The Dold-Kan Correspondence and Coalgebra Structures

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Introduction

Motivation

Model categories, first introduced by Quillen in [Qui67], are an axiomatic approach to do homotopy theory. A model category structure on a category consists of three distinguished classes of morphisms called weak equivalences, fibrations and cofibrations that satisfy certain axioms. The homotopy category on a model category is obtained by inverting weak equivalences. If a model category is equipped with a monoidal product, then, one obtains the notion of a monoidal model category structure. One requires in addition, that the monoidal product satisfies some compatibility conditions with respect to the model structure.

A monoidal category is related to the subcategory of its monoids via an adjoint pair of functors. In many cases, the forgetful functor from the category of monoids to the monoidal category has a left adjoint. In [SS00], Schwede and Shipley point out some sufficient conditions for extending a model category structure to the associated category of monoids over a monoidal model category. Whenever these conditions are fulfilled, an application of a transfer theorem along the adjoint pair of functors guarantees a model structure for the category of monoids.

Later in [SS03], Schwede and Shipley investigate another general question. They consider two monoidal model categories related via a weak monoidal Quillen equivalence. Then they discuss to what extend these Quillen equivalences are preserved for the categories of monoids. A motivating example of a weak monoidal Quillen equivalence is provided by the Dold-Kan correspondence.

The Dold-Kan correspondence, discovered independently by Dold and Kan (see [Dol58]), establishes an equivalence of categories between simplicial objects and non-negatively graded differential objects in every abelian category. Simplicial structures are relatively difficult to handle: objects are subject to many combinatorial relations. Hence, the Dold-Kan correspondence is of great importance: it provides a proper understanding of simplicial structures since working with differential graded objects is often more familiar.

The categories of simplicial abelian groups and non-negatively graded differential abelian groups are both endowed with a monoidal model category structure. Although these categories are equivalent via the Dold-Kan correspondence, their respective categories of monoids are not equivalent. This failure is roughly due to the difference between the monoidal products considered on these two monoidal categories. However, the functors involved in the Dold-Kan correspondence give rise to two weak monoidal Quillen equivalences. These Quillen equivalences are then lifted along the adjoint pairs of functors. As a consequence, Schwede and Shipley derive that the homotopy categories of the categories of monoids are equivalent.

After this overview of the work of Schwede and Shipley, a natural question arises: what can be said for the categories of comonoids over monoidal model categories? In this work, we restrict to the category of simplicial vector spaces and to the category of non-negatively graded differential vector spaces over a fixed field K. Their respective comonoids consist of the category of simplicial coalgebras and of the category of non-negatively graded differential coalgebras. Moreover, these categories of comonoids support model structures proven in [Goe95] and in [GG99]. Using the techniques pioneered by Schwede and Shipley, we discuss the comparison of the homotopy categories of these comonoids. However, on the one hand, we observe that the dualization of the main result of Schwede and Shipley turns out to be fruitless. On the other hand, a direct approach through a criterion given in [Hov99, Corollary 1.3.16] remains difficult: computing the homology of complexes that involve cofree coalgebras functors leads to cumbersome problems as we will see later.

Outline

The first chapter is devoted to some generalities about monoidal model categories. It is subdivided into three parts. In the first part, we recall the basic concept of monoidal categories with a special emphasis on the categories of monoids and comonoids. In the second part, we present the main tools used in model category structures. These tools include the Quillen (co)small object argument and the transfer theorem which is a powerful result for creating new model structures. The third part discusses the requirements that make a monoidal product compatible with a model structure. This is given by the pushout product axiom. We also state the main result of Schwede and Shipley in [SS03] that motivates the present work.

Since we are interested in comonoids, we dedicate the second chapter to the study of coalgebra structures. In [Swe69], Sweedler constructs a cofree functor that is right adjoint to the forgetful functor from the category of coalgebras to the category of vector spaces. In fact, the cofree coalgebra on a vector space V is a certain subcoalgebra of the coalgebra obtained via the dual of the tensor algebra on the dual space V^* (see [Swe69, Theorem 6.4.1]). While the tensor algebra is well described, the cofree coalgebra functor is hard to characterize: it is essentially obtained by a result of existence, but one would like to achieve concrete computations. To this end we use an equivalent approach suggested by Getzler and Goerss in [GG99] and we compute the cofree coalgebra on a one-dimensional vector space. However, dimensions higher than one lead to some difficulties left unaddressed in this work.

The third chapter deals with differential graded objects. First, we recall the standard monoidal model structure on the category of differential graded vector spaces. This model category structure is in addition cofibrantly generated. We adress the question whether it is fibrantly generated. We could find a generating set of acyclic fibrations, but our candidate for a generating set of fibrations fails: the category of differential graded vector spaces does not have cosmall objects except its terminal object. We provide a counterexample that clarifies this point. We also consider the category of differential graded algebras and explain how its model structure can be recovered by the method of Schwede and Shipley. However, similar considerations do not hold when one deals with the category of differential graded coalgebras. We point out the various problems that arise.

In chapter four, we give a brief review of the Dold-Kan correspondence. This correspondence is realized by the normalization functor N and its inverse Γ which we describe with some examples. Then we consider the Alexander-Whitney map AW and its homotopical inverse given by the shuffle map ∇. These maps are (co)monoidal transformations and serve to define comultiplications for the comonoids. We end this chapter with the description of a model structure on simplicial coalgebras proven in [Goe95].

In chapter five, we dualize the work of Schwede and Shipley in [SS03]. Namely, we prove that the normalization functor N defines a functor from the category of simplicial coalgebras to the category of non-negatively differential graded coalgebras. Similar considerations hold for the functor Γ that goes in the opposite direction. However, as in [SS03], we point out that both functors are neither adjoint nor inverse to each other. Then, we construct functors that are right adjoint to the functors N and Γ on the level of comonoids. The adjoint pairs of functors obtained in this way turn out to be Quillen pairs. It follows that the homotopy category of simplicial coalgebras and the homotopy category of non-negatively graded differential coalgebras are adjoint. But improving these Quillen pairs into Quillen equivalences remains an unsolved task: since the category of differential graded vector spaces fails to be fibrantly generated, we are not able to apply the main result of Schwede and Shipley. In order to get around this problem, we use a criterion proven in [Hov99, Corollary 1.3.16]. Once again, we are facing some difficulties: the involved cofree coalgebra functors do not behave well with respect to homology. Nevertheless, we exhibit some differential graded coalgebras that verify the criterion given in [Hov99, Corollary 1.3.16].

Notations

Throughout this work, we denote by K a given field. Vectors spaces, algebras, coalgebras and tensor products are taken over K. We adopt a mnemonic notation for the various categories: for instance, DG-K-**Vct** stands for the category of differential graded K-vector spaces.

Chapter 1

Preliminaries on Monoidal Model Structures

1.1 Monoidal structures

In this section we recall the concept of monoidal categories. This concept is studied for instance in [Bor94b] or [Mac98]. After focussing on monoids and comonoids in a given monoidal category we describe monoidal functors. When two monoidal categories are adjoint it is possible under some suitable conditions to construct an adjunction between their respective categories of monoids. We state such results at the end of this section.

1.1.1 Monoidal categories

Definition 1.1.1. A *monoidal* category $\mathbf{C} = (\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$ consist of

(i) a category $\mathbf{C}$,

(ii) a bifunctor $\otimes\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$,

(iii) a unit object $I \in \mathbf{C}$,

(iv) and three isomorphisms

$$\alpha_{A,B,C}\colon A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

$$\lambda_A\colon I \otimes A \cong A$$

$$\rho_A\colon A \otimes I \cong A$$

that are natural for all $A, B, C \in \mathbf{C}$

subject to the following coherence conditions:

- the pentagonal diagram

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[rr, "id_A \otimes \alpha_{B,C,D}"] \arrow[d, "\alpha_{A,B,C\otimes D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[dd, "\alpha_{A,B\otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A\otimes B,C,D}"'] & & \\
((A \otimes B) \otimes C) \otimes D & & (A \otimes (B \otimes C)) \otimes D \arrow[ll, "\alpha_{A,B,C} \otimes id_D"]
\end{tikzcd}$$

  commutes for all $A, B, C, D \in \mathbf{C}$;

- the triangular diagram

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[r, "\alpha_{A,I,B}"] \arrow[d, "id_A \otimes \lambda_B"'] & (A \otimes I) \otimes B \arrow[d, "\rho_A \otimes id_B"] \\
A \otimes B \arrow[r, equal] & A \otimes B
\end{tikzcd}$$

  commutes for all $A, B \in \mathbf{C}$;

- and finally $\lambda_I = \rho_I \colon I \otimes I \to I$.

Definition 1.1.2. A monoidal category $(\mathbf{C}, \otimes, I_{\mathbf{C}})$ is said to be *symmetric* if it is equipped with isomorphisms

$$\gamma_{A,B} \colon A \otimes B \cong B \otimes A$$

natural in $A, B \in \mathbf{C}$ with

$$\gamma_{A,B} \circ \gamma_{B,A} = \mathrm{Id}, \qquad \rho_B = \lambda_B \circ \gamma_{B,I} \colon B \otimes I \cong B$$

and such that the diagram

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "id_A \otimes \gamma_{B,C}"'] & (A \otimes B) \otimes C \arrow[r, "\gamma_{A\otimes B,C}"] & C \otimes (A \otimes B) \arrow[d, "\alpha_{C,A,B}"] \\
A \otimes (C \otimes B) \arrow[r, "\alpha_{A,C,B}"'] & (A \otimes C) \otimes B \arrow[r, "\gamma_{A,C} \otimes id_B"'] & (C \otimes A) \otimes B
\end{tikzcd}$$

commutes.

Example 1.1.3. In [Mac98, VII.3] MacLane gives a non-exhaustive list of examples. We recall that every category with finite products is monoidal with its terminal object as unit object. Likewise every category with finite coproducts is monoidal with its initial object as unit object. In this way one gets a large range of monoidal categories. We will consider in this work the monoidal category of vector spaces over a field K, $(K\text{-}\mathbf{Vct}, \otimes_K, K)$.

Definition 1.1.4. A *closed* category $\mathbf{C}$ is a symmetric monoidal category in which each functor $- \otimes C \colon \mathbf{C} \to \mathbf{C}$ has a specified right adjoint $(-)^C \colon \mathbf{C} \to \mathbf{C}$.

Example 1.1.5. The monoidal category $(K\text{-}\mathbf{Vct}, \otimes_K, K)$ is closed. If Y is a K-vector space then the right adjoint of the functor $- \otimes_K Y$ is given by the functor $K\text{-}\mathbf{Vct}(Y,-)$.

We now describe some specific objects in a given monoidal category.

Definition 1.1.6. Let $\mathbf{C}$ be a monoidal category.

1. A *monoid* M in a monoidal category is an object $M \in \mathbf{C}$ together with two arrows $\mu_M \colon M \otimes M \to M$ and $\eta_M \colon I \to M$ such that the diagrams

$$\begin{tikzcd}
M \otimes (M \otimes M) \arrow[r, "\alpha_{M,M,M}"] \arrow[d, "id_M \otimes \mu_M"'] & (M \otimes M) \otimes M \arrow[r, "\mu_M \otimes id_M"] & M \otimes M \arrow[d, "\mu_M"] \\
M \otimes M \arrow[rr, "\mu_M"'] & & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
I \otimes M \arrow[r, "\eta_M \otimes id_M"] \arrow[dr, "\lambda_M"'] & M \otimes M \arrow[d, "\mu_M"] & M \otimes I \arrow[l, "id_M \otimes \eta_M"'] \arrow[dl, "\rho_M"] \\
& M &
\end{tikzcd}$$

are commutative.

2. A *morphism of monoids* $f \colon (M, \mu_M, \eta_M) \longrightarrow (N, \mu_N, \eta_N)$ is a morphism $f \colon M \to N$ in $\mathbf{C}$ which satisfies

$$\mu_N(f \otimes f) = f\mu_M \quad \text{and} \quad f\eta_M = \eta_N.$$

With these morphisms, the monoids in $\mathbf{C}$ constitute a category that we denote by **C-Monoid**. The assignment $(M, \mu_M, \eta_M) \longmapsto M$ defines a forgetful functor $U \colon \textbf{C-Monoid} \longrightarrow \mathbf{C}$

Definition 1.1.7. Let $\mathbf{C}$ be a monoidal category.

1. A *comonoid* C is an object $C \in \mathbf{C}$ together with two arrows $\Delta_C \colon C \to C \otimes C$ and $\varepsilon_C \colon C \to I$ such that the diagrams

$$\begin{tikzcd}
C \arrow[rr, "\Delta_C"] \arrow[d, "\Delta_C"'] & & C \otimes C \arrow[d, "\Delta_C \otimes id_C"] \\
C \otimes C \arrow[r, "id_C \otimes \Delta_C"'] & C \otimes (C \otimes C) \arrow[r, "\alpha_{C,C,C}"'] & (C \otimes C) \otimes C
\end{tikzcd}$$

and

$$\begin{tikzcd} I \otimes C & C \otimes C \arrow[l, "\epsilon_C \otimes id_C"'] \arrow[r, "id_C \otimes \epsilon_C"] & C \otimes I \\ & C \arrow[ul, "\lambda_C^{-1}"] \arrow[u, "\Delta_C"] \arrow[ur, "\rho_C^{-1}"'] & \end{tikzcd}$$

are commutative.

2. A *morphism of comonoids* $f\colon (C, \Delta_C, \varepsilon_C) \longrightarrow (D, \Delta_D, \varepsilon_D)$ is a morphism $f\colon C \to D$ which satisfies

$$(f \otimes f)\Delta_C = \Delta_D f \quad \text{and} \quad \varepsilon_D f = \varepsilon_C.$$

With these morphisms, the comonoids in $\mathbf{C}$ constitute a category that we denote by **C-coMonoid**. The assignment $(C, \Delta_C, \varepsilon_C) \longmapsto C$ defines a forgetful functor $U\colon$ **C-coMonoid** $\longrightarrow \mathbf{C}$.

It might be useful to formally view comonoids of a monoidal category $\mathbf{C}$ as monoids in the opposite category $\mathbf{C}^{\mathrm{op}}$.

Lemma 1.1.8. *Let $(\boldsymbol{C}, \otimes, I_{\boldsymbol{C}})$ be a monoidal category. Then comonoids in $\boldsymbol{C}$ are monoids in the opposite category $\boldsymbol{C}^{\mathrm{op}}$. More precisely*

$$(\boldsymbol{C\text{-}coMonoid})^{\mathrm{op}} = \boldsymbol{C}^{\mathrm{op}}\boldsymbol{\text{-}Monoid}.$$

We refer for instance to [Por08, Section 2.2] where this topic is discussed.

1.1.2 Monoidal functors

Definition 1.1.9. A *lax monoidal functor* $F\colon (\mathbf{C}, \otimes, I_{\mathbf{C}}) \to (\mathbf{D}, \widehat{\otimes}, I_{\mathbf{D}})$ between monoidal categories consists of the following data:

(i) a functor $F\colon \mathbf{C} \to \mathbf{D}$;

(ii) for objects $A, B \in \mathbf{C}$ morphisms $\Psi_{A,B}\colon F(A)\widehat{\otimes}F(B) \to F(A \otimes B)$ in $\mathbf{D}$ which are natural in A and B;

(iii) for the units, a morphism $\psi\colon I_{\mathbf{D}} \to F(I_{\mathbf{C}})$ in $\mathbf{D}$

such that the following diagrams

$$\begin{tikzcd} F(A)\widehat{\otimes}(F(B)\widehat{\otimes}F(C)) \arrow[rr, "\alpha_{\mathbf{D}}"] \arrow[d, "id\otimes\Psi"'] & & (F(A)\widehat{\otimes}F(B))\widehat{\otimes}F(C) \arrow[d, "\Psi\otimes id"] \\ F(A)\widehat{\otimes}(F(B \otimes C)) \arrow[d, "\Psi"'] & & (F(A \otimes B)\widehat{\otimes}F(C)) \arrow[d, "\Psi"] \\ F(A \otimes (B \otimes C)) \arrow[rr, "F(\alpha_{\mathbf{C}})"'] & & F((A \otimes B) \otimes C), \end{tikzcd}$$

$$\begin{tikzcd}
F(A)\widehat{\otimes} I_{\mathbf{D}} \arrow[r, "\rho_{\mathbf{D}}"] \arrow[d, "id\otimes\psi"'] & F(A) \\
F(A)\widehat{\otimes} F(I_{\mathbf{C}}) \arrow[r, "\Psi"'] & F(A\otimes I_{\mathbf{C}}) \arrow[u, "F(\rho_{\mathbf{C}})"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
I_{\mathbf{D}}\widehat{\otimes} F(A) \arrow[r, "\lambda_{\mathbf{D}}"] \arrow[d, "\psi\otimes id"'] & F(A) \\
F(I_{\mathbf{C}})\widehat{\otimes} F(A) \arrow[r, "\Psi"'] & F(I_{\mathbf{C}}\otimes A) \arrow[u, "F(\lambda_{\mathbf{C}})"']
\end{tikzcd}$$

are commutative.

Remark 1.1.10. A lax monoidal functor is *strong monoidal* respectively *strict monoidal* if the coherence morphisms are invertible respectively identities.

Definition 1.1.11. A monoidal functor F between symmetric monoidal categories is said to be *symmetric* if the diagram

$$\begin{tikzcd}
F(A)\widehat{\otimes} F(B) \arrow[r, "\gamma_{F(A),F(B)}"] \arrow[d, "\Psi_{A,B}"'] & F(B)\widehat{\otimes} F(A) \arrow[d, "\Psi_{B,A}"] \\
F(A\otimes B) \arrow[r, "F(\gamma_{A,B})"'] & F(B\otimes A)
\end{tikzcd}$$

commutes for all A, B.

Definition 1.1.12. A *monoidal natural transformation*

$$\Theta\colon (F,\Phi,\phi)\to(G,\Gamma,\gamma)$$

between two monoidal functors is a natural transformation between the underlying functors $\Theta\colon F\to G$ such that the diagrams

$$\begin{tikzcd}
F(A)\widehat{\otimes} F(B) \arrow[r, "\Psi_F"] \arrow[d, "\Theta_A\widehat{\otimes}\Theta_B"'] & F(A\otimes B) \arrow[d, "\Theta_{A\otimes B}"] \\
G(A)\widehat{\otimes} G(B) \arrow[r, "\Psi_G"'] & G(A\otimes B)
\end{tikzcd}$$

$$\begin{tikzcd}
I_{\mathbf{D}} \arrow[r, "\psi_F"] \arrow[d, equal] & F(I_{\mathbf{C}}) \arrow[d, "\Theta_{I_{\mathbf{C}}}"] \\
I_{\mathbf{D}} \arrow[r, "\psi_G"'] & G(I_{\mathbf{C}})
\end{tikzcd}$$

commute for all A, B.

There is also the dual notion of a *lax comonoidal functor* and a *comonoidal natural transformation.*

1.1.3 Monads

Definition 1.1.13. Let $(\mathbf{C}^{\mathbf{C}}, \circ, \mathrm{Id})$ be the monoidal category of endofunctors of a category $\mathbf{C}$. Consider a triple $\mathbb{T} = (T, \eta, \mu)$ where $T\colon \mathbf{C} \to \mathbf{C}$ is an endofunctor and $\eta\colon \mathrm{Id} \to T, \mu\colon T \circ T \to T$ are two natural transformations. Then $\mathbb{T}$ is a *monad in* $\mathbf{C}$ if it is a monoid in $(\mathbf{C}^{\mathbf{C}}, \circ, \mathrm{Id})$. Notice that the transformations η and μ are respectively the unit and the multiplication of the monoid $\mathbb{T}$.

Definition 1.1.14. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad in a category $\mathbf{C}$.

1. A $\mathbb{T}$-*algebra* in $\mathbf{C}$ is a pair (C, h) where $C \in \mathbf{C}$ and $h\colon T(C) \to C$ is a morphism in $\mathbf{C}$ making the diagrams

$$\begin{tikzcd} TTC \arrow[r, "Th"] \arrow[d, "\mu_C"'] & TC \arrow[d, "h"] \\ TC \arrow[r, "h"'] & C \end{tikzcd}$$

and

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "id_C"'] & TC \arrow[d, "h"] \\ & C \end{tikzcd}$$

commute.

2. A *morphism* $\alpha\colon (C, h) \to (D, g)$ of $\mathbb{T}$-algebras is a morphism $\alpha\colon C \to D$ in $\mathbf{C}$ such that the following diagram

$$\begin{tikzcd} TC \arrow[r, "h"] \arrow[d, "T\alpha"'] & C \arrow[d, "\alpha"] \\ TD \arrow[r, "g"'] & D \end{tikzcd}$$

commutes.

With these morphisms, $\mathbb{T}$-algebras constitute a category $\mathbf{C}^{\mathbb{T}}$ and $(C, h) \longmapsto C$ defines a forgetful functor $U\colon \mathbf{C}^{\mathbb{T}} \to \mathbf{C}$.

