# Essays on Bargaining and Voting Power

Andreas Nohn

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## Thesis Committee

Chairman: Prof. Dr. Anke Gerber

First Examiner: Prof. Dr. Manfred J. Holler

Second Examiner: Prof. Dr. Hannu Vartiainen

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# Contents

# Introduction

Power is one of the most important concepts in the social sciences. According to the often used definition of Max Weber, it is an individual's ability to enforce his own interest against the resistance of others (Weber, 1947, p. 28).[1] This general definition, however, can have different instances that apply in different contexts and frameworks. The present work investigates bargaining and voting situations, and thus relevant are the two specific notions of bargaining and voting power. In a bargaining situation that is purely concerned with the division of a unit prize, nothing but the share an individual player gains can be suggested as defining his power. On the other hand, a player's power in a voting situation is commonly defined as his decisiveness, which is his ability, or the likelihood, of turning a losing coalition into a winning one (see, for instance, Felsenthal and Machover, 1998). The two notions are not, however, purely unambiguous – most technical frameworks considered in this work can be interpreted as both bargaining and voting.

Subject of the present work are specific approaches to the measurement of power, its properties, and possible foundations. The thesis consists of five articles. While not all of these are necessarily independent, every single paper is fully self-contained and can be read on its own. The first paper, 'The Public Good Index with Threats in A Priori Unions', is co-authored with Manfred J. Holler from the University of Hamburg, Germany and Public Choice Research Centre in Turku, Finland. It is published in *Essays in Honor of Hannu Nurmi, Vol. I, Homo Oeconomicus* 26(3/4), 2009. A follow-up paper, 'Axiomatizations of Public Good Indices with A Priori Unions' has been published in *Social Choice and Welfare* 35(3), 2010. Besides Manfred J. Holler, it is co-authored with José M.

[1] In the original German version, Weber uses the word 'Chance' for what I interpret as 'ability'. While 'Chance' could also be translated by 'probability', or simply by 'chance', Holler and Nurmi (2010) argue that it should rather be seen as 'possibility' or 'potential', an interpretation that is more in line with the term 'ability' used here.

Alonso-Meijide (University of Santiago de Compostela, Spain), Balbina Casas-Méndez (University of Santiago de Compostela, Spain), and Gloria Fiestras-Janeiro (University of Vigo, Spain). The third article, 'Monotonicity of Power in Weighted Voting Games with Restricted Communication' has been written together with previously mentioned José M. Alonso-Meijide and Stefan Napel from University of Bayreuth, Germany and Public Choice Research Centre in Turku, Finland. It has been presented at the Mika Widgrén Memorial Workshop in Turku, Finland (September 2009) and at the Annual Meeting of the European Public Choice Society in Izmir, Turkey (April 2010). The fourth paper is a single-authored note, 'Veto Players and Non-Cooperative Foundations of Power in Coalitional Bargaining'. It is available as PCRC working paper, and also submitted to and currently under review for Games and Economic Behavior. 'Coalitional Bargaining with Markovian Proposers' is the last paper, which is single-authored as well and also available as PCRC working paper. It has been presented at and improved from numerous comments at five occasions, most notably the Annual Meeting of the Public Choice Society in Monterey, California (March 2010).

The five articles can be seen as falling into three parts. The first two papers deal with specific measurement of power as they introduce and lay axiomatic foundations for variants of the Public Good Index for simple games with a priori unions. Standing alone as second part, the third article identifies and investigates monotonicity concepts for power in weighted voting games with restricted communication, a different variant of simple games. As, in a sense, non-cooperative counterparts, the fourth and fifth paper discuss possibilities of finding non-cooperative support for power indices in the Baron-Ferejohn model and a wide generalization.

'The Public Good Index with Threats in A Priori Unions' introduces four new variants of the Public Good Index (Holler, 1982) for simple games that have an a priori coalition structure. In these games, first considered by Owen (1977), there are unions of players which always vote as a block, such that either all members of a union, or none of them, belong to a coalition. The first variant, the Union Public Good Index, is closest in spirit to the original Public Good Index (PGI). It builds on the two assumptions that the coalitional value is a public good and only those winning coalitions that are minimal with respect to both the game and the coalition structure are relevant. The remaining three variants, called Threat Public Good Indices, distribute power in a two-step way. First, power is distributed between unions according to the PGI of the quotient game. Power within unions is then distributed proportional to the players' threat power, which is their power in case they leave their union. Each of the three indices considers one of three canonical coalition structures, or threat games, to arise upon the withdrawal of some player from his union.

'Axiomatizations of Public Good Indices with A Priori Unions' provides axiomatic characterizations for the Union PGI and Threat PGIs that build on arbitrary threat games. In addition, it offers two new characterizations for two coalitional extensions of the Public Good Index that have been previously established in Alonso-Meijide et al. (2008b), the Solidarity Public Good Index and the Owen Extended Public Good Index. The Union PGI, characterized in a rather elementary way, differs from the (original) axiomatization of the Solidarity PGI only in one axiom of symmetry and one of mergeability. The (new) characterizations of the other five variants are particularly suited to allow for comparisons betwen them. While all five indices share their being a coalitional PGI and that they satisfy quotient game property, the characterizations are completed and distinguish themselves by 'inner' axioms that regulate the distribution of power inside unions. The Solidarity PGI satisfies solidarity, all members of one union receive equal power. The power a union member is assigned by the Owen Extended PGI is proportional to his total power in a collection of simple games defined by the essential coalitions of the original game. As follows straight from the definition, the Threat PGIs distribute power proportional to the power players have in the threat game. I would like to add that, instead of only providing extensions of the PGI and corresponding characterizations, the particular form of the axiomatizations presented here allows to directly obtain analogous extensions for any other power index. To do so simply requires to consider a coalitional variant of a given baseline index, and then impose quotient game property and any set of desired inner axioms.

The third paper investigates monotonicity of power in a different variation of simple games, weighted voting games with restricted communication. As introduced by Myerson (1977), cooperation between players is restricted to those coalitions in which all members can, directly or at least indirectly, communicate with each other. In contrast to standard simple games, monotonicity of power in the voting weights of players might be plausibly violated in situations where better communication possibilities of one player outweigh another player's advantage in voting weight. 'Monotonicity of Power in Weighted Voting Games with Restricted Communication' hence sets out to, firstly, identify suitable monotonicity concepts for this framework. Various monotonicy concepts suggest themselves for both available dimensions, the voting weights of players as well as their communication possibilities. These monotonicities can deal with either local comparisons of two players in the same game, or global comparisons of one and the same player in two different games. Secondly, the paper investigates the monotonicity properties of solution concepts such as, most prominently, the Myerson value (Myerson, 1977), the restricted Banzhaf value (Owen, 1986), the position value (Borm et al., 1992), and the average tree solution (Herings et al., 2008, 2010). The results of which values satisfy or violate the various monotonicities allow to discriminate between the solution concepts, and also provide some interesting insights in the mechanisms they are based on. The Myerson value and restricted Banzhaf value satisfy all monotonicities in both weights and communication possibilities. The position value satisfies none of the monotonicity concepts, and the average tree solution satisfies all weight monotonicities but none of the communication monotonicities.

'Veto Players and Non-Cooperative Foundation of Power in Coalitional Bargaining', the note contained in chapter 4, discusses the possibilities of non-cooperative support for power indices in the Baron-Ferejohn bargaining model (Baron and Ferejohn, 1989). The discussion is based on the result that veto players either hold all power in this framework and then share it proportional to their recognition probabilities, or hold no power at all. Thus, non-cooperative support is restricted to power indices that either assign all or no power to veto players. While the core and related single-valued solution concepts such as the nucleolus can hence be given foundations (Montero, 2006), this is not the case for popular power indices such as the Shapley-Shubik index (Shapley and Shubik, 1954). This impossibility highlights a misinterpretation of results of Laruelle and Valenciano (2008a, and also 2008b, chapter 4.4) who seem to support the Shapley-Shubik index or, as they say, ‘any reasonable power index’, in the Baron-Ferejohn model. This interpretation is problematic, however, since arbitrary voting situations are modeled as a unanimity bargaining game. While the whole point of a non-unanimity voting situation seems to be that, quite simply, unanimity is not required, the result about veto players also shows that this incongruence can matter rather significantly.

The fifth paper, ‘Coalitional Bargaining with Markovian Proposers’, provides an extension of the Baron-Ferejohn model to non-independent proposers. The non-independence of proposers significantly widens the scope of possible proposer dynamics, for instance it allows for likely continued offers, alternating proposers (Calvó-Armengol, 2001a,b), and even deterministic protocols such as clockwise rotating proposers (Herrero, 1985). The generalization is motivated by a comparison of two canonical proposer dynamics applied to the particular bargaining situation in which one veto player can create a unit surplus with either of two minor players. If all players in all rounds have equal probability to be the proposer (and are thus chosen independently), the veto player holds all power. He has, however, only half of the power if proposers alternate, that is, it is always the respondent of a round that becomes proposer in the next round. This shows not only a quantitative but a qualitative difference between these two proposer dynamics. In addition, it indicates the possibility of non-cooperative support for power indices that do not assign all power to veto players – if power behaved ‘sufficiently continuous’ in the proposer dynamics, support, for instance for the Shapley-Shubik index, could be found somewhere ‘between’ the independent and the alternating protocol. However, analysis of bargaining situations with veto players shows the results found in the note are widely stable. Conversely, that the alternating protocol does not assign all power to the single veto player in the introductory example seems to be a mere singularity. Thus, the results show a great stability of support for the core and single-valued concepts such as the nucleolus. At the same time, the possibilities of non-cooperative support for indices that do not assign all or no power to veto players, such as the Shapley-Shubik index, are further limited to, at most, singular protocols.

# Chapter 1

# The Public Good Index with Threats in A Priori Unions*

*Abstract* We consider four variants of the Public Good Index for games with a priori unions. The first variant extends the original idea of Holler (1982) assuming that the coalitional value is a public good and only minimal winning coalitions are relevant. The remaining three variants assign power in a two-step way. In the first step, power is distributed between unions according to the PGI of the quotient game. On the member stage, the indices take into account the possibilities of players to threaten their partners through leaving their union. We discuss the four indices and compare them to two extensions previously introduced in Alonso-Meijide et al. (2008b). Theoretical reasoning and numerical examples demonstrate the various measures may differ substantially.

*Keywords* simple game, coalition structure, Public Good Index

*JEL Classification* C71

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* Published in Homo Oeconomicus 26(3/4), 2009. Co-authored with Manfred J. Holler (University of Hamburg, Germany and Public Choice Research Centre, Turku, Finland).

## 1.1 Introduction

Owen (1977) proposed an extension of the Shapley value to games with a priori unions, i.e., a partition of the set of players which describes an a priori coalition structure. In the first step, this measure distributes the total value of a winning coalition to the a priori unions in accordance with the Shapley value. In the second step, once again applying the Shapley value, the total reward of a union is allocated among its members, taking into account the possibility of their joining other unions. Using similar reasoning, Owen (1982) proposed a version of the Banzhaf value for measuring power in TU games with a priori unions. More recently, Alonso-Meijide et al. (2009b) extended the Deegan-Packel index to games with a priori unions. In a second paper, Alonso-Meijide et al. (2008b) discuss two characterizations of the Public Good Index for games with a priori unions. The first alternative, the Solidarity PGI, stresses the public good property which suggests that all members of a winning coalition derive equal power. The second one follows Owen (1977, 1982) and refers to essential coalitions when allocating power shares.

The following paper considers further variants of applying the PGI to games with a priori unions which differ with respect to the threat potential a player has by his or her possibility to join players or sets of players outside the a priori union that defines the membership in the game. The first variant builds on Holler (1982) assuming that the coalition is a public good and only minimal winning coalitions are relevant when it comes to measuring power. Here threats have no impact on the distribution of the coalition value as the coalition value is a pure public good and exclusion and rivalry in consumption do not apply. The remaining three variants assign power in a two-step way. In the first step, power is distributed between unions according to the PGI of the quotient game. On the member stage, the indices take into account the possibilities of players to threaten their partners through leaving their union.

Obviously, the discussed variants constitute different solution concepts for coalition games with a priori unions. The discussion will demonstrate that 'different solution concepts can...be thought of as results of choosing not only which properties one likes, but also which examples one wishes to avoid' (Aumann, 1977, p. 471).

The paper is organized as follows. In section 2, we provide the analytical tools such as simple games, power indices, and coalition structures. Section 3 introduces the four new coalitional Public Good Indices. In section 4, we investigate basic properties and the relationships of the overall six coalitional PGIs.

## 1.2 Preliminaries

### 1.2.1 Simple Games and the Public Good Index

A *simple game* is a pair $(N, W)$ of the finite *set of players* $N = \{1, \ldots, n\}$ and the *set of winning coalitions* $W$ satisfying $\emptyset \notin W$, $N \in W$ and $S \in W \Rightarrow T \in W$ for all $S \subseteq T$. A coalition $S \subseteq N$ is called *winning* or *losing* according to whether $S \in W$ or $S \notin W$. A winning coalition $S$ is a *minimal winning coalition* (MWC) if each proper subset $T \subset S$ is a losing coalition. We denote the *set of minimal winning coalitions* by $M$. Since $M$ contains all relevant information and is more suitable for what follows, we mainly denote the game $(N, W)$ by the equivalent description $(N, M)$ throughout this work.

A *power index* is a mapping $f$ assigning each simple game $(N, M)$ an $n$-dimensional real valued vector $f(N, M) = (f_1(N, M), \ldots, f_n(N, M))$. Based on the assumptions that coalitional values are public goods and only minimal winning coalitions are relevant when it comes to power, the *Public Good Index* (PGI) proposed by Holler (1982) assigns power proportional to the number of minimal winning coalitions a player belongs to. Denoting $M_i$ as the set of minimal winning coalitions containing $i$, the PGI $\delta$ is given by

$$\delta_i(N, M) = \frac{|M_i|}{\sum_j |M_j|}, \quad i = 1, \ldots, n. \tag{1.1}$$

Holler and Packel (1983) characterize the PGI as the unique power index satisfying efficiency, symmetry, null player, and PGI-mergeability, the axioms being defined as follows. An index $f$ satisfies *efficiency* if $\sum_i f_i(N, M) = 1$ for all simple games $(N, M)$. Two players $i$ and $j$ are *symmetric* if $S \cup \{i\} \in W \Leftrightarrow S \cup \{j\} \in W$ for all $S \not\ni i, j$. A player $i$ is called a *null player* if $S \backslash \{i\} \in W$ for all coalitions $S \in W$. Power index $f$ satisfies *symmetry* if $f_i(N, M) = f_j(N, M)$ for all symmetric players $i$ and $j$ and *null player* if it holds $f_i(N, M) = 0$ for all null players $i$. Two simple games $(N, M)$ and $(N, M')$ are *mergeable* if $S \in M$ implies $S \notin W'$ and $S \in M'$ implies $S \notin W$. In particular, the sets of minimal winning coalitions $M$ and $M'$ are disjoint. The *merged game* $(N, M \oplus M')$ of two mergeable games $(N, M)$ and $(N, M')$ is defined by $M \oplus M' = M + M'$. Now, a power index $f$ satisfies *PGI-mergeability* if for all mergeable games $(N, M)$ and $(N, M')$ it holds $\sum_i |M_i + M'_i| f(N, M \oplus M') = \sum_i |M_i| f(N, M) + \sum_i |M'_i| f(N, M')$.

### 1.2.2 Simple Games with Coalition Structures

For a set of players $N$, a *coalition structure* is a partition $P = \{P_1, \ldots, P_p\}$ of $N$, that is, a set of nonempty and mutually disjoint subsets of $N$ whose union coincides with $N$. We also use $P$ as the mapping assigning each player $i$ the *union* $P(i) \in P$ he is a member of. A *simple game with coalition structure* is a triplet $(N, W, P)$, that is, a set of players $N$, a set of winning coalitions $W$, and a coalition structure $P$ on $N$.

Given such a game, the corresponding *quotient game* is the simple game $(P, W^P)$ with player set $P$ and set of winning coalitions $W^P$. A coalition $R \subseteq P$ in the quotient game is winning if and only if the coalition of represented unions $\bigcup_{Q \in R} Q$ is winning in $(N, W)$. We denote the set of minimal winning coalitions in the quotient game by $M^P$ and by $M^P_Q$ the set of minimal winning coalitions containing union $Q \in P$.

In analogy to simple games, we denote simple games with coalition structures by $(N, M, P)$ and the corresponding quotient game by $(P, M^P)$. Thus, a union $Q$'s power in the quotient game, measured by the PGI, amounts to

$$\delta_Q(P, M^P) = \frac{|M^P_Q|}{\sum_k |M^P_{P_k}|}, \quad Q = P_1, \ldots, P_p. \tag{1.2}$$

A *coalitional power index* is a mapping $f$ assigning each simple game with coalition structure $(N, M, P)$ an $n$-dimensional real valued vector $f(N, M, P) = (f_1(N, M, P), \ldots, f_n(N, M, P))$.

### 1.2.3 Previous Extensions of the PGI for A Priori Unions

Alonso-Meijide et al. (2008b) introduce two variations of the PGI for a priori unions, the

*Solidarity Public Good Index* and the *Owen Extended Public Good Index*. Both indices distribute power in two steps. In the first step, they assign power to each union equal to their PGI in the quotient game (thus satisfying *quotient game property*). In the second step, they use alternative methods to distribute the union's power among its members. The Solidarity Public Good Index $\Upsilon$ does so giving each union member equal power, that is,

$$\Upsilon_i(N, M, P) = \delta_{P(i)}(P, M^P)\frac{1}{|P(i)|}, \quad i = 1, \dots, n. \tag{1.3}$$

Following an idea of Owen (1977), the Owen Extended Public Good Index splits power within unions according to the possibilities that the subsets of this union have to form winning coalitions with other unions. For this purpose, a subset $S \subseteq Q$ of union $Q \in P$ is an *essential part* with respect to coalition $R \in M_Q^P$ if $S \cup \bigcup_{Q' \in R \setminus \{Q\}} Q'$ is a winning coalition in $(N, M)$ and $T \cup \bigcup_{Q' \in R \setminus \{Q\}} Q'$ is losing for all $T \subset S$. Denoting the *set of essential parts* with respect to $R$ containing player $i$ by $E_i^R(N, M, P)$, the Owen Extended Public Good Index $\Gamma$ is defined as

$$\Gamma_i(N, M, P) = \delta_{P(i)}(P, M^P) \sum_{R \in M_{P(i)}^P} \frac{1}{|M_{P(i)}^P|} \frac{|E_i^R(N, M, P)|}{\sum_{j \in P(i)} |E_j^R(N, M, P)|}, \quad i = 1, \dots, n. \tag{1.4}$$

## 1.3 New Variations of the PGI for A Priori Unions

### 1.3.1 The Union Public Good Index

The first variation for a priori unions we introduce is the *Union Public Good Index* $\Lambda$. As close as possible to the original spirit of the PGI, it is based on the two assumptions that the coalitional value is a public good and only minimal winning coalitions are relevant. The latter assumption does, however, apply to coalitions being minimal also with respect to the coalition structure. A player's power is hence proportional to the number of minimal winning coalitions his union is a member of in the quotient game, that is,

$$\Lambda_i(N, M, P) = \frac{|M_{P(i)}^P|}{\sum_k |P_k||M_{P_k}^P|}, \quad i = 1, \dots, n. \tag{1.5}$$

As with the Solidarity PGI, it is obviously the case that all members of the same union have equal power, that is, the Union PGI as well satisfies *solidarity*. However, the Union PGI is the only of the overall six extensions not assigning power to unions on the basis of the PGI in the corresponding quotient game.

### 1.3.2 Power Distribution Based on Threats

There are different approaches to the allocation of power inside unions taking into account the players' threat power in case that they leave their union. Several possibilities arise concerning how the remaining union structure behaves upon withdrawal of a player from a union. While the union might break up into any possible partition, or even the whole union structure might change, we consider the three canonical cases in which (i) the respective union is not more stable than other unions, (ii) all other members of the respective union and (iii) all other unions are treated symmetrically. In either case, at the union level, power is distributed according to the PGI of the quotient game. At the member stage, players receive a share proportional to their power in the the *threat game*, that is, the game induced by their withdrawal from their union.

Assuming the least possible degree of stability, the first variant considers that all unions break up into singletons upon a player's withdrawal from his union. Power inside unions is split by the *Threat PGI* $T^1$ proportional to the players' power in the game without any unions. This approach turns a blind eye on how the union structure behaves in the long run and, at the same time, takes the reasoning behind the Owen Extended PGI one step further. Now, as it comes to intra-union allocation of power, subsets of a union are not allowed to cooperate with other unions only, but also with subsets of other unions. We thus define

$$T_i^1(N, M, P) = \delta_{P(i)}(P, M^P) \frac{\delta_i(N, M)}{\sum_{j \in P(i)} \delta_j(N, M)}, \quad i = 1, \ldots, n, \tag{1.6}$$

whenever $\sum_{j \in P(i)} \delta_j(N, M) > 0$ and $T_i^1(N, M, P) = 0$ otherwise.

Presupposing a greater degree of stability, the second variant considers a breakup of the leaving player's union into singletons while all other unions endure. For union

$Q \in P$, denote $P/Q = P \backslash \{Q\} \cup \{\{i\} | i \in Q\}$ as the union structure where $Q$ breaks up into singletons $\{i\}$, $i \in Q$, and define the *Threat PGI* $T^2$ by

$$T_i^2(N, M, P) = \delta_{P(i)}(P, M^P) \frac{\delta_{\{i\}}(P/P(i), M^{P/P(i)})}{\sum_{j \in P(i)} \delta_{\{j\}}(P/P(i), M^{P/P(i)})}, \quad i = 1, \ldots, n, \quad (1.7)$$

whenever $\sum_{j \in P(i)} \delta_{\{j\}}(P/P(i), M^{P/P(i)}) > 0$ and $T_i^2(N, M, P) = 0$ otherwise.

Finally, the third variant assumes that neither the cooperation of the rest of the union members nor that of other unions is affected by one member's leaving his union. By $P/i = P \backslash \{P(i)\} \cup \{\{i\}, P(i) \backslash \{i\}\}$ for a player $i$ and partition $P$ we denote the union structure in which $i$ separates from his union $P(i)$ and plays on his own. Define the *Threat PGI* $T^3$ by

$$T_i^3(N, M, P) = \delta_{P(i)}(P, M^P) \frac{\delta_{\{i\}}(P/i, M^{P/i})}{\sum_{j \in P(i)} \delta_{\{j\}}(P/j, M^{P/j})}, \quad i = 1, \ldots, n, \quad (1.8)$$

whenever $\sum_{j \in P(i)} \delta_{\{j\}}(P/j, M^{P/j}) > 0$, and $T_i^3(N, M, P) = \delta_{P(i)}(P, M^P) \frac{1}{|P(i)|}$ otherwise.[1]

## 1.4 Basic Properties of Coalitional PGIs

Alonso-Meijide et al. (2008b) provide axiomatizations of the Solidarity PGI $\Upsilon$ and the Owen Extended PGI $\Gamma$. While this does not lie within the scope of our work, we still want to comment on the basic properties and relationships of the coalitional extensions of the Public Good Index discussed here.

To begin with, all PGI extensions are efficient values. While all values but the Union PGI $\Lambda$ satisfy symmetry among unions, this holds for $\Lambda$ for equally sized unions only.

[1] Here, a distinction for whether the denominator is 0 or not is necessary. Consider for instance $N = \{1, 2, 3, 4\}$ and $M = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$. With coalition structure $P = \{\{1\}, \{2, 3, 4\}\}$, it is $M^P = \{\{\{1\}, \{2, 3, 4\}\}\}$ and thus $\delta_{\{2,3,4\}}(P, M^P) = \frac{1}{2}$. However, all members $i \in \{2, 3, 4\}$ constitute a null union $\{i\}$ with a power of 0 in their respective threat game $P/i$. Note this distinction was not necessary in the case of indices $T^1$ and $T^2$. For both indices it holds that, if all members $i \in Q$ constitute null unions $\{i\}$ in the corresponding threat games, then union $Q$ is a null union itself. Thus, in case the denominator is 0, this also holds for $\delta_Q(P, M^P)$ and hence for the overall expression.

Symmetry inside unions does, however, always apply. While all indices give a power of zero to null unions, all but the Union PGI $\Lambda$ and Solidarity PGI $\Upsilon$ necessarily give zero power to null players – the latter two indices satisfy solidarity property because of which null players have positive power if their union is not a null union. In addition, all indices but the Union PGI $\Lambda$ distribute power in a two-step way such that unions receive overall power as much as assigned by the PGI in the corresponding quotient game (quotient game property).

In order to investigate similarities and differences of the overall six PGI extensions, we start by considering the trivial coalition structures $P^n = \{\{1\}, \ldots, \{n\}\}$ and $P^N = \{\{1, \ldots, n\}\}$. In games where the coalition structure is given by singletons ($P^n$), all coalitional PGIs coincide with the PGI of the game without union structure. However, in the case $P^N$ with one grand union, the Union PGI $\Lambda$ and the Solidarity PGI $\Upsilon$ amount to the egalitarian power distribution. While the Owen Extended PGI $\Gamma$ and Threat PGI $T^1$ coincide with the PGI of the simple game without union structure, no statements can be made concerning the behavior of threat indices $T^2$ and $T^3$. In addition, the following can be said about trivial coincidences of the various extensions.

The Union PGI $\Lambda$ and the Solidarity PGI $\Upsilon$ coincide whenever all unions have equal size.

If at most one union contains more than one member, Threat PGIs $T^1$ and $T^2$ coincide.

If at most one union contains two members and all other unions are singletons, threat indices $T^1$, $T^2$, and $T^3$ coincide.

However, we can find significant differences between the indices as in the following examples.

**Example 1.1** *The possible gap between the Union PGI and the other coalitional PGIs becomes apparent in large, star-formed games* $(N, M)$ *with set of players* $N = \{1, \ldots, n\}$ *and set of minimal winning coalitions* $M = \{\{1, 2\}, \ldots, \{1, n\}\}$*. In the corresponding game with coalition structure* $P = \{\{1\}, \{2, \ldots, n\}\}$*, the only winning coalition is the*

| $N$ | $\delta$ | $\Lambda$ | $\Upsilon$ | $\Gamma$ | $T^1$ | $T^2$ | $T^3$ |
|---|---|---|---|---|---|---|---|
| 1 | $\frac{4}{13}$ | $\frac{1}{6}$ | $\frac{2}{18}$ | $\frac{6}{24}$ | $\frac{4}{18}$ | $\frac{6}{30}$ | $\frac{2}{6}$ |
| 2 | $\frac{2}{13}$ | $\frac{1}{6}$ | $\frac{3}{18}$ | $\frac{4}{24}$ | $\frac{3}{18}$ | $\frac{5}{30}$ | $\frac{1}{6}$ |
| 3 | $\frac{2}{13}$ | $\frac{1}{6}$ | $\frac{3}{18}$ | $\frac{4}{24}$ | $\frac{3}{18}$ | $\frac{5}{30}$ | $\frac{1}{6}$ |
| 4 | $\frac{3}{13}$ | $\frac{1}{6}$ | $\frac{6}{18}$ | $\frac{8}{24}$ | $\frac{6}{18}$ | $\frac{10}{30}$ | $\frac{2}{6}$ |
| 5 | $\frac{1}{13}$ | $\frac{1}{6}$ | $\frac{2}{18}$ | $\frac{1}{24}$ | $\frac{1}{18}$ | $\frac{2}{30}$ | 0 |
| 6 | $\frac{1}{13}$ | $\frac{1}{6}$ | $\frac{2}{18}$ | $\frac{1}{24}$ | $\frac{1}{18}$ | $\frac{2}{30}$ | 0 |

Table 1.1: Indices for example 1.2.

