Infinite graphs with a treelike structure

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CHAPTER 1

Introduction

We investigate several classes of infinite graphs that turn out to have a tree-like structure, each in their own way. Although this thesis consists of three parts, there is a connection from each part to its succeeding one. Before it comes to the three parts, we recall in Chapter 2 several definitions and aspects of infinite graphs, hyperbolic graphs, the structure tree theory, and dimension concepts of topological and metric spaces.

Part 1 deals with hyperbolic graphs. After Chapter 3, in which we discuss several known connections between hyperbolic graphs (together with their hyperbolic boundary) and trees (together with their boundary) and which does not contain new results, we show in Chapter 4 that every locally finite hyperbolic graph G whose hyperbolic boundary has finite Assouad dimension has a rooted spanning tree T with several properties:

- Every ray starting at the root is quasi-geodetic for some global constants.
- The main part of the hyperbolic graphs fits snugly around the described rays in the sense that for some constant d every geodetic ray of the graph lies eventually d-close to the tree.
- The identity map from T to G extends to a continuous map from $T \cup \partial T$, the tree with its boundary, to $G \cup \partial G$, the graph with its hyperbolic boundary, such that every element of ∂G has at least one but only boundedly many preimages under this extension.

This result extends a result of Gromov [**46**] which says that from every hyperbolic graph with bounded degrees one can construct a tree outside the graph with a continuous surjection from the ends of the tree onto the hyperbolic boundary such that the surjection is finite-to-one.

We remark that an $\mathbb{R}$-tree as in the result of Chapter 4 also exists in the more general situation of proper hyperbolic geodetic spaces, see [**52**]. But to keep this thesis graph theoretical, we write the chapter in the notations of graphs.

We continue in Part 2 with the investigation of graphs whose automorphism groups act with certain additional properties on the boundary of the graphs. The first chapter of Part 2 (Chapter 5) introduces to the whole topic and particularly to a property, called the fixed set property, that reveals a relation between the group action on the graph and that on the boundary of the graph. In that chapter we summarize some known results for the better understanding of the new results in Chapter 6, 7, and 8.

In Chapter 6, we prove that there is no locally finite planar hyperbolic graph with precisely one end such that a subgroup of its automorphism group acts transitively on the graph and fixes a hyperbolic boundary point. This gives a partial answer to a question of Kaimanovich and Woess [**61**].

We continue in Chapter 7 with characterizing connected graphs, not necessarily locally finite, that have infinitely many ends and whose automorphism group has a subgroup that acts transitively on the graph itself but fixes an end of a certain kind, called a non-local end. These graphs are quasi-isometric to semi-regular trees as in the case of locally finite graphs, see [**80**]. This answers a problem of Woess [**101**].

If we assume that the automorphism group of a connected graph with infinitely many non-local ends acts transitively on these ends, we obtain in Chapter 8— also as an answer to a problem of Woess [**101**]—that there is a metrically almost transitive subgraph that is quasi-isometric to a semi-regular tree and whose deletion leaves a rayless graph.

In Part 3, we look at graphs that satisfy various kinds of symmetry conditions. These conditions lie somewhere between transitivity of the automorphism groups on the vertices and homogeneity of the graphs. In the first chapter of Part 3 (Chapter 9) we shall discuss known results for some symmetry conditions: We start with homogeneous graphs and then weaken the assumptions to obtain other natural symmetry conditions. We also discuss the corresponding known results in the case of directed graphs and how the new results of Chapter 10, 11, and 12 correspond to the known results.

The main result of Chapter 10 is the classification of the connected distance-transitive graphs with more than one end. This is a generalization of the results in [**76, 81**] to graphs with arbitrary degree.

In Chapter 11 we look at connected k-CS-transitive graphs and classify, for $k \geq 3$, those that have at least two ends. Thereby, we do not only generalize a result from Gray [**42**] from locally finite graphs to graphs with arbitrary degree but also from the case $k = 3$ to arbitrary $k \geq 3$.

Finally, in Chapter 12, we study connected-homogeneous digraphs. The investigation of these digraphs was started by Gray and Möller [**45**] where they classified the two-ended connected such digraphs and gave a list of examples with infinitely many ends. We classify the connected such digraphs with more than one end and arbitrary degree, and also the locally finite connected such digraphs with precisely one end and the finite connected ones. This, then, completes the classification of the connected-homogeneous digraphs, either finite or infinite, whose vertices have finite degree. This is the only piece of this thesis in which we study finite (di-)graphs.

Let us say a word about the techniques we use for the proofs in several chapters whose corresponding theorems for locally-finite graphs are all based on Dunwoody's *structure trees* corresponding to finite edge-cuts that are invariant under the action of the automorphism group of the graph. This structure tree theory is described in the book of Dicks and Dunwoody [**25**], see also [**31**]. We refer to [**83, 82, 94**] for introductions to this topic. Since those edge-cuts must be finite, these structure trees can in general only be applied to locally finite graphs.

Recently, Dunwoody and Krön [**32**] developed a similar structure tree theory based on vertex cuts, providing a similarly powerful tool for the investigation of graphs that are not necessarily locally finite. We use this new theory in several of our proofs.

CHAPTER 2

Definitions and preliminary results

In this chapter we give the main definitions that we use throughout the thesis. Furthermore, we summarize some known results. In the last section of this chapter, we define some (di-)graphs that play a role in the various classification results of Part 3. Throughout this thesis we use the graph theoretic terms and notations from [**26**] if not stated otherwise.

2.1. Ends of graphs

We introduce some notations for infinite graphs. A *ray* is a one-way infinite path. Two rays in a graph G are *equivalent* if there is no finite vertex set S in G such that the two rays lie eventually in distinct components of $G - S$. The equivalence of rays is an equivalence relation whose classes are the *(vertex) ends* of G. If we just talk of ends of graphs we always think of vertex ends. All other end types—which we shall define in a moment—will be stated concretely.

By replacing the finite vertex set S in the definition of ends by a finite edge set one obtains *edge ends*. Obviously for every graph there is a canonical map from its ends to its edge ends which is surjective but in general not injective.

A *metric ray* is a ray such that no infinite subset of its vertices has finite diameter. An end is *global* if every ray in that end is a metric ray. If conversely there is no metric ray in an end this is a *local* end. If an end is not local, then it is a *non-local* end. So an end is non-local if it contains a metric ray.

Two metric rays are *metrically equivalent* if for every vertex set S of finite diameter both rays lie eventually in the same component of $G-S$. The equivalence classes of metrically equivalent metric rays are the *metric ends* of a graph. Just as the ends are a refinement of edge ends, the metric ends are a refinement of non-local ends. A group acts *metrically almost transitively* on a graph G if there is an $r \in \mathbb{N}$ such that for every $x \in VG$ there is $G(x, r) = G$. See [**62, 66, 67**] for more details on metric ends and metrically almost transitive graphs.

A vertex $x \in VG$ *dominates* an end ω if there is a ray R in ω and an infinite set of (except for x) pairwise disjoint x-R-paths and an end ω is *thin* if there is an $n \in \mathbb{N}$ such that there are at most n disjoint rays in ω.

The proof of the following lemma due to König can be found for example in [**26**, Lemma 8.1.2].

LEMMA 2.1. *Let $V_0, V_1, \ldots$ be an infinite sequence of disjoint non-empty finite sets, and let G be a graph with $VG = \bigcup_{i \in \mathbb{N}} V_i$. Assume that every vertex v in a set V_n with $n \geq 1$ has a neighbor $f(v)$ in V_{n-1}. Then G contains a ray $v_0 v_1 \ldots$ with $v_n \in V_n$ for all n.* $\square$

2.2. Group action on graphs

A group Γ *acts* on a graph G if Γ acts on the set VG and if for every $xy \in EG$ and for every $\alpha \in \Gamma$ the image $x^\alpha y^\alpha$ of xy is an edge in G again. Remark that we denote with x^α the image of x under α. If Γ acts transitively on VG the G is a *transitive* graph.

Let Γ be a group acting on a digraph D and let $U \subseteq VD$. We denote with Γ_U the *(pointwise) stabilizer* of U, that is the subgroup of Γ that fixes each element of U. The same notion holds for an edge $e \in ED$ or a single vertex $x \in VD$. If Γ fixes the set U setwise, then we denote with Γ^U all the automorphisms of U that are obtained by restricting elements of Γ on U.

For the following well-known proposition see for example [**99**] or [**69**, 3.1.2].

PROPOSITION 2.2. *Every subgroup of S_n with $n \in \mathbb{N}$ is equal to A_n or has an index of at least n.* $\square$

For infinite graphs, we have the following theorem which is well-known in the locally finite case.

THEOREM 2.3. [**27**, Corollary 4] *Connected infinite transitive graphs have either* 1, 2*, or infinitely many ends.* $\square$

But the case of precisely two ends only happens in locally finite graphs as the next theorem shows.

THEOREM 2.4. [**27**, Theorem 7] *Every connected transitive graph with precisely two ends is locally finite.* $\square$

We call a graph *end-transitive* if its automorphism group acts transitively on its ends.

An automorphism α of a graph G is a *translation* if there is no finite vertex set fixed by α. A Γ-*congruence* π is a Γ-invariant equivalence relation. Its *congruence classes* are the equivalence classes of that relation.

2.3. Planar graphs

Planar graphs are graphs that have an embedding of the graph as a 1-complex into the Euclidean plane. Such embeddings are also called *planar* embeddings. Whitney [**98**] proved the following result for finite graphs. Later it was extended by Imrich [**57**] and Thomassen [**93**] to infinite graphs.

THEOREM 2.5. *Planar* 3*-connected graphs have a unique plane embedding.* $\square$

The next result is due to Babai and Watkins [**7**], see also [**6**, Lemma 2.4].

Lemma 2.6. [**7**, Theorem 1] *Let G be a locally finite connected transitive graph that has precisely one end. Let d be the degree of any of its vertices. Then the graph has connectivity at least* $3(d+1)/4$. □

For a planar graph G and a vertex $x \in VG$, we consider the embedding of the edges incident with x and define the *spin* of x to be the cyclic order of the set $\{xy \mid y \in N(x)\}$ where xz is a successor of xy whenever the edge xz is the next one after the edge xy in the clockwise order around x.

We deduce from Lemma 2.6 and Theorem 2.5 that every locally finite transitive planar graph with precisely one end has a unique embedding in the Euclidean plane. So we know that every vertex of a locally finite transitive planar graph with precisely one end has either a fixed spin or its reversing spin.

An automorphism α of a graph G that is uniquely embeddable into the Euclidean plane is called *spin-preserving* if for a vertex $x \in VG$ the spin of x^α is the same as the image of the spin of x under α. If the spins are not the same, then the spin of x^α must be the reverse of the image of the spin of x under α because of the unique embedding of G. In this latter case, the automorphism is *spin-reversing*.

2.4. Digraphs

A digraph $D = (VD, ED)$ consists of a non-empty set VD, its set of *vertices*, and an asymmetric (i.e. irreflexive and anti-symmetric) binary relation ED over VD, its set of *edges*.

We write xy for an edge $(x, y) \in ED$ and say that xy is directed *from x to y*. For $x \in VD$ we define its *out-neighborhood* as $N^+(x) := \{y \in VD \mid xy \in ED\}$, its *in-neighborhood* as $N^-(x) := \{z \in VD \mid zx \in ED\}$ and finally its *neighborhood* as $N(x) := N^+(x) \cup N^-(x)$. Two vertices are called *adjacent* if one is in the other's neighborhood. For a vertex set $X \subseteq VD$ the neighborhood of X is defined as $N(X) := \left(\bigcup_{x\in X} N(x)\right) \setminus X$ and $N^+(X), N^-(X)$ are defined analogously. For all $x \in VD$ we denote with $d^+(x), d^-(x)$ the cardinality of $N^+(x), N^-(x)$, respectively. The *degree* $d(x)$ is the cardinality of $N(x)$. So we have $d(x) = d^+(x) + d^-(x)$.

A *(k)-arc* is a directed path (of length k). An *ancestor* (*descendant*) of a vertex x is any vertex y for which there exists an arc from y to x (from x to y). The *descendant-digraph* (*ancestor-digraphs*) of x is the subdigraph $\mathrm{desc}(x) \subseteq D$ (the subdigraph $\mathrm{anc}(x) \subseteq D$) that is induced by the set of all its descendants (its ancestors, respectively).

If $x_0x_1 \dots x_n$ is a sequence of vertices such that any two subsequent vertices are adjacent then it is called a *walk* and a walk of pairwise distinct vertices is called a *path*. A path that is also an arc is called a *directed path*. A digraph is called *connected* if any two vertices are joined by a path.

A walk $x_0x_1 \dots x_n$ such that $x_i \in N^+(x_{i+1}) \Leftrightarrow x_{i+1} \in N^-(x_{i+2})$ is called *alternating*. If $e = xy$ and $e' = x'y'$ are contained in a common alternating walk then they are called *reachable* from each other. This clearly defines an equivalence relation, the *reachability relation*, on ED which we denote by $\mathcal{A}$, and for $e \in ED$

we refer to the equivalence class that contains e by $\mathcal{A}(e)$. We call the subdigraph $D[\mathcal{A}(e)]$ induced by $\mathcal{A}(e)$ the *reachability digraph* of D that contains e.

The reachability digraph of an edge e is a *bipartite reachability digraph* if it is bipartite, if one class of this bipartition has empty in-neighborhood in $D[\mathcal{A}(e)]$ and if the other class has empty out-neighborhood.

Let $\sim$ be an equivalence relation on a digraph D. With $D_\sim$ we denote the digraph whose vertex set is the set of equivalence classes and with edges XY whenever there are representatives $x \in X$, $y \in Y$ such that $xy \in ED$. This is not a digraph in our restrictive meaning, because it may have loops or for an edge xy there might also exist the edge yx. However, we just consider such equivalence relations that makes $D_\sim$ to a digraph, that means its adjacency relation is irreflexive and anti-symmetric.

The *ends* of a digraph are the ends of the underlying undirected graph.

2.4.1. Group action on digraphs. A group acts on a digraph if it acts on the underlying undirected graph and, furthermore, respects the directions of the edges, that is, if xy is an edge and γ a group element, then $x^\gamma y^\gamma$ is also an edge.

In an *edge-transitive* digraph, that is a digraph such that its automorphism group acts transitively on its edges, all reachability digraphs $\Delta_e := D[\mathcal{A}(e)]$ with $e \in ED$ are isomorphic, so we may denote a representative of their isomorphism type by $\Delta(D)$. Furthermore Cameron, Praeger and Wormald proved the following proposition on the reachability relation in edge-transitive digraphs.

PROPOSITION 2.7. [**19**, Proposition 1.1] *Let D be a connected edge-transitive digraph. Then $\Delta(D)$ is edge-transitive and connected. Further, either*

(a) *$\mathcal{A}$ is the universal relation on ED and $\Delta(D) = D$, or*
(b) *$\Delta(D)$ is a bipartite reachability digraph.* □

2.5. Structure trees

In this section we introduce the terms of cuts and structure trees that were developed in [**32**] and have their applications, apart from this thesis, in [**64**]. Compared with Dunwoody and Krön [**32**] we use a different notation for the cut systems in order to indicate the relation of cut systems with the well-known graph theoretic concept of separations, see [**26**].

Let G be a connected graph and let $A, B \subseteq V(G)$ be two vertex sets. The pair (A, B) is a *separation* of G if $A \cup B = V(G)$ and $E(G[A]) \cup E(G[B]) = E(G)$. The *order* of a separation (A, B) is the cardinality of its *separator* $A \cap B$ and the subgraphs $G[A \setminus B]$ and $G[B \setminus A]$ are the *wings* of (A, B). With $(A, \sim)$ we refer to the separation $(A, (V(G) \setminus A) \cup N(V(G) \setminus A))$. A *cut* is a separation (A, B) of finite order with non-empty wings such that the wing $G[A \setminus B]$ is connected and such that no proper subset of $A \cap B$ separates the wings of (A, B). A *cut system* of G is a non-empty set $\mathcal{S}$ of separations (A, B) of G satisfying the following three properties.

1. If $(A, B) \in \mathcal{S}$ then there is an $(X, Y) \in \mathcal{S}$ with $X \subseteq B$.
2. Let $(A, B) \in \mathcal{S}$ and C be a component of $G[B \setminus A]$. If there is a separation $(X, Y) \in \mathcal{S}$ with $X \setminus Y \subseteq C$, then the separation $(C \cup N(C), \sim)$ is also in $\mathcal{S}$.
3. If $(A, B) \in \mathcal{S}$ with wings X, Y and $(A', B') \in \mathcal{S}$ with wings X', Y' then there are components C in $X \cap X'$ and D in $Y \cap Y'$ or components C in $Y \cap X'$ and D in $X \cap Y'$ such that both C and D are wings of separations in $\mathcal{S}$.

Two separations $(A_0, A_1), (B_0, B_1) \in \mathcal{S}$ are *nested* if there are $i, j \in \{0, 1\}$ such that one wing of $(A_i \cap B_j, \sim)$ does not contain any connected component C with $(C \cup N(C), \sim) \in \mathcal{S}$ and $A_{1-i} \cap B_{1-j}$ contains $(A_0 \cap A_1) \cup (B_0 \cap B_1)$. A cut system is *nested* if each two of its cuts are nested.

Remark 2.8. *The following two assertions hold.*

1. *If, for two $\mathcal{C}$-cuts $(A_0, A_1), (B_0, B_1)$, the separator $A_0 \cap A_1$ contains vertices of both wings of (B_0, B_1), then the two cuts are not nested.*
2. *In any transitive graph G with an $\mathrm{Aut}(G)$-invariant cut system $\mathcal{C}$, any two nested cuts (A_0, A_1) and (B_0, B_1) with $(A_0 \cap A_1) \cup (B_0 \cap B_1) \subseteq A_{1-i} \cap B_{1-j}$ have the property that $A_i \cap B_j$ is empty by* [**32**, Lemma 3.5].

A cut in a cut system $\mathcal{S}$ is *minimal* if its order in $\mathcal{S}$ is minimal. A *minimal cut system* is a cut system all whose cuts are minimal and thus have the same order.

Let us describe two minimal cut systems one of which was introduced by Dunwoody and Krön [**32**, Example 2.2]. Both will be used in our proofs.

Example 2.9. Let G be a connected infinite graph with at least two ends (two non-local ends). Let n be the smallest cardinality of a finite vertex set X such that there are at least two components in $G - X$ that contain a ray (a metric ray) each. Let $\mathcal{S}$ be the set of all cuts (A, B) with order n such that both $G[A]$ and $G[B]$ contain a ray (a metric ray). Then $\mathcal{S}$ is a minimal cut system.

An $\mathcal{S}$-*separator* is a vertex set S that is a separator of some separation in $\mathcal{S}$. Let $\mathcal{W}$ be the set of $\mathcal{S}$-separators. An $\mathcal{S}$-*block* is a maximal induced subgraph X of G such that

(i) for every $(A, B) \in \mathcal{S}$ there is $V(X) \subseteq A$ or $V(X) \subseteq B$ but not both;
(ii) there is some $(A, B) \in \mathcal{S}$ with $V(X) \subseteq A$ and $A \cap B \subseteq V(X)$.

Let $\mathcal{B}$ be the set of $\mathcal{S}$-blocks. For a nested minimal $\mathrm{Aut}(G)$-invariant cut system $\mathcal{S}$ let $\mathcal{T}$ be the graph with vertex set $\mathcal{W} \cup \mathcal{B}$. Two vertices X, Y of $\mathcal{T}$ are adjacent if and only if either $X \in \mathcal{W}$, $Y \in \mathcal{B}$, and $X \subseteq Y$ or $X \in \mathcal{B}$, $Y \in \mathcal{W}$, and $Y \subseteq X$. Then $\mathcal{T} = \mathcal{T}(\mathcal{S})$ is called the *structure tree* of G and $\mathcal{S}$.

Lemma 2.10. [**32**, Lemma 6.2] *Let G be a connected graph, and let $\mathcal{S}$ be a nested minimal cut system. Then the structure tree of G and $\mathcal{S}$ is a tree.* $\square$

An $\mathcal{S}$-*slice* is the induced subgraph $G[Z]$ of a component Z of $G - (A \cap B)$ with $(A, B) \in \mathcal{S}$ such that $(Z \cup (A \cap B), \sim) \notin \mathcal{S}$.

A separation $(A, B) \in \mathcal{S}$ *separates* two vertex sets, two $\mathcal{S}$-blocks, two ends of G, or an $\mathcal{S}$-block and an end of G *properly* if the blocks intersect non-trivially with distinct wings of (A, B), if each two rays of distinct ends lie eventually in distinct wings of (A, B), or if each ray eventually lies in that (A, B)-wing that intersects with the block trivially. If one separator S separates two vertices of another separator S' the separators S and S' *cross*. It is a consequence of Lemma 3.3 and Theorem 3.5 of [**32**] that two minimal separations are nested if and only if their corresponding separators do not cross.

A cut system $\mathcal{S}$ of a connected graph G is *basic* if $\mathcal{S}$ is minimal, nested, $\mathrm{Aut}(G)$-invariant, if $\mathcal{S}$ is a subsystem of the minimal cut system given in Example 2.9 and if all separators $A \cap B$ with $(A, B) \in \mathcal{S}$ belong to the same $\mathrm{Aut}(G)$-orbit.

We state here that part of Theorem 7.2 of [**32**] that we shall use here.

Theorem 2.11. *For every graph G with at least two ends (two non-local ends) there is a basic cut system $\mathcal{S}$ of G.* □

A ray R *corresponds* to a vertex X of $\mathcal{T}$ if X is a block and $R \cap X$ is infinite. A ray R *corresponds* to an end ω of $\mathcal{T}$ if for any ray P in ω and for every $\mathcal{S}$-separator S on P all but finitely many vertices of R lie in the same component of $G - S$ as that $\mathcal{S}$-block which is in $\mathcal{T}$ adjacent to S and which separates S from ω in G. Obviously a ray of G corresponds either to a vertex of $\mathcal{T}$ or to an end of $\mathcal{T}$. As all rays in the same end have to correspond to the same vertex or end of $\mathcal{T}$, we also say that the end *corresponds* to that end or vertex of $\mathcal{T}$.

For a cut (A, B) and a minimal cut system $\mathcal{S}$ let $m_{\mathcal{S}}(A, B)$ denote the number of distinct $\mathcal{S}$-separators S such that there is one $\mathcal{S}$-separation that is not nested with (A, B) and that has S as its separation. By [**32**, Theorem 3.5, Lemma 4.1] the value $m_{\mathcal{S}}(A, B)$ is finite.

Lemma 2.12. *Let G be a connected graph. Let $\mathcal{C}$ be a minimal cut system. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two basic subsystems of $\mathcal{C}$. Suppose that there are separations of $\mathcal{S}_1$ and $\mathcal{S}_2$ that are not nested. Then there is a basic subsystem $\mathcal{S}$ of $\mathcal{C}$ such that $\mathcal{S} \cup \mathcal{S}_2$ is a nested cut system and $m_{\mathcal{S}_1}(A, B) < m_{\mathcal{S}_1}(A', B')$ for all $(A, B) \in \mathcal{S}, (A', B') \in \mathcal{S}_2$.*

Proof. Let $(A_1, B_1) \in \mathcal{S}_1$ such that (A_1, B_1) is not nested with all $(A, B) \in \mathcal{S}_2$. We choose $(A_2, B_2) \in \mathcal{S}_2$ such that the intersection X of one wing of (A_2, B_2) with $A_1 \cap B_1$ is minimal but not empty and such that the $\mathcal{S}_2$-block containing X is in the structure tree $\mathcal{T}_2$ adjacent to $A_2 \cap B_2$. We may assume that $X \subseteq A_2$. Then there is a component C of $G - (A_1 \cap B_1) - (A_2 \cap B_2)$ such that $X \subseteq NC$ and $(C \cup NC, \sim)$ is a minimal cut. Let $\mathcal{S}$ be the set of all those cuts such that their separator is NC^α for any $\alpha \in \mathrm{Aut}(G)$. We just have to prove that $\mathcal{S}$ fulfills the claims of the lemma, so we have to prove that $\mathcal{S}$ is a nested cut system, that $\mathcal{S}_2 \cup \mathcal{S}$ is nested and that $m_{\mathcal{S}_1}(A, B) < m_{\mathcal{S}_1}(A', B')$ for all $(A, B) \in \mathcal{S}, (A', B') \in \mathcal{S}_2$.

By the minimal choice of X it follows that $\mathcal{S}_2 \cup \mathcal{S}$ is nested. So it remains to prove the inequality and that $\mathcal{S}$ is nested. Let us first prove the inequality. Since each $\mathcal{S}_1$-separation which is nested with (A_2, B_2) also has to be nested with

$(C \cup NC, \sim)$ by the minimal choice of X, the inequality holds with $\leq$ instead of $<$, namely $m_{\mathcal{S}_1}(C \cup NC, \sim) \leq m_{\mathcal{S}_1}(A_2, B_2)$. But on the other hand there is the $\mathcal{S}_1$-separation (A_1, B_1) that is nested with $(C \cup NC, \sim)$ but not nested with (A_2, B_2) and hence the inequality is strict. Let us finally show that $\mathcal{S}$ is nested. Let $S := A \cap B$ and let $\alpha \in \mathrm{Aut}(G)$ with S^α in the same component of $G - S$ in which X lies. By the choice of S and X we know that S^α does not cross $A_1 \cap B_1$. Thus there is a component D of $G - NC$ such that $S^\alpha \subseteq D \cup ND$. The separator $(A_1 \cap B_1)^\alpha$ crosses S^α and thus it has to lie in the same component of $G - (A \cap B)$ as S^α does. By a similar argument as before we know that S does not separate X^α from S^α and thus both S^α and X^α do not intersect with the component of $G - (C \cup NC)$ that intersects with X non-trivially. Thus there are two $\mathcal{S}$-separations with corresponding separators NC and NC^α that are nested and as mentioned before this implies by arguments of [**32**] that NC and NC^α do not cross. Thus $\mathcal{S}$ is a nested cut system. □

Lemma 2.13. [**32**, Lemma 4.1] *For any k, every pair of vertices in a connected graph is separated properly by only finitely many distinct separators of order k.* □

Lemma 2.14. *Let G be a graph and let $\mathcal{C}$ be a nested cut system of G such that no $\mathcal{C}$-separator contains any edge. For any path P that has both its end vertices in the same $\mathcal{C}$-separator S, there is a $\mathcal{C}$-block with maximal distance to S in $\mathcal{T}(\mathcal{C})$ that contains edges of P. This $\mathcal{C}$-block contains at least two edges of P.*

Proof. Any two vertices that are not in a common $\mathcal{C}$-block, are separated by some $\mathcal{C}$-separator. So we conclude that for each edge of G there is a unique $\mathcal{C}$-block that contains this edge, as it is not contained in any $\mathcal{C}$-separator. The path P has only finitely many edges, so there are just finitely many $\mathcal{C}$-blocks that contain edges of P and we may pick one, X say, with maximal distance to S in $\mathcal{T}(\mathcal{C})$. Let xy be an edge on P that lies in X. Then either x or y does not lie in that $\mathcal{C}$-separator S' that separates X from S and lies in X. We assume that this is y. Let z be the other neighbor of x on P. The edge yz cannot lie further away from S than X in the structure tree, but since $y \notin S'$, we have $yz \in EX$. So X contains two edges of P. □

In the context of a digraph D all concepts introduced in this section are related to the underlying undirected graph G of D except for one definition: We call a cut system $\mathcal{C}$ for a digraph D *basic* if it has the following properties.

(i) $\mathcal{C}$ is non-empty, minimal, nested and $\mathrm{Aut}(D)$-invariant.
(ii) $\mathrm{Aut}(D)$ acts transitively on $\mathcal{S}$.
(iii) For each $\mathcal{C}$-cut (A, B) both A and B contain an end of D and there is no separation of smaller order that has this property.

Then Theorem 2.11 does not only hold for any graph but also for any digraph by the results in [**32**]. We have to define the property of being basic differently, because we know in general only that we may consider $\mathrm{Aut}(D)$ as a subgroup of $\mathrm{Aut}(G)$, but we do not know whether it is a proper subgroup or not. Thus, our cut system could have more than one $\mathrm{Aut}(D)$-orbit of separators which would be more difficult to deal with.

2.6. Hyperbolic graphs

In this section we define hyperbolic graphs and various other related objects. For a more detailed introduction to hyperbolicity, we refer to [**1, 23, 40, 46, 87**] as well as [**12**, Chapter III.H] and [**103**, Chapter 22].

Let $G = (VG, EG)$ be a graph. A *geodesic* is a path between two vertices x and y with length $d(x, y)$ and denoted by $[x, y]$. A *triangle* is a set of three vertices (not necessarily distinct)—called *corners* of the triangle—together with paths between each two of these vertices. These paths are called *sides* of the triangle. The triangle is *geodetic* if all sides of the triangle are geodesics. We write $[x, y, z]$ for a geodetic triangle with corners x, y and z.

We are investigating G from a topological point of view, so that every edge of G can be understood as an isometric image of the real interval $[0, 1]$.

The graph G is called *δ-hyperbolic* for a $\delta \geq 0$ if for every geodetic triangle $[x, y, z]$ each of its sides lies in a δ-neighborhood of the other two sides and G is called *hyperbolic* if there exists a $\delta \geq 0$ such that G is δ-hyperbolic.

Let o be a vertex in G. The *Gromov-product* (with respect to its *base-point* o) for two vertices x and y is $(x, y)_o := \frac{1}{2}(d(x, o) + d(y, o) - d(x, y))$. If it is obvious by the context that we use o as the base-point for the product, we simply write (x, y). An easy proposition is due to Gromov.

Proposition 2.15. [**46**, 1.1B] *Let G be a graph and $o \in VG$. If*

$$(x, y)_o \geq \min\{(x, z)_o, (y, z)_o\} - \delta$$

for all $x, y, z \in VG$, then there is

$$(x, y)_w \geq \min\{(x, z)_w, (y, z)_w\} - 2\delta$$

for every $w \in VG$. □

Another definition of hyperbolicity uses the Gromov-product. So one might expect that this definition depends on the vertex o, but Proposition 2.15 has shown us that this is not the case. See for example [**1**, Proposition 2.1] for a proof of the following proposition.

Proposition 2.16. *A locally finite graph G is hyperbolic if and only if there is a vertex o and some $\delta \in \mathbb{R}_{\geq 0}$ with $(x, y)_o \geq \min\{(x, z)_o, (y, z)_o\} - \delta$ for all $x, y, z \in VG$.* □

Since we are looking at the hyperbolic boundary from distinct viewpoints, we state here three different definitions of the hyperbolic boundary all of which are equivalent. A *geodetic ray* is a ray $\pi = x_0 x_1 \ldots$ with $d(x_i, x_j) = |i - j|$ for all $i, j \geq 0$, and a double ray $\ldots x_{-1} x_0 x_1 \ldots$ is a *geodetic double ray* if $d(x_i, x_j) = |i - j|$

for all $i, j \in \mathbb{Z}$. Two geodetic rays π_1, π_2 are *equivalent* if for any sequence $(x_n)_{n\in\mathbb{N}}$ of vertices on π_1 we have $\liminf_{n\to\infty} d(x_n, \pi_2) \leq M$ for an $M < \infty$. A well-known fact is the following proposition.

PROPOSITION 2.17. [**103**, (22.12)] *The equivalence of geodetic rays in hyperbolic graphs is an equivalence relation.* □

Hence we are able to define the *hyperbolic boundary* of a hyperbolic graph: A *hyperbolic boundary point* is an equivalence class of geodetic rays. Let ∂G be the set of hyperbolic boundary points, and let $\widehat{G}$ be $G \cup \partial G$.

We are also giving a second topological definition of the hyperbolic boundary: A sequence $(x_i)_{i\geq 0}$ *converges* to a vertex x if $\lim\limits_{i\to\infty}(x_i, x) = 0$. A sequence $(x_i)_{i\geq 0}$ *converges* to ∞ if $\lim\limits_{i,j\to\infty}(x_i, x_j) \to \infty$. Like above, it is independent of the choice of o, so we just wrote (x_i, x_j) instead of $(x_i, x_j)_o$. Two sequences $(x_i)_{i\geq 0}$, $(y_j)_{j\geq 0}$ are *equivalent* if $\lim\limits_{i,j\to\infty}(x_i, y_j) = \infty$. In hyperbolic graphs this equivalence is indeed an equivalent relation. The hyperbolic boundary can also be defined as equivalence classes of this equivalence relation. A sequence $(x_i)_{i\geq 0}$ *tends* to a boundary point if it is in its equivalence class. In [**40**] the equivalence of these definitions is shown.

A third way to define the boundary is by defining a metric d_ε on G and then defining $\widehat{G}$ as the completion of G induced by d_ε. Let $\varepsilon > 0$ with $\varepsilon' := \exp(\varepsilon\delta) - 1 \leq \sqrt{2} - 1$. Let

$$\varrho_\varepsilon(x, y) := \exp(-\varepsilon(x, y)),$$

$$\varrho_\varepsilon(x_0, \ldots, x_n) := \sum_{i=1}^{n} \varrho_\varepsilon(x_{i-1}, x_i)$$

and

$$d_h(x, y) := \inf\{\varrho_\varepsilon(c) \mid c \text{ chain between } x \text{ and } y\}.$$

It is easy to check that d_ε is a metric on G.

An important theorem about the hyperbolic boundary is the following.

THEOREM 2.18. [**40**, Proposition 7.2.9] *If G is a locally finite hyperbolic graph, then $(\widehat{G}, d_\varepsilon)$ is a compact metric space.* □

We will now define a topology on G, which is compatible with the topology of $\widehat{G}$ which is induced by d_ε. For two vertices and/or hyperbolic boundary points a and b we define the *Gromov-product* (once more):

$$(a, b) := \sup \liminf_{i,j\to\infty} (x_i, y_j)$$

where the supremum is taken over all sequences $(x_i)_{i\geq 0} \to a$ and $(y_i)_{i\geq 0} \to b$. Obviously it is just the same as the previous definition for vertices, so we were allowed to use the same symbol. Let $N_k(x) := \{y \in \widehat{G} | (x, y) > k\}$ for every $x \in \partial G$ and every $k \in \mathbb{R}_{\geq 0}$ and let $B_r(x) = \{y \in VG | d(x, y) < r\}$ for every $x \in VG$ and $r \in \mathbb{R}_{\geq 0}$.

PROPOSITION 2.19. [**1**, Proposition 4.8] *Let G be a locally finite hyperbolic graph. The union of the sets $B_r(x)$ for all $x \in VG$ and all $r \in \mathbb{R}_{\geq 0}$ and $N_k(x)$ for all $x \in \partial G$ and all $k \in \mathbb{R}_{\geq 0}$ form a basis for a topology on $\widehat{G}$.* □

This topology is compatible with the metrics d_ε, which makes the boundary to a compact metric space by Proposition 2.20.

PROPOSITION 2.20. [**40**, Proposition 7.3.10] *Let G be a locally finite hyperbolic graph. There exists a metric d_ε on $\widehat{G}$ such that $(\widehat{G}, d_\varepsilon)$ is a compact metric space and such that the metric is compatible with the just defined topology in the sense that*

$$\varepsilon' \cdot \exp(-\varepsilon \cdot (\eta, \nu)) \leq d_\varepsilon(\eta, \nu) \leq \exp(-\varepsilon \cdot (\eta, \nu))$$

for all $\eta, \nu \in \partial G$ and for $\varepsilon' = \exp(\varepsilon\delta) - 1$.

In addition every ε with $\varepsilon' \leq \sqrt{2} - 1$ has this property. □

Proposition 2.20 is the reason why it is possible that we will use the metric in some place, the topology in some other place, and sometimes use them both together.

PROPOSITION 2.21. [**40**, Proposition 7.5.17] *Let G be a locally finite hyperbolic graph. There exists a continuous surjection from the hyperbolic boundary of G to its set of ends whose fibres are the connected components of ∂G.* □

We state some further propositions that we shall need later.

PROPOSITION 2.22. [**103**, (22.11) and (22.15)] *Let G be a locally finite hyperbolic graph with two distinct boundary points η and ν. Let o be a vertex in G, $(x_i)_{i\in\mathbb{N}}$ a geodetic ray converging to η, and $(y_j)_{j\in\mathbb{N}}$ a geodetic ray converging to ν. Then the following two properties holds:*

(i) *There is a geodetic ray in G starting at o and having only finitely many vertices different from $(x_i)_{i\in\mathbb{N}}$.*
(ii) *There is a geodetic double ray having only finitely many vertices different from $(x_i)_{i\in\mathbb{N}}$ and $(y_j)_{j\in\mathbb{N}}$. One side of the geodetic double ray converges to η, the other to ν.* □

PROPOSITION 2.23. [**103**, Proposition 22.12] *If $\pi = x_1x_2\dots$ and $\pi' = y_1y_2\dots$ are equivalent geodetic rays, then there is a $k \in \mathbb{Z}$ such that $d(x_n, y_{n-k}) \leq 2\delta$ for all but finitely many n.* □

Let $\gamma > 1, c \geq 0$. A (γ, c)-*quasi-isometry* from a metric space X to another metric space Y is a map $f : X \to Y$ with

$$\gamma^{-1} d_X(x, y) - c \leq d_Y(f(x), f(y)) \leq \gamma d_X(x, y) + c$$

for all $x, y \in X$ and with $\sup\{d_Y(y, f(X)) \mid y \in Y\} \leq b$. Then X is *quasi-isometric* to Y.

A (double) ray R in G is (γ, c)-*quasi-geodetic* if it is the image of a (γ, c)-quasi-isometry from $\mathbb{Z}_{\geq 0}$ ($\mathbb{Z}$, respectively) to R. Hence a (double) ray is geodetic, if it is a $(1,0)$-quasi-geodetic (double) ray. If the constants γ, c are not important then we just speak of *quasi-geodesics*.

The next proposition shows that in every locally finite hyperbolic graph the geodesics and quasi-geodesics lie close to each other, see also [**1**, Proposition 3.3], [**46**, 7.2.A], [**23**, 3.1.3], and [**40**, 5.6, 5.11].

Proposition 2.24. [**87**, Theorem 2.31], [**23**, Théorème 3.1.4] *Let G be a locally finite δ-hyperbolic graph. For all $\gamma_1 \geq 1, \gamma_2 \geq 0$ there is a constant $\kappa = \kappa(\delta, \gamma_1, \gamma_2)$ such that for every two vertices $x, y \in VG$ every (γ_1, γ_2)-quasi-geodesic between them lies in a κ-neighborhood around every geodesic between x and y and vice versa.*

Furthermore, this extends to (γ_1, γ_2)-quasi-geodetic and geodetic rays as well as double rays. □

The following is from [**1**, Proposition 3.2] (see also [**46**, 8.1.D] and [**40**, 8.21]).

Proposition 2.25. *Let G be a locally finite transitive δ-hyperbolic graph. Let $x \in VG$ and $\alpha \in \mathrm{Aut}(G)$ such that the orbit of x under α is infinite. Then the set $\{\ldots, x^{\alpha^{-1}}, x, x^{\alpha}, \ldots\}$ lies on a (κ, λ)-quasi-geodetic double ray for constants $\kappa \geq 1, \lambda \geq 0$ that depend only on δ and $d(x, x^{\alpha})$.* □

Proposition 2.26. [**103**, (22.4)] *Let G be a locally finite δ-hyperbolic graph. Then for all $x, y, z \in VG$ we have*

$$(x, y)_z \leq d(z, [x, y]) \leq (x, y)_z + 2\delta.$$ □

Combining Proposition 2.24 and [**16**, Lemma 2.2.2] we obtain the following proposition.

Proposition 2.27. *Let G be a locally finite δ-hyperbolic graph, let $\eta, \nu \in \partial G$, and let $o \in VG$. For all geodetic double rays π from η to ν we have*

$$(\eta, \nu)_o \leq d(o, \pi) \leq (\eta, \nu)_o + 4\delta.$$ □

Proposition 2.28. *Let G be a locally finite hyperbolic graph with a metric d_ε as in Theorem 2.18 with $\varepsilon > 0$ and $\varepsilon' := \exp(\varepsilon\delta) - 1 \leq \sqrt{2} - 1$. Let $o \in VG$ be the base-point for the Gromov-product of G. Then, for every $q > 0$, there exists a $\beta = \beta(q, \varepsilon) > 0$ such that for all $\eta_1, \eta_2, \mu_1, \mu_2 \in \partial X$ with $\frac{1}{q} \leq d_\varepsilon(\eta_1, \mu_1)/d_\varepsilon(\eta_2, \mu_2) \leq q$ there is $|d(o, [\eta_1, \mu_1]) - d(o, [\eta_2, \mu_2])| \leq \beta$.*

Proof. By Theorem 2.18 there is

$$\varepsilon' \exp(-\varepsilon(\eta_1, \mu_1)) \leq d_\varepsilon(\eta_1, \mu_1) \leq q d_\varepsilon(\eta_2, \mu_2) \leq q \exp(-\varepsilon(\eta_2, \mu_2)).$$

As a consequence we have by symmetry:

$$|(\eta_1, \mu_1) - (\eta_2, \mu_2)| \leq \frac{1}{\varepsilon} \ln(\frac{q}{\varepsilon'})$$

The claim follows immediately with Proposition 2.27. □

Let G be a hyperbolic graph and H a hyperbolic subgraph of G. If the identity map $\iota : H \to G$ extends to a continuous map $\hat{\iota} : \widehat{H} \to \widehat{G}$ then we say that the *canonical map* $\partial H \to \partial G$ *exists* and call the restriction of $\hat{\iota}$ to ∂H the *canonical map* $\partial H \to \partial G$.

2.7. Topology

In this section we introduce the Assouad dimension, which is the main dimension concept in Chapter 4, as well as the asymptotic dimension and the topological dimension, and we compare these dimension concepts. For a more detailed introduction to the Assouad dimension we refer to [**5**] and in particular to [**75**, Appendix A].

Let X be a metric space. For $\alpha, \beta > 0$ let $S(\alpha, \beta)$ be the maximal cardinality of a subset V of X such that each two distinct elements of V have distance at least α and at most β. Let n be the infimum of all $s \geq 0$ such that there is a $C \geq 0$ with $S(\alpha, \beta) \leq C(\frac{\beta}{\alpha})^s$ for all $0 < \alpha \leq \beta$. Then n is called the *Assouad dimension* of the metric space X (notation: $\dim_A(X) = n$).

A metric space X is *doubling* if there exists a $\kappa \geq 1$ such that every ball of radius r can be covered by at most 2^κ balls of radius at most $\frac{r}{2}$. With $\dim_2(X)$ we denote the infimum of all these κ. A subset Y of X has *diameter* $\sup\{d(x, y) \mid x, y \in Y\}$ (notation: $\operatorname{diam}(Y)$), and a set $\mathcal{Y} \subseteq \mathcal{P}(X)$ has *diameter* $\operatorname{diam}(\mathcal{Y}) = \sup\{\operatorname{diam}(Y) \mid Y \in \mathcal{Y}\}$. The *radius* of a subset Y of X is $\operatorname{rad}(Y) := \inf\{\sup\{d(x, y) \mid x \in Y\} \mid Y \in Y\}$ and the *radius* of a set $\mathcal{Y} \subseteq \mathcal{P}(X)$ is $\operatorname{rad}(\mathcal{Y}) := \sup\{\operatorname{rad}(Y) \mid Y \in \mathcal{Y}\}$. For every $r \geq 0$, a family $\mathcal{B} = (B_i)_{i \in I}$ of subsets of X has *r-multiplicity* at most n if every subset of X with diameter at most r intersects with at most n members of the family. A point $x \in X$ has *r-multiplicity* at most n in $\mathcal{B}$ if $\bar{B}_r(x)$ intersects with at most n members of the family $\mathcal{B}$ non-trivially.

Our main assumption is that the Assouad dimension of the hyperbolic boundary of our hyperbolic space is finite. It is easier to use the doubling property instead. The following theorem guarantees that we treat the same spaces.

THEOREM 2.29. [**75**, Theorem A.3] *Let X be a metric space. Then, X is doubling if and only if it has finite Assouad dimension.* $\square$

It is easy to adapt the proof of [**73**, Lemma 2.3] for Lemma 2.30, see [**50**, Lemma 3.2] for details.

LEMMA 2.30. *Let X be a doubling metric space, let $N = 2^{\dim_2(X)}$, and let $r > 0$. Then X has a covering $\mathcal{B}$ of closed balls of radius r such that $\mathcal{B}$ is the disjoint union of at most N^2 subsets $\mathcal{B}_i$ of $\mathcal{B}$ each of which has r-multiplicity at most 1; so $\mathcal{B}$ has r-multiplicity at most N^2.*

Furthermore, it is possible to choose a given subset Y of X with $d(x, y) > r$ for all $x, y \in Y$ so that Y is a subset of the set of centers of the balls in $\mathcal{B}$ and such that each two centers of these balls have distance at least r and each center has $3r$-multiplicity at most N^2 in $\mathcal{B}$. $\square$

Let us briefly compare the Assouad dimension with another dimension concept. A metric space X has *asymptotic dimension* n (notation: $\operatorname{asdim}(X) = n$) if n is the smallest natural number such that for every $\varrho > 0$ there exists an open cover $\mathcal{U}$ of X such that every $x \in X$ lies in at most $n+1$ elements of $\mathcal{U}$, such that $\sup_{U\in\mathcal{U}} \operatorname{diam}(U) < \infty$, and such that

$$\inf_{x\in X} \sup_{U\in\mathcal{U}} d(x, X \setminus U) \geq \varrho.$$

In the main theorems (Theorem 4.13 and Theorem 4.14) we are talking about proper hyperbolic geodetic spaces whose hyperbolic boundary has finite Assoud dimension. Since the hyperbolic boundary is a doubling space, we conclude from [**16**, Corollary 10.2.4] that the hyperbolic space itself has finite asymptotic dimension. We refer to [**16**] for a broader overview of the distinct dimension concepts for hyperbolic spaces and to [**8, 47**] for more about the asymptotic dimension.