Proposition 1.1.15. *Consider the following situation*

$$\begin{tikzcd}
(\boldsymbol{C}, \otimes, I_{\boldsymbol{C}}) \arrow[r, shift left, "L"] \arrow[d, shift left, "T_{\boldsymbol{C}}"] & (\boldsymbol{D}, \widehat{\otimes}, I_{\boldsymbol{D}}) \arrow[l, shift left, "R"] \arrow[d, shift left, "T_{\boldsymbol{D}}"] \\
\boldsymbol{C\text{-}Monoid} \arrow[u, shift left, "U_{\boldsymbol{C}}"] & \boldsymbol{D\text{-}Monoid} \arrow[u, shift left, "U_{\boldsymbol{D}}"] \arrow[l, "\widetilde{R}"]
\end{tikzcd}$$

where

(i) the monoidal categories $(\boldsymbol{C}, \otimes, I_{\boldsymbol{C}})$ *and* $(\boldsymbol{D}, \widehat{\otimes}, I_{\boldsymbol{D}})$ *are closed and cocomplete;*

(ii) the functor R *has a left adjoint* L*;*

(iii) and $\widetilde{R}$ *is a functor between the categories of monoids such that*

$$R \circ U_{\boldsymbol{D}} = U_{\boldsymbol{C}} \circ \widetilde{R}.$$

Then $\widetilde{R}$ *has a left adjoint.*

Proof. The assumption (i) guarantees the existence of left adjoint functors $T_{\mathbf{C}}$ and $T_{\mathbf{D}}$ for the forgetful functors $U_{\mathbf{C}}$ and $U_{\mathbf{D}}$ by [Mac98, VII, 3, Theorem 2]. The category of monoids in $\mathbf{C}$ has colimits and the left adjoint, $T_{\mathbf{C}}$, called the free monoid functor, is given for an object $C \in \mathbf{C}$ by

$$T_{\mathbf{C}}(C) = \coprod_{n \geq 0} C^{\otimes n} \quad \text{with} \quad C^{\otimes 0} = I_{\mathbf{C}}.$$

The multiplication is given by concatenation and a similar construction holds for the free monoid functor $T_{\mathbf{D}}$. In this way, the category of monoids in $\mathbf{C}$ can be interpreted as the category of $\mathbb{T}$-algebras with $\mathbb{T} = U_{\mathbf{C}} \circ T_{\mathbf{C}}$, similarly for the category of monoids in $\mathbf{D}$. Now the claim is an application of the adjoint lifting theorem proven in [Bor94b, 4.5]. $\square$

1.2 Model structures

In this section we give an overview on model category structures. This notion, first introduced by Quillen in [Qui67], is an axiomatic approach to do homotopy theory. We first recall the different axioms and their immediate consequences. Then we describe Quillen functors which compare categories endowed with model structures. In practice it is difficult to endow a category with a model structure. In order to reduce these difficulties, some useful techniques are available. These techniques include the concept of locally presentable categories and Quillen's small object argument. We are also interested in (co)fibrantly generated model categories: these model categories are characterized by sets of generating (co)fibrations and generating acyclic (co)fibrations. Having these (co)fibrant generators on a model category leads to many advantages. The most important one is the transfer theorem.

1.2.1 Axioms

Definition 1.2.1. Let $\mathbf{C}$ be a category.

1. A map $f\colon A \to B$ in $\mathbf{C}$ is a *retract* of a map $g\colon C \to D$ in $\mathbf{C}$ if there is a commutative diagram of the following form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd}$$

   where the horizontal composites are identities.

2. A map $i\colon A \to B$ has the *left lifting property* with respect to $p\colon X \to Y$ if for any commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

   there is a lift $h\colon B \to Y$ such that $hi = f$ and $ph = g$. Dually the map p is said to have the *right lifting property* with respect to i.

Definition 1.2.2. A *model category* is a category $\mathbf{C}$ endowed with three classes of morphisms, *weak equivalences, fibrations, and cofibrations* such that the following axioms hold.

MC1: All limits and colimits exist in $\mathbf{C}$, that is, $\mathbf{C}$ is complete and cocomplete.

MC2: If f and g are maps in $\mathbf{C}$ such that the composite gf is defined and if two of the three maps f, g, gf are weak equivalences, then so is the third.

MC3: If f is a retract of g and g is a fibration, cofibration, or a weak equivalence, then so is f.

MC4: Given a commutating solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"'] & X \arrow[d, two heads, "p"] \\ B \arrow[r] \arrow[ur, dotted, "l"] & Y \end{tikzcd}$$

with i a cofibration and p a fibration, a lift l exists in either of the following two situations

(i) the map p is a weak equivalence or

(ii) the map i is a weak equivalence.

MC5: Any map f can be factored in two ways

(i) $f = pi$ with p a acyclic fibration and i a cofibration or

(ii) $f = pi$ with p a fibration and i an acyclic cofibration.

Remark 1.2.3. If a category $\mathbf{C}$ has a model structure then its opposite category $\mathbf{C}^{\mathrm{op}}$ inherits a model structure. Indeed let $f\colon A \to B$ be a map in $\mathbf{C}$ and $f^{\mathrm{op}}\colon B \to A$ its corresponding opposite map in $\mathbf{C}^{\mathrm{op}}$. Then f^{op} is

- a weak equivalence in $\mathbf{C}^{\mathrm{op}}$ if f is a weak equivalence in $\mathbf{C}$;
- a cofibration in $\mathbf{C}^{\mathrm{op}}$ if f is a fibration in $\mathbf{C}$;
- a fibration in $\mathbf{C}^{\mathrm{op}}$ if f is a cofibration in $\mathbf{C}$.

Definition 1.2.4. Let $\mathbf{C}$ be a model category, $\emptyset$ its initial object and $*$ its terminal object. An object $C \in \mathbf{C}$ is *cofibrant* if the map $\emptyset \to C$ is a cofibration and *fibrant* if the map $C \to *$ is a fibration.

For each object C of a model category $\mathbf{C}$, applying axiom **MC5**(i) to the map $\emptyset \to C$ yields an acyclic fibration $(C)^{\mathrm{cof}} \to C$ with $(C)^{\mathrm{cof}}$ cofibrant. The object $(C)^{\mathrm{cof}}$ is called the *cofibrant replacement of* C. Dually using axiom **MC5**(ii) and the map $C \to *$ one defines the *fibrant replacement of* C denoted by $(C)^{\mathrm{fib}}$.

Definition 1.2.5. A *cylinder object* for $C \in \mathbf{C}$ is an object $C \wedge I$ of $\mathbf{C}$ together with a diagram

$$C \coprod C \xrightarrow{i} C \wedge I \xrightarrow{p} C$$

which factors the canonical map $C\coprod C \to C$ and such that p is a weak equivalence. A cylinder object $C \wedge I$ is called *good cylinder object* if i is a cofibration and *very good cylinder object* if in addition the weak equivalence p is a fibration.

Definition 1.2.6. A *path object* for $C \in \mathbf{C}$ is an object C^I of $\mathbf{C}$ together with a diagram

$$C \xrightarrow{i} C^I \xrightarrow{p} C\prod C$$

which factors the canonical map $C \to C\prod C$ and such that i is a weak equivalence. A path object C^I is called *good path object* if p is a fibration and *very good path object* if in addition the weak equivalence i is a cofibration.

1.2.2 Quillen functors

Definition 1.2.7. Let $\mathbf{C}$ and $\mathbf{D}$ be model categories.

1. A functor $L\colon \mathbf{C} \to \mathbf{D}$ is a *left Quillen functor* if it is a left adjoint and preserves cofibrations and trivial cofibrations.

2. A functor $R\colon \mathbf{D} \to \mathbf{C}$ is a *right Quillen functor* if it is a right adjoint and preserves fibrations and trivial fibrations.

3. An adjoint pair of functors (L, R) is a *Quillen pair* if F is a left Quillen adjoint or equivalently if R is a right Quillen functor.

Definition 1.2.8. A Quillen pair (L, R) is a *Quillen equivalence* if and only if, for every cofibrant object $C \in \mathbf{C}$ and fibrant object $D \in \mathbf{D}$ a map $f\colon LC \to D$ is a weak equivalence in $\mathbf{D}$ if and only if its adjoint map $C \to RD$ is a weak equivalence in $\mathbf{C}$.

Proving that a Quillen pair is a Quillen equivalence is not an easy task in general. The following criteria might help. Recall that a functor F is said to reflect some property of morphisms if, given a morphism f, if Ff has the property so does f.

Proposition 1.2.9. *Consider a Quillen pair $L : \boldsymbol{C} \rightleftarrows \boldsymbol{D} : R$. Then the following are equivalent:*

(a) The pair (L, R) is a Quillen equivalence.

(b) The left adjoint L reflects weak equivalences between cofibrant objects and for every fibrant object $D \in \boldsymbol{D}$ the map $L(RD)^{\mathrm{cof}} \to D$ is a weak equivalence.

(c) The right adjoint R reflects weak equivalences between fibrant objects and for every cofibrant object $C \in \boldsymbol{C}$ the map $C \to R(LC)^{\mathrm{fib}}$ is a weak equivalence.

See for instance [Hov99, Corollary 1.3.16] for a proof.

1.2.3 Smallness and cosmallness

Definition 1.2.10. Let $\mathbf{C}$ be a complete and cocomplete category, $\mathbf{D}$ be a subcategory of $\mathbf{C}$ and λ be an ordinal.

1. A *λ-sequence* in $\mathbf{C}$ is a functor $X\colon \lambda \to \mathbf{C}$, that is, a diagram
$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots \qquad (\beta < \lambda),$$
such that the induced morphism $\operatorname{colim}_{\beta<\lambda} X_\beta \to X_\gamma$ is an isomorphism for every limit ordinal $\gamma < \lambda$.

2. The *composition of a λ-sequence* is the morphism $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$.

3. A *transfinite composition* of $\mathbf{D}$-morphisms is the composition in $\mathbf{C}$ of a λ-sequence such that $X_\beta \to X_{\beta+1}$ are morphisms in $\mathbf{D}$ for all $\beta < \lambda$.

4. A *λ-tower* in $\mathbf{C}$ is a functor $X\colon \lambda \to \mathbf{C}^{\mathrm{op}}$ such that the natural map $X_\alpha \to \lim_{\beta<\alpha} X_\beta$ is an isomorphism for every limit ordinal $\alpha < \lambda$. There is also the dual notion of *composition of a λ-tower*.

Definition 1.2.11. Let $\mathbf{D}$ be a subcategory of $\mathbf{C}$.

1. An object $A \in \mathbf{C}$ is *small relative to* $\mathbf{D}$ if there is a cofinal set S of ordinals such that for all $\lambda \in S$ and for all λ-sequence $X\colon \lambda \to \mathbf{D}$, the induced map of sets
$$\operatorname*{colim}_{\beta<\lambda} \mathbf{C}(A, X_\beta) \longrightarrow \mathbf{C}(A, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$
is a bijection.

2. Dually an object $A \in \mathbf{C}$ is *cosmall relative to* $\mathbf{D}$ if there is a cofinal set S of ordinals such that for all $\lambda \in S$ and for all λ-tower $X\colon \lambda \to \mathbf{D}^{\mathrm{op}}$, the induced map of sets
$$\operatorname*{colim}_{\beta<\lambda} \mathbf{C}(X_\beta, A) \longrightarrow \mathbf{C}(\lim_{\beta<\lambda} X_\beta, A)$$
is a bijection.

There is a fruitful concept for studying smallness in a given category. The following gives a short survey to the concept of locally presentable categories. A thorough study can be found in [Bor94b] or [AR94].

Definition 1.2.12. Let $\mathbf{C}$ be a category and λ be an ordinal.

1. An object $C \in \mathbf{C}$ is *λ-presentable* if the functor $\mathbf{C}(C, -)\colon \mathbf{C} \to \mathbf{Set}$ preserves λ-directed colimits. An object is *presentable* if it is λ-presentable for some λ.

2. The category $\mathbf{C}$ is *locally λ-presentable* if it is cocomplete and has a set of λ-presentable objects such that every object is a λ-directed colimit of objects from this set. Finally $\mathbf{C}$ is *locally presentable* if it is locally λ-presentable for some λ.

When a category is known to be locally presentable, the result below reduces drastically the study of the smallness.

Proposition 1.2.13. *Each object in a locally presentable category is small relative to the whole category.*

See for instance [AR94, 1.13, 1.16, 1.17] and [Bor94b, 5.2.10] for the proof.

Definition 1.2.14. A symmetric monoidal category $(\mathbf{C}, \otimes, I)$ with $\mathbf{C}$ locally presentable is *admissible* if for each $C \in \mathbf{C}$ the endofunctor $C \otimes - : \mathbf{C} \to \mathbf{C}$ preserves directed colimits.

Notice that closedness induces admissibility: the functor above in this case is a left adjoint and therefore preserves all colimits. Since we are interested in comonoids the following result due to Porst has many useful consequences.

Proposition 1.2.15. *If $\mathbf{C}$ is an admissible monoidal category then the category of comonoids $\mathbf{C}$ is locally presentable.*

See [Por08, Section 2.7.] for the proof.

1.2.4 Quillen's small and cosmall object argument

Definition 1.2.16. Let I be a class of maps in a category $\mathbf{C}$.

1. A map is *I-injective* if it has the right lifting property with respect to every map in I. The class of I-injective maps is denoted I-inj.

2. A map is *I-projective* if it has the left lifting property with respect to every map in I. The class of I-projective maps is denoted I-proj.

Definition 1.2.17. Let I be a set of maps in a complete and cocomplete category $\mathbf{C}$.

1. A relative *I-cell complex* is a transfinite composition of pushouts of elements of I. The collection of relative I-cell complexes is denoted by I-cell.

2. Dually a relative *I-cocell complex* is a transfinite composition of pullbacks of elements of I. The collection of relative I-cocell complexes is denoted by I-cocell.

The following important results are known as respectively the small object argument and the cosmall object argument. They ensure that the factorizations axioms **MC5**(i) and **MC5**(ii) hold in a functorial way.

Theorem 1.2.18. *Let $\boldsymbol{C}$ be category and I a set of maps in $\boldsymbol{C}$.*

1. *Suppose that $\boldsymbol{C}$ is cocomplete and that the domains of the maps of I are small relative to I-cell. Then every morphism of $\boldsymbol{C}$ can be functorially factored as a map in I-cell followed by a map in I-inj.*

2. *Suppose that $\boldsymbol{C}$ is complete and that the codomains of the maps of I are cosmall relative to I-cocell. Then every morphism of $\boldsymbol{C}$ can be functorially factored as a map in I-cocell followed by a map in I-proj.*

See [Hov99, Theorem 2.1.14] for a proof of the first statement. The second statement is the categorical dual of the first one.

1.2.5 Cofibrant and fibrant generation

We briefly recall the concept of (co)fibrant generation on a model category. The basic idea is to find some sets that generate the entire model category structure. In this case, many results about model structures can be proved by means of these generating sets.

Definition 1.2.19. A model category $\mathbf{C}$ is *cofibrantly generated* if there are sets I and J of maps such that:

(i) the domain of every map in I is small relative to I-cell;

(ii) the domain of every map in J is small relative to J-cell;

(iii) the fibrations are maps which have the right lifting property with respect to the maps in J;

(iv) the trivial fibrations are maps which have the right lifting property with respect to the maps in I.

Definition 1.2.20. Dually a model category $\mathbf{C}$ is *fibrantly generated* if its dual $\mathbf{C}^{\mathrm{op}}$ is cofibrantly generated. That is, there are sets I and J of maps such that:

(i) the codomain of every map in I is cosmall relative to I-cocell;

(ii) the codomain of every map in J is cosmall relative to J-cocell;

(iii) the cofibrations are maps which have the left lifting property with respect to the maps in J;

(iv) the trivial cofibrations are maps which have the left lifting property with respect to the maps in I.

1.2.6 The transfer theorem

The transfer theorem is a powerful result for creating a new model structure from an old one. Let **C** be a category with a cofibrantly generated model structure. Suppose in addition that **C** is left adjoint to a category **D** that is complete and cocomplete. Then, under further assumptions, the category **D** inherits a cofibrantly generated model category from that of **C**. We refer for instance to [Cra95, Theorem 3.3], [Bla96] or [Rez96] for thorough details and applications of this result. For specific purposes, we give a version of the transfer theorem proved by Schwede and Shipley. Here, the cofibrantly generated model category **C** is equipped with a monad $\mathbb{T}$ and one would like to endow the category of $\mathbb{T}$-algebras with a model structure.

Proposition 1.2.21. *Assume that the underlying functor of $\mathbb{T} = (T, \eta, \mu)$ commutes with filtered colimits. Let I be a set of generating cofibrations and J be a set of generating acyclic cofibrations for the cofibrantly generated model category $\boldsymbol{C}$. Let $T(I)$ and $T(J)$ be the images of these sets under the free $\mathbb{T}$-algebra functor. Assume that the domains of $T(I)$ and $T(J)$ are small relative to $T(I)$-cell and $T(J)$-cell respectively. Suppose that*

(i) every $T(J)$-cell is a weak equivalence, or

(ii) every object of $\boldsymbol{C}$ is fibrant and every $\mathbb{T}$-algebra has a good path object.

Then the category of $\mathbb{T}$-algebras is a cofibrantly generated model category with $T(I)$ as generating set of cofibrations and $T(J)$ as generating set of acyclic cofibrations.

The proof of this proposition is given in [SS00, Lemma 2.3]. We mention that the various applications of this result are available in [SS00, Theorem 4.1]. Note that in the transfer theorem the place of the involved categories is important. In an adjunction $L : \mathbf{C} \rightleftarrows \mathbf{D} : R$, the lift is made from the left side to the right one. But in our work we would like to achieve it in the opposite direction, that is from right to left. This explains why we consider categorical dualizations. As we will see later, this dualizing process does not completely solve the issue: for instance it leads to some difficulties concerning cosmallness conditions.

1.3 Monoidal model structures

We gather in this section the basic notions of monoidal model structures. A more thorough treatment of these notions is given in [Hov99] and [SS00] for instance. The rough idea is to define a compatibility condition on a category that is simultaneously endowed with a model structure and a closed symmetric monoidal structure. We then introduce the appropriate functors which compare such categories. We postpone examples and applications to the coming chapters. We end this section with an important result that establishes a Quillen equivalence for categories of monoids. This result is due to Schwede and Shipley (see [SS03, Theorem 3.12]) and constitutes the main motivation of this work.

1.3.1 Axioms

Definition 1.3.1. A model category $\mathbf{C}$ is a *monoidal model category* if it has a closed symmetric monoidal structure with product $\otimes$ and unit object $I_{\mathbf{C}}$ and satisfies the following axioms.

Pushout product axiom: Let $A \to B$ and $K \to L$ be cofibrations in $\mathbf{C}$. Then the dotted map in the pushout square

$$\begin{tikzcd}
A \otimes K \arrow[r] \arrow[d] & A \otimes L \arrow[d] \arrow[ddr, bend left] & \\
B \otimes K \arrow[r] \arrow[drr, bend right] & A \otimes L \coprod_{A \otimes K} B \otimes K \arrow[dr, dotted] & \\
 & & B \otimes L
\end{tikzcd}$$

is also a cofibration. If in addition one of the former maps is a weak equivalence, so is the latter map.

Unit axiom: For every cofibrant object $C \in \mathbf{C}$ the morphism

$$(I_{\mathbf{C}})^{\mathrm{cof}} \otimes C \to I_{\mathbf{C}} \otimes C \cong C$$

is a weak equivalence.

1.3.2 Quillen monoidal functors

Let $(\mathbf{C}, \otimes, I_{\mathbf{C}})$ and $(\mathbf{D}, \widehat{\otimes}, I_{\mathbf{D}})$ be monoidal categories and consider a lax monoidal functor $R\colon (\mathbf{D}, \widehat{\otimes}, I_{\mathbf{D}}) \to (\mathbf{C}, \otimes, I_{\mathbf{C}})$. As seen in Definition 1.1.9, the functor R comes equipped with monoidal structure maps $\Psi_{X,Y}\colon RX \otimes RY \to R(X \widehat{\otimes} Y)$ and $\psi\colon I_{\mathbf{C}} \to R(I_{\mathbf{D}})$. Besides suppose that the functor R has a left adjoint $L\colon \mathbf{C} \to \mathbf{D}$ with η as unit and ϵ as counit of the adjunction (L, R).

Lemma 1.3.2. *The functor $L\colon \boldsymbol{C} \to \boldsymbol{D}$ as in the above situation is lax comonoidal.*

Proof. The comonoidal structure maps $\Phi_{A,B}\colon L(A\otimes B) \to LA\widehat{\otimes}LB$ are given by the composition:

$$\begin{tikzcd} L(A\otimes B) \arrow[r, "L(\eta_A\otimes\eta_B)"] \arrow[drr, "\Phi_{A,B}"'] & L(RLA\otimes RLB) \arrow[r, "L(\Psi_{LA,LB})"] & LR(LA\widehat{\otimes}LB) \arrow[d, "\epsilon_{LA,LB}"] \\ & & LA\widehat{\otimes}LB \end{tikzcd}$$

for all A and B in $\mathbf{C}$. The map $\phi\colon L(I_{\mathbf{C}}) \to I_{\mathbf{D}}$ is defined as the adjoint of ψ. We shall check that Φ satisfied the comonoidal axioms, that is, the coherence diagrams commute. Notice that the map $\Phi_{A,B}$ is adjoint to the composite map

$$A\otimes B \xrightarrow{\eta_A\otimes\eta_B} RLA\otimes RLB \xrightarrow{\Psi_{LA,LB}} R(LA\widehat{\otimes}LB).$$

Since this latter map is monoidal by hypothesis it follows by adjointness arguments that Φ is comonoidal. $\square$

Definition 1.3.3. Suppose that $(\mathbf{C},\otimes,I_{\mathbf{C}})$ and $(\mathbf{D},\widehat{\otimes},I_{\mathbf{D}})$ are monoidal model categories and that the pair (L,R) is a Quillen adjunction with the functor R lax monoidal as above. Then the pair (L,R) is *a weak monoidal Quillen pair* if the following conditions are satisfied:

1. For all cofibrant objects A and B in $\mathbf{C}$ the comonoidal map
$$\Phi_{A,B}\colon L(A\otimes B) \longrightarrow LA\widehat{\otimes}LB$$
is a weak equivalence in $\mathbf{D}$.

