierarchical structure is essentially one of the presence or absence of a decision rule. The character of these rules is such that they create constraints on the choices that the players can make, taking into account two components: *a 'pure' decision rule* and a so called *permission structure*. On the whole, a decision rule in a hierarchical organization describes how an outcome can be produced by defining which subsets of players are able to do this. While the 'pure' decision rule only contains the information about which players are necessary to approve a proposal for a *non-hierarchical structure*, i.e. in absence of the dominance relations, the permission structure takes into account dominance relations.

Formally, we can represent a hierarchical structure as a *directed graph* (or *digraph*), where the vertices represent the players and the directed edges connecting the vertices indicate the dominance relations between the players. Let $N = \{1, ..., n\}$ be a finite set of vertices (players). Then a hierarchical structure can be represented by a digraph $(N, \mathcal{D})$, where $\mathcal{D} \subset N \times N$ is a binary relation on N. Since we have fixed the set N, we can represent a digraph just by its relation $\mathcal{D}$. If $(i, j)\in\mathcal{D}$ then player j is called a *successor* of player i, and i is called a *predecessor* of j in $\mathcal{D}$. By $\mathcal{S}(i)$ and $\mathcal{S}^{-1}(i)$, respectively, we denote the set of all successors and predecessors of $i\in N$ in $\mathcal{D}$; i.e. $\mathcal{S}(i) = \{j\in N \mid (i, j)\in\mathcal{D}\}$ and $\mathcal{S}^{-1}(i) = \{j\in N \mid (j, i)\in\mathcal{D}\}$. We denote the collection of all hierarchical structures on N by $\mathcal{S}_H^N$.

Further, we restrict our attention to hierarchical structures that are acyclic, i.e. $i\notin\hat{\mathcal{S}}(i)$ for all $i\in N$. By $\hat{\mathcal{S}}$ we denote the transitive closure of the hierarchical structure $\mathcal{S}$, i.e. $j\in\hat{\mathcal{S}}(i)$ iff there exists a sequence of players $(h_1, ..., h_l)$ such that $h_1 = i$, $h_{k+1}\in\mathcal{S}(h_k)$ for all $1 \leq k \leq l\text{-}1$, and $h_l = j$. The players in $\hat{\mathcal{S}}(i)$ are called the *subordinates* of i in S, and the players in $\hat{\mathcal{S}}^{-1}(i) = \{j\in N \mid i\in\hat{\mathcal{S}}(j)\}$ are called the *superiors* of i in S.

For an illustration, consider the following three simple examples:

Example 1 Assume $N = \{a, b\}$ and $\mathcal{S}\in\mathcal{S}_H^N$ given by $\mathcal{S}(a) = \{b\}$ and $\mathcal{S}(b) = \emptyset$. (For a graphical representation see figure 1a.)

Example 2 Assume $N = \{a, b, c\}$ and $\mathcal{S}\in\mathcal{S}_H^N$ given by $\mathcal{S}(a) = \{b, c\}$ and $\mathcal{S}(b) = \mathcal{S}(c) = \emptyset$. (For a graphical representation see figure 1b.)

Figure 1: *Hierarchies with a Unique Top-Player I*

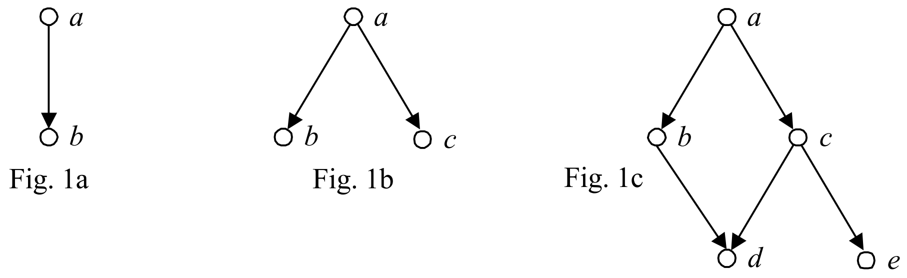

Fig. 1a Fig. 1b Fig. 1c

Example 3 Assume $N = \{a, b, c, d, e\}$ and $\mathcal{S} \in \mathcal{S}_H^N$ given by $\mathcal{S}(a) = \{b, c\}$, $\mathcal{S}(c) = \{d, e\}$, $\mathcal{S}(b) = \{d\}$, and $\mathcal{S}(d) = \mathcal{S}(e) = \varnothing$. (For a graphical representation see figure 1c.)

Additionally, an often used assumption (which we also made implicitly for Examples 1-3) is that a hierarchical structure has one unique top-player, i.e. there exists an $i \in N$ such that $\hat{\mathcal{S}}(i) = N \setminus \{i\}$ and thus $\hat{\mathcal{S}}^{-1}(i) = \varnothing$. Brink (2001) calls this *quasi-strongly connectedness*. It should be noted that this is a pleasant theoretical assumption, it is very restrictive because it is often the case that hierarchical organizations are characterized by a board structure at the top.

Examples of a hierarchy with a board structure at the top are:

Example 4 Assume $N = \{a, b, c, d, e, f, g, h, i\}$ and $\mathcal{S} \in \mathcal{S}_H^N$ given by $\mathcal{S}(a) = \mathcal{S}(b) = \mathcal{S}(c) = \{d, e\}$, $\mathcal{S}(d) = \{f, g\}$, $\mathcal{S}(e) = \{h, i\}$, and $\mathcal{S}(f) = \mathcal{S}(g) = \mathcal{S}(h) = \mathcal{S}(i) = \varnothing$. (For a graphical representation see figure 2a.)

Example 5 Assume $N = \{a, b, c, d, e, f, g, h\}$ and $\mathcal{S} \in \mathcal{S}_H^N$ given by $\mathcal{S}(a) = \mathcal{S}(b) = \mathcal{S}(c) = \{d\}$, $\mathcal{S}(d) = \{e, f\}$, and $\mathcal{S}(e) = \mathcal{S}(f) = \varnothing$. (For a graphical representation see figure 2b.)

While the information given by a hierarchical structure as described above is sufficient to analyse relational power in hierarchical organizations[7], the analysis of voting power requires the existence of a rule that stipulates how decisions are

[7] An example of such a measure that is familiar in graph theory is the *score measure* (or *Copeland score*). According to this measure the power value of a position only depends on the number of positions which it dominates directly while it does not depend on the other relations of these dominated positions. See Brink and Gilles (1994) or Brink (1994) for further information on relational power measures.

Figure 2: *Hierarchies with a Board-Structure at the Top I*

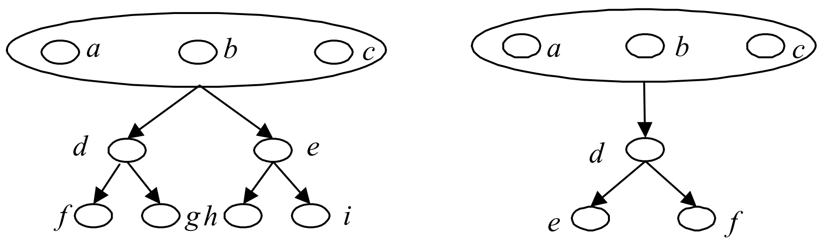

Fig. 2a

Fig. 2b

made within the organization. In order that a decision rule 'obeys' the dominance relations ('superior to' relations) of a hierarchical organization, Brink, Gilles and Owen define what they call a *game with a permission structure*. This is composed of two elements: a *permission structure* which is simply a set of subsets of positions determined by the edges of the digraph and a 'rule' which defines which of these subsets is 'winning' or is entitled to approve an action (for the latter see section 3).

For their analysis they have introduced two natural types of permission structures: the *conjunctive* and the *disjunctive permission structure*. Under the *conjunctive permission structure* a player $i \in N$ needs the permission of all his or her predecessors $\boldsymbol{S}^{-1}(i)$. Formally the collection containing all allowed subsets of players under a conjunctive permission structure is given by:

$$\mathcal{C}^{c}_{N,\boldsymbol{S}} = \{S \subseteq N \mid \boldsymbol{S}^{-1}(i) \subseteq S \text{ for all } i \in S\}.$$

Under the *disjunctive permission structure* which is formulated for *quasi-strongly connected* hierarchical structures player i needs the permission of at least one of his or her predecessors. Consequently, player i needs the permission from all his or her predecessors following at least one 'permission path' from his or her position to the top-player i_0. Formally the collection containing all allowed subsets of players under a disjunctive permission structure is given by

$$\mathcal{C}^{d}_{N,\boldsymbol{S}} = \left\{ S \subseteq N \;\middle|\; \begin{array}{l} \text{for every } i \in S \text{ there is a sequence of players } (h_1,\ldots,h_t) \\ \text{such that } h_1 = i_0,\ h_k + 1 \in \boldsymbol{S}(h_k) \text{ for all } 1 \le k \le l\text{-}1, \\ \text{and } h_l = i \end{array} \right\}.$$

Note, however, that this definition of a disjunctive permission structure can easily extended to *non-quasi-strongly connected* hierarchical structures, i.e. to structures with no unique top-player, saying that player i needs the permission from all his or her predecessors following at least one 'permission path' from his or her position to a certain number of top-players. We will make use of this more general definition of a disjunctive permission structure for our studies.

If we impose a disjunctive permission structure on Example 5 we can identify a special property of this hierarchical structure being that player d is what we call a gate-keeper. A player $i \in N$ is a gate-keeper if all 'permission paths' in order to reach a top-player must contain player i. In this context Brink (2001) says such a player i *dominates* all $j \in \boldsymbol{\mathcal{S}}(i)$ *completely* where *complete domination* is that property that says that player i dominates player j completely if all 'permission paths' from the top-player i_0 to player j contain player i. We denote the set of players that player i dominates completely by $\bar{\boldsymbol{\mathcal{S}}}(i)$, i.e.

$$\bar{\boldsymbol{\mathcal{S}}}(i) = \left\{ j \in \hat{\boldsymbol{\mathcal{S}}}(i) \;\middle|\; \begin{array}{l} i \in \{i_0, i_1, ..., i_{l-1}\} \text{ for every sequence of players} \\ i_0, i_1, ..., i_l \text{ such that } i_l = j \text{ and } i_k \in \boldsymbol{\mathcal{S}}(i_{k-1}), \\ k \in \{1,...,l\} \end{array} \right\}$$

3. Modelling Decision-Making in Hierarchical Organisations

In order to discuss Brink, Gilles and Owen's approach to model decision-making in hierarchical organizations, this section introduces their second component: simple games. We state the basic definitions and notations from the theory of simple games that we will require for our analysis. We refer the reader to Shapley (1962b), Felsenthal and Machover (1998), and Taylor and Zwicker (1999) for additional background and results.

