On a differential equation with state-dependent delay: A global center-unstable manifold bordered by a periodic orbit

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Contents

Preface

This dissertation has its origin in an important link between mathematics and other branches of sciences, namely, in mathematical modeling. The main idea of mathematical modeling is a simplified mathematical representation and subsequent analysis of systems, processes or phenomenons with the aim to describe the behavior of these systems, enhance the understanding of the underlying mechanisms or even enable predictions and use them for making decisions. There are various forms of models, involving many different areas of mathematics such as probability and game theory, dynamical systems, differential equations, optimization or numerics, to name but a few fields of mathematics. But mathematical modeling also shaped the development of different mathematical branches. So it was the abundance of applications generated by mathematical models in various sciences which mostly impelled a rapid development of the general theory of differential equations with delayed argument in the second half of the last century. Today, large parts of the theory of delay differential equations, at least in the case of constant delays, are as well understood as the theory of ordinary differential equations (henceforth abbreviated by ODE, respectively ODEs). We find delay differential equations in numerous models across many branches of sciences such as physics, mechanics, engineering, biology, medicine, economics or control theory. Many of these models are discussed in the monographs [**12, 13**] of Kolomanovskii and Myshkis and in the work [**11**] of Hartung et al. in detail.

The motivation for this thesis is a slight modification of a mathematical model from economics to describe short-term fluctuations of exchange rates in a floating exchange rate regime. The original model was, as far as known by the author, constituted by Erdélyi in his diploma thesis [**7**] and is represented by the one-parameter family

$$\dot{x}(t) = a\left[x(t) - x(t-1)\right] - |x(t)|\,x(t) \tag{$\star$}$$

of scalar differential equations with constant delay one and $a > 0$. The main ideas behind this model are explained in detail in the preprint [**4**] of Brunovský et al. and they are briefly recapitulated below.

The Original Model. Let us denote by $S = S(t)$ the exchange rate in direct quotation, that is, the price of a unit of a selected foreign currency in terms of units of the reference domestic one, as a function of time t. Then the principal assumption of the model proposed by Erdélyi in [**7**] is that there is a kind of 'natural' exchange rate S^*, specified by macroeconomic fundamentals, which is constant within the considered time range. This assumption should be consistent with empirical literature on the behavior of exchange rates so far as that literature confirms that if, for short times, the movement of exchange rate depends on any macroeconomic fundamental, so it does only insignificantly.

The second important hypothesis of the model is the assumption that economic agents anticipate the rate S^* in some way but do not know its precise value. Consider now the scaled deviation $y := (S - S^*)/S^*$ of the actually exchange rate S from the 'natural' exchange rate S^*. As agents intend to draw profit from their trading, it is supposed that they attempt to anticipate the changes of the exchange rate in the nearest future by taking into account the following two aspects:

(i) the development of the exchange rate in the immediate past, which is represented by the difference $y(t) - y(t - h)$ under the assumption that agents observe the exchange rate at equidistant time instants of short distance $h > 0$, and

(ii) a perception of the current position of the exchange rate compared to S^*, which is factored in by $y(t)$.

Both aspects above should influence the expectation of agents in different way as described in Brunovský et al. [**4**].

On the one hand, in case of a rising exchange rate in the immediate past, that is, when $y(t) - y(t - h) > 0$, the agents should expect that this trend will persist in the near future. Thus a rising exchange rate in the immediate past should result in an increasing demand for foreign currency and so in an increase of the exchange rate. Accordingly, in the situation of a falling exchange rate in the past, that is, when $y(t) - y(t - h) < 0$, a symmetric reaction is assumed. The impact of this effect on the evolution of the exchange rate is modeled linearly with respect to its source, $y(t) - y(t - h)$, and, therefore, assumed to be equal to $b_1\,(y(t) - y(t - h))$ for some constant $b_1 > 0$.

On the other hand, the more the actual exchange rate should increase and, thus, differ from the 'natural' exchange rate, the more agents should await a turning back of the trend. Consequently, a higher actual rate should lead to an increased supply and, hence, to a fall of the foreign currency reflected by a decrease of the exchange rate. A symmetrical behavior is supposed in the case if the actual exchange rate is lower than the 'natural' one. Again, the impact of this model on the evolution of the exchange rate is assumed to be linear with respect to its source, resulting in the term $-B_2\, y(t)$ with $B_2 > 0$.

With respect to the influence on the exchange rate, the two expectation mechanisms described above differ essentially. While an increase or decrease of the exchange rate in the preceding rate is well-known and has the same value for each economic agent, the value S^* is subject to the individual anticipation of an agent. However, the agents should exhibit a 'collective wisdom' to the effect that the number of agents discerning the deviation of the actual rate from S^* should increase with the magnitude of the deviation. The hypothesis of the model here is once more that in the first approximation this effect is linear, that is, $B_2 = B_2(y) = b_2\,|y|$ for a constant $b_2 > 0$.

By assuming, finally, a uniform distribution of the agents in time and therefore an acting of the corresponding fraction of agents in a short time interval $\triangle t$, the change of the exchange rate in the time interval $\triangle t$ then reads

$$y(t + \triangle t) = y(t) + b_1 \big[y(t) - y(t - h)\big] \triangle t - b_2\, |y(t)|\, y(t) \triangle t + o(\triangle t).$$

Carrying out the limit process $\triangle t \to 0$, we get the scalar differential equation

$$\dot{y}(t) = b_1 \big[y(t) - y(t - h)\big] - b_2\, |y(t)|\, y(t)$$

with reals $b_1, b_2 > 0$ and a constant delay $h > 0$. Consequently, the normalization

$$x(t) := b_2 \cdot h \cdot y(th)$$

results in the one-parameter family ($\star$) of delay differential equations with the sensitivity parameter $a > 0$, which is more convenient for mathematical analysis.

Dynamical Aspects of the Original Model. From an economic point of view, due to the simplicity of the above model, one surely may not expect an adequate description of short-term fluctuations of exchange rates. However, the solutions of the delay differential equation ($\star$) possess interesting dynamics from the mathematical standpoint. So, the numerical experiments in the initial work [**7**] of Erdélyi indicate the following behavior of solutions of Equation ($\star$).

(i) For parameters $0 < a \leq 1$, the only equilibrium $x(t) = 0$, $t \in \mathbb{R}$, of Equation ($\star$) is globally asymptotically stable, whereas for $a > 1$ it is unstable.

(ii) For $a > 1$ there is a globally stable periodic orbit. The period of this orbit tends to infinity as $a \to 1$.

In Brunovský et al. [**3**] a part of these numerical observations is rigorously established. The authors of [**3**] proved asymptotic stability of the trivial solution for parameters $0 < a < 1$, instability for $a > 1$, and the existence of a periodic solution in the situation $a > 1$. Thereby, it is worth to mention that these periodic solutions are not born in a Hopf bifurcation as the spectrum of the linearization at the trivial solution has the only imaginary eigenvalue $0 \in \mathbb{C}$, whose multiplicity is two at the critical parameter $a = 1$ and one otherwise. Rather, for a parameter above the critical one, a 2-dimensional global center-unstable manifold W connects the trivial solution to a periodic orbit, which turns out to coincide with the boundary $\overline{W} \setminus W$. An important step towards this conclusion is the study of so-called slowly oscillating solutions characterized by the property that for any two zeros $z < z'$ there holds $z < z' + 1$. The periodic orbits obtained for $a > 1$ are slowly oscillating and their minimal periods are given by three consecutive zeros.

Walther analyzes the delay differential equation ($\star$) for parameters $a > 0$ in two further papers [**27, 28**]. In [**27**] he derives the asymptotic shape of the slowly oscillating periodic orbits described above for $a \to \infty$. Additionally, he notes that, starting with a slowly oscillating periodic solution for $a > 1$, one easily obtains other periodic solutions for Equation ($\star$) for other parameter values $a > 1$ and without the property of slow oscillations.

In Walther [**28**], the bifurcation occurring at the critical parameter $a = 1$ is studied. The main result establishes that for every $\varepsilon > 0$ there exists a parameter $1 - \varepsilon < a < 1 + \varepsilon$ such that Equation ($\star$) has a slowly oscillating periodic solution $p_\varepsilon : \mathbb{R} \to (-a\,\varepsilon, a\,\varepsilon)$, whose minimal period $\tau_\varepsilon > 0$ is given by three consecutive zeros and converges to infinity as $\varepsilon \to 0$.

A Modification of the Original Model and the Main Issue of this Thesis. A possible modification of the model is to replace the constant delay by a state-dependent one. Then the corresponding family of differential equations is given by

$$(\star\star) \qquad \dot{x}(t) = a\big[x(t) - x(t - r)\big] - |x(t)|\, x(t)$$

with the parameter $a > 0$ and the state-dependent delay $r = r(x(t)) > 0$. It seems to be more realistic and hence to be an enhancement of the original model that the agents do not observe the exchange rate at equidistant time instants, but rather at instants depending on the difference between the actual and the 'natural' exchange rate represented by $x(t)$. The reason for such an assumption is that the more the actual exchange rate should differ from the 'natural' one, the more precisely the agents should pursue the development of the exchange rate and thus the shorter the time instants where agents observe the exchange rate should be. So, in view of the modeling aspects, the usage of a state-dependent delay with a delay function $r : \mathbb{R} \longrightarrow \mathbb{R}$, which is positive, takes a maximum value $r_0 > 0$ at $x = 0$, satisfies $r(x) = r(-x)$ for all $x \in \mathbb{R}$ and decreases monotonically to 0 as $|x| \to 0$, appears to be more appropriate than a constant delay to describe short-term fluctuations of the exchange rate in a free floating regime.

Apart from the modeling feature, the replacement of the constant delay by a state-dependent delay is also mathematically interesting itself. The adjustment of the delay, namely, immediately raises the question how the dynamics of the differential equation ($\star\star$) with state-dependent delay change in contrast to the differential equation ($\star$) with constant delay. The principal goal of the present work is to address the question whether periodic orbits still exist for parameters $a > 1$. We will establish the existence under mild conditions on the delay function of similar type as considered above in connection with the modeling aspect. Moreover, under the imposed conditions on r, we will generalize the main result in Brunovský et al. [**3**] to the case of a state-dependent delay and prove that in the situation $a > 1$, a center-unstable manifold connects the trivial solution with a periodic orbit of Equation ($\star\star$).

Outline

This thesis consists of two more or less self-contained parts. The first one is intended to give a brief exposition of some basic aspects of the general theory for differential equations with state-dependent delay. Here, we review some of the recent results but also establish the existence of so-called local C^1-smooth center-unstable manifolds in a general setting. In the second part, we study the model equation ($\star\star$) under certain assumptions on the delay function and prove our main result on the existence of periodic solutions for parameter $a > 1$. Below, the content of the particular chapters is briefly sketched.

The first chapter begins with a short recapitulation of some basic facts on delay differential equations and their representation in the more abstract form of functional differential equations. This discussion leads to a recent theorem on existence of a smooth semiflow for a class of functional differential equations. In particular, this theorem seems to be often applicable in cases where the corresponding functional differential equation represents a delay differential equation with state-dependent delay. Afterwards, we summarize without proofs the relevant material on linearization at a stationary point and close the first chapter with a proof of the so-called principle of linearized instability.

In the second chapter, we formulate and prove our main result of the first part of this thesis, namely, the existence of local center-unstable manifolds for the semiflow considered in Chapter 1. Additionally, we establish the C^1-smoothness of these manifolds.

Chapter 3 is the beginning of our analysis of the model equation $(\star\star)$ with parameter $a > 0$ and a state-dependent delay $r = r(x(t)) > 0$. Making certain standing assumptions on the delay functions under consideration, in a first step we study some basic properties such as existence and continuous dependence on initial values of solutions of Equation $(\star\star)$. As a simple consequence of these consideration, we construct a suitable compact state space depending on the involved parameter $a > 0$, such that solutions of the model equation constitute a continuous semiflow. Next, the differences of solutions are examined. As a result, we will be able to introduce a discrete Lyapunov functional for counting zeros of solutions. This counting tool turns out to be very useful for our analysis of Equation $(\star\star)$. At the end of the third chapter we study the so-called slowly oscillating solutions and prove that all globally defined slowly oscillating solutions form a global attractor of a semiflow.

The fourth chapter is devoted to some aspects of local dynamics of the model equation at the trivial solution. We consider the linearization and its spectrum, and we show that the trivial solution of Equation $(\star\star)$ is unstable for $a > 1$. Moreover, in this case we derive an asymptotic ODE describing the induced flow of the model equation on local center-unstable manifolds. From this ODE we infer a repellent behavior of the trivial stationary point in local center-unstable manifolds.

In the final chapter our main result is stated and proved. Thereto, we choose a sufficiently small neighborhood of the trivial solution in a local center-unstable manifold for the delay differential equation $(\star\star)$ and extend it to a global center-unstable manifold W. Using the discrete Lyapunov functional introduced in Chapter 3, we conclude that W and its closure $\overline{W}$ are both invariant for the semiflow generated by solutions of Equations $(\star\star)$ and contain, apart from the trivial stationary point, only slowly oscillating solutions for the model equation. Afterwards, we proceed with the study of the zero sets of solutions in $\overline{W}$ by applying an auxiliary projection. This finally leads us to our main result and its proof.

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Part 1

Some Theoretical Aspects of Differential Equations with State-Dependent Delay

CHAPTER 1

A Smooth Semiflow

1. Introduction

The main goal of this chapter is to provide a detailed exposition of the framework developed in Walther [**24, 25, 26**] and Hartung et al. [**11**] on the existence of a continuous semiflows for a class of functional differential equations, which is designed for applications to differential equations with state-dependent delay. In addition, we establish the so-called *principle of linearized instability* for stationary points of the considered class of functional equations.

The organization of the present chapter is as follows. In the next section we begin with the introduction of a rather general class of equations, namely the so-called *retarded functional differential equations* (abbreviated by RFDE, respectively RFDEs). By discussing some of the primary properties, we briefly indicate that under certain conditions, the solutions for the associated initial value problems generate smooth semiflows. Afterwards, we point out that these results may also be used to obtain continuous semiflows in case of differential equations with constant delays. On the other hand, it can easily be seen that the basic approach of the theory of RFDEs does generally not apply in presence of a state-dependent delay.

In Section 3 we motivate a slightly modified type of functional differential equations, and present, without proof, a major theorem on the existence of smooth semiflows with continuously differentiable time-t-maps. Then a first example indicates the applicability of these results to differential equations with state-dependent delay.

Finally, in the last section of this chapter we discuss in some detail the concept of linearization at stationary points, touch some parts of the associated spectral theory, and prove the above-mentioned *principle of linearized instability* for the motivated type of functional differential equations.

2. Retarded Functional Differential Equations

Following the classical lines in Hale and Verduyn Lunel [**10**], below we briefly sketch some basic aspects of the theory of RFDEs which has been developed since the beginning of the second half of the last century. Thereby we indicate the applicability of these results to delay differential equations (abbreviated by DDE, respectively DDEs), and point out the difficulties arising in presence of a non time-invariant delay as presented, for instance, in Walther [**24**].

Basic Terminology and Results. Let $h > 0$ and $n \in \mathbb{N}$. For abbreviation, let us denote by C the set of all continuous functions from the interval $[-h, 0]$ into $\mathbb{R}^n$, equipped with the norm $\|\varphi\|_C := \max_{s\in[-h,0]} \|\varphi(s)\|_{\mathbb{R}^n}$ of uniform convergence. Analogously, we write C^1 for the Banach space of all continuously differentiable functions $\varphi : [-h, 0] \longrightarrow \mathbb{R}^n$, provided with the norm $\|\varphi\|_{C^1} := \|\varphi\|_C + \|\varphi'\|_C$.

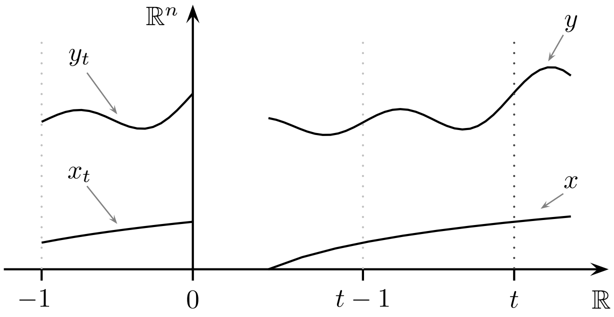

FIGURE 1.1. Segments of functions

For a given function $x : I \longrightarrow \mathbb{R}^n$ defined on some interval $I \subseteq \mathbb{R}$, and $t \in \mathbb{R}$ with $[t-h,t] \subset I$, the **segment** x_t **of** x **at** t is defined by the relation $x_t(\vartheta) := x(t+\vartheta)$, $\vartheta \in [-h,0]$; that is, by x_t we restrict the function x to $[t-h,t]$ and shift it back to $[-h,0]$ as illustrated in Figure 1.1. In particular, if the function x is continuous, then clearly $x_t \in C$.

Suppose that D is an open subset of $\mathbb{R} \times C$, and let a function $f : D \longrightarrow \mathbb{R}^n$ be given. Then we call an equation of the form

$$\dot{x}(t) = f(t, x_t)\,, \tag{1.1}$$

where " $\dot{}$ " means the (right-hand side) derivative of x with respect to t, a **retarded functional differential equation** on D. Equation (1.1) is a rather general type of differential equation classified by the right-hand side. For example, we call an RFDE (1.1) **linear**, if the function f can be represented as $f(t,\varphi) = L(t,\varphi) + q(t)$, $(t,\varphi) \in D$, with mappings L and q, where L is linear in its second argument. Analogously, an RFDE (1.1) is **autonomous**, if the function f is independent of the time variable t; that means, there is $\tilde{f}$ such that $f(t,\varphi) = \tilde{f}(\varphi)$ for all $(t,\varphi) \in D$.

The concept of solutions for retarded functional differential equations is the following. A function x is a **solution** of the RFDE (1.1) on the interval $[t_0 - h, t_+)$, if there are $t_0 \in \mathbb{R}$ and $t_+ > t_0$ such that $x : [t_0 - h, t_+) \longrightarrow \mathbb{R}^n$ is continuous, $(t, x_t) \in D$ and x satisfies Equation (1.1) for all $t_0 \leq t < t_+$. In this case, the **initial value of** x **at** t_0 is the segment $\varphi := x_{t_0} \in C$ of x. Accordingly, given $\varphi \in C$ and $t_0 \in \mathbb{R}$ with $(t_0, \varphi) \in D$ we call a function x a **solution of** (1.1) **with initial value** φ **at** t_0, if there is $t_+ > t_0$ such that x is a solution of RFDE (1.1) on $[t_0 - h, t_+)$ with initial value φ at t_0. Solutions on unbounded intervals $(-\infty, t_+)$, $t_+ \in \mathbb{R}$, are defined in an analogous way.

REMARK 1.1. To avoid any misunderstandings and to ensure consistency, we mention that in the situation of an autonomous RFDE

$$\dot{x}(t) = \tilde{f}(x_t)$$

with a function $\tilde{f} : V \longrightarrow \mathbb{R}^n$, $V \subseteq C$ open, the above definitions of solutions are understood as solutions of Equation (1.1) with $D := \mathbb{R} \times V$ and the right-hand side $f : D \ni (t,\varphi) \longmapsto \tilde{f}(\varphi) \in \mathbb{R}^n$. In particular, this means that a function x is a solution of the above autonomous RFDE with initial value $\varphi \in V$ at $t_0 \in \mathbb{R}$, if there is a real $t_+ > t_0$ such that $x : [t_0 - h, t_+) \longrightarrow \mathbb{R}^n$ is continuous, $x_{t_0} = \varphi$, $x_t \in V$ and x satisfies the differential equation for all $t_0 \leq t < t_+$.

An **initial value problem** (abbreviated by IVP) for the RFDE (1.1) has the form

$$\begin{cases} \dot{x}(t) = f(t, x_t) \\ x_{t_0} = \varphi \end{cases}$$

with given initial data $(t_0, \varphi) \in D$. Provided that the function f is continuous, this problem has a solution $x : [t_0 - h, t_+) \longrightarrow \mathbb{R}^n$, $t_+ > t_0$, for arbitrary $(t_0, \varphi) \in D$. These solutions can be extended in the forward direction of the independent variable t to a maximal interval of existence in D. If, in addition, the function f is (locally) Lipschitz continuous in its second argument, that is, if f satisfies the condition

(locLip) For each point $(t, \psi) \in D$ there is an open neighborhood N of (t, ψ) in D and a constant $\widehat{K} > 0$ such that

$$\|f(s, \psi_1) - f(s, \psi_2)\|_{\mathbb{R}^n} \leq \widehat{K} \|\psi_1 - \psi_2\|_C$$

for all $(s, \psi_1), (s, \psi_2) \in N$.

in D, then for every $(t_0, \varphi) \in D$ the solution x of the associated IVP is unique. Moreover, x depends continuously on t_0, φ, and f.

In case of an autonomous RFDE, the right-hand side of the equation is independent of the variable t. This implies the invariance of solutions under translations. More precisely, consider the autonomous RFDE

$$\dot{x}(t) = f(x_t) \tag{1.2}$$

with $f : V \longrightarrow \mathbb{R}^n$, $V \subseteq C$ open, and suppose that $x : [t_0 - h, t_+) \longrightarrow \mathbb{R}^n$ is a solution of (1.2) with initial value $\varphi = x_{t_0}$ at t_0. Then for each $\tau \in \mathbb{R}$ the translated function

$$y : [\tau + t_0 - h, \tau + t_+) \ni s \longmapsto x(s - \tau) \in \mathbb{R}^n$$

defines also a solution of Equation (1.2), but with initial value φ at $\tau + t_0$. Therefore, it is sufficient to consider only initial data given by $\varphi \in V$ at $t = 0$. If the function f in (1.2) is Lipschitz continuous, then every φ in V uniquely determines a maximal solution $x^\varphi : [-h, t_+(\varphi)) \longrightarrow \mathbb{R}^n$ with initial value φ at $t = 0$. The segments of these solutions generate a continuous semiflow on the metric space V, defined by the map

$$F : \Omega \ni (t, \varphi) \longmapsto x_t^\varphi \in V$$

with domain $\Omega := \{(t, \varphi) \in [0, \infty) \times V |\ 0 \leq t < t_+(\varphi)\}$. In this way we obtain a continuous semi-dynamical system, including the corresponding terms such as trajectories, stationary points, and periodic orbits.

All the results on RFDEs described above are well known, and for a deeper discussion including the absent proofs we refer the reader to the monographs Hale and Verduyn Lunel [**10**] and Diekmann et al.[**6**].

Representation of DDEs as RFDEs. At a first view it may be not apparent but different DDEs with constant as well as with time- or state-dependent delay can be rewritten in the more abstract form (1.1) as an RFDE. Accordingly, after carrying out such a transformation, one may ask whether basic or even far-reaching results for RFDEs may be used to study the initial differential equation. It turns out that the solution of this question is essentially dependent on the involved delays of the considered DDE.

For instance, the differential equation

$$\dot{x}(t) = g\big(x(t - h)\big)$$

with a function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ can be transformed to an autonomous RFDE by the right-hand side $f : C \ni \varphi \longmapsto g(\varphi(-h)) \in \mathbb{R}^n$. Under the assumption that g is continuous, the function f is clearly also continuous. Furthermore, Lipschitz continuity of g implies that of f, even with the same Lipschitz-constant as g. Hence, under such hypothesis on g, we may conclude the existence and uniqueness of solutions of the above DDE for all initial functions $\varphi \in C$. Additionally, in this situation the segments of the maximal solutions generate a continuous semiflow on C. Analogously, the assumption of C^1-smoothness on g leads to C^1-smoothness of f and enables us to apply further results that rest on continuous differentiability.

On the contrary, the transformation of a differential equation with retarded argument into an RFDE may lead to a significant lack of smoothness. This is in general the case in the presence of a non-trivial state-dependent delay. In particular, consider the equation

$$\dot{x}(t) = g\big(x(t - r(x_t))\big)$$

given by a function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and a delay functional $r : C \longrightarrow [0, h]$. Introducing the evaluation map

$$\mathrm{ev} : C \times [-h, 0] \ni (\varphi, s) \longmapsto \varphi(s) \in \mathbb{R}^n$$

and defining $f := g \circ \mathrm{ev} \circ (\mathbb{1} \times -r)$, we may rewrite the previous equation with state-dependent delay in the more abstract form of an RFDE. The evaluation map is obviously continuous so that the continuity of g and r implies the one of f. On the other hand, ev is not differentiable, and thus the function f is in general not C^1-smooth despite the smoothness of g and r. Moreover, except for uninteresting cases, f does not satisfy a Lipschitz condition, not even a local one, since the Lipschitz continuity of ev would imply that of $\varphi \in C$. Accordingly, independent of the smoothness of g and r, the resulting function f is in general not sufficiently smooth for applications of basic results such as uniqueness and continuous dependence on initial data for solutions of RFDEs.

3. The Semiflow on the Solution Manifold

In the last section we concluded that the theory of retarded functional differential equations is in general not suitable to study differential equations with state-dependent delays. In the following we will consider a slightly modified class of functional differential equations and discuss a fundamental theorem on existence of a smooth semiflow. Interestingly, the conditions of this result are often satisfied in cases where the corresponding abstract differential equation represents a DDE with a state-dependent delay. The underlying concept as well as the theorem itself go back to the work [**25, 24, 26**] of Walther. We recall only some of the key ideas summarized in Hartung et al. [**11**].

The Continuous Semiflow. We begin with the observation that the lack of smoothness described in the last section disappears if we replace the Banach space C by the smaller space C^1. Indeed, the appropriate evaluation map

$$\mathrm{Ev} : C^1 \times [-h, 0] \ni (\varphi, s) \longmapsto \varphi(s) \in \mathbb{R}^n \tag{1.3}$$

is not only continuous, it also continuously differentiable with partial derivatives

$$D_1 \,\mathrm{Ev}(\varphi, s)\, \chi = \mathrm{Ev}(\chi, s)$$

and

$$D_2\,\mathrm{Ev}(\varphi,s)\,1=\varphi'(s).$$

Hence, for continuously differentiable $g:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ and $r:U\longrightarrow[0,h]$, $U\subset C^1$ open, the resulting map $f:U\ni\varphi\longmapsto g\big(\varphi(-r(\varphi))\big)\in\mathbb{R}^n$ is also continuously differentiable.

On the other hand, the IVP associated with a differential equation of the form

$$\dot{x}(t)=f(x_t)$$

with a continuously differentiable function $f:U\longrightarrow\mathbb{R}^n$, $U\subseteq C^1$ open, is in general not well-posed on U. If we suppose that $x:[-h,t_+)\longrightarrow\mathbb{R}^n$, $t_+>0$, is a solution of the last equation with initial value $\varphi\in U$ at $t=0$, then x would have continuously differentiable segments $x_t\in U\subseteq C^1$ for all $0\le t<t_+$. Consequently, this would imply the continuous differentiability of x on $[-h,t_+)$, and thus the continuity of the curve $[0,t_+)\ni t\longmapsto x_t\in C^1$, which finally results in the equation

$$\dot{\varphi}(0)=\dot{x}(0)=\lim_{t\searrow 0}\dot{x}(t)=\lim_{t\searrow 0}f(x_t)=f(x_0)=f(\varphi)$$

for the initial value $\varphi\in U$. As this condition is not necessarily satisfied on all of U, the IVP associated with the differential equation given by f is in fact not well-posed on U.

The last consideration suggests to study the abstract differential equation

$$\dot{x}(t)=f(x_t) \tag{1.4}$$

defined by a function $f:U\longrightarrow\mathbb{R}^n$, $U\subseteq C^1$ open, only on the closed subset

$$X_f:=\big\{\varphi\in U\,|\;\varphi'(0)=f(\varphi)\big\} \tag{1.5}$$

of U. Under additional smoothness hypothesis on f, this idea succeeds in obtaining a continuous semiflow on X_f, generated by the segments of the solutions of Equation (1.4). The following assumptions on the function f are needed:

(S 1) f is continuously differentiable, and

(S 2) each derivative $Df(\varphi)$, $\varphi\in U$, extends to a linear map

$$D_ef(\varphi):C\longrightarrow\mathbb{R}^n,$$

and the induced map

$$U\times C\ni(\varphi,\chi)\longmapsto D_ef(\varphi)\,\chi$$

is continuous.

These conditions seem to be often fulfilled in cases where f represents the right-hand side of a DDE with state-dependent delay in the more abstract form (1.4). In particular, note that condition (S 2) does not require the continuity of the map $U\ni\varphi\longmapsto D_ef(\varphi)\in\mathcal{L}(C,\mathbb{R}^n)$, where $\mathcal{L}(C,\mathbb{R}^n)$ denotes the Banach space of all bounded linear maps from C into $\mathbb{R}^n$. Contrary to (S 2), the continuity of the last map is not satisfied in various examples of differential equations with state-dependent delays, as emphasized in Walther [**26**] and in Hartung et al. [**11**].

Before formulating the main result, we modify our understanding of solutions for the functional differential equation (1.4) in the sense of C^1-smoothness. More precisely, we call a function $x:[t_0-h,t_+)\longrightarrow\mathbb{R}^n$ with $t_0\in\mathbb{R}$ and $t_+>t_0$ a **solution** of Equation (1.4), if x is continuously differentiable on $[t_0-h,t_+)$, for all $t_0\le t<t_+$ the segments x_t of x belong to U, and x satisfies the differential

equation (1.4) for $t_0 < t < t_+$. Solutions for a given initial value $\varphi \in U$ at $t_0 \in \mathbb{R}$ as well as solutions on unbounded intervals $(-\infty, t_+)$, $t_+ \in \mathbb{R}$, are defined in the obvious way.

THEOREM 1.2 (Theorem 3.2.1 in Hartung et al. [**11**]). *Suppose that $U \subseteq C^1$ is open, the function $f : U \longrightarrow \mathbb{R}^n$ satisfies the smoothness conditions (S 1) – (S 2), and $X_f \neq \emptyset$. Then the following is true.*

(i) *The set X_f defined by* (1.5) *is a continuously differentiable submanifold of U with codimension n.*

(ii) *Each function $\varphi \in X_f$ uniquely defines a constant $t_+(\varphi) > 0$ and a non-continuable solution $x^\varphi : [-h, t_+(\varphi)) \longrightarrow \mathbb{R}^n$ of Equation* (1.4) *with initial value φ at $t = 0$. All the solution segments x_t^φ, $0 \leq t < t_+(\varphi)$, belong to X_f, and define by*

$$F : \Omega \ni (t, \varphi) \longmapsto x_t^\varphi \in X_f$$

a continuous semiflow on the metric space X_f with domain

$$\Omega := \Big\{ (t, \varphi) \in [0, \infty) \times X_f \Big|\ 0 \leq t < t_+(\varphi) \Big\}.$$

(iii) *For every $t \geq 0$ the solution map at time t, that is, the map*

$$F_t : \{\psi \in X_f |\ 0 \leq t < t_+(\psi)\} \ni \varphi \longmapsto F(t, \varphi) \in X_f,$$

is continuously differentiable, and for all $(t, \varphi) \in \Omega$ and all $\chi \in T_\varphi X_f$, where $T_\varphi X_f$ denotes the tangent space of the submanifold X_f at φ, the equation

$$DF_t(\varphi)\, \chi = v_t^{\varphi, \chi}$$

holds with the solution $v^{\varphi,\chi} : [-h, t_+(\varphi)) \to \mathbb{R}^n$ of the linear initial value problem

$$\begin{cases} \dot{v}(t) = Df(F(t, \varphi))\, v_t \\ \quad v_0 = \chi \end{cases}$$

in $T_\varphi X_f$.

(iv) *At each point (t, φ) with $\varphi \in X_f$ and $h < t < t_+(\varphi)$ the partial derivative of F with respect to the first argument exists and satisfies*

$$D_1 F(t, \varphi)\, 1 = \dot{x}_t^\varphi.$$

(v) *The restriction of the semiflow F to $\{(t, \varphi) \in \Omega |\ h < t\}$ is continuously differentiable.*

PROOF. For the complete proof we refer the reader to Walther [**25, 26**], whereas a detailed outline is also given in Hartung et al. [**11**]. □

The subset X_f of C^1 is called the **solution manifold** of Equation (1.4) on U. Even in the case of a continuous semiflow generated by the segments of the solutions of an autonomous RFDE, the corresponding set X_f contains the main dynamical structures as described in the next remark.

REMARK 1.3. If $\tilde{f} : V \longrightarrow \mathbb{R}^n$, $V \subseteq C$ open, is a (locally) Lipschitz continuous function and $x : [-h, t_+) \longrightarrow \mathbb{R}^n$, $t_+ > h$, is a solution of the autonomous RFDE

$$\dot{x}(t) = \tilde{f}(x_t),$$

then for $t \geq h$ we have $x_t \in V \cap C^1$ and $\dot{x}(t) = \tilde{f}(x_t)$; that is,

$$x_t \in X_{\tilde{f}} := \Big\{\varphi \in V \cap C^1 \,\big|\, \varphi'(0) = \tilde{f}(\varphi)\Big\}$$

for $t \geq h$. Hence, this means that every positive semi-orbit through a point $\varphi \in U$ enters and stays in $X_{\tilde{f}}$, provided it exists for sufficiently long time intervals. In particular, stationary points, periodic orbits, as well as (global) attractors of the above RFDE are contained in the set $X_{\tilde{f}}$.

A further important point concerning the last theorem is the fact that a part of the statements remains true under a slightly weaker hypothesis on the smoothness of $f : U \longrightarrow \mathbb{R}^n$, $U \subseteq C^1$ open. If the assumption (S 2) is replaced by

(S 2.1) Every derivative $Df(\varphi)$, $\varphi \in U$, of f extends to a bounded linear map $D_e f(\varphi) : C \longrightarrow \mathbb{R}^n$.

and

(S 2.2) For every $\varphi \in U$ there exists an open neighborhood $V \subset U$ of φ in U and a constant $K_{\mathrm{Lip}} > 0$ such that

$$\|f(\psi_1) - f(\psi_2)\|_{\mathbb{R}^n} \leq K_{\mathrm{Lip}} \|\psi_1 - \psi_2\|_C$$

for all $\psi_1, \psi_2 \in V$.

then at least the first two parts and the continuous differentiability of all solution maps F_t with non-empty domains are still valid, as can be found in Walther [**25**]. In this context, note that the above condition (S 2.2) is not a consequence of the C^1-smoothness of f, because the inequality in (S 2.2) involves the norm $\|\cdot\|_C$ of C and not $\|\cdot\|_{C^1}$ of C^1. However, for the proof of assertions (iv) and (v) in Walther [**26**] the stronger condition (S 2) is assumed, which particularly implies both (S 2.1) and (S 2.2).

Example. As an illustration of Theorem 1.2, let us consider the scalar differential equation

$$\dot{x}(t) = a\big[x(t) - x(t-r)\big] - |x(t)|\, x(t) \tag{1.6}$$

with parameter $a > 0$ and state-dependent delay $r = r(x(t)) > 0$ given by a continuously differentiable function $r : \mathbb{R} \longrightarrow [0, h]$. Subsequently, we transform this equation to the conceptional form (1.4) and show that the resulting equation satisfies the conditions of the theorem on the existence of a continuous semiflow. To this end, we adopt our notation by setting $n = 1$ in the definitions of C and C^1 from now on until the end of this section.

To rewrite the above DDE in the more abstract manner (1.4), we introduce the real valued function

$$g : \mathbb{R}^2 \ni (\zeta, \xi) \longmapsto a\big[\zeta - \xi\big] - |\zeta|\, \zeta \in \mathbb{R},$$

which is C^1-smooth with continuous partial derivatives

$$\frac{\partial g(\zeta, \xi)}{\partial \zeta} = a - 2\,|\zeta|$$

and

$$\frac{\partial g(\zeta, \xi)}{\partial \xi} = -a.$$

Then Equation (1.6) takes the form of the functional differential equation (1.4) with right-hand side

$$f : C^1 \ni \varphi \longmapsto g\big(\mathrm{Ev}(\varphi, 0), \mathrm{Ev}(\varphi, -r(Ev(\varphi, 0)))\big) \in \mathbb{R}, \tag{1.7}$$

involving the evaluation map (1.3). As a composition of C^1-smooth functions, f is also continuously differentiable, and hence satisfies smoothness condition (S 1). Furthermore, note that X_f is not empty. Indeed, the trivial function $x(t) = 0$, $t \in \mathbb{R}$, is a solution of the functional differential equation induced by f, and this implies $0 \in X_f$.

In order to verify the left condition (S 2) for f given by Equation (1.7), observe that a straightforward calculation yields

$$Df(\varphi)\,\psi = a\cdot\big[\psi(0)-\psi(-r(\varphi(0)))\big]+a\cdot r'(\varphi(0))\cdot\varphi'(-r(\varphi(0)))\cdot\psi(0)-2\,|\varphi(0)|\cdot\psi(0)$$

for all $\varphi, \psi \in C^1$. Since, independent of $\varphi \in C^1$, the right-hand side of this equation is also well-defined for $\psi \in C$, the existence of a linear extension $D_e f(\varphi)$ of $Df(\varphi)$ to the larger Banach space C is obvious as well as the definition of the linear operator $D_e f(\varphi)$ itself. Hence, it remains to confirm the continuity of the map

$$C^1 \times C \ni (\varphi, \psi) \longmapsto D_e f(\varphi)\,\psi \in \mathbb{R}. \tag{1.8}$$

Fix a point $(\varphi_1, \psi_1) \in C^1 \times C$ and let a constant $\varepsilon > 0$ be given. By continuity of the real valued functions

$$\mathbb{R} \ni s \longmapsto r'(s) \in \mathbb{R},$$
$$\mathbb{R} \ni s \longmapsto \psi_1(-r(s)) \in \mathbb{R},$$

and

$$\mathbb{R} \ni s \longmapsto \varphi_1'(-r(s)) \in \mathbb{R},$$

we find a constant $\delta_1 > 0$ such that for all $s \in \mathbb{R}$ with $|\varphi_1(0) - s| < \delta_1$ we have

$$|r'(\varphi_1(0)) - r'(s)| < \frac{\varepsilon}{4a\cdot(\|\varphi_1\|_{C^1}+1)\cdot(\|\psi_1\|_C+1)}\,,$$
$$|\psi_1(-r(\varphi_1(0))) - \psi_1(-r(s))| < \frac{\varepsilon}{4a}\,,$$

and

$$|\varphi_1'(-r(\varphi_1(0))) - \varphi_1'(-r(s))| < \frac{\varepsilon}{4a\cdot|r'(\varphi_1(0))|\cdot(\|\psi_1\|_C+1)}\,.$$

Choosing $0 < \delta < \min\{\delta_1, \delta_2, 1\}$, where the second constant is given by

$$\delta_2 := \frac{\varepsilon}{4\Big[2\|\varphi_1\|_{C^1}+\|\psi_1\|_C+2a+2+a\cdot|r'(\varphi_1(0))|\cdot(\|\varphi_1\|_{C^1}+\|\psi_1\|_C+1)\Big]}\,,$$

we claim that

$$|D_e f(\varphi_1)\,\psi_1 - D_e f(\varphi)\,\psi| < \varepsilon$$

holds for all $(\varphi, \psi) \in C^1 \times C$ with $\|\varphi_1 - \varphi\|_{C^1} + \|\psi_1 - \psi\|_C < \delta$. The proof of this assertion is straightforward by applying the triangle inequality in combination with mere addition of zeros. More precisely, for $(\varphi, \psi) \in C^1 \times C$ we have

$$|D_e f(\varphi_1)\,\psi_1 - D_e f(\varphi)\,\psi| \le \sum_{i=1}^{4}|A_i|$$

with

$$\begin{aligned}
A_1 &:= a\big[\psi_1(0) - \psi(0)\big],\\
A_2 &:= a\big[\psi(-r(\varphi(0))) - \psi_1(-r(\varphi_1(0)))\big]\\
&= a\big[\psi(-r(\varphi(0))) - \psi_1(-r(\varphi(0)))\big] + a\big[\psi_1(-r(\varphi(0))) - \psi_1(-r(\varphi_1(0)))\big],\\
A_3 &:= a\big[r'(\varphi_1(0)) \cdot \varphi_1'(-r(\varphi_1(0))) \cdot \psi_1(0) - r'(\varphi(0)) \cdot \varphi'(-r(\varphi(0))) \cdot \psi(0)\big]\\
&= a\big[r'(\varphi_1(0)) \cdot \varphi_1'(-r(\varphi_1(0))) \cdot \psi_1(0) - r'(\varphi_1(0)) \cdot \varphi_1'(-r(\varphi_1(0))) \cdot \psi(0)\big]\\
&\quad + a\big[r'(\varphi_1(0)) \cdot \varphi_1'(-r(\varphi_1(0))) \cdot \psi(0) - r'(\varphi_1(0)) \cdot \varphi_1'(-r(\varphi(0))) \cdot \psi(0)\big]\\
&\quad + a\big[r'(\varphi_1(0)) \cdot \varphi_1'(-r(\varphi(0))) \cdot \psi(0) - r'(\varphi_1(0)) \cdot \varphi'(-r(\varphi(0))) \cdot \psi(0)\big]\\
&\quad + a\big[r'(\varphi_1(0)) \cdot \varphi'(-r(\varphi(0))) \cdot \psi(0) - r'(\varphi(0)) \cdot \varphi'(-r(\varphi(0))) \cdot \psi(0)\big],
\end{aligned}$$

and

$$\begin{aligned}
A_4 &:= 2\big[|\varphi(0)| \cdot \psi(0) - |\varphi_1(0)| \cdot \psi_1(0)\big]\\
&= 2\big[|\varphi(0)| \cdot \psi(0) - |\varphi(0)| \cdot \psi_1(0)\big] + 2\big[|\varphi(0)| \cdot \psi_1(0) - |\varphi_1(0)| \cdot \psi_1(0)\big].
\end{aligned}$$

Applying the triangle inequality, one easily derives the following estimates for the absolute values of the last three quantities A_i:

$$\begin{aligned}
|A_2| \leq{} & a\,|\psi_1(-r(\varphi(0))) - \psi_1(-r(\varphi_1(0)))| + a\|\psi_1 - \psi\|_C\,,\\
|A_3| \leq{} & a\,|r'(\varphi_1(0))| \cdot \|\varphi_1\|_{C^1} \cdot \|\psi_1 - \psi\|_C\\
& + a\,|r'(\varphi_1(0))| \cdot \big[\|\psi_1\|_C + \|\psi_1 - \psi\|_C\big] \cdot |\varphi_1'(-r(\varphi_1(0))) - \varphi_1'(-r(\varphi(0)))|\\
& + a\,|r'(\varphi_1(0))| \cdot \big[\|\psi_1\|_C + \|\psi_1 - \psi\|_C\big] \cdot \|\varphi_1 - \varphi\|_{C^1}\\
& + a\,\big[\|\varphi_1\|_{C^1} + \|\varphi_1 - \varphi\|_{C^1}\big] \cdot \big[\|\psi_1\|_C + \|\psi_1 - \psi\|_C\big] \cdot |r'(\varphi_1(0)) - r'(\varphi(0))|\,,
\end{aligned}$$

and

$$|A_4| \leq 2\big[\|\varphi_1 - \varphi\|_{C_1} + \|\varphi_1\|_{C^1}\big] \cdot \|\psi_1 - \psi\|_C + \|\psi_1\|_C \cdot \|\varphi_1 - \varphi\|_{C^1}\,.$$

Assuming that $\|\varphi_1 - \varphi\|_{C^1} + \|\psi_1 - \psi\|_C < \delta < 1$ and using the above inequalities finally leads to

$$|D_e f(\varphi_1)\,\psi_1 - D_e f(\varphi)\,\psi| < \varepsilon$$

as claimed. This yields the continuity and thus property (S 2) of the map f defined by Equation (1.8).

By application of Theorem 1.2, we conclude that the set

$$X_f = \big\{\varphi \in C^1 \mid \varphi'(0) = f(\varphi)\big\}$$

with f from Equation (1.7) is a continuously differentiable submanifold of C^1 of codimension one. For each $\varphi \in X_f$ we have a uniquely determined (maximal) solution $x^\varphi : [-1, t_+(\varphi)) \longrightarrow \mathbb{R}$, $t_+(\varphi) > 0$, of Equation (1.6) with segments in X_f, and the totality of the segments of all these solutions constitutes a semiflow on X_f with properties as stated in Theorem 1.2. We will return to this subject in the second part of the work.

4. The Linearization at Stationary Points

For different types of non-linear differential equations an important technique to analyze the qualitative behavior of solutions near stationary points is the so-called *linearization*. The general motivation for this approach can be simply illustrated via an ordinary differential equation defined by a smooth vector field in $\mathbb{R}^n$. If in such a situation $x_0 \in \mathbb{R}^n$ is a critical point of the vector field, in other words, if x_0 is a stationary point of the underlying flow, then it is natural to expand the vector field in a Taylor series around x_0 for investigations of the ODE in a sufficiently small neighborhood of x_0. In cases where x_0 is not degenerated, the first non-zero term of the Taylor series expansion is linear and near x_0 quantitatively dominating compared to the terms of higher order. Therefore, the reduction of the expansion to the first order term leads to a linear ODE with stationary point x_0, and one expects that the behavior of this linear equation is closely related to that of the original non-linear ODE near x_0. In this context, the obtained linear ODE is called the associated *linearized problem* of the original differential equation, or more precisely, of the generated flow at the stationary point x_0. Under appropriate conditions on the vector field of the ODE, it is indeed so that different dynamical aspects in a neighborhood of a stationary point such as local stability or instability can be determined by its linearization.

For a long time it was almost mysterious how to linearize semiflows generated by differential equations with state-dependent delays. As long as this problem had not been solved, heuristical methods based on formal linearization were used for considerations as local stability and instability of stationary points. The work [**5**] of Cooke and Huang is indicative for such an approach. However, Theorem 1.2 solves the difficulties concerning the linearization of DDEs with state-dependent delays as we clarify in the sequel, following Hartung et al. [**11**, Section 3.4]. Afterwards we will summarize some spectral properties for the linearization and finally close this chapter with a proof of the *principle of linearized instability* for the semiflow from Theorem 1.2.

The Linearization. Suppose that $f : U \longrightarrow \mathbb{R}^n$, $U \subseteq C^1$ open, has properties (S 1) and (S 2) formulated on page 7 and that $\varphi_0 \in X_f$ is a stationary point of the semiflow F generated by the differential equation

$$\dot{x}(t) = f(x_t) \tag{1.9}$$

according to Theorem 1.2 in the last section. Then the tangent space of the solution manifold X_f at φ_0 is given by

$$T_{\varphi_0}X_f = \big\{\chi \in C^1 \,|\, \chi'(0) = Df(\varphi_0)\,\chi\big\},$$

and forms a Banach space with the norm $\|\cdot\|_{C^1}$ of C^1. The **linearization** of the semiflow F at a stationary point φ_0 is the family $T := \{T(t)\}_{t\geq 0}$ of bounded linear operators $T(t) := D_2F(t,\varphi_0)$, $t \geq 0$, on $T_{\varphi_0}X_f$. According to part (iii) of Theorem 1.2, for any $t \geq 0$ the action of $T(t)$ on an element $\chi \in T_{\varphi_0}X_f$ is determined by the relation $T(t)\,\chi = v_t^\chi$, where $v^\chi : [-h,\infty) \longrightarrow \mathbb{R}^n$ is the unique solution of the IVP

$$\begin{cases} \dot{v}(t) = Df(\varphi_0)\,v_t \\ \quad v_0 = \chi \end{cases}$$

in $T_{\varphi_0}X_f$. Clearly, we have $T(0) = \mathbb{1}$. Also while considering the unique solvability of the above IVP for each initial value in $T_{\varphi_0}X_f$, we see at once the semigroup property $T(t)\,T(s) = T(t+s)$ for all $t, s \geq 0$. Moreover, for all $\chi \in T_{\varphi_0}X_f$ we get $\|T(t)\,\chi - \chi\|_{C^1} \to 0$ as $t \searrow 0$, since the solutions v^χ are continuously differentiable. Therefore, the family T forms a strongly continuous semigroup of operators on the Banach space $T_{\varphi_0}X_f$. The infinitesimal generator G of T is given by the linear operator

$$G : \mathcal{D}(G) \ni \chi \longmapsto \chi' \in T_{\varphi_0}X_f$$

with domain

$$\mathcal{D}(G) := \bigl\{\chi \in C^2 \bigm| \chi'(0) = Df(\varphi_0)\,\chi, \quad \chi''(0) = Df(\varphi_0)\,\chi'\bigr\},$$

where C^2 denotes the set of all twice continuously differentiable functions from $[-h, 0]$ into $\mathbb{R}^n$.

On the other hand, the assumptions on f imply the existence of another strongly continuous semigroup as follows. By condition (S 2), the bounded linear operator $Df(\varphi_0) \in \mathcal{L}(C^1, \mathbb{R}^n)$ from the Banach space C^1 into $\mathbb{R}^n$ may be extended to a bounded linear operator $D_ef(\varphi_0)$ on the larger space C. The operator $D_ef(\varphi_0)$ induces the linear autonomous RFDE

$$\dot{w}(t) = D_ef(\varphi_0)\,w(t)$$

in C. The associated IVP

$$\begin{cases} \dot{w}(t) = D_ef(\varphi_0)\,w_t \\ \;w_0 = \chi \end{cases} \tag{1.10}$$

has, for each initial function $\chi \in C$, a unique solution $w^\chi : [-h, \infty) \longrightarrow \mathbb{R}^n$ as shown, for instance, in Diekmann et al. [**6**, Theorem 2.12 in Chapter I.2]. In the same way as for T, we see that the family $T_e := \{T_e(t)\}_{t\geq 0}$ of operators $T_e(t)$ defined by $T_e(t)\,\chi = w_t^\chi$ for all $t \geq 0$ and $\chi \in C$ is a strongly continuous semigroup on C. For the infinitesimal generator G_e of T_e we easily derive

$$G_e : \mathcal{D}(G_e) \ni \chi \longmapsto \chi' \in C$$

and

$$\mathcal{D}(G_e) = \Bigl\{\chi \in C^1 \bigm| \chi'(0) = D_ef(\varphi_0)\,\chi\Bigr\}.$$

Thus, the domain $\mathcal{D}(G_e)$ of the operator G_e coincides with the tangent space $T_{\varphi_0}X_f$ of the solution manifold X_f at the stationary point φ_0, and we have $T(t)\,\varphi = T_e(t)\,\varphi$ for all $\varphi \in \mathcal{D}(G_e)$ and $t \geq 0$.

As reflected in Hartung et al. [**11**], these relationships between T and T_e and their generators are not surprising in view of the fact that a strongly continuous semigroup induces in a natural way a strongly continuous semigroup on the domain of its generator. More precisely, suppose $S = \{S(t)\}_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators $S(t) : X \longrightarrow X$ on a (complex) Banach space X equipped with norm $\|\cdot\|_X$. Then the infinitesimal generator $A : \mathcal{D}(A) \longrightarrow X$ of S is a closed, densely defined operator so that the linear space $\mathcal{D}(A)$ forms a Banach space provided with the graph norm $\|x\|_{gr} := \|x\|_X + \|A\,x\|_X$, $x \in \mathcal{D}(A)$. Using the invariance of $\mathcal{D}(A)$ under S in combination with the fact that A commutes with $S(t)$ for all $t \geq 0$, we conclude the continuity of all linear operators

$$S_{gr}(t) : \mathcal{D}(A) \ni x \longmapsto S(t)\,x \in \mathcal{D}(A)$$

with respect to the graph norm $\|\cdot\|_{gr}$ on $\mathcal{D}(A)$. Moreover, $S_{gr} := \{S_{gr}(t)\}_{t\geq 0}$ trivially satisfies the properties of a strongly continuous semigroup and the associated infinitesimal generator A_{gr} is given by

$$A_{gr} : \mathcal{D}(A_{gr}) \ni x \longmapsto A\,x \in \mathcal{D}(A)$$

and

$$\mathcal{D}(A_{gr}) = \big\{x \in \mathcal{D}(A) \,|\, A\,x \in \mathcal{D}(A)\big\}.$$

Returning to the strongly continuous semigroup $T_e = \{T_e(t)\}_{t\geq 0}$ and noting that the graph norm $\|\cdot\|_{gr}$ of G_e is just the $\|\cdot\|_{C^1}$-norm on $\mathcal{D}(G_e)$, we see at once the identities $G = (G_e)_{gr}$, $T = (T_e)_{gr}$ and $T_{\varphi_0}X_f = \mathcal{D}(G_e)$.

Spectral Properties of the Linearization. A crucial point of the above relation between the semigroups T on $T_{\varphi_0}X_f$ and T_e on C, and their infinitesimal generators G and G_e, respectively, is the fact that it also entails a close connection for the associated spectra and spectral projections. In order to clarify this, let $\sigma(G_e)$, $\sigma(G)$ denote the spectrum of the complexification of G_e, G, respectively. As G_e is the infinitesimal generator of the solution semigroup for the linear autonomous equation (1.10), the spectrum $\sigma(G_e)$ is given by the solutions of a transcendental characteristic equation, is discrete and contains only eigenvalues with finite-dimensional generalized eigenspaces. These results in conjunction with the discussed relation between the strongly continuous semigroups T and T_e imply

$$\sigma(G) = \sigma(G_e)$$

as proven in Hartung et al. [**11**]. Moreover, for the spectral projections $P(\lambda)$, $P_e(\lambda)$ and the generalized eigenspaces $\mathcal{M}(\lambda)$, $\mathcal{M}_e(\lambda)$ associated with an eigenvalue $\lambda \in \sigma(G_e) = \sigma(G)$ we have

$$\mathcal{M}(\lambda) = \mathcal{M}_e(\lambda)$$

and

$$P(\lambda)\,\chi = P_e(\lambda)\,\chi$$

for all χ in the complexification $(T_{\varphi_0}X_f)_{\mathbb{C}}$ of the Banach space $T_{\varphi_0}X_f$. By realification, we obtain the corresponding relations for the realified generalized eigenspaces of G, G_e and their spectral projections.

Denote by $\sigma_u(G_e)$, $\sigma_c(G_e)$ and $\sigma_s(G_e)$ the subsets of the spectrum $\sigma(G_e)$, consisting of eigenvalues with positive, zero, and negative real part, respectively. As proven in Hale and Verduyn Lunel [**10**] or in Diekmann et al. [**6**], for each constant $\beta \in \mathbb{R}$ the half-plane $\{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda > \beta\}$ of $\mathbb{C}$ contains at most a finite number of elements of $\sigma(G_e)$, so that the spectral parts $\sigma_u(G_e)$, $\sigma_c(G_e)$ are empty or finite. Hence, the associated realified generalized eigenspaces C_u and C_c, which are called the **unstable** and the **center space** of G_e, respectively, are finite dimensional subspaces of C. In contrast, the **stable space** $C_s \subset C$ of G_e, that is, the realified generalized eigenspace associated to the spectral part $\sigma_s(G_e)$, is infinite-dimensional. The subspaces C_u, C_c and C_s are closed, invariant under $T_e(t)$, $t \geq 0$, and provide a decomposition

$$C = C_u \oplus C_c \oplus C_s$$

of C. As the restriction of T_e to the finite dimensional spaces C_u, C_c has a bounded generator, T_e may be extended to a one-parameter group in each case. The action of T_e on the above decomposition is characterized as follows: there are real numbers $K \geq 1$, $c_s < 0 < c_u$ and $c_c > 0$ with $c_c < \min\{-c_s, c_u\}$ such that

$$\begin{aligned} \|T_e(t)\,\varphi\|_C &\leq K e^{c_u t}\|\varphi\|_C, \quad t \leq 0,\ \varphi \in C_u, \\ \|T_e(t)\,\varphi\|_C &\leq K e^{c_c |t|}\|\varphi\|_C, \quad t \in \mathbb{R},\ \varphi \in C_c, \\ \|T_e(t)\,\varphi\|_C &\leq K e^{c_s t}\|\varphi\|_C, \quad t \geq 0,\ \varphi \in C_s. \end{aligned} \tag{1.11}$$

The relations between generalized eigenspaces and spectral projections of G_e and G imply that the unstable and center spaces of G coincide with C_u and C_c, respectively, and that the stable space of G is given by $C_s \cap \mathcal{D}(G_e)$. Subsequently, we obtain the spectral decomposition

$$Y = C_u \oplus C_c \oplus Y_s \tag{1.12}$$

for the Banach space $Y = T_{\varphi_0} X_f$, where $Y_s := C_s \cap \mathcal{D}(G_e)$. All the spaces C_u, C_c and Y_s are invariant under the semigroup T, and T forms a one-parameter group on each of the both finite-dimensional subspaces C_u and C_c. Using the exponential trichotomy (1.11), it is easy to check the analogous estimates

$$\begin{aligned} \|T(t)\,\varphi\|_{C^1} &\leq K e^{c_u t}\|\varphi\|_{C^1}, \quad t \leq 0,\ \varphi \in C_u, \\ \|T(t)\,\varphi\|_{C^1} &\leq K e^{c_c |t|}\|\varphi\|_{C^1}, \quad t \in \mathbb{R},\ \varphi \in C_c, \\ \|T(t)\,\varphi\|_{C^1} &\leq K e^{c_s t}\|\varphi\|_{C^1}, \quad t \geq 0,\ \varphi \in Y_s, \end{aligned} \tag{1.13}$$

describing the asymptotic behavior of T on the decomposition of Y, which is of particular importance in the next subsection.

The Principle of Linearized Instability. The remainder of this chapter is devoted to prove the *principle of linearized instability.* This result enables us to infer the instability of a stationary point with respect to a given non-linear semiflow of Theorem 1.2 from the instability of that point with respect to the associated linearization under certain conditions. More precisely, we will establish the following result.

Proposition 1.4 (The Principle of Linearized Instability). *Suppose the function $f : U \longrightarrow \mathbb{R}^n$, $U \subset C^1$ open, satisfies (S 1) and (S 2), and $\varphi_0 \in X_f$ is a stationary point of the associated semiflow from Theorem 1.2. If $Re\,\lambda > 0$ for some eigenvalue $\lambda \in \sigma(G_e)$, then φ_0 is unstable for the semiflow F.*

Here, the stationary point φ_0 is called **unstable** for F whenever there is $\varepsilon > 0$ such that for all $\delta > 0$ we find $\varphi \in X_f$ and $t > 0$ with $\|\varphi - \varphi_0\|_{C^1} < \delta$ but $\|F(t,\varphi) - \varphi_0\|_{C^1} > \varepsilon$. The main idea of the proof is to reduce the question on the instability of the stationary point $\varphi_0 \in X_f$ for the semiflow F to that for the discrete semi-dynamical system induced by a time-t-map of F. For smooth semiflows generated by a certain class of differential equations on open subsets of Banach spaces, especially for those induced by autonomous RFDEs, this approach is presented in detail by Diekmann et al. [**6**, Chapter VII]. We adopt this technique and need only minor changes, related to the fact that F is a semiflow on a Banach manifold. On that account we begin with the construction of local coordinates for the semiflow F in a neighborhood of φ_0, following Hartung et al. [**11**].

As the tangent space $Y = T_{\varphi_0} X_f$ of X_f at a stationary point φ_0 is a closed subspace of C^1 with codimension n, we find a closed linear subspace $E \subset C^1$ of dimension n which is complementary to Y in C^1; that is, $C^1 = Y \oplus E$. In particular, the projection P of C^1 along E onto Y is continuously differentiable, and the equation

$$Q(\varphi) = P(\varphi - \varphi_0)$$

defines a manifold chart for X_f on an open neighborhood $V \subset X_f$ of the stationary point φ_0. Thereby, the image $Y_0 := Q(V)$ of V under Q forms an open neighborhood of $0 = Q(\varphi_0)$ in the Banach space Y equipped with norm $\|\cdot\|_{C^1}$. The inverse of Q is given by a continuously differentiable map $R : Y_0 \to C^1$, and the derivative $DQ(\varphi_0)$ of Q at φ_0 as well as the derivative $DR(0)$ of R at $0 \in Y_0$ is the identity operator on Y in each case. Therefore we may assume that there is a constant $L_R > 0$ with

$$\|R(\chi_1) - R(\chi_2)\|_{C^1} \leq L_R \|\chi_1 - \chi_2\|_{C^1} \tag{1.14}$$

for all $\chi_1, \chi_2 \in Y_0$.

Let now $a > 0$ be given. By compactness of the interval $[0, a]$ together with the continuity of the map

$$(\mathbb{R} \times V) \cap \Omega \ni (t, \chi) \longmapsto F(t, R(\chi)) \in X_f,$$

we find an open neighborhood Y_a of 0 in Y_0 such that $F(t, R(\chi))$ is well-defined for all $(t, \chi) \in [0, a] \times Y_a$ and that $F([0, a], R(Y_a)) \subset V$. As a consequence, we are able to represent the semiflow F in local coordinates, namely by the map

$$H^a : [0, a] \times Y_a \ni (t, \chi) \longmapsto Q(F(t, R(\chi))) \in Y. \tag{1.15}$$

Obviously, we have $H^a(t, 0) = 0$ and $H^a(t, Y_a) \subset Y_0$ for all $0 \leq t \leq a$. The function H^a is also continuous. Moreover, for each $0 \leq t \leq a$ the induced map

$$H^a_t : Y_a \ni \chi \longmapsto H^a(t, \chi) \in Y$$

is continuously differentiable with derivative given by

$$DH^a_t(0) = DQ(\varphi_0) \circ D_2F(t, \varphi_0) \circ DR(0) = D_2F(t, \varphi_0) = T(t).$$

Suppose that we have $H^a(s, \chi) \in Y_a$ for a fixed $(s, \chi) \in [0, a] \times Y_a$. Then all values $H^a(t, H^a(s, \chi))$, $0 \leq t \leq a$, are well-defined. Accordingly, by setting $H^a(t + s, \chi) := H^a(t, H^a(s, \chi))$ for $0 \leq t \leq a$ we may represent the positive semi-orbit of the semiflow F through $R(\chi)$ in the local coordinates constructed above at least on the interval $[0, s + a]$. Additionally, in this case we see at once that the semigroup property of F implies

$$H^a(t + s, R(\chi)) = H^a(t, H^a(s, \chi)) = Q(F(t + s, R(\chi)))$$

for $0 \leq t \leq a$. Therefore, we may extend the domain for the local representation H^a of the semiflow F to the set

$$\Big\{(t, \chi) \in [0, \infty) \times Y_a \,\Big|\, Q(F(\lfloor t/a \rfloor\, a, R(\chi))) \in Y_a\Big\},$$

where $\lfloor t/a \rfloor$ denotes the integer part of the real t/a. For instance, $[0, \infty) \times \{0\}$ belongs to this extended domain of the map H^a and the stationary point $\varphi_0 \in X_f$ can obviously be represented by $0 \in Y_0$ for all $t \geq 0$.

After the above construction of local coordinates for the semiflow F we are now in the situation to prove the statement of Proposition 1.4.

PROOF OF PROPOSITION 1.4. The proof will be divided into four small parts.

1. Consider the decomposition (1.12) of Y and the associated trichotomy given by (1.13). Defining $Y_{cs} := C_c \oplus Y_s$ and $c_{cs} := c_c$, we obtain the decomposition

$$Y = C_u \oplus Y_{cs}$$

with the exponential estimates

$$\begin{aligned} \|T(t)\,\chi\|_{C^1} &\le K e^{c_u t}\|\varphi\|_{C^1}, \quad t \le 0,\ \chi \in C_u, \\ \|T(t)\,\chi\|_{C^1} &\le K e^{c_{cs} t}\|\varphi\|_{C^1}, \quad t \ge 0,\ \chi \in Y_{cs}. \end{aligned} \tag{1.16}$$

Fix $0 < q < 1$ with $1/q > K \ge 1$. Then for all $t \ge 0$ and $\chi \in C_u$ we have

$$\|\chi\|_{C^1} = \|T(-t)\,T(t)\,\chi\|_{C^1} \le K e^{-c_u t}\|T(t)\,\chi\|_{C^1} \le \frac{1}{q}\, e^{-c_u t}\|T(t)\,\chi\|_{C^1},$$

that is,

$$q\, e^{c_u t}\|\chi\|_{C^1} \le \|T(t)\,\chi\|_{C^1}. \tag{1.17}$$

Since $c_{cs} < c_u$, there is a constant $t_0 > 0$ such that $\vartheta_1 := q\, e^{c_u t_0} - K\, e^{c_{cs} t_0} > 0$. Hence the estimates (1.16) and (1.17) for the linear operator $L := T(t_0)$ imply the two inequalities

$$\begin{aligned} \|L\,\chi\|_{C^1} &\ge (\vartheta_1 + \vartheta_2)\|\chi\|_{C^1}, && \chi \in C_u, \\ \|L\,\chi\|_{C^1} &\le \vartheta_2\|\chi\|_{C^1}, && \chi \in Y_{cs}, \end{aligned} \tag{1.18}$$

where $\vartheta_2 := K\, e^{c_{cs} t_0} > 1$.

2. Let $\widehat{P}_u : Y \longrightarrow Y$ denote the projection of Y along Y_{cs} onto the unstable space C_u of the operator G. Using the continuity of $\widehat{P}_u$, it is easily seen that

$$\|\varphi\|_u := \|\widehat{P}_u\,\varphi\|_{C^1} + \|(\mathbb{1} - \widehat{P}_u)\,\varphi\|_{C^1}$$

defines a norm on Y. In particular, the norm $\|\cdot\|_u$ is equivalent to $\|\cdot\|_{C^1}$ on Y. Consider now the time-t_0-map $g := H^{t_0}(t_0, \cdot) : Y_{t_0} \longrightarrow Y$ of the semiflow F in local coordinates. Since Y_{t_0} is an open neighborhood of the origin in Y, and L is the derivative of g at $\chi = 0$, we find a sufficiently small $\varepsilon_1 > 0$ such that for all $\chi \in Y$ with $\|\chi\|_u < \varepsilon_1$ we have $\chi \in Y_{t_0}$ and

$$\|g(\chi) - L\,\chi\|_u \le \frac{1}{4}\,\vartheta_1\,\|\chi\|_u. \tag{1.19}$$

Suppose for $\chi \in Y$ with $\|\chi\|_u < \varepsilon_1$ there holds $\|(\mathbb{1} - \widehat{P}_u)\,\chi\|_{C^1} \le \|\widehat{P}_u\,\chi\|_{C^1}$. Then we claim that the value $g(\chi)$ satisfies the same cone condition as χ; that is,

$$\|(\mathbb{1} - \widehat{P}_u)(g(\chi))\|_{C^1} \le \|\widehat{P}_u(g(\chi))\|_{C^1}.$$

To see this, note first that the above assumptions on χ immediately imply the inequality $\|\chi\|_u \le 2\|\widehat{P}_u\,\chi\|_{C^1}$. Therefore the invariance of the spaces C_u, Y_{cs} for L

and the estimates (1.18), (1.19) yield

$$\begin{aligned}
\|\widehat{P}_u\, g(\chi)\|_{C^1} &\geq \|\widehat{P}_u\, L\,\chi\|_{C^1} - \|\widehat{P}_u(g(\chi) - L\,\chi)\|_{C^1} \\
&= \|L\,\widehat{P}_u\,\chi\|_{C^1} - \|\widehat{P}_u(g(\chi) - L\,\chi)\|_{C^1} \\
&\geq (\vartheta_1 + \vartheta_2)\|\widehat{P}_u\,\chi\|_{C^1} - \|g(\chi) - L\,\chi\|_u \\
&\geq (\vartheta_1 + \vartheta_2)\|\widehat{P}_u\,\chi\|_{C^1} - \frac{1}{4}\,\vartheta_1\|\chi\|_u \\
&\geq (\vartheta_1 + \vartheta_2)\|\widehat{P}_u\,\chi\|_{C^1} - \frac{1}{2}\,\vartheta_1\|\widehat{P}_u\,\chi\|_{C^1} \\
&\geq \left(\vartheta_2 + \frac{1}{2}\,\vartheta_1\right)\|\widehat{P}_u\,\chi\|_{C^1}
\end{aligned}$$

and

$$\begin{aligned}
\|(\mathbb{1} - \widehat{P}_u)\, g(\chi)\|_{C^1} &\leq \|(\mathbb{1} - \widehat{P}_u)\, L\,\chi\|_{C^1} + \|(\mathbb{1} - \widehat{P}_u)(g(\chi) - L\,\chi)\|_{C^1} \\
&= \|L\,(\mathbb{1} - \widehat{P}_u)\,\chi\|_{C^1} + \|(\mathbb{1} - \widehat{P}_u)(g(\chi) - L\,\chi)\|_{C^1} \\
&\leq \vartheta_2\|(\mathbb{1} - \widehat{P}_u)\,\chi\|_{C^1} + \|g(\chi) - L\,\chi\|_u \\
&\leq \vartheta_2\|(\mathbb{1} - \widehat{P}_u)\,\chi\|_{C^1} + \frac{1}{4}\,\vartheta_1\|\chi\|_u \\
&\leq \vartheta_2\|(\mathbb{1} - \widehat{P}_u)\,\chi\|_{C^1} + \frac{1}{2}\,\vartheta_1\|\widehat{P}_u\,\chi\|_{C^1} \\
&\leq \left(\vartheta_2 + \frac{1}{2}\,\vartheta_1\right)\|\widehat{P}_u\,\chi\|_{C^1},
\end{aligned}$$

which proves the claim. Thus for all $\chi \in Y$ with $\|\chi\|_u < \varepsilon_1$ we have the implication

$$\|(\mathbb{1} - \widehat{P}_u)\,\chi\|_{C^1} \leq \|\widehat{P}_u\,\chi\|_{C^1} \quad \Longrightarrow \quad \|(\mathbb{1} - \widehat{P}_u)(g(\chi))\|_{C^1} \leq \|\widehat{P}_u(g(\chi))\|_{C^1}.$$

3. *Proof of the instability of $\chi = 0$ for g.* Let $0 < \varepsilon_2 < \varepsilon_1/(\|\widehat{P}_u\| + \|\mathbb{1} - \widehat{P}_u\|)$ be given. Assume that for every sufficiently small $\chi \in Y$ with $\|\chi\|_u < \varepsilon_1$ and $\|(\mathbb{1} - \widehat{P}_u)\,\chi\|_{C^1} \leq \|\widehat{P}_u\,\chi\|_{C^1}$ there holds

$$\|g^k(\chi)\|_{C^1} < \varepsilon_2$$

for all $k \in \mathbb{N}$, where g^k denotes the k-th iteration of the map g. Then we would have

$$\begin{aligned}
\|g^k(\chi)\|_u &= \|\widehat{P}_u(g^k(\chi))\|_{C^1} + \|(\mathbb{1} - \widehat{P}_u)(g^k(\chi))\|_{C^1} \\
&\leq \|\widehat{P}_u\|\,\|g^k(\chi)\|_{C^1} + \|\mathbb{1} - \widehat{P}_u\|\,\|g^k(\chi)\|_{C^1} \\
&\leq (\|\widehat{P}_u\| + \|\mathbb{1} - \widehat{P}_u\|)\,\varepsilon_2 \\
&< \varepsilon_1,
\end{aligned}$$

and hence by the part above

$$\|\widehat{P}_u(g^k(\chi))\|_{C^1} \geq \left(\vartheta_2 + \frac{1}{2}\,\vartheta_1\right)^k \|\widehat{P}_u\,\chi\|_{C^1}$$

for all $k \in \mathbb{N}$. Subsequently, in consideration of $\vartheta_1 > 0$ and $\vartheta_2 > 1$, this would imply

$$\|\widehat{P}_u(g^k(\chi))\|_{C^1} \to \infty$$

for $k \to \infty$ whenever $\chi \neq 0$. But by the hypothesis $\dim C_u \geq 1$ of the proposition, we see at once the existence of any desired small $\chi_u \in Y\backslash\{0\}$ satisfying

$$\|(\mathbb{1} - \widehat{P}_u)\,\chi_u\|_{C^1} \leq \|\widehat{P}_u\,\chi_u\|_{C^1} \leq \|\chi_u\|_u < \varepsilon_1,$$

which leads to a contradiction to our assumption on boundedness for the iterations of g. Therefore the stationary point $\chi = 0$ is unstable for g.