Let us define a third dimension, the topological dimension. A *refinement* $\mathcal{U}$ of an open cover $\mathcal{V}$ of X is an open cover of X such that for every $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ with $U \subseteq V$. X has *topological dimension at most* n if every open cover has a refinement such that each $x \in X$ lies in at most $n+1$ elements of the refinement, and X has *topological dimension* n if it has topological dimension at most n but not topological dimension at most $n-1$. If there exists no $n \in \mathbb{N}$ such that X has topological dimension at most n then X has *infinite topological dimension*. Remark that we always have $\dim X \leq \dim_A X$ by [**75**, Facts 3.3].

2.8. Some classes of graphs

A tree T is *semi-regular* if all the vertices in each set of its natural bipartition have the same degree. Let $VT = A \dot\cup B$ be the partition. If all the vertices in A have degree k and all the vertices in B have degree l, then we denote the tree with $T_{k,l}$. If we consider directed trees we denote with $T_{k,l}$ a tree whose underlying undirected graph is a semi-regular tree and all whose edges are directed from the vertices in the set A to the vertices in the set B. We also call this digraph a *semi-regular tree.*

With $X_{\kappa,\lambda}$ we denote a graph with connectivity 1 such that every *block*, that is a maximal 2-connected subgraph, is a complete graph on κ vertices and every vertex lies in λ distinct blocks.

In the following we describe some classes of digraphs that occur during the investigation of locally finite C-homogeneous digraphs (Chapter 12). In the context of digraphs, we usually denote with C_m directed cycles of length m. But if it is obvious from the context that we are considering a subdigraph of a bipartite reachability digraph, then we also use C_m to denote a cycle in that reachability digraph. Cycles of length 3 are *triangles*.

For two digraphs D, D' we denote with $D[D']$ the *lexicographic product* of D and D', that is the digraph with vertex set $VD \times VD'$ and edge set

$$\{(x,y)(x',y') \mid xx' \in ED \text{ or } (x = x' \text{ and } yy' \in ED')\}.$$

The *complete bipartite* digraph is that bipartite digraph that contains all edges from A to B for the bipartition $A \cup B$. The *(directed) complement of a perfect matching* CP_k is the digraph obtained from the complete bipartite digraph where a perfect matching between A and B is removed.

Let Y_k be the digraph with vertex set $V_1 \cup V_2 \cup V_3$ where the V_i denote pairwise disjoint sets of the same cardinality k. There are no edges xy with $xy \in V_i$ for $i = 1, 2, 3$ and the subdigraphs $D[V_i, V_{i+1}]$ (for $i = 1, 2, 3$ with $V_4 = V_1$) are isomorphic to complements of perfect matchings such that all edges are directed from V_i to V_{i+1} and such that the tripartite complement of D is the disjoint union of copies of C_3, where the *tripartite complement* of D is the digraph

$$(VD, (\bigcup_{i=1,2,3} (V_i \times V_{i+1})) \setminus ED).$$

The *complete bipartite digraph* $K_{\kappa,\lambda}$ is the digraph with bipartition $A \cup B$ such that $|A| = \kappa$, $|B| = \lambda$ and all edges point from A to B, and the *directed complement of a perfect matching* CP_κ is the digraph obtained from $K_{\kappa,\kappa}$ by removing a perfect matching.

We call a bipartite graph G with bipartition $X \cup Y$ *generic bipartite* if it has the following property: For any finite disjoint subsets U and W of X (of Y) there is a vertex v in Y (in X) such that $U \subseteq N(v)$ and $W \cap N(v) = \emptyset$. Any generic bipartite graph contains any countable bipartite graph as an induced subgraph, and thus up to isomorphism there is a unique countable generic bipartite graph (cp. [**26**, p. 213] and [**35**, p. 98]). A *generic bipartite digraph* is a digraph D whose underlying undirected graph G is generic bipartite with bipartition $A \cup B$ and such that all edges of D are directed from A to B.

With H we denote the digraph depicted in Figure 1.

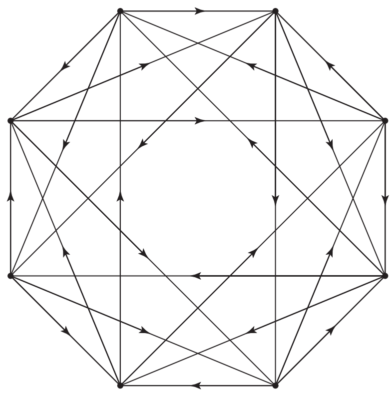

FIGURE 1. The digraph H

A *tournament* is an orientated complete graph. We shall state Lachlan's classification theorem of the homogeneous tournaments[1], but first, we have to define a countable tournament $\mathcal{P}$ to be the digraph with the rationals in the intervall $[-\pi, \pi]$ as vertex set and direct the edge from x to y if

$$x - y \leq \pi \mod 2\pi$$

and from y to x otherwise. The generic countable tournament is the unique (cp. [**26**, p. 213], and [**35**, p. 98]) countable homogeneous tournament that embeds all finite tournaments.

THEOREM 2.31 ([**21**, Theorem 3.6]). *There are up to isomorphism only* 5 *countable homogeneous tournaments: the trivial tournament on one vertex, the directed triangle, the generic tournament on ω vertices, the tournament that is isomorphic to the rationals with the usual order, and the tournament $\mathcal{P}$ described above.* □

For a homogeneous tournament T let $X_\lambda(T)$ denote the digraph where each vertex is a cut vertex and lies in λ distinct copies of T.

Given an edge-transitive bipartite digraph Δ with bipartition $A \cup B$ such that every edge is directed from A to B we define $DL(\Delta)$ to be the unique connected digraph such that each vertex separates the digraph, lies in exactly two copies of Δ, and has both in- and out-neighbors (cp. [**19, 45**]).

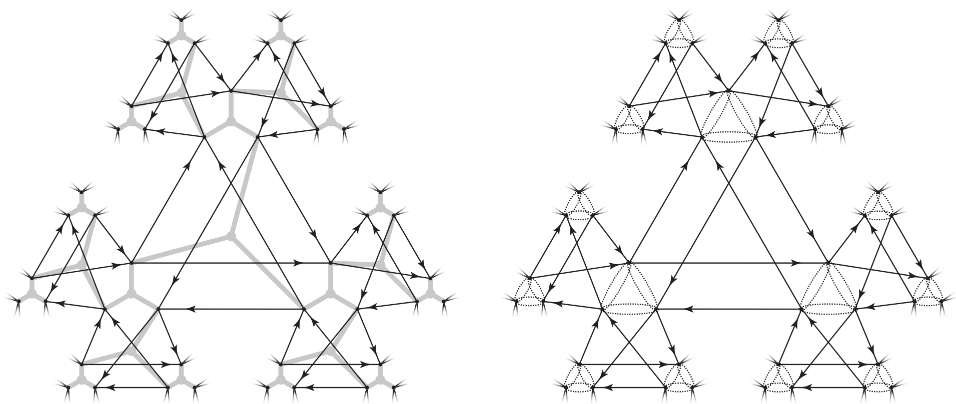

FIGURE 2. The digraph $M(3, 3)$

Now, we define a class of digraphs with connectivity 2 and reachability digraph CP_κ. Given $2 \leq m \in \mathbb{N}$ and a cardinal $\kappa \geq 3$ consider the tree $T_{\kappa,m}$ and let $U \cup W$ be its natural bipartition such that the vertices in U have degree m. Now subdivide each edge once and endow the neighborhood of each $u \in U$ with a cyclic order. Then for each new vertex y let u_y be its unique neighbor in U and denote by

[1] A tournament is *homogeneous* if every isomorphism between subtournaments extends to an automorphism of the whole tournament. For a more detailed introduction to homogeneity, we refer to Chapter 9.

$\sigma(y)$ the successor of y in $N(u_y)$. Then for each $w \in W$ and each $x \in N(w)$ we add an edge directed from x to all $\sigma(y)$ with $y \in N(w) - x$. Finally we delete the edges and vertices of the $T_{\kappa,m}$ to obtain the digraph $M(\kappa, m)$. The locally finite subclass of this class of digraphs coincides with those digraphs $M(k, n)$ for $k, n \in \mathbb{N}$ that are described in [**45**, Section 5]. In Figure 2 the digraph $M(3, 3)$ is shown: once with its construction tree and once with its set of $\mathcal{C}$-separators.

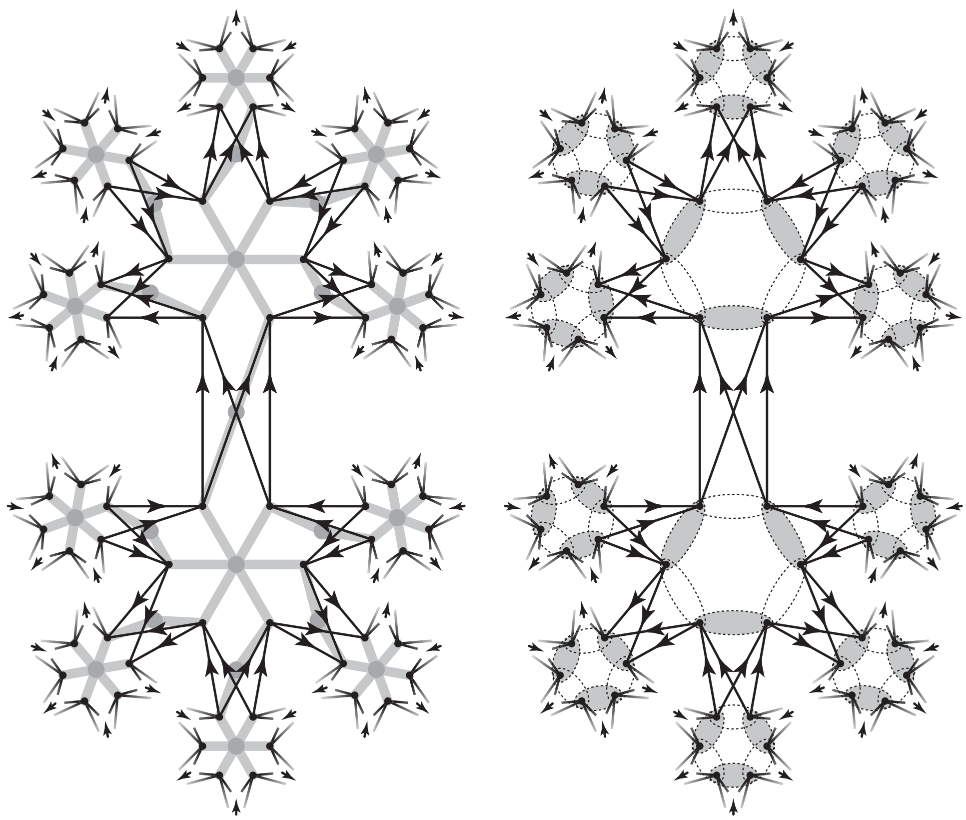

FIGURE 3. The digraph $M'(6)$

Another class of digraphs with connectivity 2, but this time with reachability digraph $K_{2,2}$ will be defined in the following. For $2 \leq m \in \mathbb{N}$ consider the tree $T_{2,2m}$ and let $U \cup W$ be its natural bipartition such that the vertices in U have degree $2m$. Now subdivide every edge once and enumerate the neighborhood of each $u \in U$ from 1 to $2m$ in such a way that the two neighbors of each $w \in W$ have distinct parity. For each new vertex x let u_x be its unique neighbor in U and define $\sigma(x)$ to be the successor of x in the cyclic order of $N(u_x)$. For any $w \in W$ we have a neighbor a_w with even index, and a neighbor b_w with odd index. Then we add edges from both a_w and $\sigma(a_w)$ to both b_w and $\sigma(b_w)$. Finally we delete the $T_{2,2m}$. With $M'(2m)$ we denote the resulting digraph. Figure 3 shows the digraph $M'(6)$:

on the left side with its construction tree and on the right side with the separators of the two possible basic cut systems.

Part 1

Tree-likeness of hyperbolic graphs

CHAPTER 3

Trees and hyperbolic graphs

A spanning tree of a graph is called *end-faithful* if the tree contains exactly one ray from each end, starting at the root. Halin [**48**] proved that every countable graph has an end-faithful spanning tree. Examples for such trees are the normal spanning trees (see [**13, 58**] and [**26**, Chapter 8]). So it is a natural question to ask—if we replace the end-compactification of a graph by other compactifications that refine the end-compactification—how we can expect a spanning tree to behave with respect to the new compactification: Is it possible that the ends of a spanning tree represent the boundary points of this compactification in a one-to-one correspondence?

In general, arbitrary hyperbolic graphs do not have spanning trees which are faithful to hyperbolic boundary points instead of ends. An easy example is the graph G of Figure 4. Its hyperbolic boundary A is homeomorphic to the real unit interval. Now suppose there is a spanning tree T of G with precisely one ray from the root to each boundary point of G. Then there is a vertex x that separates T into at least two infinite components, call them $C_1, \ldots, C_n$. For each i let $A_i \subseteq A$ be the set of boundary points to which there is a ray in C_i. As G is locally finite, it is not hard to see that the A_i are closed in A, hence they have to intersect, since A is connected and $\bigcup A_i = A$.

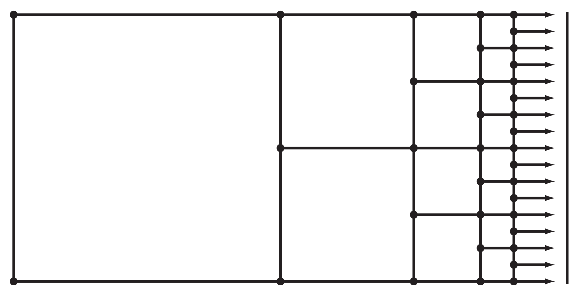

FIGURE 4. A hyperbolic graph with its boundary

On the other hand, it is easier to see that for a given hyperbolic graph G there is not always a tree such that the end space of the tree is homeomorphic to the hyperbolic boundary of G: In [**15**, Section 7] (see also [**11, 33**] or [**16**, Chapter 6]) it is shown that for every compact metric space X there is a locally finite hyperbolic graph constructed so that its hyperbolic boundary is homeomorphic to X. As the end space of any tree is a totally disconnected topological space (see [**59**]) there cannot be any spanning tree such that the induced map from the boundary of the tree to the boundary of the graph is a homeomorphism.

In fact, we shall prove in Section 4.4 that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that any spanning tree T of a locally finite hyperbolic graph G, for which the canonical map $\partial T \to \partial G$ exists, has a hyperbolic boundary point with at least $f(n)$ preimages, where n is the topological dimension of ∂G.

Whenever the identity of a hyperbolic graph G extends to a homeomorphism from $G \cup \partial G$ to G with its ends, any normal spanning tree—or more generally any end-faithful spanning tree—is faithful with respect to the hyperbolic boundary points.

Hyperbolic graphs in which the notion of hyperbolic boundary points and ends coincide are for example all locally finite graphs quasi-isometric to a tree (see [**67**]) or—more generally, compare with [**67**, Theorem 2.8]—locally finite graphs in which any end is a thin end in the sense of [**26**, Chapter 8], as any end of a locally finite hyperbolic graph that consists of more than one hyperbolic boundary point consists of uncountably many hyperbolic boundary points since this set of hyperbolic boundary points is a connected set [**40**, Proposition 7.5.17]. Thus in locally finite hyperbolic graphs, the hyperbolic boundary is a refinement of the set of its ends and it is furthermore a compact metric space [**40**, Proposition 7.2.9]. This is not the case for arbitrary graphs: neither the hyperbolic boundary has to be compact in hyperbolic graphs that are not locally finite nor it is a refinement of the set of ends of such graphs. Because of this we restrict our point of view to locally finite graphs.

Instead of spanning trees that are faithful to boundary points, we may perhaps hope that we get spanning trees that have only a finite number of distinct paths from the root to each boundary point such that the set of these numbers is bounded. This is indeed true if the boundary has finite Assouad dimension (Theorem 4.14).

The most important examples of locally finite hyperbolic graphs whose hyperbolic boundaries have finite Assouad dimension are graphs with bounded degree (see [**10**, Theorem 9.2]). These are in particular all Cayley graphs of finitely generated groups with respect to a finite set of generators.

There are various other results that investigate the relation between trees and the hyperbolic boundary of locally finite hyperbolic graphs. Several ideas for constructions of trees that capture the nature of the hyperbolic boundary of locally finite hyperbolic graphs can already be found in Gromov's article [**46**, Sections 7.6, 8.5.B, and 8.5.C]. They have been elaborated on in [**24**, Chapter 5]. For each of these trees there is a continuous map from their own boundary onto that of the graph. If the hyperbolic graph has bounded degree, then some of these maps are finite-to-one. However, these trees are not necessarily subtrees of the hyperbolic graph.

On the other hand, there are results that construct for a given hyperbolic space (not only for locally finite graphs) an $\mathbb{R}$-tree whose local structure resembles the local structure of the hyperbolic space. The best known among these are results attributed to Gromov (see [**23**, Chapitre 8] and [**40**, Section 2.2]) that construct for a finite subset of the completion of a δ-hyperbolic space an $\mathbb{R}$-tree in the space whose completion contains the given set and such that all of its geodesics between elements of the finite set are quasi-geodesics in the hyperbolic space for constants that depend only on the size of the set and on δ. There is also a result by Benjamini and Schramm [**9**, Theorem 1.5] for locally finite hyperbolic graphs which states that there exists a subtree with exponential growth such that the embedding is a bilipschitz map.

In the Chapter 4, we combine these two approaches. Inside every locally finite hyperbolic graph G whose boundary has finite Assouad dimension we construct a rooted spanning tree T such that

- the rays from the root are all quasi-geodetic rays (for the same global constants) and
- a continuous finite-to-one map from the boundary of the tree onto the one of the hyperbolic space exists where the bound only depends on the Assouad dimension of the hyperbolic boundary.

If the hyperbolic graph is visual—that is roughly speaking that every vertex has bounded distance to a geodetic double ray (see Section 4.2 for more details)—then every vertex of the graph has distance at most some constant κ from some ray of T that starts at the root (Theorem 4.13). If we consider an arbitrary locally finite hyperbolic graph, then a κ exists with the property that every geodesic outside a κ-ball around the rays of T that starts at the root has finite length (Theorem 4.14).

Remark that the results of Chapter 4 do not only hold in the case of locally finite hyperbolic graphs but also in the case of proper hyperbolic geodetic spaces (see [**50**] and in particular [**52**] for details).

There are also different approaches exhibiting the tree-likeness of hyperbolic spaces than constructing $\mathbb{R}$-trees include quasi-isometric embeddings of visual hyperbolic spaces into the product of $\mathbb{R}$-trees, see Buyalo et al. [**15**], or the sub-cones at infinity, see [**40**, Proposition 2.1.11] and [**88**, Lemme 5.6].

CHAPTER 4

Spanning trees of hyperbolic graphs

In this chapter, we prove the graph theoretic version of a theorem of [**52**] which is a strengthened form of a theorem in [**50**]. In the first part of this chapter (Section 4.1 to Section 4.3) we show that for any locally finite graph G, there exists a spanning tree T such that the canonical map $\partial T \to \partial G$ exists, is onto and for each $\eta \in \partial G$, the number of its preimages are bounded in terms of the Assouad dimension of the hyperbolic boundary. Furthermore, the spanning tree will have the additional property that it represents the hyperbolic graphs itself in a good way. There will be constants $c_1 \geq 1$, $c_2 \geq 0$ such that every ray in the spanning tree is eventually a (c_1, c_2)-quasi-geodetic ray. With these two properties, the spanning tree satisfies the two distinct approaches of showing the tree-likeness of hyperbolic graphs that we discussed in Chapter 3.

In the second part (Section 4.4) we show that for any locally finite hyperbolic graph G there is a function $f : \mathbb{N} \to \mathbb{N}$ depending only on the topological dimension of ∂G such that for any spanning tree T of G the canonical map $\partial T \to \partial G$ exists and such that there is a boundary point of the graph with at least $f(\dim(\partial G))$ many preimages und the canonical map.

4.1. Construction of the spanning tree

In this section we construct a rooted spanning tree T inside a locally finite hyperbolic graph G whose hyperbolic boundary has finite Assouad dimension and whose hyperbolic constant is not 0.

Let $d_h = d_\varepsilon$ be a metric such that ε satisfies the assumptions as in Theorem 2.18 and hence such that $(\widehat{G}, d_h)$ is a compact metric space. By [**10**, Sections 6 and 9] the property of G having finite Assouad dimension does not depend on the particular choice of ε. That means if ∂G has finite Assouad dimension for one metric d_ε, then this holds for all these metrics. That is the reason why we are able just to say that ∂G has finite Assouad dimension. By Theorem 2.29, ∂G is a doubling metric space. So let $N = 2^{\dim_2(\partial G)}$.

The rooted tree T that we shall construct will have the following properties.

(1) Every ray in T converges to a point in the hyperbolic boundary of G;
(2) for every boundary point η of G there is a ray in T converging to η;
(3) for every boundary point η of G there are at most $N^{2+\log_2(8N^2)}$ distinct rays in T that start at the root of T and converge to η.

We construct the rooted tree T recursively. So let $r \in VG$ be the base-point of the Gromov-product which we used for the definition of the metric d_ε. The vertex r will be the root of T. For the construction of T we construct a strictly descending sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ in $\mathbb{R}_{>0}$, two sequences $(S_j)_{j\in\mathbb{N}}, (Y_j)_{j\in\mathbb{N}}$ of subsets of ∂G, a sequence $(\mathcal{U}_j)_{j\in\mathbb{N}}$ of open covers of ∂G, a sequence $(\mathcal{B}_j)_{j\in\mathbb{N}}$ of closed covers of ∂G, and a sequence $(T_j)_{j\in\mathbb{N}}$ of subtrees of G. Our final tree T will be the union of all the T_j. The other sequences will help us in the construction of the trees T_j and they will satisfy the assertions (a) to (k) for every j.

(a) $\varepsilon_j = \frac{a}{8N^2}$ with $a = \frac{\varepsilon_{j-1}}{4\cdot 16}$, so $\varepsilon_j = \frac{\varepsilon_{j-1}}{512N^2}$;
(b) there is $S_{j-1} \subseteq Y_j \subseteq S_j$;
(c) $d_h(\eta,\mu) \geq \varepsilon_j$ for all $\eta \neq \mu \in S_j$;
(d) the set S_j has $\frac{\varepsilon_{j-1}}{16}$-multiplicity at most $N^{\log_2(8N)}$;
(e) $d_h(\eta,\mu) \geq \frac{\varepsilon_{j-1}}{4\cdot 16}$ for all $\eta \neq \mu \in Y_j$;
(f) The open cover $\mathcal{U}_j$ consists of precisely the open ε_j-balls around the elements of S_j;
(g) the set $\mathcal{B}_j$ consists of all closed balls of radius $\frac{\varepsilon_{j-1}}{4\cdot 16}$ around the elements of Y_j and it has $\frac{\varepsilon_{j-1}}{4\cdot 16}$-multiplicity at most N^2;
(h) every $\eta \in Y_j$ has $(3\cdot\frac{\varepsilon_{j-1}}{4\cdot 16})$-multiplicity at most N^2 in $\mathcal{B}_j$;
(i) $T_{j-1} \subseteq T_j$;
(j) every ray in T_j converges to an element of S_j and to each one converges precisely one ray that starts at the root;
(k) every ray in T_j is eventually geodetic, in particular, there is a constant c depending only on ε_j such that every ray in $T_j \setminus \bar{B}_c(x)$ is a geodetic ray.

Before we start the recursion step, we first define the elements of all sequences for $j=0$. Let $\mu^0 \in \partial G$, $S_0 = Y_0 = \{\mu^0\}$ and $\varepsilon_0 = \sup\{d_h(\mu^0,\eta) \mid \eta \in \partial G\}$—recall that ∂G is bounded by Theorem 2.18. Let $\mathcal{B}_0 = \mathcal{U}_0 = \partial G$ and let T_0 be a geodetic ray from r to μ^0 which exists by Proposition 2.22. Then all properties are satisfied for $j=0$.

For the recursion step we set ϵ_j as in (a). Lemma 2.30 shows that there is a closed covering $\mathcal{B}_j$ of ∂G with balls of radius $\frac{\varepsilon_{j-1}}{4\cdot 16}$ such that this covering has $\frac{\varepsilon_{j-1}}{4\cdot 16}$-multiplicity at most N^2 and such that the set Y_j of centers of these balls contains S_{j-1} and such that every $\eta \in Y_j$ has $(3\cdot\frac{\varepsilon_{j-1}}{4\cdot 16})$-multiplicity at most N^2 in $\mathcal{B}_j$. Then (e), (g), (h), and the first part of (b) hold.

Let S_j be a subset of ∂G with $Y_j \subseteq S_j$ such that $d_h(\mu,\nu) \geq \varepsilon_j$ for all $\mu,\nu \in S_j$, such that S_j has $\frac{\varepsilon_{j-1}}{16}$-multiplicity at most $N^{\log_2(8N)}$, and such that $\mathcal{U}_j := \{B_{\varepsilon_j}(\mu) \mid \mu \in S_j\}$ is an open cover of ∂G. This set S_j exists by applying the doubling definition $\log_N(N^{\log_2(8N)}) = \log_2(8N)$ times to the sets in $\mathcal{B}_j$ and it is finite because ∂G is doubling. As a consequence we have (c), (d), (f), and the remaining part of (b). The only element of any of the sequences that remains to be constructed is the subtree T_j.

We construct the subtree T_j recursively. Let $T_j^{0,1} = T_{j-1}$. We enumerate the set $S_j \setminus S_{j-1}$ in the following way. Let $\mu_1^1, \mu_2^1, \ldots$ be the elements with $8\varepsilon_{j-1}$-multiplicity 1 in $\mathcal{B}_{j-1}$, let $\mu_1^2, \mu_2^2, \ldots$ be the elements with $(2 \cdot 8\varepsilon_{j-1})$-multiplicity at most 2 in $\mathcal{B}_{j-1}$, and so on. As the set $\mathcal{B}_{j-1}$ has $\frac{\varepsilon_{j-2}}{4 \cdot 16}$-multiplicity at most N^2 and $8N^2\varepsilon_{j-1} = \frac{\varepsilon_{j-2}}{4 \cdot 16} \leq \mathrm{rad}(\mathcal{B}_{j-1})$, there are no μ_k^i with $i > N^2$ by (g).

The tree $T_j^{i,k}$ will be the union of the tree $T_j^{i-1,k}$ and an eventually geodetic ray from $T_j^{i-1,k}$ to the hyperbolic boundary point μ_k^i, where we denote with $T_j^{0,k}$ the union of all $T_j^{a,k-1}$. So let $\mu_k^i \in S_j \setminus S_{j-1}$ and assume that we have already constructed the subtree $T_j^{i-1,k}$. There is a $\mu \in S_{j-1}$ with $d_h(\mu_k^i, \mu) \leq \varepsilon_{j-1}$. Let R be a geodetic double ray from μ_k^i to μ. Let Q denote the largest distance from r to any geodetic double ray between two boundary points of distance at most ε_{j-1} and at least ε_j and let q denote the smallest distance of from r to such a double ray. Then we know that there is $\beta \geq Q - q$ for the constant β from Proposition 2.28.

Let us first consider the case that there is a common vertex of R and $T_j^{i-1,k}$ that has distance at most $Q + 5\delta$ to r. Then there is a first common vertex x of R and $T_j^{i-1,k}$ such that we have $V(Rx) \cap VT_j^{i-1,k} = \{x\}$ for Rx, the subray of R from x to μ_k^i. In this case we just add the subray Rx to the boundary point μ_k^i to the tree $T_j^{i-1,k}$ to obtain the tree $T_j^{i,k}$. By the choice of x, $T_j^{i,k}$ is indeed a tree. In preparation of the proof of Lemma 4.8 we denote in this case with π_R the vertex x and for Claim 4.1 we set $x_P := x$. If x lies during the construction on a geodetic double ray P, then we say that we have *connected* μ_k^i to that limit point η of P that has smaller distance to μ_k^i. If x lies on some π_P for a double ray P, then we have *connected* μ_k^i either to the boundary point η we constructed a new ray to with P or inductively to one of the possible boundary points we connected η to, depending which one has the smallest distance to μ_k^i. Since the hyperbolic boundary has the doubling property, the described relation is well-defined. If the hyperbolic boundary point η to which μ_k^i is connected lies in S_{j-1}, then μ_k^i is *eventually connected* to η. If this is not the case, then μ_k^i is *eventually connected* to that hyperbolic boundary point to which η is eventually connected.

Now we look at the case that R is totally distinct from $T_j^{i-1,k}$ in the ball with center r and radius $Q + 5\delta$. There is a geodetic ray P in $T_j^{i-1,k}$ converging to μ whose first vertex x_P has distance $Q + 5\delta$ to r. Then $d(o, x_P) \leq Q + 5\delta$. We consider a geodesic $\tilde{\pi}_R$ from R to P that has length at most δ with the additional property that for a vertex z on $R \cap B_Q(r)$ we have that $d_{R \cup \pi_R}(z, x_P)$ is minimal. This exists because every vertex y on P with $d(r, y) = Q + 3\delta$ is δ-close to a vertex on R. As it lies in the ball with center r and radius $Q + 6\delta$, there is a smallest connected subpath π_R of $\tilde{\pi}_R$ that contains a vertex of R and a vertex of $T_j^{i-1,k}$. Let x_R denote the intersection point of π_R and R. Then we add the subray of R from π_R to μ_k^i together with π_R to $T_j^{i-1,k}$ to obtain the new tree $T_j^{i,k}$. The property for μ_k^i of being *connected* is defined analog to the first case.

Let $T_j := \bigcup_{i,k} T_j^{i,k}$. Since the union of a finite sequence of trees $T_j^{i,k}$ with $T_j^{i-1,k} \subseteq T_j^{i,k}$ is a tree again, we know that T_j is a tree. We remark that the tree T_j satisfies the properties (i), (j), and (k) for $c = Q$.

We just have defined all sequences as claimed. Set

$$T' := \bigcup_{j \in \mathbb{N}} T_j.$$

We want to show that T' is a tree. As each of the trees T_j is connected and $T_j \subseteq T_{j+1}$, we know that T' is connected. So we just have to show that T does not contain any cycle. As a cycle is a finite sequence of vertices such that each vertex is adjacent to its succeeding vertex and the last is adjacent to the first vertex, such a cycle has to lie in a tree T_j which is impossible. Hence, T' is a tree.

The tree T' is not a spanning tree so far: we still have to add all the vertices in $G - T'$ with appropriate paths to T' to obtain another tree T that will be a spanning tree of G. Thereafter, we have to prove the assertions (1) to (3).

Let us add all vertices of $G - T'$ to T' recursively to get a new tree T which will be a spanning tree of G. In order not to increase the number of rays to each boundary point, we connect the new vertices without creating any new ray. This can be done as follows. First we can easily extend the tree by adding all finite components of $G - T'$ to T'. Then we add every vertex with distance $d(r, G - T')$ to T' by a path lying outside $B_{d(r,G-T')}(r)$. There might be vertices for which there exist no such path. We do not add these. Let T'_1 be the new tree. If there is a vertex in $G - T'_1$ with distance $d(r, G - T'_1)$ that does not lie in any finite component of $G - T'_1$, then there is a path from such a vertex to $T'_1 - B_{d(r,G-T')}(r)$ that intersects with T'_1 trivially except for its endvertex, because the hyperbolic boundary is a refinement of the ends in the locally finite case. This path has a last vertex with distance $d(r, G - T'_1)$ to r. But this is a contradiction, since this vertex had to be added with a path to T'. So all the vertices in $G - T'_1$ with distance $d(r, G - T'_1)$ lie in finite components of $G - T'_1$. For the following step we keep in mind the largest distance d_1 from r to a vertex lying in $T'_1 - T'$. In the step of recursion we add all finite components of $G - T'_i$. Then we add all paths from vertices x with $d(r, x) = d_{i-1} + 1$ to T'_i that are lying completely outside $B_{d_1}(r)$. Once more there might be vertices x with $d(r, x) = d_{i-1} + 1$ that cannot be connected to T'_i in such a way. These will be treated at the beginning of the next step of the recursion as before.

Let $T = \bigcup_{i \in \mathbb{N}} T'_i$. Obviously T is a spanning tree of G and in $T - T'$ there is no ray. Thus to prove the assertions (1) to (3) for T, it suffices to prove them for T'.

In order to prove the assertions (1) to (3) we shall prove several claims in which we use the notation from the construction step (j, k, i).

CLAIM 4.1. *There is* $d_{T_j^{i,k}}(R, x_P) \leq \delta$.

Proof of Claim 4.1. By induction, we know that the corresponding statement holds for all previous trees. If π_R does not meet $T_j^{i-1,k}$ except for x_P, then the assertion holds trivially, so we may assume that π_R meets some other R' or $\pi_{R'}$ (these correspond to R or π_R in a previous step). Suppose first, that it meets some $\pi_{R'}$. Then this $\pi_{R'}$ has to have distance at most δ to x_P, because otherwise the corresponding vertex $x_{P'}$ lies at distance at most 2δ from x_P and thus for the two hyperbolic boundary points to which P and P' converge, ξ and ξ', respectively, any geodetic double ray between them lies in a 3δ-neighborhood of $P \cup \pi_R \cup \pi_{R'} \cup P'$, so at least $Q + \delta$ away from r and hence we have $d_h(\xi, \xi') < \varepsilon_j$ which is impossible as soon as $\xi \neq \xi'$. Let us now suppose that π_R meets some R'. Then we have chosen $\pi_{R'}$ so that every vertex on it has distance at most δ to x_P in the tree by the same contradiction as above. By the minimality of $d_{R' \cup \pi_{R'}}(z', x_P)$ for the vertex z' that corresponds to z for R' instead of R, we know that the claim must hold. $\square$

Claim 4.2. *Let μ_k^n and μ_l^n be two elements of $S_j \setminus S_{j-1}$ with $d_h(\mu_k^n, \mu_l^n) \leq 8\varepsilon_{j-1}$ for an $n \leq N^2$. Then for any $B \in \mathcal{B}_{j-1}$ with $d_h(\mu_k^n, B) \leq n8\varepsilon_{j-1}$ there is $d_h(\mu_l^n, B) \leq n8\varepsilon_{j-1}$.*

Proof of Claim 4.2. The $((n-1)8\varepsilon_{j-1})$-multiplicity of μ_k^n and the one of μ_l^n in $\mathcal{B}_{j-1}$ has to be n. Thus for every hyperbolic boundary point η in Y_{j-1} with distance at most $n8\varepsilon_{j-1}$ to μ_k^n we have $d_h(\eta, \mu_k^n) \leq (n-1)8\varepsilon_{j-1}$ and hence also $d_h(\eta, \mu_l^n) \leq n8\varepsilon_{j-1}$. $\square$

Claim 4.3. *Let μ_{i+1}^k be connected to $\mu \in S_j$. Then $d_h(\mu, \mu_{i+1}^k) \leq 8\varepsilon_{j-1}$. If μ_{i+1}^k is eventually connected to $\eta \in S_{j-1}$ in T_j, then*

$$d_h(\eta, \mu_{i+1}^k) \leq 8N^2\varepsilon_{j-1} + \mathrm{rad}(\mathcal{B}_{j-1}) = 16N^2\varepsilon_{j-1}.$$

Furthermore, η lies in some $B \in \mathcal{B}_{j-1}$ with $d_h(\mu_{i+1}^k, B) \leq 8N^2\varepsilon_{j-1}$.

Proof of Claim 4.3. Let us first prove $d_h(\mu_{i+1}^k, \mu) \leq 8\varepsilon_{j-1}$. An immediate consequence of Claim 4.1 is, if we inserted a non-trivial geodesic π_R, that then the boundary point we connected μ_{i+1}^k to has distance at most ε_{j-1} to μ_{i+1}^k.

So we assume in the following that π_R is only one vertex. Then R meets some other double ray R' or a geodesic $\pi_{R'}$ where R' and $\pi_{R'}$ are as in the proof of Claim 4.1. If R meets some other double ray R', then μ is the limit point of R' and any geodetic double ray $[\mu_{i+1}^k, \mu]$ lies in a δ-neighborhood of $R \cup R'$, so it has distance at least $q - \delta$ to r. Then we have with

$$\delta' := \exp(6\varepsilon\delta) \leq (\sqrt{2})^6 = 8$$

for any $\mu' \in \partial X$ with $\varepsilon_j \leq d_h(\mu_{i+1}^k, \mu') \leq \varepsilon_{j-1}$:

$$\begin{array}{rcl} d_h(\mu_k^i, \mu) & \leq & \exp(-\varepsilon(\mu_k^i, \mu)) \\ & \leq & \exp(-\varepsilon(\mu_k^i, \mu') + 5\varepsilon\delta) \\ & \leq & \frac{\delta'}{\varepsilon'} \exp(-\varepsilon(\mu_k^i, \mu')) \\ & \leq & 8 d_h(\mu_k^i, \mu') \\ & \leq & 8\varepsilon_{j-1} \end{array}$$

Now we assume the last case, that is, R meets some $\pi_{R'}$. Then any vertex on $R\cap\pi_{R'}$ has distance at most δ to $x_{P'}$, where $x_{P'}$ denotes the vertex for R' that x_P denotes for R. Let P' be the ray for R' that is P for R. We conclude that there is a hyperbolic boundary point μ' such that $[\mu^k_{i+1},\mu']$ lies in a 2δ-neighborhood of $R\cup\pi_{R'}\cup P'$. This gives us the following inequality for any $\nu\in\partial X$ with $\varepsilon_j\leq d_h(\mu^k_{i+1},\nu)\leq\varepsilon_{j-1}$:

$$\begin{array}{rcl} d_h(\mu^i_k,\mu) & \leq & \exp(-\varepsilon(\mu^i_k,\mu)) \\ & \leq & \exp(-\varepsilon(\mu^i_k,\nu)+5\varepsilon\delta) \\ & \leq & \frac{\delta'}{\varepsilon'}\exp(-\varepsilon(\mu^i_k,\nu)) \\ & \leq & 8d_h(\mu^i_k,\nu) \\ & \leq & 8\varepsilon_{j-1} \end{array}$$

Let m be minimal such that the $((m-1)\cdot 8\varepsilon_{j-1})$-multiplicity of μ is not $m-1$ but such that the $(m\cdot 8\varepsilon_{j-1})$-multiplicity of μ is m. If $m=1$, then the two boundary points μ^k_{i+1} and μ lie in the same ball $B\in\mathcal{B}_{j-1}$. We conclude that all three hyperbolic boundary points μ^k_{i+1}, μ, and η lie in a common ball $B\in\mathcal{B}_{j-1}$ and hence that

$$d_h(\eta,\mu^k_{i+1})\leq \mathrm{rad}(\mathcal{B}_{j-1})\leq 8N^2\varepsilon_{j-1}.$$

Let us now assume that $m\neq 1$. We may assume that $\mu\neq\eta$, that is $\mu\in S_j\setminus S_{j-1}$. By induction we know that η lies in one of the elements of $\mathcal{B}_{j-1}$, say in B, that is responsible for the $(n\cdot 8\varepsilon_{j-1})$-multiplicity of at most n of μ where n denotes the corresponding value for μ that is m for μ^k_{i+1}. As μ^k_{i+1} is connected to μ, we have $n\leq m$. Thus $d_h(\mu^k_{i+1},B)\leq m\cdot 8\varepsilon_{j-1}$ and hence there is

$$d_h(\mu^k_{i+1},\eta)\leq m\cdot 8\varepsilon_{j-1}+\mathrm{rad}(\mathcal{B}_{j-1})\leq m\cdot 8\varepsilon_{j-1}+8N^2\varepsilon_{j-1}.$$

Since every element of $S_j\setminus S_{j-1}$ has $(8N^2\varepsilon_{j-1})$-multiplicity at most N^2 in $\mathcal{B}_{j-1}$, we have $d_h(\mu^k_{i+1},\eta)\leq 16N^2\varepsilon_{j-1}$. $\square$

By the construction of the trees T_j, we have the following property.

(∗) In every step and for every closed ball $B\in\mathcal{B}_k$ a boundary point in B can only be eventually connected to elements of at most N^2 different balls in $\mathcal{B}_k$. Furthermore, there are at most $N^{\log_2(8N^2)}$ distinct boundary points in B that are eventually connected to elements of the same ball of $\mathcal{B}_k$.

Now we are ready to prove the assertions (1) to (3) for the tree T'. For a closed ball $B\in\mathcal{B}_k$ let B' be the union of B and all other (at most N^2) closed balls in $\mathcal{B}_k$ with distance at most $8N^2\varepsilon_k$ to B.

Because of (j) we just have prove that any ray that we created without our knowledge in the limit step converges to some hyperbolic boundary point. Let π be such a ray in T'. That is, there is an infinite subset $I\subseteq\mathbb{N}$ such that, for every $i\in I$, we have $\pi\cap T_i\neq\pi\cap T_{i+1}$. Since $\widehat{G}$ is compact, the ray π has at least one limit point η in ∂G. Thus we have to prove that there exists no second limit point. Let $B_k\in\mathcal{B}_k$ be one of the closed balls in step k which contains η. Any second boundary point must lie—like η does—in $\bigcap_{k\in\mathbb{N}}B'_k$ by Claim 8.17. Since $\bigcap_{k\in\mathbb{N}}B'_k$

is a set with at most one element, π has precisely one accumulation point. Thus, we have proved (1).

For the proof of (2), let $\eta \in \partial G$. In every construction step k there is at least one closed ball $B_k \in \mathcal{B}_k$ with $\eta \in B_k$, because $\mathcal{B}_k$ is a cover of ∂G. Hence there is in each step a boundary point $\eta_k \in S_k \cap B_k$ with $d_h(\eta_k, \eta) \leq \varepsilon_k$ such that T_k contains a ray to η_k. Let π_k be a ray from r to η_k in T_k. For every $\varrho \in \mathbb{N}$ there is a path in $T_k \cap \bar{B}_\varrho(r)$ that is contained in infinitely many of the π_k, because G is locally finite. Thus there is a ray π in G such that every vertex on π lies on infinitely many of the rays π_k. Because of Claim 4.3 and the choice of the rays π_k, the hyperbolic boundary point η has to be an accumulation point of π. As (1) holds, π has precisely one accumulation point, η, and thus π converges to η.

For every $B \in \mathcal{B}_k$ in the step k there are at most N^2 closed balls in the step $k-1$ such that a boundary point in $(B \cap S_k) \setminus S_{k-1}$ is eventually connected to a hyperbolic boundary point of such a ball. Furthermore, for each of these balls there are at most $N^{\log_2(8N^2)}$ many hyperbolic boundary points to which our new ones are eventually connected. Thus we know that the number of rays to one boundary point is bounded by $N^2 \cdot N^{\log_2(8N^2)}$ and hence bounded by a function depending only on $\dim_2(\partial G)$. Thus, we have also proved the remaining assertion (3).

4.2. Visual hyperbolic graphs

As in [**15**], we call a hyperbolic graph G *visual* if for some $o \in VG$ there is a $D > 0$ such that for every $x \in VG$ there is an $\eta \in \partial G$ with

$$d(o, x) \leq (x, \eta)_o + D.$$

Remark that the property for hyperbolic spaces to be visual is independent of the choice of o.

An observation is that the definition of visual hyperbolic graphs is equivalent to the following. For some (and hence every) $o \in VG$ there is a $D' > 0$ such that for every $x \in VG$ there is an $\eta \in \partial G$ such that any geodesic between o and x lies in a D'-neighborhood of a geodetic ray from o to η.

Remark that by [**16**, Corollary 1.3.5.] hyperbolicity is preserved by quasi-isometries and it is not hard to see that the same holds for visual hyperbolicity.

4.2.1. Hyperbolic approximations of metric spaces. In [**16**], see also [**11, 15, 33**], the authors construct for every metric space X a hyperbolic graph Y whose hyperbolic boundary is homeomorphic to X. The hyperbolic graph Y is called a *hyperbolic approximation of* X. That Y is indeed a hyperbolic graph is shown in [**16**, Proposition 6.2.10]. If we assume that the space X is compact, then the hyperbolic approximation is a locally finite graph (compare [**16**, Exercise 6.4.4]). Furthermore one can see from the construction that Y is visual hyperbolic, since any vertex of Y lies on an infinite geodetic ray that starts at the root of the hyperbolic approximation.

PROPOSITION 4.4. [**16**, Proposition 6.2.10] *A hyperbolic approximation Y of any metric space X is a visual hyperbolic graph with $\partial Y \cong X$.* □

If we restrict the metric space X to be doubling, then the degrees of all the vertices in a hyperbolic approximation of X are uniformly bounded which is shown in [**16**, Proposition 8.3.3]. We combine this result with Proposition 4.4 and obtain the following proposition.

PROPOSITION 4.5. *For any doubling metric space X, its hyperbolic approximation Y is a locally finite visual hyperbolic graph with $\partial Y \cong X$ and with bounded degree.* □

4.2.2. Rough similarities. We cite a result by Buyalo and Schroeder [**16**]. In order to do that we have to make a further definition.

Let G, H be two graphs. If there are a map $f : VG \to VH$ and constants $k, \lambda > 0$ such that

$$|\lambda d_X(x, y) - d_Y(f(x), f(y))| \leq k$$

holds for all $x, y \in VG$ and $\sup_{y \in VH} d_Y(y, f(VG)) \leq k$, then G is (λ, k)-*roughly similar* to H, or just *roughly similar* to H, and we call f a (λ, k)-*rough similarity*, or just a *rough similarity*.

In particular, every graph H that is roughly similar to a graph G is also quasi-isometric to G. As (visual) hyperbolicity is preserved by quasi-isometries, it is also preserved by rough similarities. The next theorem guarantees that every hyperbolic graph contains a visual hyperbolic graph with the same hyperbolic boundary.

THEOREM 4.6. [**16**, Corollary 7.1.5.] *Every visual hyperbolic graph G is roughly similar to a subgraph of a hyperbolic graph H with the same hyperbolic boundary, $\partial G = \partial H$.* □

We deduce the following corollary from the previous theorem.