*grand coalition made up by the two unions $\{1\}$ and $\{2, \ldots, n\}$. The Union PGI $\Lambda$ gives all players in this grand coalition equal power and hence amounts to the egalitarian power distribution (all players having power $\frac{1}{n}$). Satisfying quotient game property as well as symmetry among and inside unions, all other indices assign power $\frac{1}{2}$ to player 1 and $\frac{1}{2(n-1)}$ to the minor players. Note that the overall power of unions thus sums up to $\frac{1}{n}$ for union $\{1\}$ and $\frac{n-1}{n}$ for union $\{2, \ldots, n\}$ in the case of $\Lambda$, while all other indices assign equal power to the two symmetric unions.*

**Example 1.2** *In general, extensions show different aspects of power. Consider for instance the simple game $(N, M)$ given by $N = \{1, \ldots, 6\}$ and $M = \{\{1,2\}, \{1,3\}, \{1,4,5\}, \{1,4,6\}, \{2,3,4\}\}$. A possible representation as a weighted voting game is $(55; 35, 20, 20, 15, 5, 5)$. So $(N, M)$ is not decisive and the PGI $\delta(N, M) = (\frac{4}{13}, \frac{2}{13}, \frac{2}{13}, \frac{3}{13}, \frac{1}{13}, \frac{1}{13})$ is not monotonic in voting weights. Table 1.1 shows that no two coalitional PGIs are the same for coalition structure $P = \{\{1,5,6\}, \{2,3\}, \{4\}\}$. Given this coalition structure, any two unions form a minimal winning coalition in the quotient game, that is, $M^P = \{\{\{1,5,6\}, \{2,3\}\}, \{\{1,5,6\}, \{4\}\}, \{\{2,3\}, \{4\}\}\}$.*

*Ignoring symmetry between unions but rather accounting for all players being in an* *equal number of minimal winning coalitions in the quotient game, the Union PGI* $\Lambda$ *distributes power in an egalitarian way. The remaining indices respect quotient game and symmetry among unions and hence split power equally between unions. While the Solidarity PGI* $\Upsilon$ *distributes this power equally among members, the remaining indices do so only for the symmetric members of union* $\{2,3\}$*. Concerning union* $\{1,5,6\}$*, the Owen Extended PGI* $\Gamma$ *accounts for the essential parts of a player, thus shifting power towards the major player* $1$*. Threat index* $T^1$ *does so to a slightly lesser extent, splitting power between members according to their PGI* $\delta$*. Again somewhat less variability is shown by threat index* $T^2$*, allocating power in union* $\{1,5,6\}$ *proportional to the members' power in the threat game with coalition structure* $P/\{1,5,6\} = \{\{1\},\{2,3\},\{4\},\{5\},\{6\}\}$*. Finally, the greatest degree of variability is shown by threat index* $T^3$ *which allocates power in union* $\{1,5,6\}$ *proportional to the members' power in the games with coalition structure* $P/1$*,* $P/5$*, and* $P/6$*, respectively. The minor players constitute null unions in the corresponding threat games such that, explaining the minor players' dependence on their union, player* $1$ *receives the whole share of power.*

# Chapter 2

# Axiomatizations of Public Good Indices with A Priori Unions*

*Abstract* We provide axiomatizations for six variants of the Public Good Index for games with a priori unions. Two such coalitional PGIs have been introduced and alternatively axiomatized in Alonso-Meijide et al. (2008b). They assign power in two steps. In the first step, power is distributed between unions according to the PGI of the quotient game. In a second step, the Solidarity PGI splits power equally among union members while the Owen Extended PGI takes into account so-called essential parts. The other four coalitional PGIs have been introduced in Holler and Nohn (2009). The first variant elaborates the original idea of Holler (1982) that the coalitional value is a public good and only minimal winning coalitions of the quotient game are relevant. The remaining three variants also use the two-step distribution where, however, on the member stage they take into account the possibilities of players to threaten their partners through leaving their union.

*Keywords* power, simple game, a priori unions, coalition structure, Public Good Index

*JEL Classification* C71

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## 2.1 Introduction

Forming coalitions is the most natural behavior in transferable utility cooperative games (TU games from now on). It is clear that coalitions among players with similar ideology or culture, or common economic interests, are more likely than others. TU games with a priori unions were first considered in Aumann and Drèze (1974) who extended the Shapley value (Shapley, 1953) to this new framework in such a manner that the game splits into subgames played by the unions isolated from each other. In this case, every player receives the payoff allocated to him by the Shapley value in the subgame he is playing within his union. A second approach was considered in Owen (1977) in which the coalitional value (or Owen value) is defined and characterized. In this case, the unions play a quotient game among themselves, and each one receives a payoff which, in turn, is shared among its players in an internal game. Both payoffs, in the quotient game for unions and within each union for its players, are given by the Shapley value. The Owen value is a coalitional Shapley value (it coincides with the Shapley value in case that all unions are singletons) and satisfies the quotient game property, that is, it coincides with the Shapley value if each a priori union contains one element only and the total power assigned to the players of an a priori union equals the power of this union in the game played by the unions.

Using similar reasoning, Owen (1982) proposed an application of the Banzhaf value (the natural extension of the Banzhaf-Coleman index to the family of TU games proposed in Owen, 1975b) to the framework of TU games with a priori unions. This measure is referred to as Banzhaf-Owen value. Here the assessments in each of the two steps are given by the Banzhaf value. The Banzhaf-Owen value is a coalitional Banzhaf value (it coincides with the Banzhaf value in case that all unions are singletons) but it does not satisfy quotient game property (the overall power of members of a union is not necessarily equal to that union's power in the quotient game). Alonso-Meijide and Fiestras-Janeiro (2002) propose a value for TU games with a priori unions which is a coalitional Banzhaf value and satisfies quotient game property. This value, the symmetric coalition Banzhaf value, reflects the result of a bargaining procedure by which, in the quotient game, each a priori union receives a payoff determined by the Banzhaf value; and, within each union, the members share this payoff in accordance with the Shapley value.

A particular class of TU games is the class of simple games. In this setting values are referred to as power indices. They are quantitative measures to express power. In the literature we can find a series of power indices: the Shapley-Shubik index (defined and characterized in Shapley and Shubik, 1954, as the restriction of the Shapley value to the family of simple games), the Banzhaf-Coleman index (Banzhaf, 1965, Coleman, 1971), the Deegan-Packel index (Deegan and Packel, 1978), the Johnston power index (Jonston, 1978), and the Public Good Index (Holler, 1978). Some of these power indices have a counterpart in the setting of simple games with a priori unions.

In this paper we focus on the Public Good Index. This power index was explicitly proposed in Holler (1982) and axiomatized in Holler and Packel (1983) and, recently, in Alonso-Meijide et al. (2008a). The main assumptions of the Public Good Index are that only minimal winning coalitions are relevant and only the number of these coalitions a player belongs to is used to measure power. Alonso-Meijide et al. (2008b) proposed and characterized two extensions of this power index to the setting of simple games with a priori unions. The first one, the Solidarity Public Good Index, stresses the public good property which suggests that all members of a winning coalition derive equal power, irrespective of their possibility to form alternative coalitions. The second extension follows the same line as that considered in Owen (1977) to extend the Shapley value. However, it employs the Public Good Index as the solution in both steps of the bargaining procedure. Holler and Nohn (2009) introduce, without yet providing axiomatic characterizations, four different extensions of the Public Good Index. The first variant, the Union Public Good Index, follows the main argument of the original Public Good Index, translated to the model with a priori unions, that is, only minimal winning coalitions of the quotient game are relevant. This index is a coalitional Public Good Index but does not satisfy quotient game property. The other three variants, the Threat Public Good Indices, satisfy both properties and take into account, for three canonical scenarios, the players' threat power in case that they leave their union.

In this paper, we provide axiomatic characterizations of the four variants of the Public Good Index proposed in Holler and Nohn (2009), and two new characterizations of the two extensions proposed in Alonso-Meijide et al. (2008b). The paper is organized as follows. In section 2, some basic definitions are introduced. In section 3, we provide characterizations of different variants of the Public Good Index for games with a priori unions. Finally, we illustrate and compare these extensions for a real-world example in section 4.

## 2.2 Preliminaries

### 2.2.1 Simple Games and the Public Good Index

A *simple game* is a pair $(N, W)$ of the finite *set of players* $N = \{1, \ldots, n\}$ and the *set of winning coalitions* $W$ satisfying $\emptyset \notin W$, $N \in W$ and $S \in W \Rightarrow T \in W$ for all $S \subseteq T$. A coalition $S \subseteq N$ is called *winning* or *losing* according to whether $S \in W$ or $S \notin W$. A winning coalition $S$ is a *minimal winning coalition* (MWC) if each proper subset $T \subset S$ is a losing coalition. We denote the *set of minimal winning coalitions* by $M$. Since $M$ contains all relevant information and is more suitable for what follows, we mainly denote the game $(N, W)$ by the equivalent description $(N, M)$ throughout this work.

A *power index* is a mapping $f$ assigning each simple game $(N, M)$ an $n$-dimensional real valued vector $f(N, M) = (f_1(N, M), \ldots, f_n(N, M))$. Based on the assumptions that coalitional values are public goods and only minimal winning coalitions are relevant when it comes to power, the *Public Good Index* (PGI) proposed by Holler (1982) assigns power proportional to the number of minimal winning coalitions a player belongs to. Denoting $M_i$ as the set of minimal winning coalitions containing $i$, the PGI $\delta$ is given by

$$\delta_i(N, M) = \frac{|M_i|}{\sum_j |M_j|}, \quad i = 1, \ldots, n. \tag{2.1}$$

Holler and Packel (1983) characterize the PGI as the unique power index satisfying efficiency, symmetry, null player, and PGI-mergeability, the axioms being defined as follows. An index $f$ satisfies *efficiency* if $\sum_i f_i(N, M) = 1$ for all simple games $(N, M)$.

Two players $i$ and $j$ are *symmetric* if $S \cup \{i\} \in W \Leftrightarrow S \cup \{j\} \in W$ for all $S \subseteq N \setminus \{i, j\}$. A player $i$ is called a *null player* if $S \backslash \{i\} \in W$ for all coalitions $S \in W$ with $i \in S$. Power index $f$ satisfies *symmetry* if $f_i(N, M) = f_j(N, M)$ for all symmetric players $i$ and $j$, and *null player* if it holds $f_i(N, M) = 0$ for all null players $i$. Two simple games $(N, M)$ and $(N, M')$ are *mergeable* if $S \in M$ implies $S \notin W'$ and $S \in M'$ implies $S \notin W$. In particular, the sets of minimal winning coalitions $M$ and $M'$ are disjoint. The *merged game* of two mergeable games $(N, M)$ and $(N, M')$ is defined by $(N, M \cup M')$. Now, a power index $f$ satisfies *PGI-mergeability* if for all mergeable games $(N, M)$ and $(N, M')$ it holds

$$f(N, M \cup M') = \frac{\sum_i |M_i|}{\sum_i (|M_i| + |M'_i|)} f(N, M) + \frac{\sum_i |M'_i|}{\sum_i (|M_i| + |M'_i|)} f(N, M').$$

### 2.2.2 Simple Games with A Priori Unions

For a set of players $N$, a *coalition structure* or set of *a priori unions* is a partition $P = \{P_1, \ldots, P_p\}$ of $N$, that is, a set of nonempty and mutually disjoint subsets of $N$ whose union coincides with $N$. There are two trivial a priori unions for player set $N$. The first is the structure where each player forms his own union, that is, $N^0 = \{\{i\} | i \in N\}$. The second one, $N^1 = \{N\}$, consists of the grand union only. We also use $P$ as the mapping assigning each player $i$ the *union* $P(i) \in P$ he is a member of. Also, $P(S) = \{Q \in P | Q \cap S \neq \emptyset\}$ for coalition $S \subseteq N$ is the set of unions having a member in coalition $S$. A *simple game with a priori unions* is a triplet $(N, W, P)$, that is, a set of players $N$, a set of winning coalitions $W$, and a coalition structure $P$ on $N$.

Given such a game, the corresponding *quotient game* is the simple game $(P, W^P)$ with player set $P$ and set of winning coalitions $W^P$. A coalition $R \subseteq P$ in the quotient game is winning if and only if the coalition of represented unions $\bigcup_{Q \in R} Q$ is winning in $(N, W)$. We denote the set of minimal winning coalitions in the quotient game by $M^P$ and by $M^P_Q$ the set of minimal winning coalitions containing union $Q \in P$.

In analogy to simple games, we denote simple games with a priori unions by $(N, M, P)$ and the corresponding quotient game by $(P, M^P)$. Thus, a union $Q$'s power in the quotient game, measured by the PGI, amounts to

$$\delta_Q(P, M^P) = \frac{|M_Q^P|}{\sum_k |M_{P_k}^P|}, \quad Q = P_1, \ldots, P_p. \tag{2.2}$$

A *coalitional power index* is a mapping $f$ assigning each simple game with a priori unions $(N, M, P)$ an $n$-dimensional real valued vector $f(N, M, P) = (f_1(N, M, P), \ldots, f_n(N, M, P))$.

### 2.2.3 Coalitional PGIs and Quotient Game Property

Given a power index $g$, a coalitional power index $f$ is a *coalitional $g$-value* if for all simple game $(N, M)$ it holds $f(N, M, N^0) = g(N, M)$, that is, if $f$ is equal to $g$ in case of any simple game $(N, M)$ and singleton a priori unions $N^0$.

The following four properties constitute the analogues of the four axioms of the PGI for coalitional power indices in the case of singleton a priori unions. A coalitional power index $f$ satisfies

*singleton efficiency* if for all simple games $(N, M)$ it holds $\sum_i f_i(N, M, N^0) = 1$.

*singleton null player* if for all simple games $(N, M)$ it holds $f_i(N, M, N^0) = 0$ for all null players $i$ in $(N, M)$.

*singleton symmetry* if for all simple games $(N, M)$ it is $f_i(N, M, N^0) = f_j(N, M, N^0)$ for all players $i$ and $j$ which are symmetric in $(N, M)$.

*singleton PGI-mergeability* if for any two mergeable simple games $(N, M)$ and $(N, M')$ it holds

$$f(N, M \cup M', N^0) = \frac{\sum_i |M_i| f(N, M, N^0) + \sum_i |M'_i| f(N, M', N^0)}{\sum_i (|M_i| + |M'_i|)}.$$

Perfectly analogous to the axiomatization of the PGI in Holler and Packel (1983), one finds the following proposition.

**Proposition 2.1** *A coalitional power index $f$ is a coalitional Public Good Index, $f(N, M, N^0) = \delta(N, M)$ for all simple games $(N, M)$, if and only if it satisfies singleton efficiency, singleton null player, singleton symmetry, and singleton PGI-mergeability.*

A coalitional power index $f$ satisfies *quotient game property* if for all simple games with a priori unions $(N, M, P)$ and all unions $Q \in P$ it holds

$$\sum_{i \in Q} f_i(N, M, P) = f_Q(P, M^P, P^0).$$

In the presence of quotient game property, three of the singleton properties are significantly enhanced. Singleton efficiency then induces

*efficiency* – it is $\sum_{i \in N} f_i(N, M, P) = 1$ for all simple games with a priori unions $(N, M, P)$.

Singleton null player implies

*null union* – for all simple games with a priori unions $(N, M, P)$ and any union $Q$ which is a null player in the quotient game $(P, M^P)$ it is $\sum_{i \in Q} f_i(N, M, P) = 0$.

It does *not*, however, also imply

*null player* – for all simple games with a priori unions $(N, M, P)$ and any null player $i$ in $(N, M)$ it is $f_i(N, M, P) = 0$.

In fact there are coalitional indices as for instance the Union PGI and the Solidarity PGI satisfying singleton null player and null union but not null player. Singleton symmetry turns into

*symmetry among unions* – for all simple games with a priori unions $(N, M, P)$ and all unions $Q, Q'$ which are symmetric players in the quotient game $(P, M^P)$ it is $\sum_{i \in Q} f_i(N, M, P) = \sum_{i \in Q'} f_i(N, M, P)$.

In analogy to singleton null player (which does not imply null player), singleton symmetry does *not* imply

*symmetry within unions* – for all simple games with a priori unions $(N, M, P)$ and all players $i, j$ which are symmetric players in $(N, M)$ and members of the same union it holds $f_i(N, M, P) = f_j(N, M, P)$.

Also, note that quotient game property does not necessarily extend singleton PGI-mergeability to any stronger version of PGI-mergeability.

For coalitional Public Good Indices which satisfy quotient game property we thus find efficiency, null union, and symmetry among unions while they do not necessarily satisfy null player, symmetry within unions or any particular form of PGI mergeability. In addition, for a union $Q$'s overall power one finds it is equal to its Public Good Index in the quotient game,

$$\sum_{i \in Q} f_i(N, M, P) = \delta_Q(P, M^P).$$

## 2.3 Axiomatizations of PGIs for A Priori Unions

### 2.3.1 The Solidarity Public Good Index

Alonso-Meijide et al. (2008b) introduce and axiomatize the *Solidarity Public Good Index* and the *Owen Extended Public Good Index.* Both indices distribute power in two steps. In the first step, they assign power to each union equal to their PGI in the quotient game (thus satisfying quotient game property). In the second step, they use a distinct method to distribute the union's power among its members.

The Solidarity Public Good Index $\Upsilon$ stresses the public good property by assigning equal power to each member of the same a priori union,

$$\Upsilon_i(N, M, P) = \delta_{P(i)}(P, M^P) \frac{1}{|P(i)|}, \quad i = 1, \ldots, n. \tag{2.3}$$

It thus satisfies

*solidarity* – for all simple games with a priori unions $(N, M, P)$ it holds $f_i(N, M, P) = f_j(N, M, P)$ for all players $i, j$ being member of the same union $P(i) = P(j)$.

Alonso-Meijide et al. (2008b) provide an axiomatization of the Solidarity PGI not using quotient game property but, among others, the following two properties:

*independence of superfluous coalitions* – for all simple games with a priori unions $(N, M, P)$ and $(N, M', P)$ with $M^P = {M'}^P$, it holds $f(N, M, P) = f(N, M', P)$.

*PGI-mergeability in the quotient game* – for all simple games with a priori unions $(N, M, P)$ and $(N, M', P)$ where the quotient games $(P, M^P)$ and $(P, {M'}^P)$ are mergeable, it holds

$$f(N, M'', P) = \frac{\sum_k |M^P_{P_k}| f(N, M, P) + \sum_k |{M'}^P_{P_k}| f(N, M', P)}{\sum_k (|M^P_{P_k}| + |{M'}^P_{P_k}|)}$$

for all sets of minimal winning coalitions $M'' \subseteq M \cup M'$ for which ${M''}^P = M^P \cup {M'}^P$.

**Proposition 2.2** (Alonso-Meijide et al., 2008b) *The Solidarity PGI $\Upsilon$ is the unique coalitional power index satisfying efficiency, null union, symmetry among unions, solidarity, independence of superfluous coalitions, and PGI-mergeability in the quotient game.*

There is, however, an obvious alternative.

**Proposition 2.3** *The Solidarity PGI $\Upsilon$ is the unique coalitional PGI satisfying quotient game property and solidarity.*

**Proof**

*(Existence.)* We prove the Solidarity PGI $\Upsilon$ satisfies all listed properties.

*Coalitional PGI.* It holds $\Upsilon_i(N, M, N^0) = \delta_{\{i\}}(N^0, M^{N^0}) = \delta_i(N, M)$ for all $i$ in all simple games $(N, M)$.

*Quotient Game.* For all simple games with a priori unions $(N, M, P)$ and unions $Q$ it is $\sum_{i \in Q} \Upsilon_i(N, M, P) = \sum_{i \in Q} \delta_Q(P, M^P) \frac{1}{|Q|} = \delta_Q(P, M^P) = \Upsilon_Q(P, M^P, P^0)$.

*Solidarity.* For all simple games with a priori unions $(N, M, P)$ it obviously holds $\Upsilon_i(N, M, P) = \delta_{P(i)}(P, M^P) \frac{1}{|P(i)|} = \delta_{P(j)}(P, M^P) \frac{1}{|P(j)|} = \Upsilon_j(N, M, P)$ if $P(i) = P(j)$.

*(Uniqueness.)* Let $(N, M, P)$ be a simple game with a priori unions and $f$ be a coalitional power index. If $f$ is a coalitional PGI satisfying quotient game property, then it holds for any union $Q$ that $\sum_{i \in Q} f_i(N, M, P) = \delta_Q(P, M^P)$. If $f$ also satisfies solidarity, we obtain $f_i(N, M, P) = \delta_{P(i)}(P, M^P) \frac{1}{|P(i)|} = \Upsilon_i(N, M, P)$ for all $i \in N$. $\square$

### 2.3.2 The Owen Extended Public Good Index

In their definition of the Owen Extended Public Good Index $\Gamma$, Alonso-Meijide et al. (2008b) follow an idea of Owen (1977) insofar as power is split within a union according to the possibilities that subsets of this union have to form winning coalitions with other unions. For this purpose, a subset $S \subseteq Q$ of union $Q \in P$ is an *essential part* with respect to coalition $R \in M^P$ if $S \cup \bigcup_{Q' \in R \setminus \{Q\}} Q'$ is a winning coalition in $(N, M)$ and $T \cup \bigcup_{Q' \in R \setminus \{Q\}} Q'$ is losing for all $T \subset S$. Denoting the *set of essential parts* with respect to $R$ containing player $i$ by $E_i^R(N, M, P)$, the Owen Extended Public Good Index $\Gamma$ is defined as

$$\Gamma_i(N, M, P) = \delta_{P(i)}(P, M^P) \frac{1}{|M_{P(i)}^P|} \sum_{R \in M_{P(i)}^P} \frac{|E_i^R(N, M, P)|}{\sum_{j \in P(i)} |E_j^R(N, M, P)|} \tag{2.4}$$

whenever $M_{P(i)}^P \neq \emptyset$ and $\Gamma_i(N, M, P) = 0$ otherwise, $i = 1, \ldots, n$.

The characterization of Alonso-Meijide et al. (2008b) requires the following three definitions. A coalitional power index $f$ satisfies

*independence of irrelevant coalitions* if for all simple games with a priori unions $(N, M, P)$ it holds $f(N, M, P) = f(N, M \setminus \{S\}, P)$ for all $S \in M$ for which $P(S) \notin M^P$.

*invariance with respect to essential parts* if for all simple games with a priori unions $(N, M, P)$ and $(N, M', P)$ for which there is $R \subseteq P$ such that $P(S) = R$ for all $S \in M \cup M'$ and $E_i^R(N, M, P) = E_i^R(N, M', P)$ for all $i$ it holds $f(N, M, P) = f(N, M', P)$.

*PGI-mergeability inside unions* if for all simple games with a priori unions $(N, M, P)$ and $(N, M', P)$ where $(N, M)$ and $(N, M')$ are mergeable and there exists $Q \in P$ such that $P(S) \subseteq Q$ for all $S \in M \cup M'$, it holds

$$f(N, M \cup M', P) = \frac{\sum_i |M_i| f(N, M, P) + \sum_i |M_i'| f(N, M', P)}{\sum_i (|M_i| + |M_i'|)}.$$

**Proposition 2.4** (Alonso-Meijide et al., 2008b) *The Owen Extended PGI $\Gamma$ is the unique coalitional power index satisfying efficiency, null player, symmetry among unions, symmetry inside unions, independence of irrelevant coalitions, invariance with respect to essential parts, PGI-mergeability in the quotient game, and PGI-mergeability inside unions.*

Also for this index, we find an alternative axiomatization using coalitional PGI and quotient game property. The two inner axioms are the following. We say a power index $f$ satisfies

*proportionality with respect to essential coalitions* if for all simple games with a priori unions $(N, M, P)$ it holds for all unions $Q \in P$ and all members $i, j \in Q$ that

$$f_i(N,M,P) \sum_{R \in M_Q^P} f_j(Q, E_Q^R, Q^0) = f_j(N,M,P) \sum_{R \in M_Q^P} f_i(Q, E_Q^R, Q^0)$$

where $E_Q^R = \cup_{k \in Q} E_k^R(N,M,P)$ denotes the set of all subcoalitions of union $Q$ which are an essential part with respect to coalition $R$ in the quotient game.

*nonnegativity* if $f(N,M,P) \geq 0$ for all simple games with a priori unions $(N,M,P)$.

**Proposition 2.5** *The Owen Extended PGI* $\Gamma$ *is the unique coalitional PGI satisfying quotient game property, proportionality with respect to essential coalitions, and nonnegativity.*

**Proof**

*(Existence.)* We prove the Owen Extended PGI $\Gamma$ satisfies all listed properties.

*Coalitional PGI.* For all $i$ and all simple games $(N,M)$ it holds that, if $M_{\{i\}}^{N^0} \neq \emptyset$,

$$\Gamma_i(N,M,N^0) = \delta_{\{i\}}(N^0, M^{N^0}) \frac{1}{|M_{\{i\}}^{N^0}|} \sum_{R \in M_{\{i\}}^{N^0}} 1 = \delta_{\{i\}}(N^0, M^{N^0}) = \delta_i(N,M).$$

If $M_{\{i\}}^{N^0} = \emptyset$ then also $M_i = \emptyset$ and thus $\Gamma_i(N,M,N^0) = 0 = \delta_i(N,M)$.

*Quotient Game.* For all simple games with a priori unions $(N,M,P)$ and unions $Q$,

$$\begin{aligned}\sum_{i \in Q} \Gamma_i(N,M,P) =& \sum_{i \in Q} \delta_{P(i)}(P, M^P) \frac{1}{|M_{P(i)}^P|} \sum_{R \in M_{P(i)}^P} \frac{|E_i^R(N,M,P)|}{\sum_{j \in P(i)} |E_j^R(N,M,P)|} \\ =& \delta_Q(P, M^P) = \Gamma_Q(P, M^P, P^0).\end{aligned}$$

*Proportionality with respect to essential coalitions.* For all simple games with a priori unions $(N,M,P)$, all unions $Q \in P$ and all members $i \in Q$ it holds

$$f_i(Q, E_Q^R, Q^0) = \frac{|E_i^R(N,M,P)|}{\sum_{k \in P(i)} |E_k^R(N,M,P)|}$$

and hence also

$$f_i(N,M,P) \sum_{R \in M_Q^P} f_j(Q, E_Q^R, Q^0) = f_j(N,M,P) \sum_{R \in M_Q^P} f_i(Q, E_Q^R, Q^0).$$

*Nonnegativity* is of course satisfied.

*(Uniqueness.)* Let $(N,M,P)$ be a simple game with a priori unions and $f$ be a coalitional PGI satisfying the listed properties. Due to quotient game property it holds for any union $Q$ that $\sum_{i \in Q} f_i(N,M,P) = \delta_Q(P, M^P)$. Now, if $M_Q^P = \emptyset$ for union $Q$, it is $\delta_Q(P, M^P) = 0$ and nonnegativity implies $f_i(N,M,P) = 0$ for all $i \in Q$. Yet, if $M_Q^P \neq \emptyset$, we find $f_i(Q, E_Q^R, Q^0) = \frac{|E_i^R(N,M,P)|}{\sum_{k \in Q} |E_k^R(N,M,P)|}$ for any $R \in M_Q^P$. Using proportionality with respect to essential coalitions yields $f_i(N,M,P) = c_Q \delta_Q(P, M^P) \sum_{R \in M_Q^P} \frac{|E_i^R(N,M,P)|}{\sum_{j \in Q} |E_j^R(N,M,P)|}$ for $i \in Q$ where $c_Q$ is a constant such that $\sum_{i \in Q} f_i(N,M,P) = \delta_Q(P, M^P)$. This is guaranteed just by $c_Q = \frac{1}{|M_Q^P|}$ and hence $f$ coincides with the Owen Extended PGI $\Gamma$. $\square$

### 2.3.3 The Union Public Good Index

Holler and Nohn (2009) introduce four variants of the PGI for a priori unions. The first variant, the *Union Public Good Index* $\Lambda$, is as close as possible to the original spirit of the PGI. It is based on the two assumptions that the coalitional value is a public good and only minimal winning coalitions are relevant. The latter assumption does, however, apply to coalitions being minimal not only with respect to the simple game but also with respect to a priori unions. A player's power is then proportional to the number of minimal winning coalitions his union is a member of in the quotient game, that is,

$$\Lambda_i(N,M,P) = \frac{|M_{P(i)}^P|}{\sum_k |P_k||M_{P_k}^P|}, \quad i = 1, \ldots, n. \tag{2.5}$$

As with the Solidarity PGI, it is obviously the case that all members of the same union have equal power, that is, the Union PGI as well satisfies solidarity. However, the Union PGI is the only of the overall six extensions not assigning power to unions on the basis of the PGI in the corresponding quotient game. We hence provide an axiomatization not directly using its being a coalitional PGI but more elementary axioms. For this sake, we say that a coalitional power index $f$ satisfies *symmetry among players of symmetric unions* if for all simple games with a priori unions $(N, M, P)$ it is $f_i(N, M, P) = f_j(N, M, P)$ for all members $i, j$ of symmetric unions $P(i)$ and $P(j)$.