Furthermore, we will introduce an equivalent representation for the modelling of decision-making situations in hierarchical organizations using what is known as compound or (composite) games.

3.1 Simple Games

The most important definition that we require is that of a (*collective*) *decision rule* which we will first formulate informally as follows. Let a n-member decision-making body be denoted by a set N. A decision rule specifies which subsets of N can ensure the acceptance of a proposal. Formally:

Let $N = \{1, 2, \ldots, n\}$ be the set of players. $\wp(N) = \{0, 1\}^n$ is the set of feasible coalitions. A *simple game* can be represented by a pair $(N,\mathcal{W})$ where $\mathcal{W} \subseteq \wp(N)$ of *winning coalitions*. $\mathcal{W}$ satisfies $\varnothing \notin \mathcal{W}$; $N \in \mathcal{W}$; and (monotonicity) if $S \in \mathcal{W}$ and $S \subseteq T$ then $T \in \mathcal{W}$. A simple game can be represented by $\mathcal{W}$ because N is uniquely determined by $\mathcal{W}$ (its largest member). Further, $\mathcal{W}$ can also be described by a characteristic function, $v\colon \wp(N) \to \{0, 1\}$ with $v(S) = 1$ iff $S \in \mathcal{W}(v)$ and 0 otherwise. By $\mathcal{G}^N$ we denote the set of all such n-person simple games.

We say that a player i who by leaving a winning coalition $S \in \mathcal{W}(v)$ turns it into a losing coalition $S \setminus \{i\} \notin \mathcal{W}(v)$ has a *swing* in S and is called a *critical member* of S. Coalitions where i has a swing are called *critical coalitions with respect to i*. Let us denote the set of crucial coalitions w.r.t i as $\mathcal{C}_i$. A concise description of v can be given by a set $\mathcal{M}(v)$, where $S \in \mathcal{W}(v)$ but no subset of S is in $\mathcal{W}(v)$, i.e. all members of S are critical. We call such a coalition a *minimal winning coalition* (MWC). Further, we denote by $\eta_i(v)$ the number of swings of player i in v. Thus, $\eta_i(v) =_{def} |\mathcal{C}_i(v)|$. A player i for which $\eta_i(v) = 0$ is called a *dummy* in v, i.e. it is never the case that i can turn a winning coalition into a losing coalition (it is easy to see that i is a dummy iff it is never a member of a MWC; and i is a *dictator* if $\{i\}$ is the sole MWC).

Three types of simple games which will be used later are *simple majority*, *unanimity*, and *minority* games. These are defined as follows. Let $I_n =_{\text{def}} \{1,\ldots,n\}$ be a 'canonical assembly'. Then for any positive integer k such that $k \leq n$, define $M_{n,k}$ as the simple game whose winning coalitions are just those subsets of I_n that have at least k members. As a matter of shorthand, we denote (i) the simple majority game as $M_{n,(n/2)+1}$, (ii) the unanimity game as $M_{n,n}$, and (iii) the minority game as $M_{n,1}$.

A further definition we need is that of the important sub-class of simple games known as *weighted voting games* (WVG). A WVG is characterized by a non-negative real vector $(w_1, w_2, \ldots, w_n)$ where w_i represents player i's voting weight and a quota q which is the quota of votes necessary to establish a winning coalition, such that *quota* $0 < q \leq \sum_{i \in N} w_i$. A weighted voting game is represented by $[q; w_1, w_2, \ldots, w_n]$.

3.2 Compound Games

A fact of simple games is that they can be composed. In this context we are then talking about *compound* (or *composite*) games.[8] Let m be a positive integer and let $\mathcal{W}^*$ be a simple game with assembly $I_m = \{1,\ldots,m\}$. For each $i \in I_m$, let $\mathcal{W}_i$ be an arbitrary simple game. We now define a simple game $\mathcal{W}^*[\mathcal{W}_1,\ldots,\mathcal{W}_m]$ called the *composite* of $\mathcal{W}_1, \ldots\mathcal{W}_m$ *under* $\mathcal{W}^*$. Next, define the assembly N of $\mathcal{W}^*[\mathcal{W}_1,\ldots,\mathcal{W}_m]$ as the union of the assemblies N_i of $\mathcal{W}_i$: $N =_{\text{def}} \bigcup_{i=1}^{m} N_i$. We refer to $\mathcal{W}^*$ as the 'top' and $\mathcal{W}_i$ as the i-th 'component' of the composite simple game $\mathcal{W}^*[\mathcal{W}_1, \ldots,\mathcal{W}_m]$; i.e. the 'top' is the game that is made up of all the component games: the decisions of each $\mathcal{W}_i$ are fed to $\mathcal{W}^*$ which is a rule for collating the m lower level decisions into a final decision; put it in other words: $\mathcal{W}^*$ is the collection of subsets of $\mathcal{W}^*[\mathcal{W}_1,\ldots,\mathcal{W}_m]$ that assures the acceptance of a proposal.

There are two types of compound simple games: *meets* and *joins*.

A meet of the $\mathcal{W}_i$ is denoted by $\mathcal{W}_1 \wedge \mathcal{W}_2 \wedge \ldots \wedge \mathcal{W}_m$ which is equal to a unanimity game $M_{n,n}$ on $[\mathcal{W}_1,\ldots,\mathcal{W}_m]$, i.e. each component game must be won.[9] If the assemblies N_i are pairwise disjoint, i.e. if the players are taken to be individual persons, and no person can be a member of more than one chamber, then a meet is called a *product* of $\mathcal{W}_i$ and is denoted by $\mathcal{W}_1 \times \mathcal{W}_2 \times \ldots \times \mathcal{W}_m$. For the special case where all N_i coincide (the players in each $\mathcal{W}_i$ are the same), then $\mathcal{W}_1 \wedge \mathcal{W}_2 \wedge \ldots \wedge \mathcal{W}_m$ is given by $\mathcal{W}_1 \cap \mathcal{W}_2 \cap \ldots \cap \mathcal{W}_m$.

A join of the $\mathcal{W}_i$ is denoted by $\mathcal{W}_1 \vee \mathcal{W}_2 \vee \ldots \vee \mathcal{W}_m$ and is which is equal to a minority game $M_{n,1}$ on $[\mathcal{W}_1,\ldots,\mathcal{W}_m]$, i.e. only one component game must be won. If the assemblies N_i are pairwise disjoint then the join is called a *sum* of $\mathcal{W}_i$ and is denoted by $\mathcal{W}_1 + \mathcal{W}_2 + \ldots + \mathcal{W}_m$. For the special case where all N_i coincide (the players in each $\mathcal{W}_i$ are the same), then $\mathcal{W}_1 \vee \mathcal{W}_2 \vee \ldots \vee \mathcal{W}_m$ is given by $\mathcal{W}_1 \cup \mathcal{W}_2 \cup \ldots \cup \mathcal{W}_m$.

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[8] For a general introduction to compound (simple) games see Shapley (1962b) and Straffin (1983). For more detailed treatment see Shapley (1962a, 1964, 1967).

[9] Such a 'structure' can be used to model a multi-cameral system, that is when a motion has to be approved by different 'chambers', 'committees' or 'houses'.

3.3 Games with Permission Structures, Compound Games and Their Relationship

As mentioned in section 2, in order that a decision-making rule 'obeys' the dominance relations of a hierarchical organization, Brink, Gilles and Owen define a *game with a permission structure*. For brevity, we will refer to those as *permission games*. On the basis of sections 2 and 3.1 we can define such a game as a triple (N, v, $\mathcal{S}$) where (N, v)$\in\mathcal{G}^N$ and $\mathcal{S}\in\mathcal{S}_H^N$. Moreover, specifying the type of the permission structure they define a '*conjunctive* (or *disjunctive*) *restriction of v on the permission structure* $\mathcal{S}$' as a game (N, v, $\mathcal{S}$) with the corresponding specification of the restriction on (N, v). For want of better terminology, we will refer to these games as *conjunctive* or *disjunctive restricted permission games*, respectively, and we will talk about *restricted permission games*, when we want to refer to both types of restrictions.

Thus, a restricted permission game transforms the characteristic function (of the simple game or a permission game) v into a modified characteristic function $r_{N,v,\mathcal{S}}(S)$, which takes account of the limited allowance of each player to exercise his or her choice if a proposal has to be approved within an hierarchical organisation, i.e. a restricted permission game reduces the set of winning coalitions to a subset of those winning coalitions that are given by the simple game (N, v). This is done by the restriction that each winning coalition must contain a MWC that is also an element of $\mathcal{C}^c_{N,\mathcal{S}}$ or $\mathcal{C}^d_{N,\mathcal{S}}$, respectively. Formally, the conjunctive and disjunctive restricted characteristic functions are given by:

$$r^c_{N,v,\mathcal{S}}(S)=\begin{cases}1 & \text{if } S\supseteq T \mid T\in\mathcal{M}(v)\cap\mathcal{C}^c_{N,\mathcal{S}}\\ 0 & \text{otherwise}\end{cases}$$

and

$$r^d_{N,v,\mathcal{S}}(S)=\begin{cases}1 & \text{if } S\supseteq T \mid T\in\mathcal{M}(v)\cap\mathcal{C}^d_{N,\mathcal{S}}\\ 0 & \text{otherwise}\end{cases}.$$

Thus, we will denote *conjunctive* and *disjunctive restricted permission games* by (N, $r^c_{N,v,\mathcal{S}}$) or (N, $r^d_{N,v,\mathcal{S}}$), respectively. However, note that the definition of $r^c_{N,v,\mathcal{S}}$ and $r^d_{N,v,\mathcal{S}}$ appears to be arbitrary in that sense that a coalition S is winning even if it contains non-critical players whose superiors are not necessarily a member of S. Brink, Gilles, and Owen, unfortunately, do not comment on this restriction. However, there is a good technical reason for their restriction being that it rescues monotonicity (as given in the definition of a simple game) for their restricted permission games.[10] This leads to the fact that every restricted permission game can also be represented as a 'plain' simple game.