4. Contrary to the statement of the proposition, suppose that the stationary point $\varphi_0 \in X_f$ is stable for the semiflow F. By definition, we find for each $\varepsilon > 0$ a constant $\delta(\varepsilon) > 0$ such that for all $\varphi \in X_f$ with $\|\varphi - \varphi_0\|_{C^1} < \delta$ and for all $t \geq 0$ we have

$$\|F(t, \varphi) - \varphi_0\|_{C^1} < \varepsilon.$$

Choose $\delta := \delta(\varepsilon_2/\|P\|) > 0$ where $\|P\|$ denotes the norm of the projection operator $P : C^1 \longrightarrow C^1$ along Y onto E involved in our construction of local coordinates for X_f. Further, define $\eta := \delta/L_R$ with constant $L_R > 0$ of the Lipschitz-condition (1.14) for the mapping R. According to the last points, we find $\chi \in Y \backslash \{0\}$ satisfying the cone condition

$$\|\widehat{P}_u\, \chi\|_{C^1} \geq \|(\mathbb{1} - \widehat{P}_u)\, \chi\|_{C^1}.$$

By the equivalence of the norms $\|\cdot\|_{C^1}$ and $\|\cdot\|_u$ on the space Y, we may assume $\|\chi\|_u < \varepsilon := \varepsilon_2/\|P\|$ and $\|\chi\|_{C^1} < \eta$. Hence we have

$$\|R(\chi) - \varphi_0\|_{C^1} = \|R(\chi) - R(0)\|_{C^1} \leq L_R \|\chi\|_{C^1} < \delta.$$

But on the other hand, in view of the inequality

$$\begin{aligned} \|g^k(\chi)\|_{C^1} &= \|H^{t_0}(k\, t_0, \chi)\|_{C^1} \\ &= \|Q(F(k\, t_0, R(\chi)))\|_{C^1} \\ &= \|P(F(k\, t_0, R(\chi)) - \varphi_0)\|_{C^1} \\ &\leq \|P\|\, \|F(k\, t_0, R(\chi)) - \varphi_0\|_{C^1} \end{aligned}$$

the arguments of the last part implies that the positive semi-orbit of the time-t_0-map through $R(\chi)$, and thus the positive semi-orbit of the semiflow F through $R(\chi)$, leaves the open ball at φ_0 of radius $\varepsilon_2/\|P\|$ in X_f; that is, there is a $t > 0$ with $\|F(t, R(\chi)) - \varphi_0\|_{C^1} > \varepsilon_2/\|P\|$, in contradiction to the stability of φ_0. Thus φ_0 is unstable, which completes the proof. □

REMARK 1.5. The counterpart of Proposition 1.4 is the so-called *principle of linearized stability*, which is valid under an analogous smoothness condition on f as proven in Hartung et al. [**11**, Section 3.6]. More precisely, if $f : U \longrightarrow \mathbb{R}^n$, $U \subset C^1$ open, satisfies (S 1) and (S 2) and if $\varphi_0 \in X_f$ is a stationary point of the resulting semiflow such that all eigenvalues of G_e have negative real parts, then φ_0 is (exponentially) asymptotically stable.

An alternative way to prove Proposition 1.4 is indicated in Krisztin [**15**]. Under the hypothesis of the proposition there exist local (fast) unstable manifolds at the stationary point φ_0, which can be used to show the existence of a solution $x : (-\infty, t_+) \longrightarrow \mathbb{R}^n$, $t_+ > 0$, for Equation (1.4) that converges to 0 as $t \to -\infty$.

CHAPTER 2

Local Center-Unstable Manifolds

1. Introduction

In the last chapter we saw that under certain conditions the stability properties of stationary points of non-linear semiflows are determined by those for the associated linearized systems. Another important concept arising from the linearization of non-linear differential equations at stationary points are the so-called *local invariant manifolds*. Roughly speaking, these manifolds may be regarded as deformations of the invariant linear spaces for the linearized semiflow under the non-linear perturbation effected by the original system. The principal significance of these manifolds is the fact that they provide one of the most potent ways to describe the qualitative behavior of non-linear semiflows in close vicinity of stationary points.

From a historical point of view, first results on such manifolds for non-linear differential equations date back to the work of Lyapunov [**19**] and Perron [**22**] on the stability of stationary points for ODEs. In the last decades, the ideas of Lyapunov and Perron were generalized and several other approaches were developed, so that today the existence of local invariant manifolds at stationary points is well-known for various types of differential equations. In this context, the most common invariant manifolds are the so-called *stable*, *center-stable*, *center*, *center-unstable* and *unstable manifolds*.

In case of differential equations with state-dependent delays, two of the first results on invariant manifolds near stationary points are due to Krishnan [**14**] and Krisztin [**15**]. In Krishnan [**14**], the author considers a certain class of DDEs with a state-dependent delay, and proves the occurrence of local unstable manifolds under a hyperbolicity condition. Also assuming hyperbolicity, Krisztin discusses in [**15**] the existence and smoothness of local unstable manifolds for more general equations representing differential equations with state-dependent delays in the abstract form of RFDEs. In particular, the results in Krisztin [**15**] are applicable to the semiflow F of Theorem 1.2 in the last chapter. It is worth pointing out that in both papers the approaches do not involve the knowledge of a semiflow, and in both cases only a heuristic technique for the linearization is used.

A proof of the existence of continuously differentiable local stable manifolds for the semiflow F considered in the last chapter is contained in Section 3 of Hartung et al. [**11**]. In this survey paper the reader may also find the construction of local center manifolds for F, where the associated conclusion of continuous differentiability is established in Krisztin [**16**]. The occurrence of continuously differentiable local center-stable manifolds at stationary points for the semiflow induced by Theorem 1.2 is confirmed by Qesmi and Walther in the recent work [**23**].

The goal of this chapter is to ensure the existence of continuously differentiable local center-unstable manifolds at stationary points for the semiflow given by Theorem 1.2. For this purpose, we first follow the approach used in Hartung et al.

[**11**] for the construction of local center manifolds, and apply a modification of the Lyapunov-Perron method contained in Diekmann et al. [**6**] to establish the existence of Lipschitz continuous local center-unstable manifolds. Hereafter, we employ the techniques from Krisztin [**16**] to prove C^1-smoothness.

This chapter is organized as follows. After stating our main result in the next section, we summarize the relevant material needed to show the assertion in order to make our exposition self-containing. In Section 4 we construct Lipschitz continuous local center-unstable manifolds and prove their C^1-smoothness in Section 5. Finally, we indicate an application of such manifolds for the study of the qualitative behavior of non-linear semiflows near stationary points.

2. The Main Result

Assume that $f : U \longrightarrow \mathbb{R}^n$, $U \subseteq C^1$ open, satisfies the two smoothness conditions (S 1) and (S 2) on page 7, and that $0 \in U$ with $f(0) = 0$. Then the set $X_f \subset U$ defined by (1.5) is obviously not empty, and according Theorem 1.2 the segments of the solutions for the differential equation

$$\dot{x}(t) = f(x_t) \tag{2.1}$$

with initial values in X_f generate a smooth semiflow F on X_f. In particular, $\varphi_0 = 0 \in X_f$ is a stationary point of F.

Let us denote by L the derivative $C^1 \ni \varphi \longmapsto Df(0)\,\varphi \in \mathbb{R}^n$ of f at $\varphi_0 = 0$, and let L_e denote the continuous linear extension $C \ni \varphi \longmapsto D_e f(0)\,\varphi \in \mathbb{R}^n$ on the larger Banach space C, which exists by condition (S 1). For each $\varphi \in C$ the IVP

$$\begin{cases} \dot{v}(t) = L_e\, v_t \\ \quad v_0 = \varphi \end{cases} \tag{2.2}$$

has a unique solution $v^{\varphi} : [-h, \infty) \longrightarrow \mathbb{R}^n$, and the segments of all these solutions define a strongly continuous semigroup $T_e = \{T_e(t)\}_{t \geq 0}$ on C with infinitesimal generator

$$G_e : \mathcal{D}(G_e) \ni \varphi \longmapsto \varphi' \in C$$

where $\mathcal{D}(G_e) = \{\varphi \in C^1 |\ \varphi'(0) = L_e\, \varphi\}$. The spectrum $\sigma(G_e) = \sigma((G_e)_{\mathbb{C}})$ of G_e defines a decomposition of C into

$$C = C_u \oplus C_c \oplus C_s, \tag{2.3}$$

where C_u, C_c and C_s denote the unstable, center and stable space of G_e, respectively. Both C_u and C_c belong to $\mathcal{D}(G_e) \subset C^1$ and are finite dimensional, whereas C_s is an infinite dimensional closed subspace of C. In addition, each of the spaces C_u, C_c, C_s is invariant under T_e and T_e can be extended to a one-parameter group of operators on both C_u and C_c.

As a consequence of the above decomposition of C we obtain also a decomposition of the smaller Banach space C^1, namely

$$C^1 = C_u \oplus C_c \oplus C_s^1 \tag{2.4}$$

with the closed subspace $C_s^1 := C_s \cap C^1$ of C^1.

The linearization $T = \{T(t)\}_{t \geq 0}$ of the semiflow F at the stationary point $\varphi_0 = 0$ forms a strongly continuous semigroup of linear operators on the Banach space $T_0 X_f = \left\{\chi \in C^1 \,|\, \chi'(0) = L\,\chi\right\}$, which in particular coincides with $\mathcal{D}(G_e)$. The infinitesimal generator G of T is defined by $G : \mathcal{D}(G) \ni \chi \longmapsto \chi' \in T_0 X_f$ on the domain $\mathcal{D}(G) = \{\chi \in C^2 \,|\, \chi'(0) = L\,\chi,\, \chi''(0) = L\,\chi'\}$. As discussed in the last chapter, the unstable, center and stable space of G are C_u, C_c, and $C_s \cap \mathcal{D}(G_e)$, respectively, and the decomposition of C leads to the decomposition

$$T_0 X_f = C_u \oplus C_c \oplus (C_s \cap \mathcal{D}(G_e))$$

of the tangent space $T_0 X_f$ of F at $\varphi_0 = 0$.

Using the notation $C_{cu} := C_c \oplus C_u$ for the **center-unstable space** of G, we state our result on the existence of local center-unstable manifolds for F at $\varphi_0 = 0$.

THEOREM 2.1 (Existence of Local Center-Unstable Manifold). *Suppose in addition to the previous assumptions on f that $\{\lambda \in \sigma(G_e) |\ Re\,\lambda \geq 0\} \neq \emptyset$ or, equivalently, $C_{cu} \neq \{0\}$. Then there are open neighborhoods $C_{cu,0}$ of 0 in C_{cu} and $C^1_{s,0}$ of 0 in C^1_s with $N_{cu} := C_{cu,0} + C^1_{s,0} \subseteq U$, and a Lipschitz continuous map $w_{cu} : C_{cu,0} \longrightarrow C^1_{s,0}$ with $w_{cu}(0) = 0$, such that the graph*

$$W_{cu} := \Big\{\varphi + w_{cu}(\varphi) \big|\ \varphi \in C_{cu,0}\Big\}$$

has the following properties.

(i) *The set W_{cu} belongs to the solution manifold X_f of Equation* (2.1)*. Moreover, W_{cu} is a k-dimensional Lipschitz submanifold of X_f where $k := \dim C_{cu}$.*

(ii) *For each solution $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ of Equation* (2.1) *on $(-\infty, 0]$, we have*

$$\big\{x_t \big|\ t \leq 0\big\} \subseteq N_{cu} \quad \Longrightarrow \quad \big\{x_t \big|\ t \leq 0\big\} \subseteq W_{cu}.$$

(iii) *The graph W_{cu} is positively invariant with respect to the semiflow F relative to N_{cu}; that is, if $\varphi \in W_{cu}$ and $t > 0$ then*

$$\big\{F(s,\varphi) \big|\ 0 \leq s \leq t\big\} \subset N_{cu} \quad \Longrightarrow \quad \big\{F(s,\varphi) \big|\ 0 \leq s \leq t\big\} \subset W_{cu}.$$

The submanifold W_{cu} of X_f is called a **local center-unstable manifold** of F at the stationary point $\varphi_0 = 0$. It is C^1-smooth and passes φ_0 tangentially to the center-unstable space C_{cu} as we shall have established by our next theorem.

THEOREM 2.2 (C^1-Smoothness of Local Center-Unstable Manifold). *The map $w_{cu} : C_{cu,0} \longrightarrow C^1_{s,0}$ obtained in Theorem 2.1 is continuously differentiable and $Dw_{cu}(0) = 0$.*

In the next three sections we prove the above theorems. Even though the proofs are quite long and at certain points technical, they are nevertheless not difficult to understand. As mentioned before, we follow the construction of local center manifolds in Hartung et al. [**11**] and apply the Lyapunov-Perron method to obtain the existence of local center-unstable manifolds as claimed in Theorem 2.1. The basic idea of this method is to transform the differential equation (2.1), or more precisely, a smoothed modification of it, into an integral equation such that the corresponding integral operator forms a parameter-dependent contraction in an appropriate Banach space of continuous functions. The fixed points of this contraction define a mapping whose graph forms the desired invariant manifold.

After the described construction, we follow the procedure in Krisztin [**16**] and show the C^1-dependence of the obtained fixed points on the parameter which leads to the continuous differentiability of the manifolds asserted in Theorem 2.2.

3. Preliminaries for the Existence Proof

For the transformation of the considered differential equation into an integral form we will employ a variation-of-constants formula, which is established in Diekmann et al. [**6**] and involves duality and adjoint semigroups. For the convenience of the reader and to make our exposition self-contained, we repeat some of the relevant material from Diekmann et al. [**6**] without proofs. Afterwards we discuss some preparatory results.

Duality and Sun-Reflexivity. Recall that for a Banach space X over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ the **dual space** X^* is the set of all continuous linear functionals on X, that is, X^* consists of all continuous linear maps from X into $\mathbb{K}$. We write x^* for elements of X^*, and for $x^* \in X^*$ and $x \in X$ we use the notation $\langle x^*, x\rangle \in \mathbb{K}$ instead of $x^*(x)$. Provided with the norm

$$\|x^*\|_{X^*} := \sup_{\|x\|_X \leq 1} |\langle x^*, x\rangle|,$$

where $\|\cdot\|_X$ denotes the norm on X, the dual space X^* becomes also a Banach space over $\mathbb{K}$.

If $A : \mathcal{D}(A) \to X$ is a linear operator defined on some dense linear subspace $\mathcal{D}(A)$ in X, then its **adjoint** A^* is defined by

$$\mathcal{D}(A^*) := \Big\{ x^* \in X^* \Big|\, \exists\, y^* \in X^* \text{ with } \langle y^*, x\rangle = \langle x^*, A\,x\rangle \text{ for all } x \in \mathcal{D}(A) \Big\}$$

and then for $x^* \in \mathcal{D}(A^*)$

$$A^* x^* = y^*.$$

If $A : X \longrightarrow X$ is a bounded linear operator, then for each $x^* \in X^*$ the induced map $X \ni x \longmapsto \langle x^*, A\,x\rangle \in \mathbb{K}$ is linear and bounded. Thus, in this case, the relations

$$\langle A^* x^*, x\rangle = \langle x^*, A\,x\rangle$$

for all elements $x \in X$ and $x^* \in X^*$ uniquely define a bounded linear operator $A^* : X^* \longrightarrow X^*$. In particular, we have $\|A\| = \|A^*\|$.

Consider now the Banach space C and the strongly continuous semigroup $T_e = \{T_e(t)\}_{t\geq 0}$ of bounded linear operators defined by the solutions of the IVP (2.2). For every $t \geq 0$ the adjoint $T_e^*(t)$ of $T_e(t)$ is a linear operator with norm $\|T_e^*(t)\| = \|T_e(t)\|$ on the dual space C^* of C and the family $T_e^* := \{T_e^*(t)\}_{t\geq 0}$ obviously constitutes a semigroup of operators on C^*. We also have $T_e^*(0)\,\varphi^* = \varphi^*$ for all $\varphi^* \in C^*$, but T_e^* is in general not a strongly continuous semigroup. Indeed, if C^* is equipped with the topology given by the norm $\|\cdot\|_{C^*}$, it is not difficult to see that for $\varphi^* \in C^*$ the induced curve

$$[0, \infty) \ni t \longmapsto T^*(t)\,\varphi^* \in C^* \tag{2.5}$$

is not necessarily continuous. However, the set of all functions $\varphi^\odot \in C^*$ for which the curve (2.5) is continuous, in other words, $\varphi^\odot \in C^*$ with the property $\|T_e^*(t)\,\varphi^\odot - \varphi^\odot\|_{C^*} \to 0$ as $t \searrow 0$, forms a closed subspace $C^\odot$ of C^*. Furthermore, $T_e^*(t)(C^\odot) \subset C^\odot$ for all $t \geq 0$ so that the family of operators

$$T_e^\odot(t) : C^\odot \ni \varphi^\odot \longmapsto T_e^*(t)\,\varphi^\odot \in C^\odot$$

constitutes a strongly continuous semigroup $T_e^\odot$ on $C^\odot$.

REMARK 2.3. It is worth to mention that the family T_e^* of linear operators on C^* is a weak* continuous semigroup, and G_e^* the associated weak* generator. More precisely, if the dual space C^* of C is equipped with the so-called *weak* topology*, that is, the coarsest topology on C^* such that for all $\varphi \in C$ the functions $C^* \ni \varphi^* \longmapsto \langle \varphi^*, \varphi \rangle \in \mathbb{R}$ are continuous, then for each $\varphi^* \in C^*$ the induced curve (2.5) is continuous. In this way, T_e^* becomes a continuous semigroup and G_e^* its generator.

Similarly, we can repeat the above process with the Banach space $C^{\odot}$ and the strongly continuous semigroup $T_e^{\odot}$. At first, we introduce again the adjoint operators $T_e^{\odot *}(t)$ of $T_e^{\odot}(t)$, $t \geq 0$, on the dual space $C^{\odot *}$ of $C^{\odot}$, and afterwards we restrict the semigroup $T_e^{\odot *} := \{T_e^{\odot *}(t)\}_{t \geq 0}$ to the closed subspace $C^{\odot\odot}$, for which the semigroup is strongly continuous.

$$\begin{tikzcd}
(C, T_e) \arrow[rr, dashed] \arrow[d, bend right, "j"'] \arrow[d, phantom, "\cong"] & & (C^*, T_e^*) \arrow[d, dashed] \\
(C^{\odot\odot}, T_e^{\odot\odot}) \arrow[u, bend right, "j^{-1}"'] & & (C^{\odot}, T_e^{\odot}) \arrow[dl, dashed] \\
& (C^{\odot *}, T_e^{\odot *}) \arrow[ul, dashed] &
\end{tikzcd}$$

FIGURE 2.1. The repeated star-sun-process and the resulting sun-reflexivity

The original Banach space C together with the strongly continuous semigroup T_e is $\odot$-reflexive in the sense that there is an isometric linear map $j : C \longrightarrow C^{\odot *}$ with $jC = C^{\odot\odot}$ and $T_e^{\odot *}(t)(j\varphi) = j(T_e(t)\,\varphi)$ for all $\varphi \in C$ and $t \geq 0$. We omit the embedding operator j of C in $C^{\odot *}$ and simply identify the Banach space C with $C^{\odot\odot}$ as usual. Schematically, the whole process can be represented as shown in Figure 2.1.

REMARK 2.4. As we had real Banach spaces as underlying structure, our discussion of the linearization and its spectral properties in the last chapter relied on the procedure of complexification and realification. Therefore, it is important to note that these procedures are also compatible with the duality and sun-reflexivity framework presented above. For more details we refer the reader to Diekmann et al. [**6**, Chapter III.7]

The spectrum $\sigma(G_e^{\odot *})$ of the generator $G_e^{\odot *}$ for the semigroup $T_e^{\odot *}$ coincides with $\sigma(G_e)$, and the decomposition (2.3) of C results in the decomposition

$$C^{\odot *} = C_u \oplus C_c \oplus C_s^{\odot *} \tag{2.6}$$

of $C^{\odot *}$, where C_u, C_c, and $C_s^{\odot *}$ are closed and invariant under $T_e^{\odot *}$. Furthermore, there are constants $K \geq 1$, $c_s < 0 < c_u$ and $c_c > 0$ with $c_c < \min\{-c_s, c_u\}$ so that the asymptotic behavior of $T^{\odot *}$ on these subspaces is given by

$$\begin{aligned}
\|T_e(t)\,\varphi\|_C &\leq K e^{c_u t}\|\varphi\|_C, && t \leq 0,\ \varphi \in C_u, \\
\|T_e(t)\,\varphi\|_C &\leq K e^{c_c |t|}\|\varphi\|_C, && t \in \mathbb{R},\ \varphi \in C_c, \\
\|T_e^{\odot *}(t)\,\varphi^{\odot *}\|_{C^{\odot *}} &\leq K e^{c_s t}\|\varphi^{\odot *}\|_{C^{\odot *}}, && t \geq 0,\ \varphi^{\odot *} \in C_s^{\odot *}.
\end{aligned} \tag{2.7}$$

The above decompositions (2.4), (2.6) of C^1 and $C^{\odot *}$ induce continuous projections P_u, P_c, P_s and analogously $P_u^{\odot *}$, $P_c^{\odot *}$, $P_s^{\odot *}$ onto subspaces C_u, C_c, C_s^1, and C_u, C_c, $C_s^{\odot *}$, respectively. Also, using the identification of C with $C^{\odot\odot}$ we see at once $C_s^1 = C^1 \cap C_s^{\odot *}$.

The Variation-of-Constants Formula. Next, we proceed with recalling the variation-of-constant formula for solutions of the inhomogeneous linear RFDE

$$\dot{x}(t) = L_e\, x_t + q(t) \tag{2.8}$$

with given function $q : I \longrightarrow \mathbb{R}^n$ on some interval $I \subset \mathbb{R}$. For this purpose, let $L^\infty([-h,0],\mathbb{R}^n)$ denote the Banach space of all measurable and essentially bounded functions from $[-h,0]$ into $\mathbb{R}^n$, provided with the norm $\|\cdot\|_{L^\infty}$ of essential least upper bound. With the norm

$$\|(\alpha,\varphi)\|_{\mathbb{R}^n\times L^\infty} := \max\{\|\alpha\|_{\mathbb{R}^n}, \|\varphi\|_{L^\infty}\},$$

the product space $\mathbb{R}^n \times L^\infty([-h,0],\mathbb{R}^n)$ becomes also a Banach space, which is in particular isometrically isomorphic to the space $C^{\odot *}$. Using the temporary notation $k : C^{\odot *} \longrightarrow \mathbb{R}^n \times L^\infty([-h,0],\mathbb{R}^n)$ for a norm-preserving isomorphism from $C^{\odot *}$ onto $\mathbb{R}^n \times L^\infty([-h,0],\mathbb{R}^n)$, we define elements $r_i^{\odot *} := k^{-1}(e_i, 0) \in C^{\odot *}$, $i = 1,\ldots,n$, where e_i is the i-th canonical basis vector of $\mathbb{R}^n$. Clearly, the family $\{r_1^{\odot *},\ldots,r_n^{\odot *}\}$ constitutes a basis of the linear subspace $Y^{\odot *} := k^{-1}(\mathbb{R}^n \times \{0\})$ of $C^{\odot *}$, and the requirement $l(e_i) = r_i^{\odot *}$ for $i = 1,\ldots,n$ uniquely determines a linear bijective mapping $l : \mathbb{R}^n \longrightarrow Y^{\odot *}$ with $\|l\| = \|l^{-1}\| = 1$.

For reals $a \le b \le c$ and a (norm) continuous function $w : [a,b] \longrightarrow C^{\odot *}$ the **weak* integral**

$$\int_a^b T_e^{\odot *}(c-\tau)\, w(\tau)\, d\tau \in C^{\odot *} \tag{2.9}$$

is defined by

$$\langle \int_a^b T_e^{\odot *}(c-\tau)\, w(\tau)\, d\tau, \varphi^\odot \rangle := \int_a^b \langle T_e^{\odot *}(c-\tau)\, w(\tau), \varphi^\odot \rangle\, d\tau$$

for $\varphi^\odot \in C^\odot$. Furthermore, set

$$\int_b^a T_e^{\odot *}(c-\tau)\, w(\tau)\, d\tau := -\int_a^b T_e^{\odot *}(c-\tau)\, w(\tau)\, d\tau$$

as usual. It turns out that, under the above condition on w, this weak* integral belongs to C (more precisely, to $C^{\odot\odot} = j(C)$). Additionally, one obtains the formulas

$$T_e^{\odot *}(t) \int_a^b T_e^{\odot *}(c-\tau)\, w(\tau)\, d\tau = \int_a^b T_e^{\odot *}(c+t-\tau)\, w(\tau)\, d\tau \tag{2.10}$$

for all $t \ge 0$,

$$P_\lambda^{\odot *} \int_a^b T_e^{\odot *}(c-\tau)\, w(\tau)\, d\tau = \int_a^b T_e^{\odot *}(c-\tau)\, P_\lambda^{\odot *}\, w(\tau)\, d\tau \tag{2.11}$$

with $\lambda \in \{s,c,u\}$, and finally the inequality

$$\left\| \int_a^b T_e^{\odot *}(c-\tau)\, w(\tau)\, d\tau \right\|_{C^{\odot *}} \le \int_a^b \|T_e^{\odot *}(c-\tau)\, w(\tau)\|_{C^{\odot *}}\, d\tau. \tag{2.12}$$

If $q : I \longrightarrow \mathbb{R}^n$ is a continuous function defined on some interval $I \subseteq \mathbb{R}$ and if $x : I + [-h, 0] \longrightarrow \mathbb{R}^n$ is a solution of the inhomogeneous RFDE (2.8), then the curve $u : I \ni t \longmapsto x_t \in C$ satisfies the abstract integral equation

$$u(t) = T_e(t-s)\,u(s) + \int_s^t T_e^{\odot *}(t-\tau)\,Q(\tau)\,d\tau \tag{2.13}$$

for all $s, t \in I$ with $s \leq t$, where $Q : [s,t] \ni \tau \longmapsto l(q(\tau)) \in Y^{\odot *}$. On the other hand, if $Q : I \longrightarrow Y^{\odot *}$ is continuous, and if $u : I \longrightarrow C$ is a solution of Equation (2.13) then there is a continuous function $x : I + [-h, 0] \longrightarrow \mathbb{R}^n$ with $x_t = u(t)$, $t \in I$, solving the differential equation (2.8) for the inhomogeneity $q : I \ni \tau \longmapsto l^{-1}(Q(\tau)) \in \mathbb{R}^n$. In this sense we have a one-to-one correspondence between solutions for Equations (2.8) and (2.13).

Preliminary Results on Inhomogeneous Linear Equations. As the last step to prepare the construction of local center-unstable manifolds for Equation (2.1), we establish the existence and some properties of special solutions of the integral equation (2.13). In doing so, we will need certain Banach spaces which are introduced below.

Let X be a Banach space with norm $\|\cdot\|_X$. For every $\eta \geq 0$ we define the linear space

$$C_\eta((-\infty,0],X) := \left\{ g \in C((-\infty,0],X) \,\middle|\, \sup_{s\in(-\infty,0]} e^{\eta s}\,\|g(s)\|_X < \infty \right\}$$

where $C((-\infty,0],X)$ denotes the Banach space of all continuous functions from the interval $(-\infty, 0]$ into X. Providing $C_\eta((-\infty,0],X)$ with the weighted supremum norm given by

$$\|g\|_{C_\eta} := \sup_{s\in(-\infty,0]} e^{\eta s}\|g(t)\|_X,$$

we obtain a one-parameter family of Banach spaces with the scaling property

$$C_{\eta_1}((-\infty,0],X) \subseteq C_{\eta_2}((-\infty,0],X)$$

for all $\eta_1 \leq \eta_2$ and

$$\|g\|_{C_{\eta_1}} \geq \|g\|_{C_{\eta_2}}$$

for all $g \in C_{\eta_1}((-\infty,0],X)$. To simplify notation, we use the abbreviations Y_η, C^0_η, and C^1_η, for the spaces $C_\eta((-\infty,0],Y^{\odot *})$, $C_\eta((-\infty,0],C)$, and $C_\eta((-\infty,0],C^1)$, respectively, which are mainly regarded in the sequel.

From now on, let us denote by $P^{\odot *}_{cu}$ the projection of $C^{\odot *}$ along $C^{\odot *}_s$ onto the center-unstable space C_{cu}, that is, $P^{\odot *}_{cu} := P^{\odot *}_u + P^{\odot *}_c$. For a given function $Q : (-\infty, 0] \longrightarrow Y^{\odot *}$ we formally introduce a mapping $\mathcal{K}^{cu}Q$ from $(-\infty, 0]$ into $C^{\odot *}$ by

$$(\mathcal{K}^{cu}Q)(t) := \int_0^t T_e^{\odot *}(t-\tau)\,P^{\odot *}_{cu}\,Q(\tau)\,d\tau + \int_{-\infty}^t T_e^{\odot *}(t-\tau)\,P^{\odot *}_s\,Q(\tau)\,d\tau \tag{2.14}$$

for $t \leq 0$. Note that the right-hand side of Equation (2.14) may not be well-defined for arbitrary Q. However, in our next result we show that for maps $Q \in Y_\eta$ with $\eta \in \mathbb{R}$ such that $c_c < \eta < \min\{-c_s, c_u\}$ the integrals in (2.14) do not only exist, but the functions $\mathcal{K}^{cu}Q$ form also solutions for the abstract integral equation (2.13).

PROPOSITION 2.5. *Let $\eta \in \mathbb{R}$ with $c_c < \eta < \min\{-c_s, c_u\}$ be given. Then Equation (2.14) induces a bounded linear map*

$$\tilde{\mathcal{K}} : Y_\eta \ni Q \longmapsto \mathcal{K}^{cu} Q \in C^0_\eta.$$

In addition, for every $Q \in Y_\eta$ the function $u := \tilde{\mathcal{K}} Q$ is a solution of the integral equation

$$u(t) = T_e(t-s)\, u(s) + \int_s^t T_e^{\odot *}(t-\tau)\, Q(\tau)\, d\tau \tag{2.15}$$

for $-\infty < s \le t \le 0$, and the only one in C^0_η satisfying $P_{cu}^{\odot *}\, u(0) = 0$.

PROOF. The proof falls naturally into three parts. In the first one, we show that, under the stated assumption on $\eta \in \mathbb{R}$, the formal expression (2.14) forms indeed a well-defined mapping $\mathcal{K}^{cu} Q$ from $(-\infty, 0]$ into C for all $Q \in Y_\eta$. Afterwards we prove that $\tilde{\mathcal{K}}$ is a bounded linear operator and finally we conclude the part of the proposition concerning the abstract integral equation. From now on to the end of the proof, we fix $\eta \in \mathbb{R}$ with $c_c < \eta < \min\{-c_s, c_u\}$.

1. In order to see $(\mathcal{K}^{cu} Q)(t) \in C$ for all $Q \in Y_\eta$ and $t \le 0$, recall that for given $Q \in Y_\eta$ and $t \le 0$ both

$$\int_0^t T_e^{\odot *}(t-\tau)\, P_{cu}^{\odot *}\, Q(\tau)\, d\tau = -\int_t^0 T_e^{\odot *}(-\tau)\, T_e^{\odot *}(t)\, P_{cu}^{\odot *}\, Q(\tau)\, d\tau$$

and

$$I(s) := \int_s^t T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau$$

with $s \le t$ belong to C. Hence, it remains to prove the convergence of $I(s)$ in C as $s \to -\infty$. To show this, we assume $\{s_k\}_{k \in \mathbb{N}} \subset (-\infty, t]$ with $s_k \to -\infty$ as $k \to \infty$. Then, by inequality (2.12) and the estimate (2.7) for the action of $T_e^{\odot *}$ on the center space,

$$\begin{aligned}
\|I(s_{k_2}) - I(s_{k_1})\|_{C^{\odot *}} &= \left\| \int_{s_{k_2}}^{s_{k_1}} T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau)\, d\tau \right\|_{C^{\odot *}} \\
&\le \int_{s_{k_2}}^{s_{k_1}} \left\| T_e^{\odot *}(t-\tau)\, P_s^{\odot *}\, Q(\tau) \right\|_{C^{\odot *}} d\tau \\
&\le K\, \|P_s^{\odot *}\| \int_{s_{k_2}}^{s_{k_1}} e^{c_s(t-\tau)}\, \|Q(\tau)\|_{C^{\odot *}}\, d\tau \\
&\le e^{c_s t}\, K\, \|P_s^{\odot *}\| \int_{s_{k_2}}^{s_{k_1}} e^{-(c_s+\eta)\tau}\, e^{\eta\tau}\, \|Q(\tau)\|_{C^{\odot *}} d\tau \\
&\le e^{c_s t}\, K\, \|P_s^{\odot *}\|\, \|Q\|_{Y_\eta} \int_{s_{k_2}}^{s_{k_1}} e^{-(c_s+\eta)\tau}\, d\tau \\
&\le \frac{-e^{c_s t}}{c_s+\eta}\, K\, \|P_s^{\odot *}\|\, \|Q\|_{Y_\eta} \left[e^{-(c_s+\eta)s_{k_1}} - e^{-(c_s+\eta)s_{k_2}} \right] \\
&\le \frac{-e^{c_s t}}{c_s+\eta}\, K\, \|P_s^{\odot *}\|\, \|Q\|_{Y_\eta}\, e^{-(c_s+\eta)s_{k_1}}
\end{aligned}$$

for all $k_1, k_2 \in \mathbb{N}$ with $s_{k_1} \ge s_{k_2}$. Thus, $\{I(s_k)\}_{k \in \mathbb{N}}$ constitutes a Cauchy sequence in C. In particular, $I := \lim_{k\to\infty} I(s_k)$ exists. Furthermore, in the same manner we see that for any another given sequence $\{\tilde{s}_k\}_{k \in \mathbb{N}} \subset (-\infty, t]$ of reals with $\tilde{s}_k \to -\infty$, we also have $\|I(\tilde{s}_k) - I\|_{C^{\odot *}} \to 0$ as $k \to \infty$. This implies the desired conclusion $I = \lim_{s \to -\infty} I(s)$. Hence, $(\mathcal{K}^{cu} Q)(t) \in C$ for all $Q \in Y_\eta$ and $t \le 0$.

2. The technical results in Diekmann et al. [**6**, Chapter III.2] on the continuous dependence of the weak* star integral on parameters and estimates (2.7) enable to show that the induced curve $(-\infty, 0] \ni t \longmapsto (\mathcal{K}^{cu}Q)(t) \in C$ is continuous for every $Q \in Y_\eta$. Consequently, Equation (2.14) defines by $Q \longmapsto \mathcal{K}^{cu}Q$ a mapping from Y_η into $C((-\infty, 0], C)$. This map is also linear. In addition, we claim $\mathcal{K}^{cu}Q \in C^0_\eta$ for all $Q \in Y_\eta$. To this end, consider the apparent inequality

$$\begin{aligned} e^{\eta t} \, \|(\mathcal{K}^{cu}Q)(t)\|_{C^{\odot *}} \leq e^{\eta t} & \left\| \int_0^t T_e^{\odot *}(t-\tau) \, P_c^{\odot *} \, Q(\tau) \, d\tau \right\|_{C^{\odot *}} \\ & + e^{\eta t} \left\| \int_0^t T_e^{\odot *}(t-\tau) \, P_u^{\odot *} \, Q(\tau) \, d\tau \right\|_{C^{\odot *}} \\ & + e^{\eta t} \left\| \int_{-\infty}^t T_e^{\odot *}(t-\tau) \, P_s^{\odot *} \, Q(\tau) \, d\tau \right\|_{C^{\odot *}} \end{aligned}$$

for fixed $Q \in Y_\eta$ and $t \leq 0$. Using the inequalities (2.12) and (2.7) as in the part above, we estimate the first term on the right-hand side by

$$\begin{aligned} e^{\eta t} \left\| \int_0^t T_e^{\odot *}(t-\tau) \, P_c^{\odot *} \, Q(\tau) \, d\tau \right\|_{C^{\odot *}} & \leq -e^{\eta t} \int_0^t \left\| T_e^{\odot *}(t-\tau) \, P_c^{\odot *} \, Q(\tau) \right\|_{C^{\odot *}} \, d\tau \\ & \leq -K \, e^{\eta t} \int_0^t e^{c_c |t-\tau|} \, \|P_c^{\odot *} \, Q(\tau)\|_{C^{\odot *}} d\tau \\ & = -K \int_0^t e^{(c_c - \eta)(\tau - t)} \, e^{\eta \tau} \, \|P_c^{\odot *} \, Q(\tau)\|_{C^{\odot *}} \, d\tau \\ & \leq -K \, \|P_c^{\odot *}\| \int_0^t e^{(c_c - \eta)(\tau - t)} \, e^{\eta \tau} \, \|Q(\tau)\|_{C^{\odot *}} d\tau \\ & \leq K \, \|P_c^{\odot *}\| \, \|Q\|_{Y_\eta} \int_t^0 e^{(c_c - \eta)(\tau - t)} \, d\tau \\ & \leq K \, \|P_c^{\odot *}\| \, \|Q\|_{Y_\eta} \, \frac{1}{\eta - c_c} \; . \end{aligned}$$

In the same manner we can see that

$$e^{\eta t} \left\| \int_0^t T_e^{\odot *}(t-\tau) \, P_u^{\odot *} \, Q(\tau) \, d\tau \right\|_{Y^{\odot *}} \leq K \, \|P_u^{\odot *}\| \, \|Q\|_{Y_\eta} \, \frac{1}{c_u + \eta}$$

and

$$e^{\eta t} \left\| \int_{-\infty}^t T_e^{\odot *}(t-\tau) \, P_s^{\odot *} \, Q(\tau) \, d\tau \right\|_{Y^{\odot *}} \leq K \, \|P_s^{\odot *}\| \, \|Q\|_{Y_\eta} \, \frac{1}{-c_s - \eta}.$$

Summarizing, we get

$$e^{\eta t} \, \|(\mathcal{K}^{cu}Q)(t)\|_{Y^{\odot *}} \leq K \, \|Q\|_{Y_\eta} \left(\frac{\|P_c^{\odot *}\|}{\eta - c_c} + \frac{\|P_u^{\odot *}\|}{c_u + \eta} - \frac{\|P_s^{\odot *}\|}{c_s + \eta} \right), \tag{2.16}$$

and thus $\mathcal{K}^{cu}Q \in C^0_\eta$. It follows that $Q \longmapsto \mathcal{K}^{cu}Q$ forms a linear mapping $\tilde{\mathcal{K}}$ from Y_η into C^0_η, which in particular is bounded as claimed.

3. Given any $Q \in Y_\eta$ define $\delta(t,s) := (\mathcal{K}^{cu}Q)(t) - T_e(t-s)\big((\mathcal{K}^{cu}Q)(s)\big)$ for all reals $-\infty < s \le t \le 0$. Then, by the linearity and formula (2.10), we get

$$\begin{aligned}
\delta(t,s) &= \int_0^t T_e^{\odot*}(t-\tau)\, P_{cu}^{\odot*}\, Q(\tau)\, d\tau + \int_{-\infty}^t T_e^{\odot*}(t-\tau)\, P_s^{\odot*}\, Q(\tau)\, d\tau \\
&\quad - T_e(t-s)\left(\int_0^s T_e^{\odot*}(s-\tau)\, P_{cu}^{\odot*}\, Q(\tau)\, d\tau + \int_{-\infty}^s T_e^{\odot*}(s-\tau)\, P_s^{\odot*}\, Q(\tau)\, d\tau \right) \\
&= \int_0^t T_e^{\odot*}(t-\tau)\, P_{cu}^{\odot*}\, Q(\tau)\, d\tau + \int_{-\infty}^t T_e^{\odot*}(t-\tau)\, P_s^{\odot*}\, Q(\tau)\, d\tau \\
&\quad - \int_0^s T_e^{\odot*}(t-\tau)\, P_{cu}^{\odot*}\, Q(\tau)\, d\tau - \int_{-\infty}^s T_e^{\odot*}(t-\tau)\, P_s^{\odot*}\, Q(\tau)\, d\tau \\
&= \int_s^t T_e^{\odot*}(t-\tau)\, P_{cu}^{\odot*}\, Q(\tau)\, d\tau + \int_s^t T_e^{\odot*}(t-\tau)\, P_s^{\odot*}\, Q(\tau)\, d\tau \\
&= \int_s^t T_e^{\odot*}(t-\tau)\, Q(\tau)\, d\tau,
\end{aligned}$$

which yields that $u := \mathcal{K}^{cu}Q$ satisfies Equation (2.15) for all $-\infty < s \le t \le 0$. Moreover, in view of Equation (2.11) for the relation of the weak* integrals and projections on the decomposition of $C^{\odot*}$, for $t = 0$ we have

$$\begin{aligned}
u(0) &= (\mathcal{K}^{cu}Q)(0) \\
&= \int_{-\infty}^0 T_e^{\odot*}(-\tau)\, P_s^{\odot*}\, Q(\tau)\, d\tau \\
&= P_s^{\odot*}\left(\int_{-\infty}^0 T_e^{\odot*}(-\tau)\, Q(\tau)\, d\tau \right)
\end{aligned}$$

implying $P_{cu}^{\odot*}\, u(0) = 0$.

So the assertion of the proposition follows if we are able to prove that u is the only solution of Equation (2.15) in C_η^0 with vanishing C_{cu} component at $t = 0$. For this purpose, suppose $v \in C_\eta^0$ is also a solution of (2.15) for $-\infty < s \le t \le 0$ with $P_{cu}^{\odot*}\, v(0) = 0$. Then the difference $w := u - v$ belongs to C_η^0, has a vanishing C_{cu} component at $t = 0$, and satisfies the equation

$$w(t) = T_e(t-s)\, w(s) \tag{2.17}$$

for all $-\infty < s \le t \le 0$. Furthermore, w can be extended by

$$t \longmapsto \begin{cases} w(t), & \text{for } t \le 0, \\ T(t)\, w(0), & \text{for } t \ge 0 \end{cases}$$

to a solution $\tilde{w} : \mathbb{R} \longrightarrow C$ of Equation (2.17) for all $-\infty < s \le t < \infty$. Since

$$\begin{aligned}
\sup_{t\ge 0} e^{-\eta t}\, \|w(t)\|_C &= \sup_{t\ge 0} e^{-\eta t}\, \|T_e(t)\, w(0)\|_C \\
&\le K \sup_{t\ge 0} e^{-\eta t}\, e^{c_s t}\, \|w(0)\|_C \\
&= K\, \|w(0)\|_C
\end{aligned}$$

due to $(c_s - \eta) < 0$ we get

$$\begin{aligned}
\sup_{t\in\mathbb{R}} e^{-\eta|t|}\, \|\tilde{w}(t)\|_C &\le \sup_{t\le 0} e^{\eta t}\, \|\tilde{w}(t)\|_C + \sup_{t\ge 0} e^{-\eta t}\, \|\tilde{w}(t)\|_C \\
&= \|w\|_{C_\eta^0} + K\, \|w(0)\|_C < \infty.
\end{aligned}$$

Now from Diekmann et al. [**6**, Lemma 2.4 in Section IX.2] it follows $w(0) \in C_u$ and $\tilde{w}(0) \in C_c$. As $w(0) = \tilde{w}(0)$ and $C_u \cap C_c = \{0\}$, we conclude $\tilde{w}(0) = w(0) = 0$, and so by Equation (2.17),

$$0 = T_e(s)\, w(0) = T_e(s)\, T_e(-s)\, w(s) = T_e(0)\, w(s) = u(s) - v(s)$$

for all $-\infty < s \leq 0$. This completes the proof. $\square$

Next, we prove a smoothing property of the integral equation (2.18). This property will be useful in combination with our preceding result.

PROPOSITION 2.6. *Suppose that $Q \in Y_\eta$ for some $\eta \geq 0$. If $u \in C^0_\eta$ satisfies the abstract integral equation*

$$u(t) = T_e(t-s)\, u(s) + \int_s^t T_e^{\odot *}(t-\tau)\, Q(\tau)\, d\tau \tag{2.18}$$

for all $-\infty < s \leq t \leq 0$, then $u \in C^1_\eta$ and

$$\|u\|_{C^1_\eta} \leq \left(1 + e^{\eta h}\, \|L_e\|\right) \|u\|_{C^0_\eta} + e^{\eta h}\, \|Q\|_{Y_\eta}.$$

PROOF. Consider the mapping $q : (-\infty, 0] \longrightarrow \mathbb{R}^n$ defined by $q(t) = l^{-1}(Q(t))$, $-\infty < t \leq 0$. Of course, $q \in C((-\infty,0],\mathbb{R}^n)$. Moreover, since

$$\begin{aligned}
\sup_{t\in(-\infty,0]} e^{\eta t}\, \|q(t)\|_{\mathbb{R}^n} &= \sup_{t\in(-\infty,0]} e^{\eta t}\, \|l^{-1}(Q(t))\|_{\mathbb{R}^n} \\
&= \sup_{t\in(-\infty,0]} e^{\eta t}\, \|Q(t)\|_{Y^{\odot *}} \\
&= \|Q\|_{Y_\eta}
\end{aligned}$$

we see at once $q \in C_\eta((-\infty,0],\mathbb{R}^n)$ with $\|q\|_{C_\eta} = \|Q\|_{Y_\eta}$.

By assumption, u satisfies Equation (2.18) such that, taking into account our discussion about the one-to-one correspondence between solutions for Equations (2.8) and (2.13), the function $x : (\infty, 0] \longrightarrow \mathbb{R}^n$ given by $x(t) = u(t)(0)$ is a solution of the differential equation

$$\dot{x}(t) = L_e\, x_t + q(t)$$

for all $-\infty < t \leq 0$. Accordingly, x is everywhere continuously differentiable, x_t belongs to C^1 for all $-\infty < t \leq 0$, and the map $(-\infty, 0] \ni t \longmapsto u(t) = x_t \in C^1$ is continuous. Furthermore, by the differential equation for x and the estimate for q, we have

$$\begin{aligned}
\|\dot{x}(t)\|_{\mathbb{R}^n} &\leq \|L_e\|\, \|x_t\|_C + \|q(t)\|_{\mathbb{R}^n} \\
&\leq \|L_e\|\, \|u(t)\|_C + e^{-\eta t}\, \|q\|_{C_\eta} \\
&\leq e^{-\eta t}\, (\|L_e\|\, \|u\|_{C^0_\eta} + \|Q\|_{Y_\eta})
\end{aligned}$$

and therefore

$$\begin{aligned}
\sup_{t\in(-\infty,0]} e^{\eta t}\, \|\dot{x}_t\|_C &= \sup_{t\in(-\infty,0]} \left(e^{\eta t} \sup_{\vartheta\in[-h,0]} \|\dot{x}(t+\vartheta)\|_{\mathbb{R}^n} \right) \\
&\leq (\|L_e\|\, \|u\|_{C^0_\eta} + \|Q\|_{Y_\eta}) \sup_{t\in(-\infty,0]} \left(e^{\eta t} \sup_{\vartheta\in[-h,0]} e^{-\eta(t+\vartheta)} \right) \\
&\leq e^{\eta h} (\|L_e\|\, \|u\|_{C^0_\eta} + \|Q\|_{Y_\eta}),
\end{aligned}$$

for all $-\infty < t \leq 0$. From this, it follows that $u \in C^1_\eta$ and

$$\begin{aligned}\|u\|_{C^1_\eta} &= \sup_{t\in(-\infty,0]} e^{\eta t}\|u(t)\|_{C^1}\\ &= \sup_{t\in(-\infty,0]} e^{\eta t}\|x_t\|_{C^1}\\ &= \sup_{t\in(-\infty,0]} e^{\eta t}(\|x_t\|_C + \|\dot{x}_t\|_C)\\ &\leq \|u\|_{C^0_\eta} + e^{\eta h}(\|L_e\|\|u\|_{C^0_\eta} + \|Q\|_{Y_\eta})\end{aligned}$$

as claimed. □

As an easy consequence of the last two results we conclude that the formal definition (2.14) generates a bounded linear mapping from the Banach space Y_η into C^1_η for $c_c < \eta < \min\{-c_s, c_u\}$.

COROLLARY 2.7. *For each $\eta \in \mathbb{R}$ with $c_c < \eta < \min\{-c_s, c_u\}$, relation* (2.14) *defines a bounded linear mapping*

$$\mathcal{K}_\eta : Y_\eta \ni Q \longmapsto \mathcal{K}^{cu}Q \in C^1_\eta$$

with

$$\|\mathcal{K}_\eta\| \leq K\,(1 + e^{\eta h}\,\|L_e\|)\left(\frac{\|P_c^{\odot*}\|}{\eta - c_c} + \frac{\|P_u^{\odot*}\|}{c_u + \eta} - \frac{\|P_s^{\odot*}\|}{c_s + \eta}\right) + e^{\eta h}.$$

Moreover, for all $Q \in Y_\eta$ the function $u := \mathcal{K}_\eta Q$ is a solution of

$$u(t) = T_e(t-s)\,u(s) + \int_s^t T_e^{\odot*}(t-\tau)\,Q(\tau)\,d\tau$$

for $-\infty < s \leq t \leq 0$, and the only one in C^1_η with $P_{cu}^{\odot*}\,u(0) = 0$.

PROOF. Apply Propositions 2.5 and 2.6, taking into account the estimate (2.16) for the bound of the linear map $\tilde{\mathcal{K}}$. □

REMARK 2.8. Observe that the bounds of the linear maps $\mathcal{K}_\eta$ in the above corollary are given by a continuous function in η. This will be a crucial point in the proof of Theorem 2.2.

4. The Construction of Local Center-Unstable Manifolds

This section is devoted to the actual proof of Theorem 2.1 about the existence of local center-unstable manifolds for Equation (2.1). Throughout the proof, we consider the differential equation (2.1) in the equivalent form

$$\dot{x}(t) = L\,x_t + r(x_t) \tag{2.19}$$

with the nonlinearity

$$r : U \ni \varphi \longmapsto f(\varphi) - L\,\varphi \in \mathbb{R}^n. \tag{2.20}$$

Obviously, r also satisfies the same smoothness conditions (S 1) and (S 2) as f and we have $r(0) = 0$ and $Dr(0) = 0$.

The proof is organized as follows. In the first part, we modify the nonlinearity r outside a small neighborhood of the origin and assign the resulting differential equation to an abstract integral equation by the variation-of-constants formula. Then, using the changes on the nonlinearity in combination with the auxiliary conclusions of the last section, we show that the associated integral operator forms a parameter-dependent contraction in C^1_η for an appropriate $\eta > 0$. In the final step, we prove that the graph of this contraction is an invariant manifold for the modified differential equation and that a part of this graph also satisfies the assertions of Theorem 2.1.

Smoothing Modification of the Nonlinearity. As the Banach space C_{cu} is finite-dimensional, there exists a norm $\|\cdot\|_{cu}$ on C_{cu} being infinitely often continuously differentiable on $C_{cu}\setminus\{0\}$. Introducing the projection $P_{cu} := P_c + P_u$ of C^1 along C^1_s onto the center-unstable space C_{cu} and defining

$$\|\varphi\|_1 := \max\left\{\|P_{cu}\,\varphi\|_{cu}, \|P_s\,\varphi\|_{C^1}\right\} \tag{2.21}$$

for $\varphi \in C^1$, we get a second norm on C^1, which is equivalent to $\|\cdot\|_{C^1}$.

Let $\varrho : [0,\infty) \longrightarrow \mathbb{R}$ be a C^∞-smooth function with $\varrho(t) = 1$ for $0 \leq t \leq 1$, $0 < \varrho(t) < 1$ for $1 < t < 2$, and $\varrho(t) = 0$ for all $t \geq 2$. Further, let the map $\hat{r} : C^1 \longrightarrow \mathbb{R}^n$ be given by

$$\hat{r}(\varphi) := \begin{cases} r(\varphi), & \text{for } \varphi \in U, \\ 0, & \text{for } \varphi \notin U. \end{cases}$$

Using these two functions, we introduce for all $\delta > 0$ the smoothing modification

$$r_\delta : C^1 \ni \varphi \longmapsto \varrho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right) \cdot \varrho\left(\frac{\|\varphi_s\|_{C^1}}{\delta}\right) \cdot \hat{r}(\varphi) \in \mathbb{R}^n$$

of the nonlinearity r, where we write φ_{cu}, φ_s for the components $P_{cu}\,\varphi$, $P_s\,\varphi$ of φ, respectively.

For every $\gamma > 0$ let $B_\gamma(0) := \{\varphi \in C^1 \,|\, \|\varphi\|_1 < \gamma\}$ denote the open ball in C^1 of radius γ with respect to the $\|\cdot\|_1$-norm and centered at the origin. Since $U \subset C^1$ is open and r continuously differentiable due to property (S 1), we find a sufficiently small $\delta_0 > 0$ with $B_{2\delta_0}(0) \subset U$, so that the restriction $r|_{B_{2\delta_0}(0)}$ of r to $B_{2\delta_0}(0)$ together with the associated derivative $Dr|_{B_{2\delta_0}(0)}$ are both bounded. Subsequently, for small reals $\delta > 0$, the modifications of r in a neighborhood of the origin are also bounded and continuously differentiable with bounded derivatives. More precisely, the following result holds.

COROLLARY 2.9. *For all reals $0 < \delta < \delta_0$ the restriction of the map r_δ to the strip $S := \left\{\psi \in C^1 \middle|\ \|\psi_s\|_1 < \delta\right\}$ in C^1 is a bounded, C^1-smooth function with bounded derivative. Moreover,*

$$r_\delta(\varphi) = \varrho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right) \cdot r(\varphi)$$

for all $\varphi \in S$.

PROOF. Given any positive constant $0 < \delta < \delta_0$ suppose that $\varphi \in S$. Then, by definition of r_δ in combination with the inequality $\|\varphi_s\|_{C^1} \leq \|\varphi_s\|_1$ we get

$$r_\delta(\varphi) = \varrho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right) \cdot \varrho\left(\frac{\|\varphi_s\|_{C^1}}{\delta}\right) \cdot \hat{r}(\varphi) = \varrho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right) \cdot r(\varphi).$$

Consequently, we have $r_\delta(\varphi) = r(\varphi)$ for all $\varphi \in S$ with $\|\varphi\|_1 \leq \delta$, and $r_\delta(\varphi) = 0$ for all $\varphi \in S$ with $\|\varphi\|_1 \geq 2\delta$. Since r, ϱ are C^1-smooth and the norm $\|\cdot\|_1$ continuously differentiable on $C_{cu}\setminus\{0\}$ by assumption, the restriction of r_δ to the strip S is clearly also continuously differentiable. Moreover, using the above expressions for r_δ on S together with the boundedness of r and Dr on $B_{2\delta_0}(0) \subset U$, we conclude that both r_δ and Dr_δ are bounded on S as claimed. $\square$

For sufficiently small $\delta > 0$, the functions r_δ are even globally bounded and Lipschitz continuous with constants continuously depending on δ, as proved next.

PROPOSITION 2.10. *[Proposition II.2 in Krisztin et al.* [**18**]*] Under the above assumptions there exists* $\delta_1 \in (0, \delta_0)$ *and a monotone increasing* $\lambda : [0, \delta_1] \longrightarrow [0, 1]$ *with* $\lambda(0) = 0$ *and* $\lambda(\delta) \searrow 0$ *as* $\delta \searrow 0$ *such that*

$$\|r_\delta(\varphi)\|_{\mathbb{R}^n} \leq \delta \cdot \lambda(\delta)$$

and

$$\|r_\delta(\varphi) - r_\delta(\psi)\|_{\mathbb{R}^n} \leq \lambda(\delta) \cdot \|\varphi - \psi\|_{C^1}$$

for all $0 < \delta \leq \delta_1$ *and* $\varphi, \psi \in C^1$.

PROOF. We follow the proof given in Krisztin et al. [**18**, Proposition II.2] and establish the assertion with respect to the $\|\cdot\|_1$-norm on C^1 first. Afterwards we use the equivalence of $\|\cdot\|_1$ and $\|\cdot\|_{C^1}$ to conclude the statement. On that account, we choose $K_1 \geq 1$ with the property $\|\varphi\|_1 \leq K_1\|\varphi\|_{C^1}$ for all $\varphi \in C^1$.

1. By our assumptions, we have $r(0) = 0$ and r is continuously differentiable on U with $Dr(0) = 0$. Combining these, we find a constant $\alpha_0 \in (0, 2\delta_0)$ and therewith a monotonous increasing function $\lambda_0 : [0, \alpha_0] \longrightarrow [0, \infty)$ with

$$\lim_{\delta \searrow 0} \lambda_0(\delta) = 0 = \lambda_0(0)$$

such that $\{\varphi \in C^1 | \ \|\varphi\|_1 < \alpha_0\} \subset U$ and

$$\|r(\varphi) - r(\psi)\|_{\mathbb{R}^n} \leq \lambda_0(\alpha) \cdot \|\varphi - \psi\|_1$$

for every $0 < \alpha \leq \alpha_0$ and all $\varphi, \psi \in B_\alpha(0)$. We may even assume

$$6 \cdot \lambda_0(\alpha_0) \cdot K_1 \cdot \max\left\{\sup_{s \geq 0}|\varrho'(s)|, 1\right\} \leq 1.$$

Set $\delta_1 := \alpha_0/4$. We claim

$$\|r_\delta(\varphi)\|_{\mathbb{R}^n} \leq \lambda_0(\delta) \cdot 2\delta$$

for all $(\delta, \varphi) \in (0, \delta_1] \times C^1$. Indeed, in case $\|\varphi\|_1 \geq 2\delta$ either $\|\varphi_{cu}\|_{cu} \geq 2\delta$ or $\|\varphi_s\|_{C^1} \geq 2\delta$ by Equation (2.21), and thus $r_\delta(\varphi) = 0$. On the other hand, if $\|\varphi\|_1 < 2\delta$ then we have $\|\varphi\|_1 < \alpha_0$, which implies $\varphi \in U$ and

$$\|r_\delta(\varphi)\|_{\mathbb{R}^n} \leq \|\hat{r}(\varphi)\|_{\mathbb{R}^n} = \|r(\varphi)\|_{\mathbb{R}^n} \leq \lambda_0(2\delta) \cdot \|\varphi\|_1 \leq \lambda_0(2\delta) \cdot 2\delta.$$

2. Suppose now $\delta \in (0, \delta_1]$ and $\varphi, \psi \in C^1$. Then, by definition, we have

$$\begin{aligned}
\|r_\delta(\varphi) - r_\delta(\psi)\|_{\mathbb{R}^n} &= \Bigg\| \varrho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right) \cdot \varrho\left(\frac{\|\varphi_s\|_{C^1}}{\delta}\right) \cdot \hat{r}(\varphi) \\
&\qquad - \varrho\left(\frac{\|\psi_{cu}\|_{cu}}{\delta}\right) \cdot \varrho\left(\frac{\|\psi_s\|_{C^1}}{\delta}\right) \cdot \hat{r}(\psi)\Bigg\|_{\mathbb{R}^n} \\
&\leq \|\hat{r}(\varphi) - \hat{r}(\psi)\|_{\mathbb{R}^n} \cdot \varrho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right) \cdot \varrho\left(\frac{\|\varphi_s\|_{C^1}}{\delta}\right) + \|\hat{r}(\psi)\|_{\mathbb{R}^n} \\
&\quad \cdot \left| \varrho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right) \cdot \varrho\left(\frac{\|\varphi_s\|_{C^1}}{\delta}\right) - \varrho\left(\frac{\|\psi_{cu}\|_{cu}}{\delta}\right) \cdot \varrho\left(\frac{\|\psi_s\|_{C^1}}{\delta}\right)\right|.
\end{aligned}$$

If $\|\varphi\|_1 \geq 2\delta$ and $\|\psi\|_1 \geq 2\delta$ then similar arguments to the first part, based on the definitions of $\|\cdot\|_1$ and ϱ, immediately imply

$$\|r_\delta(\varphi) - r_\delta(\psi)\|_{\mathbb{R}^n} = 0.$$

In case $\|\varphi\|_1 \geq 2\delta$ and $\|\psi\|_1 \leq 2\delta$ we have either $\|\varphi_{cu}\|_{cu} \geq 2\delta$ or $\|\varphi_s\|_{C^1} \geq 2\delta$. Both subcases result in the estimate

$$\begin{aligned}
\|r_\delta(\varphi) - r_\delta(\psi)\|_{\mathbb{R}^n} &\leq \|\hat{r}(\psi)\|_{\mathbb{R}^n} \cdot \Bigg| \varrho\Big(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\Big) \cdot \varrho\Big(\frac{\|\varphi_s\|_{C^1}}{\delta}\Big) \\
&\qquad\qquad - \varrho\Big(\frac{\|\psi_{cu}\|_{cu}}{\delta}\Big) \cdot \varrho\Big(\frac{\|\psi_s\|_{C^1}}{\delta}\Big) \Bigg| \\
&\leq \lambda_0(2\delta) \cdot 2\delta \cdot \Bigg| \varrho\Big(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\Big) \cdot \varrho\Big(\frac{\|\varphi_s\|_{C^1}}{\delta}\Big) \\
&\qquad\qquad - \varrho\Big(\frac{\|\psi_{cu}\|_{cu}}{\delta}\Big) \cdot \varrho\Big(\frac{\|\psi_s\|_{C^1}}{\delta}\Big) \Bigg| \\
&= \lambda_0(2\delta) \cdot 2\delta \cdot \Bigg| \varrho\Big(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\Big) \cdot \Big(\varrho\Big(\frac{\|\varphi_s\|_{C^1}}{\delta}\Big) - \varrho\Big(\frac{\|\psi_s\|_{C^1}}{\delta}\Big) \Big) \\
&\qquad + \Big(\varrho\Big(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\Big) - \varrho\Big(\frac{\|\psi_{cu}\|_{cu}}{\delta}\Big) \Big) \cdot \varrho\Big(\frac{\|\psi_s\|_{C^1}}{\delta}\Big) \Bigg| \\
&\leq \lambda_0(2\delta) \cdot 2\delta \cdot \Bigg(\varrho\Big(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\Big) \cdot \Big| \varrho\Big(\frac{\|\varphi_s\|_{C^1}}{\delta}\Big) - \varrho\Big(\frac{\|\psi_s\|_{C^1}}{\delta}\Big) \Big| \\
&\qquad + \varrho\Big(\frac{\|\psi_s\|_{C^1}}{\delta}\Big) \cdot \Big| \varrho\Big(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\Big) - \varrho\Big(\frac{\|\psi_{cu}\|_{cu}}{\delta}\Big) \Big| \Bigg) \\
&\leq \lambda_0(2\delta) \cdot 2\delta \cdot \Bigg(\sup_{s\geq 0} |\varrho'(s)| \cdot \frac{|\, \|\varphi_s\|_{C^1} - \|\psi_s\|_{C^1}|}{\delta} \\
&\qquad\qquad + \sup_{s\geq 0} |\varrho'(s)| \cdot \frac{|\, \|\varphi_{cu}\|_{cu} - \|\psi_{cu}\|_{cu}|}{\delta} \Bigg) \\
&\leq 4\lambda_0(2\delta) \cdot \sup_{s\geq 0} |\varrho'(s)| \cdot \|\varphi - \psi\|_1.
\end{aligned}$$

In the situation $\varphi, \psi \in B_{2\delta}(0)$, by combining the same methods as above we get the inequality

$$\|r_\delta(\varphi) - r_\delta(\psi)\|_{\mathbb{R}^n} \leq \lambda_0(2\delta) \cdot \|\varphi - \psi\|_1 + 4\lambda_0(2\delta) \cdot \sup_{s\geq 0} |\varrho'(s)| \cdot \|\varphi - \psi\|_1.$$

Summarizing, we conclude

$$\|r_\delta(\varphi) - r_\delta(\psi)\|_{\mathbb{R}^n} \leq \Big(\lambda_0(2\delta) + 4\lambda_0(2\delta) \cdot \sup_{s\geq 0} \cdot |\varrho'(s)| \Big) \cdot \|\varphi - \psi\|_1$$

for every $0 < \delta \leq \delta_0$ and all $\varphi, \psi \in C^1$.

3. Defining the function $\lambda : [0, \delta_1] \longrightarrow [0, \infty)$ by

$$\lambda(\delta) := 6\lambda_0(\delta) \cdot K_1 \cdot \max\Big\{ \sup_{s\geq 0} |\varrho'(s)|, 1 \Big\}$$

and applying the preliminary steps, one easily proves the properties of λ as given in the assertion. This completes the proof. □

Using the modification r_δ of the nonlinearity r, we introduce for each $0 < \delta \leq \delta_1$ the retarded functional differential equation

$$\dot{x}(t) = L\, x_t + r_\delta(x_t), \qquad -\infty < t \leq 0, \tag{2.22}$$

and the associated abstract integral equations

$$u(t) = T_e(t-s)\, u(s) + \int_s^t T_e^{\odot *}(t-\tau)\, l(r_\delta(u(\tau)))\, d\tau, \quad -\infty < s \leq t \leq 0. \tag{2.23}$$

We have now a one-to-one correspondence in the following sense: If the function $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ is a continuously differentiable solution of RFDE (2.22), then $u : (-\infty, 0] \longmapsto x_t \in C^1$ is a solution of Equation (2.23). On the other hand, for a continuous mapping $u : (-\infty, 0] \longrightarrow C^1$ satisfying integral equation (2.23), the function $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ defined by $x(t) = u(t)(0)$, $-\infty < t \leq 0$, forms a continuously differentiable solution of (2.22).

Center-Unstable Manifolds of the Smoothed Equation. Until the end of this section fix $\eta \in \mathbb{R}$ satisfying the estimate

$$c_c < \eta < \min\{-c_s, c_u\}. \tag{2.24}$$

Then we find a constant $0 < \delta < \delta_1$ with

$$\|\mathcal{K}_\eta\| \, \lambda(\delta) < \frac{1}{2} \tag{2.25}$$

where the mappings $\mathcal{K}_\eta$ and λ are defined in Corollary 2.7 and Proposition 2.10, respectively. Below, we construct a parameter-dependent contraction on the Banach space C^1_η, such that the fixed points will form solutions for the abstract integral equation (2.23). For this purpose, we assign to Equation (2.23) an integral operator. We begin with the non-linear part.

COROLLARY 2.11. *Let R denote the map, which assigns to $u \in C((-\infty, 0], C^1)$ the mapping $(-\infty, 0] \ni s \longmapsto l(r_\delta(u(s))) \in Y^{\odot*}$ in $C((-\infty, 0], Y^{\odot*})$. Then R maps C^1_η into Y_η, and the induced mapping $R_{\delta\eta} : C^1_\eta \ni u \longmapsto R(u) \in Y_\eta$ satisfies*

$$\|R_{\delta\eta}(u)\|_{Y_\eta} \leq \delta \, \lambda(\delta) \tag{2.26}$$

and

$$\|R_{\delta\eta}(u) - R_{\delta\eta}(v)\|_{Y_\eta} \leq \lambda(\delta) \, \|u - v\|_{C^1_\eta} \tag{2.27}$$

for all $u, v \in C^1$.

PROOF. First, note that R indeed assigns a continuous function from $(-\infty, 0]$ into $Y^{\odot*}$ to a function $u \in C((-\infty, 0], C^1)$, as the mappings l and r_δ are continuous. Given $u \in C^1_\eta$, Proposition 2.10 implies

$$\begin{aligned}
\sup_{t \in (-\infty, 0]} e^{\eta t} \, \|R(u)(t)\|_{Y^{\odot*}} &= \sup_{t \in (-\infty, 0]} e^{\eta t} \, \|l(r_\delta(u(t)))\|_{Y^{\odot*}} \\
&= \sup_{t \in (-\infty, 0]} e^{\eta t} \, \|r_\delta(u(t))\|_{\mathbb{R}^n} \\
&\leq \sup_{t \in (-\infty, 0]} e^{\eta t} \, \delta \, \lambda(\delta) \\
&= \delta \, \lambda(\delta).
\end{aligned}$$

This shows $R(C^1_\eta) \subset Y_\eta$ and in particular the boundedness of $R_{\delta\eta}$ by $\delta\lambda(\delta)$ as claimed. Using the Lipschitz continuity of r_δ from Proposition 2.10, we also see that $R_{\delta\eta}$ is Lipschitz continuous with Lipschitz constant $\lambda(\delta)$, and the corollary follows. $\square$

REMARK 2.12. The mapping $R : C((-\infty, 0], C^1) \longrightarrow C((-\infty, 0], Y^{\odot*})$ in the last result is called the **substitution** or also the **Nemitsky operator** of the mapping $C^1 \ni \varphi \longmapsto l(r_\delta(\varphi)) \in Y^{\odot*}$ on $(-\infty, 0]$.

Next, we consider the linear part of the integral equation (2.23) and prove that it constitutes a bounded linear operator from the center-unstable space into C^1_η.

COROLLARY 2.13. *For each $\varphi \in C_{cu}$, the curve $(-\infty, 0] \ni t \longmapsto T_e(t)\,\varphi \in C^1$ belongs to C^1_η, and $S_\eta : C^1 \supset C_{cu} \longrightarrow C^1_\eta$ defined by $(S_\eta\,\varphi)(t) = T_e(t)\,\varphi$ for $\varphi \in C_{cu}$ and $t \leq 0$ is a bounded linear operator with*

$$\|S_\eta\| \leq K\left(\|P_c^{\odot *}\| + \|P_u^{\odot *}\|\right). \tag{2.28}$$

PROOF. To start with, recall that T_e defines a group on $C_{cu} \subset C^1$ and coincides with T. Thus, for all $\varphi \in C_{cu}$, the curve $(-\infty, 0] \ni t \longmapsto T_e(t)\,\varphi \in C_{cu}$ takes values in C^1 and is in fact a continuous map from $(-\infty, 0]$ into C^1. Furthermore, we have

$$\|T_e(t)\,\varphi\|_{C^1} = \|T_e(t)\,\varphi\|_C + \left\|\tfrac{d}{dt}T_e(t)\,\varphi\right\|_C$$

and

$$\tfrac{d}{dt}(T_e(t)\,\varphi) = T_e(t)\,G_e\,\varphi = T_e(t)\,\varphi'$$

for $\varphi \in C_{cu}$. Hence, by the exponential trichotomy under our assumption (2.24), it follows

$$\begin{aligned}
\sup_{t\in(-\infty,0]} e^{\eta t}\,\|T_e(t)\,\varphi\|_{C^1} &= \sup_{t\in(-\infty,0]} e^{\eta t}\left(\|T_e(t)\,\varphi\|_C + \|T_e(t)\,\varphi'\|_C\right)\\
&\leq \sup_{t\in(-\infty,0]} e^{\eta t}\Big(\|T_e(t)\,P_c^{\odot *}\,\varphi\|_C + \|T_e(t)\,P_u^{\odot *}\,\varphi\|_C\\
&\qquad + \|T_e(t)\,P_c^{\odot *}\,\varphi'\|_C + \|T_e(t)\,P_u^{\odot *}\,\varphi'\|_C\Big)\\
&\leq \sup_{t\in(-\infty,0]} e^{\eta t}\left(\|T_e(t)\,P_c^{\odot *}\,\varphi\|_C + \|T_e(t)\,P_c^{\odot *}\,\varphi'\|_C\right)\\
&\quad + \sup_{t\in(-\infty,0]} e^{\eta t}\left(\|T_e(t)\,P_u^{\odot *}\,\varphi\|_C + \|T_e(t)\,P_u^{\odot *}\,\varphi'\|_C\right)\\
&\leq K \sup_{t\in(-\infty,0]} e^{-(c_c-\eta)t}\left(\|P_c^{\odot *}\,\varphi\|_C + \|P_c^{\odot *}\,\varphi'\|_C\right)\\
&\qquad + K \sup_{t\in(-\infty,0]} e^{(\eta+c_u)t}\left(\|P_u^{\odot *}\,\varphi\|_C + \|P_u^{\odot *}\,\varphi'\|_C\right)\\
&\leq K\,\|P_c^{\odot *}\|\left(\|\varphi\|_C + \|\varphi'\|_C\right) +\\
&\qquad K\,\|P_u^{\odot *}\|\left(\|\varphi\|_C + \|\varphi'\|_C\right)\\
&= K\left(\|P_c^{\odot *}\| + \|P_u^{\odot *}\|\right)\|\varphi\|_{C^1}.
\end{aligned}$$

Accordingly, $S_\eta\,\varphi \in C^1_\eta$ for $\varphi \in C_{cu}$, and thus S_η is well-defined. In addition, the mapping S_η is obviously linear by definition, and

$$\|S_\eta\varphi\|_{C^1_\eta} \leq K(\|P_c^{\odot *}\| + \|P_u^{\odot *}\|)$$

for $\|\varphi\|_{C^1} \leq 1$. Therefore, inequality (2.28) holds and this completes the proof. $\square$

Using Corollaries 2.7, 2.11, and 2.13 to guarantee the well-definedness, we introduce the mapping $\mathcal{G}_\eta$ from the product space $C^1_\eta \times C_{cu}$ into C^1_η given by

$$\mathcal{G}_\eta(u, \varphi) = S_\eta\,\varphi + \mathcal{K}_\eta \circ R_{\delta\eta}(u). \tag{2.29}$$

In the next proposition we prove that each function $\varphi \in C_{cu}$ uniquely determines a solution of $u = \mathcal{G}_\eta(u, \varphi)$ in C^1_η.

