Conjugacy Classes of Non-Connected Semisimple Algebraic Groups

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Introduction

We consider non-connected algebraic groups $\tilde{G}$ over an algebraically closed field k, whose unit component G is semisimple. The aim of this work is to understand the conjugacy classes of the adjoint G action on an exterior component of $\tilde{G}$. Non-connected semisimple groups appear naturally as centralizers of non-simply connected, connected, semisimple groups.

For a connected algebraic group G, the description of its conjugacy classes is well known, see [35, 38]. We give a brief description of it:
The action of G on itself by conjugacy will be called the adjoint action and denoted by Ad:

$$\begin{aligned} G \times G &\to G \\ (g,\, h) &\mapsto \quad Ad(g)(h) = g\, h\, g^{-1}. \end{aligned}$$

We will call the corresponding quotient $G /\!\!/ Ad(G) := \operatorname{Spec} k[G]^{Ad(G)}$ the adjoint quotient and denote the quotient map by $\pi : G \to G /\!\!/ Ad(G)$. The quotient admits the following properties:

- For a maximal torus T of G and the Weyl group $\mathcal{W} := N_G(T)/T$, we also have a quotient $T/\mathcal{W}$. (Here $w \in \mathcal{W}$ acts on T by conjugacy with a representative $n_w \in N_G(T)$.) Now the inclusion $T \hookrightarrow G$ induces an isomorphism on the quotients:

$$T/\mathcal{W} \xrightarrow{\sim} G /\!\!/ Ad(G).$$

- Each fibre of π consists of only finitely many conjugacy classes, where the closed class is exactly the semisimple one. Furthermore, we have the explicit description of the reduced fibre $\pi^{-1}(\pi(t))_{red}$ for $t \in T$ as an associated bundle:

$$\pi^{-1}(\pi(t))_{red} \cong G \times^{C_G(t)} V(t),$$

  where $V(t)$ denotes the unipotent variety of $C_G(t)$.

- If G is simply connected, even more is true:

- The adjoint quotient $G/\!/Ad(G)$ is an affine space $\mathbb{A}^r$, where r is the rank of G and π is given by:

$$\pi = (X^{\lambda_1}, ..., X^{\lambda_r}).$$

  Here the X^{λ_i} are the characters of the fundamental representations of G.

- π admits a cross section:

$$C : T/\mathcal{W} \to G,$$

  i.e. a morphism, such that $Im\, C$ is closed in G and such that $\pi|_{Im\, C}$ is an isomorphism. This cross section meets every fibre in exactly one element, a regular one. (An element of G is called regular, if its centralizer has minimal possible dimension, which is r.)

- The fibres of π are reduced and normal, and π itself is flat.

The main ideas of Steinberg's to obtain these results are:

- *Use of representation theory:* Exploitation of the fact, that the characters of the irreducible representations $\{X^\lambda, \lambda \text{ dominant weight}\}$ form a basis of $k[G]^{Ad(G)}$ and their restrictions to T one of $k[T]^{\mathcal{W}}$.

- *The explicit construction of C:* Let $\Pi = \{\alpha_1, ..., \alpha_r\}$ be the set of simple roots, $X_{\alpha_i} : k \to G$ the corresponding root groups and $n_{\alpha_i} \in N_G(T)$ representatives of the simple reflexions $s_{\alpha_i} \in \mathcal{W}$. Then $C : \mathbb{A}^r \to G$ is given by:

$$C(c_1, ..., c_r) = X_{\alpha_1}(c_1)n_{\alpha_1}...X_{\alpha_r}(c_r)n_{\alpha_r}.$$

This description of the conjugacy action of G was used by Slodowy in [27] (following Brieskorn [4]) to construct a connection between the theory of simple algebraic groups and that of simple singularities, which goes as follows:
Take a transversal slice S in G to a unipotent subregular element u (i.e. one whose centralizer is of second minimal possible dimension). If we consider the intersection $S \cap V$, where V is the unipotent variety of G, then $S \cap V$ has a simple singularity of the same type as G in u. Furthermore, the restriction $\sigma = \pi|_S : S \to T/\mathcal{W}$ provides the semiuniversal deformation of this singularity.

We want to derive similar results in the case of the adjoint action of the unit component G of a non-connected semisimple algebraic group $\tilde{G}$ on one of its exterior components. In this work, we assume $\tilde{G}$ to be a semidirect product:

$$\tilde{G} = G \rtimes \Gamma,$$

where Γ is a subgroup of the group of diagram automorphisms of the Dynkin diagram $\Delta(G)$ of G (lifted into $Aut(G)$). We require the characteristic of k not to divide the order of Γ and to be $\neq 2$.
Take a nontrivial element $\tau \in \Gamma$ and consider the the adjoint G action on $G\tau := G \times \{\tau\}$:

$$\begin{array}{rcl} G \times G\tau & \to & G\tau \\ (g,\, h\tau) & \mapsto & g\, h\tau\, g^{-1}. \end{array}$$

Observe that, using the identification of $G\tau$ with G by right multiplication with τ^{-1}, we end up with a "τ-twisted" G action on itself:

$$\begin{array}{rcl} G \times G\tau & \to & G \\ (g,\, h\tau) & \mapsto & g\, h\, \tau(g^{-1}). \end{array}$$

The main questions at the origin of this work can now be formulated as follows:

- How does the corresponding adjoint quotient $G\tau/\!\!/G$ look like?
- What is the structure of the fibres of the corresponding quotient map $\pi : G\tau \to G\tau/\!\!/G$?
- Does this quotient admit a kind of cross section similar to the one of Steinberg's in the connected case (under certain additional assumptions on the fundamental group of G)?

Before we summarize the results obtained in this work, let us illustrate this twisted action by an example:
Consider $G = SL_n(k)$ as a matrix group, then we have for the exterior automorphism τ (the one that reverses the Dynkin diagram):

$$\tau(X) = {}^tX^{-1}, \quad X \in SL_n(k).$$

Thus, the orbits of the "τ-twisted" conjugation are isomorphism classes of bilinear forms of discriminant one on k^n, where isomorphism is defined in terms of base changes in $SL_n(k)$. Then, $SL_n(k)\tau/\!\!/SL_n(k)$ is the corresponding quotient. This situation was already investigated by Spaltenstein in [29] in a slightly more general situation:

$$\begin{array}{rcl} G \times M & \to & M \\ (X,\, M) & \mapsto & X\, M\, {}^tX, \end{array}$$

where M is either $M_n(k)$ or $GL_n(k)$ and G either $GL_n(k)$ or $SL_n(k)$. (Note that τ can apparently be lifted to $GL_n(k)$.) He obtained the following results:

$$\begin{array}{rcl} M_n(k)\tau /\!\!/ GL_n(k) & \cong & \{pt\} \\ M_n(k)\tau /\!\!/ SL_n(k) & \cong & k^{[\frac{n}{2}]+1} \\ GL_n(k)\tau /\!\!/ GL_n(k) & \cong & k^{[\frac{n}{2}]} \\ GL_n(k)\tau /\!\!/ SL_n(k) & \cong & k^{[\frac{n}{2}]} \times k^*, \end{array}$$

where $[r]$ denotes the biggest integer N with $N \leq r$. The approach used by Spaltenstein was direct calculation.

A more thorough conceptual understanding of this situation will be presented in this work.
If we take a generic exterior automorphism $\tau \in \Gamma$ of G, then we argue, that we can find an integer m and a decomposition of τ into elements of $\tau_i \in \Gamma$, i.e. $\tau = \tau_1 ... \tau_m$, and a decomposition of G into normal subgroups $G = G_1 ... G_m$, with $(G_i, G_j) = \{e\}$ for $i \neq j$, such that each τ_i acts nontrivially only on G_i, where it acts as an exterior automorphism of one of the following two kinds:
(i) τ_i acts as a permutation of isomorphic simple normal subgroups of G_i, or
(ii) G_i is simple, and τ_i is a nontrivial diagram automorphism of its Dynkin diagram.

Therefore, we only need to to restrict to the two cases above. Since our results show, that the first possibility reduces more or less to Steinberg's classical theory, we shall emphasize the second case.