Using this fact and the fact that simple games can be composed, one can model each set of winning coalitions given by a restricted permission game as a join (of simple games) where each winning coalition is a unanimity game which itself is one component of the join. Therefore, we can represent every restricted permission game as a compound game. For an illustration, we use the five examples from section 2 by assuming a decision rule for each hierarchical structure. Note, that the subscripts used in the representations of WVGs denote the players.

Example 1 Assume $N = \{a, b\}$, $\boldsymbol{\mathcal{S}} \in \boldsymbol{\mathcal{S}}_H^N$ given by $\boldsymbol{\mathcal{S}}(a) = \{b\}$, and $\boldsymbol{\mathcal{S}}(b) = \varnothing$ and $v(\mathrm{S}) = 1$ if $b \in S$ and 0 otherwise. Then $(N, r^d_{N,v,\boldsymbol{\mathcal{S}}})$ and as well $(N, r^c_{N,v,\boldsymbol{\mathcal{S}}})$ can be represented by $r^d_{N,v,\boldsymbol{\mathcal{S}}}(S) = r^c_{N,v,\boldsymbol{\mathcal{S}}}(S) = 1$ if $S \in \{\{a, b\}\}$ and 0 otherwise, i.e. $\mathcal{W} = \{\{a, b\}\}$ which can also be represented by the unanimity game $M_{2,2}$.

Example 2 Assume $N = \{a, b, c\}$, $\boldsymbol{\mathcal{S}} \in \boldsymbol{\mathcal{S}}_H^N$ given by $\boldsymbol{\mathcal{S}}(a) = \{b, c\}$, and $\boldsymbol{\mathcal{S}}(b) = \boldsymbol{\mathcal{S}}(c) = \varnothing$ and $v(\mathrm{S}) = 1$ if b or $c \in S$ and 0 otherwise. Then $(N, r^d_{N,v,\boldsymbol{\mathcal{S}}})$ and as well $(N, r^c_{N,v,\boldsymbol{\mathcal{S}}})$ can be represented by $r^d_{N,v,\boldsymbol{\mathcal{S}}}(S) = r^c_{N,v,\boldsymbol{\mathcal{S}}}(S) = 1$ if $S \in \{\{a, b\}, \{a, c\}, \{a, b, c\}\}$ and 0 otherwise, i.e. $\mathcal{W} = \{\{a, b\}, \{a, c\}, \{a, b, c\}\}$ which can also be represented by the join $[2; 1_a, 1_b] \vee [2; 1_a, 1_c] \vee [3; 1_a, 1_b, 1_c]$ which can be merged to the WVG $[3; 2_a, 1_b, 1_c]$.

Example 3 Assume $N = \{a, b, c, d, e\}$, $\boldsymbol{\mathcal{S}} \in \boldsymbol{\mathcal{S}}_H^N$ given by $\boldsymbol{\mathcal{S}}(a) = \{b, c\}$, $\boldsymbol{\mathcal{S}}(c) = \{d, e\}$, $\boldsymbol{\mathcal{S}}(b) = \{d\}$, and $\boldsymbol{\mathcal{S}}(d) = \boldsymbol{\mathcal{S}}(e) = \varnothing$ and $v(\mathrm{S}) = 1$ if d or $e \in S$ and 0 otherwise. Then $(N, r^d_{N,v,\boldsymbol{\mathcal{S}}})$ can be represented by $r^d_{N,v,\boldsymbol{\mathcal{S}}}(S) = 1$ if $S \in \{\{a, b, d\}, \{a, c, d\}, \{a, c, e\}, \{a, c, d, e\}, \{a, b, c, d\}, \{a, b, d, e\}, \{a, b, c, e\}, \{a, b, c, d, e\}\}$ and 0 otherwise, i.e. $\mathcal{W} = \{\{a, b, d\}, \{a, c, d\}, \{a, c, e\}, \{a, c, d, e\}, \{a, b, c, d\}, \{a, b, d, e\}, \{a, b, c, e\}, \{a, b, c, d, e\}\}$ which is also represented by the join $[3; 1_a, 1_b, 1_d] \vee [3; 1_a, 1_c, 1_d] \vee [3; 1_a, 1_c, 1_e] \vee [4; 1_a, 1_c, 1_d, 1_e] \vee [4; 1_a, 1_b, 1_c, 1_d] \vee [4; 1_a, 1_b, 1_d, 1_e] \vee [4; 1_a, 1_b, 1_c, 1_e] \vee [5; 1_a, 1_b, 1_c, 1_d, 1_e]$ which can be merged to $[3; 1_a, 1_b, 1_d] \vee [5; 2_a, 2_c, 1_d, 1_e]$. $(N, r^c_{N,v,\boldsymbol{\mathcal{S}}})$ can be represented by $r^c_{N,v,\boldsymbol{\mathcal{S}}}(S) = 1$ if $S \in \{\{a, c, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, c, d, e\}, \{a, b, c, d, e\}\}$ and 0 otherwise, i.e. $\mathcal{W} = \{\{a, c, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, c, d, e\}, \{a, b, c, d, e\}\}$ which can also be represented by the join $[3; 1_a, 1_c, 1_e] \vee [4; 1_a, 1_b, 1_c, 1_d] \vee [4; 1_a, 1_b, 1_c, 1_e] \vee [4; 1_a, 1_c, 1_d, 1_e] \vee [5; 1_a, 1_b, 1_c, 1_d, 1_e]$ which can be merged to the WVG $[10; 4_a, 1_b, 4_c, 1_d, 2_e]$.

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[10] Otherwise, a restricted permission game would be what is called a hypergraph; see, e.g., Taylor and Zwicker (1999, p. 3).

Note, that in Example 3 under the disjunctive restriction the coalition $\{a, b, d, e\}$ is an example of a coalition which is winning under $(N, r^{d}_{N,v,\mathcal{S}})$ as $\{a, b, d\} \in \mathcal{M}(v) \cap \mathcal{C}^{d}_{N,\mathcal{S}}$ and contains e as a non-critical player, but not his superior c.

For sake of brevity, for Examples 4 and 5 we could make use of the fact that $\mathcal{M}$ could be used as concise description of a simple game in a characteristic function form, i.e. we will only state $\mathcal{M}(r^{d}_{N,v,\mathcal{S}})$ and $\mathcal{M}(r^{c}_{N,v,\mathcal{S}})$, respectively, and the merged join of each restricted permission game.

Example 4 Assume $N = \{a, b, c, d, e, f, g, h, i\}$ and $\mathcal{S} \in \mathcal{S}^{N}_{H}$ given by $\mathcal{S}(a) = \mathcal{S}(b) = \mathcal{S}(c) = \{d, e\}$, $\mathcal{S}(d) = \{f, g\}$, $\mathcal{S}(e) = \{h, i\}$, and $\mathcal{S}(f) = \mathcal{S}(g) = \mathcal{S}(h) = \mathcal{S}(i) = \varnothing$ and $v(S) = 1$ if f, g, h, or $i \in S$ and $\{a, b\}$, $\{a, c\}$ or $\{b, c\} \subset S$ and 0 otherwise. Then $(N, r^{d}_{N,v,\mathcal{S}})$ can be represented by $\mathcal{M}(r^{d}_{N,v,\mathcal{S}}) = \{\{a, b, d, f\}, \{a, b, d, g\}, \{a, \mathrm{c}, d, f\}, \{a, \mathrm{c}, d, g\}, \{b, c, d, f\}, \{b, c, d, g\}, \{a, b, e, h\}, \{a, b, e, i\}, \{a, \mathrm{c}, e, h\}, \{a, \mathrm{c}, e, i\}, \{b, c, e, h\}, \{b, c, e, i\}\}$ which can also represented by the product game $[2; 1_a, 1_b, 1_c] \times ([4; 3_d, 1_f, 1_g] \vee [4; 3_e, 1_h, 1_i])$. $(N, r^{c}_{N,v,\mathcal{S}})$ can be represented by $\mathcal{M}(r^{c}_{N,v,\mathcal{S}}) = \{\{a, b, c, d, f\}, \{a, b, c, d, g\}, \{a, b, c, e, h\}, \{a, b, c, e, i\}\}$ which is also represented by the join $[13; 3_a, 3_b, 3_c, 3_d, 1_f, 1_g] \vee [13; 3_a, 3_b, 3_c, 3_e, 1_h, 1_i]$.

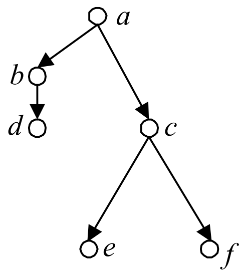
Example 5 Assume $N = \{a, b, c, d, e, f\}$ and $\mathcal{S} \in \mathcal{S}^{N}_{H}$ given by $\mathcal{S}(a) = \mathcal{S}(b) = \mathcal{S}(c) = \{d\}$, $\mathcal{S}(d) = \{e, f\}$, and $\mathcal{S}(e) = \mathcal{S}(f) = \varnothing$ and $v(S) = 1$ if e or $f \in S$ and $\{a, b\}$, $\{a, c\}$ or $\{b, c\} \subset S$ and 0 otherwise. Then $(N, r^{d}_{N,v,\mathcal{S}})$ can be represented by $\mathcal{M}(r^{d}_{N,v,\mathcal{S}}) = \{\{a, b, d, e\}, \{a, b, d, f\}, \{a, c, d, e\}, \{a, c, d, f\}, \{b, c, d, e\}, \{b, c, d, f\}\}$ which can also be represented by the product game $[2; 1_a, 1_b, 1_c] \times [3; 2_d, 1_e, 1_f]$. $(N, r^{c}_{N,v,\mathcal{S}})$ can be represented by $\mathcal{M}(r^{c}_{N,v,\mathcal{S}}) = \{\{a, b, c, d, e\}, \{a, b, c, d, \mathrm{f}\}\}$ which can also be represented by the WVG $[13; 3_a, 3_b, 3_c, 3_d, 1_e, 1_f]$.