*PGI-mergeability among unions* if for any pair $(N, M, P)$ and $(N, M', P)$ where the quotient games $(P, M^P)$ and $(P, {M'}^P)$ are mergeable it holds that

$$f(N, M'', P) = \frac{\sum_k |P_k||M^P_{P_k}|f(N, M, P) + \sum_k |P_k||{M'}^P_{P_k}|f(N, M', P)}{\sum_k |P_k|(|M^P_{P_k}| + |{M'}^P_{P_k}|)}$$

for all sets of minimal winning coalitions $M'' \subseteq M \cup M'$ for which ${M''}^P = M^P \cup {M'}^P$.

**Proposition 2.6** *The Union Public Good Index $\Lambda$ is the unique coalitional power index satisfying efficiency, null union, symmetry among players of symmetric unions, independence of superfluous coalitions, and PGI-mergeability among unions.*

**Proof**

*(Existence.)* We prove the Union PGI $\Lambda$ satisfies all listed properties.

*Efficiency.* It holds

$$\sum_i \Lambda_i(N, M, P) = \sum_k \sum_{i \in P_k} \frac{|M^P_{P_k}|}{\sum_l |P_l||M^P_{P_l}|} = \sum_k \frac{|P_k||M^P_{P_k}|}{\sum_l |P_l||M^P_{P_l}|} = 1.$$

*Null union.* By definition it is $M^P_Q = \emptyset$ for any null union $Q$ and thus $\Lambda_i(N, M, P) = 0$ for all $i \in Q$.

*Symmetry among players of symmetric unions.* For players $i$ and $j$ in symmetric unions $P(i)$ and $P(j)$, it holds $|M^P_{P(i)}| = |M^P_{P(j)}|$ and thus $\Lambda_i(N, M, P) = \Lambda_j(N, M, P)$.

*Independence of superfluous coalitions.* Given $(N, M, P)$ and $(N, M', P)$ with $M^P = {M'}^P$ it obviously holds $\Lambda(N, M, P) = \Lambda(N, M', P)$.

*PGI-mergeability among unions.* Given two simple games with a priori unions $(N, M, P)$ and $(N, M', P)$ where $(P, M^P)$ and $(P, {M'}^P)$ are mergeable it holds for all $i$ and all sets of minimal winning coalitions $M'' \subseteq M \cup M'$ for which ${M''}^P = M^P \cup {M'}^P$

$$\begin{aligned}
&\Big(\sum_k |P_k|(|M^P_{P_k}| + |{M'}^P_{P_k}|)\Big)\Lambda_i(N, M'', P) \\
= \;& |M^P_{P(i)}| + |{M'}^P_{P(i)}| = \sum_k |P_k||M^P_{P_k}|\Lambda_i(N, M, P) + \sum_k |P_k||{M'}^P_{P_k}|\Lambda_i(N, M', P).
\end{aligned}$$

*(Uniqueness.)* We prove that any coalitional power index $f$ satisfying the listed properties coincides with the Union PGI $\Lambda$.

First, consider a unanimity game with a priori unions $(N, \{S\}, P)$, $S \subseteq N$. Since $f$ satisfies efficiency, null union, and symmetry among players of symmetric unions, it amounts to

$$f_i(N, \{S\}, P) = \begin{cases} \frac{1}{\sum_{Q \in P(S)} |Q|} & \text{if } P(i) \in P(S) \\ 0 & \text{otherwise} \end{cases}$$

for all $i = 1, \ldots, n$.

Now, let $(N, M, P)$ be any simple game with a priori unions. Since $f$ satisfies independence of superfluous coalitions, we can assume that $P(S) \in M^P$ for all $S \in M$ and $P(S) \neq P(T)$ for all distinct $S, T \in M$. Thus, unanimity games $(N, \{S\}, P)$, $S \in M$, are mergeable in the quotient game. Then, using PGI-mergeability among unions and $\sum_k |P_k||\{S\}^P_{P_k}| = \sum_{Q \in P(S)} |Q|$, we find for all $i = 1, \ldots, n$ that

$$f_i(N, M, P) = \sum_{S \in M} \frac{\sum_k |P_k||\{S\}^P_{P_k}|}{\sum_k |P_k||M^P_{P_k}|} f_i(N, \{S\}, P) = \frac{|M^P_{P(i)}|}{\sum_k |P_k||M^P_{P_k}|} = \Lambda_i(N, M, P). \quad \square$$

### 2.3.4 Threat PGIs

There are different approaches to the allocation of power inside unions taking into account the players' threat power in case that they leave their union. The union might break up into any possible partition, or even the whole a priori union structure might change. Holler and Nohn (2009) elaborate three canonical cases concerning how the a priori union structure changes upon withdrawal of a player from his union. These are characterized by the assumptions that the respective union is not more stable than other unions and that all other members of the respective union and all other unions are treated symmetrically. For any such case, Holler and Nohn (2009) define a coalitional PGI distributing power inside unions proportional to the players' power in the respective threat game.

To take a more general approach, we consider Threat PGIs for any possible way a priori unions can behave upon withdrawal of a player from his union. We call a mapping *TP* a *threat partition* if it assigns a pair $(P, i)$, where $P$ is a partition of the set of players $N$ and $i \in N$ a player, another partition $TP(P, i)$ of $N$ such that $\{i\} \in TP(P, i)$. Partition $TP(P, i)$ can be seen to result when player $i$ leaves his union $P(i)$ in $P$ and plays on his own.

Now, given arbitrary threat partition $TP$, we define the corresponding *Threat PGI* $T^{TP}$ as

$$T_i^{TP}(N, M, P) = \delta_{P(i)}(P, M^P) \frac{\delta_{\{i\}}(TP(P,i), M^{TP(P,i)})}{\sum_{j \in P(i)} \delta_{\{j\}}(TP(P,j), M^{TP(P,j)})} \tag{2.6}$$

whenever $\sum_{j \in P(i)} \delta_{\{j\}}(TP(P,j), M^{TP(P,j)}) > 0$, and $T_i^{TP}(N, M, P) = \delta_{P(i)}(P, M^P) \frac{1}{|P(i)|}$ otherwise, $i = 1, \ldots, n$.

The three particular cases considered by Holler and Nohn (2009) are the following. Assuming the least possible degree of stability, *Threat PGI* $T^1$ uses threat games $N^0$ where all unions break up into singletons upon a player's withdrawal from his union. This approach turns a blind eye on how the a priori union structure behaves in the long run and, at the same time, takes the reasoning behind the Owen Extended PGI one step further. Now, as it comes to intra-union allocation of power, subsets of a union are not only allowed to cooperate with other unions, but also with subsets of other unions.[1] The second *Threat PGI* $T^2$ presupposes a greater degree of stability by considering a breakup of the leaving player's union into singletons while all other unions endure. That is, the applied threat partition is given by $P/P(i) = P \backslash \{P(i)\} \cup \{\{j\} | j \in P(i)\}$. The third *Threat PGI* $T^3$ assumes that neither the cooperation of the rest of the union members nor that of other unions is affected by one member's leaving his union. The respective threat partition in which $i$ separates from his union $P(i)$ and plays on his own is $P/i = P \backslash \{P(i)\} \cup \{\{i\}, P(i) \backslash \{i\}\}$.

Note the following basic statements about the three threat indices $T^1$, $T^2$, and $T^3$.

If at most one union contains more than one member, Threat PGIs $T^1$ and $T^2$ coincide.

If no union contains more than two members, Threat PGIs $T^2$ and $T^3$ coincide.

[1] An analogous coalitional power index based on the Shapley-Shubik index is introduced by Alonso-Meijide and Carreras (2009).

$T^1$ typically assigns power greater than 0 to any player who is no null player in $(N, M)$ and whose union is not a null union in $(N, M, P)$. Threat PGIs $T^2$ and $T^3$ possibly assign power equal to 0 to a player who is no null player in $(N, M)$ and whose union is no null union in $(N, M, P)$.

It holds $T^2_i(N, M, P) = 0$ if and only if $\Gamma_i(N, M, P) = 0$. This comes from the fact that a player $i$ constitutes a null union $\{i\}$ in $(N, M, P/P(i))$ if and only if he is no member of any essential part.

$T^2_i(N, M, P) = 0$ implies $T^3_i(N, M, P) = 0$ because $i$ constitutes a null union $\{i\}$ in $(N, M, P/i)$ whenever it is a null union in $(N, M, P/P(i))$.

We now provide a general characterization for any possible Threat PGI – axiomatizations for the three Threat PGIs $T^1$, $T^2$ and $T^3$ are special instances. Given threat partition $TP$, we say a coalitional power index $f$ satisfies

*TP proportionality within unions* if for all simple games with a priori unions $(N, M, P)$ it holds for all members $i$ and $j$ of the same union $Q \in P$ that

$$f_i(N, M, P) f_j(N, M, TP(P, j)) = f_j(N, M, P) f_i(N, M, TP(P, i)).$$

*TP empty threats* if for all simple games with a priori unions $(N, M, P)$ it holds $f_i(N, M, P) = f_j(N, M, P)$ for all members $i$ and $j$ of the same union $Q \in P$ in case of $\sum_{k \in Q} f_k(N, M, TP(P, k)) = 0$.

*TP* proportionality within unions is the main inner axiom determining that power within unions is split proportional to the members' power in their respective threat game.[2] The relatively weak property of *TP* empty threats demands that members of a union share its power equally whenever they all have zero threat power. For many threat partitions as for instance $N^0$ and $P/P(i)$ (in cases of Threat PGIs $T^1$ and $T^2$, respectively) it holds that a union has zero power in the quotient game whenever all members have zero

[2] Note the particular instance of $N^0$ proportionality within unions is introduced in Alonso-Meijide and Carreras (2009) to characterize their $T^1$-analogue based on the Shapley-Shubik index.

threat power. In these cases, all members are assigned a power of 0 (and it is possible to replace *TP* empty threats by nonnegativity). However, this is not necessarily true for all threat partitions. For instance, with threat partition $P/i$ (in case of Threat PGI $T^3$) it is possible that a union is no null union while all members have a threat power of zero.[3]

Note the combination of *TP* proportionality within unions and *TP* empty threats implies symmetry within unions. Also, certain instances of the two properties as the ones with threat partitions $N^0$ or $P/P(i)$ do imply null player. However, this does not hold in general. For instance, $T^3$ does not satisfy null player but satisfies $P/i$ proportionality within unions as well as $P/i$ empty threats.[4]

**Proposition 2.7** *For any threat partition TP, the corresponding Threat PGI $T^{TP}$ is the unique coalitional PGI satisfying quotient game property, TP proportionality within unions, and TP empty threats.*

**Proof**

*(Existence.)* We show that $T^{TP}$ is a coalitional PGI satisfying quotient game property, *TP* proportionality within unions, and *TP* empty threats.

*Coalitional PGI.* For any simple game $(N, M)$ it is $T_i^{TP}(N, M, N^0) = \delta_{\{i\}}(N^0, M^{N^0}) = \delta_i(N, M)$.

*Quotient game property.* In any simple game with a priori unions $(N, M, P)$ it holds for any union $Q$ that $\sum_{i \in Q} T_i^{TP}(N, M, P) = \delta_Q(P, M^P) = T_Q^{TP}(P, M^P, P^0)$.

*TP proportionality within unions.* Let $i$ and $j$ be members of the same union $Q \in P$.

[3] Let $N = \{1, 2, 3, 4\}$ and $M = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$. With a priori unions $P = \{\{1\}, \{2, 3, 4\}\}$, it is $M^P = \{\{\{1\}, \{2, 3, 4\}\}\}$ and thus $\delta_{\{2,3,4\}}(P, M^P) = \frac{1}{2}$. However, all members $i \in \{2, 3, 4\}$ constitute a null union $\{i\}$ with a power of 0 in their respective threat game $(P/i, M^{P/i})$.

[4] Consider the game from footnote 3 extended by one null player who joins the union of minor players. So $N = \{1, 2, 3, 4, 5\}$, $M = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$, and $P = \{\{1\}, \{2, 3, 4, 5\}\}$. Then $M^P = \{\{\{1\}, \{2, 3, 4, 5\}\}\}$ and thus $\delta_{\{2,3,4,5\}}(P, M^P) = \frac{1}{2}$. Also in this case, all members $i \in \{2, 3, 4, 5\}$ constitute a null union $\{i\}$ with a power of 0 in their respective threat game $(P/i, M^{P/i})$. Thus, despite being a null player in $(N, M)$, player 5 is assigned a power of $\frac{1}{8}$ by $T^3$.

Then

$$
\begin{aligned}
&T_i^{TP}(N,M,P)T_j^{TP}(N,M,TP(P,j))\\
&=\delta_Q(P,M^P)\frac{\delta_{\{i\}}(TP(P,i),M^{TP(P,i)})}{\sum_{k\in Q}\delta_{\{k\}}(TP(P,k),M^{TP(P,k)})}\delta_{\{j\}}(TP(P,j),M^{TP(P,j)})\\
&=\delta_Q(P,M^P)\frac{\delta_{\{j\}}(TP(P,j),M^{TP(P,j)})}{\sum_{k\in Q}\delta_{\{k\}}(TP(P,k),M^{TP(P,k)})}\delta_{\{i\}}(TP(P,i),M^{TP(P,i)})\\
&=T_j^{TP}(N,M,P)T_i^{TP}(N,M,TP(P,i)).
\end{aligned}
$$

*TP empty threats* follows directly from the definition of $T^{TP}$.

*(Uniqueness.)* We show that any coalitional power index $f$ satisfying the listed properties coincides with $T^{TP}$.

Because $f$ is a coalitional PGI satisfying quotient game property, it holds for all simple games with a priori unions $(N,M,P)$ that $\sum_{i\in Q} f_i(N,M,P) = f_Q(P,M^P,P^0) = \delta_Q(P,M^P)$ for all unions $Q\in P$.

Since $f$ also satisfies *TP* proportionality within unions, it holds for $i\in Q$ that

$$
\begin{aligned}
&f_i(N,M,P)\sum_{j\in Q}\delta_{\{j\}}(TP(P,j),M^{TP(P,j)}) = f_i(N,M,P)\sum_{j\in Q} f_j(N,M,TP(P,j))\\
&=\sum_{j\in Q} f_j(N,M,P)f_i(N,M,TP(P,i)) = \delta_Q(P,M^P)\delta_{\{i\}}(TP(P,i),M^{TP(P,i)}).
\end{aligned}
$$

In case of $\sum_{j\in Q}\delta_{\{j\}}(TP(P,j),M^{TP(P,j)})>0$ this implies

$$
f_i(N,M,P)=\delta_Q(P,M^P)\frac{\delta_{\{i\}}(TP(P,i),M^{TP(P,i)})}{\sum_{j\in Q}\delta_{\{j\}}(TP(P,j),M^{TP(P,j)})}=T_i^{TP}(N,M,P).
$$

In case of $\sum_{j\in Q}\delta_{\{j\}}(TP(P,j),M^{TP(P,j)}) = 0$, *TP* empty threats applies such that $f_i(N,M,P)=\delta_Q(P,M^P)\frac{1}{|P(i)|}=T_i^{TP}(N,M,P)$. □

| Party | Label | Seats | $\delta$ |
|---|---|---|---|
| EAJ/PNV | 1 | 22 | 0.2143 |
| PSE-EE/PSOE | 2 | 18 | 0.1429 |
| PP | 3 | 15 | 0.2143 |
| EHAK/PCTV | 4 | 9 | 0.1429 |
| EA | 5 | 7 | 0.1429 |
| EB/IU | 6 | 3 | 0.0714 |
| Aralar | 7 | 1 | 0.0714 |

Table 2.1: The Public Good Index of simple game $(N, M)$

## 2.4 An Example

In this section we conclude with a real world example of power measurement with Public Good Indices for a priori unions. We consider the Parliament of the Basque Country with 75 seats distributed according to the elections held in April 14th, 2005. The distribution of seats was the following: 22 seats for the Basque nationalist conservative party EAJ/PNV (1); 18 seats for the Spanish socialist party PSE-EE/PSOE (2); 15 seats for the Spanish conservative party PP (3); 9 seats for the Basque left-wing party EHAK/PCTV (4); 7 seats for the Basque nationalist social democrat party EA (5); 3 seats for the Spanish left-wing party EB/IU (6); and, 1 seat for the moderated left-wing party Aralar (7).

Modelling the situation as a weighted voting game, we denote the set of players as $N = \{1, 2, 3, 4, 5, 6, 7\}$. The quota in the parliament being 38, we have the representation as a weighted voting game $[38; 22, 18, 15, 9, 7, 3, 1]$ and the set of minimal winning coalitions amounts to

$$\begin{aligned} M = \{&\{1,2\}, \{1,3,4\}, \{1,3,5\}, \{1,3,6\}, \{1,3,7\}, \{1,4,5\}, \\ &\{2,3,4\}, \{2,3,5\}, \{2,4,5,6,7\}\}. \end{aligned}$$

| Union | Label | Seats | $\delta$ |
|---|---|---|---|
| EAJ/PNV, EA, EB/IU | $P_1$ | 32 | 0.3333 |
| PSE-EE/PSOE | $P_2$ | 18 | 0.2222 |
| PP | $P_3$ | 15 | 0.2222 |
| EHAK/PCTV, Aralar | $P_4$ | 10 | 0.2222 |

Table 2.2: The Public Good Index of quotient game $(P, M^P)$

Notice that there is no null player. Table 2.1 depicts the structure of the Parliament and, for each party, the value of the Public Good Index $\delta$ in simple game $(N, M)$. Note that here the PGI exhibits non-monotonicity of power: although 2 has more seats than 3, the latter party is member of two more minimal winning coalitions and hence has more power.

Taking into account that EAJ/PNV, EA, and EB/IU constituted the government before the elections, we consider these three parties forming a union $P_1 = \{1, 5, 6\}$. PSE-EE/PSOE on the one hand and PP on the other have no strong ties to any other party, so they are assumed to form unions on their own ($P_2 = \{2\}$ and $P_3 = \{3\}$, respectively). The two nationalist parties EHAK/PCTV and Aralar unite as $P_4 = \{4, 7\}$. The a priori union structure thus is $P = \{P_1, P_2, P_3, P_4\}$ and the corresponding quotient game amounts to the 4-player apex game $(P, M^P)$ with

$$M^P = \{\{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\}, \{P_2, P_3, P_4\}\}.$$

Major union $P_1$ forms a minimal winning coalition with any of the three minor unions $P_2$, $P_3$, and $P_4$, and all three minor unions together also constitute a minimal winning coalition. Table 2.2 shows the corresponding Public Good Index $\delta$. Interestingly, compared to the singleton a priori union structure, all the minor unions benefit, at least slightly, from this particular a priori union structure. Major union $P_1$ is the only union with less power than the total power of its members in simple game $(N, M)$.

Table 2.3 displays all six Public Good Indices with a priori unions considered in this

| Party | Label | $\Lambda$ | $\Upsilon$ | $\Gamma$ | $T^1$ | $T^2$ | $T^3$ |
|---|---|---|---|---|---|---|---|
| EAJ/PNV | 1 | 0.1765 | 0.1111 | 0.2222 | 0.1667 | 0.1515 | 0.1884 |
| PSE-EE/PSOE | 2 | 0.1176 | 0.2222 | 0.2222 | 0.2222 | 0.2222 | 0.2222 |
| PP | 3 | 0.1176 | 0.2222 | 0.2222 | 0.2222 | 0.2222 | 0.2222 |
| EHAK/PCTV | 4 | 0.1176 | 0.1111 | 0.2222 | 0.1481 | 0.2222 | 0.2222 |
| EA | 5 | 0.1765 | 0.1111 | 0.0833 | 0.1111 | 0.1212 | 0.1449 |
| EB/IU | 6 | 0.1765 | 0.1111 | 0.0278 | 0.0556 | 0.0606 | 0 |
| Aralar | 7 | 0.1176 | 0.1111 | 0 | 0.0741 | 0 | 0 |

Table 2.3: PGIs with a priori unions of simple game with a priori unions $(N, M, P)$

paper. No two indices coincide, they all highlight different aspects in the distribution of power in $(N, M, P)$. Notice that the Union PGI $\Lambda$, the only of the six variants not satisfying quotient game property, assigns to the members of major union $P_1 = \{1, 5, 6\}$ power greater than that of any other player. Thus, union $P_1$ receives a total power greater than its power given by the Public Good Index in the quotient game. According to the Solidarity PGI $\Upsilon$, most power is assigned to PSE-EE/PSOE (2) and PP (3) which form a singleton union each and do not have to share their union's power as given by the PGI of the quotient game. The Owen Extendend PGI $\Gamma$ assigns power according to the essential parts a player is a member of, and thus gives player Aralar (7) a power of 0 despite him being no null player in $(N, M)$. EHAK/PCTV (4) alone is sufficient for any of its union's minimal winning coalitions. The same is shown by threat indices $T^2$ and $T^3$, yet not by $T^1$.The Owen Extended PGI and the three Threat PGIs however differ in the distribution of power inside union $P_1$. In particular, $T^3$ more than any other index measures a player's dependency on his union and thus is the only index assigning 0 power to EB/IU (6).

# Chapter 3

# Monotonicity of Power in Weighted Voting Games with Restricted Communication*

*Abstract* Indices which evaluate the distribution of power in simple games are commonly required to be monotonic in players' voting weights when the game represents a voting body such as a shareholder meeting, parliament, etc. The standard notions of local or global monotonicity are bound to be violated, however, if players' cooperation is restricted to coalitions connected by a communication graph. This paper proposes new monotonicity concepts for power in games with communication structure and investigates the monotonicity properties of the Myerson value, the restricted Banzhaf value, the position value, and the average tree solution.

*Keywords* power measurement, weighted voting, restricted communication, monotonicity, centrality, Myerson value, position value, average tree solution

*JEL Classification* C71

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## 3.1 Introduction

Power is one of the most important concepts in the social sciences, but difficult to quantify. Attempts to measure specific aspects of it in particular contexts – notably committees or voting bodies that take 'yes'-or-'no' decisions – have inspired numerous studies ever since Shapley and Shubik (1954) started to investigate political applications of the Shapley value. Some power indices, like the Shapley-Shubik index, have been obtained as specializations of cooperative TU-game values to *simple games*, in which the game's characteristic function $v$ only assumes values in $\{0, 1\}$. Others have been introduced on the domain of simple games and later been extended to general TU-games, see for instance Banzhaf (1965) and Owen (1975a), or Holler (1982) and Holler and Li (1995).[1] And because many contexts make it expedient to incorporate more information than merely the partition of all subsets of players into winning and losing coalitions into a decision body's model, various specialized indices have been derived from baseline solutions such as the Shapley or Banzhaf value. Such a relevant piece of information could be, for instance, that players belong to *a priori unions*, which either support or reject proposals as a block. This case has been investigated by Aumann and Drèze (1974), Owen (1977, 1982), or Alonso-Meijide and Fiestras-Janeiro (2002). Another one is that coalitions can only form between players that can 'communicate' with another; namely, between those players who are connected in a graph that reflects, e.g., their ideological, social, or spatial proximity. The latter kind of situations define games with *restricted communication* or *communication structures*. They are the focus of this paper. Values specifically defined for these situations include the Myerson value (Myerson, 1977), the restricted Banzhaf value (Owen, 1986), the position value (Borm et al., 1992) and the average tree solution (Herings et al., 2008, 2010).

Identification of which value or power index is particularly suitable amongst the many conceivable ones is not easy, even when one restricts attention to situations which can plausibly be modeled as simple games without additional information. In determining

[1] See Felsenthal and Machover (2006) for a brief historical survey. Comprehensive introductions to power measurement are given by Felsenthal and Machover (1998) and Laruelle and Valenciano (2008b).

whether an index is suitable in a given context (or more suitable than another candidate), the respective axiomatic characterizations, probabilistic foundations, and possible interpretations play an important role. Moreover, the *monotonicity properties* of a power index are commonly regarded as a major criterion. They provide a first test of whether a candidate index fits one's basic intuition about power in the contemplated context. Specifically, consider an $n$-player simple game that is defined by a vector of voting weights $(w_1, \ldots, w_n)$ and a quota $q$ such that a coalition $S \subseteq N \equiv \{1, \ldots, n\}$ of players wins ($v(S) = 1$) iff the combined weight of $S$'s members exceeds or equals $q$. It seems quite compelling to require that a power index is *monotonic* in the following sense: if player $i$ has weakly greater *voting weight* than player $j$, then a plausible index $f$ should indicate weakly greater *voting power* for $i$, i.e., for all simple games $(N, v)$ with weighted voting representation $[q; w_1, \ldots, w_n]$ the implication $w_i \geq w_j \Rightarrow f_i(N, v) \geq f_j(N, v)$ holds.

Allingham (1975) used this property of *local monotonicity* in order to characterize a family of power indices. Satisfaction of local monotonicity – and a related *global monotonicity* property that compares a given player's power across games rather than power in a given game across players – are by many regarded as a *sine qua non* for sensible power measures. For instance, Felsenthal and Machover (1998, 245f.) are explicit that any a priori measure of power that violates local monotonicity is, in their view, pathological.

The default identification of more voting weight with more voting power is problematic, however, whenever additional structure such as procedural rules, a priori unions, or communication restrictions affect agents' decision making. This may pertain even to constitutional a priori analysis, which abstracts from issue-specific preferences but, for instance, needs to account for different decision making protocols. It is easy to construct examples in which a player's procedural advantage (for instance, as an agenda setter), his membership of a particularly strong a priori union or cooperative planning conference (Myerson, 1980), or his central position in a communication graph dominates the usually detrimental ramifications of low voting weight. The straightforward monotonicity requirements alluded to above hence need to be adapted if information on top of the quota-and-weights summary $[q; w_1, \ldots, w_n]$ is relevant.[2] On the one hand, weight monotonicity requirements should be confined to players and games that, at least, are *comparable* in any non-weight dimensions – if not identical. On the other hand, new non-weight monotonicity requirements should plausibly complement the weight-based ones for players who have identical weight but can naturally be ordered in other respects.

A detailed investigation of alternative monotonicity definitions that account for the particular decision structure captured by games with a priori unions is carried out by Alonso-Meijide et al. (2009a). There, the distinction between monotonicity requirements at two different levels – namely, *between unions* and *within unions* – sheds light on the relationship of several indices for games with a priori unions and their traditional foundations, and helps to discriminate between them. The present paper proposes a similar adaptation of the standard notions of local and global monotonicity to power indices for *games with restricted communication.* Such games can reflect a variety of social, legal, or technological constraints on the support which is necessary in order for a motion or project to be successful. By allowing the formation only of connected coalitions, the graph may literally describe who can communicate the merits of a proposal to whom or it could represent hierarchical restrictions. It could reflect ideological distances, which can only be reconciled in a connected fashion, or physical restraints that, e.g., forbid the bypassing of an agent who represents the only (indirect) link between others. Specific examples with linear communication structures include *sequencing games*, where customers in a queue with heterogeneous time preferences can switch places in order to minimize total costs (Curiel et al., 1989), and *river games*, where neighboring countries coordinate their use of a river's water (Ambec and Sprumont, 2002). Real and virtual social networks have gained enormous scientific attention recently (for overviews, see Goyal, 2007, Vega-Redondo, 2007, Jackson, 2008), and attest to implicit or explicit restrictions in people's cooperation possibilities in a variety of contexts. They also provide motivation for value concepts which do *not* follow simply from restricting an established

[2] See Holler and Napel (2004a,b) for a general discussion of problems that arise if local monotonicity is regarded as a characteristic of power *per se*.

cooperative solution to connected coalitions – which further complicates one's selection between various suggested power measures.

When adapting the baseline notions of local and global monotonicity to values for games with restricted communication, it is not hard to come up with a long list of alternative conditions for when two players or games might be deemed 'comparable' in their communication possibilities, so that monotonicity of power in weight becomes an insightful requirement again. Nor is it overly complicated to impose new monotonicity demands that relate to the power of players with equal voting weight but different communication possibilities. The difficulty, however, is to define notions of comparability that are, first, not too restrictive and therefore trivial (e.g., requiring local monotonicity of power in weight only for players that have absolutely identical communication possibilities). Second, they should not be so permissive as to render all of the established indices, which are already distinguished by meaningful interpretations and a convincing axiomatization, non-monotonic (e.g., calling for monotonicity in weight as soon as two players have the same number of links). Similar concerns apply to partial orderings of players' communication possibilities, which are the basis of corresponding monotonicity conditions. Finally, it seems desirable that new monotonicity notions preserve at least some of the structural relations that connect local and global monotonicity properties on the domain of unrestricted simple games (see Turnovec, 1998) or for games with a priori unions (Alonso-Meijide et al., 2009a).