2. The composite map
$$L\big((I_{\mathbf{C}})^{\mathrm{cof}}\big) \longrightarrow L(I_{\mathbf{C}}) \xrightarrow{\phi} I_{\mathbf{D}}$$
is a weak equivalence in $\mathbf{D}$.

A *strong monoidal Quillen pair* is a weak monoidal Quillen pair for which the comonoidal maps Φ and ϕ are isomorphisms. A Quillen equivalence pair (L,R) leads naturally to the notions of *weak monoidal Quillen equivalence* and *strong monoidal Quillen equivalence*.

In Proposition 1.1.15, we have seen that under some conditions, it is possible to lift the adjunction (L, R) on the level of categories of monoids. Let us denote by $(L^{\mathrm{mon}}, \widetilde{R})$ the resulting adjoint pair. We state now the main result obtained by Schwede and Shipley in [SS03].

Definition 1.3.4. Consider a functor $R\colon \mathbf{D} \to \mathbf{C}$ to a model category $\mathbf{C}$ with a left adjoint $L\colon \mathbf{C} \to \mathbf{D}$. An object of $\mathbf{D}$ is a *cell object* if it can be obtained from the initial object as a (possibly transfinite) composition of pushouts along morphisms of the form Lf, for f a cofibration in $\mathbf{C}$.

Definition 1.3.5. A functor R as above *creates a model structure* for the category $\mathbf{D}$ if

(i) the category $\mathbf{D}$ supports a model structure in which a morphism f is a weak equivalence respectively a fibration, if and only if the morphism Rf is a weak equivalence, respectively a fibration, in $\mathbf{C}$ and

(ii) every cofibrant object in $\mathbf{D}$ is a retract of a cell object.

Theorem 1.3.6. *Consider the following situation*

$$\begin{tikzcd}
(\boldsymbol{C}, \otimes, I_{\boldsymbol{C}}) \arrow[r, shift left, "L"] \arrow[d, shift left, "T_{\boldsymbol{C}}"] & (\boldsymbol{D}, \widehat{\otimes}, I_{\boldsymbol{D}}) \arrow[l, shift left, "R"] \arrow[d, shift left, "T_{\boldsymbol{D}}"] \\
\boldsymbol{C}\text{-}\boldsymbol{Monoid} \arrow[r, shift left, "L^{\mathrm{mon}}"] \arrow[u, shift left, "U_{\boldsymbol{C}}"] & \boldsymbol{D}\text{-}\boldsymbol{Monoid} \arrow[l, shift left, "\widetilde{R}"] \arrow[u, shift left, "U_{\boldsymbol{D}}"]
\end{tikzcd}$$

where

(i) the functor R is the right adjoint of a weak monoidal Quillen equivalence;

(ii) the unit objects $I_{\boldsymbol{C}}$ and $I_{\boldsymbol{D}}$ are both cofibrant;

(iii) the forgetful functors create model structures for the categories of monoids.

Then the adjoint pair $(L^{\mathrm{mon}}, \widetilde{R})$ is a Quillen equivalence.

We refer to [SS03, 5.1] for the proof. Applying this result to the category of simplicial abelian groups and to the category of non-negatively differential graded abelian groups, Schwede and Shipley deduce that the category of simplicial rings and the category of non-negatively graded differential rings are Quillen equivalent ([SS03, Theorem 1.1.]). More generally, they establish that for a given commutative ring A, there is a Quillen equivalence between the categories of simplicial A-algebras and non-negatively differential graded A-algebras. This latter result motivates the present work: we would like to achieve similar properties for simplicial coalgebras and non-negatively graded differential coalgebras over a field. The natural attempt is to use a categorical dualization of the above theorem.

Chapter 2

Preliminaries on Coalgebra Structures

2.1 Algebras and coalgebras

In this section we recall some basics about coalgebra structures. A thorough introduction can be found in the book of Sweedler [Swe69] or in the book of Abe [Abe77]. We briefly review the duality between algebras and coalgebras and state the fundamental theorem on coalgebras.

Definition 2.1.1. The *category of unital associative K-algebras*, denoted by K-**Alg**, is the category of monoids in the monoidal category $(K\text{-}\mathbf{Vct}, \otimes_K, K)$.

Definition 2.1.2. Dually the *category of counital coassociative coalgebras* denoted K-**coAlg** is the category of comonoids in the monoidal category $(K\text{-}\mathbf{Vct}, \otimes_K, K)$.

Proposition 2.1.3. *If $(C, \Delta_C, \epsilon_C)$ is a K-coalgebra then its linear dual $C^* = K\text{-}\boldsymbol{Vct}(C, K)$ is a K-algebra.*

This result has been proven by [Swe69] or [Abe77]. We only give the structure maps of the algebra C^*. First consider vector spaces X and Y and define the map $\rho\colon X^* \otimes Y^* \longrightarrow (X \otimes Y)^*$ by

$$\rho(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$$

for each $(f, g) \in X^* \times Y^*$ and $(x, y) \in X \times Y$. Then the multiplication μ_{C^*} is defined by the composite

$$C^* \otimes C^* \xrightarrow{\rho} (C \otimes C)^* \xrightarrow{\Delta_C^*} C^*$$

and the unit map η_{C^*} by $K \cong K^* \xrightarrow{\epsilon_C^*} C^*$.

Remark 2.1.4. Conversely, the linear dual of an algebra does not need to be a coalgebra. Nevertheless, if the algebra (A, μ_A, η_A) is finite dimensional as a K-vector space, then the map ρ is an isomorphism and A^* becomes a coalgebra with comultiplication Δ_{A^*} given by

$$A^* \xrightarrow{\mu_A^*} (A \otimes A)^* \xrightarrow{\rho^{-1}} A^* \otimes A^*$$

and counit map ϵ_{A^*} by $A^* \xrightarrow{\eta_A^*} K^* \cong K$.

We now give the fundamental theorem on coalgebras which is stated after recalling the following definition.

Definition 2.1.5. A *subcoalgebra* V of a coalgebra $(C, \Delta_C, \epsilon_C)$ is a subvector space V of C that satisfies

$$\Delta_C(V) \subseteq V \otimes_K V.$$

The structure maps of V are given by the restrictions $\Delta_C|_V$ and $\epsilon_C|_V$.

Theorem 2.1.6. *Consider a coalgebra C and a given element c in C. Then the subcoalgebra generated by c is finite dimensional.*

See for instance [Swe69, Theorem 2.2.1] for the proof.

Corollary 2.1.7. *Any coalgebra C is the union of its finite-dimensional subcoalgebras C_α. In other words a coalgebra C can be written as the filtered colimit of its finite-dimensional subcoalgebras.*

2.2 The functor $(-)^\circ$

We have already mentioned that the linear dual of an algebra does not need to be a coalgebra. The functor, $(-)^\circ$, considered in this section detects the right subspace of the linear dual that possesses a coalgebra structure.

Definition 2.2.1. Let A be a K-algebra. An ideal I of A is *cofinite* if the dimension over K of the quotient algebra A/I is finite.

Example 2.2.2. Consider $K[X]$, the polynomial algebra in one variable, X. Then every non-zero ideal in $K[X]$ is cofinite. Indeed $K[X]$ is a principal ideal domain and therefore each ideal is generated by some monic polynomial $q(X)$ in $K[X]$. Finally the dimension over K of the quotient $K[X]/\big(q(X)\big)$ is the degree of the polynomial $q(X)$.

Proposition 2.2.3. *Let (A, μ_A, η_A) be an algebra. Consider A° to be the subspace*

$$\Big\{f \in A^* \mid \ker f \textit{ contains a cofinite ideal } I \textit{ of } A\Big\}.$$

*Then A° is a coalgebra. Moreover the assignment $A \longmapsto A^\circ$ defines a contravariant functor from K-**Alg** to K-**coAlg**.*

A detailed proof can be found in [Swe69]. We only specify the structure maps on the coalgebra A°. The comultiplication Δ_{A° is defined as the restriction of the dual map $\mu_A^*\colon A^* \longrightarrow (A \otimes_K A)^*$ to $A^\circ \subseteq A^*$, that is $\Delta_{A^\circ} = \mu_A^*|_{A^\circ}$. The counit map $\epsilon_{A^\circ}\colon A^\circ \longrightarrow K$ is given by $\epsilon_{A^\circ}(f) = f(1_K)$ for $f \in A^\circ \subseteq A^*$.

Example 2.2.4. It might be difficult to compute A° for an arbitrary algebra A. We give two easy cases.

- If the algebra A is finite-dimensional over K then the coalgebra A° is the dual coalgebra A^* as one should expect.
- Recall that an algebra is called *simple* if it does not contain a non-trivial two sided ideal. For instance, the field of fractions $K(X)$ of the polynomial algebra $K[X]$ is simple. If a simple algebra A is infinite-dimensional over K then $A^\circ = 0$.

Proposition 2.2.5. *The functors $(-)^\circ$ and $(-)^*$ define an adjunction between the categories K-**coAlg** and $(K\text{-}\boldsymbol{Alg})^{\mathrm{op}}$. Precisely if A is an algebra and C a coalgebra, then*

$$K\text{-}\boldsymbol{Alg}(A, C^*) \cong K\text{-}\boldsymbol{coAlg}(C, A^\circ).$$

Proof. Consider the canonical map $\lambda\colon C \to C^{**}$ from C to its bidual C^{**}. One shows that $\lambda(C) \subseteq C^{*\circ}$ and that there exists a coalgebra morphism $\lambda_C\colon C \to C^{*\circ}$. With this in hand, define a map

$$\Phi\colon K\text{-}\mathbf{Alg}(A, C^*) \longrightarrow K\text{-}\mathbf{coAlg}(C, A^\circ)$$

by the assignment $\Phi(\phi) = \phi^\circ \circ \lambda_C$ for any algebra morphism $\phi\colon A \to C^*$. Furthermore consider the algebra morphism $\lambda_A\colon A \to A^{\circ *}$ given by $\lambda_A(a)\big(f\big) = f(a)$ for any $a \in A$ and $f \in A^\circ$. Then the map

$$\Psi\colon K\text{-}\mathbf{coAlg}(C, A^\circ) \longrightarrow K\text{-}\mathbf{Alg}(A, C^*)$$

defined by $\Psi(\psi) = \psi^* \circ \lambda_A$ is an inverse of Φ. □

Notice that both functors above do not carry out an anti-equivalence of categories between K-**coAlg** and K-**Alg**. To see this, consider for instance $K(X)$, the field of fractions of the polynomial algebra $K[X]$. Then one obtains $K(X)^\circ = 0$ and hence $K(X)^{\circ *} \neq K(X)$.

2.3 Profinite algebras

This section is concerned with the category of profinite algebras. As remarked in [Wit79, Theorem 1] or in [AW02], this category is equivalent to the opposite of the category of coalgebras. In the next section, we will use this useful result to construct a cofree coalgebra functor.

Definition 2.3.1. Let K be a given field.

1. A *profinite K-algebra* is a topological K-algebra which is isomorphic to the inverse limit of an inverse system of finite-dimensional K-algebras.

2. A *morphism of profinite K-algebras* is a morphism of K-algebras which is continuous with respect to the inverse limit topology.

Together with these morphims profinite K-algebras form a category that we denote by Pro-K-**Alg**.

Definition 2.3.2. Consider a K-algebra A and let $\big(I_\alpha\big)_\alpha$ be the set of its cofinite two-sided ideals. Define a partial order $\alpha \leq \beta$ if $I_\beta \subseteq I_\alpha$. For $\alpha \leq \beta$ consider the canonical K-algebra morphism $f_{\alpha\beta} \colon A/I_\beta \longrightarrow A/I_\alpha$. In this way the family $\big(A/I_\alpha\big)_\alpha$ forms an inverse system of algebras. The inverse limit of this inverse system is called the *profinite completion* of A and is denoted $\widehat{A} = \lim_\alpha A/I_\alpha$.

Example 2.3.3. Let $K[X]$ be the polynomial algebra and consider two polynomials $r(X)$ and $q(X)$ in $K[X]$. If $r(X)$ divides $q(X)$ then we have an inclusion of ideals $\big(r(X)\big) \supseteq \big(q(X)\big)$. Thereby we get an inverse system of quotients. The profinite completion of the polynomial algebra $K[X]$ is given by the set of residue classes

$$\Big(\big[f_{q(X)}\big]\Big) \in \prod_{q(X)} K[X]/\big(q(X)\big)$$

such that if $r(X)$ divides $q(X)$ then $f_{r(X)} \equiv f_{q(X)} \bmod r(X)$.

Proposition 2.3.4. *Profinite completion defines a functor from K-**Alg** to Pro-K-**Alg**. Moreover this functor is left adjoint to the forgetful functor.*

It is well-known that the linear dual functor takes colimits to limits. However the converse is not true. This fact leads to many issues when considering dualizing processes. The following lemma shows that the continuous dual functor is more flexible.

Lemma 2.3.5. *The continuous dual functor*

$$(-)^{*\mathrm{cont}} \colon \textit{Pro-}K\textit{-}\boldsymbol{Alg} \to K\textit{-}\boldsymbol{coAlg}$$

from profinite algebras to coalgebras carries limits to colimits.

See for instance [Bru66, Lemma A.3] for a proof.

Theorem 2.3.6. *The linear dual functor and the continuous dual functor define an anti-equivalence of categories between the category of coalgebras K-**coAlg** and the category of profinite algebras Pro-K-**Alg**.*

Proof. Let C be an object of K-**coAlg**. Then write C as $\operatorname{colim} C_\alpha$ with C_α its finite-dimensional subcoalgebras. With help of the previous lemma one has the following

$$\begin{aligned}(C^*)^{*\text{cont}} &\cong \left(\left(\operatorname*{colim}_\alpha C_\alpha\right)^*\right)^{*\text{cont}} \\ &\cong \left(\lim_\alpha C_\alpha^*\right)^{*\text{cont}} \\ &\cong \operatorname*{colim}_\alpha \left(\left(C_\alpha^*\right)^{*\text{cont}}\right) \\ &\cong \operatorname*{colim}_\alpha C_\alpha \\ &\cong C.\end{aligned}$$

Notice that $\left(C_\alpha^*\right)^{*\text{cont}} \cong C_\alpha$ since the C_α are finite-dimensional. Conversely consider $A \in$ Pro-K-**coAlg** and write A as $\lim_\alpha A_\alpha$ by definition. Then a similar argument as above shows that $\left(A^{*\text{cont}}\right)^* \cong A$. □

Corollary 2.3.7. *The category of coalgebras, K-**coAlg** is complete and cocomplete.*

Proof. The category of profinite algebras is cocomplete. Therefore the above anti-equivalence of categories ensures the completeness of K-**coAlg**. Moreover, the forgetful functor from the category of coalgebras to the category of vector spaces creates colimits. This asserts that the category K-**coAlg** is cocomplete. □

2.4 Cofree coalgebras

Cofree coalgebras are the dual notions of free algebras. In this section we investigate the construction of a cofree coalgebra on a given vector space. There are many approaches to this question. We will state the approach given by Sweedler and then give another one suggested by Getzler and Goerss. Before defining cofree coalgebras we briefly recall the standard construction of free algebras given by the tensor algebra functor. These properties follow from the proof of Proposition 1.1.15.

Let V be a K-vector space. The tensor algebra on V is the vector space

$$T(V) = K \oplus \bigoplus_{n\geq 1} V^{\otimes n}$$

equipped with the associative multiplication

$$\mu_{T(V)}\big((v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n)\big) = v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n$$

given by concatenations. The unit map, $\eta_{T(V)}$, is given by the canonical embedding $K \to T(V)$. The tensor algebra defines a functor T from K-**Vct** to K-**Alg** and is left adjoint to the functor $U\colon K\text{-}\mathbf{Alg} \to K\text{-}\mathbf{Vct}$ which forgets the algebra structure.

Definition 2.4.1. Let V be a K-vector space. A pair (C, π) with $C \in K$-**coAlg** and $\pi\colon C \to V$ a map of vector spaces is called a *cofree coalgebra on V* if for any coalgebra D and any map $\nu\colon D \to V$ of vector spaces there exists a unique coalgebra morphism g such that the diagram

$$\begin{tikzcd} V & D \arrow[l, "\nu"'] \arrow[dl, dotted, "\exists!\, g"] \\ C \arrow[u, "\pi"] & \end{tikzcd}$$

commutes. In other words, the pair (C, π) is couniversal among the pairs (D, ν). Notice that the coalgebra C when it exists is unique up to an isomorphism of coalgebras.

Lemma 2.4.2. *Suppose that there is a cofree coalgebra (C, π) on $X \in K$-**Vct** and let Y be a subspace of X. Consider D the sum of all subcoalgebras E of C such that $\pi(E) \subseteq X$ and $\rho = \pi|_D$. Then the pair (D, ρ) is a cofree coalgebra on Y.*

For a proof, see for instance [Abe77, Lemma 2.4.3].

The following result, due to Sweedler, constructs a cofree coalgebra for every K-vector space. As a consequence, it follows that the functor that forgets coalgebra structures has a right adjoint.

Theorem 2.4.3. *There exists a cofree coalgebra on any $V \in K$-**Vct**.*

Proof. We consider the tensor algebra on the dual vector space of V, $T(V^*)$. Then the canonical embedding $i\colon V^* \to T(V^*)$ yields the following composite map $\pi\colon T(V^*)^\circ \longrightarrow T(V^*)^* \xrightarrow{i^*} T(V^*)^*$ where the first map is the obvious inclusion . We prove that pair $(T(V^*)^\circ, \pi)$ is a cofree coalgebra on the bidual V^{**}. For any object $D \in K$-**coAlg** one has

$$\begin{aligned} K\text{-}\mathbf{Vct}(D, V^{**}) &\cong K\text{-}\mathbf{Vct}(V^*, D^*) \\ &\cong K\text{-}\mathbf{Alg}(T(V^*), D^*) \\ &\cong K\text{-}\mathbf{coAlg}(D, T(V^*)^\circ) \end{aligned}$$

Notice that the two last isomorphisms come respectively from the adjunction between the categories K-**Vct** and K-**Alg** and the adjunction in Proposition

2.2.5. Hence $(T(V^*)^\circ, \pi)$ is the cofree coalgebra on the bidual V^{**} as desired. Now, the previous lemma supplies the answer for the vector space V since $V^{**} \supseteq V$. $\square$

Corollary 2.4.4. *The forgetful functor U from K-**coAlg** to K-**Vct** has a right adjoint.*

Proof. This follows from the couniversality of the cofree coalgebra. $\square$

We describe now an approach given in [GG99]. Basically this approach uses the anti-equivalence of the categories Pro-K-**Alg** and K-**coAlg**. This property gives a categorical construction for a right adjoint functor from K-**Vct** to K-**coAlg**.

Proposition 2.4.5. *Let V be in K-**Vct**. If V is finite-dimensional over K then the cofree coalgebra on V is given by*

$$S(V) = \left(\widehat{T(V^*)}\right)^{*_{\text{cont}}}.$$

*For general $V \in K$-**Vct** the cofree coalgebra is defined by*

$$S(V) = \operatorname*{colim}_{\alpha} S(V_\alpha)$$

with V_α running over the finite-dimensional subvector spaces of V.

Proof. If V is finite-dimensional using the idenfication $V \cong V^{**}$ and the following bijections with any $D \in K$-**coAlg**

$$\begin{aligned} K\text{-}\mathbf{Vct}(D, V^{**}) &\cong K\text{-}\mathbf{Vct}(V^*, D^*) \\ &\cong K\text{-}\mathbf{Alg}(T(V^*), D^*) \\ &\cong \text{Pro-}K\text{-}\mathbf{Alg}(\widehat{T(V^*)}, D^*) \\ &\cong K\text{-}\mathbf{coAlg}(D, (\widehat{T(V^*)})^{*_{\text{cont}}}). \end{aligned}$$

give the desired result. Notice that the third bijection comes from the adjunction between the category of algebras and the category of profinite algebras. The last isomorphism is the anti-equivalence established in Theorem 2.3.6.

Now consider a general $V \in K$-**Vct**. Any object $D \in K$-**coAlg** can be written as the colimit of its finite-dimensional subcoalgebras, $D = \operatorname{colim}_\beta D_\beta$. Using [GG99, Lemma 1.9], we notice that the natural map

$$\operatorname*{colim}_{\alpha} K\text{-}\mathbf{coAlg}(D_\beta, S(V_\alpha)) \longrightarrow K\text{-}\mathbf{coAlg}(D_\beta, \operatorname*{colim}_{\alpha} S(V_\alpha))$$

is a bijection. Then the following

$$\begin{aligned} K\text{-}\mathbf{coAlg}(D, S(V)) &\cong \lim_{\beta} K\text{-}\mathbf{coAlg}(D_\beta, S(V)) \\ &\cong \lim_{\beta} \operatorname*{colim}_{\alpha} K\text{-}\mathbf{coAlg}(D_\beta, S(V_\alpha)) \\ &\cong \lim_{\beta} \operatorname*{colim}_{\alpha} K\text{-}\mathbf{Vct}(D_\beta, V_\alpha) \\ &\cong K\text{-}\mathbf{Vct}(D, V) \end{aligned}$$

completes the proof. $\square$

Corollary 2.4.6. *Consider V a finite dimensional K-vector space. Then*

$$T(V^*)^\circ \cong \left(\widehat{T(V^*)}\right)^{*\mathrm{cont}}$$

Proof. Any two right-adjoint of a functor are naturally isomorphic (see [Mac98, IV. Corollary 1.]), therefore the claim follows. $\square$

2.5 Cofree coalgebra on a one-dimensional space

We use the approach suggested by Getzler and Goerss in order to compute the cofree coalgebra on a one-dimensional vector space. The cofree coalgebra involves the profinite completion of the polynomial algebra $K[X]$. In this case things are a bit easier since we know that every non-zero ideal in $K[X]$ is cofinite. As we will see, the arguments mimic somehow the standard results in number theory about the cofinite completion of the integers.