Now let us give a brief summary of the results obtained:
A major step in understanding the conjugacy classes of an exterior component is to develop a notion that generalizes that of a maximal torus in the connected case. This is accomplished by the concept of Cartan subgroup, see Section 2.1, a subgroup of $\tilde{G}$, which is generated by a semisimple element and which has finite index in its normalizer in $\tilde{G}$. We will see that, for given τ, such a Cartan subgroup is given by $C := T_0^\tau \rtimes \Gamma'$, where T_0^τ is the unit component of the fixed point torus of the τ action on the maximal torus T and where $\Gamma' < \Gamma$ is the subgroup generated by τ. The group $\widetilde{\mathcal{W}} := N_G(C)/T_0^\tau$ is called the outer Weyl group and is, in this case, a finite abelian extension of the fixed point Weyl group $\mathcal{W}^\tau$, see Section 2.3. With this notion we are able to derive the following results in analogy to the connected case:

- The elements w of the outer Weyl group $\widetilde{\mathcal{W}}$ act on $T_0^\tau\tau$ by conjugation with a representative $n_w \in N_G(C)$. The corresponding quotient is denoted by $T_0^\tau\tau/\widetilde{\mathcal{W}}$. Then the inclusion $T_0^\tau\tau \hookrightarrow G\tau$ induces an isomorphism on the quotient spaces, see Section 2.5:

$$T_0^\tau \tau/\widetilde{\mathcal{W}} \xrightarrow{\sim} G\tau/\!\!/G.$$

This isomorphism is obtained by use of the representation theory of $\tilde{G}$, see Section 2.4.

- The reduced fibres of the quotient map $\pi : G\tau \to G\tau/\!\!/G$ have the following structure of associated bundles:

$$\pi^{-1}(\pi(t\tau))_{red} \cong G \times^{C_G(t\tau)} V(t\tau),$$

where $V(t\tau)$ denotes the unipotent variety of $C_G(t\tau)$. In particular, they consist of only finitely many conjugacy classes, where the closed class is exactly the semisimple one, see Chapter 3.

- If the character lattice $\chi(T)$ of a maximal torus T of G satisfies the condition $\chi(T)^\tau = \Lambda(R(G))^\tau$, where $\Lambda(R(G))$ denotes the character lattice of the root system of G and where the upper right τ denotes the sublattices of τ fixed points, we even can say more:

  - The adjoint quotient $G\tau/\!\!/Ad(G)$ is an affine space $\mathbb{A}^s$, where s is the rank of G^τ and π is explicitly given by:

  $$\pi = (X_1^{\lambda_1'}|_{G\tau}, ..., X_1^{\lambda_s'}|_{G\tau}).$$

  The $X_1^{\lambda_i'}$ are the characters of irreducible $\tilde{G}$ representations corresponding to the fundamental weights $\lambda_i' \in \Lambda(R(G))^\tau$. (The subscript is needed, because these representations are no longer unique for given λ_i'.) This is proven in Sections 2.4 and 2.5.

  - π admits a cross section, see Chapter 5:

  $$C : T_0^\tau/\widetilde{\mathcal{W}} \to G\tau.$$

  C is given explicitly by the following formula:

  $$C(c_1, ..., c_s) = X_{\alpha_1}(c_1)n_{\alpha_1}...X_{\alpha_s}(c_s)n_{\alpha_s}...\tau,$$

  where the $\alpha_i \in \Pi$ are a collection of simple roots of $R(G)$, exactly one representative α_i chosen for each τ orbit on Π. The $X_{\alpha_i} : k \to G$ are once more the root groups and the $n_{\alpha_i} \in N_G(T)$ representatives of the simple reflexion to $s_{\alpha_i} \in \mathcal{W}$.
  This cross section has the following properties:
  (i) Its image $Im\, C$ is closed in $G\tau$,
  (ii) $\pi|_{Im\, C}$ is an isomorphism and
  (iii) $Im\, C$ meets each fibre in exactly one element, a regular one. (Here an element $x \in G\tau$ is regular, if $C_G(x)$ has minimal possible dimension, which is s.)

  - The fibres of π are reduced and normal and π itself is flat, see Chapter 5.

- For simple G we can derive a connection to the theory of simple singularities under reasonable further restrictions on the characteristic of k, see Chapter 6:
  Choose $t\tau \in T_0^\tau \tau$, such that $C_G(t\tau)$ has an irreducible root system of rank s. We take Slodowy's transversal slice S to a subregular unipotent element $u \in C_G(t\tau)$, see [27]. Then $St\tau \cap V(t\tau)t\tau$ has a simple singularity in $ut\tau$ which is of the same type as $C_G(t\tau)$. Furthermore, the map $\sigma := \pi|_{St\tau} : St\tau \to G\tau/\!\!/G$ is the semiuniversal deformation of this singularity.

This work is organized as follows:
In Chapter 1, we summarize the facts about root systems R needed in this work, in particular about the folding of R by means of a diagram automorphism τ. Furthermore, we discuss the circumstances under which these exterior automorphisms lift to a semisimple group G having R as its roots system. We also give an account on the characteristics of k, such that the orbit maps of G on $G\tau$ are separable.
The major part of this work is contained in Chapters 2,3 and 5.
In Chapter 2, we first present the definition of Cartan subgroups , the analogue of maximal tori for non-connected algebraic groups, and of outer Weyl groups, the groups of "connected components" of the normalizer in G of these Cartan subgroups. Afterwards, properties of Cartan subgroups and the structure of outer Weyl groups are discussed. Then we develop the representation theory of $\tilde{G}$, which, for cyclic Γ, is essentially determined by that of its unit component G. With these preparations, we can prove the first main result, the isomorphism $G\tau/\!\!/G \cong T_0^\tau \tau/\widetilde{\mathcal{W}}$. This chapter closes with the investigation of the structure of these quotient spaces in dependence on the structure of the fundamental group of G.
We determine the structure of the fibres in Chapter 3. Furthermore, we classify the types of the centralizers of semisimple elements of the exterior component $G\tau$. In characteristic zero, their Dynkin diagrams will be proper subdiagrams of the τ-twisted affine diagram corresponding to $\Delta(G)$.
Chapter 4 contains a description of the set of irregular elements of the exterior component $G\tau$. Here, we also verify that the semisimple irregular elements form a dense subset in the set of all irregular elements.
The proof that the cross section defined above fulfills all the indicated properties, under the condition $\chi(T)^\tau = \Lambda(R(G))^\tau$, is carried out in Chapter 5. Furthermore, this cross section allows us to derive the reducedness and normality of the schematic fibres in these cases.
In Chapter 6, we establish a link to the theory of simple singularities. We also provide an account on the theories of singularities and the Slice Theorem, which play a crucial role in our construction.

We conclude this work with Chapter 7 by a summary and an outlook.
The appendix provides examples for elements $x \in G\tau$, which are not G-conjugate into $G_0^\tau\tau$, in the case where G is simple.

Chapter 1

Preliminaries

In this chapter we shall compile the basic facts used in the sequel. We first start with basic facts about root systems and exterior automorphisms. Furthermore, we investigate the circumstances under which the orbit maps considered in this work are separable.

1.1 Root Systems

Let us start with basic facts about root systems which may be found e.g. in [3, 14].
In this section, V denotes a real vector space.