The remainder of the paper is organized as follows. Section 3.2 introduces our notation, defines standard power indices and formalizes the conventional notions of symmetry and monotonicity. Section 3.3 introduces simple games with restricted communication and several value concepts that serve as power indices for these games. Then section 3.4 devises and investigates monotonicity requirements concerning the weight dimension of games with communication restrictions, before section 3.5 proposes formalizations of the intuitive requirement that greater power be indicated for players with better communication possibilities. Section 3.6 concludes with a discussion relating solution concepts for restricted communication to the measurement of *centrality* in social networks.

## 3.2 Preliminaries

A (monotone) *simple game* is a pair $(N, v)$ where $N = \{1, \dots, n\}$ is the non-empty and finite *set of players* and the *characteristic function* $v\colon 2^N \to \{0, 1\}$ defines whether any coalition $S \subseteq N$ is *winning* ($v(S) = 1$) or *losing* ($v(S) = 0$). It is required that (i) the empty coalition $\varnothing$ is losing ($v(\varnothing) = 0$), (ii) the grand coalition $N$ is winning ($v(N) = 1$), and (iii) $v$ is monotone ($S \subseteq T \Rightarrow v(S) \leq v(T)$). A simple game is called *proper* if any two winning coalitions have a non-empty intersection, i.e., $v(S) = 1$ implies $v(N \backslash S) = 0$. We call a winning coalition $S$ a *minimal winning coalition* (MWC) if every proper subcoalition $T \subset S$ is losing, and denote the set of all minimal winning coalitions for given $(N, v)$ by $M^v$. Players who do not belong to any MWC are known as *dummy* or *null players*.

A particular type of simple game, with real-world applications that range from shareholder meetings and hiring committees to the US Electoral College or the IMF's Board of Governors, is a *weighted voting game* (WVG): it is a simple game that can be represented by a pair $[q; w]$, which consists of *voting weights* $w = (w_1, \dots, w_n)$ and a *quota* $q \leq \sum w_i$ such that $v(S) = 1 \Leftrightarrow \sum_{i \in S} w_i \geq q$. Throughout the paper we assume that $q > \frac{1}{2} \sum_i w_i$, which ensures properness. Not every simple game allows for a representation $[q; w]$, written as $(N, v) = [q; w]$, whilst those which do have many equivalent representations.[3] We denote the set of all weighted voting games by $\mathcal{W}$.

A power index $f$ is a mapping that assigns an $n$-dimensional real-valued vector $f(N, v) = (f_1(N, v), \dots, f_n(N, v))$ to each simple game $(N, v)$, where $f_i(N, v)$ is interpreted as player $i$'s power in game $(N, v)$. The two most prominent power indices are the *Shapley-Shubik index* (SSI) (Shapley and Shubik, 1954) and the *Banzhaf index* (BI) (Banzhaf, 1965) defined by

$$SSI_i(N, v) \equiv \sum_{S \subseteq N \backslash \{i\}} \frac{s!(n - s - 1)!}{n!} \left( v(S \cup \{i\}) - v(S) \right), \quad i = 1, \dots, n, \tag{3.1}$$

[3] Taylor and Zwicker (1999) provide a characterization of those simple games which are weighted voting games.

where $s$ denotes the cardinality of $S$, and

$$BI_i(N,v) \equiv \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i\}} \left(v(S \cup \{i\}) - v(S)\right), \quad i = 1, \ldots, n. \tag{3.2}$$

They are restrictions of particular *semivalues* for general TU games – the Shapley value (Shapley, 1953) and the Banzhaf value (Owen, 1975a), respectively – to simple games. Semivalues are weighted averages of players' marginal contributions to coalitions

$$f_i(N,v) = \sum_{S \subseteq N \setminus \{i\}} p_S \left(v(S \cup \{i\}) - v(S)\right), \quad i = 1, \ldots, n, \tag{3.3}$$

where the weights $p_S$ depend on a coalition $S$ only via its cardinality $s$ and define a probability distribution, i.e., $p_S = p_s \geq 0$ with $\sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1$ (see Weber, 1988). Other power indices that we will comment on are the Deegan-Packel index $DPI$ (Deegan and Packel, 1978) and the Public Good Index $PGI$ (Holler, 1982) given by

$$DPI_i(N,v) \equiv \frac{1}{|M^v|} \sum_{S \in M_i^v} \frac{1}{|S|}, \quad i = 1, \ldots, n, \tag{3.4}$$

and

$$PGI_i(N,v) \equiv \frac{|M_i^v|}{\sum_j |M_j^v|}, \quad i = 1, \ldots, n, \tag{3.5}$$

where $M_i^v$ denotes the set of MWCs containing player $i$.

Some properties of power indices are widely considered as desirable on the domain of simple or weighted voting games. For instance, a power index $f$ satisfies the *null player property* if it assigns zero power to null players in any simple game $(N,v)$. Another property of a power index pertains to possible symmetries: for given $(N,v)$, players $i$ and $j$ are called *symmetric* if there exists a permutation $\pi$ on $N$ which (i) maps $i$ to $j$ and (ii) under which $v$ is invariant, i.e., $v(S) = 1 \Leftrightarrow v(\pi(S)) = 1$. Identical marginal contributions of $i$ and $j$, i.e., $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \not\ni i, j$, are sufficient for this but not in general necessary.[4] However, in the case of weighted voting

[4] In order to see the latter, consider $(N,v)$ with minimal winning coalitions $M^v = \{\{1,2,3\}, \{2,3,4\}, \{3,4,5\}\}$. Applying the permutation $\pi(1,2,3,4,5) = (5,4,3,2,1)$, one can see that players 1 and 5 are symmetric; but they have different marginal contributions, e.g., to coalition $\{2,3\}$.

games, symmetry of $i$ and $j$ is equivalent to existence of a representation $(N, v) = [q; w]$ for which $w_i = w_j$. A power index $f$ is called *symmetric* if it assigns equal power $f_i(N, v) = f_j(N, v)$ to symmetric players $i$ and $j$ in any given simple game $(N, v)$. On the domain of weighted voting games, a power index $f$ is said to be *locally monotonic* if $w_i \geq w_j$ implies $f_i(N, v) \geq f_j(N, v)$ for all $(N, v) = [q; w]$. It is called *globally monotonic* if $f_i(N, v) \geq f_i(N, v')$ for any $(N, v) = [q; w]$ and $(N, v') = [q; w']$ with $w_i \geq w'_i$, $w_j \leq w'_j$ for all $j \neq i$, and $\sum_j w_j \geq \sum_j w'_j$. If a power index $f$ is symmetric, global monotonicity of $f$ implies local monotonicity of $f$. Semivalues, including the SSI and the BI, satisfy all four properties; DPI and PGI satisfy the null player property and symmetry, but are examples of indices that are neither locally nor globally monotonic.[5]

## 3.3 Power in Weighted Voting Games with Restricted Communication

A *simple game with communication structure* is a triplet $(N, v, g)$ where $(N, v)$ is a simple game and $g \subseteq g^N \equiv \{\{i, j\} \mid i, j \in N, i \neq j\}$ is an *unweighted and undirected graph* on $N$. In what follows we pay particular attention to *weighted voting games with communication structures* or *restricted communication*, i.e., those cases where $(N, v) \in \mathcal{W}$. We denote the collection of all such games by $\mathcal{W}^{\mathcal{G}}$. Two players $i$ and $j$ are able to cooperate directly or to *communicate* in $(N, v, g)$ iff the *link* $\{i, j\}$ between these two players is a member of graph $g$. Those players $j$ that $i$ can communicate with are collected in the *set of neighbors* of player $i$, $N_i(g) = \{j \mid \{i, j\} \in g\}$.

Two players $i, j \in S$ are *connected in $S$ by $g$* if either $i = j$ or if there exists a *path* in $g$ from $i$ to $j$ which stays in $S$, i.e., there are players $k_0, \ldots, k_l \in S$ such that $k_0 = i$, $k_l = j$, and $\{k_0, k_1\}, \ldots, \{k_{l-1}, k_l\} \in g$. Coalition $S$ is *connected by $g$* if every two players $i, j \in S$ are connected in $S$ by $g$. Subset $T \subseteq S$ is a *component* of $S$ in $g$ if it is connected by $g$ and no $T'$ with $T \subset T' \subseteq S$ is connected by $g$. Thus, $g$ induces a unique partition

[5] Consider $(N, v) = [51; 35, 20, 15, 15, 15] \in \mathcal{W}$ with MWCs $M^v = \{\{1, 2\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4, 5\}\}$. Player 2 is a member of fewer MWCs than, e.g., player 3 – implying $PGI_2(N, v) < PGI_3(N, v)$ despite $w_2 > w_3$. Also, $DPI_2(N, v) = \frac{9}{60} < \frac{11}{60} = DPI_3(N, v)$.

$S/g$ (say '$S$ divided by $g$') of any coalition $S$ into its components,

$$S/g \equiv \{\ \{j \,|\, i \text{ and } j \text{ connected in } S \text{ by } g\} \mid i \in S\}. \tag{3.6}$$

It is $S/g = \{S\}$ iff $S$ is connected by $g$. The *full graph* $g^N$ and the *empty graph* $\varnothing$ induce the trivial partitions $S/g^N = \{S\}$ and $S/\varnothing = \{\{i\} \,|\, i \in S\}$, respectively.

The assumption that two players $i$ and $j$ can cooperate directly in $(N, v, g)$ only if they can communicate, and hence can cooperate indirectly in a coalition only if that coalition is connected, leads to the *restricted game* $(N, v/g)$. Its characteristic function $v/g$ is defined by

$$v/g(S) \equiv \sum_{T \in S/g} v(T), \quad S \subseteq N, \tag{3.7}$$

i.e., any coalition is split into its components and the coalition's worth is the total worth of these subcoalitions.[6] In those cases where no winning coalition of $(N, v)$ is connected by $g$, in particular also not the grand coalition $N$, the restricted game $(N, v/g)$ is a *null game* with $v/g \equiv 0$. In all other cases, the restricted game $(N, v/g)$ induced by $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$ is a simple game, in which a coalition is winning iff it contains a winning component in $g$. A communication structure can thus be implicit in the specification of a standard simple game. Note, however, that the simple game $(N, v/g)$ need *not* be a weighted voting game.[7] In case of the full graph $g^N$, the restricted game induced by $(N, v, g^N)$ coincides with $(N, v)$.

We call two players $i$ and $j$ *symmetric* in $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$ if, for some representation $(N, v) = [q; w]$, there is a permutation $\pi$ on $N$ which (i) maps $i$ to $j$ and (ii) leaves weights $w$ and graph $g$ invariant, i.e., for the permuted weights $\pi(w) \equiv (w_{\pi^{-1}(k)})_{k \in N}$ and the permuted graph $\pi(g) \equiv \{\{\pi(k), \pi(l)\} \,|\, \{k, l\} \in g\}$ we have $\pi(w) = w$ and $\pi(g) = g$. This

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[6] In non-proper simple games, several disjoint subcoalitions $T \subset S$ might be winning. Since we have ruled out this case by looking at quotas $q > \sum_i w_i/2$ only, we could replace $\sum v(T)$ by $\max v(T)$.

[7] Consider, for instance, $(N, v) = [4; 2, 1, 1, 1, 2]$ and $g = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$. The corresponding restricted game $(N, v/g)$ has $\{1, 2, 3\}$ and $\{3, 4, 5\}$ as its MWCs. If there existed a representation $[q; w]$, we would have $w_1 + w_2 + w_3 \geq q$ and $w_3 + w_4 + w_5 \geq q$. This implies $w_3 + \max(w_1, w_2) + \max(w_4, w_5) \geq q$ and hence existence of a third MWC – a contradiction.

necessitates, of course, that players $i$ and $j$ have identical weights, i.e., $w_i = w_j$. Two players who are symmetric in a weighted voting game with communication structure are necessarily symmetric in the corresponding restricted game according to the definition in section 3.2.[8]

A *power index for (games with) communication structures* or *restricted communication* is a mapping $f$ that assigns an $n$-dimensional real-valued vector $f(N,v,g) = (f_1(N,v,g),\ldots,f_n(N,v,g))$ to each simple game with communication structure $(N,v,g)$, where $f_i(N,v,g)$ is interpreted as player $i$'s power in game $(N,v,g)$. Such a power index $f$ satisfies *symmetry (SYM)* if $f_i(N,v,g) = f_j(N,v,g)$ whenever players $i$ and $j$ are symmetric in $(N,v,g)$. It satisfies *component efficiency (CE)* if $\sum_{i\in S} f_i(N,v,g) = v(S)$ for each $S \in N/g$ in any given $(N,v,g)$.[9]

The literature on simple games often distinguishes explicitly between the Banzhaf *value* and the Banzhaf *index*, or the Shapley value and its restriction to simple games, the Shapley-Shubik index. Here, we prefer to make no such distinction and usually refer to a 'value', even though the corresponding mapping is only considered on the domain of *simple* games with communication structure. The first value concept that has been proposed and axiomatized specifically for situations with restricted communication is

[8] Requiring only that two players are symmetric in the restricted game $(N, v/g)$ according to section 3.2's definition would be a strictly weaker notion of symmetry. It would have the disadvantage that players' asymmetries in weight and communication possibilities might be 'traded off' against each other: in the 3-player game $(N,v,g)$ with $(N,v) = [2;1,0,1]$ and $g = \{\{1,2\},\{2,3\}\}$, player 1 has more weight than player 2, but player 2 has more communication links. They are hence not symmetric in $(N,v,g)$ according to our definition, but happen to be symmetric in the restricted game $(N,v/g)$ (in which only $N$ is winning). Another weaker alternative to our symmetry definition would require that $i$ and $j$ are symmetric in $(N,v)$ *and* that they are symmetric in $g$, whilst allowing different permutations $\pi$ and $\pi'$ for weights $w$ and graph $g$. This would be satisfied by players 1 and 5 in the game with $(N,v) = [4;1,2,1,1,1]$ and $g = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\}\}$. However, player 5 is a dummy in $(N,v/g)$ whilst player 1 is not; so one could barely call them symmetric. (Note that having the higher-weight neighbor is not always an advantage: player 1 would become a dummy, whilst player 5 would cease to be one, if $w_3$ were raised from 1 to 2.)

[9] Myerson (1977) introduced this property as the defining characteristic of an *allocation rule* for arbitrary TU games with restricted communication.

the *Myerson value* $MV$ (Myerson, 1977). Used as a power index for games with communication structure $(N, v, g)$, it assigns the Shapley-Shubik index of the corresponding restricted game $(N, v/g)$ to each player, i.e.,

$$MV(N, v, g) \equiv SSI(N, v/g). \tag{3.8}$$

In the special case of $(N, v)$ being a *unanimity game*, i.e., $v(N) = 1$ and $v(S) = 0$ for all $S \subset N$, $MV$ indicates the power distribution $(1/n, \ldots, 1/n)$ if the grand coalition is connected by $g$, and $(0, \ldots, 0)$ otherwise. In case of the full graph, $MV(N, v, g^N)$ coincides with $SSI(N, v)$.

Analogously, the *restricted Banzhaf index* $rBI$ (Owen, 1986) maps any given game $(N, v, g)$ to the Banzhaf index of the corresponding restricted game $(N, v/g)$, i.e.,

$$rBI(N, v, g) \equiv BI(N, v/g). \tag{3.9}$$

Any other value or power index for unrestricted (simple) games, such as DPI or PGI, could in this fashion be adapted in order to account for exogenously given communication structures.[10] We will refer to the resulting power index for communication structures as the *restricted version* of the corresponding index: e.g., the Myerson value is the restricted SSI. In particular, we will also comment on the restricted DPI and the restricted PGI. The Myerson value and the restricted DPI and PGI are component efficient, whilst the restricted Banzhaf index is not. Symmetry of a power index implies symmetry of its restricted version; hence all four mentioned restricted indices are symmetric.

Operating on restricted game $(N, v/g)$, rather than on $(N, v, g)$ itself, can entail a significant loss of information. Consider, for instance, a large unanimity game $(N, v)$ and a graph $g$ in which player 1 is the center of a star. Then the restricted game is a unanimity game, too, and all players are symmetric in $(N, v/g)$. Therefore, none of the indices above can distinguish between player 1 in the center and those on the periphery: $MV$, $rBI$, etc. indicate identical power for all players. But all communication which is

[10] A small caveat is that the restricted game $(N, v/g)$ might be the null game, for which a baseline value or index could be undefined (because $M^v = \varnothing$). We set the power of all players to 0 in this case.

necessary in order for the grand coalition to be connected involves player 1. This fact might plausibly give him greater power.

The *position value* (Borm et al., 1992) follows this line of reasoning.[11] It measures agents' power in two steps: first, it considers their links as the 'players' in an auxiliary game, the *link game*. The importance of links is picked up by computing the SSI in the link game, i.e., assigning SSI values to all links of $g$. Second, the respective two players involved in any of the links share its power equally. More specifically, let $(g, v/N)$ denote the null or simple game played by the links according to the characteristic function $v/N$ given by

$$v/N(h) \equiv v/h(N), \quad h \subseteq g. \tag{3.10}$$

So, in $(N, v, g)$'s link game $(g, v/N)$, the worth of a coalition $h$ (of links) is equal to the worth of the grand coalition $N$ (of agents) in the respective restricted game $(N, v/h)$. A coalition $h$ of links, therefore, is winning iff it connects a winning coalition of $(N, v)$. The position value $PV$ is then defined by[12]

$$PV_i(N, v, g) \equiv v(\{i\}) + \frac{1}{2} \sum_{\{i,j\} \in g} SSI_{\{i,j\}}(g, v/N), \quad i = 1, \ldots, n. \tag{3.11}$$

The position value, also in general TU games, can be viewed as arising from a scenario where communication is established link by link in a random order, until all players cooperate to the highest degree allowed for by the communication technology. Players start with their respective stand-alone value $v(\{i\})$ (which is zero except if $i$ is a dictator); as communication possibilities are activated randomly, one after another, the two players that are connected by any new link share the generated worth $v/N(h \cup \{\{i, j\}\}) - v/N(h)$ equally. Then, whenever there is no dictator, $PV_i$ is equivalent to player $i$'s expected

[11] See van den Brink (2009) for an instructive comparative axiomatization of the Myerson value, the restricted Banzhaf value, the position value, and also the average tree solution introduced below (for cycle-free communication situations).

[12] We slightly generalize the definition of Borm et al. (1992) in order to allow for simple games with a dictator. Borm et al. restricted attention to zero-normalized games, in which $v(\{i\}) = 0$ for all $i \in N$.

gains or, in simple games, half of the probability that one of his links turns a losing coalition into a winning one.[13]

The position value satisfies component efficiency and symmetry. For a unanimity game $(N, v)$ in which $N$ is connected by a cycle-free graph $g$, players' position values $PV_i(N, v, g)$ equal half their relative share of total links in $g$. The position value does not reduce to any standard power index for the full graph $g^N$: in particular, null players are assigned a positive position value in $(N, v, g^N)$ whenever there is no dictator in $(N, v)$.[14] Despite its focus on agents' links instead of agents themselves, the position value is still a rather close relative of the Shapley value. In particular, it considers all random orderings of links (rather than players) as equally likely.

The *average tree solution ATS* – introduced first for cycle-free graphs by Herings et al. (2008) and later extended to arbitrary communication structures connecting the grand coalition by Herings et al. (2010) – takes a different approach.[15] It assumes that communication is established in a bottom-up fashion in a random hierarchical structure. For any sequence of link activations which reflects a particular hierarchy of information transmission, sequential endorsements, etc., power is ascribed to the unique player who turns the coalition of previous, hierarchically lower subscribers to the considered motion,

[13] Due to $v/g(S \cup \{i\}) - v/g(S) = v(\{i\}) + v/g|_{S\cup\{i\}}(N) - v/g|_S(N)$, where $g|_S = \{\{i,j\} \in g \,|\, i, j \in S\}$ is the restriction of $g$ to $S$, the Myerson value can also be written as

$$MV_i(N, v, g) = v(\{i\}) + \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} \left( v/g|_{S\cup\{i\}}(N) - v/g|_S(N) \right).$$

$MV$ can thus be seen as capturing players' expected gains analogously to the position value but when communication spreads in a lumpy way: players start with their stand-alone values, then one random player after another brings in *all* of the communication possibilities connecting him to the players who have already joined in, and receives the added value $v/g|_{S\cup\{i\}}(N) - v/g|_S(N)$. Similar formal relationships between $MV$ and $PV$ have recently been pointed out by Casajus (2007) and Kongo (2010).

[14] A star $h = \{\{i, j\} \,|\, j \neq i\}$ with null player $i$ at its center connects the grand coalition, and is a winning coalition in the link game. Some of $i$'s links therefore have a positive marginal contribution in the link game, which implies $PV_i(N, v, g^N) > 0$.

[15] Herings et al. (2010) choose one particular class of rooted spanning trees for their extension of the average tree solution. Baron et al. (2010) provide and axiomatize the average tree solution for *any* class of rooted spanning trees.

project, etc. into a winning one by joining.

More specifically, for a given graph $g$ on $N$, we call a subgraph $t \subseteq g$ a *spanning tree* on connected coalition $S$ if $S$ is connected by $t$ but not by any $t' \subset t$. A spanning tree thus represents a minimal sufficient set of communication links whose activation allows all members of $S$ to cooperate. By its minimality, a spanning tree $t$ does not contain links connected to any player outside of $S$. Also, it cannot contain any *cycle*, i.e., there do not exist distinct players $k_1, \ldots, k_l \in S$, $l \geq 3$, such that $\{k_1, k_2\}, \ldots, \{k_{l-1}, k_l\} \in t$ and $\{k_1, k_l\} \in t$. Therefore, $t$ can be *rooted* at any $j \in S$, which gives links an orientation and results in a *rooted spanning tree* $(t, j)$. A given rooted spanning tree $(t, j)$ reflects a particular order in which the communication links that suffice to bring all members of $S$ together might be activated. The set of neighbors of root player $j$ in $(t, j)$ are naturally called $j$'s *successors*, and inductively one obtains the (possibly empty) set of successors $suc_i(t, j) \equiv \{k \mid \{i, k\} \in t \text{ and } i \notin suc_k(t, j)\}$ for all $i \neq j \in S$. By $sub_i(t, j)$ we denote the set containing player $i$ and his *subordinates*, i.e., $i$'s successors, all their successors, and all subsequent successors. If communication links are activated in a given rooted spanning tree $(t, j)$ on $S$ in a bottom-up fashion, i.e., from terminal players towards the root player $j$ level-by-level, then the *marginal contribution* of player $i \in S$ in $(N, v)$ with rooted spanning tree $(t, j)$ on $S$ is

$$m_i^v(t, j) \equiv v(sub_i(t, j)) - \sum_{k \in suc_i(t,j)} v(sub_k(t, j)), \tag{3.12}$$

i.e., it is 1 iff all coalitions comprising a single successor of $i$ and all that successor's subordinates are individually losing but turn winning when they all become connected by player $i$ joining and merging them.

Given graph $g$, a rooted spanning tree $(t, j)$ on $S$ is called *admissible* if for all $i \in S$ and all distinct $k, k' \in suc_i(t, j)$ it holds that $sub_k(t, j) \cup sub_{k'}(t, j)$ is not connected by $g$. So any two players having the same predecessor are not allowed to be connected –

directly or by their subordinates in $(t,j)$ – in the original graph $g$.[16,17] The average tree solution $ATS$ now identifies $i$'s power in $(N,v,g)$ with the average marginal contribution of player $i$, computed over all admissible rooted spanning trees that can be constructed on the component of $N$ which contains $i$. Formally, denoting the set of *all admissible* rooted spanning trees of $g$ on $S$ by $T_{S,g}$, the average tree solution is defined by[18]

$$ATS_i(N,v,g) \equiv \frac{1}{|T_{S_i,g}|} \sum_{(t,j)\in T_{S_i,g}} m_i^v(t,j), \quad i = 1,\ldots,n, \tag{3.13}$$

where $S_i \in N/g$ denotes the component containing $i$. $ATS$ satisfies component efficiency and symmetry. In case of a unanimity game $(N,v)$ and any graph connecting the grand coalition, it is always the root having a marginal contribution of 1 such that $ATS$ assigns power proportional to the number of spanning trees that can admissibly be rooted in a player; hence, for a cycle-free graph, $ATS$ assigns equal power to each player. In case of the full graph $g^N$, the average tree solution coincides with the Shapley-Shubik index of $(N,v)$.[19]

We conclude the section with a simple example that illustrates the differences in

[16] Though the technical frameworks are slightly different, one easily verifies that our alternative definition of an admissible rooted spanning tree is equivalent to the one in Herings et al. (2010). Rooted spanning trees as induced by *admissible sets of subsets* in their definition obviously satisfy our condition. Conversely, given an admissible rooted spanning tree $(t,j)$ on $S$ satisfying our condition, the corresponding sets of subordinates $(sub_i(t,j))_{i\in S}$ constitute an admissible set of subsets of $S$ in their definition, and the rooted spanning tree induced by this coincides with $(t,j)$.

[17] Herings et al. (2010) provide a mainly technical motivation for this condition. First, only admissible spanning trees $(t,j)$ guarantee that the marginal contribution of a player in an admissible rooted spanning tree equals the marginal contribution when he joins the coalition of all his subordinates in the restricted game, i.e., $m_i^v(t,j) = v/g(sub_i(t,j)) - v/g(sub_i(t,j)\backslash\{i\})$. Second, the class of admissible rooted spanning trees is the largest class of rooted spanning trees such that the ATS is a so-called *Harsanyi solution* (see Baron et al., 2010).

[18] We slightly generalize the definition of Herings et al. (2010): every component is considered separately in order to allow for graphs that do not connect the grand coalition $N$. However note that, since we restrict attention to proper games, at most one component is winning.

[19] Given the full graph, any player can have at most one successor in an admissible rooted spanning tree, i.e., any admissible rooted spanning tree is a linear graph. Then $T_{S_i,g}$ represents the set of all orderings on $N$.

the power ascriptions made by the above solution concepts. Consider simple majority voting by three players located on a left-right scale, i.e., let $(N, v) = [2; 1, 1, 1]$ and $g = \{\{1, 2\}, \{2, 3\}\}$. The Myerson value and restricted Banzhaf index correspond, respectively, to the Shapley-Shubik and Banzhaf indices of the restricted game $(N, v/g)$ with $M^{v/g} = \{\{1, 2\}, \{2, 3\}\}$; hence, $MV(N, v, g) = \left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right)$ and $rBI(N, v, g) = \left(\frac{1}{4}, \frac{3}{4}, \frac{1}{4}\right)$. In contrast, the position value infers power from the (non-proper) link game $(g, v/N)$ with $M^{v/N} = \{\{\{1, 2\}\}, \{\{2, 3\}\}\}$. Both links in $g$, $\{1, 2\}$ and $\{2, 3\}$, are symmetric and have a Shapley-Shubik index of $\frac{1}{2}$ in $(g, v/N)$. The respective two players involved in each link are assigned half of its power, which yields $PV(N, v, g) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$. Finally, the average tree solution measures power based on admissible rooted spanning trees. In this case, where the graph is cycle-free, the admissible rooted spanning trees are graph $g$ rooted in any of the three players. In all three, it is the central player 2 who turns the coalition comprising him and his subordinate(s) winning, and so $ATS(N, v, g) = (0, 1, 0)$.

## 3.4 Monotonicity with Respect to Weights

We will now define two notions of monotonicity for power indices $f\colon \mathcal{W}^{\mathcal{G}} \to \mathbb{R}^n$ which relate to the voting weights of players. The subsequent section will look at monotonicity related to players' communication possibilities.

The first notion of weight monotonicity, *local* monotonicity with respect to weights, concerns the comparison of two players in a single game $(N, v, g)$. Roughly speaking, it postulates that whenever two players are interchangeable apart from one having a greater voting weight than the other in some representation $(N, v) = [q; w_1, \ldots, w_n]$, the former player's power should be at least as high as that of the latter:

**Definition 3.1** *A power index for communication structures $f$ satisfies* local monotonicity with respect to weights (LW) *or is* LW-monotonic *if for each $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$ with $(N, v) = [q; w_1, \ldots, w_n]$,*

$$f_i(N, v, g) \geq f_j(N, v, g)$$

*holds for any players $i$ and $j$ with $w_i \geq w_j$ which are symmetric in $(N, v', g)$ with $(N, v') = [q; w'_1, \ldots, w'_n]$, arbitrary $w'_i = w'_j$, and $w'_k = w_k$ for all $k \neq i, j$.*

An index is thus required to respect a weight advantage for player $i$ over player $j$, provided that both have equivalent communication possibilities in the sense that they would become symmetric if their weights were equalized (for instance, by setting both to $\frac{1}{2}(w_i + w_j)$).