Moreover, by the representation of a decision-making situation in a hierarchical organization via a compound game we are also able to model 'real life' decision-making situations which cannot be represented by restricted permission games, because the decision rules are not in line with the permission structure.

Example 6 Assume $N = \{a, b, c, d, e, f\}$, $\mathcal{S} \in \mathcal{S}^{N}_{H}$ given by $\mathcal{S}(a) = \{b, c\}$, $\mathcal{S}(b) = d$, $\mathcal{S}(c) = \{e, f\}$, and $\mathcal{S}(d) = \mathcal{S}(e) = \mathcal{S}(f) = \varnothing$, and a join $[10; 3_a, 3_b, 1_c, 1_d, 3_e] \vee [10; 3_a, 3_b, 1_c, 1_d, 3_f]$. (For a graphical representation of $\mathcal{S}$ see figure 3). Neither $(N, r^{d}_{N,v,\mathcal{S}})$ nor $(N, r^{c}_{N,v,\mathcal{S}})$ is capable to represent such a decision-making situation.

Although we could impose a dominance relation on the players e and f, i.e. $\mathcal{S}(d) = \{e, f\}$ so as to model the above game as a restricted permission game, this

Figure 3: Hierarchies with a Unique Top-Player II



is questionable because it distorts the real situation. Player *d* is never responsible for the decisions of the players *e* and *f*; *d* only participates in the decision-making process obeying the permission of his or her superiors given by the hierarchical structure. Thus, *d* is only responsible for the outcome of the decision and the compliance with the permission of his or her superiors, but not for the choice of *e* or *f*.[11]

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[11] Referring to Fn. 3 we may say, that there exists what Myerson (1977) calls a *com munication link* between *d* and *e*, *d* and *f* as well as between *b* and *c*, i.e. an undirected edge that is not a part of a hierarchical structure. The decision rule in Example 6 takes into account those links, which could, but - as shown by this example – must not have a corresponding dominance relation. An explanation for such a decision rule may be the aim to implement an error reduction or control mechanism within the decision rule which reduces the mistakes that could be made by different sections of an organization. Therefore, the undirected edges in this context probably might be better called *procedural links*. However, note, that example 6 is not entirely abstract as there exists a real correspondence to it for which also the above mentioned explanation applies. As an incidence we refer the reader to decision-making situations over risky projects in a firm, which in general involve decision-makers in positions with conflicting incentives between each other. As an example of quite practical importance, let us take our example for granting a risky loan in a bank from section 1; a bank in legal terms is defined as 'an institution whose current operations consist in granting loans and receiving deposits from the public' (Freixas and Rochet, 1997). One may think of players *e* and *f* as customer clients in the bank and of player *c* as their executive, while player *b* could be the head of the central risk management department of the bank, *d* the head of the credit department and *a* the manager of the bank. Now if a customer wants to apply for a loan assume that he or she has to get in contact with one of the customer clients. For approving the loan then *e* or *f* (following example 6) would require the consent of at least their executive *c*, the head of the central risk management department *b*, and manager *a* or, alternatively, instead of the consent of their executive the consent of the head of the credit department *d*, who is not their superior at all. This alternative gives *e* and *f* an outside option, i.e. it gives them the opportunity to go to the credit department to convince them, who have - given the incentive structure- per se a lower probability to approve a loan than *c*. If *e* or *f* is able to convince *d* it seems to be legitimate that the bank should grant the loan, if additionally *a* approves it. Note that this, due to the incentive structure, in general should not be the case, but could happen if *c* errs as *c* has, for instance, not the time or competence to evaluate the proposal properly.

4. Straffin's Approach to the Measurement of A Priori Voting Power

If we wish to calculate the voting power of the players for a decision making situation in a hierarchical organization, the question that naturally arises is which among the various measures should be used. Well known measures are, namely, the Shapley-Shubik (1954) index and the Penrose (1946)/Banzhaf (1965) measures, and those measures based upon Straffin's (1977, 1988) probabilistic *partial homogeneity* approach. What all these measures have in common is being that they are a mapping $\xi : \mathcal{G}^N \rightarrow \mathbb{R}_+^n$ that assigns to each player $i \in N$ a number ξ_i that indicates i's power in $\mathcal{W}(v)$ and that power is represented as the ability of a player i to change the outcome of a play of the game, i.e. by his or her swings.

Given that hierarchical organizations are different from *non-hierarchized* political bodies such as national legislatures in that players are not only part of an hierarchical structured organisation but also, in general, have a *damatis personae*: they are the bearers of predetermined attributes and modes of behaviour. The reason for being that is that players in a hierarchical organization often play predetermined *roles* which, as already Radner (1972) points out, are equipped with a bundle of incentive structures imposed in order to meet the goals of the organization due to the behaviour of the players. If it comes to a decision-making situation one has to take into account this fact, because it is more likely that players with same incentive structure act in the same way than players with an opposing incentive structure, i.e. one has to partition the players into subsets containing players with the same incentive structure or put in other words: with the same *a priori* standard of behaviour.

In order to consider the existence of such incentives structures that partition the players according to behavioural standards, it seems appropriate to choose a measure of voting power that allows us to include this information.[12] In this regard it should be noted that if we take into account incentive structures and that players have a damatis personae we are assuming that the players will in fact follow these incentives, i.e. that the principal-agent problem is solved and all players behave rationally. Given that the incentive structures affect the possibilities of coalition formation in an organization, Straffin's probabilistic *partial homogeneity* approach to the calculation of voting power appears to be the most appropriate

[12] That is to say, the measure that we need to use is no longer based solely on the classical 'minimalist' structure of a simple game.

methodology because we can apply this structure as an a priori structure of the decision-making situation.[13]

Straffin's approach is a particular interpretation and extension of Owen's (1972, 1975) *multilinear extension* (MLE) of a game v. Straffin applies the MLE as a probability model for answering the question, 'What is the probability that player i's vote will make a difference to the outcome?' Instead of deterministic coalitions $S \subseteq N$ that correspond to corner points $s \in \{0, 1\}^n$ of the n-dimensional unit cube, one considers random coalitions $\mathcal{S}$ represented by the points $p_i \in [0, 1]^n$ anywhere in the cube. Each p_i is interpreted as the probability of a player i deciding in favour of a random proposal or participating in a random coalition; p_i is also known as a player's *acceptance rate*.

Assuming that acceptance decisions are independent, the probability **P** of a given coalition $S \subseteq N$ is $\mathbf{P}(\mathcal{S}=S)=\prod_{i\in S} p_i \prod_{j\notin S}(1-p_j)$. If we extend the characteristic function v of a simple game by weighting each $v(S)$ with the respective probability of formation, we obtain the MLE $f:[0, 1]^n \rightarrow [0, 1]$ of a game v:

$$\begin{aligned} f(p_1,\dots,p_n) &= \sum_{S\subseteq N}\prod_{i\in S} p_i \prod_{j\notin S}(1-p_j)v(S) \\ &= \sum_{S\in \mathcal{W}(v)}\prod_{i\in S} p_i \prod_{j\notin S}(1-p_j) \end{aligned}$$

For fixed acceptance rates, the MLE gives the probability that a winning coalition S will form in v, and thus the expected value of v. The partial derivative $\partial f/\partial p_i$ of v's MLE w.r.t to p_i is called by Straffin (1977, 1988) a player's *power polynomial*, which we denote by f_i.

$f_i(p_1, \dots p_n)$ is, then, the probability of i having a swing (i.e. having power in the generic sense) in a random coalition in a game v. If player's acceptance rates are themselves random variables with a joint distribution P, the expectation $Ef_i = \int f_i(p_1, \dots p_n)dP$ is i's power in a game $\mathcal{W}(v)$. The probabilistic *measure of power* $Ef_i(v)$ coincides with the classical measures $\xi_i(v)$ under different probability models. For example:

[13] In contrast, Straffin (1978) himself argues that partial homogeneity assumptions are by their nature ad hoc. See Braham and Steffen (2002c) for a further discussion.

Independence $p_i \sim U(0, 1)\ \forall\ i \in N$ (A1)
i.e. the decision of i has nothing to do with decision of j.[14]

Homogeneity $t \sim U(0, 1), p_i = t\ \forall\ i \in N$ (A2)
i.e. each i approves or rejects a proposal with the same probability t but t varies from proposal to proposal.

It is a well-known result from Straffin that applying (A1) we obtain the Banzhaf measure (Bz measure), which is commonly known as non-normalized or absolute Banzhaf index, and applying (A2) we obtain the Shapley-Shubik index (S-S index).

It is easy to see that this probability model is extremely flexible and allows us to create families of power measures that lie between the extremes of (A1) and (A2) by mixing these assumptions.

What is important for our purposes is that we can derive a *partial homogeneity structure* from this approach. This is the partitioning of the set of players into subsets whose members are either (a) homogeneous among themselves, but behave independently from the members of the other subsets or (b) independent.[15] Formally:

A *partial homogeneity structure* on N is a partition $\mathcal{P} = \{G_1, \ldots, G_m\}$ of N into disjoint subsets:

Partial homogeneity (A3)

$$\mathcal{P} = \{G_1, \ldots, G_m\}$$
$$G_k \cap G_l = \varnothing \text{ if } k \neq l, \bigcup G_k = N$$
$$t_k \sim U(0, 1), p_i = t_k\ \forall\ i \in G_k,\ k = 1, \ldots m$$

If $\mathcal{P}$ is the discrete partition of N into one-player subsets we have (A1); if $\mathcal{P}$ is the indiscrete partition $\mathcal{P} = \{N\}$, we have (A2).

In other words, (A3) allows us to build in information about the 'prescribed' relations or 'standards' of players that exist in a differentiated structure like a hierarchical organization. The reason for their existence goes back to the fact that, in general, a player's position in a hierarchical organization is equipped with specific incentives of behaviour related to the type of decision-making situation.

[14] Actually one does not necessarily need the uniform distribution. Leech (1990) has shown that distribution must only have a mean of 0.5.