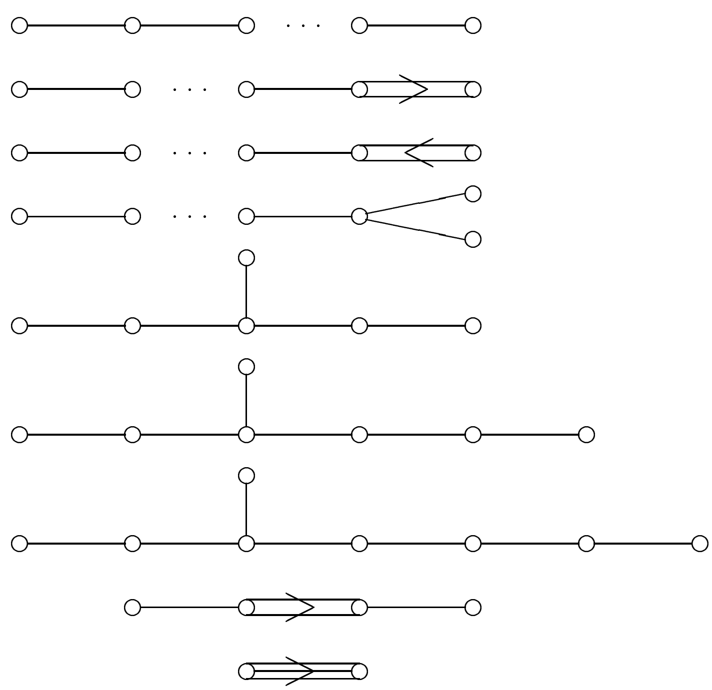
Definition 1.1 *A finite subset $R \subset V$ is called a root system, iff*
(i) R spans V,
(ii) for every $\alpha \in R$, there exists $\check{\alpha} \in V^*$, such that $\check{\alpha}(\beta)$ is an integer for all $\alpha, \beta \in R$,
(iii) the group generated by all the reflexions $r_\alpha \in GL(V)$, defined by $r_\alpha(\beta) := \beta - \check{\alpha}(\beta)\alpha$, stabilizes R.
If in addition the following property holds, the root system is called reduced, otherwise not reduced:
(iv) for every $\alpha \in R$ the only multiples in R are $\pm\alpha$.
In any case, the group generated by all the $r_\alpha, \alpha \in R$, is called the Weyl group and will be denoted by $\mathcal{W}$. The elements of R will be called roots.

It should be noted, that one can introduce a $\mathcal{W}$-invariant scalar product $(.,.)$ on V, such that, under the corresponding identification of V and V^*, we have $\check{\alpha} = \frac{2\alpha}{(\alpha,\alpha)}$.
Furthermore we need the definition of a basis of a root system:

Definition 1.2 *A subset $\Pi \subset R$ is called a basis of R, iff*
(i) Π is a basis of V, and
(ii) every root $\alpha \in R$ can be written as a sum $\alpha = \sum_{\beta \in \Pi} n_\beta \beta$ with all the n_β either non-negative or non-positive.

By the *height of a root* α, $ht\,\alpha$, we denote the sum of the coefficients of α with respect to a given basis. A theorem in the theory of root systems states that bases of root systems exist and all possible bases are conjugate under the Weyl group. A classification of reduced root systems is given by so called Dynkin diagrams, denoted by Δ, the vertices corresponding to the elements in Π, each pair α, β of them connected by $\check{\alpha}(\beta)\check{\beta}(\alpha)$ lines with an arrow pointing to the shorter root.
A reduced root system is called irreducible if its Dynkin diagram is connected. The list of irreducible reduced root systems is given in the following table:

A_n $(n \geq 1)$:
B_n $(n \geq 2)$:
C_n $(n \geq 3)$:
D_n $(n \geq 3)$:
E_6 :
E_7 :
E_8 :
F_4 :
G_2 :



Another notion in the theory of root systems is the notion of the *dual root system* $\check{R}$ of a given root system R which is given by the set $\check{R} := \{\check{\alpha}, \alpha \in R\} \subset V^*$. The *root lattice* $\mathbb{Z}(R)$ will be the $\mathbb{Z}$-span of R in V. Furthermore, we introduce the *weight lattice* $\Lambda(R)$ as the lattice $\mathbb{Z}(\check{R})^* \subset V$. Its elements will be called *weights*.

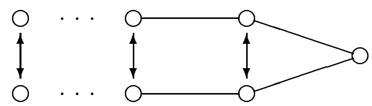
1.2 Diagram Automorphisms and Folded Root Systems

In this section, we will describe the automorphisms of reduced root systems and introduce the notion of folded root systems. An element $\sigma \in GL(V)$ is called an *automorphism of* R, iff $(\sigma(\alpha))\check{}(\sigma(\beta)) = \check{\alpha}(\beta)$ for all $\alpha, \beta \in R$. An automorphism of R is called an automorphism of the Dynkin diagram Δ, if it stabilizes a given basis Π of R. The sets of automorphisms of R and Δ will be denoted by $Aut(R)$ and $Aut(\Delta)$ respectively. A well known fact in the theory of root systems is that we have the following semidirect product:

$$Aut(R) = Aut(\Delta) \ltimes \mathcal{W} \tag{1.1}$$

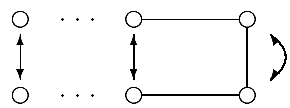
The group $Aut(\Delta)$ is trivial for irreducible root systems except in the cases A_n, D_n and E_6. A pictorial description of the group $Aut(\Delta)$ in these cases will be given in the following table:

A_{2n+1} :



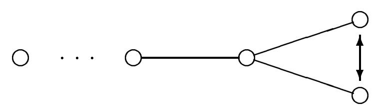
$Aut(\Delta) \cong \mathbb{Z}/2\mathbb{Z}$,

A_{2n} :



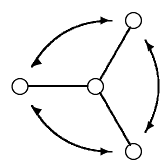
$Aut(\Delta) \cong \mathbb{Z}/2\mathbb{Z}$,

D_n $(n > 4)$:



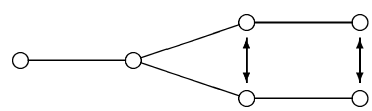
$Aut(\Delta) \cong \mathbb{Z}/2\mathbb{Z}$,

D_4 :



$Aut(\Delta) \cong S_3$,

E_6 :



$Aut(\Delta) \cong \mathbb{Z}/2\mathbb{Z}$.

With these notions, we can now define the folded root system corresponding to an automorphism $\tau \in Aut(\Delta)$:
Let V^τ be the fixed point vector space of V under τ, i.e. $V^\tau := \{v \in V, \tau(v) = v\}$. Then, define a projection map $p : V \to V^\tau$ by:

$$v \mapsto p(v) := \frac{1}{ord\,\tau} \sum_{i=0}^{ord\,\tau} \tau^i(v) \tag{1.2}$$

Set ${}^\tau R^1 := p(R)$ and $\mathcal{W}^\tau := \{w \in \mathcal{W}, \tau w = w\tau\}$. Then we get the following result:

Theorem 1.1 *${}^\tau R^1$ is a root system in V^τ with Weyl group $\mathcal{W}^\tau$.*
For irreducible R, the type of ${}^\tau R^1$ is given by the following table:

R	A_{2n-1}	A_{2n}	$D_{n+1}, \tau^2 = 1$	E_6	$D_4, \tau^3 = 1$
${}^{\tau}R^1$	C_n	BC_n	B_n	F_4	G_2

Proof: Cf. [7], Proposition 13.2.2., p. 220, or [36], examples to Theorem 32, p. 175f. ♡.

Remark: If R is reducible, such that it is the union of n copies of an irreducible root system $\tilde{R}$, and τ a permutation of its irreducible parts, then the type of ${}^{\tau}R^1$ is that of a union of k copies of $\tilde{R}$, where k is the number of cycles of τ.

We call ${}^{\tau}R^1 := p(R)$ the *folded root system of R with respect to τ*. Similarly, we define ${}^{\tau}\Lambda(R)^1 = p(\Lambda(R))$ and call it the *folded weight lattice*. A natural question to ask is, what the relation between ${}^{\tau}\Lambda(R)^1 = p(\Lambda(R))$ and the weight lattice $\Lambda({}^{\tau}R^1)$ is.

(Since ${}^{\tau}R^1 := p(R)$ is not reduced, if R is of type A_{2n}, we modify the question posed above in this case: We compare the weight lattice of the B_n root subsystem consisting of all short and intermediate roots of ${}^{\tau}R^1 := p(R)$ with the folded weight lattice ${}^{\tau}\Lambda(R)^1$). The answer is given in the following lemma:

Lemma 1.1 *(i) If R is of type A_{2n-1}, D_n and E_6, we have: ${}^{\tau}\Lambda(R)^1 = \Lambda({}^{\tau}R^1)$.*
(ii) If R is of type A_{2n} we have: ${}^{\tau}\Lambda(R)^1$ is a sublattice of the weight lattice $\Lambda(B_n)$ of index 2. In this case, ${}^{\tau}\Lambda(R)^1$ is the weight lattice of $SO_{2n+1}(k)$.