Definition 2.5.1. Let A be a commutative ring and I a proper ideal of A. If $n \geq m$ then we have the canonical ring morphism $A/I^n \to A/I^m$. The *I-adic completion* of A is the inverse limit $\lim_n A/I^n$.

Example 2.5.2. Let $K[X]$ be the polynomial algebra in one variable X and $p(X)$ be an irreducible polynomial in $K[X]$. We denote by $\big(p(X)^r\big)$ the two-sided ideal generated by $p(X)^r$. The following inclusions

$$\cdots \subseteq \big(p(X)^{r+1}\big) \subseteq \big(p(X)^r\big) \subseteq \cdots \subseteq \big(p(X)\big)$$

yield an inverse system of canonical projections

$$\cdots \to K[X]/(p(X)^{r+1}) \to K[X]/(p(X)^r) \to \cdots \to K[X]/(p(X)).$$

Then the $\big(p(X)\big)$-adic completion of $K[X]$ is given by the following sequences of residue classes

$$\Big\{\Big([f_r]\Big)_{r\geq 1} \in \prod_r K[X]/(p(X)^r) \mid \forall\, r, f_{r+1} \equiv f_r \mod p(X)^r\Big\}.$$

Definition 2.5.3. Let A be a commutative ring. A *formal power series* in one variable X with coefficients $a_n \in A$ is an expression of the form

$$\sum_{n=0}^{\infty} a_n X^n = a_0 + a_1 X + a_2 X^2 + \cdots .$$

Formal power series with coefficients in A form a ring denoted by $A\big[[X]\big]$.

In number theory it is well-known that the p-adic completion of $\mathbf{Z}$ is isomorphic to the ring of p-adic integers usually denoted by $\mathbf{Z}_p$. Similarly the (X)-adic completion of the polynomial algebra $K[X]$ is the power series ring $K\big[[X]\big]$ (see [Eis04, 7.1 Example]). The following proposition generalizes this result.

Proposition 2.5.4. *Let $p(X)$ be an irreducible polynomial in $K[X]$ and a be a root of $p(X)$ in the algebraic closure of the field K. Then the $\big(p(X)\big)$-adic completion of $K[X]$ is isomorphic to the ring of formal power series with coefficents in the field $K(a)$ obtained by adjoining the root a to K.*

Proof. Recall that if $p(X)$ is an irreducible polynomial over K and if a is a root of $p(X)$ in the algebraic closure $\bar{K}$ of K, then the quotient $K[X]/\big(p(X)\big)$ is a field isomorphic to the field $K(a)$ obtained by adjoining the root a to K. Moreover if b is another root in $\bar{K}$ then $K(b) \cong K(a)$. Since the natural map from the inverse limit to $K[X]/\big(p(X)\big)$ is a surjection, one deduces that the inverse limit must contain a sequence which contains the root a. For all r the maps $K(a)\big[[X]\big] \longrightarrow K[X]/(p(X)^r)$ which send a power series with coefficients in $K(a)$ to its residue class in $K[X]/(p(X)^r)$ give rise to a map

$$\Psi\colon K(a)\big[[X]\big] \longrightarrow \lim_{r} K[X]/(p(X)^r)$$

by universality of inverse limits. The inverse of Ψ is given by

$$\Phi\colon \lim_{r} K[X]/(p(X)^r) \longrightarrow K(a)\big[[X]\big]$$

which send a sequence of residue classes $\Big(\big[f_r\big]\Big)_{r\geq 1}$ to the formal power series $f_1 + (f_2 - f_1) + (f_3 - f_2) + \cdots$. ☐

In number theory the isomorphism $\widehat{\mathbf{Z}} \cong \prod_p \mathbf{Z}_p$ compares the profinite completion of $\mathbf{Z}$ and the p-adic completions for p running through all prime number. The proposition below is a natural analogue.

Proposition 2.5.5. *The profinite completion of the polynomial algebra is given by*

$$\widehat{K[X]} \cong \prod_{p(X)} K(a)\big[[X]\big]$$

where $p(X)$ runs over all irreducible monic polynomials in $K[X]$ and a is a root of $p(X)$ in the algebraic closure $\bar{K}$ of the field K.

Proof. Without loss of generality, we may identify polynomials in $K[X]$ that differ by a unit. Consider a polynomial $q(X) \in K[X]$. Its prime decomposition yields

$$q(X) = \prod_{p(X)} p(X)^{p_n}$$

with $p(X)$ running through all the irreducible polynomials of $K[X]$ and p_n is the integer indicating the multiplicity of $p(X)$. Then, by the Chinese Remainder Theorem, we obtain

$$K[X]/\big(q(x)\big) \cong \prod_{p(X)} K[X]/\big(p(X)^{p_n}\big).$$

Now let us fix an irreducible polynomial $p(X)$ and $a_{p(X)}$ one of its roots in the algebraic closure $\bar{K}$. By the previous proposition we obtain natural projections

$$K(a_{p(X)})\big[[X]\big] \longrightarrow K[X]/\big(p(X)^{p_n}\big).$$

These projections maps give rise to a product map

$$\prod_{p(X)} K(a_{p(X)})\big[[X]\big] \longrightarrow \prod_{p(X)} K[X]/\big(p(X)^{p_n}\big) \cong K[X]/\big(q(X)\big).$$

Thus, by the universality of inverse limits, we have an induced map

$$\Pi\colon \prod_{p(X)} K(a_{p(X)})\big[[X]\big] \longrightarrow \widehat{K[X]} = \lim_{q(X)} K[X]/\big(q(X)\big).$$

It remains to define an inverse map Θ for Π. For this consider the following composite of projection maps

$$\widehat{K[X]} = \lim_{q(X)} K[X]/\big(q(X)\big) \longrightarrow K[X]/\big(q(X)\big) \longrightarrow K[X]/\big(p(X)^r\big).$$

By the universality of inverse limits and the previous proposition, there exists a map

$$\widehat{K[X]} \longrightarrow \lim_{r} K[X]/\big(p(X)^r\big) \cong K(a_{p(X)})\big[[X]\big].$$

Finally the desired inverse map

$$\Theta\colon \widehat{K[X]} \longrightarrow \prod_{p(X)} K(a_{p(X)})\big[[X]\big]$$

is given by the universal property of products. $\square$

Recall that if V is an n-dimensional K-vector space then the tensor algebra $T(V)$ is isomorphic to the non-commutative polynomial algebra on n variables over K. The following result is suggested by Getzler and Goerss in [GG99, Example 1.13].

Corollary 2.5.6. *Let V be a one-dimensional K-vector space. Then the cofree coalgebra on V is given by*

$$S(V) \cong \left(\prod_{p(X)} K\big(a_{p(X)}\big)\big[[X]\big] \right)^{*\mathrm{cont}}$$

Proof. Using $K \cong K^*$ we identify the tensor algebra $T(K^*)$ with the polynomial algebra $K[X]$. Then, the preceding proposition supplies the desired result. $\square$

Remark 2.5.7. We complete this section with some observations. As seen above, the approach of Getzler and Goerss yields an identification of the cofree coalgebra on a one-dimensional K-vector space. However, whenever a vector space V is of dimension larger than one it might be hard to compute the cofree coalgebra $S(V)$. As noted above, we identify the tensor algebra $T(V)$ with the polynomial algebra in non-commuting variables. Thus, if for instance $\dim_K V = 2$ then $T(V) = K\{X, Y\}$. But even characterizing cofinite ideals in the algebra $K\{X, Y\}$ remains a difficult task.

While the construction of free algebras is relatively easy, that of cofree coalgebras is cumbersome. We notice that there are many other interpretations of the cofree coalgebra on a vector space. We refer to [Taf72] and [BL85] for different approaches. We also mention that recent papers such as [Haz03] still investigate this question.

Chapter 3

Differential Graded Objects

3.1 Differential graded vector spaces

This section deals with the categorical and monoidal model structure of the category of differential graded vector space. The main categorical properties of differential graded vector spaces appear in [Mac67] and [Wei94] for instance. The monoidal model structure is described in [Hov99]. We also address the question whether the opposite category of differential graded vector spaces is cofibrantly generated.

3.1.1 Categorical structure

Definition 3.1.1. Let K-**Vct** be the category of K-vector spaces.

1. A *differential graded K-vector space* X is a family $(X_n)_{n\in\mathbb{N}}$ of objects in K-**Vct** equipped with maps of vector spaces $d_n\colon X_n \to X_{n-1}$ such that $d_n \circ d_{n+1} = 0$.

2. A morphism $f\colon (X, d_X) \to (Y, d_Y)$ of differential graded vector space is a family $f_n\colon X_n \to Y_n$ of maps of vector spaces satisfying $f_{n-1} \circ d_X = d_Y \circ f_n$ for all n.

Together with these morphisms, differential graded vector spaces form a category that we denote by DG-K-**Vct**.

Definition 3.1.2. Given an object (X, d_X) of DG-K-**Vct** the n^{th} *homology module of* (X, d_X) is given by

$$H_n(X) = \ker d_n / \operatorname{im} d_{n+1}.$$

Moreover an object (X, d_X) will be said to be *acyclic* if $H_n(X) = 0$ for all n.

In the category DG-K-**Vct**, the differential graded vector space 0 with trivial vector spaces and trivial differentials is the zero object. Here are also countably many objects in DG-K-**Vct** that are relevant in this work.

1. The *n-sphere* denoted by $\mathbb{S}^n$ is the object of K-**Vct** which has $(\mathbb{S}^n)_k = K$ for $k = n$ and is trivial in all other degrees. All differentials are trivial, hence
$$H_k(\mathbb{S}^n) = \begin{cases} K & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

2. The *n-disk* denoted by $\mathbb{D}^n$ is the object of K-**Vct** which has $(\mathbb{D}^n)_k = K$ for $k = n, n-1$ and is trivial in all other degrees. The identity on K is its only non-trivial differential and therefore $H_k(\mathbb{D}^n)$ is 0 for all integer k.

Definition 3.1.3. Let $f\colon X \to Y$ be a map of differential graded vector spaces. Then the *mapping cone on f* denoted by $\mathrm{cone}(f)$ is the object of DG-K-**Vct** whose degree n part is given by $X_{n-1} \oplus Y_n$. The differential in $\mathrm{cone}(f)$ is given by $d(x, y) = (-d(x), d(y) - f(x))$ for all $x \in X_{n-1}$ and $y \in Y_n$. When f is the identity on an object X, the cone is denoted by $\mathrm{cone}(X)$.

Proposition 3.1.4. *The category of differential graded vector spaces DG-K-**Vct** is complete and cocomplete.*

This result is well-known. We refer to the book of Weibel [Wei94] for a detailled description of limits and colimits in this category: they are defined degreewise. However, in sight of the next section, we shall give a description of pushouts in the category DG-K-**Vct**. Consider the following pushout square of differential graded vector spaces

$$\begin{tikzcd}
(Z, d_Z) \arrow[r, "f"] \arrow[d, "g"'] & (X, d_Y) \arrow[d] \\
(Y, d_Y) \arrow[r] & (X, d_X) \coprod_{(Z, d_Z)} (Y, d_Y).
\end{tikzcd}$$

Then the pushout object is given by

$$(X, d_X) \coprod_{(Z, d_Z)} (Y, d_Y) = (X, d_X) \coprod (Y, d_Y)/\sim$$

where the relation $\sim$ identifies $f(z)$ and $g(z)$ for each $z \in Z$. Notice that when the maps f and g are monomorphisms then the pushout object is the union $(X, d_X) \coprod (Y, d_Y)$.

3.1.2 Monoidal model structure

Proposition 3.1.5. *The category of differential graded vector spaces DG-K-**Vct** is equipped with a cofibrantly generated model structure.*

In [Qui67, Chapter I, Example B], Quillen provides the category of differential graded vector spaces with a model structure. We refer to [DS95, Chapter 7] where Dwyer and Spalinski prove with thorough details that the category of non-negatively differential graded vector spaces has a cofibrantly generated model structure.
A map f in the category DG-K-**Vct** is a

- weak equivalence if $H_* f$ is an isomorphism,
- cofibration if for each $n \geq 0$, f_n is injective,
- fibration if for each $n \geq 1$, f_n is surjective.

The generating acyclic cofibrations are given by the maps $\left\{0 \to \mathbb{D}^n \mid n \geq 1\right\}$ and the generating cofibrations by $\left\{\mathbb{S}^{n-1} \to \mathbb{D}^n \mid n \geq 1\right\}$.

Remark 3.1.6. If K was just a commutative ring, the cofibrations would be required in addition to have degreewise projective cokernels. This becomes superfluous under the assumption that K is a field.

Definition 3.1.7. Consider two objects X and Y in the category DG-K-**Vct**. Then define the monoidal product $\otimes$ by

$$(X \otimes Y)_n = \bigoplus_{p+q=n} X_p \otimes_K Y_q$$

with differential

$$d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy.$$

The unit of this monoidal product is $\mathbb{S}^0$, the differential graded vector space concentred in degree 0. Sometimes we will denote $\mathbb{S}^0$ by $K[0]$. Notice that the monoidal product $\otimes$ is symmetric.

Example 3.1.8. The monoidal product of a n-sphere $\mathbb{S}^n$ and a m-sphere $\mathbb{S}^m$ is the $(n+m)$-sphere $\mathbb{S}^{n+m}$. The monoidal product of a n-sphere $\mathbb{S}^n$ and a m-disk $\mathbb{D}^m$ is isomorphic to an $(n+m)$-disk $\mathbb{D}^{n+m}$.

Lemma 3.1.9. *The category DG-K-**Vct** endowed with the monoidal product $\otimes$ is closed.*

Proof. Given differential graded vector spaces (X, d_X) and (Y, d_Y) we let $\mathrm{Hom}(X, Y)$ be the object of DG-K-**Vct** described as follow:

$$\mathrm{Hom}(X, Y)_n = \prod_{p \geq 0} K\text{-}\mathbf{Vct}(X_p, Y_{p+n})$$

with differential d_H for any map $f = \{f_p \colon X_p \to Y_{p+n}\}_{p\geq 0}$ given by

$$(d_H f)_p(x) = d_Y(f_p x) + (-1)^{n+1} f_{p-1}(d_X x), \quad x \in X_p.$$

The specified right adjoint of the functor $-\otimes Y \colon \text{DG-}K\text{-}\mathbf{Vct} \to \text{DG-}K\text{-}\mathbf{Vct}$ is then given by the functor $\mathrm{Hom}(Y, -) \colon \text{DG-}K\text{-}\mathbf{Vct} \to \text{DG-}K\text{-}\mathbf{Vct}$. $\square$

Proposition 3.1.10. *The category of differential graded vector space has a monoidal model category structure.*

Proof. We check that the monoidal product $\otimes$ is compatible with the model structure defined above. Since the category DG-K-**Vct** is cofibrantly generated, it suffices to use [Hov99, Corollary 4.2.5] for proving that the pushout product axiom holds. The pushout product of two generating cofibrations $\mathbb{S}^{n-1} \to \mathbb{D}^n$ and $\mathbb{S}^{m-1} \to \mathbb{D}^m$ yields the following pushout diagram

$$\begin{tikzcd}
\mathbb{S}^{n-1} \otimes \mathbb{S}^{m-1} \arrow[r] \arrow[d] & \mathbb{D}^n \otimes \mathbb{S}^{m-1} \arrow[d] \arrow[ddr, bend left] & \\
\mathbb{S}^{n-1} \otimes \mathbb{D}^m \arrow[r] \arrow[drr, bend right] & \mathbb{S}^{m-1} \otimes \mathbb{D}^n \coprod_{\mathbb{S}^{m-1}\otimes \mathbb{S}^{n-1}} \mathbb{S}^{n-1} \otimes \mathbb{D}^m \arrow[dr, dotted] & \\
& & \mathbb{D}^n \otimes \mathbb{D}^m
\end{tikzcd}$$

The dotted arrow becomes

$$\mathbb{D}^{n+m-1} \coprod_{\mathbb{S}^{n+m-2}} \mathbb{D}^{n+m-1} \longrightarrow \mathbb{D}^n \otimes \mathbb{D}^m$$

which is a cofibration by the remarks given above. Finally similar considerations show that the pushout product of a generating cofibration $\mathbb{S}^{n-1} \to \mathbb{D}^n$ with a generating acyclic cofibration $0 \to \mathbb{D}^m$

$$\begin{tikzcd}
0 \otimes \mathbb{S}^{n-1} \arrow[r] \arrow[d] & 0 \otimes \mathbb{D}^n \arrow[d] \arrow[ddr, bend left] & \\
\mathbb{D}^m \otimes \mathbb{S}^{n-1} \arrow[r] \arrow[drr, bend right] & \mathbb{S}^{n-1} \otimes \mathbb{D}^m \arrow[dr, dotted] & \\
& & \mathbb{D}^m \otimes \mathbb{D}^n
\end{tikzcd}$$

gives the map $\mathbb{D}^{n+m-1} \longrightarrow \mathbb{D}^n \otimes \mathbb{D}^m$. This map is a weak equivalence: its domain is acyclic and the codomain is as well acyclic by the Künneth formula (see [Mac67, Theorem 10.2]). $\square$

3.1.3 On the opposite category of DG-K-Vct

Recall that when a category $\mathbf{C}$ is endowed with a model structure then its opposite category $\mathbf{C}^{\mathrm{op}}$ inherits a model structure. However, if $\mathbf{C}$ is cofibrantly generated, then it is not obvious whether $\mathbf{C}^{\mathrm{op}}$ is cofibrantly generated. We investigate this issue for the opposite of the category of differential graded vector spaces. We were able to find two sets that generate fibrations and acyclic fibrations. But some cosmallness difficulties appear. Roughly these difficulties are due to the fact that opposite categories are rarely locally presentable.

Proposition 3.1.11. *Define Q to be the set $\{\mathbb{D}^n \to 0 \mid n \geq 1\}$. A chain map $f\colon X \to Y$ is injective if and only if f has the left lifting property with respect to any map in Q.*

Proof. One direction is obvious since maps $\mathbb{D}^n \to 0$ are acyclic fibrations in the standard model structure of DG-K-**Vct**.
For the other direction suppose f has the left lifting property with respect to Q. Assume that there is an $x_n \in X_n$ with $x_n \neq 0$ and $f_n(x_n) = 0$. Since $x_n \neq 0$ there is a linear map $\alpha_n\colon X_n \to K$ such that $\alpha_n(x_n) = 1$.

(i) If x_n is *not* a cycle then define $\alpha_{n-1}(dx_n) = 1$. This gives a chain map $\alpha\colon X \to \mathbb{D}^n$ and by assumption a lift ζ exits in the following diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & \mathbb{D}^n \arrow[d, "\sim"] \\ Y \arrow[r, "\beta"'] \arrow[ur, dotted, "\zeta"] & 0 \end{tikzcd}$$

Therefore $0 = \zeta_n(0) = \zeta_n \circ f_n(x_n) = \alpha_n(x_n) = 1$, which is the desired contradiction. Therefore $f_n(x_n) \neq 0$ and the chain map f is injective.

(ii) Let x_n be a cycle and a boundary. That is, $d(x_n) = 0$ and there is a $x_{n+1} \in X_{n+1}$ such that $d(x_{n+1}) = x_n$. Then define $\alpha_{n+1}(x_{n+1}) = 1$. Note that x_{n+1} is not a boundary. This gives a chain map $\alpha\colon X \to \mathbb{D}^{n+1}$ and a contradiction follows as in case (i).

(iii) Let x_n be a cycle and *not* a boundary. Then x_n generates a sphere sub-complex $\mathbb{S}^n$ of X. This sub-complex $\mathbb{S}^n$ can be mapped to $\mathbb{D}^{n+1}$ and the above argument works again.

$\square$

Proposition 3.1.12. *Define P to be the set $\{\mathbb{D}^n \to 0, \mathbb{D}^n \to \mathbb{S}^n \mid n \geq 1\}$. A chain map f is in P-proj if and only if it is injective and induces an isomorphism in homology.*

Proof. One direction is obvious since every map in P is a fibration. For the other direction, injectivity comes from P-proj $\subset Q$-proj (since $Q \subset P$) and the previous lemma. We show that every map in P-proj induces isomorphism in homology. Let $f\colon A \to B$ be a map in P-proj. As f is injective, we have the short exact sequence $0 \to A \to B \to \mathrm{coker} f \to 0$. As this sequence induces a long exact sequence in homology, it suffices to show that coker$f$ has no homology. The object coker$f$ which is built degreewise is also a pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\ B \arrow[r] & \mathrm{coker} f \end{tikzcd}$$

We deduce that the map $0 \to \mathrm{coker} f$ is also in P-proj since P-proj is closed under pushouts. Let us now assume that coker$f$ is *not* acyclic. Then there is an element $x_n \in (\mathrm{coker} f)_n$ with $d(x_n) = 0$ and x_n not a boundary. Note that x_n generates a subcomplex $\mathbb{S}^n$ of coker$f$ and that $\mathrm{coker} f \cong \mathbb{S}^n \oplus \mathrm{coker} f/\mathbb{S}^n$ as chain complexes. But the map $0 \to \mathbb{S}^n$ is not in P-proj since

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{D}^n \arrow[d] \\ \mathbb{S}^n \arrow[r, equal] & \mathbb{S}^n \end{tikzcd}$$

does not admit a lift. Therefore the map $0 \to \mathrm{coker} f$ can not be in P-proj either which is the desired contradiction. □

3.1.4 A counterexample

Both propositions in the preceding paragraph tell us that the sets Q and P are good candidates for, respectively, generating acyclic fibrations and generating fibrations in the category DG-K-**Vct**. What remains is the cosmallness of the codomains of the sets Q and P. For the set Q using the fact that every terminal object in a category is cosmall relative to that category, we deduce the cosmallness of 0 relative to Q-cocell.

