Unifying Constructions in Toric Mirror Symmetry

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To my parents

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Contents

Chapter 1

Introduction

Mirror symmetry is an ample and far reaching phenomenon originating from string theory which continues to have a deep impact on many areas of mathematics to this day. It originated from the observation that string theory suitably compactified on two different Calabi-Yau threefolds may nevertheless induce the same superconformal field theory. Since the appearance of the seminal paper by Candelas, de la Ossa, Green and Parkes [CaOsGrPa], which allowed to deduce previously unknown enumerative aspects for quintic threefolds, mathematicians got attracted by mirror symmetry as a mathematical discipline on its own.

1.1. Classical toric mirror symmetry and natural generalizations

Much has been done since then, but Batyrev's work on hypersurfaces in toric varieties [Ba] is not only largely considered the first purely mathematical manifestation of the phenomenon, but also one of the most popular and recognized approaches, supposedly because of its combinatorial nature and the fact that it is so handy to compute with. Around the same time

Berglund and Hübsch [BeHu] presented another explicit mirror construction which remained almost unnoticed by mathematicians for many years. Although the spaces in which they consider Calabi-Yau hypersurfaces are toric varieties as well, most of them have more complicated singularities than those in the work of Batyrev. Recently there have been more and more papers devoted to the construction of Berglund and Hübsch, see for instance [Bori], [ChRu1], [ChRu2] or [Kra]. However, there is a certain controversy in the current literature about what the relation between the two constructions is. Some authors that a 'vast range of cases' is not covered by Batyrev's construction [ChRu1, p.2], while others note that there might be a generalized setup in which they can both be understood [Bori]. This suggests that mirror symmetry for hypersurfaces in toric varieties is not fully understood yet and it is one of the aims of this thesis to clarify the relation between known approaches.

On the other hand, mirror symmetry has been suggested both by mathematicians and physicists to extend to a correspondence between Fano varieties and Landau-Ginzburg models, see for instance [ChOh], [FOOO1], [Gi], [HoVa]. Purely mathematically a Landau-Ginzburg model is a non-compact Kähler manifold with a holomorphic function called the superpotential. The majority of literature deals with toric varieties, where an explicit construction of the mirror was known for a long time. The work of Auroux, Katzarkov and Orlov on mirror symmetry for del Pezzo surfaces [AuKaOr], where a mirror is constructed by an ad hoc construction, presents a rare exception to this. However, the program proposed by Gross and Siebert in [GrSi1],[GrSi3] presents a framework for mirror symmetry that vastly exceeds the realm of toric geometry, but has not yet been adapted to incorporate a Fano/Landau-Ginzburg correspondence. Another goal of this thesis is to introduce a generalized approach to Landau-Ginzburg mirror symmetry within this program.

1.2. Main results of thesis

1.2.1. Calabi-Yau varieties in $\mathbb{Q}$-Gorenstein spaces and mirror symmetry. Given a lattice polytope Ξ, a well known theorem of Batyrev asserts that a Ξ-regular anti-canonical hypersurface X in the toric variety $\mathbb{P}_\Sigma$ associated to the normal fan Σ of Ξ is Calabi-Yau if and only if Ξ is a reflexive polytope. In that case $\mathbb{P}_\Sigma$ is a Gorenstein Fano variety. Ξ-regularity means that X intersects every toric stratum smoothly in codimension one and its Newton polytope equals Ξ. However, X can be Calabi-Yau even if it intersects strata non-smoothly or has a smaller Newton polytope.

Almost reflexive polytopes and singular Calabi-Yau varieties. We will carefully analyze the case when $\mathbb{P}_\Sigma$ is only $\mathbb{Q}$-Gorenstein and study various properties of such toric varieties. It turns out that the right notion to consider is that of an *almost reflexive polytope*. An almost reflexive polytope Θ is defined by the property that the integral points of its polar polytope Θ^*, a $\mathbb{Q}$-lattice polytope in general, span a reflexive polytope. Using this notion, we prove the following generalization of Batyrev's theorem above.

THEOREM 1.1 (Theorem 2.25). *Let Θ be a lattice polytope and denote its fan of cones over faces by Σ_Θ. Moreover, let $X \subset \mathbb{P}_{\Sigma_\Theta}$ be a general anti-canonical hypersurface in the $\mathbb{Q}$-Gorenstein toric variety $\mathbb{P}_{\Sigma_\Theta}$ associated to Σ_Θ. Then X is a Calabi-Yau variety if and only if Θ is almost reflexive, that is if and only if the integral points of the anti-canonical polytope of $\mathbb{P}_\Sigma$ span a reflexive polytope.*

Let us explain the relationship between this theorem and the above mentioned result by Batryrev. If Θ is almost reflexive, it follows that it is canonically embedded into a reflexive polytope $\overline{\Theta}$, called the associated reflexive polytope. The fan of cones over faces of $\overline{\Theta}$ is denoted $\Sigma_{\overline{\Theta}}$. Choosing a maximal projective triangulation of $\overline{\Theta}$ yields a fan $\widetilde{\Sigma}$ that is a common refinement of both Σ_Θ and $\Sigma_{\overline{\Theta}}$. Thus we get the following diagram of toric morphisms.

$$\begin{tikzcd} & \mathbb{P}_{\widetilde{\Sigma}} \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ \mathbb{P}_{\Sigma_\Theta} & & \mathbb{P}_{\Sigma_{\overline{\Theta}}}. \end{tikzcd}$$

By Batryrev's result a $\overline{\Theta}^*$-regular anti-canonical hypersurface $Y \subset \mathbb{P}_{\Sigma_{\overline{\Theta}}}$ is Calabi-Yau and it is known that ψ is crepant, so $\psi^*(Y)$ is Calabi-Yau as well. By our theorem the image $\phi(\psi^*(Y))$ is Calabi-Yau as well. On the other hand, we will see that ϕ is crepant on anti-canonical hypersurfaces, so for X as in the theorem $\psi(\phi^*(X))$ is also a Calabi-Yau variety. Thus Theorem 1.1 does not give genuinely new Calabi-Yau varieties, but instead shows precisely to which extend we can torically blow down $\mathbb{P}_{\widetilde{\Sigma}}$ such that a general anti-canonical hypersurface remains Calabi-Yau.

Theorem 1.1 turns out to have some impact on toric mirror symmetry and we will give two major applications of it.

Berglund-Hübsch mirror symmetry in the Batyrev setup. Firstly, we show how to incorporate the approach of Berglund-Hübsch into the framework of Batyrev mirror symmetry. A considerable step towards this has also been achieved by [Bori], whose work is independent of ours.

A polynomial of Berglund-Hübsch type W is a quasi-smooth anti-canonical hypersurface in an n-dimensional weighted projective space $\mathbb{P}(w)$ having precisely $n+1$ monomials. Denoting the zero set of W in $\mathbb{P}(w)$ by X, the results of [BeHu] show that X is a Calabi-Yau variety. $\mathbb{P}(w)$ is a $\mathbb{Q}$-Gorenstein toric variety with fan Σ_w in some lattice $N_{\mathbb{Q}}$. By Theorem 1.1 we thus know that X can only be Calabi-Yau if the span Θ of ray generators of Σ_w is an almost reflexive polytope. As before, denote the reflexive polytope associated to Θ by $\overline{\Theta}$ and the fan of cones over faces of a maximal triangulation of $\overline{\Theta}$ by $\widetilde{\Sigma}$.

Next, choose a group $\overline{G}$ of automorphisms of W acting diagonally on the coordinates of $\mathbb{P}(w)$. A central result of Berglund and Hübsch says that the space $X/\overline{G}$ is a singular Calabi-Yau and admits a full crepant resolution

$X_W := \widetilde{X/\overline{G}}$ for $n = 4$. While it is well known that dividing by certain group actions corresponds to passing to sup-lattices in toric geometry, see [Ba], this knowledge was never applied to the Berglund-Hübsch setting. We show that the choice of $\overline{G}$ corresponds to a sup-lattice $\overline{N}$ of N, so that $X/\overline{G}$ is given by viewing Σ_w as a fan in $\overline{N}$. We can now use the toric morphism $\phi : \mathbb{P}_{\widetilde{\Sigma}} \to \mathbb{P}_{\Sigma_\Theta} = \mathbb{P}(w)/\overline{G}$ from above to pull back $X/\overline{G}$ to $\mathbb{P}_{\widetilde{\Sigma}}$ which turns out to be the Calabi-Yau manifold X_W.
From W and $\overline{G}$ one can explicitly construct a dual polynomial W^* and a dual group $\overline{G}^*$. W^* defines a Calabi-Yau X^* in another weighted projective space $\mathbb{P}(w^*)$ with fan Σ_{w^*}, whose span of ray generators will be an almost reflexive polytope Ξ by the same arguments as above. The Calabi-Yau manifold $X_{W^*} := \widetilde{X^*/\overline{G}^*}$ will then analogously be the pull-back of $X^*/\overline{G}^*$ via a morphism $\phi^* : \mathbb{P}_{\widetilde{\Sigma}^*} \to \mathbb{P}_{\Sigma_\Xi}$ for some fan $\widetilde{\Sigma}^*$. The main theorem of [BeHu] asserts that X_W and X_{W^*} are mirror Calabi-Yau. By observing that the associated reflexive polytope $\overline{\Xi} \supset \Xi$ is the polar polytope of $\overline{\Theta} \supset \Theta$ we obtain the following theorem.

THEOREM 1.2 (Theorem 3.19). *A Berglund-Hübsch mirror pair X_W and X_{W^*} is an explicit choice of* special *hypersurfaces in $\mathbb{P}_{\widetilde{\Sigma}}$ and $\mathbb{P}_{\widetilde{\Sigma}^*}$, that is in toric varieties associated to maximal projective triangulations of four-dimensional reflexive polytopes $\overline{\Theta}$ and $\overline{\Xi} := \overline{\Theta}^*$. Moreover, there are polynomial deformations Y of X_W and Y^* of X_{W^*} that form a mirror pair in the sense of Batyrev [Ba].*

As an immediate corollary of this theorem, which is not clear from the original construction, we see that the mirror of a Berglund-Hübsch Calabi-Yau $\widetilde{X/\overline{G}}$ does, up to polynomial deformations, not depend on the choice of W. Another implication is that when W is of Fermat type, then by Theorem 1.2 the Berglund-Hübsch mirror is *precisely the same* as the Batyrev mirror, that is for Gorenstein Fano $\mathbb{P}(w)$ and their quotients $\mathbb{P}(w)/\overline{G}$ the two constructions coincide.

Mirrors for generalized Borcea-Voisin threefolds. As a second application of Theorem 1.1 we present a general scheme for finding mirrors of the following class of Calabi-Yau threefolds. Let E be an elliptic curve with automorphism ι of order p and let X be a $K3$ surface with automorphism σ of order p, acting as -1 on a generator of $H^{2,0}(X)$, for $p = 2, 3, 4, 6$. Then there is a Calabi-Yau resolution

$$Y \longrightarrow X \times E/(\sigma \times \iota),$$

which is called a *generalized Borcea-Voisin threefold*. For $p = 2$ this construction was discovered independently by Borcea [Borc] and Voisin [Vo]. Both autors showed that, except for eleven special cases, the mirror of a Borcea-Voisin threefold is again such a threefold.

While for $p = 4, 6$ it is an active field of research pursued by M. Artebani, S. Boissiére and A. Sarti, to find all possible pairs (X, σ) as above, for $p = 3$ this classification is known by the results of [ArSa]. The work of Dillies [Di] shows that not a single generalized Borcea-Voisin threefold for $p = 3$ can be mirror to another such threefold. Moreover, Garbagnati and van Geemen [GavGe] and Rohde [Roh] show that for $p = 3$ there are examples without mirror.

So this case substantially differs from the construction for $p = 2$, and to our knowledge so far no one has proposed a mirror construction for these generalized Borcea-Voisin threefolds. We show that often it is possible to find a singular model for a pair (X, σ) as a hypersurface in a $\mathbb{Q}$-Gorenstein weighted projective space of dimension three. Whenever this holds, Y can also be realized as hypersurface in a toric variety. To make this statement more precise denote by $S^\sigma := \{x \in H^2(X, \mathbb{Z}) | \sigma^*(x) = x\}$ the fixed lattice in the $K3$-lattice, where σ^* is the induced action on cohomology. We have to define the following two discrete invariants to state our theorem. The first is defined by $m := (22 - \text{rank}(S^\sigma))/2$. For the second invariant note that from the dual lattice $(S^\sigma)^* := \mathrm{Hom}_{\mathbb{Z}}(S^\sigma, \mathbb{Z})$ we can construct the so called discriminant group $(S^\sigma)^*/S^\sigma$, which is known to be of the form $(S^\sigma)^*/S^\sigma = (\mathbb{Z}/2\mathbb{Z})^a$ for some a. Using this notation, we have the following theorem.

THEOREM 1.3 (Theorem 3.30). *Let X be a $K3$-surface with non-symplectic automorphism σ of order $p = 3$ and discrete invariants (m, a) as in Table 1* *in section 3.3.2, then the generalized Borcea-Voisin threefold Y associated to X is given by a hypersurface in a toric variety. A mirror for Y can therefore be obtained by applying the mirror symmetry construction of Batyrev.*

1.2.2. A geometric framework for Landau-Ginzburg models. Next we present a way to naturally incorporate the Fano/Landau-Ginzburg correspondence into the mirror symmetry program proposed by Gross and Siebert [GrSi1],[GrSi3]. With some minor changes the program can be applied to toric degenerations of varieties with effective anti-canonical bundle. Doing so gives a non-compact variety as mirror right away and the key point is to construct the superpotential. We will sketch this construction briefly.

Broken lines and superpotentials up to order k. Let $(\breve{\pi} : \breve{\mathfrak{X}} \to T, \breve{\mathfrak{D}})$ be a toric degeneration of Calabi-Yau pairs over the spectrum T of a discrete valuation $\Bbbk$-algebra, such that the generic fibre $(\breve{\mathfrak{X}}_\eta, \breve{\mathfrak{D}}_\eta)$ consists of a complete variety $\breve{\mathfrak{X}}_\eta$ and a reduced effective anti-canonical divisor $\breve{\mathfrak{D}}_\eta \subset \breve{\mathfrak{X}}_\eta$. Furthermore assume that the toric degeneration is polarized and denote by $(\breve{B}, \breve{\mathscr{P}}, \breve{\varphi})$ the polarized intersection complex as described in [GrSi1]. In this situation we can apply the discrete Legendre transformation, which is at the heart of the mirror symmetry construction by Gross and Siebert. Denote the discrete Legendre dual data by $(B, \mathscr{P}, \varphi)$. As $\breve{B}$ is compact with boundary in this situation, the dual base B will be non-compact. Choosing gluing data for $(B, \mathscr{P}, \varphi)$, by methods described in [GrSi3] one can construct a scheme X_0 from this set of tropical data. The *superpotential W_0 of X_0 to order zero* is then defined as follows. Let $\sigma \in \mathscr{P}$ be an unbounded maximal cell. Then for each edge $\omega \subset \sigma$ it can be shown that there is a unique monomial z^{m_ω}, subject to certain conditions, that points in the unbounded direction of ω. Then the sum

$$W^0(\sigma) := \sum z^{m_\omega}$$

over all such monomials defines a function on the component $X_\sigma \subset X_0$ and glues to a regular function $W^0 \in \mathcal{O}(X_0)$. We have the following result for W^0:

THEOREM 1.4 (Proposition 4.9). *W^0 is proper if and only if the induced morphism $\check{\mathfrak{D}} \to T$ is a toric degeneration of Calabi-Yau varieties. In this case $\partial \check{B}$ is a smooth affine manifold and all unbounded one-dimensional strata of B are parallel.*

To define the superpotential to higher orders, recall that under certain maximal degeneracy assumptions one can canonically construct a dual toric degeneration $\pi : \mathfrak{X} \to \operatorname{Spec} \Bbbk[[t]]$ from $(B, \mathscr{P}, \varphi)$, which exhibits X_0 as central fibre, by an explicit algorithm found in [GrSi3]. More precisely we obtain a sequence of compatible structures $(\mathscr{S}_k)_{k\geq 0}$ and k-th order deformations $X_k \to \operatorname{Spec} \Bbbk[t]/(t^{k+1})$ with limit π. For a given structure $\mathscr{S}_k$ a *broken line* morally speaking is a proper continuous map

$$\beta : (-\infty, 0] \to B$$

with endpoint $p = \beta(0)$ that allows to trace a monomial z^m that comes in from infinity. Each time β changes chambers of $\mathscr{S}_k$ it possibly changes direction and picks up a coefficient in a specific way that respects the structure. β is allowed to have finitely many such "break points". The direction β hits the point p from is denoted by m_β, the respective coefficient by a_β. Now for general p in a chamber $\mathfrak{u}$ the *superpotential W^k up to order k* can locally be defined by the following sum over all broken lines ending in p

$$W^k := \sum a_\beta z^{m_\beta}.$$

This is well-defined, as we can show that this definition is independent of the choice of p and compatible with changing strata and chambers of $\mathscr{S}_k$. Hence we get a global regular function $W^k \in \mathcal{O}_{X_k}$ and thus a superpotential $W := \lim_k W_k \in \mathcal{O}(\check{\mathfrak{X}})$ for $\check{\mathfrak{X}}$. The pair

$$(\mathfrak{X} \to \operatorname{Spec} \Bbbk[[t]], W)$$

is what we call the Landau-Ginzburg model of the toric degeneration $(\check{\pi} : \check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$. To compute such Landau-Ginzburg models in practice can be very hard to achieve. However, if B has parallel unbounded one-cells and a finite scattering diagram locally on bounded cells, we can prove a key lemma suggested by Gross, that greatly reduces the number of broken lines one has to consider.

Reflexive polytopes and proper superpotentials. We present two applications of our construction of Landau-Ginzburg models that are directly related to reflexive polytopes.

First, let $\Theta \subset N_{\mathbb{R}}$ be a full-dimensional reflexive polytope such that the toric variety $\mathbb{P}_{\Sigma_\Theta}$ associated to the fan Σ_Θ of cones over proper faces of Θ is a smooth toric variety. Recall that the so called Hori-Vafa superpotential [HoVa] of $\mathbb{P}_{\Sigma_\Theta}$ with its anti-canonical polarization is

$$W(x_1, \ldots, x_n) = \sum_{\rho \in \Sigma_\Theta(1)} x^{n_\rho},$$

where x_i are coordinates on the torus $(\mathbb{C}^*)^n$, n_ρ denotes the generator of the ray ρ and x^{n_ρ} is the usual multi-index notation. Then to any such Θ we can construct a polarized tropical affine manifold $(B, \mathscr{P}, \varphi)$ such that all unbounded one-dimensional cells of B are parallel . By running the reconstruction algorithm from [GrSi3] for $(B, \mathscr{P}, \varphi)$ we therefore get a toric degeneration $\mathfrak{X} \to \operatorname{Spec} \Bbbk[[t]]$, which has a proper superpotential W by Theorem 1.4. Moreover, we have the following result

THEOREM 1.5 (Theorem 5.4). *Let $(\mathfrak{X} \to \operatorname{Spec} \Bbbk[[t]], W)$ be the Landau-Ginzburg model associated to a base $(B, \mathscr{P}, \varphi)$ obtained from an n-dimen sional reflexive polytope Θ. Then there is an open subset*

$$U \cong \operatorname{Spec} \Bbbk[[t]][x_1, \ldots, x_n] \subset \mathfrak{X}$$

such that

$$W|_U = \Big(\sum_{\rho \in \Sigma_\Theta(1)} x^{n_\rho} \Big) \cdot t.$$

Thus $W|_U$ is the Hori-Vafa mirror of the anti-canonically polarized toric variety $\mathbb{P}_{\Sigma_\Theta}$ times t.

In a second application we make a first step towards understanding the project pursued by Coates, Corti, Galkin, Golyshev and Kasprzyk in [CoCo] within the LG-model framework presented here. In this project the authors give an algorithm that produces a Laurent polynomial W from certain three-dimensional reflexive polytopes Θ that we call *fully decomposable*. In most cases such a W corresponds to a Fano manifold X. However, this procedure does not have an underpinning geometric construction, but works purely algebraically. We sketch a geometric procedure in terms of the Gross-Siebert program that conjecturally recovers their results and verify this in an explicit example.

Toric degenerations of del Pezzo surfaces. Next, we study toric degenerations for del Pezzo surfaces. Denote by dP_k the del Pezzo surface obtained from blowing up $\mathbb{P}^2$ in k general points. We call $(\check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ a *distinguished* toric degeneration of del Pezzo surfaces if it is simple, irreducible, $\check{\mathfrak{D}}$ is relatively ample over T and the generic fibre $\check{\mathfrak{D}}_\eta$ is an anti-canonical divisor in the Gorenstein surface $\check{\mathfrak{X}}_\eta$. Then we have the following uniqueness result.

THEOREM 1.6 (Theorem 5.19). *If $(\pi : \check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ is a distinguished toric degeneration of del Pezzo surfaces with non-singular generic fibre, then the associated intersection complex $(\check{B}, \check{\mathscr{P}})$ is unique up to isomorphism.*

Moreover, we explicitly study the unique bases $(\check{B}, \check{\mathscr{P}})$. Note that by definition the generic fibre $\check{\mathfrak{X}}_\eta$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or dP_k for $k \leq 3$, so this is a statement about toric degenerations of toric del Pezzo surfaces. To show that our approach is not limited by toric geometry we compute examples of Landau-Ginzburg models for toric degenerations whose generic fibre is isomorphic to dP_k for $k \geq 4$, where interesting phenomena can be observed. In particular we find tropical manifolds that correspond to the ad hoc construction for mirrors of dP_k found in [AuKaOr].

Landau-Ginzburg models for semi-Fano toric and Hirzebruch surfaces. Let $\Theta \subset N_{\mathbb{R}}$ be a two-dimensional reflexive polytope and choose a maximal projective triangulation $\mathcal{T}$ of it. This induces a maximal refinement $\widetilde{\Sigma_\Theta} \to \Sigma_\Theta$. Moreover, any such $\mathcal{T}$ comes with a strictly convex function $h_{\mathcal{T}}$ that is piecewise linear on cones of $\widetilde{\Sigma_\Theta}$. Furthermore, $\mathbb{P}_{\widetilde{\Sigma_\Theta}}$ is a semi-Fano toric surface. For any such Θ with a choice of triangulation $\mathcal{T}$

we construct a tropical manifold $(\widetilde{B}^{\mathcal{T}}, \widetilde{\mathscr{P}}^{\mathcal{T}}, \widetilde{\varphi}\mathcal{T})$ such that $\widetilde{B}^{\mathcal{T}}$ has parallel unbounded one-cells. The Landau-Ginzburg superpotential we get from this is therefore proper and locally has the following structure

$$W = \sum_{\nu \in \Theta \cap N} a_\nu \cdot t^{h_{\mathcal{T}}(\nu)} x^\nu,$$

where the coefficients a_ν can be described explicitly. Moreover, for the same choice of $\mathcal{T}$ as in [Ch] we get the same superpotentials as in this paper.

As a last application of our framework we compute tropical affine bases for toric degenerations of Hirzebruch surfaces $\mathbb{F}_m$. In the case of $\mathbb{F}_2$ and $\mathbb{F}_3$ we explicitly compute the full superpotential, which coincides with the computations in [Au].

1.3. Outline of thesis

The first two chapters start with comparatively classical material within the realm of toric geometry, whereas chapters 4 and 5 rely heavily on more sophisticated techniques and deeper results. This is intended and displays in a time-lapse the fast development mirror symmetry as a mathematical discipline has undergone in the last two decades.

On the one hand, Chapter 2 is intended to introduce necessary notation from toric geometry and state mostly classical results we need throughout the text. Section 2.1 states some general properties of singularities and explains what we mean by a Calabi-Yau variety. Section 2.2 collects various results about affine and projective toric varieties and contains the main results of the chapter, while Section 2.3 properly introduces basic properties of weighted projective spaces and shows how the results of the chapter apply to this special class of toric varieties. On the other hand, Chapter 2 aims at the proof of Theorem 2.25. The experienced reader, who wants to get to the proof of this main theorem fast, is advised to jump right to the central notion of almost reflexive polytopes given in Definition 2.11, follow the running example starting with Example 2.13 and work through Lemma 2.21 and Lemma 2.19 needed for Theorem 2.25.

In the following Chapter 3, we will start with a very short review of Batyrev's mirror construction followed by an extended introduction to the setup of Berglund-Hübsch and related work in Section 3.1. After that, Section 3.2 will be concerned with explaining how the approach of Berglund-Hübsch can be fit into a toric setup with the work of Batyrev, resulting in Theorem 3.19. In Section 3.3 we first review the Borcea-Voisin construction, then show how to generalize it and present Theorem 3.30, which demonstrates how to mirror partners for the generalized version in many cases.

Chapter 4 is devoted to the introduction of the technical data needed to properly handle Landau-Ginzburg mirror symmetry within the construction of Gross and Siebert. To this end we quickly review the main ingredients of this approach in Section 4.1, trying to keep the technicalities at a minimum. Having done so, we present the rather new tool of broken lines to deal with Landau-Ginzburg models within the Gross-Siebert program in Section 4.2, which also contains a very explicit example of a toric degeneration of $\mathbb{P}^2$ and its mirror.

The last chapter uses the previously developed machinery to derive theorems about Landau-Ginzburg models. In Section 5.1 we show how to obtain proper superpotentials from reflexive polytopes, thereby improving the situation known from toric geometry. We then devote Sections 5.2 and 5.3 to explicitly describe the situation in two dimensions, that is we deal with del Pezzo surfaces, semi-Fano toric and Hirzebruch surfaces in depth.

Throughout the thesis we will try to be as self contained as possible. However, as a premise, we expect the reader to have a solid working knowledge in toric geometry to the extend of [Od] and classical algebraic geometry as presented in [Ha1]. Moreover, for Chapters 4 and 5, it will be helpful to be familiar with the basic notions of the mirror symmetry program by Gross and Siebert [GrSi1],[GrSi3].

Chapter 2

Calabi-Yau in non-Gorenstein toric varieties

Let $\Bbbk$ be an algebraically closed field of characteristic 0. For most of the applications presented here it will be enough to assume $\Bbbk = \mathbb{C}$. Whenever we talk of varieties we mean integral quasi-projective $\Bbbk$-schemes.

2.1. Calabi-Yau varieties and singularities

To properly discuss the construction of Calabi-Yau hypersurfaces in $\mathbb{Q}$-Gorenstein projective toric varieties we first have to recall some results from singularity theory.

2.1.1. Singularities. Recall that a variety X has a *dualizing sheaf* in the sense of [Ha2] ω_X if and only if X is Cohen-Macaulay. This sheaf is always reflexive of rank one. In case that X is normal, we will need an explicit description of ω_X, which requires some notation and results.

For a Weil divisor D on a normal variety X denote the associated sheaf by $\mathcal{O}_X(D)$. Moreover, note that a coherent sheaf on a normal variety X is reflexive of rank one if and only if it is isomorphic to $\mathcal{O}_X(D)$ for some

Weil divisor D. Define the cotangent sheaf Ω^1_X locally via the sheaf of differentials of the structure sheaf relative to the ground field, that is

$$\Omega^1_X(U) := \Omega_{\mathcal{O}_X(U)/\Bbbk},$$

for $U \subset X$ open affine. It is classically known that a variety is smooth if and only if Ω^1_X is locally free. For $p \geq 1$ we define the sheaf of differential p-forms by taking the exterior product, that is $\Omega^p_X := \Lambda^p \Omega^1_X$. The restriction of Ω^p_X to the regular locus X_{reg} of X is locally free and it is known that $(\Omega^p_X)^{\vee\vee} \cong j_* \Omega^p_{X_{reg}}$. The following result shows an equivalence of properties of coherent sheaves $\mathcal{F}$ on X, which will be useful in the definition following it, where it can be applied to $\mathcal{F} = \Omega^p_X$.

PROPOSITION 2.1. *Let $\mathcal{F}$ be a coherent sheaf on a normal variety X, $j : U \to X$ an open embedding with codim$(X \setminus U) \geq 2$. Then the following statements hold:*

(1) *$\mathcal{F}^\vee$ is reflexive and hence $\mathcal{F}^{\vee\vee}$ is reflexive.*
(2) *If $\mathcal{F}$ is reflexive, then $\mathcal{F} \cong j_*(\mathcal{F}|_U)$.*
(3) *If $\mathcal{F}|_U$ is locally free, then $\mathcal{F}^{\vee\vee} \cong j_*(\mathcal{F}|_U)$.*

PROOF. Statements (1) and (2) are the content of Corollary 1.2 and Proposition 1.6 of [Ha3], while (3) is a direct corollary of both, whose proof is carried out explicitly in [CoLiSc, Proposition 8.0.1]. □

DEFINITION 2.2. Let X be a d-dimensional normal variety with regular locus $j : X_{reg} \to X$. For each $p \geq 1$ define the *sheaf of Zariski p-forms* to be

$$\hat{\Omega}^p_X := (\Omega^p_X)^{\vee\vee} \cong j_* \Omega^p_{X_{reg}}.$$

This sheaf is reflexive of rank $\binom{d}{p}$ and $\omega_X := \hat{\Omega}^d_X$ is a dualizing sheaf. □

REMARK 2.3. Let Y be a $(d+1)$-dimensional normal variety and consider the following d-dimensional subvariety. Let $X := \{s' = 0\} \subset Y$ defined by the zero set of a section $s' \in H^0(X, -\omega_Y) \setminus \{0\}$ and assume that X is normal. Then we can use the adjunction formula on the regular part of X to obtain $\Omega^d X_{reg} \cong \mathcal{O}_{X_{reg}}$. If we denote the inclusion $X_{reg} \to X$ by j_X, then we have $\omega_X \cong (j_X)_* \mathcal{O}_{X_{reg}}$. As $\mathcal{O}_X$ is reflexive, by Proposition 2.1 $(3) \Rightarrow (2)$ we get $\mathcal{O}_X \cong \omega_X$. Note, however that it is sufficient to assume that X is *regular in codimension one*, as then we can still define the dualizing sheaf ω_X by $(j_X)_*\mathcal{O}_{X_{reg}}$ and the implication $\mathcal{O}_X \cong \omega_X$ still holds. □

Next, for any section $s \in H^0(Y, \omega_X) \setminus \{0\}$ we can define the canonical divisor class by $[\mathrm{div}_X(s)]$ and we choose a representative K_X which we call the canonical divisor. Now, X is called $\mathbb{Q}$-*Gorenstein*, if rK_X is Cartier for some $r \in \mathbb{N}$, $r \geq 1$, and *Gorenstein* if we can set $r = 1$. For the minimal r with this property we will say that X is a $\mathbb{Q}$-Gorenstein variety of *index* r. In fact, ω_X is a ($\mathbb{Q}$-)bundle if and only if X is ($\mathbb{Q}$)-Gorenstein. The reader interested in properties of singularities beyond the following two basic definitions is referred to [Re2].

DEFINITION 2.4. X has *canonical singularities* if and only if it is $\mathbb{Q}$-Gorenstein of index r and for every resolution $f : \widetilde{X} \to X$ we have

$$rK_{\widetilde{X}} = f^*(rK_X) + \sum_{i \in I} a_i E_i \quad \text{with } a_i \geq 0,$$

where $\{E_i\}_{i \in I}$ denotes the family of exceptional prime divisors. If $a_i > 0$ for all $i \in I$ we say that X has *terminal singularities*. Moreover, we call the $\mathbb{Q}$-divisor $1/r \sum_{i \in I} a_i E_i$ the *discrepancy* of f and say that f is *crepant* if the discrepancy is 0. □

DEFINITION 2.5. A variety X is called

(1) *factorial* if every Weil divisor D on X is Cartier,
(2) $\mathbb{Q}$-*factorial* if every Weil divisor D has an integer multiple that is Cartier and
(3) *quasi-smooth* if X has only finite quotient singularities. □

2.1.2. The Calabi-Yau condition. In this section we will define what we mean by Calabi-Yau variety. Among many other equivalent definitions a *Calabi-Yau manifold* may be defined as a compact complex projective manifold with trivial canonical bundle and vanishing first Betti number. However, as we would like to allow for singular spaces as well, we will have to relax this definition.

DEFINITION 2.6. A *Calabi-Yau variety* X, or simply *Calabi-Yau*, is a d-dimensional compact normal projective variety subject to the following three conditions.

(1) X has at most canonical singularities.
(2) The dualizing sheaf $\hat{\Omega}_X^d$ of X is trivial, that is $\mathcal{O}_X \cong \hat{\Omega}_X^d$. This implies that X has Gorenstein singularities.
(3) $H^i(X, \mathcal{O}_X) = \{0\}$ for all $i = 1, \ldots, d-1$.

In case X has only $\mathbb{Q}$-factorial terminal singularities, we will say that X is a *minimal* Calabi-Yau variety. □

In the next sections we will study toric varieties whose dualizing sheaf is only a $\mathbb{Q}$-bundle, but which admit anti-canonical hypersurfaces X that have dualizing line bundles.

2.2. Hypersurfaces in non-Gorenstein toric varieties

We start with a minimum of notation of toric geometry used throughout the thesis. Let N be a free abelian group of rank n and let $M = \mathrm{Hom}(N, \mathbb{Z})$ be its dual lattice. For any field $\Bbbk$ denote by $N_\Bbbk := N \otimes_\mathbb{Z} \Bbbk$ and $M_\Bbbk := M \otimes_\mathbb{Z} \Bbbk$ the natural $\Bbbk$-linear extensions of these lattices. The induced non-degenerate pairing of $\mathbb{Q}$-vector spaces is denoted by

$$\langle \cdot, \cdot \rangle : N_\mathbb{Q} \times M_\mathbb{Q} \to \mathbb{Q}.$$

By a *cone* σ in $N_\mathbb{Q}$ we will mean a rational convex polyhedral cone in $N_\mathbb{Q}$. From any such cone $\sigma \subset N_\mathbb{Q}$ we can construct its *dual cone* $\sigma^*_M \subset M_\mathbb{Q}$. If the ambient dual lattices N and M are clear from the context, we will usually drop the index and write $\sigma^* := \sigma^*_M$. Define the *affine toric variety* associated to $\sigma \subset N_\mathbb{Q}$

$$U_\sigma := \mathrm{Spec}\Bbbk\big[\sigma^* \cap M\big].$$

An r-dimensional cone σ spans an r-dimensional subvectorspace $N(\sigma)_\mathbb{Q}$ of $\mathrm{N}_\mathbb{Q}$ and so defines a sublattice $N(\sigma) \subset N$ with dual lattice $M(\sigma)$. We denote the affine toric variety of $\sigma \subset N(\sigma)_\mathbb{Q}$ by

$$U_{\sigma, N(\sigma)} = \mathrm{Spec}\Bbbk\big[\sigma^*_{M(\sigma)} \cap M(\sigma)\big].$$

As we have $U_\sigma = U_{\sigma,N(\sigma)} \times (\Bbbk^*)^{n-r}$ this description will especially become useful when studying singularities. The fans Σ we are working with will always be complete. We denote by $\Sigma(i)$ the set of i-dimensional cones in Σ for all $i = 0, \ldots, n$, which we will mainly use to refer to *rays* $\Sigma(1)$ and *maximal cones* $\Sigma(n)$. Unless stated otherwise, we will further assume that the toric variety associated to Σ is projective and denote it by $\mathbb{P}_\Sigma$. Given a polytope $\Xi \subset M_\mathbb{Q}$ we denote the characters corresponding to elements $m \in \Xi \cap M$ by z^m. For $m \in l \cdot \Xi$ we introduce formal elements $t^l z^m$ and define a multiplication by

$$t^l z^m \cdot t^{l'} z^{m'} := t^{l+l'} z^{m+m'}.$$

The $\Bbbk$-algebra generated by this operation is denoted S_Ξ and the associated toric variety is called $\mathbb{P}_\Xi := \mathrm{Proj}(S_\Xi)$. For a face $\tau \subset \Xi$ we define the cone over τ by

$$\sigma_\tau = \{\mu \cdot (m - m') | m \in \Xi, m' \in \tau, \mu \in \mathbb{Q}_{\geq 0}\}.$$

The *fan of cones over proper faces* σ_τ in $M_\mathbb{Q}$ is denoted by Σ_Ξ, whereas the *normal fan* is denoted by Σ^*_Ξ. The toric varieties associated to these fans will be denoted by $\mathbb{P}_{\Sigma_\Xi}$ and $\mathbb{P}_{\Sigma^*_\Xi}$, respectively. From [Ba, Proposition 2.1.5] we know that $\mathbb{P}_\Xi \cong \mathbb{P}_{\Sigma^*_\Xi}$.