PROPOSITION 2.14. *For each $\varphi \in C_{cu}$, the mapping $\mathcal{G}_\eta(\,\cdot\,, \varphi) : C^1_\eta \longrightarrow C^1_\eta$ has exactly one fixed point $u = u(\varphi)$. Moreover, the associated solution operator*

$$\tilde{u}_\eta : C_{cu} \ni \varphi \longmapsto u(\varphi) \in C^1_\eta \tag{2.30}$$

of $u = \mathcal{G}_\eta(u, \varphi)$ is (globally) Lipschitz continuous.

PROOF. We begin with the claim that, for given $\varphi \in C_{cu}$, $\mathcal{G}_\eta(\,\cdot\,,\varphi)$ maps sufficiently large closed balls centered at the origin into themselves. Indeed, for fixed $\varphi \in C_{cu}$ we find a positive real $\gamma > 0$ with $2\,\|S_\eta\|\,\|\varphi\|_{C^1} \le \gamma$ so that both estimates (2.25) and (2.27) together imply

$$\begin{aligned}\|\mathcal{G}_\eta(u,\varphi)\|_{C^1_\eta} &= \|S_\eta\,\varphi + \mathcal{K}_\eta \circ R_{\delta\eta}(u)\|_{C^1_\eta}\\ &\le \|S_\eta\,\varphi\|_{C^1_\eta} + \|\mathcal{K}_\eta \circ R_{\delta\eta}(u)\|_{C^1_\eta}\\ &\le \|S_\eta\|\,\|\varphi\|_{C^1} + \lambda(\delta)\,\|\mathcal{K}_\eta\|\,\|u\|_{C^1_\eta}\\ &\le \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma\end{aligned}$$

for all $u \in C^1_\eta$ with $\|u\|_{C^1_\eta} \le \gamma$. Hence, $\mathcal{G}_\eta(\,\cdot\,,\varphi)$ maps $\{u \in C^1_\eta|\ \|u\|_{C^1_\eta} \le \gamma\}$ into itself. The mapping $\mathcal{G}_\eta(\,\cdot\,,\varphi)$, $\varphi \in C_{cu}$, is also a contraction since, by application of (2.25) and (2.27),

$$\begin{aligned}\|\mathcal{G}_\eta(u,\varphi) - \mathcal{G}_\eta(v,\varphi)\|_{C^1_\eta} &= \|\mathcal{K}_\eta \circ R_{\delta\eta}(u) - \mathcal{K}_\eta \circ R_{\delta\eta}(v)\|_{C^1_\eta}\\ &\le \|\mathcal{K}_\eta\|\,\|R_{\delta\eta}(u) - R_{\delta\eta}(v)\|_{Y_\eta}\\ &\le \lambda(\delta)\,\|\mathcal{K}_\eta\|\,\|u-v\|_{C^1_\eta}\\ &\le \frac{1}{2}\,\|u-v\|_{C^1_\eta}\end{aligned}$$

for all $u, v \in C^1_\eta$. Consequently, using the Banach contraction principle, we find a unique $u(\varphi) \in C^1_\eta$ satisfying $u = \mathcal{G}_\eta(u,\varphi)$.

To see the global Lipschitz continuity of $\tilde{u}_\eta : C_{cu} \ni \varphi \longmapsto u(\varphi) \in C^1_\eta$, assume $\varphi, \psi \in C_{cu}$. Using the two inequalities (2.25) and (2.27) once more, we see

$$\begin{aligned}\|\tilde{u}_\eta(\varphi) - \tilde{u}_\eta(\psi)\|_{C^1_\eta} &= \|\mathcal{G}_\eta(\tilde{u}_\eta(\varphi),\varphi) - \mathcal{G}_\eta(\tilde{u}_\eta(\psi),\psi)\|_{C^1_\eta}\\ &= \|S_\eta(\varphi-\psi) + \mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}_\eta(\varphi)) - \mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}_\eta(\psi))\|_{C^1_\eta}\\ &\le \|S_\eta\|\,\|\varphi-\psi\|_{C^1} + \|\mathcal{K}_\eta\|\,\|R_{\delta\eta}(\tilde{u}_\eta(\varphi)) - R_{\delta\eta}(\tilde{u}_\eta(\psi))\|_{Y_\eta}\\ &\le \|S_\eta\|\,\|\varphi-\psi\|_{C^1} + \lambda(\delta)\,\|\mathcal{K}_\eta\|\,\|\tilde{u}_\eta(\varphi) - \tilde{u}_\eta(\psi)\|_{C^1_\eta}\\ &\le \|S_\eta\|\,\|\varphi-\psi\|_{C^1} + \frac{1}{2}\,\|\tilde{u}_\eta(\varphi) - \tilde{u}_\eta(\psi)\|_{C^1_\eta}.\end{aligned}$$

Therefore

$$\|\tilde{u}_\eta(\varphi) - \tilde{u}_\eta(\psi)\|_{C^1_\eta} \le 2\,\|S_\eta\|\,\|\varphi-\psi\|_{C^1},$$

which completes the proof. □

For all $\varphi \in C_{cu}$, the associated fixed point $\tilde{u}(\varphi)$ of the last proposition forms a solution of Equation (2.23) in C^1_η with the property that its component in the center-unstable space at $t = 0$ is just given by φ, as shown in the following.

COROLLARY 2.15. *For all $\varphi \in C_{cu}$ the mapping $\tilde{u}_\eta(\varphi)$ is a solution of the abstract integral equation* (2.23) *with $P_{cu}(\tilde{u}_\eta(\varphi)(0)) = \varphi$.*

PROOF. The proof is straightforward. Given $\varphi \in C_{cu}$ define $z := \tilde{u}_\eta(\varphi) - S_\eta\,\varphi$. By Corollary 2.7, we have

$$z(t) = T_e(t-s)\,z(s) + \int_s^t T_e^{\odot *}(t-\tau)\,R_{\delta\eta}(\tilde{u}_\eta(\varphi))(\tau)\,d\tau, \quad -\infty < s \le t \le 0,$$

and $P_{cu}\, z(0) = P_{cu}^{\odot *}\, z(0) = 0$. From this we conclude

$$\begin{aligned}
\tilde{u}_\eta(\varphi)(t) - T_e(t)\,\varphi &= \tilde{u}_\eta(\varphi)(t) - (S_\eta\,\varphi)(t) \\
&= z(t) \\
&= T_e(t-s)\, z(s) + \int_s^t T_e^{\odot *}(t-\tau)\, R_{\delta\eta}(\tilde{u}_\eta(\varphi))(\tau)\, d\tau \\
&= T_e(t-s)\,\tilde{u}_\eta(\varphi)(s) - T_e(t-s)\,(S_\eta\,\varphi)(s) \\
&\qquad + \int_s^t T_e^{\odot *}(t-\tau)\, R_{\delta\eta}(\tilde{u}_\eta(\varphi))(\tau)\, d\tau \\
&= T_e(t-s)\,\tilde{u}_\eta(\varphi)(s) - T_e(t)\,\varphi + \int_s^t T_e^{\odot *}(t-\tau)\, R_{\delta\eta}(\tilde{u}_\eta(\varphi))(\tau)\, d\tau
\end{aligned}$$

for all $-\infty < s \leq t \leq 0$ and

$$\begin{aligned}
P_{cu}(\tilde{u}_\eta(\varphi)(0)) - \varphi &= P_{cu}(\tilde{u}_\eta(\varphi)(0)) - P_{cu}\,\varphi \\
&= P_{cu}(\tilde{u}_\eta(\varphi)(0)) - P_{cu}((S_\eta\,\varphi)(0)) \\
&= P_{cu}\, z(0) = 0
\end{aligned}$$

Adding $T_e(t)\,\varphi$ and φ, respectively, yields the assertion. □

By the discussed one-to-one correspondence of solutions for the differential equation (2.22) and the associated abstract integral equation (2.23), the above corollary shows that for all $\varphi \in C_{cu}$ there exists a continuously differentiable function $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ satisfying $x_t = \tilde{u}(\varphi)(t)$ for $-\infty < t \leq 0$ and solving Equation (2.23) on $(-\infty, 0]$. The set W^η consisting of all segments of these solutions at time $t = 0$, that is, the set

$$W^\eta := \Big\{ \tilde{u}_\eta(\varphi)(0) \big|\ \varphi \in C_{cu} \Big\},$$

is called the **global center-unstable manifold** of RFDE (2.22) at the stationary point $0 \in C^1$. Note that W^η can also be represented as the graph of the operator

$$w^\eta : C_{cu} \ni \varphi \longmapsto P_s(\tilde{u}_\eta(\varphi)(0)) \in C^1_s.$$

Indeed, applying Corollary 2.15, we see at once

$$W^\eta = \big\{ \varphi + w^\eta(\varphi) \big|\ \varphi \in C_{cu} \big\}.$$

We close this subsection with the conclusion that the values of every solution $v \in C^1_\eta$ of the abstract integral equation (2.23) belong to the global center-unstable manifold W^η.

PROPOSITION 2.16. *Suppose that $v \in C^1_\eta$ is a solution of Equation* (2.23). *Then*

$$v(t) \in W^\eta$$

for all $t \leq 0$.

PROOF. Assuming $v \in C^1_\eta$ satisfies the abstract integral equation

$$u(t) = T_e(t-s)\, u(s) + \int_s^t T_e^{\odot *}(t-\tau)\, l(r_\delta(u(\tau)))\, d\tau$$

for $-\infty < s \leq t \leq 0$, we begin with the claim that $v(0) \in W^{\eta}$. In order to see this, let $z : (-\infty, 0] \longrightarrow C^1$ be defined by $z(t) := v(t) - T_e(t)\, P_{cu}\, v(0)$. As

$$\begin{aligned}
\sup_{t\in(-\infty,0]} e^{\eta t}\, \|z(t)\|_{C^1} &= \sup_{t\in(-\infty,0]} e^{\eta t}\, \|v(t) - T_e(t)\, P_{cu}\, v(0)\|_{C^1} \\
&\leq \sup_{t\in(-\infty,0]} e^{\eta t}\, \|v(t)\|_{C^1} \\
&\qquad + \sup_{t\in(-\infty,0]} e^{\eta t}\, \|T_e(t)\, P_{cu}\, v(0)\|_{C^1} \\
&\leq \|v\|_{C^1_\eta} + \sup_{t\in(-\infty,0]} e^{\eta t}\, \|T_e(t)\, P_c\, v(0)\|_{C^1} \\
&\qquad + \sup_{t\in(-\infty,0]} e^{\eta t}\, \|T_e(t)\, P_u\, v(0)\|_{C^1} \\
&\leq \|v\|_{C^1_\eta} + K \sup_{t\in(-\infty,0]} e^{-(c_c-\eta)t}\, \|P_c\, v(0)\|_{C^1} \\
&\qquad + K \sup_{t\in(-\infty,0]} e^{(c_u+\eta)t}\, \|P_u\, v(0)\|_{C^1} \\
&\leq \|v\|_{C^1_\eta} + K\, \|P_c\|\, \|v(0)\|_{C^1} + K\, \|P_u\|\, \|v(0)\|_{C^1} \\
&\leq \big(1 + K\, \|P_c\| + K\, \|P_u\|\big)\, \|v\|_{C^1_\eta} < \infty,
\end{aligned}$$

we have $z \in C^1_\eta$. Moreover, for all $s \leq t \leq 0$, we have

$$\begin{aligned}
z(t) &= v(t) - T_e(t)\, P_{cu}\, v(0) \\
&= T_e(t-s)\, v(s) + \int_s^t T_e^{\odot *}(t-\tau)\, l(r_\delta(v(\tau)))\, d\tau - T_e(t)\, P_{cu}\, v(0) \\
&= T_e(t-s)\, v(s) - T_e(t-s)\, T_e(s)\, P_{cu}\, v(0) + \int_s^t T_e^{\odot *}(t-\tau)\, l(r_\delta(v(\tau)))\, d\tau \\
&= T_e(t-s)\, z(s) + \int_s^t T_e^{\odot *}(t-\tau)\, l(r_\delta(v(\tau)))\, d\tau.
\end{aligned}$$

Since furthermore $R_{\delta\eta}(v) \in Y_\eta$ by Corollary 2.11 and $P_{cu}^{\odot *}\, z(0) = P_{cu}\, z(0) = 0$, we obtain $z = \mathcal{K} \circ R_{\delta\eta}(v)$ due to Corollary 2.7. Hence, by definition

$$v(t) = z(t) + T_e(t)\, P_{cu}\, v(0) = (\mathcal{K}_\eta \circ R_{\delta\eta}(v))(t) + T_e(t)\, P_{cu}\, v(0)$$

for all $t \leq 0$, or equivalently,

$$v = \mathcal{K}_\eta \circ R_{\delta\eta}(v) + S_\eta(P_{cu}\, v(0)) = \mathcal{G}(v, P_{cu}\, v(0)).$$

This implies $v(0) = \mathcal{G}(v, P_{cu}\, v(0))(0) = \tilde{u}_\eta(P_{cu}\, v(0))(0) \in W^\eta$ as claimed.

The proof of $v(t) \in W^\eta$ as $t < 0$ may now be reduced to the above claim as follows. For given $t_0 < 0$ consider the translation

$$\hat{v} : (-\infty, 0] \ni s \longmapsto v(t_0 + s) \in C^1.$$

Obviously, we have $\hat{v} \in C^1_\eta$ and $\hat{v}$ is a solution of Equation (2.23). Therefore $v(-t_0) = \hat{v}(0) \in W^\eta$ by the above claim. This completes the proof. ☐

Remark 2.17. Note that by application of the above result we easily deduce the identity

$$\tilde{u}_\eta(\varphi)(t) = \tilde{u}_\eta(P_{cu}\, \tilde{u}_\eta(\varphi)(t))(0)$$

for all $\varphi \in C_{cu}$ and $t \leq 0$.

Proof of Theorem 2.1. In this final part of the present section we complete the proof of Theorem 2.1 on the existence of Lipschitz continuous local center-unstable manifolds. We conclude that in a neighborhood of the origin, the global center-unstable manifold W^η of Equation (2.22) has the properties asserted in Theorem 2.1.

Our proof starts with the following series of definitions depending on the constant $\delta > 0$ from condition (2.25):

$$\begin{aligned} C_{cu,0} &:= \Big\{\varphi \in C_{cu} \;\Big|\; \|\varphi\|_1 < \delta\Big\}, \\ C^1_{s,0} &:= \Big\{\varphi \in C^1_s \;\Big|\; \|\varphi\|_1 < \delta\Big\}, \\ N_{cu} &:= C_{cu,0} + C^1_{s,0}, \\ w_{cu} &:= w^\eta\big|_{C_{cu,0}}, \end{aligned}$$

and

$$W_{cu} := \Big\{\varphi + w_{cu}(\varphi) \;\Big|\; \varphi \in C_{cu,0}\Big\}.$$

Given an open neighborhood V of 0 in X_f, note that one may choose $\delta > 0$ with $W_{cu} \subset V$. Applying Corollary 2.7 and estimate (2.26) of Corollary 2.11, we obtain for all $\varphi \in C_{cu,0}$

$$\begin{aligned} \|w_{cu}(\varphi)\|_1 &= \|w^\eta(\varphi)\|_1 \\ &= \|P_s(\tilde{u}_\eta(\varphi)(0))\|_{C^1} \\ &= \|\tilde{u}_\eta(\varphi)(0) - P_{cu}(\tilde{u}_\eta(\varphi)(0))\|_{C^1} \\ &= \|\mathcal{G}_\eta(\tilde{u}_\eta(\varphi),\varphi)(0) - P_{cu}(\mathcal{G}_\eta(\tilde{u}_\eta(\varphi),\varphi)(0))\|_{C^1} \\ &= \|(S_\eta\,\varphi)(0) + (\mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}(\varphi)))(0) - P_{cu}((S_\eta\,\varphi)(0)) \\ &\qquad - P_{cu}((\mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}(\varphi)))(0))\|_{C^1} \\ &= \|(\mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}(\varphi)))(0)\|_{C^1} \\ &\le \|\mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}(\varphi))\|_{C^1_\eta} \\ &\le \|\mathcal{K}_\eta\|\,\|R_{\delta\eta}(\tilde{u}(\varphi))\|_{Y_\eta} \\ &\le \|\mathcal{K}_\eta\|\,\delta\,\lambda(\delta), \end{aligned} \tag{2.31}$$

and thus, $w_{cu}(C_{cu,0}) \subset C^1_{s,0}$ by assumption (2.25). The mapping w_{cu} is also Lipschitz continuous, because for all $\varphi, \psi \in C_{cu,0}$ we have

$$\begin{aligned} \|w_{cu}(\varphi) - w_{cu}(\psi)\|_{C^1} &= \|w^\eta(\varphi) - w^\eta(\psi)\|_{C^1} \\ &= \|P_s(\tilde{u}_\eta(\varphi)(0)) - P_s(\tilde{u}_\eta(\psi)(0))\|_{C^1} \\ &\le \|P_s\|\,\|\tilde{u}_\eta(\varphi)(0) - \tilde{u}_\eta(\psi)(0)\|_{C^1} \\ &\le \|P_s\|\,\|\tilde{u}_\eta(\varphi) - \tilde{u}_\eta(\psi)\|_{C^1_\eta} \end{aligned}$$

and the operator $\tilde{u}_\eta$ is (globally) Lipschitz continuous due to Proposition 2.14. Moreover, since $\mathcal{G}_\eta(0,0) = 0$ by definition, we have $\tilde{u}_\eta(0) = 0$ and hence $w_{cu}(0) = 0$. Consequently, Theorem 2.1 follows if we verify properties (i) - (iii) for W_{cu}, which is done below.

Proof of Assertion (ii). Assuming that $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ is a solution of the differential equation (2.1) with $x_t \in N_{cu}$, $t \le 0$, we have to show $x_t \in W_{cu}$ for all $t \le 0$. To this end, notice that by definition $\|P_{cu}\,x_t\|_1 < \delta$ and $\|P_s\,x_t\|_1 < \delta$

so that Corollary 2.9 yields $r(x_t) = r_\delta(x_t)$ for all $t \leq 0$. Therefore x satisfies the smoothed differential equation (2.22) as well. Setting $u(t) := x_t$, $t \leq 0$, we consequently obtain a solution of the smoothed abstract integral equation (2.23). In particular, as u is bounded on $(-\infty, 0]$, we conclude that $u \in C^1_\eta$, and hence $u(t) \in W^\eta$, $t \leq 0$, by Proposition 2.16. This implies $x_t \in W_{cu}$ for all $t \leq 0$, which is the desired conclusion. □

Proof of Assertion (iii). Assume that for $\varphi \in W_{cu}$ and $t_N > 0$ we have $\{F(t,\varphi) \,|\, 0 \leq s \leq t_N\} \subset N_{cu}$. To deduce $\{F(t,\varphi) \,|\, 0 \leq s \leq t_N\} \subset W_{cu}$ from this, consider the function

$$v(t) := \begin{cases} \tilde{u}_\eta(P_{cu}\,\varphi)(t_N + t), & \text{for } t \leq -t_N, \\ F(t_N + t, \varphi), & \text{for } -t_N \leq t \leq 0, \end{cases}$$

where $\tilde{u}_\eta(P_{cu}\,\varphi) \in C^1_\eta$ is the solution of Equation (2.23) with $\tilde{u}_\eta(P_{cu}\,\varphi)(0) = \varphi$ from Corollary 2.15. As v takes values in C^1, it is continuous at the questionable point $t = -t_N$ in view of the limits

$$\lim_{t \nearrow -t_N} v(t) = \lim_{t \nearrow -t_N} \tilde{u}_\eta(P_{cu}\,\varphi)(t_N + t) = \tilde{u}_\eta(P_{cu}\,\varphi)(0) = \varphi$$

and

$$\lim_{t \searrow -t_N} v(t) = \lim_{t \searrow -t_N} F(t_N + t, \varphi) = F(0, \varphi) = \varphi.$$

In addition, v is bounded in the $\|\cdot\|_{C^1_\eta}$-norm due to

$$\sup_{t \in (-\infty, 0]} e^{\eta t}\, \|v(t)\|_{C^1} \leq \max\left\{ \|\tilde{u}_\eta(P_{cu}\,\varphi)\|_{C^1_\eta},\ \max_{t \in [0, t_N]} \|F(t,\varphi)\|_{C^1} \right\} < \infty,$$

we have $v \in C^1_\eta$. Moreover, we claim that v is also a solution of Equation (2.23). Indeed, suppose $s, t \in (-\infty, 0]$ with $s \leq t$. Then the cases $s \leq t \leq -t_N < 0$ and $-t_N \leq s \leq t \leq 0$ are obvious, whereas in the situation $s \leq -t_N \leq t \leq 0$, we get

$$\begin{aligned}
v(t) - T_e(t-s)\,v(s) &= v(t) - T_e(t + t_N)\,T_e(-t_N - s)\,v(s) \\
&= T_e(t + t_N)\,v(-t_N) + \int_{-t_N}^{t} T_e^{\odot *}(t - \tau)\,l(r_\delta(v(\tau)))\,d\tau \\
&\qquad - T_e(t + t_N)\,T_e(-t_N - s)\,v(s) \\
&= T_e(t + t_N)\Big(v(-t_N) - T_e(-t_N - s)\,v(s)\Big) \\
&\qquad + \int_{-t_N}^{t} T_e^{\odot *}(t - \tau)\,l(r_\delta(v(\tau)))\,d\tau \\
&= T_e(t + t_N)\int_{s}^{-t_N} T_e^{\odot *}(-t_N - \tau)\,l(r_\delta(v(\tau)))\,d\tau \\
&\qquad + \int_{-t_N}^{t} T_e^{\odot *}(-t_N - \tau)\,l(r_\delta(v(\tau)))\,d\tau \\
&= \int_{s}^{-t_N} T_e^{\odot *}(t - \tau)\,l(r_\delta(v(\tau)))\,d\tau + \int_{-t_N}^{t} T_e^{\odot *}(t - \tau)\,l(r_\delta(v(\tau)))\,d\tau \\
&= \int_{s}^{t} T_e^{\odot *}(t - \tau)\,l(r_\delta(v(\tau)))\,d\tau\,.
\end{aligned}$$

Thus, v is a solution of Equation (2.23) in C^1_η as claimed.

Now Proposition 2.16 shows $v(t) \in W^\eta$ for all $t \le 0$. Consequently, for constants $0 \le t \le t_N$ we have

$$F(t,\varphi) = v(t - t_N) \in N_{cu} \cap W^\eta,$$

and hence $F(t,\varphi) \in W_{cu}$, which proves our assertion. □

PROOF OF ASSERTION (I). It remains to prove that W_{cu} is contained in the solution manifold X_f of Equation (2.1), and that W_{cu} forms a Lipschitz submanifold of dimension $\dim C_{cu}$. For the first part, let $\varphi \in W_{cu}$ be given. Then from Corollary 2.15 it follows that the equations $x_t = \tilde{u}_\eta(P_{cu}\,\varphi)(t)$, $t \le 0$, define a continuously differentiable function $x : (-\infty, 0] \longrightarrow \mathbb{R}^n$ satisfying the smoothed differential equation (2.23) on $(-\infty, 0]$ and $x_0 = \varphi$. In particular, $\dot{\varphi}(0) = L\,\varphi + r_\delta(\varphi)$. As $\varphi \in W_{cu} \subset N_{cu}$ and in addition $r_\delta = r$ on N_{cu} due to Corollary 2.9 we conclude

$$\dot{\varphi}(0) = L\,\varphi + r(\varphi) = f(\varphi) \in X_f.$$

This proves $W_{cu} \subset X_f$.

To see the second part of the assertion, we consider an n-dimensional complementary space E of $Y = T_0 X_f$ in the Banach space C^1. We claim that there is no loss of generality in assuming $E \subset C^1_s$. In fact, let $\{e_1, \ldots, e_n\}$ denote a basis of E. Then by the decomposition $C^1 = C_{cu} \oplus C^1_s$ according to Equation (2.4) we get for each $i = 1, \ldots, n$

$$e_i = u_i + s_i$$

with uniquely determined $u_i \in C_{cu}$ and $s_i \in C^1_s$. As the center-unstable space C_{cu} is contained in Y, we conclude that $s_i \notin Y$ for all $i = 1, \ldots, n$. Define vectors $\hat{e}_i := e_i - u_i$ for $i = 1, \ldots, n$ and suppose we have

$$\sum_{i=1}^{n} \lambda_i\,\hat{e}_i = 0$$

with reals λ_i, $i = 1, \ldots, n$. Using the definition of $\hat{e}_i$, we obtain

$$E \ni \sum_{i=1}^{n} \lambda_i\,e_i = \sum_{i=1}^{n} \lambda_i\,u_i \in C_{cu}.$$

Since $C_{cu} \cap E = \{0\}$ it follows $\lambda_i = 0$ for all $i \in \{1, \ldots, n\}$. Thus, the elements $\hat{e}_i$, $i = 1, \ldots, n$, generate an n-dimensional subspace $\hat{E}$ of C^1, which is complementary to Y in C^1. In particular, $\hat{E} \subset C^1_s$.

In view of the above, we suppose now that indeed $E \subset C^1_s$, which leads to

$$C^1_s = E \oplus (C^1_s \cap Y),$$
$$Y = C_{cu} \oplus (C^1_s \cap Y),$$

and

$$C^1 = E \oplus (C^1_s \cap Y) \oplus C_{cu} = E \oplus Y.$$

Let $P_Y : C^1 \longrightarrow C^1$ denote the projection operator of the Banach space C^1 onto Y along E. Then we find an open neighborhood V of 0 in X_f such that the restriction of P_Y to V forms a manifold chart of X_f with a C^1-smooth inverse mapping from $Y_0 := P_Y(V)$ onto V. Additionally, we may assume that $\delta > 0$ is sufficient small such that $W_{cu} \subset V$ and $P_Y\,W_{cu} \subset Y_0$. Consequently, we shall have established the assertion if we prove that $P_Y\, W_{cu}$ is an $\dim C_{cu}$-dimensional Lipschitz submanifold of the Banach space Y. But this is clear, since

$$P_Y W_{cu} = \big\{ P_Y(\varphi + w_{cu}(\varphi)) \,|\, \varphi \in C_{cu,0} \big\} = \big\{ \varphi + P_Y\, w_{cu}(\varphi) \,|\, \varphi \in C_{cu,0} \big\}$$

and $w_{cu}(\varphi) \in C^1_s$ for all $\varphi \in C_{cu,0}$. Therefore, for every $\varphi \in C_{cu,0}$ we obviously have $P_Y\, w_{cu}(\varphi) \in C^1_s \cap Y$, so that $P_Y\, W_{cu}$ is the graph of the map

$$\big\{ \varphi \in C_{cu} \,|\, \|\varphi\|_1 < \delta \big\} \ni \chi \longmapsto P_Y\, w_{cu}(\chi) \in C^1_s \cap Y.$$

In particular, the above map is Lipschitz continuous. This finishes the proof of the assertion (i) and so of Theorem 2.1 as a whole. □

5. The C^1-Smoothness of Local Center-Unstable Manifolds

Having proved the existence of local center-unstable manifolds in the last section, below we establish Theorem 2.2, asserting the C^1-smoothness of these manifolds. For this purpose, we follow very closely the procedure in the proof of smoothness of local center manifolds in Krisztin [**16**] and show that the technique also works in our situation.

Auxiliary Results. The main idea of the proof for Theorem 2.2 is to employ the following abstract lemma stating under which conditions the fixed points of a parameter-dependent contraction form a C^1-smooth mapping of the involved parameter.

LEMMA 2.18 (Lemma II.8 in Krisztin et al. [**18**]). *Let X, Λ denote two Banach spaces over $\mathbb{R}$, let $\mathcal{P} \subset \Lambda$ be open, and let a map $\xi : X \times \mathcal{P} \longrightarrow X$ and a real $\kappa \in [0,1)$ be given satisfying*

$$\|\xi(x,p) - \xi(\tilde{x},p)\|_X \leq \kappa\, \|x - \tilde{x}\|_X$$

for all $x, \tilde{x} \in X$ and all $p \in \mathcal{P}$. Consider a convex subset $\mathcal{M}$ of X and a map $\Phi : \mathcal{P} \longrightarrow \mathcal{M}$ with the property that for every $p \in \mathcal{P}$, the element $\Phi(p)$ is the unique fixed point of the induced map $\xi(\cdot,p) : X \longrightarrow X$. Furthermore, suppose that the following hypotheses hold.

(i) *The restriction $\xi_0 := \xi\big|_{\mathcal{M}\times\mathcal{P}}$ of the mapping ξ has a partial derivative $D_2\xi_0 : \mathcal{M} \times \mathcal{P} \longrightarrow \mathcal{L}(\Lambda, X)$, and $D_2\xi_0$ is continuous.*

(ii) *There exist a Banach space X_1 over $\mathbb{R}$ and a continuous injective map $j : X \longrightarrow X_1$ such that the composed map $k := j \circ \xi_0$ is continuously differentiable with respect to $\mathcal{M}$ in the sense that there is a continuous map*

$$B : \mathcal{M} \times \mathcal{P} \longrightarrow \mathcal{L}(X, X_1)$$

such that for every $(x,p) \in \mathcal{M} \times \mathcal{P}$ and every $\varepsilon^* > 0$ one finds a real $\delta^* > 0$ guaranteeing

$$\big\| k(\tilde{x},p) - k(x,p) - B(x,p)\big(\tilde{x} - x\big) \big\|_{X_1} \leq \varepsilon^*\, \|\tilde{x} - x\|_X$$

for all $\tilde{x} \in \mathcal{M}$ with $\|\tilde{x} - x\|_X \leq \tilde{\delta}$.

(iii) *There exist maps*

$$\xi^{(1)} : \mathcal{M} \times \mathcal{P} \longrightarrow \mathcal{L}(X, X)$$

and

$$\xi_1^{(1)} : \mathcal{M} \times \mathcal{P} \longrightarrow \mathcal{L}(X_1, X_1)$$

such that

$$B(x,p)\,\tilde{x} = \big(j \circ \xi^{(1)}(x,p)\big)(\tilde{x}) = \big(\xi_1^{(1)}(x,p) \circ j\big)(\tilde{x})$$

for all $(x,p,\tilde{x}) \in \mathcal{M} \times \mathcal{P} \times X$ *and*

$$\left\|\xi^{(1)}(x,p)\right\| \leq \kappa$$

as well as

$$\left\|\xi_1^{(1)}(x,p)\right\| \leq \kappa$$

on $\mathcal{M} \times \mathcal{P}$.

(iv) *The map*

$$\mathcal{M} \times \mathcal{P} \ni (x,p) \longmapsto j \circ \xi^{(1)}(x,p) \in \mathcal{L}(X, X_1)$$

is continuous.

Then the map $j \circ \Phi : \mathcal{P} \longrightarrow X_1$ *is continuously differentiable and its derivative satisfies*

$$D(j \circ \Phi)(p) = \xi_1^{(1)}(\Phi(p), p) \circ D(j \circ \Phi)(p) + j \circ D_2\xi_0(\Phi(p), p)$$

for all $p \in \mathcal{P}$.

To verify the hypotheses of the last lemma in our situation, we will need another auxiliary result on some smoothness properties of Nemitsky operators between scaled Banach spaces. This result is a negligible modification of Lemma II.6 in Krisztin et al. [**18**] and Lemma 3.1 in Krisztin [**16**].

LEMMA 2.19. *Given any two Banach spaces* E, F *over* $\mathbb{R}$, *consider for a real* $\eta \geq 0$ *the scaled Banach spaces* $E_\eta := C_\eta((-\infty,0],E)$ *and* $F_\eta := C_\eta((-\infty,0],F)$. *Further, let* $q : U \longrightarrow F$ *be a continuous and bounded map defined on some subset* $U \subset E$ *and let* $\mathfrak{M}((-\infty,0],U)$, $\mathfrak{M}((-\infty,0],F)$ *denote the sets of all mappings from the interval* $(-\infty,0]$ *into* U, F, *respectively. Then for the induced substitution operator*

$$\tilde{q} : \mathfrak{M}((-\infty,0],U) \longrightarrow \mathfrak{M}((-\infty,0],F)$$

defined by

$$\tilde{q}(u)(t) := q(u(t))$$

for all $u \in \mathfrak{M}((-\infty,0],U)$ *and* $t \leq 0$ *the following holds.*

(i) *If* $\eta, \tilde{\eta} \geq 0$, *then* $\tilde{q}(\mathfrak{M}((-\infty,0],U) \cap E_\eta) \subset F_{\tilde{\eta}}$.

(ii) *If* U *is open, if* q *is continuously differentiable with a bounded derivative* Dq *and* $0 \leq \eta \leq \tilde{\eta}$, *then, for all* $u \in C((-\infty,0],U)$, *the linear map*

$$A(u) : \mathfrak{M}((-\infty,0],E) \longrightarrow \mathfrak{M}((-\infty,0],F),$$

given by

$$A(u)(v)(t) := Dq(u(t))\,v(t)$$

for $v \in \mathfrak{M}((-\infty,0],E)$ *and* $t \leq 0$, *satisfies*

$$A(u)(E_\eta) \subset F_{\tilde{\eta}}$$

and

$$\sup_{\|v\|_{E_\eta} \leq 1} \|A(u)(v)\|_{F_{\tilde{\eta}}} \leq \sup_{x \in U} \|Dq(x)\|,$$

the induced linear maps

$$A_{\eta\tilde{\eta}}(u) : E_\eta \longrightarrow F_{\tilde{\eta}}$$

are continuous and in case $\eta < \tilde\eta$, the map

$$A_{\eta\tilde\eta} : \big(C((-\infty,0],U)\cap E_\eta\big) \ni u \longmapsto A_{\eta\tilde\eta}(u) \in \mathcal{L}(E_\eta, F_{\tilde\eta})$$

is continuous as well.

(iii) *If additionally to the hypothesis stated above there holds $\eta < \tilde\eta$ and the set U is convex, then for every $\tilde\varepsilon > 0$ and $u \in C((-\infty,0],U)\cap E_\eta$ there exists $\tilde\delta > 0$ such that for every $v \in C((-\infty,0],U)\cap E_\eta$ with $\|v-u\|_{E_\eta} < \tilde\delta$ we have*

$$\|\tilde q(v) - \tilde q(u) - A_{\eta\tilde\eta}(u)(v-u)\|_{F_{\tilde\eta}} \le \tilde\varepsilon\,\|v-u\|_{E_\eta}.$$

Proof. We adopt the proof of Lemma 3.1 in Krisztin [**16**] which falls naturally into three steps.

1. *The proof of (i).* Assuming $u \in (\mathfrak{M}((-\infty,0],U)\cap E_\eta)$, we see at once that the continuity of u and q implies the one of

$$(-\infty,0] \ni t \longmapsto \tilde q(u)(t) = q(u(t)) \in F.$$

Moreover, the boundedness of q leads to

$$\sup_{t\in(-\infty,0]} e^{\tilde\eta t}\,\|q(u(t))\|_F \le \sup_{t\in(-\infty,0]} e^{\tilde\eta t} \sup_{t\in(-\infty,0]} \|q(u(t))\|_F \le \sup_{x\in U}\|q(x)\|_F < \infty,$$

and thus $\|\tilde q(u)\|_{F_{\tilde\eta}} < \infty$. Consequently, we have $\tilde q(u) \in F_{\tilde\eta}$, which is the desired conclusion.

2. *The proof of (ii).* We begin with the observation that for all elements $u \in C((-\infty,0],U)$ the map $A(u)$ is well-defined, linear and that under the stated assumption the image $A(u)\,v \in \mathfrak{M}((-\infty,0],F)$ of an element $v \in E_\eta$, that is, the map

$$[0,\infty) \ni t \longmapsto Dq(u(t))\,v(t) \in F,$$

is continuous. As in this situation we also have

$$\begin{aligned} e^{\tilde\eta t}\,\|Dq(u(t))\,v(t)\|_F &\le e^{(\tilde\eta-\eta)t}\,e^{\eta t}\,\|v(t)\|_E \sup_{x\in U}\|Dq(x)\| \\ &\le \sup_{t\in(-\infty,0]} e^{\eta t}\|v(t)\|_E \sup_{x\in U}\|Dq(x)\| \\ &\le \|v\|_{E_\eta} \sup_{x\in U}\|Dq(x)\| < \infty \end{aligned}$$

due to the boundedness of Dq on U, we conclude $A(u)(E_\eta) \subset F_{\tilde\eta}$ and additionally

$$\sup_{\|v\|_{E_\eta}\le 1} \|A(u)\,v\|_{F_{\tilde\eta}} \le \sup_{x\in U}\|Dq(x)\|.$$

In particular, this shows the continuity of the maps $A_{\eta\tilde\eta} : E_\eta \longmapsto F_{\tilde\eta}$.

The only point remaining of assertion (ii) concerns the continuity of the map

$$A_{\eta\tilde\eta} : C((-\infty,0],U)\cap E_\eta \ni u \longmapsto A_{\eta\tilde\eta}(u) \in \mathcal{L}(E_\eta, F_{\tilde\eta})$$

in case $\eta < \tilde\eta$. To see this, choose $u \in C((-\infty,0],U)\cap E_\eta$ and let $\tilde\varepsilon > 0$ be given. As $\eta < \tilde\eta$ and Dq is bounded on U, there clearly is a real $t_0 < 0$ satisfying

$$2\,e^{(\tilde\eta-\eta)t} \sup_{x\in U}\|Dq(x)\| < \tilde\varepsilon$$

for all $t \le t_0$. Furthermore, in view of the continuity of u and Dq we find a constant $\tilde\delta > 0$ such that

$$B_t(u) := \Big\{ y \in E \,\Big|\, \|y - u(t)\|_E < \tilde\delta\, e^{-\eta t_0} \Big\} \subset U$$

as $t_0 \le t \le 0$ and such that additionally

$$\|Dq(y) - Dq(u(t))\| < \tilde{\varepsilon}$$

holds for all $y \in B_t$. Consequently, if $\tilde{u} \in C((-\infty, 0], U) \cap E_\eta$ with $\|\tilde{u} - u\|_{E_\eta} < \tilde{\delta}$, and if $v \in E_\eta$ with $\|v\|_{E_\eta} \le 1$, then the above estimates yield

$$e^{\tilde{\eta}t} \left\| \big(Dq(\tilde{u}(t)) - Dq(u(t))\big)\, v(t) \right\|_F \le \tilde{\varepsilon}$$

for all $t \le 0$. Indeed, in case $t \le t_0$ we see

$$\begin{aligned} e^{\tilde{\eta}t} \left\| \big(Dq(\tilde{u}(t)) - Dq(u(t))\big)\, v(t) \right\|_F &\le 2\, e^{(\tilde{\eta}-\eta)t}\, e^{\eta t}\, \|v(t)\|_E \sup_{x \in U} \|Dq(x)\| \\ &\le 2\, e^{(\tilde{\eta}-\eta)t}\, \|v\|_{E_\eta} \sup_{x \in U} \|Dq(x)\| \\ &< \tilde{\varepsilon}, \end{aligned}$$

whereas, for $t_0 < t \le 0$, we first conclude

$$\|\tilde{u}(t) - u(t)\|_E < \tilde{\delta}\, e^{-\eta t} < \tilde{\delta}\, e^{-\eta t_0}$$

and hence

$$\begin{aligned} e^{\tilde{\eta}t} \left\| \big(Dq(\tilde{u}(t)) - Dq(u(t))\big)\, v(t) \right\|_F &\le e^{(\tilde{\eta}-\eta)t}\, e^{\eta t}\, \|v(t)\|_E\, \|Dq(\tilde{u}(t)) - Dq(u(t))\| \\ &\le \|v\|_{E_\eta}\, \|Dq(\tilde{u}(t)) - Dq(u(t))\| \\ &< \tilde{\varepsilon}. \end{aligned}$$

This shows

$$\|A_{\eta\tilde{\eta}}(\tilde{u}) - A_{\eta\tilde{\eta}}(u)\| \le \tilde{\varepsilon},$$

and the continuity of $A_{\eta\tilde{\eta}}$ is proved.

3. *The proof of (iii).* Note that from the additional assumption on the convexity of the open set U in E it is easy to check that the set $C((-\infty, 0], U) \cap E_\eta$ is convex as well. Hence, for all $u, v \in C((-\infty, 0], U) \cap E_\eta$ and all $t \le 0$ we have

$$\begin{aligned} & e^{\tilde{\eta}t} \left\| q(v(t)) - q(u(t)) - Dq(u(t))\big(v(t) - u(t)\big) \right\|_F \\ &\quad = e^{\tilde{\eta}t} \left\| \int_0^1 \Big(Dq\big(s\, v(t) + (1-s)\, u(t)\big) - Dq\big(u(t)\big)\Big)\big(v(t) - u(t)\big)\, ds \right\|_F \\ &\quad \le e^{(\tilde{\eta}-\eta)t}\, e^{\eta t}\, \|v(t) - u(t)\|_E \\ &\qquad\qquad \cdot \max_{s \in [0,1]} \left\| Dq\big(s\, v(t) + (1-s)\, u(t)\big) - Dq\big(u(t)\big) \right\| \\ &\quad \le e^{(\tilde{\eta}-\eta)t}\, \|v - u\|_{E_\eta} \\ &\qquad\qquad \cdot \max_{s \in [0,1]} \left\| Dq\big(s\, v(t) + (1-s)\, u(t)\big) - Dq\big(u(t)\big) \right\|. \end{aligned} \tag{2.32}$$

Fix $u \in C((-\infty, 0], E) \cap E_\eta$ and $\tilde{\varepsilon} > 0$. Then, using $\eta < \tilde{\eta}$, we find constants $t_0 < 0$ and $\tilde{\delta} \ge 0$ as in the last part. Let now an arbitrary $v \in C((-\infty, 0], U) \cap E_\eta$ with $\|v - u\|_{E_\eta} < \tilde{\delta}$ be given. Then, in the situation $t \le t_0$, the estimate (2.32) and the choice of the real t_0 yield

$$\begin{aligned} & e^{\tilde{\eta}t} \left\| q(v(t)) - q(u(t)) - Dq(u(t))\, \big(v(t) - u(t)\big) \right\|_F \\ &\qquad \le e^{(\tilde{\eta}-\eta)t}\, \|v - u\|_{E_\eta} \\ &\qquad\qquad \cdot \max_{s \in [0,1]} \left\| Dq\big(s\, v(t) + (1-s)\, u(t)\big) - Dq\big(u(t)\big) \right\| \\ &\qquad \le 2\, e^{(\tilde{\eta}-\eta)t} \max_{x \in U} \|Dq(x)\|\, \|v - u\|_{E_\eta} \\ &\qquad < \tilde{\varepsilon}\, \|v - u\|_{E_\eta} \end{aligned}$$

On the other hand, if $t_0 < t \leq 0$, then we have

$$\|v(t) - u(t)\|_E \leq \tilde{\delta}\, e^{-\eta t} < \tilde{\delta}\, e^{-\eta t_0}.$$

This implies $s\, v(t) + (1-s)\, u(t) \in B_t(u)$ for all $0 \leq s \leq 1$ and hence, by inequality (2.32), we get again

$$\begin{aligned}
e^{\tilde{\eta} t}\, & \big\|\, q(v(t)) - q(u(t)) - Dq(u(t))\, \big(v(t) - u(t)\big) \big\|_F \\
& \leq e^{(\tilde{\eta}-\eta)t}\, \|v - u\|_{E_\eta} \\
& \qquad \cdot \max_{s\in[0,1]} \big\| Dq\big(s\, v(t) + (1-s)\, u(t)\big) - Dq\big(u(t)\big) \big\| \\
& < \tilde{\varepsilon}\, e^{(\tilde{\eta}-\eta)t}\, \|v - u\|_{E_\eta} \\
& < \tilde{\varepsilon}\, \|v - u\|_{E_\eta}.
\end{aligned}$$

Combining these yields

$$\big\|\, \tilde{q}(v) - \tilde{q}(u) - A_{\eta\tilde{\eta}}(u)\, \big(v - u\big) \big\|_{F_{\tilde{\eta}}} \leq \tilde{\varepsilon}\, \|v - u\|_{E_\eta},$$

and the proof is complete. □

Proof of Theorem 2.2. After the preparatory results above, we return to the local center-unstable manifolds from the last section and prove Theorem 2.2.

We start our proof with the observation that an important, but probably inconspicuous point of our construction of the invariant manifolds in the foregoing section was the choice of a constant $\eta > 0$ satisfying condition (2.24), that is,

$$c_c < \eta < \min\{-c_s, c_u\},$$

and hereafter the choice of a second constant $0 < \delta < \delta_1$ satisfying condition (2.25), that is,

$$\|\mathcal{K}_\eta\|\, \lambda(\delta) < \frac{1}{2}.$$

Now, recall from Corollary 2.7 that $\mathcal{K}_\eta$ is a bounded linear map from the Banach space Y_η into C^1_η. Moreover, the bound of $\mathcal{K}_\eta$ satisfies the inequality

$$\|\mathcal{K}_\eta\| < c(\eta) \tag{2.33}$$

with the continuous map $c : (c_c, \min\{-c_s, c_u\}) \longrightarrow [0, \infty)$ given by

$$c(\eta) := K\left(1 + e^{\eta h}\, \|L_e\|\right)\left(\frac{\|P_c^{\odot *}\|}{\eta - c_c} + \frac{\|P_u^{\odot *}\|}{c_u + \eta} - \frac{\|P_s^{\odot *}\|}{c_s + \eta}\right) + e^{\eta h}.$$

Hence, fixing a constant $\eta_1 > 0$ with $c_c < \eta_1 < \min\{-c_u, c_s\}$ and additionally a constant $0 < \delta < \delta_1$ with

$$c(\eta_1)\, \lambda(\delta) < \frac{1}{2},$$

we clearly find a real $c_c < \eta_0 < \eta_1$ such that the estimate

$$c(\eta)\, \lambda(\delta) < \frac{1}{2} \tag{2.34}$$

is fulfilled for all $\eta_0 \leq \eta \leq \eta_1$. As an immediate consequence, we see that for any $\eta_0 \leq \eta \leq \eta_1$ the pair (η, δ) satisfies both conditions (2.24), (2.25), and thus the construction in the last section works for any such choice of constants.

Below, we show the assertion of Theorem 2.2 for the map w^{η_1}. Hereby, remember that w^{η_1} may be also written as the composition

$$w^{\eta_1} = P_s \circ \mathrm{ev}_0 \circ \tilde{u}_{\eta_1}$$

with the projection operator P_s of C^1 along the center-unstable space C_{cu} onto C^1_s, the evaluation map

$$\mathrm{ev}_0 : C^1_{\eta_1} \ni u \longmapsto u(0) \in C^1$$

and the fixed point operator $\tilde{u}_{\eta_1} : C_{cu} \longrightarrow C^1_{\eta_1}$ defined by (2.30). Since P_s and ev_0 are both bounded linear maps, for a conclusion on the C^1-smoothness of w^{η_1} we are obviously reduced to proving the continuous differentiability of $\tilde{u}_{\eta_1}$ on C_{cu}. By application of Lemmata 2.18, 2.19, we show that $\tilde{u}_{\eta_1}$ is indeed continuously differentiable on C_{cu} in the following.

Consider the open neighborhood

$$O_\delta := \left\{\psi \in C^1 \mid \|P_s\,\psi\|_1 < \delta\right\}$$

of the origin in C^1. The set O_δ is clearly convex, and from Corollary 2.9 and Proposition 2.10 we see that the restriction of the function r_δ to O_δ is bounded, C^1-smooth and has a bounded derivative with

$$\sup_{\varphi \in O_\delta} \|Dr_\delta(\varphi)\| \le \lambda(\delta).$$

Additionally, we claim

$$\left\{\tilde{u}_\eta(\varphi)(t) \,\middle|\, \varphi \in C_{cu}, t \le 0\right\} \subset O_\delta$$

for all $\eta_0 \le \eta \le \eta_1$. Indeed, combining the inequalities (2.26), (2.33) and (2.34) yields

$$\begin{aligned}
\|w^\eta(\varphi)\|_1 &= \|P_s\,\tilde{u}_\eta(\varphi)(0)\|_{C^1} \\
&= \left\|\left(\mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}_\eta(\varphi))\right)(0)\right\|_{C^1} \\
&\le \left\|\left(\mathcal{K}_\eta \circ R_{\delta\eta}(\tilde{u}_\eta(\varphi))\right)\right\|_{C^1_\eta} \\
&\le \|\mathcal{K}_\eta\|\; \|R_{\delta\eta}(\tilde{u}_\eta(\varphi))\|_{Y_\eta} \\
&\le c(\eta)\,\delta\,\lambda(\delta) \\
&< \delta
\end{aligned}$$

as $\varphi \in C_{cu}$ and $\eta_0 \le \eta \le \eta_1$. Thus, in view of Remark 2.17 we obtain

$$\left\|P_s\,\tilde{u}_\eta(\varphi)(t)\right\|_1 = \|P_s\,\tilde{u}_\eta(P_{cu}\,\tilde{u}_\eta(\varphi)(t))(0)\|_1 = \left\|w^\eta(P_{cu}\,\tilde{u}_\eta(\varphi)(t))\right\|_1 < \delta$$

for all $(\varphi, \eta, t) \in C_{cu} \times [\eta_0, \eta_1] \times (-\infty, 0]$, as claimed. Now, setting $E = C^1$, $F = Y^{\odot *}$, $O = O_\delta$, $q = l \circ r_\delta$, $\eta = \eta_0$, $\tilde{\eta} = \eta_1$ and applying Lemma 2.19, we conclude that the linear maps

$$A(u) : \mathfrak{M}((-\infty, 0], C^1) \longrightarrow \mathfrak{M}((-\infty, 0], Y^{\odot *})$$

define a continuous map $A_{\eta_0\eta_1}$ from the convex set

$$\mathcal{M} := \left\{u \in C^1_{\eta_0} \,\middle|\, u(t) \in O_\delta \text{ for all } t \in (-\infty, 0]\right\}$$

into the Banach space $\mathcal{L}(C^1_{\eta_0}, Y_{\eta_1})$. In addition, we see that $A_{\eta_0\eta_1}$ has the property that for every point $u \in \mathcal{M}$ and every real $\tilde{\varepsilon} > 0$ there is a constant $\tilde{\delta}(\tilde{\varepsilon}) > 0$ such that for all $v \in \mathcal{M}$ with $\|v - u\|_{C^1_{\eta_0}} \le \tilde{\delta}$ we have $R_{\delta\eta_1}(u), R_{\delta\eta_1}(v) \in Y_{\eta_1}$ and

$$\left\|R_{\delta\eta_1}(u) - R_{\delta\eta_1}(v) - A_{\eta_0\eta_1}(u)\left(v - u\right)\right\|_{Y_{\eta_1}} \le \tilde{\varepsilon}\,\|v - u\|_{C^1_{\eta_0}}. \tag{2.35}$$

Next, we are going to employ Lemma 2.18. To this end, we regard the inclusion map

$$j_{\eta_0\eta_1} : C^1_{\eta_0} \ni u \longmapsto u \in C^1_{\eta_1}.$$

As $\eta_0 < \eta_1$, this map obviously is well-defined and is trivially linear and bounded. Moreover, for all $\varphi \in C_{cu}$, $j_{\eta_0\eta_1}$ maps the fixed point $\tilde{u}_{\eta_0}(\varphi)$ of $\mathcal{G}_{\eta_0}(\,\cdot\,,\varphi)$ defined in Proposition 2.14 onto the fixed point $\tilde{u}_{\eta_1}(\varphi)$ of $\mathcal{G}_{\eta_1}(\,\cdot\,,\varphi)$. Indeed, since for a given $\varphi \in C_{cu}$ we have

$$\begin{aligned}\mathcal{G}_{\eta_1}(j_{\eta_0\eta_1}(\tilde{u}_{\eta_0}(\varphi)),\varphi) &= S_{\eta_1}\,\varphi + \mathcal{K}_{\eta_1}\circ R_{\delta\eta_1}(j_{\eta_0\eta_1}(\tilde{u}_{\eta_0}(\varphi)))\\ &= T_e(\,\cdot\,)\,\varphi + \mathcal{K}^{cu}R(\tilde{u}_{\eta_0}(\varphi))\\ &= j_{\eta_0\eta_1}\big(S_{\eta_0}\,\varphi + \mathcal{K}_{\eta_0}\circ R_{\delta\eta_0}(\tilde{u}_{\eta_0}(\varphi))\big)\\ &= j_{\eta_0\eta_1}\big(\mathcal{G}_{\eta_0}(\tilde{u}_{\eta_0}(\varphi),\varphi)\big)\\ &= j_{\eta_0\eta_1}(\tilde{u}_{\eta_0}(\varphi)),\end{aligned}$$

$j_{\eta_0\eta_1}(\tilde{u}_{\eta_0}(\varphi))$ is a fixed point of $\mathcal{G}_{\eta_1}(\,\cdot\,,\varphi) : C^1_{\eta_1} \longrightarrow C^1_{\eta_1}$ and from the uniqueness of the fixed point there actually follows

$$j_{\eta_0\eta_1}(\tilde{u}_{\eta_0}(\varphi)) = \tilde{u}_{\eta_1}(\varphi).$$

Set $X = C^1_{\eta_0}$, $X_1 = C^1_{\eta_1}$, $\Lambda = \mathcal{P} = C_{cu}$, $\xi = \mathcal{G}_{\eta_0}$, $j = j_{\eta_0\eta_1}$ and $\kappa = 1/2$. Then we see at once that $\tilde{u}_{\eta_0}(P) \subset \mathcal{M}$, and this implies that the unique fixed point of $\xi(\,\cdot\,,\varphi) : X \longrightarrow X$ is given by the value $\varPhi(\varphi)$ of the map

$$\varPhi : \mathcal{P} \ni \varphi \longmapsto \tilde{u}_{\eta_0}(\varphi) \in \mathcal{M}.$$

Additionally, for each $\varphi \in C_{cu}$ the map $\xi(\,\cdot\,,\varphi) = \mathcal{G}_{\eta_0}(\,\cdot\,,\varphi)$ is Lipschitz continuous with Lipschitz constant κ due to the proof of Proposition 2.14. Thus, for an application of Lemma 2.18 with the above choice of spaces, maps and reals it remains to confirm conditions (i) - (iv). This point is done below in detail.

Verification of hypothesis (i). Observe that for the restriction ξ_0 of the map ξ to $\mathcal{M} \times \mathcal{P}$ we have

$$\xi_0(u,\varphi) = \mathcal{G}_{\eta_0}(u,\varphi) = S_{\eta_0}\varphi + \mathcal{K}_{\eta_0}\circ R_{\delta\eta_0}(u).$$

Consequently, ξ_0 is partially differentiable with respect to the second variable, and for every $(u,\varphi) \in \mathcal{M} \times \mathcal{P}$ its derivative $D_2\xi_0(u,\varphi) \in \mathcal{L}(\Lambda, X)$ is given by

$$D_2\xi_0(u,\varphi)\,\psi = S_{\eta_0}\,\psi$$

for all $\psi \in C_{cu}$. Obviously, $D_2\xi_0 : \mathcal{M} \times \mathcal{P} \longrightarrow \mathcal{L}(\Lambda, X)$ is a constant map and thus in particular continuous. This shows hypothesis (i) of Lemma 2.18.

Verification of hypothesis (ii). The mapping $k := j \circ \xi_0$ reads

$$k(u,\varphi) = S_{\eta_1}\varphi + \mathcal{K}_{\eta_1}\circ R_{\delta\eta_1}(j(u)),$$

and the map

$$B : \mathcal{M} \times \mathcal{P} \ni (u,\varphi) \longmapsto \mathcal{K}_{\eta_1}\circ (A_{\eta_0\eta_1}(u)) \in \mathcal{L}(X, X_1)$$

is of course continuous as $\mathcal{K}_{\eta_1}$, $A_{\eta_0\eta_1}$ are so. Consider next an arbitrary point $(u,\varphi) \in \mathcal{M} \times \mathcal{P}$ and $\varepsilon^* > 0$. Choosing

$$\delta^* = \tilde{\delta}\left(\frac{\varepsilon^*}{1 + \|\mathcal{K}_{\eta_1}\|}\right)$$

with the constant $\tilde{\delta}$ from estimate (2.35), we find that for all points $v \in \mathcal{M}$ with $\|v-u\|_{C^1_{\eta_0}} < \delta^*$ we have

$$\begin{aligned}
\big\|k(v,\varphi) - k(u,\varphi) - B(u,\varphi)\,(v-u)\big\|_{X^1} \\
&= \big\|\mathcal{K}_{\eta_1}(R(v)) - \mathcal{K}_{\eta_1}(R(u)) - \mathcal{K}_{\eta_1}\big(A_{\eta_0\eta_1}(u)\,(v-u)\big)\big\|_{C^1_{\eta_1}} \\
&\le \|\mathcal{K}_{\eta_1}\|\; \big\|R(v) - R(u) - A_{\eta_0\eta_1}(u)\,\big(v-u\big)\big\|_{Y_{\eta_1}} \\
&\le \|\mathcal{K}_{\eta_1}\|\; \frac{\varepsilon^*}{1+\|\mathcal{K}_{\eta_1}\|}\, \|v-u\|_{C^1_{\eta_0}} \\
&\le \varepsilon^*\, \|v-u\|_{C^1_{\eta_0}}.
\end{aligned}$$

Thus, condition (ii) is satisfied.

Verification of hypothesis (iii). Next we note that for every $u \in \mathcal{M}$ and all $v \in X$ we have

$$\begin{aligned}
A(u)(v)(t) &= Dq(u(t))\,v(t) \\
&= D(l \circ r_\delta)(u(t))\,v(t) \\
&= Dl(r_\delta(u(t))) \circ Dr_\delta(u(t))\,v(t) \\
&= l \circ Dr_\delta(u(t))\,v(t)
\end{aligned}$$

for $t \le 0$. Since $\sup_{\varphi\in O_\delta}\|Dr_\delta(\varphi)\| \le \lambda(\delta)$ and $\|\mathcal{K}_{\eta_0}\| \le c(\eta_0)$, and $\|l\| = 1$, it is obvious that for every $u \in \mathcal{M}$, the induced map

$$\mathcal{K}_{\eta_0} \circ (A_{\eta_0\eta_0}(u)) \in \mathcal{L}(X,X)$$

satisfies

$$\|\mathcal{K}_{\eta_0} \circ (A_{\eta_0\eta_0}(u))\| \le c(\eta_0)\,\lambda(\delta).$$

In the same manner we see that for all $u \in \mathcal{M}$

$$\mathcal{K}_{\eta_1} \circ (A_{\eta_1\eta_1}(u)) \in \mathcal{L}(X_1,X_1)$$

with

$$\|\mathcal{K}_{\eta_0} \circ (A_{\eta_1\eta_1}(u))\| \le c(\eta_1)\,\lambda(\delta).$$

Define

$$\xi^{(1)} : \mathcal{M}\times\mathcal{P} \ni (u,\varphi) \longmapsto \mathcal{K}_{\eta_0} \circ (A_{\eta_0\eta_0}(u)) \in \mathcal{L}(X,X)$$

and

$$\xi^{(1)}_1 : \mathcal{M}\times\mathcal{P} \ni (u,\varphi) \longmapsto \mathcal{K}_{\eta_1} \circ (A_{\eta_1\eta_1}(u)) \in \mathcal{L}(X_1,X_1).$$

Then, for all $(u,\varphi,v) \in \mathcal{M}\times\mathcal{P}\times X$, we get

$$\begin{aligned}
B(u,\varphi)\,v &= \big(\mathcal{K}_{\eta_1} \circ (A_{\eta_0\eta_1}(u))\big)(v) \\
&= \mathcal{K}^{cu}(A(u)\,v) \\
&= j\big(\xi^{(1)}(u,\varphi)\,v\big) \\
&= \xi^{(1)}_1(u,\varphi)\big(j(v)\big).
\end{aligned}$$

Moreover, in view of the choice of η_0, η_1 and δ due to Equation (2.34) we have

$$\Big\|\xi^{(1)}(u,\varphi)\Big\|_X \le \kappa$$

and

$$\Big\|\xi^{(1)}_1(u,\varphi)\Big\|_{X_1} \le \kappa$$

for all $(u, \varphi) \in \mathcal{M} \times \mathcal{P}$. This shows that hypothesis (iii) is valid too.

Verification of hypothesis (iv). Finally, we find that the map

$$\mathcal{M} \times \mathcal{P} \ni (x, p) \longmapsto j \circ \xi^{(1)}(x, p) \in \mathcal{L}(X, X_1)$$

satisfies

$$j\big(\xi^{(1)}(u, \varphi)\, v\big) = \big(j \circ \mathcal{K}_{\eta_0} \circ (A_{\eta_0\eta_0}(u))\big)(v) = \mathcal{K}^{cu}(A(u)\, v) = B(u, \varphi)\, v$$

for all $(u, \varphi, v) \in \mathcal{M} \times \mathcal{P} \times X$. As B is continuous, the continuity of the map $\mathcal{M} \times \mathcal{P} \ni (x, p) \longrightarrow j \circ \xi^{(1)}(x, p) \in \mathcal{L}(X, X_1)$ follows, and this is precisely condition (iv) of Lemma 2.18.

As by the above all assumptions of Lemma 2.18 are fulfilled, we conclude that the map

$$\tilde{u}_{\eta_1} = j \circ \Phi : C_{cu} \longrightarrow C^1_{\eta_1}$$

is in fact continuously differentiable. So, if we prove that additionally we have $Dw_{cu}(0) = 0$, the assertion of Theorem 2.2 follows. But this is easily seen in consideration of the formula

$$D\tilde{u}_{\eta_1}(\varphi) = \xi_1^{(1)}(\tilde{u}_{\eta_0}(\varphi), \varphi) \circ D\tilde{u}_{\eta_1}(\varphi) + j \circ D_2\xi_0(\tilde{u}_{\eta_0}(\varphi), \varphi)$$

for the derivative of $\tilde{u}_{\eta_1}$ at $\varphi \in C_{cu}$. Indeed, by $Dr_\delta(0) = 0$, we first obtain $A(0) = 0$ and $\xi_1^{(1)}(0, 0) = 0$. Thus, in consideration of $\tilde{u}_{\eta_0}(0) = 0$ we get

$$D\tilde{u}_{\eta_1}(0)\, \psi = j \circ D_2\xi_0(0, 0)\, \psi = S_{\eta_1}\, \psi$$

for all $\psi \in C_{cu}$. This implies

$$Dw^{\eta_1}(0)\, \psi = \big(P_s \circ \mathrm{ev}_0 \circ D\tilde{u}_{\eta_1}(0)\big)(\psi) = P_s\, \psi = 0$$

on C_{cu}. Consequently, we get

$$Dw^{\eta_1}(0) = 0$$

and this completes the proof of Theorem 2.2.

6. The Dynamics on Local Center-Unstable Manifolds

Subsequently, we demonstrate the possibility to reduce the smooth semiflow F generated by the solutions of the differential equation (2.1) to a local flow on a fixed manifold W_{cu}. The advantage of such a reduction lies in the fact that one obtains a system of ordinary differential equations, describing the dynamics on W_{cu}. In our exposition we follow the discussion and notation of Diekmann et al. [**6**, Section VI.2], where the reader may also find a thorough treatment of the necessary results of the spectral theory.

Consider once more the differential equation (2.1) under the conditions of Theorem 2.1 and let $\{\varphi_1, \dots, \varphi_d\}$, $d \in \mathbb{N}$, denote a basis of the center-unstable space C_{cu} at the trivial stationary point. Introducing the row vector

$$\Phi_{cu} := \big(\varphi_1, \dots, \varphi_d\big)$$

and using the invariance of C_{cu} under G_e, we find a matrix $B_{cu} \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ such that

$$G_e\, \Phi_{cu} = \Phi_{cu}\, B_{cu}.$$

Here $G_e\, \Phi_{cu}$ denotes the row vector $(G_e\, \varphi_1, \dots, G_e\, \varphi_n)$ and $\Phi_{cu}\, B_{cu}$ the usual multiplication of a row vector with a matrix. The matrix B_{cu} has exactly the same eigenvalues as the restriction of the operator G_e to C_{cu} and the last equation implies that the action of the group T_e on Φ_{cu} is given by

$$T_e(t)\,\Phi_{cu} = \Phi_{\,cu}\, e^{tB_{cu}}$$

for $t \in \mathbb{R}$. In particular, this relation also defines the action of T_e on the whole center-unstable space C_{cu}. Indeed, each function $\varphi \in C_{cu}$ has an unique expression

$$\varphi = \sum_{j=1}^{d} \hat{c}_j(\varphi)\,\varphi_j$$

with coefficients $\hat{c}_j(\varphi) \in \mathbb{R}$ and basis elements φ_j of C_{cu}. Writing the coefficient $\hat{c}_j(\varphi)$ as a column vector $c(\varphi) := (\hat{c}_1(\varphi), \ldots, \hat{c}_d(\varphi))^T$ in $\mathbb{R}^d$, we obtain the representation

$$\varphi = \Phi_{cu}\, c(\varphi) \tag{2.36}$$

and for the action of T_e on φ the formula

$$T_e(t)\,\varphi = T_e(t)\,\Phi_{cu}\, c(\varphi) = \Phi_{cu}\, e^{tB_{cu}}\, c(\varphi)$$

for $t \in \mathbb{R}$.

Suppose now that $x : I + [-h, 0] \longrightarrow \mathbb{R}^n$, where I is some interval in $\mathbb{R}$, is a solution of the differential equation (2.1) on the local center-unstable manifold W_{cu}; that is, x is a solution with $x_t \in W_{cu}$ for all $t \in I$. Then the function $u : I \ni t \longmapsto x_t \in W_{cu} \subset C^1$ satisfies, for all $s, t \in I$, $s \le t$, the abstract integral equation

$$u(t) = T_e(t-s)\,u(s) + \int_s^t T_e^{\odot *}(t-\tau)\, l(r(u(\tau)))\, d\tau.$$

In addition, as $u(t) \in W_{cu}$ we have the representation $u(t) = P_{cu}\,u(t) + w_{cu}(P_{cu}\,u(t))$. Set

$$v : I \ni t \longmapsto P_{cu}^{\odot *}\, u(t) \in C_{cu}.$$

From the continuity of the mapping $I \ni t \longmapsto l(r(u(t)))) \in C^{\odot *}$, the identity $P_{cu}^{\odot *}\,\varphi = P_{cu}\,\varphi$ on C^1 and the above integral equation for u, it follows that

$$\begin{aligned}
v(t) &= P_{cu}^{\odot *}\, u(t) \\
&= P_{cu}^{\odot *}\Big(T_e(t-s)\,u(s) + \int_s^t T_e^{\odot *}(t-\tau)\, l(r(u(\tau)))\, d\tau\Big) \\
&= P_{cu}^{\odot *}\, T_e(t-s)\,u(s) + P_{cu}^{\odot *} \int_s^t T_e^{\odot *}(t-\tau)\, l(r(P_{cu}\,u(\tau) + w_{cu}(P_{cu}\,u(\tau))))\, d\tau \\
&= T_e(t-s)\, P_{cu}^{\odot *}\, u(s) + \int_s^t T_e^{\odot *}(t-\tau)\, P_{cu}^{\odot *}\, l(r(v(\tau) + w_{cu}(v(\tau))))\, d\tau \\
&= T_e(t-s)\, v(s) + \int_s^t T_e(t-s)\, P_{cu}^{\odot *}\, g(v(\tau))\, d\tau
\end{aligned}$$

with the continuously differentiable function

$$g : C_{cu,0} \ni \varphi \longmapsto l(r(\varphi + w_{cu}(\varphi))) \in C^{\odot *}.$$

Thus, the spectral projection v of the solution u on the finite dimensionale space C_{cu} satisfies the abstract integral equation

$$v(t) = T_e(t-s)\, v(s) + \int_s^t T_e(t-\tau)\, P_{cu}^{\odot *} g(v(\tau)) d\tau \tag{2.37}$$

for all $s, t \in I$, $s \le t$.