[15] See Kirman and Widgrén (1995, p. 458) for a formal representation.

For an illustration to the application of Straffin's approach we refer to the six examples from section 3 on which we impose (A1), (A2), and a version of (A3) in order to illustrate the effect of *a priori* standards of behaviour.

Example 1 Assume $N = \{a, b\}$, $\boldsymbol{\mathcal{S}} \in \boldsymbol{\mathcal{S}}_H^N$ given by $\boldsymbol{\mathcal{S}}(a) = \{b\}$ and $\boldsymbol{\mathcal{S}}(b) = \varnothing$, and $v(S) = 1$ if $b \in S$ and $v(S) = 0$ otherwise. Then $(N, r^d_{N,v,\boldsymbol{\mathcal{S}}})$ and as well $(N, r^c_{N,v,\boldsymbol{\mathcal{S}}})$ can be represented by the unanimity game $M_{n,n}$.

Assume (A1) for all players. We have $Ef_a = 0.50$, $Ef_b = 0.50$ (This is the Bz measure).

Assume (A2) for all players. We have $Ef_a = 0.50$, $Ef_b = 0.50$ (This is the S-S index).

Assume (A3) as follows: player a forms a standard t and b a standard (1-t). We have $Ef_a = Ef_b = 0.5$.

Example 2 Assume $N = \{a, b, c\}$, $\boldsymbol{\mathcal{S}} \in \boldsymbol{\mathcal{S}}_H^N$ given by $\boldsymbol{\mathcal{S}}(a) = \{b, c\}$, and $\boldsymbol{\mathcal{S}}(b) = \boldsymbol{\mathcal{S}}(c) = \varnothing$ and $v(S) = 1$ if b or $c \in S$ and 0 otherwise. Then $(N, r^d_{N,v,\boldsymbol{\mathcal{S}}})$ and as well $(N, r^c_{N,v,\boldsymbol{\mathcal{S}}})$ can be represented by the WVG $[3; 2_a, 1_b, 1_c]$.

Assume (A1) for all players. We have $Ef_a = 0.75$, $Ef_b = Ef_c = 0.25$. (This is the Bz measure.)

Assume (A2) for all players. We have $Ef_a = 0.67$, $Ef_b = Ef_c = 0.17$. (This is the S-S index.)

Assume (A3) as follows: player a behaves independently, player b forms a standard t and c a standard (1-t). We have $Ef_a = 0.83$, $Ef_b = Ef_c = 0.25$.

Example 3 Assume $N = \{a, b, c, d, e\}$, $\boldsymbol{\mathcal{S}} \in \boldsymbol{\mathcal{S}}_H^N$ given by $\boldsymbol{\mathcal{S}}(a) = \{b, c\}$, $\boldsymbol{\mathcal{S}}(c) = \{d, e\}$, $\boldsymbol{\mathcal{S}}(b) = \{d\}$, and $\boldsymbol{\mathcal{S}}(d) = \boldsymbol{\mathcal{S}}(e) = \varnothing$ and $v(S) = 1$ if d or $e \in S$ and 0 otherwise. Then $(N, r^d_{N,v,\boldsymbol{\mathcal{S}}})$ can be represented by the join $[3; 1_a, 1_b, 1_d] \vee [5; 2_a, 2_c, 1_d, 1_e]$ and $(N, r^c_{N,v,\boldsymbol{\mathcal{S}}})$ by the WVG $[10; 4_a, 1_b, 4_c, 1_d, 2_e]$.

Assume (A1) for all players. For $(N, r^d_{N,v,\boldsymbol{\mathcal{S}}})$ we have $Ef_a = 0.50$, $Ef_b = Ef_e = 0.13$, and $Ef_c = Ef_d = 0.25$ and for $(N, r^c_{N,v,\boldsymbol{\mathcal{S}}})$ we have $Ef_a = Ef_c = 0.31$, $Ef_b = Ef_d = 0.06$, and $Ef_e = 0.19$. (This is the Bz measure.)

Assume (A2) for all players. For $(N, r^d_{N,v,\boldsymbol{\mathcal{S}}})$ we have $Ef_a = 0.50$, $Ef_b = Ef_e = 0.08$, and $Ef_c = Ef_d = 0.17$ and for $(N, r^c_{N,v,\boldsymbol{\mathcal{S}}})$ we have $Ef_a = Ef_c = 0.38$, $Ef_b = Ef_d = 0.05$, and $Ef_e = 0.13$. (This is the S-S index.)

Assume (A3) as follows: player a behaves independently, player b forms a standard t and c, d, and e a standard (1-t). For $(N, r^d_{N,v,\boldsymbol{\mathcal{S}}})$ we have $Ef_a = 0.50$, Ef_b

$= Ef_e = 0.08$, and $Ef_c = Ef_d = 0.25$ and for $(N, r^c_{N,v,\mathcal{S}})$ we have $Ef_a = 0.37$, $Ef_b = Ef_d = 0.04$, $Ef_c = 0.29$, and $Ef_e = 0.21$.

Example 4 Assume $N = \{a, b, c, d, e, f, g, h, i\}$ and $\mathcal{S} \in \mathcal{S}^N_H$ given by $\mathcal{S}(a) = \mathcal{S}(b) = \mathcal{S}(c) = \{d, e\}$, $\mathcal{S}(d) = \{f, g\}$, $\mathcal{S}(e) = \{h, i\}$, and $\mathcal{S}(f) = \mathcal{S}(g) = \mathcal{S}(h) = \mathcal{S}(i) = \varnothing$ and $v(S) = 1$ if f, g, h, or $i \in S$ and $\{a, b\}$, $\{a, c\}$ or $\{b, c\} \subset S$ and 0 otherwise. Then $(N, r^d_{N,v,\mathcal{S}})$ can be represented by the product game $[2; 1_a, 1_b, 1_c] \times ([4; 3_d, 1_f, 1_g] \vee [4; 3_e, 1_h, 1_i])$ and $(N, r^c_{N,v,\mathcal{S}})$ by the join $[13; 3_a, 3_b, 3_c, 3_d, 1_f, 1_g] \vee [13; 3_a, 3_b, 3_c, 3_e, 1_h, 1_i]$.

Assume (A1) for all players. For $(N, r^d_{N,v,\mathcal{S}})$ we have $Ef_a = Ef_b = Ef_c = 0.30$, $Ef_d = Ef_e = 0.23$, and $Ef_f = Ef_g = Ef_h = Ef_i = 0.08$ and for $(N, r^c_{N,v,\mathcal{S}})$ we have $Ef_a = Ef_b = Ef_c = 0.15$, $Ef_d = Ef_e = 0.06$, and $Ef_f = Ef_g = Ef_h = Ef_i = 0.02$. (This is the Bz measure.)

Assume (A2) for all players. For $(N, r^d_{N,v,\mathcal{S}})$ we have $Ef_a = Ef_b = Ef_c = 0.19$, $Ef_d = Ef_e = 0.14$, and $Ef_f = Ef_g = Ef_h = Ef_i = 0.04$ and for $(N, r^c_{N,v,\mathcal{S}})$ we have $Ef_a = Ef_b = Ef_c = 0.28$, $Ef_d = Ef_e = 0.05$, and $Ef_f = Ef_g = Ef_h = Ef_i = 0.01$. (This is the S-S index.)

Assume (A3) as follows: players a, b, and c form a standard t, d, f, and g behave independently, e, h, and i form a standard (1-t). For $(N, r^d_{N,v,\mathcal{S}})$ we have $Ef_a = Ef_b = Ef_c = 0.21$, $Ef_d = 0.30$, $Ef_e = 0.15$, $Ef_f = Ef_g = 0.10$, and $Ef_h = Ef_i = 0.05$ and for $(N, r^c_{N,v,\mathcal{S}})$ we have $Ef_a = Ef_b = Ef_c = 0.16$, $Ef_d = 0.17$, $Ef_e = 0.05$, and $Ef_f = Ef_g = 0.06$, $Ef_h = Ef_i = 0.02$.

Example 5 Assume $N = \{a, b, c, d, e, f\}$ and $\mathcal{S} \in \mathcal{S}^N_H$ given by $\mathcal{S}(a) = \mathcal{S}(b) = \mathcal{S}(c) = \{d\}$, $\mathcal{S}(d) = \{e, f\}$, and $\mathcal{S}(e) = \mathcal{S}(f) = \varnothing$ and $v(S) = 1$ if e or $f \in S$ and $\{a, b\}$, $\{a, c\}$ or $\{b, c\} \subset S$ and 0 otherwise. Then $(N, r^d_{N,v,\mathcal{S}})$ can be represented by the product game $[2; 1_a, 1_b, 1_c] \times [3; 2_d, 1_e, 1_f]$ and $(N, r^c_{N,v,\mathcal{S}})$ by the WVG $[13; 3_a, 3_b, 3_c, 3_d, 1_e, 1_f]$.

Assume (A1) for all players. For $(N, r^d_{N,v,\mathcal{S}})$ we have $Ef_a = Ef_b = Ef_c = 0.19$, $Ef_d = 0.38$, and $Ef_e = Ef_f = 0.13$, and for $(N, r^c_{N,v,\mathcal{S}})$ we have $Ef_a = Ef_b = Ef_c = Ef_d = 0.09$, and $Ef_e = Ef_f = 0.03$. (This is the Bz measure.)

Assume (A2) for all players. For $(N, r^d_{N,v,\mathcal{S}})$ we have $Ef_a = Ef_b = Ef_c = 0.13$, $Ef_d = 0.43$, and $Ef_e = Ef_f = 0.08$ and for $(N, r^c_{N,v,\mathcal{S}})$ we have $Ef_a = Ef_b = Ef_c = Ef_d = 0.23$, and $Ef_e = Ef_f = 0.03$. (This is the S-S index.)

Assume (A3) as follows: players a, b, and c form a standard t, d behaves independently, and e and f form a standard (1-t). For $(N, r^d_{N,v,\mathcal{S}})$ we have $Ef_a =$

$Ef_b = Ef_c = 0.12$, $Ef_d = 0.23$, and $Ef_e = Ef_f = 0.18$ and for $(N, r^c_{N,v,\mathcal{S}})$ we have $Ef_a = Ef_b = Ef_c = 0.07$, $Ef_d = 0.08$, and $Ef_e = Ef_f = 0.10$.