2.2.1. Affine toric varieties and their singularities. All properties of singularities of general toric varieties we will need can be read off locally from cones. For an element $m \in M_\mathbb{Q}$ we define the supporting hyperplane in $N_\mathbb{Q}$ at integral distance one associated to it by

$$H_m := \{\nu \in N_\mathbb{Q} | \langle \nu, m \rangle = -1\}.$$

Using this definition, we can now state the following lemma.

LEMMA 2.7. [Re1, Proposition 4.3, Remark 1.9]. *Let $\sigma \subset N_\mathbb{Q}$ be an r-dimensional cone with ray generators $n_i \in N$, for $i = 1, \ldots, s$. Then the following holds:*

(1) *$U_{\sigma,N(\sigma)}$ is $\mathbb{Q}$-Gorenstein if and only if there is an $m_\sigma \in M_\mathbb{Q}$ such that all of the n_i are contained in the hyperplane H_{m_σ} and U_σ is Gorenstein if and only if $m_\sigma \in M$.*
(2) *$U_{\sigma,N(\sigma)}$ is $\mathbb{Q}$-factorial if and only if σ is simplicial.*

□

For maximal cones σ we have $N(\sigma) = N$ and can therefore drop the index in this case. From this lemma it follows that a toric variety $\mathbb{P}_\Sigma$ is $\mathbb{Q}$-Gorenstein if and only if the element m_σ as above is well-defined for all its cones $\sigma \subset \Sigma$. It will prove useful to collect this information for maximal cones.

DEFINITION 2.8. Let $\mathbb{P}_\Sigma$ be an n-dimensional $\mathbb{Q}$-Gorenstein toric variety. For each maximal cone σ denote by $m_\sigma \in M_\mathbb{Q}$ the unique element from Lemma 2.7. We will call

$$\mathbf{m} := \mathbf{m}_\Sigma := \{m_\sigma\}_{\sigma\in\Sigma(n)}$$

the $\mathbb{Q}$-*Gorenstein support vectors* of $\mathbb{P}_\Sigma$. □

Note that support vectors of a $\mathbb{Q}$-Gorenstein toric variety with fan Σ define a piecewise linear *height function* $h_\mathbf{m} : N_\mathbb{Q} \to \mathbb{Q}$ on Σ that is given by $\langle\_, m_\sigma\rangle$ on each $\sigma \in \Sigma(n)$. Using this definition one can detect when singularities of affine toric varieties are terminal or canonical.

LEMMA 2.9. [Re1, Remark 1.11 (ii), (ii')]. *Let $\sigma \subset N_\mathbb{Q}$ be a full-dimensional cone with ray generators $n_1, \dots, n_s$, and let U_σ be $\mathbb{Q}$-Gorenstein with support vector m_σ. Then*

(1) *U_σ has at most terminal singularities if and only if*

$$\sigma \cap N \cap \{\nu \in N_\mathbb{Q} | \langle \nu, m_\sigma\rangle \geq -1\} = \{0, n_1, \dots, n_s\}$$

(2) *and at most canonical singularities if and only if*

$$N \cap \sigma \cap \{\nu \in N_\mathbb{Q} | \langle \nu, m_\sigma\rangle > -1\} = \{0\}.$$

□

This lemma poses strong restrictions on the fan Σ if we want $\mathbb{P}_\Sigma$ to have canonical singularities. However, it also implies that Gorenstein toric singularities are canonical, see [Ba, Corollary 2.2.5].

2.2.2. Polytopes and resolutions. Fixing dual lattices N and M, whenever we talk about lattice polytopes in $N_\mathbb{Q}$ or $M_\mathbb{Q}$, we refer to polytopes whose vertices lie in N, respectively M. With minor exceptions, for example in proofs, polytopes in $N_{\mathbb{Q}}$ will be denoted by Θ, whereas polytopes in $M_{\mathbb{Q}}$ are denoted by Ξ throughout the text. We will generally assume that polytopes contain the origin of their respective lattice. Recall that the *polar polytope* to a polytope $\Xi \subset M_{\mathbb{Q}}$ is defined by

$$\Xi^* := \{\nu \in N | \langle \nu, m\rangle \geq -1, \forall\, m \in \Xi\}.$$

To any toric variety $\mathbb{P}_\Sigma$ associated to a fan Σ with a choice of toric divisor, one can naturally define the following polytope.

DEFINITION 2.10. Let $\Sigma \subset N_{\mathbb{Q}}$ be a fan. For each ray $\rho \in \Sigma(1)$ denote by D_ρ the corresponding torus-invariant divisor on $\mathbb{P}_\Sigma$ and by $n_\rho \in N$ its generator. Let $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ be a toric divisor, then we can associate to it the convex polyhedron

$$\Xi_D := \{\nu \in N_{\mathbb{R}} | \langle \nu, n_\rho\rangle \geq -a_\rho, \forall \rho \in \Sigma(1)\},$$

called the *Newton polyhedron*. □

As we aim at constructing Calabi-Yau hypersurfaces, we will mainly use this notation in the special case of the anti-canonical divisor

$$D = -K_{\mathbb{P}(\Sigma)} = \sum_{\rho\in\Sigma(1)} D_\rho$$

. Recall that for every toric divisor D, its space of sections is given by

$$H^0(\mathbb{P}_\Sigma, D) = \bigoplus_{m\in\Xi_D\cap M} \mathbb{C}z^m. \tag{2.1}$$

Moreover, recall that a lattice polytope Ξ is called *reflexive* if and only if its polar polytope is a lattice polytope as well. An important alternative definition is that Ξ has a unique interior lattice point and all supporting hyperplanes of facets have integral distance one from this point. By definition the polar Ξ^* of a reflexive polytope Ξ is reflexive as well and we will refer to (Ξ, Ξ^*) as a *reflexive pair* in this situation.

The following definition slightly relaxes the definition of reflexive polytopes and will play a crucial role in almost all results in this and the next chapter.

DEFINITION 2.11. A lattice polytope $\Theta \subset N_{\mathbb{Q}}$ containing 0 as an inner point is called *almost reflexive* if $\overline{\Theta}^* := \mathrm{conv}(\Theta^* \cap M)$ is reflexive, that is if the integral points of its polar span a reflexive polytope. The polar polytope of $\overline{\Theta}^*$, which we denote by $\overline{\Theta}$, thus contains Θ. This situation is summarized in the following diagram

$$\begin{tikzcd}
\overline{\Theta} \arrow[ddr, leftrightarrow] & \\
\Theta \arrow[u] \arrow[r, leftrightarrow] & \Theta^* \\
 & \overline{\Theta}^* . \arrow[u]
\end{tikzcd}$$

In this situation we call $\overline{\Theta}$ the reflexive polytope associated to Θ. □

REMARK 2.12. 1) In a recent preprint [Ma] Mavlyutov defines a polytope $\Xi \subset M_{\mathbb{Q}}$ to be quasi-reflexive if the following holds

$$\operatorname{conv}((\operatorname{conv}(\Xi \cap M))^* \cap N) = \Xi^*.$$

Thus, there is no direct connection to reflexive polytopes as for almost reflexive polytopes defined above. The author of [Ma] describes a generalization of nef-partitions for this class of polytopes combinatorially. We refer the reader to [BaBo1], [BaBo2] for an introduction to nef-partitions.

2) An almost reflexive polytope $\Theta \subset N_{\mathbb{Q}}$ has no interior lattice point apart from the origin, as it is contained in the reflexive polytope $\overline{\Theta}$. Note, however, that Definition 2.11 is not equivalent to saying that Θ is a lattice polytope that has exactly one interior point, see for instance the following example. □

EXAMPLE 2.13. We will come back to the following two polytopes several times in this chapter.

1) Let Θ_1 be the polytope spanned by $(-1, 0, 0)$, $(0, -1, 0)$, $(0, 0, -1)$ and $(5, 6, 8)$, which has the origin 0 as unique interior point. The vertices of Θ_1^* are $(1, 1, 1)$, $(1, 1, -3/2)$, $(1, -7/3, 1)$ and $(-3, 1, 1)$. However, it is easily checked that the polytope $\overline{\Theta}_1^* = \operatorname{conv}(\Theta_1^* \cap M)$ has no interior points, as the origin 0 lies at the boundary. Thus Θ_1 is not almost reflexive.

2) An example of a non-reflexive but almost reflexive polytope is the convex hull Θ_2 of the vertices $(-1, 0, 0)$, $(0, -1, 0)$, $(0, 0, -1)$ and $(1, 2, 3)$. Its polar Θ_2^* has vertices $(1, 1, 1)$, $(-6, 1, 1)$, $(1, -5/2, 1)$ and $(1, 1, -4/3)$.

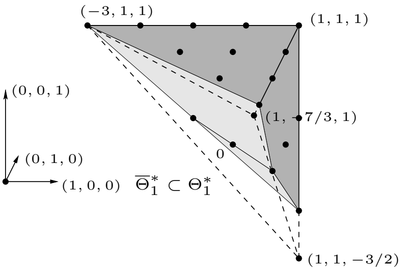

FIGURE 2.1. The polar polytope Θ_1^* of Θ_1. The span of integral points $\overline{\Theta}_1^*$ has 0 as a boundary point.

One can verify that the vertices of $\overline{\Theta}_2^* := \operatorname{conv}(\Theta_2^* \cap M)$ this time are given by

$$(1, 1, 1), (-6, 1, 1), (1, 1, -1), (0, 1, -1),$$
$$(1, -1, 0), (1, -2, 1), (0, -2, 1)$$

and that the origin 0 is an interior point of it. Thus, $\overline{\Theta}_2^*$ is a reflexive polytope whose polar $\overline{\Theta}_2$ has vertices $(-1, 0, 0)$, $(0, -1, 0)$, $(0, 0, -1)$, $(1, 2, 3)$, $(0, 1, 1)$ and $(0, 1, 2)$, so Θ_2 is contained in $\overline{\Theta}_2^*$. Moreover, the fan $\Sigma_{\overline{\Theta}_2}$ defines a refinement of Σ_{Θ_2} and a little computation shows that $\mathbb{P}_{\Sigma_{\overline{\Theta}_2}}$ is smooth, as all maximal cones are. The induced morphism $\mathbb{P}_{\Sigma_{\overline{\Theta}_2}} \to \mathbb{P}_{\Sigma_{\Theta_2}}$ therefore resolves all singularities of the $\mathbb{Q}$-Gorenstein toric variety $\mathbb{P}_{\Sigma_{\Theta_2}}$. $\square$

In this section we have so far focussed on general properties of polytopes and arbitrary toric varieties. We will now turn to properties of $\mathbb{Q}$-Gorenstein toric varieties, for which one can naturally define the following two polytopes.

DEFINITION 2.14. Let $\mathbb{P}_\Sigma$ be an n-dimensional $\mathbb{Q}$-Gorenstein $\mathbb{Q}$-Fano projective toric variety with fan Σ in $N_\mathbb{Q}$ and $\mathbb{Q}$-Gorenstein support vectors

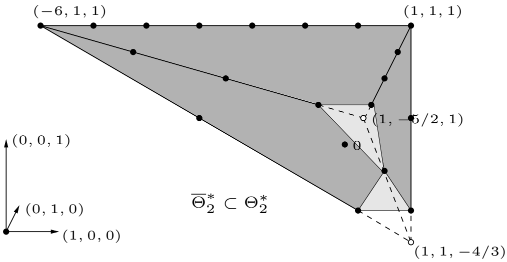

FIGURE 2.2. Θ_2^* and the reflexive polytope $\overline{\Theta}_2^*$ contained in it.

$\mathbf{m}_\Sigma$. Define

$$\begin{aligned}\Xi_\Sigma :=& \Xi_{\mathbf{m}_\Sigma} := \bigcap_{\sigma\in\Sigma(n)} (m_\sigma + \sigma^*)\\ &= \{m \in M_\mathbb{Q} | \langle n_\rho, m\rangle \geq -1 \; \forall \rho \in \Sigma(1)\} \subset M_\mathbb{Q}\\ \Theta_\Sigma :=& \Theta_{\mathbf{m}_\Sigma} := \{\nu \in N_\mathbb{Q} | \langle \nu, m_\sigma\rangle \geq -1 \; \forall \sigma \in \Sigma(n)\} \subset N_\mathbb{Q}\end{aligned}$$ □

REMARK 2.15. Note that by definition $\Xi_\Sigma = \Xi_{-K_{\mathbb{P}_\Sigma}}$ and Θ_Σ is the span of ray generators of Σ. The latter of which of course also makes sense if $\mathbb{P}_\Sigma$ is not $\mathbb{Q}$-Gorenstein, but can not be defined using $\mathbb{Q}$-Gorenstein support vectors in this case. Moreover, by definition of $\mathbb{Q}$-Gorenstein support vectors, we see that $\Xi_\Sigma = \Theta_\Sigma^*$.

EXAMPLE 2.16. We continue with Example 2.13. Denote the fans of cones over faces of Θ_1 and Θ_2 by Σ_{Θ_1} and Σ_{Θ_2}, respectively. Then we immediately get that $\Theta_{\Sigma_{\Theta_i}} = \Theta_i$ and $\Xi_{\Sigma_{\Theta_i}} = \Theta_i^*$ for $i = 1, 2$. □

Various global properties of $\mathbb{P}_\Sigma$ can be computed from these two polyhedra and their interrelation. Recall that a normal variety X is called *Fano* if $-K_X$ is ample and $\mathbb{Q}$-*Fano* if an integral multiple of $-K_X$ is ample. Furthermore, we call X *semi-Fano*, if $-K_X$ is nef. If $X = \mathbb{P}_\Sigma$ is a $\mathbb{Q}$-Gorenstein projective toric variety with $\mathbb{Q}$-Gorenstein support vectors $\mathbf{m}_\Sigma$, then X is (semi-)Fano if and only if the associated height function $h_{\mathbf{m}_\Sigma}$ is (strictly) convex on the fan Σ. The next proposition collects the most important results for later applications.

PROPOSITION 2.17. *Let $\mathbb{P}_\Sigma$ be an n-dimensional $\mathbb{Q}$-Gorenstein projective toric variety. Then*

(1) *$\mathbb{P}_\Sigma$ is $\mathbb{Q}$-Fano if and only if the vertices of Ξ_Σ are in one-to-one correspondence with maximal cones of Σ. In this case $\mathbb{P}_\Sigma$ is Fano if and only if Ξ_Σ is a lattice polytope.*
(2) *$\mathbb{P}_\Sigma$ is $\mathbb{Q}$-Fano if and only if $\mathbb{P}_\Sigma = \mathbb{P}_{\Xi_\Sigma}$.*
(3) *$\mathbb{P}_\Sigma$ has $\mathbb{Q}$-factorial terminal singularities if and only if for every cone σ in Σ the polytope*

$$\Theta_\sigma := \sigma \cap \{\nu \in N_\mathbb{Q} | \langle \nu, m_\sigma \rangle \leq 1\}$$

is an elementary simplex.

PROOF. This is essentially the content of [Ba, Proposition 2.2.23], but we include a short proof since in the reference there is none. $\mathbb{P}_\Sigma$ is $\mathbb{Q}$-Fano if and only if the height function h_Σ constructed from $\mathbf{m}_\Sigma$ is strictly convex. This property in turn simply means that the individual m_σ, for maximal cones $\sigma \in \Sigma(n)$, are precisely the vertices of the polytope $\Xi_\Sigma = \bigcap_{\sigma \in \Sigma(n)} (m_\sigma + \sigma^*)$. From Lemma 2.7 (2) we know that $m_\sigma \in N$ for all $\sigma \in \Sigma(n)$ is equivalent to $\mathbb{P}_\Sigma$ being Gorenstein and therefore Fano in this situation, which proves (1). For the second statement note that by the property $\Xi_\Sigma = \{m \in M_\mathbb{Q} | \langle n_\rho, m \rangle \geq -1 \forall \rho \in \Sigma(1)\}$ we see that the rays $\rho \in \Sigma(1)$ are normal to the facets of Ξ_Σ if and only if $\mathbb{P}_\Sigma$ is $\mathbb{Q}$-Fano. Thus the normal fan on Ξ_Σ yields the fan we started with in this case. Property (3) follows immediately from the local situation in Lemma 2.9. $\square$

It is essential to know when a birational morphism between toric varieties is crepant, which we will now study for $\mathbb{Q}$-Gorenstein varieties. The following crucial Lemma, first proved by Gelfand, Kapranov and Zelevinski, gives maximal triangulations for *arbitrary integral polyhedra* Θ, whether they give rise to a Gorenstein $\mathbb{P}_\Theta$ or not.

LEMMA 2.18. [GeKaZe, Proposition 3]. *Let $\Theta \subset N_\mathbb{Q}$ be a lattice polytope. Then there exists a maximal projective triangulation $\mathcal{T}_\Theta$ of Ξ, where projective means that we can choose a strictly convex height function $h_\Theta : N_\mathbb{Q} \to \mathbb{Z}$ that is linear on each simplex $\tau \in \mathcal{T}_\Theta$.*

$\square$

In the latter, whenever we speak of a *maximal projective triangulation* of a lattice polytope Θ, we will mean the choice of a pair $(\mathcal{T}_\Theta, h_\Theta)$ as above. Any triangulation $(\mathcal{T}_\Theta, h_\Theta)$ also defines an *induced* triangulation on the boundary $\partial\Theta$ and the fan $\widetilde{\Sigma_\Theta}$ of cones over faces of this triangulation defines a toric morphism $\mathbb{P}_{\widetilde{\Sigma_\Theta}} \to \mathbb{P}_{\Sigma_\Theta}$. By [Ba, Theorem 2.2.24] one knows that if $\mathbb{P}_\Sigma$ is a Gorenstein Fano toric variety, then this morphism is a projective and crepant partial resolution of singularities. These resolutions are maximal in the sense that any further toric resolution would be discrepant and we will call these morphisms *MPCP resolutions* in what follows.

The next lemma shows that the toric variety $\mathbb{P}_{\Sigma_\Theta}$ associated to an almost reflexive polytope Θ is always $\mathbb{Q}$-Fano and furthermore closely related to $\mathbb{P}_{\Sigma_{\overline{\Theta}}}$, where $\overline{\Theta}$ is the associated reflexive polytope of Θ. We will heavily make use of this lemma and the notation introduced there in this and the next chapter.

LEMMA 2.19. *Let $\Theta \subset N_\mathbb{Q}$ be an n-dimensional almost reflexive polytope with associated reflexive polytope $\overline{\Theta}$. Then the following holds:*

(1) *$\mathbb{P}_{\Sigma_\Theta}$ is a $\mathbb{Q}$-Fano toric variety.*
(2) *There is a common refinement $\widetilde{\Sigma}$ of Σ_Θ and $\Sigma_{\overline{\Theta}}$, inducing toric morphisms*

$$\begin{tikzcd} & \mathbb{P}_{\widetilde{\Sigma}} \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ \mathbb{P}_{\Sigma_\Theta} & & \mathbb{P}_{\Sigma_{\overline{\Theta}}}, \end{tikzcd}$$

where $\widetilde{\Sigma}$ is the fan of cones over faces of a maximal projective triangulation of $\overline{\Theta}$.

PROOF. The first part of the lemma follows from Proposition 2.17 (1), as $\mathbb{P}_{\Sigma_\Theta}$ is defined by the fan of cones over faces of Θ and the fact that Θ is the polar polytope of the anti-canonical polytope of $\mathbb{P}_{\Sigma_\Theta}$, that is $\Theta^* = \Xi_{\Sigma_\Theta}$.

For the second part, choose a maximal projective triangulation $(\mathcal{T}_{\overline{\Theta}}, h_{\overline{\Theta}})$ of $\overline{\Theta}$, so that $(\mathcal{T}_{\overline{\Theta}}, h_{\overline{\Theta}})$ refines the triangulation the faces that Θ shares with $\overline{\Theta}$ induce. This defines an induced triangulation of the boundary of $\overline{\Theta}$. The fan $\widetilde{\Sigma}$ of cones over faces of this induced triangulation defines a morphism $\psi : \mathbb{P}_{\widetilde{\Sigma}} \to \mathbb{P}_{\Sigma_{\overline{\Theta}}}$. However, we have to discuss why $\widetilde{\Sigma}$ is a refinement of Σ_Θ. Apriori, if we torically blow down all rays in $\widetilde{\Sigma}(1)$ not contained $\Sigma_\Theta(1)$ in some given order, we might arrive at a fan structure that is different from Σ_Θ.

Note that by restricting to $\Theta \subset \overline{\Theta}$, we also get a subdivision of Θ and thus also on $\partial\Theta$. Note that this subdivision is not a triangulation in the strict sense, as vertices of it will not always correspond to lattice points. However, this is just used to construct the following fan. Denote the fan of cones over faces of the subdivision of $\partial\Theta$ by $\widetilde{\Sigma_\Theta}$. Then by construction $\widetilde{\Sigma_\Theta} \to \Sigma_\Theta$ is a map of fans. Moreover, note that the closure of $\overline{\Theta} \setminus \Theta$ is a union of n-dimensional polytopes meeting along lower-dimensional strata, for otherwise at least one integral boundary point of Θ would be an interior point of the reflexive polytope $\overline{\Theta}$. These polytopes are spanned by exactly one facet τ of Θ and vertices of $\overline{\Theta}$ not contained in Θ. Denote these accordingly by $\overline{\Theta}_\tau$ for each facet $\tau \subset \Theta$. Note that the integral points of $\overline{\Theta}_\tau \setminus \tau$ lie all in the interior of the cone over τ. Moreover, any such τ can not have interior integral points, as then $\overline{\Theta}$ would not be reflexive. Now, as $\mathcal{T}_\Theta$ is induced from $\mathcal{T}_{\overline{\Theta}}$, they agree on $\partial\tau$ for all facets $\tau \subset \Theta$. The fan $\widetilde{\Sigma}$ therefore respects the fan structure of $\widetilde{\Sigma_\Theta}$, that is $\widetilde{\Sigma} \to \widetilde{\Sigma_\Theta}$ is a map of fans. The composition $\widetilde{\Sigma} \to \widetilde{\Sigma_\Theta} \to \Sigma_\Theta$ therefore defines the toric morphism $\phi : \mathbb{P}_{\widetilde{\Sigma}} \to \mathbb{P}_{\Sigma_\Theta}$, which finishes the proof. □

2.2.3. Discrepancies. Next, we study how toric resolutions of toric varieties affect the canonical bundle of the variety and hypersurfaces therein. Before we prove a technical lemma used in our central result Theorem 2.25, let us recall a classical lemma and draw some immediate conclusions from it. For notational clarity we state this lemma only in the local situation, although it applies to complete toric varieties as well.

Let σ be a full-dimensional cone in a lattice N of rank n such that the associated affine toric variety U_σ is $\mathbb{Q}$-Gorenstein with support vector m_σ. Let $\nu \in \sigma$ be primitive and denote by Σ_ν the *star subdivision* of σ by the ray generated by ν. Thus, considering σ as fan consisting of one cone, we get a refinement of fans $\Sigma_\nu \to \sigma$, which induces a partial toric resolution of singularities $\pi : \mathbb{P}_{\Sigma_\nu} \to U_\sigma$. If we denote the toric divisor associated to the ray generated by ν by D_ν, then we have the following result characterizing the discrepancy of π.

LEMMA 2.20. [Re2, Section 4]. *There is the following equality of $\mathbb{Q}$-Cartier divisor classes in $\mathbb{P}_{\Sigma_\nu}$*

$$\pi^*(K_{U_\sigma}) = K_{\mathbb{P}_{\Sigma_\nu}} + (\langle \nu, m_\sigma \rangle + 1) \cdot D_\nu. \qquad \square$$

As an immediate corollary we see that a proper birational morphism $\phi : \mathbb{P}_{\Sigma'} \to \mathbb{P}_\Sigma$ of $\mathbb{Q}$-Gorenstein toric varieties is *crepant* if and only if for all maximal cones $\sigma \in \Sigma(n)$ all rays $\tau' \in \Sigma'$ mapping to σ are generated by primitive elements from

$$N \cap H_{m_\sigma} = N \cap \{\nu \in N_\mathbb{Q} | \langle \nu, m_\sigma \rangle = 1\}.$$

Thus, if we work with reflexive polytopes Θ we know that resolutions from subdivisions of Θ are crepant, since every integral point apart from zero is on the boundary of Θ. Elements $\nu \in \Theta \cap N$ with $\nu \notin N \cap H_{m_\sigma}$ in turn always define *discrepant* morphisms. The following technical lemma, however, ensures us that the anti-canonical polytope remains unaffected if we use interior points of Θ to define resolutions.

LEMMA 2.21. *Let $\mathbb{P}_\Sigma$ be a $\mathbb{Q}$-Gorenstein toric variety of dimension n and let $0 \neq \nu \in \Theta_\Sigma$ be a primitive, integral* interior *point of Θ_Σ. The fan Σ_ν obtained from refining Σ by star subdivision with a new ray generated by ν therefore induces a morphism $\pi : \mathbb{P}_{\Sigma_\nu} \to \mathbb{P}_\Sigma$ and we have*

$$\Xi_{-K_{\mathbb{P}_\Sigma}} = \Xi_{-\pi^*(K_{\mathbb{P}_\Sigma})},$$

that is the anti-canonical polytopes before and after the resolution are the same. In particular, the integral points corresponding to anti-canonical sections are the same.

PROOF. If $\mathbb{P}_\Sigma$ is $\mathbb{Q}$-Gorenstein, then so is $\mathbb{P}_{\Sigma_\nu}$. Assume that ν is contained in the interior of a maximal cone σ. If ν lies in a lower-dimensional stratum the argument remains basically unchanged. Let $n_1, \ldots, n_l$ be the ray generators of σ. Then the $\mathbb{Q}$-Gorenstein support vector m_σ associated to σ is defined by the equations $\langle n_i, m_\sigma \rangle = -1$ for all $i = 1, \ldots, l$. By the star subdivision σ is decomposed into l subcones

$$\sigma_i := \mathrm{conv}\{\nu, n_1, \ldots \widehat{n_i}, \ldots, n_l\}$$

with $\mathbb{Q}$-Gorenstein support vectors m_{σ_i}. As the anti-canonical polytope of $\mathbb{P}_\Sigma$ is defined by

$$\begin{aligned}\Xi_{-K_{\mathbb{P}_\Sigma}} &= \bigcap_{\sigma\in\Sigma(n)} (m_\sigma + \sigma^*)\\ &= \{m \in M_\mathbb{Q} | \langle n_\rho, m\rangle \geq -1, \forall \rho \in \Sigma(1)\}\end{aligned}$$

it suffices to show that $m_\sigma + \sigma^* = \bigcap_{i=1}^{l}(m_{\sigma_i} + \sigma_i^*)$. However, as clearly

$$\nu \in \{\nu' \in N_\mathbb{Q} | \langle \nu', m_{\sigma_i}\rangle \geq -1\}$$

for all i, from the assumption that ν is an interior point of Θ_Σ, the inclusion $m_\sigma + \sigma^* \subseteq \bigcap_{i=1}^{l}(m_{\sigma_i} + \sigma_i^*)$ follows. Moreover, note that $\sigma^* = \bigcap_i \sigma_i^*$. To see that we do not get strictly more, we have to show that the vertex of the cone $\bigcap_{i=1}^{l}(m_{\sigma_i} + \sigma_i^*)$ is m_σ. But $\langle n_j, m_{\sigma_i}\rangle$ is -1 for all $i \neq j$, so $m_{\sigma_i} = m_\sigma + m_i$ where $m_i \in M_\mathbb{Q}$ is parallel to the $(n-1)$-cell

$$\{m \in M_\mathbb{Q} | \langle n_j, m\rangle = -1, \forall j \neq i\}.$$

Thus, from this description it follows that m_σ is indeed the vertex of

$$\bigcap_{i=1}^{l}(m_{\sigma_i} + \sigma_i^*),$$

which finishes the proof. $\square$

EXAMPLE 2.22. Consider the two-dimensional polytope $\Theta \subset N_\mathbb{Q} \cong \mathbb{Z}^2$ spanned by $(-1,0)$, $(0,-1)$ and $(1,3)$. The fan Σ_Θ therefore defines a $\mathbb{Q}$-Gorenstein toric variety $\mathbb{P}_{\Sigma_\Theta}$ with support vectors $(1,1)$, $(-4,1)$ and $(1,-2/3)$. The anti-canonical polytope of it, that is Ξ_{Σ_Θ}, is spanned by these three vectors. As the corresponding height function $h_{\mathbf{m}_{\Sigma_\Theta}}$ is strictly convex, but not integral, we see that $\mathbb{P}_{\Sigma_\Theta}$ is $\mathbb{Q}$-Fano. Indeed, $\mathbb{P}_{\Sigma_\Theta}$ is just the weighted projective space $\mathbb{P}(1,1,3)$.

Θ has precisely one non-zero interior integral point, namely $\nu := (0,1)$. Star subdividing Σ_Θ by ν yields a fan Σ_ν with four smooth maximal cones, that is $\mathbb{P}_{\Sigma_\nu}$ is smooth and in fact the non-Fano Hirzebruch surface $\mathbb{F}_3$. The support vectors of these four maximal cones are $(1,1)$, $(-4,1)$, $(1,-1)$ and $(2,-1)$. The intersection of the duals of these four cones translated by

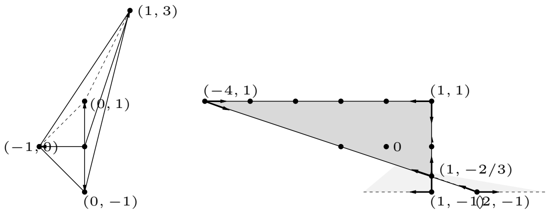

FIGURE 2.3. The polytope Θ with polar $\Xi_{\Sigma_\nu} = \Xi_{\Sigma_\Theta}$.

their respective support vectors is the anti-canonical polytope of $\mathbb{P}_{\Sigma_\nu}$, that is

$$\Xi_{\Sigma_\nu} = \bigcap_{\sigma \in \Sigma_\nu(2)} (m_\sigma + \sigma^*).$$

As seen from Figure 2.3 or directly checked by hand we see that $\Xi_{\Sigma_\nu} = \Xi_{\Sigma_\Theta}$, as expected from Lemma 2.21. □

2.2.4. Anti-canonical sections and Calabi-Yau varieties. Before stating and proving the main result of this chapter, we will shortly recall another key definition and a central result of [Ba], which our result will generalize.

DEFINITION 2.23. Let $\mathbb{P}_{\Sigma^*_\Xi}$ be the projective toric variety associated to the normal fan Σ^*_Ξ of a polytope $\Xi \subset M_\mathbb{Q}$. Denote by $\mathcal{L}(\Xi)$ the space of Laurent polynomials with Newton polytope Ξ and define $\mathcal{F}(\Xi)$ to be the subspace of $\mathcal{L}(\Xi)$ consisting of sections f such that the vanishing set of f on each stratum of $\mathbb{P}_{\Sigma^*_\Xi}$ is either smooth of codimension one or empty. $\mathcal{F}(\Xi)$ is called the *space of* Ξ*-regular hypersurfaces*. □

In case Ξ is a lattice polytope, that is if it has vertices in M, it is known that $\mathcal{F}(\Xi)$ is a Zariski dense open subset of $\mathcal{L}(\Xi)$, see [Ba, Proposition 3.1.3]. We will generalize the following theorem to the $\mathbb{Q}$-Gorenstein setting.

THEOREM 2.24. [Ba, Theorem 4.1.9]. *Let $\Xi \subset M_\mathbb{Q}$ be an n-dimensional lattice polytope and let $\mathcal{F}(\Xi)$ denote the family of Ξ-regular hypersurfaces $X = V_{\mathbb{P}_{\Sigma^*_\Xi}}(f)$ in $\mathbb{P}_{\Sigma^*_\Xi}$. Then the following are equivalent:*

(1) $\mathcal{F}(\Xi)$ *consists of Calabi-Yau varieties.*

(2) *The ample invertible sheaf* $\mathcal{O}_{\mathbb{P}_\Xi}$ *is anti-canonical, that is* $\mathbb{P}_\Xi$ *is Gorenstein Fano.*

(3) Ξ *is a reflexive polytope.* □

This theorem is already quite general, but we can consider a more general case. Given a polytope $\Xi \subset M_{\mathbb{Q}}$ with normal fan Σ, a general anti-canonical section may yield a Calabi-Yau variety without being Ξ-regular. That is, if an anti-canonical section is *singular* along a toric stratum, it may still yield canonical singularities. Another limitation of the definition of Ξ-regularity is that there may be sections with good regularity properties that have a Newton polytope which is strictly smaller than Ξ. In the following theorem, whenever we speak of a hypersurface $X = V(f)$ for a Laurent polynomial f on a toric variety, we mean the set of all points in which the section f vanishes on the toric variety.

THEOREM 2.25. *Let* $\mathbb{P}_\Sigma$ *be a* $\mathbb{Q}$*-Gorenstein projective toric variety associated to the fan* Σ *in* $N_{\mathbb{Q}}$*. Then the following are equivalent:*

(1) *Any general anti-canonical hypersurface* X *is a Calabi-Yau variety.*

(2) Θ_Σ *is an almost reflexive polytope.*

PROOF. Recall that $\Theta := \Theta_\Sigma$ is almost reflexive if and only if the integral points of the anti-canonical polytope $\Theta^* = \Xi_{-K_{\mathbb{P}_\Sigma}}$ span a reflexive polytope, that is if $\overline{\Xi} := \mathrm{conv}(\Theta^* \cap M)$ is reflexive. The associated reflexive polytope $\overline{\Theta}$ of Θ is just the polar polytope of $\overline{\Xi}$.

$(2) \Rightarrow (1)$: Assume that Θ is almost reflexive and let X be a general anti-canonical hypersurface in $\mathbb{P}_\Sigma$. From Lemma 2.19 (2) we know that there are morphisms

$$\begin{tikzcd} & \mathbb{P}_{\widetilde{\Sigma}} \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ \mathbb{P}_\Sigma & & \mathbb{P}_{\Sigma_{\overline{\Theta}}}, \end{tikzcd}$$

where $\mathbb{P}_{\widetilde{\Sigma}}$ is the toric variety associated to the fan of cones over faces of an MPCP subdivision of $\overline{\Theta}$. As the Newton polytope of $\mathbb{P}_{\widetilde{\Sigma}}$ is just $\overline{\Xi}$, we know that there is an anti-canonical hypersurface $\widetilde{X}$ of $\mathbb{P}_{\widetilde{\Sigma}}$ such that $\phi(\widetilde{X}) =$

X. As X is assumed to be general, we know that $\widetilde{X}$ is $\overline{\Xi}$-regular, so by Theorem 2.24 $\widetilde{X}$ is a Calabi-Yau variety.

As X is general, we know that it is a subvariety of $\mathbb{P}_\Sigma$ that is regular in codimension one. If it weren't regular in codimension one, it would have to contain a toric codimension two stratum of the ambient space. Hence the pull-back $\widetilde{X}$ of $\mathbb{P}_{\widetilde{\Sigma}}$ would also contain this stratum, which contradicts the assumption of $\widetilde{X}$ being general. So by Remark 2.3 we can apply the adjunction formula to the regular part X_{reg} of X to obtain $\omega_{X_{reg}} \cong \mathcal{O}_{X_{reg}}$ and can push forward via the inclusion $j_X : X_{reg} \to X$ to get $\omega_X \cong \mathcal{O}_X$. In the special case that $\mathbb{P}_\Sigma$ is a weighted projective space this part of the proof follows from [Do, Theorem 3.3.4]. Furthermore, by applying the Leray spectral sequence we get

$$H^i(X, \mathcal{O}_X) = 0 \quad \forall i = 1, \ldots n-1,$$

as we know that these cohomology groups for $\widetilde{X}$ vanish. Thus it remains to show that X has canonical singularities. However, we know that $\widetilde{X}$ as Calabi-Yau variety has canonical singularities, so by definition every resolution of singularities $f : Y \to \widetilde{X}$ has the property that $f^*(K_{\widetilde{X}}) = K_Y + \sum_{i \in I} a_i E_i$ with $a_i \geq 0$, where E_i denote the exceptional divisors with index set I. But since we just checked that

$$\phi|_{\widetilde{X}} : \widetilde{X} \to X$$

is a crepant morphism, the composition $f \circ \phi|_{\widetilde{X}} : Y \to X$ is a resolution that also has non-negative coefficients a_i. Hence X is a Calabi-Yau variety.

