Permutation equivariant ribbon categories from modular functors

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Prof. Dr. Vicente Cortés
Leiter des Fachbereichs Mathematik

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INTRODUCTION

Modular tensor categories have appeared in a variety of contexts and applications. The categories of representations of the Drinfeld double of a finite dimensional Hopf algebra, a more general class of weak Hopf algebras ([NTV03]) or rational vertex operator algebras ([Hua08]) are modular. The latter are a central source of examples for the TFT-approach to two-dimensional conformal field theory ([FRS02]).

Furthermore, modular tensor categories are an essential ingredient in the construction of Reshetikhin-Turaev invariants of knots and three-manifolds ([RT91]). This construction generalizes to the notion of a $\mathcal{C}$-extended three-dimensional topological field theory, where $\mathcal{C}$ is a modular tensor category.

The Reshetikhin-Turaev construction establishes a connection between algebraic structures (modular categories) and three-dimensional geometric objects (three-dimensional cobordisms). A similar connection in a lower dimension, which is by now a classical result (for a review see [Koc04]), states that the notions of a two-dimensional topological field theory and of a commutative Frobenius algebra are essentially equivalent. This is made precise in the statement that the category of two-dimensional topological field theories and the category of commutative Frobenius algebras are equivalent as monoidal categories.

A modular tensor category might be thought of as a higher-dimensional analogue of a commutative Frobenius algebra. This is indicated by several observations:

Recent work ([Lur09]) on extended topological field theory in the sense of higher category theory shows that a fully extended TFT is entirely determined by its value on the point viewed as a zero-dimensional manifold. On 0-1-2-extended TFT one makes the following observation: Such a theory assigns ([FHLT09]) a semi-simple Frobenius algebra A to the point and the center of A, being a commutative (semi-simple) Frobenius algebra, to the circle. Now whatever kind of tensor category $\mathcal{C}$ a 0-1-2-3-extended TFT assigns to the point, one expects that it assigns to the circle the Drinfeld center of $\mathcal{C}$, which is ([Müg03]) a modular tensor category. This is supported by [Fre09], where the idea is formulated that a modular tensor category determines a 1-2-3-extended topological field theory and vice versa. In this correspondence the modular category is the structure that the TFT assigns to the circle.

A structure that is related to three-dimensional topological field theories and modular categories is the one of a $\mathcal{C}$-extended two-dimensional modular functor (for a review see [BK01]). This is an assignment of a $\mathcal{V}ect_{\Bbbk}$-valued functor to every two-dimensional manifold with boundary. Given any $\mathcal{V}ect_{\Bbbk}$-enriched semi-simple abelian category $\mathcal{C}$, a genus zero $\mathcal{C}$-extended modular functor (i.e. a functor for every surface of genus zero) is equivalent to the structure of a weakly ribbon category ([BK01, Def. 5.3.5]) on $\mathcal{C}$. A $\mathcal{C}$-extended modular functor for arbitrary genus is equivalent to the structure of a modular tensor category on $\mathcal{C}$, provided that the induced weakly ribbon structure on $\mathcal{C}$ is ribbon.

This correspondence is in agreement with the idea that a modular category should be thought of as the categorified analogue of a commutative Frobenius algebra. The relation of modular categories to two-dimensional surfaces explains the a priori mysterious appearance of the name-giving representation of $\mathrm{SL}(2,\mathbb{Z})$ on the underlying vector-space of the Verlinde algebra $\mathcal{V}(\mathcal{C})$ of $\mathcal{C}$: The action of $\mathrm{SL}(2,\mathbb{Z})$ on the torus induces natural morphisms on the vector space $\mathcal{V}(\mathcal{C})$. Similarly, representations of mapping class groups of other surfaces naturally occur due to their action on the respective manifolds.

Given a finite group G, many of the above notions and structures may be generalized to an equivariant setting, that means one replaces manifolds by principal G-covers. In [Tur99] the notion of homotopy field theory was introduced, which as a special case contains the notion of G-equivariant topological field theory for a finite group G. It turns out that in dimension two, the monoidal categories of G-equivariant topological field theories (see definition 1.2 in this thesis) and of G-Frobenius algebras (see definition 1.6 in this thesis) are equivalent.

We now turn to an equivariant generalization of the higher-dimensional analogue of Frobenius algebras, i.e. to categorical structures. In [Tur00], a G-equivariant version of Reshetikhin-Turaev invariants was presented, together with the notion of a G-equivariant fusion category (see definition 1.21 in this thesis). These categories were also studied in [Kir04, Kir02], where the motivating example is the category of G-twisted representations of a vertex algebra $\mathcal{V}$ with finite group G of automorphisms. Given any G-equivariant fusion category $\mathcal{C}$, one may form its orbifold category $\mathcal{C}/G$. The central result of [Kir04] is that $\mathcal{C}$ is G-modular, if and only if $\mathcal{C}/G$ is modular.

The relation between G-equivariant modular functors and G-equivariant fusion categories was studied in [KP08]. Given an abelian G-equivariant category $\mathcal{C}$, the structure of a genus zero G-equivariant 2-dimensional modular functor is equivalent to the structure of a G-equivariant weakly ribbon category on $\mathcal{C}$. However, a similar result that relates G-modular categories and higher genus G-modular functors is not yet available in the literature.

We will now go into more detail on G-modular functors; for a precise definition, we refer to [KP08] and definition 1.18. G will always be a finite group, $\mathcal{C}$ a G-equivariant abelian category, i.e. an abelian category with a G-grading $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ and an action of G via functors R_g that cover the adjoint action of G on itself. All abelian categories in this thesis are enriched over the category $\mathcal{V}ect_{\Bbbk}$ of finite-dimensional vector spaces over an algebraically closed field $\Bbbk$ of characteristic zero.

A G-equivariant modular functor assigns to every principal G-cover $(P \to E)$ of a two-dimensional manifold E a functor

$$\tau^G(P\to E) : \boxtimes_{a\in A(E)} \mathcal{C}_{m_a^{-1}} \to \mathcal{V}ect_{\Bbbk} \,,$$

where $A(E)$ is the set of boundary components of E and $m_a \in G$ the monodromy of the cover $(P\to E)$ around the boundary a. To every diffeomorphism $\varphi : (P\to E) \xrightarrow{\cong} (P'\to E')$ of G-bundles, the G-equivariant modular functor assigns an isomorphism of functors

$$\varphi_* : \tau^G(P\to E) \Rightarrow \tau^G(P'\to E') \,.$$

These assignments have to satisfy a number of axioms, e.g. they have to be compatible with gluing of surfaces.

At this point a comment on the connection between the property of $\mathcal{C}$ to be G-modular and the possibility of defining a $\mathcal{C}$-extended G-equivariant modular functor of higher genus is in order. Given a G-equivariant fusion category $\mathcal{C}$, we choose representatives $(V_i)_{i\in\mathcal{I}}$ for the isomorphism classes of simple objects of $\mathcal{C}$. We write the action of G on objects as ${}^gV := R_g(V)$. Now define the extended Verlinde algebra $\tilde{\mathcal{V}}$ as in [Kir04]:

$$\tilde{\mathcal{V}}(\mathcal{C}) := \bigoplus_{g\in G, i\in\mathcal{I}} \mathrm{Hom}(V_i, {}^gV_i) \,.$$

The vector space splits as a direct sum

$$\tilde{\mathcal{V}}(\mathcal{C}) \cong \bigoplus_{\substack{g,h\in G\\ gh=hg}} \tilde{\mathcal{V}}_{g,h}(\mathcal{C})$$

with

$$\tilde{\mathcal{V}}_{g,h}(\mathcal{C}) = \bigoplus_{i\in\mathcal{I}_h} \mathrm{Hom}(V_i, {}^gV_i)\,,$$

where $\mathcal{I}_h$ is the set of isomorphism classes of simple objects in the component $\mathcal{C}_h$.

The category $\mathcal{C}$ is called G-modular, if a certain endomorphism $s : \tilde{\mathcal{V}}(\mathcal{C}) \to \tilde{\mathcal{V}}(\mathcal{C})$ is invertible. For the case $G = \{1\}$, this definition amounts to the usual definition of modularity for a ribbon category. For $G = \{1\}$ it was shown ([BK01, Section 5.5]) that invertibility of the endomorphism s is sufficient to define a modular functor for manifolds of arbitrary genus g, by gluing several appropriate manifolds of genus zero and using the gluing property of modular functors. Conversely,if a consistent $\mathcal{C}$-extended modular functor for arbitrary genus manifolds exists, s is the morphism that is assigned to the action of an element of the mapping class group of the torus and hence must be invertible

One expects that these arguments generalize to the equivariant setting: The vector spaces $\tilde{\mathcal{V}}_{g,h}(\mathcal{C})$ are the value $\tau^G(T_{g,h}{\to}T)$ of the G-modular functor that corresponds to $\mathcal{C}$ on specific covers $(T_{g,h}{\to}T)$ with monodromies g, h along a basis of fundamental cycles of the torus T. For any group G the action of the mapping class group of the torus can be lifted to an action on the collection of covers of the form $(T_{g,h}{\to}T)$. If a coherent $\mathcal{C}$-extended G-equivariant modular functor for higher genus manifolds exists, one expects that this action induces the morphism s, which then has to be invertible. Conversely, if $\mathcal{C}$ is G-modular, one would expect to get functors for arbitrary G-covers by gluing covers of genus zero surfaces.

In the above discussion of a $G = \{1\}$ $\mathcal{C}$-extended modular functor, one notices that even if the category is not modular, it is still possible to assign functors to higher genus manifolds. However if $\mathcal{C}$ is not modular, a consistent modular functor cannot be defined, precisely because it is not possible to assign natural isomorphisms to all elements of the mapping class group of the torus. So one expects that a generalization of the correspondence between G-equivariant genus zero modular functors and G-equivariant ribbon categories to higher genus and G-modular categories depends on a better understanding of the mapping class groups of G-bundles of higher genus manifolds.

We now turn to the results of this thesis. It is our aim to construct a certain class of G-equivariant categories that as a special case contain the so-called permutation equivariant categories, i.e. S_N-equivariant fusion categories that have $\mathcal{C}^{\boxtimes N}$ as neutral component with the induced permutation action of S_N. The construction will be geometric, i.e. we will give the G-equivariant modular functor that corresponds to the permutation equivariant categories. The data that enter this construction are a finite group G, a finite G-set $\mathcal{X}$ and a modular category $\mathcal{C}$.

The orbifolds of permutation equivariant categories are of particular interest: For a modular category $\mathcal{C}$ (i.e. for $G = \{1\}$), the kernel of the $\mathrm{SL}(2,\mathbb{Z})$-representation that is induced by the action of $\mathrm{SL}(2,\mathbb{Z})$ on the torus is a congruence subgroup ([NS08]). This has been conjectured for some time. Various approaches ([Ban02, Ban03]) on this problem use data of the permutation orbifolds.

As a first step towards the definition of a permutation equivariant modular functor, we construct a functor

$$\mathcal{F}_{\mathcal{X}} : \mathbf{Gcob(2)} \to \mathbf{cob(2)}$$

from the category of G-covers of two-dimensional cobordisms to the category of two-dimensional cobordisms by taking the total space of the associated bundle:

$$\mathcal{F}_{\mathcal{X}}(P \to E) := \mathcal{X} \times_G P = \mathcal{X} \times P/(g^{-1}x, p) \sim (x, gp) \tag{0.1}$$

This functor allows us to pull back a given two-dimensional topological field theory to a permutation equivariant TFT. The G-Frobenius algebra that corresponds to this topological field theory then gives enough insight to find an ansatz for the G-equivariant abelian category $\mathcal{C}^{\mathcal{X}}$, which is necessary to define a G-equivariant modular functor $\tau^{\mathcal{X}}$. The category $\mathcal{C}^{\mathcal{X}}$ is determined by the structure of orbits of the G-set $\mathcal{X}$. In this categorified case, the topological field theory is replaced by the modular functor τ that corresponds to the modular category $\mathcal{C}$. We are now ready to formulate one of our main theorems that is explained in more detail in section 2.2:

Theorem. *Let G be a finite group, $\mathcal{X}$ a finite ordered G-set and $\mathcal{C}$ a $\Bbbk$-linear modular category. Denote by $\mathcal{C}^{\mathcal{X}}$ the G-equivariant category determined by* *the orbit-structure of $\mathcal{X}$. Then the functor $\tau^{\mathcal{X}}$ defined by*

$$\tau^{\mathcal{X}}(P\to M) := \tau(\mathcal{F}_{\mathcal{X}}(P\to M))$$

is a $\mathcal{C}^{\mathcal{X}}$-extended G-equivariant modular functor.

We illustrate the situation by the following diagram:

$$\begin{tikzcd}
\text{modular category } \mathcal{C} \arrow[r] & \mathcal{C}\text{-extended modular functor } \tau \arrow[l, bend right, dashed] \arrow[d, "\mathcal{F}_{\mathcal{X}}\text{-construction of this thesis}"] \\
G\text{-modular category } \mathcal{C}^{\mathcal{X}} \arrow[r] & \mathcal{C}^{\mathcal{X}}\text{-extended modular functor } \tau^{\mathcal{X}} \arrow[l, bend left, dashed]
\end{tikzcd}$$

The upper arrow that points to the left is dashed, since a $\mathcal{C}$-extended modular functor endows an abelian category $\mathcal{C}$ only with a weak duality [BK01] while on a modular tensor category one has a strong duality. The lower arrow pointing to the left is dashed not only for this reason; as mentioned before the algebraic structure corresponding to higher genus G-equivariant modular functors has not yet been worked out.

This finishes the geometric construction. As a step to the permutation equivariant categories, we then use these geometric results to compute the structural functors and morphisms of a $\mathbb{Z}/2$-equivariant ribbon category, where $\mathbb{Z}/2$ acts on a set $\mathcal{X}$ with two elements by permutation. In this case it is possible to give the full structure of a $\mathbb{Z}/2$-equivariant monoidal category, which turns out to be ribbon, rather than only weakly ribbon. This finishes in a geometric setting a program that was started in a purely algebraic approach in [BFRS10a].

For arbitrary finite groups the situation is far more involved. In the correspondence between G-equivariant modular functors and G-equivariant weakly ribbon categories, the monoidal product comes from appropriate G-covers over the pair-of-pants, i.e. over a sphere with three non-intersecting discs removed. The main problem in the computation of the monoidal product is to handle associated covers of these G-bundles: If the total space of the associated bundle has non-zero genus, its geometry becomes complicated to work with.

As a step towards the monoidal structure on $\mathcal{C}^{\mathcal{X}}$, one notices that the axioms of a G-equivariant monoidal category imply that the summands $\mathcal{C}^{\mathcal{X}}_g$ are module categories (see [Ost03]) over the neutral component $\mathcal{C}^{\mathcal{X}}_1$, which is a braided monoidal category. Even if the G-equivariant weakly ribbon categories $\mathcal{C}^{\mathcal{X}}$ for arbitrary groups are too complicated, it is still possible to compute the module categories $\mathcal{C}^{\mathcal{X}}_g/\mathcal{C}^{\mathcal{X}}_1$, which by the results of [ENO09] determine this structure up to an element of a torsor over the group $\mathrm{H}^3(G,\Bbbk^*)$.

In addition we compute the modular invariant partition functions that come with these module categories ([Ost03, Section 5.2]): A module category $\mathcal{M}$ over a fusion category $\mathcal{D}$ comes ([Ost03]) with the two α-induction functors $\alpha^{\pm} : \mathcal{D} \to \mathcal{E}nd_{\mathcal{D}}(\mathcal{M})$. An important quantity is following matrix with non-negative integers as entries:

$$Z(\mathcal{M}/\mathcal{D})_{i,j} := \dim_{\Bbbk}\mathrm{Hom}_{\mathcal{E}nd_{\mathcal{D}}(\mathcal{M})}(\alpha^+_i, \alpha^-_j)\,,$$

where i,j label the simple objects of $\mathcal{D}$. The entries of this matrix are the coefficients of the modular invariant partition function.

The main geometric tool in this thesis is the Lego-Teichmüller Game, or LTG for short. The LTG is a systematic way to describe the action of diffeomorphisms on surfaces: A marking graph is drawn on the surfaces and a set of simple moves between such graphs is introduced. These moves implement the action of the mapping class group and the operation of gluing of surfaces.

The correspondence between the modular category $\mathcal{C}$ and the modular functor τ provides us with a dictionary that translates LTG-moves into morphisms in $\mathcal{C}$ (see appendix A). This dictionary turns out to be of invaluable use in the computation of the structure morphisms in the categories $\mathcal{C}^{\mathcal{X}}$, which are entirely expressed by the structure morphisms of $\mathcal{C}$.

Outline

In chapter 1 we will collect a number of definitions and basic properties of G-equivariant topological field theories and modular functors. In more detail, section 1.1 recalls the definition of a G-equivariant topological field theory and then focusses on the case of two-dimensional theories. Here the correspondence between G-equivariant TFTs and G-Frobenius algebras is explained. Section 1.2 is a detailed review of the Lego-Teichmüller Game, a formalism that aims at a description of mapping class groups of surfaces. In particular the connection between the LTG-moves and the mapping class group of a surface is explained. Section 1.3 recalls the concept of G-equivariant modular functors, which are related to G-equivariant weakly ribbon categories in section 1.4.

Chapter 2 contains our main construction: In section 2.1.1 we introduce the cover functor (0.1), which ist the central idea for the construction, and use it to construct permutation equivariant topological field theories. Then we give a survey of those surfaces that will be of importance in the sequel and compute the G-Frobenius algebras that correspond to our permutation equivariant TFTs. These algebras provide us with enough insight to define the permutation equivariant modular functors in section 2.2.

In chapter 3 we turn to concrete calculations: In section 3.1 we will derive in detail the full structure of a $\mathbb{Z}/2$-equivariant ribbon structure from the G-equivariant modular functor $\tau^{\mathcal{X}}$. This makes heavy use of the Lego-Teichmüller Game and is based on a careful analysis of $\mathbb{Z}/2$-covers of certain manifolds. In section 3.2 we turn to arbitrary groups and compute the module category structures for $\mathcal{C}_g^{\mathcal{X}}$ over $\mathcal{C}_1^{\mathcal{X}}$ that are part of the larger structure of a G-equivariant monoidal category. The modular invariant partition functions for these module categories are computed in 3.3. This shows that every permutation modular invariant is physical.

In appendix A we will give a short overview over the application of the Lego-Teichmüller Game to modular functors. This will include a dictionary that relates LTG-moves to the structure morphisms of a ribbon category.

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- T. Barmeier and C. Schweigert. A Geometric Construction for Permutation Equivariant Categories from Modular Functors. *Arxiv preprint arXiv:1004.1825, submitted to Transformation Groups*, 2010
- T. Barmeier. Permutation Modular Invariants from Modular Functors. *Arxiv preprint arXiv:1006.3938, submitted to Selecta Mathematica*, 2010

Another publication that is independent of this thesis is

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1. MODULAR FUNCTORS AND TFTS

We will always assume that G is a finite group and $\Bbbk$ an algebraically closed field of characteristic zero.

In this chapter we will recall the definitions of a G-equivariant topological field theory and a G-Frobenius algebra. We will investigate the category $\mathbf{Gcob(2)}$ of principal G-covers of two-dimensional cobordisms and explain how to derive a G-Frobenius algebra from a two-dimensional G-equivariant topological field theory.

We then have a closer look at the category **ESurf** of extended surfaces and introduce the Lego-Teichmüller Game. The LTG will be used in chapter 3.

Then we recall the concept of G-equivariant modular functors and of G-equivariant (weakly) ribbon categories and relate these notions.

1.1 Topological field theories and Frobenius algebras

1.1.1 Equivariant topological field theories

In this subsection we recall the notion of a G-equivariant topological field theory. We first define a category $\mathbf{Gcob(d)}$ of cobordisms with G-covers.

An object $(P{\to}\Sigma, \{e_i\})$ of $\mathbf{Gcob(d)}$ consists of a smooth $(d-1)$-dimensional closed oriented manifold Σ, together with a principal left G-bundle P on Σ. For technical reasons, we also fix a marked point e_i on each connected component of P. For any oriented manifold M, the manifold with the opposite orientation will be denoted by $\overline{M}$. It is sufficient to choose orientations of the base manifolds:

Lemma 1.1. *Let $\pi : P \to M$ be a discrete cover of an oriented manifold M. Then the orientation of M canonically induces an orientation on P.*

Proof. Since $\pi : P \to M$ is a discrete cover, π is a local diffeomorphism. The global orientation of M can be represented by a global section of the orientation bundle on M, which can be pulled back along the local diffeomorphism π to a global section of the orientation bundle of P. □

The morphisms in $\mathbf{Gcob(d)}$ from $(P_1{\to}\Sigma_1, e_1)$ to $(P_2{\to}\Sigma_2, e_2)$ are diffeomorphism classes of cobordisms of the base and the total space of the principal bundles. More precisely, consider pairs (M, E), consisting of a left principal G-bundle E over a smooth d-dimensional oriented manifold M and an orientation preserving diffeomorphism of G-bundles from the restriction $\partial E \to \partial M$ to the G-bundles on the boundary, i.e. diffeomorphisms

$$\partial M \xrightarrow{\cong} \Sigma_1 \sqcup \overline{\Sigma_2} \ , \qquad E|_{\Sigma_1} \xrightarrow{\cong} P_1 \text{ and } E|_{\overline{\Sigma_2}} \xrightarrow{\cong} \overline{P_2} \ .$$

The morphisms in $\mathbf{Gcob(d)}$ are now obtained by modding out diffeomorphisms of E. We write $(E{\to}M)$ for a representative.

The composition of morphisms is gluing along the boundary of both the base and the total space: Let $(P_i{\to}\Sigma_i, e_i)$ for $i = 1, 2, 3$ be objects in $\mathbf{Gcob(d)}$ and $(E{\to}M) : (P_1{\to}\Sigma_1, e_1) \to (P_2{\to}\Sigma_2, e_2)$ and $(E'{\to}M') : (P_2{\to}\Sigma_2, e_2) \to (P_3{\to}\Sigma_3, e_3)$ be cobordisms. Then the manifold $M \sqcup_{\Sigma_2} M'$ obtained by gluing the base spaces is naturally equipped with the G-principal bundle $E \sqcup_{P_2} E'$. The identity on $(P{\to}\Sigma, e)$ is the diffeomorphism class of the cylinder over Σ with trivial G-cover.

The category $\mathbf{Gcob(d)}$ has a natural structure of a symmetric monoidal category, with tensor product given by disjoint union of manifolds and bundles. The empty set with empty G-bundle is the tensor unit.

We comment on the role of the marked points on the G-cover P of an object $(P{\to}\Sigma, \{e_i\})$ which we have chosen as an auxiliary datum. For simplicity we assume that P is connected, so we only consider a single marked point $e \in P$. Its projection on Σ determines a base point $x \in \Sigma$. Moreover, it determines an identification of the fibre P_x over x with G. The marked points on the objects $(P{\to}\Sigma, e)$ induce marked points on the boundaries of cobordisms, but we did not impose any conditions on marked points on cobordisms. Thus different choices for $e \in P$ give objects that are isomorphic in $\mathbf{Gcob(d)}$, with the isomorphism being the cylinder with the trivial G-cover. In the case of the trivial group $G = 1$, one can forget the marked point and obtains an equivalence of categories of $\mathbf{Gcob(d)}$ to the usual category $\mathbf{cob(d)}$ of manifolds and cobordisms.

Definition 1.2. The category **GTFT** of *d-dimensional G-equivariant topological field theories* (or G-TFTs for short) is the category of symmetric monoidal functors

$$\mathrm{tft}^G : \mathbf{Gcob(d)} \to \mathcal{V}ect_{\Bbbk}. \tag{1.1}$$

with monoidal natural transformations as morphisms.

1.1.2 The category $\mathbf{Gcob(2)}$

From now on, we restrict our attention to two-dimensional cobordisms. We will at first recall basic definitions concerning the two-dimensional case and introduce some notation for the main objects of this thesis.

In the face of the technical condition in subsection 1.1.1 that the (one-dimensional) manifolds serving as objects in $\mathbf{Gcob(2)}$ have distinguished points on their connected components, the appropriate notion for morphisms in $\mathbf{Gcob(2)}$ is that of G-covers of extended surfaces. Thus from [BK00, Section 2] recall the following

Definition 1.3. An *extended surface* is a compact oriented smooth two-dimensional manifold E, possibly with boundary, together with a choice of a marked point on each connected component of the boundary ∂E. The set of boundary components of E is denoted by $A(E)$ and we write extended surfaces as $(E, \{p_a\}_{a\in A(E)})$. A morphism of extended surfaces is a diffeomorphism that preserves marked points. The category of extended surfaces together with morphisms of extended surfaces will be denoted by **ESurf**.

If $(E, \{p_a\}_{a\in A(E)})$ is an extended surface and $\alpha, \beta \in A(E)$ with $\alpha \neq \beta$, then we can glue E along the boundary components α and β. The glued surface is denoted by $\sqcup_{\alpha,\beta} E$.

We will discuss extended surfaces and particularly their automorphisms in section 1.2 in greater detail. From [Pri07, Section 2] recall the definition of a G-cover of an extended surface:

Definition 1.4. A *G-cover of an extended surface* $(E, \{p_a\}_{a\in A(E)})$ is a pair $(P\to E, \{\tilde{p}_a\}_{a\in A(E)})$, where $(P\to E)$ is a principal G-bundle and $\tilde{p}_a \in P$ is a choice of a marked point in the fiber of p_a for every $a \in A(E)$.

A morphism of G-covers of extended surfaces is a diffeomorphism between principal G-covers that preserves marked points. We do not assume that morphisms of G-covers of extended surfaces restrict to the identity on the base space. The category of G-covers of extended surfaces is denoted by **GESurf**.

The morphisms in $\mathbf{Gcob(2)}$ are thus diffeomorphism classes of G-covers of extended surfaces. We will now introduce notation for representatives of certain objects and morphisms in $\mathbf{Gcob(2)}$.

Covers of the circle

For any element $g \in G$, we introduce the principal G-bundle $(P_g \to S^1)$ with total space

$$P_g := \mathbb{R} \times G/(t + 2\pi, h) \sim (t, hg) \tag{1.2}$$

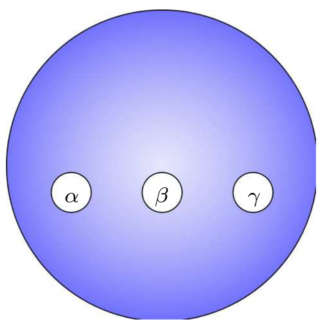
and distinguished point $[0_{\mathbb{R}}, 1_G]$. With respect to this point, the monodromy of the bundle is given by g. One easily checks that every principal G-bundle over S^1 is isomorphic to P_g for some $g \in G$.

Covers of the n-punctured sphere

We now fix a standard model for a connected extended surface of genus zero with n boundary components. These surfaces and particularly their G-covers will play a crucial role in this thesis.

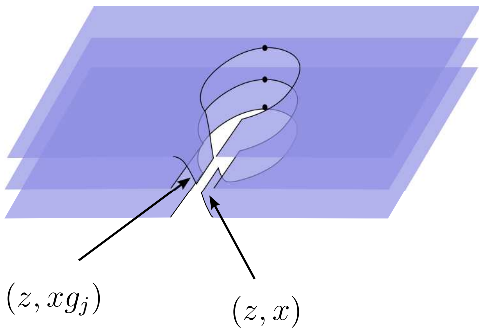
Definition 1.5. For every $n \in \mathbb{N}$ the *standard sphere* (or n-punctured sphere) S_n is the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with standard orientation and with discs of radius $\frac{1}{3}$ centered at the first n positive integers removed. As marked points on the boundary components of S_n, we choose $k - \frac{i}{3}$ for $k = 1, \dots, n$.

As an example, the standard sphere S_3 looks as follows



(1.3)

We need G-covers of the standard sphere S_n; as standard models for these covers, we use the so called *standard blocks* [Pri07]. To construct the standard blocks, we remove from S_n the straight lines connecting the points $k + \frac{i}{3}$ to the point ∞. The resulting manifold $S_n \backslash$cuts is contractible, hence it only has the trivial G-cover $((S_n\backslash\text{cuts}) \times G \to S_n)$. For any n-tuple $g_1, \dots g_n$ of elements in G whose product is the neutral element, we obtain a G-cover of S_n by gluing the two sides of the j-th cut in $(S_n\backslash\text{cuts}) \times G$ with the action of $g_j \in G$. The following picture shows the situation with a view in the direction of the negative imaginary axis:



(1.4)

with $(z, x) \in (S_n\backslash\text{cuts}) \times G$.

We finally have to specify marked points on the boundary components of $((S_n\backslash\text{cuts}) \times G)/\text{gluing}$. To this end, we choose another n-tuple $h_1, \dots, h_n$ of elements in G and take as marked points $[k - \frac{i}{3}, h_k]$ for $k = 1, \dots n$. We write $S_n(g_1, \dots, g_n; h_1, \dots, h_n)$ for the total space of these marked G-covers over S_n. In fact, any G-cover over an n-punctured sphere is diffeomorphic to one G-cover of the form $(S_n(g_1, \dots, g_n; h_1, \dots, h_n) \to S_n)$. These covers have monodromies $h_i g_i^{-1} h_i^{-1}$ around the i-th boundary circle of S_n. The restriction of the cover to the i-th boundary circle is then diffeomorphic to $(P_{h_i g_i^{-1} h_i^{-1}} \to S^1)$.

In the definition of standard blocks, the orientation of a boundary component depends on whether we consider the component as ingoing or outgoing. For example, for the pair-of-pants with one outgoing circle, the third circle is given a clockwise orientation. The cover of S_3 that is most important in the following discussion is $S_3(g_1, g_2, (g_1g_2)^{-1}; 1, 1, 1)$. When orienting the third boundary component as outgoing, this cover has monodromies g_1^{-1} and g_2^{-1} at the ingoing components and $(g_1g_2)^{-1}$ at the outgoing component.

1.1.3 From G-equivariant TFTs to G-Frobenius algebras

In this subsection, we explain the equivariant generalization of the correspondence [Koc04] between two-dimensional topological field theories and commutative Frobenius algebras to set the stage for the discussion of G-equivariant modular functors.

We start by recalling definitions from [MS06, Tur99]:

Definition 1.6. A (G-commutative) *G-Frobenius algebra* (or crossed G-Frobenius algebra, or Turaev algebra) is a G-graded associative unital algebra $A = \bigoplus_{g\in G} A_g$ together with a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$ such that

1. The G-action is compatible with the G-grading via the adjoint action of G on itself, $\alpha_h : A_g \to A_{hgh^{-1}}$.
2. The restriction of α_h to A_h is the identity.
3. A is twisted commutative: For all $a \in A, b \in A_h$ we have $\alpha_h(a)b = ba$.
4. There is a G-invariant trace $\varepsilon : A_1 \to \Bbbk$ such that the induced pairing $A_g \otimes A_{g^{-1}} \xrightarrow{m} A_1 \xrightarrow{\varepsilon} \Bbbk$ is non-degenerate.
5. For all $g, h \in G$ we have
$$\sum \alpha_h(\xi_i)\xi^i = \sum \eta_i \alpha_g(\eta^i) \in A_{hgh^{-1}g^{-1}} \tag{1.5}$$
where (ξ_i, ξ^i) and (η_i, η^i) are pairs of dual bases of $A_g, A_{g^{-1}}$ and $A_h, A_{h^{-1}}$ respectively.

We call A_g the g-graded component and A_1 the neutral component of A. A morphism of G-Frobenius algebras is a morphism of unital algebras that respects the trace, the G-action and the grading.

The following theorem [MS06, Tur99] holds:

Theorem 1.7. *The symmetric tensor categories of G-TFTs and G-Frobenius algebras are equivalent.*

Instead of reviewing the complete proof (see [Tur99] or [MS06] with slightly different conventions), we recall how to extract the data of a G-Frobenius algebra from a two-dimensional G-TFT tft^G.

- For $g \in G$, the g-graded component is defined as a vector space by
$$A_g := \mathrm{tft}^G(P_{g^{-1}} \to S^1, [0, 1_G]) \ ,$$
where $P_{g^{-1}}$ is the principal G-bundle on S^1 introduced subsection 1.1.2.

- For any pair of group elements $g, h \in G$ consider the standard three-point block
$$(S_3(g, h, (gh)^{-1}; 1, 1, 1) \to S_3)\,,$$
with clockwise orientation around the third boundary component, as a cobordism
$$(P_{g^{-1}} \to S^1, [0, 1_G]) \sqcup (P_{h^{-1}} \to S^1, [0, 1_G]) \longrightarrow (P_{(gh)^{-1}} \to S^1, [0, 1_G]) \quad (1.6)$$
Its image under the functor tft^G is a morphism $m_{g,h} : A_g \otimes A_h \to A_{gh}$ which yields an associative product $\sum_{g,h \in G} m_{g,h}$ on the G-graded vector space $A = \bigoplus_{g \in G} A_g$.

- Unit η and counit ε of A are obtained from cobordisms with the topology of a disc D. Since the disc is contractible, it only admits the trivial cover $(D \times G \to D)$ which restricts unit and counit to be trivial on A_g for $g \neq 1$.

  The unit η of A is obtained as $\mathrm{tft}^G(D \times G \to D)$ with the disc viewed as a cobordism from the empty set to $(P_1 \to S^1, [0, 1_G])$. Hence $\eta : \Bbbk \to A_1$. Similarly we define the counit as $\mathrm{tft}^G(D \times G \to D)$, where this time the disc is seen as a cobordism from $(P_1 \to S^1, [0, 1_G])$ to the empty set.

- We finally obtain the action of $h \in G$ on A_g from the cover
$$(S_2(g, g^{-1}; 1, h) \to S_2)$$
of the cylinder where the marked point over the outgoing boundary has been shifted by $h \in G$. The discussion in subsection 1.1.2 shows that the boundaries are isomorphic to the bundles $(P_{g^{-1}} \to S^1, [0, 1_G])$ and $(P_{hg^{-1}h^{-1}} \to S^1, [0, 1_G])$ respectively, hence α_h has the correct domain and target.

We refer to [MS06] for a detailed calculation, which shows that these data endow $A = \bigoplus_{g \in G} A_g$ with the structure of a G-Frobenius algebra.

1.2 The Lego-Teichmüller Game

In this section we will recall basic properties of extended surfaces and introduce the Lego-Teichmüller Game (or LTG for short). We will use the terminology of [BK00]. The LTG is a collection of simple moves between graphs on extended surfaces that describe the action of their respective mapping class groups.

These moves will play a crucial role in the discussion of modular functors and the construction of equivariant monoidal categories. In particular in sections 3.1 and 3.2 we will make heavy use of the LTG. In this section we will only consider the Lego-Teichmüller Game for extended surfaces (definition 1.3) and not for G-covers of extended surfaces as in [Pri07]. The G-equivariant Lego-Teichmüller Game, which is unfortunately only available for G-covers of surfaces of genus zero, only plays a role in the proof of 1.23. For the results of this thesis, in particular in sections 3.1 and 3.2, we only need the LTG for extended surfaces.

1.2.1 Basic definitions

Let us start by introducing one of the central objects of this section.

Definition 1.8. The *mapping class group* $\Gamma(E)$ of an extended surface $(E, \{p_a\}_{a\in A(E)})$ is the group of homotopy classes of morphisms $E \xrightarrow{\cong} E$ of extended surfaces.

By definition, the elements of the mapping class group are equivalence classes of morphisms. When we speak about elements of the mapping class group, we will use the same symbol for both the equivalence class and for a representative of this class to simplify notation.

The main purpose of this section is to give a description of the action of the mapping class group on the surface E, that is similar to a presentation by generators and relations.

Definition 1.9.

- A *cut* on an extended surface $(E, \{p_a\}_{a\in A(E)})$ is a smooth closed simple curve in the interior of E with a distinguished point on the curve.
- A *cut system* on E is a finite set C of non-intersecting cuts, such that each connected component of $E\backslash C$ is a surface of genus 0.

- The set of all cut systems on E up to isotopy is denoted by $C(E)$.

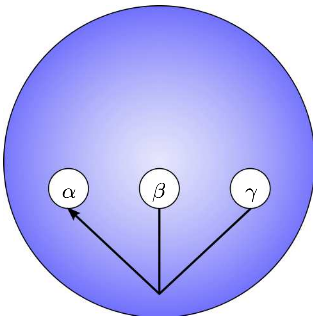
The distinguished points on the cuts turn $E \backslash C$ into an extended surface.

Definition 1.10. A *parametrization* of an extended surface $(E, \{p_a\}_{a\in A(E)})$ is a pair $(C, \{\psi_i\})$ of a cut system C on E and for every connected component E_i of $E \setminus C$ a homotopy class of a diffeomorphism $\psi_i : E_i \to S_{n_i}$ to the standard sphere S_{n_i} (see definition 1.5) for some appropriate n_i.

The mapping class group $\Gamma(E)$ of an extended surface $(E, \{p_a\}_{a\in A(E)})$ acts on the set of all parametrizations by

$$\psi \cdot (C, \{\varphi_i\}) := (\psi(C), \{\varphi_i \circ \psi^{-1}\})$$

As a tool to visualize parametrizations and to keep track of the action of the mapping class group of an extended surface E, we will now introduce *markings* on extended surfaces. As a first step we consider the standard sphere S_n. The *standard marking* on S_n is a graph m_{S_n}, that has one vertex in the lower halfplane at $-2i$, and straight edges that connect this vertex to the distinguished points $k - \frac{i}{3}$ of S_n. We use this to fix a distinguished edge (the one connected to the leftmost boundary component) and an order on the set of edges of m_{S_n}. See the following picture of the standard marking for $n = 3$; the distinguished edge is marked by an arrow:



(1.7)

We call the vertex at $-2i$ the *internal vertex* of the marking graph. It is the only vertex that does not belong to a boundary component.

Definition 1.11.

- A *marking without cuts* on an extended surface $(E, \{p_a\}_{a\in A(E)})$ of genus zero is a graph m on E, such that there is a diffeomorphism $\varphi : E \xrightarrow{\cong} S_n$ for $n := |A(E)|$ with $m = \varphi^{-1}(m_{S_n})$.

- A *marking* on an arbitrary extended surface $(E, \{p_a\}_{a\in A(E)})$ is a pair (C, m), where C is a cut system and m is a graph on E that gives a marking without cuts on every connected component of $E\setminus C$. An example of genus one with two boundary components can be seen in (1.32).

Isotopic markings are identified. The set of isotopy classes of markings on E is denoted by $M(E)$.

The marking on the standard sphere gives a distinguished edge and an ordering of edges on every marking without cuts on an extended surface of genus zero. Similarly on a general extended surface with a marking with cuts, the standard marking induces a distinguished edge and an ordering of the edges on all the markings on the connected components of $E\setminus C$. In some drawings we will mark the distinguished edge by an arrow, but more often the arrow is suppressed for simplicity. We also get an internal vertex in the induced markings on every connected component of $E\setminus C$. Note that an extended surface surface of genus zero can be endowed with a marking with cuts.

The mapping class group $\Gamma(E)$ of an extended surface $(E, \{p_a\}_{a\in A(E)})$ acts on the set $M(E)$ of isotopy classes of markings by

$$\psi \cdot (C, m) := (\psi(C), \psi(m)) \,.$$

Lemma 1.12. *Let $(E, \{p_a\}_{a\in A(E)})$ be an extended surface. There is a bijection between the set of all parametrizations of E and the set of all markings on E.*

This immediately follows from the definitions.

Remark 1.13. In addition to the action of the mapping class group, we have the operations of disjoint union and gluing of extended surfaces. We endow the disjoint union and a glued surface with the following marking graphs:

- *disjoint union*: Given two extended surfaces E_1, E_2 with markings $(C_1, m_1), (C_2, m_2)$, we endow the disjoint union $E_1 \sqcup E_2$ with the marking $(C_1 \sqcup C_2, m_1 \sqcup m_2)$.

- *gluing*: Let $(E, \{p_a\}_{a\in A(E)})$ be an extended surface with marking (C, m) and let $\alpha, \beta \in A(E)$ with $\alpha \neq \beta$. Then we endow the glued surface $\sqcup_{\alpha,\beta} E$ with the marking $\sqcup_{\alpha,\beta}(C, m) := (C \sqcup \{\alpha\}, \sqcup_{\alpha,\beta} m)$.

1.2.2 The complex of markings

In this subsection we want to associate to any extended surface $(E,\{p_a\}_{a\in A(E)})$ a 2-dimensional CW-complex (in fact a 2-groupoid) $\mathcal{M}(E)$ that describes the set of possible markings (up to isotopy) on E and the relations between these markings. To this end we introduce a set of simple moves that interchange two markings and relations between sequences of these moves.

As vertices of $\mathcal{M}(E)$ we take the set $M(E)$ of isotopy classes of possible markings on E. The edges will be the moves to be defined below, and 2-cells relations between these moves.