Example 6 Assume $N = \{a, b, c, d, e, f\}$, $\mathcal{S} \in \mathcal{S}^N_H$ given by $\mathcal{S}(a) = \{b, c\}$, $\mathcal{S}(b) = d$, $\mathcal{S}(c) = \{e, f\}$, and $\mathcal{S}(d) = \mathcal{S}(e) = \mathcal{S}(f) = \varnothing$ and a join $[10; 3_a, 3_b, 1_c, 1_d, 3_e] \vee [10; 3_a, 3_b, 1_c, 1_d, 3_f]$.

Assume (A1) for all players. We have $Ef_a = Ef_b = 0.28$, and $Ef_c = Ef_d = Ef_e = Ef_f = 0.09$. (This is the Bz measure.)

Assume (A2) for all players. We have $Ef_a = Ef_b = 0.37$, and $Ef_c = Ef_d = Ef_e = Ef_f = 0.07$. (This is the S-S index.)

Assume (A3) as follows: player a behaves independently, players b and d form a standard t and c, e, and f a standard $(1\text{-}t)$. We have $Ef_a = 0.20$, $Ef_b = 0.28$, $Ef_c = 0.06$, $Ef_d = 0.07$ and $Ef_e = Ef_f = 0.14$.[16]

5. On the Nature of A Priori Voting Power in Hierarchical Organizations

An important fact is that if a priori standards are considered we are likely to encounter violations of *local monotonicity*. Local Monotonicity is that property of a measure of voting power that ranks a player's voting weight (number of votes) in the same order as his or her voting power, i.e. local monotonicity is fulfilled if whenever the weight of player i is at least as great as that of player j, then i has at least as much power as j.[17] Formally:

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[16] Going back to our bank interpretation of Example 6 in Fn. 11 under (A3) we could say that c, e, and f are responsible for expanding credit and thus have the same standard t, b and d are responsible for managing risk and thus have an opposing standard $(1\text{-}t)$; and manager a has to consider both aspects of granting a loan and thus behaves independently.

[17] See Young (1985) and Turnovec (1998) for general introductions to the concept of monotonicity and Braham and Steffen (2002c) and Felsenthal and Machover (1995, 1998) for discussion of local monotonicity as a postulate of voting power. As a number of authors have pointed out (Felsenthal and Machover, 1995, 1998; Freixas and Gambarelli, 1997), local monotonicity is a special case of the *desirability* (also called *dominance*) relation, $\succeq$, which is a pre-ordering (i.e. it is transitive and reflexive) of the players in a simple game $\mathcal{W}(v)$. The desirability relation was first introduced by Isbell (1958) and later generalized by Maschler and Peleg (1966). See also Taylor and Zwicker (1999, pp. 86–92). The idea is that we can order the players in terms of their contribution to a coalition. Formally, $i \succeq j$ iff $S \cup \{j\} \in \mathcal{W}(v)$ implies $S \cup \{i\} \in \mathcal{W}(v)$. In words, player i is at least as desirable as j in coalition S in a game v if interchanging i and j does not change S from winning to losing. If we have $i \succeq j$ but not $j \succeq i$, then $i \succ j$, i.e. player i is strictly more desirable than player j, which says that whatever j can contribute to the passing of a bill i can do as well (is at least as desirable) and in some cases more (is more desirable). Thus, $i \succ j$ then $x_i(v) > x_j(v)$. It is also easy to see that if players i and j in v are interchangeable, then by symmetry $x_i(v) = x_j(v)$, and, $i \succeq j$ then $x_i(v) \geq x_j(v)$. For a WVG it clearly follows that if $w_i \geq w_j$ then $i \succeq j$, i.e. anything that w_j can do, w_i can also do because a winning coalition cannot become a losing coalition if it gains more weight (but it does not necessarily follow that if $w_i > w_j$ then

Local monotonicity (P1)

For every $\mathcal{W}(v)\in\mathcal{G}^N$ that can be represented as a WVG, if $w_i \geq w_j$ then $\xi_i(v) \geq \xi_j(v)$.

The reason for the violation of local monotonicity is due to the fact that under the partial homogeneity structure the winning coalitions do not occur with equal probability. Such violations are easy to find. [18]

Example 7 Assume $N = \{a, b, c\}$ playing a simple majority game $M_{n,(n/2)+1}$. If (i) a and b form a standard t and c behaves independently, we have $Ef_a = Ef_b = 50$, and $Ef_c = 0.33$, but if (ii) a and b form opposing standards (say t and $1–t$) and c behaves independently, then we have $Ef_a = Ef_b = 50$, and $Ef_c = 0.67$.

Note, however, that in more general terms, monotonicity refers to the *resources* or to use Dahl's (1957) terminology, the 'base of power', of which voting weights are only one component given by the 'bare' decision rule. As argued in Braham and Steffen (2002c) the underlying data of a voting game is made up of at least two components: (i) the voting weights and (ii) the players positions within the decision-making structure. While under (A1) or (A2) the structure is 'flat' so that resources and weight happen to coincide, under a differentiated structure such as (A3) this is no longer the case.

Taking this into account, in Example 7, in the first part players a and b in a sense have 'more' resources due to their joint standard of behaviour than player c, and therefore, the power values do not violate such a general definition of local monotonicity, while the second part is a bit more tricky. Taking into account that a and b may be acting against each other, this reduces the value of their resources as such, and therefore, the greater voting power of c fulfils local monotonicity (generally defined).

Now, referring to the examples given in section 4 we can find violations of (P1), for instance, in Examples 3 and 5 under (A3) and the conjunctive restriction. In Example 3 the players a and c have both four votes, but a is more powerful than c, while in Example 5 the players a, b, c, and d have all three votes, but d is more powerful than the other three players. The reason is easy to see by looking at the partitioning of the players under (A3). In both examples the more powerful player is assumed to behave independently, while the other players have opposing

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$i \succ j$). It is therefore straightforward that if $w_i \geq w_j$ then $x_i(v) \geq x_j(v)$, viz. precisely local monotonicity as in (P1).

[18] See also Braham and Steffen (2002c) for further examples.

standards like in Example 7 (ii), which reduces the value of their resources as such, and therefore the greater voting power of *a* in Example 3 and *d* in Example 5 fulfils local monotonicity (generally defined).

The fact that under (A3) winning coalitions do not occur with equal probability, has also further important implications on the nature of power in hierarchical organizations as it can cause violations of other monotonicity properties of voting power in hierarchical organizations.

In particular, it can lead to a violation of *weak structural monotonicity* and *structural monotonicity*. *Weak structural monotonicity* (Brink, 2001) states that the power ranking in any restricted permission game should rank a player *j* not before a player *i* if *i* dominates player *j* completely, i.e. by $j \in \bar{\mathcal{S}}(i)$. Formally:

Weak structural monotonicity (P2)

For every $\mathcal{W}(v) \in \mathcal{G}^N$ and $\mathcal{S} \in \mathcal{S}_H^N$, if $i \in N$ and $j \in \bar{\mathcal{S}}(i)$ then $\xi_i(N, r^c_{N,v,\mathcal{S}}) \geq \xi_j(N, r^c_{N,v,\mathcal{S}})$ and $\xi_i(N, r^d_{N,v,\mathcal{S}}) \geq \xi_j(N, r^d_{N,v,\mathcal{S}})$.

A stronger version known as *structural monotonicity* (Brink and Gilles, 1996) extends the above property to the case that *i* dominates *j*, i.e. $j \in \mathcal{S}(i)$, which is said to be satisfied only under the conjunctive restriction. Formally:

Structural monotonicity (P3)

For every $\mathcal{W}(v) \in \mathcal{G}^N$ and $\mathcal{S} \in \mathcal{S}_H^N$, if $i \in N$ and $j \in \mathcal{S}(i)$ then $\xi_i(N, r^c_{N,v,\mathcal{S}}) \geq \xi_j(N, r^c_{N,v,\mathcal{S}})$.

Thus, (P2) and (P3) more or less say that a player in a hierarchy who dominates another player should have at least as much voting power as the dominated player. But then it is clear that (P2) and (P3) can only be fulfilled under very specific restricted permission games, for it requires rules which guarantee that a superior is always critical in more coalitions than a subordinate, or if in less, the partial homogeneity structure gives more weight to the winning coalitions where the superior is a critical member.

For an example for a violation of (P2) and (P3) take, for instance, again Example 5 under (A3) and the conjunctive restriction. Even though player *d* dominates the players *e* and *f* completely, *d* has less power than his or her subordinates *e* and *f* for the same reason as for the violation of local monotonicity. For an additional example for a violation of (P3) see also Example 4 under (A3) and the conjunctive restriction. Even though player *d* is dominated by the board consisting of players a, b, and c, he or she has more power than these players, while player e, who compared to d has a symmetric position in the hierarchical structure $\mathcal{S}$, has less power than a, b, and c and thus than d. The reason is that the board on the top is assumed to behave according to a common standard t and e and his or her subordinates h and i are assumed to have an opposing standard (1-t), while d and his or her subordinates are assumed to behave independently from which d gains resources in comparison to the players with opposing standards, e.g., it is more likely that d is a member of a winning coalition than e.

This result, i.e. the violation of (P2) and (P3) under (A3) is of particular significance for the study of power in hierarchical organizations because it implies that contrary to common belief there is no necessary correlation between power and the rank of a player in a hierarchical structure.

Another interesting case is Example 6 under (A3). Here players e and f have more power than their superior c and player b has more power than his or her superior a. The reason is that (i) the partition of the players, i.e. that player a behaves independently, while the players b and d form a standard t and c, e, and f form a standard (1-t) and (ii) that players e and f can form a winning coalition together with b and d but without c.