Unfortunately the codomains $\mathbb{S}^n$ in the set P are not cosmall. Indeed consider a cofiltered sequence of maps in the category DG-K-**Vct**

$$\cdots \to Y_{\alpha+1} \to Y_\alpha \to \cdots \to Y_1 \to Y_0$$

such that each $Y_{\alpha+1} \to Y_\alpha$ is in P-cocell. Then the following canonical map

$$\operatorname*{colim}_\alpha \text{DG-}K\text{-}\mathbf{Vct}(Y_\alpha, \mathbb{S}^n) \longrightarrow \text{DG-}K\text{-}\mathbf{Vct}(\lim_\alpha Y_\alpha, \mathbb{S}^n)$$

does not need to be a bijection. To that end, we give a counterexample that shows that the n-sphere $\mathbb{S}^n$ is not cosmall. We define Y_α to be $\mathbb{S}^n$ and all maps $Y_\alpha \to Y_{\alpha-1}$ to be trivial. Then on the one hand, using the fact that all maps are trivial we have

$$\begin{aligned}
\operatorname*{colim}_\alpha \text{DG-}K\text{-}\mathbf{Vct}\Big(Y_\alpha, \mathbb{S}^n\Big) &\cong \operatorname*{colim}_\alpha K\text{-}\mathbf{Vct}\big(K, K\big) \\
&\cong \bigoplus_\alpha K
\end{aligned}$$

and on the other hand we obtain

$$\begin{aligned}
\text{DG-}K\text{-}\mathbf{Vct}\Big(\lim_\alpha Y_\alpha, \mathbb{S}^n\Big) &\cong \text{DG-}K\text{-}\mathbf{Vct}\Big(\prod_\alpha Y_\alpha, \mathbb{S}^n\Big) \\
&\cong K\text{-}\mathbf{Vct}\Big(\prod_\alpha K, K\Big) \\
&= \Big(\prod_\alpha K\Big)^*
\end{aligned}$$

Now we compare the duals of the two resulting vector spaces. Since the linear dual carries colimits to limits, we obtain that $\Big(\bigoplus_\alpha K\Big)^*$ is isomorphic to $\prod_\alpha K$. We conclude by observing that $\prod_\alpha K$ as an infinite dimensional vector space is not isomorphisc to its bidual $\Big(\prod_\alpha K\Big)^{**}$. Hence the n-sphere $\mathbb{S}^n$ is not cosmall.

Notice that one could also take Y_α to be the differential graded vector spaces that have in degree n the vector space $K[X]/(X^{\alpha+1})$ and that are trivial in other degrees, that is

$$\cdots \to 0 \to K[X]/(X^{\alpha+1}) \to 0 \to \cdots$$

and maps from Y_α to $Y_{\alpha-1}$ are projections $K[X]/(X^{\alpha+1}) \to K[X]/(X^\alpha)$. Since the inverse limits of the $(Y_\alpha)_\alpha$ is the vector space $K\big[[X]\big]$ of power series, one obtains another counterexample by observing that $K\big[[X]\big]$ and its bidual are not isomorphic.

We also stress another issue occuring in the study of the opposite category $(\text{DG-}K\text{-}\mathbf{Vct})^{\text{op}}$. The category $(\text{DG-}K\text{-}\mathbf{Vct})^{\text{op}}$ inherits the monoidal structure $\otimes$ defined above. However this category is not closed anymore. The functor

$$- \otimes Y \colon (\text{DG-}K\text{-}\mathbf{Vct})^{\text{op}} \longrightarrow (\text{DG-}K\text{-}\mathbf{Vct})^{\text{op}}$$

does not have a right adjoint. Otherwise $- \otimes Y$ would be a right adjoint this time on DG-K-$\mathbf{Vct}$ and this is not true since $- \otimes Y$ does not commute with arbitrary limits for instance.

3.2 Differential graded algebras

In this section we investigate the category of differential graded algebras. This category has a model structure. We present a proof suggested by Schwede and Shipley. Viewing differential graded algebras as monoids in the monoidal category of differential graded vector space, they could lift the model structure along the appropriate adjoint functors. This gives a first application of the useful transfer theorem given in Paragraph 1.2.6.

3.2.1 Categorical structure

Definition 3.2.1. The category of unital associative differential graded algebras denoted by DG-K-**Alg** is the category of monoids in the monoidal category (DG-K-**Vct**,$\otimes$,$K[0]$).

Let (X, d_X) be an object in DG-K-**Vct**. The differential graded tensor algebra $\big(T(X), d_{T(X)}\big)$ is defined by

$$T_0(X) = K \oplus \bigoplus_{p=1}^{\infty} (X_0)^{\otimes p}$$

$$\text{and for } n > 0, \qquad T_n(X) = \bigoplus X_{d_1} \otimes \cdots \otimes X_{d_t}$$

where the second sum is taken over all d_i with $d_1 + \cdots + d_t = n$. The differential is given by

$$d_{T(X)}(x_1 \otimes \cdots \otimes x_p) = \sum_{i=1}^{p} (-1)^{\eta_i} x_1 \otimes \cdots \otimes dx_i \otimes \cdots \otimes x_p$$

where $\eta_i = \deg x_1 + \cdots + \deg x_{i-1}$. It follows from the proof of Proposition 1.1.15 that the assignment $(X, d_X) \longmapsto \big(T(X), d_{T(X)}\big)$ defines a covariant functor from DG-K-**Vct** to DG-K-**Alg** and this functor is left adjoint to the functor $U\colon$ DG-K-**Alg** $\to$ DG-K-**Vct** which forgets the algebra structure.

The following result is standard. We review its proof in order to describe filtered colimits of differential graded algebras explicitly.

Lemma 3.2.2. *The forgetful functor U : DG-K-**Alg** $\longrightarrow$ DG-K-**Vct** creates filtered colimits.*

Proof. Suppose $(A_i)_{i \in I}$ is a family of differential graded algebras where I is a filtered indexing category. The colimit of $U(A_i)$ in DG-K-**Vct** is given in a standard fashion by

$$\operatorname*{colim}_i U(A_i) = \bigoplus_i U(A_i) / \sim$$

where $\sim$ is the equivalence relation described as follows:

$$(x_i \in U(A_i)) \sim (y_j \in U(A_j))$$

if and only if there exists $k \geq i, j$ and chain maps

$$f_{ik} \colon A_i \to A_k \text{ and } g_{jk} \colon A_j \to A_k \text{ such that } f_{ik}(x_i) = g_{jk}(y_j).$$

Now we provide an algebra structure $\star$ on this differential graded vector space: given two classes $[x_i] \in \operatorname{colim}_i U(A_i)$ and $[y_j] \in \operatorname{colim}_i U(A_j)$, define

$$[x_i] \star [y_j] = [f_{ik}(x_i) \bullet g_{jk}(y_j)]$$

where $\bullet$ is the algebra structure of the $(A_i)_i$. This product $\star$ is well-defined. Indeed it is independent of the choice of k since I is filtered. Moreover $(\operatorname{colim}_i U(A_i), \star)$ is *universal* among cocones of DG-K-**Alg**. Indeed given another cocone (Z, h) in DG-K-**Alg** there exists a unique K-vector space chain map $\alpha \colon \operatorname{colim}_i U(A_i) \to Z$. Since

$$\alpha([x_i] \star [y_j]) = h(f_{ik}(x_i) \bullet g_{jk}(y_j)) = h(f_{ik}(x_i)) \bullet h(g_{jk}(y_j)) = \alpha([x_i]) \star \alpha([y_j])$$

one deduces that α is an algebra map. Thus $(\operatorname{colim}_i U(A_i), \star)$ corresponds to $\operatorname{colim}_i A_i$ in DG-K-**Alg**. $\square$

3.2.2 Model structure

In order to apply the Transfer Result, we need the following lemma which constructs good path objects for each differential graded algebra.

Lemma 3.2.3. *Every differential graded algebra has a good path object.*

Proof. Let (I, d_I) denotes the following differential graded vector space concentrated in degrees 0 and 1:

$$\cdots 0 \to 0 \to K\langle a \rangle \xrightarrow{d} K\langle b, c \rangle \to 0 \cdots$$

with $d(a) = c - b$. Setting $\Delta(b) = b \otimes b$, $\Delta(c) = c \otimes c$ and $\Delta(a) = b \otimes a + a \otimes c$ yields a coassociative counital coalgebra structure on the differential graded vector space (I, d_I). Then the differential graded vector space $\operatorname{Hom}(I, A)$ as defined in the proof of Lemma 3.1.9 becomes a differential graded algebra with multiplication

$$f \cdot g = \mu_A \circ (f \otimes g) \circ \Delta$$

for each f and g in $\operatorname{Hom}(I, A)$. It remains to check that the differential graded algebra $\operatorname{Hom}(I, A)$ is a path object for a given differential graded algebra A. To see this, consider the following map of differential graded algebras

$$A \to \operatorname{Hom}(I, A) \to A \times A.$$

The first map is a weak equivalence, that is, it induces isomorphism in homology. Indeed applying the homotopy classification theorem (see [Mac67, Chapter III, Theorem 4.3.]) yields

$$H_n(\mathrm{Hom}(I, A)) \cong \prod_{p=-\infty}^{+\infty} K\text{-}\mathbf{Vct}(H_p(I), H_{p+n}(A)).$$

But since $H_0(I) \cong K$ and $H_p(I) = 0$ for $p \neq 0$, we deduce the needed result from the fact that K-$\mathbf{Vct}(K, X) \cong X$ for a given vector space X. □

Proposition 3.2.4. *The category of differential graded algebras DG-K-**Alg** supports a cofibrantly generated model category structure.*

Proof. The strategy is to transfer the model structure on differential graded algebras by means of the adjunction

$$T : \text{DG-}K\text{-}\mathbf{Vct} \rightleftarrows \text{DG-}K\text{-}\mathbf{Alg} : U.$$

Since the category DG-K-**Vct** is cofibrantly generated by $I = \{\mathbb{S}^{n-1} \to \mathbb{D}^n \mid n \geq 1\}$ and $J = \{0 \to \mathbb{D}^n \mid n \geq 1\}$ it suffices to check the criterion given in Proposition 1.2.21. Notice first that the functor T is a left adjoint and therefore commutes with filtered colimits. Now we check that the domains of $T(I)$, $T(\mathbb{S}^n)$ are small relative to the whole category DG-K-**Alg**. We have to prove that the natural map

$$\operatorname*{colim}_{\alpha} \text{DG-}K\text{-}\mathbf{Alg}(T(\mathbb{S}^n), X_\alpha) \longrightarrow \text{DG-}K\text{-}\mathbf{Alg}(T(\mathbb{S}^n), \operatorname*{colim}_{\alpha} X_\alpha)$$

is an isomorphism for any filtered sequence of maps in DG-K-**Alg**

$$X_0 \to X_1 \to \cdots \to X_\alpha \to X_{\alpha+1} \to \cdots .$$

On the one hand, using the adjunction above gives

$$\operatorname*{colim}_{\alpha} \text{DG-}K\text{-}\mathbf{Alg}(T(\mathbb{S}^n), X_\alpha) \cong \operatorname*{colim}_{\alpha} \text{DG-}K\text{-}\mathbf{Vct}(\mathbb{S}^n, U(X_\alpha))$$

On the other hand, Lemma 3.2.2 yields

$$\begin{aligned} \text{DG-}K\text{-}\mathbf{Alg}(T(\mathbb{S}^n), \operatorname*{colim}_{\alpha} X_\alpha) &\cong \text{DG-}K\text{-}\mathbf{Vct}(\mathbb{S}^n, U(\operatorname*{colim}_{\alpha} X_\alpha)) \\ &\cong \text{DG-}K\text{-}\mathbf{Vct}(\mathbb{S}^n, \operatorname*{colim}_{\alpha} U(X_\alpha)). \end{aligned}$$

Finally the desired isomorphism is deduced from the fact that the n-sphere $\mathbb{S}^n$ is small in the category DG-K-**Vct**. The tensor algebra $T(0)$ on the trivial differential graded vector space 0 is the 0-circle $\mathbb{S}^0$. By similar argument as above, it is even easier to check that $T(0)$ is also small.

Notice that every object of the category DG-K-**Vct** is fibrant. As the previous lemma provides the category DG-K-**Alg** with good path objects, the required result follows. □

In order to reduce the amount of work, we could also put foward the argument that the category of differential graded algebras is locally presentable. Since in a locally presentable category each object is small relative to the whole category smallness of the algebra $T(\mathbb{S}^n)$ and $T(0)$ follow immediately.

3.3 Differential graded coalgebras

This section is devoted to the study of the category of differential graded coalgebras. Our main source will be the preprint of Getzler and Goerss [GG99]. First they prove that the main properties of coalgebras remain valid in the differential graded context. Then they derive a cofibrantly generated model structure for the category of differential graded coalgebras. We elaborate on their arguments by showing that the category of differential graded coalgebras is locally presentable. Viewing coalgebras as comonoids of vector spaces, it is natural to ask if the Getzler-Goerss model structure comes from applying the Transfer Result as in the case of algebras.

3.3.1 Categorical structure

Definition 3.3.1. The category of counital coassociative differential graded coalgebras that we denote by DG-K-**coAlg** is the category of comonoids in the monoidal category (DG-K-**Vct**,$\otimes$,$K[0]$).

Proposition 3.3.2. *Every differential graded coalgebra is the filtered colimit of its finite dimensional subcoalgebras.*

Definition 3.3.3. Let K be a field.

1. A *profinite differential graded K-algebra* is a topological differential graded K-algebra which is isomorphic to the inverse limit of an inverse system of finite-dimensional differential graded K-algebras.

2. A *morphism of profinite differential graded K-algebra* is a morphism of differential graded K-algebras that is continuous with respect to the profinite topology.

Together with these morphims profinite differential graded K-algebras form a category, which we denote by Pro-DG-K-**Alg**.

Proposition 3.3.4. *There is an anti-equivalence of categories between the category of differential graded coalgebras DG-K-**coAlg** and the category of profinite non-positively graded differential algebras Pro-DG-K-**Alg**.*

Corollary 3.3.5. *The category of differential graded coalgebras is complete and cocomplete.*

Proof(Sketch). While colimits in differential graded coalgebras are created by the forgetful functor from DG-K-**coAlg** to DG-K-**Vct**, the existence of limits is more subtil. To see this, the idea is to prove that the category of profinite differential graded algebras Pro-DG-K-**Alg** is cocomplete and then to use the anti-equivalence stated above to derive completeness for DG-K-**coAlg** (see [GG99, Proposition 1.8]). $\square$

Proposition 3.3.6. *The forgetful functor U_d from the category of differential graded coalgebras DG-K-**coAlg** to the category of differential graded vector spaces DG-K-**Vct** has a right adjoint S_d.*

Proof. The arguments are exactly the same as in Proposition 2.4.5. We shall simply recall how cofree coalgebras are constructed. Let V be a differential graded vector space. If V is degreewise finite-dimensional, then the cofree coalgebra on V is given by

$$S_d(V) = \left(\widehat{T(V^*)}\right)^{*\mathrm{cont}}.$$

For general $V \in K$-**Vct** the cofree coalgebra is defined by

$$S_d(V) = \operatorname{colim} S_d(V_\alpha)$$

with V_α running over the finite-dimensional subvector spaces of V. $\square$

Example 3.3.7. Let C be a finite dimensional differential graded coalgebra and (V, ∂) be a differential graded vector space which is concentrated in non-negative degrees and finite dimensional in each degree. We recall from [GG99, Lemma 1.12] that

(i) if $V_0 = 0$ then $\big(C \times S_d(V)\big)^* \cong T_{C^*}\big(C^* \otimes V^* \otimes C^*\big)$ and

(ii) if $V_0 = K$ with generator X then

$$\big(C \times S_d(V)\big)^* \cong \lim_{q(X)} \Big(T_{C^*}(C^* \otimes V^* \otimes C^*)/\big(q(X), \partial q(X)\big)\Big)$$

with $q(X)$ running over all monic polynomials in $K[X]$.

In particular we can choose in (i) the differential graded vector space V to be the n-sphere $\mathbb{S}^n$ with $n \geq 1$ and C to be the one dimensional differential graded coalgebra $K[0]$ concentrated in degree zero. Note that $K[0]$ is the terminal object in the category DG-K-**coAlg** and therefore the product $K[0] \times X$ with any object X in DG-K-**coAlg** is isomorphic to X. Thus we obtain that

$$\big(S_d(\mathbb{S}^n)\big)^* \cong T_{K[0]^*}\big((\mathbb{S}^n)^*\big) \cong T_{K[0]}\big(\mathbb{S}^{-n}\big).$$

Moreover, applying (ii) to the 0-sphere $\mathbb{S}^0$ and to the one dimensional coalgebra $K[0]$ yields

$$\big(S_d(\mathbb{S}^0)\big)^* \cong \lim_{q(X)} \Big(T_{K[0]^*}\big((\mathbb{S}^0)^*\big)/\big(q(X)\big)\Big) \cong \lim_{q(X)} \Big(K[X]/\big(q(X)\big)\Big).$$

In this way we recover the example studied earlier in Section 2.5.

3.3.2 Model structure

Getzler and Goerss have provided a model structure on the category of differential graded coalgebras DG-K-**coAlg**: this paragraph summarizes their main arguments (see [GG99, Section 2]).

Definition 3.3.8. Consider a map f in the category DG-K-**coAlg**. Then define f to be a

- *weak equivalence* if $H_* f$ is an isomorphism,
- *cofibration* if it is a degree-wise injection of graded K-vector spaces,
- *fibration* if it has the right lifting property with respect to trivial cofibrations.

The following result and its corollary allow functorial factorizations of coalgebra morphisms into cofibrations followed by trivial fibrations.

Proposition 3.3.9. *Let C be a differential graded coalgebra. Then for all non-negative integers n, the natural projection*

$$C \times S_d(\mathbb{D}^n) \to C$$

induces isomorphism in homology.

See [GG99, Theorem 2.1] for the proof.

Corollary 3.3.10. *Let V be a non-negatively graded differential vector space. Suppose moreover that V is acyclic. Then the natural projection*

$$C \times S_d(V) \to C$$

induces isomorphism in homology.

See [GG99, Proposition 2.2] for the proof.

Proposition 3.3.11. *Let $f\colon C \to D$ be a morphism in the category of differential graded coalgebras DG-K-**coAlg**. Choose V an acyclic differential graded vector space containing C. Then the morphism f can be factored*

$$C \xrightarrow{i} D \times S_d(V) \xrightarrow{p} D$$

with i a cofibration and p an acyclic fibration.

Proof. We give with more details the proof given by Getzler and Goerss. In light of Corollary 3.3.10, the trick is to factor the coalgebra map $f\colon C \to D$ through a coalgebra $D \times S_d(V)$ where V is a suitable acyclic differential graded vector space. First forget the coalgebra structure on C by considering $U_d(C) \in$ DG-K-**Vct**. Then define V to be $\mathrm{cone}(U_d(C))$. The differential graded vector space $\mathrm{cone}(U_d(C))$ is acyclic and comes with a canonical embedding $j\colon U_d(C) \to \mathrm{cone}(U_d(C))$. Thus we define p to be the projection map $D \times S_d(V) \to D$. Then notice that to give a morphism in DG-K-$\mathbf{coAlg}\big(C, D \times S_d(V)\big)$ amouts to give a pair of morphisms in

$$\text{DG-}K\text{-}\mathbf{coAlg}(C, D) \times \text{DG-}K\text{-}\mathbf{coAlg}(C, S_d(V)).$$

But using the adjunction between the categories of coalgebras and vector spaces, the previous product is equivalent to

$$\text{DG-}K\text{-}\mathbf{coAlg}(C, D) \times \text{DG-}K\text{-}\mathbf{Vct}(U_d(C), V).$$

In this way, the pair (f, j) yields a coalgebra morphism i. Now consider $\pi\colon D \times S_d(V) \to S_d(V)$ the projection onto $S_d(V)$ and the map $\epsilon\colon S_d(V) \to V$ coming from the counit of the adjunction between coalgebras and vector spaces. Then the following composition

$$C \xrightarrow{i} D \times S_d(V) \xrightarrow{\pi} S_d(V) \xrightarrow{\epsilon} V = \mathrm{cone}(U_d(V))$$

is the embedding j and therefore insures that the coalgebra morphism i is a cofibration as required. $\square$

Finally let us point out that the composite map

$$C \coprod C \to C \times S_d\Big(\mathrm{cone}\big(U_d(C \coprod C)\big)\Big) \to C$$

yields a natural construction of a very good cylinder object for a given object C in the category DG-K-**coAlg**.

Lemma 3.3.12. *A morphism of differential graded coalgebras is a fibration if and only if it has the right lifting property with respect to all acyclic cofibrations $A \to B$ such that B has a countable homogeneous basis.*

See [GG99, Lemma 2.6] for the proof.

Lemma 3.3.13. *The category of differential graded coalgebra DG-K-**coAlg** is a locally presentable category.*

Proof. This lemma is a consequence of the more general result stated in Proposition 1.2.15. The category of differential graded vector spaces DG-K-**Vct** is locally presentable by. Moreover the monoidal category (DG-K-**Vct**,$\otimes$,$K[0]$) is admissible since it is closed. Hence the category of its comonoids DG-K-**coAlg** is locally presentable as well. $\square$

Corollary 3.3.14. *Any morphism in the category of differential graded coalgebras DG-K-**coAlg** can be factored by an acyclic cofibration followed by a fibration.*

Proof. Lemma 3.3.12 above supplies a good candidate for the cofibrant generating set and Lemma 3.3.13 guarantees the small object argument. □

Theorem 3.3.15. *The category of differential graded coalgebra DG-K-**coAlg** supports a cofibrantly generated model structure.*

The next result asserts that the pair of functors (S_d, U_d) is a Quillen pair.