COROLLARY 4.7. *Let G be a locally finite hyperbolic graph whose hyperbolic boundary is doubling. Let $\gamma_1 \geq 1, \gamma_2 \geq 0$ be constants. Then there is a subgraph H of G such that the following statements hold for H.*

(1) *H is a proper visual hyperbolic geodetic space;*
(2) *every (γ_1, γ_2)-quasi-geodetic ray of G lies eventually in H;*
(3) *the identity $\iota : H \to G$ extends to a homeomorphism $\hat{\iota} : \widehat{H} \to \widehat{G}$ such that $\hat{\iota}(\partial H) = \partial G$.*

PROOF. Let Z be a visual hyperbolic locally finite graph that is a hyperbolic approximation of ∂G. Let Z' be a graph of G that is (λ, k)-roughly similar to Z for some constants $\lambda \geq 1, k \geq 0$. This subgraph exists by Theorem 4.6. Let H be the subgraph of G that is induced by Z' and all vertices with distance at most $\kappa(\delta, \gamma_1, \gamma_2) + 2\,\kappa(\delta, \lambda, k)$ to any vertex of Z' for the constants $\kappa(\delta, \gamma_1, \gamma_2), \kappa(\delta, \lambda, k)$ of Proposition 2.24. As G is locally finite, the same holds for Z' and H. As Z is visual hyperbolic and this is a property that is preserved by quasi-isometries, assertion

(1) holds for Z' and thus also for H as the identity from Z' to H is a quasi-isometry by the choice of H. The assertion (2) holds because of Proposition 2.24 and the last one, (3), is obvious, because quasi-isometries between locally finite hyperbolic graphs can be extended to quasi-isometries between their hyperbolic compactifications. □

4.3. Tree-likeness of hyperbolic graphs

In this section we shall prove that every ray in the tree, that was constructed in Section 4.1 and that starts at the root, is a quasi-geodetic ray for some global constants.

LEMMA 4.8. *Let G be a locally finite hyperbolic graph whose hyperbolic boundary has finite Assouad dimension and let T be the spanning tree with root r that was constructed in Section 4.1. Then there exist constants $\gamma_1 \geq 1, \gamma_2 \geq 0$ such that every ray in T starting at the root is a (γ_1, γ_2)-quasi-geodetic ray in G.*

PROOF. We use all the assumptions and notations as in the construction step j of Section 4.1. By Proposition 2.28 there is a constant β depending only on the quotient $\frac{\varepsilon_j}{\varepsilon_{j-1}}$ and not depending on the particular j such that for every four boundary points $\eta_1, \eta_2, \eta_3, \eta_4$ with

$$\varepsilon_{j-1} \geq d_h(\eta_1, \eta_2), d_h(\eta_3, \eta_4) \geq \varepsilon_j$$

there is

$$|d(r, [\eta_1, \eta_2]) - d(r, [\eta_3, \eta_4])| \leq \beta.$$

Recall that $\beta \geq Q - q$ with Q, q as defined in Section 4.1. In the first step of the proof we shall prove that for every two vertices w, y with $y \in T_j^{i,k} \setminus T_j^{i-1,k}$, $w \in T_j^{i,k} \cap [r, y]_T$ there is

$$d_Y(w, y) \leq d(w, y) + (M + n)(75\delta + 4\beta)$$

with $Y := T_j^{i,k}$ for an $n < M := N^{2+\log_2(8N^2)}$ that represents the number how often we have already enlarged the tree T_{j-1} by additional rays whose intersection with $[r, y]_T$ is not empty. As we have proved, n is bounded by M and since we add just in this step a ray, we have $n < M$.

Let $y \in Y$ and let R be the geodetic double ray, as in the recursion step and let P be that geodetic ray with $P \subseteq R$ that we added together with π_R to $T_j^{i-1,k}$ to obtain $T_j^{i,k}$. Let x be the unique vertex in $T_j^{i-1,k} \cap \pi_R$, let x' be the unique vertex in $\pi_R \cap P$, and let a be a vertex on R with minimal distance to r. By the choice of π_R we have $d(x, x') \leq \delta$ as we already saw in Section 4.1.

Let $\eta := \mu_k^i$ and let μ be the other limit point of R. Let $b \in R$ with $d(b, [r, \eta]) \leq \delta$ and $d(b, [r, \mu]) \leq \delta$.

CLAIM 4.9. $d(a, b) \leq 4\delta$.

Proof of Claim 4.9. Let $c \in [r, \eta]$ and $c' \in [r, \mu]$ each with minimal distance to b. By the choice of b we have $d(b, c), d(b, c') \leq \delta$.

Because of the hyperbolicity of G the geodetic double ray R is contained in the δ-neighborhood of the subset $Z := [\eta, c] \cup [c, b] \cup [b, c'] \cup [c', \mu]$ of G. In particular we have $d(a, Z) \leq \delta$. Thus, there is a vertex on Z with distance at most $d(r, a) + \delta$ to r. Let $a' \in Z$ with $d(a, a') \leq \delta$. Then there is $d(r, a') \leq d(r, a) + \delta$. By symmetry we may assume that $a' \in [\eta, c] \cup [c, b]$. If $a' \in [c, b]$, then we have $d(a', c) \leq \delta$. Otherwise, since c is the vertex on $[r, \nu] \cap Z$ with minimal distance to r, and since $d(r, c), d(r, c') \geq d(r, a) - \delta$, we have $d(a', c) \leq 2\delta$. The inequality

$$d(a, b) \leq d(a, a') + d(a', c) + d(c, b) \leq \delta + 2\delta + \delta = 4\delta$$

proves Claim 4.9. □

For any another vertex $\widehat{a}$ on R with distance $d(r, a)$ to r we conclude from Claim 4.9 that $d(a, \widehat{a}) \leq 8\delta$.

Claim 4.10. *$d(a, x') \leq \beta + 15\delta$.*

Proof of Claim 4.10. There is a unique vertex x'' with $d(r, x'') = d(r, a) - \delta$ that lies on $[r, x']$. In particular we have $d(x', x'') \leq \beta + 6\delta$. Since G is hyperbolic, there is a vertex on $[r, a] \cup [a, x']$ with distance at most δ to x''. If this vertex lies on $[r, a]$, then $d(x'', a) \leq 3\delta$, and if it lies on $[a, x']$, then it has the same distance to r as a and hence distance at most 8δ to a. Thus $d(x'', a) \leq 9\delta$. Hence we proved Claim 4.10. □

Let a_w be a vertex on R with $d(w, a_w) = d(w, R)$.

Claim 4.11. *$d(w, y) \geq d(w, a_w) + d(a_w, y) - 6\delta$.*

Proof of Claim 4.11. Let y' a vertex on $[w, y]$ with distance at most δ to both $[w, a_w]$ and $[a_w, y]$. Let y_1 be such a vertex on $[w, a_w]$ and y_2 such a vertex on $[a_w, y]$. Then $d(w, y_2) \geq d(w, a_w)$ and hence $d(y_1, a_w) \leq 2\delta$. This immediately implies $d(w, y') \geq d(w, a_w) - 3\delta$ and also $d(y, y') + 3\delta \geq d(a_w, y)$. Hence we proved Claim 4.11 □

Claim 4.12. *$d(a, a_w) \leq 19\delta + \beta$.*

Proof of Claim 4.12. Since G is a hyperbolic graph, we conclude that $[a, a_w]$ lies in a 2δ-neighborhood of $[a, r] \cup [r, w] \cup [w, a_w]$. But as $d(r, a) = d(r, R)$ and $d(w, a_w) = d(w, R)$, a subpath of length at most 4δ of $[a, a_w]$ lies in the 2δ-neighborhood of $[r, a]$ and a subpath of length at most 4δ lies in the 2δ-neighborhood of $[w, a_w]$. The vertex w lies in $[r, y] \cap T_j^{i-1,k}$ and hence $d(r, w) \leq d(r, a) + \beta + 5\delta$. So there is a subpath of length at most $11\delta + \beta$ of $[a, a_w]$ in a 2δ-neighborhood of $[r, w]$. Thus, $d(a, a_w)$ is at most $19\delta + \beta$. This proves Claim 4.12. □

Let a'_w be a vertex on P with minimal distance to w. By analog arguments as in Claim 4.10 we have $d(a_w, a'_w) \leq \beta + 15\delta$. Thus, we conclude

$$\begin{aligned}
d_Y(w,y) &= d_Y(w,a'_w) + d_Y(a'_w,y) \\
&\leq d_Y(w,x) + d(x,x') + d(x',a) + d(a,a_w) + d(a_w,a'_w) \\
&\quad + d(a'_w,y) \\
&\leq d_Y(w,x) + \delta + \beta + 15\delta + \beta + 19\delta + d(a_w,y) \\
&\leq d_Y(w,x) + 34\delta + 2\beta + d(a_w,y) \\
&\leq d(w,x) + ((j-(j'+1))M+n)(\alpha_1\delta + \alpha_2\beta) \\
&\quad + 34\delta + 2\beta + d(a_w,y) \\
&\leq d(w,a_w) + d(a_w,a) + d(a,x) + d(a_w,y) + 34\delta + 2\beta \\
&\quad ((j-(j'+1))M+n)(\alpha_1\delta + \alpha_2\beta) \\
&\leq d(w,a_w) + d(a_w,y) + 19\delta + \beta + \beta + 16\delta + 34\delta + 2\beta \\
&\quad ((j-(j'+1))M+n)(\alpha_1\delta + \alpha_2\beta) \\
&\leq d(w,y) + 6\delta + 69\delta + 4\beta + ((j-(j'+1))M+n)(\alpha_1\delta + \alpha_2\beta) \\
&\leq d(w,y) + ((j-(j'+1))M+(n+1))(\alpha_1\delta + \alpha_2\beta)
\end{aligned}$$

with $\alpha_1 = 75$ and $\alpha_2 = 4$. And in particular we have

$$d_Y(w,y) \leq d(w,y) + ((j-(j'+1))M+(n+1))(75\delta + 4\beta).$$

Let π be a ray in T that starts at r. Since every step affects at most its previous and its successive step directly, there are constants c_1, c_2 (independent from the choice of π) such that π is a (c_1, c_2)-quasi-geodetic ray. For example, we may choose $c_1 = (M+1)\cdot(75\delta+4\beta)$ and $c_2 = 0$. $\square$

This lemma enables us to prove the main results of this part of the thesis. We shall prove them in two different cases. First, we prove the result for locally finite visual hyperbolic graphs and then for arbitrary locally finite hyperbolic graphs.

4.3.1. The case: visual hyperbolic graphs. Visual hyperbolic graphs seem to have a treelike-structure, because there is a maximal distance from each point to any ray that starts in one particular vertex. This in fact is the main reason why spanning trees similar to those constructed in Section 4.1 point out the tree-likeness of locally finite visual hyperbolic graphs. This is specified in Theorem 4.13.

THEOREM 4.13. *Let G be a locally finite visual hyperbolic graph whose hyperbolic boundary has finite Assouad dimension. Then there is a rooted spanning tree T in G such that the canonical map γ from ∂T to ∂G exists and has the following properties.*

(i) *It is surjective;*
(ii) *there is a constant $M < \infty$ depending only on the Assouad dimension of ∂G such that $\gamma^{-1}(\eta)$ has at most M elements for each $\eta \in \partial G$;*
(iii) *there is a constant $\Delta < \infty$ depending only on δ and on the Assouad dimension of ∂G such that every vertex of G has distance at most Δ to a ray in T that starts at the root of T.*

Proof. Let T be the spanning tree of G constructed in Section 4.1 and let r be its root. A direct consequence of the assertions (1) to (3) in Section 4.1 is, that T has the properties (i) and (ii). For the remainder of this proof we remember from Lemma 4.8 that there exist constants c_1, c_2 such that each ray in T that starts at the root is a (c_1, c_2)-quasi-geodetic ray. Because G is visual hyperbolic, there is a $D > 0$ such that for every $x \in VG$ there is an $\eta \in \partial G$ with $d(x, \pi) \leq D$ for all geodetic rays π from r to η. Let π_x be a vertex on π with $d(x, \pi_x) \leq D$. In T there is a ray π_T from r converging to η. We know from Proposition 2.24 that there is a vertex x_T on π_T with $d(\pi_x, x_T) \leq \kappa$ for a constant κ that depends only on δ, c_1, and c_2. Hence we have $d(x, x_T) \leq \kappa + D$ and we have proved the remaining assertion (iii). $\square$

4.3.2. The case: hyperbolic graphs. The final aim of this section is to demonstrate the tree-likeness of locally finite hyperbolic graphs in terms of spanning trees. For that we combine the result for the locally finite visual hyperbolic graphs with the theorems from Section 4.2.

Before we can state the result, we have to make a further definition. A subset X of the vertex set of a hyperbolic graph G has *finite geodetic out-spread* in G if every geodesic in $G - X$ has finite length.

Theorem 4.14. *Let G be a locally finite hyperbolic graph whose hyperbolic boundary has finite Assouad dimension. Then there is a rooted spanning tree T of G such that the canonical map γ from ∂T to ∂G exists and has the following properties.*

(i) *It is surjective;*
(ii) *there is a constant $M < \infty$ depending only on the Assouad dimension of ∂G such that $\gamma^{-1}(\eta)$ has at most M elements for each $\eta \in \partial G$.*

Furthermore, there is a constant Δ such that, for the subtree $\mathcal{T}$ of T that consists of all rays of T starting at the root, $\bar{B}_\Delta(\mathcal{T})$ has finite geodetic out-spread in G.

Proof. By Corollary 4.7, there is a locally finite visual hyperbolic subgraph H of G which has the property that every geodesic in $G-H$ has finite length, so H has finite geodetic out-spread in G. In H there is a spanning tree T' as in Theorem 4.13. For this tree the canonical map $\partial T' \to \partial G$ exists and is surjective and continuous by property (i) of Theorem 4.13. Furthermore, (ii) also holds because it holds for T' and H. Now, we can add all the vertices in $G - H$ with appropriate paths to T' as in Section 4.1 to obtain a spanning tree T of G all whose rays lie eventually in T'. Thus, the spanning tree T has the properties (i) and (ii).

The remaining part is a consequence of the fact that H has already finite geodetic out-spread in G, so for the tree $\mathcal{T}$ obtained from T' by taking all rays starting at the root, we conclude that $\bar{B}_{\Delta'}(\mathcal{T})$ (with the constant Δ' from Theorem 4.13) has finite geodetic out-spread in G, because $G - H \supseteq G - \bar{B}_\Delta(\mathcal{T})$. $\square$

4.4. Spanning trees and the topological dimension of the boundary

Before we compare the topological dimension of the hyperbolic boundary of a locally finite hyperbolic graph with any spanning tree of the graph, we prove a lemma on quotient spaces of totally disconnected compact metric spaces.

LEMMA 4.15. *Let X be a compact metric space such that there exists a totally disconnected compact metric space Y and an equivalence relation $\sim$ on Y with at most $M < \infty$ elements in each equivalence class such that X and $Y/\sim$ are homeomorphic. Then X has topological dimension at most $M-1$.*

PROOF. Let $\mathcal{U}$ be a finite critical open cover of X. Let $\mathcal{U}'$ be that open cover of Y that is induced by $\mathcal{U}$, that is a $U \in \mathcal{U}$ corresponds to precisely one $U' \in \mathcal{U}'$ and $y \in U'$ if and only if $[y] \in U$ (where we assume that $X = Y/\sim$). As Y has topological dimension 0, there is a finite open cover $\mathcal{V}'$ of $\mathcal{U}'$ with pairwise disjoint elements, since it is a well-know fact, that any totally disconnected compact metric space has topological dimension 0. For any $V' \in \mathcal{V}'$ let V be the set of all $[y]$ with $y \in V'$. Let $\mathcal{V}$ be the set of all such sets V for $V' \in \mathcal{V}'$. Any V is an open set and thus $\mathcal{V}$ is an open cover of X. By the construction $\mathcal{V}$ is also a refinement of $\mathcal{U}$ and has multiplicity at most M. Thus the topological dimension of X is at most $M-1$. □

Now we can easily deduce the main result of this section from Lemma 4.15.

THEOREM 4.16. *Let G be a locally finite hyperbolic graph and let T be a spanning tree of G such that the canonical map from ∂T to ∂G exists and is surjective and such that there is an $M < \infty$ such that any boundary point of G has at most M preimages. Then the topological dimension of ∂G is at most $M-1$.*

PROOF. Since ∂T and ∂G are compact metric spaces and ∂T is totally disconnected, the theorem is a direct consequence of Lemma 4.15. □

4.5. Remarks on trees in hyperbolic graphs

Our two main results of this chapter (Theorem 4.14 and Theorem 4.16) depend on distinct dimension concepts. We know that $\dim(X) \le \dim_A(X)$ for any metric space X and that there are compact metric spaces X with $\dim(X) < \dim_A(X)$ (that are the *fractals*, compare with the introduction of [**75**]) In particular, there are compact metric spaces with infinite Assouad dimension but finite topological dimension. This is for example the closure of H^2 equipped with the hyperbolic metric. This is because on the one hand, the topological dimension of H^2, and thus also the one of its closure, is 2 and on the other hand the Assouad dimension of H^2 is ∞ (this is proved in [**75**, Example 3.5.2.]) and the one of its closure is the same (see [**75**, Theorem A.5.(2)]).

Because of this different behavior of the two dimensions for some compact metric spaces, it would be nice if there is one particular dimension such that we have the two following properties for locally finite hyperbolic graphs G whose hyperbolic boundary has finite dimension.

- There is a spanning tree such that Theorem 4.14 holds for this dimension instead of the Assouad dimension.
- For every spanning tree T there is a function $f : \mathbb{N} \to \mathbb{N}$ such that whenever the canonical map $\partial T \to \partial G$ exists and the dimension of the hyperbolic boundary is n, then there is a hyperbolic boundary point of G with at least $f(n)$ many preimages under the canonical map.

Part 2

Group action on boundaries of graphs

CHAPTER 5

The fixed set property

In this part of the thesis we discuss several aspects of the so-called fixed set property. It is a property that has been of interest not only for trees with their ends [**85, 95**] but also for locally finite graphs with their end-compactification [**82, 100**], for arbitrary infinite graphs with their ends [**60, 65**] and even for proper metric spaces with an appropriate compactification, see [**102**]. Let us define this property: A group Γ has the *fixed set property* if there is a graph G with boundary ∂G such that Γ acts on $G \cup \partial G$ and such that one of the following assertions holds.

(a) There is a finite vertex set of G fixed by Γ.
(b) There is a unique element of ∂G fixed by Γ.
(c) There is a unique pair of elements of ∂G invariant under Γ that are the directions of a hyperbolic element and its inverse.

In analogy to the statements (a), (b) and (c), we define distinct classes of elements of the automorphism group of a graph G, where ∂G is again an appropriate boundary for G.

(i) An automorphism is called *elliptic* if it fixes a finite set of vertices.
(ii) An automorphism α is called *hyperbolic* if it is not elliptic and fixes precisely two boundary points η, ξ and if $(x^{\alpha^n})_{n\in\mathbb{N}}$ converges to η and $(x^{\alpha^{-n}})_{n\in\mathbb{N}}$ converges to ξ for every $x \in VG$.
(iii) An automorphism α is called *parabolic* if it is not elliptic and fixes precisely one boundary point η and if $(x^{\alpha^n})_{n\in\mathbb{N}}$ and $(x^{\alpha^{-n}})_{n\in\mathbb{N}}$ both converge to η for every $x \in VG$.

If α is a hyperbolic element, then we call the boundary point to which all the sequences (x^{α^n}), $x \in VG$, converge the *direction* of α.

The following theorem holds for the ends of G as boundary and also in the case that G is a locally finite hyperbolic graph for the hyperbolic boundary as ∂G. Woess [**102**] offers a proof in the hyperbolic case. The case for arbitrary graphs is due to Jung [**60**, Corollary 1.3 and Theorem 1.4].

Theorem 5.1. *Let G be a graph with boundary ∂G and let Γ act on G. Then either one of* (a)*,* (b)*,* (c) *holds or there are two hyperbolic elements that generate a free subgroup of* Γ. $\square$

Considering just single elements instead of the whole automorphism group, in all above described cases for G and ∂G we have also the following theorem, see [**49, 95, 102**].

THEOREM 5.2. *Let G be a graph with boundary ∂G and let Γ act on G. Then each element of Γ is either elliptic, hyperbolic, or parabolic.* $\square$

The fixed set property will be discussed in three different situations. In Chapter 6, we investigate the case that G is a locally finite hyperbolic graph and for a subgroup Γ of $\mathrm{Aut}(G)$ that acts transitively on G, the statement (b) holds as in [**39**]. The case that G has more than one end turns out to be already solved [**80**], so we just have to look at locally finite one-ended hyperbolic graphs. It is a problem of Kaimanovich and Woess [**61**] whether such a graph exists. We make some progress by showing that no planar such graphs exists (Theorem 6.3).

In Chapter 7, we generalize the mentioned result by Möller [**80**] by characterizing the infinitely-ended graph with arbitrary degree and with a subgroup of their automorphism group that acts transitively on the graph and satisfies the assertion (b). We show that they are—in analogy to the locally finite case—quasi-isometric to semi-regular trees (Theorem 7.1), see also [**53**].

The situation in Chapter 8 is a different one compared with the other two. We suppose—just like in [**53**]—that for an infinitely-ended graph (again with arbitrary degree) its automorphism group satisfies neither (b) nor (c). Particularly, we assume the automorphism group to act transitively on the ends of the graph. Then we conclude that either a generalization of (a) holds, that is that a vertex set of finite diameter is invariant under the automorphism group, or that the graph is again similar to a semi-regular tree (Theorem 8.1).

CHAPTER 6

Fixed set property in hyperbolic graphs

In this chapter we describe our partial solution [**39**] to the following problem of Kaimanovich and Woess:

Problem. [**61**, Section 6.4] *Does there exist a one-ended locally finite hyperbolic graph and a group acting transitively on the graph and fixing precisely one hyperbolic boundary point?*

The assumption having just one end seems to be a huge restriction. But the stabilizer of the hyperbolic boundary point also stabilizes the end that contains the hyperbolic boundary point. Such graphs with more than one end have infinitely many or precisely two ends, see Theorem 2.3. Locally finite graphs with infinitely many ends and with an end whose stabilizer acts transitively on the vertices of the graph are characterized in [**80**] before the above question arose. It was shown that they are quasi-isometric to trees (see Chapter 7 for more details). So, by Theorem 2.3, the only other case is that the graph has precisely two ends. But it is well-know that two-ended transitive graphs are quasi-isometric to the double ray. Hence, the only case that has to be discussed is the one-ended case as it is posed in the problem.

In this chapter we give some further results towards a general solution of the above problem. The first known result regarding the situation of that problem can be found in [**102**, Section 4.D]. It is shown there that, for a locally finite Cayley graph of a finitely generated hyperbolic group, the group itself does not have the property (b) on its Cayley graph.

We shall prove that every planar locally finite hyperbolic graph answers Question 6 in the negative. Before we directly attack the question in the situation of planar graphs, we prove a general lemma, which might help to find a solution in the general case for a negative answer to the question. But, so far, it still remains open.

Lemma 6.1. *Let G be a locally finite hyperbolic graph. Then* $\mathrm{Aut}(G)$ *contains no parabolic element.*

Proof. Suppose that $\mathrm{Aut}(G)$ contains a parabolic element α. Then there is an $\omega \in \partial G$ such that for any $x \in VG$ the sequences $(x^{\alpha^i})_{i\in\mathbb{N}}$ and $(x^{\alpha^{-i}})_{i\in\mathbb{N}}$ both converges to ω and they lie on a (c_1, c_2)-quasi-geodetic double ray by Proposition 2.25 for some $c_1 \geq 1, c_2 \geq 0$. This double ray lies in a $\kappa(\delta, c_1, c_2)$-neighborhood around a geodetic double ray for the constant $\kappa(\delta, c_1, c_2)$ as in Proposition 2.24. Since any hyperbolic boundary point is the equivalence class of geodetic rays, no geodetic double ray can have both of its sides converging to the same hyperbolic boundary point. Thus, we conclude that no parabolic element of $\mathrm{Aut}(G)$ exists. $\square$

LEMMA 6.2. *Let G be a locally finite one-ended δ-hyperbolic graph and Γ be a group acting transitively on G such that Γ fixes a hyperbolic boundary point ω of G. Then the following statements hold.*

(i) *For every two vertices $x, y \in VG$ with $d(x, y) > 2\delta$ that lie on a common geodetic double ray between ω and another hyperbolic boundary point, there exists a hyperbolic element h in Γ with $x^h = y$.*
(ii) *There exists a non-trivial elliptic element in Γ that fixes a vertex of G.*
(iii) *There exist two non-trivial distinct elliptic elements in Γ whose product is also non-trivial and elliptic and such that all these three automorphisms fix a common vertex of G.*

PROOF. To prove (i) let x, y lie on a common geodetic double ray π as in the assertion with $d(x, y) = 2\delta + d$ for a $d > 0$ and let π_0 be the subray of π that starts at y and converges to ω. As Γ acts transitively on G, there is an automorphism $\alpha \in \Gamma$ with $x^\alpha = y$. We deduce from Theorem 5.2 and Lemma 6.1 that α is either a hyperbolic or an elliptic element. So we just have to exclude the case that α is an elliptic element. Let us suppose the contrary, that is, that α is an elliptic element. Then the orbit of x under α is finite. Let $n > 0$ be minimal with $x^{\alpha^{n+1}} = x$. We consider the rays $\pi_0^{\alpha^i}$ with $i = 0, \ldots, n$. Each of these rays converges to ω and contains the vertices $x^{\alpha^{i+1}}$ and $x^{\alpha^{i+2}}$. For x^{α^i}, there is a vertex z_i on $\pi_0^{\alpha^{i+1}}$ with $d(x^{\alpha^i}, z_i) \leq \delta$, as we are δ-hyperbolic. Because of $d(x^{\alpha^i}, x^{\alpha^{i+1}}) = 2\delta + d$ we have $d(x^{\alpha^{i+1}}, z_i) \geq \delta + d$. Furthermore, z_i does not lie on $x^{\alpha^{i+2}}\pi_0^{\alpha^{i+2}}$. Inductively, $x^{\alpha^{i+1}}$ lies in an $(n\delta)$-neighborhood of a vertex on $\pi_0^{\alpha^i}$ whose distance on that ray is at least $n(\delta + d) > n\delta$, a contradiction. Hence, α has to be a hyperbolic element.

For the proof of (ii), let α be a hyperbolic element in Γ. Then α fixes ω and precisely one further boundary point η_0. We assume that the direction of α is η_0. For any $x_0 \in VG$, there are constants $c_1 \geq 1, c_2 \geq 0$ such that the vertices $x_0^{\alpha^i}$, $i \in \mathbb{Z}$, lie on a (c_1, c_2)-quasi-geodetic double ray π by Proposition 2.25. Remark that c_1 and c_2 depend only on δ, and $d(x_0, x_0^\alpha)$. If x_0 lies on a geodetic double ray π_0 between ω and η_0, then we may choose the two constants independently from x_0. Let $\kappa = \kappa(\delta, c_1, c_2)$ be the constant from Proposition 2.24, that is, every (c_1, c_2)-quasi-geodesic lies in a κ-neighborhood of a geodesic with the same endpoints and vice versa.

Let $x_1 \in VG$ with $d(x_1, \pi_0) > 2\kappa$ and let $\gamma_1 \in \Gamma$ with $x_0^{\gamma_1} = x_1$. Then $\pi_1 := \pi_0^{\gamma_1}$ cannot lie 2κ-close to a geodetic double ray between ω and η. Thus, we have $\eta_1 := \eta_0^{\gamma_1} \neq \eta_0$. Since α is a hyperbolic element, so is $\alpha_1 := \gamma^{-1}\alpha\gamma$. Recursively, we find a sequence $(x_i)_{i\in\mathbb{N}}$ of vertices in G, a sequence $(\pi_i)_{i\in\mathbb{N}}$ of (c_1, c_2)-quasi-geodetic double rays, a sequence $(\eta_i)_{i\in\mathbb{N}}$ of hyperbolic boundary points, and a sequence $(\alpha_i)_{i\in\mathbb{N}}$ of hyperbolic elements of Γ such that the orbit of x_i under α_i lies on π_i, such that the subrays of π_i converge either to ω or to η_i, and such that x_i has distance more than 2κ to all π_j with $j < i$.

Consider an infinite sequence $(B_i)_{i\in\mathbb{N}}$ of balls of radius 2κ around elements of π_0 that converge to ω. Because Γ acts transitively on VG, there is a finite number n such that each of these balls consists of n vertices. Hence there is a ball B_m such that some vertex $b \in B_m$ lies on two distinct double rays π_i, π_j (with $i < j$) and such that it has the property $b^{\alpha_i^{-1}} = b^{\alpha_j^{-1}}$ and such that b and x_i, b and x_j lie in the same α_i-orbit, the same α_j-orbit, respectively. Let us look at the automorphism $\alpha_i^{-1}\alpha_j$. This automorphism obviously fixes b, so it is an elliptic element and it is non-trivial, because $\alpha_i \neq \alpha_j$.

It remains to prove (iii). We continue with the same notation as in the proof of (ii). Let $\gamma := \alpha_i^{-1}\alpha_j$ be the elliptic element we constructed in the proof of (ii). Then, for each $k \in \mathbb{N}$, $\gamma_k := \alpha^k\gamma\alpha^{-k}$ is an elliptic element that is not trivial but acts trivially on $b\alpha^{-k}$. Each such elliptic element has to act on the set of (c_1, c_2)-quasi-geodetic rays from $b\alpha^{-k}$ to ω. Let us consider the sequence of balls $(D_k)_{k\in\mathbb{N}}$ with center $b\alpha^{-k}$ and radius 2κ. As in the proof of (ii), there is an $m \in \mathbb{Z}$ such that two distinct γ_k, γ_l, with $k \neq l$, both fix a vertex $y \in B_m$. Then $\gamma_k^{-1}\gamma_l$ also fixes y and it is again non-trivial because $\gamma_k \neq \gamma_l$. So $\gamma_k^{-1}\gamma_l$ satisfies the assertion (iii). $\square$

THEOREM 6.3. *There exists no planar locally finite one-ended hyperbolic graph G and a subgroup Γ of the automorphism group of G such that Γ acts transitively on G and fixes a hyperbolic boundary point ω of G.*

PROOF. Let us suppose that there is a planar locally finite one-ended hyperbolic graph and a subgroup Γ of $\mathrm{Aut}(G)$ acting transitively on G and fixing a hyperbolic boundary point ω. Let δ be the hyperbolicity constant of G and let d be the degree of any vertex of G. Then we have $d \geq 3$ and, thus, so we know from Lemma 2.6 and Theorem 2.5 that G has a unique planar embedding. Hence, every automorphism of the graph can be extended to an automorphism of the plane and we have the notions of spin-preserving and spin-reversing automorphisms. Let uvw be a subpath of a path P. We say that a vertex $x \in N(v) \setminus \{u, w\}$ lies *to the right* of P if in the spin of v, we have vx between vw and vu. If x does not lie to the right of P then it lies *to the left* of P.

There is a non-trivial elliptic element $\varphi \in \Gamma$ by Lemma 6.2. Let us assume that its homeomorphic extension to the plane is a rotation, that is, it is a spin-preserving automorphism of the graph. Let y be a vertex with $y^\varphi \neq y$. As φ is elliptic, there is a minimal $n \in \mathbb{N}$ such that $y^{\varphi^n} = y$. For all $i = 1, \ldots, n$, let g_i be a geodesic from $y_{i-1} := y^{\varphi^{i-1}}$ to y_i such that $g_i^\varphi = g_{i+1}$ and let π_i be a geodetic ray from y^{φ^i} converging to ω such that $\pi_i^\varphi = \pi_{i+1}$. Then $C := g_0 \ldots g_{n-1}$ is a cycle of length $n \cdot l(g)$. We distinguish between two cases: Either π_i and π_{i+1} intersect infinitely many often or in at most finitely many vertices. If they have at most finitely many common vertices, then we may choose a vertex y' on π_0 instead of y such that the corresponding rays $y_i\pi_i$ and $y_{i+1}\pi_{i+1}$ have no common vertex. So we assume that this holds for y.

For every vertex z_0 on π_0 with distance larger than $l(g_1)+\delta$ to y_0 there is a vertex z_1 on π_1 with distance at most δ to z_0. We fix a geodesic between z_0 and z_1 whose intersection with π_0, π_1, respectively, is a connected subpath. Then we may assume that the geodesics from all these vertices to π_1 lie eventually on the same side of π_0 (that is its first vertex not on π_0 lies either to the right or to the left and for all the vertices it is the same), because otherwise we may assume that infinitely many of these geodesics lie eventually to the right and infinitely many lie eventually to the left of π_0. This implies that all the rays of G converges to ω or that G has at least two ends each of which is a contradiction by assumption and because any transitive locally finite hyperbolic graph has at least two hyperbolic boundary points. We may assume that all the above described geodesics lie eventually to the right of π_0. Let V_0 be a subset of VG consisting of all the vertices from the paths π_0, π_1, g_0 and from all the paths from π_0 whose first vertex lie to the right of π_0 and that has only vertices not in $\pi_0\cup\pi_1\cup g_0$ except for its first vertex. Then any ray in $G[V_0]$ has to converge to ω. Similarly we find $V_1,\ldots,V_{n-1}$, always taking the vertices to the right of π_i to obtain V_i, because α acts on the plane as a rotation. We conclude that any other hyperbolic boundary point is separated by C from ω which is impossible, since G has at most one end.

Thus, the only case left is that π_0 and π_1 have infinitely many common vertices. By Proposition 2.23 there is a $k\in\mathbb{N}$ such that for all but finitely many vertices x on π_0 we have $d(x,x^\varphi)\leq k+2\delta$. Let us first suppose that there are also infinitely many vertices in $\pi_0-\pi_1$. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of pairwise distinct vertices on $\pi_0\cap\pi_1$ such that the predecessor of x_i on π_0 lies not in π_1 and such that π_1 comes at x_i from the same side to π_0, say from the right. Then there is an $M\in\mathbb{N}$ such that we have for all $i\geq M$ that $x_i\pi_1x_i\varphi\pi_2\ldots x_i^{\varphi^{n-1}}\pi_0x_i$ is a cycle separating C from ω. Since G has precisely one end, this contradicts the transitivity of G as all these cycles have length at most $2n(k+2\delta)$, as $d(x_i,x_i^{\varphi^{n-1}})\leq n(k+2\delta)$, and we find a sequence of these cycle that are pairwise disjoint.

Thus, there are only finitely many vertices in $\pi_0-\pi_1$ and we may assume by replacing y by another suitable vertex on π_0, that all the vertices of π_0 lie on π_1 or vice versa. But then either

$$\pi_n = y_0\pi_ny_{n-1}\cdots y_1\pi_1y_0\pi_0$$

or

$$\pi_0 = y_0\pi_0y_1\pi_1\ldots y_{n-1}\pi_{n-1}y_0\pi_n$$

is no geodetic ray, contrary to the assumption. $\square$

CHAPTER 7

Fixed set property in graphs with infinitely many ends

In [**101**], Woess posed the problem whether there is a classification of the locally finite connected graphs with infinitely many ends such that the stabilizer of one end acts transitively on the vertices of the graph. He conjectured that such graphs are quasi-isometric to trees, which was subsequently proved by Möller [**80**]. This was generalized by Krön [**63**] to graphs of arbitrary cardinality with infinitely many edge ends. We prove here the corresponding theorem from [**53**] for vertex ends:

THEOREM 7.1. *Let G be a connected graph with infinitely many ends and with automorphism group Γ such that for some end ω of G its stabilizer Γ_ω acts transitively on the vertices of G. Then G is quasi-isometric to a semi-regular tree with minimum degree 2. Furthermore, every non-local end of G is a thin global end of G.*

A further result of Möller [**80**] is that in graphs such that the stabilizer of an end acts transitively on the vertices this stabilizer also acts transitively on the other ends. In graphs that are not necessarily locally finite this is not the case. But every orbit of the ends (other than the fixed end) is dense in the end space:

THEOREM 7.2. *Let G be a connected graph with infinitely many ends and with automorphism group Γ. If Γ_ω for some end ω acts transitively on VG, then for any end $\omega' \neq \omega$ the Γ_ω-orbit of ω' is dense in the set of all ends of G.*

We mention here that—with the above notions—on one side the regular trees are examples of graphs with $\Gamma_\omega \neq \Gamma$, but on the other hand Soardi and Woess gave an example [**92**, Example 2] of a graph with $\Gamma_\omega = \Gamma$ for some end ω.

7.1. The proofs of Theorems 7.1 and 7.2

For the Lemmas 7.3 to 7.7, let G be a connected graph with infinitely many non-local ends such that the stabilizer of some end ω acts transitively on the vertices of G. Let $\Gamma := \mathrm{Aut}(G)$, let $\mathcal{S}$ be a Γ_ω-invariant, not necessarily Γ-invariant, basic cut system such that every $\mathcal{S}$-separation separates ends of G, and let $\mathcal{T}$ be the structure tree of G and $\mathcal{S}$. We remark that the end ω has to be a global end since otherwise it would be dominated by some vertex x and thus by every vertex as Γ_ω acts transitively on the vertices of G.

LEMMA 7.3. *The end ω of G corresponds to an end of $\mathcal{T}$ and not to a vertex of $\mathcal{T}$.*

Proof. Let us suppose that ω corresponds to a block vertex X of $\mathcal{T}$. Let $x \in A \cap B$ for some separation $(A, B) \in \mathcal{S}$ with $A \cap B \subseteq X$. Let y be some vertex of G not in $A \cap B$ which is separated by $A \cap B$ from ω. Since Γ_ω acts transitively on VG, there is some $\alpha \in \Gamma_\omega$ with $y^\alpha = x$. Then $(A \cap B)^\alpha$ is a separator separating $x = y^\alpha$ from ω, a contradiction to the choice of $A \cap B$ and x. Hence ω corresponds to an end of $\mathcal{T}$. $\square$

Lemma 7.4. *The diameters of all $\mathcal{S}$-blocks are globally bounded. Furthermore each vertex of an $\mathcal{S}$-block X has distance at most* 1 *to that separator that separates X from ω.*

Proof. Let X be some $\mathcal{S}$-block and let (A, B) be a separation in $\mathcal{S}$ with $A \cap B \subseteq X$ such that $A \cap B$ separates X from ω. We will prove that any vertex $x \in X \setminus (A \cap B)$ has distance 1 to $A \cap B$.

Suppose $d(x, A \cap B) = 2$. Let y be a vertex in X with $d(y, A \cap B) = 1$. There is some $\alpha \in \Gamma_\omega$ with $y^\alpha = x$. Then $(A \cap B)^{\alpha^{-1}}$ is a separator with distance 1 to x such that x and ω are separated by $(A \cap B)^{\alpha^{-1}}$. By the choice of x and (A, B) this is a contradiction. Thus we know that $\operatorname{diam}(X) \leq \operatorname{diam}(A \cap B) + 2$. Since all $\mathcal{S}$-separations have the same order, the claim follows. $\square$

A vertex x in G *determines* an $\mathcal{S}$-block (with respect to the end ω) if for the unique $\mathcal{S}$-separator S that separates X from ω and is contained in X there is $x \notin S$ and no $\mathcal{S}$-separator separates x from S.

Lemma 7.5. *For every $\mathcal{S}$-block X there is a vertex x which determines the block X (with respect to the end ω). Additionally for every vertex x there is a unique $\mathcal{S}$-separator S separating x from ω with $x \notin S$, and such that no $\mathcal{S}$-separator separates x from S.*

Proof. These are direct consequences of Section 7 from [**32**]. $\square$

Lemma 7.6. *The group Γ_ω acts transitively on the $\mathcal{S}$-blocks.*

Proof. Let X and Y be $\mathcal{S}$-blocks. By Lemma 7.5 there are vertices $x \in X$ and $y \in Y$ such that x and ω determine X and y and ω determine Y. Let $\alpha \in \Gamma_\omega$ with $x^\alpha = y$. Then $X^\alpha = Y$. $\square$

Let us define the tree order on the vertices of $\mathcal{T}$ with respect to the end $\omega_\mathcal{T}$ of $\mathcal{T}$ that corresponds to ω: $x \leq y$ if and only if x separates y from $\omega_\mathcal{T}$. Let X, Y be the blocks or separators corresponding to x, y, respectively. Then $x \leq y$ if and only if X separates Y from ω in G. As for rooted trees let $\lfloor x \rfloor$ denote all vertices y in $\mathcal{T}$ with $y \geq x$.

Lemma 7.7. *Any global end of G corresponds to an end of $\mathcal{T}$ and vice versa.*

Proof. By Lemma 7.4 any global end ω' of G corresponds to an end of $\mathcal{T}$. Suppose that there is some end of $\mathcal{T}$ whose corresponding end ω' in G is not a global end. Then the end ω' must be dominated by some vertex. Let $S_1, S_2, \ldots$

be a sequence of separators such that S_i separates S_{i-1} from S_{i+1}, S_1 separates S_2 from ω, and every S_i separates ω' from ω. Then there is no infinite pairwise disjoint subsequence of the S_i and hence there is some vertex x contained in infinitely many S_i. We may assume that $x \in S_i$ for all $i \in \mathbb{N}$ and that S_1 is an $\mathcal{S}$-separator that contains x and that is minimal in the tree order with this property. Let $S := S_1$. Since Γ_ω acts transitively on $\mathcal{S}$, the stabilizer of S in Γ_ω acts transitively on the vertices in $\lfloor S \rfloor$ with equal distance to S. There is a ray in $\lfloor S \rfloor$ such that every vertex contains x. Thus every vertex in $\lfloor S \rfloor$ contains an element of S. Hence for every ray in $\lfloor S \rfloor$ that starts in S the intersection of all vertices on that ray is not empty and thus contains an element of S. As Γ_ω acts transitively on VG, every vertex y that is separated from ω by S lies in the intersection of the vertices of a ray of $\mathcal{T}$. Hence all $\mathcal{S}$-separators $S' \in \lfloor S \rfloor$ with $d(S, S') \geq 2i$ have to contain a vertex from the finite set

$$\bigcup \{\overline{S} \in \lfloor S \rfloor \mid d(S, \overline{S}) < 2i\} \setminus \bigcup \{\overline{S} \in \lfloor S \rfloor \mid d(S, \overline{S}) < 2(i-1)\}.$$

But then all $\mathcal{S}$-separators S' with $d(S, S') \geq 2|S|$ have to contain more than $|S|$ vertices which is impossible. $\square$

Let us now prove Theorem 7.2.

Proof of Theorem 7.2. Let $\mathcal{S}$ be a basic cut system and let $\mathcal{T}$ be the structure tree of G and $\mathcal{S}$. By the Lemmas 7.3 and 7.7 all non-local ends are global ends and also $|S|$-thin ends for all $\mathcal{S}$-separators S. Thus it suffices to prove the denseness condition for G. We will first prove this condition for $\mathcal{T}$. Let us identify the unique end in $\mathcal{T}$ that corresponds to the end ω of G with ω. Let $\widetilde{\omega}, \widehat{\omega}$ be any two distinct ends of $\mathcal{T}$ that are both different from ω. We have to show that in any open neighborhood of $\widetilde{\omega}$ there is some Γ_ω-image of $\widehat{\omega}$. It suffices to show this for any neighborhood of the form $\lfloor t \rfloor$ for some $t \in V\mathcal{T}$.

Let x be a vertex on the unique double ray in $\mathcal{T}$ between $\widetilde{\omega}$ and $\widehat{\omega}$ such that x is minimal in the tree order. As Γ_ω acts on $V\mathcal{T}$ with precisely two orbits, there is an automorphism $g \in \Gamma_\omega$ such that either $x^g = t$ or $d(x^g, t) = 1$ and $x^g \in \lfloor t \rfloor$. Since g fixes ω, the end $\widehat{\omega}^g$ has to lie in $\lfloor t \rfloor$. As any open neighborhood of ω contains an end of $\mathcal{T}$ that is different to ω, any neighborhood of ω contains some $\widehat{\omega}^g$ with $g \in \Gamma_\omega$. Thus $\widehat{\omega}^{\Gamma_\omega}$ is dense in the set of non-local ends.

Let $\widetilde{\omega}$ be a local end of G. As for every $\mathcal{S}$-separators S the group Γ_ω acts transitively on those $\mathcal{S}$-separators that have distance 2 in $\mathcal{T}$ to S and that are separated from ω by S, each ray R in $\widetilde{\omega}$ meets at least infinitely many of those $\mathcal{S}$-separators. There is a block vertex $X \in V\mathcal{T}$ such that $\widetilde{\omega}$ corresponds to X. Then X must have infinitely many neighbors in $\mathcal{T}$ and also in G. As $\widehat{\omega}^{\Gamma_\omega}$ is dense in $\Omega\mathcal{T}$, for each finite vertex set S of G there is an end $\widehat{\omega}^g$ with $g \in \Gamma$ that is not separated from X by S and thus that is also not separated from $\widehat{\omega}$.

Now let $\widehat{\omega}$ be a local end of G. By Lemma 7.7 there is an $\mathcal{S}$-block X such that $\widehat{\omega}$ corresponds to X. Let first $\widetilde{\omega}$ be a global end of G. Then for each $\mathcal{S}$-separator S there is an X^g with $g \in \Gamma_\omega$ in the same component of $G - S$ in which $\widetilde{\omega}$ lies. So let

$\widetilde{\omega}$ be a local end of G. Analog to $\widehat{\omega}$, there is an $\mathcal{S}$-block Y such that $\widetilde{\omega}$ corresponds to the $\mathcal{T}$-vertex Y. Since Γ_ω acts transitively on the $\mathcal{S}$-blocks, every finite vertex set can separate Y only from finitely many X^g with $g \in \Gamma_\omega$ and thus $\widetilde{\omega}$ lies in the closure of $\widehat{\omega}$. $\square$

The following theorem immediately implies Theorem 7.1.