Next, we rewrite Equation (2.37) in coordinates on C_{cu}, with respect to the fixed basis $\{\varphi_1, \dots, \varphi_d\}$ of the center-unstable space. For this purpose consider the mapping Γ_{cu}, which assigns to each element $\varphi \in C_{cu}$ the coefficient vector $c(\varphi) \in \mathbb{R}^d$ due to relation (2.36). We see at once that Γ_{cu} is a bounded linear map. Moreover, for all $\varphi \in C_{cu}$ we have the identity $\varphi = \Phi_{cu}\, c(\varphi) = \Phi_{cu}\, \Gamma_{cu}\, \varphi$ and for all $c \in \mathbb{R}^d$ we get $c = \Gamma_{cu}\, \Phi_{cu}\, c$. Combining this with the integral equation (2.37) for v, we conclude that the coordinate function

$$y : I \ni t \longmapsto \Gamma_{cu}\, v(t) = \Gamma_{cu}\, P_{cu}^{\odot *}\, u(t) \in \mathbb{R}^d$$

of v satisfies the integral equation

$$y(t) = e^{(t-s)B_{cu}}\, y(s) + \int_s^t e^{(t-\tau)B_{cu}}\, \Gamma_{cu}\, P_{cu}^{\odot *}\, g(\Phi_{cu}\, y(\tau))\, d\tau \tag{2.38}$$

in $\mathbb{R}^d$ for all reals $s, t \in I$ with $s \leq t$. In fact, a straightforward argument shows

$$\Gamma_{cu} \int_s^t T_e(t-\tau)\, P_{cu}^{\odot *}\, g(v(\tau))\, d\tau = \int_s^t \Gamma_{cu}\, T_e(t-\tau)\, P_{cu}^{\odot *}\, g(v(\tau))\, d\tau \tag{2.39}$$

so that Equation (2.37) results in the variation-of-constants formula

$$\begin{aligned}
y(t) &= \Gamma_{cu}\, v(t) \\
&= \Gamma_{cu} \left(T_e(t-s)\, v(s) + \int_s^t T_e(t-\tau)\, P_{cu}^{\odot *}\, g(v(\tau))\, d\tau \right) \\
&= \Gamma_{cu}\, T_e(t-s)\, v(s) + \Gamma_{cu} \int_s^t T_e(t-\tau)\, P_{cu}^{\odot *}\, g(\Phi_{cu}\, \Gamma_{cu}\, v(\tau))\, d\tau \\
&= \Gamma_{cu}\, T_e(t-s)\, \Phi_{cu}\, \Gamma_{cu}\, v(s) + \int_s^t \Gamma_{cu}\, T_e(t-s)\, \Phi_{cu}\, \Gamma_{cu}\, P_{cu}^{\odot *}\, g(\Phi_{cu}\, y(\tau))\, d\tau \\
&= \Gamma_{cu}\, \Phi_{cu}\, e^{(t-s)B_{cu}}\, y(s) + \int_s^t e^{(t-\tau)B_{cu}}\, \Gamma_{cu}\, P_{cu}^{\odot *}\, g(\Phi_{cu}\, y(\tau))\, d\tau \\
&= e^{(t-s)B_{cu}}\, y(s) + \int_s^t e^{(t-\tau)B_{cu}}\, \Gamma_{cu}\, P_{cu}^{\odot *}\, g(\Phi_{cu}\, y(\tau))\, d\tau
\end{aligned}$$

for the coordinate function y.

Remark 2.20. Note the distinction of the both integrals involved in Equation (2.39). On the left hand-side we have a weak* integral for a continuous function from $[s,t]$ into $C_{cu} \subset C^{\odot *}$, whereas on the right it is the usual integral for a continuous curve from $[s,t]$ into $\mathbb{R}^d$.

By differentiation of Equation (2.38) with respect to t, we obtain the ordinary differential equation

$$\dot{y}(t) = B_{cu}\, y(t) + Q(y(t)) \tag{2.40}$$

in $\mathbb{R}^d$ with the matrix B_{cu} and the nonlinearity $Q(\,\cdot\,) := \Gamma_{cu}\, P_{cu}^{\odot *}\, g(\Phi_{cu}\,\cdot\,)$, which is obviously continuously differentiable in y. This equation describes the solutions of the differential equation (2.1) on the local center-unstable manifold W_{cu} at the stationary point $\varphi_0 = 0$.

Part 2

A Model for Currency Exchange Rates

CHAPTER 3

Basic Properties and Slowly Oscillating Solutions

1. Introduction

Here, we begin the study of the scalar differential equation

$$\dot{x}(t) = a\big[x(t) - x(t-r)\big] - |x(t)|\,x(t) \tag{3.1}$$

with a parameter $a > 0$ and a state-dependent delay $r = r(x(t)) > 0$. As already mentioned, this equation is a slight modification of a mathematical model for a currency exchange rate, which is represented by the DDE

$$\dot{x}(t) = a\big[x(t) - x(t-1)\big] - |x(t)|\,x(t) \tag{3.2}$$

and extensively analyzed by Erdélyi, Brunovský, and Walther [**7, 3, 4, 27, 28**].

For our investigations of the model equation (3.1), we suppose the following hypotheses on the delay function $r : \mathbb{R} \longrightarrow \mathbb{R}$:

(DF 1) $r \in C^1(\mathbb{R}, \mathbb{R})$,
(DF 2) $0 < r(s) \leq r(0) =: r_0$ for all $s \in \mathbb{R}$,
(DF 3) $r(s) = r(-s)$ for all $s \in \mathbb{R}$, and
(DF 4) $r_0 = 1$.

Condition (DF 1) is a mild smoothness assumption, whereas (DF 2) guarantees that the retardation does not vanish. In addition, (DF 2) ensures that all solutions forget their history prior to a maximal value r_0. The motivation for (DF 3) is the reflection symmetry of the original equation with constant delay. Under the assumption (DF 3) a function $x : [t_0 - r_0, t_+) \longrightarrow \mathbb{R}$, $t_0 < t_+$, satisfies DDE (3.1) for $t_0 < t < t_+$ if and only if $-x$ does so. For the sake of simplicity, we assume condition (DF 4) to hold, and this can be achieved by scaling. Indeed, assuming $r : \mathbb{R} \longrightarrow \mathbb{R}$ has the properties (DF 1) - (DF 3) with a maximal value $r_0 > 0$, and introducing the new time-variable $s := t/r_0$, the new delay function

$$\tau : \mathbb{R} \ni u \longmapsto \frac{1}{r_0}\, r(u/r_0) \in [0,1],$$

and the new state-variable $y(s) := r_0\, x(t)$, we obtain

$$\begin{aligned}
\dot{y}(s) &= \frac{d}{ds}\big(r_0\, x(t)\big) \\
&= r_0^2\, \dot{x}(t) \\
&= a\, r_0\big[r_0\, x(t) - r_0\, x(t - r(x(t)))\big] - |r_0\, x(t)|\,(r_0\, x(t)) \\
&= a\, r_0\big[y(s) - y(t/r_0 - r(x(t))/r_0)\big] - |y(s)|\, y(s) \\
&= a\, r_0\big[y(s) - y(s - \tau(r_0\, x(t)))\big] - |y(s)|\, y(s) \\
&= \tilde{a}\big[y(s) - y(s - \tau(y(s)))\big] - |y(s)|\, y(s),
\end{aligned}$$

where $\tilde{a} := a\, r_0 > 0$ and τ obviously satisfies (DF 1) - (DF 4). Simple examples for delay functions with the stated properties are given by

$$r_1 : \mathbb{R} \ni s \longmapsto e^{-c\, s^{2k}} \in \mathbb{R}$$

and

$$r_2 : \mathbb{R} \ni s \longmapsto \frac{1}{1 + c\, s^{2k}} \in \mathbb{R}$$

with constants $c \geq 0$ and $k \in \mathbb{N}$. For the choice $c = k = 1$, both are sketched in Figure 3.1.

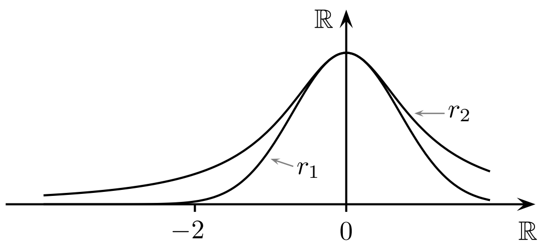

FIGURE 3.1. Examples of delay functions satisfying (DF 1) - (DF 4)

REMARK 3.1. Note that the stated assumptions on the delay function r are in particular satisfied in case of the constant delay 1, which reduces Equation (3.1) to (3.2). As a consequence, under conditions (DF 1) - (DF 4) all results for Equation (3.1) are also valid for the DDE (3.2). On the other hand, different statements and techniques from Brunovský et al. [**3**] on Equation (3.2) may be elementarily generalized to the situation of a state-dependent delay with properties (DF 1) - (DF 4), as we will see throughout this part.

Under the above hypothesis on r, Equation (3.1), or more precisely its transformation to the abstract form (1.4), fulfills the conditions of Theorem 1.2, as was seen in the adjacent example on page 9. In particular, this guarantees the existence of a continuous semiflow with respect to the underlying concept of C^1-smooth solutions. Despite this fact, our study of the DDE (3.1) begins with the classical idea of solutions for an RFDE; that is, until further notice, we call a real valued function x a solution of Equation (3.1) on $[t_0 - 1, t_+)$, if there are $t_0 \in \mathbb{R}$ and $t_+ > t_0$ such that $x : [t_0 - 1, t_+) \longrightarrow \mathbb{R}$ is continuous, continuously differentiable on $(t_0, t_0 + t_+)$, and satisfies Equation (3.1) for all $t_0 \leq t < t_0 + t_+$, where $\dot{x}(t_0)$ denotes the right-hand side derivative.

This temporary arrangement on solutions enables us to consider also an only continuous function as an initial value for the associated IVP of the DDE (3.1). On the other hand, in Chapter 1 we clarified that such a selection of initial data may raise difficulties concerning the uniqueness and continuous dependence of solutions. Using the technique in Krisztin and Arino [**17**], in the first instance we discuss the basic properties as existence and uniqueness of solutions and construct a continuous semiflow of Equation (3.1) on an appropriate subset of all continuous functions from $[-1, 0]$ into $\mathbb{R}$.

By postulating a further property of the delay function in Section 3 we will conclude that a weighted difference of any two solutions, which are globally defined and contained in the constructed state-space, satisfy a linear DDE. This fact will be crucial in the present work. It will allow us to apply a Lyapunov functional defined in Krisztin and Arino [**17**] as a counting tool for sign changes of solutions for Equation (3.1).

The last section of this chapter is devoted to so-called *slowly oscillating* solutions characterized by the property that the distance of any two consecutive zeros is greater than 1, the maximal "memory" of the DDE (3.1) due to the two conditions (DF 2) and (DF 3). After proving the existence of such solutions, we close this chapter with a characterization of all global slowly oscillating solutions.

Before we proceed recall the abbreviations C, C^1 in Part 1 and the associated norm $\|\cdot\|_C$, $\|\cdot\|_{C^1}$, respectively. Subsequently they are used for the underlying situation $h = 1$ and $n = 1$.

2. A Continuous Semiflow on a Compact State Space

The main goal of this section is the construction of a continuous semiflow on an appropriate state-space in C for solutions of Equation (3.1) by adapting the method contained in Krisztin and Arino [**17**]. For this purpose, we begin with the discussion of basic results concerning the existence and uniqueness of solutions. Hereafter, a boundedness result, which will be established completely similarly to the case of constant delay from Brunovský et al. [**3**], will allow us to restrict our attention to a subset of C and to derive a continuous semiflow.

The Existence and Uniqueness of Solutions. Introduce the mapping

$$f_e : C \ni \varphi \longmapsto a\big[\varphi(0) - \varphi(-r(\varphi(0)))\big] - |\varphi(0)|\,\varphi(0) \in \mathbb{R} \tag{3.3}$$

so that the differential equation (3.1) reads in the more abstract form

$$\dot{x}(t) = f_e(x_t) \tag{3.4}$$

of an RFDE. In order to prove the existence of non-continuable solutions we establish two auxiliary observations about the complete continuity of f_e and the boundedness of solutions.

COROLLARY 3.2. *The function f_e is completely continuous; that is, f_e is continuous and maps bounded sets into relative compact sets.*

PROOF. For the continuity of f_e, we notice the representation of f_e as

$$f_e(\varphi) = g(\mathrm{ev}_0(\varphi),\, \mathrm{ev}(\varphi, -\widehat{R}(\varphi)))$$

with the function $g : \mathbb{R}^2 \ni (\zeta, \xi) \longmapsto a\big[\zeta - \xi\big] - |\zeta|\,\zeta \in \mathbb{R}$, the evaluation maps

$$\mathrm{ev}_0 : C \ni \varphi \longmapsto \varphi(0) \in \mathbb{R}$$

and

$$\mathrm{ev} : C \times [-1, 0] \ni (\varphi, s) \longmapsto \varphi(s) \in \mathbb{R},$$

and the delay function $\widehat{R} : C \ni \varphi \longmapsto r(\varphi(0)) \in [0, 1]$. Obviously, each of these functions is continuous. Hence, f_e is also continuous as their composition.

For the remaining part of the proof, assume $U \subset C$ is bounded. Then we find a constant $K_U > 0$ such that $\|\varphi\|_C \le K_U$ for all $\varphi \in U$. Defining the constant

$M := 2a\,K_U + K_U^2 > 0$ leads to

$$\begin{aligned}|f_e(\varphi)| &= |g(\varphi(0),\, \varphi(-r(\varphi(0)))| = \big|\big[a - |\varphi(0)|\big]\,\varphi(0) - a\,\varphi(-r(\varphi(0)))\big| \\ &\le |\,a - |\varphi(0)|\,|\; K_U + a\,K_U \le a\,K_U + K_U^2 + a\,K_U = M\end{aligned}$$

for $\varphi \in U$. Thus $f_e(U) \subset \mathbb{R}$ is bounded, and subsequently $\overline{f_e(U)}$ compact. □

Next, we show the boundedness of solutions of Equation (3.1). The proof is similar to the situation of constant delay in Brunovský et al. [**3**].

PROPOSITION 3.3. *Suppose that* $x : [t_0, t_+) \longrightarrow \mathbb{R}$, $t_0 < t_+ - 1$, *is a solution of the differential equation* (3.1). *Then* x *is bounded.*

PROOF. Assume the assertion is false. Then there is a value $t_0 < s < t_+$ such that $2a < |x(s)|$, but $|x(t)| < |x(s)|$ for all $t_0 - 1 \le t < s$. Hence, for $x(s) > 2a > 0$ we obtain

$$\begin{aligned}0 &\le \dot{x}(s) \\ &= a\big[x(s) - x(s - r(x(s)))\big] - |x(s)|\,x(s) \\ &< 2a\,x(s) - |x(s)|\,x(s) \\ &\le 2a\,x(s) - 2a\,x(s) \\ &= 0,\end{aligned}$$

and in the situation $x(s) < -2a < 0$

$$\begin{aligned}0 &\ge \dot{x}(s) \\ &= a\big[x(s) - x(s - r(x(s)))\big] - |x(s)|\,x(s) \\ &> 2a\,x(s) - |x(s)|\,x(s) \\ &\ge 2a\,x(s) - 2a\,x(s) \\ &= 0,\end{aligned}$$

a contradiction to the existence of such a value s. Hence, x is bounded. □

Now we state the existence of (in forward direction) non-continuable solutions for the differential equation (3.1). This follows as a straightforward application of basic theorems on existence and continuation of solutions for RFDEs in view of the last auxiliary results.

PROPOSITION 3.4. *Let* $\varphi \in C$.

(i) *There exists a solution* $x : [-1, t_+) \longrightarrow \mathbb{R}$, $t_+ > 0$, *of Equation* (3.1) *with initial value* $x_0 = \varphi$.

(ii) *If* $x : [-1, t_+) \longrightarrow \mathbb{R}$, $t_+ > 0$, *is a (in forward direction) non-continuable solution of Equation* (3.1) *with* $x_0 = \varphi$, *then* $t_+ = \infty$.

PROOF. We understand Equation (3.1) as the autonomous RFDE (3.4). Since f_e is continuous, the existence theorem on solutions for RFDEs as stated, for instance, in Hale and Verduyn Lunel [**10**, Theorem 2.1 in Chapter 2] can be applied, and implies the existence of a solution $x : [-1, t_+) \longrightarrow \mathbb{R}$, $t_+ > 0$, with initial value φ at $t = 0$. This proves the first assertion.

For the second part of the proposition, suppose $x : [-1, t_+) \longrightarrow \mathbb{R}$, $t_+ > 0$, is a non-continuable solution of the RFDE given by f_e with initial value $x_0 = \varphi$, and $t_+ < \infty$. The continuity result on f_e from Corollary 3.2 in combination with elementary results on continuation of solutions as is, for instance, contained in Hale and Verduyn Lunel [**10**, Theorem 3.2 in Chapter 2] imply $\|x_t\|_C \to \infty$ as $t \to t_+$. But this contradicts Proposition 3.3 on boundedness of the solution x, and the proof is complete. □

Recall that we may not expect, due to the presence of a state-dependent delay, the uniqueness of solutions of Equation (3.1) for arbitrary initial data in C, even if the delay function r is smoother than C^1. However, a Lipschitz continuous initial function in C provides a solution that is uniquely determined (in the forward time direction), as shown in the next result.

PROPOSITION 3.5. *Suppose that $\varphi \in C$ is Lipschitz continuous, and suppose $x : [-1,\infty) \longrightarrow \mathbb{R}$ and $\tilde{x} : [-1,\infty) \longrightarrow \mathbb{R}$ are solutions of Equation* (3.1) *with $x_0 = \varphi = \tilde{x}_0$. Then it follows that $x(t) = \tilde{x}(t)$ for all $-1 \le t < \infty$.*

PROOF. According to Proposition 3.3, both x and $\tilde{x}$ are bounded. Hence, there are constants $0 < m_\varphi, M_\varphi$ with $x(t), \tilde{x}(t) \in (-m_\varphi, M_\varphi)$ for all $-1 \le t < \infty$. Now the maps $[0,\infty) \ni s \longmapsto x_s \in C$ and $[0,\infty) \ni s \longmapsto \tilde{x}_s \in C$ are both continuous and therefore the solutions x and $\tilde{x}$ may be represented by

$$x(t) = \varphi(0) + \int_0^t f_e(x_s)\, ds$$

and

$$\tilde{x}(t) = \varphi(0) + \int_0^t f_e(\tilde{x}_s)\, ds$$

for all $t \ge 0$. Combining this with the complete continuity of f_e due to Corollary 3.2, we find an additional constant $M_L \ge \max\{1, a\}$ such that $x, \tilde{x}$ are Lipschitz continuous in $[-1,\infty)$ with Lipschitz constant M_L. Moreover, we may also assume that the map $[-m_\varphi, M_\varphi] \ni \zeta \longmapsto |\zeta|\zeta \in \mathbb{R}$ as well as the function r are Lipschitz continuous in $[-m_\varphi, M_\varphi]$, even with the same Lipschitz constant M_L. Define now $y(t) := x(t) - \tilde{x}(t)$, $\eta(t) := t - r(x(t))$, and $\tilde{\eta}(t) := t - r(\tilde{x}(t))$. Then we have

$$\begin{aligned}
|\dot{y}(t)| &= |\dot{x}(t) - \dot{\tilde{x}}(t)| \\
&\le a\,|x(t) - \tilde{x}(t)| + a\,|x(\eta(t)) - \tilde{x}(\tilde{\eta}(t))| + |\,|\tilde{x}(t)|\,\tilde{x}(t) - |x(t)|\,x(t)\,| \\
&\le M_L\,|y(t)| + M_L\,|x(\eta(t)) - x(\tilde{\eta}(t))| + M_L\,|x(\tilde{\eta}(t)) \\
&\qquad - \tilde{x}(\tilde{\eta}(t))| + M_L\,|x(t) - \tilde{x}(t)| \\
&\le M_L\,|y(t)| + M_L^2\,|\eta(t) - \tilde{\eta}(t)| + M_L\,|y(\tilde{\eta}(t))| + M_L\,|y(t)| \\
&\le M_L\,|y(t)| + M_L^3\,|y(t)| + M_L\,|y(\tilde{\eta}(t))| + M_L\,|y(t)|
\end{aligned}$$

for all $t \ge 0$. Hence, defining

$$z(t) := \max_{s \in [-1,t]} |x(s) - \tilde{x}(s)|,$$

we obtain the estimate

$$|y(t)| \le \int_0^t \Big(3M_L + M_L^3\Big)\, z(s)\, ds \le \int_0^\tau \Big(3M_L + M_L^3\Big)\, z(s)\, ds$$

as $0 \le t \le \tau$. Thus for all $\tau \ge 0$ we have

$$z(\tau) \le \int_0^\tau \Big(3M_L + M_L^3\Big)\, z(s)\, ds,$$

and hence by application of Gronwall's inequality, compare for instance Hale [**8**, Chapter I.6], it follows

$$0 \leq z(\tau) \leq 0 \cdot \exp\left(\int_0^\tau (3M_L + M_L^3)\, ds \right) = 0.$$

Consequently, $x = \tilde{x}$ as asserted. □

The Continuous Semiflow. After the above discussion of existence and uniqueness of solutions for DDE (3.1), we proceed with the construction of a semiflow by restriction of initial data. As a first step, we show that each solution with an initial value bounded by $2a$ remains bounded by $2a$.

PROPOSITION 3.6. *Suppose that $\varphi \in C$ with $\|\varphi\|_C \leq 2a$, and that the function $x^\varphi : [-1, \infty) \longrightarrow \mathbb{R}$ is a solution of (3.1) with $x_0 = \varphi$. Then $|x(t)| \leq 2a$ for all $t \geq -1$. Furthermore, $\|\varphi\|_C < 2a$ implies $|x(t)| < 2a$ for $t \geq -1$.*

PROOF. We follow the proof in Brunovský et al. [**3**] for the case of constant delay, and prove the second part of the statement first. For this purpose, assume there is a function $\varphi \in C$ with $\|\varphi\|_C < 2a$ such that, contrary to our claim, for a solution $x : [-1, \infty) \longrightarrow \mathbb{R}$ of Equation (3.1) with $x_0 = \varphi$ the inequality $|x(t)| < 2a$ does not hold for all $t \geq -1$. Then there is a $s > 0$ such that $|x(s)| \geq 2\,a$. Especially, we find a $t_0 > 0$ such that $|x(t_0)| = 2a > |x(t)|$ for all $-1 \leq t < t_0$. But this leads to a contradiction since in case $x(t_0) = 2a$ we would have

$$\begin{aligned}
0 &\leq \dot{x}(t_0) \\
&= a\big[x(t_0) - x(t_0 - r(x(t_0)))\big] - |x(t_0)|\, x(t_0) \\
&= a\big[2\,a - x(t_0 - r(2a))\big] - 4a^2 \\
&= -2a^2 - a\, x(t_0 - r(2a)) \\
&< -2a^2 + 2a^2 \\
&= 0,
\end{aligned}$$

and a similar argument applies also to the situation $x(t_0) = -2a$. Consequently, $|x(t)| < 2a$ for all $t \geq -1$, which shows the second statement of the proposition.

For the proof of the first assertion of the proposition, let us consider a solution $x : [-1, \infty) \longrightarrow \mathbb{R}$ of Equation (3.1), and assume the existence of $s > 0$ such that $|x(s)| > 2a$ holds. By the former argument, it follows $\|\varphi\|_C \geq 2a$ where $\varphi := x_0$. Hence, it is sufficient to show that the case $\|\varphi\|_C = 2a$ leads to a contradiction. For this purpose, define

$$t_0 := \max\Big\{ t \in [-1, \infty) \,\Big|\, |x(t)| = 2a \text{ and } |x(s)| \leq 2a \text{ for all } -1 \leq s \leq t \Big\}$$

under the assumption $\|\varphi\|_C = 2a$. Obviously, we have $0 \leq t_0 < s$, and there is a constant $\varepsilon > 0$ such that either $x(t) > 2a$ for $t_0 < t < t_0 + \varepsilon$ and x is monotonically increasing in $(t_0, t_0 + \varepsilon)$, or analogously $x(t) < -2a$ for $t_0 < t < t_0 + \varepsilon$ and x is monotonically decreasing. Additionally, in view of hypothesis (DF 2) on the delay function r and the continuity of the map $[0, \infty) \ni t \longmapsto t - r(x(t)) \in \mathbb{R}$ we may assume that $t - r(x(t)) \leq t_0$ for all $t_0 < t < t_0 + \varepsilon$. Consider now the situation $x(t_0) = 2a$ first. Fixing $t_1 \in (t_0, t_0 + \varepsilon)$, by assumptions we find a constant $\delta > 0$

with $x(t_1) = 2a + \delta$. But this leads to

$$\begin{aligned}
0 &\le \dot{x}(t_1) \\
&= a\big[x(t_1) - x(s - r(x(t_1)))\big] - |x(t_1)|\,x(t_1) \\
&= a\big[(2a+\delta) - x(t_1 - r(x(t_1)))\big] - (2a+\delta)^2 \\
&= -2a^2 - 3a\delta - \delta^2 - a\,x(t_1 - r(x(t_1)) \\
&\le -2a^2 - 3a\delta - \delta^2 + 2a^2 \\
&= -3a\delta - \delta^2 \\
&< 0,
\end{aligned}$$

a contradiction. A similar reasoning applies to the case $x(t_0) = -2a$. Therefore, the situation $\|\varphi\|_C = 2a$ is impossible as claimed, and the proof is complete. $\square$

Subsequently, we merge the previous results on solutions for the delay differential equation (3.1) to obtain an appropriate state-space and a smooth semiflow. On that account we define the number

$$K_a := 4a^2 > 0, \tag{3.5}$$

where $a > 0$ is the parameter involved in the differential equation (3.1). Additionally, we introduce the real-valued function

$$g : [-2a, 2a] \times [-2a, 2a] \ni (\zeta, \xi) \longmapsto a\big[\zeta - \xi\big] - |\zeta|\,\zeta \in \mathbb{R}. \tag{3.6}$$

Then Equation (3.1) reads

$$\dot{x}(t) = g(x(t), x(t - r(x(t))),$$

and it is a simple matter to conclude $|g(\zeta, \xi)| \le K_a$ for all $\zeta, \xi \in [-2a, 2a]$ as also indicated in Figure 3.2. For a given $\varphi \in C$ satisfying the boundedness condition

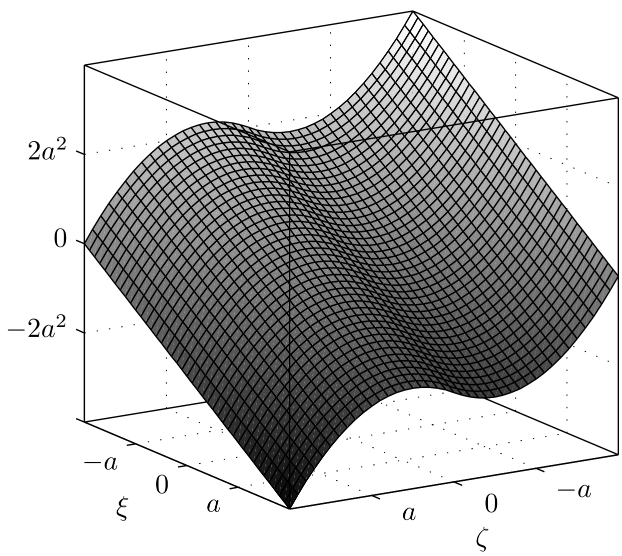

FIGURE 3.2. Behavior of g on the square $[-2a, 2a] \times [-2a, 2a]$

$\|\varphi\|_C \le 2a$ as well as the Lipschitz condition $|\varphi(s) - \varphi(\tilde{s})| \le K_a\,|s - \tilde{s}|$, we make the following observation:

(a) There is a unique solution $x^\varphi : [-1,\infty) \longrightarrow \mathbb{R}$ of Equation (3.1) with $x_0 = \varphi$ due to Propositions 3.4 and 3.5 on existence and uniqueness of solutions.
(b) All values $x^\varphi(t)$, $t \geq -1$, of the solution x^φ belong to $[-2a, 2a]$ by the last proposition on boundedness. In particular, $\|x_t^\varphi\|_C \leq 2a$ for $t \geq 0$.
(c) Every segment x_t^φ, $t > 0$, of the solution x^φ is again Lipschitz continuous with constant K_a. Indeed, observe that for all $s, \tilde{s} \in \mathbb{R}$ with $s \geq \tilde{s} \geq 0$ we have

$$\begin{aligned} |x^\varphi(s) - x^\varphi(\tilde{s})| &= \left| \int_{\tilde{s}}^{s} g(x^\varphi(t), x^\varphi(t - r(x^\varphi(t)))\, dt \right| \\ &\leq \int_{\tilde{s}}^{s} |g(x^\varphi(t), x^\varphi(t - r(x^\varphi(t))))|\, dt \\ &\leq \int_{\tilde{s}}^{s} K_a\, dt \\ &= K_a\, |s - \tilde{s}|, \end{aligned}$$

and for all $s, \tilde{s} \in \mathbb{R}$ with $s \geq 0 \geq \tilde{s} \geq -1$

$$\begin{aligned} |x^\varphi(s) - x^\varphi(\tilde{s})| &\leq |x^\varphi(s) - x^\varphi(0)| + |x^\varphi(0) - x^\varphi(\tilde{s})| \\ &= |x^\varphi(s) - x^\varphi(0)| + |\phi(0) - \phi(\tilde{s})| \\ &\leq K_a\, |s - 0| + K_a\, |\tilde{s}| \\ &= K_a\, |s - \tilde{s}|. \end{aligned}$$

The above conclusions motivate to consider the subset

$$L_a := \Big\{ \varphi \in C \,\Big|\, \|\varphi\|_C \leq 2a,\ |\varphi(s) - \varphi(\tilde{s})| \leq K_a\, |s - \tilde{s}| \ \text{ for } s, \tilde{s} \in [-1, 0] \Big\} \tag{3.7}$$

of C as a choice for a possible state space and the mapping

$$F : [0,\infty) \times L_a \ni (t, \varphi) \longmapsto x_t^\varphi \in L_a \tag{3.8}$$

for the associated semiflow.

The subset L_a of the Banach space C is of course not linear, but convex, as is easily seen. Furthermore, L_a is trivially bounded, closed and equicontinuous. Therefore it is compact by the Arzelà-Ascoli theorem, see Brown and Page [**2**, Theorem 4.6.2 in Chapter 4].

The function F is obviously well-defined, and by using the definition of F in combination with the uniqueness of solutions for initial values in L_a at $t = 0$, we obtain the identity $F(0, \varphi) = \varphi$ as well as the semigroup property

$$F(s + t, \varphi) = F(s, F(t, \varphi))$$

for all $\varphi \in L_a$ and all $s, t \geq 0$. Moreover, for each $\varphi \in L_a$ the induced map

$$[0,\infty) \ni t \longmapsto F(t, \varphi) = x_t^\varphi \in L_a$$

is continuous. The next result implies also the remaining property of a semiflow for F, namely, the continuity of the so-called time-t-maps, which are defined by

$$L_a \ni \varphi \longmapsto F(t, \varphi) \in L_a$$

for $t \geq 0$. The proof is patterned after a similar one in Krisztin and Arino [**18**].

PROPOSITION 3.7. *Suppose that $\varphi \in L_a$ and that $\{\varphi_n\}_{n\in\mathbb{N}}$ is a sequence in L_a such that $\varphi_n \to \varphi$ as $n \to \infty$. Let $x : [-1,\infty) \longrightarrow \mathbb{R}$ and $x^n : [-1,\infty) \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$, denote the uniquely determined solutions of Equation (3.1) with $x_0 = \varphi$ and $x_0^n = \varphi_n$, respectively. Then for every fixed $T > 0$ the following holds.*

(i) *The sequence $\{x^n\}_{n\in\mathbb{N}}$ converges uniformly to x in $[-1,T]$ as $n \to \infty$.*
(ii) *The sequence $\{\dot{x}^n\}_{n\in\mathbb{N}}$ converges uniformly to $\dot{x}$ in $[0,T]$ as $n \to \infty$.*

PROOF. The proof of part (i) is provided by a contradiction, whereas the statement (ii) will be deduced from the first one.

1. Assume that assertion (i) is false. Then we find a constant $\delta > 0$ and a subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\sup_{-1\le t\le T} |x^{n_k}(t) - x(t)| \ge \delta$$

for all $k \in \mathbb{N}$. Let M denote the set of all continuous functions ψ from $[-1,T]$ into $[-2a,2a]$, which are Lipschitz continuous with constant K_a. By application of the Arzelà-Ascoli theorem, we conclude that M is a compact subset of the Banach space $C([-1,T],\mathbb{R})$ equipped with the supremum norm. Accordingly, we find an element $y \in M$ and a subsequence $\{n_{k_l}\}_{l\in\mathbb{N}}$ of $\{n_k\}_{k\in\mathbb{N}}$ such that $x^{n_{k_l}} \to y$ as $l \to \infty$. Notice

$$y \neq x\big|_{[-1,T]}$$

by construction. Now we claim that $y : [-1,T) \longrightarrow \mathbb{R}$ is also a solution of Equation (3.1) with initial value φ at $t = 0$.

Proof of the claim. For every $0 \le t < T$ and $l \in \mathbb{N}$ we have

$$x^{n_{k_l}}(t) = \varphi_{n_{k_l}}(0) + \int_0^t f_e\big(x_s^{n_{k_l}}\big)\,ds,$$

and hence in consideration of $|f_e\big(x_s^{n_{k_l}}\big)| \le K_a$ for $0 \le s \le t$ and $l \in \mathbb{N}$

$$\begin{aligned} y(t) &= \lim_{l\to\infty} x^{n_{k_l}}(t) \\ &= \lim_{l\to\infty}\left(\varphi_{n_{k_l}}(0) + \int_0^t f_e\big(x_s^{n_{k_l}}\big)\,ds\right) \\ &= \lim_{l\to\infty}\varphi_{n_{k_l}}(0) + \lim_{l\to\infty}\int_0^t f_e\big(x_s^{n_{k_l}}\big)\,ds \\ &= \lim_{l\to\infty}\varphi_{n_{k_l}}(0) + \int_0^t \lim_{l\to\infty} f_e\big(x_s^{n_{k_l}}\big)\,ds \\ &= \lim_{l\to\infty}\varphi_{n_{k_l}}(0) + \int_0^t f_e\Big(\lim_{l\to\infty} x_s^{n_{k_l}}\Big)\,ds \\ &= \varphi(0) + \int_0^t f_e(y_s)\,ds\,. \end{aligned}$$

By fundamental theorem of calculus we obtain

$$\dot{y}(t) = f_e(y_t)$$

for $0 < t < T$. Since

$$y|_{[-1,0]} = \lim_{l\to\infty}\big(x^{n_{k_l}}|_{[-1,0]}\big) = \lim_{l\to\infty}\varphi_{n_{k_l}} = \varphi,$$

$y : [-1,T) \longrightarrow \mathbb{R}$ is a solution of Equation (3.1) with initial value φ at $t = 0$, as claimed.

Returning to the proof of part (i), we have solutions x and y of Equation (3.1) with $x_0 = \varphi = y_0$, but

$$x|_{[-1,T)} \neq y|_{[-1,T)},$$

in contradiction to the uniqueness property of solutions from Proposition 3.5. Hence, $x^n(t) \to x(t)$ as $n \to \infty$ uniformly for t in $[-1, T]$.

2. Let now $L_g \geq 0$ denote the Lipschitz constant of the continuously differentiable function $[-2a, 2a] \ni \zeta \longmapsto |\zeta|\,\zeta \in \mathbb{R}$, and $L_r \geq 0$ the one of the function $\mathbb{R} \ni \zeta \longmapsto r(\zeta) \in \mathbb{R}$ on the interval $[-2a, 2a]$. Then we get

$$\begin{aligned}
|\dot{x}^n(t) - \dot{x}(t)| &= \Big| a\big[x^n(t) - x^n(t - r(x^n(t)))\big] - |x^n(t)|\, x^n(t) \\
&\qquad - a\,\big[x(t) - x(t - r(x(t)))\big] + |x(t)|\, x(t)\Big| \\
&\leq a\,|x^n(t) - x(t)| + a\,|x(t - r(x(t))) - x^n(t - r(x^n(t)))| \\
&\qquad + \big|\, |x(t)|\, x(t) - |x^n(t)|\, x^n(t)\big| \\
&\leq a\,|x^n(t) - x(t)| + a\,\big|x(t - r(x(t))) - x(t - r(x^n(t)))\big| \\
&\qquad + a\,\big|x(t - r(x^n(t))) - x^n(t - r(x^n(t)))\big| + L_g\,|x(t) - x^n(t)| \\
&\leq (a + L_g)\big|x^n(t) - x(t)\big| + a\,K_a\,\big|r(x(t)) - r(x^n(t))\big| \\
&\qquad + a\,\big|x(t - r(x^n(t))) - x^n(t - r(x^n(t)))\big| \\
&\leq (a + L_g)\,|x^n(t) - x(t)| + a\,K_a\,L_r\,|x(t) - x^n(t)| \\
&\qquad + a\,\big|x(t - r(x^n(t))) - x^n(t - r(x^n(t)))\big| \\
&= (a + L_g + a\,K_a\,L_r)\,\big|x^n(t) - x(t)\big| \\
&\qquad + a\,\big|x(t - r(x^n(t))) - x^n(t - r(x^n(t)))\big| \\
&\leq (2a + L_g + a\,K_a\,L_r) \sup_{s \in [-1,T]} |x^n(s) - x(s)|
\end{aligned}$$

for all $0 \leq t \leq T$. Accordingly, the uniform convergence of $\{x^n\}_{n\in\mathbb{N}}$ to x on the interval $[-1, T]$ as $n \to \infty$ implies that $\{\dot{x}^n\}_{n\in\mathbb{N}}$ converges to $\dot{x}$ on $[0, T]$ as $n \to \infty$. This completes the proof. □

We summarize the fact that the segments of solutions for Equation (3.1) with initial values in L_a generate a semiflow and prove its continuity in the next corollary.

Corollary 3.8. *Let $a > 0$ be given. Then the mapping F given by Equation* (3.8) *defines a continuous semiflow on the compact state space L_a from* (3.7).

Proof. In view of the previous results, the only point to prove is the continuity of the mapping F on the product space $[0, \infty) \times L_a$. Now, recall that for each fixed $t \geq 0$ the map

$$L_a \ni \varphi \longmapsto F(t, \varphi) \in L_a$$

and for each fixed $\varphi \in L_a$ the map

$$[0, \infty) \ni t \longmapsto F(t, \varphi) \in L_a$$

are continuous. Moreover, the last map is continuous, uniformly with respect to $\varphi \in L_a$. Indeed, for all $\varphi \in L_a$ and arbitrary $t_1, t_2 \geq 0$ we have

$$\begin{aligned}\|F(t_1,\varphi) - F(t_2,\varphi)\|_C &= \|x^{\varphi}_{t_1} - x^{\varphi}_{t_2}\|_C \\ &= \sup_{s\in[-1,0]} |x^{\varphi}_{t_1}(s) - x^{\varphi}_{t_2}(s)| \\ &= \sup_{s\in[-1,0]} |x^{\varphi}(t_1+s) - x^{\varphi}(t_2+s)| \\ &\leq \sup_{s\in[-1,0]} K_a|t_1 - t_2| \\ &= K_a|t_1 - t_2|\end{aligned}$$

due to the Lipschitz continuity of solutions with the Lipschitz constant $K_a > 0$. Applying Lemma 8.1 in Amann [**1**, Chapter 8], we conclude the continuity of F as a mapping from $[0,\infty) \times L_a$ into L_a, and this shows the assertion of the lemma. □

In the remainder of this chapter we primarily consider this semiflow. Nevertheless one may ask about the behavior, especially the long-time behavior, of solutions for initial values $\varphi \in C$ with $\|\varphi\|_C > 2a$. A starting point for solving this question could be the next result adapted from the situation with constant delay in Brunovský et al. [**3**].

Proposition 3.9. *If $x : [-1,\infty) \longrightarrow \mathbb{R}$ is a solution of Equation* (3.1)*, then*

$$-2a \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq 2a.$$

Proof. We begin with the claim that for every $t_0 > 0$ with $x(t_0) > 2a$, or $x(t_0) < -2a$, there is $s > t_0$ such that $x(s) = 2a$, or $x(s) = -2a$, respectively.

Proof of the claim: To begin with, suppose $x(t_0) > 2a$ for some $t_0 > 0$. If our claim is false, then $x(t) > 2a$ for all $t_0 \leq t < \infty$. Hence we obtain

$$\begin{aligned}\dot{x}(t) &= \big[a - |x(t)|\big]\, x(t) - a\, x(t - r(x(t))) \\ &\leq \big[a - 2a\big]\, 2a - 2a^2 \\ &= -4a^2 \\ &< 0\end{aligned}$$

for $t_0 + 1 \leq t < \infty$, a contradiction. Consequently, there is $t \geq t_0$ with $x(t) = 2a$. In case $x(t_0) < 2a$ for some $t_0 > 0$, it is clear that an application of the above arguments to the reflected solution $-x$ yields the assertion, which completes the proof.

We return now to the proof of the proposition and assume $\|x_t\|_C > 2a$ for all $t \geq 0$ since otherwise the assertion is clear due to Proposition 3.6. Then we find a sequence $\{t_j\}_{j\in\mathbb{N}} \subset [0,\infty)$ such that $|x(t_j)| > 2a$ for all $j \in \mathbb{N}$, and $t_j \to \infty$ as $j \to \infty$. Without loss of generality we may assume that either $x(t_j) > 2a$ for all $j \in \mathbb{N}$, or conversely $x(t_j) < -2a$ for all $j \in \mathbb{N}$. Subsequently, we divide the remaining part of the proof and consider these two cases separately.

1. In the situation $x(t_j) > 2a$ for $j \in \mathbb{N}$, the above implies the existence of a sequence $\{M_j\}_{j\in\mathbb{N}} \subset [0,\infty)$ of local maxima of x such that $M_j \to \infty$ and

$$2a < x(M_j) \to \limsup_{t\to\infty} x(t)$$

as $j \to \infty$. For every $j \in \mathbb{N}$ we have

$$0 = \dot{x}(M_j) = \big[a - |x(M_j)|\big]\, x(M_j) - a\, x(M_j - r(x(M_j))).$$

In particular, the inequality $0 < 2a < x(M_j)$ implies the estimate

$$\begin{aligned} a\,x(M_j - r(x(M_j))) &= \big[a - |x(M_j)|\big]\, x(M_j) \\ &= a\,x(M_j) - \big(x(M_j)\big)^2 \\ &< -a\,x(M_j) \\ &< 0 \end{aligned}$$

for $j \in \mathbb{N}$. Set $s_j := M_j - r(x(M_j))$. Then, combining $x(M_j) > 2a$ and the last inequality, we see $x(s_j) < -2a$ for $j \in \mathbb{N}$, and $s_j \to \infty$ as $j \to \infty$. Accordingly, the claim proved above shows the existence of a sequence $\{s_j^*\}_{j\in\mathbb{N}} \subset [0, \infty)$ such that $s_j^* > s_j$ and $x(s_j^*) = -2a$ for all $j \in \mathbb{N}$. Hence, we now find a sequence $\{m_j\}_{j\in\mathbb{N}} \subset [0, \infty)$ of local minima of the solution x with the property $m_j \to \infty$ and

$$-2a > x(m_j) \to \liminf_{t\to\infty} x(t)$$

as $j \to \infty$. Analogously to the sequence $\{M_j\}_{j\in\mathbb{N}}$ of local maxima, it follows that

$$0 = \dot{x}(m_j) = \big[a - |x(m_j)|\big]\, x(m_j) - a\,x(m_j - r(x(m_j)))$$

and thus, in view of $x(m_j) < -2a < 0$, that

$$\begin{aligned} a\,x(m_j - r(x(m_j))) &= \big[a - |x(m_j)|\big]\, x(m_j) \\ &= a\,x(m_j) + \big(x(m_j)\big)^2 \\ &> -a\,x(m_j) \\ &> 0 \end{aligned}$$

for all $j \in \mathbb{N}$. By Proposition 3.3, the two limits

$$c := \limsup_{t\to\infty} x(t) \geq 2a$$

and

$$d := \liminf_{t\to\infty} x(t) \leq -2a$$

are finite, and the inequalities $x(M_j - r(x(M_j))) < -x(M_j)$, $j \in \mathbb{N}$, yield $d \leq -c$. Analogously, the inequalities $x(m_j - r(x(m_j))) > -x(m_j)$, $j \in \mathbb{N}$, imply $-c \leq d$. Therefore, we have $-c = d$ and

$$\begin{aligned} -c &= d \\ &= \liminf_{t\to\infty} x(t) \\ &\leq \liminf_{j\to\infty} x(M_j - r(x(M_j))) \\ &\leq \limsup_{j\to\infty} x(M_j - r(x(M_j))) \\ &= \limsup_{j\to\infty} \left(x(M_j) - \frac{1}{a}\big(x(M_j)\big)^2 \right) \\ &= c - c^2/a, \end{aligned}$$

which finally results in $2a \geq c$. In particular,

$$2a \geq \limsup_{t\to\infty} x(t) \geq \liminf_{t\to\infty} x(t) \geq -2a,$$

and this proves the assertion in the situation $x(t_j) > 2a$ for $j \in \mathbb{N}$.

2. In the case $x(t_j) < 2a$ for all $j \in \mathbb{N}$, the argumentation of the first part applied to the reflected solution $-x$ of Equation (3.1) implies

$$-2a \leq \liminf_{t\to\infty} -x(t) \leq \limsup_{t\to\infty} -x(t) \leq 2a,$$

and therewith

$$2a \geq \limsup_{t\to\infty} x(t) \geq \liminf_{t\to\infty} x(t) \geq -2a.$$

This completes the proof. □

3. Differences of Solutions and a Discrete Lyapunov Functional

Under an additional hypothesis on the delay function, the weighted differences of solutions of Equation (3.1) satisfy linear differential equations with time-dependent coefficients and delay. This feature will allow us to follow the idea of Krisztin and Arino [**17**] and to introduce a discrete, integer-valued functional for counting sign changes of solutions of DDE (3.1). Such a counting tool will become crucial for our considerations of so-called *slowly oscillating* solutions in the course of this work.

Weighted Differences. Until further notice we assume in addition to the hypotheses (DF 1) - (DF 4) formulated on page 57 that the considered delay function $r : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies also the following further condition:

(DF 5) $|r'(s)| < 1/K_a$ for all $-2a \leq s \leq 2a$.

This appended postulation on the delay will ensure that along a solution x the model equation (3.1) "forgets" for all $t > t_0$ its history prior to $t_0 - r(x(t_0))$ in the sense that we have $t - r(x(t)) > t_0 - r(x(t_0))$.

Remark 3.10. An easy computation shows that by a suitable choice of constant $c > 0$ for fixed $k \in \mathbb{N}$ and given parameter $a > 0$ the additional assumption (DF 5) can also be guaranteed for our examples r_1 and r_2 of a delay function on page 58.

Next we prove the announced strongly increasing property along solutions of the delayed argument involved in Equation (3.1).

Corollary 3.11. *If $x : [t_0 - 1, \infty) \longrightarrow [-2a, 2a]$ is a solution of Equation (3.1) then*

$$\frac{d}{dt}\Big(t - r(x(t))\Big) > 0$$

for all $t > t_0$. Moreover, in case of a globally defined solution $x : \mathbb{R} \longrightarrow [-2a, 2a]$, the above inequality is valid for all $t \in \mathbb{R}$.

Proof. Suppose $x : [t_0 - 1, \infty) \longrightarrow [-2a, 2a]$ is a solution of Equation (3.1). Combining condition (DF 5) on r with the fact that the function g defined by (3.6) is bounded on the square $[-2a, 2a] \times [-2a, 2a]$ by K_a, we obtain

$$\begin{aligned} \frac{d}{dt}\Big(t - r(x(t))\Big) &= 1 - r'(x(t))\,\dot{x}(t) \\ &\geq 1 - |r'(x(t))\,\dot{x}(t)| \\ &> 1 - \frac{1}{K_a}\,|g(x(t),\, x(t - r(x(t)))| \\ &\geq 1 - \frac{1}{K_a}\,K_a \\ &= 0 \end{aligned}$$

for all $t > t_0$. The same reasoning applied for a solution $x : \mathbb{R} \longrightarrow [-2a, 2a]$ completes the proof. $\square$

We can now formulate and establish the result about the differences of solutions for the differential equation (3.1).

PROPOSITION 3.12. *Suppose $x, y : \mathbb{R} \longrightarrow [-2a, 2a]$ are solutions of Equation (3.1). Then there exist constants $m_a \leq M_a < 0$, both depending only on parameter $a > 0$, and continuous functions $u : \mathbb{R} \longrightarrow \mathbb{R}$ and $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ such that*

(i) *$\alpha(\mathbb{R}) \subseteq [m_a, M_a]$,*
(ii) *u is bounded, and*
(iii) *the function*

$$v : \mathbb{R} \ni t \longmapsto \big[x(t) - y(t)\big] \cdot \exp\!\left(- \int_0^t u(s)\, ds \right) \in \mathbb{R} \tag{3.9}$$

satisfies for all $t \in \mathbb{R}$ the time-dependent delay differential equation

$$\dot{v}(t) = \alpha(t)\, v(t - r(x(t)))\,.$$

PROOF. Throughout the proof, let us define the constants u_m, m_a and M_a by $u_m := 6a$, $m_a := -a\, e^{u_m}$, and $M_a := -a\, e^{-u_m}$. Additionally, we note that x, y are both continuously differentiable as global solutions of Equation (3.1). Let now the functions $z, u : \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$z(t) = x(t) - y(t),$$

and

$$\begin{aligned} u(t) = a \int_0^1 \dot{y}\Big((1-s)\big[t - r(x(t))\big] + s\big[t - r(y(t))\big]\Big) ds \int_0^1 r'\Big((1-s)\, y(t) + s\, x(t)\Big)\, ds \\ - 2 \int_0^1 |(1-s)\, y(t) + s\, x(t)|\, ds + a, \end{aligned}$$

respectively. According to the assertions (i) to (iii), the proof will be divided into three steps, but in a different order.

1. The function u is obviously continuous. By the triangle inequality, the monotonicity property of the integral and the boundedness of g by K_a in the square $[-2a, 2a] \times [-2a, 2a]$, we obtain

$$\begin{aligned} |u(t)| &\leq a \int_0^1 \sup_{\tilde{s} \in \mathbb{R}} |\dot{y}(\tilde{s})|\, ds \int_0^1 \max_{\tilde{s} \in [-2a, 2a]} |r'(\tilde{s})|\, ds + 2 \int_0^1 2a\, ds + a \\ &= 5a + a \max_{\tilde{s} \in [-2a, 2a]} |r'(\tilde{s})| \cdot \max_{\tilde{s} \in \mathbb{R}} |g(y(\tilde{s}),\, y(\tilde{s} - r(y(\tilde{s}))))| \\ &\leq 5a + a\, K_a \max_{\tilde{s} \in [-2a, 2a]} |r'(\tilde{s})| \end{aligned}$$

for $t \in \mathbb{R}$. Hence, hypothesis (DF 5) on r implies $|u(t)| \leq u_m$, $t \in \mathbb{R}$, which in particular yields the boundedness of u claimed in part (ii).

2. According to the C^1-smoothness of solutions x, y of Equation (3.1), the function z is also continuously differentiable on $\mathbb{R}$. Moreover, from

$$\begin{aligned}
\dot{x}(t) - \dot{y}(t) &= a\big[x(t) - x(t - r(x(t)))\big] - |x(t)|\, x(t) \\
&\qquad - a\big[y(t) - y(t - r(y(t)))\big] + |y(t)|\, y(t) \\
&= a\big[x(t) - y(t)\big] - a\big[x(t - r(x(t))) - y(t - r(x(t)))\big] \\
&\qquad - a\big[y(t - r(x(t))) - y(t - r(y(t)))\big] - |x(t)|\, x(t) + |y(t)|\, y(t) \\
&= a\, z(t) - a\, z(t - r(x(t))) - a\big[y(t - r(x(t))) - y(t - r(y(t)))\big] \\
&\qquad - 2\int_0^1 |(1-s)\, y(t) + s\, x(t)| ds\ (x(t) - y(t)) \\
&= a\, z(t) - a\, z(t - r(x(t))) - 2\int_0^1 |(1-s)\, y(t) + s\, x(t)| ds\ (x(t) - y(t)) \\
&\qquad + a\int_0^1 \dot{y}\Big((1-s)\,[t - r(x(t))] + s\,[t - r(y(t))]\Big)\, ds \\
&\qquad\quad \cdot \int_0^1 r'\Big((1-s)\, y(t) + s\, x(t)\Big) ds\ (x(t) - y(t))
\end{aligned}$$

for $t \in \mathbb{R}$, we see that the difference z of solutions x and y satisfies the DDE

$$\dot{z}(t) = u(t)\, z(t) - a\, z(t - r(x(t))).$$

As a consequence of this equation, it now easily follows that the function v defined by (3.9) fulfills for all $t \in \mathbb{R}$ the differential equation

$$\dot{v}(t) = -a\, \exp\!\left(-\int_{t-r(x(t))}^{t} u(s)\, ds \right) v(t - r(x(t))).$$

Hence, defining

$$\alpha : \mathbb{R} \ni t \longmapsto -a\, \exp\!\left(-\int_{t-r(x(t))}^{t} u(s)\, ds \right) \in \mathbb{R},$$

we get assertion (iii).

3. To see the conclusion (i), that is, $\alpha(\mathbb{R}) \subset [m_a, M_a]$, recall that $|u(t)| \le u_m$, $t \in \mathbb{R}$, from the first part of the proof, and that $0 < r(s) \le 1$ by our assumptions. Consequently, we have

$$u_m \ge \int_{t-r(x(t))}^{t} u_m\, ds \ge -\int_{t-r(x(t))}^{t} u(s)\, ds \ge -\int_{t-r(x(t))}^{t} u_m\, ds \ge -u_m,$$

which finally yields $\alpha(\mathbb{R}) \subset [m_a, M_a]$ and completes the proof. □

In general, a solution of a delay differential equation may not be continued in the backward time direction in contrast to an ordinary differential equation defined by a smooth vector field; and if a backward solution of a DDE exists, the uniqueness may fail. As a simple corollary of the last result on the difference of solutions for Equation (3.1) we show that globally defined solutions are unique.

Corollary 3.13. *If both* $x : \mathbb{R} \longrightarrow [-2a, 2a]$ *and* $y : \mathbb{R} \longrightarrow [-2a, 2a]$ *are solutions of Equation* (3.1), *and if there is* $s \in \mathbb{R}$ *such that* $x_s = y_s$, *then* $x(t) = y(t)$ *for all* $t \in \mathbb{R}$.

PROOF. Under the above assumptions, for each $t \in \mathbb{R}$ the segments x_t and y_t belong to L_a. Therefore, Proposition 3.5 implies $x(t) = y(t)$ for all $t \geq s - 1$. Defining

$$t_0 := \inf \{t \in \mathbb{R} \mid x(\tilde{s}) = y(\tilde{s}) \text{ for all } \tilde{s} \geq t\},$$

we only need to show $t_0 = -\infty$. Contrary to that, let us assume that $t_0 > -\infty$. As $v(t) = 0$ for $t \geq t_0$, where v is defined by (3.9), the last propositions implies

$$v(t - r(x(t))) = 0$$

for all $t \geq t_0$. Denote by $m_r > 0$ the minimum of r in $[-2a, 2a]$. Then we obtain $v(t) = 0$ as $t \geq t_0 - m_r$, in contradiction to the definition of t_0. This yields $t_0 = -\infty$, and accordingly the assertion $x(t) = y(t)$ for all $t \in \mathbb{R}$. □

A Discrete Lyapunov Functional. The remainder of this section will be devoted to the presentation of a discrete functional for counting sign changes along solutions of Equation (3.1). This construction goes back to the work [**17**] of Krisztin and Arino and forms a modification of a corresponding concept for DDEs with constant delay, which is used in Mallet-Paret [**20**] and Mallet-Paret and Sell [**21**].

Let φ be a real-valued continuous function defined on a subset $I \subseteq \mathbb{R}$. For a subinterval $[a, b] \subset I$, $a < b$, with $\varphi|_{[a,b]} \neq 0$ the **number of sign changes** $\mathrm{sc}(\varphi, [a, b])$ **of** φ **on** $[a, b]$ is 0 if φ is nonnegative or nonpositive on $[a, b]$. Otherwise $\mathrm{sc}(\varphi, [a, b])$ is defined by the quantity

$$\begin{aligned} \mathrm{sc}(\varphi, [a, b]) := \sup \big\{ k \geq 0 \big| & \text{ there exist } s^i \in [a, b] \text{ for } i = 0, 1, \ldots, k, \\ & \text{such that } s^{i-1} < s^i \text{ and } \varphi(s^{i-1}) \cdot \varphi(s^i) < 0 \text{ for } 1 \leq i \leq k \big\}. \end{aligned}$$

In other words, $\mathrm{sc}(\varphi, [a, b]) = k \geq 0$, if and only if there exists a partition of $[a, b]$ into $(k + 1)$ subintervals such that $\varphi \geq 0$ or $\varphi \leq 0$ in each subinterval, φ does not vanish in any subinterval, and the sign of φ is different in neighboring subintervals. Accordingly, the value $\mathrm{sc}(\varphi, [a, b])$ is either a nonnegative integer, or infinity. Subsequently, we establish an auxiliary result about the behavior of sign changes for continuously differentiable functions under certain conditions.

PROPOSITION 3.14. *Let $a, b \in \mathbb{R}$ with $a < b$.*

(i) *If $\varphi \in C^1([a, b], \mathbb{R})$ and if the number of zeros of φ in $[a, b]$ is not finite, then there is some $s \in [a, b]$ with*

$$\varphi(s) = \varphi'(s) = 0.$$

(ii) *Suppose that $\varphi \in C^1([a - \delta, b + \delta], \mathbb{R})$, $\delta > 0$, has finitely many zeros in $[a, b]$, that all zeros of φ are simple, and that $\varphi(a) \neq 0, \varphi(b) \neq 0$. Then there is some $\gamma \in (0, \delta)$ such that $|a - c| < \gamma$, $|b - d| < \gamma$, and $\psi \in C^1([c, d], \mathbb{R})$ with $\|\psi - \varphi\|_{C^1([c,d],\mathbb{R})} < \gamma$ implies*

$$\mathrm{sc}(\psi, [c, d]) = \mathrm{sc}(\varphi, [a, b]).$$

PROOF. As the proof is a simple exercise in calculus, we only repeat the arguments for the first part of the assertion.

Let $x_1 \in [a, b]$ be a zero of φ. If $\varphi'(x_1) = 0$, then the claim is clear. Otherwise, the function φ has infinitely many zeros in at least one of the intervals $[a, x_1]$ and $[x_1, b]$. Let us at first assume that x has infinitely many zeros in the interval $[x_1, b]$. Then we find a zero $x_2 \in (x_1, b]$ of φ with $\varphi(s) \neq 0$ as $x_1 < s < x_2$. Again, if $\varphi'(x_2) \neq 0$ then there exists an element $x_3 \in (x_2, b]$ with $\varphi(x_3) = 0$ but $\varphi(s) \neq 0$

for all $x_2 < s < x_3$. Successively, we construct a strictly monoton increasing sequence $\{x_n\}_{n\in\mathbb{N}} \subset [a,b]$ of simple zeros of φ. Obviously, there is an $x \in [a,b]$ with $x_n \nearrow x$ as $n \longrightarrow \infty$, and the continuity of φ implies $\varphi(x) = 0$. By application of the mean value theorem, we find a sequence $\{\xi_n\}_{n\in\mathbb{N}}$ of reals in $[a,b]$ such that $\xi_n \in (x_n, x_{n+1})$ and $\varphi'(\xi_n) = 0$ for $n \in \mathbb{N}$. Hence, $\xi_n \nearrow x$ as $n \longrightarrow \infty$, and the continuous differentiability of φ implies $\varphi'(x) = 0$. In particular, $\varphi(x) = \varphi'(x) = 0$. Since an analogous reasoning applies to the situation in which the function φ has infinitely many zeros on $[a, x_1]$, this finishes the proof of (i). □

Consider once more reals $a < b$ and a real-valued continuous function φ defined on the interval $[a,b]$. Using the number of sign changes $\mathrm{sc}(\varphi,[a,b])$ of φ on $[a,b]$, we define the quantity

$$V(\varphi,[a,b]) := \begin{cases} \mathrm{sc}(\varphi,[a,b]) & \text{if } \mathrm{sc}(\varphi,[a,b]) \text{ is odd or infinite,} \\ \mathrm{sc}(\varphi,[a,b]) + 1 & \text{if } \mathrm{sc}(\varphi,[a,b]) \text{ is even.} \end{cases}$$

Obviously, $V(\varphi,[a,b])$ is either a nonnegative odd integer or infinity. Let $H_{[a,b]}$ denote the subset

$$\begin{aligned} H_{[a,b]} := \Big\{ \varphi \in C^1([a,b],\mathbb{R}) \Big|\ & \varphi(b) \neq 0 \text{ or } \varphi(a)\cdot\varphi'(b) < 0, \text{ and } \varphi(a) \neq 0 \text{ or} \\ & \varphi'(a)\cdot\varphi(b) > 0, \text{ and all zeros of } \varphi \text{ in } (a,b) \text{ are simple} \Big\} \end{aligned} \tag{3.10}$$

of $C^1([a,b],\mathbb{R})$. The next result contains some basic properties of the quantity V, especially for functions in $H_{[a,b]}$.

Proposition 3.15 (Lemma 4.1 in Krisztin and Arino [**17**]). *Let reals $a, b \in \mathbb{R}$ with $a < b$ be given.*

(i) *V is lower semicontinuous in the following sense: If $\varphi : [a,b] \longrightarrow \mathbb{R}$ is a non-zero continuous function and if $\varphi^n : [a_n, b_n] \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$, is a sequence of non-zero continuous functions defined on some intervals $[a_n, b_n] \subset \mathbb{R}$ such that*

$$\max_{s\in[a,b]\cap[a_n,b_n]} |\varphi^n(s) - \varphi(s)| \to 0, \quad a_n \to a, \quad b_n \to b$$

as $n \to \infty$, then

$$V(\varphi,[a,b]) \leq \liminf_{n\to\infty} V(\varphi^n,[a_n,b_n]).$$

(ii) *If $\varphi \in H_{[a,b]}$, then $V(\varphi,[a,b]) < \infty$.*

(iii) *If $\varphi \in C^1([a-\delta, b+\delta],\mathbb{R})$ for some $\delta > 0$ and $\varphi|_{[a,b]} \in H_{[a,b]}$, then there is a constant $\gamma \in (0,\delta)$ such that conditions $|a-c| < \gamma$, $|b-d| < \gamma$, and $\psi \in C^1([c,d],\mathbb{R})$ with $\|\psi - \varphi\|_{C^1([c,d],\mathbb{R})} < \gamma$ imply*

$$V(\psi,[c,d]) = V(\varphi,[a,b]).$$

Proof. Subsequently, we follow the proof given in Krisztin and Arino [**17**].

1. *The proof of (i).* If $V(\varphi,[a,b]) = \infty$, then assertion (i) is clear. Therefore assume $V(\varphi,[a,b]) < \infty$. Then we find a constant $\gamma \in (0,(b-a)/4)$ such that φ has no sign change in the intervals $[a, a+2\gamma]$ and $[b-2\gamma, b]$. By the assumptions $a_n \to a$ and $b_n \to b$ as $n \to \infty$, it is also clear that there is a constant $N_1 \in \mathbb{N}$ with $[a+\gamma, b-\gamma] \subset [a_n, b_n]$ for $n \geq N_1$. Moreover, from

$$\max_{s\in[a,b]\cap[a_n,b_n]} |\varphi(s) - \varphi^n(s)| \to 0$$

as $n \to \infty$ we conclude the existence of $N_2 \in \mathbb{N}$, $N_2 \geq N_1$, with

$$V(\varphi^n, [a+\gamma, b-\gamma]) \geq V(\varphi, [a+\gamma, b-\gamma])$$

for all $n \geq N_2$. Consequently, as $n \geq N_2$ we have

$$V(\varphi^n, [a_n, b_n]) \geq V(\varphi^n, [a+\gamma, b-\gamma]) \geq V(\varphi, [a+\gamma, b-\gamma]) = V(\varphi, [a,b]),$$

which implies the statement.

2. *The proof of (ii).* This part of the proposition is an immediate conclusion of Corollary 3.14. Indeed, if $V(\varphi, [a,b]) = \infty$ for a function $\varphi \in C^1([a,b], \mathbb{R})$ then there is an $s \in [a,b]$ with $\varphi(s) = \varphi'(s) = 0$, which in turn implies $\varphi \notin H_{[a,b]}$.

3. *The proof of (iii).* Under the additional assumption that $\varphi(a) \neq 0$ and $\varphi(b) \neq 0$, the assertion is obvious due to the last part in combination with Corollary 3.14. On the other hand, assume that $\varphi(b) = 0$ and $\varphi(a) \cdot \varphi'(b) < 0$. Then we have either $\varphi(a) > 0$ and $\varphi'(b) < 0$, or viceversa. However, in both cases the number of the sign changes of φ in $[a,b]$ is even. Furthermore, it is clear that there is a constant $0 < \varepsilon < \min\{\delta, (b-a)/2\}$ such that φ' has no zero in $[b-\varepsilon, b+\varepsilon]$ and thus φ is strictly monotonous in $[b-\varepsilon, b+\varepsilon]$. In particular, we have $\varphi(b-\varepsilon) \neq 0$ and $\operatorname{sc}(\varphi, [a,b]) = \operatorname{sc}(\varphi, [a, b-\varepsilon])$. Applying part (ii) of Corollary 3.14, we find a constant $0 < \gamma_1 < \varepsilon$ such that for all $c, d \in \mathbb{R}$ and all $\psi \in C^1([c,d], \mathbb{R})$ satisfying $|a-c| < \gamma_1$, $|b-\varepsilon-d| < \gamma_1$ and $\|\varphi - \psi\|_{C^1([c,d],\mathbb{R})} < \gamma_1$, we have

$$\operatorname{sc}(\varphi, [a, b-\varepsilon]) = \operatorname{sc}(\psi, [c,d]).$$

Choose $0 < \gamma < \min\{\gamma_1, |\varphi(b-\varepsilon)|, m\}$, where $m := \min\big\{|\varphi'(s)| \,|\, s \in [b-\varepsilon, b+\varepsilon]\big\} > 0$, and suppose that $c, d \in \mathbb{R}$ with $c < d$ and a function $\psi \in C^1([c,d], \mathbb{R})$ fulfill $|a-c| < \gamma$, $|b-d| < \gamma$ and $\|\varphi - \psi\|_{C^1([c,d],\mathbb{R})} < \gamma$. Then it is easy to check that $\psi(b-\varepsilon) \neq 0$ and that the function ψ is strictly monotonous in $[b-\varepsilon, d]$. Consequently, ψ can have at most one sign change in $[b-\varepsilon, d]$. Therefore, we get

$$\operatorname{sc}(\varphi, [a,b]) = \operatorname{sc}(\varphi, [a, b-\varepsilon]) = \operatorname{sc}(\psi, [c, b-\varepsilon]) \leq \operatorname{sc}(\psi, [c,d])$$

and

$$\operatorname{sc}(\psi, [c,d]) = \operatorname{sc}(\psi, [c, b-\varepsilon]) + \operatorname{sc}(\psi, [b-\varepsilon, d]) \leq \operatorname{sc}(\varphi, [a,b]) + 1.$$

Combining these inequalities yields

$$\operatorname{sc}(\varphi, [a,b]) \leq \operatorname{sc}(\psi, [c,d]) \leq \operatorname{sc}(\varphi, [a,b]) + 1,$$

and establishes the formula $V(\varphi, [a,b]) = V(\psi, [c,d])$ in consideration of the definition of V.

The same conclusion can be drawn for the case $\varphi(a) = 0$ and $\varphi'(a) \cdot \varphi(b) > 0$, and this completes the proof. $\square$

In our next result, we consider functions $\alpha, \tau : I \longrightarrow \mathbb{R}$ defined on some interval $I = [c,d] \subset \mathbb{R}$ with $\alpha(t) < 0$ and $\tau(t) > 0$ for all $t \in I$, and we assume that the induced function $\eta : I \ni t \longmapsto t - \tau(t) \in \mathbb{R}$ is strictly increasing. We further suppose that there is an integer $k > 1$ and reals $c_j \in I$, $j = 1, \ldots, k$, satisfying $c_1 = c$ and $\eta(c_j) = c_{j-1}$ for $j = 2, \ldots, k$. Additionally, let the function $\eta^0 : I \longrightarrow \mathbb{R}$ be defined by $\eta^0(t) = t$ and then inductively for each $j \in \{1, \ldots, k\}$ the function $\eta^j : [c_j, d] \longrightarrow \mathbb{R}$ by the relation $\eta^j(t) = \eta(\eta^{j-1}(t))$. Compare here Figure 3.3 which indicates how the reals $c_1, \ldots, c_k$ behave under the functions η^j for $j = 1, \ldots, k$. Note also that by assumption the interval $\{t - \tau(t) \mid t \in I\}$ contains the real $c_1 = c \in I$ and therefore the union $J := \{t - \tau(t) \mid t \in I\} \cup I$ is an interval as well.

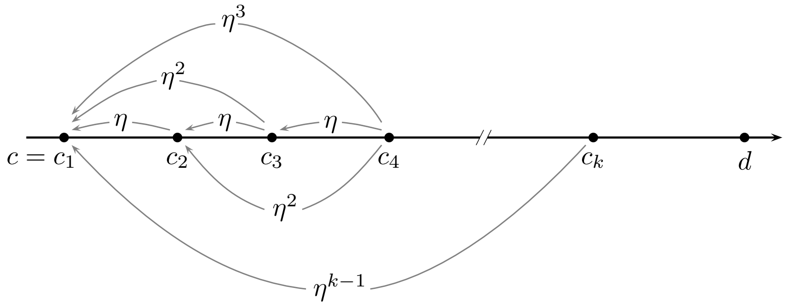

FIGURE 3.3. Sample Construction for Proposition 3.16

PROPOSITION 3.16 (Lemma 4.2 in Krisztin and Arino [**17**]). *Suppose that additionally to I, J, α, τ, η, k as above, there is a continuous function $v : J \longrightarrow \mathbb{R}$ with the property that v is continuously differentiable on I, for each $t \in I$ the restriction $v_{[\eta(t),t]}$ is not identically zero and v satisfies the differential equation*

$$\dot{v}(t) = \alpha(t)\, v(t - \tau(t)) \tag{3.11}$$

as $t \in I$. Then the following holds.

(i) *If $t_1, t_2 \in I$ and $t_1 < t_2$, then $V(v, [\eta(t_1), t_1]) \geq V(v, [\eta(t_2), t_2])$.*
(ii) *If $k \geq 3$, and if there is some $t \in [c_3, d]$ with $v(t) = v(\eta(t)) = 0$, then either $V(v, [\eta(t), t]) = \infty$ or $V(v, [\eta(t), t]) < V(v, [\eta^3(t), \eta^2(t)])$.*
(iii) *If $k \geq 4$, and if there is $t \in [c_4, d]$ satisfying*

$$V(v, [\eta(t), t]) = V(v, [\eta^4(t), \eta^3(t)]) < \infty,$$

then $v|_{[\eta(t),t]} \in H_{[\eta(t),t]}$.

PROOF. We reproduce the arguments from Krisztin and Arino [**17**] for the completeness.

1. *The proof of (i).* We begin with the claim that we shall have established assertion (i) if for each $t \in I$ we show the existence of a constant $\varepsilon_0 = \varepsilon_0(t) > 0$ so that for all $0 \leq \varepsilon \leq \varepsilon_0$ with $t + \varepsilon \in I$

$$V(v, [\eta(t), t]) \geq V(v, [\eta(t + \varepsilon), t + \varepsilon]). \tag{3.12}$$

Proof of the claim: Suppose that $t_1, t_2 \in I$ with $t_1 < t_2$ and that for each $t \in I$ we find an $\varepsilon_0 = \varepsilon_0(t) > 0$ such that for all $0 \leq \varepsilon \leq \varepsilon_0$ with $t + \varepsilon \in I$ the above inequality for V is fulfilled. Define

$$t^* := \sup \big\{ s \in [t_1, t_2] \big|\ V(v, [\eta(t_1), t_1]) \geq V(v, [\eta(u), u]) \text{ for all } t_1 \leq u \leq s \big\}.$$

By assumption, we have $t_1 < t^* \leq t_2$. In view of the definition of t^* there is a sequence $\{s_n\}_{n \geq 0} \subset [t_1, t^*]$ with $s_n \to t^*$ as $n \to \infty$ and

$$V(v, [\eta(t^1), t^1]) \geq V(v, [\eta(s_n), s_n])$$

for all $n \in \mathbb{N}$. The continuity of η leads to the convergence $\eta(s_n) \to \eta(t^*)$ as $n \to \infty$, and hence part (i) of Proposition 3.15 implies

$$V(v, [\eta(t_1), t_1]) \geq \liminf_{n \to \infty} V(v, [\eta(s_n), s_n]) \geq V(v, [\eta(t^*), t^*]).$$

If now $t^* < t_2$ were true, then due to our assumption we would find a constant $0 < \varepsilon_0(t^*) \le t_2 - t^*$ such that

$$V(v, [\eta(t^*), t^*]) \ge V(v, [\eta(t^* + \varepsilon), t^* + \varepsilon])$$

for all $0 \le \varepsilon \le \varepsilon_0(t^*)$, in contradiction to the definition of t^*. Therefore $t^* = t_2$, which establishes our claim.