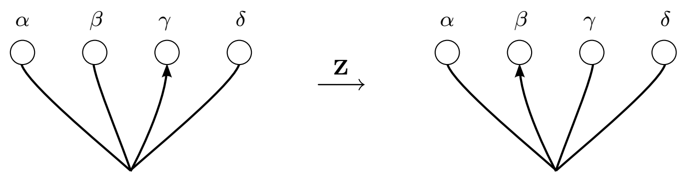
Moves

We define a set of moves that translate certain marking graphs into each other. These moves are in some cases motivated by the action of the mapping class group $\Gamma(E)$, i.e. we introduce moves that translate the marking (C,m) into $(\varphi(C),\varphi(m))$ for certain $\varphi\in\Gamma(E)$. Given two vertices in $\mathcal{M}(E)$, i.e. two marking graphs on E, we add one oriented edge between these vertices for any of the following moves described below that transforms the two corresponding markings into each other. We also add edges for pullbacks of moves along diffeomorphisms of extended surfaces. The edges are directed and for markings $(C,m),(C',m')$ on E we write an edge $\mathbf{E}$ connecting these markings as $\mathbf{E}:(C,m)\to(C',m')$. If we travel the edge $\mathbf{E}$ backwards, we write $\mathbf{E}^{-1}:(C',m')\to(C,m)$.

When we introduce these moves, we will in most cases only draw the markings and not the surfaces, to provide a clearer view on the marking. At first we give a list of genus zero move, of which there are the **Z**-, the **B**- and the **F**-move. Then we give one genus one move (the **S**-move) and finally explain how to generate more moves out of these four basic moves.

<u>Genus 0 moves</u>
Fix an extended surface $(E,\{p_a\}_{a\in A(E)})$ of genus 0.

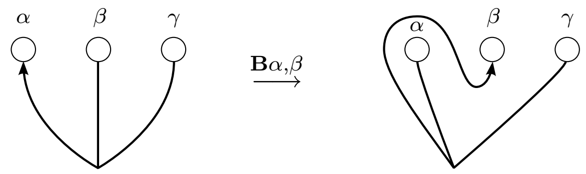
The **Z***-move:* For any marking $(\emptyset,m)$ without cuts on E we introduce the *rotation move* or **Z***-move* that moves the distinguished edge of $(\emptyset,m)$ as in the following example for a marking on S_4:



(1.8)

Observe that neither of these markings is the standard marking on S_4. The **Z**-move is motivated by the element of the mapping class group that is rotation of the standard sphere S_n around the equator.

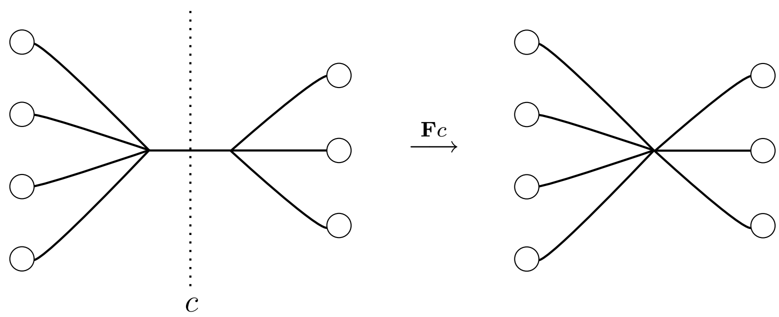
The **B**-*move:* For the standard sphere S_3 define the *braiding move* or **B**-*move*:



(1.9)

The **B**-move is motivated by rotation of the first two boundary components around each other. See subsection 1.2.4 for a discussion of this motivation.

The **F**-*move:* Let $(\{c\}, m)$ be a marking on E with only one cut. This gives markings on both connected components of $E \backslash c$. Then we introduce the *fusion move* or **F**-*move* that contracts the marking along the factorizing link, which is the edge of the marking graph that intersects the cut:



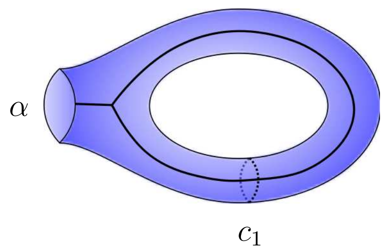
(1.10)

Here the dotted line indicates the cut c, which in fact is a closed curve, but we only draw a section of the whole marking. Note that this in fact is what is called the "generalized **F**-move" in [BK00]. In $\mathcal{M}(E)$, the **F**-move connects the markings $(\{c\}, m)$ and $(\emptyset, m')$, where m' is the contracted graph.

The **F**-move is motivated by the process of gluing of extended surfaces.

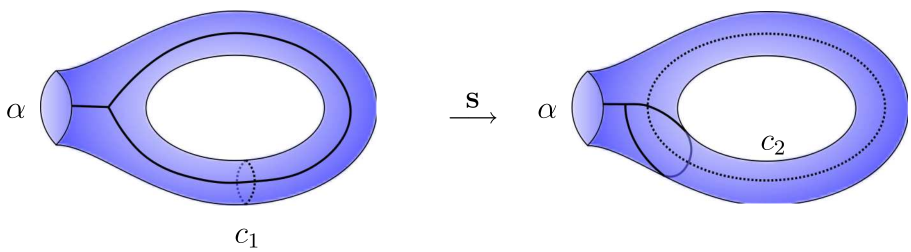
<u>*Genus* 1 *moves*</u>

The **S**-*move:* Let T_1 be a torus with one boundary component α and consider the marking



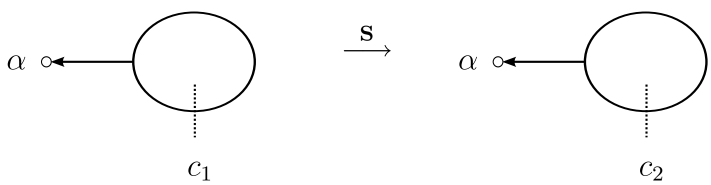
(1.11)

with one cut c_1. We introduce the **S**-*move*:



(1.12)

which connects the two indicated markings in $\mathcal{M}(T_1)$. When we simplify these pictures to show marking graphs without drawing surfaces, the **S**-move reads



(1.13)

In this simplified picture it is important to notice that the label of the cut changes.

The **S**-move is motivated by the diffeomorphism of the torus T_1 that exchanges two cycles that are canonical generators of $H_1(T_1, \mathbb{Z})$.

Generating more moves

For every extended surface $(E, \{p_a\}_{a\in A(E)})$ we add the following edges in $\mathcal{M}(E)$ in addition to the moves above:

Disjoint union: If $E = E_1 \sqcup E_2$ and $\mathbf{E} : (C_1, m_1) \to (C_1', m_1')$ is an edge in $\mathcal{M}(E_1)$ and (C_2, m_2) a marking on E_2, then we add an edge

$$\mathbf{E} \sqcup \mathrm{id}_{C_2,m_2} : (C_1, m_1) \sqcup (C_2, m_2) \to (C_1', m_1') \sqcup (C_2, m_2)\,. \tag{1.14}$$

Gluing: If $E = \sqcup_{\alpha,\beta}\tilde{E}$ with $\alpha, \beta \in A(\tilde{E})$, $\alpha \neq \beta$, and $\mathbf{E} : (C, m) \to (C, m')$ is an edge in $\mathcal{M}(\tilde{E})$, we add an edge

$$\sqcup_{\alpha,\beta}\, \mathbf{E} : \sqcup_{\alpha,\beta}(C, m) \to \sqcup_{\alpha,\beta}(C', m') \tag{1.15}$$

in $\mathcal{M}(E)$.

Distinguished moves

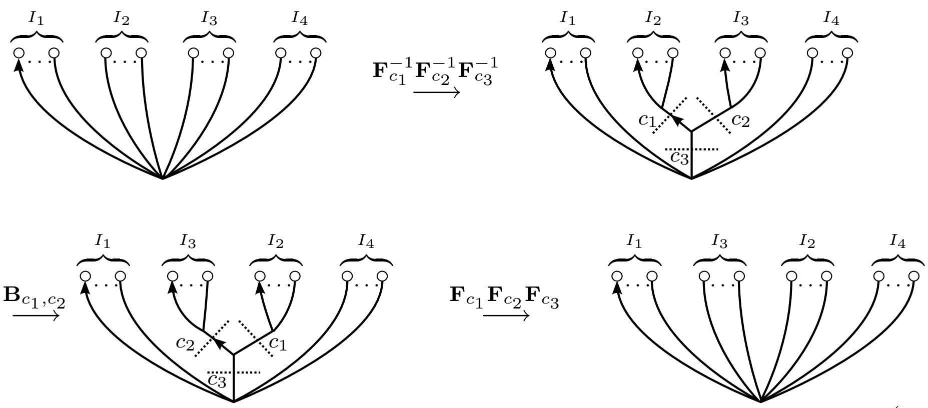
We introduce three further generalized moves. These are not necessary for the following discussion, as they are already given by sequences of the previous moves. We will only introduce these to simplify notation, they do not give edges in the complex $\mathcal{M}(E)$.

If $(C, m), (C', m'), (C'', m'')$ are three markings on E and $\mathbf{E} : (C, m) \to (C', m')$ and $\mathbf{E}' : (C', m') \to (C'', m'')$ are moves, we agree to write

$$\mathbf{E}'\mathbf{E} : (C, m) \to (C'', m'') \tag{1.16}$$

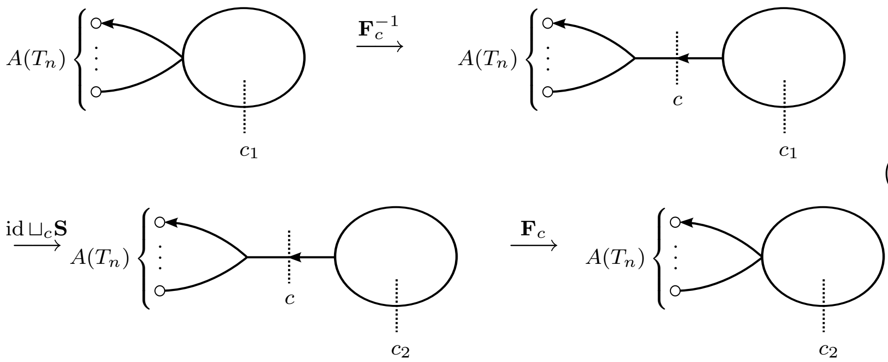
for the composition of these moves.

The generalized **B***-move:* Let $(E, \{p_a\}_{a\in A(E)})$ be an extended surface of genus 0 and $(\emptyset, m)$ a marking without cuts on E. Assume that $A(E)$ is a disjoint union $A(E) = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4$ with $I_1 < I_2 < I_3 < I_4$, where the order is induced by the order of the edges of m. Some of the sets I_k may be empty. Then the generalized braiding move $\mathbf{B}_{I_2,I_3}$ is given by



(1.17)

The generalized **S***-move:* Let T_n be a torus with n punctures. We define the generalized **S**-move as the following composition:



(1.18)

The Dehn-twist move: For any extended surface $(E, \{p_\alpha\}_{\alpha \in A(E)})$ with marking (C, m) let $\alpha \in A(E)$ be a boundary component such that the distinguished edge of m ends at α. For arbitrary markings this can be achieved by applying appropriate powers of the **Z**-move. Let c be a circle homotopic to α that does not intersect any cuts in C. Then if we cut E along c, we find that the resulting surface is diffeomorphic to a disjoint union of E with marking (C, m) and S_2 with standard marking. We use the abbreviation

$$\mathbf{T}_\alpha := \mathbf{F}_c((\mathbf{B}_{\alpha,c}^{-1}\mathbf{Z}) \sqcup_c \mathrm{id}_m)\mathbf{F}_c^{-1} \tag{1.19}$$

where $(\mathbf{B}_{\alpha,c}^{-1}\mathbf{Z})$ is meant to be applied to S_2. Here the inverse **F**-move is used to cut off a tubular neighborhood of α, i.e. a cylinder. Then the move

$\mathbf{B}_{\alpha,c}^{-1}\mathbf{Z}$ implements the Dehn-twist on this cylinder, finally the $\mathbf{F}$-move is used to glue the cylinder to the surface again. We call $\mathbf{T}_\alpha$ the *Dehn-twist move* around the boundary α.

If $(E, \{p_\alpha\}_{\alpha\in A(E)})$ is an extended surface with marking (C, m) and $c \in C$ is a cut on E, then we introduce the Dehn twist $\mathbf{T}_c$ around the cut c by applying (1.19) to one side of the cut c. The relations introduced in the next section will ensure that this is independent of the side of c on which $\mathbf{T}_c$ is applied.

Relations

We will now give a set of relations that is partly motivated by relations in the mapping class group $\Gamma(E)$. Given any two sequences of edges in $\mathcal{M}(E)$ that connect the same two vertices, we glue in a 2-cell, if any of these relations below holds between the two sequences of moves.

We first give a set of general relations, then relations between genus zero moves and then higher genus relations.

General relations

Functoriality: Assume that E is a disjoint union, $E = E_1 \sqcup E_2$, $\mathbf{E} : (C_1, m_1) \to (C_1', m_1')$. Consider the two edges $\mathbf{E}' : (C_1', m_1' \to (C_1'', m_1'')$ in $\mathcal{M}(E_1)$ and a marking (C_2, m_2) on E_2. Then for the disjoint union we impose the relation

$$(\mathbf{E}' \sqcup \mathrm{id}_{C_2,m_2})(\mathbf{E} \sqcup \mathrm{id}_{C_2,m_2}) = (\mathbf{E}'\mathbf{E} \sqcup \mathrm{id}_{C_2,m_2}) \tag{1.20}$$

and for gluing the relation

$$\sqcup_{\alpha,\beta}(\mathbf{E}'\mathbf{E}) = (\sqcup_{\alpha,\beta}\mathbf{E}')(\sqcup_{\alpha,\beta}\mathbf{E}) \tag{1.21}$$

for any $\alpha, \beta \in A(E)$ with $\alpha \neq \beta$.

Associativity:

- If $\mathbf{E}$ is an edge in $\mathcal{M}(E_1)$ and $(C_2, m_2), (C_3, m_3)$ are markings on E_2, E_3 respectively, then

$$(\mathbf{E} \sqcup \mathrm{id}_{C_2,m_2}) \sqcup \mathrm{id}_{C_3,m_3} = \mathbf{E} \sqcup \mathrm{id}_{(C_2,m_2)\sqcup(C_3,m_3)}\ . \tag{1.22}$$

- If $\alpha, \beta, \gamma, \delta \in A(E)$ are pairwise different boundary components of E, then the order of gluing does not matter, we impose the relation

$$\sqcup_{\alpha,\beta} \sqcup_{\gamma,\delta} \mathbf{E} = \sqcup_{\gamma,\delta} \sqcup_{\alpha,\beta} \mathbf{E}\,. \tag{1.23}$$

- If $E = E_1 \sqcup E_2$, $\alpha, \beta \in A(E_1)$ and $\mathbf{E} : (C, m) \to (C', m')$ is an edge in $\mathcal{M}(E_1)$, we add the relation

$$\sqcup_{\alpha,\beta} (\mathbf{E} \sqcup \mathrm{id}) = \sqcup_{\alpha,\beta}(\mathbf{E}) \sqcup \mathrm{id}\ . \tag{1.24}$$

Commutativity of disjoint union: If $E = E_1 \sqcup E_2$ and $\mathbf{E}_i$ is an edge in $\mathcal{M}(E_i)$ for $i = 1, 2$, then

$$(\mathbf{E}_1 \sqcup \mathrm{id})(\mathrm{id} \sqcup \mathbf{E}_2) = (\mathrm{id} \sqcup \mathbf{E}_2)(\mathbf{E}_1 \sqcup \mathrm{id}) \tag{1.25}$$

<u>*Genus 0 relations*</u>

Rotation: If $(\emptyset, m)$ is a marking without cuts on E and $|A(E)| = n$, then $\mathbf{Z}^n = \mathrm{id}$.

Symmetry of the **F***-move:* If $(\{c\}, m)$ is a marking on E with only one cut and $E \backslash c = E_1 \sqcup E_2$, then with $n_1 := |A(E_1)|$

$$\mathbf{Z}^{n_1 - 1} \mathbf{F}_c = \mathbf{F}_c (\mathbf{Z}^{-1} \sqcup \mathbf{Z})\,. \tag{1.26}$$

Associativity of the **F***-move:* If E is connected and $(\{c_1, c_2\}, m)$ is a marking on E with two cuts, then

$$\mathbf{F}_{c_1} \mathbf{F}_{c_2} = \mathbf{F}_{c_2} \mathbf{F}_{c_1} \tag{1.27}$$

Cylinder axiom: Let S_2 be the standard sphere with two holes, i.e. a cylinder with the standard marking $(\emptyset, m_{S_2})$ and boundary components $A(S_2) = \{\alpha, \beta\}$. Let $(E, \{p_a\}_{a \in A(E)})$ be an extended surface with marking (C, m) and $\gamma \in A(E)$. For every edge $\mathbf{E} : (C, m) \to (C', m')$ in $\mathcal{M}(E)$ we add a relation

$$\mathbf{E} \mathbf{F}_\gamma = \mathbf{F}_\gamma (\mathbf{E} \sqcup_{\gamma,\alpha} \mathrm{id}_{\emptyset, m_{S_2}}) \tag{1.28}$$

between moves $(C \sqcup \{\gamma\}, m \sqcup_{\gamma,\alpha} m_{S_2}) \to (C', m')$. Here the left hand side of (1.28) contains the homeomorphism $\varphi : E \sqcup_{\gamma,\alpha} S_2 \to E$, which is the identity outside a neighborhood of S_2 (contraction of a collar around the boundary).

Braiding axiom: Let $(E, \{p_a\}_{a\in A(E)})$ be an extended surface with four boundary components $\alpha, \beta, \gamma, \delta \in A(E)$ and $(\emptyset, m)$ a marking without cuts on E. Then

$$\begin{aligned} \mathbf{B}_{\alpha,\beta\gamma}(\emptyset, m) &= \mathbf{B}_{\alpha,\gamma}\mathbf{B}_{\alpha,\beta}(\emptyset, m)\,, \\ \mathbf{B}_{\alpha\beta,\gamma}(\emptyset, m) &= \mathbf{B}_{\alpha,\gamma}\mathbf{B}_{\beta,\gamma}(\emptyset, m)\,. \end{aligned} \tag{1.29}$$

All occurring moves are generalized braidings.

Dehn-twist axiom: Let $(E, \{p_a\}_{a\in A(E)})$ be a cylinder with boundary components α and β and let $(\emptyset, m)$ be a marking without cuts and with distinguished vertex at the boundary component α. Then the Dehn twists around either boundary of the cylinder coincide:

$$\mathbf{Z}\mathbf{B}_{\alpha,\beta}(\emptyset, m) = \mathbf{B}_{\beta,\alpha}\mathbf{Z}(\emptyset, m) \tag{1.30}$$

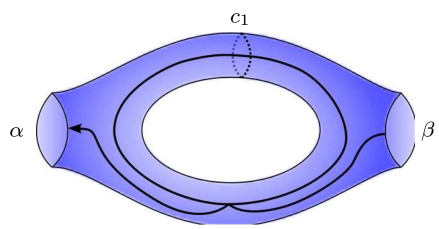
Higher genus relations

We give two more relations for surfaces of genus 1.

One-punctured torus relation: Let T_1 be a torus with one boundary component α and marking (1.11). Then we add the following relations to $\mathcal{M}(T_1)$ that resemble relations in the mapping class group of T_1:

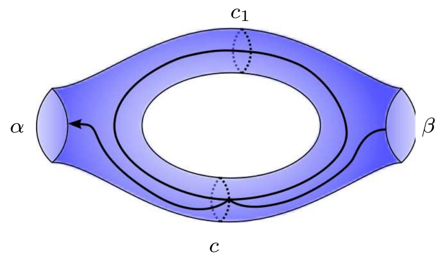
$$\mathbf{S}^2 = \mathbf{Z}^{-1}\mathbf{B}_{\alpha,c_1}\,, \quad (\mathbf{S}\mathbf{T}_{c_1})^3 = \mathbf{S}^2\,. \tag{1.31}$$

Two-punctured torus relation: Let T_2 be the torus with two boundaries α and β. Consider the marking



(1.32)

Now let c be a curve on T_2 around the lower part of T_2 at the internal vertex of the marking graph as in



$$(1.33)$$

Then we add the relation

$$\mathbf{B}_{\alpha,\beta}\mathbf{F}_{c_1}\mathbf{F}_c^{-1} = \mathbf{S}^{-1}\mathbf{B}_{c_2,\beta}^{-1}\mathbf{B}_{\beta,c_2}^{-1}\mathbf{T}_\beta^{-1}\mathbf{S} \tag{1.34}$$

where the $\mathbf{S}$-moves are generalized moves.

Propagation of relations: For every relation $\mathbf{E} = \mathbf{E}'$ we add relations on disjoint unions and gluings.

$$\mathbf{E} \sqcup \mathrm{id} = \mathbf{E}' \sqcup \mathrm{id}\,, \quad \sqcup_{\alpha,\beta}\mathbf{E} = \sqcup_{\alpha,\beta}\mathbf{E}' \tag{1.35}$$

1.2.3 Structure of the complex $\mathcal{M}(E)$

The complex $\mathcal{M}(E)$ introduced in 1.2.2 has a very simple topology. The following theorem is from [BK00]:

Theorem 1.14. *Let $(E, \{p_a\}_{a\in A(E)})$ be an extended surface. Then the complex $\mathcal{M}(E)$ is connected and simply connected.*

1.2.4 Relation to the mapping class group

We will now discuss the connection between the mapping class group $\Gamma(E)$ of an extended surface $(E, \{p_a\}_{a\in A(E)})$ and the complex $\mathcal{M}(E)$. This connection will be of crucial importance in this thesis and is the main reason for the discussion of $\mathcal{M}(E)$ in section 1.2.2.

Fix a diffeomorphism $\varphi : E \xrightarrow{\cong} E$, i.e. a representative of an element of $\Gamma(E)$. As $\Gamma(E)$ acts on the set of markings $M(E)$, for every marking (C, m) on E, we get a second marking $(\varphi(C), \varphi(m))$. Now theorem 1.14 implies that

- $(\varphi(C), \varphi(m))$ and (C, m) can be connected by a finite sequence of the LTG-moves in section 1.2.2, since $\mathcal{M}(E)$ is connected, and

- these moves are unique up to the relations in section 1.2.2, since $\mathcal{M}(E)$ is simply connected.

However, a word of warning is due. This does not mean that the action of the mapping class group on the set $M(E)$ of isotopy classes of markings is transitive. A counterexample is given by any two markings with a different number of cuts. Two such markings are connected by a finite sequence of LTG-moves (particularly by **F**-moves, which change the number of cuts), but not by a diffeomorphism $\varphi : E \to E$, which preserves the number of cuts.

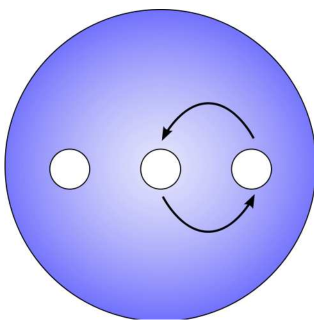
From lemma 1.12 one deduces

Lemma 1.15. *Let $(E, \{p_a\}_{a\in A(E)})$ be an extended surface and $(C, m) \in M(E)$ a marking on E. If for $\varphi \in \Gamma(E)$ we have $\varphi \cdot (C, m) = (C, m)$, then $\varphi = \mathrm{id}$.*

This means that an element $\varphi \in \Gamma(E)$ is uniquely determined by its action on any marking graph on E. So any finite sequence of LTG-moves other than the **F**-move uniquely determines an element of the mapping class group.

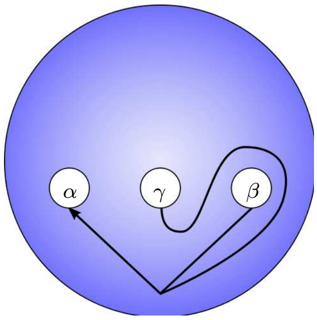
We will now discuss an example, as the translation of an element in the mapping class group into a sequence of LTG-moves will become very important in section 3.1 and 3.2.

Let $\varphi_{\mathbf{B}} \in \Gamma(S_3)$ be the element of the mapping class group that turns the second and third boundary circle counterclockwise around each other as in



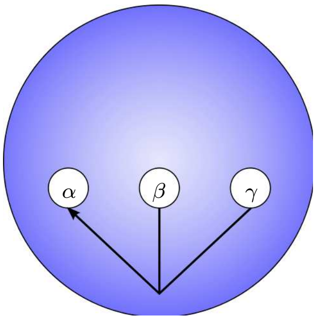
(1.36)

We call $\varphi_{\mathbf{B}}$ the *braiding diffeomorphism*. If we apply $\varphi_{\mathbf{B}}$ to the standard marking $m \equiv (\emptyset, m_{S_3})$, we get the marking

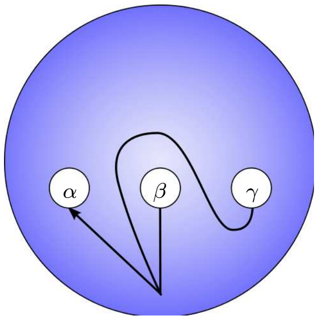


(1.37)

We can instead apply the LTG-move $\mathbf{B}$ (the generalized $\mathbf{B}$-move) to the marking m, by definition of $\mathbf{B}$ we get the marking $\varphi_{\mathbf{B}}^{-1}(m)$ on S_3:

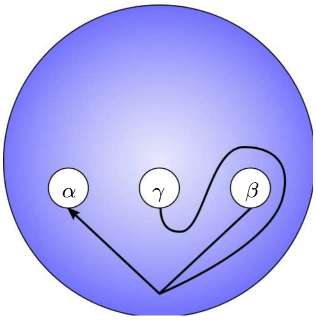


$\xrightarrow{\mathbf{B}_{\beta,\gamma}}$

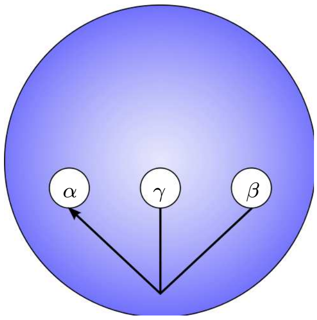


(1.38)

Now if we apply the diffeomorphism $\varphi_{\mathbf{B}}$ to both sides of (1.38), we obtain



$\xrightarrow{\mathbf{B}_{\beta,\gamma}}$



(1.39)

We see that the action of the diffeomorphism $\varphi_{\mathbf{B}}$ can be expressed by the LTG-move $\mathbf{B}$: The markings $\varphi_{\mathbf{B}}(m)$ and m are connected via the edge $\mathbf{B}$ in $\mathcal{M}(S_3)$. Conversely, if we are only given the $\mathbf{B}$-move, we find the diffeomorphism $\varphi_{\mathbf{B}}$ (up to isotopy) that interchanges the two markings that are connected by $\mathbf{B}$ in $\mathcal{M}(E)$.

1.3 G-equivariant modular functors

A structure that is related to three-dimensional topological field theory and that plays an important role in this thesis is given by the notion of a modular functor. We will discuss the more general structure of a G-equivariant modular functor, where G is a finite group.

Most of this section is from [KP08].

Definition 1.16. A *G-equivariant category* is an abelian category $\mathcal{C}^G$ with the following structure:

- A decomposition $\mathcal{C}^G \cong \bigoplus_{g\in G} \mathcal{C}^G_g$ into full abelian subcategories.
- A G-action covering the adjoint action of G on itself.
  In more detail, we have for any group element $g \in G$ a functor $R_g : \mathcal{C}^G \to \mathcal{C}^G$ and for any pair $g, h \in G$ of group elements isomorphisms $\alpha_{g,h} : R_g \circ R_h \Rightarrow R_{gh}$ such that $R_1 = \mathrm{Id}_{\mathcal{C}^G}$, $R_g(\mathcal{C}^G_h) \subset \mathcal{C}^G_{ghg^{-1}}$ and
  $$\alpha_{g_1g_2,g_3} \circ (\alpha_{g_1,g_2} * \mathrm{id}_{R_{g_3}}) = \alpha_{g_1,g_2g_3} \circ (\mathrm{id}_{R_{g_1}} * \alpha_{g_2,g_3}) \tag{1.40}$$
  as natural isomorphisms $R_{g_1} \circ R_{g_2} \circ R_{g_3} \Rightarrow R_{g_1g_2g_3}$. Here $*$ denotes the horizontal composition of 2-morphisms in the 2-category of categories.

As a shorthand, we introduce the notation ${}^gV \equiv R_g(V)$ for $g \in G$ and V in $\mathcal{C}^G$.

Definition 1.17. Let $\mathcal{C}^G$ be a G-equivariant category. We assume from now on that $\mathcal{C}^G$ is enriched over the category of finite-dimensional $\Bbbk$-vector spaces. We denote by $\boxtimes$ the Deligne tensor product of $\Bbbk$-linear categories. An object $\mathcal{R} \in \mathcal{C}^G \boxtimes \mathcal{C}^G$ is called a *gluing object* if

- $\mathcal{R}$ is of vanishing total degree, $\mathcal{R} \in \bigoplus_h \mathcal{C}^G_h \boxtimes \mathcal{C}^G_{h^{-1}}$.
- $\mathcal{R}$ is symmetric, i.e. $\mathcal{R} \cong \mathcal{R}^{op}$. Here $\mathcal{R}^{op}$ is obtained by exchanging the two factors.
- $\mathcal{R}$ is G-invariant: For every group element $g \in G$ there is an isomorphism $(R_g \boxtimes R_g)(\mathcal{R}) \cong \mathcal{R}$.
- These isomorphisms are compatible with each other.

We write $\mathcal{R}_h$ for the component of $\mathcal{R}$ in $\mathcal{C}^G_h \boxtimes \mathcal{C}^G_{h^{-1}}$; sometimes we use the Sweedler-like notation $\mathcal{R} = \mathcal{R}^{(1)} \boxtimes \mathcal{R}^{(2)}$.

We are now in a position to give the definition [KP08] of a G-equivariant modular functor. To make the notation less cumbersome, we sometimes use the abbreviation $P \equiv (P{\rightarrow}M, \{\tilde{p}_a\}_{a\in A(M)})$ for G-covers of extended surfaces.

Definition 1.18. Let $\mathcal{C}^G$ be a G-equivariant category enriched over the category of finite-dimensional $\Bbbk$-vector spaces and $\mathcal{R}$ be a gluing object for $\mathcal{C}^G$. A $\mathcal{C}^G$*-extended G-equivariant modular functor* consists of the following data:

1. Functors for G-covers:
   For every G-cover $(P{\rightarrow}M, \{\tilde{p}_a\}_{a\in A(M)})$ of an extended surface a functor
   $$\tau^G(P{\rightarrow}M, \{\tilde{p}_a\}_{a\in A(M)}) : \boxtimes_{a\in A(M)} \mathcal{C}^G_{m_a^{-1}} \rightarrow \mathcal{V}ect_{\Bbbk} \ , \tag{1.41}$$
   where m_a is the monodromy of P around the a-th boundary component of M. We will often write $\tau^G(P; \{V_a\})$ for the value of the functor on a family $\{V_a\}$ of suitable objects.

2. Functorial isomorphisms for morphisms of G-covers:
   For every isomorphism
   $$f : (P{\rightarrow}M, \{\tilde{p}_a\}_{a\in A(M)}) \rightarrow (P'{\rightarrow}M', \{\tilde{p}'_a\}_{a\in A(M')}) \tag{1.42}$$
   of G-covers of extended surfaces an isomorphism of functors
   $$f_* : \tau^G(P{\rightarrow}M, \{\tilde{p}_a\}_{a\in A(M)}) \rightarrow \tau^G(P'{\rightarrow}M', \{\tilde{p}'_a\}_{a\in A(M')})$$
   that depends only on the isotopy class of f.

3. Isomorphisms $\tau^G(\emptyset) \cong \Bbbk$ and $\tau^G(P \sqcup P') \cong \tau^G(P) \otimes_{\Bbbk} \tau^G(P')$ for all G-covers P, P'.

4. Functorial gluing isomorphisms:
   Let $(P{\rightarrow}M, \{\tilde{p}_a\}_{a\in A(M)})$ be a G-cover of an extended surface and let $\alpha, \beta \in A(M)$, $\alpha \neq \beta$, such that the respective monodromies are inverse, $m_\alpha = m_\beta^{-1}$. Then gluing of P along the boundary components over α and β is well defined. We have functorial *gluing isomorphisms*
   $$G_{\alpha,\beta} : \tau^G(P; \{V_a\}, \mathcal{R}_{\alpha,\beta}) \xrightarrow{\cong} \tau^G(\sqcup_{\alpha,\beta}P; \{V_a\}) \tag{1.43}$$
   where $\mathcal{R}_{\alpha,\beta}$ indicates that the summand $\mathcal{R}_{m_\alpha}$ of $\mathcal{R}$ is assigned to the boundary components α and β respectively. This is well defined by

vanishing total degree of $\mathcal{R}$, by symmetry of $\mathcal{R}$ and by the assumption $m_\alpha = m_\beta^{-1}$ on the monodromies of the boundaries. Here $\sqcup_{\alpha,\beta}M$ is again the surface with the boundary components α and β glued and $\sqcup_{\alpha,\beta}P$ again denotes the G-cover of $\sqcup_{\alpha,\beta}M$ that is obtained by gluing the corresponding boundary components over α and β.

5. Equivariance under the G-action:
For any G-cover $(P \to M, \{p_a\}_{a\in A(M)})$ and any tuple of group elements $\mathbf{g} = (g_a)_{a\in A(M)} \in G^{A(M)}$ we have functorial isomorphisms

$$T_{\mathbf{g}} : \tau^G(P \to M, \{\tilde{p}_a\}_{a\in A(M)}; \{V_a\}) \xrightarrow{\sim} \tau^G(P \to M, \{g_a\tilde{p}_a\}_{a\in A(M)}; \{{}^{g_a}V_a\}) \ . \tag{1.44}$$

These data are subject to the following conditions:

- $(f \circ g)_* = f_* \circ g_*$ and $\mathrm{id}_* = \mathrm{id}$.
- All structural morphisms are functorial in $(P \to M, \{\tilde{p}_a\}_{a\in A(M)})$ and compatible with each other.
- When identifying $\mathcal{R} \cong \mathcal{R}^{op}$ we have $G_{\alpha,\beta} = G_{\beta,\alpha}$.
- Normalization: $\tau^G(S^2 \times G \to S^2) \cong \Bbbk$.

Remark 1.19. Specializing to the trivial group, $G = \{1\}$ and suppressing the morphisms $T_{\mathbf{g}}$ from (1.44) which implement equivariance, we recover the usual definition of a modular functor, see e.g. [BK01].

Definition 1.20. A $\mathcal{C}^G$-extended G-equivariant modular functor is called *non-degenerate*, if for every non-zero object V in $\mathcal{C}^G$ there is a G-cover $(P \to M, \{\tilde{p}_a\}_{a\in A(M)})$ of an extended surface and a collection of objects $\{V_a\}$, such that $\tau^G(P \to M; V, \{V_a\})$ is non-zero.

A genus zero $\mathcal{C}^G$-extended G-equivariant modular functor is a G-equivariant modular functor that is only defined for G-covers $(P \to M, \{\tilde{p}_a\}_{a\in A(M)})$ where M has genus zero and the gluing isomorphisms (1.43) are only defined for pairs α, β of boundary components for which α and β lie in different connected components of M.

1.4 Modular functors and ribbon categories

For our purposes, the notion of a G-equivariant monoidal structure [Tur00, Kir04, KP08] will be important.

Definition 1.21.

1. A *G-equivariant monoidal category* is a semisimple G-equivariant category $\mathcal{C}^G$ with a monoidal structure that is compatible with the grading, i.e. $X \otimes Y \in \mathcal{C}^G_{gh}$ for $X \in \mathcal{C}^G_g, Y \in \mathcal{C}^G_h$ and for which the functors R_g implementing equivariance are endowed with the structure of tensor functors.

2. A G-equivariant monoidal category is called *braided*, if for any pair of objects $X \in \mathcal{C}^G_g, Y \in \mathcal{C}^G_h$ there are isomorphisms
$$C_{X,Y} : X \otimes Y \to {}^gY \otimes X \tag{1.45}$$
that satisfy two G-equivariant hexagon axioms.

3. An object V in a G-equivariant monoidal category $\mathcal{C}^G$ has a *weak dual* if there is an object $V^* \in \mathcal{C}^G$ representing the functor $\mathrm{Hom}_{\mathcal{C}^G}(\mathbf{1}, V \otimes ?)$. This amounts to the existence of functorial isomorphisms $\mathrm{Hom}_{\mathcal{C}^G}(\mathbf{1}, V \otimes T) \cong \mathrm{Hom}_{\mathcal{C}^G}(V^*, T)$ for all $T \in \mathcal{C}^G$. With $T = V^*$, the preimage of the identity id_{V^*} under this isomorphism gives the (right) coevaluation $\mathrm{coev}_V : \mathbf{1} \to V \otimes V^*$.

4. A G-equivariant monoidal category is called *weakly rigid* if every object has a weak dual. It is called *rigid*, if there are compatible duality morphisms (i.e. evaluation and coevaluation).

5. A G-equivariant monoidal category is called *weakly ribbon* if it is weakly rigid, braided and for every object $V \in \mathcal{C}^G_g$ there is a functorial isomorphism $\Theta_V : V \to {}^gV$ (the *ribbon twist*), satisfying certain coherence conditions spelled out in [Kir04, Section 2]. A weakly ribbon category is called a *ribbon category*, if it is rigid rather than only weakly rigid.

Observe that the morphism in part 2 of definition 1.21 does not give a braiding in the ordinary sense on $\mathcal{C}^G$ because of the g-twist in (1.45). The g-twist ist necessary for grading reasons and becomes natural in the discussion of theorem 1.22.

The discussion in sections 3.1 and 3.2 is based on the following theorem from [KP08]:

Theorem 1.22. *Let $\mathcal{C}^G$ be a semi-simple G-equivariant category that has only finitely many isomorphism classes of simple objects. A non-degenerate genus 0 $\mathcal{C}^G$-extended G-modular functor τ^G is equivalent to the structure of a G-equivariant weakly ribbon category on $\mathcal{C}^G$.*

Instead of giving the full proof, we will only explain how to obtain the structure of a G-equivariant weakly fusion category on $\mathcal{C}^G$ out of the data given by the G-equivariant modular functor τ^G. For the converse statement and detailed proof that this indeed satisfies the axioms of definition 1.21 see [KP08, Section 7.2].

Since our categories are, by assumption, semi-simple with finitely many isomorphism classes of simple objects, every $\mathcal{V}ect_{\Bbbk}$-valued functor is representable ([BK01, Lemma 5.3.1]). This allows us to adopt the following strategy:

- Objects that are needed for the definition of a G-equivariant ribbon category, i.e. the dual of an object and the tensor product of two objects, will be defined as objects representing certain functors that are induced by τ^G.

- Structure morphisms, such as the associativity constraints or the braiding, are then found by applying certain diffeomorphisms to appropriate G-covers. By the definition of a G-equivariant modular functor, these diffeomorphisms give isomorphisms of functors, which themselves by the Yoneda-lemma induce morphisms on the representing objects in the category $\mathcal{C}^G$.

One can express these diffeomorphisms by LTG-moves of G-covers of extended surfaces ([Pri07, Section 4]), which then are related to certain morphisms in the category $\mathcal{C}^G$. The relations between LTG-moves then give relations in $\mathcal{C}^G$. Like expressing diffeomorphisms by LTG-moves, one can express the process of gluing two G-covers by the G-equivariant counterpart of the **F**-move. Gluing also gives an isomorphism between certain functors. So the axioms of a G-equivariant modular functor provide us with a dictionary that translates LTG-moves into $\mathcal{C}^G$-morphisms. We give this dictionary for the non-equivariant case in appendix A.

In section 1.2 we have only introduced the Lego-Teichmüller Game for extended surfaces, not for G-covers of these. But the outcome is similar: Every diffeomorphism of G-covers (of genus zero) can be uniquely (up to relations) translated into a finite sequence of moves on markings, where in this case a marking consists of a marking graph on the base surface as in section 1.2.1 and elements of $G \times G$ attached to all edges of this graph.

For the value of the equivariant modular functor τ^G on the standard blocks of subsection 1.1.2, we introduce the shorthand notation

$$\langle V_1, \ldots, V_n \rangle_G := \tau^G(S_n(g_1, \ldots, g_n; 1, \ldots, 1) \to S_n; V_1, \ldots, V_n). \tag{1.46}$$

with objects $V_i \in \mathcal{C}^G_{g_i}$.

Now given the $\mathcal{C}^G$-extended G-equivariant modular functor τ^G, one begins by defining the **dual object** for an object V in $\mathcal{C}^G_g$. This will be an object V^* representing the functor

$$\begin{array}{ll} \mathcal{C}^G & \to \mathcal{V}ect_{\Bbbk} \\ T & \mapsto \langle V, T \rangle_G. \end{array} \tag{1.47}$$

For grading reasons, this functor takes non-zero values only if T is an object in $\mathcal{C}^G_{g^{-1}}$, hence $V^* \in \mathcal{C}^G_{g^{-1}}$. Since the object V^* is defined to be a representing object, it is determined up to canonical isomorphism by a functorial isomorphism $\mathrm{Hom}(V^*, T) \cong \langle V, T \rangle_G$. By the general theory of representing objects, this means that every functor $\mathcal{C}^G \to \mathcal{V}ect_{\Bbbk}$ is isomorphic to the functor (1.47) for some V and V is up to canonical isomorphism uniquely determined by the realization (1.47) of the hom-functor that corresponds to V.