Proposition 3.3.16. *The cofree coalgebra functor*

$$S_d \colon \text{DG-}K\text{-}\mathbf{Vct} \longrightarrow \text{DG-}K\text{-}\mathbf{coAlg}$$

preserves fibrations and weak equivalences.

Proof. Notice first that every differential graded vector space in the model category DG-K-**Vct** is fibrant and cofibrant. Furthermore, the forgetful functor $U_d \colon \text{DG-}K\text{-}\mathbf{coAlg} \longrightarrow \text{DG-}K\text{-}\mathbf{Vct}$ preserves cofibrations and weak equivalences. Hence U_d is a left Quillen functor and the claim follows. □

3.3.3 Problems with the transfer of model category structures

We make some comments on whether the Getzler-Goerss model structure on DG-K-**coAlg** can be recovered by means of the transfer theorem. Since we have the adjunction

$$U_d : \text{DG-}K\text{-}\mathbf{coAlg} \rightleftarrows \text{DG-}K\text{-}\mathbf{Vct} : S_d$$

we cannot apply the transfer theorem: the involved categories and functors are in the wrong places. To get around this problem it is natural to consider the following formal dualization provided by Lemma 1.1.8:

$$(\text{DG-}K\text{-}\mathbf{Vct})^{\text{op}} \underset{U_d^{\text{op}}}{\overset{S_d^{\text{op}}}{\rightleftarrows}} (\text{DG-}K\text{-}\mathbf{Vct})^{\text{op}}\text{-}\mathbf{Monoid} = (\text{DG-}K\text{-}\mathbf{coAlg})^{\text{op}}.$$

But some problems still occur:

- As observed in Paragraph 3.1.4, the category $((\text{DG-}K\text{-}\mathbf{Vct})^{\text{op}}, \otimes)$ is not closed. It follows that the opposite category $(\text{DG-}K\text{-}\mathbf{Vct})^{\text{op}}$ is not a monoidal model category. A similar argument is used by Hovey to show that the category of topological spaces fails to be a monoidal model category (see [Hov99, page 110]). The fact is that one needs the closedness in the definition of monoidal model categories in order to derive further properties for the homotopy category associated to a monoidal model category: under this assumption, the homotopy category inherits a closed monoidal structure (see [Hov99, Theorem 4.3.2]). In [SS00], Schwede and Shipley require the closedness of the monoidal structure in order to guarantee that the monoidal product commutes with colimits and hence to ensure that the categories of monoids have colimits whenever the monoidal categories are cocomplete. In the case of vector spaces and coalgebras, however, we can obtain cocompleteness via an alternative line of argument.

- We have seen that Proposition 3.1.11 provides the opposite category of differential graded vector spaces, (DG-K-**Vct**)$^{\mathrm{op}}$ with a set of generating acyclic cofibrations Q^{op}. However the set P^{op} in Proposition 3.1.12 is not a set of generating cofibrations since its domain $\mathbb{S}^n$ is not small in the opposite category (DG-K-**Vct**)$^{\mathrm{op}}$ as proven in Paragraph 3.1.4. The result of this observation is that the opposite category (DG-K-**Vct**)$^{\mathrm{op}}$ is not cofibrantly generated. One needs the cofibrant generation in the monoidal model category in order to guarantee the factorization axiom **MC5** for the associated category of monoids: the crucial point is to be able to apply Quillen's small object argument. Unfortunately, the dual of Quillen's small object argument is inefficient since cosmall objects are rare in this case.

To conclude, although the opposite category (DG-K-**coAlg**)$^{\mathrm{op}}$ inherits a model structure in the usual way, this model structure does not arise from an application of the transfer theorem.

Chapter 4

Simplicial Objects

4.1 The Dold-Kan correspondence

We collect from Weibel [Wei94] and Mac Lane [Mac67] some basics about simplicial methods. We focus on the Dold-Kan correspondence which establishes an equivalence of categories between simplicial objects and differential graded objects in every abelian category. This correspondence is concretely given by the normalization functor N and its inverse Γ. Following Weibel, we briefly recall the construction of the inverse functor Γ at the end of this section.

4.1.1 Generalities

Definition 4.1.1. Let $[n]$ be the ordered set $\{0 < 1 < \cdots < n\}$. A map $f\colon [n] \to [m]$ is non-decreasing if $f(i) \geq f(j)$ whenever $i > j$. The *simplicial category* Δ is the category with objects $[n]$ for $n \geq 0$ and with non-decreasing maps as morphisms.

Definition 4.1.2. Let Δ be the simplicial category.

1. The *face maps* $\epsilon_i\colon [n-1] \to [n]$ are the injections in Δ which miss $i, (0 \leq i \leq n)$.
2. The *degeneracy maps* $\eta_i\colon [n+1] \to [n]$ are the surjections in Δ which send both i and $i+1$ to i.

Proposition 4.1.3. *The face and degeneracy maps satisfy the following relations*

$$\begin{aligned} \epsilon_j\epsilon_i &= \epsilon_i\epsilon_{j-1} \quad \textit{if} \quad i < j, \\ \eta_j\eta_i &= \eta_i\eta_{j+1} \quad \textit{if} \quad i \leq j, \end{aligned}$$

$$\eta_j\epsilon_i = \begin{cases} \epsilon_i\eta_{j-1} & \textit{if} \quad i < j, \\ id_{[n]} & \textit{if} \quad i = j \textit{ or } i = j+1 \\ \epsilon_{i-1}\eta_j & \textit{if} \quad i > j+1. \end{cases}$$

Proposition 4.1.4. *Every morphism $f\colon [n] \to [m]$ in Δ has a unique epimonic factorization*

$$f = \epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_r}\eta_{j_1}\eta_{j_2}\cdots\eta_{j_s}$$

such that $i_1 < i_2 < \cdots < i_r$ and $j_1 < j_2 < \cdots < j_s$ with $m = n - s + r$.

See for instance [Wei94, Lemma 8.1.2] for a proof.

Definition 4.1.5. Let $\mathbf{C}$ be a category.

1. A *simplicial object in* $\mathbf{C}$ is a functor $X\colon \Delta^{\mathrm{op}} \to \mathbf{C}$.

   A *face operator* is a morphism $X(\epsilon_i)\colon X_n \to X_{n-1}$ and a *degeneracy operator* is a morphism $X(\eta_i)\colon X_n \to X_{n+1}$.

2. Dually a *cosimplicial object in* $\mathbf{C}$ is a functor $X\colon \Delta \to \mathbf{C}$.

   A *coface operator* is a morphism $X(\epsilon_i)\colon X^{n-1} \to X^n$ and a *codegeneracy operator* is a morphism $X(\eta_i)\colon X^{n+1} \to X^n$.

Example 4.1.6. A standard construction which can be found for instance in [Bou70, 7.7] gives rise to cosimplicial objects whenever two categories are adjoint. Indeed for arbitrary categories $\mathbf{C}$ and $\mathbf{D}$ we consider a functor $L\colon \mathbf{C} \longrightarrow \mathbf{D}$ left adjoint to $R\colon \mathbf{D} \longrightarrow \mathbf{C}$. If we denote by $\varphi\colon 1_{\mathbf{C}} \to RL$ and $\psi\colon LR \to 1_{\mathbf{D}}$ the adjunction maps and by μ the composite

$$R\psi_L\colon R(LR)L \longrightarrow R1_{\mathbf{D}}L,$$

then $\big(RL, \varphi, \mu\big)$ defines a monad on $\mathbf{C}$. Now let C be in $\mathbf{C}$. If we set

$$(RL)^0C = C \quad \text{and} \quad (RL)^nC = (RL)(RL)^{n-1}C,$$

then the functor $X_C\colon \Delta \to \mathbf{C}$ given by

$$X_C^n = (RL)^{n+1}C, \quad n \geq 0$$

with

$$\begin{aligned} d^i &= (RL)^i\varphi(RL)^{n-i}\colon (RL)^nC \longrightarrow (RL)^{n+1} \\ s^i &= (RL)^i\mu(RL)^{n-i}\colon (RL)^{n+2}C \longrightarrow (RL)^{n+1} \end{aligned}$$

define a cosimplicial object called the *cosimplicial resolution* over the category $\mathbf{C}$. One obtains an augmentation by considering the map $\varphi_C\colon C \longrightarrow X_C^{-1} = RLC$. Dually the composite functor LR defines a comonad on $\mathbf{D}$ and one obtains a simplicial object called the *simplicial resolution* over the category $\mathbf{D}$. We refer to [Mac98, VII.6] for details on simplicial resolutions.

4.1.2 The normalization functor N

Definition 4.1.7. Let A be a simplicial object in an abelian category $\mathbf{C}$.

1. The *differential graded object associated to* A is denoted by CA and is given by $(CA)_n = A_n$ with differential the alternating sum of the face operators:
$$d = \sum_{i=0}^{n} (-1)^i d_i \colon (CA)_n \to (CA)_{n-1}$$

2. The *differential graded object of degenerate simplices associated to* A, denoted by DA is the subcomplex of CA with
$$(DA)_0 = 0 \quad \text{and} \quad (DA)_n = s_0 A_{n-1} + \cdots + s_{n-1} A_{n-1} \quad \text{for} \quad n \geq 1.$$

Definition 4.1.8. The *normalized differential graded object associated to* A, denoted by NA is the quotient of CA by its subcomplex DA, that is
$$NA = CA/DA.$$

The *normalization functor* N is the functor that associates to a simplicial object its normalized differential graded object.

We shall also mention the alternative definition of the normalization functor N given by
$$(NA)_n = \bigcap_{i=0}^{n-1} \ker(d_i)$$
with differential $(-1)^n d_n \colon (NA)_n \longrightarrow (NA)_{n-1}$.

The result below, called the Dold-Kan correspondence, establishes an equivalence of categories between the category of simplicial objects and the category of differential graded objects in every abelian category.

Theorem 4.1.9. *Let $\mathbf{C}$ be an abelian category. Then, the normalization functor N is an equivalence of categories between the category of differential graded objects in $\mathbf{C}$ and the category of simplicial objects in $\mathbf{C}$.*

We refer to [Wei94, 8.4.4] for a proof of the Dold-Kan correspondence. We shall now give a description of the inverse of the normalization functor. If (C, d) is a differential graded object in $\mathbf{C}$, then the inverse functor denoted by Γ is given by
$$(\Gamma C)_n = \bigoplus_{p=0}^{n} \bigoplus_{\eta} C_p[\eta]$$
where η ranges over all surjections $[n] \to [p]$ in Δ and $C_p[\eta]$ denotes a copy of C_p. Note that there are $\binom{n}{p}$ surjections $\eta \colon [n] \to [p]$ in Δ.

Now consider a morphism $\alpha\colon [m] \to [n]$ in Δ. Then the morphism $\Gamma(\alpha)\colon (\Gamma C)_n \to (\Gamma C)_m$ is defined by its restrictions $\Gamma(\alpha, \eta)\colon C_p[\eta] \to (\Gamma C)_m$ as follows: let $\eta\colon [m] \to [n]$ be a surjection and $\eta\alpha\colon [m] \to [p]$ the resulting composite map with α. Using Proposition 4.1.4 we can factor $\eta\alpha$ as a composite of an epimorphism η' followed by a monomorphism ϵ:

$$\begin{tikzcd}
[m] \arrow[r, "\alpha"] \arrow[d, "\eta'"'] & [n] \arrow[d, "\eta"] \\
[q] \arrow[r, "\epsilon"'] & [p].
\end{tikzcd}$$

The morphism $\Gamma(\alpha, \eta)$ is then defined according to the outcome of the above epi-monic factorization:

- if $p = q$ then $\Gamma(\alpha, \eta)$ is the natural identification of $C_p[\eta]$ with the summand $C_p[\eta']$ of $(\Gamma C)_m$.
- if $p = q + 1$ and $\epsilon = \epsilon_p$ then $\Gamma(\alpha, \eta)$ is the differential

$$C_p[\eta] = C_p \xrightarrow{d} C_{p-1} = C_q[\eta'] \subseteq (\Gamma C)_m.$$

- In the other cases $\Gamma(\alpha, \eta)$ is the trivial map.

Example 4.1.10. The algorithm described above yields for instance

$$\begin{aligned}
(\Gamma C)_0 &= C_0\big[id_{[0]}\big], \\
(\Gamma C)_1 &= C_0\big[\eta_0\big] \oplus C_1\big[id_{[1]}\big], \\
(\Gamma C)_2 &= C_0\big[\eta_0\eta_0\big] \oplus C_1\big[\eta_0\big] \oplus C_1\big[\eta_1\big] \oplus C_2\big[id_{[2]}\big].
\end{aligned}$$

In order to obtain the face operator $(\Gamma C)(\epsilon_0)\colon (\Gamma C)_1 \to (\Gamma C)_0$, we have to consider the compositions with ϵ_0. Thus, we obtain with help of the simplicial identities of Proposition 4.1.3

$$\begin{tikzcd}
[0] \arrow[r, "\epsilon_0"] \arrow[d] & [1] \arrow[d, "\eta_0"] \\
[0] \arrow[r, "id_{[0]}"'] & [0]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
[0] \arrow[r, "\epsilon_0"] \arrow[d] & [1] \arrow[d, "id_{[1]}"] \\
[0] \arrow[r, "\epsilon_0"'] & [1]
\end{tikzcd}$$

and the required face operator is given by the matrix $(\Gamma C)(\epsilon_0) = \begin{pmatrix} id & 0 \end{pmatrix}$.

Similar computations show that the other face operator $(\Gamma C)(\epsilon_1)\colon (\Gamma C)_1 \to (\Gamma C)_0$ is given by the matrix $(\Gamma C)(\epsilon_1) = \begin{pmatrix} id & d \end{pmatrix}$.

4.2 Simplicial vector spaces

After the generalities of the preceding section, we focus on the category of simplicial K-vector spaces. This category is equivalent to that of differential graded K-vector spaces via the Dold-Kan correspondence. We use this correspondence to give some examples of simplicial vector spaces. Moreover, we explain how to derive from the Dold-Kan theorem a cofibrantly generated model structure for the category of simplicial vector spaces.

4.2.1 Categorical structure

Definition 4.2.1. Let K-**Vct** be the category of K-vector spaces.

1. A *simplicial vector space* X is a simplicial object in the category of vector spaces K-**Vct**, that is a functor $X\colon \Delta^{\mathrm{op}} \to K\text{-}\mathbf{Vct}$.

2. A morphism $f\colon X \to Y$ of simplicial vector spaces is a natural transformation of functors $X, Y\colon \Delta^{\mathrm{op}} \to K\text{-}\mathbf{Vct}$.

Together with these morphisms, simplicial vector spaces form a category that we denote by S-K-**Vct**.

The category of K-vector spaces, K-**Vct** is abelian. Therefore the Dold-Kan correspondence is an equivalence between the category of differential graded vector spaces, DG-K-**Vct**, and the category of simplicial vector spaces, S-K-**Vct**,

$$N : \text{S-}K\text{-}\mathbf{Vct} \rightleftarrows \text{DG-}K\text{-}\mathbf{Vct} : \Gamma.$$

It follows from [Mac98, IV.4 Theorem 1] that the normalization functor N is an adjoint equivalence of categories. The unit $\varepsilon\colon N\Gamma \to Id$ and the counit $\eta\colon Id \to \Gamma N$ of this adjoint equivalence are both natural isomorphisms. Later, we shall exploit these properties to define a suitable comultiplication for simplicial vector spaces.

We use the inverse functor Γ to give some examples of simplicial vector spaces.

Example 4.2.2. Recall that the n-sphere $\mathbb{S}^n$ is the differential graded vector space which consists of K concentrated in degree n. We describe the image under the inverse functor Γ of the vector space $\mathbb{S}^n$ for $n \geq 1$ with help of the algorithm given in Section 4.1.2. If l is an index with $l < n$, then $\big(\Gamma(\mathbb{S}^n)\big)_l$ is the zero vector space 0. For an index $l \geq n$, the vector space $\big(\Gamma(\mathbb{S}^n)\big)_l$ consists of $\binom{l}{n}$ copies of K. Hence, for instance the diagram of face operators in the simplicial vector space $\Gamma(\mathbb{S}^1)$ is

$$0 \leftleftarrows K \;\overset{\longleftarrow}{\leftleftarrows}\; K^{\oplus 2} \;\leftleftarrows\!\!\leftleftarrows\; K^{\oplus 3} \;\overset{\longleftarrow}{\leftleftarrows\!\!\leftleftarrows}\; K^{\oplus 4} \quad \dots$$

We may represent the face operators involved in $\Gamma(\mathbb{S}^1)$ with matrices where the entries 1 and 0 denote respectively the identity map on K and the trivial map. We shall give here some of these face operators. The face operators $\partial_i \colon K \longrightarrow 0$ are all trivial. The face operators $\partial_i \colon K^{\oplus 2} \longrightarrow K$ are given by

$$\begin{aligned} \partial_0 &= \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad \partial_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}, \\ \partial_2 &= \begin{pmatrix} 0 & 1 \end{pmatrix}. \end{aligned}$$

The face operators from $\partial_i \colon K^{\oplus 3} \longrightarrow K^{\oplus 2}$ are given by

$$\partial_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \partial_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \partial_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We may also recover in a similar fashion the degeneracy operators of $\Gamma(\mathbb{S}^1)$, but we restrict to the face operators since only they appear in the differential graded vector space associated to $\Gamma(\mathbb{S}^1)$.

Notice that the simplicial vector space $\Gamma(\mathbb{S}^0)$ is easier to describe: it has K in each degree and all the maps are the identity of the field K. In other words, $\Gamma(\mathbb{S}^0)$ is the constant simplicial vector space at K.

Definition 4.2.3. Let X be a simplicial vector space. We define the *n-th homotopy groups* of X, $\pi_n(X)$ to be:

$$\pi_n(X) = H_n(NX) \cong H_n(CX).$$

With this definition, computing homotopy groups of a given simplicial vector space amouts to computing the homology groups of the differential graded vector space associated to it.

Recall that a category is called *small* if its objects and morphisms are sets. Let **C** and **D** be two categories. If in addition the category **C** is small, then we denote by **Funct(C,D)** the category of functors from **C** to **D**.

Lemma 4.2.4. *Let **I** be a small category. If a given category **C** is complete and cocomplete, then **Funct(I,C)** is complete and cocomplete.*

See for instance [Bor94a, Theorem 2.15.2] for a proof.

Proposition 4.2.5. *The category of simplicial vector spaces, S-K-**Vct** is complete and cocomplete.*

Proof. The category of vector spaces is complete and cocomplete. Since the category S-K-**Vct** is by definition **Funct(**Δ^{op}**,** K**-Vct)**, the proposition is a consequence of the preceding lemma. □

4.2.2 Monoidal model structure

Proposition 4.2.6. *The category of simplicial vector spaces, S-K-**Vct** is equipped with a cofibrantly generated model structure.*

The model structure comes from defining a map f in the category S-K-**Vct** to be a

- weak equivalence if $\pi_* f$ is an isomorphism,
- cofibration if it is a level-wise injection,
- fibration if it has the right lifting property with respect to trivial cofibrations.

Since the category of differential graded vector spaces DG-K-**Vct** is cofibrantly generated, one deduces the cofibrant generation of the category S-K-**Vct** by applying the transfer result to the adjoint pair (Γ, N) provided by the Dold-Kan correspondence. Hence in the category S-K-**Vct**, the generating acyclic cofibrations are given by the maps $\left\{0 \to \Gamma(\mathbb{D}^n) \mid n \geq 1\right\}$ and the generating cofibrations by $\left\{\Gamma(\mathbb{S}^{n-1}) \to \Gamma(\mathbb{D}^n) \mid n \geq 1\right\}$.

Definition 4.2.7. Let X and Y be two objects in the category S-K-**Vct**. Then define the monoidal product $\widehat{\otimes}$ by

$$(X \widehat{\otimes} Y)_n = X_n \otimes_K Y_n.$$

The unit of the monoidal product $\widehat{\otimes}$ is the simplicial vector space that has K in each degree and identity maps on K as face and degeneracy operators. We denote this unit by $I(K)$. The monoidal product $\widehat{\otimes}$ is symmetric.

Lemma 4.2.8. *The category of simplicial vectors spaces, S-K-**Vct**, endowed with the monoidal product $\widehat{\otimes}$ is closed.*

Lemma 4.2.9. *The category of simplicial vector spaces, S-K-**Vct**, is locally presentable.*

Proof. Following [Bor94a, 5.3.3, 5.7.5], every category of functors from a small category to a locally presentable category is a locally presentable. Since the category of vector spaces K-**Vct** is locally presentable, the claim follows. □

Proposition 4.2.10. *The category of simplicial vectors spaces, S-K-**Vct**, has a monoidal model structure.*

We recall from [Qui67] or [GJ99] that a *simplicial model category* is a model category with an additional axiom which is called axiom **SM7**. In [Qui67, 4.11] it is proven that the category S-K-**Vct** is in fact a simplicial model category. Therefore, the above proposition follows because the axiom **SM7** ensures the compatibility of the monoidal product $\widehat{\otimes}$ with the model structure.

4.3 The Eilenberg-Zilber theorem

In this section, we recall the basic properties of the Alexander-Withney map and the shuffle map from [Mac67] and [May67]. In fact these maps provide the normalization functor N with respectively a monoidal and a comonoidal structure. Hence, in the next chapter, we shall make use of these observations to derive a comonoidal structure for the inverse functor Γ.