Now, we only need to show that for each $t \in I$ there is an $\varepsilon_0 = \varepsilon_0(t) > 0$ such that for all $0 \le \varepsilon \le \varepsilon_0$ with $t + \varepsilon \in I$ inequality (3.12) holds. As in the case $V(v, [\eta(t), t]) = \infty$ this assertion is obviously true, we may assume $V(v, [\eta(t), t])$ is finite. Furthermore, we need only consider the situation $v(t) = 0$, because under the condition $v(t) \neq 0$ the assertion is clear from the strong monotonicity of function η. Hence, suppose that $V(v, [\eta(t), t]) < \infty$ and $v(t) = 0$. Then $\eta(t)$ is not an accumulation point of a sequence of sign changes of v in $(\eta(t), t]$, and thus we find a sufficiently small $0 < \delta < \tau(t)$ such that v has no sign change in $[\eta(t), \eta(t) + \delta]$.

Consider first the situation $v(s) \ge 0$ for all $\eta(t) \le s \le \eta(t) + \delta$. Using the continuity and strong monotonicity of η, we see at once the existence of $\varepsilon_0 > 0$ such that $t \le s \le t + \varepsilon_0$ implies $\eta(t) \le \eta(s) \le \eta(t) + \delta$. Therefore, Equation (3.11) shows

$$\dot{v}(s) = \alpha(s)\, v(s - \tau(s)) = \alpha(s)\, v(\eta(s)) \le 0,$$

and thus $v(s) \le 0$ for all $t \le s \le t + \varepsilon_0$ due to our assumption $v(t) = 0$. If v is identically zero in $[t, t+\varepsilon_0]$, then we clearly obtain the claimed inequality (3.12) for all $0 \le \varepsilon \le \varepsilon_0$. Otherwise, $v(s) < 0$ for some $t < s \le t + \varepsilon$ such that the differential equation (3.11) implies $0 < v(\eta(\tilde{t}))$ for some $t < \tilde{t} < s$ with $\eta(t) \le \eta(\tilde{t}) \le \eta(t) + \delta$. Accordingly, we find a constant $0 < \gamma < t - \eta(t) = \tau(t)$ such that $v|_{[t-\gamma,t]} \neq 0$ and either $v(s) \ge 0$ or $v(s) \le 0$ for all $t - \gamma \le s \le t$. In case $v \ge 0$ in $[t - \gamma, t]$, we obtain

$$\mathrm{sc}(v, [\eta(t), t + \varepsilon]) \le \mathrm{sc}(v, [\eta(t), t]) + 1,$$

and hence the claimed inequality (3.12) for all $0 \le \varepsilon \le \varepsilon_0$, because $\mathrm{sc}(v, [\eta(t), t+\varepsilon])$ is even due to the fact that the value of v on the right of $\eta(t)$ and the left of t is positive. On the other hand, if $v \le 0$ in $[t - \gamma, t]$, then

$$\mathrm{sc}(v, [\eta(t), t + \varepsilon]) = \mathrm{sc}(v, [\eta(t), t])$$

such that in this case (3.12) is also satisfied for all $0 \le \varepsilon \le \varepsilon_0$. This establishes inequality (3.12) under the assumption $v(s) \ge 0$ for $\eta(t) \le s \le \eta(t) + \delta$. As the differential equation (3.11) is linear, the above argumentation applies also in the situation $-v(s) \ge 0$ for $\eta(t) \le s \le \eta(t) + \delta$, which completes the proof of the statement (i).

2. *The proof of (ii).* As in the situation $V(v, [\eta(t), t]) = \infty$ there is nothing to prove, we assume that in case of $k \ge 3$ there is $c_3 \le t \le d$ with $v(t) = v(\eta(t)) = 0$ and $V(v, [\eta(t), t]) < \infty$. Set $m := \mathrm{sc}(v, [\eta(t), t])$. Then by definition we find a subset $\{t_i |\ i = 0, 1, \ldots, m+1, m+2\}$ of $[\eta(t), t]$ with

$$\eta(t) = t_{m+2} < t_{m+1} < \cdots < t_1 < t_0 = t$$

and $v(t_i)\, v(t_{i+1}) < 0$ for all $i = 1, 2, \ldots, m$. In particular, it follows that for each $i \in \{0, 1, \ldots, m+1\}$ we have $v(t_i) - v(t_{i+1}) \neq 0$. Additionally, the mean value theorem implies the existence of $\xi_i \in (t_{i+1}, t_i)$ satisfying

$$v(t_i) - v(t_{i+1}) = \dot{v}(\xi_i)\,(t_i - t_{i+1}) = \alpha(\xi_i)\, v(\eta(\xi_i))\,(t_i - t_{i+1})$$

where the last equality holds due to Equation (3.11). Now define $\hat{t}_i := \eta(\xi_i)$ for all $i = 0, 1, \ldots, m+1$. As η is strongly increasing we get

$$\eta(t_{m+2}) = \eta^2(t) < \hat{t}_{m+1} < \hat{t}_m < \cdots < \hat{t}_0 < \eta(t) = \eta(t_0).$$

Moreover, using the above equation for the differences $v(t_i) - v(t_{i+1})$ and the assumption $v(t_0) = v(t_{m+2}) = 0$, we immediately see

$$v(\hat{t}_i)\, v(\hat{t}_{i+1}) < 0$$

for all $i \in \{0, 1, \ldots, m\}$. Thus $\mathrm{sc}(v, [\eta^2(t), \eta(t)]) \geq m+1$. If m is odd, then we have

$$V(v, [\eta^2(t), \eta(t)]) \geq m + 2 > m = V(v, [\eta(t), t]),$$

and thus part (i) implies, in view of $\eta^2(t) < \eta(t)$,

$$V(v, [\eta^3(t), \eta^2(t)]) \geq V(v, [\eta^2(t), \eta(t)]) > V(v, [\eta(t), t]),$$

as claimed.

Otherwise the integer m is even, and the values $v(\hat{t}_0)$ and $v(\hat{t}_{m+1})$ consequently have different signs. Since $v(\eta(t)) = 0$ and $v(\hat{t}_0) \neq 0$, there is a real $t^* \in (\hat{t}_0, \eta(t))$ such that

$$\mathrm{sign}(v(t^*)) = \mathrm{sign}(v(\hat{t}_0))$$

and

$$\mathrm{sign}(\dot{v}(t^*)) = -\,\mathrm{sign}(v(\hat{t}_0)).$$

Therefore, the differential equation (3.11) for v leads to

$$\mathrm{sign}(v(\eta(t^*))) = -\,\mathrm{sign}(\dot{v}(t^*)) = \mathrm{sign}(v(\hat{t}_0)) = -\,\mathrm{sign}(v(\hat{t}_{m+1})),$$

whereby the above in combination with $\eta(t^*) < \eta^2(t) < \hat{t}_{m+1}$ results in

$$\mathrm{sign}(v, [\eta(t^*), t^*]) \geq m + 2.$$

Using $\eta(t^*) > \eta^3(t)$ and applying part (i), we obtain

$$V(v, [\eta^3(t), \eta^2(t)]) \geq V(v, [\eta(t^*), t^*]) \geq m + 3 > m + 1 = V(v, [\eta(t), t]).$$

This completes the proof of assertion (ii).

3. *The proof of (iii).* Suppose that in the case of $k \geq 4$ there is a real $c_4 \leq t \leq d$ with $V(v, [\eta(t), t]) = V(v, [\eta^4(t), \eta^3(t)]) < \infty$. As for each $\eta(t) \leq s \leq t$ we have $\eta^3(t) \leq \eta^2(s) \leq s \leq t$, the first part of the proposition implies

$$V(v, [\eta^4(t), \eta^3(t)]) \geq V(v, [\eta^3(s), \eta^2(s)]) \geq V(v, [\eta(s), s]) \geq V(v, [\eta(t), t]),$$

and hence

$$V(v, [\eta(s), s]) = V(v, [\eta^3(s), \eta^2(s)]) < \infty \tag{3.13}$$

as $\eta(t) \leq s \leq t$. Applying the second part, we conclude

$$(v(s), v(\eta(s))) \neq 0 \in \mathbb{R}^2,$$

such that the differential equation (3.11) for v yields

$$(v(s), \dot{v}(s)) \neq 0 \in \mathbb{R}^2$$

for all $\eta(t) \leq s \leq t$. Therefore, every zero of v in $[\eta(t), t]$ is simple. In particular, $v(t) = 0$ implies $0 \neq \dot{v}(t) = \alpha(t)\, v(\eta(t))$ and $\dot{v}(t)\, v(\eta(t)) < 0$ as $\alpha(t) < 0$. Consequently, it remains to prove that either $v(\eta(t)) \neq 0$ or $\dot{v}(\eta(t))\, v(t) > 0$. For this purpose, suppose we have $v(\eta(t)) = 0$. Then from the second part of the proposition in combination with the above we get $v(t) \neq 0$ and $v(\eta^2(t)) \neq 0$, so Equation (3.11) yields $\dot{v}(\eta(t))\, v(t) \neq 0$.

Assuming in the following that $\dot{v}(\eta(t))\, v(t) < 0$ is true, we conclude that the integer $m := \mathrm{sc}(v, [\eta(t), t])$ is odd. By definition, there is a set $\{t_i | i = 0, 1, \dots, m{+}1\}$ of points in $[\eta(t), t]$ with

$$\eta(t) = t_{m+1} < t_m < \cdots < t_1 < t_0 = t$$

and

$$v(t_i)\, v(t_{i+1}) < 0$$

as $i = 0, 1, \dots, m-1$. We can now proceed analogously to the proof of part (ii) and find $m+1$ sign changes of v in $[\eta^2(t), \eta(t)]$ by application of the mean value theorem and using $\dot{v}(\eta(t))\, v(t) < 0$. Accordingly, in this way we conclude

$$V(v, [\eta^2(t), \eta(t)]) \geq m + 2 > m = \mathrm{sc}(v, [\eta(t), t]) = V(v, [\eta(t), t]),$$

and hence by the first part,

$$V(v, [\eta^4(t), \eta^3(t)]) > V(v, [\eta(t), t])$$

in contradiction to (3.13). This shows $\dot{v}(\eta(t))\, v(t) > 0$ under the assumption of $v(\eta(t)) = 0$ and completes the proof for $v|_{[\eta(t),t]} \in H_{[\eta(t),t]}$. □

We close this section with a remark concerning the applicability of the last result for the study of solutions of the differential equation (3.1).

Remark 3.17. Proposition 3.12 on the weighted differences of global solutions enables us to use the last result for the study of Equation (3.1). Indeed, assuming $x, y : \mathbb{R} \longrightarrow [-2a, 2a]$ are two solution of the DDE (3.1), from Proposition 3.12 we obtain the existence of bounded functions $\alpha, u : \mathbb{R} \longrightarrow \mathbb{R}$ with $\alpha(t), u(t) < 0$ for $t \in \mathbb{R}$ such that

$$v : \mathbb{R} \ni t \longmapsto \big[x(t) - y(t)\big] \cdot \exp\!\left(-\int_0^t u(s)\, ds\right) \in \mathbb{R}$$

satisfies the linear DDE

$$\dot{v}(t) = \alpha(t)\, v(t - \tau(t))$$

where $\tau(t) := r(x(t)) > 0$ and the induced function $\eta : \mathbb{R} \in t \longmapsto t - \tau(t) \in \mathbb{R}$ is strictly increasing by Corollary 3.11. Hence we may apply the last proposition for considerations of the weighted difference of x and y introduced above.

4. Slowly Oscillating Solutions

The discussion of the existence and structure of so-called *slowly oscillating* solutions for Equation (3.1) with values in $[-2a, 2a]$ is the main intention of this section. Thereby a real valued function $x : \mathbb{R} \supseteq I \longrightarrow \mathbb{R}$ is called **slowly oscillating** if $|z - \tilde{z}| > 1$ for any pair $z, \tilde{z} \in I$ of zeros of x with $z \neq \tilde{z}$. We will soon see that the differential equation (3.1) provides quite a few slowly oscillating solutions, and that the set of all global slowly oscillating solutions, which additionally are bounded by $2a$, can be characterized by the discrete Lyapunov functional introduced in the last section.

Existence of Slowly Oscillating Solutions. Suppose $x : I \longrightarrow [-2a, 2a]$ defined on $I = [-1, \infty)$ or $I = (-\infty, \infty)$ is a slowly oscillating solution of Equation (3.1) with initial value $x_0 \in L_a$. Then the segments of x belong to the subset

$$Z_a := \{\varphi \in L_a | \ \varphi \text{ has at most one zero}\} \tag{3.14}$$

of L_a. The closure S_a of Z_a with respect to the topology induced by the norm of C consists of all functions $\varphi \in L_a$ with no more than one sign change in $[-1, 0]$; that is, $\varphi \in S_a$ if and only if $\varphi \in L_a$ and φ is either nonnegative, or nonpositive, or there is a $z \in [-1, 0]$ such that $\varphi(s) \geq 0$ for $-1 \leq s \leq z$, and $\varphi(s) \leq 0$ for $z \leq s \leq 0$, or viceversa. By compactness of L_a, we conclude that S_a is also compact, but in contrast to L_a the set S_a is not convex, although $\lambda \varphi \in S_a$ for all $-1 \leq \lambda \leq 1$ and $\varphi \in S_a$.

Remark 3.18. If $x : [-1, \infty) \longrightarrow [-2a, 2a]$ is a slowly oscillating solution for the delay differential equation (3.1), and if $z \geq 0$ is a zero of x, then z is simple. In fact, the assumption $x(z) = \dot{x}(z) = 0$ leads to

$$0 = \dot{x}(z) = \big[a - |x(z)|\big]\, x(z) - a\, x(z - r(x(z))) = -a\, x(z - 1).$$

Consequently, we obtain $x(z - 1) = 0$. But this contradicts our assumption that x is slowly oscillating. Therefore, z is a simple zero of x as claimed. Analogously, all zeros of global slowly oscillating solution of (3.1) are simple.

The question on the existence of slowly oscillating solution of Equation (3.1) is solved by our next proposition. This results particularly yields that every non-trivial initial data $\varphi \in S_a$ at $t = 0$ provides a solution of (3.1) that becomes slowly oscillating in finite time. It is exactly the same conclusion as in the situation of the differential equation (3.2) with constant delay, which was established in the work [**3**, Proposition 3.1] of Brunovský et al. and which we carry over to the situation of a delay function with properties (DF 1) - (DF 5) in the following.

Proposition 3.19. *If $\varphi \in S_a \setminus \{0\}$, and $x : [-1, \infty) \longrightarrow [-2a, 2a]$ denotes the uniquely determined solution of the differential equation* (3.1) *with $x_0 = \varphi$, then*

(i) *$x_t \in S_a \setminus \{0\}$ for $0 \leq t \leq 5$,*
(ii) *there is a time $t_* \in [0, 5]$ such that $0 \notin x([t_* - 1, t_*])$, and*
(iii) *$x_t \in Z_a$ for $t \geq 5$.*

The proof of the above statement is based on three small technical propositions, which we establish below. We will make use of the observation that in case of a non-zero $\varphi \in S_a$ either $x = \varphi$ or $x = -\varphi$ possesses one of the following properties.

(A) There is a zero $z \in (-1, 0]$ of x, and $t_+ \in (-1, z)$ such that $x(t) \geq 0$ in $[-1, z]$, $x(t_+) > 0$, and $x(t) \leq 0$ in $[z, 0]$.
(B) $x(t) \geq 0$ for $-1 \leq t \leq 0$ and $x(0) > 0$.

The main idea is now to compare solutions of Equation (3.1) with solutions of appropriate initial value problems for an induced ordinary differential equation. On that account, we recall that a function $k : I \times \mathbb{R} \longrightarrow \mathbb{R}$ where I denotes an interval in $\mathbb{R}$ is called locally Lipschitz continuous in the second argument, if k satisfies the condition

(locLip) For each $(t, \eta) \in I \times \mathbb{R}$ there is an open neighborhood D of (t, η) in $I \times \mathbb{R}$ and a constant K_D such that

$$|k(t, \eta_1) - k(t, \eta_2)| \leq K_D |\eta_1 - \eta_2|$$

for all $(t,\eta_1),(t,\eta_2) \in D$.

in $I \times \mathbb{R}$. We begin with an auxiliary result about solutions of an ODE.

PROPOSITION 3.20 (Proposition 2.1 in Brunovský et al. [**3**]). *Suppose that for given $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$, the function $k : [t_0, t_1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, and satisfies a (local) Lipschitz condition in the second argument. If there exists a constant $\eta_0 > 0$ such that $\eta\, k(t,\eta) < 0$ for all $|\eta| \geq \eta_0$ and for all $t \in [t_0, t_1]$, then for every $w_0 \in \mathbb{R}$ the maximal solution of the initial value problem*

$$\begin{cases} \dot{w}(t) = k(t, w(t)) \\ w(0) = w_0 \end{cases} \tag{3.15}$$

is defined on $[t_0, t_1]$.

PROOF. See the proof of Proposition 2.1 in Brunovský et al. [**3**]. □

As a simple consequence we obtain the following corollary, which will be essential to establish Proposition 3.19.

COROLLARY 3.21. *Let $\varphi \in L_a$ and $t_1 > t_0 \geq 0$ be given. Further, denote $x : [-1, \infty) \longrightarrow [-2a, 2a]$ the uniquely determined solution of Equation (3.1) with initial value φ at $t = 0$, and let the function $k_c : [t_0, t_1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by*

$$k_c(t,\eta) := \big[a - |\eta|\big]\, \eta - a\, x(t - r(\eta)) + c$$

with parameter $c \in \mathbb{R}$. Then for each $w_0 \in \mathbb{R}$ the initial value problem

$$\begin{cases} \dot{w}(t) = k_c(t, w(t)) \\ w(0) = w_0 \end{cases}$$

has a unique solution defined for all $t \in [t_0, t_1]$.

PROOF. Obviously, it is sufficient to confirm the conditions of Proposition 3.20 for the function k_c. Furthermore, as the continuity of k_c is clear, we are left to establish the locally Lipschitz continuity of k_c in the second variable, and the existence of $\eta_0 > 0$ having the property $\eta\, k_c(t,\eta) < 0$ for all $(t,\eta) \in [t_0, t_1] \times \mathbb{R}$ with $|\eta| \geq \eta_0$.

For the proof of the local Lipschitz continuity, let $\eta \in \mathbb{R}$ be given. Then we find constants $\varepsilon > 0$, $L_r \geq 0$ and $L_{|\cdot|} \geq 0$ such that

$$\big|\, |\eta_1|\, \eta_1 - |\eta_2|\, \eta_2 \,\big| \leq L_{|\cdot|}\, |\eta_1 - \eta_2|$$

and

$$|r(\eta_1) - r(\eta_2)| \leq L_r\, |\eta_1 - \eta_2|$$

for all $\eta_1, \eta_2 \in (\eta - \varepsilon, \eta + \varepsilon)$. Using these in combination with the Lipschitz continuity of x, we obtain

$$\begin{aligned} |k_c(t,\eta_1) - k_c(t,\eta_2)| &= \Big| \big[a - |\eta_1|\big]\eta_1 - a\, x(t - r(\eta_1)) - \Big(\big[a - |\eta_2|\big]\eta_2 - a\, x(t - r(\eta_2)) \Big) \Big| \\ &\leq a\, |\eta_1 - \eta_2| + \big|\, |\eta_1|\, \eta_1 - |\eta_2|\, \eta_2 \,\big| \\ &\qquad + a\, |x(t - r(\eta_1)) - x(t - r(\eta_2))| \\ &\leq (a + L_{|\cdot|})\, |\eta_1 - \eta_2| + a\, K_a\, |r(\eta_2) - r(\eta_1)| \\ &\leq (a + L_{|\cdot|} + a\, K_a\, L_r)\, |\eta_1 - \eta_2|. \end{aligned}$$

for $(t,\eta_1),(t,\eta_2)\in[t_0,t_1]\times(\eta-\varepsilon,\eta+\varepsilon)$. Since $\eta\in\mathbb{R}$ was arbitrary, this shows the local Lipschitz continuity of function k_c with respect to the second argument.

The final assertion on the product $\eta\,k_c(t,\eta)$ is an immediate consequence of the inequality

$$\big[a-|\eta|\big]\,\eta-a\max_{s\in[-1,t_1]}x(s)+c\le k_c(t,\eta)\le\big[a-|\eta|\big]\,\eta-a\min_{s\in[-1,t_1]}x(s)+c$$

for all $(t,\eta)\in[t_0,t_1]\times\mathbb{R}$. As

$$\big[a-|\eta|\big]\,\eta-a\min_{s\in[-1,t_1]}x(s)+c\to-\infty$$

for $\eta\to\infty$, and analogously

$$\big[a-|\eta|\big]\eta-a\max_{s\in[-1,t_1]}x(s)+c\to\infty$$

for $\eta\to-\infty$, it becomes clear that we find a constant $\eta_0>0$, which is in particular independent of t, such that

$$\eta\,k_c(t,\eta)<0$$

for all $(t,\eta)\in[t_0,t_1]\times\mathbb{R}$ with $|\eta|\ge\eta_0$. □

Our next result describes the short time behavior of solutions for Equation (3.1) with initial values in S_a and property (A) on page 79.

PROPOSITION 3.22. *Suppose $x:[-1,\infty)\longrightarrow[-2a,2a]$ is a solution of Equation* (3.1) *with $x_0\in S_a$ and x_0 has property* (A) *on page 79; that is, there is a constant $z\in(-1,0]$ such that $x(s)\ge0$ for $s\in[-1,z]$, $x(t_+)>0$ for a real $t_+\in(-1,z)$, and $x(s)\le0$ on $[z,0]$. Then $x_t\in S_a\setminus\{0\}$ for all $t\in[0,1]$, and there exists $t_*\in[0,4]$ with $0\notin x([t_*-1,t_*])$.*

PROOF. The complete proof will be divided into a sequence of five steps, which we will establish one after another.

1. *Proof of $x(t)\le0$ for all $t\in[0,z+1]$.* Consider the initial value problem

$$\begin{cases}\dot w(t)=\big[a-|w(t)|\big]\,w(t)-a\,x(t-r(w(t)))\\ w(0)=0\end{cases}$$

having a maximal solution w on $[0,z+1]$ by Corollary 3.21. Since the restriction of x to $[0,z+1]$ is a solution of the ODE involved in the last IVP, and solutions for different initial values can not cross due to uniqueness, the inequality $x(0)\le0=w(0)$ implies $x(t)\le w(t)$ for all $0\le t\le z+1$. Therefore, it is sufficient to show $w(t)\le0$, $t\in[0,z+1]$. For this purpose, we regard solutions $w_\varepsilon:[0,z+1]\longrightarrow\mathbb{R}$ of the IVPs

$$\begin{cases}\dot w_\varepsilon(t)=\big[a-|w_\varepsilon(t)|\big]\,w_\varepsilon(t)-a\,x(t-r(w_\varepsilon(t)))-\varepsilon\\ w_\varepsilon(0)=0\end{cases}$$

with parameter $\varepsilon>0$, which exist for all $t\in[0,z+1]$ due to Corollary 3.21. We claim that for every $\varepsilon>0$ and $t\in[0,z+1]$ we have $w_\varepsilon(t)\le0$. If this was false, then, according to

$$w_\varepsilon(0)=0>-\varepsilon\ge0-a\,x(0-1)-\varepsilon=\dot w_\varepsilon(0),$$

we would find $t_0\in(0,z+1)$ with $w_\varepsilon(t_0)=0$ but $\dot w_\varepsilon(t_0)\ge0$, in contradiction to

$$\dot w_\varepsilon(t_0)=0-a\,x(t_0-r(w_\varepsilon(t_0)))-\varepsilon=-a\,x(t_0-1)-\varepsilon\le-\varepsilon<0.$$

Hence, $w_\varepsilon(t) \leq 0$ for all $t \in [0, z+1]$ and all $\varepsilon > 0$. Since the results on continuous dependence of solutions of ODEs on parameters, compare, for instance, Hale [**8**, Chapter 1.3], imply $w_\varepsilon(t) \to w(t)$ as $\varepsilon \searrow 0$ for all $0 \leq t \leq z+1$, we finally conclude $w(t) \leq 0$ on $[0, z+1]$.

2. *Proof on existence of* $t \in (0, z+1]$ *with* $x(t) < 0$. Assume this assertion is false. Then $x(t) = 0$ for all $t \in [0, z+1]$ due to the last part, and the hypothesis on $t_+ \in (-1, z)$ leads to the contradiction

$$\begin{aligned}
0 &= \dot{x}(t_+ + 1) \\
&= \big[a - |x(t_+ + 1)|\big]\, x(t_+ + 1) - a\, x(t_+ + 1 - r(x(t_+ + 1))) \\
&= 0 - a\, x(t_+ + 1 - r(x(t_+ + 1))) \\
&= -a\, x(t_+ + 1 - r(0)) \\
&= -a\, x(t_+) \\
&< 0.
\end{aligned}$$

Hence, there is a real $t \in (0, z+1]$ with $x(t) < 0$.

3. *Claim:* $x(t) < 0$ *for* $t \in (t_0, z+1]$*, where* $t_0 := \inf\{t \in [0, z+1] |\, x(t) < 0\}$. By the definition of t_0 in combination with the above, we have either $t_0 = 0$ and $x(t_0) = x(0) < 0$, or $0 \leq t_0 < z+1$ and additionally $x(t) = 0$ for all $0 \leq t \leq t_0$. However, in both cases there is a sequence $\{s_j\}_{j \in \mathbb{N}} \subset (t_0, z+1]$ such that $x(s_j) < 0$ and $s_j \searrow t_0$ as $j \to \infty$. Thus, it is sufficient to show $x(t) < 0$ on $[s_j, z+1]$ for all $j \in \mathbb{N}$. To obtain this conclusion, we apply for a given $j \in \mathbb{N}$ Corollary 3.21 once more and compare the resulting solution $w : [s_j, z+1] \longrightarrow \mathbb{R}$ of the IVP

$$\begin{cases} \dot{w}(t) = \big[a - |w(t)|\big]\, w(t) - a\, x(t - r(w(t))) \\ w(s_j) = 0 \end{cases}$$

with x. The inequality $x(s_j) < 0$ implies $x(t) < w(t)$ on $[s_j, z+1]$. We can now proceed analogously to the first part and derive $w(t) \leq 0$, $t \in [s_j, z+1]$. For this reason, $x(t) < 0$ for all $s_j \leq t \leq z+1$ and this implies the claim, since $j \in \mathbb{N}$ was arbitrary.

4. Suppose $x(t) < 0$ for all $t \in (t_0, t_0+1]$, where t_0 is defined as in the last part. Then it is evident that the segments x_t, $t \in [0,1]$, are non-zero and belong to S_a. For a sufficiently small $\varepsilon > 0$, the solution x has no zeros in $[t_0+\varepsilon, t_0+1+\varepsilon] \subset [0, 2]$. In particular, $x_{t_0+1+\varepsilon} \in Z_a$. Hence, under the additional assumption $x(t) < 0$ on $(t_0, t_0+1]$ the assertion of the proposition is clear.

5. For cases where the additional assumption of the last step is not satisfied, there is a smallest zero z_1 of x in $(z+1, t_0+1]$. We have $x(t) \leq 0$ for $t \in [z_1 - 1, z_1]$, $x(t) < 0$ for $t \in (t_0, z_1)$ and $z_1 - 1 \leq t_0 < z_1$. Similar arguments to those of parts (i) and (ii) imply $x(t) \geq 0$ in $[z_1, z_1+1]$ and the existence of $t \in (z_1, z_1+1]$ with $x(t) > 0$. Define $t_1 := \inf\{t \in [z_1, z_1+1] |\, x(t) > 0\}$. Then we have $z_1 \leq t_1 < z_1 + 1$ and $x(t) = 0$ for $z_1 \leq t \leq t_1$. Furthermore, using the techniques of the third step, we find $x(t) > 0$ on $(t_1, z_1+1]$. Subsequently, we distinguish the subcases $z_1 = t_1$ and $z_1 < t_1 < z_1 + 1$.

In the situation $z_1 = t_1$, it is clear that $x_t \in S_a \setminus \{0\}$ for $0 \leq t \leq 1 \leq z_1 + 1$. Additionally, we find a constant $\varepsilon > 0$ such that $z_1 + 1 + \varepsilon \leq t_0 + 2 + \varepsilon < 3$ and the restriction of x to $[z_1+\varepsilon, z_1+1+\varepsilon]$ has no zero. On the other hand, in the subcase $z_1 < t_1 < z_1 + 1$, the application of the methods used in the first three steps leads to $x(t) > 0$ on $(t_1, t_1+1]$. Subsequently, this implies $x_t \in S_a \setminus \{0\}$ for $t \in [0,1]$, and we find again a sufficiently small constant $\varepsilon > 0$ such that x has no zeros in $[t_1 + \varepsilon, t_1 + 1 + \varepsilon]$ and

$$t_1 + 1 + \varepsilon < z_1 + 2 + \varepsilon < t_0 + 3 + \varepsilon < 4.$$

□

Next, we consider the short time behavior of the solutions of Equation (3.1) with initial values in S_a having the property (B) on page 79.

PROPOSITION 3.23. *Assume that $x : [-1, \infty) \longrightarrow [-2a, 2a]$ is a solution of the DDE (3.1) with $x_0 \in S_a$, and that x_0 has property (B); that is, $x(t) \geq 0$ in $[-1, 0]$, and $x(0) > 0$. Then $x_t \in S_a \setminus \{0\}$ for $t \in [0, 1]$, and there is a real $t_* \in [0, 5]$ with $0 \notin x([t_* - 1, t_*]$.*

PROOF. By assumption we have either $x(t) > 0$ for all $t \in [0, 1]$ or we find a smallest zero $z \in (0, 1]$ of x. In the first case the assertion of the proposition follows immediately. Thus, the only point remaining concerns the behavior of x in the situation where there is a smallest zero $z \in (0, 1]$ of x. On that account, we consider the translated solution $y : [-1, \infty) \ni t \longmapsto x(z + t) \in [-2a, 2a]$ of Equation (3.1). Then $y(t) \geq 0$ for $t \in [-1, 0]$ and $y(-z) > 0$. Accordingly, $y_0 \in S_a$ has property (A) and hence Proposition 3.22 shows $y_t \in S_a \setminus \{0\}$ for $t \in [0, 1]$ and the existence of $t_* \in [0, 4]$ with $0 \notin y([t_* - 1, t_*])$. In particular, $x_t \in S_a \setminus \{0\}$ for $t \in [0, 1]$, and $0 \notin x([t_* - 1 + z, t_* + z])$ as claimed. □

The last auxiliary result for the proof of Proposition 3.19 will allow us to make conclusions from the short-time behavior on the long-time behavior of solutions of Equation (3.1) with non-trivial initial values $\varphi \in S_a$ at $t = 0$.

PROPOSITION 3.24. *Let $x : [-1, \infty) \longrightarrow \mathbb{R}$ be a solution of Equation (3.1) with initial value $x_0 \in L_a$. Suppose $z \geq 0$ is a zero of x and $0 \notin x([z - 1, z))$. Then*

$$\operatorname{sign}(x(t)) = -\operatorname{sign}(x(z - 1)) \neq 0$$

for $z < t \leq z + 1$.

PROOF. In addition to the assumption, suppose $x(t) > 0$ for $z - 1 \leq t < z$ so that we have to show $x(t) < 0$ on $(z, z + 1]$. Then, in consideration of

$$\dot{x}(z) = \big[a - |x(z)|\big]\, x(z) - a\, x(z - r(x(z))) = -a\, x(z - 1) < 0,$$

we find a constant $c \in (0, 1]$ with $x(t) < 0$ in $(z, z + c)$. Set $t_0 := z + c/2$. Applying Corollary 3.21 and comparing the resulting solution $w : [t_0, z + 1] \longrightarrow \mathbb{R}$ of

$$\begin{cases} \dot{w}(t) = \big[a - |w(t)|\big]\, w(t) - a\, x(t - r(w(t))) \\ w(t_0) = 0 \end{cases}$$

with the restriction of x to $[t_0, z + 1]$, we obtain $x(t) < w(t)$ for all $t_0 \leq t \leq z + 1$. Therefore, it is sufficient to prove $w(t) \leq 0$, $t_0 \leq t \leq z + 1$, in this case. On this purpose, we consider the IVPs

$$\begin{cases} \dot{w}_\varepsilon(t) = \big[a - |w_\varepsilon(t)|\big]\, w_\varepsilon(t) - a\, x(t - r(w_\varepsilon(t))) - \varepsilon \\ w_\varepsilon(t_0) = 0 \end{cases}$$

with the associated solutions $w_\varepsilon : [t_0, z+1] \longrightarrow \mathbb{R}$ due to Corollary 3.21. As in the proof of Proposition 3.22, we claim $w_\varepsilon(t) \leq 0$ for all $t \in [t_0, z+1]$ and all $\varepsilon > 0$. If this was false, then, in view of

$$w_\varepsilon(t_0) = 0 > -\varepsilon \geq -a\,x(t_0 - 1) - \varepsilon = \dot{w}_\varepsilon(t_0),$$

we would find a smallest $t \in (t_0, z+1)$ with $w_\varepsilon(t) = 0$ and $\dot{w}_\varepsilon(t) \geq 0$, in contradiction to

$$\dot{w}_\varepsilon(t) = 0 - a\,x(t - r(0)) - \varepsilon = -a\,x(t-1) - \varepsilon < 0.$$

Thus, $w_\varepsilon(t) \leq 0$ for all $\varepsilon > 0$ and all $t \in [t_0, z+1]$ and the convergence of $w_\varepsilon(t) \to w(t)$ for $\varepsilon \searrow 0$ implies $w(t) \leq 0$ for $t \in [t_0, z+1]$. In particular, we have $x(t) < 0$ for all $t \in (z, z+1]$.

In the case $x(z) = 0$ and $x(t) < 0$ in $[z-1, z)$, note that $-x$ is a solution of Equation (3.1), which satisfies the conditions of the proposition as well as the additional assumption. Subsequently, the above arguments are applicable to $-x$, and this completes the proof. □

We can now prove Proposition 3.19 in particular containing the statement that every non-trivial initial value in S_a provides a solution of (3.1) which becomes slowly oscillating in finite time.

Proof of Proposition 3.19. Under the stated assumption it follows that either the initial segment of the solution $y = x$ or the one of $y = -x$ has property (A) or (B) formulated on page 79. In case (A) the statement of Proposition 3.22, whereas in case (B), the statement of Proposition 3.23 implies $y_t \in S_a \setminus \{0\}$ for $t \in [0,1]$ and $0 \notin y([t_* - 1, t_*])$ for at least one $t_* \in [0,5]$. By induction, we find $y_t \in S_a \setminus \{0\}$ for all $t \geq 0$. Hence, assertions (i) and (ii) of Proposition 3.19 are clear.

For the last part of the statement, it is sufficient to prove the existence of a sequence $\{s_n\}_{n\in\mathbb{N}} \subset [5, \infty)$, which converges to infinity as $n \to \infty$ and additionally has the property that for every $n \in \mathbb{N}$ the segments x_t, $5 \leq t \leq s_n$, belong to Z_a. For this purpose, choose $s_0 \in [0,5]$ with $0 \notin x([s_0 - 1, s_0])$. If solution x has no zeros in (s_0, ∞), define $s_1 := s_0 + 1$. Obviously, we have $x_t \in Z_a$ for $s_0 \leq t \leq s_1$ in this situation. In the other case, we find a smallest zero $z \in (s_0, \infty)$ of x. Application of Proposition 3.24 implies

$$0 \notin x([z-1, z) \cup (z, z+1]).$$

Hence, there is a sufficiently small $\varepsilon > 0$ such that for $s_1 := z + 1 + \varepsilon > s_0 + 1$, all solution segments x_t, $s_0 \leq t \leq s_1$, are contained in Z_a and $0 \notin x([s_1 - 1, s_1])$.

Next, consider the restriction of x to (s_1, ∞). This has again either no zeros or a smallest zero z in (s_1, ∞). In the first situation, that is, where x has no zero in the interval (s_1, ∞), the definition $s_2 := s_1 + 1$ results in $x_t \in Z_a$ for $s_1 \leq t \leq s_2$. Otherwise, repeated application of Proposition 3.24 shows again

$$0 \notin x([z-1, z) \cup (z, z+1])$$

and therefore we find a sufficiently small $\varepsilon > 0$ such that for $s_2 := z + 1 + \varepsilon > s_1 + 1$ we get $0 \notin x([s_2 - 1, s_2])$ and $x_t \in Z_a$ as $s_1 \leq t \leq s_2$. Hence, in both cases we clearly have $0 \notin x([s_2 - 1, s_2])$ and $x_t \in Z_a$ for all $5 \leq t \leq s_2$. Successively, we construct in this way a sequence $\{s_n\}_{n\in\mathbb{N}} \subset [5, \infty)$ with the claimed properties, which implies the remaining part of the statement. □

A direct consequence of Proposition 3.19 is the positive invariance of S_a under the semiflow F from Corollary 3.8, that is, $F(t, S_a) \subset S_a$ for all $t \geq 0$.

COROLLARY 3.25. *The closure S_a of the set Z_a given by* (3.14) *is positively invariant under the continuous semiflow F defined in Corollary 3.8. In particular, the restriction*

$$F_S : [0,\infty) \times S_a \ni (t,\varphi) \longmapsto F(t,\varphi) \in S_a$$

of F to S_a induces a continuous semiflow on the compact metric space S_a.

Observe that Proposition 3.19 does not show the existence of slowly oscillating solutions of Equation (3.1) with infinitely many zeros. In Chapter 5 we will prove that under the stated assumptions (DF 1) - (DF 5) on the delay function such solutions indeed exists for parameters $a > 1$. In the situation of the DDE (3.2) with constant delay, this conclusion is proved in Brunovský et al. [**3**]. However, let us first adopt the ideas of Brunovský et al. [**3**, Proposition 3.2] and see that slowly oscillating solutions with a bounded set of zeros are not interesting and their long-time behavior is simple.

PROPOSITION 3.26. *Suppose $x : [-1,\infty) \longrightarrow [-2a, 2a]$ is a solution of Equation* (3.1)*, and suppose that x has no zeros. Then*

$$x(t) \to 0$$

as $t \to \infty$.

PROOF. If the solution $x : [-1,\infty) \longrightarrow [-2a, 2a]$ of Equation (3.1) has no zeros, then in particular we have either $x(0) > 0$ or $x(0) < 0$. Without loss of generality we may assume $x(0) > 0$ in the following, because in the situation $x(0) < 0$ the function $y = -x : [-1,\infty) \longrightarrow [-2a, 2a]$ is also a solution of Equation (3.1) but with $y(0) > 0$. Define $c := \limsup_{t\to\infty} x(t)$ and $d := \liminf_{t\to\infty} x(t)$. From Proposition 3.9 we get

$$0 \leq d \leq c \leq 2a.$$

We claim $c = d$, and establish this by contradiction for both cases $d > 0$ and $d = 0$ separately.

Proof of the claim $c = d$. Consider at first the situation $d > 0$. Under this assumption, we find an $\varepsilon > 0$ sufficiently small such that

$$2a\,\varepsilon - (d-\varepsilon)^2 < 0.$$

Furthermore, the definition of d implies the existence of $t \geq 0$ with

$$d - \varepsilon < x(s)$$

for all $s \geq t$. Suppose now, contrary to our claim, that $d < c$. Then there is a local minimum $t_m > t + 1$ such that $x(t_m) < d + \varepsilon$ holds. But this leads to

$$\begin{aligned} 0 &= \dot{x}(t_m) \\ &= a\big[x(t_m) - x(t_m - r(x(t_m)))\big] - \big(x(t_m)\big)^2 \\ &\leq a\big[d + \varepsilon - (d-\varepsilon)\big] - (d-\varepsilon)^2 \\ &= 2a\,\varepsilon - (d-\varepsilon)^2 < 0, \end{aligned}$$

a contradiction, which yields $c = d$ under the assumption $d > 0$.

On the other hand, in the situation $d = 0$ the hypothesis $d < c$ implies the existence of a real $0 < m < c$ with $x(t) > m$ for all $-1 \leq t \leq 0$. Consequently, there is a $t_m > 0$ such that $x(t_m) = m$ and $x(s) > m$ for all $0 \le s < t_m$. In particular, $\dot{x}(t_m) \le 0$. As $c > m$ we find a smallest zero $\xi \ge t_m$ of $\dot{x}$, and we have either $\xi = t_m$, or $\xi > t_m$ and $\dot{x}(s) < 0$ for all $t_m \le s < \xi$. In any case, the above yields $0 < x(\xi) < x(\xi - r(x(\xi)))$, and thus results in the contradiction

$$0 = \dot{x}(\xi) = a\big[x(\xi) - x(\xi - r(x(\xi)))\big] - \big(x(\xi)\big)^2 < -\big(x(\xi)\big)^2 < 0.$$

This clearly shows $c = d$ in the situation $d = 0$ and completes the proof of the claim $c = d$.

From the above we get $x(t) \to c$ as $t \to \infty$. Consequently, we see

$$\begin{aligned}\lim_{t\to\infty} \dot{x}(t) &= \lim_{t\to\infty} \Big(a\big[x(t) - x(t - r(x(t)))\big] - \big(x(t)\big)^2\Big) \\ &= a\big[c - c\big] - c^2 \\ &= -c^2,\end{aligned}$$

which leads to a contradiction for $c > 0$. Therefore $c = 0$, and hence $x(t) \to 0$ as $t \to \infty$, which is the assertion. □

Global Slowly Oscillating Solutions. Having discussed the existence of slowly oscillating solutions for DDE (3.1) with initial values in S_a, in this subsection we prove that the segments of all these solutions that in addition are globally defined form a global attractor for the restricted semiflow F_S from Corollary 3.25. We begin with the result that F_S has indeed a global attractor.

Proposition 3.27. *Let $\mathcal{A}$ denote the set*

$$\mathcal{A} := \bigcap_{t\ge 0} F(t, S_a).$$

(i) *$\mathcal{A}$ is a maximal compact set in S_a, which is invariant for F, that is, $F(t,\mathcal{A}) = \mathcal{A}$ for all $t \ge 0$, and attracts each bounded set $B \subset S_a$. In other words, $\mathcal{A}$ is the global attractor of the semiflow F_S on S_a from Corollary 3.25.*

(ii) *The map $F_{\mathcal{A}} : \mathbb{R} \times \mathcal{A} \ni (t,\varphi) \longmapsto x_t^{\varphi} \in \mathcal{A}$ constitutes a continuous flow on the compact metric space $\mathcal{A}$.*

(iii) *$\mathcal{A}$ is connected.*

Proof. 1. The first assertion of the proposition follows immediately from Theorem 3.4.2 in Hale [**9**] on existence of global attractors for smooth dissipative semiflows, provided that S_a is nonempty and attracts compact subsets of S_a. But this is clear, because $0 \in S_a$, and S_a is positively invariant under the semiflow F due to Corollary 3.25 and thus attracts all subsets of S_a.

2. Part (ii) is also a direct consequence of Hale [**9**, Theorem 3.4.2], since all restricted time-t-maps $F(t,\cdot)|_{\mathcal{A}}$, $t \ge 0$, are injective. Indeed, consider $\varphi, \psi \in \mathcal{A}$ and suppose the existence of $t \ge 0$ with $F(t,\varphi) = F(t,\psi)$. As $\mathcal{A}$ is invariant for the semiflow F we find solutions $x, y : \mathbb{R} \longrightarrow [-2a, 2a]$ of Equation (3.1) with segments contained in $\mathcal{A}$ and initial values $x_0 = \varphi$ and $y_0 = \psi$, respectively. By assumption, we have

$$x_t = F(t,\varphi) = F(t,\psi) = y_t,$$

so Corollary 3.13 implies $x(s) = y(s)$ for all $s \in \mathbb{R}$. In particular, $\varphi = \psi$. Hence, for each $t \ge 0$, the restricted time-t-map $F(t,\cdot)|_{\mathcal{A}}$ is in fact injective.

3. The proof of the last statement of the proposition is by contradiction. For this purpose, suppose $\mathcal{A}$ is not connected. Then, by definition, there are open disjoint sets $V_1, V_2 \subset S_a$ such that $\mathcal{A} \subset V_1 \cup V_2$, and both $V_1 \cap \mathcal{A}$ and $V_2 \cap \mathcal{A}$ are nonempty. As $\mathcal{A}$ attracts S_a we obviously find a $t \geq 0$ with

$$F(t, S_a) \subset V_1 \cup V_2.$$

Now the definition of $\mathcal{A}$ and the invariance of $\mathcal{A}$ for F give $\mathcal{A} = F(t, \mathcal{A}) \subset F(t, S_a)$. Accordingly, we obtain

$$F(t, S_a) \cap V_1 \supset \mathcal{A} \cap V_1 \neq \emptyset$$

and

$$F(t, S_a) \cap V_2 \supset \mathcal{A} \cap V_2 \neq \emptyset,$$

that is, $F(t, S_a)$ is not connected. On the other hand, for each real $-1 \leq \lambda \leq 1$ we have $\lambda S_a \subset S_a$, which implies that for every pair $\varphi, \psi \in S_a$ the mapping $p_{\varphi \to \psi} : [0,1] \longrightarrow S_a$, defined by

$$p_{\varphi \to \psi}(\lambda) := \begin{cases} (1 - 2\lambda)\, \varphi, & \text{for } 0 \leq \lambda \leq 1/2, \\ (2\lambda - 1)\, \psi, & \text{for } 1/2 \leq \lambda \leq 1, \end{cases}$$

forms a path from φ to ψ in S_a. Consequently, S_a is arcwise connected and so its continuous image $F(t, S_a)$. This forms a contradiction to the above, since each arcwise connected subset of a topological space is connected. $\square$

Our next proposition provides some criteria for solutions $x : \mathbb{R} \to [-2a, 2a]$ of Equation (3.1) to be slowly oscillating. In particular it shows that beside the zero function, $\mathcal{A}$ contains the segments of all globally defined slowly oscillating solutions of Equation (3.1) with values in $[-2a, 2a]$.

PROPOSITION 3.28. *The following statements are equivalent.*

(i) $\varphi \in \mathcal{A} \setminus \{0\}$.
(ii) *There is a solution* $x : \mathbb{R} \longrightarrow [-2a, 2a]$ *of Equation* (3.1) *satisfying* $x_0 = \varphi$ *and* $x_t \in S_a \setminus \{0\}$ *for all* $t \in \mathbb{R}$.
(iii) *There is a slowly oscillating solution* $x : \mathbb{R} \longrightarrow [-2a, 2a]$ *of Equation* (3.1) *with initial value* $x_0 = \varphi$.
(iv) *There is a non-trivial solution* $x : \mathbb{R} \longrightarrow [-2a, 2a]$ *of Equation* (3.1) *with* $x_0 = \varphi$ *such that* $x|_{[t-r(x(t)),t]} \neq 0$ *and* $V(x, [t - r(x(t)), t]) = 1$ *for* $t \in \mathbb{R}$.

PROOF. 1. *Proof of (i) ⇒ (ii).* Let $\varphi \in \mathcal{A} \setminus \{0\}$ be given. By the invariance of $\mathcal{A}$ for F from the last result, we find a solution $x : \mathbb{R} \longrightarrow [-2a, 2a]$ of the differential equation (3.1) with initial value φ at $t = 0$ and segments contained in $\mathcal{A}$. Moreover, as the restriction $F_{\mathcal{A}}$ of F to $\mathcal{A}$ constitutes a continuous flow, and as $F_{\mathcal{A}}(t, 0) = 0$ for all $t \in \mathbb{R}$, the solution x is uniquely determined by φ and we have $x_t \in \mathcal{A} \setminus \{0\} \subset S_a \setminus \{0\}$ for $t \in \mathbb{R}$. Hence, statement (i) implies (ii).

2. *Proof of (ii) ⇒ (iii).* Suppose $x : \mathbb{R} \longrightarrow [-2a, 2a]$ is a solution of Equation (3.1) such that $x_t \in S_a \setminus \{0\}$ for $t \in \mathbb{R}$ and $x_0 = \varphi$. If we prove that $x_t \in Z_a$ as $t \in \mathbb{R}$, the assertion follows. For this purpose, consider a fixed $t \in \mathbb{R}$. In case $t \geq 5$ the statement of Proposition 3.19 shows $x_t \in Z_a$. Otherwise, $t < 5$ and we find a constant $\xi \in \mathbb{R}$ with $\xi < t - 5 < 0$. Define $y : \mathbb{R} \ni s \longmapsto x(s + \xi) \in [-2a, 2a]$. Then

y is obviously also a solution of Equation (3.1), and we have $y_s = x_{s+\xi} \in S_a \setminus \{0\}$, $s \in \mathbb{R}$. Therefore by application of Proposition 3.19 to y we get

$$y_s \in Z_a \setminus \{0\}$$

for all $s \geq 5$. Especially, $x_t = y_{t-\xi} \in Z_a \setminus \{0\}$ as $t - \xi > 5$. This completes the proof of the implication $(ii) \Rightarrow (iii)$.

3. *Proof of* $(iii) \Rightarrow (iv)$. This part is immediate since the segments of a slowly oscillating solution $x : \mathbb{R} \longrightarrow [-2a, 2a]$ of Equation (3.1) have by definition at most one zero in $[t-1, t]$, and thus in particular at most one sign change in $[t - r(x(t)), t] \subset [t-1, t]$ for all $t \in \mathbb{R}$.

4. *Proof of* $(iv) \Rightarrow (i)$. Assume that $x : \mathbb{R} \longrightarrow [-2a, 2a]$ is a non-trivial solution of the differential equation (3.1) with $x|_{[t-r(x(t)),t]} \neq 0$ and $V(x, [t - r(x(t)), t]) = 1$ for $t \in \mathbb{R}$. To conclude $x_0 \in \mathcal{A} \setminus \{0\}$, we begin with the claim that the orbit

$$\mathfrak{O} := \{x_t |\ t \in \mathbb{R}\}$$

of the solution x is contained in S_a, that is, for each $t \in \mathbb{R}$ the associated segment x_t has at most one sign change. Let $t \in \mathbb{R}$ be given. We may assume that there is a largest zero $\xi_0 \in [t-1, t]$ of x in the interval $[t-1, t]$ since otherwise there is nothing to show. Then $r(x(\xi_0)) = r(0) = 1$ and it suffices to consider only the subcase $\xi_0 < t$, because in the situation $\xi_0 = t$ we have

$$\mathrm{sc}(x, [t-1, t]) \leq V(x, [t-1, t]) = V(x, [t - r(x(t)), t]) = 1.$$

Now, if $\xi_0 \in [t-1, t)$ is the largest zero of x in $[t-1, t]$, then the inequality

$$\xi_0 - r(x(\xi_0)) = \xi_0 - 1 < t - 1$$

in combination with the monotonicity of $\mathbb{R} \ni s \longmapsto s - r(x(s)) \in \mathbb{R}$ from Corollary 3.11 implies the existence of $\xi_1 \in (\xi_0, t)$ with $[t-1, \xi_0] \subset (\xi_1 - r(x(\xi_1)), \xi_1]$. By the definition of ξ_0, $x(s)$ is either strictly positive or strictly negative for all $\xi_0 < s \leq t$. Hence the solution x has no sign change in the interval $(\xi_0, t]$. Therefore we obtain

$$\mathrm{sc}(x, [t-1, t]) = \mathrm{sc}(x, [t-1, \xi_1]) \leq \mathrm{sc}(x, [\xi_1 - r(x(\xi_1)), \xi_1]) = 1,$$

which implies that x has at most one sign change in $[t-1, t]$. This proves our claim $\mathfrak{O} \subset S_a$.

The proof of the implication $(iv) \Rightarrow (i)$ follows now easily from the above. We have $F(t, \mathfrak{O}) = \mathfrak{O}$ for all $t \in \mathbb{R}$ and hence,

$$\mathfrak{O} = \bigcap_{t \geq 0} F(t, \mathfrak{O}) \subset \bigcap_{t \geq 0} F(t, S_a) = \mathcal{A}.$$

In particular, $x_0 \in \mathcal{A}$. As it also holds $x_0 \neq 0$, the above leads to the desired conclusion $x_0 \in \mathcal{A} \setminus \{0\}$. $\square$

The property (iv) of global slowly oscillating solutions in the last result will prove extremely useful in Chapter 5.

Remark 3.29. Note that the last two propositions show that the compact set $\mathcal{A}$ contains all segments of slowly oscillating solutions $x : \mathbb{R} \to [-2a, 2a]$, but they do not contain a result whether Equation (3.1) actually has globally defined slowly oscillating solutions; that is, the global attractor $\mathcal{A}$ for F_S may also consists only of the zero function. However, in Chapter 5 we will see that for parameters $a > 1$, the compact set $\mathcal{A}$ has indeed more elements than the trivial one.

We close this chapter with a sharpened version of continuous dependence on initial data for solutions of Equation (3.1) in $\mathcal{A}$, which is formulated and proven in the two corollaries below.

COROLLARY 3.30. *Suppose that* $\varphi \in \mathcal{A}$ *and that* $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ *is a sequence with* $\varphi_n \to \varphi$ *as* $n \to \infty$. *Then the family* $\{x^{\varphi_n}\}_{n\in\mathbb{N}}$ *of the uniquely determined globally defined solutions* x^{φ_n} *for Equation* (3.1) *with initial conditions* $x_0^{\varphi_n} = \varphi_n$, $n \in \mathbb{N}$, *converges uniformly on each compact interval in* $\mathbb{R}$ *to the solution* x^{φ} *of* (3.1) *with* $x_0^{\varphi} = \varphi$ *as* $n \to \infty$.

PROOF. Let the above assumptions be fulfilled, and let a compact interval $I \subset \mathbb{R}$ be given. Consider an integer $k \in \mathbb{N}$ with $I \subset [-k, k]$. As the restriction of F to $\mathcal{A}$ constitutes a continuous flow on $\mathcal{A}$ due to Proposition 3.27, we conclude that for every fixed $j \in \mathbb{Z}$ the induced map $F : (j, \cdot\,) : \mathcal{A} \longrightarrow \mathcal{A}$ is continuous. Subsequently, for each $\varepsilon > 0$ there is $\delta_j = \delta_j(\varepsilon) > 0$ such that the implication

$$\|\varphi_n - \varphi\|_C < \delta_j \qquad \Longrightarrow \qquad \|F(j, \varphi_n) - F(j, \varphi)\|_C < \varepsilon$$

is true. Set now $\delta := \min\{\delta_j |\ j \in \mathbb{Z},\ 1 \leq |j| \leq k\}$. If $\|\varphi_n - \varphi\|_C < \delta$, then we obviously have

$$\sup\left\{\|x_j^{\varphi_n} - x_j^{\varphi}\|_C \,\middle|\, j \in \mathbb{Z},\ 1 \leq |j| \leq k\right\} < \varepsilon,$$

and hence,

$$\sup_{t\in I} |x^{\varphi_n}(t) - x^{\varphi}(t)| < \varepsilon.$$

Subsequently, this shows the uniform convergence of $\{x^{\varphi_n}\}_{n\in\mathbb{N}}$ to x^{φ} on interval I for $n \to \infty$ and the assertion follows. $\square$

In the situation of the above corollary, not only the solutions $\{x^{\varphi_n}\}_{n\in\mathbb{N}}$ of Equation (3.1) do uniformly converge on compact intervals of $\mathbb{R}$ to the solution x^{φ}, but also their derivatives to the one of x^{φ}. More precisely, the following holds.

COROLLARY 3.31. *The topologies on* $\mathcal{A}$ *induced by the norms of the Banach spaces* C *and* C^1 *are equivalent.*

PROOF. We only need to prove that for every point $\varphi \in \mathcal{A}$ and all sequences $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ with $\|\varphi_n - \varphi\|_C \to 0$ as $n \to \infty$ we also have the convergence of $\|\varphi_n' - \varphi'\|_C \to 0$ as $n \to \infty$. But if $\|\varphi_n - \varphi\|_C \to 0$ for $n \to \infty$, then the last corollary implies

$$\|x_{-1}^{\varphi_n} - x_{-1}^{\varphi}\|_C \to 0$$

for $n \to \infty$. Applying Proposition 3.7 about continuous dependence of solutions for Equation (3.1) on initial values to the sequence

$$\{x_{-1}^{\varphi_n}\}_{n\in\mathbb{N}} \subset \mathcal{A}$$

and the function $x_{-1}^{\varphi} \in \mathcal{A}$, we conclude the uniform convergence of $\{y^n\}_{n\in\mathbb{N}}$ to y on $[0, 1]$ as $n \to \infty$, where $y^n(s) := \dot{x}^{\varphi_n}(-1 + s)$ and $y(s) := \dot{x}^{\varphi}(-1 + s)$, $s \in \mathbb{R}$. Consequently, $\|y_1^n - y_1\|_C \to 0$, or equivalently, in view of $y_1^n = \varphi_n'$ and $y_1 = \varphi'$, $\|\varphi_n' - \varphi'\|_C \to 0$ for $n \to \infty$. This completes the proof. $\square$

CHAPTER 4

Some Aspects of the Local Behavior at the Trivial Solution

1. Introduction

In the last chapter we discussed some basic properties and slowly oscillating solutions of the model equation (3.1). Here, we continue the study of DDE (3.1) by analyzing its local behavior near the trivial solution. To this end, we consider the model equation (3.1) in the more abstract form

$$\dot{x}(t) = f(x_t) \tag{4.1}$$

with right-hand side given by

$$f : C^1 \ni \varphi \longmapsto a\left[\varphi(0) - \varphi(-r(\varphi(0)))\right] - |\varphi(0)|\,\varphi(0) \in \mathbb{R}.$$

As seen in Section 3 of Chapter 1, the functional f satisfies the smoothness conditions (S 1) and (S 2) of Theorem 1.2 on the existence of a continuously differentiable semiflow; that is, f is continuously differentiable and each derivative $Df(\varphi)$, $\varphi \in C^1$, extends to a linear functional $D_e f(\varphi) : C \longrightarrow \mathbb{R}$ so that the induced map $C^1 \times C \ni (\varphi, \chi) \longmapsto D_e f(\varphi)\,\chi$ is continuous. Thereby recall from page 10 that the functional $D_e f(\varphi) \in \mathcal{L}(C, \mathbb{R})$, $\varphi \in C^1$, is given by

$$D_e f(\varphi)\,\psi = a\cdot\left[\psi(0) - \psi(-r(\varphi(0)))\right] + a\cdot r'(\varphi(0))\cdot\varphi'(-r(\varphi(0)))\cdot\psi(0) - 2\cdot|\varphi(0)|\cdot\psi(0)$$

for all $\psi \in C$.

From now on and until the end of this work, we shall only deal with C^1-smooth solutions as considered in Theorem 1.2. More precisely, in the following we call $x : [t_0 - 1, t_+) \longrightarrow \mathbb{R}$, $t_+ > t_0$, a solution of Equation (4.1), if $x \in C^1([t_0 - 1, t_+), \mathbb{R})$ and if x satisfies (4.1) for all $t_0 < t < t_+$. In the same way we define solutions on intervals of the form $(-\infty, t_+)$, $t_+ \in \mathbb{R}$. As the zero function is a solution of the model equation (4.1) in the sense defined above, the set

$$X_f := \left\{\varphi'(0) = f(\varphi) \mid \varphi \in C^1\right\}$$

is not empty. Subsequently, we are able to apply Theorem 1.2, which implies that each $\varphi \in X_f$ uniquely defines a non-continuable solution $x^\varphi : [-1, t_+(\varphi)) \longrightarrow \mathbb{R}$ of Equation (4.1) with $x_0^\varphi = \varphi$ and $x_t^\varphi \in X_f$ for all $0 \le t < t_+(\varphi)$. Moreover, the equation

$$F(t, \varphi) = x_t^\varphi$$

for $\varphi \in X_f$ and $0 \le t < t_+(\varphi)$, defines a domain $\Omega \subset \mathbb{R} \times X_f$ and a continuous semiflow F on the solution manifold X_f with continuously differentiable solution maps F_t, $t \ge 0$. Since every C^1-smooth solution of Equation (4.1) as introduced above is of course as well a solution of Equation (3.1) in the sense of the last chapter, we see at once that $t_+(\varphi) = \infty$ holds for all $\varphi \in X_f$ in view of Propositions 3.4 and 3.5.

REMARK 4.1. Note that it is suggestive to use the same notation for the semiflow as in the last chapter, namely F, and it shall not cause any confusions to the reader, because, as already pointed out above, each C^1-smooth solution of (4.1) is also a solution of Equation (3.1) in the sense used in the foregoing chapter. In particular, for each $\varphi \in X_f \cap L_a$ and all $0 \leq t < \infty$ we have $x_t^\varphi \in X_f \cap L_a$.

This adaption of the semiflow enables us to use the techniques of linearization and local invariant manifolds to study the local behavior of solutions for Equation (4.1) in close vicinity of the trivial stationary point. The application of these methods is the main goal of this chapter, which is arranged as follows. In the next section we discuss the linearization of F at $\varphi_0 = 0$ and its spectral properties. After that we consider the question of stability of the trivial solution in Section 3. Finally, we close this chapter by proving that in the case $a > 1$ the trivial solution is repelling within the local center-unstable manifolds. This fact will be essential for our investigation of the model equation in the last part of the work.

2. The Linearization

Our local analysis of the continuously differentiable semiflow F at the trivial solution begins with the study of the linearization and its spectral properties. For this purpose, we follow the outline in Chapter 1 on the linearization of a semiflow obtained by Theorem 1.2 and study the associated linear RFDE.

The Associated Linear RFDE. Consider the derivative $Df(0) \in \mathcal{L}(C^1, \mathbb{R})$ of f at the stationary point $\varphi_0 = 0$, which we denote by L in the following. As a trivial verification shows, L acts by

$$L\,\psi = a\big[\psi(0) - \psi(-1)\big]$$

on $\psi \in C^1$. Subsequently, the tangent space $T_0 X_f$ of the solution manifold X_f at the trivial stationary point is given by the subspace

$$T_0 X_f = \Big\{\psi \in C^1 \,\Big|\, \psi'(0) = a\,\big[\psi(0) - \psi(-1)\big]\Big\}$$

of C^1 and the linearization of the semiflow F at $0 \in X_f$ is the strongly continuous semigroup $T = \{T(t)\}_{t \geq 0}$ of bounded linear operators $T(t) = D_2 F(t, 0)$ on $T_0 X_f$. The action of $T(t)$, $t \geq 0$, on $\psi \in T_0 X_f$ is determined by the relation $T(t)\,\psi = v_t^\psi$ with the unique solution $v^\psi : [-1, \infty) \longrightarrow \mathbb{R}$ of the IVP

$$\begin{cases} \dot{v}(t) = L\,v_t \\ \quad v_0 = \psi \end{cases}$$

in $T_0 X_f$. Let now $L_e \in \mathcal{L}(C, \mathbb{R})$ denote the linear extension $D_e f(0)$ of the operator $L = Df(0)$ to C, that is, the mapping

$$L_e : C \ni \psi \longmapsto a\big[\psi(0) - \psi(-1)\big] \in \mathbb{R}. \tag{4.2}$$

Then L_e continues the above IVP induced by $Df(0)$ in a natural way to an IVP for the linear RFDE

$$\dot{z}(t) = L_e\, z_t \tag{4.3}$$

on the larger Banach space C. For every $\psi \in C$ this equation has a unique solution $z^\psi : [-1, \infty) \longrightarrow \mathbb{R}$. The corresponding solution semigroup $T_e = \{T_e(t)\}_{t \geq 0}$ is defined by

$$T_e(t) : C \ni \psi \longmapsto z_t^\psi \in C,$$

and the spectral properties of the associated infinitesimal generator G_e of T_e, that is, the linear operator

$$G_e : \mathcal{D}(G_e) \ni \psi \longmapsto \psi' \in C$$

with domain

$$\mathcal{D}(G_e) = \big\{\psi \in C^1 |\ \psi'(0) = L_e\, \psi\big\},$$

are closely related to those of the infinitesimal generator G of T, as explained in detail in Chapter 1. Therefore, we study the linear RFDE (4.3), its solution semigroup T_e and the associated generator G_e in the following.

The Spectrum and Spectral Subspaces. Using the ansatz $z(t) = e^{\lambda t}\, c$ with a scalar λ for a solution of Equation (4.3), we obtain the transcendental characteristic equation

$$\triangle(\lambda) = 0 \tag{4.4}$$

with

$$\triangle(\lambda) := \lambda - a\big[1 - e^{-\lambda}\big]$$

of RFDE (4.3). The solutions of Equation (4.4) in $\mathbb{C}$ coincide with the spectrum of the complexified operator $(G_e)_{\mathbb{C}}$. Thereby, the order of a root $\lambda \in \mathbb{C}$ of Equation (4.4) agrees with the dimension of the generalized eigenspace of $(G_e)_{\mathbb{C}}$ related to λ.

Our next result specifies the location of the roots of the transcendental equation (4.4) in the complex plane by taking into account the parameter $a > 0$.

PROPOSITION 4.2 (Proposition 4.1 in Brunovský et al. [**3**]).

(i) *If $a > 0$ and if $a \neq 1$, then $\lambda = 0$ is a single root of Equation* (4.4). *In case $a = 1$, $\lambda = 0$ is a double root of* (4.4).

(ii) *For all $a > 0$ with $a \neq 1$, Equation* (4.4) *has an unique non-trivial root $\lambda = \kappa \in \mathbb{R}$; κ is simple and $\kappa < 0$ for $0 < a < 1$, $\kappa > 0$ for $a > 1$.*

(iii) *Except for the real roots described above, the solutions of Equation* (4.4) *occur in conjugate complex pairs $\mu_k \pm i\nu_k$, $k \in \mathbb{N}$, with $\nu_k \neq 0$ and the real parts μ_k, $k \in \mathbb{N}$, satisfy $\mu_k < \kappa$ for $a < 1$, whereas $\mu_k < 0$ for $a \geq 1$.*

PROOF. Analogously to the proof in Brunovský [**3**], we begin with the observation that $\lambda \in \mathbb{C}$ is a root of Equation (4.4) for the parameter $a > 0$ if and only if $\zeta := \lambda - a$ is a solution of the equation

$$\zeta + \alpha\, e^{-\zeta} = 0 \tag{4.5}$$

for $\alpha = ae^{-a}$. We use this fact and deduce statements (i) - (iii) for Equation (4.4) from solutions of (4.5), following Wright [**29**, Theorem 5].

1. Consider the positive function $g : (0, \infty) \ni s \longmapsto s\, e^{-s} \in (0, \infty)$. In view of the derivative $g'(s) = (1 - s)\, e^{-s}$ of g, it is obvious that g is strictly monotonically increasing in $(0, 1)$, has a strict global maximum $1/e$ at $s = 1$, and is strictly monotinically decreasing in $(1, \infty)$, as shown in Figure 4.1. Therefore, the equality $\alpha = g(a)$, $a > 0$, implies that it is sufficient to consider Equation (4.5) only for parameters $0 < \alpha \leq 1/e$.

2. Let $0 < \alpha \leq 1/e$ be given. Then Equation (4.5) may have only negative real roots. Moreover, $\zeta \in \mathbb{R}$ is a root of (4.5) if and only if $g(-\zeta) = \alpha$. Combining this with

$$\lim_{s \searrow 0} g(s) = 0 = \lim_{s \nearrow \infty} g(s)$$

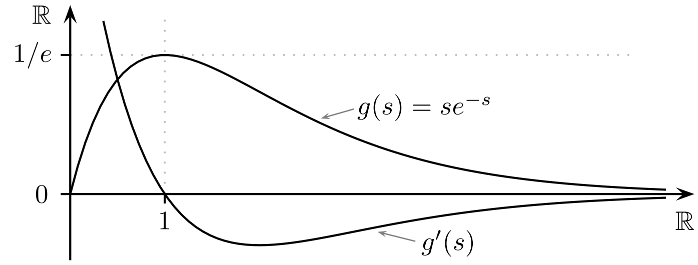

FIGURE 4.1. Relevant parameters for Equation (4.5)

and the properties of g described in the first part, we see that, as illustrated in Figure 4.2, in case $0 < \alpha < 1/e$ Equation (4.5) has exactly two real roots $\zeta_1, \zeta_2 \in \mathbb{R}$, both simple and satisfying

$$0 < -\zeta_1 < 1 < \ln(1/\alpha) < -\zeta_2,$$

whereas for $\alpha = 1/e$ there exists only the real root $\zeta = -1$ of (4.5) but of order

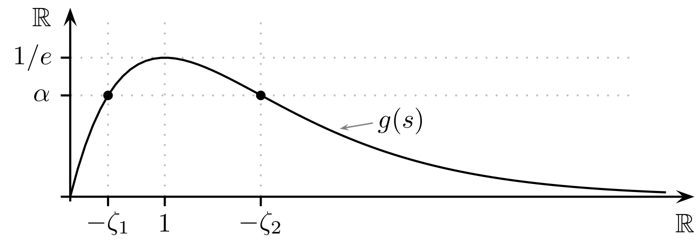

FIGURE 4.2. Real roots of Equation (4.5) for parameters $0 < \alpha \leq 1/e$

two.

Accordingly, in the situation $0 < a \neq 1$, Equation (4.4) has exactly two real roots, namely the two simple roots $\lambda_1 := \zeta_1 + a$, $\lambda_2 := \zeta_2 + a$ satisfying the inequality

$$a > \lambda_1 > a - 1 > a + \ln(a\, e^{-a}) > \lambda_2,$$

and in case $a = 1$ we only find the double root $\lambda = 0$ of Equation (4.4) in $\mathbb{R}$. Using these and taking into account that $\lambda = 0$ is a root of Equation (4.4) for all $a > 0$, we immediately conclude the first two statements of the proposition.