Proof: In order to compare the two lattices, we first have to compute the basis of the root system dual to ${}^{\tau}R^1$, respectively of the B_n subsystem of ${}^{\tau}R^1 = BC_n$ in case $R = A_{2n}$.
Let $\Pi = \{\alpha_1, ..., \alpha_n\}$ be a basis of R and $\check{\Pi} = \{\check{\alpha}_1, ..., \check{\alpha}_n\}$ and $\{\lambda_1, ..., \lambda_n\}$ the corresponding bases of $\check{R}$ and $\Lambda(R)$ respectively. Then we get by [7], Proposition 13.2.2, p. 220: $p(\Pi) = \{p(\alpha_1), ..., p(\alpha_n)\}$ and $\{p(\lambda_1), ..., p(\lambda_n)\}$ are a basis of ${}^{\tau}R^1$ (the B_n subsystem in the A_{2n} case) and ${}^{\tau}\Lambda(R)^1$ respectively. Now we have to calculate the corresponding basis $(p(\Pi))\check{}$. Apparently, $(p(\alpha_i))\check{}$ is a κ-fold multiple of $\sum_{j=1}^{ord\,\tau} \tau^j(\check{\alpha}_i)$, for a suitable $\kappa \in \mathbb{R}$. Three cases have to be distinguished:
Recall that we require $(p(\alpha_i))\check{}(p(\alpha_i)) = 2$.
(i) $\tau(\alpha_i) = \alpha_i$. Then we have $(p(\alpha_i))\check{} = \check{\alpha}_i$.
(ii) $\tau(\alpha_i) \neq \alpha_i$ and $\check{\alpha}_i(\alpha_i) = 0$: Then we have

$$(p(\alpha_i))\check{}(p(\alpha_i)) = \kappa \sum_{j=1}^{ord\,\tau} \tau^j(\check{\alpha}_i) \left(\frac{1}{ord\,\tau} \sum_{l=1}^{ord\,\tau} \tau^l(\alpha_i) \right) = 2\frac{\kappa}{ord\,\tau} ord\,\tau, \qquad (1.3)$$

forcing $\kappa = 1$. Thus we get $(p(\alpha_i))\check{} = \sum_{j=1}^{ord\,\tau} \tau^j(\check{\alpha}_i)$.
(iii) $\tau(\alpha_i) \neq \alpha_i$ and $\check{\alpha}_i(\alpha_i) \neq 0$ (which only appears in the A_{2n}-case): Similarly to (ii) we calculate

$$(p(\alpha_i))\check{}(p(\alpha_i)) = \kappa(\check{\alpha}_i + \tau(\check{\alpha}_i))\left(\frac{1}{2}(\alpha_i + \tau(\alpha_i))\right) = \frac{\kappa}{2}(2+2-1-1), \quad (1.4)$$

forcing $\kappa = 2$. Therefore we have $(p(\alpha_i))\check{} = 2(\check{\alpha}_i + \tau(\check{\alpha}_i))$.
As a next step, we evaluate the generators $p(\lambda_i)$ of ${}^\tau\Lambda(R)^1$ on the elements of $(p(\Pi))\check{}$. A simple calculation gives $p(\lambda_i)((p(\alpha_j))\check{}) = 0$, iff α_i and α_j are not in the same τ orbit. Otherwise this value is 1 in the cases (i) and (ii) above and 2 in case (iii). Since in the A_{2n}-case the situation described in (iii) shows up for only one element in $p(\Pi)$, the lemma holds. ♡.
Remark: If R is the union of n copies of an irreducible root system $\tilde{R}$ and if τ is a permutation of the irreducible parts, then again one can show $\Lambda({}^\tau R^1) = {}^\tau\Lambda(R)^1$. The proof is analogous to case (ii), but we have to take care of nontrivial powers of τ which stabilize certain roots α. If we denote by Γ the cyclic subgroup of $Aut(R)$, which is generated by τ, then we have $(p(\alpha_i))\check{} = \frac{1}{|\Gamma_{\alpha_i}|}\sum_{j=1}^{ord\,\tau} \tau^j(\alpha_i)$.

The relation between root length of elements in ${}^\tau R^1$ and the behaviour of the inverse image under the projection p, defined in Equation 1.2, will be dealt with in the following lemma, which will be used later:

Lemma 1.2 *(i) For R of type A_{2n-1}, D_n or E_6 the following holds:
If $\tau(\alpha) = \alpha$, then $p(\alpha)$ is a long root of ${}^\tau R^1$, and $\mathbb{R}\alpha \cap R$ is of type A_1.
If $\tau(\alpha) \neq \alpha$, then $p(\alpha)$ is a short root of ${}^\tau R^1$, and $(\sum_{i=0}^{ord\,\tau} \mathbb{R}\tau^i(\alpha)) \cap R$ is of type $(A_1)^{ord\,\tau}$.
(ii) For R of type A_{2n}, we have:
If $\tau(\alpha) = \alpha$, then $p(\alpha)$ is a long root of BC_n and $(\mathbb{R}\alpha) \cap R$ is of type A_1.
If $\tau(\alpha) \neq \alpha$ and $\check{\alpha}(\tau(\alpha)) = 0$, then $p(\alpha)$ is a root of intermediate length of BC_n and $(\sum_{i=0}^{ord\,\tau} \mathbb{R}\tau^i(\alpha)) \cap R$ is of type $(A_1)^2$.
If $\tau(\alpha) \neq \alpha$ and $\check{\alpha}(\tau(\alpha)) \neq 0$, then $p(\alpha)$ is a short root of BC_n and $(\sum_{i=0}^{ord\,\tau} \mathbb{R}\tau^i(\alpha)) \cap R$ is of type A_2.*

Proof: The proof is given in [36], Corollary to Theorem 32, p. 177 and [39], Lemma 2.4, p. 9. ♡.
Remark: Let us again consider the case of R being the union n copies of an irreducible root system $\tilde{R}$ and τ being a permutation of the irreducible parts. Denote by Γ the subgroup of $Aut(R)$, which is generated by τ. Then we get analogously that $\sum_{i=0}^{ord\,\tau} \mathbb{R}\tau^i(\alpha) \cap R$ is of type $(A_1)^k$, where k is the number of roots in the Γ-orbit of α.

For the representation theory of our non-connected groups, we need to introduce another root system, called R', which again is a root system in V^τ: The construction recipe is the following:

(i) In the cases A_{2n-1}, D_n and E_6: For $\alpha = \tau(\alpha)$ set $\alpha' = \alpha$,
for $\alpha \neq \tau(\alpha)$, $\check{\alpha}(\tau(\alpha)) = 0$ set $\alpha' = \sum_{i=1}^{ord\,\tau} \tau^i(\alpha)$.
(ii) In the A_{2n}-case: For $\alpha \neq \tau(\alpha)$, $\check{\alpha}(\tau(\alpha)) = 0$ set $\alpha' = (\alpha + \tau(\alpha))$,
for $\alpha \neq \tau(\alpha)$, $\check{\alpha}(\tau(\alpha)) \neq 0$ set $\alpha' = 2(\alpha + \tau(\alpha))$.
For $\alpha = \tau(\alpha)$, set $\alpha' = 2\,\alpha$. (Note that $\alpha = \tau(\alpha)$ implies the existence of a root β with $\alpha = \beta + \tau(\beta)$ in this case. Then $\alpha' = \beta'$.)
(iii) If R is reducible, the union of n copies of an irreducible root system $\tilde{R}$ and τ a permutation of the irreducible parts, and if we denote by Γ the subgroup of $Aut(R)$ generated by τ, then define R' in the following way: Let $R' := \{\alpha',\, \alpha \in R\}$, where $\alpha' = \frac{1}{|\Gamma_\alpha|} \sum_{i=1}^{ord\,\tau} \tau^i(\alpha)$.