4.3.1 The Alexander-Whitney map AW

Definition 4.3.1. Let X and Y be simplicial vector spaces. The *Alexander-Whitney map* is given by

$$AW\colon C(X\widehat{\otimes}Y) \longrightarrow C(X) \otimes C(Y)$$

which is defined degreewise by

$$\begin{aligned} AW_n\colon X_n \otimes Y_n &\longrightarrow \bigoplus_{p+q=n} X_p \otimes Y_q \\ x \otimes y &\longmapsto \sum_{i=0}^{n} \widetilde{d}^{n-i}x \otimes d_0^i y \end{aligned}$$

with $\widetilde{d}^{n-i}x = d_{i+1}\cdots d_n x$.

Proposition 4.3.2. *The Alexander-Whitney map, AW, turns C into a lax comonoidal functor.*

The claim follows from [Mac67, Lemma 8.2, Thorem 8.5, Proposition 8.7]. Notice that the above proposition has a normalized version

$$AW_N\colon N(X\widehat{\otimes}Y) \longrightarrow NX \otimes NY$$

given in [Mac67, Corollary 8.6]. Thus, the normalized Alexander-Whitney map turns the functor N into a lax comonoidal functor

$$\big(N, AW_N\big)\colon (\text{S-}K\text{-}\mathbf{Vct}, \widehat{\otimes}, I(K)) \longrightarrow (\text{DG-}K\text{-}\mathbf{Vct}, \otimes, K[0]).$$

4.3.2 The shuffle map ∇

Definition 4.3.3. Let p and q be non-negative integers. A (p,q)*-shuffle* (μ,ν) is a partition of the set $[p+q-1]$ of integers into two disjoint subsets $\mu_1 < \cdots < \mu_p$ and $\nu_1 < \cdots < \nu_q$ of p and q integers respectively.

1. A (p,q)-shuffle is also defined as a permutation σ of the set of integers $\{1,\cdots,p+q\}$ such that $\sigma(i) < \sigma(j)$ whenever $i < j \leq p$ or $p < i < j$. In this way, μ_i is given by $\sigma(i)-1$ and ν_j by $\sigma(p+j)-1$.

2. The *signature of a shuffle* (μ, ν) is the integer

$$\epsilon(\mu) = \sum_{i=1}^{p} \mu_i - (i-1).$$

Example 4.3.4. Let S_3 be the group of permutations of the set $\{1,2,3\}$. Then the $(1,2)$-shuffles are the identity, the 2-cycle (12) and the 3-cycle (132) of S_3.

Definition 4.3.5. Let X and Y be simplicial vector spaces. The *shuffle map* is given by

$$\nabla \colon C(X) \otimes C(Y) \longrightarrow C(X \widehat{\otimes} Y)$$

which is defined degreewise by

$$\begin{aligned} \nabla_n \colon \bigoplus_{p+q=n} X_p \otimes Y_q &\longrightarrow X_n \otimes Y_n \\ x_p \otimes y_q &\longmapsto \sum_{(\mu,\nu)} (-1)^{\epsilon(\mu)} (s_{\nu_q} \cdots s_{\nu_1} x_p \otimes s_{\mu_p} \cdots s_{\mu_1} y_q) \end{aligned}$$

where the sum is taken over all (p,q)-shuffles (μ,ν).

Proposition 4.3.6. *The shuffle map ∇ is a symmetric monoidal natural transformation.*

We refer to [Mac67, Theorem 8.8] for the proof. The shuffle map also has a normalised version

$$\nabla_N \colon NX \otimes NY \longrightarrow N(X \widehat{\otimes} Y)$$

given in [Mac67, Corollary 8.9]. Hence the normalized shuffle map induces a lax symmetric monoidal structure on the normalization functor

$$\big(N, \nabla_N\big) \colon (\text{S-}K\text{-}\mathbf{Vct}, \widehat{\otimes}, I(K)) \longrightarrow (\text{DG-}K\text{-}\mathbf{Vct}, \otimes, K[0]).$$

The Alexander-Whitney map AW and the shuffle map ∇ as well as their normalized versions AW_N and ∇_N satisfy the following property due to Eilenberg and Zilber.

Theorem 4.3.7. *For simplicial vector spaces X and Y there are natural chain homotopy equivalences*

$$C(X \widehat{\otimes} Y) \underset{\nabla}{\overset{AW}{\rightleftarrows}} CX \otimes CY$$

$$N(X \widehat{\otimes} Y) \underset{\nabla_N}{\overset{AW_N}{\rightleftarrows}} NX \otimes NY$$

In other words, the Alexander-Whitney map, the shuffle map and their normalized versions satisfy the homotopical relations:

$$\begin{aligned} \nabla \circ AW &\simeq \mathrm{Id}, \\ AW \circ \nabla &\simeq \mathrm{Id}, \\ \nabla_N \circ AW_N &\simeq \mathrm{Id}. \end{aligned}$$

Furthermore, following [May67, Corollary 29.10] or [SS03, Section 2.3], we note the identity $AW_N \circ \nabla_N = \mathrm{Id}$.

4.4 Simplicial coalgebras

In this section we investigate the category of simplicial K-coalgebras. As a category of functors, simplicial coalgebras inherit most of their properties from K-**coAlg**. In [Goe95] Goerss endows the category of simplicial coalgebras with a model structure. We briefly recall his construction and we point out that the category of simplicial coalgebras is locally presentable.

4.4.1 Categorical structure

Definition 4.4.1. The category of simplicial coalgebras, denoted by S-K-**coAlg**, is the category of comonoids in the monoidal category (S-K-**Vct**,$\widehat{\otimes}$,$I(K)$).

Lemma 4.4.2. *The category of simplicial coalgebras, S-K-**coAlg** is complete and cocomplete.*

Proof. The category S-K-**coAlg** is the category of functors **Funct(**Δ^{op}**,** K**-coAlg)**. Lemma 4.2.4 ensures the desired result. □

Lemma 4.4.3. *Suppose that $\boldsymbol{I}$ is a small category. Then every adjunction $L : \boldsymbol{C} \rightleftarrows \boldsymbol{D} : R$ gives rise to an adjunction **Funct(I,C)** $\rightleftarrows$ **Funct(I,D)** on the level of categories of functors.*

See for instance [Bor94a, Proposition 3.2.4] for the proof.

Proposition 4.4.4. *The forgetful functor U_s from the category of simplicial coalgebras, S-K-**coAlg**, to the category of simplicial vector spaces, S-K-**Vct**, has a right adjoint S_s.*

Proof. Since the category of simplicial objects in a category **C** is by definition **Funct(**Δ^{op}**, C)**, applying the previous lemma to the adjunction $U : K$-**coAlg** $\rightleftarrows K$-**Vct** $: S$ yields the result. □

4.4.2 Model structure

The category of simplicial coalgebras, S-K-**coAlg**, is endowed with a model structure. This model structure was established by Goerss in [Goe95, Section 3]. We shall mention that Goerss considers *cocommutative* simplicial coalgebras. But one can adapt his arguments to the non-cocommutative case as well.

Definition 4.4.5. A map f in S-K-**coAlg** is a

- weak equivalence if $\pi_* f$ is an isomorphism,
- cofibration if it is a level-wise inclusion,
- fibration if it has the right lifting property with respect to trivial cofibrations.

The main arguments of Goerss remain unchanged for the case of non-cocommutative simplicial coalgebras. The only slight difference is concerned with [Goe95, Lemma 3.5] recalled below.

Lemma 4.4.6. *Let $f\colon C \to D$ be a morphism of coalgebras. Then, f can be factored as $f = p \circ i$*

$$C \xrightarrow{i} X \xrightarrow{p} D$$

where i is a cofibration and p is an acyclic fibration.

In the proof of this lemma, we may replace the cocommutative cofree functor by its non-cocommutative version $S_s\colon \text{S-}K\text{-}\mathbf{Vct} \longrightarrow \text{S-}K\text{-}\mathbf{coAlg}$ that we consider here. Therefore, as suggested in [Goe95, Lemma 3.5], the factorization is given by the simplicial coalgebra

$$X = D \times \Big(\prod_{\alpha} S_s\big(\Gamma(\mathbb{D}^{n_\alpha})\big)\Big)$$

where the product is taken in the category S-K-**coAlg** and

$$\alpha \in \bigcup_{n \geq 0} \text{S-}K\text{-}\mathbf{coAlg}\Big(C, S_s\big(\Gamma(\mathbb{D}^n)\big)\Big).$$

The map p is the projection and the map i is taken as the product of f and all α. In this way, the arguments of Goerss hold since only the cofreeness property is required.

Proposition 4.4.7. *The category of simplicial coalgebras, S-K-**coAlg**, is a locally presentable category.*

Proof. The symmetric monoidal category $(\text{S-}K\text{-}\mathbf{Vct}, \widehat{\otimes}, I(K))$ is closed and locally presentable. Hence, the category of its comonoids is locally presentable by Proposition 1.2.15. $\square$

We notice that [Goe95, Lemma 3.7] provides the category of simplicial coalgebras, S-K-**coAlg** with generating acyclic cofibrations. Let c be an infinite regular cardinal which is at least equal to the cardinality of the field K.

Lemma 4.4.8. *A map $f\colon A \to B$ in S-K-**coAlg** is a fibration if and only if f has the right lifting property with respect to trivial cofibrations $C \to D$ such that the cardinality of a homogeneous basis of D is smaller than c.*

In view of the preceding proposition, Quillen small object argument gives the factorization axiom **MC5**(ii).

Theorem 4.4.9. *The category of simplicial coalgebras, S-K-**coAlg**, supports a model structure.*

Here is an analogue of Proposition 3.3.16 for simplicial vector spaces and simplicial coalgebras.

Proposition 4.4.10. *The cofree coalgebra functor*

$$S_s\colon \text{S-}K\text{-}\boldsymbol{Vct} \longrightarrow \text{S-}K\text{-}\boldsymbol{coAlg}$$

preserves fibrations and weak equivalences.

Chapter 5

The Dold-Kan Correspondence and Coalgebras

5.1 On coalgebras

In [SS03] Schwede and Shipley prove that the normalization functor, N, and its inverse, Γ, induce functors on the level of monoids. Then they show that the resulting functors do not form an adjunction. Mimicking their ideas, we obtain dual results for coalgebras. We define comultiplications for simplicial and differential graded vector spaces. As a consequence, the functors N and Γ give rise to functors on the categories of coalgebras. However, as for algebras, we show that the resulting coalgebra-valued functors are not adjoint.

5.1.1 The functors N and Γ on the categories of coalgebras

Proposition 5.1.1. *Consider a simplicial coalgebra $(A, \Delta_A, \varepsilon_A)$. Then $(NA, \Delta_{NA}, \varepsilon_{NA})$ is a differential graded coalgebra with a comultiplication given by the composition*

$$NA \xrightarrow{N(\Delta_A)} N(A \widehat{\otimes} A) \xrightarrow{AW_{A,A}} NA \otimes NA$$

and counit given by $N(\varepsilon_A)$.

Proof. In the diagram

$$\begin{tikzcd}
NA \otimes NA \arrow[r, "N(\Delta_A)\otimes Id"] & N(A\widehat{\otimes}A)\otimes NA \arrow[r, "AW_{A,A}\otimes Id"] & NA\otimes NA\otimes NA \\
N(A\widehat{\otimes}A) \arrow[u, "AW_{A,A}"'] \arrow[r, "N(\Delta_A\widehat{\otimes}Id)"] & N(A\widehat{\otimes}A\widehat{\otimes}A) \arrow[u, "AW_{A\widehat{\otimes}A,A}"'] \arrow[r, "AW_{A,A\widehat{\otimes}A}"] & NA\otimes N(A\widehat{\otimes}A) \arrow[u, "Id\otimes AW_{A,A}"'] \\
NA \arrow[u, "N(\Delta_A)"'] \arrow[r, "N(\Delta_A)"] & N(A\widehat{\otimes}A) \arrow[u, "N(Id\widehat{\otimes}\Delta_A)"'] \arrow[r, "AW_{A,A}"] & NA\otimes NA \arrow[u, "Id\otimes N(\Delta_A)"']
\end{tikzcd}$$

each small square commutes. To see this it suffices to use the fact that Δ_A is coassociative and that the Alexander-Whitney map AW is turns N into a lax comonoidal functor. Hence the composite square commutes as well and this means that Δ_{NA} is coassociative.

The counitary property comes from the naturality of the map AW and the fact that $K[0] \cong NI(K)$. □

Now let $f\colon (A,\Delta_A) \to (B,\Delta_B)$ be a map of simplicial coalgebras. Then the following composite diagram

$$\begin{tikzcd}
NA \arrow[r, "N(\Delta_A)"] \arrow[d, "N(f)"] & N(A\widehat{\otimes}A) \arrow[r, "AW_{A,A}"] \arrow[d, "N(f\widehat{\otimes}f)"] & NA\otimes NA \arrow[d, "N(f)\otimes N(f)"] \\
NB \arrow[r, "N(\Delta_B)"'] & N(B\widehat{\otimes}B) \arrow[r, "AW_{B,B}"'] & NB\otimes NB
\end{tikzcd}$$

commutes. Indeed the map f satisfies the identity $(f\widehat{\otimes}f)\circ\Delta_A = \Delta_B \circ f$ and therefore applying the functor N yields that the left diagram commutes. The right square commutes since the Alexander-Whitney map AW turns N into a lax comonoidal functor. This observation ensures that the normalization functor N induces a functor from S-K-**coAlg** to DG-K-**coAlg**.

Similar considerations show that the functor Γ also gives rise to a functor from the category of differential graded coalgebras, DG-K-**coAlg**, to the category of simplicial coalgebras, S-K-**coAlg**.

Proposition 5.1.2. *Consider a differential graded coalgebra $(B,\Delta_B,\varepsilon_B)$. Then $(\Gamma B, \Delta_{\Gamma B}, \varepsilon_{\Gamma B})$ is a simplicial coalgebra with a comultiplication $\Delta_{\Gamma B}$ given by the following composition*

$$\begin{tikzcd}
\Gamma B \arrow[r, "\Gamma(\Delta_B)"] \arrow[rrdd, "\Delta_{\Gamma B}"'] & \Gamma(B \otimes B) \arrow[r, "\Gamma(\varepsilon_B^{-1} \otimes \varepsilon_B^{-1})"] & \Gamma(N\Gamma B \otimes N\Gamma B) \arrow[d, "\Gamma(\nabla_{\Gamma B,\Gamma B})"] \\
 & & \Gamma N(\Gamma B \widehat{\otimes} \Gamma B) \arrow[d, "\eta^{-1}_{\Gamma B \widehat{\otimes} \Gamma B}"] \\
 & & \Gamma B \widehat{\otimes} \Gamma B
\end{tikzcd}$$

and counit given by $\Gamma(\varepsilon_B)$

Proof. Note that the functor N is an equivalence of categories and therefore gives rise to an adjoint equivalence with unit $\eta\colon \mathrm{Id} \longrightarrow \Gamma N$ and counit $\varepsilon\colon N\Gamma \longrightarrow \mathrm{Id}$ which are both natural isomorphisms. As for the previous proposition the coassociativity and the counitary property come from the fact that the shuffle map ∇ is a natural monoidal transformation. $\square$

Remark 5.1.3. We denote by $\psi_{X,Y}$ the composition of natural maps

$$\psi_{X,Y} = \eta^{-1}_{\Gamma X \widehat{\otimes} \Gamma Y} \circ \Gamma(\nabla_{\Gamma X,\Gamma Y}) \circ \Gamma(\varepsilon_X^{-1} \otimes \varepsilon_Y^{-1})$$

Since the composition $AW \circ \nabla$ is the identity, we deduce that ∇ is injective and therefore $\psi_{X,Y}$ is also injective for any X and Y in DG-K-**coAlg**. Moreover observe that

$$\psi_{B,B} = \eta^{-1}_{\Gamma B \widehat{\otimes} \Gamma B} \circ \Gamma(\nabla_{\Gamma B,\Gamma B}) \circ \Gamma(\varepsilon_B^{-1} \otimes \varepsilon_B^{-1})$$

in the definition of $\Delta_{\Gamma B}$.

5.1.2 Comonoidal properties of the adjunction counit and unit

Lemma 5.1.4. *The adjunction counit* $\varepsilon\colon N\Gamma \longrightarrow \mathrm{Id}$ *is a comonoidal transformation. The diagram*

$$\begin{tikzcd}
N\Gamma(X \otimes Y) \arrow[r, "N(\psi_{X,Y})"] \arrow[d, "\varepsilon_{X \otimes Y}"'] & N(\Gamma X \widehat{\otimes} \Gamma Y) \arrow[r, "AW_{\Gamma X,\Gamma Y}"] & N\Gamma X \otimes N\Gamma Y \arrow[d, "\varepsilon_X \otimes \varepsilon_Y"] \\
X \otimes Y \arrow[rr, equal] & & X \otimes Y
\end{tikzcd}$$

commutes for every X,Y *in DG-*K*-**coAlg**.*

Proof. The naturality of the inverse $\varepsilon^{-1}\colon \mathrm{Id} \longrightarrow N\Gamma$

$$\begin{tikzcd}
N(\Gamma X \widehat{\otimes} \Gamma Y) \arrow[r, "\varepsilon^{-1}_{N(\Gamma X \widehat{\otimes} \Gamma Y)}"] \arrow[d, "AW_{\Gamma X, \Gamma Y}"'] & N\Gamma N(\Gamma X \widehat{\otimes} \Gamma Y) \arrow[d, "N\Gamma(AW_{\Gamma X, \Gamma Y})"] \\
N\Gamma X \otimes N\Gamma Y \arrow[r, "\varepsilon^{-1}_{N\Gamma X \otimes N\Gamma Y}"'] & N\Gamma(N\Gamma X \otimes N\Gamma Y)
\end{tikzcd}$$

yields the identity

$$\varepsilon^{-1}_{N\Gamma X \otimes N\Gamma Y} \circ AW_{\Gamma X, \Gamma Y} = N\Gamma(AW_{\Gamma X, \Gamma Y}) \circ \varepsilon^{-1}_{N(\Gamma X \widehat{\otimes} \Gamma Y)}.$$

Since the composite $N\eta_A^{-1} \circ \varepsilon^{-1}_{NA}$ is the identity for every object A we deduce that $\varepsilon^{-1}_{N(\Gamma X \otimes \Gamma Y)}$ is inverse to $N(\eta^{-1}_{\Gamma X \widehat{\otimes} \Gamma Y})$ and the previous identity becomes

$$\varepsilon^{-1}_{N\Gamma X \otimes N\Gamma Y} \circ AW_{\Gamma X, \Gamma Y} \circ N(\eta^{-1}_{\Gamma X \widehat{\otimes} \Gamma Y}) = N\Gamma(AW_{\Gamma X, \Gamma Y}).$$

Using the fact that $AW \circ \nabla = Id$ we obtain

$$\begin{aligned}
&\varepsilon^{-1}_{N\Gamma X \otimes N\Gamma Y} \circ AW_{\Gamma X, \Gamma Y} \circ N(\eta^{-1}_{\Gamma X \widehat{\otimes} \Gamma Y}) \circ N\Gamma(\nabla_{\Gamma X, \Gamma Y}) \\
&\quad = \quad N\Gamma(AW_{\Gamma X, \Gamma Y}) \circ N\Gamma(\nabla_{\Gamma X, \Gamma Y}) \\
&\quad = \quad N\Gamma(AW_{\Gamma X, \Gamma Y} \circ \nabla_{\Gamma X, \Gamma Y}) \\
&\quad = \quad \mathrm{Id}.
\end{aligned}$$

Composing with $N\Gamma(\varepsilon_X^{-1} \otimes \varepsilon_Y^{-1})$ yields

$$\varepsilon^{-1}_{N\Gamma X \otimes N\Gamma Y} \circ AW_{\Gamma X, \Gamma Y} \circ N(\eta^{-1}_{\Gamma X \widehat{\otimes} \Gamma Y}) \circ N\Gamma(\nabla_{\Gamma X, \Gamma Y}) \circ N\Gamma(\varepsilon_X^{-1} \otimes \varepsilon_Y^{-1})$$
$$= N\Gamma(\varepsilon_X^{-1} \otimes \varepsilon_Y^{-1})$$

and then

$$\varepsilon^{-1}_{N\Gamma X \otimes N\Gamma Y} \circ AW_{\Gamma X, \Gamma Y} \circ N(\eta^{-1}_{\Gamma X \widehat{\otimes} \Gamma Y} \circ \Gamma(\nabla_{\Gamma X, \Gamma Y}) \circ \Gamma(\varepsilon_X^{-1} \otimes \varepsilon_Y^{-1}))$$
$$= N\Gamma(\varepsilon_X^{-1} \otimes \varepsilon_Y^{-1}).$$

Using the definition of the map ψ we can deduce that

$$\varepsilon^{-1}_{N\Gamma X \otimes N\Gamma Y} \circ AW_{\Gamma X, \Gamma Y} \circ N(\psi_{X,Y}) = N\Gamma(\varepsilon_X^{-1} \otimes \varepsilon_Y^{-1}).$$

The map ε is natural, therefore the diagram

$$\begin{tikzcd}
N\Gamma(X \otimes Y) \arrow[rr, "\varepsilon_{X\otimes Y}"] \arrow[d, "N\Gamma(\varepsilon_X^{-1}\otimes\varepsilon_Y^{-1})"'] & & X \otimes Y \arrow[d, "\varepsilon_X^{-1}\otimes\varepsilon_Y^{-1}"] \\
N\Gamma(N\Gamma X \otimes N\Gamma Y) \arrow[rr, "\varepsilon_{N\Gamma X\otimes N\Gamma Y}"'] & & N\Gamma X \otimes N\Gamma Y
\end{tikzcd}$$

commutes and we conclude that

$$\begin{aligned}
&\varepsilon_X \otimes \varepsilon_Y \circ AW_{\Gamma X,\Gamma Y} \circ N(\psi_{X,Y}) \\
&\qquad = \quad \varepsilon_X \otimes \varepsilon_Y \circ \varepsilon_{N\Gamma X\otimes N\Gamma Y} \circ N\Gamma(\varepsilon_X^{-1} \otimes \varepsilon_Y^{-1}) \\
&\qquad = \quad \varepsilon_{X\otimes Y}.
\end{aligned}$$

□

Proposition 5.1.5. *The functor*

$$\Gamma\colon \mathit{DG}\text{-}K\text{-}\boldsymbol{coAlg} \longrightarrow \mathit{S}\text{-}K\text{-}\boldsymbol{coAlg}$$

is full and faithful and respects coalgebra structures. Moreover, the composite endofunctor $N\Gamma$ is naturally isomorphic to the identity functor on the level of categories of comonoids.