$(1) \Rightarrow (2)$: Now, let Θ be a polytope that is not almost reflexive. Then we know that Ξ is not a reflexive polytope. We first exclude the case that $0 \in \partial\overline{\Xi}$ and then consider the case that it is an interior point of $\overline{\Xi}$.

Assume that 0 is a boundary point of Ξ. Choose a maximal face σ of Ξ which contains 0. It follows that σ is *not* a face of $\Xi_{-K_{\mathbb{P}_\Sigma}}$, as this polytope does have the origin as an interior point. Then the inward pointing normal vector n_σ associated to σ defines a refinement Σ' of the fan Σ and by construction we get $\mathrm{conv}(\Xi_{-K_{\mathbb{P}_{\Sigma'}}} \cap M) = \overline{\Xi}$. Denote the pullback of X by the morphism $\mathbb{P}_{\Sigma'} \to \mathbb{P}_\Sigma$ by X'. The normal vector n_σ corresponds to a ray of Σ' and therefore to a toric divisor D_σ of $\mathbb{P}_{\Sigma'}$. Recall that we can express $\mathbb{P}_{\Sigma'}$ and the hypersurface X' in Cox coordinates, see [Co].

Denote the Cox coordinate corresponding to D_σ by x_σ. As the integral distance of σ from the origin in M is zero, we see that every monomial z^m corresponding to an integral point $m \in \overline{\Xi} \cap M$ is of the form $x_\sigma \cdot z^{m'}$ for some $m' \in M$. Thus $X' = D_\sigma \cup X''$ is reducible and therefore not Calabi-Yau. Hence X cannot be a Calabi-Yau variety.

Next, assume that 0 is an interior point of Ξ. Then the normal fan $\Sigma^*_{\overline{\Xi}}$ to $\overline{\Xi}$ has a refinement $\widetilde{\Sigma}'$ that is also a refinement of Σ. From this we get induced morphisms

$$\phi' : \mathbb{P}_{\widetilde{\Sigma}'} \to \mathbb{P}_\Sigma \text{ and } \psi' : \mathbb{P}_{\widetilde{\Sigma}'} \to \mathbb{P}_{\Sigma^*_{\overline{\Xi}}}.$$

Recall that lattice polytopes which have precisely one interior point and are polar to each other form a reflexive pair. As $\overline{\Xi}$ is a lattice polytope contained in $\Xi_{-K_{\mathbb{P}_\Sigma}}$ and the polar polytope of $\Xi_{-K_{\mathbb{P}_\Sigma}}$ is Θ, we see that $\Theta \subset \overline{\Xi}^*$. Since $\overline{\Xi}^{**} = \overline{\Xi}$ and $\overline{\Xi}$ is not reflexive, $\overline{\Xi}^* \subset N_\mathbb{Q}$ must have integral interior points apart from 0. The map of fans $\widetilde{\Sigma} \to \Sigma^*_{\overline{\Xi}}$ introduces rays with ray generators corresponding to boundary and *interior* points of Θ. However, by Lemma 2.21 we know that ψ' does not affect the anti-canonical polytope of $\mathbb{P}_{\Sigma_{\overline{\Xi}}}$, that is we have

$$\Xi_{-K_{\mathbb{P}_{\widetilde{\Sigma}}}} = \Xi_{-(\psi')^*(K_{\mathbb{P}_{\Sigma_{\overline{\Xi}}}})}. \tag{2.2}$$

By Theorem 2.24 we know that $\overline{\Xi}$-regular hypersurfaces $\overline{X}$ in $\mathbb{P}_{\Sigma_{\overline{\Xi}}}$ are *not* Calabi-Yau. From equation (2.2) we see that there is an anti-canonical section $\widetilde{X}'$ such that $\psi'(\widetilde{X}') = \overline{X}$. As ψ' is a partial resolution of singularities with Newton polytope $\overline{\Xi}$ we see that $\widetilde{X}'$ is not Calabi-Yau. Thus $\phi'(\widetilde{X}')$ can not be a Calabi-Yau variety. We know that the Newton polytopes of $\mathbb{P}_\Sigma$ and $\mathbb{P}_{\Sigma_{\overline{\Xi}}}$ have the same integral points. Thus we are done, since X' was an arbitrary $\overline{\Xi}$-regular hypersurface. □

If a lattice polytope $\Theta \subset N_\mathbb{Q}$ has an interior lattice point other than 0, then we know that $\mathbb{P}_{\Sigma_\Theta}$ has non-canonical singularities by Lemma 2.9. From the proof of Theorem 2.25 we see that in this case a general anti-canonical hypersurface X of $\mathbb{P}_{\Sigma_\Theta}$ must inherit the non-canonical singularities of its ambient space. Moreover, we see that for any toric resolution of $\mathbb{P}_{\widetilde{\Sigma}} \to \mathbb{P}_\Sigma$ such that $\mathbb{P}_{\widetilde{\Sigma}}$ has canonical singularities, the strict transform of X under this morphism will be reducible.

2.3. Quotients of weighted projective spaces

Weighted projective spaces and their quotients form an important class of toric varieties, when it comes to mirror symmetry. We will start with giving the most important definitions and results in the context of these spaces and then apply the results of the last section to describe Calabi-Yau hypersurfaces in weighted projective spaces. For the rest of this chapter we will work over $\mathbb{k} = \mathbb{C}$, as we will do in chapter 3.

DEFINITION 2.26. Let $w = (w_0, \ldots, w_n)$ be an $(n+1)$-tuple of positive integers called *weights* and define a grading on the ring $S(w) := S(w_0, \ldots, w_n) = \mathbb{C}[x_0, \ldots, x_n]$ by $\deg(x_i) = w_i$ for all $i = 0, \ldots, n$. We define the weighted projective space $\mathbb{P}(w)$ with weights w as

$$\mathbb{P}(w) := \mathbb{P}(w_0, \ldots, w_n) := \operatorname{Proj}(S(w_0, \ldots, w_n)).$$

Alternatively one can define $\mathbb{P}(w)$ as the quotient of the following $\mathbb{C}^*$-action

$$\lambda \cdot (a_0, \ldots, a_n) := (\lambda^{w_0} a_0, \ldots, \lambda^{w_n} a_n),$$

that is $\mathbb{P}(w) = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ for coordinates $(a_0, \ldots, a_n)$ on $\mathbb{C}^{n+1}$ and $\lambda \in \mathbb{C}^*$. To see from this description that $\mathbb{P}(w)$ is a normal projective toric variety, note that the above $\mathbb{C}^*$-action restricts to $(\mathbb{C}^*)^{n+1}$ to give the n-dimensional torus $T := (\mathbb{C}^*)^{n+1}/\mathbb{C}^*$. This torus naturally acts on $\mathbb{P}(w)$ via

$$((t_0, \ldots, t_n), (a_0, \ldots, a_n)) \mapsto (t_0^{w_0} a_0, \ldots, t_n^{w_n} a_n)$$

and embeds it as a dense open subset of $\mathbb{P}(w)$. The fan of $\mathbb{P}(w)$ is now easily described as follows. Denote by $(e_0, \ldots, e_n)$ the standard basis of $\mathbb{Z}^{n+1}$ and by $(e_0^w, \ldots, e_n^w)$ its image in the lattice

$$N_w := \mathbb{Z}^{n+1}/\mathbb{Z}(w_0, \ldots, w_n) \cong \mathbb{Z}^n.$$

By construction there is the single relation $\sum_{i=0}^n w_i e_i^w = 0$. The ray generators n_{ρ_i} span a simplex $\Theta_w \subset N_{\mathbb{R}}$ whose fan Σ_w of cones over faces is the fan of the weighted projective space with weights w, that is $\mathbb{P}(w) = \mathbb{P}_{\Sigma_w}$. □

From the above description one immediately sees that the polytopes Θ_1 and Θ_2 from Example 2.13 correspond to the weighted projective spaces $\mathbb{P}(1, 5, 6, 8)$ and $\mathbb{P}(1, 1, 2, 3)$.

REMARK 2.27. For any set of weights $w = (w_0, \ldots, w_n)$ the space $\mathbb{P}(w)$ is a $\mathbb{Q}$-Gorenstein, $\mathbb{Q}$-Fano toric variety with fan $\Sigma_w \subset N_{\mathbb{Q}}$. Moreover, it is well known that $\mathbb{P}(w)$ is Gorenstein Fano if and only if there is an anti-canonical section with $w_j \,|\, \sum_{i=0}^n w_i$ for all $j = 0, \ldots, n$. Indeed, in (1) of Proposition 2.17 we have seen that $\mathbb{P}(w)$ being Gorenstein Fano is equivalent to $\Xi_{-K_{\mathbb{P}(w)}} \cong \Xi_{\Sigma_w}$ being a lattice polytope with vertices corresponding to maximal faces of $\Theta_{\Sigma_w} = \Theta_w$, which is a lattice polytope in $N_{\mathbb{Q}}$. As $\Theta^*_{\Sigma_w} \cong \Xi_{\Sigma_w}$ it follows that both of these polytopes are reflexive. Moreover, the vertices of the lattice simplex $\Xi_{-K_{\mathbb{P}(w)}}$ correspond to monomials of the form $x_i^{\lambda_i}$. Since this defines an anti-canonical section, we see that $\lambda_i \cdot w_i = \sum_i w_i$. □

DEFINITION 2.28. If $gcd(w_0, \ldots, \widehat{w_i}, \ldots, w_n) = 1$ for all $i = 0, \ldots, n$, then a weighted projective space $\mathbb{P}(w)$ is called *well-formed*. □

For two different sets of weights, say w and w', we want to be able to decide when $\mathbb{P}(w)$ and $\mathbb{P}(w')$ are isomorphic. The next lemma gives a partial answer to this question and shows that every $\mathbb{P}(w)$ is isomorphic to a well-formed weighted projective space.

LEMMA 2.29. [Ia, Lemma 5.5, Lemma 5.7, Corollary 5.9]. *Let $\lambda \in \mathbb{N}$ and $w = (w_0, \ldots, w_n)$ be weights. Then*

(1) $\mathbb{P}(w) \cong \mathbb{P}(\lambda \cdot w)$.
(2) *If $gcd(w_0, \ldots, w_n) = 1$ and $gcd(w_1, \ldots, w_n) = \lambda$, then $\mathbb{P}(w_0, \ldots, w_n)$ is isomorphic to $\mathbb{P}(w_0, w_1/\lambda, \ldots, w_n/\lambda)$.*
(3) *$\mathbb{P}(w)$ is isomorphic to $\mathbb{P}(w') = \mathbb{P}(w'_0, \ldots, w'_n)$, where w' is a set of weights with $gcd(w'_0, \ldots, \widehat{w'_i}, \ldots, w'_n) = 1$ for all $i = 0, \ldots, n$.* □

REMARK 2.30. Spaces $\mathbb{P}(w)$ that are not well-formed give rise to stacks, which we will not comment on here. However, the third part of the lemma justifies to neglect these phenomena without loss of generality. □

2.3.1. Hypersurfaces. In this section we will study hypersurfaces X in $\mathbb{P}(w)$ and their singularities and then specialize to the case where X has degree $d := \sum_i w_i$, thus potentially giving Calabi-Yau varieties. We compare the results from the last section with the properties of X and study an explicit example of hypersurfaces in a space $\mathbb{P}(w)$, which shows the implications of Theorem 2.25.

DEFINITION 2.31. Let $\mathbb{P}(w)$ be an n-dimensional weighted projective space with weights $w = (w_0, \ldots, w_n)$ and $X \subset \mathbb{P}(w)$ a hypersurface.

(1) X is called *quasi-smooth*, if its affine cone $\mathcal{C}_X \subset \mathbb{C}^{n+1}$ is smooth outside the origin.
(2) X is called *well-formed* if $\mathbb{P}(w)$ is well-formed and X contains no singular strata of codimension 2. In terms of weights this simply means

$$gcd(w_0, \ldots, \widehat{w_i}, \ldots w_n) = 1 \quad \text{and}$$
$$gcd(w_0, \ldots, \widehat{w_i}, \ldots, \widehat{w_j}, \ldots w_n) | \sum_{k=0}^{n} w_k$$

for all $i, j = 0, \ldots, n$. □

If a hypersurface $X = V_{\mathbb{P}(w)}(W) \subset \mathbb{P}(w)$ is quasi-smooth, it has only finite quotient singularities due to the $\mathbb{C}^*$-action. So in particular it has canonical singularities. Being quasi-smooth is a strong assumption, but it is often sufficient to require less, as the following theorem due to Dolgachev indicates.

LEMMA 2.32. [Do, Theorem 3.3.4]. *Let X be a well-formed degree d hypersurface in $\mathbb{P}(w)$. Then the adjunction formula for X holds, that is*

$$\omega_X \cong \mathcal{O}_X(d - \sum_i w_i). \qquad \square$$

We now state a criterion to check when a given hypersurface is quasi-smooth that entirely depends on the weights of the ambient space.

LEMMA 2.33. [Ia, Theorem 8.1]. *A general degree d hypersurface $X = V_{\mathbb{P}(w)}(W) \subset \mathbb{P}(w)$ is quasi-smooth if and only if one of the following cases holds.*

(1) *X is a linear cone, that is $W = x_i$ for some $i = 0, \ldots, n$.*
(2) *For all index sets $\emptyset \neq I = \{i_0, \ldots, i_k\} \subseteq \{0, \ldots, n\}$ there either exists a monomial in W of the form*

$$x_I^m = x_{i_0}^{m_0} \cdot \ldots \cdot x_{i_k}^{m_k}$$

of degree d or for each $l = 0, \dots, k$ there is a degree d monomial

$$x_I^{m_l} \cdot x_{e_l} = x_{i_0}^{m_{0,l}} \cdot \dots \cdot x_{i_k}^{m_{k,l}} \cdot x_{e_l},$$

where the e_l are distinct elements. □

In the next chapter we will mainly be concerned with threefolds realized as hypersurfaces in four-dimensional toric varieties, so let us state the following corollary to the above lemma.

COROLLARY 2.34. [Ia, Corollary 8.6] *A general degree $d := \sum_i w_i$ hypersurface*

$$X = V_{\mathbb{P}(w)}(f) \subset \mathbb{P}(w) = \mathbb{P}(w_0, w_1, w_2, w_3, w_4)$$

is a quasi-smooth Calabi-Yau variety if and only if for all $i \neq j = 0, \dots, 4$ one of the following is true:

(1) *there is a degree d monomial $x_i^m \cdot x_{e_i}$.*
(2) *There either exists a monomial $x_i^{m_i} x_j^{m_j}$ or monomials of the form $x_i^{m_{i,0}} x_j^{m_{j,0}} x_{e_0}$ and $x_i^{m_{i,1}} x_j^{m_{j,1}} x_{e_1}$ of degree d with $e_0 \neq e_1$.*
(3) *There is a degree d monomial which neither contains x_i nor x_j.* □

We will now discuss an example of anti-canonical hypersurfaces in a $\mathbb{Q}$-Gorenstein weighted projective space from the point of view of Theorem 2.25.

EXAMPLE 2.35. Let $\mathbb{P}(1,1,1,2) = \mathrm{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3])$ and take as fan the complete fan $\Sigma \subset N_{\mathbb{Q}} \cong \mathbb{Q}^3$ generated by

$$(-1,0,0), (0,-1,0), (0,0,-1) \text{ and } (1,1,2).$$

By Remark 2.27 we know that $\mathbb{P}(1,1,1,2)$ is non-Gorenstein, as $w_3 \nmid \sum_{i=0}^3 w_i$. Another way to see this is from the cone

$$\sigma = \mathrm{conv}((-1,0,0), (0,-1,0), (1,1,2)),$$

which has $\mathbb{Q}$-Gorenstein support vector $m_\sigma = (1, 1, -3/2)$. The other three maximal cones are smooth. Moreover, let

$$W \in H^0(\mathbb{P}(1,1,1,2), -K_{\mathbb{P}(1,1,1,2)})$$

be an *arbitrary* anti-canonical section. Then $V_{\mathbb{C}^4}(W)$ passes through the point $(0,0,0,1)$, which accounts for the fact that there is no monomial of the form $x_3^{5/2}$, and this point corresponds to a singular point in $\mathbb{P}(1,1,1,2)$. So even if we take W to be quasi-smooth, $X := V_{\mathbb{P}(1,1,1,2)}(W)$ will still be singular.

Now, the span of ray generators $\Theta := \Theta_\Sigma$, is an almost reflexive polytope and one can easily check that the associated reflexive polytope $\overline{\Theta}$ is spanned by the vertices of Θ and the point $(0,0,1)$. Thus, we know that there is a canonical resolution of $\mathbb{P}(1,1,1,2)$ by introducing a new ray in direction $(0,0,1)$ and star subdividing σ in three *smooth* cones. The corresponding map of fans $\widetilde{\Sigma} \to \Sigma$ therefore induces a resolution $\phi : \mathbb{P}_{\widetilde{\Sigma}} \to \mathbb{P}(1,1,1,2)$, and $\phi^*(X)$ will be smooth whenever X is quasi-smooth. So there is in fact no need to impose further regularity conditions such as $\overline{\Xi}$-regularity, where $\overline{\Xi} := \operatorname{conv}(\Xi_{-K_{\mathbb{P}(1,1,1,2)}} \cap M)$. Moreover, the Newton polytope of W can be very small compared to $\overline{\Xi}$, W only has to fulfill the combinatorial requirements of 2.33. $\square$

REMARK 2.36. While for $n = 3$ a general degree $d = \sum_i w_i$ hypersurface

$$X = V_{\mathbb{P}(w)}(W) \subset \mathbb{P}(w) = \mathbb{P}(w_0, w_1, w_2, w_3)$$

is a $K3$ surface if and only if it is quasi-smooth, see [CoGo, Theorem 1.13], in higher dimensions it is rare that hypersurfaces are quasi-smooth. For instance there are 184062 different weights $w = (w_0, \dots, w_4)$ that yield Calabi-Yau hypersurfaces, but only 7555 among them are quasi-smooth, see [Kre]. However, note that whenever it is possible to find one quasi-smooth hypersurface $X \subset \mathbb{P}(w)$, then a general hypersurface of the same degree will be quasi-smooth as well. $\square$

The next proposition connects the above results to what we have observed in the last sections.

PROPOSITION 2.37. *Let $X \subset \mathbb{P}(w)$ be a general anti-canonical section of $\mathbb{P}(w)$, that is $X = V_{\mathbb{P}(w)}(W)$ for some quasi-homogeneous polynomial W of degree $d = \sum_i w_i$. Choose a fan $\Sigma \subset N_\mathbb{R}$ for $\mathbb{P}(w)$. Then the following are equivalent:*

(1) *X is Calabi-Yau.*

(2) Θ_Σ *is almost reflexive.*
(3) X *is a well-formed hypersurface with canonical singularities.*

PROOF. The equivalence of (1) and (2) was shown in Theorem 2.25. It suffices to show that the defining conditions for well-formedness for X, that is

$$gcd(w_0, \dots, \widehat{w_i}, \dots w_n) = 1 \text{ and } gcd(w_0, \dots, \widehat{w_i}, \dots, \widehat{w_j}, \dots w_n)|d,$$

for all $i, j = 0, \dots, n$, are fulfilled and X has canonical singularities if and only if $\overline{\Xi} = \operatorname{conv}(\Xi_{-K_{\mathbb{P}(w)}} \cap M)$ is a reflexive polytope. If X is well-formed, by Theorem 2.32, we can apply the adjunction formula. As X has canonical singularities it is Calabi-Yau and therefore Θ_Σ is almost reflexive.

Conversely, if X has non-canonical singularities it is in particular not Calabi-Yau. So assume that X has canonical singularities but is not well-formed. Since we always assume the ambient space $\mathbb{P}(w)$ to be well-defined, without loss of generality let

$$gcd(\widehat{w_0}, \widehat{w_1}, w_2, \dots, w_n) \nmid d,$$

that is there is no degree d monomial of the form $x_2^{\lambda_2} \dots x_n^{\lambda_n}$, neither do degree d monomials x_I^λ with $I \subsetneq \{2, \dots, n\}$ exist. That in turn means that the toric stratum $\{x_0 = x_1 = 0\}$ is contained in X. If $\overline{\Xi}$ were reflexive, its normal fan $\Sigma^*_{\overline{\Xi}}$ would define an MPCP resolution $\mathbb{P}_{\Sigma^*_{\overline{\Xi}}} \to \mathbb{P}(w)$. But then the toric stratum corresponding to the pull-back of $\{x_0 = x_1 = 0\}$ would be contained in the pull-back of X. However, this is a contradiction to the fact that a general anti-canonical section in $\mathbb{P}_{\Sigma^*_{\overline{\Xi}}}$ is $\overline{\Xi}$-regular. Hence Ξ cannot be reflexive and so X is not a Calabi-Yau variety. □

2.3.2. Toric quotients of weighted projective spaces. We have seen that each weighted projective space $\mathbb{P}(w)$ comes with a fan Σ_w and a simplex $\Theta_w \subset \mathrm{N}_\mathbb{R}$ that is reflexive if and only if $\mathbb{P}(w) = \mathbb{P}_{\Sigma_w}$ is Fano. If we start with a simplex $\Theta \subset N_\mathbb{Q}$ with vertices in N, then the associated toric variety constructed from the fan Σ_Θ of cones over faces will in general be a finite quotient of a weighted projective space. This quotient is induced by a map of fans. As we will need these quotients in the next chapter, we will make this construction more precise.

Let Σ be a complete fan in $\mathrm{N}_{\mathbb{Q}}$ with exactly $n+1$ ray generators

$$n_{\rho_0}, \ldots, n_{\rho_n},$$

so that Θ_Σ is a simplex. Then clearly there is a unique relation $\sum_{i=0}^{n} w_i n_{\rho_i}$ for an $(n+1)$-tuple $w := (w_0, \ldots, w_n)$ of positive integers. Denote by Σ_w the fan of the weighted projective space $\mathbb{P}(w)$ with these given weights w in the lattice

$$N_w := \mathbb{Z}^{n+1}/\mathbb{Z} \cdot (w_0, \ldots, w_n).$$

The ray generators e_i^w of the fan are the images of the standard basis vectors e_i of $\mathbb{Z}^{n+1}$. The map $\mathbb{Z}^{n+1} \to N$ defined by $e_i \mapsto n_{\rho_i}$ therefore induces a map of lattices

$$\psi : N_w \to N$$

which gives a finite map of fans $\Sigma_w \to \Sigma$. Hence $\mathbb{P}(w) \to \mathbb{P}_\Sigma$ is a finite quotient of toric varieties. The last result we will need for later applications is the following.

LEMMA 2.38. [Ba, Theorem 5.3.1]. *Let $\mathbb{P}(w) \to \mathbb{P}_\Sigma$ be a finite toric quotient induced by a map $\psi : N_w \to N$ of lattices. Let $W \in H^0(\mathbb{P}_\Sigma, -K_{\mathbb{P}_\Sigma})$ be an anti-canonical section and $X := V_{\mathbb{P}_\Sigma}(W)$. Then there is a section $W_w \in H^0(\mathbb{P}(w), -K_{\mathbb{P}(w)})$ such that for $X_w := V_{\mathbb{P}(w)}(W_w)$ we have*

$$X \cong X_w/\big(N/\psi(N_w)\big). \qquad \square$$

To summarize, we have seen that the fan of cones over faces of a full-dimensional simplex $\Theta \subset N_{\mathbb{Q}} \cong \mathbb{Q}^n$ defines a quotient $\mathbb{P}(w)/\overline{G}$, where $\overline{G}$ is the quotient of a lattice by a sublattice. Moreover, G-invariant hypersurfaces of $\mathbb{P}(w)$ give hypersurfaces in the quotient. The set of almost reflexive simplices thus consists of all sub-simplices of reflexive polytopes that give rise to a well-formed weighted projective space and quotients of such. The classification of reflexive polytopes in dimension three and four found at [Kre] therefore allows to produce a complete list of almost reflexive polytopes in these dimensions.

Chapter 3

Applications to CY-CY mirror symmetry

Now that we know necessary and sufficient conditions for a general anti-canonical hypersurface in a $\mathbb{Q}$-Gorenstein toric variety to be Calabi-Yau, we move on to questions of mirror symmetry related to such Calabi-Yau varieties. After shortly reviewing the construction of Berglund and Hübsch, we will show that it is intimately related to Batyrev's work. We close with a discussion of mirror symmetry for generalized Borcea-Voisin threefolds by finding singular models in $\mathbb{Q}$-Gorenstein toric varieties for them.

For the whole chapter we will fix the following notation.

$$\mathbb{P}(w) = \mathbb{P}(w_0, \ldots, w_n)$$

is a weighted projective space with fan $\Sigma_w \subset N_{\mathbb{R}}$ as defined in Definition 2.26 and denote by Θ_w the span of ray generators of Σ. Moreover, we will write $d := \sum_{i=0}^{n} w_i$ for the anti-canonical degree and throughout the whole chapter we will work over $\Bbbk = \mathbb{C}$.

3.1. Results on $\mathbb{Q}$-Gorenstein mirror symmetry

Given a four-dimensional reflexive polytope $\Xi \subset M_{\mathbb{Q}}$ recall that the mirror symmetry construction introduced by Batyrev in [Ba] starts with a Ξ-regular anti-canonical hypersurface X in the toric variety $\mathbb{P}_\Xi$ associated to the normal fan Σ^*_Ξ. An MPCP subdivision of Ξ then yields an MPCP resolution $Y \to X$ and by [Ba, Theorem 4.4.2, Theorem 4.4.3] we get

$$h^{1,1}(Y) = l(\Xi^*) - 5 - \sum_{\sigma^*} l^*(\sigma^*) + \sum_{\tau^*} l^*(\tau^*)l^*(\check{\tau}^*)$$
$$h^{1,2}(Y) = l(\Xi) - 5 - \sum_{\sigma} l^*(\sigma) + \sum_{\tau} l^*(\tau)l^*(\check{\tau}).$$

Here $l(\_)$ stands for integral points and $l^*(\_)$ for *interior* integral points of the respective polytope. Moreover, σ and σ^* refer to codimension one, τ and τ^* codimension two faces of Ξ, respectively Ξ^* and $\check{\tau}$ denotes the face of the polar polytope dual to τ.

This result implies that, if we choose a Ξ^*-regular anti-canonical hypersurface X^* in $\mathbb{P}_{\Xi^*}$ and an MPCP subdivision for Ξ^* with induced MPCP resolution $Y^* \to X^*$, then Y and Y^* form a *topological mirror pair*, that is

$$h^{1,1}(Y) = h^{1,2}(Y^*), \quad h^{1,1}(Y^*) = h^{1,2}(Y). \tag{3.1}$$

In this situation we will say that Y and Y^* form a *Batyrev mirror pair*.

Having stated this result about *general* hypersurfaces in Gorenstein toric varieties, we will now turn to a construction due to Berglund and Hübsch that covers *special* hypersurfaces in possibly $\mathbb{Q}$-Gorenstein weighted projective spaces. While this approach was not particularly present in the mathematical literature for a long time, recently more and more papers are devoted to this topic, see for instance [Bori], [ChRu1], [ChRu2] or [Kra].

3.1.1. The Berglund-Hübsch construction.

DEFINITION 3.1. Let $w = (w_0, \ldots, w_n)$ be weights of the weighted projective space $\mathbb{P}(w)$. Define the *charges of* $\mathbb{P}(w)$ to be $q_i := w_i/d$ for all $i = 0, \ldots, n$. By the *charge* of $\mathbb{P}(w)$ we refer to the $(n+1)$-tuple $q = (q_0, \ldots, q_n)$. □

DEFINITION 3.2. Let $w = (w_0, \ldots, w_n)$ be weights. Then a polynomial $W : \mathbb{C}^{n+1} \to \mathbb{C}$ of the form

$$W = W(x_0, \ldots, x_n) = \sum_{i=0}^{n} a_i \prod_{j=1}^{n} x_j^{\lambda_{ij}}$$

is called a *polynomial of Berglund-Hübsch type*, or a *BH-type polynomial* if it is quasi-homogeneous of degree d with respect to the charges q_i and smooth outside the origin, that is if $\sum_j \lambda_{ij} q_j = 1$ and $V_{\mathbb{P}(w)}(W) \subset \mathbb{P}(w)$ is a quasi-smooth hypersurface. We furthermore assume, without loss of generality, that $a_i = 1$ for all $i = 0, \ldots, n$ and call $\Lambda_W := (\lambda_{ij})_{i,j}$ the *matrix associated to* W. □

Berglund and Hübsch in [BeHu] call quasi-smoothness *nondegeneracy* and the polynomial W *nondegenerate* if $V_{\mathbb{P}(w)}(W) \subset \mathbb{P}(w)$ is quasi-smooth. For convenience, we will make use of this terminology in the rest of this chapter.

In [KrSk] Kreuzer and Skarke give a complete classification of all polynomials of Berglund-Hübsch type by combinatorially describing necessary and sufficient conditions on the monomials of W. Their result can be summarized as follows.

LEMMA 3.3. *A quasi-homogeneous polynomial* $W : \mathbb{C}^{n+1} \to \mathbb{C}$ *of degree* d *having* $n+1$ *terms defines a quasi-smooth hypersurface* $X = V_{\mathbb{P}(w)} \subset \mathbb{P}(w)$ *if and only if it can be decomposed as direct sum of polynomials of the following types:*

$$W = x_i^{m_i}$$

$$W = x_{i_0}^{m_{i_0}} x_{i_1} + x_{i_1}^{m_{i_1}} x_{i_2} + \ldots + x_{i_{n'-1}}^{m_{i_{n'-1}}} x_{i_{n'}} + x_{i_{n'}}^{m_{i_{n'}}}$$

$$W = x_{i_0}^{m_{i_0}} x_{i_1} + x_{i_1}^{m_{i_1}} x_{i_2} + \ldots + x_{i_{n'-1}}^{m_{i_{n'-1}}} x_{i_{n'}} + x_{i_{n'}}^{m_{i_{n'}}} x_{i_0}$$

The first type is referred to as Fermat type, the second by Chain type and the last by Loop type.

PROOF. This was first proved in [KrSk, Theorem 1], but is written down slightly differently. The result as we stated it can be found in [ChRu1, Remark 3]. □

If W is a BH-type polynomial it follows that Λ_W is an invertible matrix, as the columns of Λ_W, corresponding to monomials of W, are linearly independent by Lemma 3.3. Let $\Lambda_W^{-1} = (\lambda^{ij})_{i,j}$ denote the inverse of Λ_W. It is easy to see that $q_j = \sum_{i=0}^n \lambda^{ij}$, that is the charges are determined by Λ_W.

CONSTRUCTION 3.4. Let $W = \sum_{i=0}^{n} \prod_{i=0}^{n} x_i^{\lambda_{ij}}$ be a nondegenerate quasi-homogeneous polynomial. Define its *dual polynomial* W^*, a polynomial in $(n+1)$ variables y_i, to be

$$W^* := W^*(y_0, \dots, y_n) = \sum_{i=0}^{n} \prod_{i=0}^{n} y_i^{\lambda_{ji}},$$

that is, W^* is the polynomial associated to the transpose Λ_W^T of the matrix associated to W. Furthermore, we can define the *dual charges* to be

$$q_j^* = \sum_{i=0}^{n} \lambda^{ji}.$$

Since $(\Lambda_W^T)^{-1} = (\Lambda_W^{-1})^T$, W^* is a quasi-homogeneous polynomial with respect to these charges. Let d^* denote the smallest integer such that $w_i^* := d^* \cdot q_i^* \in \mathbb{N}$ for all $i = 0, \dots, n$. Then $V_{\mathbb{P}(w^*)}(W^*)$ is a well-defined hypersurface of degree d^* in $\mathbb{P}(w^*) = \mathrm{Proj}(\mathbb{C}[y_0, \dots, y_n])$ with *dual weights* $w^* = (w_0^*, \dots, w_n^*)$. $\square$

An immediate corollary of the explicit description of nondegenerate quasi-homogeneous polynomials in Lemma 3.3 is that W is nondegenerate if and only if W^* is nondegenerate, as W can be decomposed into the three types if and only if W^* does.

EXAMPLE 3.5. For the weighted projective space

$$\mathbb{P}(1,1,2,3) = \mathrm{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3]),$$

that is with weights $w = (1,1,2,3)$, degree $d = 7$ and charge $q = (1/7, 1/7, 2/7, 3/7)$ we choose the following nondegenerate polynomial

$$W = W(x_0, x_1, x_2, x_3) = x_0^7 + x_1^5 x_2 + x_2^2 x_3 + x_3^2 x_1.$$

The first term is of Fermat type, the other three form a loop. Furthermore, the associated matrix Λ_W and its inverse Λ_W^{-1} are given as follows

$$\Lambda_W = \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 5 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \qquad \Lambda_W^{-1} = \begin{pmatrix} \frac{1}{7} & 0 & 0 & 0 \\ 0 & \frac{4}{21} & \frac{1}{21} & -\frac{2}{21} \\ 0 & -\frac{2}{21} & \frac{10}{21} & \frac{1}{21} \\ 0 & \frac{1}{21} & -\frac{5}{21} & \frac{10}{21} \end{pmatrix}.$$

Thus we recover the charges q from the sum of column entries of Λ_W^{-1}. Moreover, the sum of row entries gives dual charges

$$q^* = (1/7, 1/7, 3/7, 2/7),$$

that is $d^* = 7$ and $w^* = (1, 1, 3, 2)$. It is easy to see that the dual polynomial

$$W^* = W^*(y_0, y_1, y_2, y_3) = y_0^7 + y_1^5 y_3 + y_1 y_2^2 + y_2 y_3^2$$

is quasi-smooth and in fact quasi-homogeneous with respect to w^*. We will come back to this example after having proved Theorem 3.19. □

Next, we will define an automorphism group of W that will be of major importance for the discussion of mirror symmetry.

DEFINITION 3.6. Let $W : \mathbb{C}^{n+1} \to \mathbb{C}$ be nondegenerate. Define the finite abelian group of *diagonal automorphisms preserving* W to be

$$Aut(W) := \{g : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} | g(x_i) = g_i \cdot x_i, g(W) = W\}.$$

where by abuse of notation we also write g for the induced action on the polynomial W. We will usually write $g = (g_0, \ldots, g_n)$ for an element of $Aut(W)$. □

REMARK 3.7. 1) The subgroup $SLAut(W) := SL_{n+1}(\mathbb{C}) \cap Aut(W)$ is characterized by the property $\prod_{i=0}^n g_i = 1$, for $g \in Aut(W)$.
2) There is a distinguished element $\mathcal{J}(W) \in Aut(W)$, called *exponential grading operator* in [Kra], defined by

$$\mathcal{J}(W)(x_j) = e^{2\pi i q_j} x_j.$$

It is immediate that $\mathcal{J}(W)$ is an element of $SLAut(W)$ in our setup. □

It will be convenient to name the rows and columns of the inverse of Λ_W, as they can be identified as elements of the automorphism group of W^*, respectively W.

DEFINITION 3.8. Let W be a BH-type polynomial with associated matrix $\Lambda_W = (\lambda_{ij})_{i,j}$ and inverse Λ_W^{-1}. Denote by c_j the j-th column and by r_j the j-th row of Λ_W^{-1}. for all $j = 0, \ldots, n$. □

By definition c_j andr_j are $(n+1)$-tuples of complex numbers for all $j = 0, \ldots, n$ and by slightly abusing notation we can define the actions

$$c_j(x_k) := e^{2\pi i (c_j)_k} \cdot x_j, \qquad r_j(y_k) := e^{2\pi i (r_j)_k} \cdot y_j.$$

As observed before, the columns c_j of Λ_W^{-1} have the property $\sum_{k=0}^{n} (c_j)_k = q_j$. From this and the fact that W is quasi-homogeneous with respect to the charges q_i, we see that $c_i \in Aut(W)$. Analogously it can be seen that $r_i \in Aut(W^*)$.

To state the mirror symmetry theorem of Berglund and Hübsch, we have to define the group G^* dual to G.

DEFINITION 3.9. Let $G \subseteq Aut(W)$ be an arbitrary subgroup. Define the *dual group* G^* as follows:

$$G^* := \left\{ \begin{matrix} \prod_{i=0}^{n} r_i^{b_i} | & (b_0, \ldots, b_n)\Lambda_W^{-1}(a_0, \ldots, a_n)^T \in \mathbb{Z}, \\ & \forall (a_0, \ldots, a_n)\colon \prod_{i=0}^{n} c_i^{a_i} \in G. \end{matrix} \right\} \qquad \square$$

The fact that $G^{**} = G$ might not be entirely clear from this description, although it is elementary. We will not give a proof right away, however, as this will become apparent in the proof of Lemma 3.16. Let us summarize the construction by introducing the following notation.