3. We are left to prove part (iii) of the proposition. For this purpose, let us apply formula (4.5) to a complex number $\sigma + i\nu \in \mathbb{C}$ with $\sigma, \nu \in \mathbb{R}$. This leads to the equality

$$-\sigma - i\nu = \alpha\, e^{-\sigma - i\nu} = \alpha\, e^{-\sigma}\big(\cos(\nu) - i\sin(\nu)\big),$$

and so to

$$\begin{cases} -\sigma = \alpha\, e^{-\sigma}\, \cos(\nu), \\ \quad \nu = \alpha\, e^{-\sigma}\, \sin(\nu). \end{cases} \tag{4.6}$$

Consequently, if $\sigma + i\nu \in \mathbb{C}$ with $\sigma, \nu \in \mathbb{R}$ and $\nu \neq 0$ is a solution of Equation (4.5), then it necessarily holds that $\nu \neq k\pi$ for all $k \in \mathbb{Z}$. In addition, a trivial manipulation of the equalities in (4.6) leads to

$$-\sigma = \nu\, \cot(\nu), \tag{4.7}$$

and therewith to

$$\alpha = \nu\, e^{-\nu \cot(\nu)}\, \csc(\nu). \tag{4.8}$$

Define the right-hand side of the last equation as a function χ, that is,

$$\chi : (0, \infty) \ni s \longmapsto s\, e^{-s \cot(s)} \csc(s) \in \mathbb{R}.$$

If $(2k-1)\,\pi < s < 2k\,\pi$, $k \in \mathbb{N}$, then we have $\chi(s) < 0$, and subsequently s does surely not satisfy Equation (4.8) for $0 < \alpha \leq 1/e$. For this reason, let us suppose in the following that $2k\,\pi < s < (2k+1)\,\pi$ for $k \geq 0$, and hence $\chi(s) > 0$. Then, using

$$\chi'(s) = e^{-s \cot(s)} \cdot \csc(s) + s \cdot e^{-s \cot(s)} \cdot \csc(s) \cdot \big(-2 \cot(s) + s \cdot \csc^2(s) \big),$$

we see

$$\begin{aligned} \frac{\chi'(s)}{\chi(s)} &= \frac{1}{s} - 2 \cdot \cot(s) + s \cdot \csc^2(s) \\ &= \frac{1 - 2 \cdot s \cdot \cot(s) + s^2 \cdot \csc^2(s)}{s} \\ &= \frac{(1 - s \cdot \cot(s))^2 + s^2}{s} > 0, \end{aligned}$$

which yields that χ is strictly monotonous increasing in $(2k\,\pi, (2k+1)\,\pi)$. Accordingly, in consideration of $\chi(s) \to \infty$ as $s \nearrow (2k+1)\,\pi$, and $\chi(s) \to 0$ as $s \searrow 2k\,\pi$ for $k \geq 1$, for every positive integer k there obviously exists exactly one real $2k\,\pi < \nu_k < (2k+1)\,\pi$ satisfying (4.8), as indicated in Figure 4.3. Observe

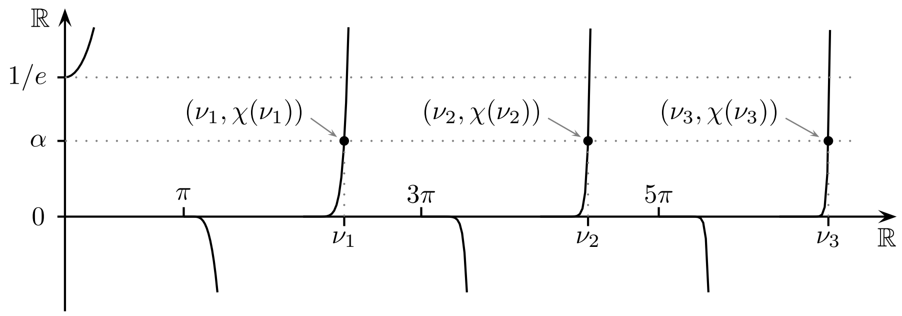

FIGURE 4.3. Solutions of Equation (4.8) for parameters $0 < \alpha \leq 1/e$

that in case $k = 0$, and thus $0 < s < \pi$, we have $\chi(s) > 1/e \geq \alpha$; this means, ν_1 is the smallest positive solution of (4.8) for $0 < \alpha \leq 1/e$. Hence, by defining $\sigma_k := -\nu_k \cot(\nu_k)$ for $k \geq 1$, in view of Equation (4.7) it becomes clear that, apart of the real roots discussed in the second part, Equation (4.5) has the complex conjugate pairs $\sigma_k \pm i\nu_k$, $k \in \mathbb{N}$, as solutions.

Next we derive an estimate for the real part of the eigenvalues $\sigma_k \pm i\nu_k$, $k \in \mathbb{N}$. For this purpose, note that $\alpha \leq \pi/2$ and $\nu_k > 2\pi$ for all $k \geq 1$. Thus, by (4.6), we easily conclude that the real parts of these roots satisfy

$$-\sigma_k = \frac{1}{2} \ln\left(\frac{\sigma_k^2 + \nu_k^2}{\alpha^2} \right) > \frac{1}{2} \ln\left(\frac{\nu_k^2}{\alpha^2} \right) > \ln\left(\frac{\nu_k}{\alpha} \right) > \ln\left(\frac{2\pi}{\frac{1}{2}\pi} \right) = \ln 4 > 1.$$

Define the sequence $\{\tilde{\sigma}_k\}_{k \in \mathbb{N}_0}$ by $\tilde{\sigma}_0 := -\zeta_2$ and $\tilde{\sigma}_k := -\sigma_k$ for all $k \geq 1$. Additionally, let $v : [0, \infty) \longrightarrow \mathbb{R}$ be given by

$$v(\tilde{\sigma}) := \alpha^2\, e^{2\tilde{\sigma}} - \tilde{\sigma}^2.$$

Using Equation (4.5) and (4.6), we see $v(\tilde{\sigma}_0) = 0$ and $v(\tilde{\sigma}_k) = \nu_k^2$ as $k \geq 1$. Now for each fixed $k \geq 1$ we have $\tilde{\sigma}_k > 1$ and this implies that for all $\tilde{\sigma} \geq \tilde{\sigma}_k$ the function v is strictly increasing in $[\tilde{\sigma}_k, \infty)$ due to

$$v'(\tilde{\sigma}) = 2\alpha^2 e^{2\tilde{\sigma}} - 2\tilde{\sigma} = 2(v(\tilde{\sigma}) + \tilde{\sigma}^2 - \tilde{\sigma}) > v(\tilde{\sigma})$$

and $v(\tilde{\sigma}_k) = \nu_k^2 > 0$. As $1 < \nu_k < \nu_{k+1}$ for all $k \geq 1$ it also follows $\tilde{\sigma}_{k+1} > \tilde{\sigma}_k$ for all $k \geq 1$. Moreover, the assumption $\tilde{\sigma}_0 \geq \tilde{\sigma}_1$ leads to $0 = v(\tilde{\sigma}_0) \geq v(\tilde{\sigma}_1) = \nu_1^2$ which is impossible as $\nu_1 > 0$. Hence, $\tilde{\sigma}_{k+1} > \tilde{\sigma}_k$ for all $k \geq 0$, and thus

$$\zeta_2 > \sigma_k > \sigma_{k+1}$$

as $k \geq 1$.

Returning to Equation (4.4) and defining $\mu_k := \sigma_k + a$ for $k \geq 1$, we conclude that $\mu_k \pm i\nu_k$, $k \in \mathbb{N}$, form complex conjugate pairs of roots for (4.4). In particular, these roots are the only ones of Equation (4.4) in $\mathbb{C}\backslash\mathbb{R}$. Finally, combining the estimates for the real roots and for σ_k, $k \in \mathbb{N}$, yields the asserted estimates for μ_k, which completes the proof. $\square$

After the description of the location of the spectrum of G_e, or more precisely of $(G_e)_{\mathbb{C}}$, in the complex plane, we proceed with the characterization of some eigenspaces. On that account, for every fixed $a > 0$ we write $C_c \subset C$ for the center space of G_e, and $M \subset C$ for the realified generalized eigenspaces given by the eigenvalues on the real line due to the last proposition. Additionally, let C_u denote the unstable space of G_e in case $a > 1$.

Corollary 4.3 (Corollary 4.1 in Brunovský et al. [**3**]). *For every real $a > 0$, $C_c \subset M$ and $\dim M = 2$. Moreover, if we define functions η_0, η_d, and in case $a \neq 1$, additionally, a function η_κ by*

$$\eta_0 : [-1, 0] \ni \vartheta \longmapsto 1 \in \mathbb{R},$$
$$\eta_d : [-1, 0] \ni \vartheta \longmapsto \vartheta \in \mathbb{R},$$

and

$$\eta_\kappa : [-1, 0] \ni \vartheta \longmapsto e^{\kappa\,\vartheta} \in \mathbb{R},$$

respectively, then

(i) *$C_c = \mathbb{R}\,\eta_0$ and $M = \mathbb{R}\,\eta_0 \oplus \mathbb{R}\,\eta_\kappa$ for $0 < a < 1$,*
(ii) *$C_c = \mathbb{R}\,\eta_0 \oplus \mathbb{R}\,\eta_d$ and $M = C_c$ in case $a = 1$, and*
(iii) *$C_c = \mathbb{R}\,\eta_0$, $C_u = \mathbb{R}\,\eta_\kappa$ and $M = C_c \oplus C_u$ for $a > 1$.*

Proof. The proof is straightforward in view of the spectrum specified in Corollary 4.3. Let us consider the case $a \neq 1$ first. The functions η_0, η_κ defined above are both continuously differentiable and satisfy

$$L_e\,\eta_0 = a\big(\eta_0(0) - \eta_0(-1)\big) = 0 = \eta_0'(0)$$

and in view of Equation (4.4)

$$L_e\,\eta_\kappa = a\big(\eta_\kappa(0) - \eta_\kappa(-1)\big) = a\big(1 - e^{-\kappa}\big) = \kappa = \eta_\kappa'(0),$$

respectively. Thus $\eta_0, \eta_\kappa \in \mathcal{D}(G_e)$. Furthermore,

$$G_e\,\eta_0 = \eta_0' = 0$$

yields $C_c = \mathbb{R}\,\eta_0$, and

$$G_e\,\eta_\kappa = \eta_\kappa' = \kappa\,\eta_\kappa$$

implies that the realified generalized eigenspace of G_e associated with the real eigenvalue κ is given by $\mathbb{R}\,\eta_\kappa$. Accordingly, we conclude $M = \mathbb{R}\,\eta_0 + \mathbb{R}\,\eta_\kappa$ for $a \neq 1$, and $C_u = \mathbb{R}\,\eta_\kappa$ in case $a > 1$.

Analogously, for $a = 1$ trivial verifications show that $\eta_0, \eta_d \in \mathcal{D}(G_e)$, $G_e\,\eta_0 = 0$ and $G_e\,\eta_d = \eta_0$, which imply $C_c = \mathbb{R}\,\eta_0 \oplus \mathbb{R}\,\eta_d$. Since $\lambda = 0$ is the only real eigenvalue of G_e, we also conclude $M = C_c$, and the proof is complete. □

Each initial value in the subspace M of C uniquely defines a global solution of Equation (4.3) which is either constant or strictly monotone and thus in particular slowly oscillating, as we show below.

COROLLARY 4.4. *Let $a > 0$ be given.*

(i) *For all $\varphi \in M$ Equation (4.3) has a unique solution $z^\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, and in the case $\varphi \neq 0$ the solution z^φ is slowly oscillating; that is, for every pair of zeros $z_1 > z_2$ of z^φ there holds $z_1 > z_2 + 1$.*

(ii) *If $a \leq 1$ and if $z : (-\infty, 0] \longrightarrow \mathbb{R}$ is a solution of Equation (4.3) such that $\|z_t\|_C \leq \|z_0\|_C$ for all $t \leq 0$, then there exists a constant $c \in \mathbb{R}$ with $z(t) = c$ as $t \leq 0$.*

(iii) *If $a > 1$ and if $z : (-\infty, 0] \longrightarrow \mathbb{R}$ is a solution of Equation (4.3) with $\|z_t\|_C \leq \|z_0\|_C$ for all $t \leq 0$ then $z_t \in M$ as $t \leq 0$.*

(iv) *If $a > 1$, and if $z : (-\infty, 0] \longrightarrow \mathbb{R}$ is a solution of Equation (4.3) such that there is a constant $\varepsilon > 0$ with $\|z_t\|_C \leq e^{(\kappa+\varepsilon)t}\|z_0\|_C$ for all $t \leq 0$, then $z(t) = 0$ as $t \leq 0$.*

PROOF. 1. *Proof of (i).* The first point of the assertion concerning the existence of a uniquely determined solution of Equation (4.3) for every initial data in M is, in consideration of the last corollary, an immediate consequence of the basic theory for RFDEs as for instance can be found in the standard work of Hale and Verduyn Lunel [**10**] or of Diekmann et al. [**6**]. For the remaining part, we note that in case $\varphi = 0$ we have $z^\varphi(t) = 0$, $t \in \mathbb{R}$, whereas for a non-trivial initial value $\varphi \in M$ we find constants $c_1, c_2 \in \mathbb{R}$ so that

$$\varphi = \begin{cases} c_1\,\eta_0 + c_2\,\eta_\kappa, & \text{for } a \neq 1, \\ c_1\,\eta_0 + c_2\,\eta_d, & \text{for } a = 1, \end{cases}$$

and the associated solution of Equation (4.3) is given by

$$z^\varphi(t) = \begin{cases} c_1 + c_2\,e^{\kappa\,t}, & \text{for } a \neq 1, \\ c_1 + c_2\,t, & \text{for } a = 1, \end{cases} \tag{4.9}$$

on $\mathbb{R}$. Each of these solutions is either constant or strictly monotonous. In particular, for $\varphi \neq 0$ the solution z^φ has clearly at most one zero and, thus, is slowly oscillating.

2. *Proof of (ii).* According to Proposition 4.2, we find a constant $\beta < 0$ such that the set $\Lambda(\beta) := \{\lambda \in \sigma(G_e) |\ \mathrm{Re}(\lambda) > \beta\}$ contains only the eigenvalue $\lambda = 0$ of G_e. Subsequently, there is, compare Diekmann et al. [**6**, Theorem 2.9 in Chapter IV.2], a complementary subspace Q for C_c in C and reals $0 < \delta < -\beta$ and $\widehat{K} \geq 1$ with $C = C_c \oplus Q$ and

$$\|T_e(t)\,\psi\|_C \leq \widehat{K}\,e^{(\beta+\delta)\,t}\|\psi\|_C \tag{4.10}$$

for all $t \geq 0$ and $\psi \in Q$. Recall thereby that the sets C_c and Q are both invariant under the solution semigroup T_e of Equation (4.3). Let now $P : C \longrightarrow C$ denote the projection operator of C along Q onto the center space C_c of G_e. Then for all $s \leq t \leq 0$ we have

$$\begin{aligned}\|(\mathbb{1} - P)\, z_t\|_C &= \|(\mathbb{1} - P)\, T_e(t-s)\, z_s\|_C \\ &= \|T_e(t-s)\,(\mathbb{1} - P)\, z_s\|_C \\ &\leq \widehat{K}\, e^{(\beta+\delta)(t-s)}\|(\mathbb{1} - P)\, z_s\|_C \\ &\leq \widehat{K}\, e^{(\beta+\delta)(t-s)}\|\mathbb{1} - P\|\, \|z_s\|_C \\ &\leq \widehat{K}\, e^{(\beta+\delta)(t-s)}\|\mathbb{1} - P\|\, \|z_0\|_C,\end{aligned}$$

and thus $\|(\mathbb{1} - P)\, z_t\|_C \to 0$ as $s \to -\infty$. Therefore $z_t \in C_c$ for all $t \leq 0$.

If $a < 1$, then part (i) of Corollary 4.3 implies $C_c = \mathbb{R}\, \eta_0$. Hence, we find a constant $c \in \mathbb{R}$ with $z_0 = c\, \eta_0$ and so obviously $z(t) = c$ for all $t \leq 0$.

On the other hand, in the case $a = 1$, we have $C_c = \mathbb{R}\, \eta_0 \oplus \mathbb{R}\, \eta_d$ due to statement (ii) of Corollary 4.3. This implies the existence of $c_1, c_2 \in \mathbb{R}$ with $z_0 = c_1\, \eta_0 + c_2\, \eta_d$ and, of course, $z(t) = c_1 + c_2\, t$ for all $t \leq 0$. Accordingly, it remains to show that $c_2 = 0$. But this is clear, since in the case $c_2 \neq 0$ we immediately find a real $t \leq 0$ with $\|z_t\|_C > \|z_0\|_C$ in contradiction to our assumptions on z. On that account, $z(t) = c_1$ for all $t \leq 0$, which is the desired conclusion.

3. *Proof of (iii).* The proof is completely analogous to the first part of the last one. Under the given assumptions we first find $\beta < 0$ with $\Lambda(\beta) = \{0, \kappa\}$ and then a subspace Q for M in C together with reals $0 < \delta < -\beta$ and $\widehat{K} \geq 1$ such that $C = M \oplus Q$ and

$$\|T_e(t)\, \psi\|_C \leq \widehat{K}\, e^{(\beta+\delta)\, t}\|\psi\|_C$$

for $\psi \in Q$ and $t \geq 0$. Then the same argumentation as we used in the proof of (ii) leads to $z_t \in M$ as $t \leq 0$.

4. *Proof of (iv).* If $a > 1$, then the spectral bound $\sup\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(G_e)\}$ of the infinitesimal generator G_e of the strongly continuous semigroup T_e is given by the real eigenvalue $\kappa > 0$. Therefore, for $0 < \delta < \varepsilon$ there is a constant $\widehat{K} \geq 1$ such that

$$\|T_e(t)\, \psi\|_C \leq \widehat{K}\, e^{(\kappa+\delta)\, t}\|\psi\|_C$$

for all $\psi \in C$ and all $t \geq 0$, as, for instance, shown in Diekmann et al. [**6**, Sections IV.2 and IV.3]. Subsequently, using the assumed estimate for z_t in combination with the above for T_e, we obtain

$$\begin{aligned}\|z_0\|_C = \|T_e(-t)\, z_t\|_C &\leq \widehat{K}\, e^{(\kappa+\delta)(-t)}\|z_t\|_C \\ &\leq \widehat{K}\, e^{(\kappa+\delta)(-t)} e^{(\kappa+\varepsilon)t}\|z_0\|_C = \widehat{K}\, e^{(\varepsilon-\delta)t}\|z_0\|_C\end{aligned}$$

for all $t \leq 0$. Choosing $t \leq 0$ small enough, namely such that $\widehat{K}\, e^{(\varepsilon-\delta)t} < 1$, yields $\|z_0\|_C = 0$, and thus $z(t) = 0$ for all $t \leq 0$ as claimed. ☐

The parts (ii) and (iii) of the above corollary particularly imply that in case of a solution $z : (-\infty, 0] \longrightarrow \mathbb{R}$ for the linear DDE (4.3), which satisfies $\|z_t\|_C \leq \|z_0\|_C$ for all $t \leq 0$, there are constants $\alpha, \beta \in \mathbb{R}$ with

$$z_t = T_e(t)\big(\alpha\, \eta_0 + \beta\, \eta_\kappa\big);$$

that is, $z(t) = \alpha + \beta\, e^{\kappa\, t}$ for all $t \le 0$. In the next result we prove that in this situation each segment z_t belongs to the subset $H_{[-1,0]}$ of C^1 defined by Equation (3.10) in the last chapter.

COROLLARY 4.5. *For given* $(\alpha,\beta) \in \mathbb{R}^2 \setminus \{0\}$ *and* $c \in \mathbb{R}$ *with* $c > 0$ *let the function* $z : (-\infty, 0] \to \mathbb{R}$ *be defined by*

$$z(t) = \alpha + \beta\, e^{c\, t}.$$

Then $V(z, [t-1, t]) = 1$ *and* $z|_{[t-1,t]} \in H_{[t-1,t]}$ *for all* $t \le 0$.

PROOF. For each choice of reals $\alpha, \beta, c \in \mathbb{R}$ with $\alpha, \beta \ne 0$ and $c > 0$, the induced function z is obviously continuously differentiable and either constant or strictly monotonous. Subsequently, z has at most one sign change on the negative real semi-axis. Accordingly, $V(z, [t-1, t]) = 1$ for all $t \le 0$.

It remains to verify that for each $t \le 0$ we have $z(t) \ne 0$ or $z(t-1) \cdot z'(t) < 0$ and also $z(t-1) \ne 0$ or $z(t) \cdot z'(t-1) > 0$. To see this, we first observe that for $\beta = 0$ this is clear since z is constant with a value $\alpha \ne 0$. Consequently, consider $\beta \ne 0$ in the following. Then z is strictly monotonous, and therefore we need only consider the subcases $z(t) = 0$ or $z(t-1) = 0$. In the situation $z(t) = 0$, we have $\alpha = -\beta\, e^{c\, t}$ and $z(t-1) \ne 0$ from the monotonicity property of z. Additionally, we see

$$\begin{aligned} z'(t)\, z(t-1) &= c\, \beta\, e^{ct} \Big(\alpha + \beta\, e^{c\,(t-1)} \Big) \\ &= -c\, \alpha \Big(\alpha - \alpha\, e^{-c} \Big) \\ &= -c\, \alpha^2 \Big(1 - e^{-c} \Big) < 0 \end{aligned}$$

as $c > 0$. On the other hand, $z(t-1) = 0$ implies $\alpha = -\beta\, e^{c\,(t-1)}$ and also $z(t) \ne 0$, which leads to

$$\begin{aligned} z'(t-1)\, z(t) &= c\, \beta\, e^{c\,(t-1)} \Big(\alpha + \beta\, e^{c\, t} \Big) \\ &= -c\, \alpha \Big(\alpha - \alpha\, e^{c} \Big) \\ &= -c\, \alpha^2 \Big(1 - e^{c} \Big) > 0. \end{aligned}$$

Thus, both situations result in $z|_{[t-1,t]} \in H|_{[t-1,t]}$. This completes the proof. $\square$

Another important consequence of Proposition 4.2 are the stability properties of the trivial solution for the linear RFDE (4.3) in dependence of the parameter $a > 0$. Even though it is a standard conclusion from the theory of linear RFDEs as, for instance, is treated in Diekmann et al. [**6**], we shall explicitly show it using the above results and ideas.

COROLLARY 4.6 (Corollary 4.2 in Brunovský et al. [**3**]). *For* $0 < a < 1$ *the trivial solution of the linear autonomous RFDE* (4.3) *is stable, while for* $a \ge 1$ *it is unstable.*

PROOF. If $a \ge 1$, then due to Equation (4.9) each neighborhood of the origin in C contains a non-trivial initial function $\varphi \in M$ such that the associated solution z^{φ}, or, more precisely, its trajectory $[0, \infty) \ni t \longmapsto z_t^{\varphi} \in C$, is unbounded. Therefore, the trivial solution is unstable.

On the other hand, suppose $0 < a < 1$ and consider the spectral decomposition $C = C_c \oplus Q$ together with the associated estimate (4.10) for the action of the solution semigroup T_e of (4.3) on Q. Writing $P : C \longrightarrow C$ for the projection of C along Q onto the center space C_c and using the invariance of the subspaces C_c, Q for T_e together with $T_e(t)\,\eta_0 = \eta_0$, we obtain

$$\begin{aligned}\|z_t^\varphi\|_C &= \|T_e(t)\,\varphi\|_C \\ &\le \|T_e(t)\,P\,\varphi\|_C + \|T_e(t)\,(\mathbb{1}-P)\,\varphi\|_C \\ &\le \|P\,\varphi\|_C + \widehat{K}\,e^{(\beta+\delta)\,t}\|(\mathbb{1}-P)\,\varphi\|_C \\ &\le \big(\|P\| + \widehat{K}\,\|\mathbb{1}-P\|\,e^{(\beta+\delta)\,t}\big)\,\|\varphi\|_C \\ &\le \big(\|P\| + \widehat{K}\,\|\mathbb{1}-P\|\big)\,\|\varphi\|_C\end{aligned}$$

for all $\varphi \in C$ and $t \ge 0$. Clearly, this implies the stability of the zero solution. □

In our next result we show that initial values $\varphi \in C$ vanishing at the boundary points but not inside of $[0,1]$ lead to solutions of Equation (4.3) on $[-1,\infty)$ which stay bounded away from zero in $[1,\infty)$.

PROPOSITION 4.7. *Suppose that $a > 0$, and let $\varphi \in C$ satisfy $\varphi(-1) = \varphi(0) = 0$ and $\varphi(t) \ne 0$ as $-1 < t < 0$. Then φ admits a unique solution $z^\varphi : [-1,\infty) \longrightarrow \mathbb{R}$ of Equation (4.3) with $z_0^\varphi = \varphi$ and there is a constant $c > 0$ with $|z^\varphi(t)| \ge c$ for all $t \ge 1$.*

PROOF. 1. Similarly to the proof of Corollary 4.4, the existence of a uniquely determined solution $z^\varphi : [-1,\infty) \longrightarrow \mathbb{R}$ of the linear RFDE (4.3) for a given initial value $\varphi \in C$ follows by the basic existence theory for RFDEs as presented in Hale and Verduyn Lunel [**10**] or in Diekmann et al. [**6**]. However, in view of the second part of the assertion it is appropriate to derive the solution z^φ at least in $[-1,1]$ by an alternate approach, the so-called *method of steps*, here.

For this purpose, note that for given $\varphi \in C$ Equation (4.3) reads

$$\dot{z}(t) = L_e z_t = a\big[z(t) - z(t-1)\big] = a\big[z(t) - \varphi(t-1)\big]$$

for all $t \in [0,1]$. Hence, it forms an ODE. By the variation-of-constants formula for ODEs as can be found in Hale [**8**], it follows that for each $c \in \mathbb{R}$

$$z_c(t) := c\,e^{at} - a\int_0^t e^{a(t-s)}\,\varphi(s-1)\,ds$$

with $0 \le t \le 1$ is a solution of the above ODE for z. Using the continuity condition $z^\varphi(0) = \varphi(0)$, we see that z^φ is given by

$$z^\varphi(t) = \begin{cases}\varphi(t), & -1 \le t \le 0 \\ \varphi(0)\,e^{at} - a\displaystyle\int_0^t e^{a(t-s)}\,\varphi(s-1)\,ds, & 0 \le t \le 1\end{cases}$$

for $-1 \le t \le 1$.

2. By assumption, we have $\varphi(-1) = \varphi(0) = 0$ and $\varphi(t) \ne 0$ as $t \in (-1,0)$. Hence, either $\varphi(t) > 0$ for all $-1 < t < 0$ or $\varphi(t) < 0$ for all $-1 < t < 0$. Consider the situation $\varphi(t) > 0$ as $-1 < t < 0$ first. We claim

$$z^\varphi(t) \le z^\varphi(1) < 0$$

for all $t \ge 1$. In order to see this, observe that z^φ is negative and strictly decreasing in $(0,1]$ since

$$z^\varphi(t) = -a\int_0^t e^{a(t-s)}\,\varphi(s-1)\,ds$$

as $0 \leq t \leq 1$ and $\varphi(s-1) > 0$ for all $0 < s < 1$. Consider now the interval $[1,2]$. By the above, for $1 \leq t \leq 2$ we have

$$a\big[y - z^\varphi(1)\big] \geq a\big[y - z^\varphi(t-1)\big] \geq ay$$

for all $y \in \mathbb{R}$. Consequently,

$$\dot{y}^1(t) \geq \dot{z}^\varphi(t)$$

and

$$y^1(t) \geq z^\varphi(t)$$

follow for all $t \in [1,2]$ where $y^1 : [1,2] \longrightarrow \mathbb{R}$ denotes the unique solution of the initial value problem

$$\begin{cases} \dot{y}(t) = a\,y(t) - a\,z^\varphi(1) \\ y(1) = z^\varphi(1) \end{cases}$$

on $[1,2]$. A simple calculation shows

$$y^1 : [1,2] \ni t \longmapsto z^\varphi(1) \in \mathbb{R}$$

and therefore

$$z^\varphi(t) \leq y^1(t) = z^\varphi(1) < 0$$

for all $t \in [1,2]$. In particular, the solution z^φ of Equation (4.3) with initial value $z_0^\varphi = \varphi$ is also monotone decreasing in the interval $[1,2]$ since $\dot{z}^\varphi(t) \leq \dot{y}^1(t) = 0$ as $1 \leq t \leq 2$. This enables us to proceed by induction. Indeed, assume for $k \in \mathbb{N}$ that the solution z^φ of (4.3) satisfies $z^\varphi(t) \leq z^\varphi(1) < 0$ as $t \in [k, k+1]$ and that z^φ is monotone decreasing in $[k, k+1]$. Then it is possible to conclude

$$a\big[y - z^\varphi(k+1)\big] \geq a\big[y - z^\varphi(t-1)\big]$$

for all $t \in [k+1, k+2]$ and all $y \in \mathbb{R}$. Hence,

$$z^\varphi(t) \leq y^{k+1}(t) \leq y^{k+1}(k+1) \leq z^\varphi(1) < 0$$

and

$$\dot{z}^\varphi(t) \leq 0 = \dot{y}^{k+1}(t)$$

for all $t \in [k+1, k+2]$ where

$$y^{k+1} : [k+1, k+2] \ni t \longmapsto z^\varphi(k+1) \in \mathbb{R}$$

is the uniquely determined solution of

$$\begin{cases} \dot{y}(t) = a\,y(t) - a\,z^\varphi(k+1) \\ y(k+1) = z^\varphi(k+1) \end{cases}$$

on $[k+1, k+2]$. This proves the proposition under the assumption $\varphi(t) > 0$ for all $-1 < t < 0$.

In the situation $\varphi(-1) = \varphi(0) = 0$ and $\varphi(t) < 0$, the above arguments applied to $-\varphi$ show $z^{-\varphi}(t) \leq z^{-\varphi}(1) < 0$ for all $t \geq 0$. By linearity of Equation (4.3), it follows that $z^\varphi = -z^{-\varphi}$ and thus

$$z^\varphi(t) = -z^{-\varphi}(t) \geq -z^{-\varphi}(1) > 0$$

for all $t \geq 1$. This completes the proof of the proposition. $\square$

Projection on the Subspace M. We close this section with the derivation of a projection operator on the two-dimensional subspace M of C^1 from Corollary 4.3 to describe the dynamics of trajectories for the semiflow F. Thereto, we consider the closed hyperplanes

$$H_{-1} := \Big\{ \varphi \in C^1 \,\Big|\, \varphi(-1) = 0 \Big\}$$

and

$$H_0 := \Big\{ \varphi \in C^1 \,\Big|\, \varphi(0) = 0 \Big\}$$

of C^1 and prove that their intersection is a complementary space of M in C^1.

COROLLARY 4.8. *Let N denote the closed subspace $H_{-1} \cap H_0$ of the Banach space C^1. Then* $\operatorname{codim} N = 2$ *and* $C^1 = N \oplus M$.

PROOF. The first part of the assertion, that is, $\operatorname{codim} N = 2$, is clear, whereas for the second one observes that each $\varphi \in M \backslash \{0\}$ has at most one zero due to statement (i) of Corollary 4.4 and hence, $\varphi \notin N$. In particular, $M \cap N = \{0\}$. As $\dim M = 2 = \operatorname{codim} N$ we conclude that M is a complementary subspace of N in C^1 as claimed. □

By the form of the functions in M, there is a function $\Phi_0 \in (H_0 \cap M) \backslash H_{-1}$ with $\Phi_0(-1) = 1$ and analogously a function $\Phi_{-1} \in (H_{-1} \cap M) \backslash H_0$ with $\Phi_{-1}(0) = 1$. As Φ_0 and Φ_{-1} are linearly independent, they form a basis of the subspace M of C^1. The projection of C^1 onto M along $H_{-1} \cap H_0$ can be represented in terms of this basis as follows.

COROLLARY 4.9. *If $P : C^1 \longrightarrow C^1$ denotes the continuous projection of C^1 onto M, then*

$$P\,\varphi = \varphi(-1)\,\Phi_0 + \varphi(0)\,\Phi_{-1}$$

for all $\varphi \in C^1$.

PROOF. Let $\varphi \in C^1$ be given. Due to $C^1 = M \oplus N$ we find an element $\psi \in N$ with $\varphi = P\varphi + \psi$. Therefore $\varphi(0) = (P\,\varphi)(0) + \psi(0) = (P\,\varphi)(0)$ and analogously $\varphi(-1) = (P\,\varphi)(-1) + \psi(-1) = (P\,\varphi)(-1)$. Now we have

$$P\,\varphi = c_0\,\Phi_0 + c_{-1}\,\Phi_{-1}$$

with reals $c_0, c_{-1} \in \mathbb{R}$ and hence,

$$(P\,\varphi)(0) = c_0 \cdot 0 + c_{-1} \cdot 1 = c_{-1}$$

and

$$(P\,\varphi)(-1) = c_0 \cdot 1 + c_{-1} \cdot 0 = c_0.$$

This establishes the asserted formula for $P\,\varphi$. □

In case $a > 1$ the elements of the kernel of the above projection P, that is, the linear space of all $\varphi \in C^1$ with $P\,\varphi = 0$, has at least three zeros as proved in the next corollary.

COROLLARY 4.10. *Suppose that $a > 1$ and that $\varphi \in C^1 \setminus \{0\}$ satisfies $P\,\varphi = 0$ where P is the projection of C^1 onto M from the last corollary. Then the function φ has a zero in $(-1, 0)$.*

PROOF. Assume the assertion is false; that is, φ is either positive in $(-1,0)$ or negative in $(-1,0)$. However, define $\psi(t) := |\varphi(t)|$ for all $t \in [-1,0]$. Then $\psi \in C^1 \setminus \{0\}$, $P\,\psi = 0$ and $\psi(t) > 0$ for all $-1 < t < 0$ by assumption. Thus, from Proposition 4.7 we see that ψ admits a unique solution $z^\psi : [-1,\infty) \longrightarrow \mathbb{R}$ of the linear RFDE (4.3) with initial value $z_0^\psi = \psi$ and that there is a constant $c > 0$ with $|z^\psi(t)| \geq c$ for all $t \geq 1$. In particular, $\|z_t^\psi\|_C \geq c$ as $t \geq 2$.

On the other hand, Corollary 4.3 implies that M coincides with the center-unstable space $C_u \oplus C_c$ of G_e as $a > 1$. Consequently, from $P\,\psi = 0$ it follows $\psi \in C_s$ where $C_s \subset C$ denotes the stable space of G_e. Now, due to the exponential trichotomy property we find real numbers $K \geq 1$ and $c_s < 0$ with

$$\|z_t^\psi\|_C = \|T_e(t)\,\psi\|_C \leq K e^{c_s t}$$

for all $t \geq 0$. Hence, we have $\|z_t^\psi\|_C \to 0$ as $t \to \infty$ and herewith a contradiction to the above conclusion $\|z_t^\psi\|_C \geq c > 0$ for all $t \geq 2$. This shows that the function $\varphi \in C^1 \setminus \{0\}$ is neither positive nor negative in $(-1,0)$. Therefore, in any case there is a real $-1 < \xi < 0$ with $\varphi(\xi) = 0$ as claimed. $\square$

3. The Stability of the Trivial Solution

After having clarified some of the spectral properties of the linearization of the semiflow F at the trivial solution, we now discuss the consequences for the local behavior of F at $\varphi_0 = 0$. In this context, an interesting question is the one on stability of the trivial stationary point in dependence of the parameter $a > 0$. Below we address this issue, even if we will not be able to solve this question for all $a > 0$.

In the remainder of this chapter we consider Equation (4.1) represented by

$$\dot{x}(t) = L\,x_t + g(x_t) \tag{4.11}$$

with the derivative $L = Df(0)$ of f at $0 \in X_f$ and the non-linear part

$$g : C^1 \ni \varphi \longmapsto f(\varphi) - L\,\varphi \in \mathbb{R}, \tag{4.12}$$

which obviously inherits the smoothness conditions (S 1) and (S 2) of f on Page 7.

An Instability Result. Our stability analysis of the semiflow F begins with an immediate consequence of the principle of linearized instability in view of the spectrum of the linearization.

COROLLARY 4.11. *For $a > 1$ the trivial solution of Equation* (4.1) *is unstable.*

PROOF. As the spectrum $\sigma(G_e)$ contains an element with positive real part, namely $\lambda = \kappa$, as proved in Proposition 4.2, the assertion follows by the principle of linearized instability from Proposition 1.4. $\square$

Geometrically speaking, the assertion of the corollary above is that in case $a > 1$ there is a neighborhood $U \subset X_f$ of the stationary point $\varphi_0 = 0$, which contains for every $\varepsilon > 0$ at least one initial value $\varphi_\varepsilon \in U$ with $\|\varphi_\varepsilon\|_{C^1} < \varepsilon$ such that the associated trajectory $[0,\infty) \ni t \longmapsto x_t^{\varphi_\varepsilon}$ for the semiflow F leaves U in finite time. This fact can also be concluded from the existence of a so-called *local unstable manifold* of Equation (4.1) at $0 \in X_f$. We show the existence of such manifolds for Equation (4.11) in the case $a > 1$ by our next result, in which we use the notation C^1_{cs} for the Banach space $C_c \oplus C^1_s$ where C^1_s is the intersection of C^1 with the stable space C_s of G_e.

PROPOSITION 4.12. *If $a > 1$, then there are open neighborhoods $C_{u,0}$ of 0 in C_u and $C_{cs,0}$ of 0 in C^1_{cs}, constants $0 < \nu < \kappa$ and $K_u > 0$, and a continuously differentiable map $w_u : C_{u,0} \longrightarrow C^1_{cs,0}$ with $w_u(0) = 0$ and $Dw_u(0) = 0$, such that the following holds.*

(i) *The graph of w_u, that is, the subset*
$$W_u := \Big\{\psi + w_u(\psi) \big| \, \psi \in C_{u,0}\Big\}$$
of C^1, belongs to X_f.

(ii) *For each $\varphi \in W_u$ there exists a unique solution $x^\varphi : (-\infty, 0] \longrightarrow \mathbb{R}$ of Equation (4.1) with $x^\varphi_0 = \varphi$ and $\|x^\varphi_t\|_{C^1} \leq K_u \, e^{\nu\, t}$ for all $t \leq 0$.*

(iii) *If $\varphi \in X_f$ with $P_u\, \varphi \in C_{u,0}$ and if there is a solution $x^\varphi : (-\infty, 0] \longrightarrow \mathbb{R}$ of Equation (4.1) with $\|x^\varphi_t\|_{C^1} \leq K_u \, e^{\nu\, t}$ for all $t \leq 0$, then $\varphi \in W_u$.*

(iv) *W_u is negative invariant in the sense, that there is an open neighborhood $N_u \subset X_f$ of $0 \in X_f$ such that $\varphi \in W_u \cap N_u$ implies $x^\varphi_t \in W_u$ for all $t \leq 0$.*

PROOF. The statement is a straightforward consequence of Theorem 4.1 in Krisztin [**15**] on the existence of local unstable manifolds for DDEs with state-dependent delay, provided Equation (4.11) satisfies the required assumptions. We verify these assumption in the following. On that account, consider the continuation of Equation (4.11) to C, that is, the RFDE
$$\dot{x}(t) = L_e\, x_t + g_e(x_t)$$
with the linear extension $L_e = D_e f(0) \in \mathcal{L}(C, \mathbb{R})$ of L from assumption (S 2) on f and the extension $g_e = f_e - L_e$ of g, where f_e is defined by Equation (3.3). By Corollary 3.2, it follows that f_e is continuous and so is g_e as a sum of continuous functions. Further, we have $g_e(0) = 0$, and the restriction of g_e to C^1, that is simply g, satisfies the smoothness conditions (S 1) and (S 2) with
$$Dg(0) = D(f - L)(0) = Df(0) - L = 0.$$
Finally, for any choice of constant $0 < \beta < \kappa$ Proposition 4.2 on the spectrum of G_e also yields the last condition (H1') in Krisztin [**15**] and enables the application of Theorem 4.1 in [**15**] to conclude the stated result. □

The set $W_u \subset X_f$ is called a **local unstable manifold** of Equation (4.1) at the stationary point $\varphi_0 = 0$. For all $a > 1$ it is a one-dimensional continuously differentiable submanifold of X_f. In view of property (ii) it becomes now clear that choosing any non-trivial $\varphi \in W_u$ we find a initial value ψ in any open ball $B_\varepsilon(0) \subset C^1$ with center $0 \in C^1$ and radius $0 < \varepsilon < \|\varphi\|_{C^1}$, such that the associated trajectory $t \longmapsto x^\psi_t \in X_f$ reaches φ and thus in particular leaves $B_\varepsilon(0)$ in finite time. Consequently, this shows once more the instability of the trivial solution in case $a > 1$.

The Center Manifold Reduction for a $\neq$ 1. In contrast to the case $a > 1$, in the situation $0 < a \leq 1$ the question on stability of the trivial solution of Equation (4.1), or equivalently of (4.11), is not solved by the stability of the trivial solution of the associated linear RFDE (4.3). Indeed, the spectrum $\sigma(G_e)$ contains a zero eigenvalue of L_e for all parameter $a > 0$ and therefore we may not apply the principle of linearized stability. For the model equation (3.2) with constant delay Brunovský et al. [**3**] solve the question on the stability by employing the classical technique, namely the so-called *center manifold reduction*, to study stability of differential equations in such a situation. The general idea of this method is to reduce the local semiflow at the stationary point to a *local center manifold*, where the restriction induces a local flow generated by an ODE and the original stationary point becomes a stationary point of this flow. Under the condition that the local center manifolds are attracting in absence of unstable directions, such a reduction allows to draw conclusions about the stability property of the stationary point for the original equation from the one of this ODE. For Equation (3.2) this approach leads to the result on local asymptotical stability of the trivial solution for $0 < a < 1$, as proved in Brunovský et al. [**3**]. One may ask whether this result is still true in case of Equation (4.1) with a state-dependent delay satisfying the assumptions (DF 1) - (DF 5).

Below we prove that the center manifold reduction of Equation (4.1), or equivalently of Equation (4.11), induces the same ODE as in case of DDE (3.2). Nevertheless, we will not be able to deduce from this fact the stability of the trivial solution for $0 < a < 1$. The reason for this is that, as far as we know, there is no result stating the attractivity of local center manifolds in absence of unstable directions for a semiflow obtained from Theorem 1.2, and an attempt for a proof of such a result exceeds the scope of this work. However, via the center manifold reduction we will see an interesting qualitative behavior of the semiflow F near the trivial solution for parameters $a > 1$. As this behavior will be crucial for our considerations of Equation (4.1), we carry out the center manifold reduction in detail.

Before we begin, let us recall the definition of the Landau symbol o characterizing the asymptotical behavior of functions. We write $h_1 = o(h_2)$ as $s \to 0$ for any given continuous functions $h_1, h_2 : U \longrightarrow \mathbb{R}$, defined on some open neighborhood U of 0 in a Banach space with norm $\|\cdot\|$, if and only if for each $c > 0$ there is a constant $\delta > 0$ such that for all $s \in U$ with $\|s\| < \delta$ we have $|h_1(s)| \leq c\,|h_2(s)|$.

We begin our discussion with a result about the existence and C^1-smoothness of local center manifolds for Equation (4.1) at the trivial solution.

PROPOSITION 4.13. *For all $a > 0$ there exist open neighborhoods $C_{c,0}$ of 0 in C_c and $C^1_{su,0}$ in $C^1_s \oplus C_u$ with $N_c = C_{c,0} + C^1_{su,0} \subset C^1$ and a continuously differentiable map $w_c : C_{c,0} \longrightarrow C^1_{su,0}$ satisfying $w_c(0) = 0$ and $Dw_c(0) = 0$ with the following properties.*

(i) *The graph*

$$W_c := \Big\{ \psi + w_c(\psi) \,\Big|\, \psi \in C_{c,0} \Big\}$$

of w_c belongs to X_f and is a continuously differentiable submanifold of X_f with the same dimension as C_c.

(ii) *If $x : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuously differentiable solution of Equation (4.1) on $\mathbb{R}$ with $x_t \in N_c$ for all $t \in \mathbb{R}$, then $x_t \in W_c$ for all $t \in \mathbb{R}$.*

(iii) *W_c is locally positively invariant with respect to the semiflow F, that is, if $\varphi \in W_c$ and $\alpha > 0$ such that $F(t,\varphi)$ is defined for all $t \in [0,\alpha)$ and $F(t,\varphi) \in N_c$ for all $t \in [0,\alpha)$, then $F(t,\varphi) \in W_c$ for all $t \in [0,\alpha)$.*

PROOF. Apply Theorem 4.1.1 in Hartung et al. [**11**] on the existence and Theorem 2.1 in Krisztin [**16**] on continuous differentiability of local center manifolds for differential equations with state-dependent delay. □

The graph W_c in the former result is called a **local center manifold** of Equation (4.1) at $\varphi_0 = 0$. As Proposition 4.2 implies $\dim C_c = 1$ for parameter $0 < a \neq 1$ and on the other hand $\dim C_c = 2$ for $a = 1$, W_c is a one-dimensional continuously differentiable submanifold of X_f in the first situation and a two-dimensional submanifold in the second one. In any case, the restriction of the non-linear part g of Equation (4.1) defined by (4.12) has the following asymptotical behavior in close vicinity of the trivial solution.

COROLLARY 4.14. *For each $a > 0$ the restriction of the non-linear continuous differentiable functional*

$$g : C^1 \ni \varphi \longmapsto a\big[\varphi(-1) - \varphi(-r(\varphi(0)))\big] - |\varphi(0)|\,\varphi(0) \in \mathbb{R}$$

to a local center manifold W_c satisfies the asymptotic formula

$$g(\psi + w_c(\psi)) = -|\psi(0)|\,\psi(0) + o(\|\psi\|^2_{C^1})$$

as $C_{u,0} \ni \psi \to 0$.

PROOF. 1. First consider the linear part of the function g for $\varphi \in C^1$, that is, the difference $a\big[\varphi(-1) - \varphi(-r(\varphi(0)))\big]$. The mean value theorem yields the existence of $0 \leq \xi \leq |\varphi(0)|$ with

$$\begin{aligned}
\varphi(-1) - \varphi(-r(\varphi(0))) &= \int_0^1 \varphi'\big(-s-(1-s)\cdot r(\varphi(0))\big)\,ds \cdot \Big(r(\varphi(0)) - 1\Big)\\
&= \int_0^1 \varphi'\big(-s-(1-s)\cdot r(\varphi(0))\big)\,ds \cdot \Big(r(\varphi(0)) - r(0)\Big)\\
&= \pm r'(\xi)\cdot\varphi(0)\cdot \int_0^1 \varphi'\big(-s-(1-s)\cdot r(\varphi(0))\big)\,ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
|\varphi(-1) - \varphi(-r(\varphi(0)))| &\leq \left|\max_{\xi\in[0,|\varphi(0)|]}|r'(\xi)|\cdot\varphi(0)\cdot\int_0^1\varphi'\big(-s-(1-s)\cdot r(\varphi(0))\big)\,ds\right|\\
&\leq \max_{\xi\in[0,\|\varphi\|_C]}|r'(\xi)|\cdot\|\varphi\|_C\cdot\int_0^1\big|\varphi'\big(-s-(1-s)\cdot r(\varphi(0))\big)\big|\,ds\\
&\leq \max_{\xi\in[0,\|\varphi\|_C]}|r'(\xi)|\cdot\|\varphi\|_C\cdot\int_0^1\|\varphi'\|_C\,ds\\
&= \max_{\xi\in[0,\|\varphi\|_C]}|r'(\xi)|\cdot\|\varphi\|_C\cdot\|\varphi'\|_C\\
&\leq k(\varphi)\cdot\|\varphi\|^2_{C^1}
\end{aligned}$$

where

$$k(\varphi) := \max_{\xi\in[0,\|\varphi\|_{C^1}]}|r'(\xi)|.$$

As the delay function is continuously differentiable and possesses a global maximum at the origin due to assumptions (DF 1) and (DF 2), the previous calculation implies

$$0 \leq \lim_{0\neq\varphi\to 0}\frac{|\varphi(-1) - \varphi(-r(\varphi(0)))|}{\|\varphi\|^2_{C^1}} \leq \lim_{0\neq\varphi\to 0}\frac{k(\varphi)\cdot\|\varphi\|^2_{C^1}}{\|\varphi\|^2_{C^1}} = \lim_{\varphi\to 0}k(\varphi) = 0,$$

that is,

$$\varphi(-1) - \varphi(-r(\varphi(0))) = o(\|\varphi\|^2_{C^1})$$

as $\varphi \to 0$.

2. Consider for arbitrary parameter $a > 0$ a local center manifold

$$W_c = \{\psi + w_c(\psi) \mid \psi \in C_{c,0}\}$$

of Equation (4.1) at the stationary point $\varphi_0 = 0$. Then for all sufficiently small $\psi \in C_{c,0}$ the mean value theorem shows

$$\|w_c(\psi)\|_{C^1} = \|w_c(\psi) - w_c(0)\| \leq \sup_{\chi \in S} \|Dw_c(\chi)\| \, \|\psi\|_{C^1}$$

where the strip S is defined by $\{\lambda\,\psi \mid \lambda \in [0,1]\}$. Combining the continuity of Dw_c with $Dw_c(0) = 0$ yields

$$w_c(\psi) = o(\|\psi\|_{C^1}) \tag{4.13}$$

as $\psi \to 0$. Next, by application of the mean value theorem in one dimension, we find for every sufficiently small $\psi \in C_{c,0}$ a constant $0 \leq \lambda \leq 1$ with

$$|(\psi + w_c(\psi))(0)|\,(\psi + w_c(\psi))(0) = |\psi(0)|\,\psi(0) + 2\,|\lambda\,\psi(0) + (1-\lambda)\,w_c(\psi)(0)|\,w_c(\psi)(0).$$

This leads to the inequality

$$\begin{aligned} |(\psi + w_c(\psi))(0)|(\psi + w_c(\psi))(0) &= |\psi(0)|\,\psi(0) + 2\,|\lambda\psi(0) + (1-\lambda)\,w_c(\psi)(0)|\,w_c(\psi)(0) \\ &\leq |\psi(0)|\,\psi(0) + 2\big(|\psi(0)| + |w_c(\psi)(0)|\big)\,|w_c(\psi)(0)| \\ &\leq |\psi(0)|\,\psi(0) + 2\big(\|\psi\|_{C^1} + \|w_c(\psi)\|_{C^1}\big)\|w_c(\psi)\|_{C^1}, \end{aligned}$$

and hence in view of Equation (4.13)

$$\big|\big(\psi + w_c(\psi)\big)(0)\big|\,\big(\psi + w_c(\psi)\big)(0) = |\psi(0)|\,\psi(0) + o(\|\psi\|_{C^1}^2)$$

as $\psi \to 0$.

3. Combining the last two parts, we immediately see

$$\begin{aligned} g(\psi + w_c(\psi)) = a\,\Big[\big(\psi + w_c(\psi)\big)(-1) - \big(\psi + w_c(\psi)\big)(-r((\psi + w_c(\psi))(0))\Big] \\ - |(\psi + w_c(\psi))(0)|\,\big(\psi + w_c(\psi)\big)(0) \\ = -|\psi(0)|\,\psi(0) + o(\|\psi\|_{C^1}^2) \end{aligned}$$

for $C_{u,0} \ni \psi \to 0$, which establishes the claimed formula. $\square$

Consider now a local center manifold W_c in the case $0 < a \neq 1$. Then, by Corollary 4.4, we have $C_c = \mathbb{R}\,\eta_0$ with the function $\eta_0 : [-1,0] \ni \vartheta \longmapsto 1 \in \mathbb{R}$ and, of course, $G_e\,\eta_0 = 0$. Applying similiar arguments to those in Section 6 of Chapter 2 on the dynamics on local center-unstable manifolds, we conclude that the local behavior of the semiflow F on W_c at the stationary point $\varphi_0 = 0 \in X_f$ is determined by the scalar first-order ODE

$$\dot{y}(t) = Q_c\big(y(t)\,\eta_0 + w_c(y(t)\,\eta_0)\big)\,, \tag{4.14}$$

where $Q_c : C_{c,0} \longrightarrow \mathbb{R}$ denotes the composition $Q_c = \Gamma_c \circ P_c^{\odot*} \circ l \circ g$ of the bounded linear operator $\Gamma_c : C_c \ni \varphi \longmapsto c(\varphi) \in \mathbb{R}$ associating $\varphi \in C_c$ with the uniquely determined real $c(\varphi)$ with $\varphi = c(\varphi)\,\eta_0$, the spectral projection $P_c^{\odot*}$ of $C^{\odot*}$ along $C_s^{\odot*} \oplus C_u$ onto the center space C_c, the linear bijective mapping $l : \mathbb{R} \longrightarrow Y^{\odot*}$ involved in the variation-of-constants formula (2.13) and finally the nonlinearity g of Equation (4.1) defined by (4.12). Observe that the right-hand side of this ODE is continuously differentiable and thus there is an open neighborhood $U \subset \mathbb{R}$ of $0 \in \mathbb{R}$ such that every $c \in U$ uniquely defines a solution $y : \mathbb{R} \supset (t_-(u), t_+(u)) \longrightarrow \mathbb{R}$ of Equation (4.14) with $y(0) = c$. In particular, these solutions generate a local flow with $y_0 = 0$ as a stationary point.

In order to obtain an explicit asymptotic formula for the right-hand side of the above ODE, we calculate the projection operator P_c of C along $C_s \oplus C_u$ onto C_c. To this end, let $\zeta : [0,1] \longrightarrow \mathbb{R}$ be defined by

$$\zeta(\vartheta) := \begin{cases} 0, & \vartheta \in \{0,1\}, \\ a, & \vartheta \in (0,1), \end{cases} \tag{4.15}$$

such that we have

$$L_e\,\varphi = \int_0^1 d\zeta(\vartheta)\,\varphi(-\vartheta)$$

for all $\varphi \in C$ with the continuous extension L_e of L on C given by (4.2). Following Chapter IV in Diekmann et al. [**6**] on spectral theory for operator semigroups, where a straightforward argument also shows the result of [**6**, Exercise 3.12 in Chapter IV.3], we obtain

$$\begin{aligned} P_c\,\varphi &= \frac{1}{1-a}\left(\varphi(0) + \int_0^1 d\zeta(\vartheta)\int_0^{\vartheta}\varphi(\sigma-\vartheta)\,d\sigma\right)\eta_0 \\ &= \frac{1}{1-a}\left(\varphi(0) - a\int_0^1 \varphi(\sigma-1)\,d\sigma\right)\eta_0 \\ &= \frac{1}{1-a}\left(\varphi(0) - a\int_{-1}^0 \varphi(-\sigma)\,d\sigma\right)\eta_0 \end{aligned} \tag{4.16}$$

for functions $\varphi \in C$ and so

$$(P_c^{\odot *}\circ l)(c) = \frac{c}{1-a}\,\eta_0 \tag{4.17}$$

for all $c \in \mathbb{R}$. Combining this with Corollary 4.14 leads to

$$\begin{aligned} Q_c(y\,\eta_0 + w_c(y\,\eta_0)) &= \big(\mathit{\Gamma}_c \circ P_c^{\odot *}\circ l\big)\big(g(y\,\eta_0 + w_c(y\,\eta_0))\big) \\ &= \big(\mathit{\Gamma}_c \circ P_c^{\odot *}\circ l\big)\big(-|y|\,y + o(|y|^2)\big) \\ &= -\frac{1}{1-a}\,|y|\,y + o\big(|y|^2\big) \end{aligned}$$

for all sufficiently small reals y. Consequently, in the case $0 < a \neq 1$ the center manifold reduction (4.14) of the semiflow F at the stationary point $\varphi_0 = 0$ has the asymptotic expansion

$$\dot{y}(t) = -\frac{1}{1-a}\,|y(t)|\,y(t) + o(|y(t)|^2) \tag{4.18}$$

in close vicinity of the stationary point $y_0 = 0$.

Remark 4.15. As already mentioned, the derived asymptotic form (4.18) of the center manifold reduction for the model equation (4.1) is just the same as in the case of Equation (3.2) with constant delay, which is discussed in Brunovský et al. [**3**] in detail and from which we also adopt the next result.

An important consequence of the obtained asymptotic expansion (4.18) is the answer on the question of the stability of the trivial solution for the differential equation (4.14).

Proposition 4.16. *For all* $0 < a < 1$ *the trivial solution of Equation* (4.14) *is locally asymptotically stable, whereas for* $a > 1$ *it is unstable.*

PROOF. As solutions of scalar first order ODEs are monotone functions, the behavior of solutions of the ODE (4.14) near the trivial solution is determined by the leading term of the asymptotic expansion (4.18). If $0 < a < 1$, then the coefficient of the leading term $|y(t)|\, y(t)$ of the asymptotic expansion (4.18) is negative and consequently the trivial stationary point is locally asymptotically stable. On the other hand, for $a > 1$ we have $-1/(1-a) > 0$ and thus the trivial stationary point is unstable. □

An analogous result for the situation of constant delay leads to the conclusion, compare Brunovský et al. [**3**, Corollary 5.1], that in the case $0 < a < 1$ the trivial solution of Equation (3.2) is asymptotically stable. But, unfortunately, we are not able to draw the same conclusion for the model equation (4.1) due to the lack of a result on the attractivity of local center manifolds for semiflows from Theorem 1.2, provided there is no unstable direction. On the other hand, observe that in the case $a > 1$, where the question of stability of the trivial solution for Equation (4.1) is clear according to Corollary 4.11, the asymptotic expansion (4.18) shows that the trivial solution is repelling in the local center manifold W_c at $\varphi_0 = 0$; that is, there is an open neighborhood U of $\varphi_0 = 0 \in X_f$ in W_c such that for every $\varphi \in U$ there exists a solution $x^\varphi : (-\infty, \infty) \longrightarrow \mathbb{R}$ of Equation (4.1) with $x_0^\varphi = \varphi$ and $x_t^\varphi \in W_c$ for $t \leq 0$, and $x_t^\varphi \to 0$ as $t \to -\infty$. This fact will be essential in the following section.

REMARK 4.17. In the case $a = 1$ the local center manifolds at the trivial solution of Equation (4.1) are two-dimensional and it is also possible to carry out a center manifold reduction, which is given by a two-dimensional first order ODE on an open neighborhood of 0 in $\mathbb{R}^2$. But since we will not discuss the case $a = 1$ in the remainder of this work, we do not develop this point here.

4. The Repellent Behavior of the Trivial Solution in Local Center-Unstable Manifolds

In our discussion of the local analysis of the model equation so far, we concluded that for $a > 1$, Equation (4.1) possesses continuously differentiable local center- and unstable manifolds at the stationary point $\varphi_0 = 0$. But besides these conclusions, the main result of Chapter 2 also implies the existence of local center-unstable manifolds for the semiflow F at the trivial solution. Furthermore, as the trivial solution $\varphi_0 = 0$ is repelling on local center manifolds by the last section, it is of course expected that the trivial solution is also repelling in these center-unstable manifolds for the semiflow F. Below we shall prove this conjecture in detail. But let us first state the result on the existence of local center-unstable manifolds at the trivial solution for the semiflow F.

COROLLARY 4.18. *If $a > 1$, then there is an open neighborhood $C_{cu,0}$ of 0 in the center-unstable space $C_{cu} = C_c \oplus C_u$ of the operator G_e, an open neighborhood $C^1_{s,0}$ of 0 in $C^1_s = C_s \cap C^1$ and a continuously differentiable map $w_{cu} : C_{cu,0} \longrightarrow C^1_{s,0}$ with $w_{cu}(0) = 0$ and $Dw_{cu}(0) = 0$, such that*

$$W_{cu} := \Big\{ \psi + w_{cu}(\psi) \,\Big|\, \psi \in C_{cu,0} \Big\}$$

satisfies the following.

(i) *The graph* W_{cu} *of* w_{cu} *is a* 2*-dimensional continuously differentiable submanifold of* X_f.
(ii) *If* $x : (-\infty, 0] \longrightarrow \mathbb{R}$ *is a solution of Equation* (4.1) *with* $x_t \in C_{cu,0} \oplus C^1_{s,0}$ *for all* $t \leq 0$*, then* $\{x_t \,|\, t \leq 0\} \subset W_{cu}$.
(iii) *The set* W_{cu} *is positively invariant under* F *relative to* $N_{cu} := C_{cu,0} \oplus C^1_{s,0}$ *in the sense that if* $\varphi \in W_{cu}$ *and* $t > 0$ *then*

$$\{F(s, \varphi) \mid 0 \leq s \leq t\} \subset N_{cu} \quad \Longrightarrow \quad \{F(s, \varphi) \mid 0 \leq s \leq t\} \subset W_{cu}.$$

PROOF. Apply Theorems 2.1 and 2.2 in view of Proposition 4.2. □

In the remainder of this chapter we assume $a > 1$ and derive, by applying the reduction principle of Section 6 in Chapter 2, an ODE describing the dynamics induced by the semiflow F on the local center-unstable manifolds. After that we will be in the position to prove the repellent behavior of $\varphi_0 = 0$.

The Reduced Differential Equation. In order to deduce the restricted differential equation describing the dynamics on the local center-unstable manifolds of the semiflow F at $\varphi_0 = 0$, recall from Proposition 4.2 that in the regarded situation the spectrum $\sigma(G_e)$ of operator G_e has, apart from the trivial one, only the simple positive real eigenvalue κ with a non-negative real part. Additionally, by Corollary 4.3 we have seen that the center-unstable space C_{cu} is spanned by the continuously differentiable functions

$$\eta_0 : [-1, 0] \ni \vartheta \longmapsto 1 \in \mathbb{R}$$

and

$$\eta_\kappa : [-1, 0] \ni \vartheta \longmapsto e^{\kappa\,\vartheta} \in \mathbb{R}.$$

Of course, we have $G_e\,\eta_0 = 0$ and $G_e\,\eta_\kappa = \kappa\,\eta_\kappa$ and thus, the matrix

$$B_{cu} := \begin{pmatrix} 0 & 0 \\ 0 & \kappa \end{pmatrix}$$

satisfies

$$G_e\,\Phi_{cu} = \Phi_{cu}\,B_{cu}$$

with the row vector $\Phi_{cu} := (\eta_0,\, \eta_\kappa)$.

Consider now a local center-unstable manifold $W_{cu} = \{\psi + w_{cu}(\psi) \,|\, \psi \in C_{cu,0}\}$ of F at $\varphi_0 = 0$ given by a continuously differentiable map $w_{cu} : C_{cu,0} \longrightarrow C^1_s$ on some open neighborhood $C_{cu,0}$ of the origin in the center-unstable space C_{cu}. Let $\Gamma_{cu} : C_{cu} \longrightarrow \mathbb{R}^2$ denote the coefficient function which associates to each $\psi \in C_{cu}$ the uniquely determined column vector $(c_1, c_2)^T \in \mathbb{R}^2$ with $\psi = c_1\,\eta_0 + c_2\,\eta_\kappa$, and let $P^{\odot*}_{cu} : C^{\odot*} \longrightarrow C_{cu}$ denote the projection of $C^{\odot*}$ along $C^{\odot*}_s$ onto C_{cu}. The operator $P^{\odot*}_{cu}$ is the sum $P^{\odot*}_c + P^{\odot*}_u$ where $P^{\odot*}_c : C^{\odot*} \longrightarrow C_c$ is the projection of $C^{\odot*}$ along $C_u \oplus C^{\odot*}_s$ onto the center space C_c and $P^{\odot*}_u : C^{\odot*} \longrightarrow C_u$ is the projection of $C^{\odot*}$ along $C_c \oplus C^{\odot*}_s$ onto the unstable space C_u. From Section 6 in Chapter 2 we see at once that the dynamics of F on the local center-unstable manifold W_{cu} is given by the two-dimensional first-order ODE

$$\dot{y}(t) = B_{cu}\,y(t) + Q_{cu}(\Phi_{cu}\,y(t) + w_{cu}(\Phi_{cu}\,y(t))) \tag{4.19}$$

with the continuously differentiable non-linear part $Q_{cu} = \Gamma_{cu} \circ P^{\odot*}_{cu} \circ l \circ g$.

Our next goal is to gain an asymptotic expansion of Q_{cu} by analogy to the center manifold reduction carried out in the last section. Thereto, we begin with the observation that the asymptotic formula for the function g obtained in Corollary 4.14 is also valid in W_{cu}.

COROLLARY 4.19. *For any $a > 1$ the restriction of the non-linear continuous differentiable functional g defined by* (4.12) *to W_{cu} is of type*

$$g(\psi + w_{cu}(\psi)) = -|\psi(0)|\,\psi(0) + o\big(\|\psi\|^2_{C^1}\big)$$

as $C_{cu,0} \ni \psi \to 0$.

PROOF. We have the same situation as in the proof of Corollary 4.14 and can thus proceed completely analogously. The details are left to the reader. □

In addition to the last result, we also need the projection operator $P_{cu}^{\odot *}$ of the Banach space $C^{\odot *}$ along $C_s^{\odot *}$ onto C_{cu} for our calculation of an asymptotic formula for the right-hand side of ODE (4.19). For this purpose, observe that $P_{cu} = P_c + P_u$ and that by Chapter IV in Diekmann et al. [**6**] we have in analogy to the projection P_c calculated in (4.16)

$$\begin{aligned} P_u\varphi &= \frac{1}{1 - a\,e^{-\kappa}}\left(\varphi(0) + \int_0^1 d\zeta(\vartheta)\int_0^{\vartheta} e^{-\kappa\,\sigma}\varphi(\sigma - \vartheta)\,d\sigma\right)\eta_\kappa \\ &= \frac{1}{1 - a\,e^{-\kappa}}\left(\varphi(0) - a\int_0^1 e^{-\kappa\,\sigma}\varphi(\sigma - 1)\,d\sigma\right)\eta_\kappa \end{aligned}$$

for all $\varphi \in C$. Combining this with Equation (4.17), we conclude

$$\begin{aligned} (P_{cu}^{\odot *} \circ l)(c) &= (P_c^{\odot *} \circ l)(c) + (P_u^{\odot *} \circ l)(c) \\ &= \frac{c}{1 - a}\,\eta_0 + \frac{c}{1 - a\,e^{-\kappa}}\,\eta_\kappa. \end{aligned}$$

for $c \in \mathbb{R}$ and hence, in view of equivalence of norm on finite dimensional spaces

$$\begin{aligned} Q_{cu}(y_1\eta_0 + y_2\,\eta_\kappa) &= \big(\Gamma_{cu} \circ P_{cu}^{\odot *} \circ l\big)\big(g(y_1\,\eta_0 + y_2\,\eta_\kappa + w_{cu}(y_1\,\eta_0 + y_2\,\eta_\kappa))\big) \\ &= \big(\Gamma_{cu} \circ P_{cu}^{\odot *} \circ l\big)\big(-|y_1 + y_2|(y_1 + y_2) + o((|y_1| + |y_2|)^2)\big) \\ &= \Gamma_{cu}\bigg(\Big(-\tfrac{1}{1-a}\,|y_1 + y_2|(y_1 + y_2) + o\big((|y_1| + |y_2|)^2\big)\Big)\,\eta_0 \\ &\qquad + \Big(-\tfrac{1}{1-a\,e^{-\kappa}}|y_1 + y_2|(y_1 + y_2) + o\big((|y_1| + |y_2|)^2\big)\Big)\,\eta_\kappa\bigg) \\ &= \begin{pmatrix} \tfrac{1}{a-1}\,|y_1 + y_2|(y_1 + y_2) + o((|y_1| + |y_2|)^2) \\ \tfrac{1}{a\,e^{-\kappa}-1}|y_1 + y_2|(y_1 + y_2) + o\big((|y_1| + |y_2|)^2\big) \end{pmatrix} \end{aligned}$$

as $y_1, y_2 \to 0$. Consequently, the reduction (4.19) of F on W_{cu} has the asymptotical behavior

$$\begin{cases} \dot{y}_1(t) = \dfrac{1}{a - 1}\,|y_1(t) + y_2(t)|(y_1(t) + y_2(t)) + o\big((|y_1(t)| + |y_2(t)|)^2\big) \\[2ex] \dot{y}_2(t) = \kappa\,y_2(t) + \dfrac{1}{a\,e^{-\kappa} - 1}|y_1(t) + y_2(t)|(y_1(t) + y_2(t)) \\ \qquad\qquad\qquad\qquad\qquad\qquad + o\big((|y_1(t)| + |y_2(t)|)^2\big) \end{cases} \tag{4.20}$$

in the close vicinity of the trivial solution $(0, 0) \in \mathbb{R}^2$, which corresponds to the stationary point $\varphi_0 = 0$ of F.

The Local Flow on $\mathbf{W_{cu}}$ at the Trivial Solution. After having derived the reduced differential equation (4.19) and the associated asymptotic expansion (4.20) for the dynamics on W_{cu}, our next goal is to establish a repellent behavior of the trivial solution for F in a sufficiently small neighborhood in W_{cu}. For this, consider the following observation concerning the vector field induced by the right-hand side of Equation (4.20).

COROLLARY 4.20. *Suppose that the functions $f_1, f_2 : U \longrightarrow \mathbb{R}$, defined on some open neighborhood $U \subset \mathbb{R}^2$ of $(0,0) \in \mathbb{R}^2$, are continuous and satisfy*

$$f_1(y_1, y_2) = \frac{1}{a-1}\,|y_1 + y_2|\,(y_1 + y_2) + o\big((|y_1| + |y_2|)^2\big)$$

and

$$f_2(y_1, y_2) = \kappa\, y_2 + \frac{1}{a\,e^{-\kappa} - 1}\,|y_1 + y_2|\,(y_1 + y_2) + o\big((|y_1| + |y_2|)^2\big)$$

as $(y_1, y_2) \to (0,0)$, and let S^1 denote the unit circle. Then there is a constant $\varrho_0 > 0$ such that

$$R(y_1, y_2; s) := f_1(s\,y_1, s\,y_2)\,y_1 + f_2(s\,y_1, s\,y_2)\,y_2 > 0 \tag{4.21}$$

for all $(y_1, y_2) \in S^1$ and all $0 < s \le \varrho_0$.

PROOF. The main idea of the proof is to estimate the value $R(y_1, y_2; s)$ for $(y_1, y_2) \in S^1$ and $s \in (0, \infty)$ with $(s\,y_1, s\,y_2) \in U$ in two different ways as $s \to 0$. As U is an open neighborhood of $0 \in \mathbb{R}^2$ it is clear that we find a constant $\varrho > 0$ such that for all $(y_1, y_2) \in S^1$ and $0 \le s \le \varrho$ we have $(s\,y_1, s\,y_2) \in U$. Throughout the proof, we write

$$\begin{aligned} G(y_1, y_2; s) := R(y_1, y_2; s) &- \frac{s^2}{a-1}\,|y_1 + y_2|\,y_1^2 - \frac{s^2}{a-1}\,|y_1 + y_2|\,y_1\,y_2 \\ &- \kappa\,s\,y_2^2 - \frac{s^2}{a\,e^{-\kappa} - 1}|y_1 + y_2|\,y_1\,y_2 - \frac{s^2}{a\,e^{-\kappa} - 1}|y_1 + y_2|\,y_2^2 \end{aligned}$$

for all $(y_1, y_2) \in S^1$ and reals $0 \le s \le \varrho$. Note that we have the asymptotic behavior $G(y_1, y_2; s) = o(s^2)$ as $[0, \varrho) \ni s \to 0$ uniform for all $(y_1, y_2) \in S^1$.