The second notion of monotonicity concerns the comparison of a single player's power across two different games, and is hence referred to as *global* monotonicity with respect to weights. It demands that whenever the underlying weighted voting game changes in a way that can unambiguously be judged favorable for player $i$ – specifically, $i$'s voting weight increases and/or voting weight is shifted from others to $i$ – then this should have a non-negative effect for player $i$'s power:

**Definition 3.2** *A power index for communication structures $f$ satisfies* global monotonicity with respect to weights (GW) *or is* GW-monotonic *if for all $(N, v, g), (N, v', g) \in \mathcal{W}^{\mathcal{G}}$ with $(N, v) = [q; w_1, \ldots, w_n]$ and $(N, v') = [q; w'_1, \ldots, w'_n]$,*

$$f_i(N, v, g) \geq f_i(N, v', g)$$

*holds whenever $w_i \geq w'_i$, $w_j \leq w'_j$ for all $j \neq i$, and $\sum_j w_j \geq \sum_j w'_j$.*

As is the case for local and global monotonicity of power indices for games without explicit communication structure, GW is a stronger requirement than LW in the presence of symmetry.

**Proposition 3.3** *If a power index for communication structures $f$ is GW-monotonic and symmetric, then $f$ is also LW-monotonic.*

**Proof** Let $f$ be a power index satisfying GW and SYM, and consider a game $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$ for which LW would require $f_i(N, v, g) \geq f_j(N, v, g)$. Specifically, let $(N, v) = [q; w_1, \ldots, w_n]$ be such that players $i$ and $j$ with $w_i \geq w_j$ are symmetric in $(N, v', g) \in \mathcal{W}^{\mathcal{G}}$ with $(N, v') = [q; w'_1, \ldots, w'_n]$ where $w'_i = w'_j = \frac{1}{2}(w_i + w_j)$ and $w'_k = w_k$ for all $k \neq i, j$.

Now, noting that $(N, v, g)$ is more favorable than $(N, v', g)$ for player $i$ in the sense of GW and that the reverse is true for player $j$, GW and SYM imply

$$f_i(N, v, g) \geq f_i(N, v', g) = f_j(N, v', g) \geq f_j(N, v, g). \qquad \square$$

Since all power indices for games with communication structure that we consider are symmetric, proposition 3.3 facilitates the checking of their monotonicity properties: proof that $f$ is GW-monotonic implies LW-monotonicity, and an example showing $f$ is not LW-monotonic rules out GW-monotonicity. The latter immediately excludes the restricted versions of non-monotonic symmetric indices for unrestricted weighted voting games, such as the DPI and PGI: they violate LW on the domain $\mathcal{W}^{g^N} \equiv \{(N, v, g^N) \in \mathcal{W}^{\mathcal{G}}\}$, i.e., when players' communication is restricted trivially by the full graph $g^N$, and thus *a forteriori* on $\mathcal{W}^{\mathcal{G}}$.

A power index's being locally or globally monotonic is not typically translated to its restricted version.[20] However, in the case of semi-values, one finds the following result.

**Proposition 3.4** *The restricted versions of all semivalues – so, in particular, the Myerson value MV and restricted Banzhaf index rBI – are LW-monotonic and GW-monotonic.*

**Proof** Let $f\colon \mathcal{W}^{\mathcal{G}} \to \mathbb{R}^n$ be the restricted version of a semivalue $\tilde{f}$, i.e., $f(N, v, g) \equiv \tilde{f}(N, v/g)$. Now consider two games $(N, v, g), (N, v', g) \in \mathcal{W}^{\mathcal{G}}$ with $(N, v) = [q; w_1, \ldots, w_n]$ and $(N, v') = [q; w'_1, \ldots, w'_n]$ such that $w_i \geq w'_i$, $w_j \leq w'_j$ for all $j \neq i$, and $\sum_k w_k \geq \sum_k w'_k$. For any $S \not\ni i$, choose $T \in (S \cup \{i\})/g$ with $T \ni i$. Then it is

[20] To see this, consider weighted voting game $(N, v) = [4; 1, 1, 0, 2, 1, 1, 1]$ and graph $g = \{\{1, 2\}, \{2, 3\}, \ldots, \{6, 7\}\}$. The corresponding restricted game has $M^{v/g} = \{\{1, 2, 3, 4\}, \{4, 5, 6\}\}$ and hence does not have a representation as a weighted voting game. Now, define a power index $f$ to be the SSI for all simple games apart from $(N, v/g)$ for which we assume $f(N, v/g) = (1/6, 1/6, 1/6, 1/2, 0, 0, 0)$. The fact that $(N, v/g)$ is not a weighted voting game ensures that $f$ does *not* violate local or global monotonicity. In addition, $f$ is even efficient and symmetric. However, $f$'s restricted version violates LW since it assigns strictly more power to 3 than to 5. It also violates GW since an increase of weight of 1 for player 3 to $(N, v') = [4; 1, 1, 1, 2, 1, 1, 1]$ would result in $f(N, v'/g) = (0, 2/15, 2/15, 7/15, 2/15, 2/15, 0)$.

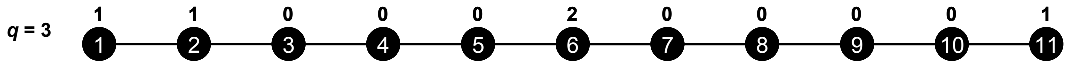


Figure 3.1: Violation of LW by the position value

$v/g(T) \geq v'/g(T)$ and $v/g(T\backslash\{i\}) \leq v'/g(T\backslash\{i\})$, and thus

$$\begin{aligned} & v/g(S \cup \{i\}) - v/g(S) = v/g(T) - v/g(T\backslash\{i\}) \\ \geq \quad & v'/g(T) - v'/g(T\backslash\{i\}) = v'/g(S \cup \{i\}) - v'/g(S). \end{aligned}$$

This and (3.3) imply $\tilde{f}_i(N, v/g) \geq \tilde{f}_i(N, v'/g)$. Hence $f_i(N, v, g) \geq f_i(N, v', g)$ and $f$ must be GW-monotonic. Because $f$, as the restricted version of a semivalue, is symmetric, it must also be LW-monotonic (proposition 3.3). □

In contrast, the position value is *not* LW-monotonic, and hence – being symmetric – *not* GW-monotonic. To see this, consider $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$ with eleven players located on a left-right scale, i.e., $g = \{\{1,2\},\{2,3\},\ldots,\{10,11\}\}$, and $(N, v) = [3; 1, 1, 0, 0, 0, 2, 0, 0, 0, 0, 1]$ (see figure 3.1). Players 2 and 10 become symmetric if $w_2$ is lowered to $w_{10} = 0$, and hence LW would require $PV_2(N, v, g) \geq PV_{10}(N, v, g)$. The corresponding link game $(g, v/N)$ has two disjoint minimal winning coalitions: one comprising the four links on the left side of the central player 6, $h_1 = \{\{2,3\},\ldots\{5,6\}\}$, and the five links on 6's right, $h_2 = \{\{6,7\},\ldots\{10,11\}\}$. In particular, $g$'s left-most link $\{1,2\}$ is never necessary for the establishment of a winning coalition, and hence it is a null player in $(g, v/N)$: player 2's weight of $w_2 = 1$ makes the inclusion of player 1 redundant. In contrast, $w_{10} = 0$ requires activation of link $\{10, 11\}$ before a losing coalition on the right can become winning. The SSI in the link game is then zero for the null link $\{1, 2\}$, $\frac{5}{36}$ for any other link $l \in h_1$ on the left side of player 6, and $\frac{4}{45}$ for any link $l \in h_2$. This results in position values $PV_2(g, v/N) = \frac{5}{72} < \frac{4}{45} = PV_{10}(g, v/N)$ – a violation of LW.

The average tree solution satisfies both weight monotonicity requirements.[21]

**Proposition 3.5** *The average tree solution ATS is both LW-monotonic and GW-monotonic.*

[21] This proposition holds for average tree solutions based on any class of rooted spanning trees.

**Proof** Consider two games $(N, v, g), (N, v', g) \in \mathcal{W}^{\mathcal{G}}$ with $(N, v) = [q; w_1, \ldots, w_n]$ and $(N, v') = [q; w'_1, \ldots, w'_n]$ such that $w_i \geq w'_i$, $w_j \leq w'_j$ for all $j \neq i$, and $\sum_k w_k \geq \sum_k w'_k$. Let $S_i \in N/g$ be the component containing player $i$. Then, for any arbitrary rooted spanning tree $(t, j)$ on $S_i$, we have $v(sub_i(t, j)) \geq v'(sub_i(t, j))$ because $i \in sub_i(t, j)$. Moreover, for all $k \in suc_i(t, j)$, $v(sub_k(t, j)) \leq v'(sub_k(t, j))$ since $i \notin sub_k(t, j)$. Thus $m_i^v(t, j) \geq m_i^{v'}(t, j)$ for any rooted spanning tree $(t, j)$ on $S_i$, and therefore $ATS_i(N, v, g) \geq ATS_i(N, v', g)$. So $ATS$ must be GW-monotonic and – applying proposition 3.3 – LW-monotonic, too. □

## 3.5 Monotonicity with Respect to Communication Possibilities

We now investigate local and global notions of monotonicity which relate to players' communication possibilities. The latter may be judged as better for player $i$ than for $j$ (or better for a given player in graph $g$ than in $g'$), for different – some more, some less compelling – reasons. These will be accounted for by introducing several distinct notions of local and global monotonicity with respect to communication possibilities. Moreover, we will suggest a plausible condition relating to the ranking of different players' gains from a given change in the communication structure.[22]

We begin with the rather straightforward requirement that whenever player $i$ would be symmetric to player $j$ if $i$'s communication possibilities were restricted (i.e., after the deletion of some of $i$'s links) and/or $j$'s possibilities were expanded (i.e., after the addition of some links for $j$), then player $i$'s power should be no smaller than $j$'s.

[22] Since all definitions in this section do not involve any voting weights, they could be extended to situations where a value $f$ is applied to general TU-games with communication structue $(N, v, g)$. All statements about relationships of the properties or their satisfaction by the mentioned solution concepts remain valid as long as the considered TU-games are superadditive.

**Definition 3.6** *A power index for communication structures $f$ satisfies* local monotonicity with respect to communication possibilities (LC) *or is* LC-monotonic *if for each $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$,*

$$f_i(N, v, g) \geq f_j(N, v, g)$$

*holds for any players $i$ and $j$ which are symmetric in $(N, v, g')$ for some $g'$ with $N_i(g') \subseteq N_i(g)$ and $N_j(g') \supseteq N_j(g)$ and $N_k(g')\backslash\{i, j\} = N_k(g)\backslash\{i, j\}$ for all $k \neq i, j$.*

Implicitly, the definition requires $i$ and $j$ to have equal voting weights. Also note that existence of some $g'$ satisfying the stated conditions is equivalent to existence of a *minimal* $g'' \subseteq g$ with $N_i(g'') \subseteq N_i(g)$ and $N_k(g'')\backslash\{i\} = N_k(g)\backslash\{i\}$ for all $k \neq i$ such that $i$ and $j$ are symmetric in $(N, v, g'')$.

A first notion of global monotonicity with respect to communication possibilities pertains to games involving two graphs $g$ and $g'$ such that $g \supseteq g'$ and all additional links in $g$ involve the same player $i$. One might plausibly require that $i$'s power is weakly greater for communication structure $g$, since the additional links improve $i$'s communication possibilities in absolute terms (but note that $i$'s neighbor's may also have improved possibilities). An alternative second notion looks at games with graphs $g \subseteq g'$ in which all links missing in $g$ do not affect $i$'s direct or indirect cooperation possibilities. Again, $i$'s power may plausibly be required to be greater for communication structure $g$, since the removed links improve $i$'s communication possibilities in relative terms: other players have lost some of their communication possibilities, whilst $i$ has not. This can be formalized as follows.

**Definition 3.7**

*(i) A power index for communication structures $f$ satisfies* global monotonicity with respect to added communication possibilities (GC$^+$) *or is* GC$^+$-monotonic *if for all $(N, v, g), (N, v, g') \in \mathcal{W}^{\mathcal{G}}$,*

$$f_i(N, v, g) \geq f_i(N, v, g')$$

*holds whenever $N_i(g) \supseteq N_i(g')$ and $N_j(g)\backslash\{i\} = N_j(g')\backslash\{i\}$ for all $j \neq i$.*

*(ii) The index satisfies* global monotonicity with respect to removed communication possibilities (GC$^-$) *or is* GC$^-$-monotonic *if for all* $(N, v, g), (N, v, g') \in \mathcal{W}^{\mathcal{G}}$,

$$f_i(N, v, g) \geq f_i(N, v, g')$$

*holds whenever* $\{S \subseteq N \mid S$ *is connected on* $g$ *and* $i \in S\} = \{S \subseteq N \mid S$ *is connected on* $g'$ *and* $i \in S\}$ *and* $N_j(g) \subseteq N_j(g')$ *for all* $j \neq i$.

It is straightforward to check that GC$^+$ is equivalent to the *stability* condition in Myerson (1977). The requirement in GC$^-$ that the sets of connected coalitions involving player $i$ are the same in $g$ and $g'$ necessitates $N_i(g) = N_i(g')$. The latter condition is also sufficient for the former when only communication possibilities that connect neighbors of $i$ are lost, i.e., if $\{j, k\} \in g' \backslash g$ implies $j, k \in N_i(g)$.[23]

Unlike in the case of weight monotonicities, if both types of global monotonicity with respect to communication possibilities are satisfied by a symmetric power index $f$, local monotonicity with respect to communication possibilities does not necessarily hold.

The restricted games $(N, v/g)$ and $(N, v/g')$ which are induced by two comparable communication structures $g$ and $g'$ preserve the notion of $g$ offering better communication possibilities for a given player than $g'$ which is underlying GC$^+$ or GC$^-$. Similarly, some player having greater communication possibilities than another one locally, i.e., in a given graph, translates into a restricted game that is better for one player than the other. Power indices for communication structures that are defined as a semivalue of the respective restricted game, therefore, satisfy all of the monotonicity notions defined above.

**Proposition 3.8** *The restricted versions of all semivalues – so, in particular, the Myerson Value MV and restricted Banzhaf index rBI – are LC-monotonic, GC$^+$-monotonic, and GC$^-$-monotonic.*

[23] Strengthening GC$^-$ to the sole requirement of $N_i(g) = N_i(g')$ and $N_j(g) \subseteq N_j(g')$ for all $j \neq i$, as a seemingly natural analogue to GC$^+$, does not provide a suitable monotonicity concept. Consider for instance 5-player simple majority voting $(N, v) = [3; 1, 1, 1, 1, 1]$ and $g = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$. Removal of link $\{2, 3\}$ should then constitute an advantage for player 1 while it, however, turns him into a null player in the corresponding restricted game.

**Proof**

(LC) Consider $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$ such that $g$ offers better communication for $i$ than for $j$ in the sense of LC. Then there is a minimal subgraph $g' \subseteq g$ with $N_i(g') \subseteq N_i(g)$ and $N_k(g')\backslash\{i\} = N_k(g)\backslash\{i\}$ for all $k \neq i$ such that $i$ and $j$ are symmetric in $(N, v, g')$ with respect to permutation $\pi$ with $\pi(i) = j$. For this it holds $v/g(S \cup \{i\}) \geq v/g(\pi(S) \cup \{j\})$ and $v/g(S) \leq v/g(\pi(S))$ for all $S \not\ni i$. Hence, $v/g(S \cup \{i\}) - v/g(S) \geq v/g(\pi(S) \cup \{j\}) - v/g(\pi(S))$ for all $S \not\ni i$. This yields the desired inequality for the semi-values of $i$ and $j$.

($\text{GC}^+$) Consider $(N, v, g), (N, v, g') \in \mathcal{W}^{\mathcal{G}}$ such that $g$ offers better communication possibilities for player $i$ in the sense of $\text{GC}^+$. Then for any $S \not\ni i$, it holds $v/g(S \cup \{i\}) \geq v/g'(S \cup \{i\})$ and $v/g(S) = v/g'(S)$. Therefore, it is $v/g(S \cup \{i\}) - v/g(S) \geq v/g'(S \cup \{i\}) - v/g'(S)$ and the desired inequality follows directly from the definition of a semivalue (see equation (3.3)).

($\text{GC}^-$) Consider $(N, v, g), (N, v, g') \in \mathcal{W}^{\mathcal{G}}$ such that $g$ offers better communication possibilities for player $i$ in the sense of $\text{GC}^-$. For any $S \not\ni i$, choose $T \in (S \cup \{i\})/g$ with $T \ni i$. For this we also have $T \in (S \cup \{i\})/g'$ and thus $v/g(T) = v/g'(T)$. Moreover, $v/g(T\backslash\{i\}) \leq v/g'(T\backslash\{i\})$. Thus, $v/g(S \cup \{i\}) - v/g(S) = v/g(T) - v/g(T\backslash\{i\}) \geq v/g'(T) - v/g'(T\backslash\{i\}) = v/g'(S \cup \{i\}) - v/g'(S)$. The desired conclusion follows. $\square$

An analogous statement is not possible for the restricted versions of indices that build on minimal winning coalitions (or, put differently, averages of marginal contributions with game-specific weights) such as the Deegan-Packel index and Public Good Index. In fact, both indices satisfy *none* of the communication monotonicities. Consider for instance the seven-player game $(N, v) = [3; 1, 1, 0, 0, 0, 0, 2]$ with graphs $g = \{\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{1,7\},\{2,3\},\{2,4\},\{2,5\},\{2,6\},\{3,7\},\{4,7\},\{5,7\},\{6,7\}\}$ and $g' = g \cup \{\{2,7\}\}$ (see figure 3.2). In $(N, v, g)$, player 1 can communicate with all players while player 2 cannot communicate with player 7 directly. Player 1 should hence be at least as powerful as player 2 according to LC. However, the MWCs of the restricted game $(N, v/g)$ are $\{1,7\}$, $\{2,3,7\}$, $\{2,4,7\}$, $\{2,5,7\}$, and $\{2,6,7\}$ such that the DPI amounts to $(\frac{3}{30}, \frac{8}{30}, \frac{2}{30}, \frac{2}{30}, \frac{2}{30}, \frac{2}{30}, \frac{11}{30})$ and the PGI amounts to $(\frac{1}{13}, \frac{4}{13}, \frac{1}{13}, \frac{1}{13}, \frac{1}{13}, \frac{1}{13}, \frac{5}{13})$ – a violation LC. In the transition from $(N, v, g)$ to $(N, v, g')$, $\text{GC}^+$ predicts a weak gain of

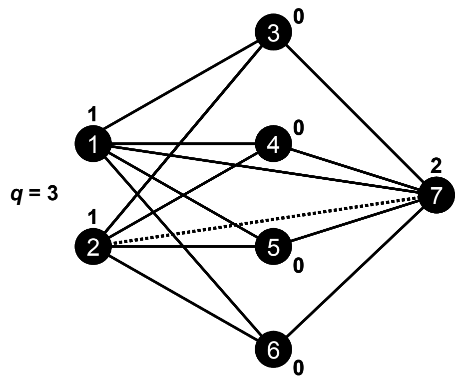


Figure 3.2: Violation of LC, GC$^+$, and GC$^-$ by the restricted DPI and PGI

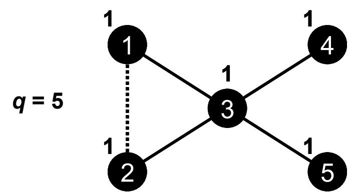


Figure 3.3: Violation of LC, GC$^+$, and GC$^-$ by the position value

power for player 2 and, at the same time, GC$^-$ predicts a weak loss of power for player 1. However, the MWCs of $(N, v/g')$ are $\{1,7\}$ and $\{2,7\}$ such that the DPI and PGI both amount to $(\frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0, \frac{1}{2})$. With both indices, player 2 loses and player 1 gains power, in violation of GC$^+$ and GC$^-$.

The position value does not satisfy *any* of the communication monotonicities either, not even in perfectly symmetric voting situations. To see this, consider five player unanimity voting $(N, v) = [5; 1, 1, 1, 1, 1]$ with graphs $g = \{\{1,2\},\{1,3\},\{2,3\},\{3,4\},\{3,5\}\}$ and $g' = g\backslash\{\{1,2\}\}$ (see figure 3.3). With graph $g$, players 1 and 2 share player 4's and 5's sole possibility to communicate with player 3 and should hence have at least as much power according to LC. However, 4's and 5's single links function as veto players in the link game while 1's and 2's communication possibilities in some sense compete with each other – the position value amounts to $PV(N, v, g) = (0.1, 0.1, 0.45, 0.175, 0.175)$

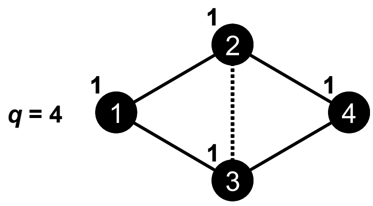


Figure 3.4: Violation of LC, GC$^+$, and GC$^-$ by the average tree solution

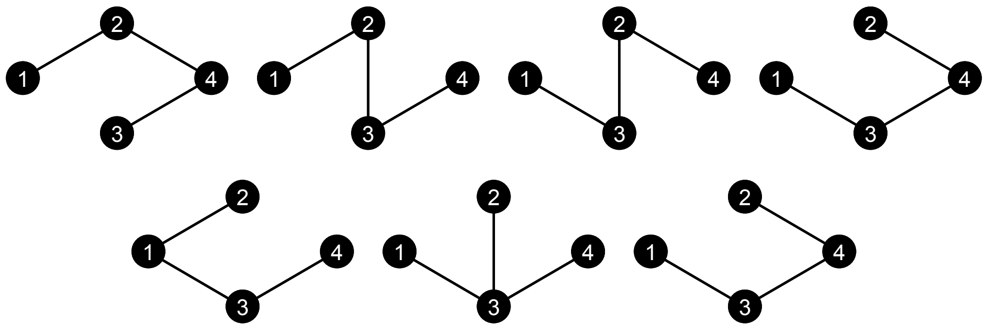

Figure 3.5: Spanning trees of the communication structure in figure 3.4 (including link $\{2,3\}$) admissibly rooted in players 1 (top row) and 2 (bottom row)

and violates both local monotonicities with respect to communication structure.[24] In transition from $g$ to $g'$, players 1 and 2 lose the communication possibility between them, which should be detrimental for these players according to GC$^+$. Also, according to GC$^-$, it should be weakly beneficial for players 4 and 5 since no cooperation possibility is lost for them. However, the position value evaluates to $PV(N, v, g') = (0.125, 0.125, 0.5, 0.125, 0.125)$, in violation of GC$^+$ and GC$^-$.

Also the average tree solution, as defined by Herings et al. (2010), does not satisfy any of the communication monotonicities. Consider four player unanimity voting $(N, v) = [4; 1, 1, 1, 1]$ with graphs $g = \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}\}$ and $g' = g \backslash \{\{2,3\}\}$ (see figure 3.4). In general, with unanimity voting, a player's power is proportional to the

[24] That the violation of monotonicities arises due to competition between the links of a player obviously suggests the introduction of some sort of a priori unions in the link game. However, and as noted before, this then results in the Myerson value; see Casajus (2007) and Kongo (2010).

number of admissible spanning trees rooted in it. For communication structure $g$, there are four admissible spanning trees with players 1 or 4 as the root whilst there are three rooted in players 2 or 3 (see figure 3.5). The average tree solution, therefore, amounts to $ATS(N, v, g) = (\frac{4}{14}, \frac{3}{14}, \frac{3}{14}, \frac{4}{14})$ – in violation of LC which requires players 2 and 3 not to be worse off than 1 and 4. Removal of link $\{2, 3\}$ makes all players symmetric such that $ATS(N, v, g') = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. This violates GC$^+$, according to which the loss of their communication possibility should not be beneficial for 2's and 3's power. It also violates GC$^-$: it requires that 1 and 4 are not harmed by a loss of communication possibilities between players which they can communicate with directly.

All of the above notions of local or global monotonicity impose requirements on power in a purely ordinal sense. But power indices come with an at least implicit presumption of cardinality (in contrast, say, to desirability relations in the tradition of Isbell, 1958). This motivates to also impose cardinal restrictions for how the addition (or removal) of links affects the power of players. For example, Myerson (1977) introduced the so-called *fairness* property that whenever a single link is added to the communication structure, the power of the two newly linked players changes equally. In the spirit of Myerson's constraint, we find an additional demand very intuitive which concerns power changes when a link is added to the current communication structure: the power of the two newly linked players should increase by at least as much as that of those whose communication possibilities improved only indirectly. This combines the local and global perspective of our earlier requirements and explicitly treats power indications as cardinal.

**Definition 3.9** *A power index for communication structures $f$ satisfies* highest gains with respect to added communication possibilities (HC$^+$) *if for all* $(N, v, g)$, $(N, v, g') \in \mathcal{W}^{\mathcal{G}}$,

$$f_i(N, v, g) - f_i(N, v, g') \geq f_j(N, v, g) - f_j(N, v, g') \quad \forall j \neq i$$

*holds for all $i$ with $N_i(g) \supseteq N_i(g')$ and $N_j(g)\backslash\{i\} = N_j(g')\backslash\{i\}$ for all $j \neq i$.*

HC$^+$ implies Myerson's fairness property. In the presence of component efficiency and symmetry, it is also stronger than GC$^+$-monotonicity and LC-monotonicity.

**Proposition 3.10** *If a power index for communication structures $f$ satisfies HC$^+$ and is symmetric, then $f$ is also LC-monotonic. If $f$ satisfies HC$^+$ and component efficiency, then it is also GC$^+$-monotonic.*

**Proof** First, let $f$ be a symmetric power index which satisfies HC$^+$. Consider $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$ with players $i$ and $j$ for which LC calls for $f_i(N, v, g) \geq f_j(N, v, g)$. Then there is $g' \subseteq g$ with $N_i(g') \subseteq N_i(g)$ and $N_k(g')\backslash\{i\} = N_k(g)\backslash\{i\}$ for all $k \neq i$ such that $i$ and $j$ are symmetric in $(N, v, g')$. By HC$^+$, $f_i(N, v, g) - f_i(N, v, g') \geq f_j(N, v, g) - f_j(N, v, g')$ and SYM implies $f_i(N, v, g') = f_j(N, v, g')$. This yields $f_i(N, v, g) = (f_i(N, v, g) - f_i(N, v, g')) + f_i(N, v, g') \geq (f_j(N, v, g) - f_j(N, v, g')) + f_j(N, v, g') = f_j(N, v, g)$.

Second, let $f$ satisfy HC$^+$ and CE. Consider $(N, v, g), (N, v, g') \in \mathcal{W}^{\mathcal{G}}$ such that $N_i(g) \supseteq N_i(g')$ and $N_j(g)\backslash\{i\} = N_j(g')\backslash\{i\}$ for all $j \neq i$. Suppose that $f$ violates GC$^+$, i.e., $f_i(N, v, g) < f_i(N, v, g')$. Since HC$^+$ implies that $f_i(N, v, g) - f_i(N, v, g') \geq f_j(N, v, g) - f_j(N, v, g')$ for all $j \neq i$, we must then have $f_j(N, v, g) < f_j(N, v, g')$ for all $j \neq i$, too. Component efficiency thus entails

$$\sum_{S \in N/g} v/g(S) = \sum_{S \in N/g} \sum_{j \in S} f_j(N, v, g) < \sum_{S \in N/g'} \sum_{j \in S} f_j(N, v, g') = \sum_{S \in N/g'} v/g'(S).$$

At the same time, for $g \supseteq g'$ and $v$ being monotone and proper, we must have $\sum_{S \in N/g} v/g(S) \geq \sum_{S \in N/g} v/g'(S)$ – a contradiction. □

Since HC$^+$ does not impose requirements when the communication possibilities of third players deteriorate, one cannot deduce GC$^-$ from it, even under the assumption of component efficiency. The natural variation of HC$^+$ which considers *removed* rather than added communication possibilities in the sense of GC$^-$ (leading to a property HC$^-$ in analogy to GC$^-$) would do so. However, the corresponding requirement is not satisfied by any of the power indices for communication structures that we consider, and is therefore omitted. The same holds for an analogous highest gains requirement concerning voting weights (namely, a property HW based on the comparability notion underlying GW).