THEOREM 7.8. *Let G be a connected graph with infinitely many ends such that for some end ω of G the stabilizer of ω in the automorphism group Γ of G acts transitively on the vertices of G. Then G is quasi-isometric to any structure tree of G and a basic cut system of G which is a semi-regular tree with minimum degree 2. Furthermore, every non-local end of G is a thin global end of G.*

PROOF. Let $\mathcal{S}$ be a basic cut system and let $\mathcal{T}$ be the structure tree of G and $\mathcal{S}$. By Lemma 7.6 the tree $\mathcal{T}$ is a semi-regular tree and by Lemma 7.4 all vertices of $\mathcal{T}$ have bounded diameter in G and thus the claim holds. $\square$

CHAPTER 8

End-transitive graphs

In [**101**], Woess asked for a classification of locally finite connected graphs with infinitely many ends and with an end-transitive automorphism group. Möller [**79**] and Nevo [**86**] independently described these graphs. The essence of their work is that they are quasi-isometric to semi-regular trees. In this chapter we prove a theorem of [**53**] which is a generalization of the results of Möller and Nevo to graphs that are not necessarily locally finite. Before we state the theorem let us briefly define an abbreviation: For a graph G, a vertex $x \in VG$, and $R \in \mathbb{N}$ let $G(x,R)$ denote the union of the balls $B_R(x^\varphi)$, where φ ranges over all automorphisms of G.

Theorem 8.1. *Let G be a connected graph with infinitely many ends such that G is end-transitive. Either* $\mathrm{Aut}(G)$ *fixes a vertex set of finite diameter or there is for every $x \in VG$ an $R \in \mathbb{N}$ such that $G(x,R)$ is quasi-isometric to a tree and $G - G(x,R)$ does not contain a ray.*

The assumptions of Theorem 8.1 (infinitely many ends, end-transitivity, no vertex set of finite diameter fixed by the automorphism group) are necessary. Whenever we omit one of them, the conclusion of Theorem 8.1 fails.

Throughout the remainder of this chapter we shall prove Theorem 8.1 with the aid of the theory of structure trees, see Section 2.5, and we shall discuss the general situation in this setting, for example that the structure tree of a basic cut system is essentially unique (Section 8.3).

8.1. Structure trees and semi-regular trees

In this section we prove that every structure tree with regard to a basic cut system of any connected graph whose automorphism group acts transitively on the non-local ends and fixes no vertex set of finite diameter is a semi-regular tree or a subdivided semi-regular tree.

Although the proof of the following lemma is similar to arguments in [**86, 92**] we proof it here because of the last part claimed.

Lemma 8.2. *Let T be a tree and $\Gamma \leq \mathrm{Aut}(T)$ such that Γ acts transitively on one set A of the natural bipartition $A \cup B$ of VT. Then for every path $x_0 \dots x_4$ of length 4 between two vertices of A there is an automorphism $g \in \Gamma$ such that g is a translation on T and either $x_0^g = x_2$ or $x_0^g = x_4$.*

Proof. There is an automorphism $g \in \Gamma$ with $x_0^g = x_2$. If $x_1^g \neq x_1$ then g is a translation as claimed. So let us assume that $x_1^g = x_1$. There is an automorphism

$h \in \Gamma$ with $x_0^h = x_4$. If $x_2^h \neq x_2$ then h is a translation as claimed. Thus let us assume that $x_2^h = x_2$. Let $f := gh$. Then $x_0^f = x_2$ and $x_1^f \neq x_1$. Hence f is a translation and the lemma is proved. □

LEMMA 8.3. *Let G be a connected graph with infinitely many non-local ends such that $\Gamma := \mathrm{Aut}(G)$ acts transitively on the non-local ends of G. Let $\mathcal{S}$ be a basic cut system such that each $\mathcal{S}$-separation separates metric rays. If Γ fixes no vertex set of finite diameter, then no end of $\mathcal{T}$, the structure tree of G and $\mathcal{S}$, corresponds to a local end of G.*

PROOF. We first remark that for every $n \in \mathbb{N}$ there is a pair of $\mathcal{S}$-separators with distance at least n as otherwise the union of all $\mathcal{S}$-separators is a vertex set of finite diameter. Let us suppose that there is an end of $\mathcal{T}$ that corresponds to a local end of G. Then there is a ray in $\mathcal{T}$ and a vertex x of G such that x lies in all the vertices of that ray as otherwise there are infinitely many disjoint $\mathcal{S}$-separators on that ray and thus the end of $\mathcal{T}$ corresponds to a non-local end of G. Similar to Lemma 8.2 there is an $\alpha \in \Gamma_x$, the stabilizer of x in Γ, that acts on $\mathcal{T}$ like a translation and thus x lies in all vertices of the uniquely determined α-invariant double ray R. If $\mathcal{T}$ has just two ends, then all separators lie on R and thus the intersection of all the separators is non-empty, of finite diameter, and Γ-invariant, but no such vertex set exists by the assumptions. Hence we know that $\mathcal{T}$ has infinitely many ends.

Let S_0, S_1, S_2 be three distinct $\mathcal{S}$-separators such that S_0 and S_1 lie on R, such that S_2 is disjoint from $S_0 \cup S_1$, and such that

$$d_{\mathcal{T}}(S_0, S_1) + d_{\mathcal{T}}(S_1, S_2) = d_{\mathcal{T}}(S_0, S_2).$$

By a similar argument as in the proof of Lemma 8.2 there is an automorphism β of G that acts on $\mathcal{T}$ like a translation either with $S_2^\beta = S_1$ or with $S_2^\beta = S_0$. Hence there is a double ray P in $\mathcal{T}$ and an $m \in \mathbb{N}$ such that each vertex of G lies in at most m vertices of P. The ends of G which contain subrays of P have to correspond to non-local ends of G.

By using the double rays P^γ and R^γ for some $\gamma \in \Gamma$, it is not hard to construct a ray Q such that the $\mathcal{S}$-separators on infinitely many subpaths of length more than m do not intersect trivially but such that every vertex of G lies in only finitely many $\mathcal{S}$-separators on Q. Thus the end that contains Q corresponds to a non-local end of G. But this end cannot be mapped by an automorphism of G onto an end that contains a subray of P. We conclude that no local end of G can correspond to an end of $\mathcal{T}$. □

By changing the cut system to a cut system $\mathcal{S}$ such that every wing of every separation just contains a ray we obtain the following corollary of the proof of Lemma 8.3.

COROLLARY 8.4. *Let G be a connected graph with infinitely many local ends such that $\Gamma := \mathrm{Aut}(G)$ acts transitively on the ends of G. Let $\mathcal{S}$ be a basic cut* *system such that each $\mathcal{S}$-separation separates rays. Then either Γ fixes no vertex set of finite diameter or $\mathcal{T}$, the structure tree of G and $\mathcal{S}$, does not contain any ray.* □

By the next lemma we show that every structure tree of a basic cut system that contains a ray is essentially a semi-regular tree.

LEMMA 8.5. *Let G be a connected graph with infinitely many non-local ends such that $\Gamma = \mathrm{Aut}(G)$ acts transitively on the non-local ends of G and fixes no vertex set of finite diameter. Let $\mathcal{S}$ be a basic cut system of G such that each $\mathcal{S}$-separation separates non-local ends and let $\mathcal{T}$ be the structure tree of G and $\mathcal{S}$. If $\mathcal{T}$ contains some ray, then the set of $\mathcal{S}$-blocks consists of at most two Γ-orbits.*

In particular then there are two different cases: either Γ has precisely two orbits on $V\mathcal{T}$, or the separator vertices in $\mathcal{T}$ have degree 2 and there are precisely three Γ-orbits on $V\mathcal{T}$.

PROOF. Let us first suppose that every $\mathcal{S}$-separator lies in at most 2 $\mathcal{S}$-blocks. Then there are at most two Γ-orbits on $\mathcal{S}$. Thus there are at most two Γ-orbits on the set of $\mathcal{S}$-blocks.

Let us now suppose that every $\mathcal{S}$-separator lies in at least 3 distinct $\mathcal{S}$-blocks. If there are at least two Γ-orbits on the set of $\mathcal{S}$-blocks, we can construct two rays R and P such that the ends ω_R and ω_P defined by R and P, respectively, are not in the same Γ-orbit: There is a ray R such that every fourth vertex lies in some Γ-orbit $\mathcal{X}$ of $\mathcal{S}$-blocks which is avoided completely by a second ray P. As the ends ω_R and ω_P of $\mathcal{T}$ corresponds uniquely to some non-local ends $\widehat{\omega}_R$ and $\widehat{\omega}_P$ of G by Lemma 8.3 and Γ acts transitively on the non-local ends of G, there is some α with $\widehat{\omega}_R^\alpha = \widehat{\omega}_P$ and thus $\omega_R^\alpha = \omega_P$. As $\mathcal{T}$ is a tree, there has to be some vertex in $\mathcal{X}$ on P, in particular every fourth vertex of P must be an element of $\mathcal{X}$. Since this is not the case, we get a contradiction. □

LEMMA 8.6. *Let G be a connected graph with infinitely many non-local ends such that $\Gamma = \mathrm{Aut}(G)$ acts transitively on the non-local ends of G. If there is no vertex set of finite diameter invariant under Γ, then the structure tree $\mathcal{T}$ of G and any basic cut system $\mathcal{S}$ has infinitely many ends.*

PROOF. We just have to prove that $\mathcal{T}$ has some end ω. If this is the case, then we know that the non-local end $\widehat{\omega}$ of G corresponding to ω has infinitely many images under Γ and thus also ω must have infinitely many images under Γ as any end of $\mathcal{T}$ corresponds to precisely one non-local end of G.

So let us suppose that $\mathcal{T}$ has no end. As Γ acts transitively on the separator vertices of $\mathcal{T}$, the diameter of $\mathcal{T}$ is at most 4. If the diameter is 2, then there is a unique $\mathcal{S}$-separator S in G and thus S is a vertex set of finite diameter invariant under Γ in contradiction to our assumption. Hence we know that $\mathcal{T}$ has diameter 4. Our aim is to show that also in this case the vertex set of all those vertices that lie in any $\mathcal{S}$-separator is a vertex set of finite diameter.

Let X be that $\mathcal{S}$-block that is in $\mathcal{T}$ adjacent to all separator vertices. We will prove that X contains some non-local end of G. If this is the case, then the assumptions, that G contains infinitely many non-local ends and that Γ acts transitively on those ends, implies that all non-local ends must lie in X since X is fixed by the whole automorphism group of $\mathcal{T}$. But as the separations are chosen so that in both wings there are non-local ends, there is some vertex in $\mathcal{T}$ different to X that contains a non-local end of G, a contradiction.

Since Γ fixes no vertex set of finite diameter, for every $\mathcal{S}$-separator S and every $r \in \mathbb{N}$ there is an $\mathcal{S}$-separator S' with $d_X(S, S') = d(S, S') \geq r$. Let us say that a component C of $X - B_r(S)$ has the property $(*)$ if

$(*)$ the $\mathcal{S}$-separators in C have unbounded distances to S.

In a first step we show that for any $r > 0$ there is a component C of $X - B_r(S)$ with property $(*)$. So let us assume that there is an $r > 0$ such that in each component of $X - B_r(S)$ all $\mathcal{S}$-separators have bounded distance to S. Let S' be an $\mathcal{S}$-separator with $d(S, S') \geq 2r$. Then $X - B_r(S')$ contains a component C with the property $(*)$ with respect to S' instead of S, a contradiction to $S' = S^\alpha$ for some $\alpha \in \Gamma$. Thus for every $r > 0$ there is a component C of $X - B_r(S)$ with property $(*)$.

If on the other hand there is an $r > 0$ such that two components C_1, C_2 of $X - B_r(S)$ have the property $(*)$, we construct a metric ray in X and thereby show that X has to contain a non-local end. Let S_0 be an $\mathcal{S}$-separator. Assuming that we have already chosen $\mathcal{S}$-separators S_j and components C_j of $X - B_r(S_j)$ with $C_j \subseteq C_{j-1}$ for $j < i$, let S_i be an $\mathcal{S}$-separator in C_{i-1} with $d(S_i, X - C_{i-1}) > r$. Then there are at least two components of $X - B_r(S_i)$ with $(*)$. One of those has to lie completely in C_{i-1}. Let C_i be that component. Fix some vertex $x_i \in S_i$ and let R_i be a path from x_{i-1} to x_i. Then there is a ray R in the union of all the R_i. This ray has to be a metric ray as there are only finitely many vertices on the R_i that have distance smaller than nr for all $n \in \mathbb{N}$. Thus X contains some non-local end.

Let us finally suppose that for all $r > 0$ there is precisely one component C_r of $X - B_r(S)$ with $(*)$. Then $C_{r+1} \subseteq C_r$ for all r. Let S_i be some $\mathcal{S}$-separator with $d(S, S_i) > i$, and let x_i be some vertex of S_i and R_i some path from x_i to x_{i+1}. Then there is a ray R in the union of all the paths R_i. Again R has to be a metric ray and thus X contains a non-local end of G.

So in all cases we either got directly a contradiction or some non-local end in X which also leads to a contradiction as indicated before. Thus the lemma is proved. $\square$

By replacing the cut system we used for Lemma 8.6 by a cut system such that each separation separates local ends we obtain the following corollary.

Corollary 8.7. *If G is a connected graph with infinitely many local ends such that the automorphism group Γ of G acts transitively on the ends of G, then either*

Γ *fixes a vertex set of finite diameter or any structure tree of* G *and of a basic cut system* $\mathcal{S}$ *such that each* $\mathcal{S}$*-separation separates ends of* G *has a ray.* $\square$

A direct consequence of the Corollaries 8.4 and 8.7 is the following theorem.

THEOREM 8.8. *Let* G *be a connected graph with infinitely many ends such that its automorphism group acts transitively on the ends of the graph. If all ends are local ends, then there is a vertex set of finite diameter that is fixed by* $\mathrm{Aut}(G)$. $\square$

Motivated by the fact that for any graph G with the assumptions on its non-local ends as in this section we have that its structure tree is either a semi-regular tree or a subdivided semi-regular tree, we show in Section 8.3 that the semi-regular tree is uniquely determined up to subdivision for each such graph.

8.2. Metric ends of end-transitive graphs

In this section we will show that for every connected graph with infinitely many non-local ends such that no vertex set of finite diameter is fixed by its automorphism group it is equivalent that its automorphism group acts transitively on the non-local ends or on the metric ends of the graph. Furthermore if the automorphism group of such a graph G is transitive on the non-local ends or on the metric ends, then all non-local ends of G are thin global ends of G.

Throughout this section let G be a connected graph with infinitely many non-local ends such that its automorphism group Γ acts transitively on the non-local ends of G and such that no vertex set of finite diameter is fixed by Γ. Furthermore let $\mathcal{S}$ be a basic cut system such that each $\mathcal{S}$-separation separates non-local ends, and let $\mathcal{T}$ be the structure tree of G and $\mathcal{S}$.

LEMMA 8.9. *Any thin global end of* G *corresponds to an end of* $\mathcal{T}$ *and vice versa. In particular all non-local ends are global ends.*

PROOF. By Lemma 8.6 the structure tree $\mathcal{T}$ has infinitely many ends. We will show that there is a sequence of separations $(A_i, B_i) \in \mathcal{S}$ such that $A_i \subseteq A_{i+1}$ and $A_i \cap B_{i+1} = \emptyset$ for all $i \in \mathbb{N}$. Suppose that this is not the case. If any two distinct $\mathcal{S}$-separators are not disjoint, then the set of all those vertices that lie in any $\mathcal{S}$-separator is a vertex set of finite diameter, its diameter is bounded by $2 \cdot \mathrm{diam}(S)$ for any $\mathcal{S}$-separator S. Thus we may assume that there are two disjoint $\mathcal{S}$-separators S_1, S_2. Let S_3 be another $\mathcal{S}$-separator such that $d_{\mathcal{T}}(S_1, S_2) = d_{\mathcal{T}}(S_2, S_3)$ and $d(S_1, S_3) = 2 \cdot d_{\mathcal{T}}(S_1, S_2)$. By a similar argument to the one of Lemma 8.2 there is an automorphism $g \in \Gamma$ that acts on $\mathcal{T}$ like a translation with $S_1^g = S_2$ or $S_1^g = S_3$. Thus the sequence $(S_1^{g^i})_{i\in\mathbb{N}}$ is a sequence of pairwise disjoint $\mathcal{S}$-separators such that each element of that sequence separates its predecessor from its successor. $\square$

We can reformulate the statement of Lemma 8.9 for the following corollary.

COROLLARY 8.10. *Let* G *be a connected graph with infinitely many non-local ends such that* $\Gamma := \mathrm{Aut}(G)$ *acts transitively on the non-local ends of* G *and fixes* *no vertex set of finite diameter. An end of G is dominated if and only if it is a local end.* $\square$

THEOREM 8.11. *Let G be a connected graph with infinitely many non-local ends and let $\Gamma := \mathrm{Aut}(G)$. Γ fixes no vertex set of finite diameter. Then Γ acts transitively on the non-local ends of G if and only if Γ acts transitively on the metric ends of G.*

PROOF. Let $\mathcal{S}$ be a basic cut system and let $\mathcal{T}$ be the structure tree of G and $\mathcal{S}$. Every global end of G must be a metric end since by the transitivity of Γ on the $\mathcal{S}$-separators the ray of $\mathcal{T}$ it corresponds to has to define precisely one metric end. By Lemma 8.9 Γ acts transitively on the metric ends of G. On the other hand for every metric end there is a unique non-local end it corresponds to. Since Γ acts transitively on the metric ends, Γ also has to be transitive on the non-local ends, as for every non-local end of G there is at least one metric end corresponding to it. $\square$

The following lemma can be found in [**89**, Corollary 2.5].

LEMMA 8.12. *For every connected graph G with a separation (A, B) of G such that $A \cap B$ is finite and with an automorphism $\alpha \in \mathrm{Aut}(G)$ with $A^\alpha \subseteq A \setminus B$ there is some power of α that fixes a geodetic double ray with one end in A and one end in B.* $\square$

8.3. Uniqueness of the structure tree

Our aim is to show that the structure of the tree $\mathcal{T}$ is essentially independent of the choice of $\mathcal{S}$. But Example 8.13 shows that in general it is not unique. The graph of the example has two different structure trees one of which is the subdivision of the other tree. But in Theorem 8.14 we show that this is always the only ambiguity that could occur.

EXAMPLE 8.13. Let T be a subdivision of a semi-regular tree T' that is not regular. We suppose that $VT' \subseteq VT$ and that $2d_{T'}(x, y) = d_T(x, y)$ for all $x, y \in VT'$. Let $A \cup B = VT'$ be be the natural bipartition of T' and let $C = VT \setminus VT'$. Then all the sets A, B, C are $\mathrm{Aut}(T)$-invariant. Let $\mathcal{A} = \{\{a\} \mid a \in A\}$ and let $\mathcal{B}$ and $\mathcal{C}$ be the corresponding sets for the sets B and C. Let $\mathcal{S}_\mathcal{A}, \mathcal{S}_\mathcal{B}, \mathcal{S}_\mathcal{C}$ be $\mathrm{Aut}(T)$-invariant cut systems such that the corresponding sets of separators are $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, respectively. Then the structure trees $\mathcal{T}_\mathcal{A}$ and $\mathcal{T}_\mathcal{B}$ are isomorphic to T' and the structure tree $\mathcal{T}_\mathcal{C}$ is isomorphic to T. Thus not all structure trees are isomorphic.

THEOREM 8.14. *Let G be a connected graph with infinitely many non-local ends such that the automorphism group Γ of G acts transitively on the non-local ends of G and fixes no vertex set of finite diameter. Then the structure trees for any two basic cut systems $\mathcal{S}_1, \mathcal{S}_2$ such that each $\mathcal{S}_i$-separation for $i = 1, 2$ separates non-local ends are either the same or one is the subdivision of the other.*

We will prove the theorem by a series of claims.

Claim 8.15. *It is sufficient to prove the theorem for any two cut systems such that their union is a nested cut system.*

Proof. Let $\mathcal{S}_1, \mathcal{S}_2$ be two distinct basic cut systems such that their union is not a nested cut system. By Lemma 2.12 there is a nested cut system $\mathcal{S}$ which is nested with $\mathcal{S}_2$ and such that $m_{\mathcal{S}_1}(A,B) < m_{\mathcal{S}_1}(A',B')$ for all $(A,B) \in \mathcal{S}, (A',B') \in \mathcal{S}_2$. By induction on the value $m_{\mathcal{S}_1}(A,B)$ the structure trees for $\mathcal{S}$ and $\mathcal{S}_1$ as well as the structure trees for $\mathcal{S}$ and $\mathcal{S}_2$ are essentially the same, that means either they are isomorphic or one is the subdivision of the other, and by Lemma 8.5 we also know that the claim holds for the structure trees of $\mathcal{S}_1$ and of $\mathcal{S}_2$. $\square$

So let $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2$ be a nested cut system and let $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2$ be the structure trees of G and $\mathcal{S}, \mathcal{S}_1, \mathcal{S}_2$, respectively.

Claim 8.16. *For each two $\mathcal{S}_1$-separators with distance 2 in $\mathcal{T}_1$ there are at most two $\mathcal{S}_2$-separators separating them.*

Proof. Suppose that this is not the case. Let A, A' be two $\mathcal{S}_1$-separators with distance 2 in $\mathcal{T}_1$ and let B_1, B_2, B_3 be three $\mathcal{S}_2$-separators such that each of them separates A and A'. By Lemma 8.2 there is a translation $\alpha \in \Gamma$ such that one of B_1, B_2, B_3 is mapped by α onto another one of those three separators. Furthermore α fixes the $\mathcal{S}_1$-block X between A and A'. Thus there is an end of $\mathcal{T}_2$, namely both ends fixed by α—compare that by Lemma 8.12 there are precisely two such ends—, and hence a corresponding global end in G by Lemma 8.9 that lies in X. This is a contradiction to the same lemma, as all non-local ends of G corresponds to ends of the structure trees $\mathcal{T}_1$ and $\mathcal{T}_2$. $\square$

So there are at most two $\mathcal{S}_2$-separators B_1 and B_2 separating A and A'.

Claim 8.17. *If there are two $\mathcal{S}_2$-separators B_1 and B_2 separating A and A', then there are two orbits on the $\mathcal{S}_2$-blocks of G in one of which all $\mathcal{S}_2$-blocks contain $\mathcal{S}_1$-separators and in one of which no $\mathcal{S}_2$-block contains any $\mathcal{S}_1$-separator.*

Proof. Let us suppose that this is not the case. It is not possible that there are several distinct Γ-orbits on the $\mathcal{S}_2$-blocks that contain $\mathcal{S}_1$-separators as Γ acts transitively on the $\mathcal{S}_1$-separators. Furthermore there cannot be distinct Γ-orbits on the $\mathcal{S}_2$-blocks that do not contain any $\mathcal{S}_1$-separator by the transitivity of Γ on the $\mathcal{S}_1$-separators and by Claim 8.16.

Let $Y_1B_1Y_2B_2$ be a path in $\mathcal{T}_2$. Then there is an $\alpha \in \Gamma$ such that $Y_1^\alpha = Y_2$. If α does not act like a translation on $\mathcal{T}_2$, then $B_1^\alpha = B_1$ and $B_2^{\alpha^{-1}} \subseteq Y_1$ and thus there are the three $\mathcal{S}_2$-separators $B_2^{\alpha^{-1}}, B_1, B_2$ separating two $\mathcal{S}_1$-separators with distance 2 in $\mathcal{T}_1$ or there is again a non-local end in X. This contradicts Claim 8.16 or Lemma 8.9 and thus α is a translation. Hence there is a unique double ray R in $\mathcal{T}_2$ invariant under α. Let ω_1, ω_2 be the ends that are defined by R. By Lemma 8.9 there are two non-local ends of G corresponding to ω_1 and ω_2 and thus there is

$\beta \in \Gamma$ with $\omega_1^\beta = \omega_2$. Then there are two infinite subrays R_1 and R_2 of R such that $R_1^\beta = R_2$. The automorphism β has to map $\mathcal{S}_2$-separators onto $\mathcal{S}_2$-separators and $\mathcal{S}_2$-blocks onto $\mathcal{S}_2$-blocks. So let $R_1 = x_0 x_1 \ldots$ and $R_2 = y_0 y_1 \ldots$ such that x_0 and y_0 are $\mathcal{S}_2$-blocks and $x_i^\beta = y_i$. We show that the double ray R has an orientation in $\mathcal{T}$ and thus the ends defined by R cannot be mapped onto each other by some $\gamma \in \Gamma$: For each x_i with odd i there is another $\mathcal{S}_2$-separator in x_{i-1} but not in x_{i+1} that is separated from x_i by no $\mathcal{S}_1$-separator. Conversely for each y_i with odd i there is another $\mathcal{S}_2$-separator in y_{i+1} but not in y_{i-1} that is separated from y_i by no $\mathcal{S}_1$-separator. Thus the ends ω_1 and ω_2 do not lie in the same Γ-orbit. This proves that no α with $Y_1^\alpha = Y_2$ exists. □

We separate the remaining part of the proof of Theorem 8.14 into three cases: In the first one there are two Γ-orbits on the $\mathcal{S}_i$-blocks for $i = 1, 2$, in the second case there is just one Γ-orbit on the $\mathcal{S}_1$-blocks but two on the $\mathcal{S}_2$-blocks, and in the third case there is just one Γ-orbit on the $\mathcal{S}_i$-blocks, for $i = 1, 2$. In each case we show the conclusion of Theorem 8.14 which we denote by $(*)$.

Let $\mathcal{G}_i$, $i = 1, 2$, be the set of all $\mathcal{S}$-blocks such that each element only contains $\mathcal{S}_i$-separators, and let $\mathcal{G}_3$ be the set of all other $\mathcal{S}$-blocks which contain both $\mathcal{S}_1$- and $\mathcal{S}_2$-separators.

Claim 8.18. *If there are two Γ-orbits on the $\mathcal{S}_1$-blocks and on the $\mathcal{S}_2$-blocks, then $(*)$ holds.*

Proof. All $\mathcal{S}$-separators must have degree 2 in $\mathcal{T}$ and there are at least three Γ-orbits on the $\mathcal{S}$-blocks. Hence we know that $\mathcal{G}_3 \neq \emptyset$.

The elements of $\mathcal{G}_3$ must have degree 2 in $\mathcal{T}$ as otherwise there would be three $\mathcal{S}_i$-separators ($i = 1$ or 2) between two $\mathcal{S}_j$-separators of distance 2 in $\mathcal{T}_j$, $i \neq j$, a contradiction to Claim 8.16. Let X_1, X_2 be $\mathcal{S}_1$-blocks of distinct Γ-orbits, let Y_1, Y_2 be distinct $\mathcal{S}_2$-blocks of distinct Γ-orbits, and let $Z_1 \in \mathcal{G}_1$, $Z_2 \in \mathcal{G}_2$. Then there is w.l.o.g. $d_{\mathcal{T}_1}(X_1) = d_{\mathcal{T}}(Z_1) = d_{\mathcal{T}_2}(Y_1)$ and $d_{\mathcal{T}_1}(X_2) = d_{\mathcal{T}}(Z_2) = d_{\mathcal{T}_2}(Y_2)$ and thus $(*)$ holds. □

Claim 8.19. *If Γ acts transitively on the $\mathcal{S}_1$-blocks but if there are two Γ-orbits on the $\mathcal{S}_2$-blocks, then $(*)$ holds.*

Proof. In this case there is $\mathcal{G}_1 = \emptyset$ and $\mathcal{G}_2, \mathcal{G}_3 \neq \emptyset$. All elements of $\mathcal{G}_2$ must have degree 2 in $\mathcal{T}$ and thus for every $\mathcal{S}_1$-separator S there is $d_{\mathcal{T}_1}(S) = d_{\mathcal{T}_2}(X_1)$ for one (and hence every) $\mathcal{S}_2$-block X_1 containing an $\mathcal{S}_1$-separator and for every $\mathcal{S}_1$-block Y there is $d_{\mathcal{T}_1}(Y) = d_{\mathcal{T}_2}(X_2)$ for one (and hence every) $\mathcal{S}_2$-block containing no $\mathcal{S}_1$-separator. Thus the claim follows. □

Claim 8.20. *If Γ acts transitively on both the $\mathcal{S}_1$- and the $\mathcal{S}_2$-blocks, then $(*)$ holds.*

Proof. In this case there is $\mathcal{G}_1 = \mathcal{G}_2 = \emptyset$ and for all $\mathcal{S}_j$-separators S_j and all $\mathcal{S}_j$-blocks X_j the equality $d_{\mathcal{T}_j}(S_j) = d_{\mathcal{T}_i}(X_i)$ with $i \neq j$ holds by Claim 8.17. □

This was the last one of the three cases by Lemma 8.5 and thus Theorem 8.14 is proved.

8.4. End-transitive graphs

Throughout this section let G be a connected graph with infinitely many non-local ends on which $\Gamma := \mathrm{Aut}(G)$ acts transitively. Furthermore Γ fixes no vertex set of G of finite diameter. Let $\mathcal{S}$ be a basic cut system and let $\mathcal{T}$ be the structure tree of G and $\mathcal{S}$.

LEMMA 8.21. *The distances between any two $\mathcal{S}$-separators in a common $\mathcal{S}$-block are bounded.*

PROOF. We have to show that there is a constant $m < \infty$ such that for all two $\mathcal{S}$-separators S_1, S_2 in a common $\mathcal{S}$-block X there is $d(S_1, S_2) \leq m$.

By Lemma 8.2 there is a translation in G such that on the double ray Q defined by that translation the distance between any two $\mathcal{S}$-separators with distance 2 in $\mathcal{T}$ has at most two distinct values. Let us suppose that no such m as conjectured exists. Then there is a ray R in $\mathcal{T}$ such that there is a sequence $(S_i)_{i\in\mathbb{N}}$ of separators on R with $d(S_i, S_i') > i$ where S_i' is that $\mathcal{S}$-separator on R following on S_i. But the two described ends, one defined by R and the other one defined by some translation, cannot be mapped onto each other. This is a contradiction and thus the lemma is proved. □

LEMMA 8.22. *There is an $M < \infty$ such that for each vertex x of G and for each ray R in T there are at most M $\mathcal{S}$-blocks in R which contain x.*

PROOF. Suppose the claim does not hold. We construct a sequence $(P_i)_{i\in\mathbb{N}}$ of finite paths with $P_i \subseteq P_{i+1}$ such that there is a subpath of length i of P_i such that the intersection of the blocks and separators on that subpath is not empty. Then for every finite path P_i of length at least i with end vertex X in $\mathcal{T}$ there is an infinite component C of $\mathcal{T} - P_i$ that is adjacent to X and in which there is a ray containing $2i + 2$ separator vertices with a non-trivial intersection as Γ acts transitively on the separator vertices of $\mathcal{T}$. The elements of that non-trivial intersection might intersect trivially with any separator of the path P_i. We may extend P_i and get a finite path P_{i+1} such that there is a sequence of at least $i + 1$ separator vertices containing a vertex $y_{i+1} \in VG$. By recursion we get a ray $R = \bigcup_{i\in\mathbb{N}} P_i$ in $\mathcal{T}$ such that for each $i \in \mathbb{N}$ there is a sequence of length i of separator vertices on R that intersect non-trivially.

By Lemma 8.9 there is a global end ω of G defined by R. By Lemma 8.2 there is an automorphism of G that acts on $\mathcal{T}$ like a translation. The unique double ray of $\mathcal{T}$ fixed by that automorphism has an infinite subray R'. Let ω' be the end of G defined by R'. Since Γ acts transitively on the non-local ends, there is some $g \in \Gamma$ with $\omega^g = \omega'$. But then we may assume that R^g has only finitely many vertices not contained in R' and thus we may also assume that $R^g \subseteq R'$.

If we finally show that on R' any vertex of G lies in only m separator vertices of R' for a constant m, then we get a contradiction and this would prove the lemma. But if there is a sequence $(x_i)_{i\in\mathbb{N}}$ such that x_i lies in at least i separators on R', then each separator must contain infinitely many vertices, as the translation maps any separator at most 2 separators apart, in contradiction to the definition of the $\mathcal{S}$-separators. □

LEMMA 8.23. *Let G be quasi-isometric to a semi-regular tree T. Then the set of all $\mathcal{S}$-blocks and all $\mathcal{S}$-slices has bounded diameter.*

PROOF. Suppose the lemma is false. Let us first assume that the set of all $\mathcal{S}$-blocks has bounded diameter. By Lemma 8.5 there is an $\mathcal{S}$-block X that has no finite diameter. Then with Lemma 8.21 there is a sequence $(x_i)_{i\in\mathbb{N}}$ in X such that $\min\{d(x_i, S) \mid S\ \mathcal{S}\text{-separator}\} \geq i$ for all $i \in \mathbb{N}$. Let x be a vertex in S for an $\mathcal{S}$-separator $S \subseteq X$ and let $t \in VT$ be the vertex with $x^\varphi = t$ for the quasi-isometry $\varphi : G \to T$. Let $(t_i)_{i\in\mathbb{N}}$ be a sequence in VT with $x_i^\varphi = t_i$. This sequence has an infinite subsequence $(t_i)_{i\in I}$ with $I \subseteq \mathbb{N}$ of pairwise distinct elements. By Lemma 8.21 there is $d(x, y) < M$ for some constant $M < \infty$ and all $y \in S'$ for some $\mathcal{S}$-separator S' in X. Thus there is an $r \in \mathbb{N}$ such that $B_r(t)$ contains the images of all the vertices $y \in S'$ for all $\mathcal{S}$-separators $S' \subseteq X$. Then there is a component C of $T - B_r(t)$ that contains at least one vertex t_i. Since T is a semi-regular tree, C contains a ray R. Let R' be a set of vertices in X such that there is an $r' \in \mathbb{N}$ with $R \subseteq B_{r'}((R')^\varphi)$. As G is quasi-isometric to T, there is at least one non-local end of G defined by R' and this has to lie in X, a contradiction to Lemma 8.9.

So let us assume that the $\mathcal{S}$-slices have unbounded diameter. Let S be an $\mathcal{S}$-separator and let $\mathcal{Y}$ be the set of all those slices which are components of $G - S$. Then $X := \bigcup \mathcal{Y}$ contains no metric ray as otherwise there would be a non-local end in X but all such ends corresponds to ends of $\mathcal{T}$ by Lemma 8.9. Thus there is no ray of T whose preimages lie in X by a similar argument as in the first case. Hence X has only finite diameter. □

LEMMA 8.24. *Assume that there is some $\delta \geq 0$ such that any vertex of G lies at distance at most δ to the union of all geodetic double rays. Then the set of all $\mathcal{S}$-blocks and all $\mathcal{S}$-slices has bounded diameter.*

PROOF. By Lemmas 8.2 and 8.12 the intersection of each $\mathcal{S}$-separator with all geodetic double rays is not empty. As all $\mathcal{S}$-separators have the same diameter and the distances between any two $\mathcal{S}$-separators in the same $\mathcal{S}$-block are bounded by Lemma 8.21, there is an upper bound on the distance from each vertex in any $\mathcal{S}$-block to any $\mathcal{S}$-separator. Additionally the distance from each vertex of any $\mathcal{S}$-slice to any $\mathcal{S}$-separator is bounded by $\delta + s$ where m denotes the diameter of any $\mathcal{S}$-separator. □

All the previously proved lemmas enable us to prove Theorem 8.25.

THEOREM 8.25. *Let G be a connected graph with infinitely many non-local ends such that $\Gamma := \mathrm{Aut}(G)$ acts transitively on the non-local ends. Either Γ fixes a vertex set of finite diameter or the following assertions hold:*

(a) *For every $x \in VG$ there is an $r \in \mathbb{N}$ such that $G(x, r)$ covers all geodetic double rays of G.*

(b) *For every $x \in VG$ there is an $R \in \mathbb{N}$ such that the graph $G - G(x, R)$ contains no metric ray of G.*

(c) *The following statements are equivalent:*
   (i) *There is an $r \in \mathbb{N}$ such that every vertex is r-close to some geodetic double ray;*
   (ii) *Γ acts metrically almost transitively on G;*
   (iii) *there is a Γ-congruence π such that a vertex set of finite diameter meets every congruence class of π;*
   (iv) *G is quasi-isometric to any structure tree of G and a basic cut system of G;*
   (v) *G is quasi-isometric to a semi-regular tree with minimum degree 2.*

(d) *All of the properties of part (c) hold for the subgraph $G(x, R)$ of property (b).*

PROOF. Let $\mathcal{S}$ be a basic cut system and let $\mathcal{T}$ be the structure tree of G and $\mathcal{S}$. Let m be the constant of Lemma 8.21 and let s be the diameter of any $\mathcal{S}$-separator. Let B be the $(2m+2s)$-ball around an arbitrary $\mathcal{S}$-separator S. Then B covers the intersection of each geodetic double ray in G with every $\mathcal{S}$-block X with $S \subseteq X$ and B covers also the intersection of each geodetic double ray with every $\mathcal{S}$-slice Y such that Y is a component of $G - S$. So if we set d as the minimum over all $d(x, S)$ for $\mathcal{S}$-separators S, then the statement of part (a) holds for $r := 2m + 2s + d$.

If we set $R := 2m+2s+d$ (where d denotes the same value as before), then any R-ball around x covers all $\mathcal{S}$-separators and since no metric ray lies in any $\mathcal{S}$-block by Lemma 8.9 and by definition no metric ray lies in any $\mathcal{S}$-slice. Thus we just proved part (b).

The assertion (d) is an immediate consequence of (b) and (c) as the graph $G(x, R)$ of (b) is by construction a metrically almost transitive graph. So we just have to prove the equivalences of (c). The equivalence of (i) and (ii) follows with Lemma 8.12 and Lemma 8.21 from the definition of metrically almost transitive graphs. The equivalence of (ii) and (iii) is just the definition of metrically almost transitive graphs.

By the Lemmas 8.22 and 8.24 the condition (i) of this theorem immediately implies (iv). So let us assume that (iv) holds, that is G is quasi-isometric to $\mathcal{T}$. Then there is some constant k such that every vertex of G lies at distance at most k to some $\mathcal{S}$-separator. By Lemma 8.12 each $\mathcal{S}$-separator meets some geodetic double ray of G and thus if the $\mathcal{S}$-separators have diameter s, then every vertex of G lies at most $s + k$ apart from any geodetic double ray.

As the last part of the proof of Theorem 8.25 we show the equivalence of (iv) and (v). If G is quasi-isometric to a semi-regular tree, then the Lemmas 8.22 and

8.23 imply that G is quasi-isometric to any structure tree of G and a basic cut system of G. So let us assume that G is quasi-isometric to a structure tree $\mathcal{T}$ with respect to a basic cut system of G. By Lemma 8.5 the structure tree $\mathcal{T}$ is quasi-isometric to a semi-regular tree and thus G is also quasi-isometric to a semi-regular tree. $\square$

If we just require Γ to act transitively on the ends of G instead of the non-local ends of G, we get Theorem 8.1 as a corollary of Theorem 8.25.

We finish this chapter with an observation: Let G be a graph as in Theorem 8.1 and let H be any rayless graph. If we add to each vertex $y \in VG$ a copy H_y of H with an additional edge, then for the constructed graph G' there is a component of $G - G(x, R)$ (where x and R are as in the theorem) that is isometric to H. So any rayless graph can occur as a component of $G - G(x, R)$.

Part 3

Transitive group actions on graphs

CHAPTER 9

Transitivity in graphs

In this chapter we shall describe various kinds of transitivity and homogeneity for graphs and digraphs. First we discuss them in the situation of graphs, and then we extend our discussion to digraphs and look at different situations that occur by orientating the edges. For example, whether for every homogeneous graph there is an orientation of its edges such that the resulting digraph is also homogeneous.

We have a rather weak form of symmetry obtained by the automorphism group of a graph if the group acts transitively on the vertices of the graph, hence, if the graph is transitive. This is a very rich class, as it contains all Cayley graphs. On the other side of symmetry, that of *homogeneous* graphs is very strong, that is every isomorphism between two finite induced subgraphs extends to an automorphism of the graph. Whereas there is no problem to classify the countable homogeneous graph—these are the C_5, the line graph of $K_{3,3}$, countably many copies of a complete graph, a generic K_n-free graph or the generic graph, see [**22, 34, 37, 72, 90**]—, the condition of transitivity is too weak to obtain a classification of such graphs.

That is the reason, why we look at kinds of symmetry that are stronger than transitivity but not as strong as homogeneity. There is the natural question of how much we can weaken the assumptions of homogeneity still being able to give a classification of such graphs.

One way to weaken homogeneity is the *k-homogeneity*, that is, to assume only isomorphisms between induced subgraphs on at most k vertices to extend to an automorphism of the graph. So 1-homogeneity is the same as transitivity and k-homogeneity for all $k \in \mathbb{N}$ is the same as homogeneity. There are not many results in this direction, for example the classification of the finite 2-homogeneous graphs is a consequence of [**74**]. We can also consider a homogeneity notion between homogeneity and k-homogeneity, the *$\leq k$-homogeneity*, that is that the graph is l-homogeneous for all $l \leq k$. There is a result due to Droste and Macpherson [**30**] that says that for every k there are uncountably many countable graphs that are $\leq k$-homogeneous but not $(k+1)$-homogeneous. Another result is due to Cameron [**17**] who proved that ≤ 5-homogeneous graphs with diameter 2 are already homogeneous.

A different way of weakening the notion of homogeneity is to suppose that for two finite induced isomorphic subgraphs there is an isomorphism that extends to an automorphism of the whole graph. This is the notion of *set-homogeneity*. In fact, for finite graphs, the notion of homogeneity and set-homogeneity coincides [**90**]. But for infinite graphs, the class of set-homogeneous graphs is larger than the class of homogeneous graphs, see [**28, 29**]. A question posed in [**29**] seems to be still open: Do there exist further countable set-homogeneous graphs except for the two described in [**29**] that are not homogeneous? As for homogeneity, we can consider *k-set-homogeneity* (also called *k-S-transitivity* in Chapter 11) and *$\leq k$-set-homogeneity* to obtain a kind of symmetry between set-homogeneity and transitivity.

Another symmetry condition is to require the automorphism group of a graph G to act transitively on the set $\{(x,y) \in VG \times VG \mid d(x,y) = i\}$ for some $i \in \mathbb{N}$. If this holds for all $i \leq k$ for a fixed $k \in \mathbb{N}$ then we say that G is *k-distance-transitive*, and if it holds for all $i \in \mathbb{N}$ then G is called *distance-transitive*. The classification of the distance-transitive graphs is close to be finished in the finite case (see [**14, 18, 96**]), but the infinite (countable) case is far from being solved. A good survey of the topic of distance-transitive graphs can be found in [**18**]. One result we have to mention is the classification of the connected locally finite distance-transitive graphs due to Macpherson [**76**]. He proved that these are precisely the graphs $X_{k,l}$ for finite k, l. Later, Möller [**81**] proved that we can relax the assumptions to 2-distance-transitive and require the graph to have at least two ends, to obtain the same class of graphs. We shall generalize their result in Chapter 10, see in particular Theorem 10.1, to arbitrary graphs with at least two ends, that is, also in this case we just obtain the graphs $X_{\kappa,\lambda}$, only for arbitrary cardinals κ and λ this time.

Instead of mapping any two vertices with the same distance onto any other two such vertices, we can assume that the same holds for any two paths of length k. For a given $k \in \mathbb{N}$, a graph is *k-(arc)-transitive* if its automorphism group acts transitively on all paths of length k and *arc-transitive* if it is k-transitive for all $k \in \mathbb{N}$. As infinite k-transitive graphs cannot have cycles of length at most k, the only connected arc-transitive graphs are the r-regular trees for an arbitrary cardinal $r \geq 2$. If we consider—as a relaxation as in the case of the distance-transitive graphs—just the locally finite 2-transitive graphs with at least two ends, then Thomassen and Woess [**94**] proved that the connected such graphs are precisely the r-regular trees for finite $r \geq 2$. We generalize their result in Chapter 10 to graphs of arbitrary degree and obtain also just the regular trees as such graphs.

If we return to the definition of homogeneity and ask the finite induced subgraphs to be connected then we obtain the notion of *connected-homogeneous* graphs, or *C-homogeneous* graphs for short. The finite and countably infinite such graphs are classified, see [**34, 38, 43, 56, 97**]. In Chapter 11 we classify the connected C-homomgeneous graphs with more than one end. This class coincides again with the class of the distance-transitive graphs with at least two ends. Our result in Chapter 11 also states that these graphs are precisely the connected *C-transitive* graphs, that are those where we replace the assumption that every automorphism between two finite induced connected subgraphs extends to an automorphism of the graph by the assumption that there is an isomorphism between these graphs that extends to an automorphism.

We can also require that, for any two isomorphic induced connected subgraphs on k vertices, every isomorphism extends to an automorphism of the graph. Then we obtain the notion of *k-CS-homogeneous* graphs. Graphs with the property that, for any two isomorphic induced connected subgraphs on k vertices, there is an automorphism of the graph mapping one of these subgraphs onto the other are called *k-CS-transitive*. Gray [**42**] classified the connected locally finite 3-CS-transitive and also the connected locally 3-CS-homogeneous graphs with at least two ends. We generalize his result in Chapter 11 not only to graphs with arbitrary degree, but also to arbitrary $k \geq 3$, that is, we classify for every $k \in \mathbb{N}_{\geq 3}$ the connected k-CS-transitive graphs with at least two ends and obtain as a corollary the classification of the connected k-CS-homogeneous graphs with at least two ends.

When we consider digraphs instead of graphs, mostly the same notion of the distinct kinds of symmetry applies. This holds in particular for the notions of (k-)homogeneity, (k-)set-homogeneity, and (k-)C-homogeneity. But instead of k-transitivity we have *k-arc-transitivity*, that is, the automorphism group of the digraph acts transitively on the arcs of length k. We call a digraph *highly-arc-transitive* if it is k-arc-transitive for every $k \in \mathbb{N}$.

Although the definitions of the kinds of symmetry do not differ much from the undirected case, the class of digraphs for each of these conditions is in most situations harder to determine and, particularly, the arc-transitive graphs—a class that consists only of the regular trees—has a directed counterpart which is far from being understood, see [**2, 3, 4, 19, 36, 45, 77, 78, 84, 91**].