1. The first estimate of the values of R is based on the simple observation that

$$R(y_1, y_2; s) \ge \frac{s^2}{a-1}\,|y_1 + y_2|\,y_1^2 + s^2\,|y_1 + y_2|\left(\frac{y_1\,y_2}{a-1} + \frac{y_1\,y_2 + y_2^2}{a\,e^{-\kappa} - 1}\right) + G(y_1, y_2; s)$$

for all $(y_1, y_2) \in S^1$ and $0 \le s \le \varrho$. Define

$$p_1 : S^1 \ni (y_1, y_2) \longmapsto \frac{1}{a-1}\,y_1\,y_2 + \frac{1}{a\,e^{-\kappa} - 1}\big(y_1\,y_2 + y_2^2\big) \in \mathbb{R}.$$

Then the inequality

$$\begin{aligned} |p_1(y_1, y_2)| &\le \left|\frac{1}{a-1} + \frac{1}{a\,e^{-\kappa} - 1}\right|\,|y_1\,y_2| + \left|\frac{1}{a\,e^{-\kappa} - 1}\right|\,y_2^2 \\ &\le \left|\frac{1}{a-1} + \frac{1}{a\,e^{-\kappa} - 1}\right|\,|y_2|\sqrt{1 - y_2^2} + \left|\frac{1}{a\,e^{-\kappa} - 1}\right|\,y_2^2 \end{aligned}$$

yields the existence of a constant $0 < \varepsilon_1 < 1$ such that

$$|p_1(y_1, y_2)| < \frac{1}{2\,(a-1)}\,(1 - y_2^2) = \frac{1}{2\,(a-1)}\,y_1^2$$

for all $(y_1, y_2) \in S^1$ with $-2\,\varepsilon_1 < y_2 < 2\,\varepsilon_1$. Thereby, we can certainly assume

$$\frac{1}{2\,(a-1)}\Big(\sqrt{1-4\,\varepsilon_1^2}-2\,\varepsilon_1\Big)\Big(1-4\,\varepsilon_1^2\Big) > 0.$$

As $G(y_1, y_2; s) = o(s^2)$ as $[0, \varrho) \ni s \to 0$, there also is a sufficiently small constant $0 < \varepsilon_2 < \min\{1, \varrho\}$ such that for all $(y_1, y_2) \in S^1$ with $y_2 \in (-2\,\varepsilon_1, 2\,\varepsilon_1)$ and all reals $0 < s < \varepsilon_2$

$$\begin{aligned}
|G(y_1, y_2; s)| &< \frac{s^2}{2\,(a-1)}\Big(\sqrt{1-4\,\varepsilon_1^2}-2\,\varepsilon_1\Big)\Big(1-4\,\varepsilon_1^2\Big)\\
&< \frac{s^2}{2\,(a-1)}\,||y_1|-|y_2||\,y_1^2\\
&\leq \frac{s^2}{2\,(a-1)}\,|y_1+y_2|\,y_1^2.
\end{aligned}$$

Combining the above inequalities for R and for the absolute values of p_1 and G gives

$$\begin{aligned}
R(y_1, y_2; s) &\geq \frac{s^2}{a-1}|y_1+y_2|\,y_1^2 + s^2\,|y_1+y_2|\left(\frac{y_1\,y_2}{a-1}+\frac{y_1\,y_2+y_2^2}{a\,e^{-\kappa}-1}\right) + G(y_1, y_2; s)\\
&\geq \frac{s^2}{a-1}|y_1+y_2|\,y_1^2 - s^2\,|y_1+y_2|\,|p_1(y_1,y_2)| + G(y_1, y_2; s)\\
&> \frac{s^2}{a-1}|y_1+y_2|\,y_1^2 - s^2\,|y_1+y_2||p_1(y_1,y_2)| - \frac{s^2}{2\,(a-1)}|y_1+y_2|\,y_1^2\\
&> \frac{s^2}{a-1}|y_1+y_2|\,y_1^2 - \frac{s^2}{2\,(a-1)}\,|y_1+y_2|\,y_1^2 - \frac{s^2}{2\,(a-1)}\,|y_1+y_2|\,y_1^2\\
&= 0
\end{aligned}$$

for all $(y_1, y_2) \in S^1$ with $|y_2| < 2\,\varepsilon_1$ and $0 < s < \varepsilon_2$.

2. On the other hand, note that the real-valued function

$$\begin{aligned}
p_2 : \mathbb{R}^2 \ni (y_1, y_2) \longmapsto\; &\frac{1}{a-1}|y_1+y_2|\,y_1^2 + \frac{1}{a-1}|y_1+y_2|\,y_1\,y_2\\
&+ \frac{1}{a\,e^{-\kappa}-1}|y_1+y_2|\,y_1\,y_2 + \frac{1}{a\,e^{-\kappa}-1}|y_1+y_2|y_2^2 \in \mathbb{R}
\end{aligned}$$

is continuous and thus takes a minimum $m \in \mathbb{R}$ on the compact set S^1. This shows

$$R(y_1, y_2; s) = s\,\kappa\,y_2^2 + s^2\,p_2(y_1, y_2) + G(y_1, y_2; s) \geq s\,\kappa\,y_2^2 + s^2\,m + G(y_1, y_2; s)$$

for all $(y_1, y_2) \in S^1$ as $0 \leq s < \varrho$. Next, in view of $\kappa > 0$ we find a sufficiently small real $0 < \varepsilon_3 < \min\{1, \varrho\}$ with

$$\frac{1}{2}\,\kappa\,\varepsilon_1^2 + s\,m > 0$$

as $0 < s < \varepsilon_3$. Additionally, as $G(y_1, y_2; s) = o(s^2)$ for $[0, \varrho) \ni s \to 0$, there is also a constant $0 < \varepsilon_4 < \varrho$ such that for all $(y_1, y_2) \in S^1$ and $0 < s < \varepsilon_4$

$$|G(y_1, y_2; s)| < \frac{\kappa\,\varepsilon_1^2}{2}\,s^2.$$

Consequently, using the last three inequalities, we conclude that for all $(y_1, y_2) \in S^1$ with $|y_2| \geq \varepsilon_1$ and for all $0 < s < \min\{\varepsilon_3, \varepsilon_4\}$

$$\begin{aligned} R(y_1, y_2; s) &\geq s\,\kappa\, y_2^2 + s^2\, m + G(y_1, y_2; s) \\ &> s\,\kappa\, y_2^2 + s^2\, m - \frac{\kappa\,\varepsilon_1^2}{2}\, s^2 \\ &\geq s\,\kappa\, y_2^2 + s^2\, m - \frac{\kappa\, y_2^2}{2}\, s^2 \\ &\geq \frac{1}{2} s\,\kappa\,\varepsilon_1^2 + s^2\, m \\ &> 0. \end{aligned}$$

3. Choosing $0 < \varrho_0 < \min\{\varepsilon_2, \varepsilon_3, \varepsilon_4\}$ and applying the last two steps implies

$$R(y_1, y_2; s) > 0$$

for all $(y_1, y_2) \in S^1$ and $0 < s \leq \varrho_0$ and this is precisely the assertion of the proposition. □

The last corollary gains in interest if we additionally assume that the functions f_1, f_2 are locally Lipschitz continuous and consider the autonomous ODE

$$\begin{cases} \dot{y}_1(t) = f_1(y_1(t), y_2(t)) \\ \dot{y}_2(t) = f_2(y_1(t), y_2(t)) \end{cases} \tag{4.22}$$

defined by f_1, f_2. Under these assumptions each initial data in U admits a uniquely determined solution of the above ODE in U. Now the continuous map

$$\mathfrak{L} : \mathbb{R}^2 \ni (y_1, y_2) \longmapsto \frac{1}{2}\big(y_1^2 + y_2^2\big) \in \mathbb{R} \tag{4.23}$$

is positive definite, that is, $\mathfrak{L}(0,0) = 0$ and $\mathfrak{L}(y_1, y_2) > 0$ for all $(y_1, y_2) \neq (0,0)$, and the last result yields that $\mathfrak{L}$ is strictly increasing along non-trivial solutions of (4.22). From this we conclude not only the instability but also the repellent behavior of the trivial solution of Equation (4.22) in sufficiently small neighborhoods.

Proposition 4.21. *Suppose that in addition to the assumptions of Corollary 4.20 the functions f_1, f_2 are both locally Lipschitz continuous. Then there exists an open neighborhood $V \subset U$ of $(0,0) \in \mathbb{R}^2$ such that for any non-trivial initial value $u \in V$ there is a solution $y : (-\infty, t_+(u)) \to \mathbb{R}^2$, $t_+(u) > 0$, of Equation (4.22) with $y(0) = u$, $y(t) \in V$ for all $t \leq 0$, $y(t) \to 0$ as $t \to -\infty$, and $y(t) \notin V$ for some $0 < s < t_+(u)$.*

Proof. As the functions $f_1, f_2 : U \longmapsto \mathbb{R}$ are locally Lipschitz continuous, the induced map $U \ni y \longmapsto (f_1(y), f_2(y)) \in \mathbb{R}^2$ is locally Lipschitz continuous too. Therefore, it is clear that for every $u \in U$ there exists a maximal solution $y : (t_-(u), t_+(u)) \longrightarrow \mathbb{R}^2$, $t_-(u) < 0 < t_+(u)$, of Equation (4.22) with initial value u at $t = 0$. Further, applying the last result, we find a sufficiently small open ball $B_{\varrho_0}(0)$, $\varrho_0 > 0$, around the origin in $\mathbb{R}^2$ with $R(u/\|u\|_2; \|u\|_2) > 0$ for all u in the closure $\overline{B_{\varrho_0}(0)}$ of $B_{\varrho_0}(0)$, where R is defined by Equation (4.21) and $\|\cdot\|_2$ denotes the Euclidean norm.

Set $V := B_\varrho(0)$ for a fixed $0 < \varrho < \varrho_0$ and suppose $y : (t_-(u), t_+(u)) \longrightarrow \mathbb{R}^2$, $t_-(u) < 0 < t_+(u)$, is a maximal solution of Equation (4.22) with $y(0) = u \in V\backslash\{0\}$. We have

$$\frac{d}{dt}\mathfrak{L}(y(t)) = \|y(t)\|_2 \cdot R\big(y(t)/\|y(t)\|_2; \|y(t)\|_2\big) > 0 \tag{4.24}$$

as long as $y(t) \in B_{\varrho_0}(0)$. This gives $0 < \mathfrak{L}(u) \le \mathfrak{L}(y(t))$ and hence

$$0 < \|u\|_2 \le \|y(t)\|_2$$

for all $0 \le t < t_+(u)$ with $y(t) \in B_{\varrho_0}(0)$. Define

$$m := \min \Big\{ \tfrac{d}{dt}\mathfrak{L}(y(t)) \,\Big|\, y(t) \in B_{\varrho_0}(0) \,\text{and}\, \mathfrak{L}(y(t)) \ge \mathfrak{L}(u) \text{ with } 0 \le t < t_+(u) \Big\}.$$

Then m exists and is positive since the norm of the considered values $y(t)$ as well as the continuous function R on the compact set $\overline{B_{\varrho_0}(0)} \setminus B_{\|u\|_2}(0)$ are both bounded from below by a positive constant. As $\|y(t)\|_2 \ge \|u\|_2$ for all $0 \le t < t_+(u)$ with $y(t) \in B_{\varrho_0}(0)$ we conclude

$$\mathfrak{L}(y(t)) = \mathfrak{L}(u) + \int_0^t \tfrac{d}{ds}\mathfrak{L}(y(s))\, ds \ge \mathfrak{L}(u) + m\, t$$

as long as $0 \le t < t_+(u)$ and $y(t)$ stays in $B_{\varrho_0}(0)$. Assume now that solution y would not reach the boundary $\partial B_\varrho(0)$ as $0 < t < t_+(u)$. From the continuation results for solutions of ODEs as presented in Hale [**8**, Chapter I.2] we would obtain $t_+(u) = \infty$ since y is a maximal solution. Hence, the above inequality for $\mathfrak{L}$ would be valid for all $t \ge 0$ and therefore the value of $\mathfrak{L}(y(t))$ would become arbitrarily large. But on the other hand, the function $\mathfrak{L}$ is bounded from above on $B_{\varrho_0}(0)$, in contradiction to the previous inequality. Consequently, the non-continuable solution y reaches the boundary $\partial B_\varrho(0)$ in finite time and hence necessarily leaves the neighborhood V for some $0 < t < t_+(u)$.

For the the remaining part of the assertion, observe that Equation (4.24) also implies $0 < \mathfrak{L}(y(t)) \le \mathfrak{L}(u)$ for $t_-(u) < t \le 0$. Therefore, the non-continuable solution y stays bounded by $\|u\|_2 > 0$ for $t \le 0$ and thus does not reach the boundary of V. In particular, we have $y(t) \in V$ for all $t_-(u) < t \le u$. Moreover, using the continuation results for solutions of ODEs, we get $t_-(u) = -\infty$. We shall have established the proposition if we additionally prove that for any given $0 < \varepsilon < \|u\|_2$ there is a $T_\varepsilon \in (-\infty, 0]$ with $\|y(t)\|_2 < \varepsilon$ for all $t \le T_\varepsilon$. To obtain this, fix for given constant $0 < \varepsilon < \min\{1, \|u\|_2\}$ a constant

$$0 < m_\varepsilon < \min \Big\{ \|w\|_2 \cdot R\big(w/\|w\|_2; \|w\|_2\big) \,\Big|\, w \in W \Big\}$$

where $W := \{w \in \mathbb{R}^2 \,|\, \varepsilon/2 \le \|w\|_2 \le \|u\|_2\}$. If $\|y(s)\|_2 < \varepsilon$ for some $s \le 0$, then in view of Equation (4.24) we have $\|y(t)\|_2 < \varepsilon$ for all $t \le T_\varepsilon$ with $T_\varepsilon := s$. Now assume there is no such a constant $s < 0$. Then we would have $y(t) \in W$ and

$$\mathfrak{L}(u) - \mathfrak{L}(y(t)) = \int_t^0 \tfrac{d}{d\tilde{s}}\mathfrak{L}(y(\tilde{s}))\, d\tilde{s} \ge -m_\varepsilon\, t,$$

that is,

$$\mathfrak{L}(y(t)) \le \mathfrak{L}(u) + m_\varepsilon\, t,$$

for all $t \le 0$. But this is impossible, because the continuous and positive definite function $\mathfrak{L}$ takes a positive minimum $m_{\mathfrak{L}} > 0$ on the compact set W and for all $t < (m_{\mathfrak{L}} - \mathfrak{L}(u))/m_\varepsilon \le 0$ we would obtain

$$\mathfrak{L}(y(t)) \le \mathfrak{L}(u) + m_\varepsilon\, t < \mathfrak{L}(u) + m_\varepsilon \frac{m_{\mathfrak{L}} - \mathfrak{L}(u)}{m_\varepsilon} = m_{\mathfrak{L}}.$$

Hence, the assumption that there does not exist some $s < 0$ with $\|y(s)\|_2 < \varepsilon$ results in a contradiction. Consequently, it follows that we find $T_\varepsilon \le 0$ with $\|y(t)\|_2 < \varepsilon$ for all $t \le T_\varepsilon$, and this completes the proof. $\square$

Obviously, in consideration of the asymptotic formula (4.20) the above result also applies to the ODE (4.19) and consequently shows that the trivial solution $\varphi_0 = 0$ of the model equation (4.1) has a repelling behavior in the local center-unstable manifold W_{cu} in the sense of Proposition 4.21. This fact will be crucial in the next chapter.

Another important point in the last part of this work will follow from our next proposition stating that a solution of the ODE (4.22), which converges to the origin as $t \to -\infty$, does so tangentially to an axis of the Cartesian coordinate system of the Euclidean plane.

PROPOSITION 4.22. *Suppose that the right-hand side of Equation* (4.22) *is locally Lipschitz continuous and satisfies the hypothesis of Corollary 4.20, and assume* $y : (-\infty, t_+) \longrightarrow \mathbb{R}^2$, $t_+ > 0$, *is a non-trivial solution of Equation* (4.22) *with* $y(t) \to 0$ *as* $t \to -\infty$. *Let* $e_1 := (1,0)^T$, $e^2 := (0,1)^T$ *denote the canonical basis vectors of the Euclidean space* $\mathbb{R}^2$. *Then*

$$\mathrm{dist}_{\|\,\cdot\,\|_2}\Big(\|y(t)\|_2^{-1}\, y(t), \{e_1, -e_1\}\Big) \to 0$$

or

$$\mathrm{dist}_{\|\,\cdot\,\|_2}\Big(\|y(t)\|_2^{-1}\, y(t), \{e_2, -e_2\}\Big) \to 0$$

as $t \to -\infty$.

PROOF. The proof of the assertion will be divided into two steps. At first we show that

$$\lim_{t \to -\infty} \frac{1}{\|y(t)\|_2}\, y(t)$$

indeed exists, followed by the conclusion that this limit or its negative coincides with one of the canonical basis vectors of the Euclidean plane.

1. In order to see the existence of the limit above, we start with the observation that there is a sufficiently small $K \leq 0$ such that the restriction of the solution y to $(-\infty, K]$ does not oscillate around $(0,0) \in \mathbb{R}^2$. More precisely, we claim the existence of a constant $K \leq 0$ and the existence of some point φ of the unit circle S^1 with the property

$$\frac{1}{\|y(t)\|_2}\, y(t) \neq \varphi$$

for all $t \leq K$. To see this claim, consider for a given $\varphi_1 \in S^1$ with components $\varphi_{11}, \varphi_{12} > 0$ and a given $\varphi_2 \in S^1$ with $\varphi_{21} < 0 < \varphi_{22}$ the lines

$$\mathcal{G}_{\varphi_1} : [0, \infty) \ni s \longmapsto s\,\varphi_1 \in \mathbb{R}^2$$

and

$$\mathcal{G}_{\varphi_2} : [0, \infty) \ni s \longmapsto s\,\varphi_2 \in \mathbb{R}^2,$$

and the associated perpendicular vectors $\psi_1 := (-\varphi_{12}, \varphi_{11})$ and $\psi_2 := (\varphi_{22}, -\varphi_{21})$, respectively. For these fixed point $\varphi_1, \varphi_2 \in S^1$ we have

$$\begin{aligned} f_1(\mathcal{G}_{\varphi_1}(s))\,\psi_{11} + f_2(\mathcal{G}_{\varphi_1}(s))\,\psi_{12} &= -\frac{s^2}{a-1}\,|\varphi_{11} + \varphi_{12}|\,(\varphi_{11} + \varphi_{12})\,\varphi_{12} + s\,\kappa\,\varphi_{12}\,\varphi_{11} \\ &\quad + \frac{s^2}{a\,e^{-\kappa} - 1}\,|\varphi_{11} + \varphi_{12}|\,(\varphi_{11} + \varphi_{12})\,\varphi_{11} + o(s^2) \\ &= s\,\kappa\,\varphi_{12}\,\varphi_{11} + o(s) \end{aligned}$$

and

$$\begin{aligned} f_1(\mathcal{G}_{\varphi_2}(s))\,\psi_{21} + f_2(\mathcal{G}_{\varphi_2}(s))\,\psi_{22} &= \frac{s^2}{a-1}\,|\varphi_{21}+\varphi_{22}|\,(\varphi_{21}+\varphi_{22})\,\varphi_{22} - \kappa\, s\,\varphi_{22}\,\varphi_{21} \\ &\quad - \frac{s^2}{a\,e^{-\kappa}-1}|\varphi_{21}+\varphi_{22}|\,(\varphi_{21}+\varphi_{22})\,\varphi_{21} + o(s^2) \\ &= -s\,\kappa\,\varphi_{22}\,\varphi_{21} + o(s) \end{aligned}$$

as $s \to 0$. Consequently, as $\varphi_{11}, \varphi_{12}, \varphi_{22} > 0$ and $\varphi_{21} < 0$, we obtain

$$f_1(\mathcal{G}_{\varphi_1}(s))\,\psi_{11} + f_2(\mathcal{G}_{\varphi_1}(s))\,\psi_{12} > 0$$

and

$$f_1(\mathcal{G}_{\varphi_2}(s))\,\psi_{21} + f_2(\mathcal{G}_{\varphi_2}(s))\,\psi_{22} > 0$$

for all sufficiently small $s > 0$. This implies that in the situation of increasing t, the solution y may cross both lines $\mathcal{G}_{\varphi_1}, \mathcal{G}_{\varphi_2}$ in close vicinity of $(0,0) \in \mathbb{R}^2$ only from bottom to top as sketched in Figure 4.4. But since $y(t) \to 0$ as $t \to -\infty$, this clearly yields the existence of a constant $K \leq 0$ such that y is not oscillating around

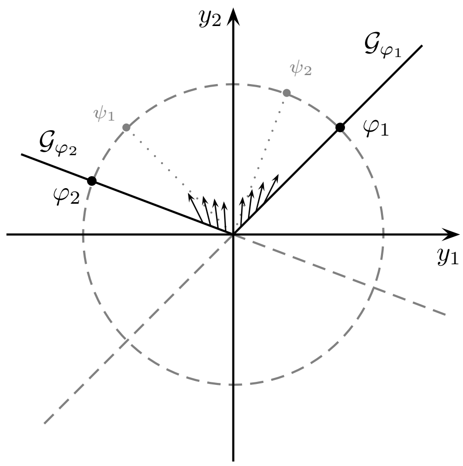

FIGURE 4.4. Scheme illustrating that non-trivial solutions of Equation (4.22) may cross the lines $\mathcal{G}_{\varphi_1}$, $\mathcal{G}_{\varphi_2}$ near $(0,0) \in \mathbb{R}^2$ only in one way, namely, from lower to upper side

$(0,0) \in \mathbb{R}^2$ for $-\infty < t \leq K$, as claimed. In particular, the former arguments show that there is no sequence $\{t_k\}_{k\in\mathbb{N}} \subset (-\infty, 0]$ with $t_k \to -\infty$ as $k \to \infty$ such that φ_1 or φ_2 is an accumulation point of the sequence $\{y(t_k)/\|y(t_k)\|_2\}_{k\in\mathbb{N}}$.

Now, in the same manner as above, we can also see that in case of points $\varphi_3, \varphi_4 \in S^1$ with coordinates $\varphi_{31}, \varphi_{32} < 0$ and $\varphi_{41} > 0 > \varphi_{42}$, respectively, and the associated lines

$$\mathcal{G}_{\varphi_3} : [0,\infty) \ni s \longmapsto s\,\varphi_3 \in \mathbb{R}^2$$

and

$$\mathcal{G}_{\varphi_4} : [0,\infty) \ni s \longmapsto s\,\varphi_4 \in \mathbb{R}^2,$$

we have

$$\mathcal{G}_{\varphi_3}(s)\,\psi_{31} + \mathcal{G}_{\varphi_3}(s)\,\psi_{32} > 0$$

and

$$\mathcal{G}_{\varphi_4}(s)\,\psi_{41} + \mathcal{G}_{\varphi_4}(s)\,\psi_{42} > 0$$

for the two vectors $\psi_3 := (-\varphi_{32}, \varphi_{31})$, $\psi_4 := (\varphi_{42}, -\varphi_{41})$ and all sufficiently small $s > 0$. Hence, it follows that near $(0,0) \in \mathbb{R}^2$ the solution y may cross any of the lines $\mathcal{G}_{\varphi_i}$, $i = 1, \dots, 4$, only in the way as indicated in Figure 4.5. Particularly, we conclude that there does not exist a sequence $\{t_k\}_{k\in\mathbb{N}} \subset (-\infty, 0]$ with $t_k \to -\infty$ as $k \to \infty$ such that any of the points $\{\varphi_i \mid i = 1, 2, 3, 4\}$ is an accumulation point of $\{y(t_k)/\|y(t_k)\|_2\}_{k\in\mathbb{N}}$.

The above gives now a simple argument for the existence of the limit

$$\lim_{t\to-\infty} \xi(t)$$

where $\xi(t) := y(t)/\|y(t)\|_2$, $-\infty < t \le t_+$. Namely, consider an arbitrary sequence

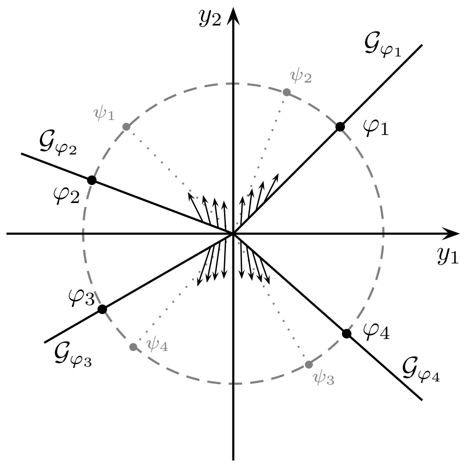

FIGURE 4.5. Indication of possible cross-directions of a non-trivial solution for Equation (4.22) along lines near the origin $(0,0) \in \mathbb{R}^2$

$\{t_k\}_{k\in\mathbb{N}} \subset (-\infty, t_+)$ with $t_k \to -\infty$ as $k \to \infty$. Since S^1 is compact and $\xi(t_k) \in S^1$ for all $k \in \mathbb{N}$, there clearly is at least one accumulation point $\xi^* \in S^1$ of $\{\xi(t_k)\}_{k\in\mathbb{N}}$. But this is also the only such point of the sequence $\{\xi(t_k)\}_{k\in\mathbb{N}}$. Indeed, assume we would find another accumulation point $\xi^{**} \in S^1 \setminus \{\xi^*\}$. Then there would exists a line $\mathcal{G}_\varphi : [0,\infty) \ni s \longmapsto s\,\varphi \in \mathbb{R}^2$ for a point $\varphi \in S^1$ with components $\varphi_1, \varphi_2 \ne 0$ and a sequence $\{\tilde{t}_k\}_{k\in\mathbb{N}} \subset (-\infty, 0]$ with $\tilde{t}_k \to -\infty$ as $k \to \infty$ such that φ would be an accumulation point of $\{\xi(\tilde{t}_k)\}_{k\in\mathbb{N}}$. But the existence of such a line and such a sequence would contradict the above conclusions. Hence, $\lim_{k\to\infty} \xi(t_k) = \xi^*$. By the same method, we see that in case of another sequence $\{s_k\}_{k\in\mathbb{N}} \subset (-\infty, t_+]$ with $s_k \to -\infty$ as $k \to \infty$ we have again $\lim_{k\to-\infty} y(s_k) = \xi^*$, because otherwise we might construct a sequence with two different accumulation points. This finally shows the existence of $\lim_{t\to-\infty} \xi(t)$.

2. Consider $\xi^* = \lim_{t\to-\infty} \xi(t)$ with $\xi(t) = y(t)/\|y(t)\|_2$, $-\infty < t < t_+$, and suppose $\xi^* \neq \pm(1,0)^T$, that is,

$$\lim_{t\to-\infty} \frac{y_2(t)}{\|y(t)\|_2} \neq 0.$$

Then we claim that

$$\lim_{t\to-\infty} \frac{y_1(t)}{y_2(t)} = 0.$$

In order to see this, observe that we have $y_1(t), y_2(t) \to 0$ due to $\|y(t)\|_2 \to 0$ as $t \to -\infty$ and thus by application of l'Hospital's rule

$$\begin{aligned}
\lim_{t\to-\infty} \frac{y_1(t)}{y_2(t)} &= \lim_{t\to-\infty} \frac{\dot y_1(t)}{\dot y_2(t)} \\
&= \lim_{t\to-\infty} \frac{f_1(y_1(t), y_2(t))}{f_2(y_1(t), y_2(t))} \\
&= \lim_{t\to-\infty} \left(\frac{f_1(y_1(t), y_2(t))}{\|y(t)\|_2} \, \frac{\|y(t)\|_2}{f_2(y_1(t), y_2(t))} \right)
\end{aligned}$$

whenever the last limit exists. But this is indeed the case since

$$\begin{aligned}
\lim_{t\to-\infty} \frac{f_1(y_1(t), y_2(t))}{\|y(t)\|_2} &= \lim_{t\to-\infty} \left(\frac{1}{a-1} \, |y_1(t) + y_2(t)| \left(\frac{y_1(t)}{\|y(t)\|_2} + \frac{y_2(t)}{\|y_2(t)\|_2} \right) \right) \\
&\qquad + \lim_{t\to-\infty} \frac{1}{\|y(t)\|_2} \, o((|y_1(t)| + |y_2(t)|)^2) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to-\infty} \frac{f_2(y_1(t), y_2(t))}{\|y(t)\|_2} &= \lim_{t\to-\infty} \left(\frac{1}{a\,e^{-\kappa} - 1} \, |y_1(t) + y_2(t)| \left(\frac{y_1(t)}{\|y(t)\|_2} + \frac{y_2(t)}{\|y(t)\|_2} \right) \right) \\
&\qquad + \lim_{t\to-\infty} \frac{1}{\|y(t)\|_2} \left(\kappa\, y_2(t) + o((|y_1(t)| + |y_2(t)|)^2) \right) \\
&= \lim_{t\to-\infty} \kappa \, \frac{y_2(t)}{\|y(t)\|_2}.
\end{aligned}$$

Consequently, the claimed formula

$$\lim_{t\to-\infty} \frac{y_1(t)}{y_2(t)} = 0$$

follows.

The rest of the proof is now trivial in view of the limit above, because we see at once

$$\xi_1^* = \lim_{t\to-\infty} \frac{y_1(t)}{\|y(t)\|_2} = \lim_{t\to-\infty} \frac{y_1(t)}{y_2(t)} \, \frac{y_2(t)}{\|y(t)\|_2} = \lim_{t\to-\infty} \frac{y_1(t)}{y_2(t)} \, \lim_{t\to-\infty} \frac{y_2(t)}{\|y(t)\|_2} = 0.$$

But $\xi^* \in S^1$ and thus we necessarily have $|\xi_2^*| = 1$, which is the desired conclusion. $\square$

The foregoing proposition suggests that solutions of Equation (4.1) in W_{cu} tending to zero as $t \to -\infty$ do so either tangentially to the center or to the unstable direction. This is indeed the case and will play a decisive role.

Corollary 4.23. *Suppose $x : (-\infty, t_+) \longrightarrow \mathbb{R}$, $t_+ > 0$, is a solution of Equation (4.1) with $x_t \in W_{cu}$ for $t \le 0$ and $x(t) \to 0$ as $t \to -\infty$. Then we have either*

$$\operatorname{dist}_{\|\cdot\|_{C^1}} \left(\|x_t\|_{C^1}^{-1} \, x_t, \left\{ \|\eta_0\|_{C^1}^{-1} \, \eta_0, \, -\|\eta_0\|_{C^1}^{-1} \, \eta_0 \right\} \right) \to 0$$

or

$$\mathrm{dist}_{\|\cdot\|_{C^1}}\Big(\|x_t\|_{C^1}^{-1}\, x_t, \Big\{\|\eta_\kappa\|_{C^1}^{-1}\, \eta_\kappa,\, -\|\eta_\kappa\|_{C^1}^{-1}\, \eta_\kappa\Big\}\Big) \to 0$$

as $t \to -\infty$.

PROOF. By assumption, we have $x_t = P_{cu}\, x_t + w_{cu}(P_{cu}\, x_t)$ for $t \leq 0$ and hence, according to Section 6 in Chapter 2, there is a continuously differentiable function $y : (-\infty, 0] \longrightarrow \mathbb{R}^2$ with $P_{cu}\, x_t = y_1(t)\, \eta_0 + y_2(t)\, \eta_\kappa$ which forms a solution of Equation (4.19). Moreover, from $x_t \to 0$ for $t \to -\infty$ we see at once $y(t) \to 0$ as $t \to -\infty$, and thus the last proposition implies that either

$$\mathrm{dist}_{\|\cdot\|_2}\Big(\|y(t)\|_2^{-1}\, y(t), \{e_1, -e_1\}\Big) \to 0$$

or

$$\mathrm{dist}_{\|\cdot\|_2}\Big(\|y(t)\|_2^{-1}\, y(t), \{e_2, -e_2\}\Big) \to 0$$

for $t \to -\infty$.

To begin with, consider the first situation, that is, $y_1(t)/\|y(t)\|_2 \to \pm 1$ and $y_2(t)/\|y(t)\|_2 \to 0$ as $t \to \infty$. Then there is a constant $T \leq 0$ with $y_1(t) \neq 0$ for all $-\infty < t \leq T$ and we have $y_2(t)/y_1(t) \to 0$ as $t \to -\infty$. Now w_{cu} is C^1-smooth, implying

$$\begin{aligned} x_t &= y_1(t)\, \eta_0 + y_2(t)\, \eta_\kappa + w_{cu}\big(y_1(t)\, \eta_0 + y_2(t)\, \eta_\kappa\big) \\ &= y_1(t)\, \eta_0 + y_2(t)\, \eta_\kappa + \int_0^1 Dw_{cu}\big(s\, y_1(t)\, \eta_0 + s\, y_2(t)\, \eta_\kappa\big)\, \big(y_1(t)\, \eta_0 + y_2(t)\, \eta_\kappa\big)\, ds \\ &= y_1(t)\bigg(\eta_0 + \tfrac{y_2(t)}{y_1(t)}\, \eta_\kappa + \int_0^1 Dw_{cu}\big(s\, y_1(t)\, \eta_0 + s\, y_2(t)\, \eta_\kappa\big)\, \Big(\eta_0 + \tfrac{y_2(t)}{y_1(t)}\, \eta_\kappa\Big)\, ds\bigg) \\ &= y_1(t)\big(\eta_0 + \xi(t)\big) \end{aligned}$$

for all $-\infty < t \leq T$, where

$$\xi(t) := \frac{y_2(t)}{y_1(t)}\, \eta_\kappa + \int_0^1 Dw_{cu}\big(s\, y_1(t)\, \eta_0 + s\, y_2(t)\, \eta_\kappa\big)\Big(\eta_0 + \tfrac{y_2(t)}{y_1(t)}\, \eta_\kappa\Big)\, ds.$$

Furthermore, since we have $Dw_{cu}(0) = 0$, $\lim_{t\to-\infty} \xi(t) = 0$ and thus we get

$$\begin{aligned} \frac{1}{\|x_t\|_{C^1}}\, x_t - \frac{y_1(T)}{|y_1(T)|\|\eta_0\|_{C^1}}\, \eta_0 &= \pm\bigg(\frac{1}{\|\eta_0 + \xi(t)\|_{C^1}}\, \big[\eta_0 + \xi(t)\big] - \frac{1}{\|\eta_0\|_{C^1}}\, \eta_0\bigg) \\ &= \pm\bigg(\frac{1}{\|\eta_0 + \xi(t)\|_{C^1}} - \frac{1}{\|\eta_0\|_{C^1}}\bigg)\, \eta_0 \\ &\qquad \pm \frac{1}{\|\eta_0 + \xi(t)\|_{C^1}}\, \xi(t) \\ &\to 0 \end{aligned}$$

as $t \to -\infty$, which is the desired conclusion.

In the case $y_1(t)/\|y_1(t)\|_2 \to 0$ and $y_2(t)/\|y_2(t)\|_2 \to \pm 1$ for $t \to -\infty$, one finds a constant $T \leq 0$ with $y_2(t) \neq 0$ for $-\infty < t \leq T$ and immediately sees $y_1(t)/y_2(t) \to 0$ as $t \to -\infty$. A computation similar to the above leads to

$$\frac{1}{\|x_t\|_{C^1}}\, x_t - \frac{y_2(T)}{|y_2(T)|\|\eta_\kappa\|_{C^1}}\, \eta_\kappa \to 0$$

for $t \to -\infty$, which finally completes the proof. $\square$

CHAPTER 5

Dynamics on the Center-Unstable Manifolds

1. Introduction

In this final chapter of the second part we establish our main result for the model equation

$$\dot{x}(t) = a\left[x(t) - x(t-r)\right] - |x(t)|\,x(t)$$

with a state-dependent delay $r = r(x(t)) > 0$ and parameter $a > 0$, which asserts the existence of a periodic orbit for $a > 1$. This orbit will be given by a slowly oscillating solution $p : \mathbb{R} \longrightarrow [2a, 2a]$ and its minimal period by three consecutive zeros of p. In the situation of the constant-delay $r \equiv 1$, that is, in case of Equation (3.2), the above assertion was proved by Brunovský et al. in [**3**], and we generalize this statement under certain conditions to the situation with a state-dependent delay.

Overall, our discussion below follows the proceedings in Brunovský et al. [**3**], and partially we arrive at analogous conclusions, but, due to the presence of a state-dependent delay, we use different techniques on several occasions, following once more the ideas in Krisztin and Arino [**17**]. For example, in the course of our analysis we will repeatedly make use of the Lyapunov functional introduced in Chapter 3, whereas the periodic solutions in [**3**] are constructed without such a counting tool for the zeros of solutions. However, the final result on the existence of a periodic solution for the above DDE for $a > 1$ will be completely the same as in case of Equation (3.2).

From now on and until the end of this chapter, we make the standing assumption that $a > 1$. Then, by Corollary 4.18, we see the existence of a local center-unstable manifold

$$W_{cu} = \{\psi + w_{cu}(\psi) \,|\, \psi \in C_{cu,0}\} \subset X_f$$

of the model equation above, or more precisely, of its abstract representation (4.1), at the stationary point $\varphi_0 = 0$. Additionally, our consideration of the model so far immediately implies that the strip

$$X_f^{2a} := \left\{\varphi \in X_f \,|\, \|\varphi\|_C < 2a\right\}$$

of the solution manifold X_f is non-empty, positively invariant for the semiflow F in view of Proposition 3.6 and $t_+(\varphi) = \infty$ for all $\varphi \in X_f^{2a}$. Furthermore, combining the above with Proposition 4.21 and Corollary 4.23 in the preceding chapter straightforwardly leads to the following repellent behavior of the trivial solution in $W_{cu} \cap X_f^{2a}$.

COROLLARY 5.1. *There is a neighborhood N^{cu} of 0 in $W_{cu} \cap X_f^{2a}$ with the following properties:*

(i) *For each non-trivial* $\varphi \in N^{cu}$ *there is a uniquely determined solution* $x^\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ *of Equation* (4.1) *satisfying* $x_t^\varphi \in N^{cu}$ *as* $t \leq 0$, $x^\varphi(t) \to 0$ *as* $t \to -\infty$ *and* $x_s^\varphi \notin N^{cu}$ *for some* $s > 0$.

(ii) *If* $\varphi \in N^{cu}$ *and* $\varphi \neq 0$, *then either*
$$\mathrm{dist}_{C^1}\left(\frac{x_t^\varphi}{\|x_t^\varphi\|_{C^1}}, \left\{\frac{\eta_0}{\|\eta_0\|_{C^1}}, \frac{-\eta_0}{\|\eta_0\|_{C^1}}\right\}\right) \to 0$$
or
$$\mathrm{dist}_{C^1}\left(\frac{x_t^\varphi}{\|x_t^\varphi\|_{C^1}}, \left\{\frac{\eta_\kappa}{\|\eta_\kappa\|_{C^1}}, \frac{-\eta_\kappa}{\|\eta_\kappa\|_{C^1}}\right\}\right) \to 0$$
as $t \to -\infty$.

Now the main idea for the construction of a periodic solution for Equation (4.1) is to study the forward extension

$$W := F([0,\infty) \times N^{cu}) \tag{5.1}$$

of the neighborhood N^{cu} of $\varphi_0 = 0$ from the above corollary and its closure $\overline{W}$.

The remaining part of this chapter is structured as follows. In the next section we show the invariance of $\overline{W}$ under the semiflow F and the slowly oscillating behavior of the non-trivial solutions in $\overline{W}$. Using these facts, in Section 3 we prove that solutions in $\overline{W} \setminus \{0\}$ are indeed oscillating in the sense that they have infinitely many zeros. From this, we finally conclude our main result which in particular asserts that $\overline{W} \setminus W$ is a slowly oscillating periodic orbit of the model equation.

2. The Slowly Oscillating Behavior of Solutions

Our study of the set $\overline{W}$ begins with the simple observation that W belongs to $X_f^{2a} \cap L_a$ with L_a given by (3.7) and is invariant for the semiflow F.

COROLLARY 5.2. *The set* W *is contained in* $X_f^{2a} \cap L_a$ *and for each* $\varphi \in W$ *there is a solution* $x^\varphi : \mathbb{R} \longrightarrow (-2a, 2a)$ *of Equation* (4.1) *with* $x_t^\varphi \in W$ *for all* $t \in \mathbb{R}$.

PROOF. Given $\varphi \in W$, observe that from the definition of W we find a function $\psi \in N^{cu}$ and a real $t_0 \geq 0$ with $\varphi = F(t_0, \psi)$. By Corollary 5.1, it is clear that the solution $x^\varphi = x^\psi(t_0 + \,\cdot\,)$ exists on all of $\mathbb{R}$ and that there is no loss of generality in assuming $t_0 > 2$ in view of $x_{t-t_0}^\varphi = x_t^\psi \in N^{cu}$ for $t \leq 0$. As we also have $\|\psi\|_C < 2a$, Proposition 3.6 yields $\|x_t^\psi\|_C < 2a$ for all $t \geq 0$, and we thus get $\varphi = F(t_0, \psi) \in X_f^{2a}$.

For the conclusion that φ is also contained in L_a, it is sufficient to show the Lipschitz continuity of φ with the constant $K_a > 0$ defined in (3.5). For this purpose, recall from Section 2 in Chapter 3 that for all $\chi \in C$ with $\|\chi\|_C \leq 2a$ the absolute value of $f(\chi)$ is bounded by K_a. Considering arbitrary $s, \tilde{s} \in [0,1]$ and using the continuous differentiability of $x^\varphi = x^\psi(t_0 + \,\cdot\,) : \mathbb{R} \longrightarrow \mathbb{R}$, we find a constant $0 < \xi < 1$ with

$$|\varphi(s) - \varphi(\tilde{s})| = |x^\varphi(s) - x^\varphi(\tilde{s})| = |\dot{x}^\varphi(\xi)||s - \tilde{s}|.$$

From $t_0 + \xi > 0$ we conclude $x_{t_0+\xi}^\psi \in X_f^{2a}$ and hence

$$|\dot{x}^\varphi(\xi)| = |\dot{x}_{t_0}^\psi(\xi)| = |f(x_{t_0+\xi}^\psi)| \leq K_a.$$

Consequently, φ is Lipschitz continuous with Lipschitz constant K_a, which yields $\varphi \in L_a$ as claimed.

It remains to prove the invariance of W under F, that is, $x_t^{\varphi} \in W$ for all $t \in \mathbb{R}$. To see this, let $s \in \mathbb{R}$ be given. Then, using $x^{\psi}(t) \to 0$ as $t \to -\infty$ once more, we find $k \leq 0$ with $t_0 + s + k \leq 0$ and $x^{\psi}_{t_0+s+k} \in N^{cu}$. Therefore

$$x_s^{\varphi} = x^{\psi}_{t_0+s} = F(-k, x^{\psi}_{t_0+s+k}) \in W,$$

and the proof is complete. □

Another simple conclusion from the construction of W is the fact that W can not contain any periodic solutions for Equation (4.1). The reason is that on the one hand, non-trivial solutions in W without zeros converge to 0 as $t \to \infty$ due to Proposition 3.26, but on the other hand the zero set of a non-trivial solution in W is bounded from below as shown next.

COROLLARY 5.3. *If $\varphi \in W \backslash \{0\}$ then there is a $t = t(\varphi) \in \mathbb{R}$ such that*

$$x^{\varphi}(s) \neq 0$$

for all $s \leq t(\varphi)$.

PROOF. Assuming $\varphi \in W$ and $\varphi \neq 0$, from the definition of the set W we find a function $\psi \in N^{cu}$ and a constant $t \geq 0$ with $\varphi = F(t, \psi)$. In particular, $x^{\varphi} = x^{\psi}(t + \cdot\,)$ and $\psi \neq 0$. Now we have

$$0 < \frac{e^{-\kappa}}{1+\kappa} = \frac{\min_{\vartheta \in [-1,0]} \eta_{\kappa}(\vartheta)}{\|\eta_{\kappa}\|_{C^1}} < 1 = \frac{\min_{\vartheta \in [-1,0]} \eta_0(\vartheta)}{\|\eta_0\|_{C^1}},$$

and part (ii) of Corollary 5.1 implies that either

$$\mathrm{dist}_{C^1}\left(\frac{x_s^{\psi}}{\|x_s^{\psi}\|_{C^1}}, \left\{\frac{\eta_0}{\|\eta_0\|_{C^1}}, \frac{-\eta_0}{\|\eta_0\|_{C^1}}\right\}\right) \to 0$$

or

$$\mathrm{dist}_{C^1}\left(\frac{x_s^{\psi}}{\|x_s^{\psi}\|_{C^1}}, \left\{\frac{\eta_{\kappa}}{\|\eta_{\kappa}\|_{C^1}}, \frac{-\eta_{\kappa}}{\|\eta_{\kappa}\|_{C}}\right\}\right) \to 0$$

for $s \to -\infty$. Consequently, in view of $\|\chi\|_C \leq \|\chi\|_{C^1}$ for all $\chi \in C^1$, in any case there is clearly a real $t_* \leq 0$ such that for all $s \leq t_*$ and all $-1 \leq \tilde{s} \leq 0$

$$\frac{1}{2\,(1+\kappa)\,e^{\kappa}} < \frac{|x_s^{\psi}(\tilde{s})|}{\|x_s^{\psi}\|_{C^1}}$$

holds. Therefore we get

$$0 \neq x^{\psi}(s) = x^{\varphi}(t+s)$$

for all $s \leq t_*$, and this is precisely the assertion of the corollary. □

Our next aim is to prove that all solutions of Equation (4.1) in $\overline{W}$ are slowly oscillating and that the difference of any two such solutions $x^{\varphi_1}, x^{\varphi_2} : \mathbb{R} \longrightarrow [-2a, 2a]$ with $\varphi_1 \neq \varphi_2$ has at most one sign change in $[t - r(x^{\varphi_1}(t)), t]$, $t \in \mathbb{R}$. In preparation for the proofs of these assertions, we will need some auxiliary results involving the discrete Lyapunov functional V introduced in Section 3 of the last chapter. The first one particularly yields that for solutions $x^{\varphi_1}, x^{\varphi_2} : \mathbb{R} \longrightarrow [-2a, 2a]$ of Equation (4.1) with initial functions

$$x_0^{\varphi_1} = \varphi_1 \neq \varphi_2 = x_0^{\varphi_2}$$

we have

$$(x^{\varphi_1} - x^{\varphi_2})\,|_{[t - r(x^{\varphi_1}(t)), t]} \neq 0$$

for all $t \in \mathbb{R}$, such that the values

$$V(x^{\varphi_1} - x^{\varphi_2}, [t - r(x^{\varphi_1}(t)), t])$$

are indeed defined.

PROPOSITION 5.4. *Let $\varphi_1, \varphi_2 \in X_f$ with $\varphi_1 \neq \varphi_2$ and $\|\varphi_1\|_C, \|\varphi_2\|_C \leq 2a$ be given, and suppose that $x^{\varphi_1}, x^{\varphi_2} : \mathbb{R} \longrightarrow [-2a, 2a]$ are associated solutions of Equation* (4.1) *with initial values $x_0^{\varphi_1} = \varphi_1$, $x_0^{\varphi_2} = \varphi_2$, respectively. Then*

$$(x^{\varphi_1} - x^{\varphi_2})|_{[t - r(x^{\varphi_1}(t)), t]} \neq 0$$

for all $t \in \mathbb{R}$. Moreover, under the same hypothesis, the following conditions are equivalent.

(i) $V(x^{\varphi_1} - x^{\varphi_2}, [t - r(x^{\varphi_1}(t)), t]) = 1$ *for all* $t \in \mathbb{R}$.
(ii) $V(x^{\varphi_2} - x^{\varphi_1}, [t - r(x^{\varphi_2}(t)), t]) = 1$ *for all* $t \in \mathbb{R}$.

PROOF. We begin with the observation that Corollary 3.13 in Section 3 of Chapter 3 implies $x_t^{\varphi_1} \neq x_t^{\varphi_2}$ and therefore

$$(x^{\varphi_1} - x^{\varphi_2})|_{[t-1, t]} \neq 0$$

for all $t \in \mathbb{R}$. Suppose now, contrary to our claim, that there is a $t \in \mathbb{R}$ with

$$(x^{\varphi_1} - x^{\varphi_2})|_{[t - r(x^{\varphi_1}(t)), t]} = 0.$$

Define

$$t_0 := \inf \left\{ s \in \mathbb{R} \mid x^{\varphi_1}(\vartheta) = x^{\varphi_2}(\vartheta) \text{ for all } s \leq \vartheta \leq t \right\}.$$

Then, by assumption, we have $t - 1 < t_0 \leq t - r(x^{\varphi_1}(t))$ and subsequently also

$$\dot{x}^{\varphi_1}(s) - \left[a - |x^{\varphi_1}(s)|\right] x^{\varphi_1}(s) = \dot{x}^{\varphi_2}(s) - \left[a - |x^{\varphi_2}(s)|\right] x^{\varphi_2}(s)$$

as well as

$$r(x^{\varphi_1}(s)) = r(x^{\varphi_2}(s))$$

for all $t_0 \leq s \leq t$. As x^{φ_1}, x^{φ_2} are both solutions of Equation (4.1) and $a \neq 0$, we see that

$$x^{\varphi_1}(s - r(x^{\varphi_1}(s))) = x^{\varphi_2}(s - r(x^{\varphi_2}(s)))$$

as $t_0 \leq s \leq t$. Consequently, the above assumption leads to the conclusion $x^{\varphi_1}(s) = x^{\varphi_2}(s)$ for all $\min\{\vartheta - r(x^{\varphi_1}(\vartheta))|\ t_0 \leq \vartheta \leq t\} \leq s \leq t$ in contradiction to the definition of t_0 in view of condition (DF 2) on the delay functional. This yields

$$(x^{\varphi_1} - x^{\varphi_2})\,|_{[t - r(x^{\varphi_1}(t)), t]} \neq 0$$

for $t \in \mathbb{R}$.

For the second part of the proposition, we initially note that the application of the first part to the situation with reversed roles of φ_1 and φ_2 shows

$$(x^{\varphi_1} - x^{\varphi_2})\,|_{[t - r(x^{\varphi_2}(t)), t]} \neq 0$$

as $t \in \mathbb{R}$. Hence, for each $t \in \mathbb{R}$, both values $V(x^{\varphi_1} - x^{\varphi_2}, [t - r(x^{\varphi_1}(t)), t])$ and $V(x^{\varphi_1} - x^{\varphi_2}, [t - r(x^{\varphi_2}(t)), t])$ are in fact defined, and it is sufficient to show the implication $(i) \Rightarrow (ii)$. Accordingly, assume that for all $t \in \mathbb{R}$

$$V(x^{\varphi_1} - x^{\varphi_2}, [t - r(x^{\varphi_1}(t)), t]) = 1,$$

and consider first the case where there is a sufficiently small constant $-\infty < M \leq 0$ such that $x^{\varphi_1}(t) - x^{\varphi_2}(t) \neq 0$ as $-\infty < t \leq M$. Then, by definition of V, we see at once

$$V(x^{\varphi_1} - x^{\varphi_2}, [t - r(x^{\varphi_2}(t), t]) = 1$$

for all $t \leq M$. Now, combining Corollary 3.11 and Proposition 3.12 yields the applicability of Proposition 3.16 on the monotonicity of V which leads to

$$1 = V(x^{\varphi_1} - x^{\varphi_2}, [M - r(x^{\varphi_2}(M)), M]) \geq V(x^{\varphi_1} - x^{\varphi_2}, t[-r(x^{\varphi_2}(t)), t]) \geq 1$$

for $t > M$. Hence, $V(x^{\varphi_1} - x^{\varphi_2}, [t - r(x^{\varphi_2}), t]) = 1$ for all $t \in \mathbb{R}$ in this case.

If there does not exist a constant $M \leq 0$ such that $x^{\varphi_1}(t) - x^{\varphi_2}(t) \neq 0$ for all reals $t \leq M$, then we find a sequence $\{t_k\}_{k\in\mathbb{N}}$ with $t_k \to -\infty$ as $k \to \infty$ and $x^{\varphi_1}(t_k) - x^{\varphi_2}(t_k) = 0$ for all $n \in \mathbb{N}$. In particular, $r(x^{\varphi_1}(t_k)) = r(x^{\varphi_2}(t_k))$ and therefore

$$V(x^{\varphi_1} - x^{\varphi_2}, [t_k - r(x^{\varphi_2}(t_k)), t_k]) = V(x^{\varphi_1} - x^{\varphi_2}, [t_k - r(x^{\varphi_1}(t_k), t_k]) = 1$$

for $k \in \mathbb{N}$. Subsequently, for each $t \in \mathbb{R}$ there is an index $k \in \mathbb{N}$ with $t_k < t$ such that the application of Proposition 3.16 on the monotonicity property of V results in

$$1 = V(x^{\varphi_1} - x^{\varphi_2}, [t_k - r(x^{\varphi_2}(t_k))]) \geq V(x^{\varphi_1} - x^{\varphi_2}, [t - r(x^{\varphi_2}(t))]) \geq 1.$$

This shows $V(x^{\varphi_1} - x^{\varphi_2}, [t - r(x^{\varphi_2}(t)), t]) = 1$ as $t \in \mathbb{R}$ and completes the proof. $\square$

The next step towards the result that $\overline{W}$ is invariant for F and every non-trivial solution of Equation (4.1) in $\overline{W}$ is slowly oscillating is the conclusion that solutions in $W \setminus \{0\}$ are slowly oscillating. This will be a simple consequence of the following proposition.

Proposition 5.5. *If $\varphi_1, \varphi_2 \in N^{cu}$ and $\varphi_1 \neq \varphi_2$ then*

$$V(\varphi_1 - \varphi_2, [-r(\varphi_1(0)), 0]) = 1.$$

Proof. As the proof is technical, we divide it into three consecutive parts.

1. Consider for given $\varphi_1, \varphi_2 \in N^{cu}$ with $\varphi_1 \neq \varphi_2$ the associated global solutions $x^{\varphi_1}, x^{\varphi_2} : \mathbb{R} \longrightarrow [-2a, 2a]$ of the differential equation (4.1) with the initial values $x_0^{\varphi_1} = \varphi_1$, $x^{\varphi_2} = \varphi_2$, respectively, which exists due to Corollary 5.1. As $x^{\varphi_1}(t), x^{\varphi_2}(t) \to 0$ for $t \to -\infty$, we find a sequence $\{t_k\}_{k\in\mathbb{N}} \subset (-\infty, 0]$ with $t_k \to -\infty$ as $k \to \infty$, satisfying

$$|x^{\varphi_1}(t_k) - x^{\varphi_2}(t_k)| = \sup\left\{|x^{\varphi_1}(t_k + t) - x^{\varphi_2}(t_k + t)| \,\middle|\, t \leq 0\right\}$$

for all $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ define the function

$$z^k : (-\infty, 0] \ni t \longmapsto \frac{x^{\varphi_1}(t_k + t) - x^{\varphi_2}(t_k + t)}{|x^{\varphi_1}(t_k) - x^{\varphi_2}(t_k)|} \in \mathbb{R}.$$

Then a computation similar to the one in the proof of Proposition 3.12 shows that for each $k \in \mathbb{N}$ the function z^k satisfies the differential equation

$$\dot{z}^k(t) = \alpha_k(t)\, z^k(t) - a\, z^k(t - r(x^{\varphi_1}(t_k + t))) \tag{5.2}$$

with $\alpha_k : (-\infty, 0] \longrightarrow \mathbb{R}$ given by

$$\begin{aligned}\alpha_k(t) =& a - \int_0^1 2\,|(1-s)\cdot x^{\varphi_2}(t_k + t) + s \cdot x^{\varphi_1}(t_k + t)|\, ds \\ &+ a \int_0^1 \dot{x}^{\varphi_2}\big((1-s)(t_k + t - r(x^{\varphi_2}(t_k + t))) + s(t_k + t - r(x^{\varphi_1}(t_k + t)))\big) ds \\ &\cdot \int_0^1 r'\big((1-s)\cdot x^{\varphi_1}(t_k + t) + s \cdot x^{\varphi_2}(t_k + t)\big) ds\end{aligned}$$

for $k \in \mathbb{N}$. Using $x^{\varphi_1}(t), x^{\varphi_2}(t) \to 0$ as $t \to -\infty$, we also see at once $\alpha_k(t) \to a$ and $r(x^{\varphi_1}(t_k + t)) \to 1$ uniformly on $(-\infty, 0]$ for $k \to \infty$.

2. Next we claim that there is a subsequence $\{z^{k_l}\}_{l\in\mathbb{N}}$ of $\{z^k\}_{k\in\mathbb{N}}$ and a continuously differentiable function $z : (-\infty, 0] \longrightarrow \mathbb{R}$ such that $z^{k_l} \to z$ uniformly on compact subsets of $(-\infty, 0]$ as $l \to \infty$ and additionally, z satisfies the differential equation

$$\dot{z}(t) = a\big[z(t) - z(t-1)\big]$$

on $(-\infty, 0]$ and its segments the inequality

$$\|z_t\|_C \leq |z(0)| = 1$$

for all $t \leq 0$.

To verify the above claim, we start with the simple observation that the sequence $\{z^k\}_{k\in\mathbb{N}}$ of real-valued functions with domain $(-\infty, 0]$ obtained in the first part is equicontinuous and equibounded. Indeed, by definition, we have $|z^k(t)| \leq 1$ for all $k \in \mathbb{N}$ and $t \leq 0$ and from the differential equations (5.2) we moreover conclude the existence of a real $\widehat{K} > 0$ satisfying $|\dot{z}^k(t)| \leq \widehat{K}$ as $k \in \mathbb{N}$ and $t \leq 0$. Thus, for each $T < 0$, the sequence $\{z^k|_{[T,0]}\}_{k\in\mathbb{N}}$ is relatively compact as a subset of the Banach space $C([T, 0], \mathbb{R})$ due to the Arzelà-Ascoli theorem. We proceed with a diagonalization process with respect to $T < 0$.

Fix $T < 0$ and define the function sequence $\{v^k\}_{k\in\mathbb{N}}$ by $v^k := z^k$, $k \in \mathbb{N}$. As $\{v^k|_{[T,0]}\}_{k\in\mathbb{N}}$ is relatively compact in $C([T, 0], [-1, 1])$ by the above, we find a subsequence $\{v^{k_l}|_{[T,0]}\}_{l\in\mathbb{N}}$ of $\{v^k|_{[T,0]}\}_{k\in\mathbb{N}}$ and a function $w^1 \in C([T, 0], [-1, 1])$ with $v^{k_l}(t) \to w^1(t)$ uniformly on $[T, 0]$ as $l \to \infty$. Define $u^1 := v^{k_1}$. For the next step, set $v^l := v^{k_l}$ as $l \in \mathbb{N}$, to simplify notation. As the sequence $\{v^l|_{[2T,0]}\}_{l\in\mathbb{N}}$ is relatively compact in the function space $C([2T, 0], [-1, 1])$, we conclude, completely analogous to our first step, the existence of a subsequence $\{v^{l_m}|_{[2T,0]}\}_{m\in\mathbb{N}}$ of $\{v^l|_{[2T,0]}\}_{l\in\mathbb{N}}$ and a continuous function $w^2 \in C([2T, 0], [-1, 1])$ with $v^{l_m}(t) \to w^2(t)$ uniformly on $[2T, 0]$ as $l \to \infty$. Clearly, we have $w^2|_{[T,0]} = w^1$. Define $u^2 := v^{l_2}$. By induction, we find a subsequence $\{u^k\}_{k\in\mathbb{N}}$ of the original sequence $\{v^k\}_{k\in\mathbb{N}}$ and a continuous function $w \in C((-\infty, 0], [-1, 1])$ with the property $u^k \to w(t)$ as $k \to \infty$, where the convergence is uniform in t on each compact subset of $(-\infty, 0]$.

Combining the above with the differential equations (5.2) for u^k, we see that w is also continuously differentiable and that the sequence $\{\dot{u}^k\}_{k\in\mathbb{N}}$ of the derivatives of functions u^k, $k \in \mathbb{N}$, converges uniformly to $\dot{w}$ on compact subsets of $(-\infty, 0]$ as $k \to \infty$. Furthermore, w satisfies the limit differential equation

$$\dot{w}(t) = a\big[w(t) - w(t-1)\big]$$

since $\alpha_k(t) \to a$ and $r(x^{\varphi_1}(t_k)) \to 1$ for $k \to \infty$. As we obviously also have $|w(0)| = 1$ and $\|w_t\|_C \leq 1$ for $t \leq 0$, the choice $z^{k_l} = u^l$, $l \in \mathbb{N}$, and $z = w$ proves our claim.

3. Consider the function $z : (-\infty, 0] \longrightarrow \mathbb{R}$ obtained in the last step. From Corollary 4.4 and the subsequent comment, we see that there are $\alpha, \beta \in \mathbb{R}$ with $z(t) = \alpha + \beta\, e^{\kappa t}$, $t \leq 0$. Moreover, $(\alpha, \beta) \neq (0, 0)$ since $z(0) \neq 0$. Accordingly, using Corollary 4.5 we get $V(z, [t-1, t]) = 1$ and $z|_{[t-1,t]} \in H_{[t-1,t]}$ for all $t \leq 0$ and thus, part (iii) of Proposition 3.15 yields the existence of $\widehat{N} \in \mathbb{N}$ implying

$$V(z^{k_l}, [-r(x^{\varphi_1}(t_{k_l})), 0]) = V(z, [-1, 0]) = 1$$

for all $l \geq \widehat{N}$. Next, by Proposition 3.12, we find bounded continuous functions $\tilde{u}, \tilde{\alpha} : \mathbb{R} \longrightarrow \mathbb{R}$ with $\tilde{\alpha}(t) < 0$ such that $v : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$v(t) := \big(x^{\varphi_1}(t) - x^{\varphi_2}(t)\big) \cdot \exp\left(-\int_0^t \tilde{u}(s)ds\right)$$

satisfies the differential equation

$$\dot{v}(t) = \tilde{\alpha}(t)\, v(t - r(x^{\varphi_1}(t))).$$

Using Proposition 3.16, we finally get

$$\begin{aligned}
1 &\leq V(\varphi_1 - \varphi_2, [-r(\varphi_1(0)), 0]) \\
&= V(v, [-r(\varphi_1(0)), 0]) \\
&\leq V(v, [t_{k_l} - r(\varphi_1(t_{k_l})), t_{k_l}]) \\
&= V(x^{\varphi_1}(t_{k_l} + \,\cdot\,) - x^{\varphi_2}(t_{k_l} + \,\cdot\,), [-r(x^{\varphi_1}(t_{k_l})), 0]) \\
&= V(z^{k_l}, [-r(x^{\varphi_1}(t_{k_l})), 0]) \\
&= V(z, [-1, 0]) = 1
\end{aligned}$$

for all sufficiently large $k \in \mathbb{N}$, which establishes the asserted formula. $\square$

As already mentioned, a simple consequence of the last proposition is the fact that all non-trivial solutions of the model equation in W are slowly oscillating, which is part of the corollary below.

COROLLARY 5.6. *Suppose that $\varphi_1, \varphi_2 \in W$ satisfy $\varphi_1 \neq \varphi_2$. Then*

$$V(x^{\varphi_1} - x^{\varphi_2}, [t - r(x^{\varphi_1}(t)), t]) = 1$$

for all $t \in \mathbb{R}$. In particular, each non-trivial solution in W is slowly oscillating.

PROOF. 1. We first prove that for given functions $\varphi_1, \varphi_2 \in W$ with $\varphi_1 \neq \varphi_2$ we have $V(\varphi_1 - \varphi_2, [-r(x^{\varphi_1}(0), 0]) = 1$. For this purpose, we choose reals $t_1, t_2 \geq 0$ and functions $\chi_1, \chi_2 \in N^{cu}$ with $\varphi_1 = F(t_1, \chi_1)$ and analogously $\varphi_2 = F(t_2, \chi_2)$ from the definition of W. As by Corollary 5.1 we have $x^{\chi_j}(t) = x^{\varphi_j}(t_j + \,\cdot\,) \to 0$ as $t \to -\infty$ and $x_t^{\chi_j} \in N^{cu}$ for all $t \leq 0$, there actually is a constant $t_0 \geq 0$ and functions $\psi_1, \psi_2 \in N^{cu}$ with $\varphi_1 = F(t_0, \psi_1)$ and $\varphi_2 = F(t_0, \psi_2)$. In particular, $\psi_1 \neq \psi_2$. From Proposition 3.12 we now find functions $\tilde{u}, \tilde{\alpha} : \mathbb{R} \longrightarrow \mathbb{R}$ with $\tilde{\alpha} < 0$ such that $v : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$v(t) := \big(x^{\psi_1}(t) - x^{\psi_2}(t)\big) \cdot \exp\left(-\int_0^t \tilde{u}(s)ds\right)$$

satisfies the differential equation

$$\dot{v}(t) = \tilde{\alpha}(t)\, v(t - r(x^{\psi_1}(t)))$$

for all $t \in \mathbb{R}$. Hence, Proposition 3.16 in combination with the last result leads to

$$\begin{aligned}
1 &= V(\psi_1 - \psi_2, [-r(\psi_1(0)), 0]) \\
&= V(x^{\psi_1} - x^{\psi_2}, [-r(\psi_1(0), 0]) \\
&= V(v, [-r(\psi_1(0)), 0]) \\
&\geq V(v, [t_0 - r(x^{\psi_1}(t_0)), t_0]) \\
&= V(x^{\psi_1} - x^{\psi_2}, [t_0 - r(x^{\psi_1}(t_0)), t_0]) \\
&= V(x^{\psi_1}(t_0 + \,\cdot\,) - x^{\psi_2}(t_0 + \,\cdot\,), [-r(x^{\psi_1}(t_0)), 0]) \\
&= V(x^{\varphi_1} - x^{\varphi_2}, [-r(x^{\varphi_1}(0)), 0]) \\
&= V(\varphi_1 - \varphi_2, [-r(\varphi_1(0)), 0]) \geq 1,
\end{aligned}$$

which implies the desired conclusion

$$V(\varphi_1 - \varphi_2, [-r(\varphi_1(0)), 0]) = 1.$$

2. To see that for given $\varphi_1, \varphi_2 \in W$ with $\varphi_1 \neq \varphi_2$ and all $t \neq 0$ we also have $V(x^{\varphi_1} - x^{\varphi_2}, [t - r(x^{\varphi_1}(t)), t]) = 1$, recall from Corollary 5.2 the invariance of W which implies $x_t^{\varphi_1}, x_t^{\varphi_2} \in W$. In particular, we have $x_t^{\varphi_1} \neq x_t^{\varphi_2}$. Therefore, the part above yields

$$\begin{aligned}
1 &= V(x_t^{\varphi_1} - x_t^{\varphi_2}, [-r(x_t^{\varphi_1}(0)), 0]) \\
&= V(x^{\varphi_1}(t + \,\cdot\,) - x^{\varphi_2}(t + \,\cdot\,), [-r(x^{\varphi_1}(t)), 0]) \\
&= V(x^{\varphi_1} - x^{\varphi_2}, [t - r(x^{\varphi_1}(t)), t])
\end{aligned}$$

and this completes the proof of the first part of the statement.

3. It remains to prove that non-trivial solutions $x : \mathbb{R} \longrightarrow [-2a, 2a]$ of the model equation with segments in W are slowly oscillating. But this is clear in view of Proposition 3.28 since the application of the part above to solutions determined by the initial values $\varphi_1 = x_0$ and $\varphi_2 = 0$ yields $V(x, [t - r(x(t)), t]) = 1$ for all $t \in \mathbb{R}$. ☐

Now we are in the position to establish the assertion that $\overline{W}$ is also invariant for the continuous semiflow F and all solutions of the model equation in $\overline{W} \setminus \{0\}$ are slowly oscillating as well.

PROPOSITION 5.7. *Consider $\mathcal{A}$ defined in Proposition 3.27. Then $W \subset \mathcal{A}$, and the closure $\overline{W} \subset \mathcal{A}$ is compact. Moreover, for every $\varphi \in \overline{W} \setminus \{0\}$ there is a uniquely determined slowly oscillating solution $x^\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ of Equation (4.1) with $x_0^\varphi = \varphi$, $x_t^\varphi \in \overline{W}$ for all $t \in \mathbb{R}$.*

PROOF. 1. Let $\varphi \in W$ be given. If $\varphi = 0$, then we clearly have $\varphi \in \mathcal{A}$. Otherwise $\varphi \neq 0$, and the last corollary yields that the uniquely defined solution $x^\varphi : \mathbb{R} \longrightarrow [-2a, 2a]$ with $x_0^\varphi = \varphi$ is slowly oscillating. Using part (iv) of Proposition 3.28, we get $\varphi \in \mathcal{A} \setminus \{0\}$. This shows $W \subset \mathcal{A}$.

2. The compactness of $\overline{W}$ is an immediate consequence of the first part as $\mathcal{A}$ is compact due to Proposition 3.27 and Corollary 3.31 on the equivalence of the both topologies on $\mathcal{A}$ induced by the norms of C and C^1.

3. From Proposition 3.28 it is now clear that for all $\varphi \in \overline{W} \setminus \{0\}$ there is a uniquely determined slowly oscillating solution $x^\varphi : \mathbb{R} \longrightarrow [-2a, 2a]$ of Equation (4.1) with initial value φ at $t = 0$. Therefore, it remains to prove that for such a solution we indeed have $x_t^\varphi \in \overline{W}$ for all $t \in \mathbb{R}$. For $\varphi = 0$ this is clear. In the situation $\varphi \neq 0$ we find a sequence $\{\varphi_k\}_{k \in \mathbb{N}} \subset W$ with $\varphi_k \to \varphi$ as $k \to \infty$. As W is

invariant for F, we get $F(t,\varphi) = \lim_{k\to\infty} F(t,\varphi_k) \in \overline{W}$ for all $t \geq 0$, which implies the positive invariance of $\overline{W}$ for F. Now the assertion $x_t^\varphi \in \overline{W}$ for all $t \in \mathbb{R}$ follows if we prove that for each $j \in \mathbb{N}$ there is a function $\psi_j \in \overline{W}$ satisfying $F(j,\psi_j) = \varphi$. To deduce this, consider for a fixed $j \in \mathbb{N}$ the functions $\psi_{j,k} := F(-j,\varphi_k) \in W$, $k \in \mathbb{N}$. Then the compactness of $\overline{W}$ implies the existence of $\psi_j \in \overline{W}$ and a subsequence $\{\psi_{j,k_l}\}_{l\in\mathbb{N}}$ with $\psi_{j,k_l} \to \psi_j$ as $l \to \infty$. Consequently, the continuity of F implies

$$F(j,\psi_j) = \lim_{l\to\infty} F(j,\psi_{j,k_l}) = \lim_{l\to\infty} F(j,F(-j,\varphi_{k_l})) = \lim_{l\to\infty} \varphi_{k_l} = \varphi,$$

and this completes the proof. □

The above result on the invariance of $\overline{W}$ under F allows to extend the statement of Corollary 5.6 on the number of sign changes of differences of solutions to the closure of W.

Corollary 5.8. *Given any $\varphi_1, \varphi_2 \in \overline{W}$ suppose that $\varphi_1 \neq \varphi_2$ holds. Then*

$$V(x^{\varphi_1} - x^{\varphi_2}, [t - r(x^{\varphi_1}(t)), t]) = 1$$

for all $t \in \mathbb{R}$.

Proof. We proceed analogously to the proof of Corollary 5.6 and first show that for given functions $\varphi_1, \varphi_2 \in \overline{W}$ with $\varphi_1 \neq \varphi_2$

$$V(\varphi_1 - \varphi_2, [-r(\varphi_1(0)), 0]) = 1$$

holds. Thereto, we choose a sequence $\{\psi_k^1\}_{k\in\mathbb{N}} \subset W$ and a sequence $\{\psi_k^2\}_{k\in\mathbb{N}}$ with $\psi_k^1 \to \varphi_1$ and $\psi_k^2 \to \varphi_2$ as $k \to \infty$. Hereby, we clearly may assume that $\psi_k^1 \neq \psi_k^2$ for all $k \in \mathbb{N}$, since $\varphi_1 \neq \varphi_2$ by assumption. Now from Corollary 5.6 it follows that

$$V(\psi_k^1 - \psi_k^2, [-r(\psi_k^1(0)), 0]) = 1$$

for all $k \in \mathbb{N}$. Hence part (i) of Proposition 3.15 gives

$$\begin{aligned} 1 &\leq V(\varphi_1 - \varphi_2, [-r(\varphi_1(0)), 0]) \\ &\leq \liminf_{k\to\infty} V(\psi_k^1 - \psi_k^2, [-r(\psi_k^1(0)), 0]) = 1; \end{aligned}$$

that is,

$$V(\varphi_1 - \varphi_2, [-r(\varphi_1(0)), 0]) = 1.$$

Let now $t \in \mathbb{R}$ be given. Then, due to the invariance of $\overline{W}$ from the last proposition, we have $x_t^{\varphi_1}, x_t^{\varphi_2} \in \overline{W}$ and in view of uniqueness, $x_t^{\varphi_1} \neq x_t^{\varphi_2}$. Thus from the above conclusion we see

$$V(x^{\varphi_1} - x^{\varphi_2}, [t - r(x^{\varphi_1}(t)), t]) = V(x_t^{\varphi_1} - x_t^{\varphi_2}, [-r(x_t^{\varphi_1}(0)), 0]) = 1.$$

□

3. The Zero Sets of Solutions

The last preliminary step smoothing the way for our proof on the existence of a periodic solution for the model equation (4.1) is the analysis of the zero sets of solutions in $\overline{W}$. Below we address this subject, following once more the ideas in Brunovský et al. [**3**].