This is important in the definition of the **tensor product** of two objects A, B in $\mathcal{C}^G$. It is defined by

$$\langle T, A \otimes B \rangle_G \cong \langle T, A, B \rangle_G. \tag{1.48}$$

So $A \otimes B$ should be thought of as an object that represents the functor $\langle ?, A, B \rangle_G$. In detail we have $\langle T, A, B \rangle_G \cong \mathrm{Hom}(T^*, A \otimes B)$. So if $A \in \mathcal{C}^G_g, B \in \mathcal{C}^G_h$ then for grading reasons the functor (1.48) only takes non-zero values if $T \in \mathcal{C}^G_{(gh)^{-1}}$. By duality, this means that $A \otimes B \in \mathcal{C}^G_{gh}$. The **associativity constraints** $(A \otimes B) \otimes C \to A \otimes (B \otimes C)$ for this tensor product are found by analyzing the covers $(S_4((ghk)^{-1}, g, h, k; 1, 1, 1, 1) \to S_4$ of the four-punctured sphere. These can be cut in two different ways, representing the tensor products $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$. This cutting procedure gives functorial isomorphisms between the corresponding $\mathcal{V}ect_{\mathbb{k}}$-valued functors. As these functors are representable with representing objects $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$, the Yoneda-lemma gives an isomorphism $(A \otimes B) \otimes C \to A \otimes (B \otimes C)$. So the associativity constraints are constructed out of the $\mathcal{C}^G$-morphisms that represent the G-equivariant **F**-move. The **pentagon axiom** for these associativity constraints follows from associativity of gluing and of the **F**-move.

The **tensor unit** is defined as the object **1** that obeys

$$\langle \mathbf{1}, T \rangle_G \cong \langle T \rangle_G \tag{1.49}$$

for every T in $\mathcal{C}^G$.

Let $A \in \mathcal{C}^G_g$ and $B \in \mathcal{C}^G_h$. Recall the braiding diffeomorphism $\varphi_{\mathbf{B}} : S_3 \to S_3$ from section 1.2.4. It turns the second and third boundary circle of S_3 counterclockwise. $\varphi_{\mathbf{B}}$ lifts to a diffeomorphism

$$\tilde{\varphi}_{\mathbf{B}} : S_3((gh)^{-1}, g, h; 1, 1, 1) \to S_3((gh)^{-1}, ghg^{-1}, g; 1, g^{-1}, 1) \tag{1.50}$$

of G-covers of S_3. This induces a functorial isomorphism

$$\begin{aligned}
&\langle T, A \otimes B \rangle_G \overset{\text{def}}{=} \langle T, A, B \rangle_G \overset{\text{def}}{=} \tau^G(S_3((gh)^{-1}, g, h; 1, 1, 1) \to S_3; T, A, B) \\
&\overset{(\tilde{\varphi}_B)_*}{\to} \tau^G(S_3((gh)^{-1}, ghg^{-1}, g; 1, g^{-1}, 1); T, B, A) \\
&\overset{T_{(1,g,1)}}{\to} \tau^G(S_3((gh)^{-1}, ghg^{-1}, g; 1, 1, 1); T, {}^gB, A) \\
&\overset{\text{def}}{=} \langle T, {}^gB, A \rangle_G \overset{\text{def}}{=} \langle T, {}^gB \otimes A \rangle_G \,,
\end{aligned} \tag{1.51}$$

where in the third line the equivariance morphisms (1.44) are used. By the Yoneda-lemma this gives an isomorphism

$$C_{A,B} : A \otimes B \to {}^gB \otimes A \tag{1.52}$$

between the representing objects in $\mathcal{C}^G$, which will serve as equivariant **braiding isomorphisms** in $\mathcal{C}^G$. The equivariant hexagon axioms are satisfied as a consequence of the braiding axiom of the G-equivariant Lego-Teichmüller Game. (1.52) shows that the g-twist in definition 1.21.2 of the equivariant braiding arises naturally.

The **ribbon twist** $\Theta_V : V \to {}^gV$ for an object $V \in \mathcal{C}^G_g$ is defined by

$$\langle V, T \rangle_G \overset{(\tilde{\varphi}_{\mathbf{B}}^{-1})_*}{\to} \langle T, {}^gV \rangle_G \overset{\tilde{\varphi}_{\mathbf{Z}}}{\to} \langle {}^gV, T \rangle_G \tag{1.53}$$

where in this case $\tilde{\varphi}_{\mathbf{B}}$ is the lift of the braiding diffeomorphism $\varphi_{\mathbf{B}} : S_2 \to S_2$ on the two-punctured sphere and $\varphi_{\mathbf{Z}} : S_2 \to S_2$ is rotation of S_2 around the equator which exchanges the two boundary components. Again the relations in the G-equivariant LTG (resembling relations in the mapping class group) ensure that the axioms for the G-equivariant ribbon twist are satisfied.

It is important to notice that the above procedure does in general give a non-strict monoidal category. This means that the associativity constraints are not necessarily identity morphisms.

In [KP08] no explicit prescription is given for how to obtain from a $\mathcal{C}^G$-extended G-modular functor τ^G the **tensoriality constraints** for the equivariance functors $R_g : \mathcal{C}^G \to \mathcal{C}^G$. We will need the explicit form of these structure morphisms and therefore explain them in some detail.

In subsection 1.1.2 we defined the standard blocks on the n-punctured sphere S_n as a quotient of $S_n\backslash\text{cuts} \times G$. By this definition, a point in the total space $S_n(g_1, \dots, g_n; h_1, \dots, h_n)$ is an equivalence class $[z, x]$ with $z \in S_n\backslash\text{cuts}$ and $x \in G$. For every group element $k \in G$ the map $[z, x] \mapsto [z, xk]$ induces an isomorphism of G-covers

$$\tilde{k} : \quad S_n(g_1, \dots, g_n; h_1, \dots, h_n) \to S_n(k^{-1}g_1k, \dots, k^{-1}g_nk; h_1k, \dots, h_nk) \tag{1.54}$$

The corresponding natural transformations enter in the construction of the tensoriality constraints. To construct these morphisms, let $h \in G$ and $A \in \mathcal{C}^G_{g_1}$ and $B \in \mathcal{C}^G_{g_2}$ be objects of $\mathcal{C}^G$. The main step is to construct a natural isomorphism between the functors $\mathcal{C}^G \to \mathcal{V}ect_{\Bbbk}$ given by

$$T \mapsto \tau^G(S_2(hg_2^{-1}g_1^{-1}h^{-1}, hg_1g_2h^{-1}; 1, 1); T, {}^h(A \otimes B)) \tag{1.55}$$

and

$$T \mapsto \tau^G(S_2(hg_2^{-1}g_1^{-1}h^{-1}, hg_1g_2h^{-1}; 1, 1); T, {}^hA \otimes {}^hB) \ . \tag{1.56}$$

The objects ${}^h(A \otimes B)$ and ${}^hA \otimes {}^hB$ are representing the functors (1.55) and (1.56) respectively. Thus, by the Yoneda lemma the image of the identity for $T^* = {}^h(A \otimes B)$ under the natural isomorphism that we will construct between the functors (1.55) and (1.56) gives an isomorphism

$$\varphi^h_{A,B} : \quad {}^h(A \otimes B) \to {}^hA \otimes {}^hB \ .$$

The natural transformation we use is given by

$$
\begin{aligned}
& \tau^G(S_2(hg_2^{-1}g_1^{-1}h^{-1}, hg_1g_2h^{-1};1,1);T,{}^h(A\otimes B)) \\
\stackrel{T_{(1,h^{-1})}}{\longrightarrow}\; & \tau^G(S_2(hg_2^{-1}g_1^{-1}h^{-1}, hg_1g_2h^{-1};1,h^{-1});T,A\otimes B) \\
\stackrel{(\tilde h)_*}{\longrightarrow}\; & \tau^G(S_2(g_2^{-1}g_1^{-1}, g_1g_2;h,1);T,A\otimes B) \\
\stackrel{G^{-1}}{\longrightarrow}\; & \tau^G(S_2(g_2^{-1}g_1^{-1}, g_1g_2;h,1);T,\mathcal{R}^{(1)}) \\
& \qquad\qquad \otimes_{\Bbbk}\tau^G(S_2(g_2^{-1}g_1^{-1}, g_1g_2;1,1);\mathcal{R}^{(2)},A\otimes B) \\
\stackrel{\text{def}}{=}\; & \tau^G(S_2(g_2^{-1}g_1^{-1}, g_1g_2;h,1);T,\mathcal{R}^{(1)}) \\
& \qquad\qquad \otimes_{\Bbbk}\tau^G(S_3(g_2^{-1}g_1^{-1}, g_1,g_2;1,1,1);\mathcal{R}^{(2)},A,B) \\
\stackrel{G}{\longrightarrow}\; & \tau^G(S_3(g_2^{-1}g_1^{-1}, g_1,g_2;h,1,1);T,A,B) \\
\stackrel{(\tilde h^{-1})_*}{\longrightarrow}\; & \tau^G(S_3(hg_2^{-1}g_1^{-1}h^{-1}, hg_1h^{-1},hg_2h^{-1};1,h^{-1},h^{-1});T,A,B) \\
\stackrel{T_{(1,h,h)}}{\longrightarrow}\; & \tau^G(S_3(hg_2^{-1}g_1^{-1}h^{-1}, hg_1h^{-1},hg_2h^{-1};1,1,1);T,{}^hA,{}^hB) \\
\stackrel{\text{def}}{=}\; & \tau^G(S_2(hg_2^{-1}g_1^{-1}h^{-1}, hg_1g_2h^{-1};1,1);T,{}^hA\otimes{}^hB)
\end{aligned}
\tag{1.57}
$$

for any T in $\mathcal{C}^G_{hg_2^{-1}g_1^{-1}h^{-1}}$. The idea in the definition is to use the equivariance isomorphisms defined in equation (1.44) to shift the G-action from objects in the category to geometric quantities. Then a factorization is applied to be able to use the definition of the tensor product and finally, the G-action is shifted back to objects.

The following observation follows from the compatibility of all occurring morphisms and the definition of the associativity constraints in $\mathcal{C}^G$:

Lemma 1.23.
The morphisms $\varphi^h_{A,B}$ endow the functor R_h with the structure of a monoidal functor.

Next, we adapt the discussion of [BK01, prop. 5.3.13] of conditions ensuring that a weakly ribbon category is a ribbon category to the G-equivariant case. Suppose $\mathcal{C}^G$ is a weakly rigid G-equivariant category in which biduals of objects can be functorially identified with the objects, $(V^*)^* \cong V$ for all V in $\mathcal{C}^G$. This condition is fulfilled for all categories with tensor structure obtained from a modular functor. The image of the identity on V^* under the functorial isomorphisms of definition 1.21.3 provides us with a morphism $i_V : \mathbf{1} \to V \otimes V^*$ for any object $V \in \mathcal{C}^G$. Consider for a simple object V of $\mathcal{C}^G$ the morphism

$$\alpha^{-1}_{V^*,V,V^*} \circ (\mathrm{id}_{V^*} \otimes i_V) : V^* \to (V^* \otimes V) \otimes V^* \tag{1.58}$$

constructed from i_V and the associativity constraint α of $\mathcal{C}^G$. Since $\mathcal{C}^G$ is semisimple, we can decompose $V^* \otimes V \cong \bigoplus_j V_j$ into a direct sum of simple objects V_j. Multiplicities can occur, but since $\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}^G}(\mathbf{1}, V^* \otimes V) = 1$ we can decompose the morphism

$$\alpha^{-1}_{V^*,V,V^*} \circ (\mathrm{id}_{V^*} \otimes i_V) = a_V \otimes \mathrm{id}_{V^*} + \sum \psi_j \tag{1.59}$$

into a morphism $a_V : \mathbf{1} \to V^* \otimes V$ in the one-dimensional vector space $\mathrm{Hom}(\mathbf{1}, V^* \otimes V)$ and certain morphisms ψ_j. By the same arguments as in [BK01], we have

Proposition 1.24. *The category $\mathcal{C}^G$ is rigid, if and only if $a_V \neq 0$ with a_V as defined by (1.59) for all simple objects V.*

2. THE COVER FUNCTOR

This chapter contains our main construction. Its aim is to give a geometric description of a specific class of G-equivariant (weakly) ribbon categories, i.e. to define a G-equivariant modular functor that satisfies our purposes. This chapter is based on the article [BS10].

To this end we fix a finite group G and a finite G-set $\mathcal{X}$. In section 2.1 we will introduce the cover functor as a functor

$$\mathcal{F}_{\mathcal{X}} : \mathbf{Gcob(d)} \to \mathbf{cob(d)}\,. \tag{2.1}$$

This functor assigns to every principal G-bundle $(P{\to}M)$ the total space of the associated cover $(\mathcal{X} \times_G P{\to}M)$.

This assignment encodes the combinatorical information about the action of G on $\mathcal{X}$ in geometrical data. For example, in the case of two-dimensional theories, which is the case of our main interest, the set of boundary components of $\mathcal{X} \times_G P$ can be expressed in terms of the orbit structure of $\mathcal{X}$.

First we will introduce the cover functor and show that it allows us to pull back d-dimensional topological field theories to G-equivariant theories. In the case of $d = 2$ we will then investigate the surfaces that are necessary to derive the G-Frobenius algebra from the G-equivariant TFT (as in theorem 1.7).

Given a commutative Frobenius algebra $(R, \eta, m, \varepsilon, \Delta)$, we will pull back the associated TFT along the cover functor and express the G-Frobenius algebra that corresponds to this G-TFT in terms of the G-set $\mathcal{X}$ and the structure of R.

Once we know this G-Frobenius algebra in dependence of R, we will categorify this structure and define a G-equivariant abelian category $\mathcal{C}^{\mathcal{X}}$, by replacing R with a modular category $\mathcal{C}$. It turns out that this is the right category to define a $\mathcal{C}^{\mathcal{X}}$-extended G-equivariant modular functor $\tau^{\mathcal{X}}$ out of the cover functor, as we will do in section 2.2.

2.1 Cover functors and two-dimensional topological field theories

2.1.1 Definition of the cover functor

One ingredient in our construction of permutation equivariant G-TFTs is a finite left G-set $\mathcal{X}$. In the construction we will need to make choices for all partitions of $\mathcal{X}$. To this end, we fix an order on $\mathcal{X}$ as an auxiliary datum.

For any object $(P \to \Sigma, e)$ of $\mathbf{Gcob(d)}$, we consider the smooth $(d-1)$-manifold $\mathcal{X} \times_G P$, where $\mathcal{X} \times_G P := (\mathcal{X} \times P)/((g^{-1}x, p) \sim (x, gp))$. Similarly, we obtain smooth d-manifolds for morphisms of $\mathbf{Gcob(d)}$. For any G-bundle $P \to M$, the manifold $\mathcal{X} \times_G P$ is the total space of an $|\mathcal{X}|$-fold cover of M, we call this cover the associated G-cover. We agree to write $[x, p] \in \mathcal{X} \times_G M$ for the equivalence class of $(x, p) \in \mathcal{X} \times M$ for any G-manifold M.

Proposition 2.1. *Let $\mathcal{X}$ be a finite ordered left G-set. The assignment $(P \to \Sigma, e) \mapsto \mathcal{X} \times_G P$ defines a symmetric monoidal functor*

$$\mathcal{F}_{\mathcal{X}} : \ \mathbf{Gcob(d)} \to \mathbf{cob(d)} \tag{2.2}$$

Proof. The proof is straightforward, including its most intricate aspect, the fact that gluing of cobordisms is respected. □

Definition 2.2. The functor $\mathcal{F}_{\mathcal{X}}$ in proposition 2.1 is called the *d-dimensional cover functor* for the G-set $\mathcal{X}$.

Corollary 2.3. *Let* $\mathrm{tft} : \mathbf{cob(d)} \to \mathcal{V}ect_{\Bbbk}$ *be a topological field theory and let $\mathcal{X}$ be a G-set. Then the composite functor*

$$\mathrm{tft}^{\mathcal{X}} : \ \mathbf{Gcob(d)} \xrightarrow{\mathcal{F}_{\mathcal{X}}} \mathbf{cob(d)} \xrightarrow{\mathrm{tft}} \mathcal{V}ect_{\Bbbk} \tag{2.3}$$

is a d-dimensional G-equivariant topological field theory in the sense of definition 1.2.

2.1.2 The two-dimensional cover functor and permutation equivariant Frobenius algebras

From now on, we specialize to dimension $d = 2$. Recall definition 1.3 of an extended surface. The cover functor $\mathcal{F}_{\mathcal{X}}$ induces a functor $\mathbf{GESurf} \to \mathbf{ESurf}$ which we will also denote by $\mathcal{F}_{\mathcal{X}}$. To see this, let $(P \to M, \{\tilde{p}_a\}_{a \in A(M)})$ be an object in **GESurf**, i.e. a G-cover of an extended surface. By definition, P is endowed with a marked point $\tilde{p}_a$ for each connected component of the boundary of M. This yields on the boundary of $\mathcal{F}_{\mathcal{X}}(P{\to}M)$ the marked points $[x, \tilde{p}_a]$ for every element $x \in \mathcal{X}$ of the G-set and every boundary component a of M. To turn $\mathcal{F}_{\mathcal{X}}(P{\to}M)$ into an extended surface, we need to choose just a single point for every connected component of the boundary of $\mathcal{F}_{\mathcal{X}}(P{\to}M)$. Here we use the auxiliary structure of an ordering on the G-set $\mathcal{X}$ to choose the point $[x, \tilde{p}_a]$ with the smallest value of $x \in \mathcal{X}$ in that boundary component.

To prepare the discussion of permutation equivariant Frobenius algebras, we will now analyze covers of the following basic manifolds:

1. The vector spaces relevant for a G-equivariant topological field theory are given by evaluations of the TFT functor on G-covers of the circle S^1.

2. The multiplicative structure on the vector spaces underlying a topological field theory comes from the 3-punctured sphere, the so-called pair-of-pants. To set the stage for the discussion in sections 3.1 and 3.2, we consider covers of the n-punctured sphere.

Given a G-cover $(P{\to}M)$, we only consider the total space $\mathcal{X} \times_G P$ of the associated cover in our construction. But in the analysis of the geometry of the manifold $\mathcal{X} \times_G P$ it will sometimes be convenient to view $\mathcal{X} \times_G P$ as a $|\mathcal{X}|$-fold cover over M. We will adopt this point of view whenever useful.

Covers of the circle

Given a finite G-set $\mathcal{X}$, we define for every $g \in G$ a $|\mathcal{X}|$-fold cover of S^1 with total space $E_g := \mathbb{R} \times \mathcal{X}/(t + 2\pi, x) \sim (t, g^{-1}x)$. The following lemma is straightforward:

Lemma 2.4. *For any $g \in G$, the covers E_g and $\mathcal{X} \times_G P_g$ over S^1 with P_g as in (1.2) are isomorphic.*

As a closed one-dimensional manifold, the total space E_g is a disjoint union of circles. We describe the connected components of E_g:

Lemma 2.5. *For any element $g \in G$ there is a one-to-one correspondence between the connected components of the manifold E_g and the orbits of $\mathcal{X}$ under the action of the cyclic group $\langle g \rangle$.*

We denote the set of orbits of $\mathcal{X}$ under the action of the cyclic group $\langle g \rangle \subset G$ by O_g; let $b_g := |O_g|$ be the number of orbits.

The following lemma follows from an easy calculation:

Lemma 2.6. *The map*

$$\begin{array}{ll} E_g & \to E_{hgh^{-1}} \\ [t, x] & \mapsto [t, hx] \end{array} \tag{2.4}$$

is an isomorphism of covers of S^1. The induced map on the sets of connected components is given by the map

$$\begin{array}{ll} O_g & \to O_{hgh^{-1}} \\ o & \mapsto ho. \end{array} \tag{2.5}$$

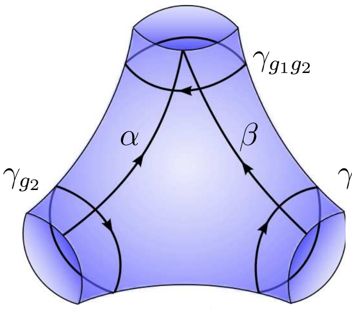
between the sets of orbits of cyclic groups.

Covers of the n-punctured sphere

We will now turn to the morphisms in **Gcob(2)** and exhibit certain covers of the n-punctured sphere that will be important in the following discussion. We begin by analyzing covers of the three-punctured sphere S_3. Sometimes we abbreviate $E_{g_1;g_2} := \mathcal{F}_{\mathcal{X}}(S_3(g_1, g_2, (g_1g_2)^{-1}; 1, 1, 1) \to S_3)$.

To analyze how the structure of $\mathcal{F}_{\mathcal{X}}(S_3(g_1, g_2, (g_1g_2)^{-1}; 1, 1, 1) \to S_3)$ depends on the group elements g_1 and g_2, we introduce the following paths in the base manifold, the pair-of-pants S_3:

- γ_{g_1}, the path with winding number one around the ingoing boundary circle which has monodromy g_1^{-1}.
- γ_{g_2} the path winding once around the ingoing boundary circle which has monodromy g_2^{-1}.
- $\gamma_{g_1g_2}$ the path winding once around the outgoing boundary circle which has monodromy $(g_1g_2)^{-1}$.
- α, β open paths connecting the base points of the ingoing circles with the base point of the outgoing circle.



(2.6)

The following lemma describes the connected components of $E_{g_1;g_2}$:

Lemma 2.7.

(i) There is a natural bijection between the connected components of

$$\mathcal{F}_{\mathcal{X}}(S_3(g_1, g_2, (g_1g_2)^{-1}; 1, 1, 1) \to S_3) = E_{g_1;g_2}$$

and orbits of the G-set $\mathcal{X}$ under the action of the subgroup $\langle g_1, g_2\rangle \subset G$ of G generated by the elements g_1 and g_2.

(ii) By lemma 2.4 the restriction of $E_{g_1;g_2}$ to the boundary with monodromy g_1^{-1} is diffeomorphic to $E_{g_1^{-1}}$ and similarly for the other boundaries. Let o be a $\langle g_1, g_2\rangle$-orbit of $\mathcal{X}$ and write $E^o_{g_1;g_2}$ for the connected component of $E_{g_1;g_2}$ corresponding to the orbit o. The boundary components of $E^o_{g_1;g_2}$ correspond to precisely those orbits of the cyclic subgroups $\langle g_1\rangle$, $\langle g_2\rangle$ and $\langle g_1g_2\rangle$ that are contained in the orbit o of the group $\langle g_1, g_2\rangle$.

(iii) In particular, the number of sheets of the cover $E^o_{g_1;g_2} \to S_3$ is $|o|$.

This is seen by choosing appropriate lifts of the paths $\gamma_{g_1}, \gamma_{g_2}, \gamma_{g_1g_2}, \alpha$ and β. We leave the details to the reader as an exercise. We write $b^o_{g_1}$ for the number of $\langle g_1\rangle$-orbits that are contained in o and similarly for g_2 and g_1g_2. We can now describe the topology of the connected components of the cover:

Lemma 2.8. *Let o be a $\langle g_1, g_2\rangle$-orbit of $\mathcal{X}$. Then the component $E^o_{g_1,g_2}$ of*

$$\mathcal{F}_{\mathcal{X}}(S_3(g_1, g_2, (g_1g_2)^{-1}; 1, 1, 1) \to S_3)$$

is a surface of genus

$$\frac{2 - b^o_{g_1} - b^o_{g_2} - b^o_{g_1g_2} + |o|}{2} \tag{2.7}$$

Proof. The cover $(E^o_{g_1,g_2}\to S_3)$ is an unramified $|o|$-fold cover of the pair-of-pants S_3 with Euler characteristic $\chi(S_3) = 2 - 3 = -1$. Hence, by the theorem of Riemann-Hurwitz, $\chi(E^o_{g_1,g_2}) = -|o|$.

As described in lemma 2.7, the number of boundary components of $E^o_{g_1;g_2}$ is $b^o_{g_1} + b^o_{g_2} + b^o_{g_1g_2}$. This implies formula (2.7) for the genus of $E^o_{g_1,g_2}$. □

Finally, we consider the special case of the cylinder S_2 with principal bundle $(S_2(g,g^{-1};1,h)\to S_2)$, where the first boundary is oriented as ingoing and the second boundary as outgoing. Then $S_2(g,g^{-1};1,h)$ has monodromies g^{-1} around the ingoing boundary and $hg^{-1}h^{-1}$ around the outgoing boundary. We identify the ingoing boundary of $(S_2(g,g^{-1};1,h)\to S_2)$ with $(P_{g^{-1}}\to S^1,[0,1_G])$ and the outgoing boundary with $(P_{g^{-1}}\to S^1,[0,h])$. The latter is isomorphic to $(P_{hg^{-1}h^{-1}}\to S^1,[0,1_G])$ under the map $[t,k]\mapsto[t,kh^{-1}]$ of bundles. Hence the boundaries of $\mathcal{F}_{\mathcal{X}}(S_2(g,g^{-1};1,h)\to S_2)$ are isomorphic to $E_{g^{-1}}$ and $E_{hg^{-1}h^{-1}}$ respectively.

Lemma 2.9. *The manifold $\mathcal{F}_{\mathcal{X}}(S_2(g,g^{-1};1,h)\to S_2)$ is a disjoint union of cylinders. These cylinders interpolate between the connected component of $E_{g^{-1}}$ corresponding to the $\langle g\rangle$-orbit o of $\mathcal{X}$ and the connected component of $E_{hg^{-1}h^{-1}}$ that corresponds to the $\langle hg^{-1}h^{-1}\rangle = \langle hgh^{-1}\rangle$-orbit ho.*

Proof. Consider the following paths in the base cylinder S_2:

- γ_g the path winding once around the ingoing boundary circle which has monodromy g^{-1}.
- $\gamma_{hg^{-1}h^{-1}}$ the path winding once around the outgoing boundary circle which has monodromy $hg^{-1}h^{-1}$.
- α a path connecting the base points of both boundary circles.

Let $x\in o$ be an element of the $\langle g\rangle$-orbit o. The point $[0,x]$ in $E_{g^{-1}}$ is connected to the points $[0,g^ix]$ of $E_{g^{-1}}$ by repeated lifts of γ_g, similar to the proof of lemma 2.7. By lifting α to a path $\hat\alpha_{g^ix}$ in $\mathcal{F}_{\mathcal{X}}(S_2(g,g^{-1};1,h)\to S_2)$ with initial point $\hat\alpha_{g^ix}(0) = [0,g^ix]\in E_{g^{-1}}$, these points are connected to the points $[0,hg^ix]\in E_{hg^{-1}h^{-1}}$, where the map from lemma 2.6 is used to identify the outgoing boundary of $\mathcal{F}_{\mathcal{X}}(S_2(g,g^{-1};1,h)\to S_2)$ with $E_{hg^{-1}h^{-1}}$. These again are connected by lifts of $\gamma_{hg^{-1}h^{-1}}$ only connected to points of the same form. By lemma 2.5, the connected component of $E_{hg^{-1}h^{-1}}$ containing these points, corresponds to the $\langle hg^{-1}h^{-1}\rangle$-orbit ho. □

Permutation equivariant Frobenius algebras

We now describe a construction that can be seen as the decategorified version of the main construction of this thesis. Suppose we are given a finite ordered G-set $\mathcal{X}$ and a commutative Frobenius algebra $(R, m, \eta, \Delta, \varepsilon)$, playing the role of a decategorification of a modular tensor category. A Frobenius algebra structure on an associative unital algebra (R, m, η) can be equivalently described ([FS08]) by a linear form ε such that the induced bilinear pairing on R is non-degenerate or by a coalgebra structure (R, Δ, ε) such that Δ is a morphism of (R, m, η)-bimodules. Here, we prefer the latter description. We want to to construct a G-Frobenius algebra A with neutral component $A_1 = \bigotimes_{\mathcal{X}} R$, where the G-action on A_1 is induced by the G-action on $\mathcal{X}$.

We start by constructing the underlying G-graded vector space:

- Composing the 2-dimensional topological field theory associated to R

$$\mathrm{tft}_R : \mathbf{cob(2)} \to \mathcal{V}ect_{\Bbbk} \tag{2.8}$$

  with the tensor functor $\mathcal{F}_{\mathcal{X}}$, we obtain by corollary 2.3 a 2-dimensional G-equivariant TFT,

$$\mathrm{tft}_R^{\mathcal{X}} := \mathrm{tft}_R \circ \mathcal{F}_{\mathcal{X}} : \mathbf{Gcob(2)} \to \mathcal{V}ect_{\Bbbk} \,.$$

- To describe the G-Frobenius algebra $A = \bigoplus_{g\in G} A_g$ that corresponds to this G-TFT, we first describe the vector spaces A_g for $g \in G$. By lemma 2.4, we have $\mathcal{F}_{\mathcal{X}}(P_{g^{-1}} \to S^1, [0, 1_G]) \cong E_{g^{-1}}$; since the functor tft_R is monoidal, we only need to know the number of connected components of $E_{g^{-1}}$ which by lemma 2.5 is the number b_g of orbits of the cyclic group $\langle g^{-1}\rangle = \langle g\rangle$ on $\mathcal{X}$. Thus, as a vector space,

$$A_g \cong \mathrm{tft}_R(E_{g^{-1}}) \cong \mathrm{tft}_R(\sqcup_{o\in O_g} S^1) \cong R^{\otimes O_g} \cong \bigotimes_{o\in O_g} R_o \,, \tag{2.9}$$

  with $R_o \cong R$ as a vector space. Hence an element of A_g is a linear combination of elements of the form $r_{o_1} \otimes \cdots \otimes r_{o_{b_g}}$ with $r_{o_i} \in R$.

The product morphisms $m_{g_1,g_2} : A_{g_1} \otimes A_{g_2} \to A_{g_1 g_2}$ are induced by the covers

$$(S_3(g_1, g_2, (g_1 g_2)^{-1}; 1, 1, 1) \to S_3)$$

of the three-punctured sphere. On these covers the cover functor $\mathcal{F}_\mathcal{X}$ gives the surfaces $E_{g_1;g_2}$.

By lemma 2.7 each $\langle g_1, g_2\rangle$-orbit o on $\mathcal{X}$ gives a connected component $E^o_{g_1;g_2}$ of $E_{g_1;g_2}$ and thus a contribution to the product morphism. We describe these contributions separately; we write generators in A_{g_1} as products of elements $r_{o'_i} \in R$ and generators in A_{g_2} as tensor products of elements $s_{o''_i} \in R$.

- First multiply all elements $r_{o'}$ and $s_{o''}$ for all $\langle g_1\rangle$-orbits o' and the $\langle g_2\rangle$-orbits o'' that are contained in the $\langle g_1, g_2\rangle$-orbit o. No choices are involved, because the Frobenius algebra R is commutative.

- By lemma 2.8, the genus of $E^o_{g_1;g_2}$ equals $p := \frac{2-b^o_{g_1}-b^o_{g_2}-b^o_{g_1g_2}+|o|}{2}$. In a second step, apply the endomorphism $(m \circ \Delta)^p$ of R to the product of the previous step.

- Let $o_1, \dots, o_k$ be those orbits of the cyclic group $\langle g_1g_2\rangle$ that are contained in the $\langle g_1, g_2\rangle$-orbit o. To the element of R obtained in the previous step, apply the k-fold coproduct of R, so we get an element in $R_{o_1} \otimes \cdots \otimes R_{o_k}$. This element is well defined by coassociativity of R.

- Map the factors R_{o_i} of the previous step to the corresponding factors of $A_{g_1g_2}$. No choices are involved, since the coproduct on the Frobenius algebra R is cocommutative.

This provides the prescription for the product on the G-Frobenius algebra A. As explained in theorem 1.7, the unit of A is obtained as the evaluation of the functor $\mathrm{tft}^\mathcal{X}_R$ on the disc with the trivial G-cover. The cover functor maps $(D \times G \to D)$ to the disjoint union $\bigsqcup_{x\in\mathcal{X}} D$ of $|\mathcal{X}|$-many copies of discs, hence the unit of the G-Frobenius algebra A is just the tensor product of the units $\bigotimes_{x\in\mathcal{X}} \eta : \Bbbk \to A_1 \cong \bigotimes_{x\in\mathcal{X}} R$ of R. Similarly we find that the counit of A is given by tensor product of the counits of R.

From lemma 2.9 we deduce that the G-action $\alpha_h : A_g \to A_{hgh^{-1}}$ is given by the permutation of factors: The connected components of the cover $\mathcal{F}_\mathcal{X}(S_2(g, g^{-1}; 1, h) {\to} S_2)$ of the cylinder are again cylinders. For any $\langle g\rangle$-orbit o of $\mathcal{X}$ the factor R_o of A_g is thus mapped to the factor R_{ho} of $A_{hgh^{-1}}$.

2.2 G-equivariant modular functors from the cover functor

In this section we will use the two-dimensional cover functor to construct a G-equivariant modular functor out of a given non-equivariant modular functor. To achieve this we categorify the results of section 2.1.2.

Let $\mathcal{C}$ be a modular category that is enriched over the category $\mathcal{V}ect_{\Bbbk}$ of finite dimensional vector spaces over the field $\Bbbk$; we assume $\mathcal{C}$ to be strict. Let $\mathcal{X}$ be a finite ordered G-set. For any pair $(\mathcal{C}, \mathcal{X})$ we will construct a G-equivariant modular functor $\tau^{\mathcal{X}}$.

We wish to use the 2-dimensional cover functor

$$\mathcal{F}_X : \mathbf{GESurf} \to \mathbf{ESurf} \tag{2.10}$$

on extended surfaces to associate to every modular tensor category $\mathcal{C}$ a G-equivariant modular functor.

We choose representatives $(U_i)_{i\in\mathcal{I}}$ for the isomorphism classes of simple objects of $\mathcal{C}$ and denote by $\theta_i \in \Bbbk$ the eigenvalue of the twist on U_i and by d_i the dimension of U_i. As usual, we introduce the scalars $p^{\pm} := \sum_{i\in\mathcal{I}} \theta_i^{\pm} d_i^2$ and assume that $p^+ = p^-$. Finally, we introduce

$$\mathcal{D} = \sqrt{p^+p^-} = \sqrt{\sum_i (d_i)^2}\ . \tag{2.11}$$

With this assumption, the modular category $\mathcal{C}$ defines (see e.g. [Tur94, BK01]) a $\mathcal{C}$-extended modular functor $\tau \equiv \tau_{\mathcal{C}}$ which is defined for standard spheres as

$$\tau(S_n; V_1, \dots, V_n) := \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_1 \otimes \cdots \otimes V_n)\ . \tag{2.12}$$

To define a G-modular functor, we need a G-equivariant category (definition 1.16) as input. Given $(\mathcal{C}, \mathcal{X})$, we define a G-equivariant category by transferring the results of subsection 2.1.1 to categorical structures. We obtain

Definition 2.10. Let G be a finite group, $\mathcal{X}$ a finite G-set and $\mathcal{C}$ a modular tensor category. Then the following data define a G-equivariant category $\mathcal{C}^{\mathcal{X}}$:

1. For $g \in G$ set
$$\mathcal{C}_g^{\mathcal{X}} := \mathcal{C}^{\boxtimes O_g} \equiv \mathcal{C}_{o_1} \boxtimes \cdots \boxtimes \mathcal{C}_{o_{b_g}}\ ,$$
where O_g is the set of $\langle g\rangle$-orbits o_i of $\mathcal{X}$ and $b_g = |O_g|$. Moreover, $\mathcal{C}_{o_i} \cong \mathcal{C}$ as abelian categories.

2. For $g, h \in G$ define a functor $R_h : \mathcal{C}_g^{\mathcal{X}} \to \mathcal{C}_{hgh^{-1}}^{\mathcal{X}}$ by permutation of factors: the factor $\mathcal{C}_o$ of $\mathcal{C}_g^{\mathcal{X}}$ is mapped to the factor $\mathcal{C}_{ho}$ of $\mathcal{C}_{hgh^{-1}}^{\mathcal{X}}$.

   The equivariance functors obey the strict identities $R_h \circ R_k = R_{hk}$ for all $h, k \in G$ which allows us to choose trivial composition constraints, $\alpha_{h,k} = \mathrm{id}$ for all $h, k \in G$.

The G-equivariant category $\mathcal{C}^{\mathcal{X}}$ is semisimple and has only finitely many isomorphism classes of simple objects. Representatives of the isomorphism classes of simple objects in $\mathcal{C}_g^{\mathcal{X}}$ are $U_{i_1} \times \cdots \times U_{i_{b_g}}$. We next define the gluing object (cf. definition 1.17)

$$\mathcal{R} \in \bigoplus_{g \in G} \mathcal{C}_g^{\mathcal{X}} \boxtimes \mathcal{C}_{g^{-1}}^{\mathcal{X}} \; ; \tag{2.13}$$

its component $\mathcal{R}_g$ in $\mathcal{C}_g^{\mathcal{X}} \boxtimes \mathcal{C}_{g^{-1}}^{\mathcal{X}}$ is

$$\mathcal{R}_g := \bigoplus_{i_1, i_2 \ldots i_{b_g}} (U_{i_{o_1}} \times \cdots \times U_{i_{o_{b_g}}}) \times (U_{i_{o_1}}^{\vee} \times \cdots \times U_{i_{o_{b_g}}}^{\vee}) \tag{2.14}$$

where the direct sum is taken over all isomorphism classes of simple objects of $\mathcal{C}$. Thus the direct sum in (2.14) is taken over representatives of all simple objects of $\mathcal{C}^{\boxtimes O_g}$. With this definition, $\mathcal{R}$ is clearly symmetric and G-invariant.

Now we are able to define a $\mathcal{C}^{\mathcal{X}}$-extended G-equivariant modular functor $\tau^{\mathcal{X}}$. Let $(P \to M, \{\tilde{p}_a\}_{a \in A(M)})$ be a G-cover of an extended surface M. Consider the surface

$$\mathcal{F}_{\mathcal{X}}(P \to M, \{\tilde{p}_a\}_{a \in A(M)}) = \mathcal{X} \times_G P \; .$$

It can be viewed as the total space of an $|\mathcal{X}|$-fold cover of M and has, by the discussion in subsection 2.1.2 the structure of an extended surface. Let $a \in A(M)$ be a boundary component of M. By lemmas 2.4 and 2.5 the restriction of $\mathcal{X} \times_G P$ to a has a connected component for every $\langle m_a \rangle$-orbit of $\mathcal{X}$, where m_a is the monodromy of P around a.

Definition 2.11. Let $(P \to M, \{\tilde{p}_a\}_{a \in A(M)})$ be a G-cover of an extended surface M. Define a functor

$$\tau^{\mathcal{X}}(P \to M, \{\tilde{p}_a\}_{a \in A(M)}) : \boxtimes_{a \in A(M)} \mathcal{C}_{m_a^{-1}}^{\mathcal{X}} \to \mathcal{V}ect_{\Bbbk} \tag{2.15}$$

by

$$\tau^{\mathcal{X}}(P \to M, \{\tilde{p}_a\}_{a \in A(M)}) := \tau(\mathcal{F}_{\mathcal{X}}(P \to M, \{\tilde{p}_a\}_{a \in A(M)})) = \tau(\mathcal{X} \times_G P) \, , \tag{2.16}$$

where τ is the modular functor (2.12) corresponding to $\mathcal{C}$.

This is indeed well-defined: from $\mathcal{C}^{\mathcal{X}}_{m_a^{-1}} = \boxtimes_{O_{m_a}} \mathcal{C}$ and lemmas 2.4 and 2.5 we conclude that

$$\boxtimes_{a\in A(M)} \mathcal{C}^{\mathcal{X}}_{m_a^{-1}} = \boxtimes_{a\in A(\mathcal{X}\times_G P)} \mathcal{C} \ .$$

Note that the definition of $\tau^{\mathcal{X}}$ also depends on the choice of marked points in $\mathcal{X} \times_G P$.

We will now describe all the additional data that is needed to turn $\tau^{\mathcal{X}}$ into a $\mathcal{C}^{\mathcal{X}}$-extended G-equivariant modular functor and prove that all axioms are satisfied.

- Let $f : (P{\to}M, \{\tilde{p}_a\}_{a\in A(M)}) \xrightarrow{\cong} (P'{\to}M', \{\tilde{p}'_a\}_{a\in A(M')})$ be an isomorphism in **GESurf**. This gives a morphism $\bar{f} := \mathcal{F}_{\mathcal{X}}(f) : \mathcal{X} \times_G P \xrightarrow{\cong} \mathcal{X} \times_G P'$ in **ESurf**. By definition, the $\mathcal{C}$-extended modular functor τ gives an isomorphism $\bar{f}_* : \tau(\mathcal{X} \times_G P) \Rightarrow \tau(\mathcal{X} \times_G P')$, hence an isomorphism $\bar{f}_* : \tau^{\mathcal{X}}(P{\to}M, \{\tilde{p}_a\}_{a\in A(M)}) \Rightarrow \tau^{\mathcal{X}}(P'{\to}M', \{\tilde{p}'_a\}_{a\in A(M')})$. This isomorphism only depends on the isotopy class of f and it obeys $\overline{(f \circ g)}_* = \bar{f}_* \circ \bar{g}_*$.