DEFINITION 3.10. Let $W = \sum_{i=0}^{n} \prod_{i=0}^{n} x_i^{\lambda_{ij}}$ be of BH-type and choose a subgroup $G \subseteq Aut(W)$ such that $\mathcal{J}(W) \in G$. Then we call (W, G) a *Berglund-Hübsch pair* and the pair (W^*, G^*) as described above its *dual pair*. $\square$

The requirement $\mathcal{J}(W) \in G$ is natural and no restriction for the setup we are working with, as we will see in Lemma 3.18.

REMARK 3.11. Let (W, G) and (W^*, G^*) be dual Berglund-Hübsch pairs. Note that G and G^* really act on $\mathbb{C}^5$, but not directly on weighted projective spaces. To this end define

$$\overline{G} := G/\langle \mathcal{J}(W) \rangle, \quad \overline{G}^* := G^*/\langle \mathcal{J}(W^*) \rangle.$$

From the definition of $\mathcal{J}(W)$ and $\mathcal{J}(W^*)$ it follows that $\overline{G}$ defines an action on $\mathbb{P}(w)$ and $\overline{G}^*$ an action on $\mathbb{P}(w^*)$. Moreover, define

$$X := V_{\mathbb{P}(w)}(W) \subset \mathbb{P}(w) \text{ and } X^* := V_{\mathbb{P}(w^*)}(W^*) \subset \mathbb{P}(w^*).$$

As $Aut(W)$ preserves W and W is quasi-smooth, we can consider the quotient $X/\overline{G}$, which is a possibly singular Calabi-Yau in $\mathbb{P}(w)/\overline{G}$ and analogously so is $X^*/\overline{G}^* \subset \mathbb{P}(w^*)/\overline{G}^*$. □

We now come to the central result of Berglund-Hübsch [BeHu] which is a statement about Calabi-Yau threefolds, that is for $n = 4$.

THEOREM 3.12. [BeHu, Section 2]. *Let (W, G) and (W^*, G^*) be dual Berglund-Hübsch pairs for $n = 4$. Then there exist crepant Calabi-Yau resolutions $X_W := \widetilde{X/\overline{G}}$ and $X_{W^*} := \widetilde{X^*/\overline{G}^*}$ that form a topological mirror pair, that is such that*

$$h^{1,1}(X_W) = h^{1,2}(X_{W^*})$$
$$h^{1,2}(X_W) = h^{1,1}(X_{W^*}).$$ □

For a choice of dual Berglund-Hübsch pairs for $n = 4$ we will refer to the Calabi-Yau three-manifolds X_W and X_{W^*} from Theorem 3.12 as the *corresponding mirror pair*.

The resolutions X_W and X_{W^*} Berglund and Hübsch use for their theorem rely on a result of Roan, see [Roa1, Roa2], we will present next. The statement itself is based on a toroidal Calabi-Yau resolution $\widetilde{X} \to X$, first constructed by Greene, Roan and Yau [GrRoYa]. The proposition we will state is interesting on its own, as it not only gives an explicit resolution for $X/\overline{G}$, but also tell us about the Hodge structure of the resolution. As we will see, the Hodge numbers of $\widetilde{X/\overline{G}}$ will only depend on the weights of the ambient space $\mathbb{P}(w)$ and the action of $\overline{G}$ on it, not on the actual equation W.

PROPOSITION 3.13. [Roa2, Theorem 1]. *Let G be a group linearly acting on $\mathbb{C}^5$ with standard basis $(e_i)_{i=0,\dots 4}$ and let $X = V_{\mathbb{P}(w)}(W)$ be a $\overline{G}$-invariant hypersurface. Moreover, for $g \in G$ define*

$$\beta_g := \frac{1}{|G|} \sum_{h \in G} \prod_{i: g(e_i) = h(e_i) = e_i} (1 - \frac{1}{q_i}) \qquad \xi_g := \#\{i | g(e_i) = e_i\}.$$

Then there is a toroidal resolution $\widetilde{X/\overline{G}}$ whose Hodge numbers are given by

$$b^3(\widetilde{X/\overline{G}}) = -\sum_{\xi_g \geq 3} \beta_g \qquad b^2(\widetilde{X/\overline{G}}) = -1 + \frac{1}{2} \sum_{\xi_g < 3} \beta_g,$$

□

so in particular they do not depend on W.

The proposition of Roan works with toroidal resolutions. However, the following result shows that most of the time we can in fact work with global toric resolutions.

COROLLARY 3.14. *Let (W, G) be a Berglund-Hübsch pair such that W is* general. *If $X = V_{\mathbb{P}(w)}(W)$, then the above resolution $\widetilde{X/\overline{G}}$ is induced by an MPCP subdivision of the reflexive polytope $\overline{\Theta_w}$ associated to Θ_w.*

PROOF. By Theorem 2.25 we know that a general X is Calabi-Yau if and only if the fan of $\mathbb{P}(w)$ is the fan of cones over faces of an almost reflexive polytope $\Theta_w \subset N_{\mathbb{Q}}$. Recall that this means that $\Xi := \mathrm{conv}(\Xi_{-K_{\mathbb{P}(w)}} \cap M)$ is a reflexive polytope and $\overline{\Theta}_w := \overline{\Xi}^* \subset N_{\mathbb{Q}}$ is a reflexive polytope containing Θ_w. Choosing an MPCP subdivision for $\overline{\Theta}_w$ therefore induces a maximal refinement $\widetilde{\Sigma_w}$ of the fan Σ_w. Thus by construction $\mathbb{P}_{\widetilde{\Sigma_w}}$ is an MPCP-resolved Gorenstein toric variety. The map of fans $\widetilde{\Sigma_w} \to \Sigma_w$ furthermore induces a resolution $\psi : \widetilde{X} \to X$. As X is quasi-smooth by assumption and ψ resolves toric singularities, we know that $\widetilde{X}$ is smooth, since the singular locus of $\mathbb{P}_{\widetilde{\Sigma}}$ is of codimension 4 by [Ba, Corollary 4.2.3] and X is general. □

Theorem 3.12 is a mirror symmetry statement for $n = 4$ Berglund-Hübsch pairs. The only thing that keeps it from holding in higher dimensions is that there might only be a *partial* resolution of $X/\overline{G}$. The construction itself, however, remains unchanged in any dimension. In [ChRu1] Chiodo and Ruan prove a higher-dimensional analogue of the result of Berglund-Hübsch that uses Chen-Ruan orbifold cohomology. We will not give a definition of this cohomology theory here, as it is not important for the rest of the text. A short introduction can be found in §3.2 of [ChRu1]. All we need to know here is that for an orbifold $[X]$ Chen-Ruan orbifold cohomology groups $H^{p,q}_{CR}([X]; \mathbb{C})$ with coefficients in $\mathbb{C}$ agree with usual cohomology groups $H^{p,q}(\widetilde{X}, \mathbb{C})$ if a crepant resolution $\widetilde{X} \to [X]$ exists.

THEOREM 3.15. [ChRu1, Theorem 14]. *Let (W, G) and (W^*, G^*) be dual Berglund-Hübsch pairs for some $n \in \mathbb{N}$ and let $X := V_{\mathbb{P}(w)}(W)$, $X^* := V_{\mathbb{P}(w^*)}(W^*)$ as before. Then the orbifolds $[X/\overline{G}]$ and $[X^*/\overline{G}^*]$ are topological mirrors with respect to Chen-Ruan orbifold cohomology,* *that is*

$$H^{p,q}_{CR}([X/\overline{G}];\mathbb{C}) \cong H^{n-2-p,q}_{CR}([X^*/\overline{G}^*];\mathbb{C}). \qquad \square$$

3.2. Geometrization of $\mathbb{Q}$-Gorenstein mirror symmetry

We will now show that Berglund-Hübsch mirror symmetry is directly related to Batyrev mirror symmetry for $n = 4$. Fix a Berglund-Hübsch pair (W, G). Let N and M be finitely generated free abelian groups with generators $(n_i)_{i=0,\dots,n}$ and $(m_i)_{i=0,\dots,n}$. With help of the invertible matrix $\Lambda_W = (\lambda_{ij})_{i,j} \in GL_{n+1}(\mathbb{Q})$ we declare an integral, nondegenerate, but *not necessarily unimodular* pairing

$$\langle \cdot,\cdot \rangle_\Lambda : N \times M \to \mathbb{Z}$$

by setting $\langle n_i, m_j\rangle_\Lambda := \lambda_{ij}$. Denote the lattices dual to N and M with respect to the given pairing by N^* and M^*. Given this data we can deduce the following lemma.

LEMMA 3.16. *For a given Berglund-Hübsch polynomial W the choice of a group $G \subset Aut(W)$ is equivalent to a choice of lattices $M^* \supseteq N_G \supseteq N$ and $N^* \supseteq M_G \supseteq M$ such that N_G and M_G are dual lattices.*

PROOF. As a general element of M^* is of the form $\sum_{j=0}^n a_j n_j$ with $a_j \in \mathbb{Q}$ and analogously for N^* we see that $M^*/N \cong Aut(W)$ as the following two maps are inverse to each other:

$$\left[\sum_{j=0}^n a_j n_j\right] \mapsto (e^{2\pi i a_j})_j$$
$$g = (g_j)_j \mapsto \left[\sum_{j=0}^n \frac{1}{2\pi i}\log(g_j) n_j\right].$$

Here the square brackets denote cosets modulo N. In exactly the same way it can be shown that $N^*/M \cong Aut(W)$. Moreover, via this automorphism we see that each column c_i of Λ_W^{-1} is in fact an element of M^*/N, so it can be represented by an element $\widetilde{c_i} \in M^*$. Analogously, each r_i can be represented by $\widetilde{r_i} \in N^*$. In fact, if we choose bases $(m_{G,i})_{i=0,\dots n}$ of M_G and $(n_{G,i})_{i=0,\dots n}$ of N_G, then $(\widetilde{c_i})_{i=0,\dots,n}$ and $(\widetilde{r_i})_{i=0,\dots,n}$, respectively, are dual bases. To conclude, note that the pairing $\langle\cdot,\cdot\rangle$ induced on $N_G \times M_G$ is in fact what we want since by construction we have $\langle c_i, r_j\rangle = \lambda^{ij}$, that is the pairing is defined by the inverse matrix Λ_W^{-1}. $\square$

REMARK 3.17. Note that as they stand the lattices in the last lemma have rank $n+1$ and do not give rise to n-dimensional weighted projective space or quotients of such. Instead we will have to work modulo the group $\langle \mathcal{J}(W)\rangle$ generated by the element

$$\mathcal{J}(W) = (e^{2\pi i q_0}, \dots, e^{2\pi i q_n}).$$

If we take for instance $G = \langle \mathcal{J}(W)\rangle$, then

$$N_G/G \cong \mathbb{Z}^{n+1}/\mathbb{Z}(w_0, \dots, w_n)$$

is by construction nothing but the lattice N_w we fixed for the weighted projective space $\mathbb{P}(w)$. By Lemma 3.16 it can be concluded that larger groups $\langle \mathcal{J}(W)\rangle \subset G'$ define quotients of $\mathbb{P}(w)$, as long as $G' \subset SLAut(W)$. $\square$

LEMMA 3.18. *Let* $W = \sum_{i=0}^n \prod_{i=0}^n x_i^{\lambda_{ij}}$ *be of BH-type and choose a subgroup* $G \subseteq Aut(W)$. *Then*

$$\langle \mathcal{J}(W)\rangle \subseteq G \subseteq SLAut(W) \Leftrightarrow \langle \mathcal{J}(W^*)\rangle \subseteq G^* \subseteq SLAut(W^*).$$

PROOF. We only prove that $\langle \mathcal{J}(W)\rangle \subseteq G$ is equivalent to $G^* \subseteq SLAut(W^*)$ as the other half of the statement is just dual to this. As $\mathcal{J}(W) \in G$ just means $(q_0, \dots, q_n) = \sum_j q_j n_j \in N_G$ and $G^* \subseteq SLAut(W^*)$ translates to $\prod_i g_j^* = 1$ for all $g^* \in G^*$, we see that

$$\begin{aligned}
\prod_j g_j^* = 1 &\iff \frac{1}{2\pi i}\log(\prod_i g_j^*) \in \mathbb{Z} \\
&\iff \sum_j \frac{1}{2\pi i}\log(g_j^*) \in \mathbb{Z} \\
&\iff \Big\langle \sum_j q_j n_j, \sum_j \frac{1}{2\pi i}\log(g_j^*) m_j \Big\rangle \in \mathbb{Z} \\
&\iff \sum_j q_j n_j \in N_G
\end{aligned}$$

and the claim follows. $\square$

Note that Lemma 3.16 and Lemma 3.18 were proved by Borisov in [Bori, Proposition 2.3.1, Corollary 2.3.5], which was available online after we gave the proofs presented here.

3.2.1. Berglund-Hübsch pairs and almost reflexive polytopes. We now come to the first main theorem of this chapter. It roughly says that Berglund-Hübsch mirror symmetry is induced by Batyrev mirror symmetry in the sense that there is a mirror pair of Calabi-Yau threefolds as in (3.1) such that the Berglund-Hübsch mirror is obtained by *polynomial deformation* of this pair.

THEOREM 3.19. *Let (W, G) and (W^*, G^*) be dual Berglund-Hübsch pairs for $n = 4$ with corresponding mirror pair X_W and X_{W^*}. Then there is a reflexive pair $(\overline{\Theta}, \overline{\Xi})$ of four-dimensional polytopes and refinements $\widetilde{\Sigma_{\overline{\Theta}}} \to \Sigma_{\overline{\Theta}}$, $\widetilde{\Sigma_{\overline{\Xi}}} \to \Sigma_{\overline{\Xi}}$ induced by MPCP subdivisions of $\overline{\Theta}$ and $\overline{\Xi}$ such that*

(1) *X_W and X_{W^*} are anti-canonical hypersurfaces in $\mathbb{P}_{\widetilde{\Sigma_{\overline{\Theta}}}}$ and $\mathbb{P}_{\widetilde{\Sigma_{\overline{\Xi}}}}$, respectively.*
(2) *There are polynomial deformations Y of X_W and Y^* of X_{W^*} such that Y and Y^* form a Batyrev mirror pair.*
(3) *Given $(\overline{\Theta}, \overline{\Xi})$ the choice of dual BH pairs (W, G) and (W^*, G^*) is equivalent to a choice of almost reflexive simplices $\Theta \subset \overline{\Theta}$ and $\Xi \subset \overline{\Xi}$.*

PROOF. Let (W, G), (W^*, G^*) be a Berglund-Hübsch pair for $n = 4$. By Lemma 3.16 there are dual lattices N_G and M_G and we may further define

$$\overline{M} := M_G / \big\langle \mathcal{J}(W^*) \big\rangle, \quad \overline{N} := N_G / \big\langle \mathcal{J}(W) \big\rangle$$

which clearly are dual as well. As observed earlier in Remark 3.11, dividing by $\langle \mathcal{J}(W) \rangle$ means descending from $\mathbb{C}^5$ to $\mathbb{P}(w)$ and analogously for $\langle \mathcal{J}(W^*) \rangle$, where one descends from $\mathbb{C}^5$ to $\mathbb{P}(w^*)$. As before, let us consider the hypersurfaces $X := V_{\mathbb{P}(w)}(W)$ and $X^* := V_{\mathbb{P}(w^*)}(W^*)$, which are by definition invariant under the action of $\overline{G}$ and $\overline{G}^*$, respectively. So by Theorem 2.25

$$X/\overline{G} \subset \mathbb{P}(w)/\overline{G} \quad \text{and} \quad X^*/\overline{G}^* \subset \mathbb{P}(w^*)/\overline{G}^*$$

are Calabi-Yau as well. Note that by construction $\mathbb{P}(w)/\overline{G}$ is just the toric variety associated to the fan Σ_w considered in the sup-lattice $\overline{N}_{\mathbb{Q}}$ instead of in $N_{w,\mathbb{Q}}$ and $\mathbb{P}(w^*)/\overline{G^*}$ is the toric variety associated to the fan Σ_{w^*}

considered as a fan in $\overline{M}_{\mathbb{Q}}$. Moreover, as $X/\overline{G}$ and $X^*/\overline{G^*}$ are Calabi-Yau the lattice polytopes

$$\Theta := \Theta_{\Sigma_w} \subset \overline{N}_{\mathbb{Q}} \quad \text{and} \quad \Xi := \Theta_{\Sigma_{w^*}} \subset \overline{M}_{\mathbb{Q}}$$

have to be almost reflexive. Thus we have $\Theta \subset \overline{\Theta}$ and $\Xi \subset \overline{\Xi}$ for the associated reflexive polytopes $\overline{\Theta}$ and $\overline{\Xi}$. By construction it is clear that $\overline{\Xi} \subset \Theta^*$ and $\overline{\Theta} \subset \Xi^*$, from which we can conclude that $\overline{\Xi} \subset \overline{\Theta}^*$ and $\overline{\Theta} \subset \overline{\Xi}^*$. Thus, as the same argument holds true with the roles of Ξ and Θ reversed, it follows that $\overline{\Xi}^* = \overline{\Theta}$, which means that $(\overline{\Theta}, \overline{\Xi})$ is a *reflexive pair*. Choosing MPCP subdivisions for $\overline{\Theta}$, respectively $\overline{\Xi}$ yields resolutions for the varieties associated to the normal fan of these polytopes and hence resolutions

$$X_W = \widetilde{X/\overline{G}} \quad \text{and} \quad X_{W^*} = \widetilde{X^*/\overline{G^*}}.$$

From here it is clear that the Calabi-Yau X_W and X_{W^*} are special anti-canonical hypersurfaces in MPCP-resolved toric varieties associated to reflexive polytopes, thus (1) follows. For (2) note that this in particular implies that there are $\overline{\Xi}$-regular, respectively $\overline{\Theta}$-regular hypersurfaces Y and Y^* in $\widetilde{\mathbb{P}_{\overline{\Theta}}}$ and $\widetilde{\mathbb{P}_{\overline{\Xi}}}$ which are polynomial deformations of X_W and X_{W^*}, having the same Hodge-numbers by Proposition 3.13. The third part of the theorem is clear from the construction we have just presented. □

COROLLARY 3.20. *If $\mathbb{P}(w) = \mathbb{P}(w_0, w_1, w_2, w_3, w_4)$ is a smooth Fano four-manifold with fan Σ_w in $N_{w,\mathbb{Q}}$ and we take the Fermat polynomial $W = \sum_i x_i^{d/w_i}$, then the Batyrev mirror of $X = V_{\mathbb{P}(w)}$ agrees with the Berglund-Hübsch mirror.*

PROOF. If $\mathbb{P}(w)$ is smooth, it follows that $\overline{\Theta} := \Theta_\Sigma$ is a reflexive simplex with polar $\overline{\Xi} := \Xi_{-K_{\mathbb{P}(w)}}$. Moreover, the vertices of $\overline{\Xi}$ simply correspond to the monomials x_i^{d/w_i}, so X is not only quasi-smooth, but also $\overline{\Xi}$-regular. In fact, reflexive simplices have the property that the vertices of their polar satisfy the same single relation as that of the simplex itself. Thus $\overline{\Theta}$ and $\overline{\Xi}$ are essentially the same polytopes, but are defined in dual lattices. Therefore the toric variety $\mathbb{P}_{\Sigma_{\overline{\Xi}}}$ is thus just a quotient $\mathbb{P}(w)/H$ of the space $\mathbb{P}(w)$ for some group H and the Batyrev mirror $\check{X}$ is just a $\overline{\Theta}$-regular hypersurface in $\mathbb{P}(w)/H$. So without loss of generality we can assume that $\check{X} = X/H$. On the other hand, from Berglund and Hübsch's construction, we have $W^* = W$ and $G = \langle \mathcal{J}(W) \rangle$. As we furthermore know that $H = \overline{G}^*$, since both are isomorphic to $\overline{M}/M$ by the proof of Theorem 3.19, this finishes the proof. $\square$

COROLLARY 3.21. *Let $\mathbb{P}(w)$ be a four-dimensional weighted projective space and choose two different BH-type polynomials W and W', that is we have $\langle \mathcal{J}(W) \rangle = \langle \mathcal{J}(W') \rangle$. Then the following holds:*

(1) *the respective threefolds X and X' in $\mathbb{P}(w)$ and their resolutions will be polynomial deformations of each other.*
(2) *If we choose the same automorphism group $G \subset Aut(W) \cap Aut(W')$ simultaneously for W and W', then $\widetilde{X/G}$ and $\widetilde{X'/G}$ are deformations of each other.*

PROOF. Both statements follow immediately from Theorem 3.19, but are not clear within the Berglund-Hübsch framework. $\square$

REMARK 3.22. Let us stress the importance of Theorem 3.19 by some more observations. We have seen that the choice of a BH pair (W, G) *is* the choice of an almost reflexive simplex Θ, as $\mathbb{P}_{\Sigma_\Theta} \cong \mathbb{P}(w)/\overline{G}$, plus the choice of an almost reflexive simplex Ξ, the span of the points in the Newton polytope of $\mathbb{P}_{\Sigma_\Theta}$ corresponding to the monomials in W. On a combinatorial level the dual pair (W^*, G^*) is obtained from *exchanging the roles of* Θ *and* Ξ in this situation. By this we mean that the space $\mathbb{P}(w)/\overline{G}^*$ is isomorphic to $\mathbb{P}_{\Sigma_\Xi}$ and the polytope Θ we started with can be interpreted as the convex hull of points in the Newton polytope of $\mathbb{P}_{\Sigma_\Xi}$ corresponding to the monomials in W^*. Moreover, if we for a moment forget all data introduced before and start with an almost reflexive simplex Θ, we can freely choose *any* almost reflexive simplex Ξ in Θ^* and will always end up with dual Berglund-Hübsch pairs. Furthermore, from the proof of Theorem 3.19 and Lemma 2.38 we can deduce that $\overline{G} \cong \overline{N}/N$ and $\overline{G}^* \cong \overline{M}/M$. $\square$

EXAMPLE 3.23. With the new insights we have gained, we will now have a closer look at Example 3.5. Although it only deals with three-dimensional polytopes, we can still see all the features of the construction. Recall that in this example we considered the $\mathbb{Q}$-Gorenstein weighted projective spaces $\mathbb{P}(w) = \mathbb{P}(1, 1, 2, 3) = \mathrm{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3])$ and

$$\mathbb{P}(w^*) = \mathbb{P}(1, 1, 3, 2) = \mathrm{Proj}(\mathbb{C}[y_0, y_1, y_2, y_3])$$

with BH-type polynomials

$$\begin{aligned} W &= W(x_0, x_1, x_2, x_3) = x_0^7 + x_1^5 x_2 + x_2^2 x_3 + x_3^2 x_1, \\ W^* &= W^*(y_0, y_1, y_2, y_3) = y_0^7 + y_1^5 y_3 + y_1 y_2^2 + y_2 y_3^2, \end{aligned}$$

but we did not choose a group G. As we have seen this choice boils down to the choice of dual lattices with an almost reflexive polytopes in each lattice. Let $\overline{N} = \mathbb{Z}^3$ with dual lattice $\overline{M}$ and consider the polytope $\Theta \subset N_{\mathbb{Q}}$ spanned by $v_0 := (1, 2, 3)$, $v_1 := (-1, 0, 0)$, $v_2 := (0, -1, 0)$ and $v_3 := (0, 0, -1)$. These four vectors have a single relation, namely $v_0 + v_1 + 2 \cdot v_2 + 3 \cdot v_3 = 0$. As furthermore $\langle v_0, v_1, v_2, v_3 \rangle_{\mathbb{Z}} = \mathbb{Z}^3$ we know that the toric variety $\mathbb{P}_{\Sigma_\Theta}$ associated to the fan of cones over faces of Θ is the $\mathbb{Q}$-Gorenstein $\mathbb{Q}$-Fano weighted projective space $\mathbb{P}(w)$. In other words, by this choice of lattice and fan we have chosen $\overline{G}$ to be trivial and we can already conclude that $\overline{G}^*$ is non-trivial. Equivalently one sees that $N \cong \mathbb{Z}^4$ and $G = \mathbb{Z} \cdot (1, 1, 2, 3)$, so the fan Σ_Θ *is* the fan Σ_w from Definition 2.26.

Now, the fact that we can choose a BH-type polynomial W tells us that Θ is almost reflexive, for otherwise $X := V_{\mathbb{P}(w)}(W)$ would not be a Calabi-Yau by Theorem 2.25. Let us study the Newton polytope of $\mathbb{P}(w)$ in order to identify the points that correspond to the monomials in W. Note that we have already considered the exact same situation in the running example in Chapter 2. The Newton polytope $\Xi_{-K_{\mathbb{P}(w)}}$ is depicted in Figure 2.2. We have seen that the integral points of the Newton polytope span a lattice polytope with vertices $(1, 1, 1)$, $(-6, 1, 1)$, $(1, 1, -1)$, $(0, 1, -1)$, $(1, -1, 0)$, $(1, -2, 1)$ and $(0, -2, 1)$. We can identify these seven points with the monomials x_1^7, x_0^7, $x_1 x_3^2$, $x_0 x_3^2$, $x_2^2 x_3$, $x_1 x_2^2$ and $x_0 x_2^2$ in this order. Thus we see that the monomials x_0^7, $x_1^5 x_2$, $x_2^3 x_3$ and $x_1 x_3^2$ that W consists of correspond to the simplex Ξ spanned by $v_0^* := (-6, 1, 1)$, $v_1^* := (1, 0, 1)$, $v_2^* := (1, -1, 0)$ and $v_3^* := (1, -1, 1)$. These four vectors again have exctly one relation, namely

$$v_0^* + v_1^* + 3 \cdot v_2^* + 2 \cdot v_3^* = 0.$$

However, the span $\langle v_0^*, v_1^*, v_2^*, v_2^* \rangle_{\mathbb{Z}}$ does not give all of $\mathbb{Z}^3$, but only a three-dimensional sublattice $\overline{M}'$ of $\overline{M}$ of some index. By Lemma 2.38 we therefore know that the fan of cones over faces of $\Xi := \operatorname{conv}(v_0^*, v_1^*, v_2^*, v_3^*)$

is $\mathbb{P}(w^*)/(\overline{M}/\overline{M}')$ and we define

$$X^* := V_{\mathbb{P}(w^*)}(W^*)/(\overline{M}/\overline{M}').$$

We will not explicitly compute $\overline{G}^*$ here, as it is rather large and the precise structure is not needed to proceed. Note however that $\overline{G}^*$ is indeed just the quotient $\overline{M}/\overline{M}'$.

Moreover, it can be checked by hand that Ξ is an almost reflexive polytope. The integral points of the polar Ξ^* span a reflexive polytope with vertices $(-1,0,0)$, $(0,-1,0)$, $(0,0,-1)$, $(1,2,3)$, $(0,1,1)$ and $(0,1,2)$ and we know from Example 2.13 that this is the reflexive polytope $\overline{\Theta}$ associated to the almost reflexive polytope Θ we started with. Furthermore, the vectors $(-1,0,0)$, $(0,-1,0)$, $(0,0,-1)$, $(1,2,3)$ that span Θ can be identified with the monomials in W^*. Thus, we see that $X/\overline{G}$ and $X^*/\overline{G}^*$ are special hypersurfaces in quotients of weighted projective spaces associated to almost reflexive polytopes $\Theta \subset \overline{N}$ and $\Xi \subset \overline{M}$. The associated reflexive polytopes $\overline{\Theta}$ and $\overline{\Xi}$ define toric resolutions as discussed and polynomial deformations of the pulled back hypersurfaces form a Batyrev mirror. $\square$

3.3. Mirrors for Borcea-Voisin manifolds

As an application of Theorem 2.25 and the realization of the Berglund-Hübsch approach in the context of Batyrev mirror symmetry we will now construct mirror partners for so called *generalized Borcea-Voisin threefolds*. To do so, we will first give a short survey of the classical construction and its generalization with an overview of the literature on this topic.

3.3.1. Borcea-Voisin threefolds revisited. Let us start with a brief discussion of Borcea-Voisin Calabi-Yau threefolds and their properties. Let X be a K3 surface and let σ be a non-symplectic involution, that is a holomorphic involution such that the induced action on cohomology σ^* acts on a generator ω of $H^{2,0}(X)$ as $\sigma^*(\omega) = -\omega$. This determines the fixed lattice

$$S^\sigma := \{x \in H^2(X,\mathbb{Z}) | \sigma^*(x) = x\} \subset H^2(X,\mathbb{Z}) = \mathrm{H}^3 \oplus (-\mathrm{E}_8)^2$$

of the K3-lattice with intersection form denoted by $\langle \_, \_ \rangle$. Here H denotes the hyperbolic plane and E_8 the root lattice corresponding to the system E_8. Moreover, denote by $r = \mathrm{rank}(S^\sigma)$ the *rank of the fixed lattice*. By

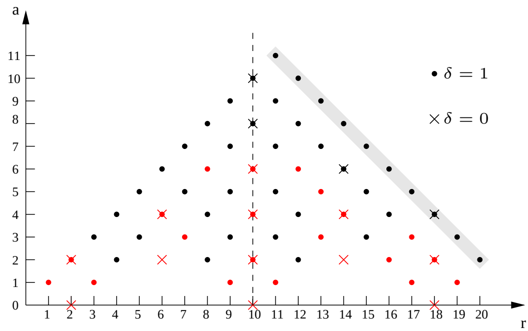

FIGURE 3.1. The complete list of triples of discrete data determining all deformation classes of $K3$ surfaces with non-symplectic involution.

abuse of notation we denote the dual pairing on the dual lattice $(S^\sigma)^* := \mathrm{Hom}_{\mathbb{Z}}(S^\sigma, \mathbb{Z})$ by $\langle\_,\_\rangle$ as well. It is known that the the *discriminant group* $(S^\sigma)^*/S^\sigma$ of such a pair (X, σ) is 2-elementary, that is $(S^\sigma)^*/S^\sigma = (\mathbb{Z}/2\mathbb{Z})^a$ for some a, see [Ni]. Furthermore, define the number δ to be 0 if for any $x \in (S^\sigma)^*$ we have $\langle x, x\rangle \in \mathbb{Z}$ and 1 otherwise. With this notation we can state the following classical theorem, whose proof large parts of [Ni] are devoted to.

THEOREM 3.24. [Ni]. *The pair (X, σ) depends up to deformation only on the triple of integers (r, a, δ). Moreover, for given (r, a, δ) there is a connected $(20 - r)$-parameter family of K3 surfaces with non-symplectic involution (X, σ) of that type.* □

Denote by $X^\sigma \subset X$ the set of fixed points of the involution. Then we know that $X^\sigma \neq \emptyset$, except for $(r, a, \delta) = (10, 10, 0)$, where the involution has no fixed points and X/σ is an Enriques surface and $(r, a, \delta) = (10, 8, 0)$, where X^σ is the union of two elliptic curves. Moreover, X^σ has the following structure:

$$X^\sigma = C_g \cup R_1 \cup \ldots \cup R_k.$$

Here C_g is a curve of genus $g = (22 - r - a)/2$ and the R_i are rational curves with $k = (r - a)/2$. See [Borc, §3] for more on this.

REMARK 3.25. In Figure 3.1 we depicted all possible triples (r, a, δ) of Nikulin's classification. Observe that if we leave out the points on the shaded $(r + a = 22)$-line and the point $(14, 6, 0)$, then the set of points becomes symmetric with respect to the dotted $(r = 10)$-axis. Moreover, one can easily check that (r, a, δ) and $(20 - r, a, \delta)$ exchange the numbers $k + 1$ and g. □

Now let E be an elliptic curve with involution ι and let $(X(r, a, \delta), \sigma)$ be a $K3$ with discrete data (r, a, δ) and non-symplectic involution σ. Then there is a crepant resolution of singularities

$$Y := Y(r, a, \delta) \mapsto (E \times X(r, a, \delta))/\iota \times \sigma$$

which is a Calabi-Yau threefold called *Borcea-Voisin threefold*, as they were first studied independently by Borcea [Borc] and Voisin [Vo]. The Hodge numbers of these threefolds are

$$\begin{aligned} h^{1,1}(Y) &= 5 + 3r - 2a = 1 + r + 4(k + 1) \\ h^{2,1}(Y) &= 65 - 3r - 2a = 1 + (20 - r) + 4g. \end{aligned}$$

as shown in [Borc, Section 4]. From this description we see that $Y(r, a, \delta)$ and $Y(20 - r, a, \delta)$ form a topological mirror pair if $r + a \neq 22$ and $(r, a, \delta) \neq (14, 6, 0)$. In [Borc] Borcea remarks that some of the $K3$ surfaces X and explicit involutions σ can be realized as singular hypersurfaces in weighted projective spaces. Figure 3.1 displays all triples (r, a, δ) for which this is possible in red, see [Borc, Section 3] for details. Whenever this is true, there is also a singular model for $Y(r, a, \delta)$ in a four-dimensional weighted projective space. We will generalize this construction in the next section and will therefore not further comment on it here.

REMARK 3.26. Although there is an almost perfect symmetry for 'red' triples (r, a, δ) and $(20 - r, a, \delta)$ in Figure 3.1 as well, Borcea correctly remarks that for $Y(r, a, \delta) \subset \mathbb{P}(w)$ there is no threefold $Y(20 - r, a, \delta) \subset \mathbb{P}(w)$ related to $Y(r, a, \delta)$ by Batyrev's mirror construction unless they are both Fermat type hypersurfaces. This observation is clear in light of Corollary 3.20. □

3.3.2. The generalized Borcea-Voisin construction and mirror symmetry. The construction just presented has the following natural generalization. Let X be a $K3$ surface, this time equipped with a *non-symplectic* *automorphism σ of order p*, that is an automorphism of order p acting as multiplication by ξ_p on a generator ω of $H^{2,0}(X,\mathbb{Z})$, where ξ_p denotes a primitive p-th root of unity. Moreover, let E be an elliptic curve with automorphism ι of order p . Then from [Di, Section 5] we know that there is a Calabi-Yau resolution

$$Y \to (X \times E)/\sigma \times \iota$$

we will refer to as *generalized Borcea-Voisin threefold*. Of course automorphisms of elliptic curves only exist for $p = 2, 3, 4, 6$, which restricts this construction quite heavily. On the $K3$ side, non-symplectic automorphisms of prime order p are completely classified in [ArSaTa]. For non-prime p this is an active field of research pursued by Artebani, Boissiére and Sarti. Thus, the only case we can fully treat here is $p = 3$, to which the paper [ArSa] is devoted. Once the cases $p = 4, 6$ are fully classified, the methods presented here can be used to find mirrors for Borcea-Voision threefolds in these cases, too.

As before, the most important part is to know the discrete invariants of the $K3$ surface and the $K3$ lattice. Recall that the *Picard lattice* S_X of X is given by

$$S_X := \{x \in H^2(X,\mathbb{Z}) | \langle x, \omega\rangle = 0\}$$

and the *transcendental lattice* $T_X := S_X^{\perp}$ by the orthogonal complement with respect to the intersection pairing. The fixed lattice of σ is again denoted by S^σ. Let $T^\sigma := (S^\sigma)^\perp$, then by [ArSaTa, Theorem 2.1] we have $S^\sigma \subset S_X$ and $T_X \subset T^\sigma$. Using the fact that the transcendental lattice has even rank by [ArSa, Lemma 1.3], for $p = 3$ it is often handier to work with m defined by rank$(T^\sigma) =: 2m$ instead of $r = 22 - 2m$. With this notation we can state the following analogon of the structure theorem for $p = 2$.

THEOREM 3.27. *Let X be a $K3$ surface with non-symplectic automorphism σ of order $p = 3$ and discriminant group of rank a. Then (X,σ) is up to deformation determined by (m,a). Furthermore, its fixed locus X^σ has the following form:*

$$X^\sigma = C_g \cup R_1 \cup \ldots \cup R_k \cup \{p_1, \ldots, p_n\}$$

where R_i is a smooth rational curve, p_i is an isolated point and C_g a curve of genus g. Moreover, the values of n, k and g are explicitly given by

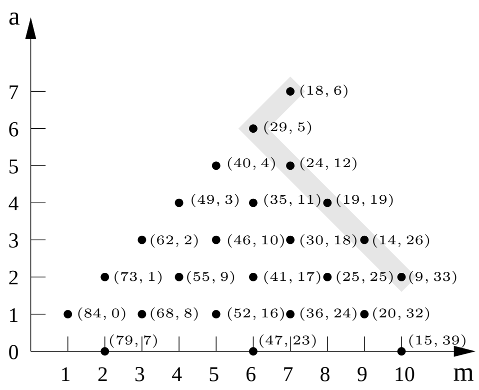

FIGURE 3.2. The complete list of pairs (m, a) parameterizing deformation classes of $K3$ surfaces with non-symplectic automorphism of order 3 and discrete invariants (m, a).