The countable homogeneous digraphs are classified in [**20, 21, 22, 70, 71**]. This class contains the homogeneous tournaments, see [**21, 71**] and compare Theorem 2.31. The finite such digraphs are the C_4, $\bar{K}_n$, $C_3[\bar{K}_n]$, $\bar{K}_n[C_3]$, and H (recall that we defined the particular digraph H in section 2.8). Here, three typical things happen when we consider digraphs instead of graphs:

1. There are homogeneous graphs that are the underlying undirected graphs of homogeneous digraphs, for example K_3 and C_3.
2. There are homogeneous graphs with no orientation of their edges such that these orientations make them a homogeneous digraph. An examples is the graph K_4, because there is no homogeneous tournament on four vertices.
3. There are homogeneous digraphs whose underlying undirected graphs are not homogeneous. For this situation, the digraph H is an example.

Similar situations occur for the other kinds of symmetry.

For graphs, the notions of set-homogeneous and homogeneous graphs coincides, but for digraphs this is not the case. But by the classification of the finite set-homogeneous digraphs [**44**], there is just one digraph that is set-homogeneous but not homogeneous: the directed cycle C_5.

Considering C-homogeneous digraphs, the only known result is due to Gray and Möller [**45**]. They classified the infinite connected two-ended digraphs and offered a list of examples for the connected locally finite C-homogeneous digraphs with infinitely many ends. In Chapter 12 we prove that their list is complete. Furthermore, we classify the finite as well as infinite locally finite connected C-homogeneous digraphs with at most one end. Thereby, we complete the classification of the (connected) locally finite C-homogeneous digraphs. It remains to classify all countable such digraphs, although it would mean to extend Cherlin's classification of the countable homogeneous digraphs, see [**22**].

CHAPTER 10

Distance-transitive graphs

In this chapter, we prove a theorem of [**55**], that is, we classify the connected (2-)distance-transitive graphs with at least two ends. Thereby, we generalize the classification results of Macpherson [**76**] and Möller [**81**] as discussed in Chapter 9.

Thereafter, we obtain the classification of the connected 2-transitive graphs with more than one end as a corollary.

THEOREM 10.1. *Let G be a connected infinite graph with more than one end. The following properties are equivalent:*

(i) G is distance-transitive;
(ii) G is 2-distance-transitive;
(iii) $G \cong X_{\kappa,\lambda}$ for some cardinals κ and λ with $\kappa, \lambda \geq 2$.

PROOF. Since the graphs $X_{\kappa,\lambda}$ are distance transitive, it suffices to prove that every connected 2-distance transitive graph with at least two ends is some $X_{\kappa,\lambda}$ (with $\kappa, \lambda \geq 2$). So let G be a connected 2-distance-transitive graph with more than one end. Let $\mathcal{S}$ be a minimal cut system of G such that the structure tree of G and $\mathcal{S}$ is basic. Then $\mathcal{S}$ is a nested cut system—in particular for every separation $(A, B) \in \mathcal{S}$ and every automorphism α of G, the cuts $(A, B), (A^\alpha, B^\alpha)$ are nested—and both wings of any cut in $\mathcal{S}$ contain a ray.

CLAIM 10.2. *All $\mathcal{S}$-blocks are complete.*

PROOF OF CLAIM 10.2. Let $(A, B) \in \mathcal{S}$, and let $a \in A \setminus B$ and $b \in B \setminus A$ be vertices with distance 2 in G. Suppose that some $\mathcal{S}$-block X is not complete. Let x, y be two non-adjacent vertices in X, and let P be a shortest x–y path in G. Let $\mathcal{T}'$ be *the* minimal subtree of $\mathcal{T}$ containing X such that its vertices cover P. All leaves of $\mathcal{T}'$ are $\mathcal{S}$-blocks. Let Y be a leaf of $\mathcal{T}'$ different from X, if available, and $Y = X$ otherwise. Then there are two vertices $a', b' \in Y$ with distance 2 in G. As G is 2-distance-transitive there is an automorphism α of G with $a^\alpha = a'$ and $b^\alpha = b'$. Then Y meets both wings of (A^α, B^α)—namely $a^\alpha, b^\alpha \in Y$—which contradicts the fact that Y is an $\mathcal{S}$-block. $\square$

CLAIM 10.3. *Any two $\mathcal{S}$-separators are equal or disjoint.*

PROOF OF CLAIM 10.3. Let S, S' be any two distinct $\mathcal{S}$-separators, $(A, B) \in \mathcal{S}$, and $\alpha \in \mathrm{Aut}(G)$ such that $S = A \cap B$ and $S^\alpha = S'$. As (A, B) and (A^α, B^α) are nested we may assume (by symmetry) that $A \cap A^\alpha \subseteq B \cap B^\alpha$, and that there is a vertex $v \in A \cap A^\alpha$—$S \cap S' = \emptyset$ otherwise. As (A, B) is a cut of minimal order and all blocks are complete there are vertices $x \in A^\alpha \setminus S^\alpha$ and $y \in A \setminus S$ with distance two. There is a vertex $x' \in S^\alpha \setminus S$, since $S \neq S^\alpha$ and both separators have the same (finite) order. As $yx' \notin E(G)$ and x, y' are adjacent to v at a time, they have distance 2. Thus there is an automorphism β of G with $x^\beta = x'$ and $y^\beta = y$. This is a contradiction according to Lemma 2.13 as there are more cuts in $\mathcal{S}$—all of the same size—separating x from y than x' from y. □

Let us show that all $\mathcal{S}$-separators have order 1. Suppose not, then there are at least two vertices in some $\mathcal{S}$-separator S and, as all $\mathcal{S}$-blocks are complete, there is an edge e in $G[S]$. On the other hand, there is an edge e' that has precisely one of its end vertices in S. Since G is 2-distance-transitive, it is 1-distance-transitive and thus there is an automorphism of G that maps e to e'. This is a contradiction, as the $\mathcal{S}$-separators intersect trivially.

As G is 1-distance-transitive any two blocks have the same order and every vertex lies in the same number of blocks. The size of an $\mathcal{S}$-block is at least 2, since there are edges in G. Every $\mathcal{S}$-separator lies in at least two different $\mathcal{S}$-blocks, as there are at least two different ends in G. Thus G is isomorphic to $X_{\kappa,\lambda}$ for some cardinals $\kappa, \lambda \geq 2$. □

Corollary 10.4. *Let G be an infinite connected graph with more than one end. Then G is 2-transitive if and only if it is a λ-regular tree for some cardinal $\lambda \geq 2$.*

Proof. A 2-transitive graph with at least two ends is also 2-distance-transitive and hence an $X_{\kappa,\lambda}$ with $\kappa, \lambda \geq 2$. If $\kappa \geq 3$ there is a path of length 2 in every block whose (adjacent) endvertices can be mapped onto vertices with distance 2. This is a contradiction and hence $\kappa = 2$. The graphs $X_{2,\lambda}$ with $\lambda \geq 2$ are precisely the λ-regular trees. □

CHAPTER 11

k-CS-transitive graphs

This chapter deals with the classification of the k-CS-transitive graphs, a result in [**55**]. We prove the following theorem. To state it, we should define a class of graphs $\mathcal{E}_{k,m,n}$ first, but we refer to Section 11.1 instead.

THEOREM 11.1. *Let $k \geq 3$. A connected graph with more than one end is k-CS-transitive if and only if it is isomorphic to one of the following graphs:*

(1) *$X_{\kappa,\lambda}(K^1)$ with arbitrary κ and λ;*
(2) *$X_{2,\lambda}(K^n)$ with arbitrary λ and $n < \frac{k}{2} + 1$;*
(3) *$X_{\kappa,2}(\overline{K^m})$ with arbitrary κ and $m < \frac{k}{3} + 1$;*
(4) *$X_{2,2}(E)$ with $E \in \mathcal{E}_{k,m,n}$, $m \leq k-2$ and $n \leq \frac{k-|E|}{2} + 1$;*
(5) *Y_κ with arbitrary κ (if k is odd);*
(6) *$Z_{2,2}(\overline{K^m}, K^n)$ with $2m + n \leq k + 1$ (if k is even);*
(7) *$Z_{\kappa,\lambda}(K^1, K^n)$ with $n < k$, arbitrary κ, λ with $\kappa = 2$ or $\lambda = 2$ (if k is even);*
(8) *$Z_{2,2}(K^1, E)$ with $E \in \mathcal{E}_{k,m,n}$, $m \leq k-2$, $n \leq \frac{k}{2} + 1$ (if k is even).*

Since neither Y_κ for $\kappa \geq 3$, nor $Z_{\kappa,\lambda}(H_1, H_2)$ for any distinct graphs H_1, H_2 or distinct cardinals κ, λ are k-homogeneous, but any k-CS-transitive $X_{\kappa,\lambda}(H)$ is k-homogeneous for homogeneous finite graphs H, Theorem 11.1 allows us to extend Gray's classification of the 3-CS-homogeneous graphs to arbitrary k-CS-homogeneous with at least two ends, as follows:

COROLLARY 11.2. *Let $k \geq 3$. A connected graph with more than one end is k-CS-homogeneous if and only if it is isomorphic to $X_{\kappa,\lambda}(H)$ for one of the following values of κ, λ and H:*

(1) *arbitrary κ and λ and $H = K^1$;*
(2) *$\kappa = 2$, arbitrary λ, $n < \frac{k}{2}$ and $H = K^n$;*
(3) *arbitrary κ, $\lambda = 2$, $m < \frac{k}{3}$ and $H = \overline{K^m}$;*
(4) *$\kappa = 2 = \lambda$, $H \in \mathcal{E}_{k,m,n}$ for $m \leq k-2$ and $n \leq \frac{k-|E|}{2} + 1$.* □

Gray and Macpherson [**43**] classified the countable C-homogeneous graphs with at least two ends, as those described in our Theorem 10.1. As a further corollary of Theorem 11.1 we can extend their classification to arbitrary graphs with more than one end.

COROLLARY 11.3. *For connected graphs with at least two ends the notions of being distance-transitive, C-transitive, or C-homogeneous coincide.* □

11.1. The local structure for some finite subgraphs

In some k-CS-transitive graphs the finite homogeneous graphs play a role as building blocks. Enomoto [**34**] gave a combinatorially characterization of these homogeneous graphs. We apply a corollary of his result [**34**, Theorem 1] in our proofs.

For a subgraph X of a graph G let $\Gamma(X) = \bigcap_{x \in V(X)} N(x)$, which is the set of all vertices in G that are adjacent to all the vertices in X. A graph G is *combinatorially homogeneous* if $|\Gamma(X)| = |\Gamma(X')|$ for any two isomorphic subgraphs X and X'. Furthermore, a graph G is *l-S-transitive* if for every two isomorphic subgraphs of order l there is an automorphism of G mapping one onto the other.

Enomoto [**34**] showed that the finite combinatorially homogeneous graphs are precisely the finite homogeneous graphs.

THEOREM 11.4 ([**34**, Theorem 1]). *Let G be a finite graph. The following properties of G are equivalent.*

(1) *G is homogeneous;*
(2) *G is combinatorially homogeneous;*
(3) *G is isomorphic to*
  (a) *a disjoint union of isomorphic complete graphs,*
  (b) *a complete t-partite graph K_r^t with r vertices in each partition class and with $2 \leq t, r$,*
  (c) *C_5, or*
  (d) *$L(K_{3,3})$ (the line graph of $K_{3,3}$).* □

Whenever we need finite homogeneous graphs as building blocks for k-CS-transitive graphs we use Corollary 11.5 to handle them.

COROLLARY 11.5. *Let $k \geq 3$, $m \leq k-2$, and $n \leq \frac{k}{2}$ be integers. Let G be a finite graph with maximum degree m that is neither complete nor the complement of a complete graph. If G is l-S-transitive for all $l \leq k-1$, any induced subgraph of G on n vertices is connected, and two non-adjacent vertices do not have $k-2$ common neighbors, then G is (combinatorially) homogeneous and isomorphic to*

(a) *t disjoint K^r with $2 \leq t$, $1 \leq r-1 \leq m$, and $tr \leq n-1$,*
(b) *K_r^t with $2 \leq t$, $2 \leq r \leq n-1$, and $(t-1)r \leq \min\{m, k-3\}$,*
(c) *C_5 with $2 \leq m$ and $4 \leq n$, or*
(d) *$L(K_{3,3})$ with $4 \leq m$ and $6 \leq n$.*

PROOF. Theorem 11.4 provides that, ignoring the boundaries, there are no other cases as (a) to (d). The specific boundaries for each case can be checked easily. For example, in case (b) the '$k-3$' in the inequality $(t-1)r \leq \min\{m, k-3\}$ ensures that K_r^t does not contain two non-adjacent vertices with $k-2$ common neigbours if $m = k-2 = (t-1)r$. □

Let $\mathcal{E}_{k,m,n}$ be the class of all those graphs that satisfy the assumptions of Corollary 11.5 with the values k, m and n.

11.2. The k-CS-transitivity for special graphs

In this section we show that any graph G from Theorem 11.1 is indeed k-CS-transitive for the specific values of k. The general idea behind the following proofs is that any connected induced subgraph of G on k vertices contains an anchor like part.

To clarify, let G be a graph and let X be a connected induced subgraph of G. A subgraph A of X is an *anchor* of X in G if for every induced subgraph Y in G isomorphic to X there is some isomorphism γ from X to Y such that the restricted map $\gamma|_A$ extends to an automorphism of G that maps X to Y.

REMARK 11.6. If every induced connected subgraph of order k of some graph G contains an anchor, then G is k-CS-transitive.

The anchors we commonly use are either induced paths of length 3 or smallest separators.

The *building blocks* of $X_{\kappa,\lambda}(H)$ and $Z_{\kappa,\lambda}(H_1, H_2)$ are the isomorphic copies of H, H_1, and H_2 that are used for the construction of these graphs.

LEMMA 11.7. *Let G and k belong to one of the classes (1) to (8) of Theorem 11.1. If some connected induced subgraph X of G on k vertices has diameter 1—i. e. X is complete, then it itself is an anchor.*

PROOF. The only graphs from Theorem 11.1 that may contain complete graphs on k vertices are isomorphic to some $X_{2,\lambda}(K^n)$, $X_{\kappa,\lambda}(K^1)$, Y_κ, $Z_{\kappa,2}(K^1, K^m)$, or $Z_{2,\lambda}(K^1, K^m)$.

- In $X_{2,\lambda}(K^n)$ any complete graph on k vertices consists of precisely two building blocks or precisely two building blocks without one vertex depending on the parity of k.
- In $X_{\kappa,\lambda}(K^1)$ and Y_κ any complete graph on k (≥ 3) vertices lies completely in some K^κ.
- In $Z_{\kappa,2}(K^1, K^m)$ and $Z_{2,\lambda}(K^1, K^m)$ any complete graph on k vertices consists of precisely two adjacent building blocks.

In all these cases every isomorphism between complete subgraphs on k vertices that respects the building blocks (or any K^k if $G \cong Y_\kappa$) extends to some automorphism of the whole graph. □

LEMMA 11.8. *Let G and k belong to one of the classes (1) to (8) of Theorem 11.1. If some connected induced subgraph X of G on k vertices has diameter 2, then it contains an anchor.*

PROOF. Let X be a connected induced subgraph of G on k vertices with diameter 2. If $G \cong Y_\kappa$ then X is isomorphic to some K^{k-1} with one edge attached. This edge is an anchor. Thus we may assume that $G \ncong Y_\kappa$.

Since all building blocks are homogeneous we may assume that X meets at least two building blocks. If X meets precisely two building blocks, then $G \cong Z_{2,2}(K^1, E)$

for some graph $E \in \mathcal{E}_{k,m,n}$ with $m \leq k-2$ and $n \leq \frac{k}{2}+1$ or $G \cong X_{2,2}(E)$ for some graph $E \in \mathcal{E}_{k,k-2,n}$ with $m \leq 2$ and $n \leq \frac{k-|E|}{2}+1$, by cardinality means. In the first case there is one vertex v with $k-1$ neighbors (the building block K^1) which is an anchor, since the maximum degree of E is at most $m \leq k-2$ and $X-v$ is connected. In the second case $E \cong C_5$ or $E \cong L(K_{3,3})$, again by cardinality. If $E \cong C_5$, then $k=10$ and X itself is an anchor. If $E \cong L(K_{3,3})$, then $15 \leq k \leq 18$. Both of the maximal subgraphs of X that lie completely in one of the building blocks are anchors, since $L(K_{3,3})$ is 4-connected and at most three vertices (for $k=15$) of these two building blocks do not lie in X.

We may assume that X meets at least three building blocks. Let B be the building block that touches all vertices of X, which exists by the small diameter of X. If a separator in X does not contain every vertex of $X \cap B$, then it must contain at least all the vertices in $X \setminus B$. In all possible cases $|X \cap B|$ is smaller than $|X \setminus B|$ and $X \cap B$ is indeed the unique smallest separator. Thus for every isomorphic induced copy Y of X in G precisely the vertices of $X \cap B$ are mapped to *the* smallest separator S in Y. We may assume that Y meets three building blocks, as it contains an anchor otherwise. Since S is a smallest separator, $S = Y \cap D$ for the unique building block D of G that touches all vertices of Y. Since the building blocks are homogeneous and B is mapped to D by some automorphism of G, every isomorphism from X to Y extends to an automorphism of G. In particular $X \cap B$ is an anchor. $\square$

LEMMA 11.9. *Let G and k belong to one of the classes (1) to (8) of Theorem 11.1. If some connected induced subgraph X of G on k vertices has diameter at least three, then it contains an anchor.*

PROOF. If $G \cong Y_\kappa$, then every minimal separator is a single vertex and an anchor.

For the other cases, let X be some connected induced subgraph of G on k vertices with diameter at least three, and let P be an (induced) path of length 3 in X that meets four building blocks of G. Such a path exists since there is no building block B such that $X \cap B$ contains an induced path of length 3 whose end vertices have distance 3 in X.

Let $\gamma : X \to Y$ be some isomorphism for an induced subgraph Y of G. We further require that v and v^γ for each vertex $v \in P$ belong to building blocks in the same orbit of the automorphism group of G. This is a legitimate request, since the number of vertices in P is even, and if P embeds into P^γ uniquely, then there are stars or triangles in X and Y that force P and P^γ to be aligned or P is a path of length $k-1$ and thus it itself is an anchor.

Let us recursively construct an automorphism of G that maps X to Y. Let α_0 be an automorphism of G with $\alpha_0|_P = \gamma$ such that vertices in the (homogeneous) building blocks containing P are mapped to Y if and only if they lie in X.

To define the automorphism α_l of G for $l \geq 1$ let α_i be defined for $i < l$. First, let W be the set of vertices in G with distance at most $l-1$ to the building blocks that contain P. The graphs X and Y induce graphs $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ with $X_j^\gamma = Y_j$ for all $1 \leq j \leq n$ in the components of $G - W$ and $G - W^{\alpha_{l-1}}$, respectively. Let α_l be an automorphism of G with $w^{\alpha_l} := w^{\alpha_{l-1}}$ for $w \in W$, that maps the component of $G - W$ containing X_j to the component of $G - W^{\alpha_{l-1}}$ containing Y_j for all $j \leq n$ such that the vertices of X adjacent to W are mapped precisely to those vertices of Y adjacent to $W^{\alpha_{l-1}}$. Since the diameter of X is less than k, the automorphism α_k of G maps X onto Y. $\square$

These three lemmas show that in all cases there are anchors as needed and, thus, every graph in Theorem 11.1 is k-CS-transitive for the specific value of k.

11.3. The global structure of k-CS-transitive graphs

The following two lemmas can be shown for $k \leq 2$ easily. As the lemmas with $k \leq 2$ do not play any role in the proof of Theorem 11.1, we do not provide the corresponding proofs for smaller k.

Lemma 11.10. *If G is a connected k-CS-transitive graph with at least two ends and $k \geq 3$, then every basic structure tree of G has no leaves.*

Proof. Let $\mathcal{S}$ be a minimal cut system of G such that the structure tree $\mathcal{T}$ of G and $\mathcal{S}$ is basic. Suppose that $\mathcal{T}$ contains a leaf. Let X be some $\mathcal{S}$-block representing a leaf in $\mathcal{T}$, and let $(A, B) \in \mathcal{S}$ be a separation with $X \subseteq A$ and $A \cap B \subseteq X$. Then $A \cap B$ is the only $\mathcal{S}$-separator in X and $X = A$. Since there is a ray in $G[A]$, the block X is infinite. There is no vertex in X that has distance $k+1$ to B, as an induced path starting in $A \cap B$ could be mapped into $X \setminus (A \cap B)$ and as this would contradict the fact that $A \cap B$ is the only $\mathcal{S}$-separator in X. Thus there are vertices of infinite degree in X. Let the vertex $x \in X$ have infinite degree and minimal distance to B with this property. Let N be the infinite set of neighbors of x with $d(v, B) > d(x, B)$ for all $v \in N$. Then there is a $K^{\aleph_0}$ or its complement in $G[N]$. As $k-2$ independent vertices in N together with x and one neigbour of x that is neither contained in N nor adjacent to any vertex in N induce a subgraph that could be mapped onto a subgraph induced by $k-1$ independent vertices in N and x, there has to be a $K^{\aleph_0}$ in $G[N]$. This yields to a contradiction, too. Let H be a complete graph on k vertices in $G[N]$, and let $v \in V(H)$. Then there is no automorphism of G that maps $H - v + x$ to H, which is a contradiction to the k-CS-transitivity of G. $\square$

Lemma 11.11. *For $k \geq 3$, every connected k-CS-transitive graph G with at least two ends has infinite diameter.*

Proof. Let $\mathcal{S}$ be a minimal cut system of G such that the structure tree $\mathcal{T}$ of G and $\mathcal{S}$ is basic. Then there is a double ray R in $\mathcal{T}$ as there is no leaf in $\mathcal{T}$ by Lemma 11.10. This ray hits infinitely many different (finite) $\mathcal{S}$-separators. Suppose that there is a vertex x in G that lies in infinitely many of these separators. Since x has neighbors in infinitely many $\mathcal{S}$-blocks on R, this results in two induced stars with $k-1$ leaves in G whose corresponding $\mathcal{S}$-blocks (regarded as vertices in $\mathcal{T}$) induce vertex sets with different diameter in $\mathcal{T}$. This is a contradiction according to Lemma 2.13, since the leaves of these two stars that are furthest away in $\mathcal{T}$ are separated by a different number of separators of the same (finite) order. Thus we conclude that there are infinitely many pairwise disjoint $\mathcal{S}$-separators on R. Two $\mathcal{S}$-separators S_1, S_2 that have n disjoint $\mathcal{S}$-separators on their S_1–S_2 path in $\mathcal{T}$ have distance at least n in G. $\square$

The separators come in very handy. With an application of Menger's Theorem one may construct order of the separators many disjoint rays in G following any ray in a basic structure tree of G. Every such ray in G induces a (connected) path in every block and contains at most one vertex from each separator. This implies that every vertex lies in some block.

Lemma 11.12. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends and let $\mathcal{S}$ be a minimal cut system of G such that the structure tree of G and $\mathcal{S}$ is basic. Let S be an $\mathcal{S}$-separator. If every $s \in S$ has for every $\mathcal{S}$-block X containing S an adjacent vertex in $X \setminus S$, then S is disjoint to any other $\mathcal{S}$-separator S'.*

Proof. Suppose that distinct $\mathcal{S}$-separators S, S' contain a common vertex s and for every $\mathcal{S}$-block X containing S there is an edge between s and $X \setminus S$. Let $(A, B), (A', B') \in \mathcal{S}$ be cuts with separators S and S', respectively. We may assume that $S' \subseteq B$ and $S \subseteq B'$ since $\mathcal{S}$ is nested.

There is an induced path P of length $k-2$ ending in s whose other vertices lie in $A \setminus B$. By assumption s has at least two neighbors x and y such that x lies in $B' \setminus A'$ and y lies in $A' \setminus A$. The paths Px and Py of length $k-1$ can be mapped onto each other with an automorphism of G by the k-CS-transitivity of G. But the endvertices of Px and of Py are separated properly by a different number of $\mathcal{S}$-separators, since any $\mathcal{S}$-separator separating the endvertices of Py properly separates also the endvertices of Px properly as $\mathcal{S}$ is nested, and on the other hand the separator S' separates only the endvertices of Px properly. This contradicts the choice of x and y. $\square$

Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a minimal cut system of G such that the basic structure tree of G and $\mathcal{S}$ is basic. There are two profoundly different cases. In the first case the graph is covered with $\mathcal{S}$-separators while in the second case there are vertices in G that do not belong to any $\mathcal{S}$-separator.

Before we begin investigating these cases we need some definitions. For an $\mathcal{S}$-block X we define the *open ($\mathcal{S}$-)block*

$$\mathring{X} := X \setminus \bigcup \{A \cap B \mid (A, B) \in \mathcal{S}\}.$$

A *k-spoon* is an induced subgraph of G that consists of a triangle and a path starting in one of its triangel vertices with all in all precisely k vertices. A spoon H *pokes* in an $\mathcal{S}$-block X, an $\mathcal{S}$-separator S, or two $\mathcal{S}$-separators S, S' if its degree 2 vertices of the triangle are contained in $\mathring{X}$, S, or one in S and one in S', respectively. A *k-fork* is another induced subgraph of G that consists of its *prongs*, a pair of two non-adjacent vertices, and of its *handle*, a path such that both prongs are adjacent to the same endvertex of the handle, and has k vertices. A fork H *pokes* in an $\mathcal{S}$-block X, an $\mathcal{S}$-separator S, two $\mathcal{S}$-blocks X, Y, or two $\mathcal{S}$-separators S, S' if its prongs are contained in $\mathring{X}$, in S, meet $\mathring{X}$ and $\mathring{Y}$, or meet S and S', respectively.

11.3.1. Empty open blocks. This is the slightly simpler case. If k is odd this is the only possible case as we will show in Lemma 11.18.

Lemma 11.13. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a minimal cut system such that the structure tree of G and $\mathcal{S}$ is basic and such that for some $\mathcal{S}$-block X its open block $\mathring{X}$ is empty. If S, S' are distinct $\mathcal{S}$-separators that both lie in X, then $ss' \in E(G)$ for all $s \in S$ and $s' \in S' \setminus S$, and any two distinct $\mathcal{S}$-separators in X are disjoint.*

Proof. In the case that all vertices lie in $\mathcal{S}$-separators, there is a path of arbitrary length (following a path in the structure tree of G and $\mathcal{S}$) such that any two vertices with distance 2 or greater do not lie in the same $\mathcal{S}$-block. There is also an induced s–s' path P whose inner vertices do not meet the component C of $G - S'$ that contains $S \setminus S'$. If the length of P is less than $k-1$ we elongate P from s into C. Thus there is an induced subpath P' of P of length $k-1$. By the k-CS-transitivity any two vertices of distance at least two on P' and hence also on P lie in different blocks. This implies $d(s, s') < 2$. With Lemma 11.12 it follows that any two distinct $\mathcal{S}$-separators in X are disjoint. $\square$

Lemma 11.14. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a minimal cut system such that the structure tree of G and $\mathcal{S}$ is basic. If every open $\mathcal{S}$-block is empty, then any two $\mathcal{S}$-blocks are isomorphic, or k is odd and there is a cardinal κ such that $G \cong Y_\kappa$.*

Proof. Suppose that there are two $\mathcal{S}$-blocks X and Y that are not isomorphic. Since there is an automorphism α of G with $X \cap Y^\alpha \neq \emptyset$, we may assume that $X \cap Y \neq \emptyset$. If $X \cap Y$ contains two distinct vertices, there is either a k-fork with both prongs in X and one k-fork with both prongs in Y, or there is a k-spoon with its triangle—the subgraph isomorphic to a K^3—in X and one k-spoon with its triangle in Y. This contradicts the k-CS-transitivity of G. Thus the $\mathcal{S}$-blocks intersect in at most one vertex, and every $\mathcal{S}$-block is complete by Lemma 11.13. If there are two $\mathcal{S}$-blocks with more than 2 vertices each, then for both these $\mathcal{S}$-blocks there is a k-spoon poking it, which implies that they are $\mathrm{Aut}(G)$-isomorphic. As every $\mathcal{S}$-block contains an edge we know that there are precisely two different kinds of $\mathcal{S}$-blocks: one $\mathcal{S}$-block is isomorphic to a K^2 and another one is isomorphic to a

K^κ for some $\kappa \geq 3$. This yields to the Y_κ. No Y_κ with $\kappa \geq 3$ is k-CS-transitive for even k since there is a path of length $k-1$ with both outermost edges in $\mathcal{S}$-blocks isomorphic to a K^2 and there is a path of length $k-1$ with both outermost edges in $\mathcal{S}$-blocks isomorphic to a K^κ with $\kappa \geq 3$. There is no automorphism of G mapping the first onto the second path. This completes the proof. $\square$

LEMMA 11.15. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a minimal cut system such that the structure tree of G and $\mathcal{S}$ is basic and every open $\mathcal{S}$-block is empty. If any two $\mathcal{S}$-blocks are isomorphic, then there are precisely two $\mathcal{S}$-separators per $\mathcal{S}$-block and every $\mathcal{S}$-separator lies in precisely two $\mathcal{S}$-blocks, or there are cardinals $\kappa, \lambda \geq 3$ and integers $2 \leq m < \frac{k}{3}+1$ and $2 \leq n < \frac{k}{2}+1$ such that $G \cong X_{2,\lambda}(\overline{K^m})$ or $G \cong X_{\kappa,2}(K^n)$ or $G \cong X_{\kappa,\lambda}(K^1)$.*

PROOF. We already know that $G \cong X_{\kappa,\lambda}(H)$ for some finite graph H and we may assume that $\kappa > 2$ or $\lambda > 2$. If there are edges in H and $\lambda \geq 3$ then there are two kinds of k-spoons: one with its triangle meeting 3 $\mathcal{S}$-separators and one meeting precisely two $\mathcal{S}$-separators. If there are two non-adjacent vertices in H and $\kappa \geq 3$ then there are two kinds of k-forks: one pokes in a single separator and one pokes in two different separators. Thus there is $G \cong X_{2,\lambda}(\overline{K^m})$, $G \cong X_{\kappa,2}(K^n)$, or $G \cong X_{\kappa,\lambda}(K^1)$ with $m, n \geq 2$. It remains to show that $m \leq \frac{k}{3}$ and $n \leq \frac{k}{2}$. Suppose that $m \geq \frac{k}{3}+1$ and let S_1, S_2 be $\mathcal{S}$-separators in different $\mathcal{S}$-blocks both adjacent to some $\mathcal{S}$-separator S_0. Let $A_i \subseteq S_i$ for $i = 0,1,2$ with $|A_0| = |A_1| = \frac{k}{3}+1$ and $|A_2| = \frac{k}{3}-2$. Let $B_i \subseteq S_i$ for $i = 0,1,2$ with $|B_i| = \frac{k}{3}+1-i$. Then there is no automorphism of G from $G[\bigcup A_i]$ to $G[\bigcup B_i]$ although both induced subgraphs are isomorphic. Thus there is $m < \frac{k}{3}+1$.

Suppose $n \geq \frac{k}{2}+1$. Let S_0, S_1 be two adjacent $\mathcal{S}$-separators. Let $A_i \subseteq S_i$ with $|A_0| = \frac{k}{2}+1$ and $|A_1| = \frac{k}{2}-1$, and let $B_i \subseteq S_i$ with $|B_i| = \frac{k}{2}$. Then there is no automorphism of G from the complete graph on k vertices $G[\bigcup A_i]$ to the complete graph on k vertices $G[\bigcup B_i]$. Thus $n < \frac{k}{2}+1$ follows. $\square$

LEMMA 11.16. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a minimal cut system such that the structure tree of G and $\mathcal{S}$ is basic, every open $\mathcal{S}$-block is empty and all $\mathcal{S}$-blocks are isomorphic. If every $\mathcal{S}$-block contains precisely two $\mathcal{S}$-separators and every $\mathcal{S}$-separator is contained in precisely two $\mathcal{S}$-blocks, then $G \cong X_{2,2}(E)$ with $E \in \mathcal{E}_{k,m,n}$, $m \leq k-2$, and $n \geq \frac{k-|E|}{2}+1$, $E \cong \overline{K^m}$ with $1 \leq m < \frac{k}{3}+1$, or $E \cong K^n$ with $1 \leq n < \frac{k}{2}+1$.*

PROOF. As the open $\mathcal{S}$-blocks are empty it is obvious that $G \cong X_{2,2}(E)$ for some finite graph E. If E is a complete graph or the complement of a complete graph, then the same proof as in Lemma 11.15 shows that $1 \leq m < \frac{k}{3}+1$ if $E \cong \overline{K^m}$ or $1 \leq n < \frac{k}{2}+1$ if $E \cong K^n$. By Theorem 11.5 it suffices to show that (a) E is l-S-transitive for all $l \leq k-1$, (b) $\Delta(E) \leq k-2$, (c) any vertex set of order $\frac{k-|E|}{2}+1$ in E is connected, and (d) no two non-adjacent vertices of E have $k-2$ common neighbors.

(a) Let $A, B \subseteq S$ be isomorphic graphs with at most $k-1$ vertices for some $\mathcal{S}$-separator S ($\cong E$). Then there is a common neighbor v of these vertices in an adjacent $\mathcal{S}$-separator. Adding a path of suitable length that starts in v and that is only in v adjacent to S, one gets two connected subgraphs of order k and an automorphism of G mapping one to the other. If $|A| \neq 1$, then this automorphism must map the vertices of A onto vertices of S and hence onto B. If $|A| = 1$, let S' be an $\mathcal{S}$-separator such that some induced path of length $k-1$ starting in A ends in S'. Let φ, φ' be the isomorphisms from E to S, S', respectively. Let $A' \subseteq S'$ be $(A^{\varphi^{-1}})^{\varphi'}$. Then we may assume that the path ends in A'. Thus there has to be an automorphism of G mapping A to B or A' to B.

(b) Let S ($\cong E$) be an $\mathcal{S}$-separator. Suppose there is a vertex v of degree at least $k-1$ in $G[S]$. Let $A \subseteq S$ contain v and $k-1$ of its neigbours. Let w be some vertex from an $\mathcal{S}$-separator that is adjacent to S. Then $G[A - v + w]$ is isomorphic to $G[A]$ but there is no automorphism of G mapping the one onto the other. Thus no vertex in S has degree $k-1$ or greater.

(c) Finally, suppose there is a vertex set $X \subseteq V(E)$ of order at least $\frac{k-|E|}{2} + 1$ that is not connected. Let S, S', S'' be three distinct $\mathcal{S}$-separators such that S' is adjacent to the other two. Let $A \subseteq S, A'' \subseteq S''$ be copies of X in S and S'' that contain the components $C, D \subseteq A$ and $C'', D'' \subseteq A''$, respectively. Let $c \in C$ and $c'' \in C''$ be two vertices such that $C - c$ and $C'' - c''$ are isomorphic. Let $d \in D$ be any vertex. Then the graphs $G[A \cup S' \cup A''] - \{c, d\}$ and $G[A \cup S' \cup A''] - \{c'', d\}$ are isomorphic but there is no G-automorphism mapping one to the other. There are k or $k+1$ vertices in these subgraphs, depending on the parity of $|E|$. Since the argument stays valid even if we ignore one of the vertices in S' there is such a CS-transitivity contradicting graph with precisely k vertices.

(d) Suppose that there are two non-adjacent vertices x, y in some $\mathcal{S}$-separator S' ($\cong E$) with $k-2$ common neighbors and let $N \subseteq S'$ be $k-2$ of these neighbors. Let S, S'' be distinct $\mathcal{S}$-separators adjacent to S' and let $s \in S$ and $s'' \in S''$. Then $G[N + x + y]$ and $G[N + s + s'']$ are isomorphic, but there is no automorphism of G mapping one onto the other. $\square$

As a corollary of Lemma 11.14, Lemma 11.15, and Lemma 11.16 we may finish the first case.

THEOREM 11.17. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a minimal cut system of G such that the structure tree of G and $\mathcal{S}$ is basic and every open block is empty. Then there are cardinals $\kappa, \lambda \geq 2$ and integers m, n such that G is isomorphic to one of the following graphs:*

(1) $X_{\kappa,\lambda}(K^1)$,
(2) $X_{\kappa,2}(K^n)$ *with* $n < \frac{k}{2} + 1$,
(3) $X_{2,\lambda}(\overline{K^m})$ *with* $m < \frac{k}{3} + 1$,

(4) $X_{2,2}(E)$ *with* $E \in \mathcal{E}_{k,m,n}$, $m \leq k-2$ *and* $n \leq \frac{k-|E|}{2}+1$,
(5) Y_κ *(if* k *is odd).* □

11.3.2. A non-empty open block. Let us discuss the connected k-CS-transitive graphs with at least two ends for $k \geq 3$ such that no orbit of any smallest $\mathcal{S}$-separator, that separates ends, covers the whole graph. In other words, there is a non-empty open $\mathcal{S}$-block. As mentioned before this case restricts k to be even.

LEMMA 11.18. *Let* $k \geq 3$ *and* G *be a connected* k*-CS-transitive graph with at least two ends and let* $\mathcal{S}$ *be a minimal cut system such that the structure tree of* G *and* $\mathcal{S}$ *is basic. If some open* $\mathcal{S}$*-block is non-empty, then* k *is even.*

PROOF. Recall that there is an induced ray and a path with k vertices whose middle vertex lies in some $\mathcal{S}$-separator if k is odd. We may map the path anywhere into the ray and, thus, we know that there are k succeeding vertices on the ray that belong to an $\mathcal{S}$-separator. Since the diameter is infinite, in every vertex starts an induced path of length $k-1$ and, thus, every vertex lies on a path all whose vertices lie in some $\mathcal{S}$-separator. Thus, if k is odd, then every vertex lies in some $\mathcal{S}$-separator. □

LEMMA 11.19. *Let* $k \geq 3$ *and let* G *be a connected* k*-CS-transitive graph with at least two ends and let* $\mathcal{S}$ *be a minimal cut system such that the structure tree of* G *and* $\mathcal{S}$ *is basic and some open* $\mathcal{S}$*-block is not empty. If vertices* $s, s' \in G$ *belong to different* $\mathcal{S}$*-separators then* s *and* s' *are not adjacent.*

PROOF. Suppose s and s' are adjacent. Then there is some induced path P of length $k-1$ in G such that the innermost (k is even) edge is ss'. Mapping this path with the edge ss' to successive edges on an induced ray, we obtain a ray all whose edges have end vertices only in $\mathcal{S}$-separators. This is a contradiction since in every vertex starts an induced path of length $k-1$ and at least one vertex lies in an open $\mathcal{S}$-block. □

LEMMA 11.20. *Let* $k \geq 3$ *and let* G *be a connected* k*-CS-transitive graph with at least two ends and let* $\mathcal{S}$ *be a minimal cut system such that the structure tree of* G *and* $\mathcal{S}$ *is basic and some open* $\mathcal{S}$*-block is not empty. If vertices* $s, x \in G$ *lie in the same* $\mathcal{S}$*-block with* s *in some* $\mathcal{S}$*-separator and* x *not, then* s *and* x *are adjacent. Furthermore, any two distinct* $\mathcal{S}$*-separators are disjoint.*

PROOF. There is a path of arbitrary length such that any two vertices with distance at least 3 on the path do not lie in the same $\mathcal{S}$-block and every other vertex on this path lies in an $\mathcal{S}$-separator. As there is some induced path between s and x which can be extended if necessary to an induced path of length $k-1$, we know that $d(x,s) < 3$. Since x does not lie in any $\mathcal{S}$-separator and every path enters and leaves $\mathcal{S}$-blocks through $\mathcal{S}$-separators $d(x,s) < 2$ holds. By Lemma 11.12 and Lemma 11.13 we may conclude that distinct $\mathcal{S}$-separators are disjoint. □

The final step to show that these k-CS-transitive graphs with non-empty open blocks resemble some $Z_{\kappa,\lambda}(H_1, H_2)$ is, that their automorphism group acts transitively on its open blocks.

LEMMA 11.21. *Let $k \geq 3$ and let G be a connected k-CS-transitive graph with at least two ends and let $\mathcal{S}$ be a minimal cut system such that the structure tree of G and $\mathcal{S}$ is basic and some open $\mathcal{S}$-block is not empty. The automorphism group of G acts transitively on the open $\mathcal{S}$-blocks.*

PROOF. Since $\mathcal{T}$ has no leaves, every induced path in G of length 2 through some $\mathcal{S}$-block X can be elongated to an induced path P of length $k-1$ such that the innermost edge (k is even) of P lies in no other $\mathcal{S}$-block than X. A similar path P' can be found for any other $\mathcal{S}$-block X' and hence there is an automorphism α of G with $P^\alpha = P'$ and thus also $X^\alpha = X'$. $\square$

Thus every connected k-CS-transitive graph for $k \geq 3$ with more than one end and some non-empty open block is isomorphic to $Z_{\kappa,\lambda}(H_1, H_2)$ for some graphs H_1 and H_2. It remains to specify the building blocks and possible values for κ and λ of these graphs.

LEMMA 11.22. *Let $k \geq 3$ and let $G \cong Z_{\kappa,\lambda}(H_1, H_2)$ be a k-CS-transitive graph and let $\mathcal{S}$ be a minimal cut system such that the structure tree of G and $\mathcal{S}$ is basic and some open $\mathcal{S}$-block is not empty. The following holds:*

(i) *At least one of κ or λ is 2.*
(ii) *If H_i contains two non-adjacent vertices, then H_j ($j \neq i$) is complete and $\kappa = \lambda = 2$.*
(iii) *If H_i contains an edge, then H_j ($i \neq j$) contains no edge.*

PROOF. Recall that $H_1 \ncong H_2$ or $\kappa \neq \lambda$ since the copies of H_1 and H_2 are not $\mathrm{Aut}(G)$-isomorphic. Suppose $\kappa, \lambda \neq 2$, then there are two k-forks. One that pokes in two distinct open $\mathcal{S}$-blocks, and one that pokes in two distinct $\mathcal{S}$-separators. But there is no automorphism of G mapping one to the other. This proves (i).

With an analog argument follows (ii): Suppose κ or λ is greater than 2. Then there is a k-fork that pokes just one copy of an H_i and one that pokes two distinct $\mathcal{S}$-separators ($\lambda > 2$) or two distinct open $\mathcal{S}$-blocks ($\kappa > 2$). Suppose on the other hand that there are two non-adjacent vertices in H_j, then there are two incompatible k-forks, too. One pokes an open $\mathcal{S}$-block and the other one an $\mathcal{S}$-separator.

For (iii), suppose that H_i as well as H_j contains edges. Then there are k-spoons that poke an open $\mathcal{S}$-block and others that poke an $\mathcal{S}$-separator. $\square$

From the previous lemma we immediately get the following corollary:

COROLLARY 11.23. *Let $k \geq 3$ and let $G \cong Z_{\kappa,\lambda}(H_1, H_2)$ be a k-CS-transitive graph and let $\mathcal{S}$ be a minimal cut system such that the structure tree of G and $\mathcal{S}$ is basic and some open $\mathcal{S}$-block is not empty. If both H_1 and H_2 have at least two* *vertices, one is a complete graph and the other one is the complement of a complete graph and $\kappa = \lambda = 2$.* □

To finish the proof in the situation that both H_1 and H_2 have at least two vertices, we will restrict the order of those graphs:

LEMMA 11.24. *Let $k \geq 3$ and let $G \cong Z_{2,2}(H_1, H_2)$ be a k-CS-transitive graph and let $\mathcal{S}$ be a minimal cut system such that the structure tree of G and $\mathcal{S}$ is basic and some open $\mathcal{S}$-block is not empty. If H_1 contains two non adjacent vertices or H_2 contains an edge, then $H_1 \cong \overline{K^m}$ and $H_2 \cong K^n$ with $2m + n \leq k + 1$.*

PROOF. By Corollary 11.23 it suffices to show that the boundaries for m and n hold. Let H_1 be the complement of a complete graph and let H_2 be a complete graph. Let S be some $\mathcal{S}$-separator, and let X and Y be two distinct $\mathcal{S}$-blocks with $S \subseteq X, Y$. Then any graph with precisely k vertices that consists of S, more than $\frac{k-n}{2}$ ($+\frac{1}{2}$ if n is odd) vertices in X and less than $\frac{k-n}{2}$ vertices in Y can be mapped onto a graph consisting of S, $\frac{k-n}{2}$ ($+\frac{1}{2}$ if n is odd) many vertices in X and $\frac{k-n}{2}$ ($-\frac{1}{2}$ if n is odd) many vertices in Y. Thus

$$2m + n \leq 1 + n + (\frac{k-n}{2} + \frac{1}{2}) + (\frac{k-n}{2} - \frac{1}{2}) \leq k + 1.$$

We seemingly loose one vertex in the case that n is even. Since $2m + n$ and k are even then, this is not a true loss. □

LEMMA 11.25. *Let $k \geq 3$ and let $G \cong Z_{\kappa,\lambda}(H_1, H_2)$ be a k-CS-transitive graph and let $\mathcal{S}$ be a minimal cut system such that the structure tree of G and $\mathcal{S}$ is basic and some open $\mathcal{S}$-block is not empty. If $|H_1| = 1$ and one of κ and λ is not 2, H_2 is a complete graph on at most $k - 1$ vertices.*

PROOF. The first part follows directly from Lemma 11.22 (ii). For the second part, suppose that H_2 has more than $k - 1$ vertices. Then every open $\mathcal{S}$-block $\mathring{X}$ contains an isomorphic copy of a K^k and there is a second isomorphic copy of a K^k with $k - 1$ vertices in $\mathring{X}$ and one vertex in some $\mathcal{S}$-separator $S \subseteq X$. Since there is no automorphism of G mapping the one onto the other, H_2 has at most $k - 1$ vertices. □

As the last part in this case of the proof (that there is some non-empty open block) we will determine the graphs H_2 if H_1 is only one vertex and the open blocks are neither complete nor complements of complete graphs.

LEMMA 11.26. *Let $k \geq 3$, let $G \cong Z_{2,2}(H_1, H_2)$ be a k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a minimal cut system of G such that the structure tree of G and $\mathcal{S}$ is basic and that some open $\mathcal{S}$-block is not empty. If $|H_1| = 1$ and H_2 is neither complete nor a complement of a complete graph, then $H_2 \in \mathcal{E}_{k,m,n}$ with $m \leq k - 2$ and $n \leq \frac{k}{2} + 1$.*

PROOF. By Corollary 11.5 it suffices to show that (a) H_2 is l-S-transitive for all $l \leq k-1$, (b) $\Delta(H_2) \leq k-2$, (c) any vertex set of order $\frac{k}{2}+1$ in H_2 is connected, and (d) no two non-adjacent vertices of E have $k-2$ common neighbors.