- The $\mathcal{C}$-extended modular functor has enough structure to provide an isomorphism of functors

$$\tau^{\mathcal{X}}(\emptyset) \stackrel{\text{def}}{=} \tau(\mathcal{X} \times_G \emptyset) = \tau(\emptyset) \cong \Bbbk \tag{2.17}$$

  and for G-covers $(P{\to}M, \{\tilde{p}_a\}_{a\in A(M)})$ and $(P'{\to}M', \{\tilde{p}'_a\}_{a\in A(M')})$

$$\begin{aligned} &\tau^{\mathcal{X}}(P \sqcup P') \stackrel{\text{def}}{=} \tau(\mathcal{X} \times_G (P \sqcup P')) \cong \tau((\mathcal{X} \times_G P) \sqcup (\mathcal{X} \times_G P')) \\ &\cong \tau(\mathcal{X} \times_G P) \otimes_{\Bbbk} \tau(\mathcal{X} \times_G P')) \stackrel{\text{def}}{=} \tau^{\mathcal{X}}(P) \otimes_{\Bbbk} \tau^{\mathcal{X}}(P'). \end{aligned} \tag{2.18}$$

- Next we describe the gluing isomorphisms. Let $(P{\to}M, \{\tilde{p}_a\}_{a\in A(M)})$ be a G-cover of M and let $\alpha, \beta \in A(M)$ be two different boundary components of M with inverse monodromies, $m_\alpha = m_\beta^{-1}$. This condition ensures that we can glue P along α and β. For all $a \in A(M) \setminus \{\alpha, \beta\}$, let W_a be an object of $\mathcal{C}^{\mathcal{X}}_{m_a^{-1}}$. Then by definition 2.14 of the gluing object $\mathcal{R}$

$$\tau^{\mathcal{X}}(P, \{W_a\}, \mathcal{R}_{\alpha,\beta}) \cong \bigoplus \tau(\mathcal{X}{\times_G}P, \{W_a\}, U_{i_{o_1}}{\times}{\cdots}{\times}U_{i_{o_k}}, U^{\vee}_{i_{o_1}}{\times}{\cdots}{\times}U^{\vee}_{i_{o_k}}). \tag{2.19}$$

Here the o_i are the $\langle m_\alpha^{-1}\rangle$-orbits of $\mathcal{X}$. Since the monodromies are inverses, $m_\beta = m_\alpha^{-1}$, these orbits are equal to the $\langle m_\beta\rangle$-orbits so that the assignment of the simple objects in the first component of $\mathcal{R}_{m_\alpha^{-1}}$ to the boundary components of $\mathcal{X} \times_G P$ over α and the simple objects in the second component of $\mathcal{R}_{m_\alpha^{-1}}$ to the boundary components over β is compatible. The direct sum is taken over all simple objects as in definition 2.14, where $\mathcal{R}$ has been introduced.

Since τ is a modular functor, we get gluing isomorphisms for all boundary components of the cover $\mathcal{X} \times_G P$ over α and β by gluing $\mathcal{X} \times_G P$ along each boundary component over α and β separately. The gluing isomorphisms of τ satisfy an associativity condition; thus the procedure does not depend on the order in which the boundary components are glued. Hence we get gluing isomorphisms

$$G_{\alpha,\beta} : \tau^{\mathcal{X}}(P, \{W_a\}, \mathcal{R}_{\alpha,\beta}) \xrightarrow{\cong} \tau^{\mathcal{X}}(\sqcup_{\alpha,\beta} P, \{W_a\}) \tag{2.20}$$

that are functorial in the objects W_a. The procedure in the definition of $G_{\alpha,\beta}$ relies on the fact that the cover functor $\mathcal{F}_{\mathcal{X}}$ respects gluing of covers.

- Next we implement G-equivariance. Let $(P \to M, \{\tilde{p}_a\}_{a\in A(M)})$ be a G-cover of an extended surface M with marked points $\{\tilde{p}_a\}_{a\in A(M)}$ and let $\mathbf{g} = (g_a)_{a\in A(M)} \in G^{A(M)}$ be a tuple of elements of G for each boundary component of M. For all $a \in A(M)$ let W_a be an object in $\mathcal{C}^G_{m_a^{-1}}$. We need to give functorial isomorphisms

  $$T_{\mathbf{g}} : \tau^{\mathcal{X}}(P \to M, \{\tilde{p}_a\}, \{W_a\}) \xrightarrow{\cong} \tau^{\mathcal{X}}(P \to M, \{g_a\tilde{p}_a\}, \{{}^{g_a}W_a\}). \tag{2.21}$$

  implementing the action of $\mathbf{g}$ on the boundary components.

  The boundary component of $(P \to M, \{\tilde{p}_a\})$ corresponding to the a-th boundary component of M is isomorphic to the cover $(P_{m_a} \to S^1, [0, 1_G])$. Now we analyze the situation on the right hand side of (2.21). The boundary of the cover $(P \to M, \{g_a\tilde{p}_a\})$ is isomorphic to to the cover $(P_{m_a} \to S^1, [0, g_a]) \cong (P_{g_a m_a g_a^{-1}} \to S^1, [0, 1_G])$. This induces precisely the map $E_{m_a} \to E_{g_a m_a g_a^{-1}}$ in lemma 2.6. After identifying boundary components and orbits, a $\langle m_a\rangle$-orbit o is mapped under this map to the $\langle g_a m_a g_a^{-1}\rangle$-orbit $g_a o$. On the other hand, the action of R_{g_a} on $\mathcal{C}^{\mathcal{X}}_{m_a^{-1}}$ permutes the objects in $\mathcal{C}^{\mathcal{X}}_{m_a^{-1}}$ in exactly the same way. Hence we can choose the equivariance isomorphisms $T_{\mathbf{g}}$ to be identity morphisms.

- It follows from functoriality of the cover functor $\mathcal{F}_\mathcal{X}$ and of τ that all isomorphisms constructed above are functorial in $(P{\to}M, \{\tilde{p}_a\}_{a\in A(M)})$. Similarly, one concludes that all isomorphisms are compatible.

- The G-equivariant modular functor $\tau^\mathcal{X}$ is normalized:

$$\tau^\mathcal{X}(S^2 \times G{\to}S^2) \stackrel{\text{def}}{=} \tau(\mathcal{X} \times_G (S^2 \times G)) \cong \tau(\mathcal{X} \times S^2) \cong \Bbbk \otimes \cdots \otimes \Bbbk \cong \Bbbk \tag{2.22}$$

We summarize these findings in the following theorem, which is one of the main results of this thesis:

Theorem 2.12. *Let G be a finite group, $\mathcal{X}$ a finite ordered G-set and $\mathcal{C}$ a $\Bbbk$-linear modular category. Denote by $\mathcal{C}^\mathcal{X}$ the G-equivariant category determined by the orbit-structure of $\mathcal{X}$ as introduced in definition 2.10. Then the functor $\tau^\mathcal{X}$ defined by*

$$\tau^\mathcal{X}(P{\to}M, \{\tilde{p}_a\}_{a\in A(M)}) := \tau(\mathcal{F}_\mathcal{X}(P{\to}M, \{\tilde{p}_a\}_{a\in A(M)})) \tag{2.23}$$

is a $\mathcal{C}^\mathcal{X}$-extended G-equivariant modular functor.

From theorem 1.22 we immediately obtain the following

Corollary 2.13.

1. *The G-equivariant modular functor $\tau^\mathcal{X}$ induces a G-equivariant monoidal structure on the G-equivariant category $\mathcal{C}^\mathcal{X}$.*

2. *The equivariant modular functor $\tau^\mathcal{X}$ induces on the G-equivariant monoidal category $\mathcal{C}^\mathcal{X}$ the structure of a weakly fusion category. Since the G-modular functor in theorem 2.12 is defined for arbitrary genus, we expect this structure to be G-modular, so that orbifold theories exist. Unfortunately, G-equivariant modular functors for higher genus are not yet well understood.*

3. *By restriction, the G-equivariant functor $\tau^\mathcal{X}$ endows for any group element $g \in G$ the abelian category $\mathcal{C}^{\boxtimes O_g}$ with the structure of a module category over the tensor category $\mathcal{C}^{\boxtimes \mathcal{X}}$.*

In section 3.1, we make this structure explicit in the case of $G = \mathbb{Z}/2$ acting non-trivially on a two-element set $\mathcal{X}$. In this case, we can show that the equivariant category is rigid rather than only weakly rigid. It is also known in this case ([BFRS10a]) that the module category structure on $\mathcal{C}$ over $\mathcal{C} \boxtimes \mathcal{C}$ describes the permutation modular invariant.

In section 3.2 we explicitely compute the monoidal product and structure morphisms for the module categories in statement 3 of corollary 2.13 for arbitrary groups G. It turns out in section 3.3 that also in the case of arbitrary groups the module categories $\mathcal{C}_g^{\mathcal{X}}$ over $\mathcal{C}_1^{\mathcal{X}}$ describe the permutation modular invariant.

For our construction, it is indispensable to require $\mathcal{C}$ to be modular. A genus 0 $\mathcal{C}^{\mathcal{X}}$-extended G-equivariant modular functor has to be defined on G-covers of extended surfaces of genus 0. According to lemma 2.8, the total space of the associated $|\mathcal{X}|$-fold cover, however, can have a non-zero genus. Hence the $\mathcal{C}$-extended modular functor τ has to be defined for any genus, and thus we can define, as in theorem 2.12, the equivariant modular functor $\tau^{\mathcal{X}}$ for any genus as well.

3. PERMUTATION EQUIVARIANT RIBBON CATEGORIES FROM MODULAR FUNCTORS

As we already pointed out, we want to define permutation equivariant ribbon categories for a finite group G, a finite G-set $\mathcal{X}$ and a modular category $\mathcal{C}$. To do this, we use the G-equivariant modular functor from section 2.2 and follow the lines of theorem 1.22.

It turns out that in contrast to the definition of the cover functor, which is very easy, it is a very difficult task to compute the structure morphisms for the G-equivariant (weakly) ribbon categories.

In the case of $G = \mathbb{Z}/2$ action on a two-element set $\mathcal{X}$ by permutation, it is possible to give the full structure of the $\mathbb{Z}/2$-equivariant monoidal category, which turns out to be ribbon. For arbitrary groups however, we can only adress the question of finding the monoidal product of two objects in the neutral component $\mathcal{C}_1^{\mathcal{X}}$ and of an object in $\mathcal{C}_1^{\mathcal{X}}$ with an object in some other component $\mathcal{C}_g^{\mathcal{X}}$. We can find the mixed associativity constraints to give the component $\mathcal{C}_g^{\mathcal{X}}$ as a module category over the monoidal category $\mathcal{C}_1^{\mathcal{X}}$.

With the methods we use, we are not able to derive the mixed tensor products for arbitrary groups. The surfaces $\mathcal{X} \times_G P$ become more complicated for groups larger that $\mathbb{Z}/2$; even for $\mathbb{Z}/2$ the definition of the tensor product for two objects in the twisted sector and the computation of the associativity constraint for three objects in the twisted sector is the most complicated part of section 3.1.

There are two reasons for these problems: If the genus of the surfaces $\mathcal{X} \times_G P$ becomes larger, the analysis will become more complicated, this already occurs in the $\mathbb{Z}/2$-case. The other reason lies in the Lego-Teichmüller Game: It becomes very difficult to draw and analyze marking graphs on surfaces of varying structure and unknown genus.

However for arbitrary groups we can compute the modular invariant partition functions that come with the module categories $\mathcal{C}_g^{\mathcal{X}}$ over $\mathcal{C}_1^{\mathcal{X}}$. Furthermore the results of [ENO09] imply that the module category structures already fix the full monoidal product on $\mathcal{C}^{\mathcal{X}}$ up to an element of a torsor over $\mathrm{H}^3(G, \Bbbk^*)$.

3.1 $\mathbb{Z}/2$-permutation equivariant fusion categories

This section is based on the article [BS10].

3.1.1 Notation and conventions

For this section, we restrict ourselves on the case where the cyclic group $G = \mathbb{Z}/2$ acts on the ordered two-element set $\mathcal{X} = \{1, 2\}$ by permutation of elements. We denote the generator of $\mathbb{Z}/2$ by g and hence write $\mathbb{Z}/2 = \{1, g\}$ with $g^2 = 1$.

We note that in this special case, the principal $\mathbb{Z}/2$-cover $P \to M$ and the cover $\mathcal{X} \times_{\mathbb{Z}/2} P \to M$ are isomorphic as covers over the manifold M. However, whereas P has only one marked point for every connected component of ∂M, the cover $\mathcal{X} \times_{\mathbb{Z}/2} P$ has one marked point in every connected component of its boundary. We will assume that one distinguished point has been chosen over each connected component of ∂P over ∂M (so that the total space P is an extended surface itself) and work with P instead of $\mathcal{X} \times_G P$.

Our construction will involve choices: Following theorem 1.22 we define the value of functors like the duality functor or the tensor product functor by objects representing functors constructed from the equivariant modular functor $\tau^{\mathcal{X}}$. This is possible, since by [BK01, lemma 5.3.1] any additive functor $F : \mathcal{D} \to \mathcal{V}ect_{\Bbbk}$ from a semisimple abelian category $\mathcal{D}$ with finitely many objects is representable. By definition of the dual object this shows that any object V of $\mathcal{C}^{\mathcal{X}}$ is determined by the functor $\tau^{\mathcal{X}}(S_2(h^{-1}, h; 1, 1) \to S_2; ?, V)$ for appropriate $h \in \mathbb{Z}/2$.

The representing objects from the Yoneda lemma will only be unique up to canonical isomorphism. Different choices of representing objects lead to different structures of ribbon categories on $\mathcal{C}^{\mathcal{X}}$, which are equivalent via a tensor functor which is the identity functor on $\mathcal{C}^{\mathcal{X}}$ and whose structure morphisms are provided the Yoneda lemma. We will indicate such choices in our discussion.

For the value of the equivariant modular functor $\tau^{\mathcal{X}}$ on the standard blocks of subsection 1.1.2, we introduce the shorthand notation

$$\langle V_1, \dots, V_n \rangle_{\mathcal{X}} := \tau^{\mathcal{X}}(S_n(g_1, \dots, g_n; 1, \dots, 1) \to S_n; V_1, \dots, V_n). \tag{3.1}$$

with objects $V_i \in \mathcal{C}^{\mathcal{X}}_{g_i}$. We use a similar shorthand for the $\mathcal{C}$-extended modular functor τ as well,

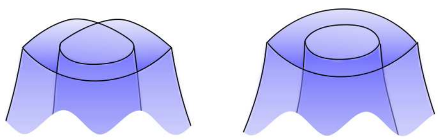
$$\langle W_1, \dots, W_n \rangle := \tau(S_n; W_1, \dots, W_n) \ , \tag{3.2}$$

where $W_1, \dots, W_n$ are objects in $\mathcal{C}$.

As an abelian category, the $\mathbb{Z}/2$-equivariant category $\mathcal{C}^{\mathcal{X}}$ is of the form $(\mathcal{C}\boxtimes\mathcal{C})\oplus(\mathcal{C})$, where $\mathcal{C}\boxtimes\mathcal{C}$ is the *neutral* and $\mathcal{C}$ the *twisted* component or sector of $\mathcal{C}^{\mathcal{X}}$. We denote objects in the neutral component by $A_1 \times A_2, B_1 \times B_2, \dots$ and objects in the twisted component by $M, N, \dots$.

The dual of an object V in $\mathcal{C}$ will be denoted by $V^\vee$. We agree to drop the tensor product symbol for objects of $\mathcal{C}$ when using the monoidal structure of $\mathcal{C}$, so we write $AB \equiv A \otimes_{\mathcal{C}} B$. The braiding of two objects in $\mathcal{C}$ will be denoted by $c_{A,B}$ and the equivariant braiding in $\mathcal{C}^{\mathcal{X}}$ by $C_{A,B}$ in capital letters. Similarly, θ_U denotes the twist, b_U the (right) coevaluation and d_U the (right) evaluation in $\mathcal{C}$, while for the corresponding morphisms Θ_A, B_A and D_A of $\mathcal{C}^{\mathcal{X}}$ capital letters are used.

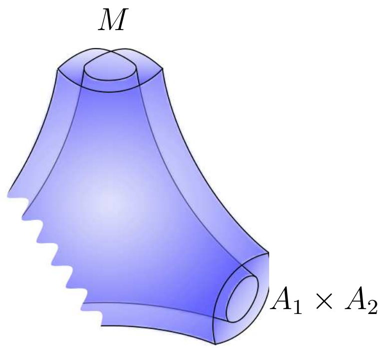
When dealing with modular functors we use a graphical notation. First note that the total spaces of $\mathbb{Z}/2$-covers over S_n can have the following two types of boundary components:



(3.3)

The first is a boundary component of non-trivial monodromy g, the second with trivial monodromy $1 \in \mathbb{Z}/2$.

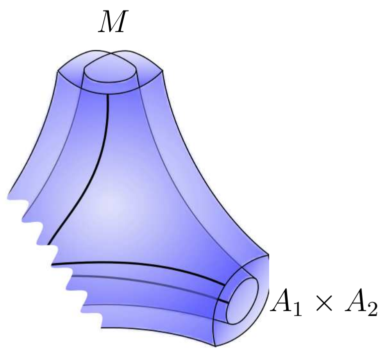
A decorated surface Σ also represents the corresponding vector space $\tau(\Sigma; V_1, \dots, V_n)$: we then write objects of the appropriate categories to the boundary components:



(3.4)

When we write an object $A_1 \times A_2$ next to a boundary embedded in $\mathbb{R}^3$, our convention is such that the first object is assigned to the outer circle and the second object to the inner circle.

To keep track of diffeomorphisms of a surface Σ, we use techniques from the Lego-Teichmüller Game (section 1.2) and use the graphical representation of these diffeomorphisms. The total space of a $\mathbb{Z}/2$-cover with a marking graph on it is depicted in



(3.5)

Recall the standard marking on the standard sphere S_n from subsection 1.2.1 that relates the marked point $k - \frac{i}{3}$ to the point $-2i$ for $k = 1, 2 \ldots n$ by n straight lines.

When we apply diffeomorphisms to surfaces we will analyze the image of markings graphs under the diffeomorphisms and relate the marking graphs by finite sequences of LTG moves, see section 1.2.4.

The LTG gives rules to translate these moves into natural isomorphisms between the corresponding vector spaces, see appendix A for this translation.

In most cases, we will suppress the LTG-move **Z** (the rotation move) in manipulations of the marking graphs; when translating the LTG-moves into morphisms in $\mathcal{C}$, we will point out at what point one has to insert **Z**-moves.

Similarly, if a surface has been obtained by gluing two boundary circles, the corresponding circle will be drawn on the surface, this circle will then give a cut in the glued marking graph.

3.1.2 Dual objects

We start by finding for every object $V \in \mathcal{C}^{\mathcal{X}}$ a candidate for the dual object. This will be the object V^* representing the functor

$$\begin{array}{ll} \mathcal{C}^{\mathcal{X}} & \to \mathcal{V}ect_{\Bbbk} \\ T & \mapsto \langle V, T\rangle_{\mathcal{X}}. \end{array} \tag{3.6}$$

We consider two cases separately.

- For V in the neutral component, $V = A_1 \times A_2 \in \mathcal{C} \boxtimes \mathcal{C}$, we consider the standard block $S_2(1,1;1,1)$ whose cover spaces is the disjoint union of two copies of S_2 and find the object representing the functor

$$\begin{array}{ll} \mathcal{C} \boxtimes \mathcal{C} & \to \mathcal{V}ect_{\Bbbk} \\ T_1 \times T_2 & \mapsto \langle A_1 \times A_2, T_1 \times T_2\rangle_{\mathcal{X}}. \end{array} \tag{3.7}$$

  Here, we restricted our attention to the value of the functor on the neutral component, since the grading implies that the functor is zero on the twisted component. We compute

$$\begin{aligned} \tau^{\mathcal{X}}(S_2(1,1;1,1) \to S_2; V, T) &= \tau(S_2 \sqcup S_2; A_1, A_2, T_1, T_2) \\ &\cong \tau(S_2; A_1, T_1) \otimes_{\Bbbk} \tau(S_2; A_2, T_2) \\ &\overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A_1 T_1) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A_2 T_2) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(A_1^\vee, T_1) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(A_2^\vee, T_2) \\ &\overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{C}}(A_1^\vee \times A_2^\vee, X_1 \times T_2) \\ &\overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}^{\mathcal{X}}}(A_1^\vee \times A_2^\vee, X_1 \times T_2). \end{aligned} \tag{3.8}$$

  Let us explain in this example the various steps in detail; in subsequent calculations, we will only explain additional new features. The first equality is by definition of the equivariant modular functor $\tau^{\mathcal{X}}$ via the cover functor $\mathcal{F}_{\mathcal{X}}$; the second isomorphy is the tensoriality of τ. The third equality follows from the definition of the modular functor τ from the modular category $\mathcal{C}$. The next isomorphism is a consequence of the duality in the category $\mathcal{C}$, while the last identities follow from the definition of the Deligne product $\mathcal{C} \boxtimes \mathcal{C}$ and the definition of $\mathcal{C}^{\mathcal{X}}$.

Hence we find that $(A_1 \times A_2)^* \cong A_1^\vee \times A_2^\vee$. A choice of the representing object and of a diffeomorphism is involved in equation (3.8); different choices ultimately lead to equivalent dualities on $\mathcal{C}^{\mathcal{X}}$.

- Let M be an object in the twisted component of $\mathcal{C}^{\mathcal{X}}$, hence $M \in \mathcal{C}$. The total space of the cover $(S_2(g,g;1,1) \to S_2)$ has only one connected component and is in fact diffeomorphic to S_2 as a smooth manifold, so we find

$$\begin{aligned} \tau^{\mathcal{X}}(S_2(g,g;1,1);M,T) &= \tau(S_2;M,T) \\ &\overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, MT) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(M^\vee, T) \end{aligned} \tag{3.9}$$

We thus find that $M^* \cong M^\vee$. Again, a choice of a representing object and of a diffeomorphism $S_2(g,g;1,1) \xrightarrow{\cong} S_2$ are involved.

So far we only found dual objects, for the further discussion on duality in $\mathcal{C}^{\mathcal{X}}$ we need a tensor product on $\mathcal{C}^{\mathcal{X}}$. For this reason, we return to more aspects of a ribbon structure in section 3.1.7.

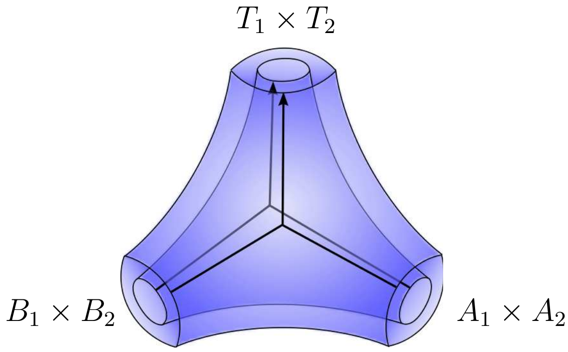
3.1.3 The tensor product

As in theorem 1.22, the tensor product of two objects $V, W \in \mathcal{C}^{\mathcal{X}}$ is defined as the object representing the functor $\langle ?, V, W\rangle_{\mathcal{X}}$, i.e. by $\langle T, V \otimes W\rangle_{\mathcal{X}} = \langle T, V, W\rangle_{\mathcal{X}}$. This again involves a choice of the representing object. The total space of the standard cover $S_3((g_1g_2)^{-1}, g_1, g_2; 1,1,1)$ is isomorphic to the n-punctured sphere S_n for a value of n that depends on the group elements g_1 and g_2. We will have to choose such a diffeomorphism as well. Different choices of this diffeomorphism lead to isomorphic tensor products, but the choice will enter in the associativity constraints; hence we have to keep track of this choice. This is done by considering the image of the standard marking graph on S_n under this diffeomorphism. We call this image the standard marking graph on $S_3((g_1g_2)^{-1}, g_1, g_2; 1,1,1)$.

- We first consider two objects $V \equiv A_1 \times A_2$ and $W \equiv B_1 \times B_2$ in the neutral component of $\mathcal{C}^{\mathcal{X}}$. Since total space of the cover $S_3(1,1,1;1,1,1)$ is just the disjoint union of two three-holed spheres, we find

$$\begin{aligned}
\langle T, V \otimes W \rangle_{\mathcal{X}} \equiv \langle T_1 \times T_2, A_1 \times A_2, B_1 \times B_2 \rangle_{\mathcal{X}}& \\
&\overset{\text{def}}{=} \tau^{\mathcal{X}}(S_3(1,1,1;1,1,1) \to S_3; T_1 \times T_2, A_1 \times A_2, B_1 \times B_2) \\
&\overset{\text{def}}{=} \tau(S_3 \sqcup S_3; T_1, T_2, A_1, A_2, B_1, B_2) \\
&\cong \tau(S_3; T_1, A_1, B_1) \otimes_{\Bbbk} \tau(S_3; T_2, A_2, B_2) \\
&\overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, T_1 A_1 B_1) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, T_2 A_2 B_2) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(T_1^{\vee}, A_1 B_1) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(T_2^{\vee}, A_2 B_2) \\
&\overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{C}}(T_1^{\vee} \times T_2^{\vee}, A_1 B_1 \times A_2 B_2)
\end{aligned} \tag{3.10}$$

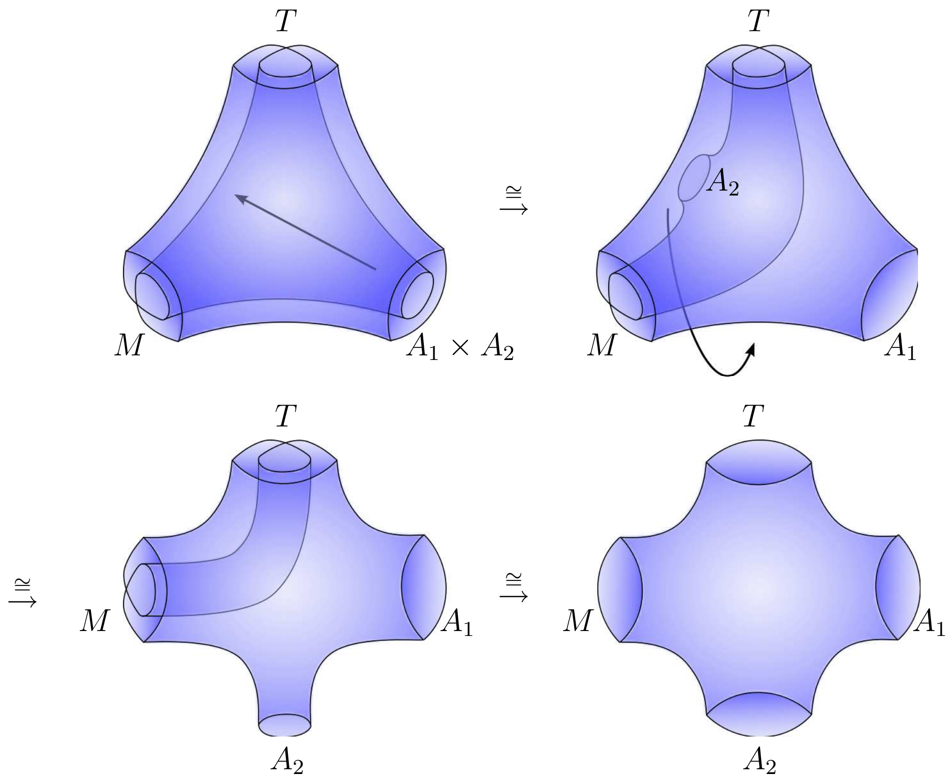
Our definition of the tensor product on objects of the untwisted component thus yields $(A_1 \times A_2) \otimes (B_1 \times B_2) := A_1 B_1 \times A_2 B_2$, i.e. the usual tensor product on $\mathcal{C} \boxtimes \mathcal{C}$. The standard marking graph on the cover $S_3(1,1,1;1,1,1)$ is then



(3.11)

- Next we consider the tensor product of an object in the untwisted component $A_1 \times A_2$ with an object M in the twisted component. The total space $S_3(g,1,g;1,1,1)$ of the relevant cover is diffeomorphic to S_4

as can be seen by the following chain of diffeomorphisms:

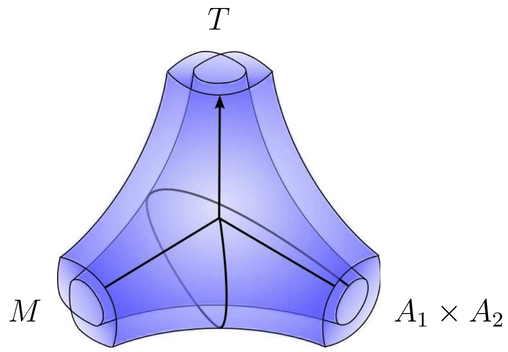


(3.12)

In the first and second step we move the inner hole labelled by A_2 around the component of the boundary with non-trivial monodromy, labelled by M. The last step is also an isomorphism of manifolds; the reader should not be confused by the fact that we have to draw two-dimensional manifolds immersed into the three-dimensional space $\mathbb{R}^3$. Hence we find that

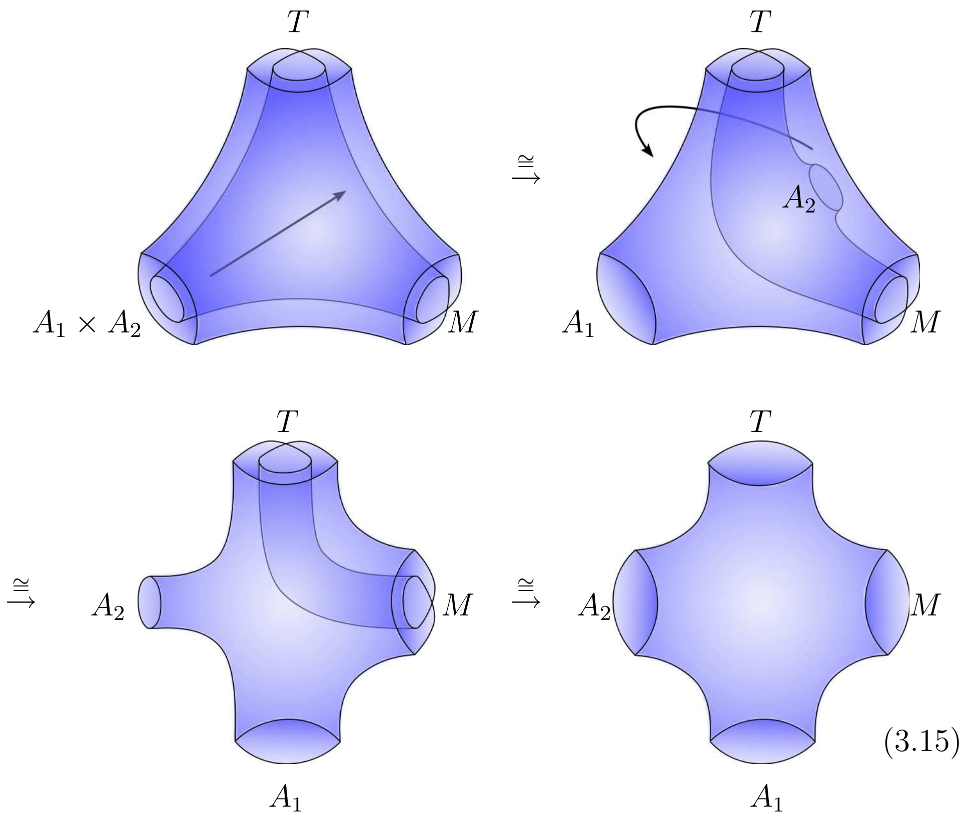
$$\begin{aligned}
\langle T, (A_1 \times A_2) \otimes M \rangle_{\mathcal{X}} &= \langle T, A_1 \times A_2, M \rangle_{\mathcal{X}} \\
&\stackrel{\text{def}}{=} \tau^{\mathcal{X}}(S_3(g,1,g;1,1,1) \to S_3; T, A_1 \times A_2, M) \\
&\stackrel{\text{def}}{=} \tau(S_3(g,1,g;1,1,1); T, A_1, A_2, M) \\
&\cong \tau(S_4; T, A_1, A_2, M) \\
&\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, T A_1 A_2 M) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(T^\vee, A_1 A_2 M).
\end{aligned} \tag{3.13}$$

We are thus lead to the definition $(A_1 \times A_2) \otimes M := A_1 A_2 M$. The image of the standard marking graph on S_4 under the diffeomorphism described in equation (3.12) gives this marking on $S_3(g, 1, g; 1, 1, 1)$:



(3.14)

- The discussion of the tensor product $M \otimes (A_1 \times A_2)$ closely parallels the preceding discussion. Again the manifold $S_3(g, g, 1; 1, 1, 1)$ is diffeomorphic to the four-punctured sphere S_4. We introduce a diffeomorphism similar to the one defined in equation (3.12):

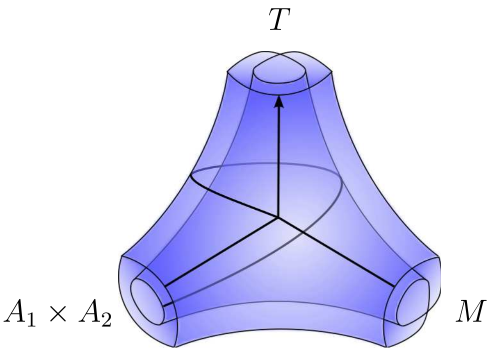


(3.15)

We find:

$$
\begin{aligned}
\langle T, M \otimes (A_1 \times A_2) \rangle_{\mathcal{X}} &\stackrel{\text{def}}{=} \langle T, M, A_1 \times A_2 \rangle_{\mathcal{X}} \\
&\stackrel{\text{def}}{=} \tau^{\mathcal{X}}(S_3(g, g, 1; 1, 1, 1) \to S_3; T, M, A_1 \times A_2) \\
&\stackrel{\text{def}}{=} \tau(S_3(g, g, 1; 1, 1, 1); T, M, A_1, A_2) \\
&\cong \tau(S_4; T, M, A_1, A_2) \\
&\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, TMA_1A_2) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(T^\vee, MA_1A_2)
\end{aligned}
\tag{3.16}
$$

so that we are lead to define $M \otimes (A_1 \times A_2) := MA_1A_2$ and have the standard marking graph on $S_3(g, g, 1; 1, 1, 1)$



(3.17)

These definitions coincide with the ad hoc definitions made in [BFRS10a].

- We now turn to the tensor product of two objects in the twisted component of $\mathcal{C}^{\mathcal{X}}$. To this end, we first note that the tensor product functor

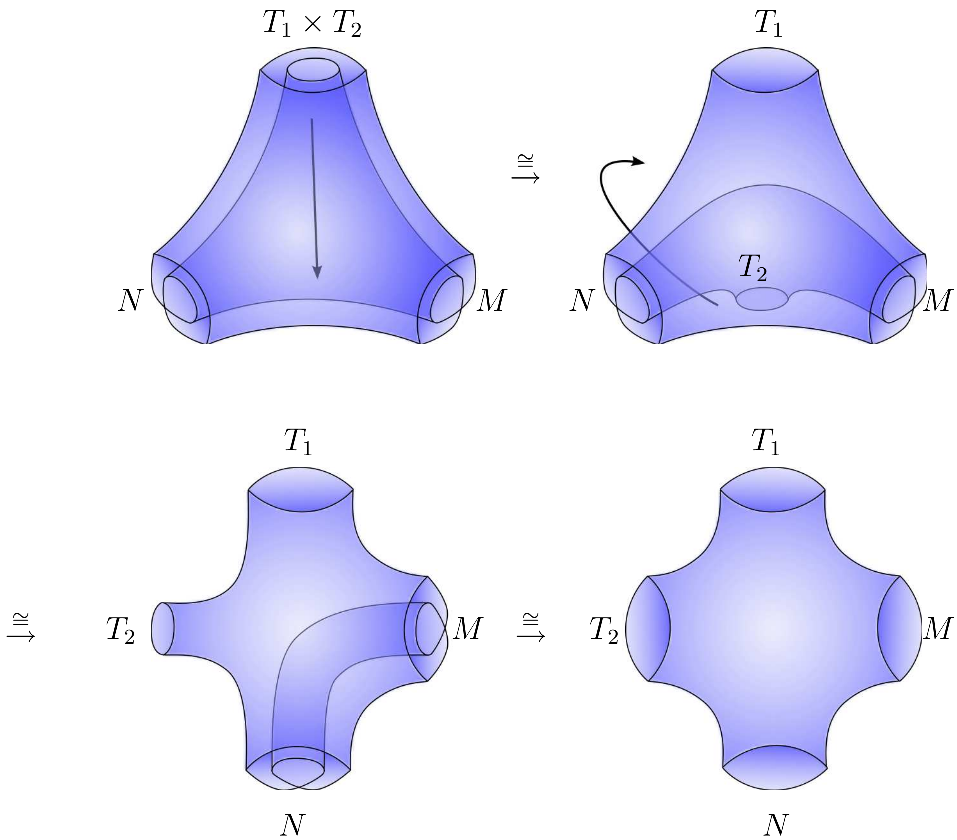
$$
\begin{aligned}
\mathcal{C} \boxtimes \mathcal{C} &\to \mathcal{C} \\
\bigoplus_l V_l \times W_l &\mapsto \bigoplus_l V_l W_l
\end{aligned}
\tag{3.18}
$$

has the right adjoint [KR09, Thm. 2.20]

$$
\begin{aligned}
R:\ \mathcal{C} &\to \mathcal{C} \boxtimes \mathcal{C} \\
V &\mapsto \bigoplus_{i \in \mathcal{I}} V U_i^\vee \times U_i \ ,
\end{aligned}
\tag{3.19}
$$

where the sum is over representatives of isomorphism classes of simple objects of $\mathcal{C}$.

We have to consider the manifold $S_3(1, g, g; 1, 1, 1)$ together with the following diffeomorphisms



(3.20)

We thus find

$$
\begin{aligned}
\langle T_1 \times T_2, M \otimes N\rangle_{\mathcal{X}} &\stackrel{\text{def}}{=} \langle T_1 \times T_2, M, N\rangle_{\mathcal{X}} \\
&\stackrel{\text{def}}{=} \tau^{\mathcal{X}}(S_3(1, g, g; 1, 1, 1) \to S_3; T_1 \times T_2, M, N) \\
&\stackrel{\text{def}}{=} \tau(S_3(1, g, g; 1, 1, 1); T_1, T_2, M, N) \\
&\cong \tau(S_4; T_2, T_1, M, N) \\
&\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, T_2 T_1 M N) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(T_1^\vee T_2^\vee, MN) \\
&\cong \mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{C}}(T_1^\vee \times T_2^\vee, R(MN))
\end{aligned}
\tag{3.21}
$$

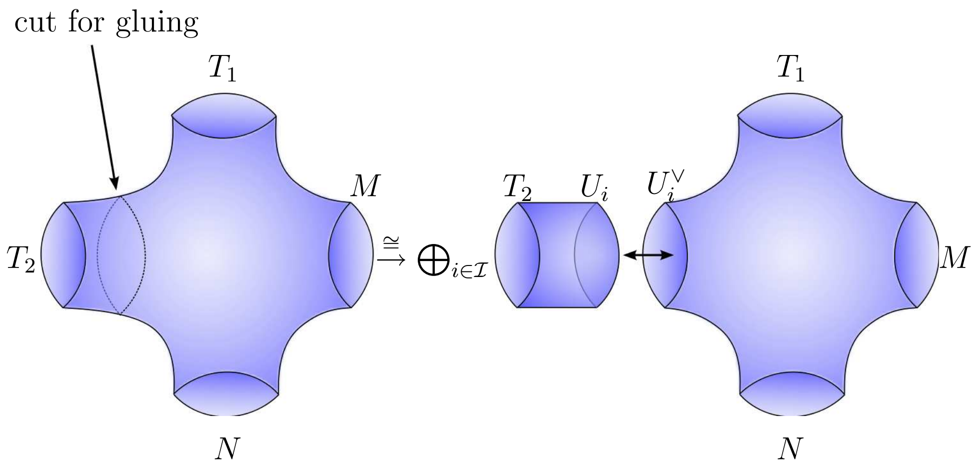
where the last isomorphism is given by the adjunction between R and the tensor product functor. Hence we set

$$M \otimes N = R(MN) = \bigoplus_{i \in \mathcal{I}} MNU_i^\vee \times U_i. \tag{3.22}$$

We note that the adjunction morphism

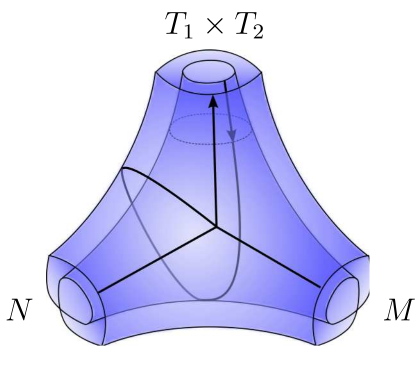
$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(T_1^\vee T_2^\vee, MN) &\cong \bigoplus_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{C}}(T_1^\vee \times T_2^\vee, MNU_i^\vee \times U_i) \\ &\cong \bigoplus_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(T_1^\vee, MNU_i^\vee) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(T_2^\vee, U_i) \end{aligned} \tag{3.23}$$

coincides with the gluing isomorphism



(3.24)

so that on the total space $S_3(1, g, g; 1, 1, 1)$ of the cover we have the standard marking graph



(3.25)

which has a cut drawn on the manifold close to the insertion of T_2. The second arrow in the marking on $S_3(1,g,g;1,1,1)$ comes from the marking on $S_2(U_i,T_2)$ and fixes the order of the objects in $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},U_iT_2)$.