$n = 10 - m, \quad k = 6 - (m + a)/2, \quad g = (m - a)/2$. *The full list of pairs (m, a) is depicted in Figure 3.2.*

PROOF. See Theorem 0.1, Theorem 1.1 and Theorem 9.1 in [ArSaTa]. □

LEMMA 3.28. *Let X be a K3 surface with order 3 non-symplectic automorphism σ and let S^σ have discrete invariants (m, a). Furthermore let E be the elliptic curve with order 3 automorphism ι. Then the Hodge numbers of the generalized Borcea-Voisin threefold $Y \to (X \times E)/\sigma \times \iota$ are given by*

$$h^{1,1}(Y) = 7 + 4r - 3a = 3 + 6k + 5n,$$
$$h^{1,2}(Y) = 43 - 2r - 3a = 27 + 6k - 7n.$$

PROOF. See [Di, Section 7.1.1] for a computation using Chen-Ruan orbifold cohomology. There is a minor mistake in the reference, where it is stated that $h^{1,2}(Y) = 38 - 2r - 3a$, which sometimes gives negative values. We recomputed this number using the techniques of Jimmy Dillies, who confirmed the error. □

REMARK 3.29. 1) Figure 3.2 shows all possible pairs (m, a) along with the respective Hodge numbers $(h^{1,1}(Y(m, a)), h^{1,2}(Y(m, a)))$ of the respective generalized Borcea-Voisin threefold. Note that there is again a symmetry about the $(m = 5)$-axis for most of the $K3$ surfaces with discrete invariants as depicted. However, unlike for involutions, among the Calabi-Yau threefolds $Y(m, a)$ we get from the generalized Borcea-Voisin construction, except for the two cases where $h^{1,1} = h^{1,2}$, namely the pairs $(19, 19)$ and $(25, 25)$, there is not a single pair $(h^{1,1}, h^{1,2})$ that mirrors another pair from the list, see Figure 3.2.

2) As far as we know Cynk and Hulek in [CyHu] were the first to consider this generalized Borcea-Voisin construction. Moreover, Rohde in [Roh] shows that the complex moduli spaces of the seven examples with Hodge-numbers $(18 + 11 \cdot l, 6 - l)$ in Figure 3.2 do not have a point of maximal unipotent monodromy. In [GavGe] the example $(73, 1)$ is studied in detail. Note that we do *not* get mirrors for any of these examples in the following theorem. □

THEOREM 3.30. *Let $X := X(m, a)$ be a $K3$ surface with non-symplectic automorphism σ of order $p = 3$ and discrete data (m, a) as in Table 1. Then there is a topological mirror for the generalized Borcea-Voisin threefold $Y := Y(m, a)$ associated to X, given by a hypersurface in a toric variety.*

PROOF. Let $\mathbb{P}(w) := \mathbb{P}(w_0, w_1, w_2, w_3)$ be well-formed with coordinates $x_0, \ldots, x_3$ such that $\sum_i w_i = 3 \cdot w_0$. By this last assumption we know that there is an anti-canonical section of the form $x_0^3 = f(x_1, x_2, x_3)$. Then the zero set $X \subset \mathbb{P}(w)$ of this section is a $K3$ surface and as non-symplectic automorphism σ we choose the one acting as $\sigma(x_0) := \xi_3 x_0$ on x_0 and trivially on x_1, x_2 and x_3. Yonemura [Yo, Table 2.2 & Table 4.6] gives a complete list of all 95 spaces $\mathbb{P}(w_0, w_1, w_2, w_3)$ which have anti-canonical $K3$ sections together with an explicit choice of such a section. In a slightly different context Reid [Re2, Section 4.5] was the first to discover this list. We now simply extract every polynomial of the form $x_0^3 = f(x_1, x_2, x_3)$ from the list. In order to find the pair (m, a) from this it is enough to determine the values k and g.

To determine g note that the possibly singular curve

$$C := \{f(x_1, x_2, x_3) = 0\} \subset \mathbb{P}(w_1, w_2, w_3)$$

is the fixed locus of the automorphism σ. By [Yo, Theorem 3.1] there exists a minimal resolution π of X, which also resolves the singularities of C. As by [Yo] X has only A_l-singularities and π is in fact a blow-up we get

$$\pi^* C = C_g \cup R_1 \cup \ldots R_k,$$

where R_k are rational curves and C_g is the unique smooth curve from Theorem 3.27 whose genus g, by [Ia, Theorem 12.2], is given by

$$g = \frac{1}{2}\Big(\frac{d^2}{w_1 w_2 w_3} - d\sum_{i>j} \frac{gcd(w_i, w_j)}{w_i w_j} + \sum_{i=1}^{2} \frac{gcd(w_i, d)}{w_i} - 1\Big).$$

Here $d := \sum_{i=1}^{4} w_i$ and $gcd(\_, \_)$ denotes the greatest common divisor of two integers.

The number k of rational components can be determined by computing which A_l singularity of X, as shown in the list [Yo, Table 4.6], lies on C, as each such singularity contributes l rational curves in the resolution.

Thus from the relations $k = 6 - (m + a)/2$, and $g = (m - a)/2$ we get the pair (m, a). In Table 1 we list, along with the respective number in Yonemura's list and the polynomial given there and all possible pairs (m, a). Note that in the table we keep the notation of Yonemura [Yo, Table 2.2], that is the polynomials have coordinate functions x, y, z, w. The coordinate x_0 from above corresponds to the underlined coordinate in each column of the table.

In order to proceed, note that there are precisely three Gorenstein weighted projective surfaces, namely $\mathbb{P}(1, 1, 1)$, $\mathbb{P}(1, 1, 2)$ and $\mathbb{P}(2, 1, 3)$. Denote the i-th canonical coordinate in each of these cases by y_i. Among these surfaces there are precisely two Fermat-type equations that yield smooth elliptic curves E with order three automorphism ι, namely

$$\{y_0^3 = y_1^3 + y_2^3\} \subset \mathbb{P}(1, 1, 1) \quad \text{and} \quad \{y_0^3 = y_1^2 + y_2^6\} \subset \mathbb{P}(2, 1, 3).$$

In each case we can now construct a rational map, similar to those found in [Borc, Section 5]. However, it suffices to study the first case here. The rational map

$$\begin{aligned} &\mathbb{P}(w_0, w_1, w_2, w_3) \times \mathbb{P}(1,1,1) \\ &\qquad\qquad \to \mathbb{P}(w_0, w_0, w_1, w_2, w_3) =: \mathbb{P}(w_0, w) \\ &\Big((x_0 : x_1 : x_2 : x_3), (y_0 : y_1 : y_2)\Big) \\ &\qquad\qquad \mapsto \left(y_1 \cdot \frac{x_0}{y_0} : y_2 \cdot \frac{x_0}{y_0} : x_1 : x_2 : x_3\right) \end{aligned}$$

restricts to a map on $X \times E$ that maps generically $3:1$ to the hypersurface Y' given by

$$z_0^3 + z_1^3 = f(z_2, z_3, z_4), \tag{3.2}$$

where $z_0, \ldots, z_4$ denote the coordinates on $\mathbb{P}(w_0, w)$. From Equation (3.2) it follows that Y' is a possibly singular Calabi-Yau hypersurface in $\mathbb{P}(w_0, w)$ and thus a singular model for the Borcea-Voisin Calabi-Yau $Y(m, a)$. Note that Equation (3.2) is of Berglund-Hübsch type if and only if f is. If f is of BH-type, we can of course just apply the Berglund-Hübsch construction. Otherwise note that we can deform f without changing r and g, which means that we can always assume that $x_0^3 + f(x_1, x_2, x_3)$ is $\Xi_{-K_{\mathbb{P}(w)}}$-regular, where $\Xi_{-K_{\mathbb{P}(w)}}$ is the anti-canonical polytope. Therefore we immediately get that Equation (3.2) gives a $\Xi_{-K_{\mathbb{P}(w_0,w)}}$-regular hypersurface, which means that we can apply Batyrev's mirror symmetry construction. $\square$

EXAMPLE 3.31. To illustrate the proof of the theorem, we will compute one explicit example, namely the first in Table 1. Consider the zero set X of the Fermat type equation $x_0^3 = x_1^4 + x_2^4 + x_3^6$ of degree $d = 12$ in the weighted projective space $\mathbb{P}(4, 3, 3, 2)$ with charge $q = (3, 4, 4, 6)$. We know that the smooth curve C_g of degree $d = 12$ defined by $\{x_1^4 + x_2^4 + x_3^6 = 0\}$ in $\mathbb{P}(3, 3, 2)$ has genus

$$g = \frac{1}{2}\Big(\frac{12^2}{3 \cdot 3 \cdot 2} - 12\Big(\frac{3}{3 \cdot 3} + \frac{1}{6} + \frac{1}{6}\Big) + \frac{3}{3} + \frac{3}{3} + \frac{2}{2} - 1\Big) = 1.$$

Furthermore, checking Table 4.6 in [Yo] shows that X has three A_1 and four A_2 singularities. However, the fix locus of the automorphism $\sigma : X \to X$

defined by $\sigma(x_0) = \xi_3 \cdot x_0$ is just the singular curve C in defined by $\left\{x_1^4 + x_2^4 + x_3^6 = 0\right\}$ in $\mathbb{P}(4,3,3,2)$ and it can be checked that this curve has an A_2-singularity at $(1,0,0,0)$. Blowing-up via the minimal resolution π in [Yo] therefore yields $\pi^*(C) = C_g \cup R_1 \cup R_2$. Thus $k = 2$ and from the formulas $k = 6 - (m+a)/2$, and $g = (m-a)/2$ we get $m = 5$ and $a = 3$. Moreover, let E be the Fermat hypersurface $\left\{y_0^3 + y_1^3 + y_2^3 = 0\right\}$ in $\mathbb{P}(1,1,1)$. Then the image of $X \times E$ via the rational map $\mathbb{P}(4,3,3,2) \times \mathbb{P}(1,1,1) \to \mathbb{P}(4,4,3,3,2)$ from Theorem 3.30 defines a singular model Y' of the generalized Borcea-Voisin threefold as the zero set of

$$z_0^3 + z_1^3 = z_2^4 + z_3^4 + z_4^6, \tag{3.3}$$

where the z_i denote the canonical coordinates on $\mathbb{P}(4,4,3,3,2)$. Note that $\mathbb{P}(4,4,3,3,2)$ is in fact a Gorenstein toric variety. Moreover, since (3.3) is of Fermat type, by Corollary 3.20 the Berglund-Hübsch and the Batyrev construction give the same mirror. □

TABLE 1. Data set for $K3$ surfaces with non-symplectic automorphism of order 3 which are hypersurfaces in some $\mathbb{P}(w)$.

No. in Yonemura	Charges	Polynomial	m	a
2	$(\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6})$	$\underline{x}^3 + y^4 + z^4 + w^6$	5	3
3	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$	$\underline{x}^3 + y^3 + z^6 + w^6$	1	2
4	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{12})$	$\underline{x}^3 + y^3 + z^4 + w^{12}$	9	3
10	$(\frac{1}{2}, \frac{1}{3}, \frac{1}{12}, \frac{1}{12})$	$x^2 + \underline{y}^3 + z^{12} + w^{12}$	10	0
11	$(\frac{1}{2}, \frac{1}{3}, \frac{1}{10}, \frac{1}{15})$	$x^2 + \underline{y}^3 + z^{10} + w^{15}$	5	1
12	$(\frac{1}{2}, \frac{1}{3}, \frac{1}{9}, \frac{1}{18})$	$x^2 + \underline{y}^3 + z^9 + w^{18}$	10	2
13	$(\frac{1}{2}, \frac{1}{3}, \frac{1}{8}, \frac{1}{24})$	$x^2 + \underline{y}^3 + z^8 + w^{24}$	6	0
14	$(\frac{1}{2}, \frac{1}{3}, \frac{1}{7}, \frac{1}{42})$	$x^2 + \underline{y}^3 + z^7 + w^{42}$	6	0
15	$(\frac{1}{3}, \frac{4}{15}, \frac{1}{5}, \frac{1}{5})$	$\underline{x}^3 + y^3 z + y^3 w + z^5 - w^5$	6	4
16	$(\frac{1}{3}, \frac{7}{24}, \frac{1}{4}, \frac{1}{8})$	$\underline{x}^3 + y^3 w + z^4 + w^8$	4	2
20	$(\frac{3}{8}, \frac{1}{3}, \frac{1}{4}, \frac{1}{24})$	$x^2 z + x^2 w^6 + \underline{y}^3 + z^4 + w^{24}$	5	1
22	$(\frac{2}{5}, \frac{1}{3}, \frac{1}{5}, \frac{1}{15})$	$x^2 z + x^2 w^3 + \underline{y}^3 + z^5 - w^{15}$	5	1
24	$(\frac{5}{12}, \frac{1}{3}, \frac{1}{6}, \frac{1}{12})$	$x^2 z + x^2 w^2 + \underline{y}^3 + z^6 + w^{12}$	9	3
25	$(\frac{4}{9}, \frac{1}{3}, \frac{1}{9}, \frac{1}{9})$	$x^2 z + x^2 w + \underline{y}^3 + z^9 - w^9$	10	2
28	$(\frac{10}{21}, \frac{1}{3}, \frac{1}{7}, \frac{1}{21})$	$x^2 w + \underline{y}^3 + z^7 + w^{21}$	9	3
46	$(\frac{1}{2}, \frac{1}{3}, \frac{1}{11}, \frac{5}{66})$	$x^2 + \underline{y}^3 + z^{11} + z w^{12}$	4	2
48	$(\frac{1}{2}, \frac{1}{3}, \frac{5}{48}, \frac{1}{16})$	$x^2 + \underline{y}^3 + z^6 w + w^{16}$	5	1
49	$(\frac{1}{2}, \frac{1}{2}, \frac{5}{42}, \frac{1}{21})$	$x^2 + \underline{y}^3 + z^8 w + w^{21}$	8	4
51	$(\frac{1}{2}, \frac{1}{3}, \frac{5}{36}, \frac{1}{36})$	$x^2 + \underline{y}^3 + z^7 w + w^{36}$	9	3
54	$(\frac{1}{3}, \frac{2}{7}, \frac{5}{21}, \frac{1}{7})$	$\underline{x}^3 + y^3 w + y z^3 + z^3 w^2 - w^7$	5	3
59	$(\frac{8}{21}, \frac{1}{3}, \frac{5}{21}, \frac{1}{21})$	$x^2 z + x^2 w^5 \underline{y}^3 + z^4 w - w^{21}$	8	4
65	$(\frac{14}{33}, \frac{1}{3}, \frac{5}{33}, \frac{1}{11})$	$x^2 z + \underline{y}^3 + z^6 w + w^{11}$	7	5

Chapter 4

Toric degenerations and Landau-Ginzburg models

Mirrors of Fano varieties are suggested to be so called *Landau-Ginzburg models* or just *LG-models*. For us such models are non-compact algebraic varieties with a holomorphic function, referred to as *superpotential*. See [ChOh], [FOOO1], [Gi], [HoVa] for some approaches and results on this topic. One of the most classical result found in the literature is the so called *Hori-Vafa mirror*, see [HoVa], of a toric variety. Let $X := \mathbb{P}_\Sigma$ be an n-dimensional smooth projective toric variety associated to a complete fan Σ in $N_\mathbb{R} \cong \mathbb{R}^n$ and choose an ample toric divisor $D := \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$. Then the so called *Hori-Vafa mirror* [HoVa] to (X, D) is the n-dimensional torus $(\mathbb{C}^*)^n$ together with the superpotential

$$W : (\mathbb{C}^*)^n \to \mathbb{C}, \quad W(x_1, \ldots, x_n) = \sum_{\rho\in\Sigma(1)} e^{a_\rho} x^{n_\rho}, \tag{4.1}$$

where x_i are coordinates on the torus, n_ρ denotes the ray generator of ρ and we adopt the usual multi-index notation for the monomial x^{n_ρ}. The choice of D in this case is equivalent to the choice of a Kähler parameter for X, which influences the complex parameters of the Landau-Ginzburg model as stated. One often works with the anti-canonical polarization $D = -K_X = \sum_{\rho\in\Sigma(1)} D_\rho$, in which case the superpotential W is up to the multiple by a scalar the sum of all monomials corresponding to toric prime divisors. The goal of this chapter is to set up all foundations to describe Landau-Ginzburg models within the Gross-Siebert program as laid out in [GrSi1] and [GrSi3]. After a quick reminder on the basic notions of this program , we will construct LG-models via deformations of a non-compact union of toric varieties and define the superpotential by extension from the central fiber using the technique of broken lines, introduced by Gross in [Gr2].

4.1. Short introduction to the Gross-Siebert program

The Gross-Siebert program developed in [GrSi1], [GrSi2] and [GrSi3] is a very general algebro-geometric approach to mirror symmetry consistent with the SYZ philosophy [StYaZa]. One of its central notion is that of a *toric degeneration of Calabi-Yau pairs*, see Definition 4.2 below. The central fiber of such a degeneration has a tropical model from which one can compute a mirror dual model by discrete methods. The mirror toric degeneration is obtained from this model by an explicit algorithm using log geometry. The program for instance reproduces the work of Batyrev and Borisov [BaBo1, BaBo2] for Calabi-Yau complete intersections in toric varieties, see [Gr1], and is expected to hold for a much larger class of Calabi-Yau varieties.

4.1.1. Toric degenerations and tropical affine manifolds. We will remind the reader of some of the central notions on the complex geometric side of the Gross-Siebert program and then shortly recall how to obtain affine geometric data from it. We roughly follow [GrSi3, §1], where more fundamentals can be found. The reader without previous knowledge of the program is referred to [GrSi4] for an introduction.

DEFINITION 4.1. ([GrSi3, Definition 1.6]) A *totally degenerate CY-pair* is a reduced variety X with a reduced divisor $D \subset X$ subject to the following conditions. Let $\nu : \widetilde{X} \to X$ be the normalization and $C \subseteq \widetilde{X}$ its conductor locus. Then we have:

(1) $\widetilde{X}$ is a disjoint union of toric varieties whose fans have convex support, i.e. *algebraically convex* toric varieties.
(2) C is reduced and $[C] + \nu^*[D]$ is the sum of all toric prime divisors.

(3) $\nu|_C : C \to \nu(C)$ is unramified and generically two-to-one.
(4) The following diagram is cartesian and co-cartesian

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\nu"'] & \widetilde{X} \arrow[d, "\nu"] \\ \nu(C) \arrow[r] & X. \end{tikzcd}$$

□

So morally speaking X is obtained from a set of toric varieties by pairwise identifying some of its toric prime divisors, while D is the union of all remaining such divisors. For the next definition recall the definition of a *log smooth morphism* $\pi : (\mathfrak{X}, X; D) \to (T, 0)$ as defined in [GrSi3, Definition 1.7].

DEFINITION 4.2. ([GrSi3, Definition 1.8]) Let T be the spectrum of a discrete valuation $\Bbbk$-algebra with closed point $0 \in T$. A *toric degeneration of CY-pairs over* T is a flat morphism $\pi : \mathfrak{X} \to T$ with a reduced divisor $\mathfrak{D} \subseteq \mathfrak{X}$, such that the following properties hold:

(1) $\mathfrak{X}$ is normal.
(2) The central fibre $X := \pi^{-1}(0)$ together with $D = \mathfrak{D} \cap \mathfrak{X}$ is a totally degenerate CY-pair.
(3) Away from a closed subset $\mathfrak{Z}$ of relative codimension two, not containing any toric stratum of X, the map $\pi : (\mathfrak{X}, X; D) \to (T, 0)$ is log smooth. □

Next, fix a lattice $M \cong \mathbb{Z}^n$ and denote its *integral affine transformations* by

$$\mathrm{Aff}(M) = M \rtimes GL(M),$$

which naturally acts on polyhedra in $M_{\mathbb{R}}$. Moreover, let $\Xi \subset M_{\mathbb{R}}$ be an m-dimensional lattice polytope. The lattice of *integral vector fields* along Ξ is denoted by $\Lambda_\Xi \cong \mathbb{Z}^m$. Recall that the category $\underline{\text{LPoly}}$, see [GrSi3, p. 9], has lattice polyhedra as objects and the identity morphism and integral affine isomorphisms onto faces as morphisms.

DEFINITION 4.3. For a category $\mathscr{P}$ with at most one morphism $\tau \to \sigma$ for all $\tau, \sigma \in \mathscr{P}$ we call a functor

$$F : \mathscr{P} \to \underline{LPoly}$$

such that for all $\Xi \in F(\mathscr{P})$ and each face $\Xi' \subset \Xi$ we have $\Xi' \in F(\mathscr{P})$ an *integral polyhedral complex.* $\square$

To an integral polyhedral complex F we can associate a topological space B as follows:

$$B = \coprod_{\sigma \in \mathscr{P}} F(\sigma) / \sim .$$

Here $p \in F(\sigma)$ and $p' \in F(\sigma')$ are considered equivalent if there are $\tau \in \mathscr{P}$, $q \in F(\tau)$ and morphisms $e : \tau \to \sigma$, $e' : \tau \to \sigma'$ such that $p = F(e)(q)$ and $p' = F(e')(q)$. We will slightly abuse notation and identify elements of $\mathscr{P}$ with their images under F and call them *cells* of $\mathscr{P}$. Furthermore we will say that $\mathscr{P}$ is an integral polyhedral complex, that is is a *polyhedral decomposition* of B and refer to the k-dimensional cells of $\mathscr{P}$ as $\mathscr{P}^{[k]}$.

For the next definition recall that a *fan structure* $S_\tau : U_\tau \to \mathbb{R}^k$ along a cell $\tau \in \mathscr{P}$ is essentially a continuous map that maps the interior of τ to the origin and the open star U_τ of τ to a finite fan Σ_τ, see [GrSi3, Definition 1.1]. Two structures S_τ, S'_τ are said to be *equivalent* if they differ by an integral affine transformation of $\mathbb{R}^k$. If $\tau \subset \sigma$, then we get a fan structure along σ *induced from* S_τ from the composition

$$U_\sigma \longrightarrow U_\tau \xrightarrow{S_\tau} \mathbb{R}^k \longrightarrow \mathbb{R}^k/\text{span}(S_\tau(\text{int}(\sigma))).$$

DEFINITION 4.4. ([GrSi3, Definition 1.2]) Let $n \in \mathbb{N}_{>0}$. An n-dimensional *integral tropical manifold* $(B, \mathscr{P})$ consists of a countable polyhedral complex $\mathscr{P}$ with associated topological space B and a fan structure $S_\nu : U_\nu \to \mathbb{R}^n$ for all $\nu \in \mathscr{P}^{[0]}$ with the following properties.

(1) For each $\nu \in \mathscr{P}^{[0]}$ the support $|\Sigma_\nu|$ is convex and has a non-empty interior.
(2) If $\nu, \nu' \in \mathscr{P}^{[0]}$ are vertices of a cell $\tau \in \mathscr{P}$, then the fan structures along τ induced from S_ν and $S_{\nu'}$ are equivalent. $\square$

Hence the manifold B as above is an n-dimensional manifold with boundary. Moreover, the interior of each cell and the fan structures at each vertex give integral affine charts for B, meaning that there is a subset $\Delta \subset B$ of codimension at least two which is a locally finite union of closed submanifolds of B called the *discriminant locus* such that $B_0 := B \setminus \Delta$ is a manifold with integral affine transition functions. There are in general different possible choices of Δ and we will in each application explicitly state what Δ is. However, we will always assume that none of the vertices of B are contained in Δ and for each maximal cell σ of $\mathscr{P}$ it holds $\Delta \cap \sigma \subset \partial\sigma$. Various important properties of $(B, \mathscr{P})$, such as *positivity* ([GrSi1, Definition 1.54]), can be read off from the monodromy imposed on B by Δ, but we will not go into details here.

Let $(B, \mathscr{P})$ be an integral tropical manifold with discriminant locus Δ. Recall the definition of *integral affine functions* and *integral PL-functions* on open sets $U \subset B$. These naturally define sheaves $\mathcal{A}ff(B, \mathbb{Z})$ and $\mathcal{PL}_{\mathscr{P}}(B, \mathbb{Z})$ on B, see [GrSi1, Definition 1.43] and can be used to polarize $(B, \mathscr{P})$ as follows.

DEFINITION 4.5. ([GrSi1, Definition 1.47], [GrSi3, Remark 1.14]) A section φ of

$$\mathcal{PL}_{\mathscr{P}}(B, \mathbb{Z})/\mathcal{A}ff(B, \mathbb{Z})$$

is called a *polarization* if it defines a strictly convex linear function on the fan Σ_τ for each cell $\tau \in \mathscr{P}$. Given a polarization φ, the triple $(B, \mathscr{P}, \varphi)$ is called a *polarized integral tropical manifold*. □

Given a toric degeneration of CY-pairs $(\pi : \mathfrak{X} \to T, \mathfrak{D})$ recall that there are basically two possibilities to construct an integral tropical manifold from it. The first is obtained from *the fan picture*, which is explicitly described in [GrSi3, Example 1.10] in our context and based on [GrSi1, §4.1]. It will be denoted by $(B, \mathscr{P})$ throughout the text. In fact $\mathscr{P}$ is the *dual intersection complex* of X. For the second construction, based on *the cone picture* presented in [GrSi3, Example 1.12] and [GrSi1, §4.2], we need to polarize the central fiber $X = \pi^{-1}(O)$ by an ample line bundle $\mathcal{L}$. This yields an integral tropical manifold $(\check{B}, \check{\mathscr{P}})$, where $\check{\mathscr{P}}$ is the *intersection complex* of $(X, \mathcal{L})$ and the polarization $\mathcal{L}$ yields a polarization $\check{\varphi}$ of $(B, \mathscr{P})$.

We will explicitly discuss an example of a toric degeneration of $\mathbb{P}^2$ in cone and fan picture in Section 4.2.2 below.

4.1.2. The discrete Legendre transform and scattering diagrams. Next, we will very briefly discuss how toric degenerations and polarized integral tropical manifolds with singularities are used to describe mirror symmetry in the Gross-Siebert program. As any attempt to discuss details would result in a text much longer than appropriate for this introduction, we will focus on the main notions and concepts and refer the interested reader to [GrSi3] for details of the construction.

Let $(B, \mathscr{P}, \varphi)$ be a polarized integral tropical manifold. Then by [GrSi1, §1.4] there is a transformation called the *discrete Legendre transform* which we denote by $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$. This transformation is one of the key constructions of the Gross-Siebert program. Assume $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ is the polarized intersection complex associated to a *polarized toric degeneration of varieties with effective anti-canonical bundle* as in [GrSi1]. A central result of the program is that $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ is the dual intersection complex of a toric degeneration, if $(\check{B}, \check{\mathscr{P}})$ is *positive* [GrSi3, Definition 1.54] and *simple* [GrSi3, Definition 1.60]. An important aspect of this result is that it is *constructive*: It gives a canonical deformation of the central fibre $\check{X}_0 := \check{X}(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ constructed from $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ in form of an algorithm [GrSi3, Section 3.]. More explicitly, the theorem gives k-th order deformations

$$\check{X}_k \to \operatorname{Spec} \Bbbk[t]/(t^{k+1})$$

from *structures* $\mathscr{S}_k$ [GrSi3, Definition 2.22], which consist of codimension one polyhedral subsets of $\check{B}$ called *slabs* [GrSi3, Definition 2.17] and *walls* [GrSi3, Definition 2.20]. The central tool to construct $\mathscr{S}_{k+1}$ from $\mathscr{S}_k$ is that of *scattering diagrams* [GrSi3, Section 3.2]. Assume $\mathscr{S}_k$ is a structure that is consistent to order k [GrSi3, Definition 2.28] and let $\mathfrak{j}$ be a joint [GrSi3, Definition 2.27] of $\mathscr{S}_k$. Let

$$\mathfrak{D} = (\mathfrak{r}_i, f_\mathfrak{c})$$

be the associated *scattering diagram*, for some vertex $v \in \sigma_\mathfrak{j}$ and $\omega \in \mathscr{P}$ with $\sigma_\mathfrak{j} \subset \omega$, as explained in [GrSi3, Construction 3.4]. Recall that for a joint $\mathfrak{j}$ we denote the minimal cell of $\mathscr{P}$ containing it by $\sigma_\mathfrak{j}$ and for any vertex $v \in \sigma_\mathfrak{j}$ denote the *normal space of the joint* $\mathfrak{j}$ by $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}} := \Lambda_{v,\mathbb{R}}/\Lambda_{\mathfrak{j},\mathbb{R}}$. For an exponent m [GrSi3, Definition 2.2] propagating in direction $\overline{m} \in \Lambda_v \setminus \Lambda_\mathfrak{j}$ we wish to define the scattering of the monomial z^m, which we think of traveling along the ray $-\mathbb{R}_{\geq 0}\overline{\overline{m}}$ into the origin of $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$. In a scattering diagram monomials move along so called *trajectories*, which we will define next. Compare this to the definition of rays in [GrSi3, Definition 3.3].

DEFINITION 4.6. A *trajectory* in $\mathcal{Q}^v_{\mathfrak{j},\mathbb{R}}$ is a triple $(\mathfrak{t}, m_\mathfrak{t}, a_\mathfrak{t})$, where $m_\mathfrak{t}$ is a monomial on a maximal cell $\sigma \ni v$ with $\pm\overline{\overline{m}}_\mathfrak{t} \in \overline{\overline{\sigma}}$ and $m \in P_x$ for any $x \in \mathfrak{j} \setminus \Delta$, $\mathfrak{t} = \pm\mathbb{R}_{\geq 0}\overline{\overline{m}}$, and $a_\mathfrak{t} \in \Bbbk$. The trajectory is called *incoming* if $\mathfrak{t} = \mathbb{R}_{\geq 0}\overline{\overline{m}}$, and *outgoing* if if $\mathfrak{t} = -\mathbb{R}_{\geq 0}\overline{\overline{m}}$. By abuse of notation we often suppress $m_\mathfrak{t}$ and $a_\mathfrak{t}$ when referring to trajectories. □

The following proposition incorporating trajectories represents the necessary generalization of the central existence and uniqueness result for scattering diagrams shown in Proposition 3.9 of [GrSi3].

PROPOSITION 4.7. [CaPuSi, Proposition 3.2]. *Let $\mathfrak{D}$ be the scattering diagram defined by $\mathscr{S}_k$ for $\mathfrak{j} \in \mathrm{Joints}(\mathscr{S}_k)$, $g : \omega \to \sigma_\mathfrak{j}$ and $v \in \omega$. Let $(\mathbb{R}_{\geq 0}\overline{\overline{m}}_0, m_0, 1)$ be an incoming trajectory and $\sigma \supset \mathfrak{j}$ a maximal cell with $\overline{\overline{m}}_0 \in \overline{\overline{\sigma}}$. For $\overline{\overline{m}} \in \mathcal{Q}^v_{\mathfrak{j},\mathbb{R}} \setminus \{0\}$ denote by*

$$\theta_{\overline{\overline{m}}} : R^k_{g,\sigma'} \to R^k_{g,\sigma}$$

the ring isomorphism defined by $\mathfrak{D}$ for a path connecting $-\overline{\overline{m}}$ to $-\overline{\overline{m}}_0$, where σ' is a maximal cell with $-\overline{\overline{m}} \in \overline{\overline{\sigma'}}$.

Then there is a set of outgoing trajectories $\mathfrak{T}$ such that

$$z^{m_0} = \sum_{\mathfrak{t} \in \mathfrak{T}} \theta_{\overline{\overline{m}}_\mathfrak{t}}(a_\mathfrak{t} z^{m_\mathfrak{t}}) \tag{4.2}$$

holds in $R^k_{g,\sigma}$. Moreover, $\mathfrak{T}$ is unique if $a_\mathfrak{t} \neq 0$ for all $\mathfrak{t} \in \mathfrak{T}$ and if $m_\mathfrak{t} \neq m_{\mathfrak{t}'}$ whenever $\mathfrak{t} \neq \mathfrak{t}'$. □

4.1.3. Adjustments to the program for LG-models. In [GrSi3] boundedness of B was assumed at the following places: the consistency in codimension zero [GrSi3, Section 3.4], in the homological argument [GrSi3, Section 3.5] and in the normalization procedure [GrSi3, Section 3.6]. This was related to the question of what the right conditions on mononmials coming in from unbounded directions should be. With a better understanding of unbounded cells via the Landau-Ginzburg setup it has become clear that in fact there should be no such monomials. They may lead to obstructions and are rather treated by the superpotential. The inductive process itself does not produce such monomials. Adding the condition of no monomials incoming from unbounded directions in the definition of structures then makes the algorithm of [GrSi3] work also for unbounded B. Details will appear in a revision of [CaPuSi].

4.2. Proper superpotentials and broken lines

4.2.1. The superpotential to order zero. This section is based on joint work with Bernd Siebert, who gave the proofs for Propositions 4.8 and 4.9 below.

From this section on we work with the following convention. Let $(\check{\pi} : \check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ be a toric degeneration of Calabi-Yau pairs over the spectrum T of a discrete valuation $\Bbbk$-algebra, such that the generic fibre $(\check{\mathfrak{X}}_\eta, \check{\mathfrak{D}}_\eta)$ consists of a complete variety $\check{\mathfrak{X}}_\eta$ and a reduced effective anti-canonical divisor $\check{\mathfrak{D}}_\eta \subset \check{\mathfrak{X}}_\eta$. Assume that $(\check{\pi} : \check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ is polarized and denote by $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ the polarized intersection complex. Then our starting point to compute Landau-Ginzburg models is the discrete Legendre dual $(B, \mathscr{P}, \varphi)$ of $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$. So, morally speaking, we take a toric degeneration on the "Fano side" with cone picture $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ and fan picture $(B, \mathscr{P}, \varphi)$ and construct the Landau-Ginzburg model from the latter.

Let $(B, \mathscr{P}, \varphi)$ as just described and let $\sigma \in \mathscr{P}$ be an unbounded maximal cell. For each unbounded edge $\omega \subset \sigma$ there is a unique monomial $z^{m_\omega} \in R^0_{\mathrm{id}_\sigma,\sigma}$ with $\mathrm{ord}_\sigma(m_\omega) = 0$ and $-\overline{m_\omega}$ a primitive generator of $\Lambda_\omega \subset \Lambda_\sigma$ pointing in the unbounded direction of ω. Denote by $\mathscr{R}(\sigma)$ the set of such monomials m_ω. Note that in $\mathscr{R}(\sigma)$ parallel unbounded edges ω, ω' only contribute one exponent $m_\omega = m_{\omega'}$. Now at any point of $\partial\sigma$ the tangent vector $-\overline{m_\omega}$ points into σ. Hence

$$W^0(\sigma) := \sum_{m \in \mathscr{R}(\sigma)} z^m$$

extends to a regular function on the component $X_\sigma \subset X_0$ corresponding to σ. For bounded σ define $W^0(\sigma) = 0$. Since the restrictions of the $W^0(\sigma)$ to lower dimensional toric strata agree they define a function $W^0 \in \mathcal{O}(X_0)$. This is what we call the *superpotential to order zero*. A motivation for this definition in terms of counts of holomorphic disks will can be found in [CaPuSi, Section 5].

PROPOSITION 4.8. [CaPuSi, Proposition 2.1]. *A necessary and sufficient condition for W^0 to be proper is the following:*

(4.3) $\forall\sigma$ *and unbounded* $\omega, \omega' \subset \sigma$ *it holds* $\Lambda_\omega = \Lambda_{\omega'}$, *as subspaces of* Λ_σ. □

Thus if one is to study LG-models via our degeneration approach, then to obtain the full picture one has to impose Condition 4.3 in Proposition 4.8. On the mirror side Condition 4.3 also has a natural interpretation. To state it, recall the notion of toric degenerations of Calabi-Yau pairs $(\check{\pi} : \check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ from Definition 4.2.

PROPOSITION 4.9. [CaPuSi, Proposition 2.2]. W^0 *is proper if and only if* $\check{\mathfrak{D}} \to T$ *is a toric degeneration of Calabi-Yau varieties.* □

This result motivates the following natural definition.