Furthermore, we may have an violation of *disjunctive* and *conjunctive fairness*. *Disjunctive fairness* (Brink, 1997) requires that deleting the relation between two players i and $j\in\mathcal{S}(i)$ (with $|\mathcal{S}^{-1}(j)| \geq 2$) changes the power value for player i and j by the same amount. Moreover, the power values of all players h that completely dominate player i, in the sense that $h\in\bar{\mathcal{S}}^{-1}(i)$ change by this same amount. Formally:

Disjunctive fairness (P4)

For every $\mathcal{W}(v)\in\mathcal{G}^N$ and $\mathcal{S}\in\mathcal{S}_H^N$, if $i\in N$ and $j\in\mathcal{S}(i)$ then $\xi_h(N,\ r^d_{N,v,\mathcal{S}}) - \xi_h(N,\ r^d_{N,v,\mathcal{S}-(i,j)}) = \xi_j(N,\ r^d_{N,v,\mathcal{S}}) - \xi_j(N,\ r^d_{N,v,\mathcal{S}-(i,j)})$ for all $h\in\{i\}\cup\bar{\mathcal{S}}^{-1}(i)$.

Conjunctive fairness (Brink, 2001) says that deleting the relation between two players i and $j\in\mathcal{S}(i)$ (with $|\mathcal{S}^{-1}(j)| \geq 2$) changes the power value for player j and any other predecessor $h\in\mathcal{S}^{-1}(j) \setminus \{i\}$ by the same amount. Moreover, the power values of all players that completely dominate the other predecessor h change by this same amount. Formally:

Conjunctive fairness (P5)

For every $\mathscr{W}(v)\in\mathscr{G}^N$ and $\boldsymbol{\mathcal{S}}\in\boldsymbol{\mathcal{S}}_H^N$, if i, j, $h\in N$ such that $i \neq h$ and $j\in\boldsymbol{\mathcal{S}}(i)\cap\boldsymbol{\mathcal{S}}(h)$, then $\xi_g(N,\ r^c_{N,v,\boldsymbol{\mathcal{S}}}) - \xi_g(N,\ r^c_{N,v,\boldsymbol{\mathcal{S}}-(i,j)}) = \xi_j(N,\ r^c_{N,v,\boldsymbol{\mathcal{S}}}) - \xi_j(N,\ r^c_{N,v,\boldsymbol{\mathcal{S}}-(i,j)})$ for all $g\in\{h\}\cup\ \bar{\boldsymbol{\mathcal{S}}}^{-1}(h)$.

Hence, (P4) and (P5) roughly stipulate that the deletion of a hierarchical relation between two players should under (P4) change their voting power and that of the superiors of the dominating player by the same amount and in the same direction, while under (P5) the voting power of the dominated player and his or her superiors should be changed by the same amount and in the same direction, i.e. Brink's fairness conditions turn out to be very specific monotonicity conditions that can only be fulfilled under very specific restricted permission games, for it requires rules that guarantee that the proportions of the numbers of swings η_i, η_j, η_h, and η_g remain the same under the new restricted permission game without the dominance relation (i, j) or, if not, that the partial homogeneity structure restores the original proportions. Consider the restricted permission game given by Example 8 which is the same as Example 3 but with (b, e) as an additional dominance relation:

Example 8 Assume $N = \{a, b, c, d, e\}$, $\boldsymbol{\mathcal{S}}\in\boldsymbol{\mathcal{S}}_H^N$ given by $\boldsymbol{\mathcal{S}}(a) = \{b, c\}$, $\boldsymbol{\mathcal{S}}(b) = \boldsymbol{\mathcal{S}}(c) = \{d, e\}$, and $\boldsymbol{\mathcal{S}}(d) = \boldsymbol{\mathcal{S}}(e) = \varnothing$ and v(S) = 1 if d or $e\in S$ and 0 otherwise. Then (N, $r^d_{N,v,\boldsymbol{\mathcal{S}}}$) can be represented by the join $[5; 2_a, 2_b, 1_d, 1_e] \vee [5; 2_a, 2_c, 1_d, 1_e]$ and (N, $r^c_{N,v,\boldsymbol{\mathcal{S}}}$) by the WVG $[10; 3_a, 3_b, 3_c, 1_d, 1_e]$.

Assume (A1) for all players. For (N, $r^d_{N,v,\boldsymbol{\mathcal{S}}}$) we have $Ef_a = 0.56$ and $Ef_b = Ef_c = Ef_d = Ef_e = 0.19$ and for (N, $r^c_{N,v,\boldsymbol{\mathcal{S}}}$) we have $Ef_a = Ef_b = Ef_c = 0.19$ and $Ef_d = Ef_e = 0.06$. (This is the Bz measure.)

Assume (A2) for all players. For (N, $r^d_{N,v,\boldsymbol{\mathcal{S}}}$) we have $Ef_a = 0.53$, $Ef_b = Ef_c = Ef_d = Ef_e = 0.12$ and for (N, $r^c_{N,v,\boldsymbol{\mathcal{S}}}$) we have $Ef_a = Ef_b = Ef_c = 0.30$ and $Ef_d = Ef_e = 0.05$. (This is the S-S index.)

Assume (A3) as follows: player a behaves independently, player b forms a standard t and c, d, and e a standard (1-t). For (N, $r^d_{N,v,\boldsymbol{\mathcal{S}}}$) we have $Ef_a = 0.55$, $Ef_b = 0.13$, and $Ef_c = Ef_d = Ef_e = 0.21$ and for (N, $r^c_{N,v,\boldsymbol{\mathcal{S}}}$) we have $Ef_a = 0.12$, $Ef_b = 0.21$, $Ef_c = 0.13$, and $Ef_d = Ef_e = 0.04$.

Now, comparing the results of Examples 3 and 8 we can find that (P4) and (P5) are satisfied under (A1) as proved in Brink (2001) and under (A2) as proved in Brink (1997) for (P4) and in Brink and Gilles (1996) for (P5). However, we can also see that (P4) and (P5) are violated under (A3). In particular, we find that under the disjunctive restriction the power values of players b, e, and a who dominates b completely decreases by 0.06 under (A1) and by 0.03 under (A2) if we delete the dominance relation (b, e), while under (A3) the power values of a and b are reduced by 0.05 while that of e decreases by 0.13, i.e. (P4) is violated. Under the conjunctive restriction we can find a similar result but with an increase of the power values. Under (A1) and (A2) the power values of player e, his or her predecessor c, and a who dominates c completely are all increasing by 0.12 and 0.08, respectively, if we delete the dominance relation (b, e), while under (A3) the power values of e increases by 0.17 and those of c and a by 0.16 and 0.25, respectively.

However, (P4) and (P5) can also be violated if the hierarchical organization does not have a unique top-player but with a board structure at the top, i.e. if we deal with non quasi-strongly connected hierarchical structures. Note, that this result contradicts Brink's (2001) speculation that 'the results in this paper could also be stated for acyclic permission structures that not necessarily are quasi-strongly connected.' E.g., consider a restricted permission game as given by Example 9.

Example 9 Assume $N = \{a, b, c, d, e, f, g\}$ and $\boldsymbol{\mathcal{S}} \in \boldsymbol{\mathcal{S}}_H^N$ given by $\boldsymbol{\mathcal{S}}(a) = \boldsymbol{\mathcal{S}}(b) = \boldsymbol{\mathcal{S}}(c) = \{d, e\}$, $\boldsymbol{\mathcal{S}}(d) = \boldsymbol{\mathcal{S}}(e) = \{f, g\}$, and $\boldsymbol{\mathcal{S}}(f) = \boldsymbol{\mathcal{S}}(g) = \varnothing$, and $v(\mathrm{S}) = 1$ if f or $g \in S$ and $\{a, b\}$, $\{a, c\}$ or $\{b, c\} \subset S$ and $v(\mathrm{S}) = 0$ otherwise. (See figure 4a for a graphical representation of the hierarchical structure). Then (N, $r^d_{N,v,\boldsymbol{\mathcal{S}}}$) can be represented by the product game $[2; 1_a, 1_b, 1_c] \times ([3; 2_d, 1_f, 1_g] \vee [3; 2_e, 1_f, 1_g])$ and (N, $r^c_{N,v,\boldsymbol{\mathcal{S}}}$) by the join $[6; 1_a, 1_b, 1_c, 1_d, 1_e, 1_f] \vee [6; 1_a, 1_b, 1_c, 1_d, 1_e, 1_g]$.

Assume (A1) for all players. For (N, $r^d_{N,v,\boldsymbol{\mathcal{S}}}$) we have $Ef_a = Ef_b = Ef_c = 0.28$, and $Ef_d = Ef_e = Ef_f = Ef_g = 0.19$ and for (N, $r^c_{N,v,\boldsymbol{\mathcal{S}}}$) we have $Ef_a = Ef_b = Ef_c = Ef_d = Ef_e = 0.05$, and $Ef_f = Ef_g = 0.02$. (This is the Bz measure.)

Assume (A2) for all players. For (N, $r^d_{N,v,\boldsymbol{\mathcal{S}}}$) we have $Ef_a = Ef_b = Ef_c = 0.18$, and $Ef_d = Ef_e = Ef_f = Ef_g = 0.11$ and for (N, $r^c_{N,v,\boldsymbol{\mathcal{S}}}$) we have $Ef_a = Ef_b = Ef_c = Ef_d = Ef_e = 0.19$, and $Ef_f = Ef_g = 0.02$. (This is the S-S index.)

Assume (A3) as follows: player a behaves independently, players b and d form a standard t and c, e, f and g a standard (1-t). For (N, $r^d_{N,v,\boldsymbol{\mathcal{S}}}$) we have $Ef_a = 0.36$, $Ef_b = Ef_c = 0.28$, $Ef_d = 0.13$, and $Ef_e = Ef_f = Ef_g = 0.21$ and for (N, $r^c_{N,v,\boldsymbol{\mathcal{S}}}$) we have $Ef_a = 0.024$, $Ef_b = Ef_d = 0.033$, $Ef_c = Ef_e = 0.025$, and $Ef_f = Ef_g = 0.083$.