We finally determine the tensor unit $\mathbf{1}^{\mathbb{Z}/2}$ of $\mathcal{C}^{\mathcal{X}}$ that is defined by

$$\langle \mathbf{1}^{\mathbb{Z}/2}, U\rangle_{\mathcal{X}} := \langle U\rangle_{\mathcal{X}} \tag{3.26}$$

for all U in $\mathcal{C}^{\mathcal{X}}$. The classification of covers of the one-holed sphere implies that $\langle U\rangle_{\mathcal{X}}$ is only defined when U is in the neutral component, $U\in\mathcal{C}^{\mathcal{X}}_1$. With $U=A_1\times A_2\in\mathcal{C}\boxtimes\mathcal{C}$, we find $\langle A_1\times A_2\rangle_{\mathcal{X}}\cong\langle A_1\rangle\otimes_{\Bbbk}\langle A_2\rangle$ and hence

$$\mathbf{1}^{\mathbb{Z}/2}=\mathbf{1}\times\mathbf{1}. \tag{3.27}$$

Since the modular tensor category $\mathcal{C}$ was supposed to be strict, the unit constraints of $\mathcal{C}^{\mathcal{X}}$ are the identity morphisms.

3.1.4 The associativity constraints

The next step is to derive the associativity constraints for the tensor product given in the previous section. As special cases, we will find the mixed associativity constraints for which ansätze were proposed in [BFRS10a].

For any choice of three elements $p,q,r\in\mathbb{Z}/2$ we need to consider the $\mathbb{Z}/2$-cover

$$S_4((pqr)^{-1},p,q,r;1,1,1,1)\to S_4$$

of the four-holed sphere S_4. This cover will be cut in two different ways, representing the tensor product $(A\otimes B)\otimes C$ and $A\otimes(B\otimes C)$, respectively. By gluing the markings that represent these tensor products, the two ways of cutting give two different markings on $S_4((pqr)^{-1},p,q,r;1,1,1,1)$ which in turn represent two different diffeomorphisms from the total space of the cover

$$f,g: S_4((pqr)^{-1},p,q,r;1,1,1,1)\xrightarrow{\cong} S \tag{3.28}$$

to the appropriate standard block S, which is either a punctured sphere, a disjoint union of punctured spheres or a surface of genus one. We then consider for all objects $T\in\mathcal{C}^{\mathcal{X}}$ the following chain of natural transformations:

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{C}^{\mathcal{X}}}(T^*,(A\otimes B)\otimes C)\xrightarrow{\cong}\tau(S;T,A,B,C)\\&\xrightarrow{f_*^{-1}}\tau(S_4((pqr)^{-1},p,q,r;1,1,1,1);T,A,B,C)\xrightarrow{g_*}\tau(S;T,A,B,C)\\&\xrightarrow{\cong}\mathrm{Hom}_{\mathcal{C}^{\mathcal{X}}}(T^*,A\otimes(B\otimes C)).\end{aligned} \tag{3.29}$$

The first and the last isomorphism in (3.29) are determined by the definition of the tensor products $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ as objects in $\mathcal{C}$ or $\mathcal{C} \boxtimes \mathcal{C}$ respectively. The Yoneda lemma implies that this natural transformation comes from a natural isomorphism $\alpha_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)$. The arguments of [KP08, BK01] then imply that these isomorphisms satisfy the pentagon axiom.

The Lego-Teichmüller Game turns out to be a powerful tool for this task: Both isomorphisms in (3.28) can be evaluated on the standard marking graph on the standard block S. The composition $g \circ f^{-1}$ gives a new marking on S that can be transformed into the standard marking by a finite sequence of LTG-moves. Together with the dictionary for LTG-moves in appendix A, these in turn give the natural transformation in (3.29).

We summarize our strategy:

- Determine the two marking graphs on $S_4((pqr)^{-1}, p, q, r; 1, 1, 1, 1)$ induced on the cover by cutting S_4 in the two ways determined by associativity and by our definition of the tensor product. Denote the marking graph representing $(A \otimes B) \otimes C$ by M_1 and the marking graph representing $A \otimes (B \otimes C)$ by M_2.

- Determine the standard manifold S that is diffeomorphic to the total space $S_4((pqr)^{-1}, p, q, r; 1, 1, 1, 1)$. Then transform the manifold $S_4((pqr)^{-1}, p, q, r; 1, 1, 1, 1)$ with the marking graph M_1 to the standard manifold S in the way prescribed by the marking M_2.

- This yields S, together with a marking graph. Determine the LTG-moves that transform this graph into the standard marking graph on S.

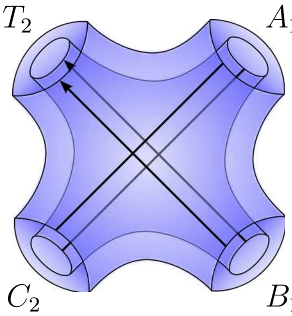
- Translate these LTG-moves into morphisms in $\mathcal{C}$ or $\mathcal{C} \boxtimes \mathcal{C}$.

Since we need to do this for any choice of $p, q, r \in \mathbb{Z}/2$, we will get eight different associativity constraints $\alpha_{A,B,C}$, depending on the sector of the objects A, B, C.

Three objects from the untwisted component

To illustrate the method, we start with the easiest case, three objects $A_1 \times A_2$, $B_1 \times B_2$ and $C_1 \times C_2$ in the untwisted component. The total space

$S_4(1,1,1,1;1,1,1,1)$ is just a disjoint union of two four-holed spheres. Both cutting procedures yield the same marking graph on $S_4 \sqcup S_4$:



(3.30)

Hence we arrive at the trivial LTG-move. Taking into account that $\mathcal{C}$ is supposed to be strict, we find that $\alpha_{A_1\times A_2, B_1\times B_2, C_1\times C_2}$ is the identity on $A_1B_1C_1 \times A_2B_2C_2$.

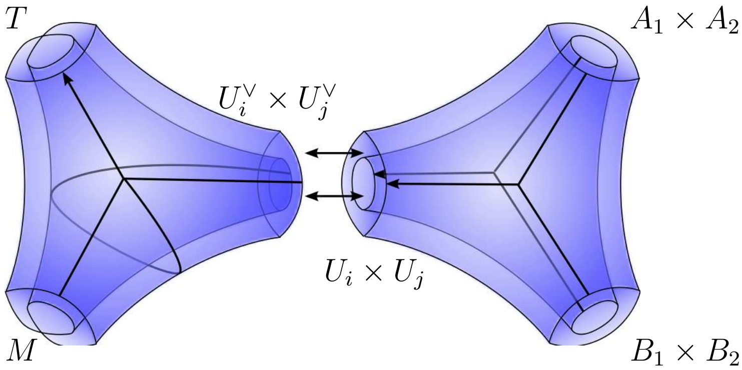
One object from the twisted component

Next we derive the associativity isomorphisms involving one object in the twisted component and two objects in the untwisted component. To explain our prescription in detail, we first discuss the associativity constraint $\alpha_{A_1\times A_2, B_1\times B_2, M}$ in greater detail.

The first cutting on the total space of $S_4(g,1,1,g;1,1,1,1)$ amounts to the gluing isomorphism in $\mathcal{C}\boxtimes\mathcal{C}$:

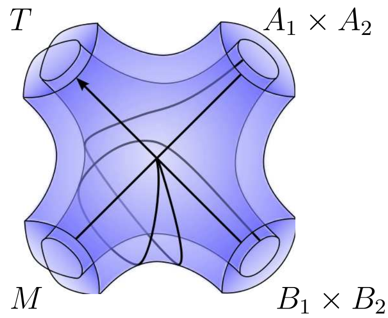
$$\begin{aligned}&\bigoplus_{i,j\in\mathcal{I}} \langle T, U_i^\vee \times U_j^\vee, M\rangle_{\mathcal{X}} \otimes_{\Bbbk} \langle U_i \times U_j, A_1 \times A_2, B_1 \times B_2\rangle_{\mathcal{X}}\\ \xrightarrow{\cong}&\langle T, A_1 \times A_2, B_1 \times B_2, M\rangle_{\mathcal{X}}\end{aligned} \tag{3.31}$$

With the standard markings on S_3-covers introduced in section 3.1.3, we arrive at the following picture:



(3.32)

By contracting the marking along the factorizing link, we get on the manifold $S_4(g,1,1,g;1,1,1,1)$ the marking

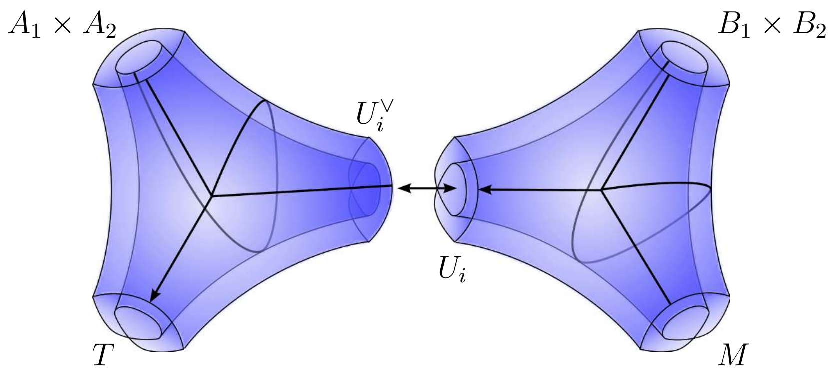


(3.33)

The second gluing procedure is the isomorphism

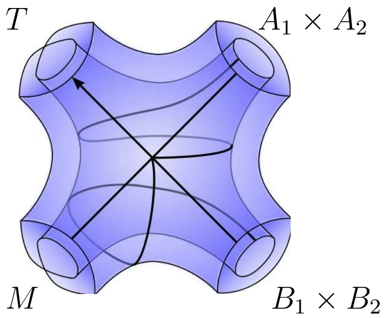
$$\begin{aligned} &\bigoplus_{i\in\mathcal{I}} \langle T, A_1 \times A_2, U_i^\vee \rangle_{\mathcal{X}} \otimes_{\Bbbk} \langle U_i, B_1 \times B_2, M \rangle_{\mathcal{X}} \\ \xrightarrow{\cong} &\langle T, A_1 \times A_2, B_1 \times B_2, M \rangle_{\mathcal{X}} \end{aligned} \tag{3.34}$$

with the pictorial description



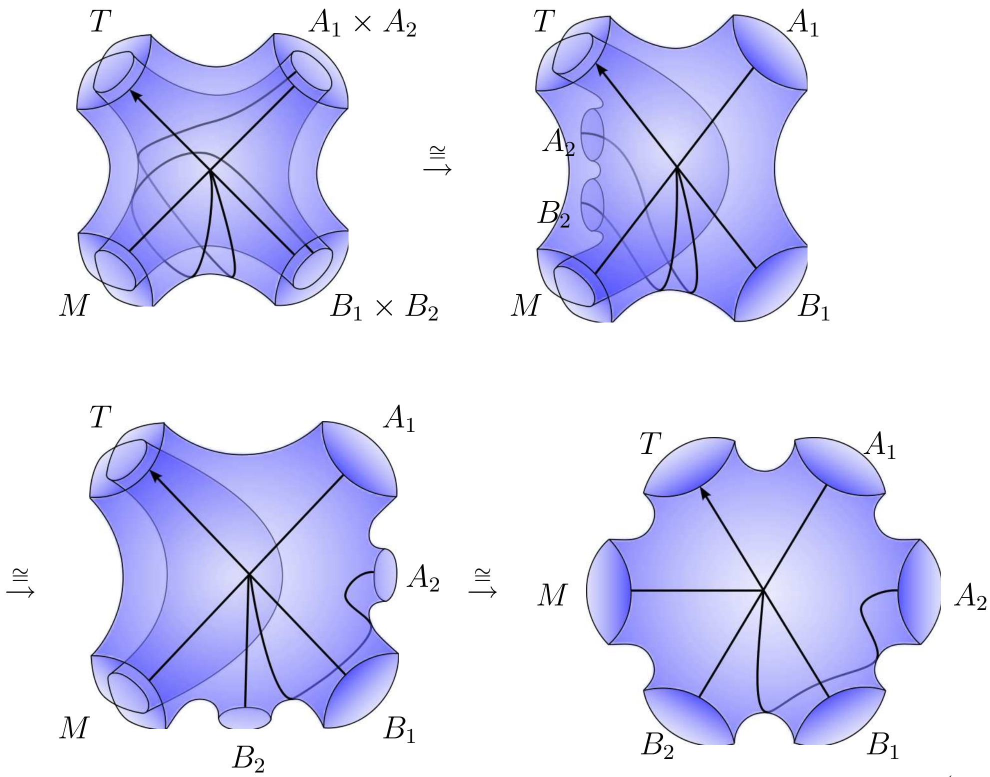
(3.35)

Again contracting the marking along the factorizing link, we get on the manifold $S_4(g,1,1,g;1,1,1,1)$ the second marking



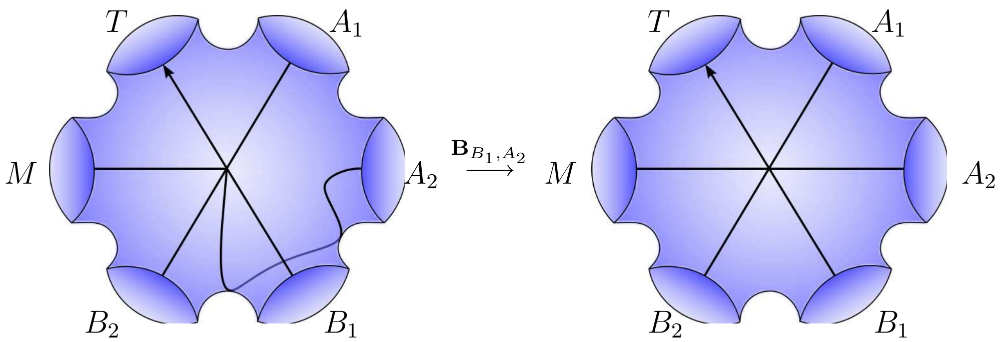
(3.36)

The surface $S_4(g,1,1,g;1,1,1,1)$ is isomorphic to a sphere S_6 with 6 punctures. We draw the graph (3.33) obtained from the first gluing procedure on $S_4(g,1,1,g;1,1,1,1)$ and use the isomorphism to S_6 encoded in the graph (3.36) obtained from the second gluing procedure:



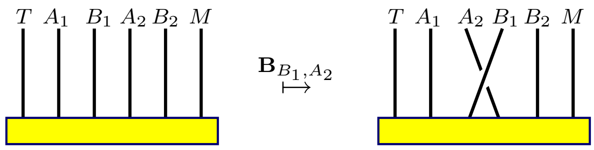
(3.37)

The LTG-moves that transform this marking into the standard marking on S_6 are easily read off:



(3.38)

The LTG-move $\mathbf{B}_{B_1,A_2}$ corresponds to the natural transformation on the functor $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, TA_1B_1A_2B_2M)$ that is induced by the braiding in $\mathcal{C}$ (see appendix A), pictorially,



(3.39)

The ribbon structure on $\mathcal{C}$ induces an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, TA_1B_1A_2B_2M) \cong \mathrm{Hom}_{\mathcal{C}}(T^\vee, A_1B_1A_2B_2M)\,. \tag{3.40}$$

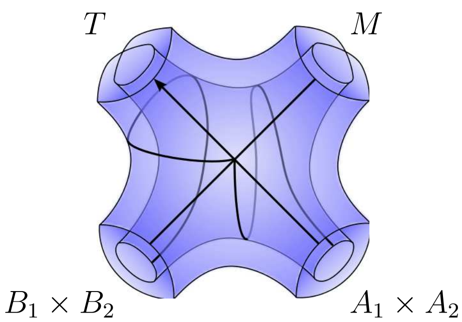
Setting $T^\vee := A_1B_1A_2B_2M$ and evaluating the natural transformation (3.39) on the identity, we obtain the morphism for the associativity constraint as $\alpha_{A_1\times A_2, B_1\times B_2, M} = \mathrm{id}_{A_1} \otimes c_{B_1,A_2} \otimes \mathrm{id}_{B_2M}$.

These are precisely the mixed associativity constraints proposed in the paper [BFRS10a]; in that paper, a whole family of mixed associativity constraints was given. Here we read off the morphisms from marking graphs on surfaces. These graphs depend on the choice of diffeomorphisms made in subsection 3.1.3. Different choices of diffeomorphisms lead to different associativity constraints, of which some were already proposed in [BFRS10a]. To be precise, for every associativity constraint in [BFRS10a] there exists a choice of parametrization of the total space $S_3(g,1,g;1,1,1)$ that induces that associativity constraint. Different choices of diffeomorphisms are related by elements of the mapping class group of the relevant surface. These group elements can be translated into morphisms in $\mathcal{C}$ which can be used to endow the identity functor on $\mathcal{C}^{\mathcal{X}}$ with the structure of an monoidal equivalence of monoidal categories.

In fact in lemma 5 of [BFRS10a] it was shown by a calculation that for the different associativity constraints the identity functor with tensoriality constraints only containing braiding and ribbon twist of $\mathcal{C}$ furnishes a monoidal equivalence. One can verify that for any two choices of associativity constraints in [BFRS10a], the tensoriality constraints in [BFRS10a, Lemma 5] are induced by the element of the mapping class group that interchanges the two parametrizations that lead to these associativity constraints.

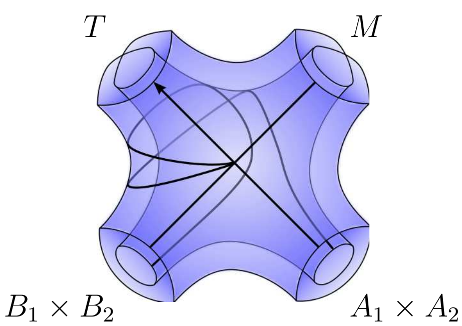
So the equivalence of the associativity constraints in [BFRS10a] is not surprising from the modular functor point of view.

For the associativity constraint $\alpha_{M,A_1\times A_2,B_1\times B_2}$, we get a similar picture. The first gluing procedure on $S_4(g,g,1,1;1,1,1,1)$ gives the marking



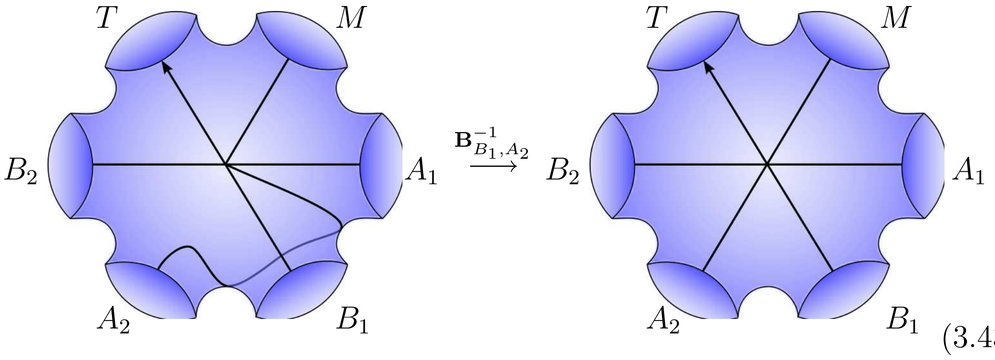
(3.41)

while the second procedure gives the marking



(3.42)

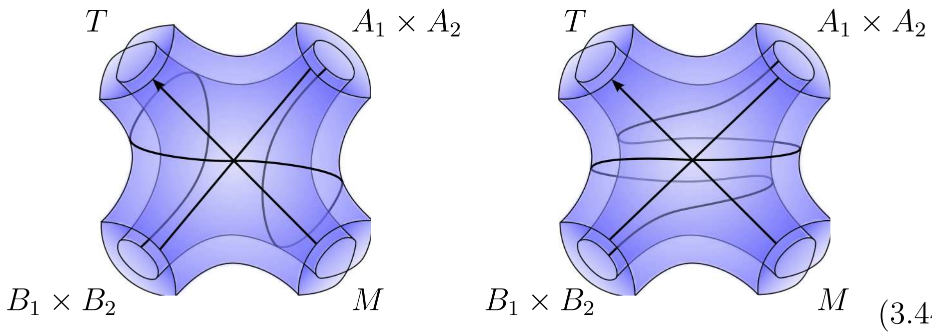
On the sphere S_6 with 6 punctures, this gives the LTG-move



(3.43)

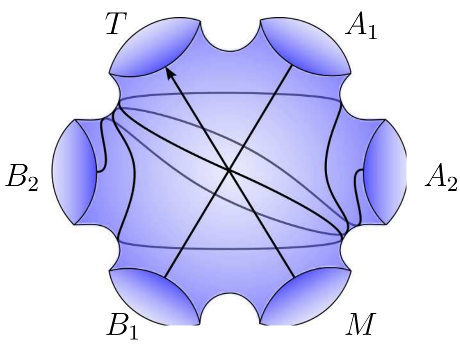
By the same reasoning as after equation (3.39), for the associativity constraints we conclude that $\alpha_{M,A_1\times A_2,B_1\times B_2} = \mathrm{id}_{MA_1} \otimes c^{-1}_{B_1,A_2} \otimes \mathrm{id}_{B_2}$. This is one of the constraints in [BFRS10a, Corollary 3].

For the associativity constraint $\alpha_{A_1\times A_2,M,B_1\times B_2}$ we consider the manifold $S_4(g,1,g,1;1,1,1,1) \cong S_6$. The two gluing procedures give the two markings



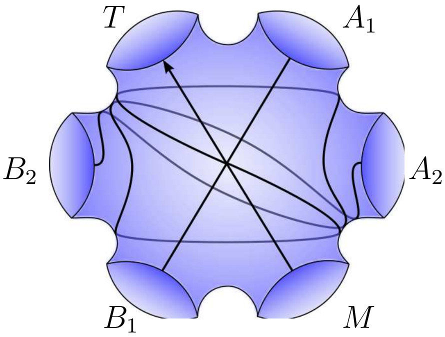
(3.44)

Our general prescription gives the following marking graph on S_6:

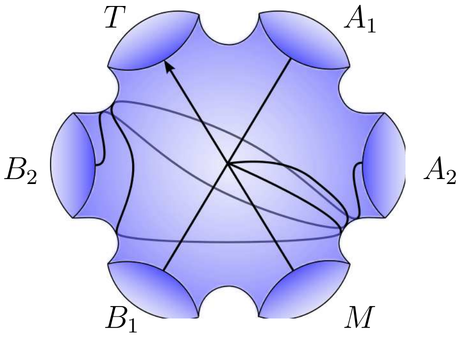


(3.45)

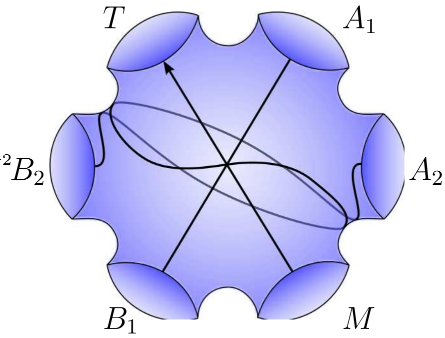
The following sequence of four LTG-moves transforms this marking graph into the standard marking graph on S_6:



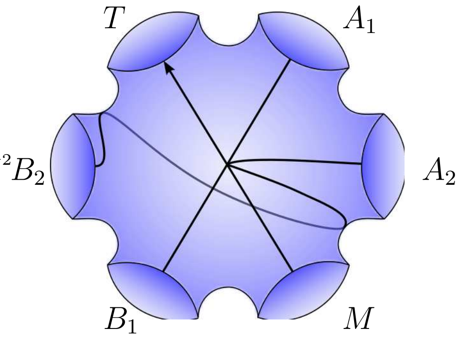
$\xrightarrow{\mathbf{B}^{-1}_{TA_1,B_2}}$



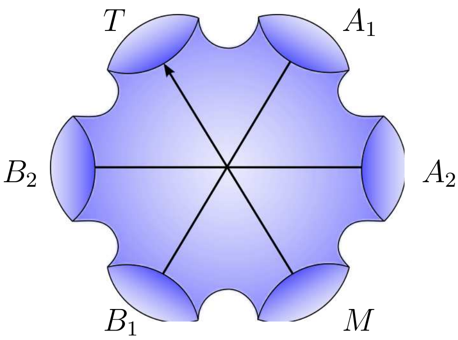
$\xrightarrow{\mathbf{B}^{-1}_{MB_1,A_2}}$



$\xrightarrow{\mathbf{B}_{B_2MB_1,A_2}}$



$\xrightarrow{\mathbf{B}_{TA_1A_2,B_2}}$



(3.46)

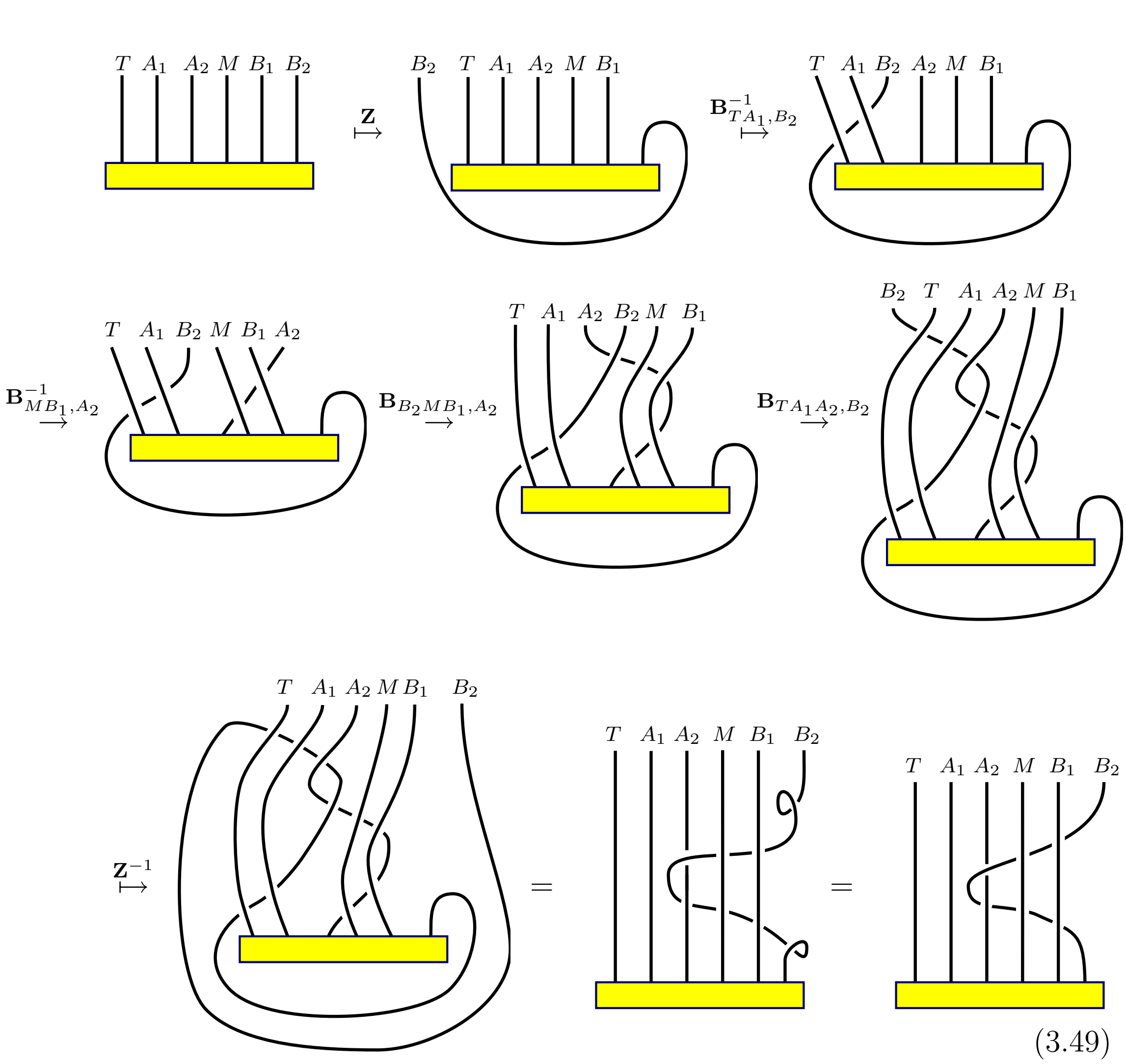
The sequence

$$\mathbf{B}_{TA_1A_2,B_2} \circ \mathbf{B}_{B_2MB_1,A_2} \circ \mathbf{B}^{-1}_{MB_1,A_2} \circ \mathbf{B}^{-1}_{TA_1,B_2} \tag{3.47}$$

of LTG-moves is translated into the natural transformation

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, TA_1A_2MB_1B_2) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, TA_1A_2MB_1B_2), \tag{3.48}$$

where we will need to insert the appropriate $\mathbf{Z}$-moves implementing cyclicity:



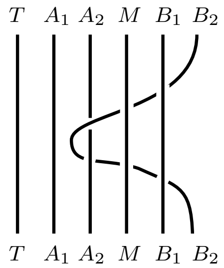
(3.49)

Using again the isomorphy

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, TA_1A_2MB_1B_2) \cong \mathrm{Hom}_{\mathcal{C}}(T^{\vee}, A_1A_2MB_1B_2) \tag{3.50}$$

implied by duality and evaluating on the identity morphism gives the associativity constraint

$$\alpha_{A_1 \times A_2, M, B_1 \times B_2} =$$

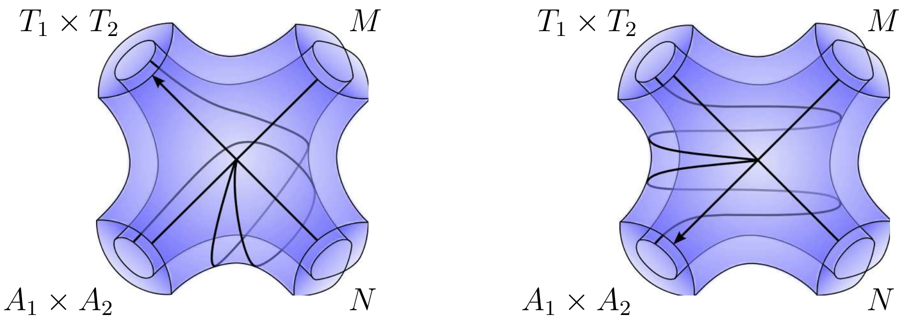


(3.51)

Two objects in the twisted component

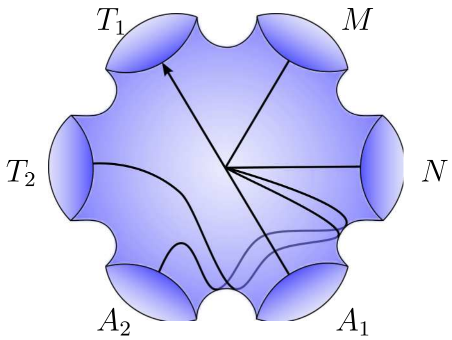
We next discuss associativity constraints involving two objects in the twisted and one in the untwisted component of $\mathcal{C}^{\mathcal{X}}$. The tensor product of three such objects lies in the untwisted component. Again the relevant marking graphs on covers of the four-punctured sphere contain cuts.

We start with the associativity constraint $\alpha_{M,N,A_1 \times A_2}$. The two gluing procedures over S_4 yield the following two markings on $S_4(1, g, g, 1; 1, 1, 1, 1)$, where we already removed the cuts and contracted the factorizing links:



(3.52)

This gives the following marking on the six-punctured sphere S_6:



(3.53)

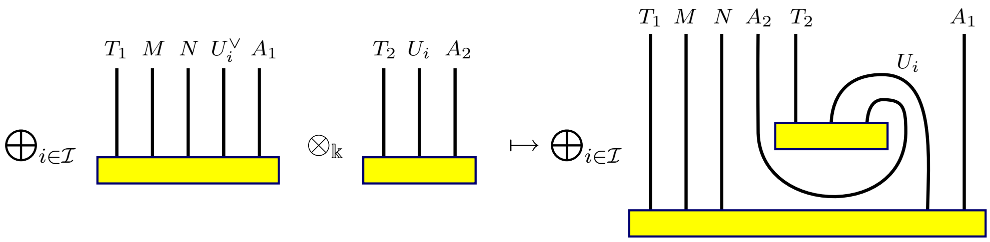
It is transformed into the standard marking on S_6 by the LTG-moves $\mathbf{B}^{-1}_{A_1,A_2} \circ \mathbf{B}^{-1}_{A_1,T_2}$. Now we need to translate this into a natural isomorphism

$$\begin{aligned} &\bigoplus_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, T_1MNU_i^\vee A_1) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, T_2U_iA_2) \\ \xrightarrow{\cong} &\bigoplus_{j\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, T_1MNA_1A_2U_j^\vee) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, T_2U_j) \end{aligned} \tag{3.54}$$

Removing the cuts in the markings (3.52) on $S_4(1, g, g, 1; 1, 1, 1, 1)$ amounts to the isomorphism

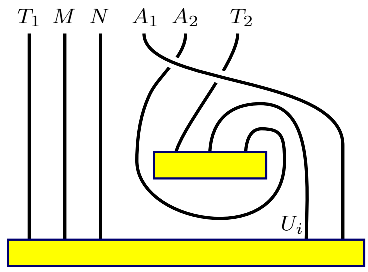
$$\bigoplus_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, T_1MNU_i^\vee A_1) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, T_2U_iA_2) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, T_1MNA_2T_2A_1) \tag{3.55}$$

which is given by the generalized $\mathbf{F}$-move:



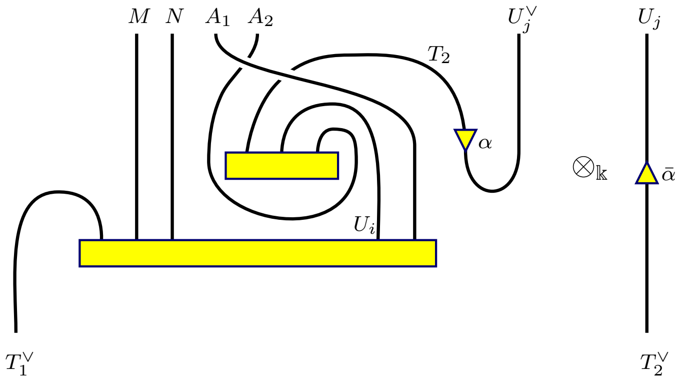
(3.56)

Then applying the LTG-moves $\mathbf{B}^{-1}_{A_1,A_2} \circ \mathbf{B}^{-1}_{A_1,T_2}$ gives

$\bigoplus_{i\in\mathcal{I}}$ 

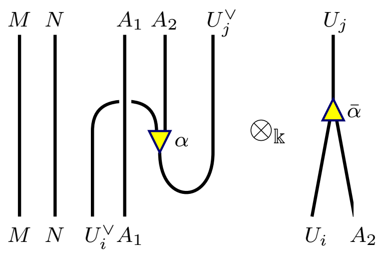
(3.57)

to which we now apply the adjunction (3.23). After applying duality morphisms, we obtain the following ribbon graph for the morphism on the right hand side of (3.54):

$\bigoplus_{i,j\in\mathcal{I}}\sum_\alpha$ 

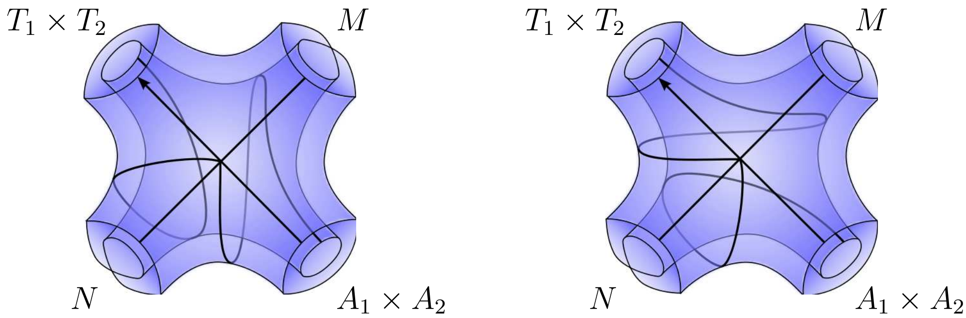
(3.58)

The sum is taken over a basis of $\mathrm{Hom}_\mathcal{C}(U_j, T_2^\vee)$ and the corresponding dual basis of $\mathrm{Hom}_\mathcal{C}(T_2^\vee, U_j)$. It is obvious that the morphism does not depend on the choice of basis. To use again the Yoneda lemma, we evaluate the corresponding natural transformation on $T_1^\vee = MNU_i^\vee A_1$ and $T_2^\vee = U_iA_2$ for the identity. We get the following explicit formula for the associativity constraint:

$\alpha_{M,N,A_1\times A_2} = \bigoplus_{i,j\in\mathcal{I}}\sum_\alpha$ 

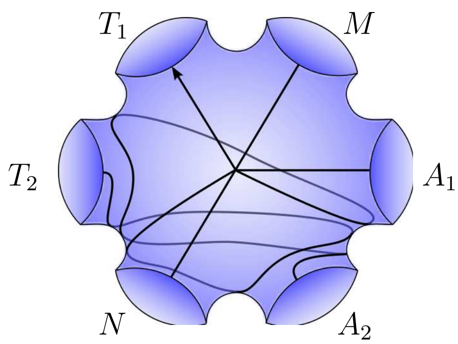
(3.59)

For the associativity constraint $\alpha_{M,A_1\times A_2,N}$ gluing $S_4(1,g,1,g;1,1,1,1)$ over the four-punctured sphere S_4 gives, after removing the cuts, the graphs



(3.60)

This results in the following marking on the 6-punctured sphere S_6:



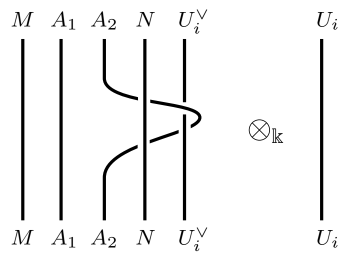
(3.61)

This marking is transformed into the standard graph on S_6 by the following chain of LTG-moves:

$$\mathbf{B}^{-1}_{T_2T_1MA_1,A_2} \circ \mathbf{B}^{-1}_{NA_2,T_2} \circ \mathbf{B}_{N,T_2} \circ \mathbf{B}_{T_1MA_1,A_2} \tag{3.62}$$

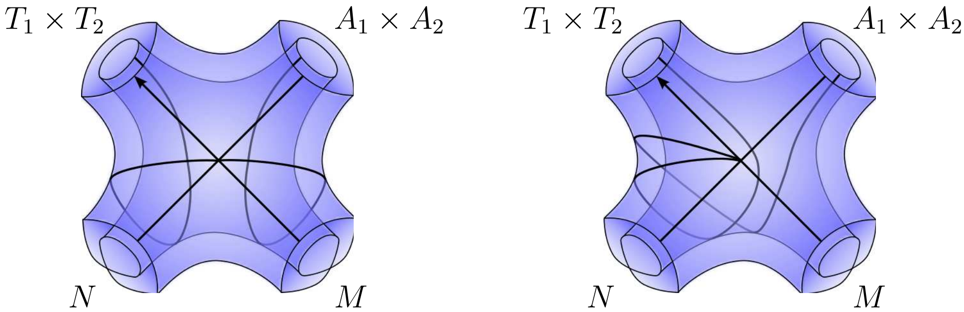
They eventually lead to the associativity isomorphism

$$\alpha_{M,A_1\times A_2,N} = \bigoplus_{i\in\mathcal{I}}$$



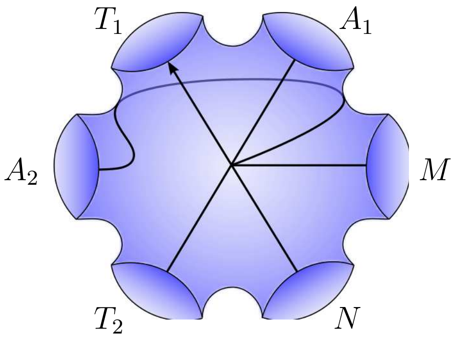
(3.63)

For the associativity constraints $\alpha_{A_1 \times A_2, M, N}$ gluing over S_4 gives the graphs



(3.64)

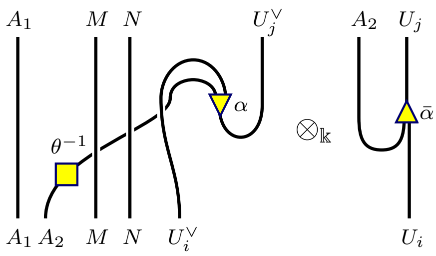
Transforming $S_4(1,1,g,g;1,1,1,1)$ to S_6 gives the following marking on S_6:



(3.65)

The transformation into the standard graph on S_6 is just given by the LTG-move $\mathbf{B}_{T_1 A_1, A_2}$. The procedure outlined before then gives the constraint

$$\alpha_{A_1 \times A_2, M, N} = \bigoplus_{i \in \mathcal{I}} \sum_{\alpha}$$

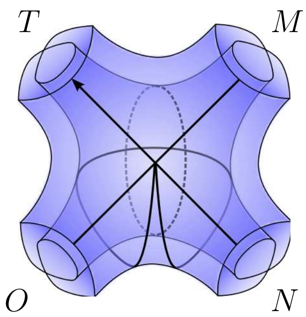


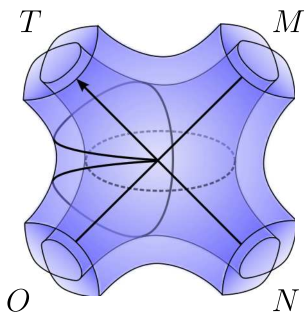
(3.66)

where the α-summation is over a basis of $\mathrm{Hom}_{\mathcal{C}}(U_j, A_2^\vee U_i)$ and the corresponding dual basis of $\mathrm{Hom}_{\mathcal{C}}(A_2^\vee U_i, U_j)$. The ribbon twist enters the morphism during the application of the various **F**- and **Z**-moves and the adjunction (3.23).