DEFINITION 4.10. A toric degeneration of Calabi-Yau pairs $(\check{\pi} : \check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ with $\check{\mathfrak{D}} \to T$ a toric degeneration of Calabi-Yau varieties is called *irreducible*. □

4.2.2. A first example: $\mathbb{P}^2$. The standard method to construct the LG-mirror for $\mathbb{P}^2$ is to start from the momentum polytope

$$\Xi = \mathrm{conv}\{(-1,-1),(2,-1),(-1,2)\}$$

of $\mathbb{P}^2$ with its anti-canonical polarization. The rays of the corresponding normal fan Σ^*_Ξ associated to this polytope are generated by

$$(1,0),(0,1),(-1,-1)$$

. Calling the monomials corresponding to the first two points x and y, respectively, we obtain the usual non-proper Landau-Ginzburg model on the big torus $(\mathbb{G}_m(\Bbbk))^2$ by the function $x + y + \frac{1}{xy}$.

To obtain a proper superpotential instead, we need to achieve the dual of 4.3, that is, make the boundary of the momentum polytope flat in affine coordinates. To do this one has to trade the corners with singular points in the interior. The most simple choice is a decomposition $\check{\mathscr{P}}$ of $\check{B} = \Xi$ into three triangles with three singular points with simple monodromy, that is, conjugate to $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$, as depicted in Figure 4.1 in the upper left picture. A minimal choice of the PL-function $\check{\varphi}$ with integral slopes takes values

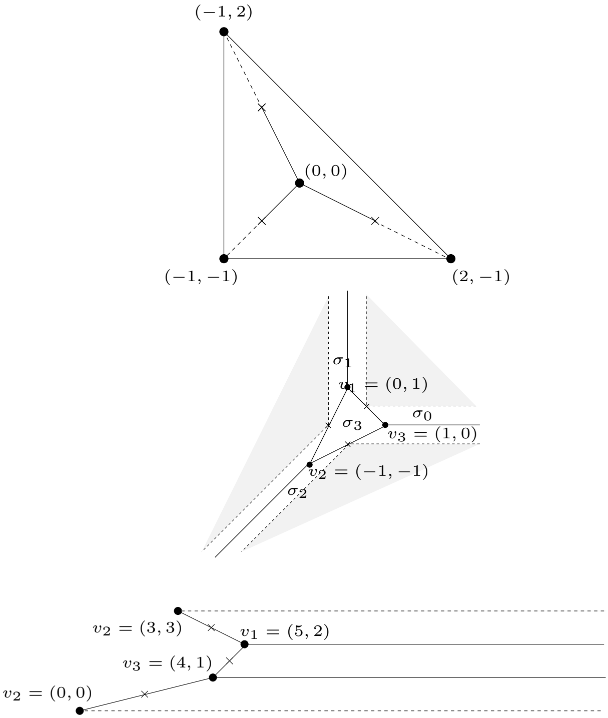

FIGURE 4.1. An intersection complex $(\check{B}, \check{\mathscr{P}})$ for $\mathbb{P}^2$ with straight boundary and its Legendre dual (B, φ) for the minimal polarization, with a chart on the complement of the shaded region and a chart showing the three parallel unbounded edges.

0 at the origin and 1 on ∂B. For this choice of $\check{\varphi}$ the Legendre dual of $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ is shown in Figure 4.1 in the upper right picture. Note that the unbounded edges are indeed parallel, so each unbounded edge comes with copies of the other two unbounded edges parallel at integral distance 1, as illustrated in the picture on the bottom in Figure 4.1.

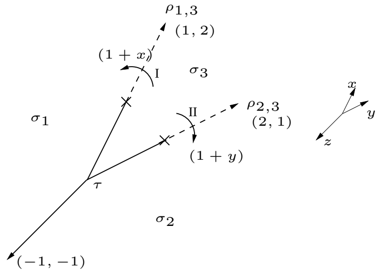

FIGURE 4.2. The lower left chart for $(B, \mathscr{P}, \varphi)$ as in Figure 4.1 showing all the identifications needed to compute the fiber product.

Next, consider the chart at v_2 of the tropical affine manifold $(B, \mathscr{P})$, as shown in Figure 4.2. The computation we carry out here can be done in the same manner for the other two charts at v_1 and v_3, despite the fact that the situation there is slightly less symmetric. We refer the reader to in [GrSi4, Section 2.3] for a first example of the kind of computation we carry out here. We use the notation as indicated in Figure 4.2. Thus we have three coordinates x, y, z pointing in directions $(-1,-1), (1,2), (2,1)$ as well as three maximal cells σ_1, σ_2 and σ_3. Crossing the slab $\rho_{1,2}$ between the first two we pick up a $(1+x)$-factor due to monodromy. We refer to this as chart II and analogously we define chart I by crossing the slab $\rho_{2,3}$ between σ_3 and σ_2. More precisely, if we denote by x_i, y_i and z_i the local coordinates on σ_i we obtain the following computations for charts I and II:

$$\begin{array}{llll} x_3 & \mapsto x_1 & x_3 & \mapsto (1+y)^3 x_2 \\ y_3 & \mapsto (1+x)^3 y_1 & y_3 & \mapsto y_2 \\ z_3 & \mapsto (1+x)^{-1} z_1 & z_3 & \mapsto (1+y)^{-1} z_2 \end{array}$$

The fiber product we are interested in is

$$R^k_{\sigma_1} \times_{\left(R^k_{\rho_{1,3},\sigma_3}\right)_{1+x}} R^k_{\sigma_3} \times_{\left(R^k_{\rho_{2,3},\sigma_3}\right)_{1+y}} R^k_{\sigma_2},$$

which we obtain by gluing along the morphisms in the following diagram.

$$\begin{tikzcd}
R^k_{\sigma_1} \arrow[d] & R^k_{\sigma_3} \arrow[d] \arrow[rr, leftrightarrow] & & R^k_{\sigma_3} \arrow[d] & R^k_{\sigma_2} \arrow[d] \\
R^k_{\rho_{1,3},\sigma_1} \arrow[r, leftrightarrow] & R^k_{\rho_{1,3},\sigma_3} \arrow[dr] & & R^k_{\rho_{2,3},\sigma_3} \arrow[dl] \arrow[r, leftrightarrow] & R^k_{\rho_{2,3},\sigma_2} \\
& & R^k_{\tau,\rho_{1,3},\rho_{2,3}} & &
\end{tikzcd}$$

Glueing coordinate functions as indicated by the charts I and II above gives rise to the following coordinates on the fiber product:

$$\begin{aligned}
X &= (x_1, x_3, (1+y_2)^3 x_2) \\
Y &= ((1+x_1)^3 y_1, y_3, y_2) \\
Z &= ((1+y_1)z_1, (1+x_3)(1+y_3)z_3, (1+x_2)z_2).
\end{aligned}$$

It can be checked that we obtain exactly one relation for them, namely $XYZ^3 = (1+X)^3 \cdot (1+Y)^3 \cdot t^3$.

Now let us describe the central fibre X_0 by means of glueing toric strata and compute the superpotential $W^0_{\mathbb{P}^2}$. The polyhedral decomposition has one bounded maximal cell σ_3 and three unbounded maximal cells $\sigma_0, \sigma_1, \sigma_2$. The bounded cell σ_3 is the momentum polytope of a toric quotient of $\mathbb{P}^2$, as the three ray generators $(-1,-1)$,$(2,-1)$ and $(-1,2)$ of its normal fan sum up to zero and span a sublattice of index six in $\mathbb{Z}^2$. Denote this quotient by X_{σ_3}. Each unbounded cell is affine isomorphic to $[0,1] \times \mathbb{R}_{\geq 0}$, the momentum polytope of $\mathbb{P}^1 \times \mathbb{A}^1 =: X_{\sigma_i}$, $i = 0,1,2$. These glue together by torically identifying pairs of $\mathbb{P}^1$'s and $\mathbb{A}^1$'s as prescribed by the polyhedral decomposition to yield the central fibre X_0. Clearly $W^0_{\mathbb{P}^2}$ vanishes identically on the compact component X_{σ_3}. Each of the unbounded components has two parallel unbounded edges, leading to the pull-back to $\mathbb{P}^1 \times \mathbb{A}^1$ of the toric coordinate function of $\mathbb{A}^1$, say z_i for the i-th copy. Thus $W^0_{\mathbb{P}^2}|_{X_{\sigma_i}} = z_i$ for $i = 0,1,2$. These functions are readily checked to be compatible with the toric gluings. So we see that $W^0_{\mathbb{P}^2}$ is in fact a compactification of the non-proper Hori-Vafa mirror $x + y + \frac{1}{xy}$.

REMARK 4.11. An interesting feature of the degeneration point of view is that the mirror construction respects the finer data related to the degeneration such as the monodromy representation of the affine structure. In particular, this poses a question of uniqueness of the Landau-Ginzburg mirror. For the anti-canonical polarization such as the chosen one in the case of $\mathbb{P}^2$, the tropical data $(\check{B}, \check{\mathscr{P}})$ is essentially unique, see Theorem 5.19 for a precise statement. For larger polarizations, thus enlarging $\check{B}$, there are certainly many more possibilities. For example, as an affine manifold with singularities one can perturb the location of the singular points transversely to the invariant directions over the rational numbers and choose an adapted integral polyhedral decomposition after appropriate rescaling. It is not clear to us if all $(\check{B}, \check{\mathscr{P}})$ leading to $\mathbb{P}^2$ can be obtained by this procedure. □

4.2.3. Broken lines. The easiest way to define the superpotential in full generality is by the method of broken lines. Broken lines haven been introduced by Mark Gross for $\dim B = 2$ in his work on mirror symmetry for $\mathbb{P}^2$ [Gr2]. We assume we are given a locally finite scattering diagram $\mathscr{S}_k$ for a polarized integral tropical manifold $(B, \mathscr{P}, \varphi)$ that is consistent to order k. The notion of broken lines is based on the transport of monomials by changing chambers of $\mathscr{S}_k$. Recall from [GrSi3], Definition 2.22, that a chamber is the closure of a connected component of $B \setminus |\mathscr{S}_k|$. This section is based on joint work with Bernd Siebert.

DEFINITION 4.12. Let $\mathfrak{u}, \mathfrak{u}'$ be neighbouring chambers of $\mathscr{S}_k$, that is, $\dim(\mathfrak{u} \cap \mathfrak{u}') = n - 1$. Let az^m be a monomial defined at all points of $\mathfrak{u} \cap \mathfrak{u}'$ and assume without loss of generality that $\overline{m}$ points from $\mathfrak{u}'$ to $\mathfrak{u}$. Let $\tau := \sigma_{\mathfrak{u}} \cap \sigma_{\mathfrak{u}'}$ and

$$\theta : R^k_{\mathrm{id}_\tau, \sigma_{\mathfrak{u}}} \to R^k_{\mathrm{id}_\tau, \sigma_{\mathfrak{u}'}}$$

be the gluing isomorphism changing chambers. Then if

$$\theta(az^m) = \sum_i a_i z^{m_i} \tag{4.4}$$

we call any summand $a_i z^{m_i}$ with $\mathrm{ord}_{\sigma_{\mathfrak{u}'}}(m_i) \le k$ a *result of transport of* az^m from $\mathfrak{u}$ to $\mathfrak{u}'$. □

Note that since the change of chamber isomorphisms commute with changing strata, the monomials $a_i z^{m_i}$ in Definition 4.12 are defined at all points of $\mathfrak{u} \cap \mathfrak{u}'$.

DEFINITION 4.13. ([Gr2, Definition 4.9].) A *broken line* for $\mathscr{S}_k$ is a proper continuous map

$$\beta : (-\infty, 0] \to B$$

with image disjoint from any joints of $\mathscr{S}_k$, along with a sequence

$$-\infty = t_0 < t_1 < \ldots < t_{r-1} \leq t_r = 0$$

for some $r \geq 0$ with $\beta(t_i) \in |\mathscr{S}_k|$, and for $i = 1, \ldots, r$ monomials $a_i z^{m_i}$ defined at all points of $\beta([t_{i-1}, t_i])$ (for $i = 1$, $\beta((-\infty, t_1])$), subject to the following conditions.

(1) $\beta|_{(t_{i-1}, t_i)}$ is a non-constant affine map with image disjoint from $|\mathscr{S}_k|$, hence contained in the interior of a unique chamber $\mathfrak{u}_i$ of $\mathscr{S}_k$, and $\beta'(t) = -\overline{m}_i$ for all $t \in (t_{i-1}, t_i)$. Moreover, if $t_r = t_{r-1}$ then $\mathfrak{u}_r \neq \mathfrak{u}_{r-1}$.
(2) $a_1 = 1$ and there exists a (necessarily unbounded) $\omega \in \mathscr{P}^{[1]}$ with $\overline{m}_1 \in \Lambda_\omega$ primitive and $\mathrm{ord}_\omega(m_1) = 0$.
(3) For each $i = 1, \ldots, r - 1$ the monomial $a_{i+1} z^{m_{i+1}}$ is a result of transport of $a_i z^{m_i}$ from $\mathfrak{u}_i$ to $\mathfrak{u}_{i+1}$ (Definition 4.12).

The *type* of β is the tuple of all $\mathfrak{u}_i$ and m_i. By abuse of notation we suppress the data t_i, a_i, m_i when talking about broken lines, but introduce the notation

$$a_\beta := a_r, \quad m_\beta := m_r.$$

For $p \in B$ the set of broken lines β with $\beta(0) = p$ is denoted $\mathfrak{B}(p)$. □

For a broken line β the endpoint $\beta(0)$ will in applications often be referred to as the *root vertex* and $\overline{m}_\beta$ is called the *root tangent vector*.

REMARK 4.14. 1) If all unbounded edges are parallel 4.3 then the condition $\overline{m}_1 \in \Lambda_\omega$ in (2) follows from (1).

2) A broken line β is determined uniquely by specifying its endpoint $\beta(0)$ and its type. In fact, the coefficients a_i are determine inductively from $a_1 = 1$ by Equation (4.4). □

According to Remark 4.14,(2) the map $\beta \mapsto \beta(0)$ identifies the space of broken lines of a fixed type with a subset of $\mathfrak{u}_r$. This subset is the interior of a polyhedron:

PROPOSITION 4.15. [CaPuSi, Proposition 4.4]. *For each type* $(\mathfrak{u}_i, m_i)$ *of broken lines there is an integral, closed, convex polyhedron* Ξ*, of dimension* n *if non-empty, and an affine immersion*

$$\Phi : \Xi \longrightarrow \mathfrak{u}_r,$$

so that $\Phi\big(\operatorname{Int}\Xi\big)$ *is the set of endpoints* $\beta(0)$ *of broken lines* β *of the given type.*

□

REMARK 4.16. A point $p \in \Phi(\partial\Xi)$ still has a meaning as an endpoint of a piecewise affine map $\beta : (-\infty, 0] \to B$ together with data t_i and $a_i z^{m_i}$, defining a *degenerate broken line*. For this not to be a broken line $\mathrm{im}(\beta)$ has to intersect a joint. By convexity of the chambers this comprises the case that there exists $t \in (-\infty, 0] \setminus \{t_0, \ldots, t_r\}$ with $\beta(t) \in |\mathscr{S}_k|$, or even that β maps a whole interval to $|\mathscr{S}_k|$. Note also the possibility that $t_{i-1} = t_i$ for some $i \in \{2, \ldots, r-1\}$, but then $\beta(t_{i-1}) = \beta(t_i)$ is contained in a joint. All other conditions in the definition of broken lines are closed.

The set of endpoints $\beta(0)$ of degenerate broken lines of a given type is the $(n-1)$-dimensional polyhedral subset $\Phi(\partial\Xi) \subset \mathfrak{u}$. The set of degenerate broken lines *not transverse* to each joint of $\mathscr{S}_k$ is polyhedral of smaller dimension. □

Any finite structure $\mathscr{S}_k$ involves only finitely many slabs and walls, and each polynomial coming with each slab or wall carries only finitely many monomials. Hence broken lines for $|\mathscr{S}_k|$ exist only for finitely many types. The following definition is therefore meaningful.

DEFINITION 4.17. A point $p \in B$ is called *general* for the given structure $\mathscr{S}_k$ if it is not contained in $\Phi(\partial\Xi)$, for any Φ as in Proposition 4.15. □

4.2.4. The superpotential to order k. This section based on joint work with Bernd Siebert. Lemmata 4.19 and 4.20 are his achievements and can be found in [CaPuSi, Section 4]. Recall from [GrSi3, §2.6] that the structure $\mathscr{S}_k$ defines a k-th order deformation of X_0 by gluing the sheaf of rings defined by $R^k_{g,\sigma_{\mathfrak{u}}}$, with $g : \omega \to \tau$ and $\mathfrak{u}$ a chamber of $\mathscr{S}_k$ with $\omega \cap \mathfrak{u} \neq \emptyset$, $\tau \subset \sigma_{\mathfrak{u}}$.

DEFINITION 4.18. Let $p \in \mathfrak{u}$ be general. The *superpotential up to order k* is defined locally as an element of $R^k_{g,\sigma_{\mathfrak{u}}}$ by

$$W^k_{g,\mathfrak{u}}(p) := \sum_{\beta \in \mathfrak{B}(p)} a_\beta z^{m_\beta}. \tag{4.5}$$ □

The existence of a canonical extension W^k of W^0 to X_k follows once we prove the following two lemmata.

LEMMA 4.19. [CaPuSi, Lemma 4.7]. *Let $\mathfrak{u}$ be a chamber of $\mathscr{S}_k$ and $g : \omega \to \tau$ with $\omega \cap \mathfrak{u} \neq \emptyset$, $\tau \subset \sigma_{\mathfrak{u}}$. Then $W^k_{g,\mathfrak{u}}(p)$ is independent of the choice of $p \in \mathfrak{u}$.* □

By Lemma 4.19 we are entitled to define $W^k_{g,\mathfrak{u}} := W^k_{g,\mathfrak{u}}(p)$ for any general choice of $p \in \operatorname{Int} \mathfrak{u}$. For the next lemma recall the basic gluing morphisms and the notions of changing chambers and strata from [GrSi3, Construction 2.24].

LEMMA 4.20. [CaPuSi, Lemma 4.9]. *The $W^k_{g,\mathfrak{u}}$ are compatible with changing strata and changing chambers.* □

In view of these lemmata we can therefore study the superpotential $W^k \in \mathcal{O}_{X_k}$ globally on X_k. Moreover, taking the direct limit $\mathfrak{X} = \lim_{k \to \infty} X_k$ we also get a limit for the superpotential up to order k, namely $W := \lim_{k\to\infty} W^k \in \mathcal{O}(\mathfrak{X})$. We will refer to the pair $(\pi : \mathfrak{X} \to \operatorname{Spec} \Bbbk[[t]], W)$ as *Landau-Ginzburg model* and to W as the *full superpotential*. Large parts of Chapter 5 are devoted to computing this superpotential for various toric degenerations.

Chapter 5

Applications to Landau-Ginzburg mirror symmetry

By using the technique of broken lines on affine tropical manifolds, we established a very general framework for Landau-Ginzburg mirror symmetry. This chapter is devoted to the task of constructing tropical manifolds within this framework and computing superpotentials in various contexts explicitly.

5.1. Reflexive polytopes and proper LG-models

The objective of this section is to show how to construct proper superpotentials for tropical affine bases constructed from a class of reflexive polytopes. In Proposition 4.9 we have seen that properness of the superpotential W^0 associated to a toric degeneration of Calabi-Yau pairs $(\breve{\pi}:\breve{\mathfrak{X}}\to T, \breve{\mathfrak{D}}\subset\breve{\mathfrak{X}})$ is equivalent to $\breve{\mathfrak{D}}\to T$ being a toric degeneration of Calabi-Yau varieties. Recall that in this case the generic fibre of $\breve{\mathfrak{D}}\to T$ is irreducible. The intersection complex $\breve{B}$ of such *irreducible* toric degenerations is characterized by the property that its boundary $\partial\breve{B}$ is a smooth affine manifold. Equivalently, by Proposition 4.8, all unbounded one-dimensional strata of the dual intersection complex B have to be parallel.

Starting from tropical data $(B, \mathscr{P}, \varphi)$ in this situation, the key tool to compute superpotentials in finitely many steps is the following lemma, suggested by Mark Gross. It greatly reduces the number of broken lines to be considered in situations fulfilling 4.3 and with a finite structure on the bounded cells.

LEMMA 5.1. *Let $\mathscr{S}$ be a structure for a non-compact, polarized tropical manifold $(B, \mathscr{P}, \varphi)$ that is consistent to all orders. We assume that there is a subdivision $\mathscr{P}'$ of $\mathscr{P}$ with vertices disjoint from Δ and with the following properties.*

(1) *Each $\sigma \in \mathscr{P}'$ is affine isomorphic to $\rho \times \mathbb{R}_{\geq 0}$ for some bounded face $\rho \subset \sigma$.*
(2) *$B \setminus \mathrm{Int}(|\mathscr{P}'|)$ is compact and locally convex at the vertices, which makes sense in an affine chart.*
(3) *If m is an exponent of a monomial of a wall (or slab) intersecting some $\sigma \in \mathscr{P}'$, $\sigma = \rho + \mathbb{R}_{\geq 0}\overline{m}_\sigma$, then $-\overline{m} \in \Lambda_\rho + \mathbb{R}_{>0} \cdot \overline{m}_\sigma$ (or $-\overline{m} \in \Lambda_\rho + \mathbb{R}_{\geq 0} \cdot \overline{m}_\sigma$ for slabs).*

Then the first break point t_1 of a broken line β with $\mathrm{im}(\beta) \not\subset |\mathscr{P}'|$ can only happen after leaving $\mathrm{Int}\,|\mathscr{P}'|$, that is,

$$t_1 \geq \inf \big\{ t \in (-\infty, 0] \,\big|\, \beta(t) \notin |\mathscr{P}'| \big\}.$$

PROOF. Assume $\beta(t_1) \in \sigma \setminus \rho$ for some $\sigma = \rho + \mathbb{R}_{\geq 0}\overline{m}_\sigma \in \mathscr{P}'$. Then $\beta|_{(-\infty,t_1]}$ is an affine map with derivative $-\overline{m}_\sigma$, and $\beta(t_1)$ lies on a wall. By the assumption on exponents of walls on σ, the result of nontrivial scattering at time t_1 only leads to exponents m_2 with $-\overline{m}_2 \in \Lambda_\rho + \mathbb{R}_{\geq 0}\overline{m}_\sigma$, the outward pointing half-space. In particular, the next break point can not lie on ρ. Going by induction one sees that any further break point in σ preserves the condition that β' does not point inward. Moreover, by the convexity assumption, this condition is also preserved when moving to a neighbouring cell in $\mathscr{P}'$. Thus $\mathrm{im}(\beta) \subset |\mathscr{P}'|$. □

5.1.1. Proper superpotentials for toric Fano manifolds. Let $\Theta \subset N_{\mathbb{R}}$ be a full-dimensional reflexive polytope with unique interior lattice point v_0. If the fan $\Sigma := \Sigma_\Theta$ of cones over faces of Θ defines a smooth toric variety $\mathbb{P}_\Sigma$, we know that all integral boundary points of Θ are vertices. A necessary and sufficient condition for $\mathbb{P}_\Sigma$ to be smooth in this case is that all cones of Σ are elementary.

We have the following construction of tropical data $(B, \mathscr{P}, \varphi)$ from Θ, that fulfills the conditions of Lemma 5.1 and therefore allows a very explicit description of the superpotential, as we will show right after the construction.

CONSTRUCTION 5.2. Let $\Theta \subset N_{\mathbb{R}} \cong \mathbb{R}^n$ be an n-dimensional reflexive polytope such that $\mathbb{P}_{\Sigma_\Theta}$ is a toric Fano manifold. Denote by $\check{B} := \Xi$ the polar polytope of Θ and by $\check{v}_0 \in \check{B}$ the unique interior integral point. Define the polyhedral decomposition $\check{\mathscr{P}}$ of $\check{B}$ with maximal cells the convex hulls of the facets of Ξ and of $\check{v}_0$. The affine chart at $\check{v}_0$ is the one defined by the affine structure of Ξ. For the other charts note that for any vertex $\check{v}$ of Ξ, the integral tangent vectors of all adjacent facets and $\check{v}$ generate the lattice M. This follows from reflexivity of Ξ, as all facets have integral distance one from the origin $\check{v}_0$, so if the above span would not be M, Ξ would have to have interior lattice points other than $\check{v}_0$. Denote the set of integral tangent vectors of all facets adjacent to $\check{v}$ by $\{\check{v}_1, \dots \check{v}_l\}$. Then the images of the $\check{v}_i$ for $i = 1, \dots, l$ under the projection $\psi_v : M \to M/\langle\check{v}\rangle$ generate $M/\langle\check{v}\rangle$. So we can define the affine structure at $\check{v}$ by the unique chart induced by ψ_v, that is compatible with the affine structure of the adjacent maximal cells and making $\partial\check{B}$ totally geodesic. By what we just said, the transition functions between charts are indeed integral affine, that is in $Aff(M)$. Moreover, $(\check{B}, \check{\mathscr{P}})$ has a natural polarization of minimal degree by defining $\check{\varphi}(\check{v}_0) = 0$ and $\check{\varphi}(\check{v}) = 1$ for any other vertex $\check{v}$, which is well-defined by reflexivity of Ξ. We have not yet specified the discriminant locus, as this is easier discribed for the discrete Legendre transform $(B, \mathscr{P}, \varphi)$.

By construction $(B, \mathscr{P}, \varphi)$ has a unique bounded cell σ_0, isomorphic to Θ. Moreover, for each m-dimensional proper face τ of Θ, there is an unbounded cell of $\mathscr{P}$ of dimension $m+1$ that we denote by $\widetilde{\tau}$. Up to addition of a global affine function the dual polarizing function $\check{\varphi}$ is the unique piecewise affine function changing slope by one along the unbounded facets. Note that in the present case $\check{\varphi}$ is single-valued. The discriminant locus Δ of B, which fixes $\check{\Delta}$ on $\check{B}$, is chosen as follows. Let σ be a facet of Θ. Then the discriminant locus on σ is defined by the first barycentric subdivision of

σ. Furthermore on each unbounded $(n-1)$-cell $\widetilde{\tau}$ of $\mathscr{P}$, corresponding to a codimension two face of Θ, declare the discriminant locus also by the first barycentric subdivision. To see why this choice of $(B, \mathscr{P})$ and Δ is indeed *simple*, note that by assumption on Θ all facets σ of Θ and hence all unbounded facets $\widetilde{\tau}$ are *elementary* simplices. In [Gr1] Gross considers toric degenerations associated to reflexive polytopes in a similar fashion. In fact, if we restrict our $(B, \mathscr{P})$ to the bounded part, that is to the cells corresponding to faces of Θ, this is exactly the situation considered in [Gr1, Definition 2.10], in particular with the same discriminant locus. Denote this restriction to the bounded part by $(B_\Theta, \mathscr{P}_\Theta)$. In [Gr1, Theorem 3.16] it is shown that $(B_\Theta, \mathscr{P}_\Theta)$ is simple. The MPCP resolution chosen in this theorem is *trivial* here, as $\mathbb{P}_{\Sigma_\Theta}$ is smooth by assumption. Moreover, by applying the argument presented in the proof of [Gr1, Theorem 3.16] to the unbounded part of $(B, \mathscr{P})$, we get simplicity for the whole affine manifold with singularities $(B, \mathscr{P})$. □

PROPOSITION 5.3. *Let $\Theta \subset N_\mathbb{R}$ be a reflexive polytope, such that $\mathbb{P}_{\Sigma_\Theta}$ is smooth. Denote by $(B, \mathscr{P}, \varphi)$ the affine base constructed from Θ by means of Construction 5.2 with unique maximal bounded cell $\sigma_0 \subset B$. Then there is neighbourhood U of the interior vertex $v_0 \in \sigma_0$ such that for any $p \in U$ there is a canonical bijection between broken lines with endpoint p and rays of Σ.*

PROOF. For each vertex v of Θ denote by U_v the connected component of $\Theta \setminus \Delta$ and define the n-dimensional set $\widetilde{U}_v := U_v + \mathbb{R} \cdot (-v)$. The common intersection of $\widetilde{U}_v$ for all vertices v of Θ is non-empty, as it contains the origin v_0. So we can choose an n-dimensional open subset U of this intersection. By construction for any point $p \in U$ the translation by p of each ray $\rho \in \Sigma(1)$ can be considered as the image of a unique broken line β_ρ ending in p. Running the reconstruction algorithm from [GrSi3] we get a structure $\mathscr{S}$ consistent to all orders and there is in fact no scattering within the interior of σ_0. Thus we can apply Lemma 5.1 in this situation, as B has parallel unbounded one-cells and by construction and the bounded cell σ_0 is convex. Moreover, observe that if a broken line has root vertex p, root tangent direction $\nu \in \partial\Theta \cap N$ and scatters non-trivially at $\partial\sigma_0$, it would have to scatter at least one more time. However, this is not possible by Lemma 5.1, so we are done. □

This proposition implies the following interesting corollary, which says that the Landau-Ginzburg superpotential W we get from applying Construction 5.2 to Θ agrees on an open part with the Hori-Vafa mirror of $(\mathbb{P}_{\Sigma_\Theta}, -K_{\mathbb{P}_{\Sigma_\Theta}})$.

THEOREM 5.4. *Let* $(\mathfrak{X} \to \operatorname{Spec} \Bbbk[[t]], W)$ *be the Landau-Ginzburg model associated to the base* $(B, \mathscr{P}, \varphi)$ *obtained from a full-dimensional reflexive polytope* $\Theta \subset N_\mathbb{R} \cong \mathbb{R}^n$ *by Construction 5.2. Moreover, denote by* Σ_Θ *the fan of cones over faces of* Θ*. Then there is an open subset* $U \cong \operatorname{Spec} \Bbbk[[t]][x_1, \ldots, x_n] \subset \mathfrak{X}$ *such that*

$$W|_U = \Big(\sum_{\rho \in \Sigma_\Theta(1)} x^{n_\rho} \Big) \cdot t,$$

where x^{n_ρ} *denotes the usual multi-index notation. So* $W|_U$ *is the usual Hori-Vafa monomial sum of the anti-canonically polarized toric variety* $\mathbb{P}_{\Sigma_\Theta}$ *times* t*.* □

REMARK 5.5. 1) Note that for a polarized toric degeneration $(\breve{\pi} : \breve{\mathfrak{X}} \to T, \breve{\mathfrak{D}})$ with intersection complex $(\breve{B}, \breve{\mathscr{P}}, \breve{\varphi})$ Legendre dual to $(B, \mathscr{P}, \varphi)$ as in Corollary 5.4 one would expect the generic fibre $\breve{X}_\eta$ to be isomorphic to $\mathbb{P}_{\Sigma_\Theta}$, as W is a partial compactification of the Hori-Vafa mirror of $\mathbb{P}_{\Sigma_\Theta}$. We do not know how to prove this statement in general, but we will check it for $\dim B = 2$ in Section 5.2.

2) The fact that we recover the Hori-Vafa mirror for the anti-canonical polarization comes from our specific choice of φ. For other polarizations φ' the terms in the superpotential receive different powers of t, just as in the Hori-Vafa proposal. □

EXAMPLE 5.6. Starting from $\mathbb{P}^3$ with its anti-canonical polarization, we obtain a model with irreducible affine base by "trading corners and edges for singularities of the affine structure", as described in the above Construction 5.2. More precisely, subdivide the anti-canonical polytope $\breve{B} := \Xi := \Xi_{-K_{\mathbb{P}^3}}$ by introducing six two-faces spanned by the origin and two distinct corners of Ξ. Then choose the discriminant locus $\breve{\Delta}$ to be defined by the first barycentric subdivision of these six affine triangles, as shown in Figure 5.1. By setting $\breve{\varphi}(\breve{v}) = 1$ for every vertex $\breve{v}$ of Ξ we arrive at the tropical affine manifold $(\breve{B}, \breve{\mathscr{P}}, \breve{\varphi})$ as depicted.

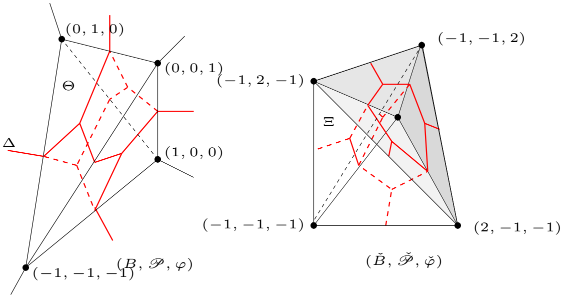

FIGURE 5.1. The affine base of an irreducible toric degeneration of $\mathbb{P}^3$ and its Legendre dual.

It can be checked that the discrete Legendre transform

$$(B, \mathscr{P}, \varphi)$$

of $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ is the one obtained from Construction 5.2, which is drawn in Figure 5.1 on the left. $(B, \mathscr{P}, \varphi)$ has four parallel unbounded rays and a discriminant locus Δ with six unbounded rays. If these rays were bounded, Δ would be homeomorphic to $\check{\Delta}$. Denote the bounded three-cell of $\mathscr{P}$ by σ_0. Every bounded two-face is subdivided into three chambers by Δ and at every vertex v of B three such chambers meet.

Now choose a general root $p \in U \subset \sigma_0$, where U is the open subset from the proof of Proposition 5.3. As p is general, it is an element of the interior of a chamber $\mathfrak{u}$. With the convention that the monomials corresponding to the vectors $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ are denoted by x, y and z, we claim that

$$W^1_{\mathbb{P}^3}(p) = \left(x + y + z + \frac{1}{xyz}\right) \cdot t.$$

We will check this result by hand, thereby confirming Corollary 5.4 for this example. To this end we only consider broken lines β that pass through one of the three chambers adjacent to $(0,0,1)$, that is, broken lines that come from infinity in direction $(0,0,-1)$. The other cases work analogously. Denote the root tangent vector of β by m. Then either $m = (0,0,1)$ and β is a straight line or it is broken at the boundary of σ_0 such that the result of scattering of m is $(0,0,1)$. By symmetry we may assume that β hits the chamber $\mathfrak{u}'$ on the face spanned by $(0,0,1)$, $(0,1,0)$ and $(-1,-1,-1)$. The monodromy invariant plane for $\mathfrak{u}'$ is spanned by the vectors $(0,-1,1)$ and $(1,1,2)$. Therefore m has to be either $(0,1,0)$ or $(-1,-1,-1)$. Thus, the set of root tangent vectors of broken lines ending in p is contained in the set of vertices of $\check{B}$. Moreover, it is easy to see that indeed every vertex occurs exactly once, by following a ray starting at p in direction m and breaking at a wall in $\partial\sigma_0$ if necessary. $\square$

5.1.2. Superpotentials for Fano varieties from admissible decompositions. In the last section we have seen how to construct proper superpotentials for smooth Fano toric varieties associated to a reflexive polytope $\Theta \subset N_{\mathbb{Q}}$. If Θ is three-dimensional and *fully decomposable*, a property we will define next, there is a procedure that associates a Laurent polynomial W to Θ that is the Landau-Ginzburg model of a not necessarily toric Fano variety in most cases. This method to find Fano varieties and their Landau-Ginzburg potentials is pursued by Tom Coates, Alessio Corti, Sergey Galkin, Vasily Golyshev and Al Kasprzyk in an ongoing project and all the relevant data can be found online, see [CoCo]. Their method is based on the combinatorics of Θ, but has no underpinning algebro-geometric construction. The goal of this section is to make a first step towards understanding the approach of [CoCo] within the Gross-Siebert programm. All definitions made in this section are inspired by and can be recovered from the data in [CoCo].

DEFINITION 5.7. Let $\Theta \subset N_{\mathbb{R}}$ be a three-dimensional reflexive polytope and $\tau \subset \Theta$ a facet. Then

(1) τ is called *admissible* if it has no interior lattice points.
(2) τ is called *irreducible* if there is no non-trivial Minkowski decomposition of τ.
(3) Θ is called *fully decomposable* if every facet of it has at least one Minkowski-decomposition into admissible and irreducible polytopes. $\square$

LEMMA 5.8. *Let τ be an admissible and irreducible two-dimensional polytope. Then τ is a triangle with at most one side containing more than two integral points.*

PROOF. Let τ be admissible and $\overline{\tau}$ be the triangle spanned by $(0,0)$, $(a,0)$ and $(0,b)$ for natural numbers $a, b \in \mathbb{N}$. Without loss of generality we can assume that there is a vertex v_0 of τ with adjacent edges ρ_1 and ρ_2 in direction $(1,0)$ and (p,q), p and q coprime, such that ρ_1 has at least one interior integral point. Consider the matrix $\psi := \left(\begin{smallmatrix} 1 & p \\ 0 & q \end{smallmatrix}\right)$ and assume that a and b are the largest integers such that the image of $\overline{\tau}$ under ψ is still contained in τ. If $a > 2$ and $b \geq 2$ or vice versa, the image of the interior point $(1,1)$ of $\overline{\tau}$ will also be an interior point of τ. Since τ is admissible, that is not possible. In case that $a = b = 2$ and $\psi(\overline{\tau}) \subset \tau$, we see that $\psi\left((1,1)\right)$ is an interior point if τ has at least four vertices, as τ is convex. So for $a = b = 2$ τ has to be a triangle, in which case $\psi(\overline{\tau}) = \tau$. So in this case τ can be Minkowski decomposed into two irreducible admissible triangles spanned by $(0,0)$, $(1,0)$ and (p,q). Note that even if $a = b = 1$ and τ has at least five vertices, we see that $\psi\left((1,1)\right)$ is always an interior integral point of τ.