Proof. The map $\Gamma_{X,Y}\colon \text{DG-}K\text{-}\mathbf{coAlg}(X,Y) \longrightarrow \text{S-}K\text{-}\mathbf{coAlg}(\Gamma X, \Gamma Y)$ is injective and surjective for any X and Y in DG-K-**coAlg** since the functor Γ is an equivalence on underlying monoidal categories. It remains to show that if $f = \Gamma(g)\colon \Gamma X \to \Gamma Y$ is a map of simplicial coalgebras then $g\colon X \to Y$ is a map of differential graded coalgebras. Suppose that $f = \Gamma(g)$ is comultiplicative. That means that in the diagram

$$\begin{tikzcd}
\Gamma X \arrow[r, "\Gamma(\Delta_X)"] \arrow[d, "\Gamma(g)"] & \Gamma(X \otimes X) \arrow[r, "\psi_{X,X}"] \arrow[d, "\Gamma(g\otimes g)"] & \Gamma X \widehat{\otimes} \Gamma X \arrow[d, "\Gamma(g)\widehat{\otimes}\Gamma(g)"] \\
\Gamma Y \arrow[r, "\Gamma(\Delta_Y)"'] & \Gamma(Y \otimes Y) \arrow[r, "\psi_{Y,Y}"'] & \Gamma Y \widehat{\otimes} \Gamma Y
\end{tikzcd}$$

the composite square commutes. Since the transformation ψ is natural, the right square commutes. Consequently, using the fact that the map $\psi_{X,X}$ is injective we deduce that the left square commutes as well. This implies that $\Gamma\big((g \otimes g) \circ \Delta_X\big) = \Gamma(\Delta_Y \circ \Gamma g)$. Therefore, using the faithfulness of Γ, we conclude that $(g \otimes g) \circ \Delta_X = \Delta_Y \circ g$, that is to say g is comultiplicative.

The counit $\varepsilon\colon N\Gamma \longrightarrow \mathrm{Id}$ is comonoidal by the previous lemma. So ε is comultiplicative and therefore it is a natural isomorphism on the level of comonoids. □

The unit $\eta\colon \mathrm{Id} \longrightarrow \Gamma N$ does not have good comonoidal properties. More precisely, there are objects X and Y in the category S-K-**coAlg** such that the diagram

$$\begin{tikzcd}
X\widehat{\otimes}Y \arrow[rr, equal] \arrow[d, "\eta_{X\widehat{\otimes}Y}"'] & & X\widehat{\otimes}Y \arrow[d, "\eta_X\widehat{\otimes}\eta_Y"] \\
\Gamma N(X\widehat{\otimes}Y) \arrow[r, "\Gamma(AW_{X,Y})"'] & \Gamma(NX\otimes NY) \arrow[r, "\psi_{NX,NY}"'] & \Gamma NX\widehat{\otimes}\Gamma NY
\end{tikzcd}$$

does not commute. Indeed consider for example $X = Y = \Gamma(\mathbf{Z}[1])$ as in [SS03, Remark 2.14].
Since N is left inverse to Γ by the previous proposition, one has

$$NX = NY = N\Gamma(\mathbf{Z}[1]) \cong \mathbf{Z}[1]$$

and

$$NX \otimes NY \cong \mathbf{Z}[1] \otimes \mathbf{Z}[1] = \mathbf{Z}[2]$$

Therefore the lower composite map in the previous diagram vanishes in degree 1 since

$$[\Gamma(NX \otimes NY)]_1 = [\Gamma(\mathbf{Z}[2])]_1 = 0.$$

But in degree 1, the right map $\eta_X\widehat{\otimes}\eta_Y$ is an isomorphism between free abelian groups of rank one since

$$[\Gamma(NY)]_1 \cong [Y]_1 \cong \mathbf{Z} \cong [X]_1 \cong [\Gamma(NX)]_1.$$

5.2 Quillen functors for coalgebras

We construct right adjoint functors for the coalgebra-valued functors defined in the preceding section. It turns out that the resulting pairs of functors are Quillen pairs. Then we investigate the question whether these Quillen pairs are Quillen equivalences. As already mentioned, the opposite category of differential graded vector spaces is not cofibrantly generated. Hence we can not apply the transfer result and the main result of Schwede and Shipley in order to derive a Quillen equivalence for coalgebras. Thus we consider a direct approach via a criterion given by Hovey. We observe that several problems, essentially due to the functor S from vector spaces to coalgebras, appear.

5.2.1 Two Quillen pairs

Let us consider the pair (N, Γ) and the induced normalization functor from the category of simplicial coalgebras S-K-**coAlg** to the category of differential graded coalgebras DG-K-**coAlg**, denoted here by $\widetilde{N}$. As seen above, the categories of vector spaces and that of coalgebras are related by adjoint pairs. We sum up these various adjunctions in the diagram below:

$$\begin{tikzcd}
(\text{S-}K\text{-}\mathbf{Vct}, \widehat{\otimes}, I(K)) \arrow[r, shift left, "N"] \arrow[d, shift right, "S_s"] & (\text{DG-}K\text{-}\mathbf{Vct}, \otimes, K[0]) \arrow[l, shift left, "\Gamma"] \arrow[d, shift right, "S_d"] \\
\text{S-}K\text{-}\mathbf{coAlg} \arrow[u, shift right, "U_s"] \arrow[r, "\widetilde{N}"] & \text{DG-}K\text{-}\mathbf{coAlg}. \arrow[u, shift right, "U_d"]
\end{tikzcd}$$

Then the following result constructs a Quillen right adjoint functor for the functor $\widetilde{N}$.

Proposition 5.2.1. *In the above situation the functor $\widetilde{N}$ has a right adjoint R^{com}. Moreover the adjoint pair $(\widetilde{N}, R^{\text{com}})$ is a Quillen pair.*

Proof. First let V be a differential graded vector space and $S_d(V)$ its differential graded cofree coalgebra. We set

$$R^{\text{com}}(S_d(V)) = S_s\Gamma(V).$$

Indeed the various adjoint pairs (U_s, S_s), (N, Γ), (U_d, S_d) and the identity $NU_s = U_d\widetilde{N}$ respectively, yield the following bijections

$$\begin{aligned}
\text{S-}K\text{-}\mathbf{coAlg}\Big(X, R^{\text{com}}S_dV\Big) &= \text{S-}K\text{-}\mathbf{coAlg}\Big(X, S_s\Gamma V\Big) \\
&\cong \text{S-}K\text{-}\mathbf{Vct}\Big(U_sX, \Gamma V\Big) \\
&\cong \text{DG-}K\text{-}\mathbf{Vct}\Big(NU_sX, V\Big) \\
&\cong \text{DG-}K\text{-}\mathbf{Vct}\Big(U_d\widetilde{N}X, V\Big) \\
&\cong \text{S-}K\text{-}\mathbf{coAlg}\Big(\widetilde{N}X, S_dV\Big).
\end{aligned}$$

This means that the functor R^{com} is right adjoint to $\widetilde{N}$ for cofree coalgebras. Now we notice that the adjunction (U_d, S_d) defines a monad S_dU_d over the category DG-K-**coAlg**. Thus if C is a differential graded coalgebra its cosimplicial resolution starts as follow:

$$\begin{tikzcd}
C \arrow[r] & S_dU_dC \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & S_dU_dS_dU_dC \dots \arrow[l]
\end{tikzcd}$$

Since the functor R^{com} should be a right adjoint it has to preserve limits. Therefore defining $R^{\text{com}}(C)$ as the equalizer of the maps $R^{\text{com}}(d^0)$ and $R^{\text{com}}(d^1)$ yields the needed right adjoint.

Finally we observe that the cofibrations and acyclic cofibrations in S-K-**Vct** and DG-K-**Vct** match with those of their respective categories of comonoids S-K-**coAlg** and DG-K-**coAlg**. Since the functor N is a left Quillen functor the identity $NU_s = U_d\widetilde{N}$ ensures that the functor $\widetilde{N}$ is a left Quillen functor. □

We could also consider the pair (Γ, N) and the induced functor $\widetilde{\Gamma}$ from the category of differential graded coalgebras DG-K-**coAlg** to the category of simplicial coalgebras S-K-**coAlg**. Gathering again the various adjunctions in the diagram below

$$\begin{tikzcd}
(\text{DG-}K\text{-}\mathbf{Vct}, \otimes, K[0]) \arrow[r, shift left, "\Gamma"] \arrow[d, shift left, "S_d"] & (\text{S-}K\text{-}\mathbf{Vct}, \widehat{\otimes}, I(K)) \arrow[l, shift left, "N"] \arrow[d, shift left, "S_s"] \\
\text{DG-}K\text{-}\mathbf{coAlg} \arrow[u, shift left, "U_d"] \arrow[r, "\widetilde{\Gamma}"] & \text{S-}K\text{-}\mathbf{coAlg} \arrow[u, shift left, "U_s"]
\end{tikzcd}$$

one obtains by a similar consideration a Quillen right adjoint functor for the functor $\widetilde{\Gamma}$. For every simplicial vector space V it suffices to set

$$R^{\mathrm{com}}(S_s(V)) = S_d N(V)$$

and then to consider the cosimplicial resolution arising from the monad $S_s U_s$ over the category S-K-**coAlg**.

We deduce the following result by [Hov99, Lemma 1.3.10].

Corollary 5.2.2. *The homotopy categories of differential graded coalgebras and simplicial coalgebras are adjoint.*

5.2.2 Hovey's criterion for coalgebras

Following Hovey, the Quillen adjunction $(\widetilde{N}, R^{\mathrm{com}})$ would be a Quillen equivalence if $\widetilde{N}$ reflected weak equivalences between cofibrant objects and if for every fibrant differential graded coalgebra Y, the map $\widetilde{N}\big(R^{\mathrm{com}}Y\big)^{\mathrm{cof}} \to Y$ was a weak equivalence in DG-K-**coAlg**. Notice that since every object in S-K-**coAlg** is cofibrant, the cofibrant replacement $\big(R^{\mathrm{com}}Y\big)^{\mathrm{cof}}$ can be taken to be $R^{\mathrm{com}}Y$. Hence we are reduced to studying the map $\widetilde{N}R^{\mathrm{com}}Y \to Y$ for fibrant coalgebras Y.

Definition 5.2.3. A fibrant differential graded coalgebra Y *satisfies the Hovey criterion* if the above map is a weak equivalence in DG-K-**coAlg**, that is

$$H_*(\widetilde{N}R^{\mathrm{com}}Y) \cong H_*(Y).$$

We investigate the Hovey's criterion on some differential graded coalgebras.

Proposition 5.2.4. *The following objects in DG-K-**coAlg**:*

1. *the terminal object $K[0]$,*

2. *the cofree coalgebra $S_d(\mathbb{S}^0)$ on the 0-sphere $\mathbb{S}^0$,*

3. the cofree coalgebra $S_d(V)$ on every acyclic vector space V

are fibrant and satisfy the Hovey's criterion.

Proof. As remarked earlier, in the category DG-K-**coAlg** the one dimensional coalgebra K concentrated in degree zero is the terminal object. This terminal object, denoted by $K[0]$, is fibrant in the model structure of DG-K-**coAlg**. Since the functor R^{com} is a right adjoint, it carries the terminal object $K[0]$ to the terminal object of S-K-**coAlg** which is $I(K)$, the simplicial coalgebra with the one dimensional coalgebra K in each degree. Therefore we obtain

$$H_*\big(\widetilde{N}R^{\mathrm{com}}(K[0])\big) \cong H_*\big(\widetilde{N}I(K)\big) \cong H_*\big(K[0]\big),$$

hence the Hovey criterion is satisfied by $K[0]$.

Recall that every object is fibrant in the model structure on the category DG-K-**Vct**. Since the functor S_d preserves fibrations, it follows that every cofree coalgebra is fibrant. In particular $S_d(\mathbb{S}^0)$ is fibrant. Now the definition of the functor R^{com} on cofree coalgebras and the Dold-Kan correspondence yield

$$\widetilde{N}R^{\mathrm{com}}S_d\big(\mathbb{S}^0\big) = \widetilde{N}S_s\Gamma\big(\mathbb{S}^0\big) \cong \widetilde{N}S_s\big(I(K)\big).$$

Since the face maps in the simplicial coalgebra $S_s\big(I(K)\big)$ are all identities on $S(K)$, their alternating sums are either 0 or $id_{S(K)}$ and the associated complex is:

$$\cdots \longleftarrow 0 \longleftarrow S(K) \xleftarrow{0} S(K) \xleftarrow{id} S(K) \xleftarrow{0} S(K) \xleftarrow{id} \cdots$$

Therefore we obtain

$$H_*\Big(\widetilde{N}S_s\big(I(K)\big)\Big) = \begin{cases} 0 & \text{if } * \neq 0 \\ S(K) & \text{if } * = 0 \end{cases}$$

Finally the Hovey criterion for the differential graded coalgebra $S_d(\mathbb{S}^0)$ follows from the fact that $S_d(\mathbb{S}^0)$ is concentrated in degree zero and from the identification of $S_d(\mathbb{S}^0)$ with $S(K)$.

Now let V be an acyclic differential graded vector space. In other words V is weakly equivalent to 0, the zero object of the category DG-K-**Vct**. Since the functors S_d, Γ and S_s preserve weak equivalences, we deduce on the one hand that $S_d(V)$ is weakly equivalent to $S_d(0) = K[0]$ and on the other hand

$$\begin{aligned} \widetilde{N}R^{\mathrm{com}}S_d(V) &= \widetilde{N}S_s\Gamma(V) \\ &\simeq \widetilde{N}S_s\Gamma(0) \\ &\simeq \widetilde{N}S_s(0) \\ &= \widetilde{N}I(K) \\ &= K[0] \end{aligned}$$

Hence we obtain the required result for acyclic differential graded vector spaces. Notice in particular that the Hovey criterion holds for the cofree coalgebra on every n-disk $\mathbb{D}^n$. $\square$

We were not able to prove that arbitary fibrant differential graded coalgebras satisfy the Hovey criterion. For instance consider the cofree coalgebra $S_d(\mathbb{S}^n)$ on a n-sphere $\mathbb{S}^n$ with $n \geq 1$. Then with help of the isomorphism

$$\big(S_d(\mathbb{S}^n)\big)^* \cong T_{K[0]}\big(\mathbb{S}^{-n}\big) = \mathbb{S}^0 \oplus \mathbb{S}^{-n} \oplus \mathbb{S}^{-2n} \oplus \mathbb{S}^{-3n} \oplus \cdots$$

provided in Example 3.3.7 we know the homology of the differential graded coalgebra $S_d(\mathbb{S}^n)$. However, computing the homology of the differential graded coalgebra

$$\widetilde{N} R^{\mathrm{com}} S_d(\mathbb{S}^n) = \widetilde{N} S_s \Gamma(\mathbb{S}^n)$$

leads to some difficulties. Let us consider for instance the case $n = 1$. In Example 4.2.2 we have already studied the simplicial vector space $\Gamma(\mathbb{S}^1)$. It follows that the complex associated to the simplicial coalgebra $S_s(\Gamma(\mathbb{S}^1))$ is:

$$0 \longleftarrow S(0) \xleftarrow{\partial_1} S(K) \xleftarrow{\partial_2} S(K^{\oplus 2}) \xleftarrow{\partial_3} S(K^{\oplus 3}) \longleftarrow \cdots$$

with differentials

$$\partial_n = \sum_{i=0}^{n} (-1)^i S(d_i)$$

arising from the face maps of the simplicial vector space $\Gamma(\mathbb{S}^1)$. But we have noted in Remark 2.5.7 that the functor $S\colon K\text{-}\mathbf{Vct} \to K\text{-}\mathbf{coAlg}$ applied to vector spaces with dimension larger than one is hard to express. Moreover the functor S involves not only the tensor algebra functor but the profinite completion functor. Hence it does not have good additivity properties that could simplifly the study of the injectivity or the surjectivity of the above differentials.

5.2.3 On non-counital coalgebras

We end this chapter with an observation on non-counital coassociative coalgebras.

Definition 5.2.5. Let V be a positively differential graded K-vector space. The *non-counital tensor coalgebra* $T'(V)$ over V is the vector space

$$T'(V) = \bigoplus_{n \geq 1} V^{\otimes n}$$

equipped with the coassociative comultiplication

$$\Delta_{T'(V)}(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n)$$

given by deconcatenations.

The result below asserts that the tensor coalgebra $T'(V)$ is couniversal and hence, defines a cofree functor that is right adjoint to the obvious forgetful functor.

Proposition 5.2.6. *Let (C, Δ_C) be a non-counital positively graded differential coalgebra. Consider $u\colon C \to V$, a map of differential graded vector spaces, and $\pi\colon T'(V) \to V$ the natural projection. Then, there exists a unique map of coalgebras $\theta\colon C \to T'(V)$ such that the diagram*

$$\begin{tikzcd} V & C \arrow[l, "u"'] \arrow[dl, dotted, "\exists!\,\theta"] \\ T'(V) \arrow[u, "\pi"] & \end{tikzcd}$$

commutes.

Proof. For $i \geq 1$, let us consider the maps

$$\Delta^{(i-1)}\colon C \to C \otimes C \to \cdots \to C^{\otimes(i)}$$

given by the i-fold iterations of the coassociative comultiplication Δ_C. For $n \geq 1$, we need to define maps

$$\theta_n\colon C_n \longrightarrow T'_n(V) = \bigoplus_{d_1+\ldots+d_t=n} V_{d_1} \otimes \ldots \otimes V_{d_t}.$$

Since the indexes d_i are positive, we notice that the direct sum is finite and therefore equivalent to a finite product. In this way, for $i \leq n$, the composite maps

$$u^{\otimes i} \circ \Delta^{(i-1)}\colon C \longrightarrow C^{\otimes i} \longrightarrow V^{\otimes i}$$

give rise to the desired maps θ_n by the universality of products. Next, by an induction argument, one obtains the identity $\big(\Delta^{(i-1)} \otimes \Delta^{(j-1)}\big) \circ \Delta_C = \Delta^{(i+j-1)}$ that guarantees that the map $\theta = (\theta_n)_{n\geq 1}$ is in fact a map of differential graded coalgebras. $\square$

The positive gradings together with the definition of the monoidal product in differential graded vector spaces force the couniversality of the tensor coalgebra $T'(V)$. The main advantage of this construction is the isomorphism $H_*\big(T'(V)\big) \cong T'\big(H_*(V)\big)$ given by the Künneth formula.

However, a similar construction does not hold for simplicial non-counital coalgebras, since in this case, the monoidal product is defined degreewise.

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Zusammenfassung

Die Dold-Kan-Korrespondenz etabliert eine Äquivalenz zwischen der Kategorie der simplizialen Vektorräume und der Kategorie der nicht-negativ graduierten differentiellen Vektorräume über einem Körper K. Beide Kategorien besitzen geschlossene monoidale Modellstrukturen. Diese Arbeit untersucht in welchem Ausmaß die an der Dold-Kan-Korrespondenz beteiligten Funktoren die Modellstrukturen der assoziierten Kategorien von Komonoiden erhalten.

Wir beweisen, dass der Normalisierungsfunktor N Teil einer Quillen-Adjunktion zwischen der Kategorie simplizialer Koalgebren und der Kategorie nicht-negativ graduierter differentieller Koalgebren ist. Weiter zeigen wir ein analoges Resultat für den zu N inversen Funktor Γ, welcher uns eine weitere Adjunktion liefert.

Wir verwenden zwei Methoden um zu überprüfen, ob diese Quillen-Adjunktionen sogar Quillen-Äquivalenzen sind: zum Einen die kategorielle Dualisierung nach Schwede und Shipley, zum Anderen ein Kriterium von Hovey. Im ersten Fall stellen wir fest, dass die opponierte Kategorie der nicht-negativ graduierten differentiellen Vektorräume keine monoidale Modellstruktur zulässt. Dies begründet sich darin, dass das monoidale Produkt in dieser Kategorie nicht geschlossen ist. Auch hat diese Kategorie nicht genug kokleine Objekte um Quillens kokleine-Objekte-Argument anzuwenden. Damit schließen wir, dass die Modellstruktur auf der opponierten Kategorie der nicht-negativ graduierten differentiellen Vektorräume nicht durch das Transfer-Theorem wiedergegeben werden kann. Im zweiten Fall betrachten wir den direkten Zugang nach Hovey. Wir können Hoveys Kriterium nicht für beliebige fasernde Koalgebren nachweisen: Betrachten wir den Komplex, der zu der kofreien Koalgebra auf der simplizialen n-Sphäre für $n > 0$ assoziiert wird. Schon die Berechnung der Homologie dieses Komplexes führt zu diversen Problemen, da der kofreie Koalgebra-Funktor unzugänglich ist und keine guten homologischen Eigenschaften hat. Allerdings finden wir Klassen differentieller graduierter Koalgebren, die Hoveys Kriterium erfüllen. Diese umfassen unter anderem die kofreien Koalgebren auf der Null-Sphäre und azyklische Vektorräume.

Lebenslauf

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