Three objects in the twisted component

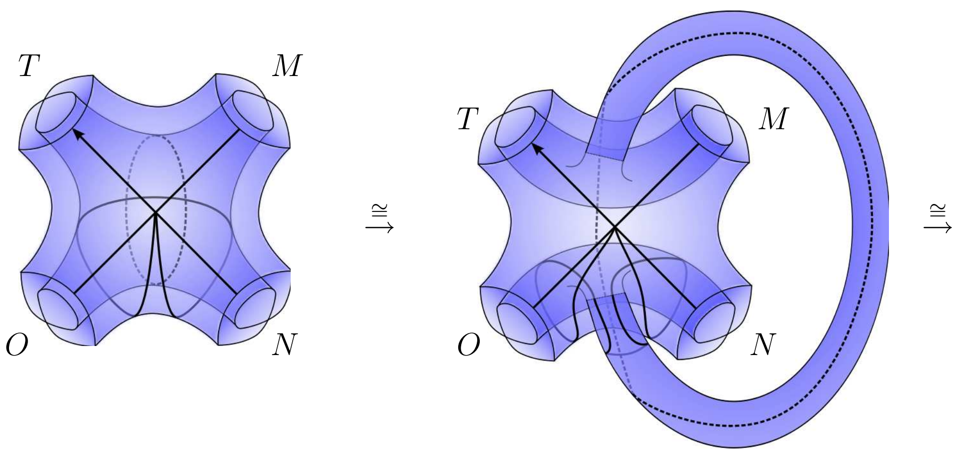
The last associativity isomorphism $\alpha_{M,N,O}$ for three objects M, N, O in the twisted component of $\mathcal{C}^{\mathcal{X}}$ is more involved: the total space of the relevant cover ist $S_4(g,g,g,g;1,1,1,1)$, which is a surface of genus one. The two gluing procedures over S_4 give the following markings:

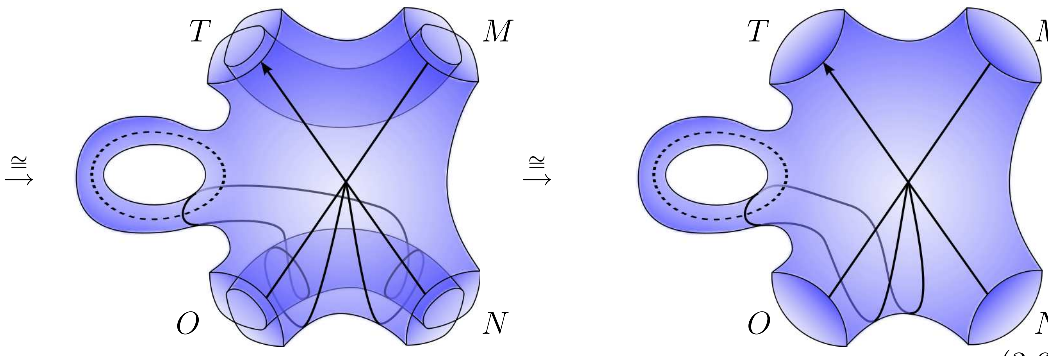




(3.67)

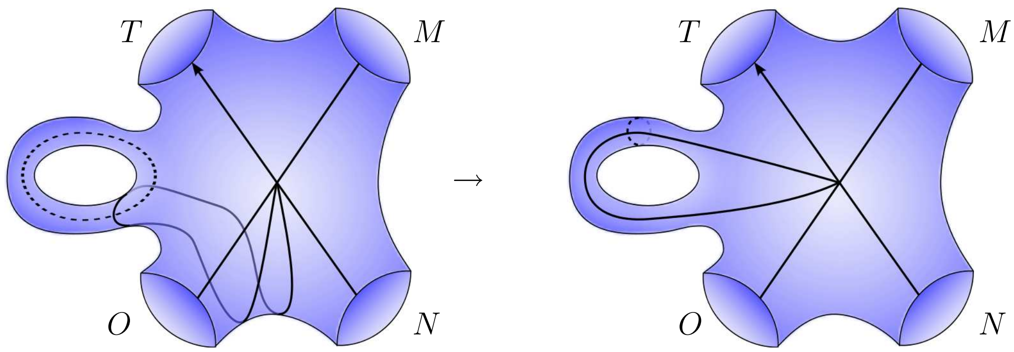
Since the cover is not a genus zero surface any longer, the rules of the LTG do not allow us to remove the cuts and to contract lines of the marking. Hence we keep lines indicating the cuts which are drawn with dotted lines. Transforming $S_4(g,g,g,g;1,1,1,1)$ into a standard block along the second marking gives the following surface of genus one with four boundary components:





(3.68)

The transformation



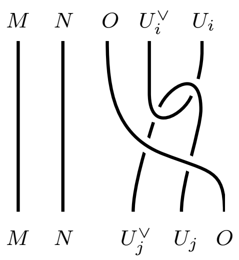
(3.69)

of the resulting marking into the standard marking on the four-holed torus is given by the following sequence of LTG-moves:

$$\mathbf{S}^{-1} \circ \mathbf{B}_{TMN,\mathcal{R}^{(1)}} \circ \mathbf{B}^{-1}_{O,\mathcal{R}^{(2)}} \tag{3.70}$$

Translating these moves into a morphism in the modular tensor category $\mathcal{C}$ gives

$$\alpha_{M,N,O} = \quad \bigoplus_{i,j\in\mathcal{I}} \frac{d_i}{\mathcal{D}}$$



(3.71)

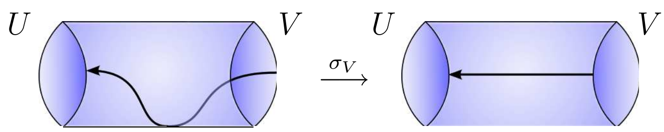
with $\mathcal{D} = \sqrt{p^+p^-}$ and $p^\pm$ defined as in section 2.2. (Note that the morphism corresponding to the $\mathbf{S}$-move in [BK01] is defined by a direct sum of morphisms $U_l U_l^\vee \to U_k U_k^\vee$; we have inserted two pairs of isomorphisms identifying for a simple object $U_k \cong U_{k^*}^\vee$ which cancel pairwise.)

These associativity constraints have to satisfy mixed pentagon axioms for any choice of four objects in the two sectors of $\mathcal{C}^{\mathcal{X}}$, yielding in total 16 different types of pentagon diagrams. Theorem 2.12 asserts that our construction yields a G-equivariant modular functor; theorem 1.22 then ensures that all associativity constraints obtained in this section satisfy the pentagon axiom. We have checked this by hand as well; only the pentagon with four objects in the twisted component is more involved.

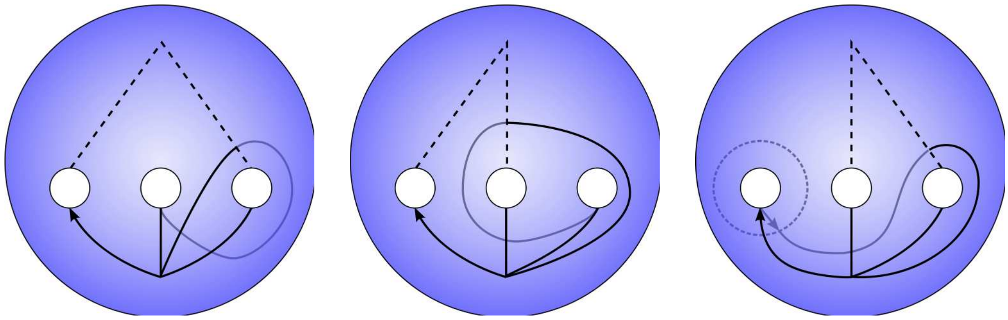
3.1.5 Tensoriality of the $\mathbb{Z}/2$-action

We next derive the isomorphisms $\varphi_{A,B} :\ {}^h(A \otimes B) \to {}^hA \otimes {}^hB$ that turn the equivariance functor R_h into a tensor functor. They have been described in general before lemma 1.23. The functor R_1 is the identity functor and the tensoriality constraints $\varphi^1_{A,B}$ are identity morphisms. The non-trivial element $g \in \mathbb{Z}/2$ acts by permutation of factors on the untwisted component $\mathcal{C} \boxtimes \mathcal{C}$ and as the identity on the twisted component $\mathcal{C}$. Hence we only compute the tensoriality constraint $\varphi^g_{A,B}$.

Before we proceed, we will have a look at the S_2 cover of the form $(S_2(g,g;1,g) \to S_2)$, together with its marking. As a smooth manifold the total space $S_2(g,g;1,g)$ is diffeomorphic to $S_2(g,g;1,1)$ by a half turn around the second hole. However this is not a map of marked covers over S_2. We choose a half turn diffeomorphism (say by rotating the second circle clockwise by π) to identify the total spaces $S_2(g,g;1,g)$ and $S_2(g,g;1,1)$. Both spaces are diffeomorphic to the two-punctured sphere S_2 and by the axioms of the $\mathcal{C}$-extended modular functor τ, the corresponding diffeomorphism of S_2 induces a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, UV) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, U\,{}^gV)$. Using the duality, we get an isomorphism $\sigma_V : V \to {}^gV = V$. We use this isomorphism to identify the respective hom-spaces. On marking graphs this introduces an additional move which we call the σ-move,



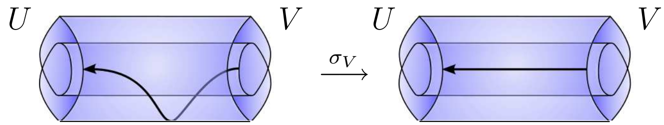
(3.72)



$(A_1 \times A_2) \otimes M$ $\quad$ $M \otimes (A_1 \times A_2)$ $\quad$ $M \otimes N$

Fig. 3.1: The marking graphs on S_3-covers that represent the non-trivial tensor products. We show three Riemann spheres $\overline{\mathbb{C}}$ from above. The arrow points to the disc where the test object T is inserted. All covers are twofold; their total space has the topology of a four-holed sphere. The dashed line is a branch cut linking the two insertions with non-trivial monodromy and indicates a self-intersection in the immersion of the total space of the cover into three-dimensional space.

on the two-punctured sphere S_2 and

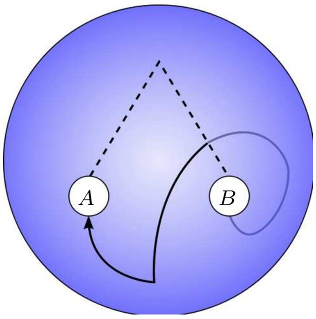


(3.73)

on the total space $S_2(g, g; 1, g)$ of the cover. Applying the half-turn diffeomorphism twice is the Dehn-twist, hence we have the identity $\sigma_V^2 = \theta_V \in \mathrm{Hom}_{\mathcal{C}}(V, V)$.

From now on we will draw the standard spheres S_n as the one-point compactification $\overline{\mathbb{C}}$ of the complex plane with n discs of radius $1/3$ centered at $1, 2, \ldots n$ removed like in definition 1.5. As in subsection 1.1.2, the standard blocks are obtained by identifying the trivial cover of the cut sphere along the cuts, i.e. as $(S_n \backslash \text{cuts} \times G)/\sim$. For instance the marking graphs on covers of S_3 that represent the tensor products are shown in figure 3.1.

We are now ready to compute the tensoriality constraint $\varphi^g_{A,B}$. We apply the σ-move introduced in picture (3.73) to the relevant cover of S_2 obtain the marking graph



$$(3.74)$$

on $S_2(g,g;1,g)$. Then we apply the sequence of transformations in (1.57) on $S_2(g,g;1,g)$ and translate them into morphisms, which have to be applied after the morphism σ.

We outline our procedure to determine the tensoriality constraint $\varphi^g_{A,B}$:

1. Determine the cover of the two-punctured sphere that is appropriate for the pair of objects (A,B).

2. If the tensor product takes its value in the twisted component, $A \otimes B \in \mathcal{C}^{\mathcal{X}}_g$, use the half-twist σ to identify the covers $S_2(g,g;1,1)$ and $S_2(g,g;1,g)$.

3. Apply the diffeomorphism $\tilde{g}$ from equation (1.54) to the cover.

4. Now we can glue in the cover of the three-punctured sphere S_3 that implements the tensor product $A \otimes B$.

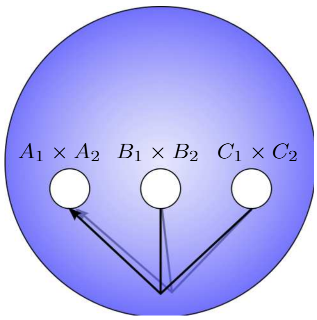
5. Apply the diffeomorphism $\widetilde{g^{-1}} = \tilde{g}$ again.

6. Transform the resulting cover of S_3 back to a standard block using the marking from the tensor product ${}^gA \otimes {}^gB$.

7. Read off the LTG-moves that transform the resulting marking into the standard marking. If $A \otimes B$ is in the untwisted component, this is the tensoriality constraint. Otherwise, if $A \otimes B \in \mathcal{C}^{\mathcal{X}}_g$, the tensoriality constraint is obtained by first applying the morphism $\sigma_{A\otimes B}$ to ${}^g(A \otimes B)$ to take into account step 2, and then the LTG-moves.

The choice of the diffeomorphism σ to identify the total spaces $S_2(g, g; 1, 1)$ and $S_2(g, g; 1, g)$ is non-canonical. Since the mapping class group of the cylinder is generated by the Dehn-twist, different choices of identifications differ by powers of the Dehn-twist, which in our conventions is the square of σ. A different choice of the identification of $S_2(g, g; 1, 1)$ and $S_2(g, g; 1, g)$ gives different markings on the cover of S_3 in step 6 of our procedure. As a consequence, the LTG-moves will differ by powers of the ribbon-twist on ${}^g(A \otimes B)$. The difference then cancels in step 7, so that the tensoriality constraint is independent of the choice of identifications of $S_2(g, g; 1, 1)$ and $S_2(g, g; 1, g)$.

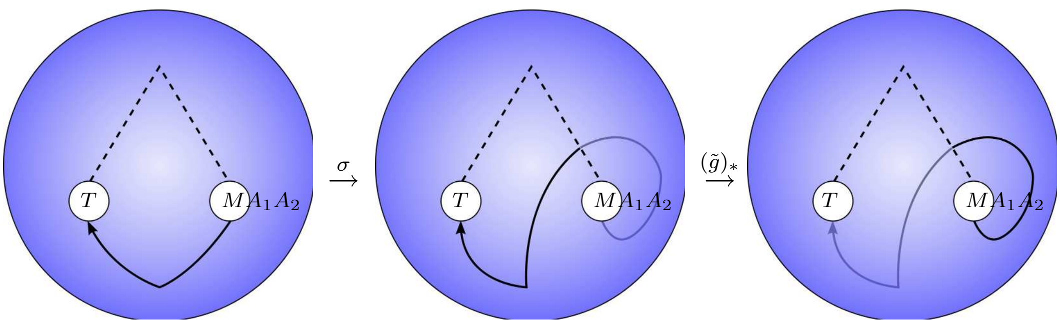
We will start by deriving the tensoriality constraint $\varphi^g_{A_1\times A_2, B_1\times B_2}$ for g on the tensor product of two objects in the untwisted component. Our procedure gives the standard marking

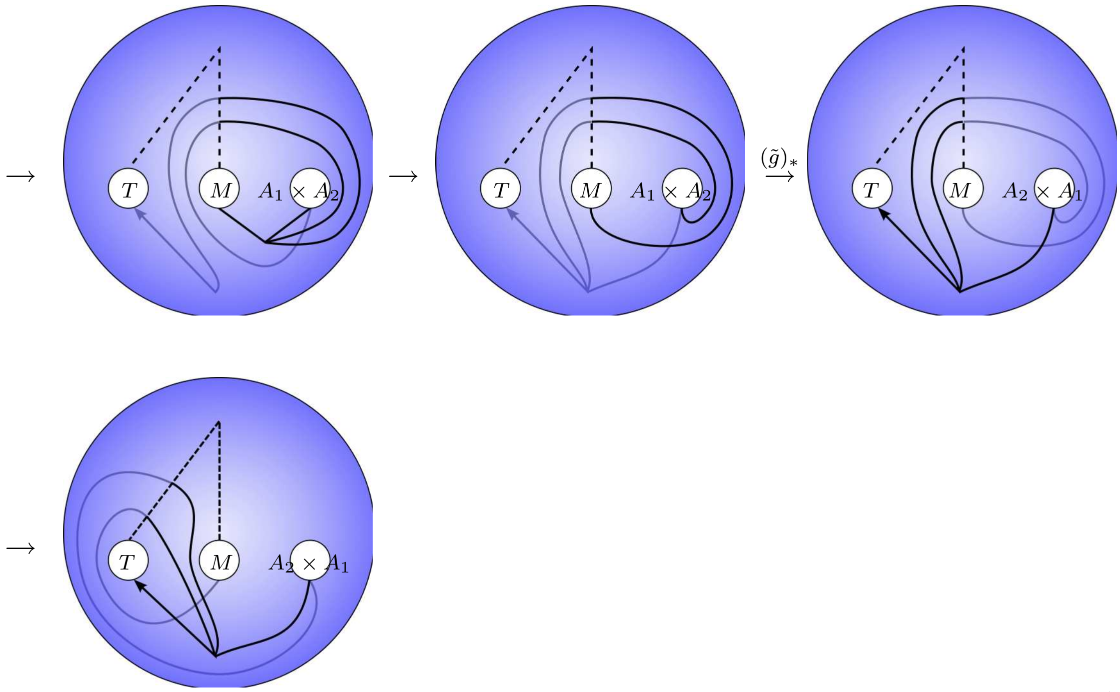


$$(3.75)$$

Hence we deduce that the constraint is the identity, $\varphi^g_{A_1^\times A_2, B_1\times B_2} = \mathrm{id}_{A_2B_2\times A_1B_1}$.

For the constraint $\varphi^g_{M,A_1\times A_2}$, we first have to use σ to identify $S_2(g, g; 1, 1)$ and $S_2(g, g; 1, g)$; then we apply the sequence of operations described in (1.57):



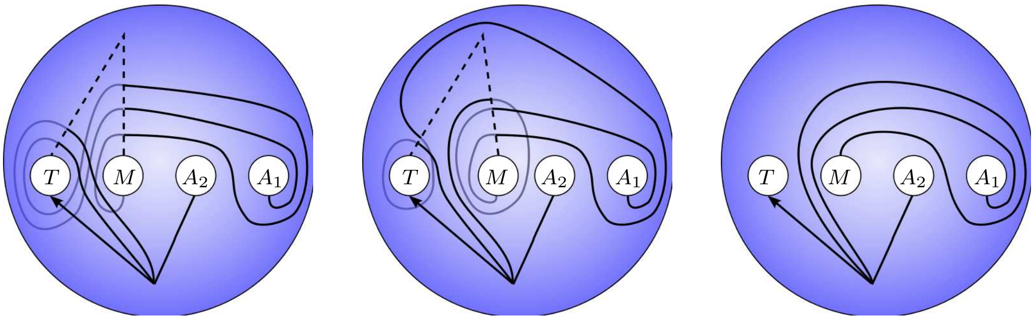


(3.76)

In the first step we apply the half-twist $\sigma_{MA_1A_2}$, in the second step the application of $\tilde{g}$ as defined in (1.54) exchanges the two sheets of the cover. In the third step we glue in the three-puncture sphere $S_3(g, g, 1; 1, 1, 1)$ with the marking representing the tensor product $M \otimes (A_1 \times A_2)$, see figure 3.1. In the fourth step we perform the gluing on the marking graph, i.e. we contract the marking along the factorizing link. In the fifth step we apply $\tilde{g}$ again, which in particular exchanges the holes labelled by A_1 and A_2. The last step is merely a simplification of the graph: we move the lines around the back side of the sphere.

We use the marking on $S_3(g, g, 1; 1, 1, 1)$ in figure 3.1 that represents the tensor product $M \otimes (A_2 \times A_1)$ to get an isomorphism to S_4. This marking instructs us to move the disc labelled by A_1 around the disc labelled by M.

This yields the left figure in the following line:



(3.77)

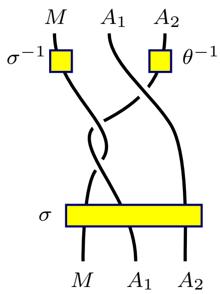
In the second picture we redraw the marking graph in a more convenient shape. The last picture is obtained by a diffeomorphism of non-embedded manifolds.

The marking graph in the third picture of (3.77) is transformed into the standard marking graph on S_4 by the following sequence of LTG-moves:

$$\mathbf{T}_{A_1}^{-1} \circ \sigma_M \circ \mathbf{B}_{MA_2,A_1}^{-1} \circ \mathbf{B}_{A_2,M} \circ \mathbf{B}_{T,M} \tag{3.78}$$

where $\mathbf{T}$ is the Dehn twist move as in section 1.2.2. When we translate the LTG moves into morphisms, several Dehn twists occur in manipulations of the ribbon graphs. They have to be combined with the morphism σ which squares to the twist. This leads to powers of σ that differ from the naive expectations, and we arrive at the tensoriality constraint

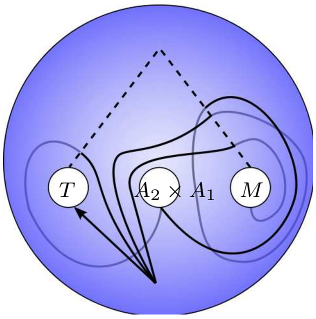
$$\varphi^g_{M,A_1\times A_2} =$$

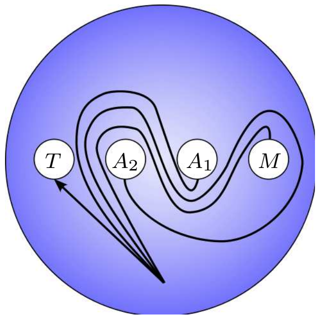


(3.79)

where we have included $\sigma_{MA_1A_2}$ according to our general prescription, step 7.

For tensoriality constraint $\varphi^g_{A_1\times A_2,M}$, we proceed in a similar way. Again we identify the total spaces of the covers $S_2(g,g;1,1)$ and $S_2(g,g;1,g)$ with the diffeomorphism σ and apply (1.57). This gives:





(3.80)

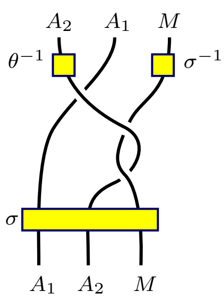
The first picture is the result of (1.57). The second picture is the result of transforming $S_3(g, 1, g; 1, 1, 1)$ to S_4 using the marking on $S_3(g, 1, g; 1, 1, 1)$ that represents the tensor product ${}^g(A_1 \times A_2) \otimes {}^gM = (A_2 \times A_1) \otimes M$.

The resulting marking graph is transformed into the standard graph by the sequence

$$\mathbf{B}^{-1}_{A_2,A_1} \circ \mathbf{B}^{-1}_{A_2,M} \circ \mathbf{T}^{-1}_{A_2} \circ \mathbf{B}^{-1}_{M,A_2} \circ \sigma^{-1}_M \tag{3.81}$$

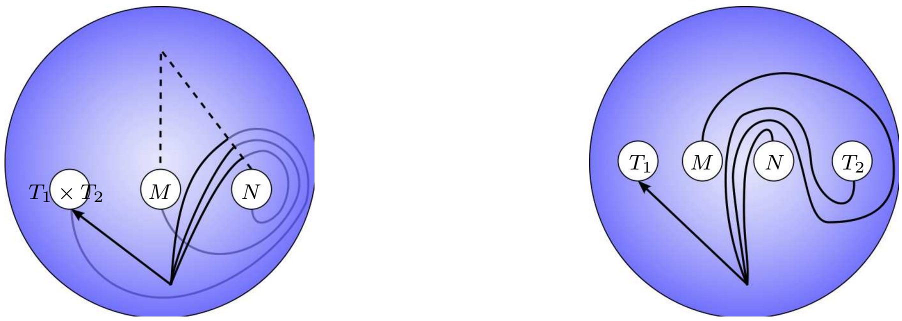
of LTG-moves. This gives the morphism

$\varphi^g_{A_1\times A_2,M} =$



(3.82)

The last tensoriality constraint to be determined is $\varphi^g_{M,N}$. We apply (1.57) to $S_2(1,1;1,1)$ and get



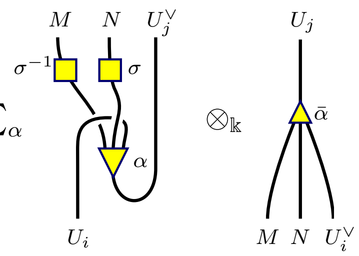
(3.83)

The first picture is again the result of (1.57), the second picture is the marking graph we obtain on S_4. As usual we glued all occurring cuts. Now this marking is transformed into the standard marking by the LTG-moves

$$\mathbf{B}_{NT_2,M} \circ \mathbf{B}^{-1}_{N,T_2} \circ \mathbf{B}^{-1}_{N,M} \circ \sigma_N^{-1} \circ \sigma_M \tag{3.84}$$

Recall that ${}^g(M \otimes N) = \bigoplus U_i \times MNU_i^\vee$. Hence, we have to apply the corresponding transformations to $\mathrm{Hom}_\mathcal{C}(\mathbf{1}, T_1U_i) \otimes_\Bbbk \mathrm{Hom}_\mathcal{C}(\mathbf{1}, T_2MNU_i^\vee) \cong \mathrm{Hom}_\mathcal{C}(\mathbf{1}, T_1T_2MN)$ and finally use the adjunction (3.23), that resembles the application of the $\mathbf{F}$-move, to take the suppressed cuts in equation (3.83) into account. We obtain

$$\varphi^g_{M,N} = \quad \bigoplus\nolimits_{i,j\in\mathcal{I}} \sum\nolimits_\alpha$$



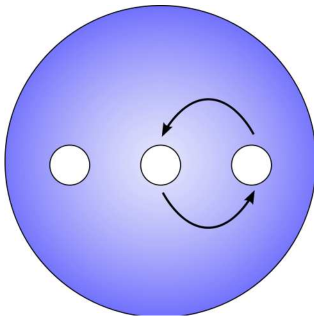
(3.85)

where the summation over α runs over a basis of $\mathrm{Hom}_\mathcal{C}(U_j, MNU_i^\vee)$ and the corresponding dual basis. Again, we had to take into account Dehn twists which shifted the powers of σ.

We have checked directly that all morphisms derived indeed satisfy all identities needed to endow the functor R_g with the structure of a tensor functor.

3.1.6 The braiding

We finally derive the braiding on the $\mathbb{Z}/2$-equivariant category $\mathcal{C}^{\mathcal{X}}$ which consists of isomorphisms $C_{U,V} : U \otimes V \xrightarrow{\cong} {}^{p}V \otimes U$ with $U \in \mathcal{C}^{\mathcal{X}}_p$ and $V \in \mathcal{C}^{\mathcal{X}}_q$. Recall from section 1.2.4 the braiding diffeomorphism



(3.86)

of the three-holed sphere S_3. We lift this morphism to appropriate covers and obtain a diffeomorphism

$$\tilde{\varphi}_{\mathbf{B}} : S_3((pq)^{-1}, p, q; 1, 1, 1) \xrightarrow{\cong} S_3((pq)^{-1}, pqp^{-1}, p; 1, p^{-1}, 1) \ . \tag{3.87}$$

As in theorem 1.22 it induces for any object $T \in \mathcal{C}^{\mathcal{X}}$ a natural isomorphism

$$\begin{aligned}&\langle T, U \otimes V\rangle_{\mathcal{X}} \overset{\text{def}}{=} \langle T, U, V\rangle_{\mathcal{X}} \overset{\text{def}}{=} \tau^{\mathcal{X}}(S_3((pq)^{-1}, p, q; 1, 1, 1); T, U, V) \overset{(\tilde{\varphi}_{\mathbf{B}})_*}{\longrightarrow} \\ &\tau^{\mathcal{X}}(S_3((pq)^{-1}, pqp^{-1}, p; 1, p^{-1}, 1); T, V, U) = \\ &\tau^{\mathcal{X}}(S_3((pq)^{-1}, pqp^{-1}, p; 1, 1, 1); T, {}^{p}V, U) \overset{\text{def}}{=} \langle T, {}^{p}V, U\rangle_{\mathcal{X}} \overset{\text{def}}{=} \langle T, {}^{p}V \otimes U\rangle_{\mathcal{X}}.\end{aligned} \tag{3.88}$$

Thus the procedure to determine braidings is analogous to the one to determine the associativity constraints:

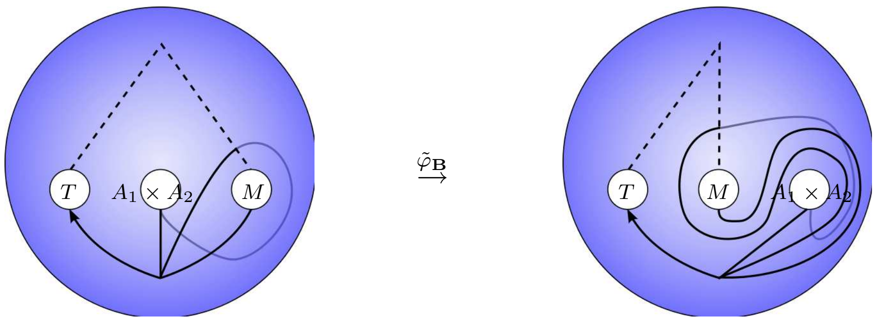
1. Start with the standard marking on the cover $S_3((pq)^{-1}, p, q; 1, 1, 1)$ that represents the tensor product $U \otimes V$.
2. Apply the diffeomorphism $\tilde{\varphi}_{\mathbf{B}}$ from (3.87).
3. The result is the cover $S_3((pq)^{-1}, pqp^{-1}, p; 1, p^{-1}, 1)$ and has to be transformed into a standard block, using the marking representing the tensor product ${}^{p}V \otimes U$.
4. Next use the LTG-moves to transform the resulting marking graph on the standard block into the standard marking graph.

5. Finally translate the LTG-moves into morphisms in $\mathcal{C}$ or $\mathcal{C}\boxtimes\mathcal{C}$, respectively.

Not surprisingly, the braiding of two objects $A_1 \times A_2$ and $B_1 \times B_2$ in the neutral component of $\mathcal{C}^{\mathcal{X}}$ turns out to be the braiding on $\mathcal{C}\boxtimes\mathcal{C}$. To see this, we lift the braiding diffeomorphism $\varphi_{\mathbf{B}}$ to $S_3(1,1,1;1,1,1)$. As a smooth manifold, this is just isomorphic to the disjoint union $S_3 \sqcup S_3$ of two three-holed spheres. The lift $\tilde{\varphi}_{\mathbf{B}}$ of the braiding isomorphism is just $\varphi_{\mathbf{B}}$ applied to both components. Hence

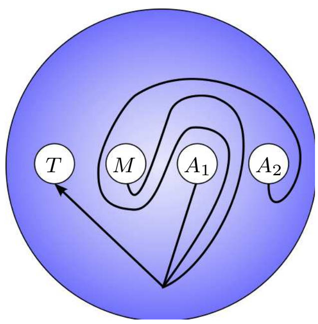
$$C_{A_1\times A_2, B_1\times B_2} = c_{A_1,B_1} \otimes_{\Bbbk} c_{A_2,B_2} \quad . \tag{3.89}$$

We now discuss the more complicated situations. For the braiding $C_{A_1\times A_2,M}$: $(A_1\times A_2)\otimes M \to M\otimes(A_1\times A_2)$, we have to consider the cover $S_3(g,1,g;1,1,1)$ of S_3 with its standard marking graph. Lifting the braiding $\varphi_{\mathbf{B}}$ gives the marking



(3.90)

on $S_3(g,g,1;1,1,1)$. Applying the diffeomorphism to S_4 that represents the tensor product ${}^1M \otimes (A_1 \times A_2) = M \otimes (A_1 \times A_2)$ gives the marking

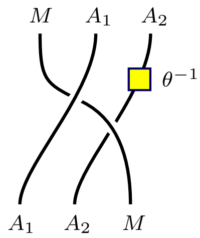


(3.91)

on S_4. It is connected to the standard marking of the four-punctured sphere S_4 by the following sequence of LTG-moves:

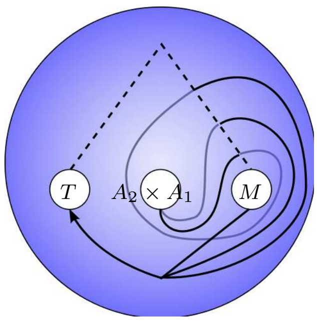
$$\mathbf{B}_{A_1,M} \circ \mathbf{B}_{T,A_2} \circ \mathbf{B}_{A_1,A_2} \tag{3.92}$$

Applied to $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, TA_1A_2M)$, this induces the following braiding isomorphism on $\mathcal{C}^{\mathcal{X}}$

$C_{A_1 \times A_2, M} =$ 

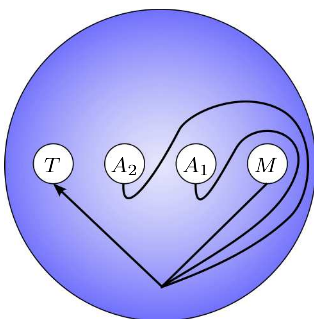
(3.93)

For the braiding $C_{M,A_1\times A_2} : M \otimes (A_1 \times A_2) \to {}^g(A_1 \times A_2) \otimes M = (A_2 \times A_1) \otimes M$ we consider the cover $S_3(g, g, 1; 1, 1, 1)$ of S_3 with its standard marking graph. We lift the braiding $\varphi_{\mathbf{B}}$ and get the marking



(3.94)

on $S_3(g, 1, g; 1, g, 1)$. Note that the insertions of A_1 and A_2 have exchanged the sheet, when they were moved pass the self-intersection. We apply the diffeomorphism to S_4 given by the marking on $S_3(g, 1, g; 1, g, 1)$ that represents the tensor product ${}^g(A_1 \times A_2) \otimes M = (A_2 \times A_1) \otimes M$. The result is the following marking on S_4:



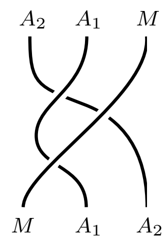
(3.95)

This marking on S_4 is transformed into the standard marking on S_4 by the LTG-moves

$$\mathbf{B}_{A_1 M, A_2} \circ \mathbf{B}_{M, A_1}. \tag{3.96}$$

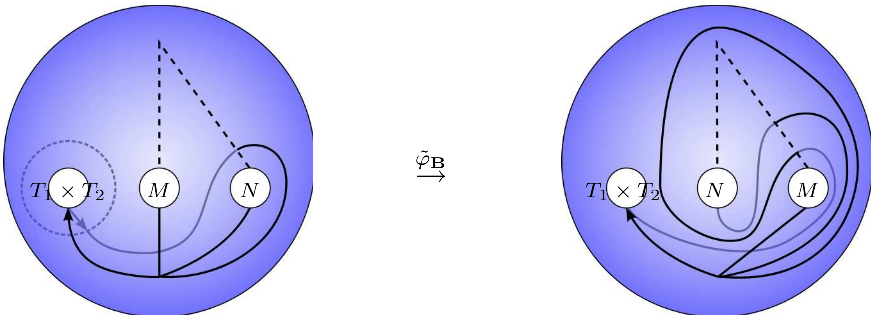
We get the braiding isomorphism on $\mathcal{C}^{\mathcal{X}}$:

$$C_{M, A_1 \times A_2} =$$



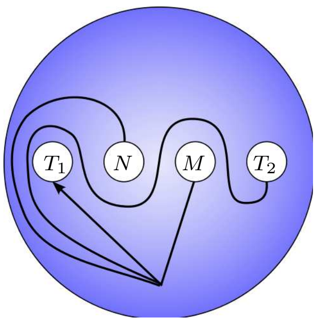
(3.97)

Finally we describe the braiding isomorphism $C_{M,N} : M \otimes N \to {}^g N \otimes M = N \otimes M$. We lift the braiding $\varphi_{\mathbf{B}}$ to $S_3(1, g, g; 1, 1, 1)$. The standard marking on $S_3(1, g, g; 1, 1, 1)$ involves a cut: we first remove this cut and then apply $\tilde{\varphi}_{\mathbf{B}}$:



(3.98)

We transform the resulting manifold into S_4 and arrive at

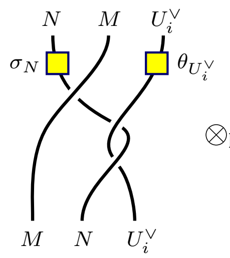


(3.99)

This marking is transformed into the standard marking on S_4 by the following sequence of moves:

$$\mathbf{B}^{-1}_{T_1,N} \circ \mathbf{B}^{-1}_{M,T_2} \circ \mathbf{B}^{-1}_{T_1,T_2} \circ \sigma^{-1}_N \tag{3.100}$$

When we apply this to $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, T_1MNT_2)$ and perform the gluing process in the adjunction (3.19), we arrive at

$$C_{M,N} = \bigoplus_{i\in\mathcal{I}}$$  $$\otimes_{\Bbbk}$$ 

(3.101)

Again powers of σ are changed by taking into account Dehn twists.

By theorem 2.12, our construction yields a G-equivariant modular functor, and by theorem 1.22, braiding morphisms obtained from a G-equivariant modular functor satisfy $\mathbb{Z}/2$-equivariant generalizations of the hexagon axioms. We have also directly verified that the morphisms presented in this subsection satisfy the hexagon axioms.

3.1.7 Equivariant ribbon structure

We now return to study the existence of a $\mathbb{Z}/2$-equivariant ribbon structure on $\mathcal{C}^{\mathcal{X}}$. The general results of [KP08] ensure that $\mathcal{C}^{\mathcal{X}}$ has a twist but do not guarantee the existence of duality morphisms like the left evaluation $\tilde{D}_U : U \otimes U^* \to \mathbf{1}$. However, in the case of $\mathbb{Z}/2$-permutation equivariant categories, there are indeed compatible duality morphisms that endow $\mathcal{C}^{\mathcal{X}}$ with a ribbon structure.