With respect to the construction of the root system ${}^{\tau}R^1$, the root system R' emerges from the root system ${}^{\tau}R^1$ by keeping the long roots and multiplying the short roots of ${}^{\tau}R^1$ with the order of τ in the A_{2n-1}, D_n and E_6-cases, and doubling the intermediate and long roots while quadrupling the short roots of ${}^{\tau}R^1$ in the A_{2n}-case. (Recall that the set of short and intermediate roots in the BC_n-case exhibit the structure of a B_n root system.) This yields:

Lemma 1.3 *(i) In the cases A_{2n-1}, D_n and E_6, the root system $R'$is the dual of the root system ${}^{\tau}R^1$.*
(ii) In the case A_{2n}, the root system R' is the dual of the reduced irreducible root system of type B_n which appears as the subsystem of roots of intermediate and short length in ${}^{\tau}R^1$.

Proof: This is immediate by construction of ${}^{\tau}R^1$ and R'. ♡.
Remark: If R is the union of n copies of an irreducible root system $\tilde{R}$ and τ a permutation of the irreducible parts, then R' is of the same type as ${}^{\tau}R^1$.

We summarize this result in the following table:

R	A_{2n-1}	A_{2n}	$D_{n+1}, \tau^2 = 1$	E_6	$D_4, \tau^3 = 1$
${}^{\tau}R^1$	C_n	BC_n	B_n	F_4	G_2
R'	B_n	C_n	C_n	F_4	G_2

The introduction of the root system R' is justified by the following lemma:

Lemma 1.4 *(i) The weight lattice $\Lambda(R')$ is equal to the lattice $\Lambda(R)^\tau$ of τ fixed points in $\Lambda(R)$.*
(ii) If R is not of type A_{2n}, we have $\mathbb{Z}(R') = \mathbb{Z}(R)^\tau$, and, if R is of type A_{2n}, we have that $\mathbb{Z}(R')$ is a sublattice of $\mathbb{Z}(R)^\tau$ of index two.

Proof: (i) Let $\Pi = \{\alpha_1, ..., \alpha_n\}$ be a basis of R stabilized by τ and $\check{\Pi} = \{\check{\alpha}_1, ..., \check{\alpha}_n\}$ the corresponding basis of $\check{R}$. Then $\Pi' := \{\alpha'_1, ..., \alpha'_n\}$ is a basis of R'. Now we have to find basis of $\check{R}'$. Clearly, $\check{\alpha}'_i$ has to be a κ-fold multiple of $\sum_{i=1}^{ord\,\tau} \tau^i(\check{\alpha}_i)$, for a suitable $\kappa \in \mathbb{R}$. For a while, we assume R to

be irreducible. Then, three cases have to be distinguished:
(a) $\tau(\alpha_i) = \alpha_i$: Then $\check{\alpha_i'} = \check{\alpha}_i$.
(b) $\tau(\alpha_i) \neq \alpha_i$ and $\check{\alpha}_i(\tau(\alpha_i)) = 0$: Then

$$\check{\alpha_i'}(\alpha_i') = \kappa \sum_{j=1}^{ord\,\tau} \tau^j(\check{\alpha}_i) \left(\sum_{l=1}^{ord\,\tau} \tau^l(\alpha_i) \right) = 2\,\kappa\, ord\,\tau, \tag{1.5}$$

which forces $\kappa = \frac{1}{ord\,\tau}$. Thus we have $\check{\alpha_i'} = \frac{1}{ord\,\tau} \sum_{j=1}^{ord\,\tau} \tau^j(\check{\alpha}_i)$.
(c) $\tau(\alpha_i) \neq \alpha_i$ and $\check{\alpha}_i(\tau(\alpha_i)) \neq 0$: This appears only in the A_{2n}-case, where $ord\,\tau = 2$. Similarly to step two we have:

$$\check{\alpha_i'}(\alpha_i') = \kappa(\check{\alpha}_i + \tau(\check{\alpha}_i))(2(\alpha_i + \tau(\alpha_i))) = 2\kappa(2 + 2 - 1 - 1), \tag{1.6}$$

forcing $\kappa = \frac{1}{2}$. Thus giving $\check{\alpha_i'} = \frac{1}{2}(\check{\alpha}_i + \tau(\check{\alpha}_i))$.
Now, we consider the weight lattice $\Lambda(R)$, and let the fundamental weights corresponding to Π be denoted by $\{\lambda_1, ..., \lambda_n\}$. Then, generators of the invariant lattice $\Lambda(R)^\tau$ are given by the collection $\{\hat{\lambda}_1, ..., \hat{\lambda}_n\}$, where $\hat{\lambda}_i$ is defined by $\hat{\lambda}_i = \lambda_i$, if $\tau(\lambda_i) = \lambda_i$, and $\hat{\lambda}_i = \sum_{i=1}^{ord\,\tau} \tau^i(\lambda_i)$ otherwise. A simple calculation, using $\lambda_i(\check{\alpha}_j) = \delta_{ij}$, yields $\hat{\lambda}_i(\check{\alpha_j'}) = 1$, iff α_j' is in the τ-orbit of α_i'. Otherwise it is zero.
Now, let R be the union of n copies of an irreducible root system $\tilde{R}$ and τ a permutation of its irreducible factors. Again, we denote by Γ the subgroup of $Aut(R)$ generated by τ. Then, a similar calculation to the one carried out in (b) yields: $\check{\alpha_i'} = \frac{1}{ord\,\tau} \sum_{j=1}^{ord\,\tau} \tau^j(\check{\alpha}_i)$ and similarly $\hat{\lambda}_i = \frac{1}{|\Gamma_{\alpha_i}|} \sum_{j=1}^{ord\,\tau} \tau^j(\lambda_i)$, again a sum over the Γ-orbit of λ_i.
(ii) Follows easily from the definition of R' and the fact that the simple roots Π form a basis of R. $\heartsuit$.
Remark: (i) By construction it is clear that our identification of $\Lambda(R')$ and $\Lambda(R)^\tau$ is $\mathcal{W}^\tau$-equivariant.
(ii) This isomorphism also respects the structure of the cone of dominant weights, i.e. if we intersect the cone of dominant weights of $\Lambda(R)$ with $\Lambda(R)^\tau$, we get exactly the cone of dominant elements of $\Lambda(R')$.

In Chapter 2 we still need a slightly sharper version of the lemma above:

Corollary 1.1 *Assume R to be irreducible. Let χ be a τ-stable sublattice of $\Lambda(R)$ containing the root lattice $\mathbb{Z}(R)$, then we have:*
(i) If R is of type A_{2n}, E_6 or D_4 with $\tau^3 = 1$, then $\chi^\tau = \Lambda(R')$
(ii) If R is of one of the other types, namely, A_{2n-1} or D_n with $\tau^2 = 1$, then χ^τ is either $\mathbb{Z}(R')$ or $\Lambda(R')$.

Proof: The chain of inclusions $\mathbb{Z}(R) \subset \chi \subset \Lambda(R)$ clearly yields a chain of its fixed point lattices $\mathbb{Z}(R)^\tau \subset \chi^\tau \subset \Lambda(R)^\tau$.
If R is of type E_6 or of type D_4 with $\tau^3 = 1$, we have that R' is of type F_4 or G_2, respectively. This implies $\mathbb{Z}(R)^\tau = \mathbb{Z}(R') = \Lambda(R') = \Lambda(R)^\tau$ by use

of the above lemma, forcing $\chi^\tau = \Lambda(R')$.
If R is of type A_{2n}, we have that R' is of type C_n and therefore $\Lambda(R')/\mathbb{Z}(R') \cong \mathbb{Z}/2\mathbb{Z}$. Now, we have a chain of inclusions $\mathbb{Z}(R') \subset \mathbb{Z}(R)^\tau \subset \chi^\tau \subset \Lambda(R)^\tau = \Lambda(R')$. Here, $\mathbb{Z}(R)^\tau/\mathbb{Z}(R') \cong \mathbb{Z}/2\mathbb{Z}$ by the lemma above, forcing $\mathbb{Z}(R)^\tau = \chi^\tau = \Lambda(R)^\tau$.
(ii) In the other cases, we have R of type A_{2n-1} or D_n with $\tau^2 = 1$. Hence R' is of type B_n or C_{n-1}, respectively. Here, $\Lambda(R')/\mathbb{Z}(R') \cong \mathbb{Z}/2\mathbb{Z}$, and hence $\Lambda(R)^\tau/\mathbb{Z}(R)^\tau \cong \mathbb{Z}/2\mathbb{Z}$, forcing χ^τ to be either $\mathbb{Z}(R')$ or $\Lambda(R')$. ♡.