The proofs of (a), (b) and (d) are analog to those of Lemma 11.16 (a), (b) and (d).

(c) We follow the argument of Lemma 11.16 (c). The boundary of n is slightly different, since there is only one vertex in an $\mathcal{S}$-separator available. Thus every set of order $\frac{k-1}{2}+1$ is connected, since k is even and since a set with at least $\frac{k-1}{2}+1$ vertices has at least $\frac{k}{2}+1$ vertices. □

These lemmas let us finish the case for non-empty open blocks.

THEOREM 11.27. *Let $k \geq 3$, let $G \cong Z_{2,2}(H_1, H_2)$ be a k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a minimal cut system of G such that the structure tree of G and $\mathcal{S}$ is basic and that some open $\mathcal{S}$-block is not empty. Then k is even and G is isomorphic to one of the following graphs:*

(6) $Z_{2,2}(\overline{K^m}, K^n)$ with $2m+n \leq k+1$ and $m \leq n$;
(7) $Z_{\kappa,\lambda}(K^1, K^n)$ with $n \leq k-1$ and cardinals κ, λ with $\kappa = 2$ or $\lambda = 2$;
(8) $Z_{2,2}(K^1, E)$ with $E \in \mathcal{E}_{k,m,n}$, $m \leq k-2$ and $n \leq \frac{k}{2}+1$. □

With Section 11.2 the Theorems 11.17 and 11.27 imply one of our main results, Theorem 11.1.

11.4. Ends of k-CS-transitive graphs

Gray [**42**] asked whether every locally finite k-CS-transitive graph (with $k \geq 3$) is end-transitive. With Theorem 11.1 we may answer his question.

THEOREM 11.28. *Let $k \geq 3$ and let G be a connected locally finite graph. If G is k-CS-transitive, then it is end-transitive.* □

Thus, this class of graphs belongs to those graphs discussed in Chapter 8. Theorem 11.28 does not extend to graphs with vertices of infinite degree. For example the graphs $X_{\kappa,\lambda}$ with $\kappa \geq \aleph_0, \lambda \geq 2$ contain fundamentally different ends: local ends and global ends. Theorem 11.1 shows that in k-CS-transitive graphs with $k \geq 3$ every end is either local or global.

THEOREM 11.29. *Let $k \geq 3$ and G be a connected k-CS-transitive graph with more than one end. Then every end of G is either local or global. The automorphism group of G acts transitively on the local ends, as well as on the global ends. G is end-transitive if and only if it has no local end.* □

Considering the metric ends, we obtain the following corollary.

COROLLARY 11.30. *If $k \geq 3$, then the automorphism group of any k-CS-transitive graph with at least two ends acts transitively on the metric ends of the graph.* □

CHAPTER 12

Connected-homogeneous digraphs

This chapter deals with C-homogeneous digraphs. As we have already written in Chapter 9, we do not only investigate the infinite C-homogeneous digraphs but also the finite such digraphs and thereby, we prove the classification theorems of [**51**] and [**54**]. The chapter is structured as follows. First, we determine the infinite connected C-homogeneous digraphs with infinitely many ends whose underlying undirected graph is a C-homogeneous graph those of *Type I* (Section 12.3), followed by those with infinitely many ends whose underlying undirected graph is not a C-homogeneous graph that are the digraphs of *Type II* (Section 12.4). Thereafter we turn our point of interest to the finite and locally finite C-homogeneous digraphs with at most one end. The first results in this part are obtained in the case that the out-neighborhood (or in-neighborhood) of any vertex is not independent (Section 12.5) and then we look at the more difficult case, where the out-neighborhood and also the in-neighborhood is independent (Section 12.6 and Section 12.7). This latter case also divides into two parts: either the directed triangle embeds into the digraph or not.

It is well known that a transitive connected locally finite graph either contains one, two, or infinitely many ends. For arbitrary transitive connected infinite graphs, this was proved by Diestel, Jung and Möller [**27**]. Since the underlying undirected graph of a transitive digraph is also transitive, the same holds for infinite transitive digraphs. As two-ended connected transitive digraphs are locally finite [**27**, Theorem 7] we refer to Gray and Möller [**45**, Theorem 6.2] for the classification result in this case. Consequently, we complete the classification of the locally finite C-homogeneous digraphs and of the connected C-homogeneous digraphs with more than one end.

12.1. C-homogeneous bipartite graphs

In this chapter we complete the classification of connected C-homogeneous bipartite graphs, which was already done for locally finite graphs, by Gray and Möller [**45**]. They already mentioned that their work should be extendable with not too much effort—and indeed this section has essentially the same structure.

The proof of the locally finite analog [**45**, Lemma 4.4] of Lemma 12.1 is self contained and does not use the local finiteness of the graph. Thus we can omit the proof here.

LEMMA 12.1. *Let G be a connected C-homogeneous bipartite graph with bipartition $X \cup Y$. If G is not a tree and has at least one vertex with degree greater than 2 then G embeds C_4 as an induced subgraph.* □

Let G be a bipartite graph with bipartition $X \cup Y$. Then for each edge $xy \in EG$ we define the neighborhood graph to be:

$$\Omega(x, y) := G[N(x) + N(y) - \{x, y\}]$$

A C-homogeneous graph G is, in particular, edge-transitive. Hence there is a unique neighborhood graph $\Omega(G)$.

LEMMA 12.2. *Let G be a connected C-homogeneous bipartite graph. Then $\Omega(G)$ is a homogeneous bipartite graph, and therefore is one of: an edgeless bipartite graph, a complete bipartite graph, a complement of a perfect matching, a perfect matching, or a homogeneous generic bipartite graph.*

PROOF. If we do not ask $\Omega(G)$ to be finite, the proof of the locally finite analogue [**45**, Lemma 4.5] carries over. Compared to the locally finite case, we only have to deal with one other 'type' of graph, due to [**41**, Remark 1.3] □

LEMMA 12.3. *Let G be a C-homogeneous generic bipartite graph. Then G is homogeneous bipartite.*

PROOF. Let $VG = A \cup B$ be the natural bipartition of G, let X and Y be two isomorphic induced finite subgraphs of G, and let $\varphi : X \to Y$ be an isomorphism. Let $a \in A \setminus X$ be a vertex adjacent to all the vertices of $X \cap B$ and let $b \in B \setminus X$ be a vertex adjacent to all the vertices of $X \cap A$ and to a. Let a', b' be the corresponding vertices for Y. Since G is bipartite, both $G[X+a+b]$ and $G[Y+a'+b']$ are connected induced subgraphs of G that are isomorphic to each other. Furthermore there is an isomorphism $\psi : G[X + a + b] \to G[Y + a' + b']$ such that the restriction of ψ to X is φ. As there is an automorphism of G that extends ψ, this automorphism also extends φ and G is homogeneous. □

THEOREM 12.4. *A connected graph is a C-homogeneous bipartite graph if and only if it belongs to one of the following classes:*

(i) $T_{\kappa,\lambda}$ *for cardinals* κ, λ;
(ii) C_{2m} *for* $m \in \mathbb{N}$;
(iii) $K_{\kappa,\lambda}$ *for cardinals* κ, λ;
(iv) CP_κ *for a cardinal* κ;
(v) *homogeneous generic bipartite graphs.*

PROOF. The nontrivial part is to show that this list is complete. So consider an arbitrary connected C-homogeneous bipartite graph G with bipartition $X \cup Y$. If G is a tree then it is obviously semi-regular and hence a $T_{\kappa,\lambda}$. So suppose G contains a cycle. Then, since G is C-homogeneous, each vertex lies on a cycle. Now G is either a cycle, which is even since G is bipartite, or at least one vertex in G

has a degree greater than 2 and G embeds a C_4, due to Lemma 12.1. Thus $\Omega(G)$ contains at least one edge and by Lemma 12.2 we have to consider the following cases:

Case 1: $\Omega(G)$ *is complete bipartite*. Suppose that there is an induced path $P = uxyv$ in G. Then $\Omega(x, y)$ gives rise to an edge between u and v, a contradiction. Hence G is complete bipartite.

Case 2: $\Omega(G)$ *is the complement of a perfect matching*. Consider $x \in X$ and $y \in Y$ such that $\{x, y\}$ is an edge of G. Since $\Omega(x, y)$ is the complement of a perfect matching and G is not a cycle, there is an index set $I \supseteq \{1, 2\}$ such that $N(x) = \{y\} \cup \{y_i | i \in I\}$, $N(y) = \{x\} \cup \{x_i | i \in I\}$ and for $i \in I$ the vertex x_i is nonadjacent to y_i but adjacent to all y_j with $j \in I \setminus \{i\}$. Since $\Omega(x, y_1)$ is also the complement of a perfect matching there is a unique vertex $a \in N(y_1) \setminus N(y)$. Since x_i with $i \neq 1$ is adjacent to y_1 it is contained in $\Omega(x, y_1)$ and therefore y_i is adjacent to a. Thus for all $i \in I$ we have $N(y_i) = N(y) - x_i + a$. Now by symmetry there is a unique vertex b adjacent to all x_i with $i \in I$ but non-adjacent to x and for all $i \in I$ there is $N(x_i) = N(x) - y_i + b$. If we look at $\Omega(x_1, y_2)$ we have $x, a \in N(y_2)$ and $y, b \in N(x_1)$ which implies $\{a, b\} \in EG$ and hence $N(a) = N(x) - y + b$ and $N(b) = N(y) - x + a$. Because G is connected we have $X = N(y) + a$ and $Y = N(x) + b$ which means that G is itself the complement of a perfect matching.

Case 3: $\Omega(G)$ *is a perfect matching*. For the same reason as for locally finite graphs this case cannot occur (cp. [**45**, Theorem 4.6]).

Case 4: $\Omega(G)$ *is homogeneous generic bipartite*. Let U and W be two disjoint finite subsets of X (of Y). Since G is connected there is a finite connected induced subgraph $H \subset G$ that contains both U and W. By genericity, we find an isomorphic copy H_Ω of H in $\Omega(G)$. Because G is C-homogeneous there is an automorphism φ of G with $H_\Omega^\varphi = H$. Now there is a vertex v in Y (in X) that is adjacent to all vertices in $U^{\varphi^{-1}}$ and non-adjacent to all vertices in $W^{\varphi^{-1}}$. Hence v^φ is adjacent to all vertices in U and none in W which implies that G is generic bipartite. Furthermore G is homogeneous bipartite by Lemma 12.3, as it is C-homogeneous. $\square$

12.2. Local structure of C-homogeneous digraphs of Type I and Type II

In this section we summarize some preliminary results of the relation between a C-homogeneous digraph and a basic cut system $\mathcal{C}$ of this digraph. In particular we investigate the local structure around $\mathcal{C}$-separators.

Lemma 12.5. *Let D be a connected C-homogeneous digraph with more than one end. Let $\mathcal{C}$ be a basic cut system and let S be a $\mathcal{C}$-separator. Then there is no edge xy in D with both vertices in S. In particular, no two $\mathcal{C}$-blocks can share an edge.*

Proof. Let $(A, B) \in \mathcal{C}$ with $A \cap B = S$ and let us suppose that there is $xy \in ED$ with $x, y \in S$. By the minimality of $\mathcal{C}$ each vertex in a $\mathcal{C}$-separator has neighbors in both wings of the corresponding separation. Let $a \in A\backslash B$ and $b \in B\backslash A$ be such neighbors of y. Then there are different possibilities for the direction of their connecting edges. Let us first consider the case that $ay, by \in ED$. Then there is an automorphism α that maps xy onto by. Then S^α lies in B, since $\mathcal{C}$ is nested and $b \in S^\alpha$, and we have either $A \subseteq A^\alpha$ or $A \subseteq B^\alpha$ by the nestedness of $\mathcal{C}$. So we have a vertex b_0, which is either a^α or b^α, that lies either in $B \cap B^\alpha$ or in $B \cap A^\alpha$ such that $b_0 y \in ED$ and $S_0 := S^\alpha$ separates a and b_0. Let $\{A_0, B_0\} = \{A^\alpha, B^\alpha\}$ such that $a \in A_0$ and $b_0 \in B_0$.

Now let α_0 be an automorphism of D such that $a^{\alpha_0} = a$ and $(by)^{\alpha_0} = b_0 y$. Hence the vertex $b_1 := b_0^{\alpha_0}$ lies in $B_1 \setminus A_1$, where $A_1 := A_0^{\alpha_0}$ and $B_1 := B_0^{\alpha_0}$, and we have $b_1 y \in ED$. Since A_0 meets A_1, $S_1 := S_0^{\alpha_0} \neq S_0$ lies in B_0, and $\mathcal{C}$ is nested, we know that A_0 is a proper subset of A_1, B_1 is a proper subset of B_0 and S_0 lies in A_1, which implies $x \in A_1$. Furthermore b_1 and a are separated from each other by both S_0 and S_1. By repeating this process recursively, we have an α_i that fixes a and y and maps b_{i-1} onto b_i and we get a further vertex $b_{i+1} = b_i^{\alpha_i} \in B_{i+1} \setminus A_{i+1}$ that is separated by $S_{i+1} := S_i^{\alpha_i}$ from $a \in A_{i+1} \setminus B_{i+1}$. And with the same argument as before we have that A_i is a proper subset of $A_{i+1} := A_i^{\alpha_i}$, that $B_{i+1} := B_i^{\alpha_i}$ is a proper subset of B_i and that $b_i \in A_{i+1}$. Hence, $b_j \in A_{i+1}$ for all $j \leq i$, which implies $b_i \neq b_j$ for all $i \neq j$.

Thus, after the step $m := |S|+1$ there has to be some $k < |S|$ such that b_k is not contained in S_m and therefore lies in $A_m \setminus B_m$. That is, b_k is also separated from b_m by S_m and $I := \{a, b_k, b_m\}$ forms an independent set. Note that by construction all elements in I have y as a common out-neighbor. Hence, due to the C-homogeneity of D, there is an automorphism β of D which interchanges a and b_k and fixes y and b_m. Note that $a \in A \setminus B \subset A_{k+1} \setminus B_{k+1}$ and $b_m \in B_m \setminus A_m \subset B_{k+1} \setminus A_{k+1}$. Thus, S_{k+1} is a separator containing b_k that separates a and b_m, which implies that S_{k+1}^β is a separator containing a that separates b_k and b_m. But due to the minimality of $\mathcal{C}$, there is a b_k-b_m-path in B_{k+1} that meets S_{k+1} only in b_k and therefore S_{k+1}^β meets both $A_{k+1} \setminus B_{k+1}$ and $B_{k+1} \setminus A_{k+1}$, contradicting the nestedness of $\mathcal{C}$ (confer Remark 2.8).

So let us suppose $by, ya \in ED$. Let α be an automorphism of D with $(xy)^\alpha = ya$ and choose $\{X, Y\} = \{A^\alpha, B^\alpha\}$ such that $b \in X \setminus Y$. Then there is a neighbor c of y in $Y \setminus X$, which is separated from b by S^α. Note that by nestedness, and since both X and Y meet A, we have that either $X \cap B$ or $Y \cap B$ is empty. So $b \in X \cap B$ yields $c \in A$. If $cy \in ED$ then we may take the vertices c, b instead of a, b and get a contradiction by the first case above. Thus we may assume that $yc \in ED$. But then we can map the digraph $D[b, y, a]$ onto $D[b, y, c]$ such that the number of separators that separate b from a without containing one of them equals the number of separators that separate b and c without containing one of them. Due to [**32**, Lemma 4.1] this number is finite. Because of the nestedness of $\mathcal{C}$, every separator that separates the vertices b and a lies entirely in X, and since a and c are joined by a path that lies except for a in $Y \setminus X$, it also separates b and c. But

S^α contains a and separates b from c, a contradiction. The case $ay, yb \in ED$ is analogous.

Let us finally suppose that $ya, yb \in ED$. By considering the digraph D^{-1} instead of D we also may assume that there are $a' \in A \setminus B$ and $b' \in B \setminus A$ with $a'x, b'x \in ED$. Let α be an automorphism of D with $(xy)^\alpha = yb$. Then there is a vertex $b'' \in B \setminus A$ that is separated by S^α from a and such that $b''y \in ED$. But then we have the situation of the previous case and thus we know that no such edge xy exists.

Let X and Y be distinct $\mathcal{C}$-blocks, then there is $x \in X \setminus Y$. Since Y, being a $\mathcal{C}$-block, is maximally $\mathcal{C}$-inseparable, there is a $\mathcal{C}$-separator S that separates x from Y. As X is also $\mathcal{C}$-inseparable, we have $X \cap Y \subseteq S$. Therefore X and Y cannot share an edge. $\square$

Lemma 12.6. *Let D be a connected C-homogeneous digraph with more than one end and let $\mathcal{C}$ be a basic cut system. Then for each 2-arc P in D we have $|P \cap S| \leq 1$ for all $\mathcal{C}$-separators S.*

Proof. Let $P = xay$ be a 2-arc in D and S a $\mathcal{C}$-separator. By Lemma 12.5 we only have to show that S cannot contain both x and y. So assume $\{x, y\} \subseteq S$. Let $(A, B) \in \mathcal{C}$ with $A \cap B = S$ and $a \in A$. Since D is transitive there is an arc zx in D. If z lies in A consider a neighbor z' of x in B. Now either zxa, zxz' or $z'xa$ is an induced 2-arc in D, which we denote by Q, with one vertex in $A \setminus B$ and one vertex in $B \setminus A$. Because D is connected-homogeneous there is an automorphism α with $P^\alpha = Q$. Then S^α contains vertices of both wings of (A, B). By Remark 2.8, this contradicts the nestedness of $\mathcal{C}$. $\square$

Lemma 12.7. *Let D be a connected C-homogeneous digraph with more than one end, let $\mathcal{C}$ be a basic cut system of D, and let S be a $\mathcal{C}$-separator. Then there is no directed path in D with both endvertices in S.*

Proof. Suppose that there is such a path P. We may choose the path such that it has minimal length. Then all of the vertices of P lie in the same $\mathcal{C}$-block X. By Lemma 12.6 the endvertices of any directed path of length 2 are separated by a $\mathcal{C}$-separator. Hence no directed path of length at least 2 can lie in any $\mathcal{C}$-block. $\square$

Lemma 12.8. *Let D be a connected C-homogeneous triangle-free digraph with more than one end, and let $\mathcal{C}$ be a basic cut system. Then for any cut $(A, B) \in \mathcal{C}$ there is no path xyz in $D[A]$ with $y \in A \cap B$.*

Proof. By Lemma 12.5 we only have to show that given a cut $(A, B) \in \mathcal{C}$ there is no 2-arc xyz in D such that $y \in S := A \cap B$ and $x, z \in A \setminus B$. So let us suppose there is such a path. Then y has a neighbor $b \in B \setminus A$. We may assume that their connecting edge is pointing towards y, since otherwise changing the direction of each edge gives a digraph D' which is C-homogeneous and has this property.

Suppose that there is a second neighbor $c \in B \setminus A$ of y. If $yc \in ED$, then there is an $\alpha \in \mathrm{Aut}(D)$ that fixes b, y, z and with $x^\alpha = c$, $c^\alpha = x$, as D is triangle-free.

But then the separations (A, B) and (A^α, B^α) are not nested. Thus we may assume that $cy \in ED$. In this situation let β be an automorphism of D that fixes x, y, b and maps z onto c and vice versa—a contradiction as before.

So b is the unique neighbor of y in B. We may assume that there is another vertex a, say, that lies in S, since otherwise we could map the 2-arc byz onto xyz, as D is C-homogeneous and triangle-free, and, thus, y would seperate x from z, contradicting the fact that x and z lie in the same component of $D - S$. Now consider a path P in D connecting a and y and let $\mathcal{T}$ denote the structure tree of D and $\mathcal{C}$. Let $\mathcal{M}$ be the set of $\mathcal{C}$-blocks containing edges of P. Since $\mathcal{C}$-separators do not contain any edge, distinct blocks cannot contain a common edge. Thus we choose a block $M \in \mathcal{M}$ whose distance to S in $\mathcal{T}$ is maximal with respect to $\mathcal{M}$.

Now each nontrivial component of $P \cap M$ has to contain exactly two edges: An isolated edge would either be contained in a separator, in contradiction to Lemma 12.5, or it would connect M to two distinct neighbors in $\mathcal{T} \cap \mathcal{M}$, contradicting the choice of M. If there is a segment of P in M with a length of at least three, then it contains either a directed subsegment, isomorphic to byz, or a subsegment isomorphic to $by \cup xy$. In each case there exists an isomorphism φ such that S^φ separates the endvertices of this subsegment, which is impossible since M is a $\mathcal{C}$-block.

Considering an arbitrary nontrivial component of $P \cap M$, its two edges have a common vertex which we denote by m. With an analogous argument as above, both edges are directed away from m. Let us denote their heads by u and v, respectively. By construction, u and v lie both in the separator $S_M \subset M$ that lies on the unique shortest path between M and S in $\mathcal{T}$. Consider an arbitrary cut with seperator S_M. Then u has a neighbor u' in the wing not containing m. Let ψ be an automorphism with $(mu)^\psi = by$ and either $(uu')^\psi = yz$, if $uu' \in ED$ or $(u'u)^\psi = xy$, if $u'u \in ED$. Since $\mathcal{C}$ is nested we have $S_M^\psi \subset B$ which means that x and z are separated from b by S_M^ψ. By relabeling $S := S_M^\psi$ and $a := v^\psi$, if neccessary, we may assume that ba is an edge.

Then there is a neighbor z' of b in $B \setminus A$, and we can find an automorphism γ with $(by)^\gamma = ba$ and either $x^\gamma = z'$ or $z^\gamma = z'$, depending on the orientation of the edge between b and z'. Again by the nestedness of $\mathcal{C}$ we have $S^\gamma \subset B$ and also $B^\gamma \subseteq B$. And since x is separated from b by S^γ we have $y \in S^\gamma$. But that implies that y and a both have b as their unique neighbor in B^γ. Hence, $S^\gamma \setminus \{y, a\} \cup \{b\}$ is a seperator in D that seperates ends and has smaller cardinality, contradicting the fact that $\mathcal{C}$ is basic. □

Lemma 12.9. *Let D be a connected C-homogeneous triangle-free digraph that is not a tree and that has more than one end, and let $\mathcal{C}$ be a basic cut system of D. Let S be a $\mathcal{C}$-separator and let $s \in S$. Then there is precisely one $\mathcal{C}$-block that contains s and all edges directed away from s, and there is precisely one $\mathcal{C}$-block that contains s and all edges directed towards s. Furthermore there is $d^+(s) > 1$ and $d^-(s) > 1$.*

Proof. By Lemma 12.8 there is at most one kind of neighbors in each $\mathcal{C}$-block. Suppose first that there is a $\mathcal{C}$-block Z with only one neighbor a of s. We may assume that $as \in ED$. By C-homogeneity, we can map each edge xs onto as. As there is by Lemma 12.5 precisely one $\mathcal{C}$-block Y that contains xs, Y contains no other neighbor of s, because the same holds for s and Z. Thus each component of each $\mathcal{C}$-block is either a single vertex or a star the edges of which are directed towards the leaves of the star. If each $\mathcal{C}$-block is a tree and every $\mathcal{C}$-separator consists of one vertex, then the digraph D has to be a tree. Since we excluded this case, there is a second vertex $t \in S$. For every component C of $D - S$, there is an (undirected) s-t-path P with all its vertices but s and t in C. Let X be a $\mathcal{C}$-block with maximal distance to S in the structure tree of G and $\mathcal{C}$ such that there are at least two edges from P in X. This $\mathcal{C}$-block exists by Lemma 2.14. As each component of X that contains edges is a star, the longest subpath of P that lies completely in X has length 2. Let xyz be such a subpath. Then due to Lemma 12.6 we have $xy, zy \in ED$ and y is the only X-neighbor of both x and z. Let S' be the $\mathcal{C}$-separator in X that separates X from S. Then, S' contains x and z. But, as in the previous lemma, $S' \setminus \{x, z\} \cup \{y\}$ would be a separator of smaller cardinality separating two ends, a contradiction.

Thus a $\mathcal{C}$-block cannot contain s together with a single neighbor of s and by C-homogeneity there has to be one $\mathcal{C}$-block that contains all in-neighbors of s and one that contains all out-neighbors of s. $\square$

Lemma 12.10. *Let D be a connected C-homogeneous triangle-free digraph that is not a tree and that has more than one end, and let $\mathcal{C}$ be a basic cut system of D. Then each $\mathcal{C}$-separator has degree two in the structure tree $\mathcal{T}$ for D and $\mathcal{C}$.*

Proof. Let S be a C-separator. Then for each component X of $\mathcal{T} - S$ the vertex set $(\bigcup X) \setminus S$ is the union of components of $D - S$. Since each $s \in S$ has a neighbor in each component of $D - S$, it also has at least one neighbor in each component of $\mathcal{T} - S$. With Lemma 12.9 we have $d_{\mathcal{T}}(S) = 2$. $\square$

If we combine Lemma 12.9 and Lemma 12.10 we get the following

Corollary 12.11. *Let D be a connected C-homogeneous triangle-free digraph that is not a tree and that has more than one end, and let $\mathcal{C}$ be a basic cut system of D. Let B be a $\mathcal{C}$-block, $S \subset B$ a $\mathcal{C}$-separator and $s \in S$. If s has no neighbor in B, then there is exactly one $\mathcal{C}$-separator $S' \subset B$ such that $s \in S' \cap S$. If s has a neighbor in B, then S is the only $\mathcal{C}$-separator in B that contains s.* $\square$

Lemma 12.12. *Let D be a connected C-homogeneous digraph with more than one end that embeds a triangle, and let $\mathcal{C}$ be a basic cut system of D. Then every $\mathcal{C}$-block that contains edges is a tournament and D has connectivity* 1.

Proof. Let S be a $\mathcal{C}$-separator and let $x \in S$. Then x has adjacent vertices in both wings of each cut $(A, B) \in \mathcal{C}$ with $A \cap B = S$. As D contains triangles, each edge lies on a triangle. We know that each wing of (A, B) contains both an in-and an out-neighbor of x, as any triangle contains a 2-arc and D is edge-transitive. Thus every induced path of length 2 in D can be mapped on a path crossing S, i.e. a path both end vertices of which lie in distinct wings of (A, B). Hence no two vertices in the same $\mathcal{C}$-block can have distance 2 from each other and, in particular, every component of every $\mathcal{C}$-block has diameter 1.

To prove that each $\mathcal{C}$-block has diameter 1 we just have to show that each $\mathcal{C}$-block is connected. So let us suppose that this is not the case. Let X be a $\mathcal{C}$-block and let P be a minimal (undirected) path in D from one component of X to another. Let Y be a $\mathcal{C}$-block with maximal distance in the structure tree of D and $\mathcal{C}$ to X that contains edges of P. By Lemma 12.5 the block Y has to contain at least two edges and there are two non-adjacent vertices in the same component of Y. This contradicts the fact that these components are complete graphs. Hence each $\mathcal{C}$-block that contains edges has precisely one component which has diameter 1.

For any $\mathcal{C}$-block X, there is a $\mathcal{C}$-separator S with $S \subseteq X$. By Lemma 12.5, S contains no edge and thus precisely one vertex. $\square$

12.3. C-homogeneous digraphs of Type I

In this section we shall completely classify the countable connected C-homogeneous digraphs of Type I with more than one end and give—apart from the classification of infinite uncountable homogeneous tournaments—a classification of uncountable such digraphs. As a part of the countable classification we apply the classification of Lachlan [**71**], see also [**21**], of the countable homogeneous tournaments (see Theorem 2.31).

The underlying undirected graph of a digraph $X_\lambda(T)$ for a homogeneous tournament T is a distance-transitive graph as described in [**55, 76, 81**]. Thus, if a digraph $X_\lambda(T)$ is C-homogeneous, then so is its underlying undirected graph.

THEOREM 12.13. *Let D be a connected digraph with more than one end. Then D is C-homogeneous of Type I if and only if one of the following statements holds:*

(1) *D is a tree with constant in- and out-degree;*
(2) *D is isomorphic to a $X_\lambda(T^\kappa)$, where κ and λ are cardinals with $\lambda \geq 2$ and κ either* 3 *or infinite and T^κ is a homogeneous tournament on κ vertices.*

PROOF. Let us first assume that D is a C-homogeneous digraph of Type I. Then the underlying undirected graph is isomorphic to a $X_{\kappa,\lambda}$ for cardinals $\kappa, \lambda \geq 2$. If $\kappa = 2$, then D is a tree with constant in- and out-degree, so we may assume $\kappa \geq 3$. As each block is a complete digraph, it is homogeneous and, thus, we conclude from Theorem 2.31 that the cardinal κ has to be either 3 or infinite. This proves the necessity-part of the statement.

Since the digraphs of part (1) are obviously C-homogeneous of Type I, we just have to assume for the remaining part that D is isomorphic to $X_\lambda(T^\kappa)$ for a cardinal $\lambda \geq 2$ and a homogeneous tournament T^κ on κ vertices for a cardinal κ that is either 3 or infinite. Let $\mathcal{C}$ be a basic cut system of D. Let X and Y

be two connected induced finite and isomorphic subdigraphs of D. Let φ be the isomorphism from X to Y. If X has no cut vertex, then X lies in a subgraph of D that is a homogeneous tournament and the same is true for Y, so φ extends to an automorphism of D. So let $x \in VX$ be a cut vertex of X. Hence x^φ is a cut vertex of Y. It is straight forward to see that for any $\mathcal{C}$-block B the image of $X \cap B$ in Y is precisely the intersection of Y with a $\mathcal{C}$-block A. Since the $\mathcal{C}$-blocks are all isomorphic homogeneous tournaments, the isomorphism from $X \cap B$ to $Y \cap A$ extends to an isomorphism from X to Y. Thus the isomorphism from X to Y easily extends to an automorphism of D. Since the underlying undirected graph is C-homogeneous by Corollary 11.3, D is C-homogeneous of Type I. □

Lachlan's theorem together with Theorem 12.13 enables us to give a complete classification of countable connected C-homogeneous digraphs of Type I and with more than one end:

Corollary 12.14. *Let D be a countable connected digraph with more than one end. Then D is C-homogeneous of Type I if and only if one of the following assertions holds:*

(1) *D is a tree with constant countable in- and out-degree;*
(2) *D is isomorphic to a $X_\lambda(Y)$, where κ is a countable cardinal greater or equal to* 2 *and Y is one of the four non-trivial homogeneous tournaments of Theorem* 2.31. □

12.4. C-homogeneous digraphs of Type II

12.4.1. Reachability and descendant digraphs. In this subsection we prove that, if a connected C-homogeneous digraph D with more than one end contains no triangles, then D is highly-arc-transitive, each reachability digraph of D is bipartite, and, if furthermore D has infinitely many ends, then the descendants of each vertex in D induce a tree. All these properties were proved to be true in the case that D is locally finite, see [**45**, Theorem 4.1].

Theorem 12.15. *Let D be a connected C-homogeneous triangle-free digraph with more than one end. Then D is highly-arc-transitive.*

Proof. Let $\mathcal{C}$ be a basic cut system. It suffices to show that each directed path is induced. Suppose this is not the case. Then there is a smallest k such that there is a k-arc $A = x_0 \dots x_k$ that is not induced. Hence there is an edge between x_0 and x_k. Consider a $\mathcal{C}$-separator S that contains x_1. By Lemma 12.7 we have $x_k \notin S$ and by Lemma 12.5 we have $x_0 \notin S$; hence x_0 and x_k lie on the same side of S. But then the same holds for x_{k-1} and so on. So finally x_0 and x_2 have to lie on the same side of S, in contradiction to Lemma 12.8. □

Theorem 12.16. *Let D be a connected C-homogeneous triangle-free digraph with more than one end. Then $\Delta(D)$ is bipartite and if D is not a tree, then each*

Δ_e with $e \in ED$ is a component of a $\mathcal{C}$-block. Furthermore, if D has infinitely many ends, then every descendant digraph $\mathrm{desc}(x)$ *with $x \in VD$ is a tree.*

Proof. Let $\mathcal{C}$ be a basic cut system. We first show that either D is a tree or any Δ_e with $e \in ED$ is a component of a $\mathcal{C}$-block. Let us assume that D is not a tree. Lemma 12.9 immediately implies that Δ_e for any $e \in ED$, cannot be separated by any $\mathcal{C}$-separator and, thus, each Δ_e lies in a $\mathcal{C}$-block. As there are induced paths of length 2 crossing some $\mathcal{C}$-separator and as D contains no triangle, a component of a $\mathcal{C}$-block X cannot contain more vertices than Δ_e with $e \in E(D[X])$ contains. Thus Δ_e is a component of a $\mathcal{C}$-block.

Suppose that $\Delta(D)$ is not bipartite. Then there is a cycle of odd length in $\Delta(D)$. Thus there has to be a directed path of length at least 2 on that cycle. By Lemma 12.6 this path lies in distinct $\mathcal{C}$-blocks. This is not possible as shown above and thus $\Delta(D)$ has to be bipartite.

Now suppose that there is $x \in VD$ such that $\mathrm{desc}(x)$ contains a cycle. So by transitivity there is a descendant y of x such that there are two x-y-arcs that are apart from x and y totally disjoint. Thus, since we are C-homogeneous, any two out-neighbors of x have a common descendant. Assume that there are two distinct $\mathcal{C}$-separators S, S' such that both $Y := S \setminus S'$ and $Y' := S' \setminus S$ contain an out-neighbor of x. Then it exists a vertex z in D with Y-z- and Y'-z-arcs. But by the Lemmas 12.7 and 12.8 the vertices x and z cannot lie on the same side of S and S', respectively, hence S and S' meet on both sides, a contradiction to the nestedness of $\mathcal{C}$. Thus there is a $\mathcal{C}$-separator S_{+1} that contains the whole out-neighborhood of x. This implies that all descendants of distance k are contained in a common $\mathcal{C}$-separator S_{+k}, since either all distinct k-arcs originated at x are disjoint, and we can apply the same argument as above, or each two of those k-arcs intersect in a vertex x' in D that has the same distance to x on both arcs by Lemma 12.7, and we are home by induction.

With a symmetric argument we get that each k-arc that ends in x has to start in a common $\mathcal{C}$-separator S_{-k}. For a path P in D that starts in x, let $\sigma(P)$ denote the difference of the number of edges in P that are directed away from x (with respect to P) minus the number of edges of the other type. Then one easily checks that the endvertex of P lies in $S_{\sigma(P)}$. Since all $\mathcal{C}$-separators have the same finite order s, say, there can be at most $2s$ rays that are eventually pairwise disjoint. Hence D has finitely many ends, which proves the last statement of the theorem. $\square$

Lemma 12.17. *Let D be a connected C-homogeneous triangle-free digraph with more than one end and let $\mathcal{C}$ be a basic cut system of D. Then for each $\mathcal{C}$-separator S of order at least 2 there is a reachability digraph Δ_e and a $\mathcal{C}$-block K such that $|S \cap \Delta_e| \geq 2$, $\Delta_e \subseteq K$, and $S \subseteq K$.*

Proof. Let S be a $\mathcal{C}$-separator with $|S| \geq 2$. Suppose that there is no reachability digraph Δ_e with $|S \cap \Delta_e| \geq 2$. Let $x, y \in S$ and let P be an x-y-path in a component of $D - S$. Let B be a $\mathcal{C}$-block that contains edges of P and such that $d_{\mathcal{T}}(S, B)$ is maximal with this property. Then the $\mathcal{C}$-separator $S_B \subseteq B$ that separates S and B in $\mathcal{T}$ has the desired property and thus each $\mathcal{C}$-separator has it, in contradiction to the assumption. $\square$

We have roughly described the global structure of C-homogeneous digraphs. To investigate the local structure of these graphs, we show that the underlying undirected graph of each reachability digraph is a connected C-homogeneous bipartite graph. Such graphs were already described in Section 12.1.

LEMMA 12.18. *Let D be a connected C-homogeneous digraph such that $\Delta(D)$ is bipartite. Then the underlying undirected graph of $\Delta(D)$ is a connected C-homogeneous bipartite graph.* $\square$

12.4.2. The classification. As a first result we prove that no connected C-homogeneous digraph of Type II with more than one end contains any triangle.

LEMMA 12.19. *Let D be a connected C-homogeneous digraph of Type II with more than one end. Then D contains no triangle.*

PROOF. Let $\mathcal{C}$ be a basic cut system and suppose that D contains a triangle. By Lemma 12.12, every $\mathcal{C}$-block of D that contains an edge is a tournament and D has connectivity 1. Hence, each $\mathcal{C}$-block contains edges and the $\mathcal{C}$-blocks have to be homogeneous tournaments. Thus, D is of Type I in contradiction to the assumption. $\square$

Now we are able to classify the connected C-homogeneous digraphs of Type I with at least two ends and connectivity 1.

LEMMA 12.20. *Let D be a connected C-homogeneous digraph of Type II with more than one end. If D has connectivity* 1*, then D is isomorphic to $DL(\Delta(D))$.*

PROOF. This is direct consequence of Lemma 12.19 and Lemma 12.9. $\square$

In the next two theorems we prove that in the cases that the reachability digraph is either isomorphic to CP_κ or to $K_{2,2}$ the digraph has connectivity at most 2. Thus, in this case it remains to determine those with connectivity exactly 2.

THEOREM 12.21. *Let D be a connected C-homogeneous digraph of Type II with infinitely many ends and with $\Delta(D) \cong CP_\kappa$ for a cardinal $\kappa \geq 3$. If D has connectivity more than one, then D is isomorphic to $M(\kappa, m)$ for an $m \in \mathbb{N}$ with $m \geq 2$.*

PROOF. By Lemma 12.19 the digraph D contains no triangle. Let $\mathcal{C}$ be a basic cut system and let $\mathcal{T}$ be the structure tree of D and $\mathcal{C}$. Let S^0 be a $\mathcal{C}$-separator, let $X^0 = \Delta_e$ for an $e \in ED$ such that $|S^0 \cap X^0| \geq 2$, and let K^0 be a $\mathcal{C}$-block with $S^0 \subseteq K^0$ and $\Delta_e \subseteq K^0$, which all exists by Lemma 12.17. Let $A \cup B$ be the natural bipartition of X^0 such that its edges are directed from A to B. For each $a \in A$ let us denote with b_a the unique vertex in B such that ab_a is not an edge in X^0. By symmetry we may assume that $A \cap S^0 \neq \emptyset$, so let $a \in A \cap S^0$.

First we will show that $X^0 \cap S^0 = \{a, b_a\}$. Since S^0 contains no edges by Lemma 12.5 it suffices to show that $A \cap S^0 = \{a\}$. So let us suppose that there is another vertex $a' \neq a$ in $A \cap S^0$. Since any two vertices in A have a common successor in B, we have $A \subseteq S^0$ by C-homogeneity. Let $a' \in A$ be distinct from a and P an induced a-a'-path whose interior is contained in $D - K^0$. Denote the unique neighbor of a on P by c. Taking into account that X^0 is a CP_κ, there is a common successor for each pair of A-vertices; let b be such a common successor of a and a'. Since S^0 separates both, b and b_a, from the interior of P, the paths cPb and cPb_a are isomorphic and,by C-homogeneity, we can map cPb onto cPb_a by an automorphism φ of D. Then a^φ is a successor of c that sends an edge to b_a. Hence a^φ lies in A and is distinct from a, contradicting the fact that desc(c) is a tree. Thus we know that $X^0 \cap S^0 = \{a, b_a\}$ for a vertex $a \in A$.

For the remainder let $X^0 \cap S^0 = \{x_0, x_1\}$. Because each vertex clearly lies in exactly two distinct reachability digraphs, there is a unique reachability digraph $X^1 \neq X^0$ that contains x_1. If $x_0 \in X^1$ then it is straight forward to see that $D \cong M(\kappa, 2)$. So assume $x_0 \notin X^1$ and let ψ be an automorphism of D mapping X^0 onto X^1 and x_0 to x_1. Let S^1, K^1 denote the image under ψ of S^0, K^0, respectively, and let $x_2 = x_1^\psi$. Since $\mathcal{C}$ is basic there is an induced x_0-x_1-path P the interior of which lies in $D - K^0$. We shall show that P contains x_2.

Suppose that P does not contain x_2 and has minimal length with this property. Let u be the neighbor of x_1 on P, which clearly lies in X^1, and let v be a neighbor of u in X^1 distinct from x_1. If v does not lie on P, then Puv is a path of the same length as P which is induced by the minimality of P and Theorem 12.16, contradicting the fact that x_0 and v cannot lie in a common reachability digraph. On the other hand, if v lies on P then consider a neighbor w of x_2 in X^1 distinct from v. Remark that since X^1 is a CP_κ there is an edge between v and x_2. Thus by the choice of P the path Pvx_2w is induced and of the same length as P, which is impossible since x_0 and w do not belong to a common reachability digraph. Hence P contains x_2.

We have just proved that $\{x_1, x_2\}$ separates x_0 from any neighbor of x_1 in X^1. Hence all $\mathcal{C}$-separators have order 2 and thus the blocks which contain edges consist each of a single reachability digraph. Now we repeat the previous construction to continue the sequences $(X^i)_{i\in\mathbb{N}}$, $(S^i)_{i\in\mathbb{N}}$, $(K^i)_{i\in\mathbb{N}}$ and $(x_i)_{i\in\mathbb{N}}$, respectively. Since Px_2 is an induced x_0-x_2-path the interior of which lies in $D - K^1$, we can apply the same argument as above to assure that P contains x_3. Hence by induction we have $x_i \in P$ for all $i \in \mathbb{N}$, and since P is finite there is an $m \in \mathbb{N}$ such that $x_m = x_0$. Furthermore we have $X^m = X^0$, $S^m = S^0$ and $K^m = K^0$. One can verify that $\{x_0, x_1, \ldots, x_{m-1}\}$ forms a maximal $\mathcal{C}$-inseparable set—a $\mathcal{C}$-block—which means that D is isomorphic to $M(\kappa, m)$. □

Theorem 12.22. *Let D be a connected C-homogeneous digraph of Type II with infinitely many ends and with $\Delta(D) \cong K_{2,2}$. If D has connectivity more than one, then D is isomorphic to $M'(2m)$ for $2 \leq m \in \mathbb{N}$.*

Proof. Lemma 12.19 implies that D contains no triangle. Let $\mathcal{C}$ be a basic cut system of D. Let S^0 be a $\mathcal{C}$-separator and let $X^0 = \Delta_e$ for an $e \in ED$ such that $|S^0 \cap X^0| \geq 2$. Such an X^0 exists by Lemma 12.17. As $\Delta(D) \cong K_{2,2}$ and as no $\mathcal{C}$-separator contains any edge by Lemma 12.5, there is $|S^0 \cap X^0| = 2$. So let x_0, x_1 be the two vertices in $X^0 \cap S^0$. Let X^1 be the other reachability digraph that contains x_1 and let x_2 be the unique vertex in X^1 that is not adjacent to x_1. Let ψ be an automorphism of D that maps X^0 onto X^1 and let S^1 be the image of S^0 under ψ.

With the same technique as in the previous proof, we can verify that $\{x_1, x_2\}$ separates D and so $S^0 = \{x_0, x_1\}$. We can continue the sequences $(x_i)_{i\in\mathbb{N}}$ and $(S^i)_{i\in\mathbb{N}}$ so that $S^1 = \{x_1, x_2\}$ and $S^i = \{x_i, x_{i+1}\}$, and there is an $n \in \mathbb{N}$ such that $x_n = x_0$. Since D has infinitely many ends we have $n \geq 3$, and as $x_i \in S^i$ only holds for all even integers i we have $n = 2m$ with $m \geq 2$. Now analog as in the proof of Theorem 12.21 $\bigcup_i S^i$ forms a $\mathcal{C}$-block that contains no edges. Hence there are precisely two $\mathrm{Aut}(D)$-orbits on the $\mathcal{C}$-blocks and D is isomorphic to $M'(2m)$. □

If we assume $\Delta(D)$ to be one of the other possibilities as described in Theorem 12.4, then the C-homogeneous digraphs have—in contrast to the other two cases—connectivity 1.

Lemma 12.23. *Let D be a connected C-homogeneous digraph of Type II with infinitely many ends and such that $\Delta(D)$ is isomorphic to a $T_{\kappa,\lambda}$ for cardinals κ, λ, a C_{2m} with $4 \leq m \in \mathbb{N}$, a $K_{\kappa,\lambda}$ for cardinals $\kappa, \lambda \geq 2$, or an infinite homogeneous generic bipartite digraph. Then D has connectivity* 1.

Proof. Since D is of Type II, it contains no triangle by Lemma 12.19. Let us suppose that D has connectivity at least 2 and let $\mathcal{C}$ be a basic cut system of D. Let S be a $\mathcal{C}$-separator and let X be a reachability digraph with $|S \cap X| \geq 2$ as in Lemma 12.17. We investigate the given reachability digraphs one by one and get in each case a contradiction and, thereby, we get a contradiction in general to the assumption that D has connectivity at least 2. So let us assume that $X \cong T_{\kappa,\lambda}$ for cardinals κ, λ. By Lemma 12.9 we know that $\kappa, \lambda \geq 2$, as D is not a tree. Let $x, y \in S \cap X$ such that $d_X(x, y)$ is maximal. Such vertices exist as S is finite. Let e_1 be the first edge on the path from x to y in X and let e_2 be another edge incident with x. There is an $\alpha \in \mathrm{Aut}(D)$ with $e_1^\alpha = e_2$. But then y^α lies in a common separator with x, as $x^\alpha = x$. By Corollary 12.11 the separator S^α has to be the same as S. But this contradicts the maximality of $d_X(x, y)$, as $d_X(y^\alpha, y) > d_X(x, y)$.