We begin with the auxiliary observation that the delay function r being under consideration is bounded from below by $1/2$ along solutions of Equation (4.1) with values in $[-2a, 2a]$.

COROLLARY 5.9. *Suppose $x : \mathbb{R} \longrightarrow [-2a, 2a]$ is a solution of Equation* (4.1). *Then*

$$r(x(t)) > 1/2$$

for all $t \in \mathbb{R}$.

PROOF. Recall our standing assumptions (DF 1) - (DF 5) on the delay function r from pages 57 and 69 and note that it is sufficient to prove $r(s) > 1/2$ for all $0 \leq s \leq 2a$ due to condition (DF 3). Hence, let $s \in [0, 2a]$ be given. Then the fundamental theorem of calculus implies

$$r(s) = \int_0^s r'(t)\,dt + r(0) = \int_0^s r'(t)\,dt + 1.$$

From the inequality $r'(t) > -1/(4a^2)$ for all $-2a \leq t \leq 2a$ due to (DF 5) and the fact $a > 1$ we obtain

$$r(s) \geq \int_0^s \frac{-1}{4a^2}\,dt + 1 = -\frac{s}{4a^2} + 1 \geq -\frac{2a}{4a^2} + 1 = \frac{(a-1)+a}{2a} > \frac{a}{2a} = \frac{1}{2}$$

as claimed. □

Using the above conclusion, we prove that $\overline{W}$ is homeomorphic to a closed subset of the two-dimensional subspace M of C^1 from Corollary 4.3. A homeomorphism hereby is simply defined by the restriction of the projection of the Banach space C^1 along the complementary space N for M in C^1 considered in Corollary 4.8 in the previous chapter.

PROPOSITION 5.10. *Let $P_M : C^1 \longrightarrow C^1$ denote the continuous projection of C^1 along N onto M from Corollary 4.8. Then the restriction of P_M to $\overline{W}$ is injective. Moreover,*

$$P : \overline{W} \ni \varphi \longmapsto P_M\,\varphi \in P_M\,\overline{W}$$

is a homeomorphism between $\overline{W}$ and $P_M\,\overline{W}$, and $\overline{P\,W} = P\,\overline{W}$.

PROOF. 1. We begin with the proof that the restriction of P_M to $\overline{W}$ is injective. This is done by contradiction. Hence, assume that there are functions $\varphi_1, \varphi_2 \in \overline{W}$ with $\psi := \varphi_1 - \varphi_2 \neq 0$ but

$$P_M\,\psi = P_M(\varphi_1 - \varphi_2) = 0.$$

Then Corollary 4.9 shows $\psi(-1) = \psi(0) = 0$ and from the hypothesis $\psi \neq 0$ we find an additional zero $\xi \in (-1, 0)$ of ψ due to Corollary 4.10. In view of the last result, it follows that

$$-r(\varphi_1(0)) \leq \xi < 0$$

or

$$\xi - r(\varphi_1(\xi)) \leq -1 < \xi.$$

On the other hand, Corollary 5.8 implies

$$V(x^{\varphi_1} - x^{\varphi_2}, [t - r(x^{\varphi_1}(t)), t]) = 1 \tag{5.3}$$

for all $t \in \mathbb{R}$ where $x^{\varphi_1}, x^{\varphi_2} : \mathbb{R} \longrightarrow [-2a, 2a]$ denote the solutions of Equation (4.1) with initial values φ_1, φ_2, respectively. Moreover, from Proposition 3.12 we conclude the existence of continuous bounded functions $\tilde{u}, \tilde{\alpha} : \mathbb{R} \longrightarrow \mathbb{R}$ with $\tilde{\alpha} < 0$ such that $v : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$v(t) := \big(x^{\varphi_1}(t) - x^{\varphi_2}(t)\big) \cdot \exp\left(- \int_0^t \tilde{u}(s) ds \right)$$

satisfies the differential equation

$$\dot{v}(t) = \tilde{\alpha}(t)\, v(t - r(x^{\varphi_1}(t)))$$

for all $t \in \mathbb{R}$. Obviously, each zero of the difference $x^{\varphi_1} - x^{\varphi_2}$ and so of ψ is also a zero of v. Applying part (iii) of Proposition 3.16 to v, we see

$$v|_{[t-r(x^{\varphi_1}(t)),t]} \in H_{[t-r(x^{\varphi_1}(t)),t]}$$

as $t \in \mathbb{R}$. This implies that all zeros of v are simple and additionally that

$$\xi \neq -r(\varphi_1(0)) = -r(x^{\varphi_1}(0))$$

as well as

$$-1 \neq \xi - r(\varphi_1(\xi)) = \xi - r(x^{\varphi_1}(\xi)).$$

Hence, as the function $\mathbb{R} \ni t \longmapsto t - r(x^{\varphi_1}(t)) \in \mathbb{R}$ is monotonically increasing and continuous we find a sufficient small constant $\varepsilon > 0$ such that

$$\varepsilon - r(x^{\varphi_1}(\varepsilon)) < \xi < 0 < \varepsilon$$

or

$$\xi + \varepsilon - r(x^{\varphi_1}(\xi + \varepsilon)) < -1 < \xi < \xi + \varepsilon\,;$$

that is, at least the interior of one of the intervals

$$I_1 := [\varepsilon - r(x^{\varphi_1}(\varepsilon)), \varepsilon]$$

and

$$I_2 := [\xi + \varepsilon - r(x^{\varphi_1}(\xi + \varepsilon)), \xi + \varepsilon]$$

contains two zeros of v. Since each zero of v leads to a sign change in view of the simplicity of zeros, we have $\mathrm{sc}(v|_{I_1}, I_1) \geq 2$ or $\mathrm{sc}(v|_{I_2}, I_2) \geq 2$. However, as

$$\mathrm{sc}(v|_{[t-r(x^{\varphi_1}(t)),t]}, [t - r(x^{\varphi_1}(t)), t]) \leq V(v, [t - r(x^{\varphi_1}(t)), t])$$

for all $t \in \mathbb{R}$ by definition, in any case we obtain a contradiction to Equation (5.3). Consequently, there are no functions $\varphi_1, \varphi_2 \in \overline{W}$ with $\varphi_1 \neq \varphi_2$ but $P_M\, \varphi_1 = P_M\, \varphi_2$. This proves the injectivity of the restriction of P_M to $\overline{W}$.

2. By the above, P is a continuous bijection from $\overline{W}$ onto $P_M\, \overline{W}$. Hence, the compactness of $\overline{W}$ implies the continuity of P^{-1}.

For the last part of the statement observe that $\overline{P\, W} \supset P\, \overline{W}$ immediately follows from the continuity of the operator P. The inverse relation is a consequence of the compactness of $P\, \overline{W}$ in combination with $\overline{P\, W} \subset \overline{P\, \overline{W}}$. ☐

Despite our analysis of solutions for the model equation (4.1) and their slowly oscillating behavior so far, one question still unanswered is whether there is a slowly oscillating solution with infinitely many zeros. Under the assumptions made on the delay functional, this is indeed the case since all solutions in $\overline{W} \setminus \{0\}$ have this property. For the proof of this statement we need the following corollary of the preceding proposition.

Corollary 5.11. *Suppose that $\varphi \in N^{cu}$ is an interior point of N^{cu}. Then $P\,N^{cu}$ forms a neighborhood of $P\,\varphi$ in M; that is, there is an open set $V \subset M$ with $P\,\varphi \in V \subset P\,N^{cu}$.*

Proof. If $\varphi \in N^{cu}$ is an interior point of N^{cu}, then in particular we have $\varphi \in W_{cu}$ and thus, $\varphi = P\,\varphi + w_{cu}(P\,\varphi)$. Suppose now that the assertion of the corollary is false. Using the openness of $C_{cu,0}$ in M and the relation

$$P\,N^{cu} \subset C_{cu,0} \subset M,$$

we find a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ with $\psi_n \in C_{cu,0}$ but $\psi_n \notin P\,N^{cu}$ for all $n \in \mathbb{N}$ such that $\psi_n \to P\,\varphi$ as $n \to \infty$. Define $\varphi_n := \psi_n + w_{cu}(\psi_n) \in W_{cu}$. Then by the continuity of w_{cu}

$$\varphi_n = \psi_n + w_{cu}(\psi_n) \to P\,\varphi + w_{cu}(P\,\varphi) = \varphi$$

for $n \to \infty$. Now φ is an interior point of N^{cu} and therefore there clearly is a sufficiently large $\widehat{N} \in \mathbb{N}$ such that

$$\varphi_n = \psi_n + w_{cu}(\psi_n) \in N^{cu}$$

for all $n \geq \widehat{N}$. But this implies

$$P\,\varphi_n = P\,\psi_n + P\,w_{cu}(\psi_n) = \psi_n + 0 = \psi_n \in P\,N^{cu}$$

for all $n \geq \widehat{N}$, in contradiction to our assumption $\{\psi_n\}_{n\in\mathbb{N}} \subset C_{cu,0} \backslash P\,N^{cu}$. Hence, PN^{cu} is a neighborhood of $P\,\varphi$ in M as claimed. □

A consequence of the above corollary is the fact that the set $P\,W$ is open in the two dimensional Banach space M. This is shown in our next corollary.

Corollary 5.12. *The set $P\,W$ is open in M.*

Proof. Recall from Corollary 5.2 that for every $\varphi \in W$ there is a unique solution $x^\varphi : \mathbb{R} \longrightarrow [-2a, 2a]$ and that for all $t \in \mathbb{R}$ the segments x_t^φ belong to W. Furthermore, observe that in view of Proposition 5.7 and Corollary 3.31 the restriction $F_{\mathcal{A}}^W$ of the continuous flow $F_{\mathcal{A}}$ from Proposition 3.27 to $\mathbb{R} \times W$ generates also a continuous flow on W, even though the topology on W is induced by the norm of the Banach space C^1. In particular, we clearly have $F_{\mathcal{A}}^W(t, \varphi) = x_t^\varphi$ for all $t \in \mathbb{R}$ and all $\varphi \in W$.

Suppose now $\varphi \in P\,W$. We have to show that there is a neighborhood $U \subset M$ of φ with $U \subset P\,W$. By assumption, we have $\xi := P^{-1}\,\varphi \in W$. Consider the associated solution $x^\xi : \mathbb{R} \longrightarrow [-2a, 2a]$. By Corollary 5.1, it follows $x_t^\xi \to 0 \in N^{cu}$ as $t \to -\infty$. Therefore, we find an open neighborhood V of 0 in M and a $T > 0$ such that

$$P \circ F_{\mathcal{A}}^W(-T, P^{-1}\,\varphi) = P \circ F_{\mathcal{A}}^W(-T, \xi) = P\,x_{-T}^\xi \in V \subset P\,N^{cu}.$$

As for each $t \in \mathbb{R}$ the mapping $P \circ F_{\mathcal{A}}^W(t, \cdot\,) \circ P^{-1} : P\,W \longrightarrow P\,W$ is continuous, we conclude that the set

$$\left(P \circ F_{\mathcal{A}}^W(-T, \cdot\,) \circ P^{-1}\right)^{-1}(V) \subset P\,W$$

is open. Additionally, we have

$$\varphi \in \left(P \circ F_{\mathcal{A}}^W(-T, \cdot\,) \circ P^{-1}\right)^{-1}(V) \subset P\,W.$$

This proves the assertion. □

We can now return to the solutions of Equation (4.1) in $\overline{W} \setminus \{0\}$ and prove, as already announced, that their zero sets are unbounded.

PROPOSITION 5.13. *Consider $\varphi \in \overline{W}$ with $\varphi \neq 0$. Then for every $t \in \mathbb{R}$ the interval $[t, \infty)$ contains a zero of the solution x^φ of Equation (4.1) with initial value φ at $t = 0$. Moreover, for each $\varphi \in \overline{W}\setminus\{0\}$ the zeros of x^φ form a strictly monotonically increasing sequence $\{z_j(\varphi)\}_{J(\varphi)}^{\infty} \subset \mathbb{R}$, where $J(\varphi) \in \mathbb{Z}$ for $\varphi \in W\setminus\{0\}$ and $J(\varphi) = -\infty$ otherwise. In both cases,*

$$z_{j-1}(\varphi) + 1 < z_j(\varphi)$$

for all $j > J(\varphi)$.

PROOF. We follow the ideas of the proof of Corollary 6.5 in Brunovský et al. [**3**] of the assertion for the case of constant delay.

1. Given a non-trivial $\varphi \in \overline{W}$, suppose that the solution $x^\varphi : \mathbb{R} \longrightarrow [-2a, 2a]$ of the differential equation (4.1) has no zeros in the interval $[t, \infty)$ for some $t \in \mathbb{R}$. Then, by Proposition 3.26, we get $x^\varphi(s) \to 0$ as $s \to \infty$. Hence, as N^{cu} is a neighborhood of 0 in $\overline{W}$ and thus $P\,N^{cu}$ a neighborhood of the origin in M due to the last corollary, we consequently find a real $s_0 \in \mathbb{R}$ with the property that for all $s \geq s_0$ the projected segments $P\,x_s$ are contained in $P\,N^{cu}$. But this implies $x_s \in N^{cu}$ for $s \geq s_0$ in consideration of Proposition 5.10 and therefore contradicts the fact that each positive semi-orbit of F through a value in $N^{cu}\setminus\{0\}$ leaves N^{cu} in finite time due to Corollary 5.1. Thus $[t, \infty)$ contains a zero of x^φ.

2. Now consider the case $\varphi \in W\setminus\{0\}$. Combining the above with Corollary 5.3, we find a smallest zero $z_0(\varphi)$ of x^φ in $\mathbb{R}$. As x^φ is slowly oscillating and its segments belong to $W \subset L_a$ we may apply Proposition 3.24, which yields

$$\operatorname{sign}(x^\varphi(t)) = -\operatorname{sign}(x^\varphi(z_0(\varphi) - 1)) \neq 0$$

for $z_0(\varphi) < t \leq z_0(\varphi) + 1$. Define $z_1(\varphi)$ as the smallest zero of the solution x^φ in $(z_0(\varphi) + 1, \infty)$ and proceed by induction. In this way, we obtain a strictly monotonous sequence $\{z_j(\varphi)\}_{j\in\mathbb{N}}$ of zeros of x^φ.

3. Suppose $\varphi \in \overline{W}\setminus W$ next. As both W and $\overline{W}$ are invariant for F, $x_t^\varphi \in \overline{W}\setminus W$ for all $t \in \mathbb{R}$ follows. We proceed with the conclusion that the set of zeros of x^φ is unbounded from below. For this purpose, assume contrary to this claim the existence of a $t_0 \in \mathbb{R}$ with $x^\varphi(t) \neq 0$ for all $t \leq t_0$ and consider the case $x^\varphi(t) > 0$ on $(-\infty, t_0]$ first. Since $x_t^\varphi \in \overline{W}\setminus W \subset \overline{W}\setminus N^{cu}$ for all $t \in \mathbb{R}$, the bijectivity of P immediately implies $P\,x_t^\varphi \notin P\,N^{cu}$ as $t \in \mathbb{R}$. But $P\,N^{cu}$ is a neighborhood of the origin in M due to Corollary 5.11, such that there clearly is a $c > 0$ with $c \leq \|x_t^\varphi\|_{C^1}$ for $t \in \mathbb{R}$. Therefore, the α-limit set $\alpha(\varphi)$, which belongs to $\overline{W}$ and is not empty due to the compactness of $\overline{W}$, consists only of nonnegative functions $\psi \in C^1$ with $\|\psi\|_{C^1} \geq c$. As $\alpha(\varphi)$ is invariant for F, there is a solution of Equation (4.1) with segments contained in $\alpha(\varphi)$. In particular, such a solution is slowly oscillating, nonnegative and does not converge to 0 as $t \to \infty$. But the last point is impossible in view of Proposition 3.26 and the fact that each zero of a slowly oscillating solution is simple. Thus we obtain a contradiction under the hypothesis that there is a $t_0 \in \mathbb{R}$ with $x^\varphi(t) > 0$ for all $t \in (-\infty, t_0]$. Under the assumption that there is a $t_0 \in \mathbb{R}$ with $x^\varphi(t) < 0$ for all $t \leq t_0$, a completely analogous argument works. Thus we see the unboundedness of the zero set of x^φ from below.

The proof of the remaining assertion now becomes clear. Fixing $t_0 \in \mathbb{R}$, we find a smallest zero $z_0(\varphi)$ of the solution x^φ in $[t_0, \infty)$. Since x^φ is slowly oscillating and $x_t^\varphi \in \overline{W} \subset (C^1 \cap L_a)$, there is a largest zero $z_{-1}(\varphi)$ of x^φ in $(-\infty, z_0(\varphi))$. An application of Proposition 3.24 shows

$$\operatorname{sign}(x(t)) = -\operatorname{sign}(x(z_{-1}(\varphi) - 1)) \neq 0$$

for $z_{-1} < t \leq z_{-1} + 1$. Next we find a largest zero $z_{-2}(\varphi)$ of x^φ in $(-\infty, z_{-1}(\varphi))$ and can inductively proceed to obtain a strongly increasing sequence $\{z_j\}_{-\infty}^{0}$ of zeros of x^φ in $(-\infty, t_0]$. On the other hand, Proposition 3.24 yields

$$\operatorname{sign}(x(t)) = -\operatorname{sign}(x(z_0(\varphi) - 1)) \neq 0$$

for $z_0(\varphi) < t \leq z_0(\varphi) + 1$. Hence, by the first part, there is a smallest zero $z_1(\varphi)$ of x^φ in $(z_0(\varphi), \infty)$, and we may apply the same argument again. Thus, by induction, we finally get a strongly increasing sequence $\{z_j(\varphi)\}_{-\infty}^{\infty}$ of zeros of x^φ with the asserted property. $\square$

For the rest of this work it is convenient to fix a numbering of the zeros of solutions for Equation (4.1) in $\overline{W}$. We do it in the following way.

REMARK 5.14. If $\varphi \in \overline{W}$ and $\varphi \neq 0$, then let $z_0(\varphi)$ denote the smallest zero of the solution x^φ for the differential equation (4.1) in $[-1, \infty)$.

The oscillation of the non-trivial solutions in $\overline{W}$ induces a winding behavior of the projected trajectories in $P\,\overline{W}$, which becomes exceedingly clear by means of the basis $\{\Phi_0, \Phi_{-1}\}$ for M selected prior to Corollary 4.9. Recall that we have $\Phi_0(0) = \Phi_{-1}(-1) = 0$ and $\Phi_0(-1) = \Phi_{-1}(0) = 1$.

COROLLARY 5.15. *Let $\varphi \in \overline{W}\backslash\{0\}$, and let x^φ be briefly denoted by x and the associated strongly increasing sequence of zeros due to Proposition 5.13 by $\{z_j\}_{J(\varphi)}^{\infty}$.*

(i) *If $0 < \dot{x}(z_j)$ for some j, then*

$$\begin{aligned}
&P\,x_{z_j} \in (-\infty, 0)\,\Phi_0, \\
&P\,x_t \in (-\infty, 0)\,\Phi_0 + (0, \infty)\,\Phi_{-1} && \text{for } z_j < t < z_j + 1, \\
&P\,x_{z_j+1} \in (0, \infty)\,\Phi_{-1}, \\
&P\,x_t \in (0, \infty)\,\Phi_0 + (0, \infty)\,\Phi_{-1} && \text{for } z_j + 1 < t < z_{j+1}, \text{ and} \\
&P\,x_{z_{j+1}} \in (0, \infty)\,\Phi_0.
\end{aligned}$$

(ii) *If $\varphi \in W\backslash\{0\}$, and if $0 < \dot{x}_{z_{J(\varphi)}}$ and $t < z_{J(\varphi)}$, then*

$$P\,x_t \in (-\infty, 0)\,\Phi_0 + (-\infty, 0)\,\Phi_{-1}.$$

(iii) *If $0 > \dot{x}(z_j)$ for some j, then*

$$\begin{aligned}
&P\,x_{z_j} \in (0, \infty)\,\Phi_0, \\
&P\,x_{z_j} \in (0, \infty)\,\Phi_0 + (-\infty, 0)\,\Phi_{-1} && \text{for } z_j < t < z_j + 1, \\
&P\,x_{z_j+1} \in (-\infty, 0)\,\Phi_{-1}, \\
&P\,x_t \in (-\infty, 0)\,\Phi_0 + (-\infty, 0)\,\Phi_{-1} && \text{for } z_j + 1 < t < z_{j+1}, \text{ and} \\
&P\,x_{z_{j+1}} \in (-\infty, 0)\,\Phi_0.
\end{aligned}$$

(iv) *If $\varphi \in W\backslash\{0\}$, and if $0 > \dot{x}_{z_{J(\varphi)}}$ and $t < z_{J(\varphi)}$, then*

$$P\,x_t \in (0, \infty, 0)\,\Phi_0 + (0, \infty)\,\Phi_{-1}.$$

(v) *If* $P\,x_{z_j} \in (0,\infty)\,\Phi_0$ *for some* j, *then*

$$\begin{aligned} &P\,x_t \in (0,\infty)\,\Phi_0 + (0,\infty)\,\Phi_{-1} && \text{for } z_j - 1 < t < z_j, \text{ and}\\ &P\,x_t \in (0,\infty)\,\Phi_0 + (-\infty,0)\,\Phi_{-1} && \text{for } z_j < t < z_j + 1. \end{aligned}$$

PROOF. The proof is a straightforward application of Corollary 4.9 in combination with the foregoing result. Indeed, Corollary 4.9 implies

$$\begin{aligned} P\,x_t &= x_t(-1)\,\Phi_0 + x_t(0)\,\Phi_{-1}\\ &= x(t-1)\,\Phi_0 + x(t)\,\Phi_{-1}, \end{aligned}$$

for all $t \in \mathbb{R}$. Now, assuming $0 < \dot{x}(z_j)$ for some j and using the preceding result, we see that $x(s) < 0$ for $z_j - 1 \leq s < z_j$ and $x(s) > 0$ for $z_j < s < z_{j+1}$ where, in particular, $z_j + 1 < z_{j+1}$. Substituting these values into the above equation for $P\,x_t$ immediately yields assertion (i). Statement (ii) is also obvious in view of the fact that the stated assumption forces $x(s) < 0$ for all $s < J(\varphi)$.

In the case $\dot{x}(z_j) > 0$, a completely analogous reasoning establishes parts (iii) and (iv). The details, as well as the trivial verification of assertion (v), are left to the reader. □

We close this section with the construction of a recurrence map for the set of all nonnegative functions $\varphi \in \overline{W} \setminus \{0\}$ with $\varphi(0) = 0$ under the semiflow F. This set is compact and coincides with the set of all $\varphi \in \overline{W}$ with $P\,\varphi \in (0,\infty)\,\Phi_0$, as shown in the next corollary.

COROLLARY 5.16. *Define the subset* H_0^+ *of the closed hyperplane* H_0 *of* C^1 *from page 102 by*

$$H_0^+ := \big\{\varphi \in H_0 \,|\, \varphi \geq 0\big\}.$$

Then $\overline{W} \cap H_0^+$ *is compact and*

$$\Big\{\varphi \in \overline{W} \,\Big|\, P\,\varphi \in (0,\infty)\,\Phi_0 \cup \{0\}\Big\} = \overline{W} \cap H_0^+. \tag{5.4}$$

PROOF. For the first part of the statement, recall that $\overline{W}$ is a compact subset of C^1 due to Corollary 5.7. As H_0^+ is obviously closed in C^1, the compactness of $\overline{W}$ clearly implies the one of the closed subset $\overline{W} \cap H_0^+$. Hence, it only remains to prove Equation (5.4). For this purpose, note that both sets obviously contain $0 \in C^1$ and that in view of Proposition 5.10 for $\varphi \in \overline{W}$ we have $P\,\varphi = 0$ if and only if $\varphi = 0$. Consider now $\varphi \in (\overline{W} \cap H_0^+)$ with $\varphi \neq 0$. Then Corollary 4.9 leads to

$$0 \neq P\,\varphi = \varphi(-1)\,\Phi_0 + \varphi(0)\,\Phi_{-1} = \varphi(-1)\,\Phi_0$$

and so $\varphi(-1) \neq 0$. As $\varphi \geq 0$ it follows $\varphi(-1) > 0$. Thus

$$P\,\varphi = \varphi(-1)\,\Phi_0 \in (0,\infty)\,\Phi_0.$$

On the other hand, if $\varphi \in \overline{W}$ and $P\,\varphi \in (0,\infty)\,\Phi_0 \subset H_0^+$ then, by definition of Φ_0, there holds $\varphi(-1) > 0$ and $\varphi(0) = 0$. As all elements in $\overline{W} \setminus \{0\}$ have at most one zero due to the last proposition, we see $\varphi \geq 0$. Thus $\varphi \in H_0^+$, and (5.4) is proved. □

Now we establish a recurrence map for the set $(\overline{W} \cap H_0^+) \setminus \{0\}$ which will be essential for our proof of the existence of a slowly oscillating periodic solution for Equation (4.1).

PROPOSITION 5.17. *The map*

$$\mathcal{W} : (\overline{W} \cap H_0^+)\backslash\{0\} \ni \varphi \longmapsto x^{\varphi}_{z_2(\varphi)} \in (\overline{W} \cap H_0^+)$$

is well-defined and maps the set $(\overline{W} \cap H_0^+)\backslash\{0\}$ *homeomorphically onto*

$$\big\{\psi \in (\overline{W} \cap H_0^+) \setminus \{0\} \,|\, J(\psi) \leq -2\big\}.$$

PROOF. We begin the proof with the observation that for $\varphi \in (\overline{W} \cap H_0^+) \setminus \{0\}$ we have indeed

$$\mathcal{W}(\varphi) \in \big\{\psi \in (\overline{W} \cap H_0^+) \setminus \{0\} \mid J(\psi) \leq -2\big\}$$

due to Corollaries 5.15 and 5.16. The remaining part falls naturally into four steps.

1. *Proof of the continuity of* $\mathcal{W}$. Given any $\varphi \in (\overline{W} \cap H_0^+)\backslash\{0\}$ suppose that for a sequence $\{\varphi_k\}_{k\in\mathbb{N}} \subset (\overline{W} \cap H_0^+)\backslash\{0\}$ we have $\varphi_k \to \varphi$ as $k \to \infty$. Using the definition of H_0^+ together with the fact that solutions $x^{\varphi}, x^{\varphi_k}$ are slowly oscillating, we conclude $z_1(\varphi), z_1(\varphi^k) > 1$ and $x^{\varphi}(1), x^{\varphi_k}(1) < 0$ for all $k \in \mathbb{N}$. By Proposition 3.7, for each $\varepsilon > 0$ and each $z_2(\varphi) < T < z_3(\varphi)$ there is a constant $k_1(\varepsilon, T) \in \mathbb{N}$ with

$$\sup\big\{|x^{\varphi_k}(t) - x^{\varphi}(t)|\big|1 \leq t \leq T\big\} < \varepsilon$$

as $k \geq k_1(\varepsilon, T)$. Now for every given $0 < \delta < \min\{1/2, T - z_2(\varphi)\}$ we find an $\varepsilon > 0$ sufficiently small such that $|x^{\varphi}(t)| \geq \varepsilon$ holds for all $t \in I$, where

$$I := \bigcap_{i=1,2} \big\{1 \leq t \leq T\big|\ |t - z_i(\varphi)| \geq \delta\big\}.$$

In particular, under this condition, we have $x^{\varphi_k}(t) \neq 0$ for all $k \geq k_1(\varepsilon, T)$ and all $t \in I$ and the solutions x^{φ_k} have exactly one zero in each interval $[z_i(\varphi)-\delta, z_i(\varphi)+\delta]$ for $i = 1, 2$. But as $\delta > 0$ may be chosen arbitrarily small, we obviously get $z_1(\varphi_k) \to z_1(\varphi)$, $z_2(\varphi_k) \to z_2(\varphi)$ as $k \to \infty$. Hence, the continuity of the semiflow F implies

$$\mathcal{W}(\varphi_k) = x^{\varphi_k}_{z_2(\varphi_k)} \to x^{\varphi}_{z_2(\varphi)} = \mathcal{W}(\varphi)$$

as $k \to \infty$. Since $\varphi \in (\overline{W} \cap H_0^+)\backslash\{0\}$ was arbitrary, this shows the continuity of $\mathcal{W}$.

2. *The proof of the injectivity of* $\mathcal{W}$. Suppose that for $\varphi, \psi \in (\overline{W} \cap H_0^+)\backslash\{0\}$ we have

$$\mathcal{W}(\varphi) = \mathcal{W}(\psi).$$

As solutions with initial values in $\overline{W}$ are unique, this implies

$$x^{\varphi}(t + z_2(\varphi)) = x^{\psi}(t + z_2(\psi)),$$

or equivalently

$$x^{\varphi}(t) = x^{\psi}(t + z_2(\psi) - z_2(\varphi)), \tag{5.5}$$

for all $t \in \mathbb{R}$.

We claim that $z_2(\varphi) = z_2(\psi)$ and thus, $x^{\varphi} = x^{\psi}$. In order to derive a contradiction, suppose that this claim is false. Then there is no loss of generality in assuming $z_2(\psi) > z_2(\varphi)$. From $z_2(\varphi) > 0$ it follows

$$0 < z_2(\psi) - z_2(\varphi) < z_2(\psi).$$

Combining this with $x^{\psi}(0) = 0$ and $0 = x^{\varphi}(0) = x^{\psi}(z_2(\psi) - z_2(\varphi))$ due to Equation (5.5), we get

$$z_1(\psi) = z_2(\psi) - z_2(\varphi).$$

On the other hand, Equation (5.5) implies

$$0 = x^{\varphi}(z_1(\varphi)) = x^{\psi}(z_1(\varphi) + z_2(\psi) - z_2(\varphi)),$$

and hence

$$z_1(\psi) = z_2(\psi) - z_2(\varphi) + z_1(\varphi)$$

in view of $0 < z_1(\varphi) + z_2(\psi) - z_2(\varphi) < z_2(\psi)$. Consequently, we obtain $z_1(\varphi) = 0$ in contradiction to $0 = z_0(\varphi) < z_1(\varphi)$. This shows $z_2(\varphi) = z_2(\psi)$ and thus, $x^{\varphi} = x^{\psi}$, which finally implies

$$\varphi = x_0^{\varphi} = x_0^{\psi} = \psi.$$

3. *The proof of the surjectivity of* $\mathcal{W}$. Let $\varphi \in (\overline{W} \cap H_0^+)\backslash\{0\}$ with $J(\varphi) \leq -2$ be given. As x^{φ} is slowly oscillating, every zero of x^{φ} is simple and thus leads to a sign change of x^{φ}. By assumption, x^{φ} has at least the three zeros

$$z_{-2}(\varphi) < z_{-1}(\varphi) < z_0(\varphi) = 0$$

in $(-\infty, 0]$. Furthermore, we obviously can conclude $0 > \dot{x}^{\varphi}(z_0(\varphi))$ and $x^{\varphi}(t) > 0$ for $z_{-1}(\varphi) > t > z_0(\varphi) = 0$. Consequently, $\dot{x}^{\varphi}(z_{-1}(\varphi)) > 0$ and $x^{\varphi}(t) < 0$ for all $z_{-2}(\varphi) > t > z_1(\varphi)$ follows such that finally we obtain $0 > \dot{x}^{\varphi}(z_{-2}(\varphi))$ and hence

$$x^{\varphi}_{z_{-2}(\varphi)} \in (\overline{W} \cap H_0^+)\backslash\{0\}$$

by Corollary 5.15 and 5.16. In particular, this implies

$$\mathcal{W}(x^{\varphi}_{z_{-2}(\varphi)}) = x_0^{\varphi} = \varphi,$$

and shows

$$\mathcal{W}\big((\overline{W} \cap H_0^+)\backslash\{0\}\big) = \big\{\psi \in (\overline{W} \cap H_0^+)\backslash\{0\} \,\big|\, J(\psi) \leq -2\big\}.$$

4. For the conclusion on the continuity of the map

$$\big\{\psi \in (\overline{W} \cap H_0^+)\backslash\{0\} |\ J(\psi) \leq -2\big\} \ni \varphi \longmapsto x^{\varphi}_{z_{-2}(\varphi)} \in (\overline{W} \cap H_0^+)\backslash\{0\}$$

recall that $\overline{W}$ is contained in $\mathcal{A}$. Hence, in consideration of Corollary 3.30 on the continuity of the flow on $\mathcal{A}$, we can proceed analogously to the first part which completes the proof. □

REMARK 5.18. The map $\mathcal{W}$ defined in the last result is called the **first return map** of $(\overline{W} \cap H_0^+)\backslash\{0\}$ under the flow F. By Corollaries 5.15 and 5.16, $\mathcal{W}(\varphi)$ is the first intersection between the set $(\overline{W} \cap H_0^+)\backslash\{0\}$ and the orbit of the trajectory $\xi : \mathbb{R} \ni s \longmapsto x_s^{\varphi} \in \overline{W}$ for some $t > 0$, provided $\xi(0) = \varphi \in (\overline{W} \cap H_0^+)\backslash\{0\}$.

4. A Slowly Oscillating Periodic Orbit

After the preliminaries of the two preceding sections, we are now in the position to show that $\overline{W} \setminus W$ is a slowly oscillating periodic orbit for the model equation (4.1). The crucial idea here is to consider the ω-limit sets of trajectories in $\overline{W}$.

PROPOSITION 5.19. *Suppose* $\varphi \in W$ *and* $\varphi \neq 0$. *Then* $0 \notin \omega(\varphi)$ *and* $\omega(\varphi)$ *is a slowly oscillating periodic orbit. Moreover, if* $\varphi \in \overline{W} \setminus W$ *then* $\omega(\varphi) = \alpha(\varphi)$ *and* $x^{\varphi} : \mathbb{R} \longrightarrow [-2a, 2a]$ *is a slowly oscillating periodic solution.*

Proof. 1. Let $\varphi \in \overline{W}$ with $\varphi \neq 0$ be given. Notice that there is no loss of generality in assuming $P\,\varphi \in (0,\infty)\,\Phi_0$. Indeed, in the case $P\,\varphi \notin (0,\infty)\,\Phi_0$, by Propositions 5.13 and Corollary 5.15 we would find a $t \in \mathbb{R}$ and a $\psi \in \overline{W}\backslash\{0\}$ such that $\psi = x_t^\varphi$ and $P\,\psi \in (0,\infty)\,\Phi_0$. As in this situation we obviously would have $\omega(\varphi) = \omega(\psi)$ for the associated ω-limit sets, it becomes clear that there is in fact no loss of generality in assuming $P\,\varphi \in (0,\infty)\,\Phi_0$ and thus, $\varphi \in (\overline{W} \cap H_0^+)\backslash\{0\}$ due to Corollary 5.16.

2. Consider the strictly monotonic sequence $\{z_j(\varphi)\}_{J(\varphi)}^\infty$ of zeros of the solution x^φ and set

$$\xi^j := P\,x^\varphi_{z_{2j}(\varphi)}$$

for all $j \in \mathbb{Z}$ with $2j \geq J(\varphi)$. In this way, we get a sequence $\{\xi^j\}_{J^\star(\varphi)}^\infty$ in $(0,\infty)\,\Phi_0$ where $J^\star(\varphi) = \infty$ if $J(\varphi) = \infty$ and $J^\star(\varphi) \in \mathbb{Z} \setminus \mathbb{N}$ otherwise. Additionally, we define a sequence $\{s^j\}_{J^\star(\varphi)}^\infty$ in $\mathbb{R}$ by

$$s^j := z_{2j}(\phi).$$

The two sequences are obviously related by

$$x^\varphi_{s^j} = P^{-1}(\xi^j),$$

and $\{s^j\}_{J^\star(\varphi)}^\infty$ is strongly increasing. Moreover, combining the facts that F is a continuous flow on $\overline{W}$ and that P maps $\overline{W}$ homeomorphic onto $P\,\overline{W}$, we obtain a monotonicity property for $\{\xi^j\}_{J^\star(\varphi)}^\infty$ in the following sense. We have either

$$0 < \|\xi^{j+1}\|_{C^1} \leq \|\xi^j\|_{C^1} \tag{5.6}$$

for all integers $j \geq J^\star(\varphi)$ or vice versa

$$\|\xi^{j+1}\|_{C^1} \geq \|\xi^j\|_{C^1} > 0 \tag{5.7}$$

for all integers $j \geq J^\star(\varphi)$. Thus we may define the values

$$\xi^- := \begin{cases} \lim\limits_{j\to-\infty} \xi^j, & \text{for } J^\star(\varphi) = -\infty, \\ 0, & \text{for } J^\star(\varphi) \in \mathbb{Z}, \end{cases}$$

and

$$\xi^+ := \lim_{j\to\infty} \xi^j,$$

since $\xi^j \in P(\overline{W} \cap H_0^+)$ for all $j \geq J^\star(\varphi)$ and $P(\overline{W} \cap H_0^+)$ is compact as a continuous image of the compact subset $\overline{W} \cap H_0^+$ of $\overline{W}$. In particular, we conclude that $\xi^-, \xi^+ \in (0,\infty)\,\Phi_0 \cup \{0\}$.

3. We now claim that if $\{\xi^j\}_{j\in\mathbb{N}}$ converges to some $\xi^+ \in (0,\infty)\,\Phi_0$ as $j \to \infty$, then $x^{P^{-1}(\xi^+)}$ is a slowly oscillating periodic solution of Equation (4.1) and

$$\omega(\varphi) = \left\{ x_t^{P^{-1}(\xi^+)} \,\middle|\, t \in \mathbb{R} \right\}.$$

For the proof of this claim, first observe that under the given assumption we have $\xi^j, \xi^+ \in P((\overline{W} \cap H_0^+)\backslash\{0\})$ and hence $P^{-1}(\xi^j), P^{-1}(\xi^+) \in (\overline{W} \cap H_0^+)\backslash\{0\}$ for all $j \in \mathbb{N}$. Now for each $j \geq 0$

$$P^{-1}(\xi^j) = x^\varphi_{s^j} = x^\varphi_{z_{2j}(\varphi)} = \mathcal{W}^{-1}(x^\varphi_{z_{2j+2}(\varphi)}) = \mathcal{W}^{-1}(x^\varphi_{s^{j+1}}) = \mathcal{W}^{-1}(P^{-1}(\xi^{j+1})).$$

Thus, by application of $\mathcal{W}$ and consideration of the last equation for $j \to \infty$, we obtain

$$P^{-1}(\xi^+) = \mathcal{W}(P^{-1}(\xi^+)) = F(z_2(P^{-1}(\xi^+)), P^{-1}(\xi^+)),$$

since P^{-1} and $\mathcal{W}$ are continuous. Consequently, the solution $x^{P^{-1}(\xi^+)}$ of Equation (4.1) is not only slowly oscillating, but also $z_2(P^{-1}(\xi^+))$-periodical since we have

$$x_0^{P^{-1}(\xi^+)} = x_{z_2(P^{-1}(\xi^+))}^{P^{-1}(\xi^+)}$$

and for all $\psi \in \overline{W}$ Equation (4.1) has a unique solution $x^\psi : \mathbb{R} \longrightarrow [-2a, 2a]$ with initial value $x_0^\psi = \psi$. In particular, $z_2(P^{-1}(\xi^+))$ is the minimal period of $x^{P^{-1}(\xi^+)}$ in view of

$$F(z_1(P^{-1}(\xi^+)), P^{-1}(\xi^+)) \in (-\infty, 0)\,\Phi_0$$

due to Corollary 5.15.

Defining

$$\mathfrak{O}^+ := \left\{ x_t^{P^{-1}(\xi^+)} \,\middle|\, 0 \leq t \leq z_2(P^{-1}(\xi^+)) \right\},$$

we shall have established the claim if we prove $\mathrm{dist}_{C^1}(x_t^\varphi, \mathfrak{O}^+) \to 0$ as $t \to \infty$. For this purpose, let $\varepsilon > 0$ be given. From Propositions 3.7 and 3.14 we find a constant $\delta(\varepsilon) > 0$ such that for all $\psi \in (\overline{W} \cap H_0^+)\backslash\{0\}$ with $\|\psi - P^{-1}(\xi^+)\|_{C^1} < \delta(\varepsilon)$ we have

$$\mathrm{dist}_{C^1}(x_s^\psi, \mathfrak{O}^+) < \varepsilon$$

as $0 \leq s \leq z_2(P^{-1}(\xi^+)) + 1$, and

$$|z_2(\psi) - z_2(\xi^+)| < 1.$$

Fix $j_0 \geq 0$ with

$$\|P^{-1}(\xi^j) - P^{-1}(\xi^+)\|_{C^1} < \delta(\varepsilon)$$

as $j \geq j_0$, and let $t > s^{j_0}$ be given. Since the sequence $\{s^j\}_{J^\star(\varphi)}^\infty$ is strongly increasing, we find an integer $j_1 \geq j_0$ such that

$$s^{j_0} \leq s^{j_1} < t \leq s^{j_1+1}.$$

In particular, $\|P^{-1}(\xi^{j_1}) - P^{-1}(\xi^+)\|_{C^1} < \delta(\varepsilon)$, and hence

$$\mathrm{dist}_{C^1}(x_s^{P^{-1}(\xi^{j_1})}, \mathfrak{O}^+) < \varepsilon$$

as $0 \leq s \leq z_2(P^{-1}(\xi^+)) + 1$, and

$$|z_2(P^{-1}(\xi^{j_1})) - z_2(P^{-1}(\xi^+))| < 1.$$

Using

$$x_{s^{j_1+1}}^\varphi = F(z_2(P^{-1}(\xi^{j_1})), x_{s^{j_1}}^\varphi) = F(z_2(P^{-1}(\xi^{j_1})), P^{-1}(\xi^{j_1})),$$

we see

$$0 < t - s^{j_1} \leq s^{j_1+1} - s^{j_1} = z_2(P^{-1}(\xi^{j_1})) < z_2(P^{-1}(\xi^+)) + 1,$$

and this implies

$$\mathrm{dist}_{C^1}\left(x_t^\varphi, \mathfrak{O}^+\right) = \mathrm{dist}_{C^1}\left(x_{t-s^{j_1}}^{P^{-1}(\xi^{j_1})}, \mathfrak{O}^+\right) < \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we conclude $\mathrm{dist}_{C^1}(x_t^\varphi, \mathfrak{O}^+) \to 0$ as $t \to \infty$. In particular, this shows $\omega(\varphi) = \mathfrak{O}^+$ and completes the proof of our claim.

4. We return to the proof of the proposition and consider the case $\varphi \in W$ with $\varphi \neq 0$ first. Then, by definition, there are $\psi \in N^{cu}$ and $s \geq 0$ with $\varphi = F(s, \psi)$ and $x^\varphi(t) = x^\psi(s + t) \to 0$ as $t \to -\infty$. Hence, $P\,x_t^\varphi \to 0$ as $t \to -\infty$. Now, recall the monotonicity property of the sequence $\{\|\xi^j\|_{C^1}\}_{J^\star(\varphi)}^\infty$. We have either the case (5.6) or the case (5.7) for all integers $j \geq J^\star(\varphi)$. Assuming that (5.6) holds, we obtain a contradiction, since the projected flowline $\mathbb{R} \ni t \longmapsto P\,x_t^\varphi \in W \setminus \{0\}$ is

injective and $P\,x_t^\varphi \to 0$ as $t \to -\infty$. Therefore, we have the situation (5.7) and thus,

$$\|\xi^j\|_{C^1} \ge \|\xi^0\|_{C^1} > 0$$

for all $j \ge 0$. As in addition $J^\star(\varphi) \in \mathbb{Z}$ due to Corollary 5.3 we obtain the inequality $\|\xi^+\|_{C^1} > 0 = \|\xi^-\|_{C^1}$ and hence the conclusion

$$\xi^+ \in (0,\infty)\,\Phi_0.$$

Subsequently, the result of the last step shows that $x^{P^{-1}(\xi^+)}$ is a slowly oscillating periodic solution and

$$\omega(\varphi) = \left\{ x_t^{P^{-1}(\xi^+)} \,\middle|\, t \in \mathbb{R} \right\}.$$

This proves the first part of the proposition.

5. Assume now $\varphi \in \overline{W} \setminus W$. By the invariance of W and $\overline{W}$ under F, we get $x_t^\varphi \in \overline{W} \setminus W$ for all $t \in \mathbb{R}$. Additionally, the zero set of the solution x^φ is unbounded from both sides. As $P\,W$ is an open neighborhood of $0 \in M$ and $\xi^j \in P(\overline{W} \setminus W) = P\,\overline{W} \setminus P\,W = \overline{P\,W} \setminus P\,W$ for all $j \in \mathbb{Z}$ due to Proposition 5.10, the sequence $\{\xi^j\}_{j\in\mathbb{Z}}$ can not converge to 0 as $j \to \infty$ or $j \to -\infty$. Moreover, we find a sufficiently small $\varepsilon > 0$ with $\|\xi^j\|_{C^1} \ge \varepsilon$ for all $j \in \mathbb{Z}$. Therefore, we obtain $\xi^+, \xi^- \in (0,\infty)\,\Phi_0$ and the third part of the proof implies that $x^{P^{-1}(\xi^+)}$ is a slowly oscillating periodic solution and

$$\omega(\varphi) = \left\{ x_t^{P^{-1}(\xi^+)} \,\middle|\, t \in \mathbb{R} \right\}.$$

Next, we claim that $x^{P^{-1}(\xi^-)}$ is also a slowly oscillating periodic solution of Equation (4.1) and

$$\alpha(\varphi) = \left\{ x_t^{P^{-1}(\xi^-)} \,\middle|\, t \in \mathbb{R} \right\}.$$

To show this claim, we proceed analogously to the third part and consider the sequence $\{\xi^j\}_{j\in\mathbb{Z}\setminus\mathbb{N}}$ in $(0,\infty)\,\Phi_0 = P((\overline{W} \cap H_0^+) \setminus \{0\})$ with $\xi^j \to \xi^- \in (0,\infty)\,\Phi_0$ as $j \to -\infty$. For all $j \le 0$ we have $P^{-1}(\xi^j), P^{-1}(\xi^-) \in (\overline{W} \cap H_0^+) \setminus \{0\}$ and

$$\mathcal{W}(P^{-1}(\xi^{j-1})) = \mathcal{W}(x^\varphi_{s^{j-1}}) = \mathcal{W}(x^\varphi_{z_{2(j-1)}(\varphi)}) = x^\varphi_{z_{2j}(\varphi)} = x^\varphi_{s^j} = P^{-1}(\xi^j).$$

Carrying out the limit $j \to -\infty$, we obtain

$$P^{-1}(\xi^-) = \mathcal{W}(P^{-1}(\xi^-)) = F(z_2(P^{-1}(\xi^-)), P^{-1}(\xi^-))$$

in view of the continuity of P^{-1} and $\mathcal{W}$. It follows that $x^{P^{-1}(\xi^-)}$ is a slowly oscillating and $z_2(P^{-1}(\xi^-))$-periodic solution. By Corollary 5.15, we have

$$F(z_1(P^{-1}(\xi^-)), P^{-1}(\xi^-)) \in (-\infty, 0)\,\Phi_0$$

so that $z_2(P^{-1}(\xi^-)$ is also the minimal period of solution $x^{P^{-1}(\xi^-)}$ of Equation (4.1). Hence, the above claim follows, if we additionally prove $\operatorname{dist}(x_t^\varphi, \mathfrak{O}^-) \to 0$ as $t \to -\infty$ where

$$\mathfrak{O}^- := \left\{ x_t^{P^{-1}(\xi^-)} \,\middle|\, 0 \le t \le z_2(P^{-1}(\xi^-)) \right\}.$$

For this purpose, observe that for a given $\varepsilon > 0$ Corollaries 3.30, 3.31 and Proposition 3.14 imply the existence of a constant $\delta(\varepsilon) > 0$ such that for all functions $\psi \in (\overline{W} \cap H_0^+) \setminus \{0\}$ with $\|\psi - P^{-1}(\xi^-)\|_{C^1} < \delta(\varepsilon)$ we have

$$\operatorname{dist}_{C^1}(x_s^\psi, \mathfrak{O}^-) < \varepsilon$$

as $0 \le s \le z_2(P^{-1}(\xi^-)) + 1$, and

$$|z_2(\psi) - z_2(\xi^-)| < 1.$$

Choose $j_0 \leq 0$ with
$$\|P^{-1}(\xi^j) - P^{-1}(\xi^-)\|_{C^1} < \delta(\varepsilon)$$
for all $j \leq j_0$, and fix $t < s^{j_0}$. Then, from the monotonicity of the sequence $\{s^j\}_{j\in\mathbb{Z}\setminus\mathbb{N}}$, we find $j_1 < j_0$ with
$$s^{j_1} \leq t < s^{j_1+1} \leq s^{j_0} \leq 0.$$
As $\|P^{-1}(\xi^{j_1}) - P^{-1}(\xi^-)\|_{C^1} < \delta(\varepsilon)$ due to the choice of j_1, we see that
$$\operatorname{dist}_{C^1}(x_s^{P^{-1}(\xi^{j_1})}, \mathfrak{O}^-) < \varepsilon$$
for all $0 \leq s \leq z_2(P^{-1}(\xi^-)) + 1$ and
$$z_2(P^{-1}(\xi^{j_1})) < z_2(P^{-1}(\xi^-)) + 1.$$
From
$$x^{\varphi}_{s^{j_1+1}} = F(z_2(P^{-1}(\xi^{j_1})), x^{\varphi}_{s^{j_1}}) = F(z_2(P^{-1}(\xi^{j_1})), P^{-1}(\xi^{j_1}))$$
we conclude
$$0 \leq t - s^{j_1} < s^{j_1+1} - s^{j_1} = z_2(P^{-1}(\xi^{j_1})) < z_2(P^{-1}(\xi^-)) + 1.$$
Consequently, we obtain
$$\operatorname{dist}_{C^1}(x_t^{\varphi}, \mathfrak{O}^-) = \operatorname{dist}_{C^1}(x_{t-s^j}^{P^{-1}}(\xi^{j_1}), \mathfrak{O}^-) < \varepsilon,$$
which completes the proof of our claim that $x^{P^{-1}(\xi^-)}$ is a slowly oscillating periodic solution of Equation (4.1) and that
$$\alpha(\varphi) = \Big\{x_t^{P^{-1}(\xi^-)} \,\Big|\, t \in \mathbb{R}\Big\}.$$

6. It remains to prove that for $\varphi \in \overline{W} \setminus W$ the solution x^{φ} of Equation (4.1) is periodic. For this purpose, consider the two periodic orbits $\mathfrak{O}^+ = \omega(\varphi)$ and $\mathfrak{O}^- = \alpha(\varphi)$ and assume that $\mathfrak{O}^+ \neq \mathfrak{O}^-$ in the following. Then the projected trajectories
$$c^+ : [0, z_2(P^{-1}(\xi^+))] \ni t \longmapsto P\, x_t^{P^{-1}(\xi^+)} \in P\,\overline{W}$$
and
$$c^- : [0, z_2(P^{-1}(\xi^-))] \ni t \longmapsto P\, x_t^{P^{-1}(\xi^-)} \in P\,\overline{W}$$
of these two orbits are continuous and not intersecting in view of the uniqueness of solutions and injectivity of P. The restriction of c^- to the interval $[0, z_2(P^{-1}(\xi^-)))$ is injective, since $z_2(P^{-1}(\xi^-))$ is the minimal period of solution $x^{P^{-1}(\xi^-)}$. Consequently, c^- is a simple closed curve from $[0, z_2(P^{-1}(\xi^-))]$ into the two-dimensional Banach space M. Applying the Jordan curve theorem, we conclude that the trace $|c^-| = P\,\mathfrak{O}^-$ of c^- decomposes M into two connected components, a bounded one, called the interior of c^- and denoted by $\operatorname{int}(c^-)$, and an unbounded one, called the exterior of c^- and denoted by $\operatorname{ext}(c^-)$. By repeating the same arguments for c^+ instead of c^-, it follows that the trace $|c^+| = P\,\mathfrak{O}^+$ also decomposes M into two connected components: $\operatorname{int}(c^+)$ and $\operatorname{ext}(c^+)$.

If $\mathfrak{O}^+ \neq \mathfrak{O}^-$ is true, then either $\|\xi^+\|_{C^1} > \|\xi^-\|_{C^1}$ or $\|\xi^+\|_{C^1} < \|\xi^-\|_{C^1}$. Consider the situation $\|\xi^+\|_{C^1} > \|\xi^-\|_{C^1}$ first. As $\xi^+ = p^+ \varPhi_0$ and $\xi^- = p^- \varPhi_0$ with uniquely determined constants $p^+, p^- \in (0, \infty)$, the inequality $\|\xi^+\|_{C^1} > \|\xi^-\|_{C^1}$ implies $p^+ > p^-$. From the fact
$$|c^-| \cap [0, \infty)\, \varPhi_0 = \{\xi^-\} = \{p^-\, \Phi_0\}$$

we conclude that the ray $(p^-,\infty)\,\Phi_0$ is either contained in the interior $\operatorname{int}(c^-)$ or contained in the exterior $\operatorname{ext}(c^-)$ of c^-. But $(p^-,\infty)\,\Phi_0$ is unbounded and this implies $(p^-,\infty)\,\Phi_0 \subset \operatorname{ext}(c^-)$. Furthermore, since the trace $|c^+|$ of c^+ is connected as a continuous image of an interval and $|c^+| \cap (p^-,\infty)\,\Phi_0 = \{p^+\,\Phi_0\}$, $|c^+| \subset \operatorname{ext}(c^-)$.

On the other hand, we claim that the bounded ray $[0,p^-)\,\Phi_0$ is contained in the interior $\operatorname{int}(c^-)$ of the trace $|c^-|$. In order to see this, observe that by Corollary 4.9 the projected trajectory c^- can be represented by

$$c^-(t) = x^{P^{-1}(\xi^-)}(t-1)\,\Phi_0 + x^{P^{-1}(\xi^-)}(t)\,\Phi_{-1}$$

for all $t \in [0, z_2(P^{-1}(\xi^-)]$. In particular, c^- is continuously differentiable and

$$(\tfrac{d}{dt}c^-)(0) = \dot{x}^{P^{-1}(\xi^-)}(-1)\,\Phi_0 + \dot{x}^{P^{-1}(\xi^-)}(0)\,\Phi_{-1}.$$

Now, 0 is a simple zero of the solution $x^{P^{-1}(\xi^-)}$ of Equation (4.1) and therefore we have $\dot{x}^{P^{-1}(\xi^-)}(0) \neq 0$. This implies that $(\frac{d}{dt}c^-)(0)$ and Φ_0 are linearly independent. Hence, it follows that there is $\varepsilon > 0$ with

$$\big(p^- - \varepsilon, p^-\big)\,\Phi_0 = \big(x^{P^{-1}(\xi^-)}(-1) - \varepsilon,\, x^{P^{-1}(\xi^-)}(-1)\big)\,\Phi_0 \subset \operatorname{int}(c^-).$$

As $[0,p^-)\,\Phi_0$ is connected, the above leads to $[0,p^-)\,\Phi_0 \subset \operatorname{int}(c^-)$, which was our claim.

Due to the compactness of $(\operatorname{int} c^- \cup |c^-|)$ and the inequality $\|\xi^+\|_{C^1} > \|\xi^-\|_{C^1}$ assumed above, we find an open neighborhood $U \subset M$ of the point ξ^+ such that $U \cap |c^-| = U \cap \operatorname{int}(c^-) = \emptyset$. Since we have $\xi^+ \in \overline{PW} \setminus PW$ and PW is open as shown in Corollary 5.12, we find a point $\xi \in U \cap PW$. In particular, $\xi \in \operatorname{ext}(c^-)$. Consider the solution $x^{P^{-1}(\xi)} : \mathbb{R} \longrightarrow [-2a, 2a]$ of Equation (4.1). This solution is slowly oscillating and its segments converge to 0 as $t \to -\infty$ due to Corollary 5.1. Consequently, $P\,x_t^{P^{-1}(\xi)} \to 0 \in M$ as $t \to -\infty$. But

$$P\,x_0^{P^{-1}(\xi)} = \xi \in \operatorname{ext}(c^-)$$

and $0 \in \operatorname{int}(c^-)$, so there would exist some $T \leq 0$ with $P\,x_T^{P^-(\xi)} \in |c^-|$, in contradiction to the uniqueness of solutions.

By assuming $\|\xi^+\|_{C^1} < \|\xi^-\|_{C^1}$ and reversing the roles of c^- and c^+, the same arguments as above leads to a contradiction. We conclude that $\|\xi^+\|_{C^1} = \|\xi^-\|_{C^1}$ holds and thus the both equalities $\xi^+ = \xi^-$ and $\mathfrak{O}^+ = \mathfrak{O}^-$. But, in view of the equality $\xi^+ = \xi^-$, the monotonicity property of the sequence $\{\xi^j\}_{j\in\mathbb{Z}}$ implies $\xi_j = \xi^+$ for all $j \in \mathbb{Z}$. This finally shows $x^\varphi = x^{P^{-1}(\xi^+)}$ and completes the proof. $\square$

The preceding proposition ensures the existence of a slowly oscillating periodic orbit for Equation (4.1), and we are thus led to the main result of the second part of this work.

THEOREM 5.20. *The set* $\overline{W}\backslash W$ *is the orbit* $\mathfrak{O} := \{p_t \mid t \in \mathbb{R}\}$ *of a slowly oscillating periodic solution* $p : \mathbb{R} \to \mathbb{R}$ *of Equation* (4.1) *whose minimal period is given by three consecutive zeros of* p. *Furthermore,*

$$\operatorname{int}(P\,\mathfrak{O}) = P\,W,$$

and for every $\varphi \in W\backslash\{0\}$ *we have*

$$\operatorname{dist}_{C^1}(F(t,\varphi), \mathfrak{O}) \to 0$$

as $t \to \infty$.

PROOF. Fix any element $\varphi \in (W \cap H_0^+)\backslash\{0\}$ and define the sequence $\{\xi^j\}_{J^\star(\varphi)}^{\infty}$ in $P(\overline{W} \cap H_0^+)$ and its limit ξ^+ as in the proof of the last proposition. Then the function $p : \mathbb{R} \to \mathbb{R}$ given by $p(t) = x^{P^{-1}(\xi^+)}(t)$, $t \in \mathbb{R}$, clearly forms a slowly oscillating periodic solution of Equation (4.1) with $p(0) = 0$, $P\,p_0 \in (0,\infty)\,\Phi_0$ and minimal period $z_2(P^{-1}(\xi^+))$. Denote by

$$\mathfrak{O} := \{p_t|\ t \in \mathbb{R}\}$$

the orbit and by

$$\sigma : [0, z_2(P^{-1}(\xi^+))] \ni t \longmapsto p_t \in C^1$$

the trajectory of p. Then the projected trajectory

$$c : [0, z_2(P^{-1}(\xi^+))] \ni t \longmapsto P\,p_t \in P\,\overline{W}$$

is continuous and its restriction to the interval $[0, z_2(P^{-1}(\xi^+)))$ is injective, since $z_2(P^{-1}(\xi^+))$ is the minimal period of the solution x^φ. Hence, c is a simple closed curve from $[0, z_2(P^{-1}(\xi^+))]$ into the two-dimensional Banach space M. Application of the Jordan curve theorem shows that the trace $|c| = |P\,\sigma| = P\,\mathfrak{O}$ of c decomposes M into two connected components $\mathrm{int}(c)$ and $\mathrm{ext}(c)$ as already described in the proof of the foregoing proposition.

1. *The proof of* $P\,W \subset \mathrm{int}(P\,\mathfrak{O})$. By Corollaries 5.11 and 5.12, $P\,W$ forms an open neighborhood of the origin in M. Furthermore, for every $\psi \in W$ we have $P\,W \ni P\,x_t^\psi \to 0$ as $t \to -\infty$ in view of the definition of W and Corollary 5.1. This implies that $P\,W$ is arcwise connected. From $\mathfrak{O} \subset \overline{W} \setminus W$ we see $P\,W \cap P\,\mathfrak{O} = \emptyset$. Hence, either $P\,W \subset \mathrm{int}(P\,\mathfrak{O})$ or $P\,W \subset \mathrm{ext}(P\,\mathfrak{O})$. Arguing as in part 6 of the proof of Proposition 5.19, we conclude $0 \in \mathrm{int}(P\,\mathfrak{O})$. Therefore, we have $0 \in P\,W \cap \mathrm{int}(P\,\mathfrak{O})$ and $P\,W \subset \mathrm{int}(P\,\mathfrak{O})$ follows.

2. *The proof of* $P\,W \supset \mathrm{int}(P\,\mathfrak{O})$. Assuming the assertion $\mathrm{int}(P\,\mathfrak{O}) \subset P\,W$ is false, we find a point $\chi \in \mathrm{int}(P\,\mathfrak{O})\backslash P\,W$ and an associated path

$$\gamma : [0,1] \longrightarrow \mathrm{int}(P\,\mathfrak{O})$$

from $\gamma(0) = 0 \in P\,W$ to $\gamma(1) = \chi \in \mathrm{int}(P\,\mathfrak{O})\backslash P\,W$. Define

$$\tilde{s} := \inf\{0 \le s \le 1 \,|\, \gamma(s) \notin P\,W\}.$$

As by Corollaries 5.11 and 5.12, the set $P\,W$ is an open neighborhood of $0 \in M$, $0 < \tilde{s} \le 1$. Furthermore, we have $\gamma(\tilde{s}) \neq 0$ since $\gamma([0,\tilde{s})) \subset P\,W$ and

$$\gamma(\tilde{s}) \in \overline{P\,W}\backslash P\,W = P\,\overline{W}\backslash P\,W = P(\overline{W}\backslash W).$$

Hence, $\psi := P^{-1}(\gamma(\tilde{s})) \in \overline{W}\backslash W$, and x^ψ is a slowly oscillating periodic solution of Equation (4.1) due to the last proposition. Since $P\,\psi = \gamma(\tilde{s}) \in \mathrm{int}(P\,\mathfrak{O})$ holds, we have $\psi \notin \mathfrak{O}$ and thus $x_t^\psi \notin \mathfrak{O}$ for all $t \in \mathbb{R}$. In particular, this implies $P\,x_t^\psi \notin P\,\mathfrak{O}$ as $t \in \mathbb{R}$. Now, by Corollary 5.15, the slowly oscillating periodic solution x^ψ has a zero $z > 0$ with $P\,x_z^\psi \in (0,\infty)\,\Phi_0$. Let $c_1, c_2 > 0$ be defined by $c_1\,\Phi_0 = \xi^+$ and $c_2\,\Phi_0 = P\,x_z^\psi$. Using the arguments contained in part 6 of the proof of the last proposition, we find $[0,c_1)\,\Phi_0 \subset \mathrm{int}(P\,\mathfrak{O})$, $(c_1,\infty)\,\Phi_0 \subset \mathrm{ext}(P\,\mathfrak{O})$ and analogously $[0,c_2)\,\Phi_0 \subset \mathrm{int}(P\,\widehat{\mathfrak{O}})$, $(c_2,\infty)\,\Phi_0 \subset \mathrm{ext}(P\,\widehat{\mathfrak{O}})$ where

$$\widehat{\mathfrak{O}} := \left\{x_t^\psi \,\middle|\, t \in \mathbb{R}\right\}.$$

From $P\,\mathfrak{O} \cap P\,\widehat{\mathfrak{O}} = \emptyset$ and $P\,x_0^{\psi} = P\,\psi = \gamma(\tilde{s}) \in \operatorname{int}(P\,\mathfrak{O}) \cap P\,\widehat{\mathfrak{O}}$ we conclude that $P\,\widehat{\mathfrak{O}} \subset \operatorname{int}(P\,\mathfrak{O})$. In particular, it follows that

$$\{c_2\,\Phi_0\} = \{P\,x_z^{\psi}\} = P\,\widehat{\mathfrak{O}} \cap [0,\infty)\,\Phi_0 \subset \operatorname{int}(P\,\mathfrak{O}) \cap [0,\infty)\,\Phi_0 = [0,c_1)\,\Phi_0$$

and thus

$$\|\xi^+\|_{C^1} = \|c_1\,\Phi_0\|_{C^1} = c_1\|\Phi_0\|_{C^1} > c_2\|\Phi_0\|_{C^1} = \|c_2\,\Phi_0\|_{C^1} = \|Px_z^{\psi}\|_{C^1}.$$

Therefore, the convergence

$$\xi^j \to \xi^+ = P\,x_0^{P^{-1}(\xi^+)} = P\,p_0 \in P\,\mathfrak{O}$$

as $j \to \infty$ shows the existence of an integer $j_0 \geq J^{\star}(\varphi)$ with

$$\|\xi^{j_0}\|_{C^1} > \|P\,x_z^{\psi}\|_{C^1} = c_2\|\Phi_0\|_{C^1}$$

and hence,

$$\xi^{j_0} \in P(W \cap H_0^+) \setminus (P\,\widehat{\mathfrak{O}} \cup \operatorname{int}(P\,\widehat{\mathfrak{O}})) = P(W \cap H_0^+) \cap \operatorname{ext}(P\widehat{\mathfrak{O}}).$$

As in the first step, we see $x_t^{P^{-1}(\xi^{j_0})} \to 0$ and thus, $P\,x_t^{P^{-1}(\xi^{j_0})} \to 0 \in \operatorname{int}(P\,\widehat{\mathfrak{O}})$ as $t \to -\infty$. Consequently, we find a $T \leq 0$ with $P\,x_T^{P^{-1}(\xi^{j_0})} \in P\,\widehat{\mathfrak{O}}$, in contradiction to the uniqueness of solutions in view of the injectivity of P on $\overline{W}$.

3. *The proof of* $\overline{W}\backslash W = \mathfrak{O}$. Using Proposition 5.10 stating $P\,\overline{W} = \overline{P\,W}$ and the relation $\mathfrak{O} \subset \overline{W}\backslash W$, we see

$$\begin{aligned} P\,\overline{W} &= \overline{P\,W} \\ &= \overline{\operatorname{int}(P\,\mathfrak{O})} \\ &= \operatorname{int}(P\,\mathfrak{O}) \cup \partial(\operatorname{int}(P\,\mathfrak{O})) \\ &\subset P\,W \cup P\,\mathfrak{O} \\ &= P(W \cup \mathfrak{O}) \subset P\,\overline{W}, \end{aligned}$$

which yields $\overline{W} = W \cup \mathfrak{O}$.

4. Given any function $\psi \in \overline{W} \setminus \{0\}$, Proposition 5.19 yields that the segments x_t^{ψ} of the associated solution for Equation (4.1) converge to an orbit of a slowly oscillating periodic solution in $\overline{W}$ as $t \to \infty$. Since the set $\mathfrak{O} = \overline{W}\backslash W$ is the only orbit of a slowly oscillating periodic solution in $\overline{W}$, we see $x_t^{\psi} \to \mathfrak{O}$ as $t \to \infty$, which completes the proof of the theorem. □

REMARK 5.21. In case of constant delay $r \equiv 1$, that is, in the situation of Equation (3.2), the theorem above has been shown by Brunovský et al. in [**3**].

List of Figures

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Index and Notations

Abstract

The main issue of this dissertation is the question whether periodic orbits of a delay differential equation still exist if a state-dependent delay is assumed instead of a constant one. Motivated by a mathematical model to describe short term fluctuations of exchange rates, a one-parameter family of scalar delay differential equations is studied, where the positive parameter involved represents a kind of sensitivity to price changes. In the case of a constant delay, this family of equations was analyzed in different works in the last years, and so it is well known that at a certain critical parameter the single non-hyperbolic equilibrium solution loses its stability and is connected to a periodic orbit by a global center-unstable manifold. Consequently, one may ask whether this is still true in the situation of a state-dependent delay.

To address the above problem, the first part of this thesis is devoted to the discussion of a class of functional differential equations that was recently developed to study differential equations with a state-dependent delay. Besides a brief summary of some well-known facts, we establish the so-called principle of linearized instability and construct local center-unstable manifolds at stationary points for this class of functional differential equations. Additionally, we prove the continuous differentiability of the constructed manifolds.

The study of the introduced one-parameter family of delay differential equations with a state-dependent delay is the main goal of the second part of this work. Analyzing the structure of solutions and particularly their local behavior at the single equilibrium solution, we finally generalize the result on existence of periodic solutions in the situation of constant delay and prove that, under certain conditions on the delay function and for parameter values above a critical one, a global two-dimensional center-unstable manifold connects the equilibrium to a periodic orbit.

Zusammenfassung

Das Hauptanliegen der vorliegenden Dissertation ist die Klärung der Frage, ob eine spezielle verzögerte Differentialgleichung noch immer periodische Orbits aufweist, wenn die involvierte Verzögerung nicht mehr als konstant, sondern als zustandsabhängig angenommen wird. In der Arbeit wird eine 1-parametrige Familie von skalaren verzögerten Differentialgleichungen, die ein mathematischen Modell zur Beschreibung von kurzfristigen Schwankung von Wechselkursen repräsentiert, studiert. Hierbei stellt der auftretende Parameter eine Art von Sensitivität dar, mit der auf die Änderung von Kursen reagiert wird. Im Fall einer konstanten Verzögerung wurde diese Familie von Differentialgleichungen in den letzten Jahren sehr ausgiebig studiert und dabei insbesondere analytisch bewiesen, dass ab einem gewissen kritischen Parameterwert das einzige auftretende Equilibrium, das zudem nicht-hyperbolisch ist, seine Stabilität verliert und stattdessen durch Trajektorien in einer globalen zentral-instabilen Mannigfaltigkeit mit einer periodischen Lösung verbunden wird. Folglich stellt sich die Frage, ob dieses Szenario auch im Falle einer zustandsabhängigen Verzögerung auftritt.

Zur Erörterung der oben beschriebenen Fragestellung beschäftigt sich der erste Teil der vorliegenden Dissertation mit einer speziellen Klasse von Funktionaldifferentialgleichungen, die in den letzten Jahren für die analytische Untersuchung von Differentialgleichung mit zustandsabhängiger Verzögerung entwickelt worden ist. Neben einem kurzen Überblick über einige bekannte Sachverhalte wird hier auch das sogenannte Prinzip der linearisierten Instabilität und die Existenz von lokalen zentral-instabilen Mannigfaltigkeiten an stationären Punkten für diese Klasse von Funktionaldifferentialgleichungen bewiesen. Zusätzlich wird die C^1-Glattheit dieser Mannigfaltigkeiten gezeigt.

Der zweite Teil der Arbeit ist dann der Untersuchung der eingeführten 1-parametrigen Familie von Differentialgleichungen mit zustandsabhängiger Verzögerung gewidmet. Durch das Studium der Struktur von Lösungen und vor allem deren lokalen Verhalten an dem einzigen auftretenden Equilibrium wird letztlich das Resultat über die Existenz von periodischen Orbits in der Situation der konstanten Verzögerung verallgemeinert und bewiesen, dass unter bestimmten Voraussetzungen an die Verzögerungsfunktion und im Falle eines über einem kritischen Wert liegenden Parameters eine globale zweidimensionale zentral-instabile Mannigfaltigkeit das Equilibrium mit einem periodischen Orbit verbindet.

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Stipendien

09/2000 - 08/2005	Lucia Pfohe Stiftung Stipendium im Rahmen des Studiums
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