1.3 Exterior Automorphisms of Algebraic Groups

In the sequel, let G denote a semisimple linear algebraic group over an algebraically closed field k of characteristic $char(k) \neq 2$. Additionally, we require that $char(k)$ does not divide the order of τ, a given diagram automorphism of the Dynkin diagram of G. We want to show, for G simple, that, except for certain cases of type D_{2n}, our diagram automorphism τ from the previous section lifts to an automorphism of the group G, and we shall provide an explicit formula for its action on the root groups $X_\alpha : k \to G, \alpha \in R$. Here, we assume knowledge of the theory of Chevalley groups, found e.g. in [36], and on linear algebraic groups, found e.g. in [13, 2, 33]

Theorem 1.2 *Let G, τ, k be as above, and let T be a maximal torus of G and $\chi(T)$ its character lattice. Assume that that $\chi(T)$ is stabilized by τ. (This is automatic for G simple, if G is not of type D_{2n}.)*
Then, τ can be lifted to an exterior automorphism τ of G such that:
(i) $\tau(X_\alpha(t)) = X_{\tau(\alpha)}(t), \forall t \in k,\ \alpha \in R$, in case G is not of type A_{2n} respectively
(ii) $\tau(X_\alpha(t)) = X_{\tau(\alpha)}((-1)^{ht\,\alpha+1}t), \forall t \in k\ \alpha \in R$, in case G is of type A_{2n}.

Proof: In our cases the existence of τ as an automorphism of a linear algebraic group G is clear by [36], Corollary to Theorem 29, p. 156f.
We only have to show that we can define our root groups in such a way that τ acts in the described manner. By the construction of the Chevalley groups, look at [36], Chapters 1,2,3, this amounts to finding a Chevalley basis $x_\alpha, \alpha \in R$ of the corresponding simple complex Lie algebra $\mathfrak{g}$, such that τ acts on $\mathfrak{g}$ by:
(i) $\tau(x_\alpha) = x_{\tau(\alpha)}, \forall t \in R$, in the case G is not of type A_{2n}, or
(ii) $\tau(x_\alpha) = (-1)^{ht\,\alpha+1} x_{\tau(\alpha)}, \forall t \in R$, in case G is of type A_{2n}.
To prove this, we first introduce a specific order relation on the set of simple roots Π, which is invariant under τ in the non A_{2n}-case and anti-invariant in the A_{2n}-case. This is done in the proof of Lemma 2.6 and Remark 2.7 of [39], p. 10. Now, we choose the $x_\alpha, \alpha \in \Pi$, in an unspecific way, and we

define $x_\beta, \beta \in R^+ - \Pi$, by multicommutators

$$x_\beta := [x_{\alpha_1}, [...[x_{\alpha_{k-1}}, x_{\alpha_k}]...], \tag{1.7}$$

respecting the order, i.e. $\alpha_i \leq \alpha_j$, for $i < j$, if possible, and such that α_k is the unique maximal element with respect to this order, if this maximal element appears nontrivially in the linear combination of β as a sum of simple roots. It is shown in the proof of Lemma 2.6 and in Remark 2.7 of [39], that this basis has the properties (i) and (ii) above. If we choose $\tilde{x}_\beta, \beta \in R$ to be a Chevalley basis of $\mathfrak{g}$ with $\tilde{x}_\alpha = x_\alpha$, for $\alpha \in \Pi$, then, since $\mathfrak{g}$ is simply-laced and henceforth the structure constants are ± 1 or 0, we have that $\tilde{x}_\beta$ corresponds, up to sign, to the above multicommutators. So by adjusting the signs, which does not change the corresponding Chevalley algebra, we can achieve the above formulae. ♡.

Remark: (i) Let the Dynkin diagram of G be not connected, i.e. let G be semisimple but not simple and τ a permutation of different, isomorphic, irreducible parts of the root system $R(G)$. Then we can lift τ to G, if and only if the character lattice $\chi(T)$ of a maximal torus T of G is τ-stable.
(ii) With this choice of τ, the order of τ, considered as an automorphism of G, is exactly the order of the corresponding diagram automorphism.

We are now interested in the fixed point group G^τ of G under the automorphism τ, respectively its unit component G_0^τ.

Proposition 1.1 *For G simple, the type of the group G_0^τ is given in the following table:*

Type G	A_{2n-1}	A_{2n}	$D_{n+1}, \tau^2 = 1$	E_6	$D_4, \tau^3 = 1$
Type G_0^τ	C_n	B_n	B_n	F_4	G_2

Additionally, G^τ / G_0^τ is finite abelian and, if G is simply connected, G^τ is connected.

Proof: Cf. [7], Section 13.3, p. 221ff, together with a remark in Section 14.4, at the bottom of p. 264. The additional assertions are proven in [37], Theorem 8.2, p. 52, and Corollary 9.4, p. 60. ♡.

Remarks: (i) If $R(G)$ is a union of n copies of an irreducible root system $\tilde{R}$ and τ a permutation of the irreducible parts thereof, then the type of G_0^τ is the one of a k-fold union of $\tilde{R}$, where k is the number of cycles of τ.
(ii) We see: The root system of G_0^τ is given by ${}^\tau R^1$, if G is not of type A_{2n} and by the B_n-subsystem, given by the roots of short and intermediate length of ${}^\tau A_{2n}^1 = BC_n$, in the A_{2n}-case.
(iii) By Lemma 1.3, the root system R' is exactly the dual of the root system $R(G_0^\tau)$, the root system of the fixed point group.

(iv) In Chapter 3 we shall give a separate proof of the proposition above, when we calculate the root systems of centralizers of semisimple elements.

We are now in a position to define the group $\tilde{G}$, the one we are dealing with in this work:
Let G be a semisimple linear algebraic group, $\Gamma < Aut(\Delta)$ a subgroup of the group of diagram automorphism corresponding to G, such that all elements of Γ can be lifted to G. Then we set $\tilde{G} := G \rtimes \Gamma$. For G simple of type $A\,D\,E$, the elements of Γ act on G in the manner given by the theorem above. Furthermore, we restrict the characteristic of k in such a way that $char(k) \neq 2$ and that it does not divide the order of any element of Γ. $\tilde{G}$ is clearly an affine algebraic group, because G and $Aut(\Delta)$ are, and the elements of $Aut(\Delta)$ act on G as morphisms. In addition, the unit-component of $\tilde{G}$ is just G. For any element $z \in \tilde{G}$, let $C_G(z)$ denote the centralizer of z in G given by $C_G(z) := \{g \in G, g\,z\,g^{-1} = z\}$.
Remarks: (i) For $z = \tau$ we have $C_G(z) = G^\tau$.
(ii) Our construction of $\tilde{G}$ involves a specific choice of a Borel subgroup B and a maximal torus $T < B$, because we use a specific basis of a root system of G. Since all pairs $T < B$ are conjugate under G, and, therefore, the corresponding diagram automorphisms are also conjugate under G, the resulting group $\tilde{G}$ does not depend on the choices.

1.4 Separability of the Orbit Map

This section will only be used in $char(k) > 0$. Let τ be a diagram automorphism. Here, conditions on the characteristic are given, such that the conjugacy action of G, the unit component of $\tilde{G}$, on a given other component $G\tau$ is separable. Our reasoning will be similar to [34], Chapter I, §5, p. E-16ff, where the connected case is treated.
We start with a lemma:

Lemma 1.5 *Let G and $\tilde{G}$ be as above and assume, furthermore, that $\tilde{G} < GL_n(k)$, such that the following conditions hold:*
(i) $\mathfrak{gl}_n = \mathfrak{g} \oplus \mathfrak{m}$
(ii) $\mathfrak{m}$ is stable under $\tilde{G} < GL_n(k)$.
Then every conjugacy class of $GL_n(k)$ meets $\tilde{G}$ in finitely many G-conjugacy classes.