In view of proposition 3.10, the violations of GC$^+$ or LC by the restricted DPI and PGI, by the position value, and by the average tree solution imply that neither of these satisfies HC$^+$. As with the ordinal communication monotonicities, however, the Myerson value and restricted Banzhaf index satisfy HC$^+$:

**Proposition 3.11** *The restricted versions of all semivalues – so, in particular, the Myerson value $MV$ and restricted Banzhaf index $rBI$ – satisfy $HC^+$.*

**Proof** Let $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$ offer better communication possibilities for player $i$ than $(N, v, g') \in \mathcal{W}^{\mathcal{G}}$ in the sense of HC$^+$. Then $v/g(S \cup \{i\}) \geq v/g'(S \cup \{i\})$ and $v/g(S) = v/g'(S)$ for all $S \not\ni i$. This implies the following three statements. First, for all coalitions $S \not\ni i$,

$$v/g(S \cup \{i\}) - v/g(S) \geq v/g'(S \cup \{i\}) - v/g'(S),$$

i.e., in a transition from $g'$ to $g$ player $i$ does not lose a swing with coalition $S$. Second, for all coalitions $S \not\ni i, j$ where $j \neq i$,

$$v/g(S \cup \{j\}) - v/g(S) = v/g'(S \cup \{j\}) - v/g'(S),$$

i.e., $j$ does not gain a swing with $S$. Third, for all coalitions $S \not\ni i, S \ni j$ and $S' = S \backslash \{j\} \cup \{i\}$ where $j \neq i$,

$$\begin{aligned} &(v/g(S \cup \{i\}) - v/g(S)) - (v/g'(S \cup \{i\}) - v/g'(S)) \\ \geq\ &(v/g(S' \cup \{j\}) - v/g(S')) - (v/g'(S' \cup \{j\}) - v/g'(S')), \end{aligned}$$

i.e., $i$ gains a swing with $S$ whenever $j$ gains a swing with $S'$. (To see the latter, note that $S \cup \{i\} = S' \cup \{j\}$, $v/g(S) - v/g'(S) = 0$ because $i \notin S$, and $v/g(S') - v/g'(S') \geq 0$ because $i \in S'$). The desired conclusion follows from considering the fact that semivalues constitute weighted averages of marginal contributions or swings where weights only depend on the size of the respective coalition. $\square$

## 3.6 Discussion

Power indices for voting games with restricted communication try to condense the two-dimensional resources of players – their weights and their communication possibilities – into a single number. This is bound to be difficult, and each of the indices investigated above comes with a different implicit model of how collective decisions emerge from agents' interaction along bilateral communication links. The implicit power tradeoffs across the weight and communication dimensions may stretch one's intuition about how power indications should respond to differences in either dimension. This, in our view, is precisely where well-defined notions of monotonicity can help, because (i) they clarify intuitions that are initially rather vague, (ii) they allow to transparently connect distinct requirements for how power indices should behave *ceteris paribus*, and (iii) they classify indices according to which intuitive better-than-relations they preserve.

The monotonicity concepts that we have defined above are unlikely to be able to play a role in axiomatic characterizations of power indices for communication structures because the same is true for the monotonicities that have been considered for unrestricted weighted voting games, or for games with a priori unions. Axiomatic characterization are, of course, very useful – in particular, they illuminate the relation between different indices. The Myerson value, restricted Banzhaf index, and average tree solution, for instance, fundamentally differ only by violating a distinct axiom out of the three-axiom list of component efficiency, collusion neutrality, and the so-called superfluous link property (for details, see van den Brink, 2009). But axioms are naturally formulated with the goal of characterization in mind; they are usually designed, first, to fix a value for the elements of a basis of a specific space of games and, second, to determine a way of aggregating different basis games. We regard such cardinal characterizations and the classifying ordinal requirements investigated in this paper as important complements.

Because power indices for communication structures evaluate both weight and communication resources, they can, trivially, also be applied for the analysis of *only weight* differences, or of *only communication* differences. Regarding weights, restricted versions of standard power indices, like the Shapley-Shubik index, simply return a known index when they are applied to communication situations captured by the full graph; others define 'new' power indices for unrestricted voting games. Analogously, following an idea of Owen (1986), one might measure the advantageousness of particular network locations, usually identified with the *centrality* of players in a communication graph, by applying a power index for communication structures to a symmetric voting situation, i.e., a voting rule such as simple majority or unanimity voting.

Table 3.1 reports the Myerson value $MV$, restricted Banzhaf index $rBI$, position value $PV$ and the average tree solution $ATS$ for the communication structure shown in figure 3.6 (adapted from Jackson, 2008, p. 38) and, respectively, simple majority voting ($q = 4$), different supermajority rules ($q = 5$ and $q = 6$), and unanimity rule ($q = 7$). It can be seen that the ranking of players depends on which symmetric voting rule is applied. In particular, ties are created or broken in case of $MV$ and $rBI$, and a strict ordering is even reversed for $ATS$.

The most established amongst the many measures of centrality in social networks – *degree centrality*, *Katz prestige*, and *Bonacich* or *eigenvector centrality* – rank players 3 and 5 as the most central, followed by player 4, and finally players 1, 2, 6, and 7 (Jackson, 2008, p. 43). This indicates one reason why we did not follow a tempting route to the possible formalization of monotonicity with respect to communication possibilities: namely, to require players' power to be weakly increasing *ceteris paribus* in a given measure of centrality. As table 3.1 indicates, none of the established indices would satisfy this, at least for the most straightforward centrality measures.[25] A second reason is that many centrality measures are very sophisticated summaries of players' communication possibilities that themselves need to be checked against various intuitive notions of when a

[25] Established measures of network centrality could also play a role in the definition of monotonicity with respect to *weights*. One could, e.g, demand that player $i$'s power is at least as great as $j$'s if $w_i \geq w_j$ and both have equal centrality. However, being highly central but surrounded by low-weight players may provide worse opportunities than being less central but having high-weight neighbors. Therefore, it is problematic to base requirements for indices which provide a combined evaluation of the weight and communication situation on independent measures of players' communication possibilities and their weights.

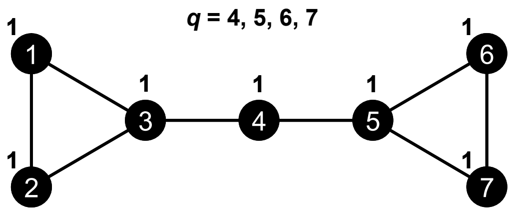


Figure 3.6: The communication structure from Jackson (2008, p. 38) with four symmetric voting rules

| $q=4$ | | | | |
|---|---|---|---|---|
| $i$ | $MV_i$ | $rBI_i$ | $PV_i$ | $ATS_i$ |
| $1,2,6,7$ | 0.06 | 0.08 | 0.08 | 0.00 |
| $3,5$ | 0.17 | 0.23 | 0.22 | 0.00 |
| 4 | 0.42 | 0.36 | 0.27 | 1.00 |

| $q=5$ | | | | |
|---|---|---|---|---|
| $i$ | $MV_i$ | $rBI_i$ | $PV_i$ | $ATS_i$ |
| $1,2,6,7$ | 0.03 | 0.05 | 0.02 | 0.00 |
| $3,5$ | 0.30 | 0.17 | 0.24 | 0.43 |
| 4 | 0.30 | 0.17 | 0.43 | 0.14 |

| $q=6$ | | | | |
|---|---|---|---|---|
| $i$ | $MV_i$ | $rBI_i$ | $PV_i$ | $ATS_i$ |
| $1,2,6,7$ | 0.07 | 0.05 | 0.04 | 0.07 |
| $3,5$ | 0.24 | 0.08 | 0.21 | 0.29 |
| 4 | 0.24 | 0.08 | 0.35 | 0.14 |

| $q=7$ | | | | |
|---|---|---|---|---|
| $i$ | $MV_i$ | $rBI_i$ | $PV_i$ | $ATS_i$ |
| $1,2,6,7$ | 0.14 | 0.02 | 0.07 | 0.14 |
| $3,5$ | 0.14 | 0.02 | 0.18 | 0.14 |
| 4 | 0.14 | 0.02 | 0.29 | 0.14 |

Table 3.1: Power for the communication structure and voting rules of figure 3.6

player should be regarded as more central, and as more influential *a priori*, than another one. Recursing to them in the formalization of monotonicity of power in the context of restricted communication would in our view be less transparent than the definitions presented above. Still, future research might make useful attempts in this direction – and perhaps come up with interesting new power indices by investigating extensions of centrality measures to the domain of cooperative games with restricted communication.

We set out in this paper to define notions of comparability for players and games that are, first, not too restrictive and therefore trivial. Second, they should not be so permissive as to render *all* of the established indices non-monotonic. And, third, they should preserve at least some of the structural relations that connect local and global monotonicity properties on the domain of unrestricted simple games and for games with a priori unions. It has turned out that, in particular, the monotonicity notions which relate to players' communication possibilities are not as straightforward to define and to relate at the local and global level as are corresponding notions for games with a priori unions. This may be regarded as slightly disappointing, but it should not come as a big surprise: as the plethora of measures which try to assess the centrality of a node in a social network can attest, the comparison of two players' respective sets of connections to other players is considerably harder than that of their weights or a fixed bloc that they belong to. This is the fundamental reason for examining several notions of local and global monotonicity with respect to communication possibilities, which cannot unfortunately be put into neat '$A$ and $B$ imply $C$'-relationships.

Regarding the monotonicity properties of established indices, we find that the Myerson value and the restricted Banzhaf index are most in line with the notions of monotonicity proposed in this paper. They satisfy all of the requirements that we have discussed here; and when they violate yet more demanding restrictions, so do their peers. That the position value – like the restricted Deegan-Packel and Public Good indices – violates *all* of the monotonicities considered in this paper should call for some caution. This is especially so since the lack of monotonicity with respect to communication possibilities extends beyond simple games with a weighted voting representation; it applies to the entire class of *superadditive TU-games* (cf. fn. 22). The position value is based on a sound evaluation of the importance of individual links, but simply allocating the contributions of links half-half to the involved nodes may produce quite puzzling player rankings.

The average tree solution does satisfy both weight monotonicities but none of the considered communication monotonicities. Again, this non-monotonicity *a fortiori* pertains to the whole class of superadditive TU-games. It is, at least partly, due to the restriction to admissible rooted spanning trees. We show in the appendix that, in contrast to the average tree solution of Herings et al. (2010), an alternative definition based on *all* possible rooted spanning trees satisfies two of the suggested requirements at least in situations in which no cycles are directly involved in player or game comparisons.

## 3.A Appendix. The Unrestricted Average Tree Solution

In the main text, we examined the average tree solution suggested by Herings et al. (2010). It restricts the averaging to a particular class of admissible rooted spanning trees. Baron et al. (2010), however, have introduced and axiomatized average tree solutions for all possible classes of rooted spanning trees. In this appendix, we examine the monotonicity properties of the 'canonical' *unrestricted* variant which is based on the class of *all* possible rooted spanning trees.

Define the *unrestricted average tree solution* $uATS$ by

$$uATS_i(N,v,g) \equiv \frac{1}{|T'_{S_i,g}|} \sum_{(t,j)\in T'_{S_i,g}} m_i^v(t,j), \quad i=1,\ldots,n, \tag{3.14}$$

where $S_i \in N/g$ is the component containing $i$ and $T'_{S_i,g}$ is the set of *all* rooted spanning trees on $S_i$ for $g$. $uATS$ satisfies component efficiency and symmetry. It is also LW and GW-monotonic (see fn. 21). In case of a unanimity game $(N,v)$ with any graph $g$ that connects the grand coalition, $uATS$ assigns equal power to each player (in contrast to $ATS$, which does not do so and even violates the examined communication monotonicities for unanimity games). The unrestricted average tree solution does not coincide with any standard power index for the graph $g^N$; in particular, it then indicates positive power for null players if there is no dictator.[26]

The unrestricted average tree solution satisfies LC at least when the communication possibilities that a player $i$ has in advance of $j$ connect him only to players which would otherwise, i.e., without the links he has in advance, belong to a different component. Similarly, it satisfies GC$^+$ at least when the additional neighbors of a player $i$ in game $g$

[26] For a given null player $i$, the star $h=\{\{i,j\}\,|\,j\neq i\}$ with $i$ at its center is a spanning tree on $N$. In rooted spanning tree $(h,i)$, $i$ has a positive marginal contribution, which implies $uATS_i(N,v,g^N)>0$.

are not a member of his component in $g'$. Before we turn to the formal result, note that the average tree solution of Herings et al. (2010) does *not* satisfy LC or GC$^+$ in these situations: consider $(N, v) = [2; 1, 1, 1, 0]$, i.e., simple majority voting amongst players 1, 2 and 3, with player 4 added as a null player. With a full graph $g = g^{\{1,2,3\}}$ on the first three players, one obtains $ATS(N, v, g) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$. Then, with link $\{1, 4\}$ added, i.e., considering $g' = g \cup \{\{1, 4\}\}$, GC$^+$ calls for player 1's power to be weakly greater with $g'$ than with $g$. Moreover, LC requires 1's power to be no less than 2's or 3's with communication structure $g'$. But $ATS(N, v, g') = (\frac{2}{8}, \frac{3}{8}, \frac{3}{8}, 0)$ in violation of GC$^+$ and LC.

**Proposition 3.12** *Denote the component $S \in N/g$ for which $S \ni i$ by $S_i(g)$.*
*(i) For each $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$,*

$$uATS_i(N, v, g) \geq uATS_j(N, v, g)$$

*holds for all players $i$ and $j$ for which there is $g'$ such that (a) $N_i(g') \subseteq N_i(g)$ and $N_k(g')\backslash\{i\} = N_k(g)\backslash\{i\}$ for all $k \neq i$, (b) $i$ and $j$ are symmetric in $(N, v, g')$, and (c) $\{k, k'\} \notin g$ for all $k \in S_i(g)\backslash S_i(g')$ and $k' \in S_i(g')\backslash\{i\}$.*
*(ii) For all $(N, v, g), (N, v, g') \in \mathcal{W}^{\mathcal{G}}$,*

$$uATS_i(N, v, g) \geq uATS_i(N, v, g')$$

*holds whenever (a) $N_i(g) \supseteq N_i(g')$ and $N_j(g)\backslash\{i\} = N_j(g')\backslash\{i\}$ for all $j \neq i$, and (b) $\{k, k'\} \notin g'$ for all $k \in S_i(g)\backslash S_i(g')$ and $k' \in S_i(g')\backslash\{i\}$.*

**Proof**

(i) First, if $S_i(g) = S_i(g')$, then $uATS_i(N, v, g) = uATS_j(N, v, g)$ by symmetry. So assume $S_i(g) \supset S_i(g')$. Let $\pi$ with $\pi(i) = j$ be a permutation such that $i$ and $j$ are symmetric in $(N, v, g')$. Without loss of generality, let $\pi(k) = k$ for $k \notin S_i(g')$. Every spanning tree $t$ on $S_i(g)$ can be uniquely partitioned into disjoint spanning trees $t'$ on $S_i(g')$ and $t''$ on $S_i(g)\backslash S_i(g') \cup \{i\}$. And, conversely, the union of disjoint spanning trees $t'$ on $S_i(g')$ and $t''$ on $S_i(g)\backslash S_i(g') \cup \{i\}$ yields a spanning tree on $S_i(g)$. Due to symmetry on $S_i(g')$, also $\pi(t') \cup t''$ is a spanning tree on $S_i(g)$ and the transition from $(t' \cup t'', k)$ to $(\pi(t') \cup t'', \pi(k))$ constitutes a 1-to-1 mapping on $T'_{S_i(g),g}$. Now the inequality

$$m_i^v(t' \cup t'', k) \geq m_i^v(t', i) = m_j^v(\pi(t'), j) \geq m_j^v(\pi(t'), i) = m_j^v(\pi(t') \cup t'', k)$$

holds for any spanning tree $(t' \cup t'', k)$ on $S_i(g)$ and $k \in S_i(g) \backslash S_i(g')$. And for $k \in S_i(g')$ we have

$$m_i^v(t' \cup t'', k) = m_j^v(\pi(t') \cup \pi(t''), \pi(k)) \geq m_j^v(\pi(t') \cup t'', \pi(k)).$$

It follows that $uATS_i(N, v, g) \geq uATS_j(N, v, g)$.

(ii) First, if $S_i(g) = S_i(g')$, then $g = g'$ such that, trivially, $uATS_i(N, v, g) = uATS_i(N, v, g')$. So assume $S_i(g) \supset S_i(g')$. Every spanning tree $t$ on $S_i(g)$ can be uniquely partitioned into disjoint spanning trees $t'$ on $S_i(g')$ and $t''$ on $S_i(g) \backslash S_i(g') \cup \{i\}$. And, conversely, the union of any disjoint spanning trees $t'$ on $S_i(g')$ and $t''$ on $S_i(g) \backslash S_i(g') \cup \{i\}$ yields a spanning tree on $S_i(g)$. Consider a given spanning tree $t = t' \cup t''$ on $S_i(g)$ and note, first, that for any $k \in S_i(g')$

$$m_i^v(t' \cup t'', k) \geq m_i^v(t', k).$$

Second, for $k \in S_i(g) \backslash S_i(g')$ we have

$$m_i^v(t' \cup t'', k) \geq m_i^v(t', i) = \max_{k \in S_i(g')} m_i^v(t', k).$$

Thus it follows that the average marginal contribution for spanning tree $t = t' \cup t''$, taken over all roots $k \in S_i(g)$, is weakly greater than the average marginal contribution for spanning tree $t'$, taken over all roots $k \in S_i(g')$. Since every spanning tree $t'$ on $S_i(g')$ corresponds with a constant number $|T'_{S_i(g) \backslash S_i(g') \cup \{i\}, g}|$ of spanning trees $t = t' \cup t''$ on $S_i(g)$, it follows that $uATS_i(N, v, g) \geq uATS_i(N, v, g)$. □

In view of the combinatorial difficulties involved with spanning trees in the presence of cycles, we do not have a conjecture regarding the satisfaction of LC or GC$^+$ by $uATS$ for arbitrary communication situations or arbitrary changes in communication possibilities, respectively. We know that the unrestricted average tree solution violates HC$^+$, as the following example shows: consider three-player simple majority voting

$(N, v) = [2; 1, 1, 1]$ with cooperation structures $g = \{\{1, 2\}\}$ and $g' = \{\{1, 2\}, \{2, 3\}\}$. Then $uATS(N, v, g) = (\frac{1}{2}, \frac{1}{2}, 0)$ and $uATS(N, v, g') = (0, 1, 0)$. Player 3 does not profit from their added communication possibility in the same way as player 2 does (in fact, player 3 does not benefit at all). This is a violation of Myerson's fairness property and, therefore, also of HC$^+$.

# Chapter 4

# Veto Players and Non-Cooperative Foundations of Power in Coalitional Bargaining*

*Abstract* In coalitional bargaining of the Baron-Ferejohn type, veto players either hold all power and share it proportional to their recognition probabilities, or hold no power at all. Hence, in this setting, it is impossible to provide non-cooperative support for power indices which do not assign all or no power to veto players. This highlights problems in the interpretation of results of Laruelle and Valenciano (2008a,b) which are taken as support for the Shapley-Shubik index and other normalized semi-values.

*Keywords* coalitional bargaining, power, veto players

*JEL Classification* C72, C78

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## 4.1 Introduction

An important and often raised question for cooperative solution concepts such as power indices is whether they can be given non-cooperative support or, equivalently, non-cooperative foundations. By this it is meant the existence of a non-cooperative game which, firstly, resembles the cooperative situation at hand in a reasonable way and, secondly, possesses equilibria whose payoffs coincide with those selected by the solution concept. As for power indices, the support of the Shapley-Shubik index (Shapley and Shubik, 1954) has received the greatest attention. Foundations for the more general Shapley value (Shapley, 1953), of which the Shapley-Shubik index is the restriction to simple games, have been provided by Hart and Mas-Colell (1996) and Pérez-Castrillo and Wettstein (2001), among others.[1]

Coalitional bargaining as introduced by Baron and Ferejohn (1989) and later extended by Eraslan (2002) and Eraslan and McLennan (2006) considers in its non-cooperative description of simple games only the most essential features of bargaining. For this reason, it is of particular interest whether power indices can be supported in this setting. Based on a result for the power of veto players, this note gives a partial, negative answer to this question. In addition, it highlights problems in the interpretation of results of Laruelle and Valenciano (2008a, and also 2008b, chapter 4.4) which are taken as support for the Shapley-Shubik index and other normalized semi-values.

The note is organized as follows. Section 2 introduces coalitional bargaining and existent results. The power of veto players is discussed in section 3. Section 4 defines suitable notions and derives a negative result for the non-cooperative support of power indices in coalitional bargaining.

[1] Gul (1989) provides another foundation for the Shapley value. However, his assumptions do not apply to simple games, and hence his results cannot be interpreted as support for the Shapley-Shubik index.

## 4.2 Coalitional Bargaining

This section presents coalitional bargaining games and existent results. Notations and formulations partly more resemble Eraslan (2002) or Montero (2006), yet only Eraslan and McLennan (2006) establish the results in this generality.

A *coalitional bargaining game* is given by a tuple $(N, W, p, \delta)$ where $(N, W)$ is a *simple game*, i.e. $N$ is the non-empty and finite set of *players* and $W \subseteq 2^N$ the set of *winning coalitions* for which $\emptyset \notin W$, $S \in W$ implies $T \in W$ for all $T \supseteq S$, and $N \in W$. The *protocol* $p = (p_i)_{i \in N}$ on $N$ with $p \geq 0$ and $\sum_{i \in N} p_i = 1$ defines the *recognition probabilities* of players, and $\delta$ with $0 \leq \delta < 1$ is the *discount factor* common to all players. Then, in every of possibly infinitely many rounds, one player $i \in N$ is randomly chosen as the *proposer* according to protocol $p$. Proposer $i$ chooses a feasible coalition $S \in W_i = \{S \in W | S \ni i\}$ and suggests a split of a unit surplus between the members of $S$. In some arbitrary order, all *respondents* $j \in S \backslash \{i\}$ vote on acceptance or rejection of this offer. Unless all respondents agree, the game proceeds to the next round and future payoffs are discounted by $\delta$. If respondents agree unanimously, the suggested split is implemented and the game ends. In addition, players are assumed to be risk-neutral, i.e. payoffs equal the (possibly discounted) shares received, and to have complete and perfect information.

Now, restrict attention to *stationary strategies* for which actions do not depend on past rounds. Thus, equilibrium strategies can be characterized by the corresponding (and as well stationary) *payoffs* $v = (v_i)_{i \in N}$ that players expect before any round. Facing the choice of a coalition, proposer $i$ chooses a coalition according to his *selection distribution* $\lambda_i = (\lambda_{iS})_{S \in W_i}$ with positive probability only on feasible coalitions $S \in W_i$ which have maximum *excess* $1 - \sum_{j \in S} \delta v_j$. Having chosen coalition $S$, proposer $i$ offers any respondent $j \in S \backslash \{i\}$ that player's *continuation value* $\delta v_j$, himself receiving his own continuation value $\delta v_i$ plus the coalition's *excess* $1 - \sum_{j \in S} \delta v_j$. Irrespective of proposer or coalition, a respondent $j$ accepts any share equal to or greater than his continuation value $\delta v_j$.

Thus, payoffs $v$ are supported by a *stationary subgame perfect equilibrium* (SSPE) with selection distributions $(\lambda_i)_{i\in N}$ if and only if, for all $i \in N$,

$$v_i = p_i \sum_{S\in W_i} \lambda_{iS}(1 - \sum_{j\in S} \delta v_j) + \sum_{j\in N} p_j \sum_{S\in W_i\cap W_j} \lambda_{jS}\delta v_i \tag{4.1}$$

and $\lambda_i = (\lambda_{iS})_{S\in W_i}$ puts positive probability only on feasible coalitions $S \in W_i$ that have maximum excess $1 - \sum_{j\in S} \delta v_j$. Eraslan and McLennan (2006) show the existence of a SSPE and the uniqueness of corresponding payoffs. As follows from above characterization of stationary equilibrium strategies, all SSPE are efficient ($\sum_i v_i = 1$) and there is no delay of agreement.

## 4.3 The Power of Veto Players

Given simple game $(N, W)$ and protocol $p$ on $N$, denote $\xi(N, W, p) = (\xi_i(N, W, p))_{i\in N}$ as the limit of SSPE payoffs in coalitional bargaining games $(N, W, p, \delta)$ as $\delta \to 1$.[2] Very commonly, $\xi_i(N, W, p)$ is interpreted as the *(bargaining) power* of player $i \in N$. A *veto player* $i \in N$ is a member of all winning coalitions, $i \in S$ for all $S \in W$. Denote the set of veto players by $V$.

**Proposition 4.1** *Let $(N, W)$ be a simple game and $p$ a protocol on $N$. Assume the set of veto players $V$ is non-empty and $p_i > 0$ for some $i \in V$. Then veto players hold all power and share it proportional to their recognition probabilities, $\xi_i(N, W, p) = p_i / \sum_{j\in V} p_j$ for all $i \in V$. In particular, all other players have no power, $\xi_i(N, W, p) = 0$ for all $i \notin V$.*[3]

**Proof** The proposition is shown in two steps, (i) $p_j \xi_i(N, W, p) = p_i \xi_j(N, W, p)$ for all $i, j \in V$ and then (ii) $\sum_{i\in V} \xi_i(N, W, p) = 1$. The two statements combined yield $\xi_i(N, W, p) = p_i / \sum_{j\in V} p_j$ for all $i \in V$.

i) For an arbitrary discount factor $\delta$, consider coalitional bargaining game $(N, W, p, \delta)$ with SSPE payoffs $v$ and selection distributions $(\lambda_i)_{i\in N}$. For a veto player $i \in V$,

[2] The theorem of the maximum ensures that the unique payoffs are continuous in $\delta$. Hence, the limit of payoffs as $\delta \to 1$ exists.

[3] Note that, in footnote 9 of her paper, Montero (2006) also claims, however does not proof, that veto players hold all power.

$\sum_{j \in N} p_j \sum_{S \in W_i \cap W_j} \lambda_{jS} = 1$ and the optimality of $i$'s selection distribution $\lambda_i$ allow to write (4.1) as $v_i = p_i \max_{S \in W}(1 - \sum_{j \in S} \delta v_j)/(1 - \delta)$. Thus, for all $i, j \in V$, $p_j v_i = p_i v_j$ and, as $\delta \to 1$, $p_j \xi_i(N, W, p) = p_i \xi_j(N, W, p)$.

ii) Fix $i \in V$ with $p_i > 0$. Without loss of generality, assume the selection distributions of players are continuous in $\delta$ and denote $\lambda_j^* = (\lambda_{jS}^*)_{S \in W}$ as the limit of selection distributions of any player $j \in N$ as $\delta \to 1$. The limit of $i$'s equation system (4.1) for $\delta \to 1$ then reads

$$\xi_i(N, W, p) = p_i \sum_{S \in W_i} \lambda_{iS}^*(1 - \sum_{j \in S} \xi_j(N, W, p)) + \sum_{j \in N} p_j \sum_{S \in W_i \cap W_j} \lambda_{jS}^* \xi_i(N, W, p).$$

Noticing $\sum_{j \in N} p_j \sum_{S \in W_i \cap W_j} \lambda_{jS}^* = 1$ yields $\sum_{S \in W_i} \lambda_{iS}^*(1 - \sum_{j \in S} \xi_j(N, W, p)) = 0$. Given this, $W_i = W$ and the optimality of $\lambda_i^*$ require $\sum_{j \in S} \xi_j(N, W, p) = 1$ for all $S \in W$. Now, for all $j \notin V$, there is $S \in W$ with $S \not\ni j$ and thus $\xi_j(N, W, p) = 0$. In particular, $\sum_{j \in V} \xi_j(N, W, p) = 1$. $\square$

Infinitely patient and able to block all proposals, veto players can play off the remaining players against each other to an extent that none of the latter can expect any positive share from bargaining. To do so, however, veto players themselves need an at least arbitrarily small possibility to make offers. In the case of $V$ being non-empty but $p_i = 0$ for all $i \in V$, the power $\xi_i(N, W, p)$ of any veto player $i \in V$ is 0.[4]

## 4.4 On the Support of Power Indices

Denote a *protocol scheme* $p$ as a mapping which assigns to every simple game $(N, W)$ a protocol $p(N, W) = (p_i(N, W))_{i \in N}$ on $N$. Thus, $\xi(N, W, p(N, W))$ is the bargaining power supported by protocol scheme $p$ for simple game $(N, W)$. A *power index* $f$ is a mapping which assigns a distribution of power $f(N, W) = (f_i(N, W))_{i \in N}$ to every simple game $(N, W)$.