Let us now assume that $X \cong C_{2m}$ for a $4 \leq m \in \mathbb{N}$ and let x, y be distinct vertices in $S \cap X$. Then there is an induced path P from x to y that lies apart from x and y in a component of $D - S$ that intersects trivially with X. We first show that we may assume that $d_X(x, y) \geq 4$. Let e_1, e_2 be the two edges in $D[X]$ that are incident with x. If $d_X(x, y) = k \leq 3$, then let $\alpha \in \mathrm{Aut}(D)$ with $e_1^\alpha = e_2$. Then there is $d_X(y, y^\alpha) = 2k$, as $m \geq 4$. Thus we have shown that there are $x, y \in S \cap X$ with $d_X(x, y) \geq 4$. Let s_1 and s_2 be the vertices in X that are adjacent to y and let t be a vertex in X that is adjacent to x. Since $d_X(x, y) \geq 4$, the graphs $txPys_i$ for $i = 1, 2$ are induced paths. Hence there is an automorphism α of D that maps $txPys_1$ onto $txPys_2$ and thus $d_X(s_1, x) = d_X(s_2, x)$ and $d_X(s_1, t) = d_X(s_2, t)$, a contradiction as X is a cycle.

For the next case let us assume that $X \cong K_{\kappa,\lambda}$ for cardinals $\kappa, \lambda \geq 2$. Let $A \cup B$ be the natural bipartition of X. Since $|S \cap X| \geq 2$, the vertices in $S \cap X$ lie in the same partition set, A say. By the C-homogeneity it is an immediate consequence that $A \subseteq S$. As the $\mathcal{C}$-separators have minimal cardinality with respect to separating ends, there is $|A| \leq |B|$. If there is a $\mathcal{C}$-separator S' with $|S' \cap B| \geq 2$, then $B \subseteq S'$. If in addition the intersection of B with another reachability digraph distinct from X is B, then it is a direct consequence that $\kappa = \lambda$ is finite and that D has two ends. Thus there are two distinct reachability digraphs X_1, X_2 that intersect with B non-trivially and that are distinct from X. Let A_1, B_1, A_2, B_2 be the natural bipartitions of X_1, X_2, respectively. Let P be an induced path from $A_1 \cap B$ to $A_2 \cap B$ in a component of $D - S'$ that intersects non-trivially with X. Let a be the vertex on P that is adjacent to the vertex in $P \cap A_1$ and let b be a vertex in $B \cap A_1$ not on P. Then there is an automorphism α of D that maps P onto baP. But this contradicts the fact that the endvertices of P lie both in B but the endvertices of baP do not lie in in any common reachability digraph as $|A_1 \cap B| = 1$. Thus we conclude that $|B \cap S'| = 1$. So let $x, y, z \in B$ be three distinct vertices. There is a shortest induced path P from x to y in that component of $D - S$ that contains B. Let $a \in A$ and let b be the vertex on P with distance 2 to y. Then there is an automorphism α of D that maps $zaxPb$ onto $yaxPb$. Thus we conclude that $d(b, z) = 2$. But then z has to have incident edges that are directed both towards or both from distinct $\mathcal{C}$-blocks. This contradicts Lemma 12.9.

Let us finally assume that X is isomorphic to an infinite homogeneous generic bipartite digraph. Let again $A \cup B$ be the natural bipartition of X. Since X is homogeneous, all vertices in the same set A or B have distance 2 to each other. We conclude that $|S \cap A| \geq 2$ immediately implies $A \subseteq S$ which contradicts the finiteness of S. Conversely we also know $|B \cap S| \leq 1$. Since D has connectivity at least 2, there is $|A \cap S| = 1 = |B \cap S|$. Let a, b be the vertices in $A \cap S, B \cap S$, respectively, and let $ab'a'b$ be a path of length 3 from a to b. This path exists because each two vertices in the same set A or B have distance 2 to each other as before. Since there are infinitely many vertices in A that are adjacent to b' but not to b, all these vertices have to lie in S, a contradiction. Thus we conclude that D has connectivity 1. $\square$

Let us summarize the conclusions of this section in the following theorem. In its proof we will finally prove that all the candidates for C-homogeneous digraphs are really C-homogeneous.

Theorem 12.24. *Let D be a connected digraph of Type II with infinitely many ends. Then D is C-homogeneous if and only if one of the following holds:*

(1) $\Delta(D) \cong CP_\kappa$ *for a cardinal* $\kappa \geq 3$ *and* $D \cong DL(\Delta(D))$.
(2) $\Delta(D) \cong C_{2m}$ *for* $2 \leq m \in \mathbb{N}$ *and* $D \cong DL(\Delta(D))$.
(3) $\Delta(D) \cong K_{\kappa,\lambda}$ *for cardinals* $\kappa, \lambda \geq 2$ *and* $D \cong DL(\Delta(D))$.
(4) $\Delta(D)$ *is isomorphic to an infinite homogeneous generic bipartite digraph and* $D \cong DL(\Delta(D))$.
(5) $\Delta(D) = CP_\kappa$ *and* $D \cong M(\kappa, m)$ *for a cardinal* $\kappa \geq 3$ *and* $2 \leq m \in \mathbb{N}$.
(6) $\Delta(D) = K_{2,2}$ *and* $D \cong M'(2m)$ *for* $2 \leq m \in \mathbb{N}$.

Proof. By the Lemmas 12.19, 12.20, and 12.23 and by the Theorems 12.21 and 12.22, it remains to show that the described digraphs are indeed C-homogeneous. Remark that the underlying undirected graph of $DL(T_{\kappa,\lambda})$ is a regular tree and thus $DL(T_{\kappa,\lambda})$ is not of Type II. It is straight forward to see that the graphs of the part (1)-(4) are C-homogeneous. So let $D \cong M(\kappa, m)$ for an $m \in \mathbb{N}$ with $m \geq 2$ and a cardinal κ. Let $\mathcal{C}$ be a basic cut system of D. Let A and B be two connected induced finite and isomorphic subdigraphs of D and let φ be an isomorphism from A to B. Let us first consider the case that A contains no 2-arc. Then both A and B lie in a reachability digraph, each. Without loss of generality we may assume that they lie in the same reachability digraph Δ of D. But, as the reachability-digraphs are obviously C-homogeneous, it is straight forward to see that the isomorphism φ from A to B first extends to an automorphism of Δ and then also to an automorphism of D. So let us assume that A contains a 2-arc. Let us consider the case that A is a k-arc for some $k \geq 2$. Let A_1, A_2 be two induced subdigraphs of A that have one common vertex, are both connected, and whose union is A. Then both are shorter arcs and, by induction, we can extend both restrictions, $\varphi|_{A_1}$ and $\varphi|_{A_2}$, to automorphisms ψ_1, ψ_2 of D, respectively. Let S be a $\mathcal{C}$-separator that contains the common vertex of A_1 and A_2. There are two possibilities for S if $m \geq 3$, and one possibility if $m = 2$. If $m = 2$, then it is an immediate consequence that $S^{\psi_1} = S^{\psi_2}$ and that we can combine the two automorphisms to one that extends φ by setting $\varphi|_{K_i} = \psi_i|_{K_i}$, where K_i is the component of $D - S$ that contains vertices of A_i, and $\varphi|_S = \psi_1|_S$. So we assume that $m \geq 3$. We choose in this case S so that it lies in a common $\mathcal{C}$-block with an edge of A_1. Let $S' = S^{\psi_1}$. As we had just two possibilities for the choice of S, the image of S under ψ_2 has to be S', too. In the same way as above, we can combine appropriate restrictions of ψ_1 and ψ_2 to an automorphism of D that extends φ.

Now let us assume that A is no k-arc. Then there is a $\mathcal{C}$-block X that contains two edges of A that have a common vertex. Let us first assume that X contains three edges of A. Then, since $\Delta \cong CP_\kappa$, we know that $X \cap A$ is connected. Thus, $(X \cap A)^\varphi$ lies in a $\mathcal{C}$-block Y of B and we have $(X \cap A)^\varphi = B \cap Y$. We have already shown that we can extend $\varphi|_{A \cap X}$ to an automorphism ψ_X of D. If each component of $D - X$ contains at most one component of A, then we have the extensions of the restriction of φ to these components and we can construct, as in the case of k-arcs, an automorphism of D. So we assume that there is at least one component C of $D - X$ such that, for the $\mathcal{C}$-separator $S \subseteq X$ that separates X and C, the digraph $A' = A \cap (C \cup S)$ contains at least two components. As the $\mathcal{C}$-separators have cardinality 2, A' consists of precisely two components. Let $Z \neq X$ be the second $\mathcal{C}$-block that contains S. If Z contains edges, that means $m = 2$, then $A \cap Z$ consists of precisely two edges that have their other incident vertices again in a common separator. Since the same must be true for $Z^{\psi_X} \cap B$, we may assume inductively that we have extended $\varphi|_{A \cap X}$ so that ψ_X coincides with $\varphi|_{Z \cap A}$ on $Z \cap A$. Thus, we can consider the case that Z does not contain any edge. There is an enumeration $z_1, \ldots, z_m$ of the vertices of Z such that $\{z_m, z_1\}$ and for all $i \leq m$ also $\{z_i, z_{i+1}\}$ are all the $\mathcal{C}$-separators in X. We may assume that $S = \{z_1, z_m\}$. Let C_i be the subdigraph induced by z_i, z_{i+1} and that component of $D - \{z_i, z_{i+1}\}$ that contains no other z_j. If $C_i \cap A$ consists of one component and contains z_i and z_{i+1}, then we can extend the restriction of φ to that component to an automorphism ψ_i of D and we may suppose that we have chosen ψ_X so that they are equal on C_i. If there is one C_i that has at least two components of $A \cap C_i$, then it is unique and we can suppose that $\psi_X|_{C_i} = \psi_i|_{C_i}$ on all C_i such that $A \cap C_i$ is connected. By induction, we can assume that the same holds also for a component C_i such that $A \cap C_i$ is not connected. So the only remaining case is if $C_i \cap A$ is connected but contains only one of the vertices z_i, z_{i+1}. But in this case we know that this situation occurs in at most one other C_j with $i \neq j$. Then $\varphi|_{A \cap C_k}$ with $k \in \{i, j\}$ extends to an automorphism ψ_k of D by induction. Because these two automorphisms exist, we know that $S_i^{\psi_k}$ contains only one vertex of B, and hence we can assume that ψ_X and ψ_k coincide on C_k. Thus, if we extend this to all the components of $D - X$, we know that ψ_X extends φ.

The final case that remains is when the block X contains only two edges. Then it might be the case that $X \cap A$ is not connected. If it is not connected, then there has to be a $\mathcal{C}$-block that contains at least three edges, so we assume that $X \cap A$ is connected. If, for the $\mathcal{C}$-block Y that contains $(X \cap A)^{\varphi}$, we have that $Y \cap B$ is connected, then we can construct an automorphism that extends φ as in the case where X contained three edges of A. On the other hand, if $Y \cap B$ is not connected, there has to be a $\mathcal{C}$-block that contains three edges of B, and the same must be true for a $\mathcal{C}$-block and A. Since we know that in this case there is an automorphism of D that extends φ, we have proved that $M(\kappa, m)$ is C-homogeneous.

In the case that $D \cong M'(2m)$ for an $m \in \mathbb{N}$ the arguments used are analog ones as in the case $D \cong M(\kappa, m)$ and therefore we omit that proof here. $\square$

12.4.3. Line digraphs of C-homogeneous digraphs. It is well known (see [**19**]) that line digraphs of highly-arc-transitive digraphs are again highly-arc-transitive. In some cases also C-homogeneity is preserved under taking the line digraph: Gray and Möller [**45**] stated that the line digraph of a $DL(C_{2m})$ is C-homogeneous. In terms of our classification:

Remark 12.25. *For each $m \in \mathbb{N}$ we have $L(DL(C_{2m})) \cong M'(2m)$.*

PROOF. Consider the digraph $D = DL(C_{2m})$ for a $m \in \mathbb{N}$. By construction the deletion of each single vertex v of D splits the digraph into two components such that v has two out-neighbors in the one and two in-neighbors in the other component. Thus the four edges that are incident with v form a $K_{2,2}$ in $L(D)$ whose independent vertex sets separate $L(D)$. Furthermore the edges of each C_{2m} in D form an independent set in $L(D)$ so that any two adjacent edges lie in a common $K_{2,2}$ in $L(D)$. One can easily verify that this digraph is indeed isomorphic to $M'(2m)$. $\square$

Interestingly, our classification of the C-homogeneous digraphs with infinitely many ends implies that C-homogeneity is not generally preserved under taking line digraphs. Indeed, for all $m \in \mathbb{N}$ the line digraph of $M'(2m)$ is triangle-free, has infinitely many ends, and has connectivity 4, hence it is not of Type II. Thus, by Theorem 12.24, we know that $L(M'(2m)) \cong L(L(DL(C_{2m})))$ is not C-homogeneous. This had remained an open question in [**45**].

12.5. The non-independent case for C-homogeneous digraphs with at most one end

Now we turn our point of view to the C-homogeneous digraphs all whose vertices have finite degree and that is, if it is an infinite digraph, at most one end. It is a straightforward argument that the out-neighborhood as well as the in-neighborhood of any vertex of a C-homogeneous digraph has to be a homogeneous digraph. We state the classification result of Lachlan in Theorem 12.26. We investigate which of the homogeneous digraphs of Theorem 12.26 may occur as a subdigraph induced by $N^+(x)$ or by $N^-(x)$ for a vertex $x \in VD$. In this section we take a look at those cases that contain an edge and show that there is precisely one such case that may occur. This case is a generalization of the digraph H that occurs in the case (v) of Theorem 12.26.

THEOREM 12.26. [**70**, Theorem 1] *A finite digraph is homogeneous if and only if it is isomorphic to one of the following digraphs:*

(i) *the C_4;*
(ii) *a $\bar{K}_n$ for an $n \geq 1$;*
(iii) *a $\bar{K}_n[C_3]$ for an $n \geq 1$;*
(iv) *a $C_3[\bar{K}_n]$ for an $n \geq 1$;*
(v) *the digraph H.* $\square$

Let x be a vertex of a connected locally finite C-homogeneous digraph. Our first aim is to show that $N^+(x)$ and $N^-(x)$ are both not isomorphic to H. Therefore, we define a *dominated* directed triangle to be a digraph that is isomorphic to a directed triangle together with a vertex that sends edges to all its vertices (Figure 5).

LEMMA 12.27. *For every connected locally finite C-homogeneous digraph D there is $N^+(x) \ncong H$ and $N^-(x) \ncong H$ for all $x \in VD$.*

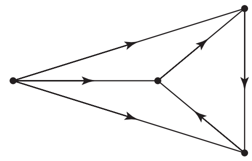

FIGURE 5. A dominated directed triangle

PROOF. Let $x \in VD$ and suppose by symmetry that $N^+(x) \cong H$. Then there is a dominated directed triangle embedded in $N^-(y)$ for all $y \in N^+(x)$. Hence we also have $N^-(x) \cong H$.

If two vertices x, y are adjacent, say $xy \in ED$, then $|N^+(x) \cap N^+(y)| \leq 3$ since $N^+(y) \cong H$. Furthermore, there exists a vertex $z \in N^-(y) \cap N^+(x)$.

CLAIM 12.28. *No neighbor of y lies in $N^+(x) \cap N^+(z)$.*

PROOF OF CLAIM 12.28. Suppose that there is a vertex $a \in N^+(x) \cap N^+(z) \cap N(y)$. By mapping $D[x, y, z]$ onto $D[x, a, z]$ by an automorphism of D, we get recursively a directed cycle in $N^+(x) \cap N^+(z)$. We already mentioned that $|N^+(x) \cap N^+(z)| \leq 3$. Hence there is a directed triangle in $N^+(x) \cap N^+(z)$. Let v_1 be a vertex in $N^+(x)$ that has two neighbors in $N^+(x) \cap N^+(z)$, let v_2 be a vertex in $N^+(x)$ with $N^+(x) \cap N^+(z) \subseteq N^+(v_2)$, and let v_3 be a vertex in $N^+(x)$ with $N^+(x) \cap N^+(z) \subseteq N^-(v_3)$. Such vertices exist because $N^+(x) \cong H$. Then either two of these vertices are adjacent to z—and hence lie in $N^-(z)$—or two of them are not adjacent to z. Let v_i, v_j $(i \neq j)$ be two vertices either both of the first or both of the second kind. Then $D[z, x, v_i] \cong D[z, x, v_j]$, and thus there is an automorphism of D mapping the first onto the second subdigraph. But this is a contradiction by the choice of v_i and v_j. $\square$

As $N^+(x) \cong H$, there are two out-neighbors of z that are adjacent to y in contradiction to Claim 12.28. Thus the lemma is proved. $\square$

The next case that we exclude is that neither the out- nor the in-neighborhood induces a subdigraph isomorphic to C_4.

LEMMA 12.29. *Let D be a connected locally finite C-homogeneous digraph and let $x \in VD$. Then $N^+(x) \not\cong C_4$ and $N^-(x) \not\cong C_4$.*

PROOF. By regarding the digraph whose edges are directed in the inverse way, if necessary, we may suppose that $N^+(x) \cong C_4$. Let us denote with $v_1, \ldots, v_4$ the four vertices in $N^+(x)$ with $v_i v_{i+1} \in ED$ for $1 \leq i \leq 3$ and $v_4 v_1 \in ED$. Since $x, v_4 \in N^-(v_1)$ and since $N^-(v_1)$ is homogeneous, there is another vertex in $N^-(v_1)$ distinct from both x and v_4.

We know by Lemma 12.27 that $N^-(v_1) \not\cong H$.

CLAIM 12.30. *There is no other vertex than x in $N^-(v_1) \cap N^-(v_2)$.*

Proof of Claim 12.30. Let us suppose that there is a vertex $y \in N^-(v_1) \cap N^-(v_2)$. Then an immediate consequence of the C-homogeneity is $N^+(x) = N^+(y)$. But then neither xy nor yx can be an edge of D. The subdigraph induced by $\{x, y, 4\}$ is a subdigraph of $N^-(v_1)$ and thus $N^-(v_1) \cong \bar{K}_n[C_3]$ with $n > 1$. Then there is $z \in N^+(x) \cap N^-(v_1)$ which is distinct from v_4. This is not possible and hence no such y exists. □

Claim 12.31. *There is no vertex in $N^-(v_1) \cap N^+(v_2)$.*

Proof of Claim 12.31. Suppose that there is a vertex $y \in N^-(v_1) \cap N^+(v_2)$. If y is neither adjacent to x nor to v_4, then $N^-(v_1)$ has to be isomorphic to $\bar{K}_n[C_3]$ with $n > 1$. Then there is an automorphism α of D with $v_4^\alpha = v_4$, $v_1^\alpha = v_1$, and $v_2^\alpha = y$ and hence $x \neq x^\alpha \in N^-(v_1) \cap N^-(v_4)$. This contradicts Claim 12.30 with v_4 and v_1 instead of v_1 and v_2. So y is adjacent to at least one of x and v_4.

If y is adjacent to x but not to v_4, then $N^-(v_1) \cong C_4$ or $N^-(v_1) \cong H$ since an induced path of length 2 embeds into $N^-(v_1)$. As we already saw, only the first case can occur and then there is an automorphism α of D with $v_4^\alpha = y$, $v_1^\alpha = v_1$, and $v_2^\alpha = v_2$. So $x \neq x^\alpha$ and x^α is a second vertex in $N^-(v_1) \cap N^-(v_2)$, which is impossible by Claim 12.30.

If y is adjacent to v_4 but not to x, then we distinguish two cases: in the first one $yv_4 \in ED$. But then by C-homogeneity applied to $D[y, x, v_4]$ and $D[y, x, v_1]$ also $v_2 \in N^+(y)$ contrary to the case we are discussing. In the second case we have $v_4y \in ED$ and thus $N^-(v_1) \cong C_4$. Then there has to be a vertex $z \in N^-(v_1) \setminus \{v_4, x, y\}$. If z is not adjacent to v_2, then there is an automorphism of D that maps $D[v_2, v_1, z]$ onto $D[v_2, v_1, v_4]$. Since this automorphism cannot fix x, the image of x also lies in $N^-(v_1) \cap N^-(v_2)$ contrary to Claim 12.30. If $v_2z \in ED$, then there is an automorphism of D that maps the cycle $D[v_2, y, v_1]$ onto $D[v_2, z, v_1]$. This is again a contradiction and the final contradiction in the case that y is adjacent to v_4 but not to x is given directly by Claim 12.30 since, if $zv_2 \in ED$, then $z \in N^-(v_1) \cap N^-(v_2)$.

Let us now consider the case that both x and v_4 are adjacent to y. By the same arguments as above there has to be $v_4y \in ED$ and not $yv_4 \in ED$. By C-homogeneity we have $yv_3 \in ED$ and since $y \notin N^+(x)$, we have $yx \in ED$. But then $D[v_1, x, v_3]$ is a subdigraph of $N^+(y)$ but this digraph cannot be embedded into a C_4 and thus we just have proved the final contradiction of this claim. □

Claim 12.32. *There is no vertex in $N^-(v_1) \cap N^+(v_4)$ that is not adjacent to v_2.*

Proof of Claim 12.32. Let us suppose that there exists $y \in N^-(v_1) \cap N^+(v_4)$ such that y is not adjacent to v_2. Then v_3 is not adjacent to y, too, and hence there is an automorphism α of D that maps $D[v_3, v_4, v_1]$ onto $D[v_3, v_4, y]$. Since $y \notin N^+(x)$, we have $x \neq x^\alpha \in N^-(v_3) \cap N^-(v_4)$ and thus a contradiction to Claim 12.30. □

Claim 12.33. *There is $N^-(v_1) \cap N^+(v_2) \neq \emptyset$ or $N^-(v_1) \cap N^+(v_4) \neq \emptyset$.*

Proof of Claim 12.33. Suppose that both intersections are empty. Let $y \in N^-(v_1)$ with $x \neq y \neq v_4$. If x and y are not adjacent, then $N^-(v_1)$ has to be isomorphic to $\bar{K}_n[C_3]$ for an $n > 1$. Hence there is $z \in N^+(v_4) \cap N^-(x) \cap N^-(v_1)$. This is a direct contradiction to the assumptions. Thus x and y have to be adjacent and hence $yx \in ED$. So there is an induced path of length 2 in $N^-(v_1)$ and thus $N^-(v_1) \cong C_4$ or $N^-(v_1) \cong H$, whereas the second case cannot occur by Lemma 12.27. So $N^-(v_1) \cong C_4$. Then both $D[v_4, v_1, v_2]$ and $D[y, v_1, v_2]$ are isomorphic subdigraphs of D and thus there is an automorphism α of D that fixes v_1 and v_2 and maps v_4 onto y. We conclude that $x^\alpha \in N^-(v_1) \cap N^-(v_2)$ which is untenable because of Claim 12.30. □

By all the claims we showed that there is no vertex in $N^-(v_1)$ distinct from x and from v_4 in contradiction to the homogeneity of $N^-(v_1)$ by Theorem 12.26. Thus we proved Lemma 12.29. □

Lemma 12.34. *Let D be a connected C-homogeneous digraph with $N^+(x) \cong \bar{K}_n[C_3]$ and $N^-(x) \cong \bar{K}_m[C_3]$ for all $x \in VD$ and for some $m, n \geq 1$. Then $m = n = 1$.*

Proof. Let $xy \in ED$. Then there exists $z \in N^-(y) \cap N^-(x)$. By regarding $N^-(y)$, we obtain an $a \in N^-(y) \cap N^+(x)$ with $az \in ED$. Let b be the third vertex of $N^+(x)$ in that isomorphic image of C_3, that contains y and a. We have neither zb nor bz in ED since otherwise there is an edge either in $N^+(x) \cap N^+(z)$ or in $N^+(x) \cap N^-(z)$ and by applying the C-homogeneity we obtain the whole isomorphic image of C_3, $D[a, b, y]$, in $N^+(x) \cap N^+(z)$ or in $N^+(x) \cap N^-(z)$ which is impossible.

Let us suppose that $n > 1$. Then there exists a vertex $y' \in N^+(x)$ that is distinct from a, b, and y. Then there is a vertex $v \in \{a, b, y\}$ such that $D[z, x, v] \cong D[z, x, y']$ and hence the isomorphic image of C_3 in $N^+(x)$ that contains y' contains a vertex of $N^+(z)$. We may suppose that $y' \in N^+(z)$. But then $D[y, x, y']$ is a digraph that cannot be embedded into $N^+(z)$. So $n \ngtr 1$. By a symmetric argument we also have $m = 1$. □

Lemma 12.35. *Let D be a connected locally finite C-homogeneous digraph and $x \in VD$. If $N^+(x) \cong C_3[\bar{K}_n]$ or if $N^-(x) \cong C_3[\bar{K}_n]$ for an $n \geq 1$, then there is $D \cong H[\bar{K}_n]$.*

Proof. We assume that $N^+(x) \cong C_3[\bar{K}_n]$ for an $n \geq 1$. Let $y \in N^+(x)$. Then x together with n vertices of $N^+(x)$ lie in $N^-(y)$ and hence $N^-(y) \cong C_3[\bar{K}_m]$ for an $m \geq n$ or $n = 1$ and $N^-(y) \cong \bar{K}_m[C_3]$ for an $m \geq 1$ which has to be equal to 1 by Lemma 12.34, so in each case $N^-(y) \cong C_3[\bar{K}_m]$ for an $m \geq n$. By symmetry we conclude that $m = n$. Then there is a vertex $z \in N^-(x) \cap N^-(y)$.

Claim 12.36. *$N^+(x) \cap N^+(z)$ is an independent set of cardinality n.*

Proof of Claim 12.36. This is a direct consequence of the fact that $N^+(z)$ is isomorphic to $C_3[\bar{K}_n]$. □

An immediate consequence of the C-homogeneity of D is $N^+(x) \cap N^-(z) \neq \emptyset$.

Claim 12.37. *$N^+(x) \cap N^-(z)$ is an independent set of cardinality n.*

Proof of Claim 12.37. We already know that the set $N^+(x) \cap N^-(z)$ is not empty. So let us suppose that there is an edge ab with both of its incident vertices in $N^+(x) \cap N^-(z)$. Then the digraphs $D[z, x, a]$ and $D[z, x, b]$ are isomorphic and hence there is an automorphism of D mapping the first onto the second one. As a consequence of Claim 12.36 both, a and b, have to be adjacent to all the vertices in $N^+(x) \cap N^+(z)$. Hence there is $y'a \in ED$ and $by' \in ED$ for all $y' \in N^+(x) \cap N^+(z)$. Thus no such automorphism can exist and we conclude that no such edge ab can exist. Since there are at least n vertices in $N^-(y)$ that lie in $N^+(x) \cap N^-(z)$ and since there are at most n vertices in $N^+(x)$ that are pairwise not adjacent, the assertion follows. $\square$

Claim 12.38. *There is $|N^+(x) \cap N^+(z)| = n = |N^+(x) \cap N^-(y)|$.*

Proof of Claim 12.38. This is a direct consequence of the fact that the subdigraph induced by $N^+(x)$ is isomorphic to $C_3[\bar{K}_n]$. $\square$

Claim 12.39. *There is an equivalence relation $\sim$ on VD whose equivalence classes have precisely n independent vertices each and such that $D_\sim$ is isomorphic to H and $D_\sim[\bar{K}_n]$ is isomorphic to D.*

Proof of Claim 12.39. Let us define a relation $\sim$ via

$$a \sim b :\Leftrightarrow N^-(a) = N^-(a) \cap N^-(b) = N^-(b).$$

Then $\sim$ is obviously an equivalence relation.

If we consider two of the equivalent classes of $\sim$, then all of the edges between these two classes must be directed in the same direction and furthermore the digraph induced by these two classes is a complete bipartite digraph. Hence D induces a C-homogeneous digraph on $D_\sim$ with $D \cong D_\sim[\bar{K}_n]$.

It is a straightforward argument to show that $D \cong H$ if $N^+(x) \cong C_3$. So if we consider $D_\sim$, then we may instead assume that $N^+(x) \cong C_3$ for all $x \in D_\sim$ and hence we obtain the isomorphism. $\square$

The lemma is a direct consequence of the previous claim. $\square$

12.6. The independent case for C-homogeneous digraphs with at most one end

In this section we consider the situation that every out-neighborhood—and hence by the results of Section 12.5 also every in-neighborhood—is independent. The first case we classify is if every vertex has in- or out-degree 1.

Lemma 12.40. *Let D be a locally finite connected C-homogeneous digraph and let $x \in VD$. If $N^+(x)$ or $N^-(x)$ consists of precisely one vertex, then D is either an infinite tree or a directed cycle.*

Proof. By symmetry we may assume that $N^+(x)$ consists of precisely one vertex. Let F be the subdigraph of D that is induced by all descendants of x. Then either this contains a directed ray or a directed cycle. If F contains a directed cycle, then, by the C-homogeneity, x has to lie on such a cycle, say C. Suppose that a vertex y exists that does not lie on C but has a successor on C. Let α be an automorphism of D with $x^\alpha = y$. Then $C^\alpha \cap C$ contains the successor of y and hence y has to lie on C since every vertex of C has its unique successor on C. So in this case we conclude that D is a directed cycle.

We now assume that no directed cycle lies in F. Let H be the digraph that is induced by all ancestors of vertices of F. As $|N^+(x)| = 1$ and as D is connected, H has to be the whole digraph D. Now let us suppose that there is an undirected cycle in D. Then there has to be a vertex on that cycle that has out-degree at least 2 since F is a ray, contrary to the assumption. Hence D is an infinite tree. $\square$

In the following we will at first suppose that our digraphs suffice the assumptions of Lemma 12.18 to apply the result of Section 12.1 and to obtain partial classification results which will be completed in Section 12.7. Recall that the assumptions of Theorem 12.18 is that the reachability digraph of the digraph is bipartite. Thereafter we shall prove in the lemmas 12.47 and 12.53 that the connected locally finite C-homogeneous digraphs indeed always satisfy these assumptions.

Lemma 12.41. *Let D be a locally finite connected C-homogeneous digraph such that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$. If $\Delta(D)$ is bipartite, then either $\Delta(D)$ is a finite digraph or C_3 embeds into D and $\Delta(D) \cong T_{2,2}$.*

Proof. Suppose that $\Delta(D)$ is not finite. Since D is locally finite, we conclude from Theorem 12.4 that $\Delta(D) \cong T_{k,l}$ for integers $k, l \geq 2$. We distinguish two cases: Either C_3 embeds into D or not. So let us first suppose that C_3 does not embed into D. Let Δ_1, Δ_2 be two distinct reachability digraphs with non-empty intersection and let us denote with d_i the distance in Δ_i between vertices of Δ_i. If Δ_1 and Δ_2 intersect in at most one vertex, let $x, y, z \in VD$ with $xz, yz \in ED$, $x, y, z \in V\Delta_1$, $z \in V\Delta_2$. Then there is a ray R in Δ_2 starting in z and such that no vertex on R except for z is adjacent to x or y because none of the out-neighbors of z is adjacent to x or y. Let $a \in V\Delta_1$ with $r := d_1(a, x) \geq 2$ and $d_1(a, x) < d_1(a, y), d_1(a, z)$. Then there is a path P outside $B_{r+2}(x)$ from a to R. Let P' be the (induced) path in $P \cup R$ from a to z. Then the subdigraphs $P' \cup \{x\}$ and $P' \cup \{y\}$ are isomorphic—we can map x, z, z_1 onto x, z, z_2 for any two successors of z and thus we may conclude that no vertex of Δ_2 except for z is adjacent to x or to y. But there is no automorphism of D mapping the first onto the second since $d_1(a, x) < d_1(a, y)$. Thus we have $|\Delta_1 \cap \Delta_2| \geq 2$ and hence there are infinitely many vertices in $\Delta_1 \cap \Delta_2$ because of the C-homogeneity of D.

If there are two vertices u, v in $\Delta_1 \cap \Delta_2$ with minimal distance $d_i(u, v)$ and with $d_i(u, v) \geq 3$ and $d_j(u, v) \geq 2$ $(i \neq j)$, then we get a contradiction by two analog paths as before.

So we conclude that for all $u, v \in V\Delta_1 \cap V\Delta_2$ with minimal distance in Δ_1 there is $d_1(u, v) = 2 = d_2(u, v)$. Now we shall construct a cycle in Δ_2. Let x_1 be the vertex in Δ_1 that is adjacent to both u and v and let x_2 be another neighbor of u in Δ_1. Let y_1 and y_2 be analog vertices in Δ_2. Then there is an automorphism of D that fixes u, y_1 and y_2 and maps x_1 onto x_2 and vice versa. Hence x_1 has another neighbor v' in Δ_1 that is also a neighbor in Δ_2 of y_2. But as $d_1(v, v') = 2$, there is a neighbor y_3 of v and v' in Δ_2. Then the digraph induced by the vertices u, v, v', y_1, y_2, y_3 induces a cycle of length 6 in Δ_2 which is impossible.

For the last case we suppose that C_3 embeds into D. Let Δ_1, Δ_2 be as above and $xz, yz \in E\Delta_1$ with $z \in \Delta_1 \cap \Delta_2$. Let us suppose that $d^+ \geq 3$ or $d^- \geq 3$. Then we obtain a contradiction similar to the first one, if there is an out-neighbor of z that is adjacent neither to x nor to y. So we may assume that there are at least two elements of $N^+(z)$ that are adjacent to x. As D is C-homogeneous, each two elements of $N^+(z)$ have a common successor, and since $\Delta(D) \cong T_{k,l}$, there is a vertex adjacent to all elements of $N^+(z)$. This vertex has to be x by the choice of x. But by C-homogeneity this also holds for y, so there is a cycle in Δ_3 which is impossible. So in this case we have $d^+ = d^- = 2$. □

Lemma 12.42. *Let D be a locally finite connected C-homogeneous digraph with at most one end such that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$. If $\Delta(D)$ is bipartite and if the intersection of any two reachability digraphs does not separate each of them, then no reachability digraph separates D.*

Proof. Suppose that there is a reachability digraph Δ_1 that separates D. Let Δ_2 be a reachability digraph with $V\Delta_1 \cap V\Delta_2 \neq \emptyset$, let $x \in V\Delta_1 \cap V\Delta_2$, y be a neighbor of x in Δ_2, and let C_i, $i = 1, 2$, be the component of $D - \Delta_i$ that contains y or is adjacent to y by an edge that lies not in $E\Delta_i$. If C_2 does not contain any vertex of Δ_1, then $C_2 \subset C_1$ with $C_2 \neq C_1$. So both C_1 and C_2 have to be infinite since they are isomorphic. Thus D has one end in $C_1 \cap C_2$ and symmetrically also another one in $(D - C_1) \cap (D - C_2)$ contrary to the assumptions. So C_2 contains a vertex of Δ_1 and $C_2 \not\subset C_1$. But then, as $\Delta_1 \setminus V\Delta_2$ is connected, there is another component C_2' of $D - \Delta_2$ that is completely contained in C_1 and contains no vertex of Δ_1. The component C_2' does not have to be isomorphic to C_1, but since there is a reachability digraph Δ_3 in C_2', we obtain a component C_3 of $D - \Delta_3$ with $C_3 \subset C_2'$ and so on. Because the degree of any vertex is finite, there are m, n such that C_m and C_n—or C_2' if m or n is 2—lead to an analog contradiction as before. □

The following lemma is the main lemma for the case that there is no isomorphic copy of C_3 in the C-homogeneous digraph. After its proof, we show in Lemma 12.47 that in the case that the out-neighborhood and the in-neighborhood each are independent sets the connected locally finite C-homogeneous digraphs that contains directed triangles always satisfy the assumptions of Lemma 12.43, so that in this case the conclusion of Lemma 12.43 holds.

LEMMA 12.43. *Let D be a locally finite connected C-homogeneous digraph that contains no directed triangle and such that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$. If $\Delta(D)$ is bipartite, then either D has at least two ends or D is isomorphic to $C_m[\bar{K}_n$ for an $m \geq 4$, $n \geq 1$.*

PROOF. Let $\Delta(D)$ be bipartite. We suppose that D contains at most one end and, by Lemma 12.40, that $d^+, d^- \geq 2$.

CLAIM 12.44. *Let Δ_1, Δ_2 be two reachability digraphs with non-trivial intersection. Then either their intersection is contained in the same side of the bipartition of Δ_1 or $\Delta(D) \cong CP_k$ for a $k \geq 3$ and the intersection consists of precisely one unmatched pair in CP_k.*

PROOF OF CLAIM 12.44. Suppose that the claim does not hold. We remember that the reachability digraphs are induced subdigraphs. So D consists of at least 3 reachability digraphs. We also conclude that $\Delta(D)$ cannot be a complete bipartite digraph, so it is either the complement of a perfect matching or a directed cycle. Let $x, y \in \Delta_1 \cap \Delta_2$ be on distinct sides of Δ_1 (and hence also of Δ_2) with minimal distance in Δ_1. If $\Delta_1 \cong C_{2m}$ for an $m \geq 4$, then we choose a minimal path P in Δ_2 from x to y. Let x' be a neighbor of x in Δ_1 and y_1, y_2 two neighbors of y in Δ_1. Then by mapping Pyy_1 onto Pyy_2 we obtain $d_{\Delta_1}(x, y) = m$ and hence the subdigraphs induced by $x'xPyy_1$ and $x'xPyy_2$ are isomorphic paths. We conclude that also $d_{\Delta_1}(x', y) = m$, a contradiction. So $\Delta(D)$ is isomorphic to CP_k for a $k \geq 3$. Then $\Delta_1 \cap \Delta_2$ consists of precisely two vertices that are not matched as claimed. $\square$

Since each vertex lies in at most two reachability digraphs, we consider the following two relations: Let $x \sim y$ for $x, y \in VD$ if x and y *lie on the same side* of a reachability digraph, that is, both have the same out-degree and the same in-degree in that reachability digraph and one of these two values is 0. Let $x \approx y$ for $x, y \in VD$ if x and y lie on the same side of two reachability digraphs.

CLAIM 12.45. *Let $x, y \in VD$. Then $x \sim y$ if and only if $x \approx y$.*

PROOF OF CLAIM 12.45. Let $x, y \in VD$. It suffices to prove that $x \sim y$ implies $x \approx y$. So let us suppose that $x \sim y$ but $x \not\approx y$ and let Δ be the reachability digraph that contains both vertices, x and y, on the same side. Since $\Delta(D)$ is finite by Lemma 12.41, we know from Theorem 12.4 that both sides of the reachability digraph have the same size. Hence there is a vertex with two successors in distinct reachability digraphs and one with two predecessors in two distinct reachability digraphs. We conclude by the C-homogeneity that for every vertex each two successors lie in precisely one common reachability digraph and the same holds for each two predecessors. So we may assume that x and y have distance 2. Let v_1 be a vertex in the same reachability digraph as x and y that is adjacent to both x and y. By symmetry we may assume that $xv_1, yv_1 \in ED$.

The next aim is to show that no reachability digraph separates D. Let us suppose that the converse holds. By Lemma 12.42 each two reachability digraphs that have at least one common vertex, have at least two common vertices. We conclude from Lemma 12.42 and Claim 12.44 that the intersection of each two reachability digraphs is contained in one side of the bipartition of each one. But then Theorem 12.4 implies that $\Delta(D)$ is a cycle of length $2m$ with $m \geq 4$. So let a, b be two vertices in the same two reachability digraphs Γ_1 and Γ_2 with minimal distance. Then there is a minimal path P between a and b in Γ_1. Let w_1, w_2 be the neighbors of b in Γ_2, let u_1 be the vertex on P that is adjacent to a, and let u_2 be a vertex in Γ_2 that is adjacent to a. Then the paths u_1Pbw_1 and u_1Pbw_2 are isomorphic and thus there is an automorphism of D that maps the first onto the second one. This automorphism has to fix a and thus the distance in Γ_2 from a to w_1 is the same as the one from a to w_2. But since $m \geq 4$, we can also map the path u_2aPbw_1 onto u_2aPbw_2. Then also u_2 and w_1 have the same distance in Γ_2 as u_2 and w_2. But this cannot be true. Thus we know that no reachability digraph can separate D.

Hence we find an (undirected) induced path R from v_1 to y whose only vertices in Δ are v_1 and y and that does not use the edge yv_1. We may choose R so that the only vertex on R that is adjacent to x is v_1 by applying the C-homogeneity to an automorphism that maps $D[x, v_1, y]$ onto $D[y, v_1, x]$. Let v_3, v_2, y be the last three vertices on R. If $v_3 \sim y$, then we conclude that $v_3 \not\sim x$. But then yv_1Rv_3 can be mapped onto xv_1Rv_3 by an automorphism of D and we obtain a contradiction by $x \sim v_3$.

So $v_3 \not\sim y$. By an analog automorphism to the one above—one that maps yv_1Rv_3 onto xv_1Rv_3—we obtain that v_3 and x have a common neighbor v_4. Let w_1 be the neighbor of v_1 on R. Since D contains no directed triangle, there is an automorphism α of D that fixes w_1Rv_2 and also v_4x pointwise, but with $y^\alpha \neq y$. But then y^α has to lie in the reachability digraph Δ as y and x which is impossible as we already saw. $\square$

We conclude from Claim 12.45 that $\sim$ and $\approx$ are equivalence relations on VD. Let Γ be a digraph on the equivalence classes of $\sim$ such that there is an edge from one class X_1 to another X_2 if and only if there are vertices $x_1 \in X_1$ and $x_2 \in X_2$ with $x_1x_2 \in ED$. By Claim 12.45 each vertex of Γ has precisely one successor and one predecessor. It is a straightforward argument that Γ is a C-homogeneous digraph. Since D has at most one end, Γ must be a directed cycle C_n for an $n \geq 3$ by Lemma 12.40.

It remains to show that the inverse images of any edge of Γ, that is the subdigraph of D induced by the equivalence classes that are incident with that particular edge of Γ, and that is precisely one reachability digraph, induces a complete bipartite digraph. Let $V_1, \ldots, V_n$ denote the equivalence classes such that $V_iV_{i+1} \in E\Gamma$ for $i < n$ and $V_nV_1 \in E\Gamma$ with $n \geq 4$.

It follows from Lemma 12.18 and Theorem 12.4 that $\Delta(D)$ is either a semi-regular tree $T_{k,l}$, a cycle C_{2m}, the complement of a perfect matching CP_k, or a complete bipartite digraph $K_{k,l}$. To prove the lemma, we have to show that none of the first three cases can occur where the first one was already excluded by Lemma 12.41.

Let us suppose that $\Delta(D) \cong C_{2m}$ for an $m \geq 4$. Let $x \in V_1$ and let a, b be its successors. Let P be a shortest a-b-path in Δ_2, the subdigraph induced by V_2 and V_3, and let $P^\circ := P - b$. Let P' be a path of the same length as P° in Δ_2 that starts in a and is, except for a, distinct from P. By mapping xP° onto xP', we obtain $d_{\Delta_2}(a, b) = m$. But then the same holds for the other predecessor $y \neq x$ of a and thus y also has to be adjacent to b and hence $m = 2$, a contradiction.

Let us now suppose that $\Delta(D) \cong CP_k$ for a $k \geq 3$. Let $x \in V_1$. Then there exists a unique vertex in V_2 that is not adjacent to x and this vertex itself has a unique vertex $y \in V_3$ it is not adjacent to. Now let X be the digraph $D[(V_3 \setminus \{y\}) \cup P]$ where P denotes a path that consists of one vertex from each V_i, $i \geq 4$ and of x such that the vertex in V_4 is the only vertex incident with all of V_3 but y. Let x' be another vertex of V_1 that is adjacent to the predecessor of x on P and let Y be the digraph $D[(VX \setminus \{x\}) \cup \{x'\}]$. Then X and Y are isomorphic subdigraphs of D but there is no automorphism of D mapping the first onto the second one since for x and y there is a unique vertex in V_2 that is not adjacent to both, but for x' and y there is no such vertex. Hence $\Delta(D) \not\cong CP_k$ and thus we conclude from Theorem 12.4, since we excluded all other cases, that $\Delta(D)$ is a complete bipartite digraph. As all equivalence classes have the same size, $\Delta(D) \cong K_{k,k}$ for a $k \geq 1$. □

The following proposition is similar to a result by Malnič et.al., see [**77**, Proposition 3.2]. Since we apply it in a situation where its original assumptions need not to be satisfied, we formulated the result with different assumptions. But the general idea of the proof of Proposition 12.46 is quite similar to the one of the proof of [**77**, Proposition 3.2]. Because our assumptions are to handle differently, we prove it here.

PROPOSITION 12.46. *Let D be a connected C-homogeneous digraph such that in-degree and out-degree of any vertex are at most a fixed integer d and such that both $N^+(x)$ and $N^-(x)$ are independent sets. Let $\Gamma = \mathrm{Aut}(D)$, $xy \in ED$ and $\Omega \subseteq N^+(x)$ with $|\Omega| = d$ such that $H = \Gamma_{xy}$ fixes Ω setwise but stabilizes no vertex of Ω.*

Then there is no alternating walk whose first edge is xy and which ends at a vertex of Ω.