Proof: Take a $GL_n(k)$-conjugacy class $\hat{C}$. Note that, since G is connected, every G-conjugacy class in $\tilde{G}$ is irreducible.
Let $Z \subset \tilde{G} \cap \hat{C}$ be an irreducible component. Since there exist only finitely many irreducible components of $\tilde{G} \cap \hat{C}$, we are done, if we can show, that Z consists of only one G-conjugacy class.
By irreducibility, Z has to be G-stable, so let $C \subset Z$ be a G-conjugacy class of $\tilde{G}$. Choose $g \in C$.

Next, we define a map

$$\begin{aligned} f : GL_n(k) &\to \hat{C}g^{-1} \\ x &\mapsto x\,g\,x^{-1}\,g^{-1}. \end{aligned} \tag{1.8}$$

We clearly have $f(e) = e$. Now, we show the following statement:
Claim:

$$(df)_e : \mathfrak{gl}_n \to T_e\hat{C}g^{-1} \quad \text{is surjective.} \tag{1.9}$$

Proof of Claim: Since we have $dim\, T_e\hat{C}g^{-1} = dim\, GL_n(k) - dim\, C_{GL_n(k)}(g)$, we must prove $dim\, ker((df)_e) = dim\, C_{GL_n(k)}(g)$. But $ker((df)_e)$ is an associative (matrix)-algebra and has the explicit form:

$$ker((df)_e) = \{X \in \mathfrak{gl}_n,\; g\,X\,g^{-1} = X\}. \tag{1.10}$$

Furthermore, $C_{GL_n(k)}(g)$ is just the group of units of this algebra $ker((df)_e)$. This group of units is an open subset of $ker((df)_e)$ (the complement of the kernel of the determinant) and therefore $ker((df)_e)$ and $C_{GL_n(k)}(g)$ have the same dimension. ♢.
Now we have

$$\begin{aligned} T_eZg^{-1} \subset T_e\hat{C}g^{-1} \cap \mathfrak{g} &= (1 - Ad(g))(\mathfrak{gl}_n) \cap \mathfrak{g} \\ &= (1 - Ad(g))(\mathfrak{g}) \\ &= (df)_e(\mathfrak{g}) \\ &\subset T_eCg^{-1} \subset T_eZg^{-1}, \end{aligned} \tag{1.11}$$

where the identifications in the first and third line follow from the definition of f and the identification in the second follows from assumption (ii). Now we conclude that C has to be an open subset of Z. Since this holds for every conjugacy class $C \subset Z$, we conclude $C = Z$ by the irreducibility of Z. ♡.

As a consequence, we get two corollaries:

Corollary 1.2 *Under the same assumptions as in the preceding lemma, we have for $g \in \tilde{G}$:*

$$\mathfrak{c}_{\mathfrak{g}}(g) = Lie\, C_G(g). \tag{1.12}$$

Proof: From the proof of Lemma 1.5 we have $dim\,(1 - Ad(g))(\mathfrak{g}) = dim\, C$, where C is the G conjugacy class of g in $\tilde{G}$. Since $dim\, \mathfrak{c}_{\mathfrak{g}}(g) = dim\,(ker(1 - Ad(g))|_{\mathfrak{g}})$, we easily calculate, using the dimension formula and the above identification:

$$\begin{aligned} \mathfrak{c}_{\mathfrak{g}}(g) &= dim\, \mathfrak{g} - dim\,(1 - Ad(g))(\mathfrak{g}) \\ &= dim\, G - dim\, C \\ &= dim\, C_G(g). \end{aligned} \tag{1.13}$$

Since we already know $Lie\, C_G(g) \subset \mathfrak{c}_{\mathfrak{g}}(g)$, the assertion follows. ♡.

Corollary 1.3 *Under the same assumptions as in the preceding lemma, we have that for every $\tau \in \Gamma$ and for every $x \in G\tau$ the orbit map*

$$\begin{aligned} G &\to G.x \subset G\tau \\ g &\mapsto g\,x\,g^{-1} \end{aligned} \tag{1.14}$$

is separable.

Proof: This is just a reformulation of Corollary 1.2. ♡.

Let us denote the universal cover of G by $\hat{G}$ and the covering map by π. Furthermore, denote the kernel of π by $C_G < C(\hat{G})$.
In addition, we need the notions of good and very good characteristic as found in [27], Section 3.13, p. 37f:

Definition 1.3 *A prime p is called good for a root system R, iff there does not exist a $\mathbb{Z}$-closed root subsystem $\tilde{R}$ of R, such that $\mathbb{Z}(R)/\mathbb{Z}(\tilde{R})$ has torsion of order p.*
A prime is called very good for R, if p is good for R and if p does not divide $n+1$, in the case that R contains a component of type A_n.

We collect the bad primes for irreducible R in the following table:

Type G	A_n	B_n	C_n	D_n	E_8	E_7	E_6	F_4	G_2
p bad	none	2	2	2	2,3,5	2,3	2,3	2,3	2,3

In particular, we see, that a very good p does not divide $|\Lambda(R)/\mathbb{Z}(R)|$.

Now, consider the non-connected group $\tilde{G} \rtimes \Gamma$. Since every diagram automorphism lifts to the universal cover $\hat{G}$ of its unit component, there is a non-connected algebraic group $\hat{G} \rtimes \Gamma$. By the construction of the Chevalley groups as given in [36], Chapters 1,2,3, C_G has to be τ invariant and the covering map $\pi : \hat{G} \to G$ to be τ equivariant. Therefore, we have a covering map

$$\begin{aligned} \hat{\pi} : \hat{G} \rtimes \Gamma &\to G \rtimes \Gamma \\ g\sigma &\mapsto \pi(g)\sigma. \end{aligned} \tag{1.15}$$

Now, we have the following statement:

Lemma 1.6 *Take $x \in G\tau$ and $\hat{x} \in \hat{G}\tau$ with $\hat{\pi}(\hat{x}) = x$ and let $char(k)$ be either zero or a very good prime for $R(G)$, then we have, for the respective orbit maps:*

$$G \to G.x \quad \textit{is separable} \quad \Leftrightarrow \quad \hat{G} \to \hat{G}.\hat{x} \quad \textit{is separable} \tag{1.16}$$

Proof: By the τ-equivariance of π we see that $\hat{\pi}$ is $\hat{G}$ equivariant where $\hat{g} \in \hat{G}$ acts on $\hat{G}\tau$ by conjugation and on $G\tau$ by conjugating with $\pi(\hat{g})$.
Since $char(k)$ does not divide $|C_G|$, by assumption, we conclude that $\hat{\pi}$ is separable. Therefore, the Lie algebras $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ are isomorphic. Now the adjoint actions of x and $\hat{x}$ on $\mathfrak{g}$ coincide and therefore we get:

$$\mathfrak{c}_{\mathfrak{g}}(\hat{x}) = \mathfrak{c}_{\mathfrak{g}}(x). \tag{1.17}$$

Furthermore it follows from $\pi(C_{\hat{G}}(\hat{x})) = C_G(x)_0$, see [37], Lemma 9.2, p. 60, that we have

$$dim\, C_{\hat{G}}(\hat{x}) = dim\, C_G(x). \tag{1.18}$$

Therefore global and infinitesimal centralizer dimensions coincide for x exactly if they do for $\hat{x}$. ♡.