[4] In any coalitional bargaining game $(N, W, p, \delta)$, the payoff of any player $i \in N$ with $p_i = 0$ is zero, $v_i = 0$ (see for instance Eraslan and McLennan, 2006). This of course translates to $i$'s power for simple game $(N, W)$ and protocol $p$.

Then, protocol scheme $p$ *supports* power index $f$ if $\xi(N, W, p(N, W)) = f(N, W)$ for every simple game $(N, W)$. Proposition 4.1 and the subsequent comment yield the following corollary.

**Corollary 4.2** *There exists no protocol scheme supporting any power index that does not assign all or no power to veto players.*

In particular, coalitional bargaining as considered here does not allow for support of the Shapley-Shubik index or the (normalized) Banzhaf index (Banzhaf, 1965), Deegan-Packel index (Deegan and Packel, 1978), or Public Good Index (Holler, 1982). However, there is the possibility that other solution concepts which do ascribe all power to veto players can be supported by suitable protocol schemes. For instance, Montero (2006) shows the nucleolus (Schmeidler, 1969) is even *self-confirming*, i.e. it is supported by the protocol scheme which assigns the nucleolus itself as the protocol.

Corollary 4.2 also highlights a misinterpretation of results of Laruelle and Valenciano (2008a, and also 2008b, chapter 4.4). They investigate non-cooperative bargaining as in this note, apart from two exceptions. Firstly, they more generally allow for non-transferable utility, and secondly, they deal with unanimity bargaining only. In this setting, they consider the Shapley-Shubik index (and other normalized semi-values) of an arbitrary simple game as the protocol and find it then also emerging as bargaining power.[5] At first sight, the result itself suggests to be taken as non-cooperative support. The authors seem to understand it this way as well and even say '[it] provides a noncooperative interpretation of any reasonable power index' (Laruelle and Valenciano, 2008a, p. 352). Only note that, in the particular case of unanimity bargaining with transferable utility, power is always given by the recognition probabilities: proposition 4.1 implies $\xi(N, \{N\}, p) = p$ for all sets of players $N$ and arbitrary protocols $p$ on $N$. So any non-negative and normalized power index, be it 'reasonable' or not, could hence be supported

[5] In the more general case of non-transferable utility, bargaining power materializes as the weights of the asymmetric Nash-solution, the latter arising as the limit of payoffs when players grow infinitely patient. In the particular case of transferable utility, bargaining power then again corresponds with payoffs exactly as in this note.

by using it as a protocol scheme. However, this approach does not seem adequate. The modeling of an arbitrary voting situation as unanimity bargaining typically bears an incongruence that, as the results of this note show, can matter significantly.

# Chapter 5

# Coalitional Bargaining with Markovian Proposers*

*Abstract* The paper generalizes coalitional bargaining of the Baron-Ferejohn type (Eraslan and McLennan, 2006) to non-independent proposers. Widening the scope of possible proposer dynamics significantly, this allows for likely continued offers, alternating offers (Calvó-Armengol, 2001b), and deterministic dynamics such as clockwise rotation of proposers (Herrero, 1985). Existence of stationary subgame perfect equilibria is shown as well as non-uniqueness of corresponding payoffs. In addition, generalizing a result of Nohn (2010), we show that veto players hold all bargaining power in case of almost all protocols.

*Keywords* coalitional bargaining, Markov process, subgame perfect equilibrium, stationary strategies, power, veto players

*JEL Classification* C72, C78

* Available as PCRC working paper. Earlier versions of this paper have been presented at the Adam Smith Seminar in Hamburg, Germany (December 2009), the PCRC seminar in Turku, Finland (December 2009), the Annual Meeting of the Public Choice Society in Monterey, California (March 2010), the summer school 'Institutions in Context' in Tampere, Finland (June 2010), and the workshop 'Voting, Manipulation and Power' in Prague, Czech Republic (September 2010). I thank all audiences for the numerous comments that helped to improve this paper significantly.

## 5.1 Introduction

In coalitional bargaining as introduced in Baron and Ferejohn (1989), multiple players bargain non-cooperatively on the division of a unit surplus that can be generated once by one of given feasible coalitions. In every of possibly infinitely many rounds, one player is assigned the role of the proposer according to some random protocol. The proposer can offer a split of the value of 1 to one of the feasible coalitions. If all respondents, which are the other members of that coalition, accept the proposer's offer, the surplus is split accordingly and the game ends. If respondents do not agree unanimously, the game proceeds to the next round and all possible future payoffs are discounted.

Baron and Ferejohn (1989) consider a simple majority rule defining the feasible coalitions, yet the model has been successively extended. Eraslan (2002) more generally allows for any kind of majority rule with symmetric voters, and Eraslan and McLennan (2006) implement arbitrary sets of feasible coalitions. Applying the Baron-Ferejohn model to apex games, Montero (2002) finds that equilibrium payoffs coincide with the kernel of the grand coalition if the probabilities for being proposer are either equal or proportional to voting weights. Montero (2006) shows that the nucleolus of a simple game is self-confirming in the respective Baron-Ferejohn bargaining game, that is, it is obtained as the equilibrium payoffs whenever it is also used as the protocol. Also, there are models building on Baron and Ferejohn (1989) in other ways than Eraslan and McLennan (2006). Banks and Duggan (2000) provide a generalization to a multidimensional spatial model. Okada (1996) considers arbitrary superadditive transferable-utility games underlying the non-cooperative bargaining game, and Miyakawa (2009) investigates non-superadditive bargaining situations.

This paper sets out to extend coalitional bargaining of Eraslan and McLennan (2006). Instead of modifying which coalitions can be feasible (which has been exhaustively done) or which values coalitions can have, we relax one key assumption common to all above models. While we still rely on, except for degenerated cases, protocols that are random, we waive the assumption that proposers are chosen independently in different rounds. In a Markovian way, the probability distribution for the choice of a proposer is allowed to

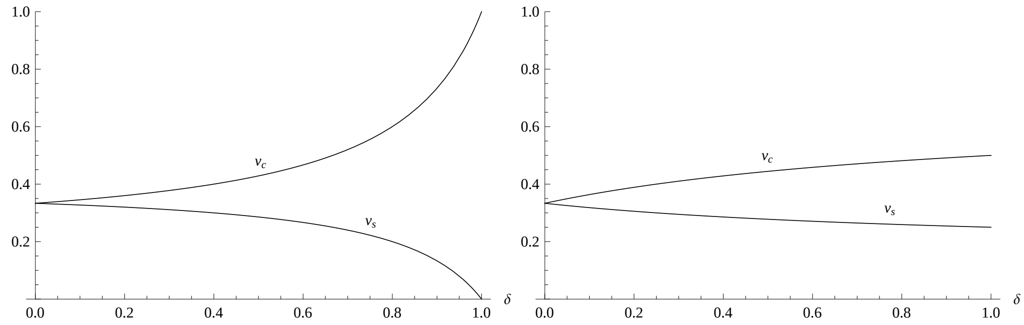


Figure 5.1: SSPE payoffs of example 5.1 with the independent protocol (left) and the alternating protocol (right)

depend on the identity of the proposer as well as the coalition of the previous round.

Compared to coalitional bargaining with independent proposers, the scope of possible dynamics is widened significantly. The non-independence of proposers for instance allows for likely continued offers in the sense that a proposer can have increased chances of making a new offer in case his current one is rejected. Not less plausible, respondents can have greater than normal probability of making an offer in the next round. Taking this to the extreme where only respondents have the chance of being proposer in the next round, our model incorporates the proposer dynamics of Calvó-Armengol (2001a,b). The two papers model bilateral bargaining among multiple players and, as such, are extensions of the alternating offer model of Rubinstein (1982). However, in contrast to our model, they only consider exogenous matching of proposer and respondent (Calvó-Armengol, 2001a) or deterministic, non-random selection of bargaining partners (Calvó-Armengol, 2001b). Finally, degenerated Markovian dynamics can implement deterministic protocols such as clockwise rotation of proposers in Herrero (1985), a multilateral extension of Rubinstein (1982) with unanimity bargaining.

To see what qualitative impact the choice of protocol can make, compare the two canonical protocols as in the following example.

**Example 5.1** *Let there be three players $c$, $s_1$, and $s_2$ such that central player $c$ can generate the unit surplus with either of the two minor players $s_1$ or $s_2$. Consider, on the* *one hand, the protocol where all players are chosen as the proposer in all rounds with equal probability (and thus, in fact, independently). On the other hand, as in Calvó-Armengol (2001a,b), let all players be proposer in the first round with equal probability and, in all subsequent rounds, let the respondent of the previous round take the role of the proposer. In addition, let all players have a common unconditional discount factor* $\delta$*,* $0 \leq \delta < 1$*. Figure 5.1 shows (stationary subgame perfect) equilibrium payoffs* $v_c$ *of player c and* $v_s$ *of players* $s_1$ *or* $s_2$ *as a function of* $\delta$*. Both equilibria are symmetric in that player c proposes to both minor players* $s_1$ *and* $s_2$ *with equal probability. In the case of the independent protocol, player c holds all power (as which payoffs are usually interpreted when players grow infinitely patient, that is, when* $\delta \to 1$*). However, he only holds half of the power if proposers are chosen according to the alternating protocol.*

The example shows a great sensitivity of the equilibrium payoffs with respect to modeling assumptions and calls for caution in the comparison of models with different proposer dynamics. Throughout the literature, one frequently encounters misleading statements such as one model (typically one that considers independent random proposers) 'extends' another one (typically Rubinstein, 1982), irrespective of whether both models are based on the same proposer dynamics.[1] The results in a model with independent random proposers are not, however, necessarily equivalent to results one would obtain from a respective extension of Rubinstein (1982) that has, for instance, alternating or clockwise rotating proposers.

In addition, the example raises the question of whether such qualitative differences persist in general and, if yes, in what way. More particularly even, it highlights the possibility of non-cooperative support for power indices in this extended framework. For the case of independent proposers, Nohn (2010) shows that veto players hold all

[1] Okada (1996) claims to extend the alternating offer bargaining model of Rubinstein (1982) while he in fact uses an independent random protocol. Similarly, Montero (2006) says that Baron and Ferejohn (1989), and thus also herself, 'build on' Rubinstein (1982). Laruelle and Valenciano (2008a) suggest they, by generalizing Binmore (1987), connect Nash (1950) and Rubinstein (1982). Binmore (1987), however, considers several variations of Rubinstein (1982), and it is only one particular variation with independent random proposers that is generalized Laruelle and Valenciano (2008a).

bargaining power and share it proportional to their recognition probabilities, or hold no power at all. Non-cooperative support for power indices in the Baron-Ferejohn model with independent proposers is hence restricted to power indices that assign all or no power to veto players. This, unfortunately, excludes support for popular indices such as the Shapley-Shubik index (Shapley and Shubik, 1954).[2,3] For the voting game underlying example 5.1, the Shapley-Shubik index amounts to 2/3 for player $c$ and 1/6 for both minor players $s_1$ and $s_2$. It could hence be supported by a suitable protocol somewhere 'between' the independent and alternating protocol if only the limit of payoffs behaved 'sufficiently continuous' in the proposer dynamics.

The remainder of the paper is organized as follows. Section 5.2 defines our extension of coalitional bargaining games to non-independent proposers. A characterization of stationary subgame perfect equilibria (SSPE), the solution concept we apply, is provided in section 5.3. Section 5.4 presents and proofs the existence of SSPE for all coalitional bargaining games and some basic proprties of the corresponding payoffs. The non-uniqueness of the payoffs, in contrast to the case of independent proposers (Eraslan and McLennan, 2006), is shown in section 5.5. Section 5.6 provides and discusses a statement that veto players hold all power with almost all protocols, and section 5.7 concludes.

## 5.2 The Coalitional Bargaining Game

Let $N$ be the non-empty and finite *set of players*. By $W \subseteq 2^N$ we denote the *set of feasible coalitions*, i.e. those coalitions able to generate a surplus of 1. We assume that the empty coalition is not feasible, $\emptyset \notin W$, and that for every player $i \in N$ his set of feasible coalitions $W_i = \{S \in W | S \ni i\}$ is non-empty.[4] We call any pair $(i, S)$ where

[2] Nohn (2010) makes this point in particular to argue that it is not reasonable to interpret the results of Laruelle and Valenciano (2008a) as non-cooperative support for the Shapley-Shubik index.

[3] This contrasts with results for models that incorporate more sophisticated features of bargaining. Hart and Mas-Colell (1996) or Pérez-Castrillo and Wettstein (2001) support the Shapley value (Shapley, 1953), and thus in particular also the Shapley-Shubik index.

[4] As in Eraslan and McLennan (2006), it is equally possible to just have non-empty sets of feasible coalitions $W_i$, $i \in N$, which satisfy $S \ni i$ for all $S \in W_i$ and all $i \in N$. These sets of different players

$i \in N$ is a player and $S \in W_i$ one of player $i$'s feasible coalitions a *match* and denote the set of matches by $M$. The *protocol* is a collection $P = (p, (p^{jT})_{(j,T) \in M})$ that defines the *recognition probabilities* for the assignment of proposers. In this, $p = (p_i)_{i \in N}$ with $p \geq 0$ and $\sum_i p_i = 1$ is the *initial protocol* for the choice of a proposer in the first round. In any subsequent round, the recognition probabilities depend on the match of the previous round. We call $p^{jT} = (p_i^{jT})_{i \in N}$ with $p^{jT} \geq 0$ and $\sum_i p_i^{jT} = 1$ the *conditional protocol* for match $(j,T) \in M$.[5] In addition, players have a common discount factor $\delta$ with $0 \leq \delta < 1$ by which future payoffs are discounted in case of a delay of one round.[6,7]

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do not need to be related in any particular way. That is, in contrast to our definition, it does not have to hold that $S \in W_j$ for all $j \in S$ for all feasible coalitions $S \in \cup_{i \in N} W_i$. Also with this less restrictive definition of feasible coalitions, all statements of sections 5.3 and 5.4 remain valid and can be proven accordingly and without any further effort.

[5] As a natural approach to non-independent protocols, consider that players attempt to make offers with rates depending on their current role in bargaining. More specifically, players independently attempt to make offers after exponentially distributed waiting times, and the player trying first initiates the next round as the proposer. For each player $i \in N$, let $(o_i, o_i^p, o_i^r, o_i^n)$ denote the nonnegative *offering rates* for the first round, for previously being proposer, respondent, or not being involved in bargaining, respectively (in case a rate is 0, the respective player does not make offers in that particular situation). Assume the initial total offering rate $\bar{o} = \sum_{i \in N} o_i$ as well as all conditional total offering rates $\bar{o}^{jT} = o_j^p + \sum_{k \in T \setminus \{j\}} o_k^r + \sum_{k \in N \setminus T} o_k^n$, $(j,T) \in M$, are positive. Then, in the first round, $p_i = o_i / \bar{o}$, $i \in N$. For all rounds following a match $(j,T) \in M$, $p_i^{jT} = o_i^p / \bar{o}^{jT}$ for previous proposer $i = j$, $p_i^{jT} = o_i^r / \bar{o}^{jT}$ for previous respondents $i \in T \setminus \{j\}$, and $p_i^{jT} = o_i^n / \bar{o}^{jT}$ for players $i \in N \setminus T$ not previously involved in bargaining.

[6] It is equally possible to assume individual and conditional discount factors. One then has a collection of discount factors $(\delta_i^{jT})_{i \in N, (j,T) \in M}$ such that the payoff of player $i \in N$ is discounted by $\delta_i^{jT}$ after any round with match $(j,T) \in M$. A collection of conditional discount factors thus allows to incorporate individual time preferences and delays that depend, for instance, on the size of the coalition. While conditional discount factors might thus in fact have an impact on the selection behavior of proposers, we do not incorporate them in the model due to a lack of respective analytical statements concerning, for instance, monotonicities. All statements of sections 5.3 and 5.4, however, can be modified accordingly and proven analogously without any further effort.

[7] With respect to footnotes 5 and 6, assume that players $i \in N$ have offering rates $(o_i, o_i^p, o_i^r, o_i^n)$ and *unit discount factors* $\delta_i$, $0 \leq \delta_i < 1$, for the discount of payoffs in a unit time interval. For convention, let $ln(0) = -\infty$ and $1/\infty = 0$. Conditional discount factors then arise as the expected discount between two rounds and amount to $\delta_i^{jT} = \bar{o}^{jT} / (\bar{o}^{jT} - ln(\delta_i))$ for all $(j,T) \in M$, $i \in N$.

A *coalitional bargaining game* is defined by the tuple $(N, W, P, \delta)$ containing the set of players $N$, the set of feasible coalitions $W$, the protocol $P$, and the discount factor $\delta$. Then, in any of possibly infinitely many rounds, a player $i \in N$ is chosen to be the *proposer* for which the recognition probabilities are given by the initial protocol $p = (p_i)_{i \in N}$ in the first round or by conditional protocols $p^{jT} = (p_i^{jT})_{i \in N}$ after rounds with match $(j, T) \in M$, respectively. Proposer $i$ selects one of his feasible coalitions $S \in W_i$. He suggests a division of the surplus of 1 between him and the *respondents* $k \in S \backslash \{i\}$. The respondents decide in some arbitrary order - which one is irrelevant - on the proposal's acceptance or rejection.[8] If at least one respondent rejects the offer, the game proceeds to the next round and all future payoffs are discounted according to the discount factor $\delta$. Unanimous acceptance, however, leads to the respective division and terminates the game.

As additional assumptions, we let all players be risk-neutral and identify their utility with the share they receive. Also, players have complete and perfect information, and everything is common knowledge.

## 5.3 Characterization of Stationary Subgame Perfect Equilibra

Since games of this kind typically have a multiplicity of equilibria and corresponding payoffs (see for instance the simple majority case in Baron and Ferejohn, 1989), we follow the literature in that we restrict attention to stationary strategies, and hence solve for stationary subgame perfect equilibria (SSPE). While the plausibility of this restriction is certainly open for debate, note that SSPE do not only constitute subgame perfect equilibria in the class of stationary strategies but, more generally, also in the class of all possible strategies.[9]

Strategies of players are *stationary* whenever actions are both *time-homogeneous* and

[8] Note that the decision order might not be irrelevant if the protocols are allowed to be conditioned on the identities of those respondents that reject on offer.

[9] This holds in general for *Markov strategies*, see Fudenberg and Tirole (1991, p. 501).

*independent* of previous rounds. In contrast to coalitional bargaining models which use independent proposers, we yet allow for actions depending on the current match. In any round, a player $i \in N$ who has been chosen as the proposer selects a feasible coalition $S \in W_i$ according to a stationary *coalition distribution* $\lambda_i = (\lambda_{iS})_{S \in W_i}$. Conditional on having selected coalition $S \in W_i$, he suggests a split $x^{iS} = (x_k^{iS})_{k \in S}$ of the surplus of 1 between the members of that coalition. The decisions about acceptance or rejection of respondents $k \in S \backslash \{i\}$ depend only on the offer itself and the current match $(i, S) \in M$.

Strategies being stationary, the payoffs players expect before any round at most depend on the match of the last round. Given current match $(j, T) \in M$, denote by $v^{jT} = (v_i^{jT})_{i \in N}$ the conditional payoffs of players before next round. By $v = (v_i)_{i \in N}$ we denote the payoffs players expect before the game. Stationarity allows to characterize the strategies by the corresponding conditional payoffs.

**Proposition 5.2** *SSPE strategies are characterized by the corresponding conditional payoffs* $(v^{jT})_{(j,T) \in M}$ *as follows.*

*(i) Given current match* $(i, S) \in M$*, respondents* $k \in S \backslash \{i\}$ *accept any share* $x_k^{iS}$ *not exceeded by their* continuation value $\delta v_k^{iS}$.[10,11]

*(ii) Having chosen coalition* $S \in W_i$*, proposer* $i \in N$ *offers any respondent* $k \in S \backslash \{i\}$ *that player's continuation value,* $x_k^{iS} = \delta v_k^{iS}$*, and his own share amounts to* $x_i^{iS} = 1 - \sum_{k \in S \backslash i} \delta v_k^{iS}$.

*(iii) Any player* $i \in N$ *uses a coalition distribution* $\lambda_i$ *with positive probability only on coalitions* $S \in W_i$ *which maximize* $1 - \sum_{k \in S \backslash \{i\}} \delta v_k^{iS}$.

[10] Here, it is crucial that respondents do not decide simultaneously but successively in some arbitrary order. Simultaneous decisions of two or more respondents allow arbitrary threats to the proposer as in Haller (1986).

[11] Note there are SSPE in which respondents decide differently. This is possible, however, only where another respondent already has, or certainly will, reject an offer. Hence, all these SSPE are equivalent in outcome and payoffs.

**Proof**

(i) Consider proposer $i \in N$ suggesting split $(x_k^{iS})_{k \in S}$ to coalition $S \in W_i$. Respondents decide in some arbitrary order on the proposal's acceptance. In case that any respondent rejects, respondent $k \in S \backslash \{i\}$ alternatively expects $\delta v_k^{iS}$ by continuation of the game in the next round. So acceptance (rejection) is a best response for respondent $k \in S \backslash \{i\}$ in case that $x_k^{iS} > \delta v_k^{iS}$ ($x_k^{iS} < \delta v_k^{iS}$) and it is a strictly best action if and only if all previous and subsequent respondents accept. Being offered his reservation value, $x_k^{iS} = \delta v_k^{iS}$, the respondent is indifferent and can well accept (and, in fact, acceptance in this case is necessary to find a best action for the proposer in (ii), and thus for the existence of a SSPE).

(ii) Consider a proposer $i \in N$ making an offer to coalition $S \in M_i$. Given respondents' behavior according to (i), offering $x_k^{iS} = \delta v_k^{iS}$ to all respondents $k \in S \backslash \{i\}$ and keeping $x_i^{iS} = 1 - \sum_{k \in S \backslash \{i\}} \delta v_k^{iS}$ for himself is the most profitable among all accepted offers. Moreover, due to $1 - \sum_{k \in S \backslash \{i\}} \delta v_k^{iS} > \delta v_i^{iS}$, making this offer is strictly better than continuation of the game in the next round due to a rejected offer.

(iii) Follows directly given later behavior as described by (i) and (ii). $\square$

Note that, unlike in the case with independent proposers, proposer $i \in N$ does not typically choose a coalition $S \in W_i$ with maximum *excess* $1 - \sum_{k \in S} \delta v_k^{iS}$. Since his continuation value $\delta v_i^{iS}$ possibly depends on $S$, maximization of the excess $1 - \sum_{k \in S} \delta v_k^{iS}$ is not necessarily equivalent to the maximization of his share $1 - \sum_{k \in S \backslash \{i\}} \delta v_k^{iS}$. Thus, depending on the protocol, it is even possible that players propose to coalitions which are not minimal.[12]

By proposition 5.2, any stationary subgame perfect equilibbrium is fully described by its conditional payoffs $(v^{jT})_{(j,T) \in M}$ and coalition distributions $(\lambda_i)_{i \in N}$. The following proposition gives a necessary and sufficient condition for when conditional payoffs and

[12] Consider $N \in W$ and a protocol $P$ with $p_i^{iN} = 1$, $i \in N$. In this case, proposing to the grand coalition would make a player be the proposer in the next round with certainty. Then being the only possible proposer when always offering to the grand coalition, any player could then expect the whole unit surplus, compare proposition 5.5. Thus, this selection behavior is subgame perfect.

coalition distributions are supported by a SSPE.

**Proposition 5.3** *A collection of conditional payoffs* $(v^{jT})_{(j,T)\in M}$ *is supported by a SSPE with coalition distributions* $(\lambda_i)_{i\in N}$ *if and only if*

*for all matches* $(j,T)\in M$ *and all* $i\in N$ *it holds*

$$v_i^{jT} = p_i^{jT} \sum_{S\in W_i} \lambda_{iS}(1 - \sum_{k\in S\backslash\{i\}} \delta v_k^{iS}) + \sum_{k\in N\backslash\{i\}} p_k^{jT} \sum_{S\in W_k : S\ni i} \lambda_{kS} \delta v_i^{kS} \tag{5.1}$$

*and for all* $i\in N$ *it holds that* $\lambda_i$ *maximizes*

$$\sum_{S\in W_i} \lambda_{iS}(1 - \sum_{k\in S\backslash\{i\}} \delta v_k^{iS}) \tag{5.2}$$

*given constraints* $\lambda_i \geq 0$ *and* $\sum_{S\in W_i} \lambda_{iS} = 1$.

**Proof**

(Only if.) Let conditional expectations $(v^{jT})_{(j,T)\in M}$ and coalition distributions $(\lambda_i)_{i\in N}$ be supported by an SSPE. Then strategies according to proposition 5.2 yield conditional payoffs as in (5.1) and coalition distributions satisfying (5.2).

(If.) Assume $(v^{jT})_{(j,T)\in M}$ and $(\lambda_i)_{i\in N}$ satisfy (5.1) and (5.2) and consider strategies as follows. Let coalition distributions be given by $(\lambda_i)_{i\in N}$. Given player $i\in N$ proposes to coalition $S\in W_i$, let him offer $\delta v_k^{iS}$ to any respondent $k\in S\backslash\{i\}$ and keep the remaining $1-\sum_{k\in S\backslash\{i\}} \delta v_k^{iS}$ for himself. Let respondents $k\in S\backslash\{i\}$ accept any share not exceeded by their reservation value $\delta v_k^{iS}$. Due to (5.1), these strategies generate conditional payoffs again given by $(v^{jT})_{(j,T)\in M}$. Thus, from (5.2) and proposition 5.2 it follows that these strategies constitute a SSPE. $\square$

Note that, for given coalition distributions, conditional payoffs correspond in a one-to-one fashion with the offers made. Also, conditional payoffs $v_i^{jT}$ where $(j,T)\in M$ and $i\notin T\backslash\{j\}$ are redundant in the equilibrium characterization and one can restrict attention to the conditional payoffs of previous respondents, $v_i^{jT}$ for which $(j,T)\in M$ and $i\in T\backslash\{j\}$. Thus, a stationary equilibrium characterization based on conditional payoffs is equivalent to a characterization based on offers as in Calvó-Armengol (2001a,b).

## 5.4 Existence of Stationary Subgame Perfect Equilibra

The propositions of the previous section allow for the following statement.

**Theorem 5.4** *In any coalitional bargaining game* $(N, W, P, \delta)$*, there exists a stationary subgame perfect equilibrium.*

**Proof** We prove the theorem in three stages. We first show, (i), that for any given collection of coalition distributions $(\lambda_i)_{i\in N}$, there is one unique solution $(v^{jT})_{(j,T)\in M}$ to equation system (5.1). Then, (ii), any such solution constitutes a collection of nonnegative and normalized vectors and is hence feasible as a collection of conditional payoffs. Finally, (iii), Kakutani's fixed-point theorem is applied to show the mutual existence of $(v^{jT})_{(j,T)\in M}$ and $(\lambda_i)_{i\in N}$ such that (5.1) and (5.2).