By theorem 1.22 we obtain the twist morphism $\Theta_U : U \to {}^pU$ for $U \in \mathcal{C}^{\mathcal{X}}_p$ by the Yoneda lemma from a natural transformation of functors

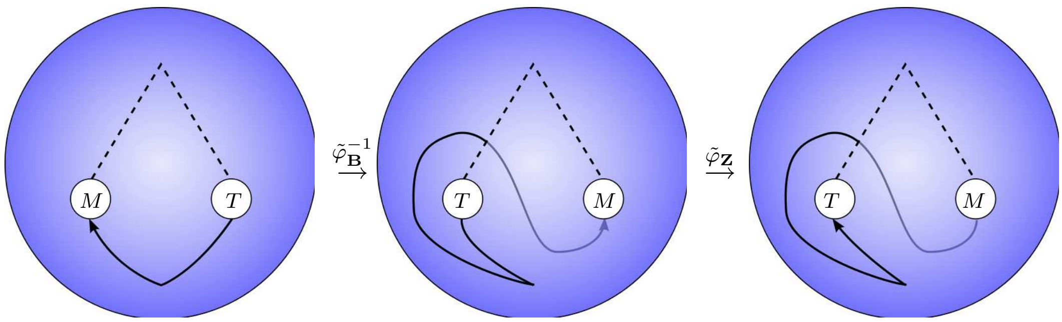
$$\begin{aligned}
\tau^{\mathcal{X}}(S_2(p,p^{-1};1,1);U,T) &\overset{(\tilde{\varphi}_{\mathbf{B}}^{-1})_*}{\longrightarrow} \tau^{\mathcal{X}}(S_2(p^{-1},p;1,p^{-1});T,U) \\
&\overset{T_{(1,p)}}{=} \tau^{\mathcal{X}}(S_2(p^{-1},p;1,1);T,{}^pU) \\
&\overset{(\tilde{\varphi}_{\mathbf{Z}})_*}{\longrightarrow} \tau^{\mathcal{X}}(S_2(p,p^{-1};1,1);{}^pU,T) \ .
\end{aligned} \tag{3.102}$$

Here $\varphi_{\mathbf{B}}$ is the braiding of the two holes of S_2 and $\tilde{\varphi}_{\mathbf{B}}$ its lift to the cover $S_2(p^{-1}, p; 1, p^{-1})$. The equivariance morphism $T_{(1,p)}$ is, in our case, the identity. Finally, $\varphi_{\mathbf{Z}}$ is the diffeomorphism of the standard sphere inducing a cyclic move of the distinguished edge, and $\tilde{\varphi}_{\mathbf{Z}}$ is its lift.

For an object $U = A_1 \times A_2$ in the untwisted component of $\mathcal{C}^{\mathcal{X}}$, the total space $S_2(1,1;1,1)$ of the cover is again just a disjoint union of two copies of S_2 with the lifts of $\varphi_{\mathbf{B}}^{-1}$ and $\varphi_{\mathbf{Z}}$ being the application of the diffeomorphisms to both components separately. Hence the twist on $\mathcal{C}_1^{\mathcal{X}}$ is just the usual twist on the category $\mathcal{C} \boxtimes \mathcal{C}$:

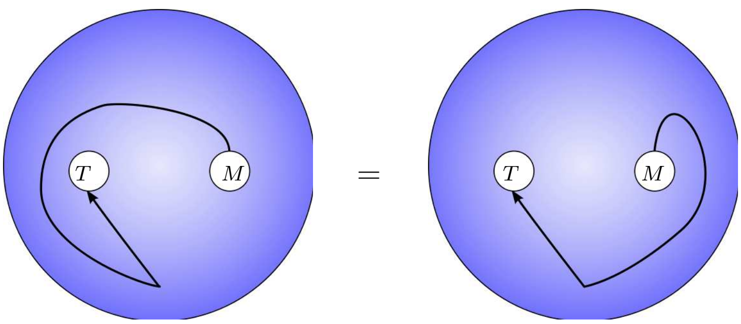
$$\Theta_{A_1 \times A_2} = \theta_{A_1} \otimes_{\Bbbk} \theta_{A_2} \tag{3.103}$$

To calculate the twist morphism for an object $U = M \in \mathcal{C}_g^{\mathcal{X}}$ in the twisted component, our moves amount to



(3.104)

Transforming the total space of this cover into the standard sphere S_2 gives the following marking:



(3.105)

Here, we redrew the figure by pulling the line connecting to M along the backside of the sphere. The last figure is transformed into the standard marking on S_2 by applying the half turn move σ_M. Hence we find

$$\Theta_M = \sigma_M \ . \tag{3.106}$$

Permutation orbifolds of rational conformal field theories have been analyzed with representation theoretic tools in [BHS97]. The formula (3.106) for the twist in the twisted component is in full agreement with formula [BHS97, (4.21)] for the conformal weights.

Up to this point, we have derived all structure on $\mathcal{C}^{\mathcal{X}}$ from the approach of $\mathbb{Z}/2$-equivariant modular functors [KP08]. In fact, we have fully exploited this ansatz and obtained the structure of a weakly rigid $\mathbb{Z}/2$-equivariant monoidal category. We next check that the condition in proposition 1.24 that ensures that the tensor category $\mathcal{C}^{\mathcal{X}}$ is even rigid is satisfied. The criterion is easy to check for simple objects $U_i \times U_j$ in the untwisted component of $\mathcal{C}^{\mathcal{X}}$. In this case the morphism

$$i_{U_i \times U_j} : \mathbf{1} \times \mathbf{1} \to U_i U_i^\vee \times U_j U_j^\vee \tag{3.107}$$

is just given by the tensor product of two right coevaluations of $\mathcal{C}$,

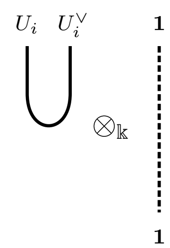
$$i_{U_i \times U_j} = b_{U_i} \otimes_{\Bbbk} b_{U_j} \ . \tag{3.108}$$

The morphisms $a_{i \times j}$ in proposition 1.24 are non-zero by rigidity of $\mathcal{C} \boxtimes \mathcal{C}$. The ribbon structure on the neutral component $\mathcal{C}_1^{\mathcal{X}}$ is then just the usual ribbon structure on $\mathcal{C} \boxtimes \mathcal{C}$.

Now we look at a simple object U_i in $\mathcal{C}_g^{\mathcal{X}}$. As we have seen before, the dual object of U_i in $\mathcal{C}^{\mathcal{X}}$ is $U_i^\vee$. The adjunction (3.23) with $T_1 = T_2 = \mathbf{1}$ and $N = U_i^\vee$ gives isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(U_i^\vee, U_i^\vee) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, U_i U_i^\vee) \cong \mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{C}}(\mathbf{1} \times \mathbf{1}, R(U_i U_i^\vee)) \tag{3.109}$$

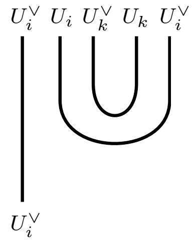
Evaluating the isomorphism on $\mathrm{id}_{U_i^\vee}$ gives a morphism in $\bigoplus_{j \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{C}}(\mathbf{1} \times \mathbf{1}, U_i U_i^\vee U_j^\vee \times U_j)$ (the right coevaluation) whose only non-vanishing component appears for $j = 0$. It reads

$i_{U_i} :=$ 

(3.110)

Now compute $\alpha^{-1}_{U_i^\vee,U_i,U_i^\vee} \circ (\mathrm{id}_{U_i^\vee} \otimes i_{U_i})$ using equation (3.71) for the associativity constraint:

$$\alpha^{-1}_{U_i^\vee,U_i,U_i^\vee} \circ (\mathrm{id}_{U_i^\vee} \otimes i_{U_i}) = \quad \bigoplus\nolimits_{k\in\mathcal{I}} \frac{d_k}{\mathcal{D}}$$



(3.111)

Here d_k is the dimension of the simple object U_k and $\mathcal{D}$ is the dimension of the category $\mathcal{C}$ introduced in (2.11). The morphism

$$a_i : \mathbf{1} \times \mathbf{1} \to \bigoplus_k U_i^\vee U_i U_k^\vee \times U_k$$

from in proposition 1.24 can only have a non-vanishing component for $k = 0$,

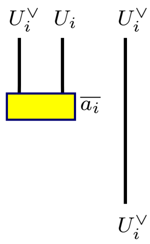
$$a_i^{(0)} : \mathbf{1} \times \mathbf{1} \to U_i^\vee U_i \times \mathbf{1} \; .$$

Since the tensor unit $\mathbf{1}$ is absolutely simple, i.e. $\mathrm{End}(\mathbf{1}) = \Bbbk\, \mathrm{id}_\mathbf{1}$, this component $a_i^{(0)}$ is of the form

$$a_i^{(0)} = \overline{a_i} \otimes_\Bbbk \mathrm{id}_\mathbf{1}$$

with $\overline{a_i} \in \mathrm{Hom}_\mathcal{C}(\mathbf{1}, U_i^\vee U_i)$. We then have

$$\alpha^{-1}_{U_i^\vee,U_i,U_i^\vee} \circ (\mathrm{id}_{U_i^\vee} \otimes i_{U_i}) =$$

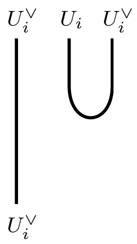


$$+ \sum\nolimits_{k\neq 0} \; \ldots$$

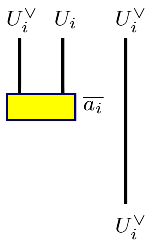
(3.112)

Hence

$$\frac{1}{\mathcal{D}}$$

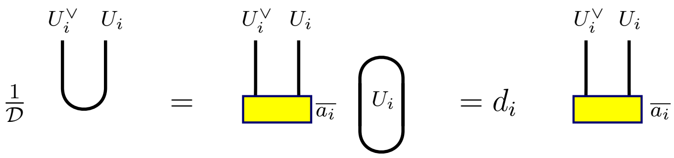


$$=$$



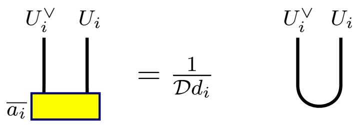
(3.113)

To determine $\overline{a_i}$, we take a partial trace on both sides an arrive at



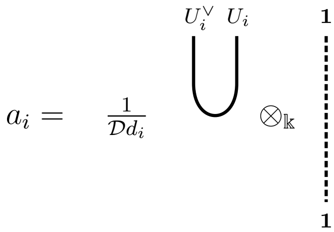
(3.114)

so that



(3.115)

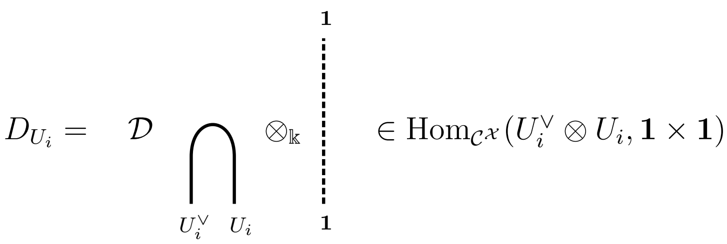
and hence



(3.116)

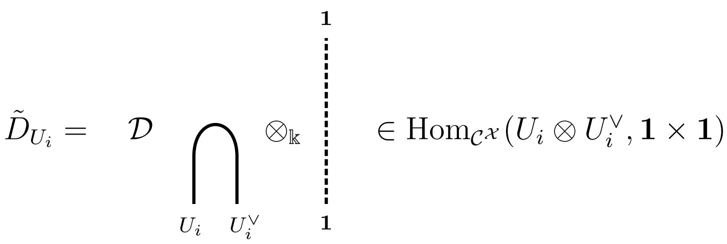
This morphism is non-zero since the left coevaluation of $\mathcal{C}$ is non-zero. Hence the $\mathbb{Z}/2$-permutation equivariant tensor category $\mathcal{C}^{\mathcal{X}}$ is rigid.

The right evaluation morphisms $D_{U_i} : U_i^\vee \otimes U_i \to \mathbf{1} \times \mathbf{1}$ are fixed by the condition $D_{U_i} \circ a_i = \mathrm{id}_{\mathbf{1}\times\mathbf{1}}$. It is easy to see that the right evaluation reads



(3.117)

Similarly we find for the left evaluation



(3.118)

3.2 Module categories from G-equivariant modular functors

In this section we want to find the module category structures that were already mentioned in corollary 2.13. To this end we fix an element $g \in G$.

By theorem 2.12 and corollary 2.13 there is the structure of a G-equivariant monoidal category on $\bigoplus_{h \in G} \mathcal{C}_h^{\mathcal{X}}$ with $\mathcal{C}_1^{\mathcal{X}} = \mathcal{C}^{\boxtimes O_1} = \mathcal{C}^{\boxtimes \mathcal{X}}$ and $\mathcal{C}_h^{\mathcal{X}} = \mathcal{C}^{\boxtimes O_h}$. We first want to find the monoidal structure on the neutral component $\mathcal{C}^{\boxtimes \mathcal{X}}$ and then for every $g \in G$ the module action functor

$$\mathcal{C}^{\boxtimes \mathcal{X}} \times \mathcal{C}^{\boxtimes O_g} \to \mathcal{C}^{\boxtimes O_g} \ .$$

3.2.1 The neutral component

We first briefly turn our attention to the monoidal structure on $\mathcal{C}_1^{\mathcal{X}}$. In this case, all relevant G-covers of extended surfaces are trivial covers. Since the cover functor $\mathcal{F}_{\mathcal{X}}$ maps trivial covers to disjoint unions of copies of the base space, the monoidal structure on $\mathcal{C}_1^{\mathcal{X}}$ is found by evaluating the modular functor τ on disjoint unions of standard n-pointed spheres for appropriate n. The occurring marking graphs are in all cases the standard marking graphs on S_n. Now the following lemma is an easy observation, similar to the computations for the neutral component in section 3.1:

Lemma 3.1. *The weakly ribbon structure on $\mathcal{C}_1^{\mathcal{X}} = \mathcal{C}^{\boxtimes \mathcal{X}}$ induced by the G-equivariant modular functor $\tau^{\mathcal{X}}$ is ribbon (even modular) and is equivalent to the standard ribbon structure on $\mathcal{C}^{\boxtimes \mathcal{X}}$. The tensoriality constraints of the permutation action of G on $\mathcal{C}^{\boxtimes \mathcal{X}}$ are identities.*

3.2.2 The module action functor

The action of $\mathcal{C}_1^{\mathcal{X}}$ on $\mathcal{C}_g^{\mathcal{X}}$ is found by evaluating the G-equivariant modular functor $\tau^{\mathcal{X}}$ on the principal G-cover $(S_3(g^{-1}, 1, g; 1, 1, 1) \to S_3)$ of the three-punctured sphere. In lemma 2.7 the connected components of the total space of the associated bundle $\mathcal{F}_{\mathcal{X}}(S_3(g^{-1}, 1, g; 1, 1, 1) \to S_3)$ where fully described. In this special case we have

Lemma 3.2.
(i) There is a natural bijection between the connected components of

$$\mathcal{F}_{\mathcal{X}}(S_3(g^{-1}, 1, g; 1, 1, 1) \to S_3) = E_{g^{-1};1}$$

and orbits of the G-set $\mathcal{X}$ under the action of the cyclic subgroup $\langle g \rangle \subset G$ of G.

(ii) The restriction of $\mathcal{F}_{\mathcal{X}}(S_3(g^{-1},1,g;1,1,1) \to S_3)$ *to the first boundary is diffeomorphic to the manifold* E_g*, the restriction to the second boundary to* E_1 *and the restriction to the third boundary to* $E_{g^{-1}}$*. Let* o *be a* $\langle g\rangle$*-orbit of* $\mathcal{X}$ *and write* $E^o_{g^{-1};1}$ *for the connected component of* $E_{g^{-1};1}$ *corresponding to the orbit* o*. The boundary components of* $E^o_{g^{-1};1}$ *over the second boundary circle of* S_3 *correspond to those elements of* $\mathcal{X}$*, that are contained in the orbit* o *of the group* $\langle g\rangle$*.* $E^o_{g^{-1};1}$ *has a single boundary component over both the first and third boundary circle of* S_3

(iii) In particular, the number of sheets of the cover $E^o_{g^{-1};1} \to S_3$ *is* $|o|$*.*

By lemma 2.8 the genus of the relevant surface $E_{g^{-1};1}$ is zero.

From now on we restrict our attention to the connected components of $E_{g^{-1};1}$ which by lemma 3.2 is the same as fixing a $\langle g\rangle$-orbit o.

In the definition of the module action functor

$$\mathcal{C}^{\boxtimes\mathcal{X}} \times \mathcal{C}^{\boxtimes O_g} \to \mathcal{C}^{\boxtimes O_g}$$

the connected component $E^o_{g^{-1};1}$ will give a contribution

$$\mathcal{C}^{\boxtimes o} \times \mathcal{C} \to \mathcal{C}\ ;$$

all these contributions are then factor-wise combined to give the full functor. By describing this contribution for all $\langle g\rangle$-orbits separately, we get the full module action functor.

We can thus restrict ourselves to one connected component. Hence we adopt from now on:

Convention 3.3. *G is a cyclic group with generator g. The ordered set $\mathcal{X}$ has a single G-orbit, its smallest element is $x_0 \in \mathcal{X}$. Let $n := |\mathcal{X}|$.*

This convention allows us to simplify notation. In particular we have $\mathcal{X} = \{g^k x_0 | k = 0, \dots, n-1\}$. Note that there is no further assumption on the action of G on $\mathcal{X}$. So $\mathcal{X}$ can be a one-element-set and hence the case $n = 1$ is allowed. From now on we only consider a functor $\mathcal{C}^{\mathcal{X}} \times \mathcal{C} \to \mathcal{C}$.

We will write objects in $\mathcal{C}_1^{\mathcal{X}}$ as $(A_x)_{x\in\mathcal{X}}$ and sometimes use the abbreviation (A_x). The order of $\mathcal{X}$ induces an order on the factors in $\mathcal{C}_1^{\mathcal{X}} = \mathcal{C}^{\boxtimes\mathcal{X}}$. When we draw pictures, we will occasionally write A^k for the factor $A_{g^k x_0}$ to provide a clearer view of the drawing.

Now let $(A_x)_{x\in\mathcal{X}}$ be an object of $\mathcal{C}^{\boxtimes\mathcal{X}}$ and M an object in $\mathcal{C}$. As in theorem 1.22 the tensor product $(A_x)_{x\in\mathcal{X}} \otimes M$ is defined to be the object of $\mathcal{C}$ that represents the functor

$$\begin{aligned} \mathcal{C} &\to \mathcal{V}ect_{\Bbbk} \\ T &\mapsto \tau^{\mathcal{X}}(S_3(g^{-1},1,g;1,1,1) \to S_3; T, (A_x)_{x\in\mathcal{X}}, M) = \tau(E_{g^{-1};1}; T, A_x, M)\ . \end{aligned} \tag{3.119}$$

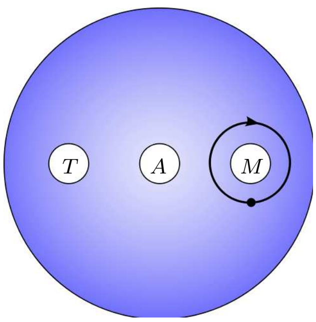
Since the oriented manifold $E_{g^{-1};1}$ is of genus zero and has $n+2$ boundary components, there is a diffeomorphism $E_{g^{-1};1} \cong S_{n+2}$. The choice of such a diffeomorphism induces a natural isomorphism

$$\tau(E_{g^{-1};1}; T, A_x, M) \xrightarrow{\cong} \tau(S_{n+2}; T, A_x, M) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, T \otimes (\bigotimes_{x\in\mathcal{X}} A_x) \otimes M)\,. \tag{3.120}$$

The order of the objects A_x in $\bigotimes_{x\in\mathcal{X}} A_x$ is determined by the diffeomorphism $E_{g^{-1};1} \cong S_{n+2}$. This gives a choice $(\bigotimes_{x\in\mathcal{X}} A_x) \otimes M$ of the object representing the functor (3.119).

To find an appropriate diffeomorphism, we will define a marking graph on the surface $E_{g^{-1};1}$. This is most conveniently done by viewing $E_{g^{-1};1}$ as the total space of a cover over S_3 and then lifting paths in S_3 to $E_{g^{-1};1}$. The marking will have one vertex for every boundary component of $E_{g^{-1};1}$ and one internal vertex. Observe that $E_{g^{-1};1}$ has one boundary component over the first and third boundary circle of S_3 respectively and $n = |\mathcal{X}|$ boundary components over the second boundary circle of S_3.

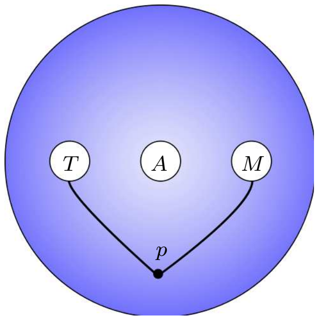
Before we turn to concrete graphs, we first have a look at the lifting properties of $E_{g^{-1};1}$. Consider the following path in S_3 that turns clockwise around the third boundary circle and has its starting point p in the lower half-plane:



(3.121)

Now we lift this closed path to an open path in $E_{g^{-1};1}$ with starting point $[x,p]$ and find that its end point is $[g^{-1}x,p]$ (see lemma 2.4). We will later use this type of path in order to connect the boundary components over the second boundary circle of S_3 to our marking graph.

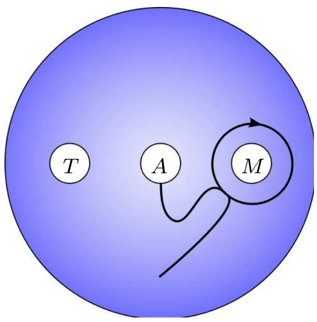
As a first step to finding a marking graph on the cover, we define a path in S_3 that connects the first and third boundary circle of S_3. The lift of this path then connects the boundary components of $E_{g^{-1};1}$ over those boundary circles. By definition of the standard block $S_3(g^{-1}, 1, g; 1, 1, 1)$, the marked point on its first boundary component is $p_1 = [1 - \frac{i}{3}, 1_G]$ and similarly for the third boundary component $p_3 = [3 - \frac{i}{3}, 1_G]$. By section 2.1.2 this gives the marked points $[x_0, p_1], [x_0, p_3] \in E_{g^{-1};1}$. Now consider the following path in S_3:



(3.122)

Lifting this path to $E_{g^{-1};1}$ such that the vertex at the first boundary circle is lifted to $[x_0, p_1]$ gives a path that connects $[x_0, p_1]$ and $[x_0, p_3]$ in $E_{g^{-1};1}$. The lift $\hat{p}$ of the point p will serve as internal vertex of the marking graph.

Over the second boundary circle of S_3 the surface $E_{g^{-1};1}$ has one boundary component in every sheet with marked points $[x, p_2]$ for every $x \in \mathcal{X}$. By convention 3.3 we have $x = g^l x_0$ for some l. Since G and $\mathcal{X}$ are finite, we can choose $x = g^{-k} x_0$ for some $k = 0, \ldots, n-1$. Now consider the following path in S_3 that winds k times clockwise around the third boundary circle:



(3.123)

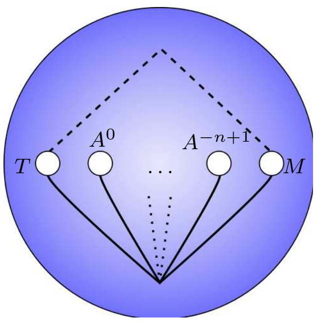
This should be read as follows: The path starts in the lower half-plane, moves near the third boundary component, winds k times clockwise around it and then connects to the marked point of the second boundary component. Obviously this path has self-intersections in S_3 but the lift to $E_{g^{-1};1}$ does not. This paths slightly differs for different choices of k to avoid self-intersections of the resulting marking graph. We will explain the differences:

As radius of the circular part of this path we choose $r_k = \frac{1}{3} + \frac{1}{10}(1 - \frac{k-1}{n-1})$. The lift of this path to $E_{g^{-1};1}$ with starting point $\hat{p}$ then has end point $[g^{-k}x_0, p_2]$. Now we draw the lift of this path on $E_{g^{-1};1}$ for every $k = 1 \dots n-1$. The assumption on the radius r_k ensures that the circular part of the graph has no intersection with the paths for different k, since the radius decreases with increasing k. Any other choice of radius with this property gives a homotopic path. The line that connects the internal vertex with the circular of the path is assumed to bend further to the right with increasing k so that again it does not intersect with the paths for different k.

For $k = 0$ we connect the internal vertex $\hat{p}$ and $[x_0, p_2]$ with a straight line. This is equivalent to the path for $k = 0$ constructed above, as the straight line is homotopic to the path that winds zero times around the third boundary circle.

In the cover $E_{g^{-1};1}$, the lifts of the paths (3.123) do not intersect with the lift of the path (3.122), that connects the internal vertex to the boundary components labeled by T and M.

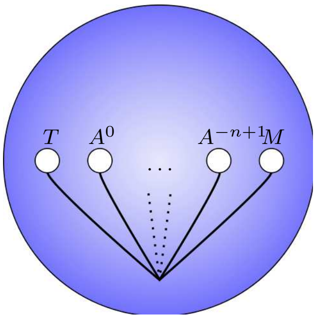
The lift of the path (3.122) and the lifts of the paths (3.123) for all k finally give a marking on $E_{g^{-1};1}$ that connects all marked points on all boundary components. Now we get a diffeomorphism to S_{n+2} by moving the boundary components of $E_{g^{-1};1}$ along the marking graph and obtain



(3.124)

where the upper dashed line marks multiple self-intersections of the immersion of the surface into three-dimensional space like in the pictures in section

3.1. As a non-embedded manifold, this is diffeomorphic to



$$(3.125)$$

This diffeomorphism induces an isomorphism

$$\begin{aligned}\tau(E_{g^{-1};1}; T, (A_x)_{x\in\mathcal{X}}, M) &\xrightarrow{\cong} \tau(S_{n+2}; T, A_{x_0}, A_{g^{-1}x_0}, \dots, A_{g^{-n+1}x_0}, M) \\ &\overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, TA_{x_0} \dots A_{g^{-n+1}x_0}M)\,.\end{aligned} \tag{3.126}$$

This shows

Lemma 3.4. *The functor*

$$\begin{aligned}\mathcal{C} &\to \mathcal{V}ect_{\Bbbk} \\ T &\mapsto \tau^{\mathcal{X}}(S_3(g^{-1}, 1, g; 1, 1, 1) \to S_3; T, (A_x)_{x\in\mathcal{X}}, M)\,.\end{aligned} \tag{3.127}$$

is represented by the object

$$(A_x)_{x\in\mathcal{X}} \otimes M := A_{x_0}A_{g^{-1}x_0} \dots A_{g^{-n+1}x_0}M\,, \tag{3.128}$$

which gives the module action functor

$$\begin{aligned}\mathcal{C}^{\boxtimes\mathcal{X}} \times \mathcal{C} &\to \mathcal{C} \\ (A_x)_{x\in\mathcal{X}} \times M &\mapsto A_{x_0}A_{g^{-1}x_0} \dots A_{g^{-n+1}x_0}M\end{aligned} \tag{3.129}$$

A comment on the order of the objects A_x is due: In $\mathcal{C}_1^{\mathcal{X}} = \mathcal{C}^{\boxtimes\mathcal{X}}$ the factors are ordered by the order of the G-set $\mathcal{X}$. In the module action, the order is by decreasing powers of the generator g.

3.2.3 Associativity constraints

We now turn to the question of finding associativity constraints

$$\psi_{(A_x),(B_x),M} : ((A_x) \otimes (B_x)) \otimes M \xrightarrow{\cong} (A_x) \otimes ((B_x) \otimes M)$$

The general procedure of reading off associativity constraints from G-equivariant modular functors was described in theorem 1.22 and especially for the case $G = \mathbb{Z}/2$ in section 3.1.4 in greater detail. When dealing with arbitrary groups, even with the restriction to cyclic groups in convention 3.3, the analysis of covers of the 4-punctured sphere is rather involved. The main downside is that we are no longer able to draw marking graphs on the manifolds $\mathcal{X} \times_G P$ themselves, but have to view them as total spaces of covers over S_4 and lift paths, as in the definition of the module action functor. Since this section is very technical, its results are summarized in theorem 3.5.

We again restrict ourselves to convention 3.3. Then the cover $(\mathcal{X} \times_G S_4(g^{-1}, 1, 1, g; 1, 1, 1, 1) \to S_4)$ has one boundary component over the first and fourth and $n = |\mathcal{X}|$ boundary components over the second and third boundary circle of S_4. In total this makes $2 + 2n$ boundary components and with the theorem of Riemann-Hurwitz we compute that the genus of the total space $\mathcal{X} \times_G S_4(g^{-1}, 1, 1, g; 1, 1, 1, 1)$ is zero. This implies that $\mathcal{X} \times_G S_4(g^{-1}, 1, 1, g; 1, 1, 1, 1)$ is isomorphic to the standard sphere S_{2n+2} as a smooth manifold.

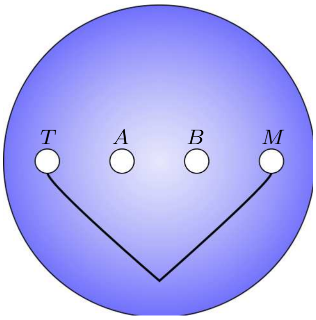
We can now proceed as in section 3.1.4:

1. Determine the two marking graphs on $\mathcal{X} \times_G S_4(g^{-1}, 1, 1, g; 1, 1, 1, 1)$ induced on the cover by cutting S_4 in the two ways determined by associativity and by our definition of the module action. Denote the marking graph representing $((A_x) \otimes (B_x)) \otimes M$ by m_1 and the marking graph representing $(A_x) \otimes ((B_x) \otimes M)$ by m_2.

2. Transform the surface $\mathcal{X} \times_G S_4(g^{-1}, 1, 1, g; 1, 1, 1, 1)$ with the marking graph m_1 on it into the standard sphere S_{2n+2} in the way prescribed by the marking m_2.

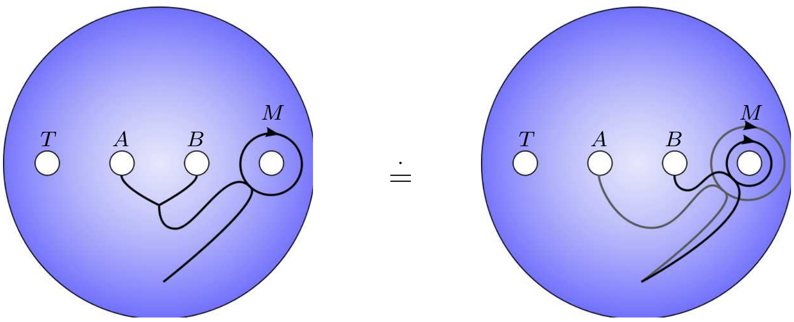
3. This yields a marking graph on S_{2n+2}. Determine the LTG-moves that transform this graph into the standard marking graph on S_{2n+2} and translate these LTG-moves into morphisms in $\mathcal{C}$.

We turn to the first cutting procedure, that represents the tensor product $((A_x) \otimes (B_x)) \otimes M$. In this case the surface $S_4(g^{-1}, 1, 1, g; 1, 1, 1, 1)$ is cut into a trivial G-cover $S_3(1, 1, 1; 1, 1, 1)$ representing the tensor product $(A_x) \otimes (B_x) \in \mathcal{C}^{\boxtimes \mathcal{X}}$ and the G-cover $S_3(g^{-1}, 1, g; 1, 1, 1)$ representing the product $(C_x) \otimes M \in \mathcal{C}$ with $(C_x) = (A_x) \otimes (B_x) = (A_x B_x)$. As the cover functor respects gluing, we can analyze the process by considering $\mathcal{X} \times_G S_4(g^{-1}, 1, 1, g; 1, 1, 1, 1)$ and the respective associated covers over S_3. We analyze the resulting marking graph on $\mathcal{X} \times_G S_4(g^{-1}, 1, 1, g; 1, 1, 1, 1)$ by considering paths in the base S_4 and lifting these to the covers. For the first and fourth boundary component we get a lift of the path



(3.130)

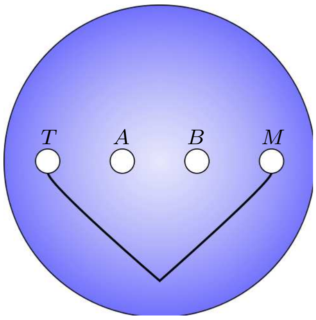
which is again lifted into the sheet corresponding the the generator $x_0 \in \mathcal{X}$. Now for the paths that connect to the boundary components over the second and third boundary circle we consider every sheet separately. For the boundary components in the sheet that corresponds to $x = g^{-k}x_0 \in \mathcal{X}$, gluing gives an edge which is a lift of



(3.131)

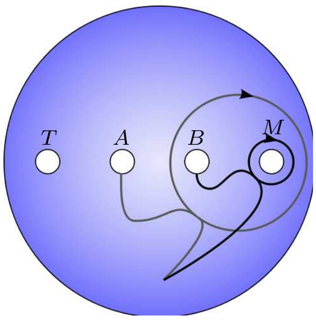
where the path turns k times around the fourth boundary circle. In the second picture we contracted the graph along the factorizing link, i.e. along the edge that connects the vertex near the A and B boundary components with the free vertex at the bottom of the picture. We draw the edges that connect to different boundary circles in different colors to avoid confusion. Note that the intersections of these paths do not give intersections in the total space of the cover as turning around the fourth boundary circle lifts to paths in the total space that connect different sheets.

In the second cutting procedure the G-cover $S_4(g^{-1},1,1,g;1,1,1,1)$ is cut into two G-covers $S_3(g^{-1},1,g;1,1,1)$, one representing the tensor product $(B_x)\otimes M$ and one representing the tensor product $(A_x)\otimes N$ with $N=(B_x)\otimes M$. In this case for the first and fourth boundary component we again get the path



(3.132)

as in the first cutting procedure. For the second and third boundary circle, we again consider all sheets separately; for the sheet corresponding to $g^{-k}x_0\in\mathcal{X}$ we get

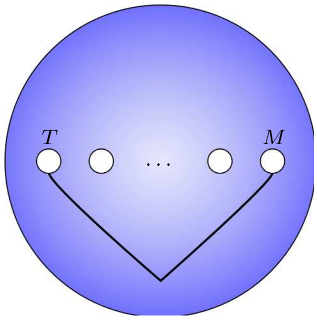


(3.133)

where the path connecting to the boundary component labeled by B turns k times around the boundary component labeled by M, whereas the path connecting to the A-boundary turns k times around both the B- and the M-boundary. Again we use different colors to distinguish the edges.

Now we use the diffeomorphism given by the second marking to transform the manifold into the standard sphere S_{n+2}. To find the image of the first marking on S_{2n+2}, we pick two boundary components of $\mathcal{X}\times_G S_4(g^{-1},1,1,g;1,1,1,1)$ and see how the corresponding edges of the marking behave relative to each other while applying the diffeomorphism to S_{2n+2}.

The first thing to notice is that the edges connecting to the first and last boundary component do not interfere with the edges of any other boundary in the application of the diffeomorphism. On S_{2n+2} this just gives

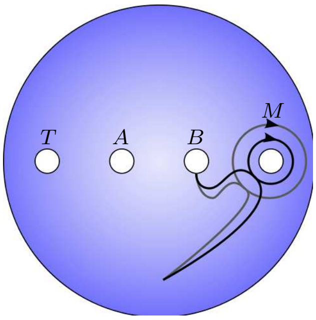


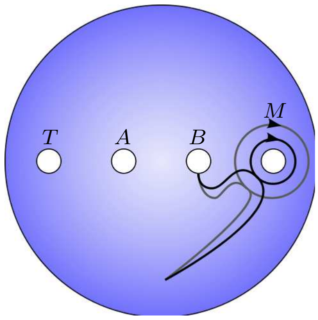
(3.134)

We now turn to the boundary components over the second and third boundary circle of S_4. When applying the diffeomorphism to S_{2n+2}, all boundary components of $\mathcal{X} \times_G S_4(g^{-1}, 1, 1, g; 1, 1, 1, 1)$ are moved simultaneously. When checking the relative behavior of two boundary components we will freely move these boundary components and the corresponding edges of the marking. If the edge comes near any other boundary component over the second and third boundary circle of S_4, we will assume that this component is already moved out of the way or is moved at the same time. This allows us to move the edge over the other boundary components over the second and third boundary circle of S_4. So the following analysis could be formulated as a recursive algorithm to transform $\mathcal{X} \times_G S_4(g^{-1}, 1, 1, g; 1, 1, 1, 1)$ into S_{2n+2}. The reader should always be aware of this procedure and should check that the simultaneous movement indeed justifies this process.

We will now distinct all possible choices of boundary components over the second and third boundary circle of S_4.

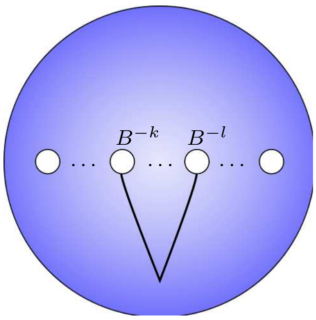
- We start by comparing two boundary components over the third boundary circle of S_4, i.e. two boundary components labeled by $B_{g^{-k}x_0}$ and $B_{g^{-l}x_0}$, where without loss of generality we assume $l > k$. In the markings obtained from the two gluing procedures, the edges corresponding to $B_{g^{-k}x_0}$ and $B_{g^{-l}x_0}$ are lifts of





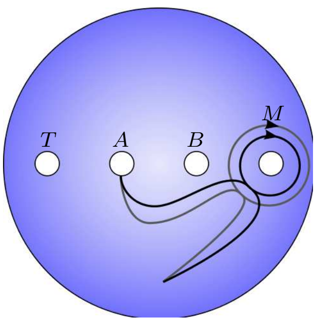
(3.135)

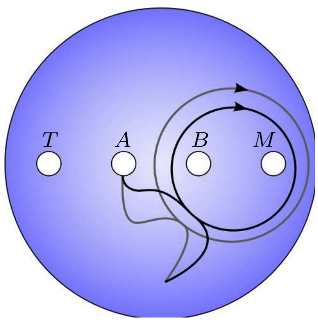
where the first picture shows the gluing for $((A_x) \otimes (B_x)) \otimes M$ and the second picture for $(A_x) \otimes ((B_x) \otimes M)$ as explained above. The darker line connects to the boundary component of $B_{g^{-l}x_0}$ and turns l times around the fourth boundary circle while the lighter line performs k turns and connects to the $B_{g^{-k}x_0}$-boundary. Obviously both markings coincide, hence on S_{2n+2} we get



(3.136)

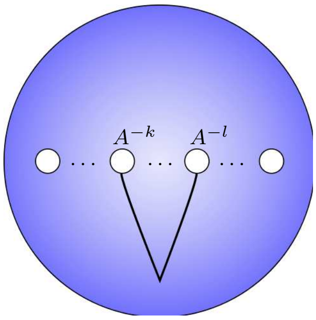
- For two boundaries over the second circle of S_4 labeled by $A_{g^{-k}x_0}$ and $A_{g^{-l}x_0}$ with $l > k$ we get a similar picture. In the gluing procedures described above we obtain edges for the respective boundaries that are lifts of





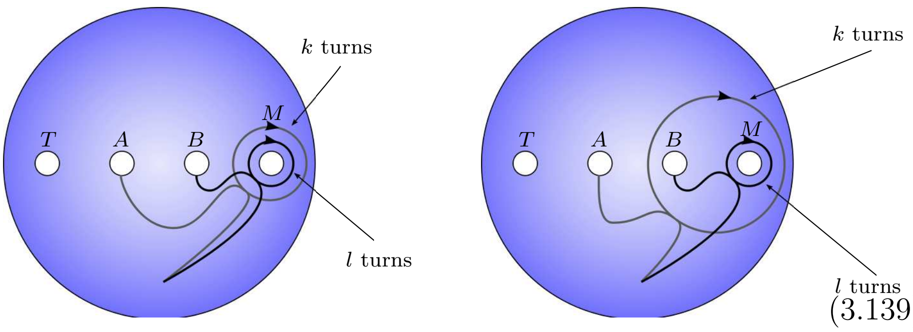
(3.137)

As we assume that the boundary components over the third boundary circle of S_4 are already moved out of the way, both edges can be transformed into each other, hence on S_{2n+2} we get:



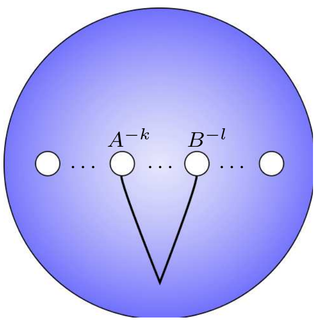
(3.138)

- Now we turn to the more complicated situations. For $k \leq l$ we compare the boundary components labeled by $A_{g^{-k}x_0}$ and $B_{g^{-l}x_0}$. The gluing procedures give us markings where the relevant edges are lifts of



(3.139)

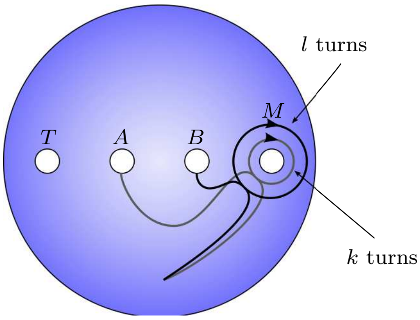
We see that under the assumption that other boundaries are already moved out of the way, again both edges coincide for $k \leq l$. Hence when applying the diffeomorphism to S_{2n+2} given by the second marking, we get

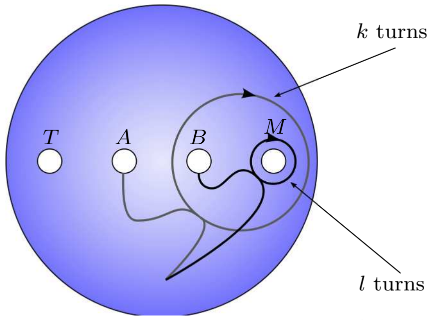


(3.140)

on S_{2n+2}.