Next, assume $a \geq 2$ and $b = 1$. If τ is a triangle, it is automatically admissible and irreducible in this case. If τ is a quadrangle it follows that the side not adjacent to ρ_1 is in fact parallel to ρ_1, as otherwise $\psi\left((1,1)\right)$ would again be an interior point. In this case τ can be Minkowski decomposed into the segment $a' \cdot (1,0)$ and an admissible and irreducible triangle spanned by $(0,0)$, $(a'',0)$ and (p,q) such that $a = a' + a''$, which finishes the proof. $\square$

Given a three-dimensional fully decomposable polytope Θ, we can thus *choose* a Minkowski-decomposition $\mathcal{M}_\Theta$ of its facets, that is for every facet τ there is a Minkowski-sum

$$\mathcal{M}_\Theta(\tau) = \bigoplus_{i=1}^{s} \tau_i.$$

where each τ_i is irreducible and admissible. We will refer to such decompositions as *full Minkowski-decompositions* in what follows.

DEFINITION 5.9. Let τ be an admissible and irreducible two-dimensional polytope. For each integral point $\nu \in \tau$ we define a *weight function* $l_\tau(\nu)$ as follows. If τ is the k-th integral point on a one-dimensional stratum of length n of τ, then

$$l_\tau(\nu) = \binom{n}{k}.$$

This clearly does not depend on the endpoint of τ we choose to determine the position of ν. $\square$

Using the weights of Minkowski-pieces we can now associate a multiplicity to each $\nu \in (\Theta \setminus \{0\}) \cap N$ which will play the role of coefficients of the monomial z^ν in the superpotential we will define below in Definition 5.12.

DEFINITION 5.10. Let $\Theta \subset N_{\mathbb{Q}}$ be a fully decomposable three-dimensional polytope and choose a full Minkowski-decomposition $\mathcal{M}_\Theta$. Define the *multiplicity of* $\mathcal{M}_\Theta$

$$mult : (\Theta \setminus \{0\}) \cap N \to \mathbb{N}$$

as follows. Choose a facet τ such that $\nu \in \tau$ and consider its full decomposition $\mathcal{M}_\Theta(\tau) = \bigoplus_{i=1}^s \tau_i$, then

$$mult_{\mathcal{M}_\Theta}(\nu) := \sum_{\nu=\nu_1+\ldots+\nu_s} \sum_{i=1}^{s} l_{\tau_i}(\nu_i),$$

where $\nu_i \in \tau_i \cap N$. $\square$

REMARK 5.11. The definition of the multiplicity involves the choice of a facet that contains ν and we have to check that this is well-defined. The multiplicity of a lattice point is just the weighted sum of all possible Minkowski-representations of the point. Hence, if ν is a vertex of Θ, then $mult(\nu) = 1$. Moreover, integral points on one-dimensional faces get binomial coefficients as in Definition 5.9 as multiplicities, even if they are Minkowski decomposed. The multiplicity of interior integral points of facets depends on the Minkowski-decomposition. Therefore we see that the definition does not depend on the facet τ we choose. $\square$

DEFINITION 5.12. Let Θ be a fully decomposable three-dimensional polytope with choice of full decomposition $\mathcal{M}_\Theta$. Define the *Landau-Ginzburg potential associated to* $\mathcal{M}_\Theta$ by

$$W(\mathcal{M}_\Theta) = \sum_{\nu \in (\Theta \setminus \{0\}) \cap N} mult_{\mathcal{M}_\Theta}(\nu) z^\nu$$

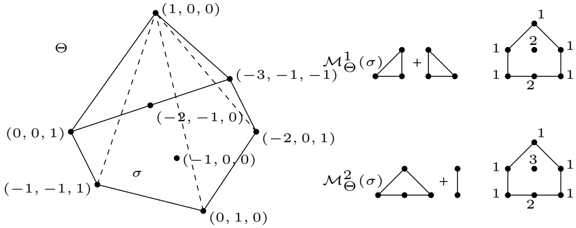

FIGURE 5.2. The two Minkowski-decompositions of the facet σ of Θ

and set $x := z^{(1,0,0)}$, $y := z^{(0,1,0)}$ and $z := z^{(0,0,1)}$ to express $W(\mathcal{M}_\Theta)$ as Laurent polynomial in x, y and z. □

EXAMPLE 5.13. Let Θ be the polytope spanned by

$$(1,0,0),(0,1,0),(0,0,1),(-2,0,-1),(-3,-1,-1),(-1,-1,1).$$

Θ has a facet σ, which is the pentagon spanned by all vertices but the first. Clearly σ has an additional integral boundary point at $(-2,-1,0)$ and an integral interior point at $(-1,0,0)$. All other facets of Θ are irreducible and admissible triangles which connect the boundary of σ with the vertex $(1,0,0)$. It is clear that there are exactly two full decompositions of Θ, that is two Minkowski-decompositions of σ. Both decompositions are depicted in Figure 5.2. Thus we obtain two different Laurent polynomials, namely

$$\begin{aligned} W(\mathcal{M}^1_\Theta) &= x + y + z + x^{-2}z^{-1} + x^{-3}y^{-1}z^{-1} + x^{-1}y^{-1}z \\ &\quad + 2\cdot x^{-2}y^{-1} + 2\cdot x^{-1} \\ W(\mathcal{M}^2_\Theta) &= x + y + z + x^{-2}z^{-1} + x^{-3}y^{-1}z^{-1} + x^{-1}y^{-1}z \\ &\quad + 2\cdot x^{-2}y^{-1} + 3\cdot x^{-1}. \end{aligned}$$

□

The crucial point in defining these potentials $W := W(\mathcal{M}_\Theta)$ is the following. To each W we can associate the *Jacobian ring*

$$Jac(W) := \mathbb{C}[x,y,z]/\big(\partial_x W, \partial_y W, \partial_z W\big)$$

and we will always consider this ring as a $\mathbb{C}$-vector space. Then there is the following construction due to Coates, Corti et al.

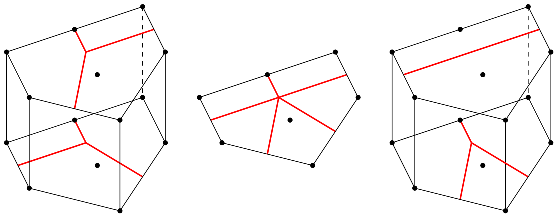

FIGURE 5.3. Different Minkowski-decompositions induce different decompositions of Δ on the pentagon σ.

CONSTRUCTION 5.14. Let Θ be a fully decomposable three-dimensional reflexive polytope, $\mathcal{M}_\Theta$ a full decomposition and $W(\mathcal{M}_\Theta)$ its associated Landau-Ginzburg potential. Then in many cases there is a Fano manifold X, polarized by $-K_X$, such that

$$Jac\big(W(\mathcal{M}_\Theta)\big) \cong QH^*(X,\mathbb{C}), \tag{5.1}$$

where $QH^*(X,\mathbb{C})$ denotes small quantum cohomology. □

REMARK 5.15. 1) The precise procedure can be found in [CoCo], where the authors compute the Jacobian ring $Jac(W(\mathcal{M}_\Theta))$ for all possible pairs $(\Theta, \mathcal{M}_\Theta)$. As $QH^*(X,\mathbb{C})$ for a Fano variety X determines X up to isomorphism and the list of all possible $QH^*(X,\mathbb{C})$ in dimension three is known, it suffices to compare to this list to get X, see for instance [IsPr]. More often than not this yields a Fano X. However, there are examples for which the authors can not identify a Fano, see for instance polytope number 14 in [CoCo].

2) This construction justifies the name Landau-Ginzburg potential, since (5.1) is in fact a mirror symmetry theorem for X, if it exists. Note, however, that the construction is not of a geometric nature, we only know X from comparing the Jacobian ring to quantum cohomology. □

Next, we will verify in an example that Construction 5.14 may in fact be geometrically described within the LG-framework in the Gross-Siebert we introduced in the last chapter.

EXAMPLE 5.16. We will continue with Example 5.13 and concentrate on the pentagon σ, on which the discriminant locus Δ is described by five segments meeting at the barycentre of σ, as shown in the middle of Figure 5.3. The two Minkowski-decompositions $\mathcal{M}^1_\Theta$ and $\mathcal{M}^2_\Theta$ from Example 5.13 correspond to different decompositions of Δ, as shown on the left and right picture of Figure 5.3. The left decomposition corresponding to $\mathcal{M}^1_\Theta$ already consists of two simple pieces, while in the right decomposition corresponding to $\mathcal{M}^2_\Theta$ we have to further subdivide the part of the discriminant locus on the lower copy of the pentagon. The result of this procedure can be seen in Figure 5.4. The bounded part of the discriminant locus $\Delta_{\mathcal{M}^2_\Theta}$ can, with a little bit effort, be seen to be homeomorphic to a cube.

Note the remarkable similarity of that situation and the following example. Let $\Sigma \subset \mathbb{R}^3$ be the complete fan generated by the rays in direction $\pm(1,0,0)$, $\pm(0,1,0)$ and $\pm(0,0,1)$. Clearly we have $\mathbb{P}_\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and we can apply Construction 5.2 to the reflexive polytope Θ spanned by these ray generators. It is easily checked that the affine base $(B, \mathscr{P}, \varphi)$ obtained from this has a discriminant locus Δ which is homeomorphic to $\check{\Delta}_{\mathcal{M}^2_\Theta}$. In fact, from Theorem 5.14, that is from the full list in [CoCo], we get that the potential

$$\begin{aligned} W(\mathcal{M}^2_\Theta) = x + y + z + x^{-2}z^{-1} + x^{-3}y^{-1}z^{-1} + x^{-1}y^{-1}z \\ + 2 \cdot x^{-2}y^{-1} + 2 \cdot x^{-1}. \end{aligned}$$

corresponds to the Fano variety $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In fact, Galkin in [Ga] shows that there is an explicit birational transformation of $W(\mathcal{M}^2_\Theta)$ to the standard superpotential of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Although details for this statement are not completely worked out, note that in the fan picture the generic fiber of a toric degeneration does not seem to depend on the polyhedral decomposition of the affine manifold, but purely on the *affine structure with singularities*, as communicated to us by Bernd Siebert [Si]. Thus, although (B, Δ) is seemingly quite different from $(B_{\mathcal{M}^2_\Theta}, \Delta_{\mathcal{M}^2_\Theta})$, both toric degenerations might have the same generic fiber. We will, however, not go into the computation to check that this is the case. □

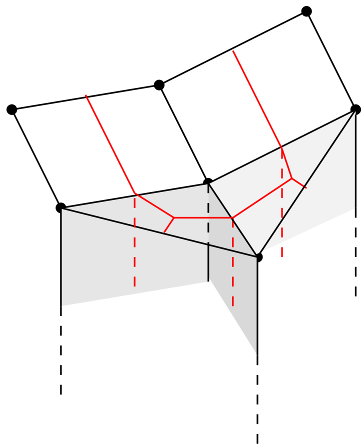

FIGURE 5.4

We conjecture that this method of "pulling apart" the discriminant locus as dictated by the given Minkowski decomposition presented in the last example, which always works locally, can be extended to a global construction for every fully admissible Θ.

CONJECTURE 5.17. *Let Θ be a fully decomposable three-dimensional reflexive polytope with full decomposition $\mathcal{M}_\Theta$. Then there is a toric degeneration with dual intersection complex $(B_{\mathcal{M}_\Theta}, \mathscr{P}_{\mathcal{M}_\Theta})$ such that the generic fibre is the Fano variety X corresponding to the potential $W(\mathcal{M}_\Theta)$ obtained from Construction 5.14, if there is such an X.* □

5.2. Del Pezzo surfaces

In this section we will compare superpotentials for different toric degenerations of del Pezzo surfaces using broken lines. Recall that apart from $\mathbb{P}^1 \times \mathbb{P}^1$ all other del Pezzo surfaces dP_k can be obtained by blowing up $\mathbb{P}^2$ in $0 \leq k \leq 8$ points. Note that dP_k for $k \geq 5$ is not unique up to isomorphism but has a $2(k-4)$-dimensional moduli space. For the anticanonical bundle to be ample the blown-up points need to be in sufficiently general position. This means that no three points are collinear, no six points lie on a conic and no eight points lie on an irreducible cubic which has a double point at one of the points. However, rather than ampleness of $-K_X$ the existence of certain toric degenerations is central to our approach. For example, our point of view naturally includes the case $k = 9$.

5.2.1. Smooth toric del Pezzo surfaces. Up to lattice isomorphism there are exactly five toric del Pezzo surfaces $\mathbb{P}_\Sigma$ whose fans Σ are depicted in Figure 5.5, namely $\mathbb{P}^2$ blown up torically in at most three points and $\mathbb{P}^1 \times \mathbb{P}^1$. To construct proper superpotentials for these surfaces we consider the following class of toric degenerations. Recall the notions of irreducibility (Definition 4.10) and simplicity ([GrSi1], §1.5) of toric degenerations.

DEFINITION 5.18. A *distinguished toric degeneration of del Pezzo surfaces* is an irreducible, simple toric degeneration $(\check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ with $\check{\mathfrak{D}}$ relatively ample over T and with generic fibre $\check{\mathfrak{D}}_\eta \subset \check{\mathfrak{X}}_\eta$ an anti-canonical divisor in a Gorenstein surface. □

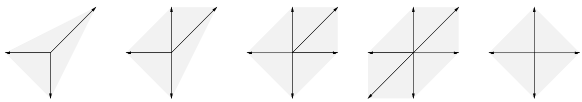

FIGURE 5.5. Fans of the five toric del Pezzo surfaces

If the general fiber of a toric degeneration as in the definition is smooth then it is a dP_k for some k, together with a smooth anti-canonical divisor. The point of this definition is both the irreducibility of the anti-canonical divisor and the fact that this divisor extends to a polarization on the central fiber. We can specialize Construction 5.2 to del Pezzo surfaces. We start from the polytopes spanned by the ray generators of the five fans depicted in Figure 5.5. The result of the construction is depicted in Figure 5.6, which shows a chart in the complement of the dotted segments. Note that the discrete Legendre transform $(B, \mathscr{P}, \varphi)$, also depicted in Figure 5.6, indeed has parallel outgoing rays. Conversely, in dimension 2 we have the following uniqueness result.

THEOREM 5.19. *If* $(\pi : \check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ *is a distinguished toric degeneration of del Pezzo surfaces with non-singular generic fibre, then the associated intersection complex* $(\check{B}, \check{\mathscr{P}})$ *is isomorphic to one listed in Figure 5.6.*

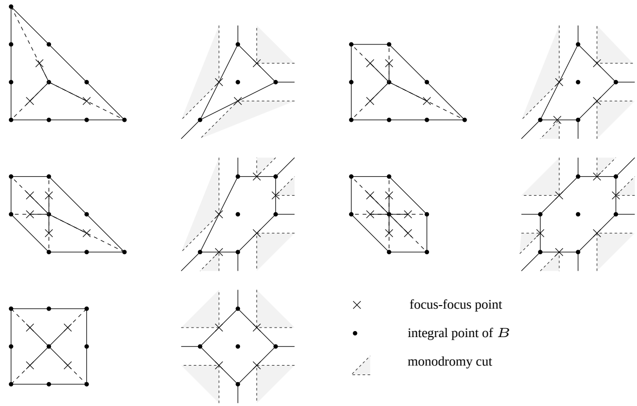

FIGURE 5.6. The intersection complexes $(\check{B}, \check{\mathscr{P}})$ of the five distinguished toric degenerations of toric del Pezzo surfaces and their Legendre duals $(B, \mathscr{P})$.

PROOF. Let $(\pi : \check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ be the given toric degeneration, then by definition the generic fibre $\check{\mathfrak{X}}_\eta$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or to a del Pezzo surface dP_k for some $0 \leq k \leq 3$.

First we determine the number of integral points of $\check{B}$. Let $\mathcal{L}$ be the polarizing line bundle on $\check{\mathfrak{X}}$. By assumption we have

(5.2)

$$h^0(\check{\mathfrak{X}}_\eta, \mathcal{L}|_{\check{X}_\eta}) = h^0(dP_k, -K_{dP_k}) = \begin{cases} 10-k, & \check{\mathfrak{X}}_\eta \simeq dP_k \\ 9, & \check{\mathfrak{X}}_\eta \simeq \mathbb{P}^1 \times \mathbb{P}^1. \end{cases}$$

Let $t \in \mathcal{O}_{T,0}$ be a uniformizing parameter, $\check{X}_n := \mathrm{Spec}\left(\Bbbk[t]/(t^{n+1})\right) \times_T \mathfrak{X}$ the n-th order neighbourhood of $\check{X}_0 := \pi^{-1}(0)$ in $\check{\mathfrak{X}}$ and write $\mathcal{L}_n := \mathcal{L}|_{\check{X}_n}$. Then for any n there is an exact sequence of sheaves on $\check{X}_0$ given by

$$0 \longrightarrow \mathcal{O}_{\check{X}_0} \longrightarrow \mathcal{L}_{n+1} \longrightarrow \mathcal{L}_n \longrightarrow 0.$$

By the analogue of [GrSi2, Theorem 4.2] for Calabi-Yau pairs, we know

$$h^1(\check{X}_0, \mathcal{O}_{\check{X}_0}) = h^1(\check{\mathfrak{X}}_\eta, \mathcal{O}_{\check{\mathfrak{X}}_\eta}) = 0.$$

Thus the long exact sequence on cohomology induces a surjection

$$H^0(\check{X}_0, \mathcal{L}_{n+1}) \twoheadrightarrow H^0(\check{X}_0, \mathcal{L}_n)$$

for each n. By the theorem on formal functions and cohomology and base change ([Ha1, Theorem 11.1 and Theorem 12.11] we therefore conclude that $\pi_*\mathcal{L}$ is locally free, with fibre over 0 isomorphic to $H^0(\check{X}_0, \mathcal{L}_0)$. In view of (5.2) we thus conclude

$$h^0(\check{X}_0, \mathcal{L}_0) = \begin{cases} 10-k, & \check{\mathfrak{X}}_\eta \simeq dP_k \\ 9, & \check{\mathfrak{X}}_\eta \simeq \mathbb{P}^1 \times \mathbb{P}^1. \end{cases}$$

Now we know that on a toric variety the dimension of the space of sections of a polarizing line bundle equals the number of integral points of its momentum polytope. Since $\check{X}_0$ is a union of toric varieties each integral point $x \in \check{B}$ provides a monomial section of $\mathcal{L}_0$ on any irreducible component $\check{X}_\sigma \subset \check{X}_0$ with $\sigma \in \mathscr{P}$ containing x. These provide a basis of sections of $H^0(\check{X}_0, \mathcal{L}_0)$.

Hence $\check{B}$ has $10-k$ integral points if $\check{\mathfrak{X}}_\eta \simeq dP_k$ and 9 integral points if $\check{\mathfrak{X}}_\eta \simeq \mathbb{P}^1 \times \mathbb{P}^1$. An analogous argument shows that the number of integral points of $\partial\check{B}$ equals

$$h^0(\check{D}_0, \mathcal{L}_0) = h^0(\check{D}_\eta, \mathcal{L}_\eta),$$

which by Riemann-Roch equals $K^2_{dP_k} = 9-k$ or $K^2_{\mathbb{P}^1\times\mathbb{P}^1} = 8$. In either case we thus have a unique integral interior point $v_0 \in \check{B}$. In particular, $\check{B}$ has the topology of a disk, and each singular point of the affine structure lies on an edge connecting v_0 to an integral point of $\partial\check{B}$.

Pushing the singular points $p_1, \ldots, p_l$ into $\partial\check{B}$, thereby trading them for corners, we arrive at an l-gon with a unique interior integral point, hence a reflexive polygon. Moreover, since the generic fibre of $\check{\mathfrak{X}} \to \check{\mathfrak{D}}$ is non-singular, at each vertex integral generators of the tangent spaces of the adjacent edges form a lattice basis. Thus by the classification of reflexive lattice polygons, up to adding some edges connecting v_0 to $\partial\check{B}$ the only configurations possible are the ones shown in Figure 5.6. □

REMARK 5.20. 1) From the proof we see that the five possible types can be distinguished by $\dim H^0(\check{\mathfrak{X}}_\eta, \mathcal{L}_\eta)$, except for $\mathbb{P}^1 \times \mathbb{P}^1$ and dP_1. Alternatively, by Proposition 5.23, one could use $H^1(\check{\mathfrak{X}}_\eta, \Omega^1_{\check{\mathfrak{X}}_\eta})$.

2) For each $(\check{B}, \check{\mathscr{P}})$ there is a discrete set of choices of $\check{\varphi}$, which determines the local toric models of $\check{\mathfrak{X}} \to \check{\mathfrak{D}}$. This reflects the fact that the base of (log smooth) deformations of the central fibre $\check{X}_0^\dagger$ as a space over the standard log point $\Bbbk^\dagger$ is higher dimensional. In fact, let r be the number of vertices on $\partial\check{B}$. Then taking a representative of $\check{\varphi}$ that vanishes on one maximal cell, $\check{\varphi}$ is defined by the value at $r-2$ vertices on $\partial\check{B}$. Convexity then defines a submonoid $Q \subset \mathbb{N}^{r-2}$ with the property that $\mathrm{Hom}(Q, \mathbb{N})$ is isomorphic to the space of (not necessarily strictly) convex, piecewise affine functions on $(\check{B}, \check{\mathscr{P}})$ modulo global affine functions. To avoid technicalities, we will from now on assume that $\mathrm{Hom}(Q, \mathbb{N})$ is in fact isomorphic to the space of *strictly* convex, piecewise affine functions. Running the construction of [GrSi3] with parameters then produces a log smooth deformation with the given central fibre $(\check{X}, \check{D})$ over the completion at the origin of $\mathrm{Spec}\, \Bbbk[Q]$. For the minimal polyhedral decompositions of Figure 5.6 with $r = l$ we have $\mathrm{rk}\, Q = l-2$, which by Remark 5.24 2) below agrees with the dimension of the space $H^1(\check{X}_0, \Theta_{\check{X}_0^\dagger/\Bbbk^\dagger})$ of infinitesimal log smooth deformations of $X_0^\dagger/\Bbbk^\dagger$. One can show that in this case the constructed deformation is in fact semi-universal. $\square$

As an immediate corollary of Proposition 5.3 for distinguished toric del Pezzo degenerations, we get the following result.

COROLLARY 5.21. *Let $(\mathfrak{X} \to \mathrm{Spec}\, \Bbbk[[t]], W)$ be the Landau-Ginzburg model mirror to a distinguished toric del Pezzo degeneration. Then there is an open subset*

$$U \cong \mathrm{Spec}\, \Bbbk[[t]][x, y] \subset \mathfrak{X}$$

such that $W|_U$ equals the usual Hori-Vafa monomial sum times t. $\square$

EXAMPLE 5.22. Let us study the distinguished toric degeneration of dP_3 with the minimal polarization $\check{\varphi}$ explicitly. The two pictures in Figure 5.7 show all broken lines for different choices of root vertex p in the bounded cell σ_0 of $\mathscr{P}$. This illustrates the invariance under the change of root vertex within a chamber of the structure proved in Lemma 4.19. The root tangent vectors in each case are $(1,0), (1,1), (0,1), (-1,0), (-1,-1)$ and $(0,-1)$. For the choice of root vertex p as indicated in the left picture of

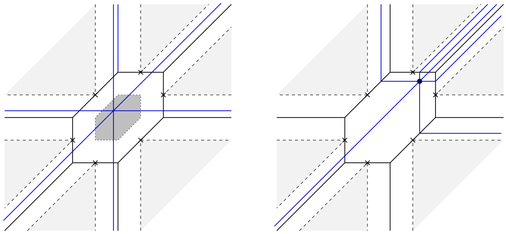

FIGURE 5.7. Broken lines in the mirror of the distinguished base of an dP_3 indicating the invariance under change of root vertex.

Figure 5.7 this gives the superpotential

$$W^1_{dP_3}(\sigma_0) = \left(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right) \cdot t,$$

which for $t \neq 0$ has six critical points. Applying Lemma 5.1 shows that this is the superpotential to all orders, that is $W_{dP_3}(\sigma_0) = W^1_{dP_3}(\sigma_0)$. For each choice of p within the shaded open hexagon U shown in the picture, none of the six broken lines has a break point. □

An analogous picture arises for the other four distinguished del Pezzo degenerations. Morally speaking the last example shows that in toric situations ray generators of the fan are sufficient to compute the superpotential, but really they should be seen as special cases of broken lines.

5.2.2. Non-toric del Pezzo surfaces. In this section we consider del Pezzo surfaces dP_k for $k \geq 4$, referred to as higher del Pezzo surfaces. Let us first determine the topology of B and the number of singular points of the affine structure.

PROPOSITION 5.23. *Let $(B, \mathscr{P})$ be the dual intersection complex of an irreducible, simple toric degeneration $(\pi : \check{\mathfrak{X}} \to T, \check{\mathfrak{D}})$ of two-dimensional log Calabi-Yau pairs. In particular, the generic fibre $\check{\mathfrak{X}}_\eta$ is a proper surface with $\check{\mathfrak{D}}$ a smooth anti-canonical divisor. Then B is homeomorphic to $\mathbb{R}^2$, and the affine structure has $l = \dim H^1(\check{\mathfrak{X}}_\eta, \Omega^1_{\check{\mathfrak{X}}_\eta}) + 2$ singular points.*

PROOF. Since the relative logarithmic dualizing sheaf

$$\omega_{\check{\mathfrak{X}}/\check{\mathfrak{D}}}(-\log\check{\mathfrak{D}})$$

is trivial, the generalization of [GrSi1], Theorem 2.39, to the case of log Calabi-Yau pairs shows that B is orientable. By the classification of surfaces with effective anti-canonical divisor we know $H^i(\check{\mathfrak{X}}_\eta, \mathcal{O}_{\check{\mathfrak{X}}_\eta}) = 0$, $i = 1, 2$. As in the proof of Theorem 5.19 this implies $H^1(\check{X}_0, \mathcal{O}_{\check{X}_0}) = 0$. Thus by the log Calabi-Yau analogue of [GrSi1], Proposition 2.37,

$$H^1(B, \Bbbk) = H^1(\check{X}_0, \mathcal{O}_{\check{X}_0}) = 0.$$

In particular, $\check{B}$ has the topology of $\mathbb{R}^2$.

Denote by $\Lambda^\perp \subset \Lambda^*$ the sheaf of $\mathbb{Z}$-cotangent vectors vanishing on unbounded rays. As for the number of singular points the generalization of [GrSi2], Theorem 3.21 and Theorem 4.2, shows that $\dim H^1(\check{\mathfrak{X}}_\eta, \Omega^1_{\check{\mathfrak{X}}_\eta})$ is related to an affine Hodge group:

$$\dim H^1(\check{\mathfrak{X}}_\eta, \Omega^1_{\check{\mathfrak{X}}_\eta}) = \dim H^1(B, i_*\Lambda^\perp \otimes_{\mathbb{Z}} \mathbb{R}).$$

To compute $H^1(B, i_*\Lambda^\perp \otimes_{\mathbb{Z}} \mathbb{R})$ we choose the following Čech cover of B. Since B is homeomorphic to $\mathbb{R}^2$ there is an open disk $U_0 \subset B$ with all singular points contained in ∂U_0. Order the singular points $p_1, \ldots, p_l \in B$ by following the circle ∂U_0. Then there exist open sets $U_1, \ldots, U_l \subset B$ with the following properties. (1) $U_i \cap \{p_1, \ldots, p_l\} = \{p_i\}$, (2) $B \setminus U_0 \subset \bigcup_{i=1}^l U_i$, (3) $U_i \cap U_0$, $U_i \cap U_{i+1}$ and $U_l \cap U_1$ are contractible and disjoint from $\{p_1, \ldots, p_l\}$, (4) For pairwise disjoint $i, j, k \geq 1$ we have $U_i \cap U_j \cap U_k = \emptyset$. Then $\mathfrak{U} := \{U_0, U_1, \ldots, U_l\}$ is a Leray covering of B for $i_*\Lambda^*_{\mathbb{R}} := i_*\Lambda^* \otimes_{\mathbb{Z}} \mathbb{R}$ (cf. [GrSi1], Lemma 5.5). The terms in the Čech complex are

$$C^0(\mathfrak{U}, i_*\Lambda^\perp_{\mathbb{R}}) = \mathbb{R}^2 \times \prod_{i=1}^l \mathbb{R}^1$$

$$C^1(\mathfrak{U}, i_*\Lambda^\perp_{\mathbb{R}}) = \prod_{i=1}^l \mathbb{R}^2 \times \prod_{i=1}^l \mathbb{R}^2$$

$$C^2(\mathfrak{U}, i_*\Lambda^\perp_{\mathbb{R}}) = \prod_{i=1}^l \mathbb{R}^2.$$

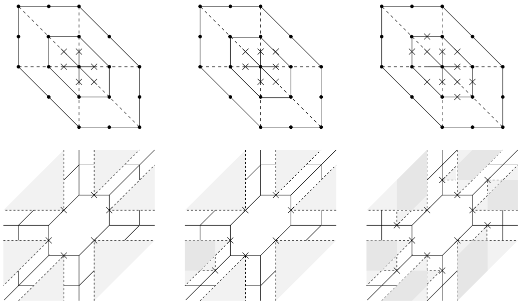

FIGURE 5.8. Straight boundary models for higher del Pezzo surfaces obtained by changing affine data for dP_3 and their Legendre duals.

It is easy to see that the Čech differential $C^0(\mathfrak{U}, i_*\Lambda_{\mathbb{R}}^{\perp}) \to C^1(\mathfrak{U}, i_*\Lambda_{\mathbb{R}}^{\perp})$ is injective while $C^1(\mathfrak{U}, i_*\Lambda_{\mathbb{R}}^{V}) \to C^2(\mathfrak{U}, i_*\Lambda_{\mathbb{R}}^{\perp})$ is surjective. Hence

$$\dim H^1(B, i_*\Lambda_{\mathbb{R}}^{\perp}) = 4l - 2l - (l+2) = l - 2$$

determines the number l of focus-focus points as claimed. □

REMARK 5.24. 1) From the analysis in Proposition 5.19 and Proposition 5.23 it is clear that for del Pezzo surfaces of degree at least four the anti-canonical polarization is too small to extend over a toric degeneration. The associated tropical manifold would simply not have enough integral points to admit the required number of singular points.

2) Essentially the same argument also computes the dimension of the space of infinitesimal deformations of the pair $(\check{\mathfrak{X}}_\eta, \check{\mathfrak{D}}_\eta)$:

$$h^1(\check{\mathfrak{X}}_\eta, \Theta_{\check{\mathfrak{X}}_\eta}(\log \check{\mathfrak{D}}_\eta)) = h^1(\check{X}_0, \Theta_{\check{X}_0^\dagger/\Bbbk^\dagger}) = h^1(B, i_*\Lambda_{\mathbb{R}}) = l - 2.$$ □

It is rather easy to write down toric degenerations of non-toric del Pezzo surfaces, since they all can be represented as hypersurfaces or complete intersections in weighted projective spaces, as for example done for dP_5 in [GrSi4, Example 4.2]. In this example dP_5 is realized as complete intersection of two quadrics in $\mathbb{P}^4$.

The most natural toric degenerations in this setup have as central fibre the toric boundary divisor of the ambient space. But because this construction gives reducible D_η such toric degenerations are never distinguished. To obtain proper superpotentials we therefore need a different approach.

CONSTRUCTION 5.25. Start from the intersection complex $(\check{B}, \check{\mathscr{P}})$ of the distinguished dP_3. The six focus-focus points in the interior of the bounded two cell make the boundary ρ straight, There is no space to introduce more singular points because all interior edges already contain a singular point. To get around this, polarize by $-2 \cdot K_{dP_3}$ and adapt $\mathscr{P}$ in the obvious way shown in Figure 5.8. This scales the affine manifold B by two, but keeps the singular points fixed. The new boundary now has 12 integral points and the union γ of edges neither intersecting the central vertex nor ∂B is a geodesic. We can then introduce new singular points as visualized in Figure 5.8. Moreover, let $\check{\varphi}$ be unchanged on the interior cells and change slope by one when passing to a maximal cell intersecting ∂B. Plugging in up to five singular points, one can show that toric degenerations obtained from the tropical data are in fact degenerations of dP_k, $4 \leq k \leq 9$ by Proposition 5.23. By σ_0 we will still refer to the same bounded cell of $\mathscr{P}$. In Figure 5.8 when two shaded regions, corresponding to the monodromy cuts of the respective focus-focus singularities, overlap, we shade them darker to indicate the non-trivial transformation there. □

Unlike in the anti-canonically polarized case, the models constructed in this way are not unique, so we can not expect to get a classification statement as Theorem 5.19. The geodesic γ is divided into six segments by $\mathscr{P}$, and the choice on which of these segments we place the singular points results in different models. Although there are other ways to define distinguished models for higher del Pezzo surfaces, for example by choosing another polarization or polyhedral decomposition, in this way we can extend the unique toric models most naturally, since all broken lines we studied before arise in these models without any change.

REMARK 5.26. Note that introducing six new points, for instance as in the rightmost picture in Figure 5.8, corresponds to a blow up of $\mathbb{P}^2$ in nine points, which is not Fano anymore, but from our point of view still has a Landau-Ginzburg mirror. From a different point of view this has already been noted in [AuKaOr], where the authors construct a compactification of the Hori-Vafa mirror as a symplectic Lefschetz fibrations as follows. Start

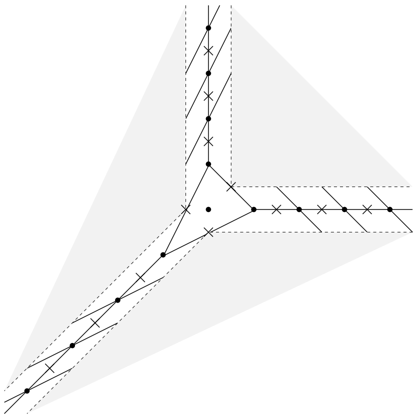

FIGURE 5.9. An alternative base for higher del Pezzo surfaces and their mirror.

with the standard potential $x + y + \frac{1}{xy}$ for $\mathbb{P}^2$ and compactify by a divisor at infinity consisting of nine rational curves. Then by a deformation argument it is possible to push k of those rational curves to the finite part and decompactify to obtain a potential for dP_k, including $k = 9$.

We can reproduce this result from our point of view by adapting the base of the distinguished toric degeneration for $\mathbb{P}^2$ instead of dP_3, as in Construction 5.25. Polarizing by $-4 \cdot K_{\mathbb{P}^2}$ scales the base $(\check{B}, \check{\mathscr{P}})$ by a factor four and we can adapt the polyhedral decomposition as in Construction 5.25. Moving rational curves from infinity to the finite part is corresponds to introducing new focus-focus points. In the present case one may put at most three focus-focus points on each unbounded ray of $(B, \mathscr{P})$ until the respective Legendre dual corner of $(\check{B}, \check{\mathscr{P}})$ becomes straight. Figure 5.9 shows nine such points, corresponding to the case $k = 9$ above. Any additional singular point would result in a concave boundary. This can be seen as an affine-geometrical explanation for why the compactification constructed by the authors in [AuKaOr] has exactly nine irreducible components. Note that it is possible to introduce more singular points when passing to larger polarizations. This leads to other toric degenerations, but it is not clear to us what the generic fibres of these degenerations would be in this case. □

We now turn to the task of determining the superpotential for toric degenerations of the non-toric del Pezzo surfaces dP_4 and dP_5. The point is that in these examples we can place the additional focus-focus singularities we introduce in Construction 5.25 in such a way, that the scattering diagram is finite in σ_0. This implies that we can use Lemma 5.1 to compute the superpotential to all orders. For toric degenerations of dP_k with $k > 6$ the scattering diagram locally in σ_0 seems to be infinite for any choice of position of focus-focus singularities and it is very hard to compute the superpotential even up to small orders. As we will see in Example 5.30, for dP_6 there are toric degenerations with finite and others with infinite scattering diagrams in σ_0, depending on the choice of focus-focus points.