(i) For a given collection of coalition distributions $(\lambda_i)_{i\in N}$, assume there are two distinct solutions $(v^{jT})_{(j,T)\in M}$ and $(w^{jT})_{(j,T)\in M}$ to equation system (5.1). For any match $(k, S) \in M$, denote $N_+^{kS} = \{i \in N | v_i^{kS} - w_i^{kS} > 0\}$ and fix one match $(j, T) \in M$ such that $\sum_{i\in N_+^{jT}}(v_i^{jT} - w_i^{jT})$ is maximal among all $\sum_{i\in N_+^{kS}}(v_i^{kS} - w_i^{kS})$, $(k, S) \in M$. In particular, it holds

$$\sum_{i\in N_+^{jT}} (v_i^{jT} - w_i^{jT}) > \sum_{i\in S\cap N_+^{jT}} \delta(v_i^{kS} - w_i^{kS})$$

for all $(k, S) \in M$. Also note that from (5.1) it follows $\sum_{i\in N} v_i^{kS} = 1$ for all $(k, S) \in M$. Thus, $\sum_{i\in N}(v_i^{kS} - w_i^{kS}) = 0$ for all $(k, S) \in M$ and hence $\sum_{i\in N_+^{jT}}(v_i^{jT} - w_i^{jT}) \geq -\sum_{i\in N\setminus N_+^{kS}}(v_i^{kS} - w_i^{kS})$ for all $(k, S) \in M$. In particular,

$$\sum_{k\in N_+^{jT}} (v_k^{jT} - w_k^{jT}) > -\sum_{k\in S\setminus N_+^{jT}} \delta(v_k^{iS} - w_k^{iS})$$

for all $(i, S) \in M$. Thus, summing up equation system (5.1) over $i \in N_+^{jT}$ for both $v$ and $w$ and considering the difference, we find the following contradiction.

$$\begin{aligned}
&\sum_{i \in N_+^{jT}} (v_i^{jT} - w_i^{jT}) \\
= &- \sum_{i \in N_+^{jT}} \sum_{S \in W_i} p_i^{jT} \lambda_{iS} \sum_{k \in S \setminus N_+^{jT}} \delta(v_k^{iS} - w_k^{iS}) \\
&+ \sum_{k \in N \setminus N_+^{jT}} \sum_{S \in W_k} p_k^{jT} \lambda_{kS} \sum_{i \in S \cap N_+^{jT}} \delta(v_i^{kS} - w_i^{kS}) \\
< \quad &\sum_{i \in N_+^{jT}} \sum_{S \in W_i} p_i^{jT} \lambda_{iS} \sum_{k \in N_+^{jT}} (v_k^{jT} - w_k^{jT}) \\
&+ \sum_{k \in N \setminus N_+^{jT}} \sum_{S \in W_k} p_k^{jT} \lambda_{kS} \sum_{i \in N_+^{jT}} (v_i^{jT} - w_i^{jT}) \\
= &\sum_{i \in N_+^{jT}} (v_i^{jT} - w_i^{jT})
\end{aligned}$$

Hence, linear equation system (5.1) always has a unique solution.

(ii) Consider any given collection of coalition distributions $\lambda = (\lambda_i)_{i \in N}$. For any match $(j, T) \in M$ and player $i \in N$, define the mapping $f_i^{jT}$ by

$$\begin{aligned}
&f_i^{jT}((x_{i'}^{j'T'})_{(j',T') \in M, i' \in N}) \\
&= p_i^{jT} \sum_{S \in W_i} \lambda_{iS}(1 - \sum_{k \in S \setminus \{i\}} \delta x_k^{iS}) + \sum_{k \in N \setminus \{i\}} p_k^{jT} \sum_{S \in W_k, S \ni i} \lambda_{kS} \delta x_i^{kS}.
\end{aligned}$$

In case that $(x_{i'}^{j'T'})_{i' \in N}$ is a nonnegative and normalized vector for any match $(j', T') \in M$, then also $(f_i^{jT}((x_{i'}^{j'T'})_{(j',T') \in M, i' \in N})_{i \in N}$ is nonnegative and normalized for any match $(j, T) \in M$. From $f$'s continuity and Brouwer's fixed-point theorem, it follows the existence of a collection $(v_i^{jT})_{(j,T) \in M, i \in N}$ of nonnegative and normalized vectors which constitutes a fixed-point of $(f_i^{jT})_{(j,T) \in M, i \in N}$ or, equivalently, which satisfies linear equation system (5.1).

(iii) For any given collection of coalition distributions $\lambda = (\lambda_i)_{i \in N}$, denote by $\bar{v}^{jT}(\lambda) = (\bar{v}_i^{jT}(\lambda))_{i \in N}$ for every match $(j, T \in M)$ the corresponding conditional payoff vector resulting from linear equation system (5.1). For any collection of conditional payoff vectors $(v^{jT})_{(j,T) \in M}$, denote by $\bar{\lambda}((v^{jT})_{(j,T) \in M})$ the set of collections of coalition distributions solving optimization problem (5.2). By the theorem of the maximum, $\bar{\lambda}((v^{jT})_{(j,T)\in M})$ is upper semi-continuous in $(v^{jT})_{(j,T)\in M}$. Since $(\bar{v}^{jT}(\lambda))_{(j,T)\in M}$ is continuous, also $\bar{\lambda}((\bar{v}^{jT}(\lambda))_{(j,T)\in M})$ is upper semi-continuous. Kakutani's fixed-point theorem ensures the existence of a collection of coalition distributions $\lambda^*$ with $\lambda^* \in \bar{\lambda}((\bar{v}^{jT}(\lambda^*))_{(j,T)\in M})$. By construction, the pair $(\bar{v}^{jT}(\lambda^*))_{(j,T)\in M}$ and $\lambda^*$ satisfy linear equation system (5.1) and optimization problem (5.2) and hence are, by proposition 5.3, supported by a SSPE. □

The (ex-ante) payoffs $v$ in a SSPE with conditional payoffs $(v^{jT})_{(j,T)\in M}$ and coalition distributions $\lambda = (\lambda_i)_{i\in N}$ then amount, for $i \in N$, to

$$v_i = p_i \sum_{S\in W_i} \lambda_{iS}(1 - \sum_{k\in S\setminus\{i\}} \delta v_k^{iS}) + \sum_{k\in N\setminus\{i\}} p_k \sum_{S\in W_k:S\ni i} \lambda_{kS}\delta v_i^{kS}. \tag{5.3}$$

This and the equilibrium condition for conditional payoffs, proposition 5.3, give rise to the following basic properties of SSPE payoffs.

**Proposition 5.5**

*(i) There is no delay of equilibrium and all SSPE are efficient,* $\sum_{i\in N} v_i = 1$.

*(ii) For all* $i \in N$, *if* $p_i > 0$, *then* $v_i > 0$.

*(iii) For all* $i \in N$, *if* $p_i = 0$ *and* $p_i^{jT} = 0$ *for all* $(j,T) \in M$, *then* $v_i = 0$.

**Proof**

(i) Follows from proposition 5.3 and by summing up equation (5.3) over all $i \in N$.

(ii) From equation (5.1), it is $\sum_{i\in N} v_i^{jT} = 1$ for all $(j,T) \in M$. Hence, $1 - \sum_{k\in S\setminus\{i\}} \delta v_k^{iS} > 0$ for all $S \in W_i$. Thus, if $p_i > 0$, then $v_i > 0$.

(iii) Let $p_i^{kS} = 0$ for all $(k,S) \in M$ and choose a particular $(j,T) \in M$ for which $v_i^{jT}$ is maximal among all $v_i^{kS}$, $(k,S) \in M$. Equation (5.1) reads $v_i^{jT} = \sum_{k\in N\setminus\{i\}} p_k^{jT} \sum_{S\in W_k:S\ni i} \lambda_{kS}\delta v_i^{kS}$ which implies $v_i^{jT} \leq \delta v_i^{jT}$. This can be satisfied only by $v_i^{jT} = 0$, and thus also $v_i^{kS} = 0$ for all $(k,S) \in M$. If in addition $p_i = 0$, then equation (5.3) yields $v_i = 0$. □

Part (i) of the property states that players reach an agreement in the first round and share the entire unit surplus. Parts (ii) and (iii) constitute the analogues to the fact that, in the case of independent proposers, a player expects a positive share from bargaining if and only if he has the chance of making offers. In our model, it is sufficient for a positive ex-ante payoff to have the chance of being proposer in the first round, even if all conditional recognition probabilities are zero. To have an ex-ante payoff of zero, however, requires more than not having the chance of being proposer in the first round – a player must also not expect a positive share from being respondent. The latter is given, for instance, whenever all conditional recognition probabilities are zero.

Other candidates for properties typically include monotonicities of SSPE payoffs. Eraslan (2002) shows that, in case of independent proposers and a symmetric voting rule defining the feasible coalitions, payoffs are weakly monotonic in the respective recognition probabilities. Finding similar monotonicities in this much wider framework is difficult for analytical reasons. However, it is also not straightforward how the protocol should typically change to favor a player, and seemingly advantageous changes might still turn out to be detrimental to the respective player if this lets other players propose to coalitions he is a member of less often.[13]

## 5.5 Non-Uniqueness of Stationary Equilibrium Payoffs

Eraslan and McLennan (2006) show that, while there may be a multiplicity of stationary subgame perfect equilibria, the corresponding (ex-ante) payoffs are unique in the case of independent proposers. In the general case of possibly dependent proposers, we find this uniqueness not to hold anymore. To see this, consider the bargaining situation from the introductory example with an arbitrary number of players and alternating protocol. In general, the *alternating protocol* can only be considered for games with no feasible singleton coalition, i.e. $\{i\} \notin W_i$ for all $i \in N$. Initially, it assigns equal probability to be

[13] Even in much simpler situations, not all intuitive monotonicities hold, for instance that players' payoffs are non-decreasing in their patience. Kawamori (2005) considers the setting of Eraslan (2002) with independent proposers bargaining under a symmetric non-unanimity voting rule. Given equal recognition probabilities and sufficiently similar discount factors, payoffs of players are proportional to their inverse discount factors. This result corrects a false proposition of Eraslan (2002).

the proposer to all players and, in all subsequent rounds, the respondents of the previous round have equal probability to be the proposer. Formally, $p_i = 1/|N|$ for all $i \in N$ and for all $(j,T) \in M$, $p_i^{jT} = 1/(|T|-1)$ for $i \in T\backslash\{j\}$ and $p_i^{jT} = 0$ for $i \notin T$.

**Example 5.6** *Let there be $1+n \geq 3$ players, $N = \{c, s_1, \ldots, s_n\}$, such that central player $c$ can create a surplus of 1 together with any single minor player $s_1, \ldots, s_n$. The feasible coalitions are hence given by $S_i = \{c, s_i\}$, $i = 1, \ldots, n$. Let proposers be assigned according to the alternating protocol and let players have a discount factor $\delta$, $0 \leq \delta < 1$. Restricting attention to the conditional payoffs of previous respondents, the particular instances of equation system (5.1) arise to*

$$v_c^{s_i S_i} = \sum_{j=1,\ldots,n} \lambda_{cS_j}(1 - \delta v_{s_j}^{cS_j}), \quad i = 1, \ldots, n, \tag{5.4}$$

$$v_{s_i}^{cS_i} = 1 - \delta v_c^{s_i S_i}, \quad i = 1, \ldots, n. \tag{5.5}$$

*Conditional payoffs thus happen to be independent of player $c$'s coalition distribution $\lambda_c = (\lambda_{cS_1}, \ldots, \lambda_{cS_n})$ and, more precisely, the conditional payoff, or share, of any proposer is that of the proposer in a 2-player Rubinstein alternating offer game (Rubinstein, 1982),*

$$v_c^{s_i S_i} = \frac{1}{1+\delta}, \quad i = 1, \ldots, n, \tag{5.6}$$

$$v_{s_i}^{cS_i} = \frac{1}{1+\delta}, \quad i = 1, \ldots, n. \tag{5.7}$$

*In particular, there is an SSPE for every possible coalition distribution $\lambda_c$ of player $c$. Unlike in the case with independent proposers, ex-ante SSPE payoffs $v$ coincide only if players are myopic,*

$$v_c = \frac{1+n\delta}{(1+n)(1+\delta)}, \tag{5.8}$$

$$v_{s_i} = \frac{1+\lambda_{cS_i}\delta}{(1+n)(1+\delta)}, \quad i = 1, \ldots, n. \tag{5.9}$$

Interestingly, the multiplicity of stationary equilibrium payoffs found for non-zero discount factors seems to be a mere singularity. If the alternating protocol is slightly perturbed, the selection probabilities of player $c$ do not drop out in equations (5.4) and (5.5) and only a single coalition distribution of player $c$ constitutes an equilibrium.

## 5.6 The Power of Veto Players

Given non-empty and finite set of players $N$, set of feasible coalitions $W$ and protocol $P$, denote by $\bar{\xi}$ the non-empty and compact set of all limit points of (ex-ante) SSPE payoffs in coalitional bargaining games $(N, W, P, \delta)$ as $\delta$ approaches 1. Very commonly, a vector $\xi = (\xi_i)_{i \in N}$ that lies in $\bar{\xi}$ is interpreted as *(bargaining) power* for the described situation. While this seems to be problematic whenever power is not unambiguous, one can still interpret it as power for a given limit of stationary equilibria. In the case of unique SSPE payoffs such as with independent proposers (Eraslan and McLennan, 2006), it follows anyway that $\bar{\xi}$ contains a single element only. We say that $\xi \in \bar{\xi}$ is *obtained* by a collection of coalition distributions $\lambda = (\lambda_i)_{i \in N}$ if there is a sequence of SSPE in games $(N, W, P, \delta)$ that have collections of coalition distributions converging to $\lambda$ and payoffs converging to $\xi$ as $\delta$ approaches 1.

A *veto player* $i \in N$ is a member of all feasible coalitions, $i \in S$ for all $S \in W$, and is thus able to block all proposals. Denote $V$ as the set of veto players. For the case of independent proposers and some veto player having positive recognition probability, Nohn (2010) shows that veto players hold all power. The following proposition generalizes this result to coalitional bargaining with non-independent proposers.[14]

**Proposition 5.7** *Let $N$ be a non-empty and finite set of players, $W$ a set of feasible coalitions, and $P$ a protocol. Assume the set of veto players $V$ is non-empty and let there be a veto player $i \in V$ such that $p_i^{jT} > 0$ for all $(j, T) \in M$. It then holds for all $\xi \in \bar{\xi}$ that $\sum_{i \in V} \xi_i = 1$ and, in particular, $\xi_i = 0$ for all $i \notin V$.*

**Proof** Consider $\xi \in \bar{\xi}$ obtained by $\lambda = (\lambda_i)_{i \in N}$ and denote by $\xi^{jT} = (\xi_i^{jT})_{i \in N}$ the corresponding limit of conditional payoffs after match $(j, T) \in M$. Let $i^*$ be a veto player such that $p_{i^*}^{jT} > 0$ for all $(j, T) \in M$. We prove the proposition in seven steps.

[14] If one does not assume a common set of feasible coalitions but possibly unrelated individual sets of feasible coalitions as mentioned in footnote 4, the proposition requires the following condition. The set of veto players $V$ is non-empty and there is a veto player $i \in V$ such that (i) $p_i^{jT} > 0$ for all $(j, T) \in M$ and (ii) for all $j \notin V$ there is $S \in W_i$ for which $S \not\ni j$.

(i) With $\xi^{jT}$, $(j,T) \in M$, and $\lambda = (\lambda_i)_{i \in N}$ arising as limits of conditional payoffs and coalition distributions of SSPE, proposition 5.3 yields for all matches $(j,T) \in M$ and all $i \in N$ that

$$\xi_i^{jT} = p_i^{jT} \sum_{S \in W_i} \lambda_{iS}(1 - \sum_{k \in S} \xi_k^{iS}) + \sum_{k \in N} p_k^{jT} \sum_{S \in W_k : S \ni i} \lambda_{kS} \xi_i^{kS}. \tag{5.10}$$

Also, for all $i \in N$ it holds that $\lambda_i$ maximizes $\sum_{S \in W_i} \lambda_{iS}(1 - \sum_{k \in S \setminus \{i\}} \xi_k^{iS})$ given constraints $\lambda_i \geq 0$ and $\sum_{S \in W_i} \lambda_{iS} = 1$.

(ii) A match $(i,S) \in M$ is said to be *directly accessible* from match $(j,T) \in M$ if $p_i^{jT} \lambda_{iS} > 0$. Due to $p_{i^*}^{jT} > 0$ for all $(j,T) \in M$, all matches $(i^*,S)$ where $S \in W$ and $\lambda_{i^*S} > 0$ are directly accessible from any other match. A match $(i,S) \in M$ is said to be *accessible* from match $(j,T) \in M$ if there is a chain of matches leading from $(i,S)$ to $(j,T)$ in which any match is directly accessible from its predecessor. Let $M^*$ be the set of all matches that are accessible from matches $(i^*,S)$, $S \in W$. All matches in $M^*$ are accessible from each other.

(iii) Consider $i \in V$ and choose $(j,T) \in M^*$ such that $\xi_i^{jT}$ is minimal among all $\xi_i^{kS}$, $(k,S) \in M^*$. Due to $\sum_{k \in N} p_k^{jT} \sum_{S \in W_k} \lambda_{kS} = 1$, equation (5.10) for $(j,T)$ and $i$ requires $\xi_i^{kS} = \xi_i^{jT}$ for all $(k,S) \in M^*$ that are directly accessible from $(j,T)$. Iteration yields $\xi_i^{kS} = \xi_i^{jT}$ for all $(k,S) \in M^*$.

(iv) Consider $i \in V$ and choose $(j,T) \in M$ such that $\xi_i^{jT}$ is minimal among all $\xi_i^{kS}$, $(k,S) \in M$. Assume $\xi_i^{jT} < \xi_i^{kS}$ for $(k,S) \in M^*$. Note that $p_{i^*}^{jT} \sum_{S \in W} \lambda_{i^*S} > 0$ and $(i^*,S) \in M^*$ for all $S \in W$ with $\lambda_{i*S} > 0$. Thus, we have $\xi_i^{jT} < \sum_{k \in N} p_k^{jT} \sum_{S \in W_k} \lambda_{kS} \xi_i^{kS}$, a contradiction to equation (5.10) for $(j,T)$ and $i$. Thus, it needs to be $\xi_i^{jT} \geq \xi_i^{kS}$ for $(k,S) \in M^*$.

(v) For $i \in V$ for which there is $(j,T) \in M^*$ with $p_i^{jT} > 0$, (iii) and equation (5.10) require $\sum_{S \in W_i} \lambda_{iS}(1 - \sum_{k \in S} \xi_k^{iS}) = 0$. Given optimality of $\lambda_i$, this is equivalent to $\sum_{k \in S} \xi_k^{iS} = 1$ for all $S \in W$.

(vi) Let $i \notin V$. By (v), $\xi_i^{i^*S} = 0$ for all $S \in W$ for which $i \notin S$. Note there is at least one such coalition $S \in W$. Equation (5.10) for $(i^*,S)$ and $i$ yields $0 \geq \sum_{k \in N} p_k^{i^*S} \sum_{S \in W_k, S \ni i} \lambda_{kS} \xi_i^{kS}$. This requires $\xi_i^{jT} = 0$ for all $(j,T) \in M^*$ which are directly accessible from $(i^*, S)$ and for which $T \ni i$. In particular, $\xi_i^{i^*T} = 0$ for all $T \in W_{i^*}$ where $(i^*, T) \in M^*$ and $T \ni i$.

(vii) From (vi), $\xi_i^{i^*S} = 0$ for all $i \notin V$, $S \in W$ where $(i^*, S) \in M^*$. Thus, $\sum_{i \in V} \xi_i^{i^*S} = 1$. Due to (iii), $\sum_{i \in V} \xi_i^{jT} = 1$ for all $(j, T) \in M^*$. By (iv), $\sum_{i \in V} \xi_i^{jT} = 1$ for all $(j, T) \in M$. Thus, for the ex-ante power of veto players, $\sum_{i \in V} \xi_i = 1$, and $\xi_i = 0$ for all $i \notin V$. □

Proposition 5.7 states that veto players hold all power if there is a veto player who always has positive conditional recognition probability. Infinitely patient and able to block all unsuitable proposals, veto players can play off the remaining players against each other such that the latter can not expect any positive share from bargaining. To do so, however, veto players need the possibility of making offers themselves – proposition 5.5 yields that if $p_i = 0$ and $p_i^{jT} = 0$ for all $(j, T) \in M$ for some $i \in N$, then $\xi_i = 0$ for all $\xi \in \bar{\xi}$. Also note that, being infinitely patient and always certainly involved in bargaining, initial recognition probability of a veto player does not matter for the proposition to hold.

Unlike in the mentioned case of independent proposers, however, we are not able to specify how exactly power is distributed among veto players. For the case of independent proposers and some veto player having positive recognition probability, Nohn (2010) shows that, as players grow infinitely patient, bargaining essentially turns into unanimity bargaining between the veto players only who then share all power proportional to their recognition probabilities. Britz et al. (2010) investigate unanimity bargaining with non-transferable utility where an irreducible and aperiodic Markov chain defines the proposers. The stationary distribution of this process then materializes as the weights of the asymmetric Nash solution to which payoffs converge as players grow infinitely patient.[15,16] For our case of transferable utility, this means that payoffs in unanimity bargaining are given by the stationary distribution of proposers. Thus, consider the following. Given protocol $P$ and a collection of coalition distributions $\lambda = (\lambda_i)_{i \in N}$, the

[15] The asymmetric Nash solution is introduced in Kalai (1977).

[16] Britz et al. (2010) actually consider a risk of breakdown instead of discount factors. Technically, the two approaches are equivalent, and a vanishing risk of breakdown corresponds with players growing infinitely patient.

probability that $j \in N$ will be proposer in the next round if $i \in N$ is the proposer in this round amounts to $p_j^i = \sum_{S \in W_i} \lambda_{iS} p_j^{iS}$. These transition probabilities, $(p_j^i)_{i,j \in N}$, constitute a Markov chain on $N$ which has, under the conditions of proposition 5.7, a *stationary protocol* $\pi = (\pi_i)_{i \in N}$ uniquely defined by $\pi_i = \sum_j \pi_j p_i^j$ for all $i \in N$ and $\pi \geq 0$.[17,18] In the long run, and given that all offers are rejected, all players $i \in N$ are proposer in a relative share of $\pi_i$ of all rounds.

**Conjecture 5.8** *Let $N$ be a non-empty and finite set of players, $W$ a set of feasible coalitions, and $P$ a protocol. Assume the conditions of proposition 5.7 to hold and let $\xi \in \bar{\xi}$ be a power distribution obtained by $\lambda = (\lambda_i)_{i \in N}$ with stationary protocol $\pi = (\pi_i)_{i \in N}$. It then holds $\xi_i = \pi_i / \sum_{j \in V} \pi_j$ for all $i \in V$, and, in particular, $\xi_i = 0$ for all $i \notin V$.*

Note that veto players do not in general hold all power if for any match there is some, while possibly different, veto player that has positive recognition probability in the next round.[19] Although it suggests itself as a natural analogue to the statements in Nohn (2010) and Britz et al. (2010), it is neither sufficient for proposition 5.7 or conjecture 5.8 to hold that veto players have the possibility of recurrently making offers in the long run: if the proposer process is aperiodic and irreducible and some veto player has positive stationary recognition probability, it may still be that veto players do not hold

[17] For general information on aperiodic and irreducible Markov chains and stationary distributions, see Karlin and Taylor (1975).

[18] Note that, under the conditions of proposition 5.7, this Markov chain possibly is aperiodic and irreducible only on a subset of players $N^* \subseteq N$. For this, it holds $\pi_i = 0$ for all $i \notin N^*$.

[19] Let $N = \{c_1, c_2, s_1, s_2\}$ with feasible coalitions $S_i = \{c_1, c_2, s_i\}$, $i = 1, 2$, and a protocol as follows. Let initial recognition be equal, $p_{c_i} = p_{s_i} = 1/4$, $i = 1, 2$. Given previous coalition $S_i$, $i = 1, 2$, let $c_i$ and $s_i$ have equal probability 1/2 to be the proposer in the next round. Then, selection probabilities $\lambda_{c_i S_i} = 1$, $i = 1, 2$, constitute an SSPE. Once a coalition $S_i$, $i = 1, 2$, has arisen in the first round, the game virtually becomes two-player unanimity bargaining between $c_i$ and $s_i$. Selecting the other feasible coalition is not an option for either veto player since the protocol then denies him the possibility of making any further offer in the future. Thus, the protocol prevents the effective use of the strategic possibilities that come with both veto players' veto. As players grow infinitely patient, power amounts to $\xi_{c_i} = \xi_{s_i} = 1/4$, $i = 1, 2$.

all power.[20,21]

Note, however, that the requirement that one veto player always has positive recognition probability is a sufficient but at the same time not a necessary condition. Nor is it necessary that there is always some, while possibly different, veto player who can be proposer. Unlike in the star with alternating protocol, where the veto player only holds half of the power (example 5.6), a single veto player can hold all power also with the alternating protocol. If he needs the support of all but one of at least 3 minor players to create the unit surplus, he is assigned all power by the alternating protocol.[22] That the veto player nevertheless holds all power in this example but not in the star – a situation in which he needs less support by the minor players – is a technical peculiarity of the alternating protocol for which we cannot, unfortunately, provide an intuitive explanation. Equally counterintuitive, that the alternating protocol does not assign all power to veto players neither translates to when the single veto player in the star is replaced by two or more veto players.[23] In this light, it seems the qualitative results of Calvó-Armengol (2001a,b) do not, at least not in general, extend beyond bilateral bargaining.

[20] Assuming aperiodicity also suggests itself because there is a period of 2 in the star with alternating protocol (example 5.6) where the single veto player does not hold all power. Marginal perturbations of the protocol simultaneously ensure both aperiodicity and that the veto player holds all power.

[21] Consider $N = \{c, s_{11}, s_{12}, s_{21}, s_{22}\}$ with feasible coalitions $S_i = \{c, s_{i1}, s_{i2}\}$, $i = 1, 2$, and alternating protocol. Let $c$ propose to $S_1$ and $S_2$ with equal probability. The corresponding chain of proposers is aperiodic and irreducible and has a stationary protocol $\pi$ with $\pi_c = 1/3$ and $\pi_{s_{ij}} = 1/6$ for $i, j = 1, 2$. So $\pi_c > 0$, but power coincides with the stationary protocol, $\xi_c = 1/3$ and $\xi_i = 1/6$ for $i, j = 1, 2$.

[22] Let $N = \{c, s_1, \ldots, s_n\}$, $n \geq 3$ with feasible coalitions $S_i = N\backslash\{s_i\}$, $i = 1, \ldots, n$ and alternating protocol. Consider symmetric selection behavior of the central player, $\lambda_{cS_i} = 1/n$ for all $i = 1, \ldots, n$, and as well of all minor players, $\lambda_{s_i S_j} = 1/(n-1)$ for $i, j = 1, \ldots, n$ where $i, j = 1, \ldots, n$ and $i \neq j$. Then $c$ holds all power, $\xi_c = 1$ and $\xi_{s_i} = 0$ for all $i = 1, \ldots, n$.

[23] Let $N = \{c_1, \ldots, c_m, s_1, \ldots, s_n\}$, $m \geq 1$ and $n \geq 2$, with feasible coalitions $S_i = \{c_1, \ldots, c_m, s_i\}$, $i = 1, \ldots, n$, and alternating protocol. Assume symmetric selection behavior of the central players, $\lambda_{c_i S_j} = 1/n$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$. If $m = 1$, then $\xi_c = 1/2$ and $\xi_{s_i} = 1/(2n)$ for all $i = 1, \ldots, n$. If $m \geq 2$, then $\xi_{c_i} = 1/m$ for $i = 1, \ldots, m$ and $\xi_{s_i} = 0$ for $i = 1, \ldots, n$.

## 5.7 Conclusion

In the Baron-Ferejohn model, protocols not assigning all power to veto players are a mere singularity also with non-independent proposers – arbitrarily small perturbations of any protocol ensure that veto players hold all power. This at last rejects the initial hope of finding support for power indices such as the Shapley-Shubik index somewhere 'between' the independent and alternating protocol. Any non-trivial convex combination of these two protocols assigns all power to veto players. Thus, the possibility of non-cooperative support for power indices that do not assign all power to veto players is further limited and restricts at most to singular cases such as the alternating protocol. However, even this does not always not assign all power to veto players, because of which such non-cooperative foundations seem even less likely. Instead, our results indicate a great stability of the support that has been found for the core (Yan, 2002) and related single-valued solution concepts such as the nucleolus (Montero, 2006).

Foundations of indices not assigning all power to veto players could, at least possibly, be found if the model was further extended. For instance, the probabilities for the next proposer might plausibly depend on the identities of those respondents that reject an offer, resulting for instance in an alternating protocol where only rejecting respondents have the chance of being proposer in the next round. In addition, the particular offer made by a proposer might play a role in his probability to make a new offer, for instance could this probability be lower the further his offer is away from an acceptable one. These generalizations, and also the questions of when SSPE payoffs are unique in this framework and whether or not conjecture 5.8 does hold, are left for future research.

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