Now, if we remove the dominance relation (e, f), the corresponding results are given by Example 10:

Example 10 Assume $N = \{a, b, c, d, e, f, g\}$ and $\mathcal{S} \in \mathcal{S}_H^N$ given by $\mathcal{S}(a) = \mathcal{S}(b) = \mathcal{S}(c) = \{d, e\}$, $\mathcal{S}(d) = \{f, g\}$, $\mathcal{S}(e) = \{g\}$, and $\mathcal{S}(f) = \mathcal{S}(g) = \varnothing$, and $v(\mathrm{S}) = 1$ if f or $g \in S$ and $\{a, b\}$, $\{a, c\}$ or $\{b, c\} \subset S$ and $v(\mathrm{S}) = 0$ otherwise. (See figure 4b for a graphical representation of the hierarchical structure). Then (N, $r^d_{N,v,\mathcal{S}}$) can be represented by the product game $[2; 1_a, 1_b, 1_c] \times ([3; 2_d, 1_f, 1_g] \vee [3; 2_e, 1_g])$ and (N, $r^c_{N,v,\mathcal{S}}$) by the join $[5; 1_a, 1_b, 1_c, 1_d, 1_f] \vee [6; 1_a, 1_b, 1_c, 1_d, 1_e, 1_g]$.

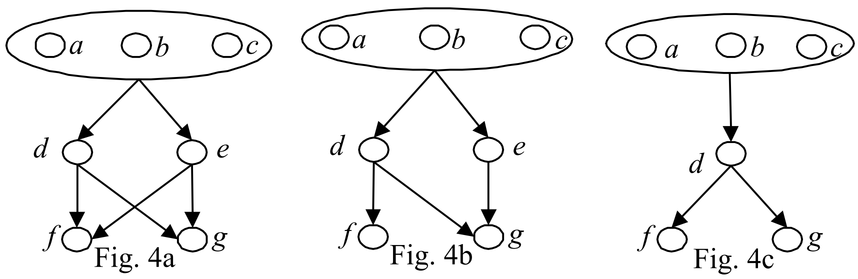
Assume (A1) for all players. For (N, $r^d_{N,v,\mathcal{S}}$) we have $Ef_a = Ef_b = Ef_c = Ef_d = Ef_g = 0.25$ and $Ef_e = Ef_f = 0.13$ and for (N, $r^c_{N,v,\mathcal{S}}$) we have $Ef_a = Ef_b = Ef_c = Ef_d = 0.08$, $Ef_e = Ef_g = 0.02$, and $Ef_f = 0.06$. (This is the Bz measure.)

Assume (A2) for all players. For (N, $r^d_{N,v,\mathcal{S}}$) we have $Ef_a = Ef_b = Ef_c = Ef_d = Ef_g = 0.17$ and $Ef_e = Ef_f = 0.08$ and for (N, $r^c_{N,v,\mathcal{S}}$) we have $Ef_a = Ef_b = Ef_c = Ef_d = 0.22$, $Ef_e = Ef_g = 0.02$, and $Ef_f = 0.07$. (This is the S-S index.)

Assume (A3) as follows: player a behaves independently, players b and d form a standard t and c, e, f and g a standard (1-t). For (N, $r^d_{N,v,\mathcal{S}}$) we have $Ef_a = 0.33$, $Ef_b = 0.18$, $Ef_c = 0.33$, $Ef_d = Ef_e = Ef_f = 0.17$, and $Ef_g = 0.33$ and for (N, $r^c_{N,v,\mathcal{S}}$) we have $Ef_a = 0.04$, $Ef_b = Ef_c = Ef_d = Ef_f = 0.05$, and $Ef_e = Ef_g = 0.01$.

Comparing the results of Example 9 with those of Example 10 in which we have deleted (e, f) we can see that (P4) and (P5) are already violated under (A1) and (A2) due to the fact that these hierarchical organizations do not obey quasi-strongly connectedness, and are still violated under (A3). In particular under the

Figure 4: *Hierarchies with a Board-Structure at the Top II*



disjunctive restriction the power value of f under (A1) decreases by 0.06 while that of his or her predecessor d increases by 0.06. Under (A2) we obtain a similar result, i.e. the power value of f decreases by 0.03 while that of d increases by 0.06. Also applying (A3) the structure of the result remains the same: the power value of f decreases by 0.04 while that of d increases by 0.04. Under the conjunctive restriction the power value of f under (A1), (A2), and (A3) increases by 0.04 while that of his or her predecessor d increases only by 0.03.

However, note, that it could be possible that (A3) just rescues (P4) or (P5) if its violation occurs due to a non-quasi-strongly connected hierarchical structure.

Finally, if we reduce the number of players in Example 10 by eliminating player e, we obtain the restricted game with a permission structure as given by Example 11.

Example 11 Assume $N = \{a, b, c, d, f, g\}$ and $\mathcal{S} \in \mathcal{S}_H^N$ given by $\mathcal{S}(a) = \mathcal{S}(b) = \mathcal{S}(c) = \{d\}$, $\mathcal{S}(d) = \{f, g\}$, and $\mathcal{S}(f) = \mathcal{S}(g) = \varnothing$, and $v(\mathrm{S}) = 1$ if f or $g \in S$ and $\{a, b\}$, $\{a, c\}$ or $\{b, c\} \subset S$ and $v(\mathrm{S}) = 0$ otherwise. (See figure 4c for a graphical representation of the hierarchical structure). Then $(N, r^d_{N,v,\mathcal{S}})$ can be represented by the product game $[2; 1_a, 1_b, 1_c] \times [3; 2_d, 1_f, 1_g]$ and $(N, r^c_{N,v,\mathcal{S}})$ by the WVG $[9; 2_a, 2_b, 2_c, 2_d, 1_f, 1_g]$.

Assume (A1) for all players. For $(N, r^d_{N,v,\mathcal{S}})$ we have $Ef_a = Ef_b = Ef_c = 0.19$, $Ef_d = 0.38$, $Ef_f = Ef_g = 0.13$ and for $(N, r^c_{N,v,\mathcal{S}})$ we have $Ef_a = Ef_b = Ef_c = Ef_d = 0.09$, and $Ef_f = Ef_g = 0.03$. (This is the Bz measure.)

Assume (A2) for all players. For $(N, r^d_{N,v,\mathcal{S}})$ we have $Ef_a = Ef_b = Ef_c = 0.13$, $Ef_d = 0.43$, $Ef_f = Ef_g = 0.08$ and for $(N, r^c_{N,v,\mathcal{S}})$ we have $Ef_a = Ef_b = Ef_c = Ef_d = 0.23$, and $Ef_f = Ef_g = 0.03$. (This is the S-S index.)

Assume (A3) as follows: player a behaves independently, players b and d form a standard t and c, f and g a standard (1-t). For $(N, r^d_{N,v,\mathcal{S}})$ we have $Ef_a = 0.15$, $Ef_b = Ef_c = 0.13$, $Ef_d = 0.33$, $Ef_f = Ef_g = 0.17$ and for $(N, r^c_{N,v,\mathcal{S}})$ we have $Ef_a = 0.05$, $Ef_b = Ef_d = 0.06$, $Ef_c = 0.07$, and $Ef_f = Ef_g = 0.03$.

Comparing the results of Example 11 with those of Examples 9 and 10 we can see that deleting player e does not necessarily lead to a shift of the players' power values in a specific direction in a hierarchical organization.

6. Conclusions

Our study of a priori voting power in hierarchical organizations using an alternative approach to Brink, Gilles and Owen has shown a number of counterintuitive results, namely, that monotonicity properties of voting power in hierarchical organizations like (weak) structural monotonicity and dis- and conjunctive fairness do not hold any longer if we take into account predetermined roles of players in a hierarchy and/or we drop the assumption of a unique top-player, i.e. the hierarchical structure is not quasi-strongly connected.

Moreover, we have illustrated that (i) every 'restricted' permission game can be represented as a compound game and (ii) that (as shown by Example 11) dropping a player belonging to an intermediate hierarchical level, does not necessarily imply that this voting power is transferred downwards to lower hierarchical levels.

We wish to tie up our conclusions with some further thoughts related to the analysis of voting power in hierarchical organizations and with an important implication that flows from it; for a more elaborated discussion on this issue we refer the reader to Braham and Steffen (2001a, 2002b).

There exists another related issue to characterize hierarchical organizations based upon the theory of simple games which is a subject to further research. This is the responsiveness of a hierarchical organization to the overall nature of a (collective) decision-making rule, i.e. the ability of a hierarchical organisation to approve projects. What we suspect from the fact that this analysis is based on the theory of simple games that 'lean' hierarchies are not necessarily more 'responsive' and innovative than 'fatter' ones. While it is true that the time it may take to reach a decision is shorter in a lean than in a fat hierarchy because it has less layers and, therefore, less committees (component games). However, this does not imply that more projects will be approved (and, therefore, the organization will be more 'responsive' and innovative).

This together with the results provided in this chapter has an important implication to two related management concepts which are known as *empowerment*[19] and *lean management*[20]. Both are based on the idea that: (i) by

[19] See Gal-Or and Amit (1998) for a summary of empowerment. For examples of the meaning and implications of empowerment in the management and organizational behaviour literature see Conger and Kanungo (1988), Pfeffer (1992), Spreitzer (1995, 1996), and Thomas and Velthouse (1990).

[20] This concept goes back to Krafcik (1988).

removing intermediate layers or parts of layers of a hierarchy power can be transferred downwards to employees on the lower levels and that (ii) such a change will lead to increased motivation due to employees having more of a say in the organization's destiny and thus, increased responsiveness and productivity gains for the organization. But as indicated above (i) is not necessarily the case if we remove layers or parts of layers. Thus, frustration and reduction of effort is likely to be the result. But even if an employee's voting power increases, this does not imply that motivation will increase. As indicated above, reducing the length and breadth of a hierarchy does not necessarily increase the probability that a project will be accepted. That is to say, in absence of a change in the decision rule and/or the partial homogeneity structure, empowerment and lean management can lead to an increase in the rejection rate and, thus, a fall in motivation.

The practical implications of this perspective is that when we come to look at the performance of organizations, it is necessary to abstract from the particular personalities that are involved. The success or failure of an organization may not be so much a matter of its 'leadership' and 'management style' – its 'corporate culture' – but of the interaction of its *behavioral incentives* and *decision-making rules* (including the *permission* and *communication structures* of the organization).[21]

[21] For a recent economic analysis of visionary leadership, see Rotemberg and Saloner (2000) who completely ignore the effects of the permission structure.

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