EXAMPLE 5.27. Figure 5.10 shows the dual intersection complex $(B, \mathscr{P})$ of a toric degeneration of dP_4 from Construction 5.25. The additional focus-focus point on the unbounded ray in direction $(-1,-1)$ changes the structure $\mathscr{S}$, as it introduces a wall τ in σ_0 and allows broken lines to scatter with the wall in direction $(1,1)$ in the bounded cell σ_0, which yields new root tangent directions. A non-straight broken line coming from infinity in direction $(\pm 1, 0)$ will produce root tangent vector $(-1 \pm 1, -1)$, whereas one with direction $(0, \pm 1)$ will take direction $(-1, -1 \pm 1)$. By construction, every broken line reaching the interior cell σ_0 will have t-order at least 2. Moreover, note that the structure we arrived at by inserting the wall τ in σ_0 locally in σ_0 is consistent to all orders. Thus, we can apply Lemma 5.1 to compute the superpotential to all orders. To this end, note that a broken line can have at most one break point once it reaches the interior of σ_0, namely by non-trivially scattering with τ, in which case the t-order increases by one. We get two generically new root tangent directions, namely $(-1,-2)$ and $(-2,-1)$, as well as possibly more contributions from directions $(0,-1)$ and $(-1,0)$. The left picture in Figure 5.10 shows several broken lines with root tangent vector $(-1,-2)$ that are related to each other by moving the root vertex to different chambers.

The other two pictures in Figure 5.10 show the images in $(B, \mathscr{P})$ of all new broken lines for different choices of root vertex, apart from the six toric ones we have already encountered in Example 5.22. Denote the root vertex of the picture in the middle by p_1 and the one in the rightmost picture by p_2. Moreover, call the chamber p_i lies in $\mathfrak{u}_i$, for $i = 1, 2$. Depending on

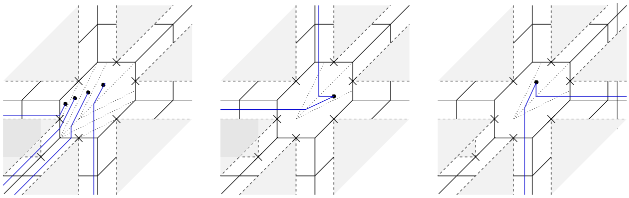

FIGURE 5.10. Mirror base to dP_4 showing new broken lines contributing to $W^2_{dP_4}$, indicating the wall crossing pheomenon and the invariance under change of endpoint within a chamber.

this choice, the superpotential W_{dP4} locally is given by

$$W_{dP4}(p_1) = W^3_{dP4}(p_1) = \Big(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\Big) \cdot t^2 + \Big(\frac{1}{x} + \frac{1}{x^2y}\Big) \cdot t^3 \quad \text{or}$$

$$W_{dP4}(p_2) = W^3_{dP4}(p_2) = \Big(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\Big) \cdot t^2 + \Big(\frac{1}{y} + \frac{1}{xy^2}\Big) \cdot t^3,$$

where as before x and y correspond to $(1,0)$ and $(0,1)$. These superpotentials are not only related by simply interchanging x and y, for symmetry reasons, but also by wall crossing along the wall τ separating the chambers $\mathfrak{u}_1$ and $\mathfrak{u}_2$. □

EXAMPLE 5.28. Attaching another singular point on the unbounded ray in direction $(0,-1)$ as shown in the four pictures in Figure 5.11 we arrive at a degeneration of dP_5. For the structure to be consistent to all orders locally in the bounded cell σ_0 it is necessary to introduce three walls as follows. Two of them are the extensions of the slabs with tangent directions $(1,1)$ and $(0,1)$ caused by the additional two focus-focus points we introduced. The third is the result of scattering of the other two, that is the wall with tangent direction $(1,2)$, indicated in red in the lower left picture of Figure 5.11. Because $(1,1)$ and $(0,1)$ form a lattice basis the scattering procedure at the origin does not produce any additional walls. Thus we can apply Lemma 5.1 to reduce the number of broken lines to be considered.

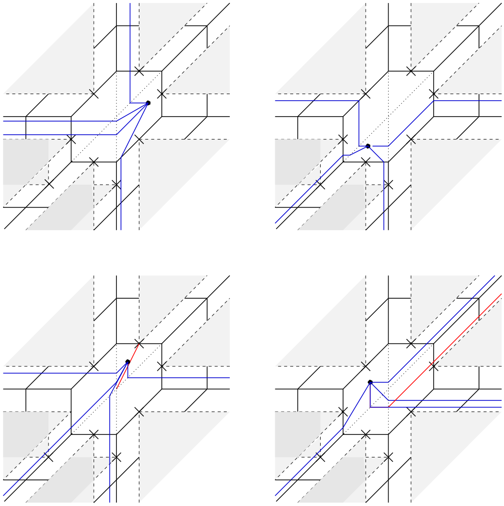

FIGURE 5.11. Mirror bases to dP_5 showing all new broken lines.

We first compute superpotentials up to order three for four choices of root vertices in different chambers, as indicated in Figure 5.11. Starting in the upper left picture, going in counter-clockwise direction, denote the choices of root vertex in the i-th picture by p_i and the corresponding chamber these roots lie in by $\mathfrak{u}_i$, for $i = 1, 2, 3, 4$. Then Figure 5.11 explicitly shows all non-toric broken lines with root p_i up to t-order three in blue and we get the following superpotentials:

$$\begin{aligned}
W^3_{dP5}(p_1) &= (x+y+xy+\frac{1}{x}+\frac{1}{y}+\frac{1}{xy})\cdot t^2 \\
&\quad +(\frac{1}{x}+\frac{1}{xy}+\frac{1}{x^2y}+\frac{1}{xy^2})\cdot t^3, \\
W^3_{dP5}(p_2) &= (x+y+xy+\frac{1}{x}+\frac{1}{y}+\frac{1}{xy})\cdot t^2 \\
&\quad +(\frac{1}{y}+\frac{1}{xy}+2\cdot\frac{1}{xy^2})\cdot t^3, \\
W^3_{dP5}(p_3) &= (x+y+xy+\frac{1}{x}+\frac{1}{y}+\frac{1}{xy})\cdot t^2 \\
&\quad +(\frac{1}{y}+x+\frac{x}{y}+\frac{1}{xy^2})\cdot t^3, \\
W^3_{dP5}(p_4) &= (x+y+xy+\frac{1}{x}+\frac{1}{y}+\frac{1}{xy})\cdot t^2 \\
&\quad +(\frac{1}{x}+x+\frac{1}{x^2y}+\frac{x}{y})\cdot t^3,
\end{aligned}$$

Of course, these superpotentials can be obtained from each other by wall crossing. Going to t-order four, there will be more broken lines. In the lower right picture of Figure 5.11 we indicated the new broken line in red which contributes as $\frac{1}{y}\cdot t^4$ to the superpotential. Note that this monomial is invariant under crossing the wall in direction $(0,1)$. In fact, we do not get any more broken lines for higher t-orders by Lemma 5.1 and thus we get the full potential

$$\begin{aligned}
W_{dP5}(p_3) = W^4_{dP5}(p_3) &= (x+y+xy+\frac{1}{x}+\frac{1}{y}+\frac{1}{xy})\cdot t^2 \\
&\quad +(\frac{1}{y}+x+\frac{x}{y}+\frac{1}{xy^2})\cdot t^3+\frac{1}{y}\cdot t^4.
\end{aligned}$$

□

EXAMPLE 5.29. We study another model of the mirror to dP_5, which differs from the last one only in the position of the second focus-focus point. Instead of placing it on the ray with generator $(0,-1)$ we move it to the ray generated by $(1,1)$, as shown in Figure 5.12 on the right. With choice p of root vertex as indicated in the picture, this more particular choice of

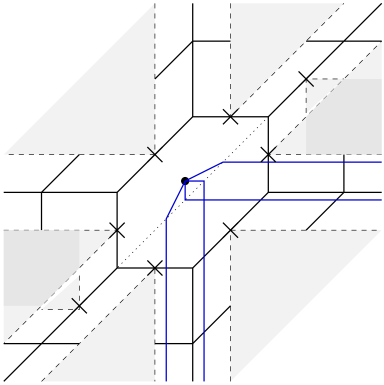

FIGURE 5.12. An alternative mirror base for dP_5

focus-focus points yields the superpotential

$$W_{dP_5}(p) = W^3_{dP_5}(p) = \left(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right) \cdot t^2 \\ + \left(\frac{1}{y} + x + x^2y + \frac{1}{xy^2}\right) \cdot t^3.$$

It is an interesting question to understand in detail the effect of particular choices of singular points and the corresponding degenerations. □

EXAMPLE 5.30. As a last example, we study broken lines in the mirror of a distinguished model of dP_6. For the choice of focus-focus points and root vertex p_1 as depicted on the left in Figure 5.13, we this time obtain the following superpotential up to order five:

$$W^5_{dP_6}(p_1) = \left(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right) \cdot t^2 \\ + \left(2 \cdot \frac{1}{xy^2} + \frac{1}{xy} + 2 \cdot \frac{1}{y}\right) \cdot t^3 + \frac{1}{y} \cdot t^4 + \frac{1}{y} \cdot t^5.$$

Again, this potential comes from a special choice of positions of singular points and root vertex. However, for this choice we can not apply Lemma 5.1, as the three walls meeting at the origin scatter infinitely often. It is already very complicated to determine all broken lines up to t-order six.

In contrast, the picture on the right of Figure 5.13 shows the mirror base of a dP_6-degeneration with a different placement of focus-focus singularities. Introducing walls by extending the slabs in direction $(1,1)$, $(-1,0)$ and $(0,-1)$ makes the scattering diagram locally in the bounded cell σ_0 finite, so we can again apply Lemma 5.1 to obtain the following superpotential for the choice of root vertex p_2 as in the picture

$$\begin{aligned} W_{dP_6}(p_2) = W^3_{dP_6}(p_2) &= \Big(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\Big) \cdot t^2 \\ &\quad + \Big(\frac{y}{x} + \frac{1}{x} + \frac{1}{xy^2} + 2 \cdot \frac{1}{y} + \frac{x}{y}\Big) \cdot t^3. \quad \square \end{aligned}$$

These examples illustrate that if we leave the realm of toric geometry, Landau-Ginzburg potentials for del Pezzo surfaces can, at least locally, still be described by Laurent polynomials, as in the toric setting. After completion of the work on this chapter, the preprint [GrHaKe] was available online. It covers much what we what we have done and goes beyond that in some regards.

5.3. Semi-Fano and Hirzebruch surfaces

Recall that in Construction 5.2 we dealt with polytopes Θ that gave rise to *smooth* Fano toric varieties $\mathbb{P}_{\Sigma_\Theta}$. Restricting to dimension two, we will now see that we can also treat the case of *semi-Fano* and even *non-Fano* toric surfaces.

5.3.1. MPCP subdivisions and smooth semi-Fano surfaces. Recall that by Lemma 2.9 and Proposition 2.17 a projective toric variety with fan Σ is Fano with at most canonical singularities if and only if the span of ray generators Θ_Σ is reflexive. There are 16 reflexive polytopes Θ in dimension two, among which the fan Σ_Θ of cones over faces gives rise to a smooth variety in precisely the five cases we have been studying in section 5.2.1. The remaining eleven polytopes are characterized by the property that Θ has at least one integral boundary point that is not a vertex. So in these eleven cases $\mathbb{P}_{\Sigma_\Theta}$ will be $\mathbb{Q}$-Gorenstein $\mathbb{Q}$-Fano. If we applied Construction 5.2 to a Θ with integral boundary points, the discriminant locus Δ would not be simple, as the facets of Θ are not elementary simplices anymore. To

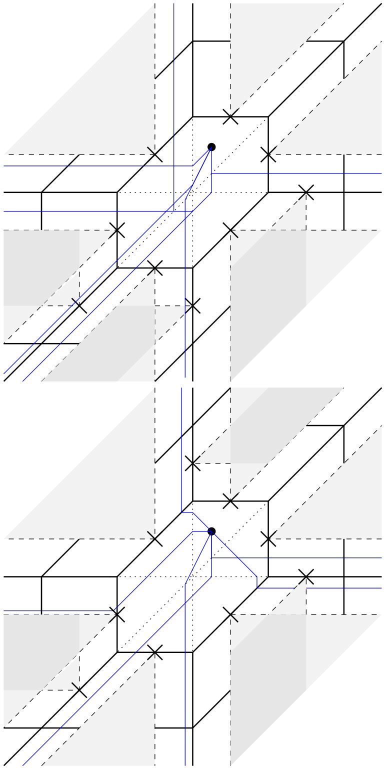

FIGURE 5.13. Two mirror bases to dP_6.

resolve this issue, let $(\mathcal{T}, h_{\mathcal{T}})$ be a maximal triangulation of Θ, which we can choose by Lemma 2.18, and denote by $\widetilde{\Sigma}$ the fan of cones over faces contained in the boundary of Θ, that is

$$\widetilde{\Sigma} = \{\sigma_\tau | \tau \in \mathcal{T}, \tau \subset \partial\Theta\},$$

where σ_τ denotes the cone over τ. Then $\mathbb{P}_{\widetilde{\Sigma}}$ is a smooth toric surface. Moreover, it is clear that $\mathbb{P}_{\widetilde{\Sigma}}$ is *semi-Fano* since the anticanonical divisor $-K_{\mathbb{P}_{\widetilde{\Sigma}}}$ defines a not convex support function on Σ that is not strictly convex.

With this notation we can now adapt Construction 5.2 apply to this semi-Fano setting.

CONSTRUCTION 5.31. Let $\Theta \subset N_{\mathbb{R}}$ be a two-dimensional reflexive polytope and choose an MPCP-subdivision $(\mathcal{T}, h_{\mathcal{T}})$ of Θ, which induces a maximal triangulation of $\partial\Theta$ and defines the fan $\widetilde{\Sigma}$. We construct a tropical manifold $(\widetilde{B}^{\mathcal{T}}, \widetilde{\mathscr{P}}^{\mathcal{T}})$ as follows.
Start from the tropical manifold $(B, \mathscr{P})$ obtained from Construction 5.2. Then the bounded cells of $\widetilde{\mathscr{P}}^{\mathcal{T}}$ are defined to be the polytopes in the triangulation $\mathcal{T}$ with induced charts, which refines $\mathscr{P}$. Moreover, for each ray $\rho \in \widetilde{\Sigma}(1) \setminus \Sigma_{\Theta}(1)$ with ray generator n_ρ add an unbounded one-cell in direction of ρ starting at n_ρ. Next we describe the discriminant locus $\widetilde{\Delta}^{\mathcal{T}}$. Put a focus-focus singularity on the barycentre of each one-cell corresponding to a polytope in $\tau \in \mathcal{T}$ with $\tau \subset \partial\Theta$. This choice makes the unbounded one-cells of $\widetilde{\mathscr{P}}^{\mathcal{T}}$ parallel, as is elementary to check in each case. To define $\widetilde{\varphi}$ we use the height function $h_{\mathcal{T}}$. For each $\nu \in \partial\Theta \cap N$ define $\widetilde{\varphi}^{\mathcal{T}}(\nu) := h_{\mathcal{T}}(\nu)$, $\widetilde{\varphi}^{\mathcal{T}}(0) = 0$ and let $\widetilde{\varphi}^{\mathcal{T}}$ change slope by one on the unbounded part. As $h_{\mathcal{T}}$ is strictly convex, so is $\widetilde{\varphi}^{\mathcal{T}}$. □

REMARK 5.32. An analogous construction should work in higher dimensions, there are however certain technical issues we can not resolve in this case, such as proving simplicity of the discriminant locus. □

EXAMPLE 5.33. Let Θ be the reflexive polytope spanned by $(-1,1)$, $(1,1)$ and $(0,-1)$. Note that Θ has $(0,1)$ as boundary point. Now, choose the MPCP subdivision $\mathcal{T}$ of Θ that assigns value 2 to $(-1,1)$ and 1 to all other lattice points of the boundary of Θ. On the two-dimensional toric variety $\mathbb{P}_{\widetilde{\Sigma}}$ associated to the complete fan $\widetilde{\Sigma}$ with ray generators $(-1,1)$, $(0,1)$, $(1,1)$ and $(0,-1)$, corresponding to toric divisors D_1, D_2, D_3 and D_4, the MPCP subdivision $\mathcal{T}$ corresponds to the ample divisor $D := 2 \cdot D_1 + D_2 + D_3 + D_4$. Note that $-K_{\mathbb{P}_{\widetilde{\Sigma}}}$ is *not* an ample divisor. Applying Construction 5.31 in this situation yields $(\widetilde{B}^{\mathcal{T}}, \widetilde{\mathscr{P}}^{\mathcal{T}}, \widetilde{\varphi}^{\mathcal{T}})$ as depicted in Figure 5.14. The polytopes in $\widetilde{\mathscr{P}}^{\mathcal{T}}$ are simplicial and the singular locus $\widetilde{\Delta}^{\mathcal{T}}$ consists of four isolated focus-focus singularities. It is in fact easy to check that $\mathbb{P}_{\widetilde{\Sigma}} \cong \mathbb{F}_2$, the only semi-Fano Hirzebruch surface. □

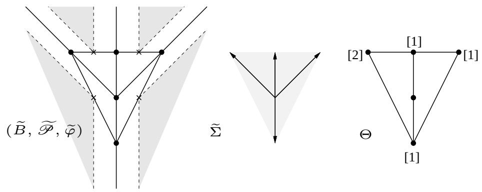

FIGURE 5.14. An irreducible tropical manifold for a semi-Fano surface degeneration

Having studied a first example, in Figure 5.15 we summarize the tropical affine manifolds

$$(\widetilde{B}_i, \widetilde{\mathscr{P}}_i, \widetilde{\varphi}_i)$$

obtained from applying Construction 5.31 to the eleven reflexive polytopes Θ_i for $i = 1, \ldots, 11$, with a particular choice of MPCP subdivision $\mathcal{T}_i$ as indicated. In the pictures Θ_i is the union of all bounded cells of $\widetilde{\mathscr{P}}_i$ for all $i = 1, \ldots, 11$. Note that we suppress the index $\mathcal{T}_i$ for this particular choice. The corresponding fans will be denoted by $\widetilde{\Sigma}_i$ and the associated toric varieties accordingly by $\mathbb{P}_{\widetilde{\Sigma}_i}$. The values $[a_\nu]$ in square brackets in Figure 5.15, defining the PL-function $\widetilde{\varphi}_i$, have the following interpretation.

Each integral boundary point $\nu \in \partial\Theta_i$ corresponds to a ray generator of a ray $\rho_\nu \in \widetilde{\Sigma}_i(1)$ and hence to a toric divisor D_ν. The value $[a_\nu]$ associated to ν is chosen such that

$$D_i := \sum_{\rho_\nu \widetilde{\Sigma}_i(1)} [a_\nu] \cdot D_\nu \tag{5.3}$$

is an ample toric divisor on $\mathbb{P}_{\widetilde{\Sigma}_i}$, as this is equivalent to saying that $\widetilde{\varphi}$ is strictly convex. Another equivalent formulation is that the polytope Ξ_{D_i} associated to D_i, see Definition 2.10, is a convex polyhedron whose facets are in one-to-one correspondence with integral points of $\partial\Theta_i$.

We will now proceed by computing superpotentials for the eleven bases. For the following lemma define a weight function l_τ, similar to Definition 5.9, as follows. Assume τ is a line segment with n integral points and ν is the k-th of these counting from one endpoint of τ, we set $l_\tau(\nu) = \binom{n}{k}$.

LEMMA 5.34. *Let $(\widetilde{B}_i^{\mathcal{T}}, \widetilde{\mathscr{P}}_i^{\mathcal{T}}, \widetilde{\varphi}_i^{\mathcal{T}})$ be the polarized tropical manifold associated to the polytope Θ_i and choice of MPCP subdivision $(\mathcal{T}_i, h_{\mathcal{T}_i})$ from*

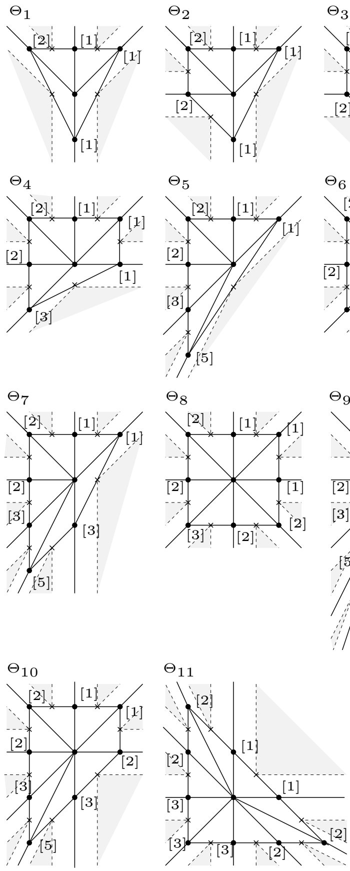

FIGURE 5.15. Dual intersection complexes $(\widetilde{B}, \widetilde{\mathscr{P}}, \widetilde{\varphi})$ for all eleven smooth semi-Fano toric surfaces .

Construction 5.31. Choose a root vertex p in a small open neighbourhood U of the origin and let β be a broken line with root p and root tangent vector $m_\beta = \nu$, where $\nu \in \partial\Theta \cap N$. Then β crosses the wall τ corresponding to

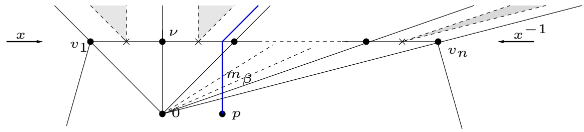

FIGURE 5.16

the facet of Θ *on which* ν *lies and we have*

$$a_\beta = l_\tau(\nu).$$

PROOF. Denote the integral points of τ by $v_1, \ldots v_n$ and assume that $\nu = v_k$ for some $k = 1, \ldots, n$. Moreover, denote the monodromy invariant direction of the $n-1$ focus-focus points on τ by x and its inverse by x^{-1}. By our convention on slab functions we have that

$$f_{\tau,v_1} = (1+x)\cdot(1+t^{e_1}x)\cdot\ldots\cdot(1+t^{e_{n-1}}x)$$

and accordingly

$$f_{\tau,v_n} = (1+x^{-1})\cdot(1+t^{e_1}x^{-1})\cdot\ldots\cdot(1+t^{e_{n-1}}x^{-1}).$$

The precise value of the e_i depends on the MPCP height function $h_{\mathcal{T}}$ we choose, as this influences $\widetilde{\varphi}^{\mathcal{T}}$. We visualized this situation in Figure 5.16. Note that we can apply Lemma 5.1, as scattering on the bounded cells of $\widetilde{\mathscr{P}}_i^{\mathcal{T}}$ will always be finite and $\widetilde{B}_i^{\mathcal{T}}$ has parallel one-dimensional strata for all $i = 1, \ldots, 11$. Thus, if p is close to the origin 0, then β passes through τ without break point. If it had a break point, its first break point would necessarily have to be outside of the bounded part of $\widetilde{\mathscr{P}}_i^{\mathcal{T}}$, which contradicts Lemma 5.1. This means that a_β is the constant term of

$$(1+x)\cdot(1+t^{e_1}x)\cdot\ldots\cdot(1+t^{e_{k-1}}x)\cdot(1+t^{e_k}x^{-1})\cdot\ldots\cdot(1+t^{e_{n-1}}x^{-1}),$$

which is the coefficient of the x^k-term in f_{τ,v_1}. This in turn is nothing but $\binom{n}{k}$, that is $a_\beta = l_\tau(v_k) = l_\tau(\nu)$. $\square$

Thus, if we choose the root vertex p to be in a small neighbourhood of 0, then we get broken lines without break point in the direction of all rays of $\widetilde{\Sigma}_i$ for all $i = 1, \ldots, 11$. Moreover, by Lemma 5.34 we know the values a_β for each such broken line β. We therefore obtain the following analogue of Theorem 5.21. Note that we get the full potential, as we can apply Lemma 5.1 once again.

COROLLARY 5.35. *Let Θ be a two-dimensional reflexive polytope, choose an MPCP subdivision $(\mathcal{T}, h_{\mathcal{T}})$ and denote the tropical manifold obtained from Construction 5.31 by $(\widetilde{B}, \widetilde{\mathscr{P}}, \widetilde{\varphi})$. Then there is an open subset $U \cong \operatorname{Spec} \Bbbk[[t]][x_1, x_2]$ such that the full superpotential W for $(\widetilde{B}, \widetilde{\mathscr{P}}, \widetilde{\varphi})$ is given by*

$$W|_U = \sum_{\nu \in \Theta \cap N} a_\nu \cdot t^{h_{\mathcal{T}}(\nu)} x^\nu,$$

where $a_\nu = l_\tau(\nu)$ for a choice of a one-dimensional face $\tau \subset \Theta$ containing ν. □

REMARK 5.36. 1) Note that if Θ is such that $\mathbb{P}_{\Sigma_\Theta}$ is smooth, then it follows that $a_\nu = 1$ for all integral boundary points of Θ. Moreover, in this case we can choose $h_{\mathcal{T}}(\nu) = 1$ for all ν. So in this case the Corollary 5.35 recovers Theorem 5.21 for del Pezzo surfaces.

2) In [ChLa], based on work in [Ch], Chan and Lau compute superpotentials for each of the eleven semi-Fano toric surfaces, without explicitly stressing the importance of the choice of MPCP subdivision, respectively height function. Our approach in fact computes compactfications of the superpotentials obtained in [ChLa] for the same choice of D_i as in (5.3). The polytopes in [ChLa, Figure 7] then correspond, up to lattice automorphism, to Ξ_{D_i}. □

5.3.2. Hirzebruch surfaces. As a last application we will study proper superpotentials for Hirzebruch surfaces $\mathbb{F}_m$. We fix the fan Σ in $N \cong \mathbb{Z}^2$ of $\mathbb{F}_m$ to be the fan generated by the four primitive vectors $\rho_0 = (0, 1)$, $\rho_1 = (-1, 0)$, $\rho_2 = (0, -1)$ and $\rho_3 = (1, m)$. Since $\mathbb{F}_m$ is only Fano in the cases $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1 = dP_1$, for $m \geq 2$ the normal fan of the anti-canonical polytope will not be the fan of a Hirzeburch surface. Denote by D_{ρ_i} the torus-invariant divisor associated to ρ_i. Instead of the anti-canonical divisor, which is not ample for $m \geq 2$, we consider a smooth divisor D_m in the ample class

$$D_{\rho_0} + D_{\rho_1} + D_{\rho_2} + m \cdot D_{\rho_3}.$$

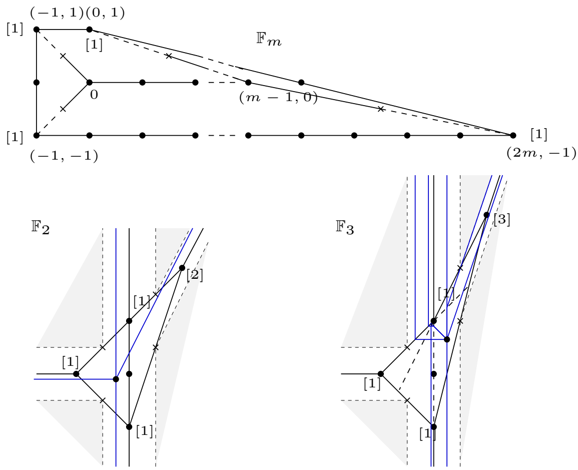

FIGURE 5.17. A straight boundary model for the Hirzebuch surfaces $\mathbb{F}_m$ with mirrors for $m = 2, 3$.

Define a tropical affine manifold $(\check{B}_m, \check{\mathscr{P}}_m, \check{\varphi}_m)$ with straight boundary as follows. $\check{B}_m$ is obtained from the Newton polytope Ξ_{D_m} by joining it with the interval $[0, m-1]$ as shown in Figure 5.17 on the top, and then introducing a single focus-focus singularity on each of the four joining one-cells. The polyhedral decomposition $\check{\mathscr{P}}_m$ is the one induced by this construction and it is again elementary to check that this makes $\partial\check{B}_m$ totally geodesic. Clearly, setting $\check{\varphi}_m(v) = 1$ for all vertices v of ∂B_m and

$$\check{\varphi}_m((0,0)) = \check{\varphi}_m((m-1,0)) = 0$$

defines a strictly convex PL function on $(\check{B}_m, \check{\mathscr{P}}_m)$. From this description it follows that the Legendre dual $(B_m, \mathscr{P}_m, \varphi_m)$ has two bounded cells

$$\sigma_0 = \operatorname{conv}(\tau_0, \tau_1, \tau_2) \quad \text{and} \quad \sigma_1 = \operatorname{conv}(\tau_0, \tau_2, \tau_3),$$

four unbounded one-cells in direction of the τ_i's, seen from a bounded cell. These one-cells are indeed parallel in a chart on the complement of the ray generated by any of the τ_i. Moreover, $\partial(\sigma_0 \cup \sigma_1)$ has precisely four integral points, namely τ_0, τ_1, τ_2 and τ_3. Furthermore set $\varphi_m(0) = \varphi_m(\tau_0) =$

$\varphi_m(\tau_1) = \varphi_m(\tau_2) = 1$, $\varphi_m(\tau_3) = m$ and let φ_m change slope by one at the boundary of $\sigma_0 \cup \sigma_1$.

Thus we see that for $m \geq 3$ the union $\sigma_0 \cup \sigma_1$ is non-convex, which causes scattering phenomena in the interior of this bounded part. This makes it more difficult to compute the potential to higher t-order in this case and distinguishes it from other situations we considered so far. We now explicitly compute the full superpotential for $\mathbb{F}_2$ and $\mathbb{F}_3$.

First, for the base $(B_2, \mathscr{P}_2)$ and the choice of root vertex $p \in \sigma_0$ as depicted on the lower left in Figure 5.17 the Landau-Ginzburg superpotential is given by

$$W(p) = W^2(p) = \Big(\frac{1}{x} + \frac{1}{y} + y\Big) \cdot t + xy^m \cdot t^2.$$

This is indeed the full potential, as there is no scattering in both σ_0 and σ_1, so we can apply Lemma 5.1.

As remarked above, for $m = 3$ the bounded part of B_3 is subject to scattering. To see this, note that at the point $(0,1)$ two walls in directions $(1,1)$ and $(-1,-2)$ with focus-focus points on them meet. Locally this is equivalent to the situation of walls in direction $(1,0)$ and $(0,1)$ meeting at the origin. An explicit computation carried out in [GrSi4] shows that this scattering diagram can be made consistent to all orders by introducing outgoing walls in directions $(1,0)$, $(0,1)$ and $(1,1)$, which translates to walls in directions $(1,1)$, $(-1,-2)$ and $(0,-1)$ in our situation, as indicated by the dashed lines in the lower right of Figure 5.17. Of course it will be necessary to insert more walls *outside* of the bounded part, but this is unessential for the computation of the superpotential. Hence scattering on the bounded part $\sigma_0 \cup \sigma_1$ is finite and we can once more apply Lemma 5.1. For the choice of root vertex p' as indicated in the lower right picture of Figure 5.17 we get the following full superpotential for $(B_3, \mathscr{P}_3)$

$$W(p') = W^3(p') = \Big(\frac{1}{x} + \frac{1}{y} + y + \frac{y}{x}\Big) \cdot t + \frac{y}{x} \cdot t^2 + \Big(xy^3 + y^2\Big) \cdot t^3.$$

Thus, neglecting t-orders for a moment, we have three new contributions that differ from the Hori-Vafa potential $\frac{1}{x} + \frac{1}{y} + y + xy^3$, namely the monomial y^2 and two times the term $\frac{y}{x}$, which come from broken lines that have a break point at the new walls emanating from $(0,1)$ in direction $(1,1)$, $(0,-1)$ and $(-1,-2)$, respectively. Note that these are precisely the terms Auroux found in [Au, Proposition 3.2], when we make the coordinate change $x \mapsto \frac{1}{x}$ and $y \mapsto \frac{1}{y}$. The computation in [Au, Proposition 3.2] is very explicit and rather long when compared with our derivation. Of course all the hard work is carefully hidden in the Gross-Siebert program, but still it is remarkably easy to compute Landau-Ginzburg models with this approach, once everything is set up.

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Zusammenfassung

Die vorliegende Arbeit behandelt algebraisch-geometrische Konstruktionen, um das Spiegelsymmetrie-Phänomen aus der Stringtheorie zu verstehen. Vornehmlich geht es darum einen gemeinsamen Rahmen für klassische wie moderne Ansätze aus der torischen Geometrie zu finden und im erweiterten Kontext des Spiegelsymmetrie-Programms nach Gross und Siebert zu erklären. Die Arbeit deckt dabei sowohl Spiegelsymmetrie für Calabi-Yau Mannigfaltigkeiten als auch sogenannte Landau-Ginzburg Modelle ab, die als Spiegel von Fano-Mannigfaltigkeiten auftreten.

Die Dissertation gliedert sich in zwei Hauptteile. Im ersten Teil werden mittlerweile als klassisch anzusehende Konstruktionen der Spiegelsymmetrie für Calabi-Yau Mannigfaltigkeiten mit Methoden der torischen Geometrie studiert. Nach einer kurzen Einführung in die hier relevanten Begriffe torischer Geometrie, werden die für diesen Teil wichtigsten Definitionen anhand von Beispielen erläutert. Im Folgenden werden die Zugänge von Batyrev und Berglund-Hübsch im Detail beleuchtet und miteinander verknüpft. Eine milde Verallgemeinerung des ersten und eine geometrische Reformulierung des zweiten Ansatzes erlauben es beide in einem gemeinsamen Kontext aufzufassen. Die Spiegelpartner bei Berglund-Hübsch sind demnach Hyperflächen sehr spezieller Struktur in torischen Varietäten, während die Spiegelpartner in Batyrevs Arbeit allgemeine Hyperflächen in den selben torischen Varietäten sind. Insbesondere sind die Spigelpartner beider Konstruktionen jeweils polynomiale Deformationen voneinander und stimmen in Spezialfällen sogar überein. Als Anwendung dieses verallgemeinerten Standpunkts werden zudem Borcea-Voisin Calabi-Yau Mannigfaltigkeiten und deren Verallgemeinerung diskutiert. Sind gewisse technische Voraussetzungen erfüllt lassen sich Spiegelpartner für diese Klasse von Mannigfaltigkeiten explizit mit den vorgestellten Techniken konstruieren.

Der zweite, weitaus technischere Teil der Arbeit behandelt die Erweiterung des Gross-Siebert Programms für torische Entartungen von Varietäten mit effektivem anti-kanonischen Bündel. Zur Konstruktion des sogenannten

Superpotentials wird das Gross-Siebert Programm in leicht verallgemeinerter Form herangezogen. Landau-Ginzburg Modelle können dadurch anhand des Rekonstruktions-Algorithmus nach Gross-Siebert induktiv definiert werden. Unter gewissen technischen Bedingungen lässt sich die Endlichkeit der Beiträge zum Superpotential beweisen, was eine vollständige Berechnung erlaubt. In diesem Kontext werden viele klassische und einige neue Beispiele detailliert behandelt. Eine Vielzahl der bislang bekannten Beispielklassen für Landau-Ginzburg Modelle wird durch diese Konstruktion abgedeckt.

In erster Anwendung wird die auf Hori und Vafa zurückzuführende Konstruktion für glatte torische Varietäten, deren Impulspolytope reflexiv sind, derart verallgemeinert, dass das Superpotential eine eigentliche Abbildung wird, die Fasern in komplexer Geometrie demnach kompakt sind. Der zweite Anwendungsbereich liegt im Studium komplexer Flächen und der expliziten Konstruktion von Spiegelpartnern. Ist die allgemeine Faser der torischen Entartung eine glatte torische Fläche, lassen sich unter Zusatzbedingungen Eindeutigkeitsaussagen über den zugehörigen Schnittkomplex der Entartung ableiten. Im Falle allgemeiner Fasern, die nicht isomporph zu torischen Flächen sind, lassen sich Spiegelpartner weiterhin explizit berechnen. Während vergleichbare Resultate in der Literatur auf häufig auf massivem Rechenaufwand beruhen, steckt die Komplexität der in der Arbeit abgeckten Methode im Gross-Siebert Programm selbst, die Berechnung der Superpotentiale ist vergleichsweise handhabbar und anschaulich. So lassen sich etwa Spiegelpartner für torische semi-Fano- und Hirzebruch-Flächen ohne weiteres im vorgestellten Rahmen berechnen.

Lebenslauf

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