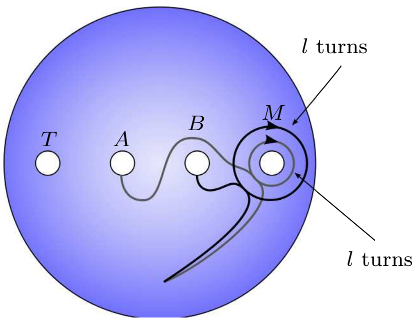
- Finally we compare the edges corresponding to boundary components labeled by $A_{g^{-k}x_0}$ and $B_{g^{-l}x_0}$ with $k > l$. In this case the edges in the markings are lifts of





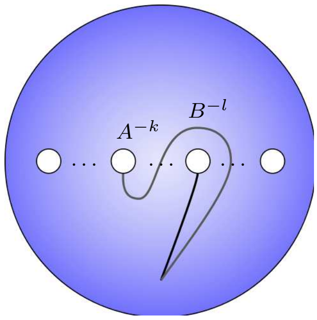
(3.141)

Observe that in the first picture the path connecting the internal vertex to the $A_{g^{-k}x_0}$-boundary turns around the fourth circle of S_4 with a smaller radius than the path connecting to the $B_{g^{-l}x_0}$-boundary, since $k > l$. Also check that all crossings in the paths in S_4 do not give self-intersections in $\mathcal{X} \times_G S_4(g^{-1}, 1, 1, g; 1, 1, 1, 1)$ as the crossing sections of the paths lift to different sheets. Now we carefully apply the diffeomorphism represented by the second marking. It instructs us to turn the $A_{g^{-k}x_0}$-boundary k times around the third and fourth boundary circle and the $B_{g^{-l}x_0}$-boundary l times around the fourth boundary circle. As a first step we turn the $A_{g^{-k}x_0}$-boundary $(k - l)$ times around. This transforms the first marking into



(3.142)

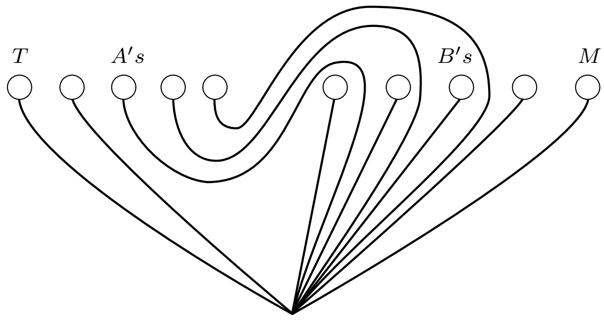
Here both paths wind l times around the fourth boundary circle. When we turned around the $A_{g^{-k}x_0}$-boundary, the edge connecting to it always passed along the $B_{g^{-l}x_0}$-boundary as they were lying in different sheets. Now we turn both boundaries around the fourth circle l times simultaneously and finally end up with the marking



(3.143)

on S_{2n+2}.

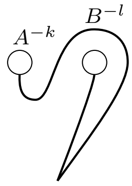
This describes the relative position of all pairs of edges of the marking we obtain on S_{2n+2}. An example of the final marking in the case of $n = 4$ is depicted in



(3.144)

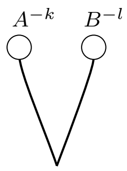
In the general case, the final marking on S_{2n+2} now has straight lines that connect the internal vertex to the T-, the M-, the B_x- and to the A_{x_0}-boundaries. The edge that connects the internal vertex to the $A_{g^{-k}x_0}$-boundary passes between the $B_{g^{-k+1}x_0}$- and the $B_{g^{-k}x_0}$-boundaries and then turns around the $B_{g^{-j}x_0}$-boundary for $j < k$ and then connects to the $A_{g^{-k}x_0}$-boundary parallel to the other A_x-edges.

We now want to transform this marking into the standard marking by a finite sequence of LTG-moves. To do so, recall that for every $k > l$ the marking is of the form



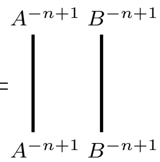
(3.145)

Hence we need to apply the LTG-move $\mathbf{B}_{B_{g^{-l}x_0}, A_{g^{-k}x_0}}$ to turn this into the marking



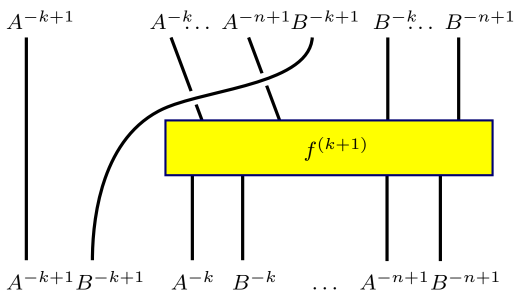
(3.146)

Now starting with the $B_{g^{-n+2}x_0}$-boundary, for $l = n-2, \ldots, 1$ we successively apply $\mathbf{B}_{B_{g^{-l}x_0}, A_{g^{-k}x_0}}$ for $k = l+1, \ldots, n-1$. This finally gives the standard marking on S_{2n+2}. To translate these LTG-moves into a morphism in $\mathcal{C}$, we introduce auxiliary morphisms $f^{(k)}$ for $k = n, \ldots, 1$ with

$f^{(n)} =$ 

(3.147)

the identity and $f^{(k)}$ for $k = n-1, \ldots, 1$ recursively

$f^{(k)} =$ 

(3.148)

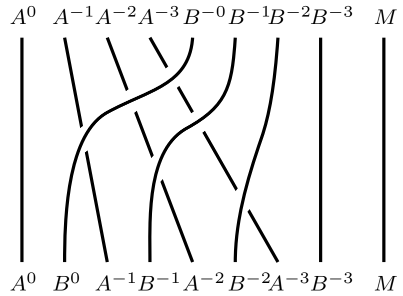
So the recursive step $f^{(l+1)} \to f^{(l)}$ resembles the application of the LTG-moves $\mathbf{B}_{B_{g^{-l+1}x_0}, A_{g^{-k}x_0}}$ for $k = l, \ldots, n-1$.

Alltogether this gives the associativity constraint

$$\psi_{(A_x),(B_x),M} = f^{(1)} \otimes \mathrm{id}_M \tag{3.149}$$

An example of this morphism in the case $n = 4$ is depicted in

$$\psi_{(A_x),(B_x),M} =$$



(3.150)

For general n, the morphism $\psi_{(A_x),(B_x),M}$ is a shuffle that moves all B-objects over the A-objects.

By theorem 1.22 or the general arguments of [KP08], these associativity constraints satisfy the mixed pentagon axiom, which can also easily be verified by hand. We summarize our findings for arbitrary groups in the following

Theorem 3.5. *For any finite group G, any finite G-set $\mathcal{X}$ and any $g \in G$ the functor*

$$\begin{aligned} \mathcal{C}^{\boxtimes \mathcal{X}} \times \mathcal{C}^{\boxtimes O_g} &\to \mathcal{C}^{\boxtimes O_g} \\ (A_x)_{x\in\mathcal{X}} \times (M_o)_{o\in O_g} &\mapsto (A_{x_o} A_{g^{-1}x_o} \cdots A_{g^{-|o|+1}x_o} M_o)_{o\in O_g} \end{aligned} \tag{3.151}$$

with x_o the smallest element in the $\langle g\rangle$-orbit o of $\mathcal{X}$, together with the associativity constraints

$$\Psi_{A,B,M} = (\psi^o_{(A_x)_{x\in o},(B_x)_{x\in o},M})_{o\in O_g}\,, \tag{3.152}$$

with the morphisms ψ^o as in equation (3.150), endows the category $\mathcal{C}^{\boxtimes O_g}$ with the structure of a module category over the tensor category $\mathcal{C}^{\boxtimes \mathcal{X}}$.

The associativity constraints that we found in this section depend on the choice of the marking on $E_{g^{-1};1}$ made in section 3.2.2. Like in the case of $G = \mathbb{Z}/2$ in section 3.1 a different choice of marking leads to an equivalent module category structure.

Similarly one can compute the bimodule category structure on $\mathcal{C}^{\boxtimes O_g}$. As we mentioned before, this already fixes a large part of the full monoidal structure of $\mathcal{C}^{\mathcal{X}}$:

In [ENO09, Section 8] it was shown that for a fusion category $\mathcal{D}$, a group homomorphism $c : G \to \mathrm{BrPic}(\mathcal{D})$ from G into the group $\mathrm{BrPic}(\mathcal{D})$ of equivalence classes of invertible module categories over $\mathcal{D}$ determines two elements in certain cohomology groups of G. A structure of a G-equivariant monoidal category with neutral component $\mathcal{D}$ and as twisted components representatives of the equivalence classes $c(g)$, exists, if and only if these two obstruction classes vanish. Equivalence classes of G-equivariant categories based on the homomorphism c then form a torsor over $H^3(G, \Bbbk^\times)$.

In our modular functor approach, the existence of a G-equivariant monoidal structure on the system of module categories $\mathcal{C}_g^{\mathcal{X}}$ is already ensured by theorem 1.22, hence in our situation both obstruction classes have to be trivial. Thus the results of this section, summarized in theorem 3.5, describe the equivalence class of the G-equivariant monoidal category up to an element of a torsor over $H^3(G, \Bbbk^\times)$.

3.3 Permutation modular invariants

For any module category $(\mathcal{M}, \otimes, \psi)$ over a monoidal category $\mathcal{D}$ the category $\mathcal{E}nd_{\mathcal{D}}(\mathcal{M})$ of $\mathcal{D}$-module endofunctors of $\mathcal{M}$ is again a monoidal category that acts on $\mathcal{M}$. If $\mathcal{D}$ is braided, recall from [Ost03, Section 5.1,5.2] the following definition:

Definition 3.6. Define two functors

$$\alpha^{\pm} : \mathcal{D} \to \mathcal{E}nd_{\mathcal{D}}(\mathcal{M}) \tag{3.153}$$

by putting

$$\alpha^{\pm}(U)(M) := U \otimes M \tag{3.154}$$

as functors and with the following module functor constraints for $\alpha^{\pm}(U)$:

$$\begin{aligned} \gamma^{U,+}_{V,M} &:= \psi_{V,U,M} \circ (c_{U,V} \otimes \mathrm{id}_M) \circ \psi^{-1}_{U,V,M} \qquad \text{and} \\ \gamma^{U,-}_{V,M} &:= \psi_{V,U,M} \circ (c^{-1}_{U,V} \otimes \mathrm{id}_M) \circ \psi^{-1}_{U,V,M} \,, \end{aligned} \tag{3.155}$$

where $c_{U,V}$ is the braiding in $\mathcal{D}$. The functors $\alpha^{\pm}$ are called the *α-induction functors.*

Remark 3.7. The associativity constraints ψ of the module category $\mathcal{M}$ endow the functors $\alpha^{\pm}$ with the structure of monoidal functors.

If U is an object of $\mathcal{D}$, we abbreviate $\alpha^{\pm}(U) \equiv \alpha^{\pm}_U$; if U_k is a simple object of $\mathcal{D}$, we write $\alpha^{\pm}(U_k) \equiv \alpha^{\pm}_k$. Now for any two simple objects U_i, U_j of $\mathcal{D}$ we define the non-negative integers

$$Z_{i,j} := \dim_{\Bbbk} \mathrm{Hom}_{\mathcal{E}nd_{\mathcal{D}}(\mathcal{M})}(\alpha^+_i, \alpha^-_j) \tag{3.156}$$

The $|\mathcal{I}| \times |\mathcal{I}|$-matrix $Z(\mathcal{M}/\mathcal{D}) := (Z_{i,j})$ then obeys the requirements on a modular invariant (see [FRS02, Theorem 5.1]). It is the aim of this section to compute the structure of this matrix in the case that the module category under consideration is given by the data of the G-equivariant modular functor $\tau^{\mathcal{X}}$.

So we fix an element $g \in G$ and examine the module category $\mathcal{C}^{\mathcal{X}}_g = \mathcal{C}^{\boxtimes O_g}$ over $\mathcal{C}^{\mathcal{X}}_1 = \mathcal{C}^{\boxtimes \mathcal{X}}$. We will continue in two steps: First we show that certain entries of the matrix $Z(\mathcal{C}^{\mathcal{X}}_g/\mathcal{C}^{\mathcal{X}}_1)$ are non-zero. Then we show that $Z(\mathcal{C}^{\mathcal{X}}_g/\mathcal{C}^{\mathcal{X}}_1)$ is a permutation matrix, i.e. it contains precicely one entry 1 in every row and column and 0 elsewhere. This already determines the whole matrix.

For the first step we will give for every object U in $\mathcal{C}^{\boxtimes\mathcal{X}}$ an invertible natural transformation

$$\Gamma^U : \alpha^+(U) \Rightarrow \alpha^-({}^gU) \tag{3.157}$$

between module functors. For $U = U_{\bar\imath}$ a simple object, this will be a non-zero element in

$$\mathrm{Hom}_{\mathcal{E}nd_{\mathcal{C}^{\boxtimes\mathcal{X}}}(\mathcal{C}^{\boxtimes O_g})}(\alpha^+_{\bar\imath}, \alpha^-_{g\bar\imath}) \tag{3.158}$$

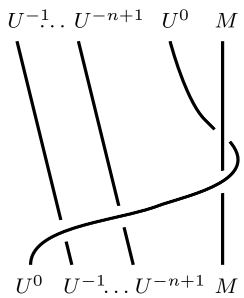
so that this vector-space is non-zero. As in section 3.2 we can restrict our discussion to only one factor of $\mathcal{C}^{\boxtimes O_g}$ by giving Γ^U for every factor separately so that we are again in the situation of convention 3.3. Hence we will consider the problem where $\mathcal{C}^{\boxtimes\mathcal{X}}$ acts on a single copy of $\mathcal{C}$.

From now on we fix an object $U = (U_x)_{x\in\mathcal{X}}$ in $\mathcal{C}^{\boxtimes\mathcal{X}}$ and write Γ instead of Γ^U. For M in $\mathcal{C}$ by theorem 3.5 and the definition of the equivariance functors R_g in definition 2.10 we find

$$\begin{aligned} \alpha^+_U(M) &= U_{x_o} U_{g^{-1}x_o} \cdots U_{g^{-n+1}x_o} M \\ \alpha^-_{{}^gU}(M) &= U_{g^{-1}x_o} \cdots U_{g^{-n+1}x_o} U_{x_o} M \end{aligned} \tag{3.159}$$

Hence we make the ansatz

$$\Gamma_M :=$$



(3.160)

In formulas:

$$\Gamma_M = \left[\mathrm{id}_{U_{g^{-1}x_o} \cdots U_{g^{-n+1}x_o}} \otimes (c_{M,U_{x_o}} \circ c_{U_{x_o},M})\right] \circ \left[c_{U_{x_o},U_{g^{-1}x_o}\cdots U_{g^{-n+1}x_o}} \otimes \mathrm{id}_M\right] \tag{3.161}$$

Lemma 3.8. Γ *is a non-zero natural transformation between the module functors* α^+_U *and* $\alpha^-_{{}^gU}$.

Proof. Obviously Γ_M is natural in M and invertible, hence non-zero. To show that Γ is a natural transformation of module functors, we have to show that for another object $V=(V_x)$ of $\mathcal{C}^{\boxtimes\mathcal{X}}$ the following compatibility with the module functor constraints (3.155) holds:

$$(\mathrm{id}_{(V_x)}\otimes\Gamma_M)\circ\gamma^{U,+}_{V,M}=\gamma^{{}^gU,-}_{V,M}\circ\Gamma_{(V_x)\otimes M} \tag{3.162}$$

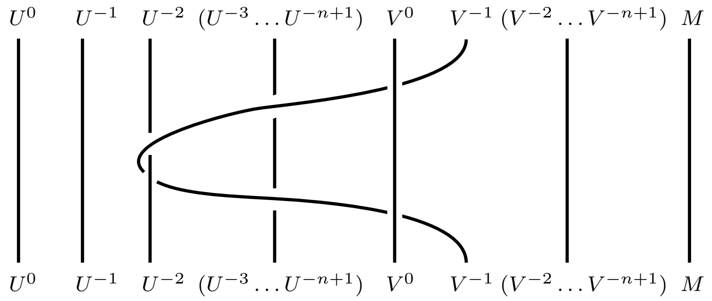
Spelling out all occurring morphisms, this amounts to

$$\begin{aligned}&\left[\mathrm{id}_{V_{x_o}\cdots V_{g^{-n+1}x_o}}\otimes\Gamma_M\right]\circ\psi_{(V_x),(U_x),M}\circ\left[c^{\mathcal{X}}_{(U_x),(V_x)}\otimes\mathrm{id}_M\right]\circ\psi^{-1}_{(U_x),(V_x),M}=\\&\psi_{(V_x),{}^g(U_x),M}\circ\left[(c^{\mathcal{X}}_{(V_x),{}^g(U_x)})^{-1}\otimes\mathrm{id}_M\right]\circ\psi^{-1}_{{}^g(U_x),(V_x),M}\circ\Gamma_{V_{x_o}\cdots V_{g^{-n+1}x_o}M}\end{aligned} \tag{3.163}$$

where $c^{\mathcal{X}}_{(U_x),(V_x)}$ denotes the braiding of two objects in $\mathcal{C}^{\boxtimes\mathcal{X}}$. This is an equality in

$$\mathrm{Hom}_{\mathcal{C}}(U_{x_o}\cdots U_{g^{-n+1}x_o}V_{x_o}\cdots V_{g^{-n+1}x_o}M,V_{x_o}\cdots V_{g^{-n+1}x_o}U_{g^{-1}x_o}\cdots U_{g^{-n+1}x_o}U_{x_o}M)\ .$$

Denote by $L^n(U_{x_o},\ldots,U_{g^{-n+1}x_o};V_{x_o},\ldots,V_{g^{-n+1}x_o};M)$ the left hand side of (3.163) and by $R^n(U_{x_o},\ldots,U_{g^{-n+1}x_o};V_{x_o},\ldots,V_{g^{-n+1}x_o};M)$ the right hand side. The endomorphism

$F=$ 

(3.164)

is obviously invertible and an easy but lengthy graphical calculation shows that it obeys

$$\begin{aligned}&L^n(U_{x_o},\ldots,U_{g^{-n+1}x_o};V_{x_o},\ldots,V_{g^{-n+1}x_o};M)\circ F=\\&L^{n-1}(U_{x_o},U_{g^{-1}x_o}U_{g^{-2}x_o},\ldots,U_{g^{-n+1}x_o};V_{x_o},V_{g^{-1}x_o}V_{g^{-2}x_o},\ldots,V_{g^{-n+1}x_o};M)\end{aligned} \tag{3.165}$$

and

$$\begin{aligned}&R^n(U_{x_o},\ldots,U_{g^{-n+1}x_o};V_{x_o},\ldots,V_{g^{-n+1}x_o};M)\circ F=\\&R^{n-1}(U_{x_o},U_{g^{-1}x_o}U_{g^{-2}x_o},\ldots,U_{g^{-n+1}x_o};V_{x_o},V_{g^{-1}x_o}V_{g^{-2}x_o},\ldots,V_{g^{-n+1}x_o};M)\end{aligned} \tag{3.166}$$

Hence (3.163) holds by induction, the case $n = 2$ is an easy calculation using only relations in the braid group on five strands. □

When we apply lemma 3.8 to a simple object $U_{\bar\imath} = (U_i)_{i\in\bar\imath}$ with $\bar\imath \in \mathcal{I}^{\mathcal{X}}$ we see

$$Z_{\bar\imath,g\bar\imath} = \dim_{\Bbbk}\mathrm{Hom}_{\mathcal{E}nd_{\mathcal{C}\boxtimes\mathcal{X}}(\mathcal{C})}(\alpha^+_{\bar\imath}, \alpha^-_{g\bar\imath}) \neq 0\,. \tag{3.167}$$

We now proceed with the second step in the computation of the matrix $Z(\mathcal{C}^{\mathcal{X}}_g/\mathcal{C}^{\mathcal{X}}_1)$, i.e. we show that it is a permutation matrix. We start with the following

Definition 3.9. An *Azumaya category* $\mathcal{M}$ over a braided monoidal category $\mathcal{D}$ is a left module category $\mathcal{M}$ over $\mathcal{D}$ for which the two monoidal functors $\alpha^{\pm}$ from $(\mathcal{D}, \otimes)$ to $(\mathcal{E}nd_{\mathcal{D}}(\mathcal{M}), \circ)$ are equivalences.
An *Azumaya algebra* A in a braided monoidal category $\mathcal{D}$ is an algebra A in $\mathcal{D}$ such that the category of right A-modules is an Azumaya category over $\mathcal{D}$.

Remark 3.10. This definition is equivalent to the definition given in [VZ98, Section 3].

By [ENO09, Theorem 6.1] every bimodule category $\mathcal{M}$ over $\mathcal{D}$, which is part of an equivariant monoidal category with neutral component $\mathcal{D}$, is invertible with respect to a tensor product of module categories. We will need the following criterion for invertibility of a module category. We assume that $\mathcal{D}$ is braided and $\mathcal{M}$ is a module category that is turned into a bimodule category by using the braiding of $\mathcal{D}$.

Lemma 3.11. *A semisimple module category $\mathcal{M}$ over a modular category $\mathcal{D}$ is invertible if and only if it is equivalent to $A-$mod, as a module category over $\mathcal{D}$, for some Azumaya algebra A in $\mathcal{D}$.*

Proof. By proposition 4.2 and section 5.4 of [ENO09] invertibility of $\mathcal{M}$ is equivalent to $\mathcal{M}$ being an Azumaya category. If $\mathcal{M}$ is invertible, by [ENO09, Corollary 4.4] it is indecomposable over $\mathcal{D}$. It follows from [Ost03, Theorem 1] that as a module category $\mathcal{M}$ is equivalent to $A-$mod for some algebra A in $\mathcal{D}$, which then is Azumaya by definition 3.9. □

Theorem 3.12. *The modular matrix $Z(\mathcal{C}^{\mathcal{X}}_g/\mathcal{C}^{\mathcal{X}}_1)$ for the module category described in theorem 3.5 reads*

$$Z(\mathcal{C}^{\mathcal{X}}_g/\mathcal{C}^{\mathcal{X}}_1)_{\bar\imath,\bar\jmath} = \delta_{\bar\jmath,g\bar\imath} \tag{3.168}$$

where $\bar\imath, \bar\jmath \in \mathcal{I}^{\mathcal{X}}$ label the simple objects of $\mathcal{C}_1^{\mathcal{X}} = \mathcal{C}^{\boxtimes \mathcal{X}}$ and $g\bar\imath$ is the multi-index $\bar\imath$ permuted by the action of the group element $g \in G$.

Proof. Since $\tau^{\mathcal{X}}$ is a G-equivariant modular functor, it induces ([KP08]) the structure of a G-equivariant category on $\bigoplus \mathcal{C}_h^{\mathcal{X}}$. The module category $\mathcal{C}_g^{\mathcal{X}}$ over $\mathcal{C}_1^{\mathcal{X}}$ is part of this larger structure, by [ENO09, Theorem 6.1] it is invertible, hence by lemma 3.11 equivalent to the category $A-$mod for some Azumaya algebra A in $\mathcal{C}_1^{\mathcal{X}}$. As A is Azumaya, the functors $\alpha^{\pm}$ are equivalences

$$\alpha^{\pm} : \mathcal{C}_1^{\mathcal{X}} \to \mathcal{E}nd_{\mathcal{C}_1^{\mathcal{X}}}(\mathcal{C}_g^{\mathcal{X}}) \cong A-\text{bimod}$$

of tensor categories. Hence $A-$bimod is semisimple and for a simple object $U_{\bar\imath}$ in $\mathcal{C}_1^{\mathcal{X}}$ the objects $\alpha^{\pm}(U_{\bar\imath})$ in A–bimod are again simple. By semisimplicity of $A-$bimod, the matrix

$$Z(\mathcal{C}_g^{\mathcal{X}}/\mathcal{C}_1^{\mathcal{X}})_{\bar\imath,\bar\jmath} := \dim_{\Bbbk}\mathrm{Hom}_{\mathcal{E}nd_{\mathcal{C}_1^{\mathcal{X}}}(\mathcal{C}_g^{\mathcal{X}})}(\alpha^+_{\bar\imath}, \alpha^-_{\bar\jmath})$$

has exactly one entry 1 in every row and every column and 0 elsewhere. By lemma 3.8 we find that in every row and column the numbers $Z_{\bar\imath,g\bar\imath}$ are non-zero. □

Remark 3.13. With techniques similar to those used in [BFRS10a, Theorem 5.1] it is possible to compute a representative of the Morita class of algebras A in $\mathcal{C}^{\boxtimes \mathcal{X}}$, such that $A-\text{mod} \cong \mathcal{C}^{\boxtimes O_g}$ as module categories over $\mathcal{C}^{\boxtimes \mathcal{X}}$. Arguments parallel to [BFRS10a, Proposition 6.1] show that this representative A carries the structure of a special symmetric Frobenius algebra. Together with theorem 3.12 this implies that the permutation modular invariant (3.168) is physical, in the sense that there exists a consistent set of correlators ([FFRS08, Definition 3.14]) with the permutation modular invariant as torus partition function.

APPENDIX

A. THE LTG-DICTIONARY

We will now give a brief overview of the connection between the moves in the (non-equivariant) Lego-Teichmüller-Game and the structure morphisms of a modular category $\mathcal{C}$. To this end we relate every LTG-move with a natural isomorphism between hom-functors of $\mathcal{C}$. We will do this for the LTG-moves on the standard spheres and the one-punctured torus.

A.1 Modular categories vs. modular functors

Our motivation is theorem 1.22, where we explained how to obtain a G-equivariant weakly ribbon category $\mathcal{C}^G$ out of the structure of a $\mathcal{C}^G$-extended G-equivariant modular functor τ^G. The objects that are needed for this structure, i.e. the dual of an object and the tensor product of two objects, were defined as objects that represent the functors $\tau(P{\to}E)$ for appropriate G-covers $(P{\to}E)$. The structure morphisms like for example the associativity constraints or the equivariant braiding, were induced by the natural isomorphisms that the modular functor τ^G assigns to appropriate diffeomorphisms of the G-covers.

The action of diffeomorphisms on G-covers can be dealt with by the equivariant Lego-Teichmüller-Game. So the proof of theorem 1.22 relies on a comparison of the LTG-moves and the structure morphisms of a G-equivariant (weakly) ribbon category $\mathcal{C}^G$ and in particular relations between these. We want to make this comparison explicit in the non-equivariant case.

If $\mathcal{C}$ is a semisimple category, we denote by $\mathcal{I}$ the set of isomorphism classes of simple objects and choose representatives U_i with $i \in \mathcal{I}$. If $\mathcal{C}$ is modular, we denote by $d_i \in \Bbbk$ the dimension of U_i and by $\theta_i \in \Bbbk$ the eigenvalue of the twist on U_i. As usual, we introduce the scalars $p^{\pm} := \sum_{i\in\mathcal{I}} \theta_i^{\pm} d_i^2$ and assume that $p^+ = p^-$, so that there exists a $\mathcal{C}$-extended modular functor.

The correspondence between modular categories and modular functors has two directions. First we discuss how to obtain parts of the structure of a modular functor from a modular category $\mathcal{C}$.

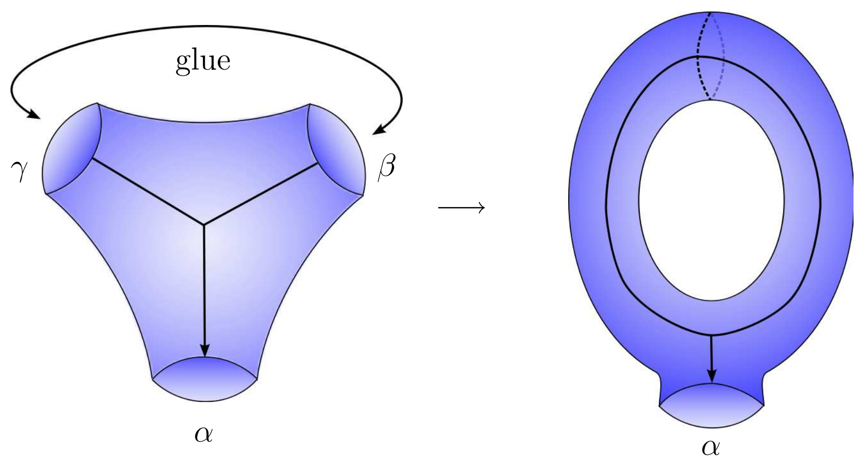
Given a modular category $\mathcal{C}$, the object $\mathcal{R} := \bigoplus_{i\in\mathcal{I}} U_i \times U_i^\vee$ in $\mathcal{C}\boxtimes\mathcal{C}$ satisfies the requirements 1.17 on a gluing object. The corresponding $\mathcal{C}$-extended modular functor $\tau \equiv \tau_{\mathcal{C}}$ is defined for the standard spheres as

$$\begin{aligned} \tau(S_n) : \mathcal{C}^{\boxtimes n} &\to \mathcal{V}ect_{\Bbbk} \\ V_1 \times \cdots \times V_n &\mapsto \tau(S_n; V_1, \ldots, V_n) := \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_1 \cdots V_n)\,, \end{aligned} \tag{A.1}$$

where again we agree to drop the tensor product symbol of objects in $\mathcal{C}$ in the notation. Another standard manifold that will be needed for the LTG-dictionary is the one-punctured torus T_1. We compute the corresponding functor

$$\begin{aligned} \tau(T_1) : \mathcal{C} &\to \mathcal{V}ect_{\Bbbk} \\ V &\mapsto \tau(T_1; V) \end{aligned} \tag{A.2}$$

by representing T_1 as a gluing of the three-punctured sphere S_3 :



(A.3)

where we draw a marking on the surfaces to illustrate the gluing process. For the functor $\tau(T_1)$, the gluing isomorphisms (1.43) give an isomorphism

$$G_{\beta,\gamma} : \tau(S_3; V, \mathcal{R}_{\beta,\gamma}) \xrightarrow{\cong} \tau(T_1; V) \tag{A.4}$$

for every object V of $\mathcal{C}$. Here $\mathcal{R}_{\beta,\gamma}$ indicates that the two factors of the gluing object $\mathcal{R}$ in $\mathcal{C}\boxtimes\mathcal{C}$ are assigned to the boundary components β and γ. If we spell out the domain of (A.4), we obtain a functorial isomorphism

$$\tau(T_1, V) \cong \tau(S_3; V, \mathcal{R}_{\beta,\gamma}) \stackrel{\text{def}}{=} \bigoplus_{i\in\mathcal{I}} \tau(S_3; V, U_i, U_i^\vee) \stackrel{\text{def}}{=} \bigoplus_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V U_i U_i^\vee)\,. \tag{A.5}$$

Hence the $\mathcal{C}$-extended modular functor corresponding to a modular category $\mathcal{C}$ associates the the one-punctured torus T_1 the functor $\tau(T_1)$ given by (A.5). Observe that up to this point we did not use the requirement on $\mathcal{C}$ to be modular; this will enter once we discuss morphisms of extended surfaces, see (A.12).

Assume we are given a semisimple abelian $\mathcal{V}ect_{\Bbbk}$-enriched category $\mathcal{C}$ with gluing object $\mathcal{R}$ and a $\mathcal{C}$-extended modular functor. The functor $\tau(S_3)$ for the standard sphere S_3 induces on $\mathcal{C}$ a tensor product of two objects of $\mathcal{C}$. More generally for the n-punctured sphere we obtain $(n-1)$-ary products as representing objects of $\tau(S_n; ?, V_1, \ldots, V_{n-1})$:

$$\tau(S_n; T, V_1, \ldots, V_{n-1}) \cong \mathrm{Hom}_{\mathcal{C}}(T^\vee, V_1 \cdots V_{n-1})\,, \tag{A.6}$$

with $T^\vee$ the dual object as in theorem 1.22. Using duality, we identify

$$\tau(S_n; V_1, \ldots, V_n) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_1 \cdots V_n)\,. \tag{A.7}$$

We used the standard sphere S_n to define $(n-1)$-ary tensor products. The tensor product obtained in this way is in general not strictly associative.

If τ is defined for objects of higher genus, it follows that $\mathcal{C}$ has only finitely many simple objects up to isomorphism. From the definition of dual objects, one can compute the gluing object $\mathcal{R}$, which turns out to be of the form $\mathcal{R} \cong \bigoplus_{i\in\mathcal{I}} U_i \times U_i^\vee$. Similarly the functor $\tau(T_1)$ for the one-punctured torus also takes the form

$$\tau(T_1, V) \cong \bigoplus_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V U_i U_i^\vee)\,. \tag{A.8}$$

A.2 The dictionary

Before giving the dictionary for the Lego-Teichmüller-Game, we pause to explain how it should be read.

1. Suppose we are given a $\mathcal{C}$-extended modular functor τ and want to find the weakly ribbon structure on $\mathcal{C}$.

   We have already explained how to find dual objects and n-ary tensor products for $\mathcal{C}$ by choosing representing objects. Now let $\varphi : E \to E'$ be a diffeomorphism of extended surfaces. We choose parametrizations of

E and E', i.e. we draw marking graphs on both surfaces. In particular the cuts in these marking graphs decompose E and E' into disjoint unions of standard spheres. Then the functors $\tau(E)$ and $\tau(E')$ can be expressed as tensor products of hom-functors of $\mathcal{C}$ as in (A.7) with appropriate insertions of gluing objects.

Now we want to find the natural isomorphism between these hom-functors that the modular functor τ assigns to φ. To this end we apply φ to the marking graph on E and obtain a new marking on E'. According to section 1.2.4 this new marking is transformed into the given marking on E' by a finite and (up to relations) unique sequence of LTG-moves. Then we take the dictionary below and look up the natural isomorphisms between hom-functors, expressed by structure morphisms of a ribbon category, that correspond to these LTG-moves. The composition of these isomorphisms is then the isomorphism that the modular functor τ assigns to φ and one can read of the structure morphisms.

However, this isomorphism still depends on the parametrizations of E and E'. In theorem 1.22 the dictionary is only used for diffeomorphisms of standard spheres, which come with a canonical parametrization. For arbitrary surfaces E, one has to keep track of these parametrizations, as we did in sections 3.1 and 3.2.

2. Suppose we are given a modular tensor category $\mathcal{C}$ and want to define a $\mathcal{C}$-extended modular functor τ.

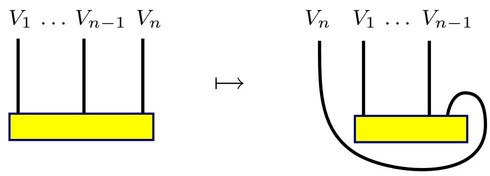
   For the standard spheres S_n, we define τ by the hom-functor as in (A.1). Now let $\varphi : S_n \to S_n$ be a diffeomorphism. To define a full modular functor, we have to assign to φ a functorial isomorphism $\varphi_* : \tau(S_n) \Rightarrow \tau(S_n)$. We apply φ to the standard marking on S_n and obtain a new marking. Afterwards, this marking is transformed back into the standard marking by a finite sequence of LTG-moves. Now we look up these LTG-moves in the dictionary and get a composition of natural transformations between the hom-functors. We assign this composition to the diffeomorphism φ.

   Defining the modular functor τ for arbitrary surfaces E is more involved. To do so, we need to relate all possible choices of parametrizations of E in a canonical way. We are then in the position to apply the above procedure and assign functorial isomorphisms to diffeomorphisms of extended surfaces. For details on this, we refer to [BK01, Section 5.4].

Recall from section 1.2 the basic LTG-moves **Z**, **B**, **F** and **S**. We now give the dictionary:

- The **Z**-move is related to the natural transformation

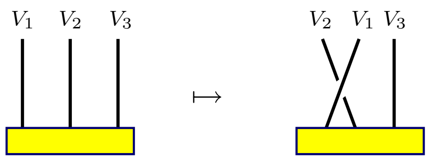
$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_1 \cdots V_n) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_n V_1 \cdots V_{n-1})$$



(A.9)

- The **B**-move is related to the natural transformation

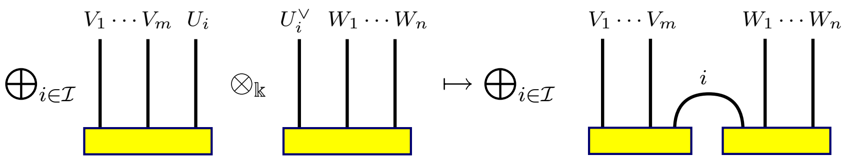
$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_1 V_2 V_3) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_2 V_1 V_3)$$



(A.10)

- For the **F**-move, one needs to assign the gluing object $\mathcal{R}$ to the boundary components that are glued, i.e. to both sides of the cut that is removed by the **F**-move. By applying appropriate powers of the **Z**-move, we can assume that we glue the last boundary component of the standard sphere S_m to the first boundary component of the standard sphere S_n for some m, n. The **F**-move is then related to the transformation

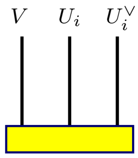
$$\begin{aligned} \textstyle\bigoplus_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_1 \cdots V_m U_i) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, U_i^{\vee} W_1 \cdots W_n) \\ \to \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_1 \cdots V_m W_1 \cdots W_n) \end{aligned}$$

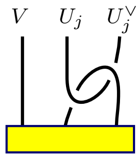


(A.11)

- Finally for the **S**-move, we recall that the functor $\tau(T_1)$ is isomorphic to the functor (A.5). Under this identification the **S**-move is related to

$$\bigoplus_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, VU_iU_i^\vee) \to \bigoplus_{j\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, VU_jU_j^\vee)$$

$\bigoplus_{i\in\mathcal{I}}$ 

$\mapsto \bigoplus_{i,j\in\mathcal{I}} \frac{d_j}{\mathcal{D}}$ 

(A.12)

  This is an isomorphism between the corresponding vector spaces, if and only if the ribbon category $\mathcal{C}$ is modular.

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Zusammenfassung

In dieser Arbeit untersuchen wir eine bestimmte Klasse G-äquivarianter Tensorkategorien, wobei G eine endliche Gruppe ist. Als Spezialfall erhalten wir für die symmetrischen Gruppen $G = S_N$ die sogenannten Permutations-äquivarianten Tensorkategorien. Die Konstruktion ist geometrisch, d.h. wir geben einen G-äquivarianten modularen Funktor an, welcher dann die Struktur einer G-äquivarianten monoidalen Kategorie induziert. Als Eingangsdaten verwenden wir eine endliche Gruppe G, eine endliche G-Menge $\mathcal{X}$ und eine modulare Tensorkategorie $\mathcal{C}$.

Die wesentliche Idee unserer Konstruktion ist die Definition des sogenannten Cover-Funktors. Dies ist der Tensorfunktor von der Kategorie $\mathbf{Gcob(d)}$ der G-Prinzipalbündel d-dimensionaler Kobordismen in die Kategorie $\mathbf{cob(d)}$ der d-dimensionalen Kobordismen, welcher einem G-Bündel den Totalraum des durch die G-Wirkung auf $\mathcal{X}$ assoziierten Bündels zuordnet.

Zunächst führen wir unsere Konstruktion dekategorifiziert durch, um einen Ansatz für die abelsche Kategorie in Abhängigkeit von $\mathcal{C}$ und $\mathcal{X}$ zu bekommen. Dazu betrachten wir eine kommutative Frobenius-Algebra R und ziehen die von R induzierte topologische Feldtheorie entlang des Cover-Funktors zu einer G-äquivarianten topologischen Feldtheorie zurück. Anschließend berechnen wir die G-Frobenius-Algebra, welche zu dieser Feldtheorie äquivalent ist.

Durch Kategorifizieren der Struktur dieser G-Frobenius-Algebra erhalten wir eine G-äquivariante abelsche Kategorie $\mathcal{C}^{\mathcal{X}}$. Dann definieren wir einen $\mathcal{C}^{\mathcal{X}}$-erweiterten G-äquivarianten modularen Funktor durch Anwenden des zu $\mathcal{C}$ gehörenden modularen Funktors τ auf Totalräume von assoziierten Bündeln.

Die anschließende Berechnung der G-äquivarianten Ribbon-Struktur auf $\mathcal{C}^{\mathcal{X}}$ erweist sich als sehr anspruchsvoll. Im Fall $G = \mathbb{Z}/2$ können wir diese Berechnung noch vollständig durchführen, für beliebige Gruppen jedoch können wir nur noch die Modulkategorie-Strukturen auf den getwisteten Komponenten über der neutralen Komponente bestimmen. Nach einem Resultat von Etingof, Nikshych und Ostrik ist dadurch aber schon ein großer Teil der monoidalen Struktur auf der G-äquivarianten Kategorie bestimmt.

Abschließend wenden wir diese Resultate an und berechnen die modularen Invarianten der genannten Modulkategorien und zeigen so, dass alle modularen Invarianten vom Permutationstyp physikalisch sind.

Lebenslauf

Till Barmeier

21.02.1982	geboren in Hannover
2001	Abitur am Albert-Einstein-Gymnasium in Hameln
2007	Diplom in Mathematik an der Universität Hamburg
2007–2010	wissenschaftlicher Mitarbeiter an der Universität Hamburg
Oktober 2007–Februar 2008	Young Researcher Fellowship des European Superstring Theory Network