PROOF. Since D is C-homogeneous, the group H acts on Ω like S_Ω, i.e. $H^\Omega \cong S_\Omega$. Let P be an alternating walk with initial edge xy. Suppose that $H_P^\Omega = H^\Omega$. Let $e \in ED$ such that Pe determines an alternating walk, and let z be the vertex incident with e but distinct from the end vertex of P. Then there are at most $d-1$

vertices in $\{z^\alpha \mid \alpha \in H_P\}$. Since $H^\Omega = H_P^\Omega$, either we have $d = 2$ or we have $|H^\Omega : H_z^\Omega| < d$. Let us first assume that $d \neq 2$. Then, by Proposition 2.2, H_z^Ω is either H^Ω or isomorphic to A_Ω. We shall now show that the latter case cannot occur. So suppose that $H_z^\Omega \cong A_\Omega$. Then H_z acts transitively on Ω but there is no automorphism fixing $|\Omega| - 2$ elements and switching the other two. Since D is C-homogeneous and Ω is independent, this is impossible. Hence $H_z^\Omega = H^\Omega$ and thus no vertex of Ω is fixed by H_z. So let us now assume that $d = 2$. But in this case we immediately deduce from the fact that the orbit of z under H contains only z, that $H = H_z$. So we conclude in each case that no vertex of Ω can lie on an alternating walk. □

LEMMA 12.47. *Let D be a connected locally finite C-homogeneous digraph such that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$ and assume that D contains no directed triangle. Then the reachability relation of D is not universal.*

PROOF. Let $xy \in ED$. By symmetry we may assume that $d^+(x) \geq d^-(x)$. Let $\Omega = N^+(y)$. By applying Proposition 12.46 we conclude that no vertex of Ω lies on an alternating path that starts with the edge xy and thus that the reachability relation of D cannot be universal. □

LEMMA 12.48. *Let D be a connected locally finite C-homogeneous digraph such that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$. If C_3 embeds into D, then we have $d^+(x) = d^-(x)$.*

PROOF. Let $y \in N^+(x)$ and $z \in N^-(x) \cap N^+(y)$. The number n_1 of directed triangles that contain xy is equal to the number of all 2-arcs from y to x. By C-homogeneity this is the same as the number of all 2-arcs from x to z which is again equal to the number n_2 of all directed triangles that contain zx. Let n_3 denote the number of all directed triangles that contain x. Then we conclude from the C-homogeneity

$$|N^+(x)|n_1 = n_3 = |N^-(x)|n_2.$$

Since $n_1 = n_2$, the claim follows. □

LEMMA 12.49. *Let D be a locally finite C-homogeneous digraph that contains a directed triangle. Then for every edge $xy \in ED$ the number of directed cycles that contains xy is either 1 or at least $(d^+ - 1)$.*

PROOF. Let d_1 be the number of elements of $N^+(y)$ that lie on a common directed triangle with xy and let d_2 be the number of elements of $N^+(y)$ for which this is not the case. Then $d = d_1 + d_2$ where $d := d^+$ which is the same as d^- by Lemma 12.48. Let Ω_1 be the set of all vertices of $N^+(y)$ that lie on a common directed triangle with xy and let $\Omega_2 = N^+(y) \setminus \Omega_1$. Let $\Omega_3 := N^+(x) \setminus \{y\}$. We consider the action of $H := \mathrm{Aut}(D)_{xy}$ on Ω_3. Because $N^+(x)$ is an independent set, H acts on Ω_3 like A_{Ω_3}. For $z \in \Omega_i$, $i = 1, 2$, we have $|H : H_z| = d_i < d^+ - 1 = |\Omega_3|$. Thus and by Proposition 2.2, H_z acts on Ω_3 either like S_{Ω_3} or like S_{Ω_3}. Let us first consider the second case. By a similar argument as in Proposition 12.46, we know that $|\Omega_3| = 2$. But then $d^+ = 3$ and the assertion trivially holds. So we assume that H_z acts on Ω_3 like S_{Ω_3}. In that case either no vertex of $N^+(y)$ lies in $N^+(y')$ for any $y \neq y' \in N^+(x)$ or every vertex of $N^+(y)$ lies in $N^+(y')$ for every $y \neq y' \in N^+(x)$. We conclude that each edge lies either on precisely one or on d distinct directed triangles. □

The following lemma is the main lemma for the case that the C-homogeneous digraph contains a directed triangle. The assumption that $\Delta(D)$ is bipartite in this case shall be verified in Lemma 12.53 and the case (iv) of the conclusions shall be investigated in Section 12.7.

LEMMA 12.50. *Let D be a locally finite connected C-homogeneous digraph that contains an isomorphic copy of C_3 and that has the property that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$. If $\Delta(D)$ is bipartite, then one of the following cases holds.*

(i) *The digraph D has at least two ends.*
(ii) *The reachability digraph $\Delta(D)$ is isomorphic to $K_{k,k}$ for a $k \geq 3$ and D is isomorphic to $C_3[\bar{K}_k]$.*
(iii) *The reachability digraph $\Delta(D)$ is isomorphic to CP_k for a $k \in \mathbb{N}$ and D is isomorphic to a Y_k for a $k \geq 3$.*
(iv) *The reachability digraph $\Delta(D)$ is isomorphic to C_{2m} for an $m \geq 4$ or to $T_{2,2}$.*

PROOF. Let us assume that the digraph D has at most one end, that case (iv) does not occur, and that $d^+, d^- \geq 2$ by Lemma 12.40. By Theorem 12.4, we know that $\Delta(D)$ is either a semi-regular tree—which is impossible in our situation because of Lemma 12.41—, a complete bipartite digraph, a CP_k, or a cycle C_{2m}—which we also excluded. Let us first assume that $\Delta(D)$ is a complete bipartite digraph $K_{k,l}$ for $k, l \in \mathbb{N}$ but not $K_{2,2}$. By Lemma 12.48, we know that $k = l$. If, for two reachability digraphs Δ, Δ', there is $|\Delta \cap \Delta'| \geq 2$, then it is a direct consequence of the C-homogeneity that $\Delta \cap \Delta'$ is a complete side of each of Δ, Δ'. Thus it is—like in the proof of Lemma 12.43—a direct consequence that (ii) holds in this case. So let us suppose that $\Delta \cap \Delta'$ has cardinality 1. If an edge lies on more than one directed triangle, then we know from Lemma 12.49 that it lies on at least $k-1$ distinct such triangles. But then, the intersection $\Delta \cap \Delta'$ has to contain at least $k-1$ elements which is a contradiction. So every edge lies on a uniquely determined directed cycle of length 3.

CLAIM 12.51. *For every four distinct reachability digraphs $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ such that $\Delta_i \cap \Delta_{i+1}$ $(i = 1, 2, 3)$ is not empty and such that $(\Delta_{i-1} \cup \Delta_{i+1}) \cap \Delta_i$ lies on the same side of Δ_i for $i = 2, 3$, $\Delta_1 \cap \Delta_4$ is not empty, too, and its intersection lies on the same side of Δ_4 as $\Delta_3 \cap \Delta_4$.*

PROOF OF CLAIM 12.51. Let us suppose that $\Delta_1 \cap \Delta_4$ is empty. Since every edge lies on a directed triangle, there has to be a vertex x with successors in Δ_1

and Δ_2. Let y be its sucessor in Δ_1, z be its successor in Δ_2 and let a be the vertex in $\Delta_2 \cap \Delta_3$, b the vertex in $\Delta_3 \cap \Delta_4$. Then there is no automorphism of D that maps a to b and fixes all of x, y, z, because $\Delta_1 \cap \Delta_2 \neq \emptyset$ but $\Delta_1 \cap \Delta_4 = \emptyset$. Hence we proved the claim. □

Now we are able to show that the whole situation cannot occur. Let x, y be two vertices on the same side of a reachability digraph such that their out-degree is 0. Let a, b be successors of x, y, respectively, such that they lie in a common reachability digraph. As $k \geq 3$ and as every edge lies on precisely one copy of C_3, there is a successor c of a and b such that neither $D[x, a, c]$ nor $D[y, b, c]$ are triangles. Furthermore, there exists a predecessor z of b such that z and c are not adjacent. The vertices a and z cannot be adjacent, because otherwise y and x have to lie in two common reachability digraphs which we supposed to be false. Then $D[x, a, c, b, y]$ and $D[x, a, c, b, z]$ are isomorphic, but there is no automorphism of D that maps one onto the other just by fixing all of x, a, c, b. Thus we showed that there are no two reachability digraphs whose intersection consists of precisely one vertex. This finishes the case $\Delta(D) \cong K_{k,l}$.

The next and final situation which we consider is $\Delta(D) \cong CP_k$ for a $k \geq 4$. Let Δ_1, Δ_2 be two distinct reachability digraphs of D with non-trivial intersection. We prove that $|\Delta_1 \cap \Delta_2| \geq 2$. So suppose that $|\Delta_1 \cap \Delta_2| = 1$. Let $b \in VD$ and let a, c be two predecessors of b. Let x, y be two predecessors of a and let v (w) be a vertex such that av, vx $(aw, wy$, respectively) lie in ED and such that cw but not cv lies in ED. Then the digraphs $D[a, b, c, x]$ and $D[a, b, c, y]$ are isomorphic but there is no automorphism of D that maps them onto each other because such an automorphism has to map v onto w but w is adjacent to c and v is not.

Claim 12.52. *The set $\Delta_1 \cap \Delta_2$ is one whole side of each of Δ_1, Δ_2.*

Proof of Claim 12.52. Let us first suppose that $\Delta_1 \cap \Delta_2$ is not contained in any of the sides of Δ_1. Then $\Delta_1 \cap \Delta_2$ consists of precisely two vertices that are adjacent in the bipartite complement of Δ_1. Let us consider the subdigraph of Δ_i with vertices a, b, c, d, with edges ba, bc, dc such that $a, d \in \Delta_1 \cap \Delta_2$. Let x, y be two predecessors of d in Δ_j with $i \neq j$ and let z be the neighbor of x in the bipartite complement of Δ_j. Since each edge lies on a directed triangle, we may assume that b, a, z form such a triangle and, since $k \geq 4$, we also may assume that c and y do not lie in a common reachability digraph. Then neither c nor y lies in a common reachability digraph with b and z. So each of the subdigraphs $D[a, b, c, d, x]$ and $D[a, b, c, d, y]$ contains precisely 4 edges and they are isomorphic to each other. Hence there is an automorphism α that fixes each of a, b, c, d and maps x onto y which is impossible because y and b do not lie in any common reachability digraph in contrast to x and b. Thus we proved that $\Delta_1 \cap \Delta_2$ is contained in one side of Δ_i, $i = 1, 2$.

The C-homogeneity directly implies that $\Delta_1 \cap \Delta_2$ is a whole side of Δ_i, $i = 1, 2$. Thus we proved the claim. □

We shall now show that $D \cong Y_k$. Let $\overline{D}$ denote the tripartite complement of D. Since $\Delta(D) \cong CP_k$, the digraph $\overline{D}$ is a union of directed cycles. We want to show that every component of $\overline{D}$ is a directed cycle of length 3. So let us suppose that this is not the case. Then there are $x, y \in V_1$ that lie on a common directed cycle of length at least 6 and have distance 3 on that cycle in $\overline{D}$. Since $k \geq 3$, there is a vertex $a \in V_2$ that is adjacent to both x and y. We conclude that for every vertex $z \in V_1$, distinct from x, x and z lie on a common cycle and have distance 3 on that cycle. It is a direct consequence that $k = 3$ and $\overline{D} \cong C_9$. But then there are edges of D that lie on precisely one copy of C_3 and some lie on two copies which contradicts the C-homogeneity. Hence we have $D \cong Y_k$. □

LEMMA 12.53. *Let D be a connected locally finite C-homogeneous digraph that contains a directed triangle. Furthermore, assume that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$. Then the reachability relation of D is not universal.*

PROOF. Let $d = d^+$. By Lemma 12.48 we have $d = d^-$. Suppose that the reachability relation of D is universal. Let D_1 be the digraph depicted in Figure 6.

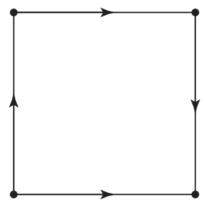

FIGURE 6. The digraph D_1

CLAIM 12.54. *D contains an isomorphic image of D_1.*

PROOF OF CLAIM 12.54. Since the reachability relation of D is universal, there is an induced cycle such that for an edge xy on that cycle the other path between x and y is an alternating path but such that the whole cycle is not alternating. Such a cycle shows that the reachability relation is not universal. To show that such a cycle exists, suppose that it is not the case. We choose a counterexample C with minimal length. Since there is always a cycle that shows that the reachability relation is not universal, we may assume that C is not induced. So there is a chord in C and hence one of the smaller cycles is a counterexample of smaller length, contrary to the assumption. Thus a cycle as described exists.

Let us first assume that such a cycle C has odd length. Then it has length at least 5. By symmetry we may assume that for the edge xy described above we have $d_C^-(x) = 1$ and $d_C^-(y) = 2$. Let z be the other vertex in $N^-(y) \cap VC$. Then there is an automorphism α of D that maps $C - x$ onto $C - y$. The digraph $D[x, y, z, x^\alpha]$ is isomorphic to D_1 because $N^-(x)$ and $N^+(x)$ are independent sets.

Let us now consider the case that C is an induced cycle of even length and let xy be again the above described edge. Let $a \neq y$ be the vertex on C adjacent to x, let $b \neq x$ be the vertex on C adjacent to y, let P_C be the path on C between a and b that contains neither x nor y, and let P_C^- denote the path inverse to P_C. Since C has odd length, we have $P_C \cong P_C^-$. Then we can map xP_C onto yP_C^- by an automorphism of D and obtain an induced subdigraph isomorphic to D_1 by the two paths of length 2 between a and y. □

CLAIM 12.55. *There is* $|N^+(y) \cap N^-(x)| = 1$ *for all edges* $xy \in ED$.

PROOF OF CLAIM 12.55. By Lemma 12.49 we know that either $|N^+(y) \cap N^-(x)| = 1$ or $|N^+(y) \cap N^-(x)| \geq d-1$. So it suffices to prove that $N^+(y) \setminus N^-(x)$ contains at least two vertices. Let u, v, a, b be the four vertices of the digraph D_1 such that u has the two predecessors a and b. Since there is an automorphism of D that fixes u and maps a onto b and vice versa, there is a directed path of length 2 from a to b and one from b to a. Since $N^+(v)$ and $N^-(v)$ are both independent sets and the same holds for v', the image of v under the described automorphism, there is no edge between v and v'. We may assume by symmetry that va is an edge in D. Then both v' and u are vertices in $N^+(a)$ that do not lie on a common directed triangle with va, so we conclude that $N^+(a) \cap N^-(v)$ contains precisely one vertex. □

Let D_2 be the digraph shown in Figure 7.

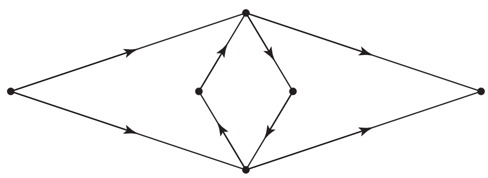

FIGURE 7. The digraph D_2

CLAIM 12.56. *There is an isomorphic image of* D_2 *in* D.

PROOF OF CLAIM 12.56. By an analog argument as in the proof of Claim 12.55, we immediately see that either D_2 is an induced subgraph or D_2 together with an edge from the vertex on the right hand to the vertex on the left hand is an induced subgraph of D. But the latter cannot be the case since the additional edge would lie on at least two copies of C_3, contrary to Claim 12.55. □

Now let D' be an isomorphic copy of D_2 in D. Let x be the vertex on the left, y the one on the right and a, b, u, v the vertices of the cycle such that x and y are adjacent to a and u. Since C_3 embeds into D, there is a vertex $a' \in N^+(a) \cap N^-(x)$. Then a' is adjacent neither to b, nor to v, nor to y, since the only directed triangle that contains aa' is $D[x, a, a']$ and since directed cycles are the only cycles of length 3 that embed into D. Then there is an automorphism of D that fixes a', x, and u, and maps v onto y. This automorphism also has to fix a, since it fixes together with x and a' the unique vertex in the directed triangle that contains the edge $a'x$. Hence such an automorphism cannot exist. □

12.7. An imprimitive case

In this section we investigate the following situation. Let D be a C-homogeneous digraph that contains directed triangles of length 3 and whose reachability digraph is either $T_{2,2}$ or C_{2m} for an $m \geq 2$. There exists a well-known such digraph, the digraph $T_2(C_3)$ that was defined in the introduction. This digraph has infinitely many ends. But although we are interested only in digraphs with at most one end, this particular digraph turns out to be very important in this case. We shall show that every digraph with the above described properties and with at most one end is a homomorphic image of $T_2(C_3)$ in a very particular way.

THEOREM 12.57. *The following two assertions are equivalent for any locally finite connected digraph D.*

(i) *The digraph D is C-homogeneous, contains directed cycles of length* 3*, and its reachability digraph is either $T_{2,2}$ or C_{2m} for an $m \geq 2$.*
(ii) *There exists a subgroup H of* $\mathrm{Aut}(T_2(C_3))$ *acting transitively on $VT_2(C_3)$ and an H-invariant equivalence relation $\sim$ on $VT_2(C_3)$ such that $T_2(C_3)_\sim$ is isomorphic to D.*

Furthermore, the digraph has at most one end if and only if each equivalence class consists of more than one element.

In the situation (ii) *we may always choose H to be the whole automorphism group of D.*

PROOF. First, let us assume that (ii) holds. We choose H so that it is maximal such that $\sim$ is H-invariant.

CLAIM 12.58. *The stabilizer H_x of any vertex x of $T_2(C_3)$ has order* 2*.*

PROOF OF CLAIM 12.58. There are two possibilities for an element of H_x. Either it fixes both directed triangles that meet x or it changes the two triangles. As every isomorphism between two directed triangles of $T_2(C_3)$ extends uniquely to an automorphism of $T_2(C_3)$, $|H_x| \leq 2$. So we have to prove that the element $\alpha \neq id$ of $\mathrm{Aut}(T_2(C_3))_x$ is also contained in H. Because H is the maximal subgroup of $\mathrm{Aut}(T_2(C_3))$ such that $\sim$ is H-invariant, we have to prove that α maps one equivalence class onto another. Let y, z be the two predecessors of x. If they lie in the same equivalence class, then this is fixed by α and the same holds for the equivalence class that contains both successors of x and this extends to all equivalence classes because of the transitivity of H on $VT_2(C_3)$. So y and z lie in distinct equivalence classes. But then α maps the equivalence class of y onto the one of z and vice versa, and the same holds for the two equivalence classes of the two successors of x. By induction on $d(x,a)$ for any vertex a of $T_2(C_3)$ its equivalence class is mapped onto the one of the unique vertex b with $d(x,b)=d(x,a)$ and for which the shortest path from x to b is isomorphic to the shortest path from x to a. So α acts on all equivalence classes and $\sim$ is α-invariant. $\square$

To show that $D \cong T_2(C_3)_\sim$ is C-homogeneous, let A, B be isomorphic induced connected subdigraphs of D and $\varphi : A \to B$ be an isomorphism. Then there are induced subdigraphs A', B' of $T_2(C_3)$ with $A \cong A'$, $B \cong B'$ and such that the equivalence classes of the vertices of A (of B) are the vertices of $T_2(C_3)_\sim$ that induce the digraph A (the digraph B, respectively). Let φ_0 be the isomorphism $A' \to B'$ that maps the equivalence class of $x \in VA$ to the equivalence class of x^φ. We may assume that A contains an edge xy. Let u, v be vertices in $T_2(C_3)$ such that $uv \in ET_2(C_3)$ and such that the equivalence class of u, v is x, y, respectively. Since H_{x^φ} has order 2 by Claim 12.58, there is an automorphism $\alpha \in H$ with $u^\alpha = u^{\varphi_0}$ and with $v^\alpha = v^{\varphi_0}$. But then the claim immediately implies $A'^\alpha = B'$ and that the canonical image α' of α maps A onto B like φ. Furthermore, α' is an automorphism of D because $\alpha \in H$.

For the other direction, let D satisfy the assumptions of (i). Let π be the map $T_2(C_3) \to T_2(C_3)_\sim$ that maps $x \in VT(2)$ onto its equivalence class. We may assume that D is not isomorphic to C_3. Let $xy \in ED$, $ab \in ET_2(C_3)$. For every vertex u in $T_2(C_3)$ there exists a unique shortest path P_1 from a to u. In D there are precisely two paths isomorphic to P_1 with the property that no two endvertices of any subpath of length 2 are adjacent. If the second vertex of the path P_1 is b or is adjacent to b, then let P_2 that one of the above described paths in D whose second vertex is y or is adjacent to y, and in the other case for P_1 let P_2 also be the other one in D. Let u_D denote the last vertex of P_2.

We are now able to define the equivalence relation. Let $u \sim v$ for two vertices $u, v \in VT_2(C_3)$ if $u_D = v_D$. This is obviously an equivalence relation. It remains to show that it is $\mathrm{Aut}(T_2(C_3))$-invariant. So let u and v be arbitrary vertices of $T_2(C_3)$ and let ψ be an automorphism of $T_2(C_3)$ with $u^\psi = v$. We have to show that the equivalence class of u is mapped onto the one of v. So let $w \sim u$. It suffices to consider the case where the shortest path from u to w does not contain any other vertex of the equivalence class that contains u. Let P be the shortest path from u to w. We look at the paths P^π and $(P^\psi)^\pi$. The path P^π starts and ends at the same vertex. We can map Q^{φ_0} for every subpath Q of P that starts in u onto $(Q^\psi)^\pi$ inductively, because on the one hand D is C-homogeneous and on the other hand for such a Q its succeeding vertex is uniquely determined in D by the two digraphs Q^π and $(Q^\psi)^\pi$. So we conclude that also $(P^\psi)^\pi$ has the same endvertices. But then u^ψ and w^ψ have to be equivalent. It is an immediate consequence that this holds also for any $z \sim u$.

The only remaining part to show is the additional claim on the multi-ended digraphs which is a direct consequence of [**45**, Theorem 7.1]. $\square$

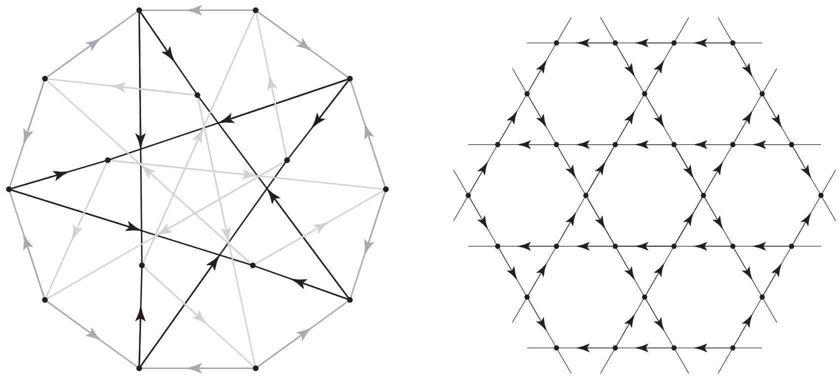

FIGURE 8. A finite and an infinite one-ended C-homogeneous digraph

Figure 8 shows two C-homogeneous digraphs that arise as factor digraphs in Theorem 12.57 one of which is finite and the other being infinite and one-ended. In the finite digraph every reachability digraph, which is isomorphic to C_{10}, is drawn in a different shade of gray. The reachability digraphs of the infinite digraph are the cycles of length 6.

As the automorphism group of $T(2)$ is a free product of the cyclic groups C_2 and C_3, it is isomorphic to the modular group. If we consider the Cayley graph of $C_2 * C_3 = \langle x \rangle * \langle y \rangle$ with respect to the two canonical generators x, y (and with directed edges) and contract the edges by the involution, then we obtain the digraph $T(2)$. Hence, instead of giving a precise list of the digraphs that may occur as quotients in Theorem 12.57 it is equivalent to describe all those subgroups of $C_2 * C_3$ that contain x. By Kurosh's Subgroup Theorem [**68**] every subgroup of the modular group is a free product of cyclic groups of orders 2, 3, or ∞. So any such subgroup with at least one generator of order 2 provides us with an example of a C-homogeneous digraph that is finite or locally finite and contains the directed triangle as a subdigraph and whose reachability digraph is either $T_{2,2}$ or C_{2m} for an $m \geq 2$. Conversely, every such digraph gives us a subgroup of the modular group that contains an involution.

12.8. The classification result for locally finite C-homogeneous digraphs with at most one end

Let us now state our main result. We shall prove it by applying all the results of the previous sections.

THEOREM 12.59. *Let D be a locally finite connected digraph with at most one end. Then D is C-homogeneous if and only if one of the following cases holds.*

(i) $D \cong C_m[\bar{K}_n]$ *for integers* $m \geq 3, n \geq 1$*;*

(ii) $D \cong H[\bar{K}_n]$ *for an integer* $n \geq 1$*;*

(iii) *$D \cong Y_k$ for an integer $k \geq 3$;*
(iv) *there exists a non-trivial* $\mathrm{Aut}(T_2(C_3))$*-invariant equivalence relation $\sim$ on $VT_2(C_3)$ such that $D \cong T_2(C_3)_\sim$.*

PROOF. Let D be a connected locally finite C-homogeneous digraph with at most one end. If the out-neighborhood (or symmetrically the in-neighborhood) of any vertex of D is not independent, then we conclude from Theorem 12.26, Lemma 12.27, Lemma 12.29, Lemma 12.34, and Lemma 12.35 that D is finite and isomorphic to an $H[\bar{K}_n]$ for an $n \geq 1$. So we may assume that the out-neighborhood of every vertex is independent. Since D is in particular 1-arc transitive, we conclude from Proposition 2.7, Lemma 12.47, and Lemma 12.53 that the reachability digraph of D is bipartite. Thus one direction of the theorem follows directly from Lemma 12.43, Lemma 12.50, and Theorem 12.57.

To prove the remaining part of Therorem 12.59 it suffices to prove that the digraphs Y_k are C-homogeneous, because it is an easy consequence of the fact that H is homogeneous, that $H[\bar{K}_n]$ is C-homogeneous, and, furthermore, an immediate consequence of the fact that $K_{n,n}$ is a homogeneous bipartite graph is, that $C_m[\bar{K}_n]$ is C-homogeneous. That the graphs in part (iv) are C-homogeneous was already proved in Theorem 12.57. To prove that the digraphs Y_k with $k \geq 3$ are C-homogeneous, let A and B be two isomorphic connected induced subgraphs of $D := Y_k$. Let V_1, V_2, V_3 be the three vertex sets as in the proof of Lemma 12.43 and let $\Delta_1, \Delta_2, \Delta_3$ be the corresponding reachability digraphs. Let α be an isomorphism from A to B. It is straightforward to see that $(VA \cap V_i)^\alpha$ is precisely the intersection of VB with a V_j. So we may assume that $(VA \cap V_i)^\alpha = VB \cap V_i$. If A and B have at most six vertices, then it is easy to see that every isomorphism from A to B extends to an automorphism of D. So we may assume that there is at least one V_i, say V_1, that contains at least three vertices of A. Then both subdigraphs $\Delta_1 \cap A$ and $\Delta_3 \cap A$ are connected subdigraphs. Let Δ_1' be a minimal subdigraph isomorphic to a CP_l with $l \leq k$ such that $A \cap \Delta_1 = A \cap \Delta_1'$. By replacing B by B^γ, for an automorphism γ of D, we may assume that also $B \cap \Delta_1 = B \cap \Delta_1'$ holds. Since CP_l is a C-homogeneous bipartite graph, we can extend every isomorphism from $\Delta_1' \cap A$ to $\Delta_1' \cap B$ to an automorphism of Δ_1' and hence, in particular, the restriction of α. Let α' be the automorphism of Δ_1' that extends the above restriction of α. Let $V_3' \subseteq V_3$ be the set of those vertices which are not adjacent to all vertices of Δ_1'. As each vertex in V_3' is uniquely determined by two non-adjacent vertices one of which lies in V_1 and the other in V_2, α' has precisely one extension β on $D' := D[V\Delta_1' \cup V_3']$. By the construction of β it is easy to see that the restriction of α to D' is again an isomorphism from $A \cap D'$ to $B \cap D'$ and is equal to the restriction of β to $A \cap D'$. Since all vertices of $A \cap (V_3 \setminus V_3')$ are adjacent to all vertices of $A \cap (V_1 \cup V_2)$ and since the same holds for B instead of A, β can be extended to an automorphism of D whose restriction to A is α. $\square$

12.9. Final remarks on C-homogeneous digraphs

Let us take a closer look at two specific kinds of digraphs that occur as 'building blocks' in our classification of the infinitely-ended C-homogeneous digraphs. The first kind are the homogeneous tournaments, which feature in our classification of the connected C-homogeneous digraphs of Type I. While Lachlan [**71**] classified the countable homogeneous tournaments, no characterization is known for the uncountable ones. The second kind of building blocks that deserve a closer look are the generic homogeneous bipartite graphs, which occur in the classification of the connected C-homogeneous digraphs of Type II. There is exactly one countable such digraph ([**41**, Fact 1.2]), but it is shown in [**41**] that the number of isomorphism types of homogeneous generic bipartite graphs with $\aleph_0$ vertices on the one side of the bipartition and $2^{\aleph_0}$ vertices on the other side is independent of ZFC. Hence, classifying the uncountable generic homogeneous bipartite graphs remains an undecidable problem.

As we mentioned in Chapter 9, we have finished the classification of the locally finite C-homogeneous digraphs. But the classification of the countable C-homogeneous digraphs is not yet complete. In fact, it might be that we just have solved a smaller part of it, because this classification would generalize the classification of the countable homogeneous digraphs, that were determined by Cherlin [**22**].

Bibliography

1. J.M. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, and H. Short, *Notes on word hyperbolic groups*, Group Theory from a Geometrical Viewpoint (Trieste, 1990) (E. Ghys, A. Haefliger, and A. Verjovsky, eds.), World Scientific, 1991, pp. 3–63.
2. D. Amato, *Descendants in infinite, primitive, highly arc-transitive digraphs*, Discrete Math. **310** (2010), no. 13-14, 2021–2036.
3. D. Amato and J.K. Truss, *Crown-free highly arc-transitive digraphs*, Forum Mathematicum (to appear).
4. ———, *Some constructions of highly arc-transitive digraphs*, Combinatorica (to appear).
5. P. Assouad, *Plongements lipschitziens dans* $\mathbb{R}^n$, Bull. Soc. Math. France **111** (1983), no. 4, 429–448.
6. L. Babai, *The growth rate of vertex-transitive planar graphs*, Proceedings of the Eighth Annual ACM-SIAM Symposium on Discrete Algorithms (New Orleans, LA, 1997), ACM, New York, 1997, pp. 564–573.
7. L. Babai and M.E. Watkins, *Connectivity of infinite graphs having a transitive torsion group action*, Arch. Math. (Basel) **34** (1980), no. 1, 90–96.
8. G. Bell and A. Dranishnikov, *Asymptotic dimension*, Topology Appl. **155** (2008), no. 12, 1265–1296.
9. I. Benjamini and O. Schramm, *Every graph with a positive Cheeger constant contains a tree with a positive Cheeger constant*, Geom. Funct. Anal. **7** (1997), no. 3, 403–419.
10. M. Bonk and O. Schramm, *Embeddings of Gromov hyperbolic spaces*, Geom. Funct. Anal. **10** (2000), no. 2, 266–306.
11. M. Bourdon and H. Pajot, *Cohomologie* ℓ_p *et espace de Besov*, J. Reine Angew. Math. **558** (2003), 85–108.
12. M.R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, 1999.
13. J.M. Brochet and R. Diestel, *Normal tree orders for infinite graphs*, Trans. Am. Math. Soc. **345** (1995), no. 2, 871–895.
14. A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-regular graphs*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 18, Springer-Verlag, Berlin, 1989.
15. S. Buyalo, A. Dranishnikov, and V. Schroeder, *Embedding of hyperbolic groups into products of binary trees*, Invent. Math. **169** (2007), no. 1, 153–192.
16. S. Buyalo and V. Schroeder, *Elements of Asymptotic Geometry*, EMS Monographs in Mathematics, EMS, Zürich, 2007.
17. P.J. Cameron, 6-*transitive graphs*, J. Combin. Theory (Series B) **28** (1980), no. 2, 168–179.
18. ———, *A census of infinite distance-transitive graphs*, Discrete Math. **192** (1998), no. 1-3, 11–26.
19. P.J. Cameron, C.E. Praeger, and N.C. Wormald, *Infinite highly arc transitive digraphs and universal covering digraphs*, Combinatorica **13** (1993), no. 4, 377–396.
20. G.L. Cherlin, *Homogeneous directed graphs. the imprimitive case*, Logic colloquium '85 (Orsay, 1985), Stud. Logic Found. Math., vol. 122, North-Holland, Amsterdam, 1987, pp. 67–88.

21. ______, *Homogeneous tournaments revisited*, Geom. Dedicata **26** (1988), no. 2, 231–239.
22. ______, *The classification of countable homogeneous directed graphs and countable homogeneous n-tournaments*, vol. 131, Mem. Amer. Math. Soc., no. 621, Amer. Math. Soc., 1998.
23. M. Coornaert, T. Delzant, and A. Papadopoulos, *Notes sur les groupes hyperboliques de Gromov*, Lecture Notes in Mathematics, vol. 1441, Springer-Verlag, 1990.
24. M. Coornaert and A. Papadopoulos, *Symbolic dynamics and hyperbolic groups*, Springer Lecture Notes, vol. 1539, Springer-Verlag, 1993.
25. W. Dicks and M.J. Dunwoody, *Groups Acting on Graphs*, Cambridge Stud. Adv. Math., vol. 17, Cambridge Univ. Press, 1989.
26. R. Diestel, *Graph Theory* (4th edition), Springer-Verlag, 2010.
27. R. Diestel, H.A. Jung, and R.G. Möller, *On vertex transitive graphs of infinite degree*, Arch. Math. (Basel) **60** (1993), no. 6, 591–600.
28. M. Droste, M. Giraudet, and D. Macpherson, *Set-homogeneous graphs and embeddings of total orders*, Order **14** (1997), no. 1, 9–20.
29. M. Droste, M. Giraudet, D. Macpherson, and N. Sauer, *Set-homogeneous graphs*, J. Combin. Theory (Series B) **62** (1994), no. 1, 63–95.
30. M. Droste and H.D. Macpherson, *On k-homogeneous posets and graphs*, J. Combin. Theory (Series A) **56** (1991), no. 1, 1–15.
31. M.J. Dunwoody, *Cutting up graphs*, Combinatorica **2** (1982), no. 1, 15–23.
32. M.J. Dunwoody and B. Krön, *Vertex cuts*, arXiv:0905.0064, 2010.
33. G. Elek, *The ℓ_p-cohomology and the conformal dimension of hyperbolic cones*, Geom. Dedicata **68** (1997), no. 3, 263–279.
34. H. Enomoto, *Combinatorially homogeneous graphs*, J. Combin. Theory (Series B) **30** (1981), no. 2, 215–223.
35. P. Erdős and J. Spencer, *Probabilistic Methods in Combinatorics*, Academic Press, New York, 1974.
36. D.M. Evans, *An infinite highly arc-transitive digraph*, European J. Combin. **18** (1997), no. 3, 281–286.
37. A. Gardiner, *Homogeneous graphs*, J. Combin. Theory (Series B) **20** (1976), no. 1, 94–102.
38. ______, *Homogeneity conditions in graphs*, J. Combin. Theory (Series B) **24** (1978), no. 3, 301–310.
39. A. Georgakopoulos and M. Hamann, *Transitivity in planar hyperbolic graphs*, manuscript, 2011.
40. E. Ghys and P. de la Harpe, *Sur les groupes hyperboliques, d'après M. Gromov*, Progress in Math., vol. 83, Birkhäuser, Boston, 1990.
41. M. Goldstern, R. Grossberg, and M. Kojman, *Infinite homogeneous bipartite graphs with unequal sides*, Discrete Math. **149** (1996), no. 1-3, 69–82.
42. R. Gray, *k-CS-transitive infinite graphs*, J. Combin. Theory (Series B) **99** (2009), no. 2, 378–398.
43. R. Gray and D. Macpherson, *Countable connected-homogeneous graphs*, J. Combin. Theory (Series B) **100** (2010), no. 2, 97–118.
44. R. Gray, D. Macpherson, C.E. Praeger, and G.F. Royle, *Set-homogeneous directed graphs*, arXiv:1011.4216, 2010.
45. R. Gray and R.G. Möller, *Locally-finite connected-homogeneous digraphs*, to appear in Discrete Math., 2009.
46. M. Gromov, *Hyperbolic Groups*, Essays in group theory (S.M. Gersten, ed.), MSRI, vol. 8, Springer, New York, 1987, pp. 75–263.
47. ______, *Asymptotic invariants of infinite groups*, London Math. Soc. Lecture Notes, vol. 182, Cambridge Univ. Press, 1993.
48. R. Halin, *Über unendliche Wege in Graphen*, Math. Ann. **157** (1964), 125–137.

49. ______, *Automorphisms and Endomorphisms of Infinite Locally Finite Graphs*, Abh. Math. Sem. Univ. Hamburg **39** (1973), no. 1, 251–283.
50. M. Hamann, *Spanning trees in hyperbolic graphs*, arXiv:0910.5605, 2009.
51. ______, *The classification of finite and locally finite connected-homogeneous digraphs*, arXiv:1101.2330, 2011.
52. ______, *On the tree-likeness of hyperbolic spaces*, preprint, 2011.
53. ______, *End-transitive graphs*, Israel J. Math. (to appear), arXiv:1003.3665.
54. M. Hamann and F. Hundertmark, *A classification of connected-homogeneous digraphs*, arXiv:1004.5273, 2010.
55. M. Hamann and J. Pott, *Transitivity conditions in infinite graphs*, arXiv:0910.5651, 2009.
56. S. Hedman and W.Y. Pong, *Locally finite homogeneous graphs*, Combinatorica **30** (2010), no. 4, 419–434.
57. W. Imrich, *On Whitney's theorem on the unique imbeddability of* 3*-connected planar graphs*, Recent Advances in Graph Theory (M. Fiedler, ed.), Academia Praha, Prague, 1975, pp. 303–306.
58. H.A. Jung, *Wurzelbäume und Kantenorientierungen in Graphen*, Math. Nachr. **36** (1968), 351–359.
59. ______, *Connectivity in infinite graphs*, Studies in Pure Mathematics (L. Mirsky, ed.), Academic Press, 1971, pp. 137–143.
60. ______, *On finite fixed sets in infinite graphs*, Discrete Math. **131** (1991), no. 1-3, 115–125.
61. V.A. Kaimanovich and W. Woess, *Boundary and entropy of space homogeneous Markov chains*, Ann. Probab. **30** (2002), no. 1, 323–363.
62. B. Krön, *End compactifications in non-locally-finite graphs*, Math. Proc. Cambridge Phil. Soc. **131** (2001), no. 3, 427–443.
63. ______, *Quasi-Isometries between Non-Locally-Finite Graphs and Structure Trees*, Abh. Math. Sem. Univ. Hamburg **71** (2001), no. 1, 161–180.
64. ______, *Cutting up graphs revisited - a short proof of Stallings' structure theorem*, Groups Complex. Cryptol. **2** (2010), no. 2, 213–222.
65. B. Krön and R.G. Möller, *Free groups and ends of graphs*, Abh. Math. Sem. Univ. Hamburg **78** (2008), no. 1, 1–15.
66. ______, *Metric ends, fibers and automorphisms of graphs*, Math. Nachr. **281** (2008), no. 1, 62–74.
67. ______, *Quasi-isometries between graphs and trees*, J. Combin. Theory (Series B) **98** (2008), no. 5, 994–1013.
68. A. Kurosch, *Die Untergruppen der freien Produkte von beliebigen Gruppen*, Math. Ann. **109** (1934), no. 1, 647–660.
69. H. Kurzweil and B. Stellmacher, *The theory of finite groups. An introduction*, Universitext, Springer-Verlag, New York, 2004.
70. A.H. Lachlan, *Finite homogeneous simple digraphs*, Proceedings of the Herbrand symposium (Marseilles, 1981) (J. Stern, ed.), Stud. Logic Found. Math., vol. 107, North-Holland, 1982, pp. 189–208.
71. ______, *Countable homogeneous tournaments*, Trans. Am. Math. Soc. **284** (1984), no. 2, 431–461.
72. A.H. Lachlan and R. Woodrow, *Countable ultrahomogeneous undirected graphs*, Trans. Am. Math. Soc. **262** (1980), no. 1, 51–94.
73. U. Lang and T. Schlichenmaier, *Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions*, Int. Math. Res. Not. **58** (2005), 3625–3655.
74. M.W. Liebeck, *The affine permutation groups of rank three*, Proc. London Math. Soc. (3) **54** (1987), no. 3, 477–516.

75. J. Luukainen, *Assouad Dimension: antifractal metrization, Porous sets, and homogeneous measures*, J. Korean Math. Soc. **35** (1998), no. 1, 23–76.
76. H.D. Macpherson, *Infinite distance transitive graphs of finite valency*, Combinatorica **2** (1982), no. 1, 63–69.
77. A. Malnič, D. Marušič, R.G. Möller, N. Seifter, V. Trofimov, and B. Zgrablič, *Highly arc transitive digraphs: reachability, topological groups*, European J. Combin. **26** (2005), no. 1, 19–28.
78. A. Malnič, D. Marušič, N. Seifter, and B. Zgrablič, *Highly arc-transitive digraphs with no homomorphism onto Z*, Combinatorica **22** (2002), no. 3, 435–443.
79. R.G. Möller, *Ends of graphs*, Math. Proc. Cambridge Phil. Soc. **111** (1992), no. 2, 255–266.
80. ———, *Ends of graphs II*, Math. Proc. Cambridge Phil. Soc. **111** (1992), no. 3, 455–460.
81. ———, *Distance-transitivity in infinite graphs*, J. Combin. Theory (Series B) **60** (1994), no. 1, 36–39.
82. ———, *Groups acting on locally finite graphs—a survey of the infinitely ended case*, Groups '93 Galway/St. Andrews, vol. 2, Cambridge Univ. Press, 1995, pp. 426–456.
83. ———, *Accessibility and ends of graphs*, J. Combin. Theory (Series B) **66** (1996), no. 2, 303–309.
84. ———, *Descendants in highly arc transitive digraphs*, Discrete Math. **247** (2002), no. 1-3, 147–157.
85. C. Nebbia, *Amenability and Kunze-Stein property for groups acting on a tree*, Pacific J. Math. **135** (1988), no. 2, 371–380.
86. A. Nevo, *A structure theorem for boundary-transitive graphs with infinitely many ends*, Israel J. Math. **75** (1991), no. 1, 1–19.
87. K. Ohshika, *Discrete groups*, Translated from the 1998 Japanese original by the author. Translations of Mathematical Monographs, vol. 207, Iwanami Series in Modern Mathematics, Amer. Math. Soc., Providence, RI, 2002.
88. F. Paulin, *Topologie de Gromov équivariante, structures hyperboliques et arbres réels*, Invent. Math. **94** (1988), no. 1, 53–80.
89. N. Polat and M.E. Watkins, *On translations of double rays in graphs*, Period. Math. Hungar. **30** (1995), no. 2, 145–154.
90. C. Ronse, *On homogeneous graphs*, J. London Math. Soc. **17** (1978), no. 3, 375–379.
91. N. Seifter, *Transitive digraphs with more than one end*, Discrete Math. **308** (2008), no. 9, 1531–1537.
92. P.M. Soardi and W. Woess, *Amenability, unimodularity and the spectral radius of random walks on infinite graphs*, Math. Zeitschr. **205** (1990), no. 3, 471–486.
93. C. Thomassen, *Duality of infinite graphs*, J. Combin. Theory (Series B) **33** (1982), no. 2, 137–166.
94. C. Thomassen and W. Woess, *Vertex-transitive graphs and accessibility*, J. Combin. Theory (Series B) **58** (1993), no. 2, 248–268.
95. J. Tits, *Sur le groupe des automorphismes d'un arbre*, Essays on topology and related topics (Mémoires dédiés à Georges de Rham), Springer, New York, 1970, pp. 188–211.
96. J. van Bon, *Finite primitive distance-transitive graphs*, European J. Combin. **28** (2007), no. 2, 517–532.
97. R. Weiss, *Glatt einbettbare Untergraphen*, J. Combin. Theory (Series B) **21** (1976), no. 3, 275–281.
98. H. Whitney, *Non-separable and planar graphs*, Trans. Am. Math. Soc. **34** (1932), no. 2, 339–362.
99. H. Wielandt, *Finite permutation groups*, Academic Press, New York-London, 1964.
100. W. Woess, *Amenable group actions on infinite graphs*, Math. Ann. **284** (1989), no. 2, 251–265.

101. ———, *Topological groups and infinite graphs*, Discrete Math. **95** (1991), no. 1-3, 373–384.
102. ———, *Fixed sets and free subgroups of groups acting on metric spaces*, Math. Z. **214** (1993), no. 3, 425–440.
103. ———, *Random walks on infinite graphs and groups*, Cambridge University Press, 2000.

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Zusammenfassung

In der vorliegenden Dissertation untersuchen wir im ersten Teil die Baumähnlichkeit hyperbolischer Graphen. Dazu konstruieren wir für jeden lokal-endlichen hyperbolischen Graphen, dessen hyperbolischer Rand eine endliche Assouad-Dimension hat, einen Spannbaum, sodass einerseits der hyperbolische Graph selbst durch den Baum gut dargestellt wird: jeder Strahl des Baumes ist schließlich quasi-geodätisch und jeder geodätische Strahl des Graphen liegt schließlich in einer konstanten Umgebung des unendlichen Gerüsts des Baumes. Andererseits gibt der Rand des Baumes uns auch eine gute Darstellung des hyperbolischen Randes des Graphens, indem sich die Einbettung des Baumes stetig auf den Rand zu einer surjektiven Abbildung fortsetzen lässt, sodass jeder Randpunkt des Graphens beschränkt viele Urbilder unter dieser Fortsetzung hat.

Im zweiten Teil der Arbeit werden Graphen studiert, die gewisse Gruppenoperationen auf ihrem Rand realisieren: zuerst zeigen wir, dass kein lokal-endlicher ein-endiger hyperbolischer plättbarer Graph existiert, auf dem eine Gruppe derart opertiert, dass sie einen seiner Randpunkte fixiert und auf seinen Knoten transitiv operiert. Danach werden zusammenhängende unendlich-endige Graphen charakterisiert, auf denen eine Gruppe transitiv operiert und gleichzeitig einen der Enden fixiert. Wir erhalten, dass diese Graphen quasi-isometrisch zu Bäumen sind. Der letzte Abschnitt des zweiten Teils charakterisiert Graphen mit unendlich vielen Enden, sodass die Automorphismgruppe des Graphen transitiv auf dessen Enden operiert. Auch in diesem Fall erhalten wir eine Bäumähnlichkeit: es existiert ein Teilgraph, der quasi-isometrisch zu einem Baum ist und dessen Löschung aus dem ursprünglichen Graphen einen strahlenlosen Graphen übrig lässt.

Im dritten Teil dieser Dissertation erhalten wir Klassifikationsresultate für Graphen, die spezielle Transitivitäts- oder Homogenitätseigenschaften besitzen. So werden zunächst mehr-endige Abstands-transitive Graphen klassifiziert und anschließend mehr-endige k-CS-transitive für $k \geq 2$. Im letzten Kapitel klassifizieren wir zusammenhängend-homogene Digraphen, die entweder endlich oder lokal-endlich oder zusammenhängend und mehr-endig sind.

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