With these preparations, we can explicitly describe the cases, where the orbit maps are separable:

Theorem 1.3 *Let G be a simple group of type A, D, E, with nontrivial group of diagram automorphisms and let τ be a nontrivial diagram automorphism, which is defined for G. Then the orbit maps $G \to G.x$ are separable for all $x \in G\tau$, if $char(k)$ is either zero or very good and does not divide the order of τ.*

Proof: Fixing R and τ, we only need to verify the statement of the theorem for G of one isomorphism class with $R(G) = R$, such that τ exists for G, by Lemma 1.6. (Recall that all isomorphism classes of algebraic groups G with $R(G) = R$ only differ by their fundamental groups.) This will be done case-by-case:
1. G is of type D_n, $n \geq 4$, $\tau^2 = 1$: In this case, we choose $G = SO(2n)$. Then we have:

$$\tilde{G} = G \rtimes \mathbb{Z}/2\mathbb{Z} = O(2n), \tag{1.19}$$

where we can choose $\tau = \begin{pmatrix} E_{2n-2} & \\ & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix}$. Now we consider the trace form on $\mathfrak{gl}_{2n}$:

$$\begin{aligned} tr : \mathfrak{gl}_{2n} \times \mathfrak{gl}_{2n} &\to k \\ (X, Y) &\mapsto tr(X\,Y). \end{aligned} \tag{1.20}$$

Then tr is $\tilde{G}$ invariant and we have, as is well known:

$$\mathfrak{g} = \{X \in \mathfrak{gl}_{2n},\, X = -{}^tX\}. \tag{1.21}$$

Now a direct summand of $\mathfrak{g}$ in $\mathfrak{gl}_{2n}$ is given by:

$$\mathfrak{m} = \{X \in \mathfrak{gl}_{2n},\, X = {}^tX\}. \tag{1.22}$$

(The direct sum property is given by the decomposition $X = \frac{1}{2}(X + {}^tX) + \frac{1}{2}(X - {}^tX)$, which holds because of $char(k) \neq 2$.) A simple calculation yields

that $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{g}$ with respect to the trace form. Therefore $\mathfrak{m}$ has to be $\tilde{G}$ stable. The statement in this case now follows from Corollary 1.3.
2. G of the other cases: Here our condition on the characteristic can be rephrased as follows:
$char(k)$ does not divide $2\,cox(R(G))$,
where $cox(R(G))$ is the Coxeter number, the order of the Coxeter element. This follows from explicit consideration of the Coxeter numbers, which are given, e.g. in [3], Planches, p. 250ff.
Here, we consider the group G of adjoint type. By [36], Chapters 1,2, we have that $\mathfrak{g} = L \otimes_{\mathbb{Z}} k$, where L is the Chevalley algebra of adjoint type of type $R(G)$. Denote the Killing forms of $\mathfrak{g}$ and L by κ and κ' respectively. By [34], Chapter I, 4.8, p. E-14, we have:

$$det\,\kappa' = (-1)^{|R(G)^+|}(2\,cox(R(G)))^{dim\,G}|\Lambda(R)/\mathbb{Z}(R)|^{-1} \tag{1.23}$$

with respect to the corresponding Chevalley basis. Now, our assumption guarantees, that κ is non-degenerate. Let us consider $G < GL(\mathfrak{g})$. Then $\mathfrak{g} = \mathfrak{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$. Recall that κ is just the restriction of the trace form on $\mathfrak{gl}(\mathfrak{g})$ to $\mathfrak{g}$. Then we choose $\mathfrak{m}$ to be the orthogonal complement of $\mathfrak{g}$ with respect to the trace form. This guarantees $\mathfrak{m}$ to be $\tilde{G}$ stable. Since restriction of the trace form on $\mathfrak{gl}(\mathfrak{g})$ to $\mathfrak{g}$ is non-degenerate, we conclude $\mathfrak{g} \cap \mathfrak{m} = 0$. Hence we have:

$$\mathfrak{gl}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{m}. \tag{1.24}$$

The statement now follows from Corollary 1.3. $\heartsuit$.

This result motivates the following definition:

Definition 1.4 *The characteristic $p = char(k)$ is called excellent for a pair (G, τ) consisting of a semisimple algebraic group G and an exterior automorphism $\tau \in Aut(\Delta(G))$, if p is very good for G and if p does not divide the order of τ.*

Chapter 2

Invariant Theory

In this chapter, we want to describe the invariant theory of the adjoint G-action on an exterior component of $\tilde{G}$ and to give a description of the corresponding categorical quotient, the so called adjoint quotient. First, we need the notion of a Cartan subgroup, generalizing the concept of a maximal torus to non-connected linear algebraic groups, followed by the development of the representation theory of $\tilde{G}$. Afterwards, we also investigate the quotient of an exterior component of such a Cartan subgroup modulo the action of its normalizer in G and show the isomorphism of this quotient with the former one.
From now on we assume, for the rest of this work, that the algebraically closed field k is not an algebraic extension of a finite field.

2.1 Cartan Subgroups

We denote a not necessarily connected algebraic group by $\tilde{G}$ and by G its unit component. Furthermore, we assume knowledge of the theory of linear algebraic groups as presented in e.g. [13, 2, 33]

Definition 2.1 *An algebraic subgroup $C < \tilde{G}$ is called a Cartan subgroup if all the following properties hold:*
(i) C is diagonalizable,
(ii) C has finite index in its normalizer (in $\tilde{G}$),
(iii) C contains an element z generating C as an algebraic group.
The finite group $\mathcal{W}(C) := N_G(C)/C_0$ is called the outer Weyl group of $(\tilde{G}, C)$ and denoted by $\mathcal{W}(C)$.

As a first consequence of the definition we get:

Lemma 2.1 *Let C be as in the definition above. Then C is abelian.*

Proof: Follows directly from the commutator formula $\overline{(H,H)} = \overline{(\overline{H},\overline{H})}$, found in [2], p. 57 and (iii) of the definition. $\heartsuit$.

Remarks: (a) Clearly we can replace (i) and (iii) in the definition by
(i)' C contains a semisimple element z generating C as an algebraic group.
(b) The existence of Cartan subgroups is not clear, but will be proven for G reductive in the sequel.
(c) The above definition is motivated by a similar notion of Cartan subgroups in the theory of non-connected compact groups, cf. [5], Section IV.4.

First we show some properties of Cartan subgroups respectively diagonalizable groups:

Lemma 2.2 *Let D be a diagonalizable group. Then D contains a generating element z iff $D = (k^*)^n \times \mathbb{Z}/l\mathbb{Z}$, for suitable integers l, n.*

Proof: " $\Leftarrow$ " : By [2], Proposition 8.8, p. 115ff, we can find an element $t = (t_1, ..., t_n) \in (k^*)^n$ generating $(k^*)^n$. Taking $u = (\sqrt[l]{t_1}, ..., \sqrt[l]{t_n})$ we have that $z = (u, 1)$ is a generating element of D. (Here the condition on k not to be an algebraic extension of a finite field enters.)
" $\Rightarrow$ " : D, being diagonalizable, is isomorphic to $(k^*)^n \times \mathbb{Z}/l_1\mathbb{Z} \times ... \times \mathbb{Z}/l_p\mathbb{Z}$ for appropriately chosen integers $n, p, l_1, ..., l_p$. Let pr_2 denote the projection to the product of the finite factors. Now let z be the generating element of D. Then $pr_2(z)$ is a generator of $\mathbb{Z}/l_1\mathbb{Z} \times ... \times \mathbb{Z}/l_p\mathbb{Z}$ which implies $\mathbb{Z}/l_1\mathbb{Z} \times ... \times \mathbb{Z}/l_p\mathbb{Z}$ to be isomorphic to $\mathbb{Z}/l\mathbb{Z}$ for a certain choice of l. $\heartsuit$.

We now prove an existence result for Cartan subgroups in the reductive case:

Proposition 2.1 *Let $\tilde{G}$ as above with G reductive. Then every semisimple element of $\tilde{G}$ is contained in a Cartan subgroup C.*

Proof: We prove the proposition first in the case of semisimple G.
Let $g \in \tilde{G}$ be a semisimple element, $C_G(g)_0$ be the unit component of its centralizer in G and $S < C_G(g)_0$ a maximal torus thereof. Denote by H the algebraic subgroup of $\tilde{G}$ generated by S and g. As in the proof of Lemma 2.1, we see that H is abelian. Since, in addition, every element of H is diagonalizable in a faithful representation of $\tilde{G}$ also H is diagonalizable.
Next observe that $S = H_0$, because clearly we have $S < H_0 < C_G(g)_0$ and H_0 is a torus. Now the finite group H/S is generated by gS yielding the existence of a generating element of H. Thus we are left to show that H has finite index in its normalizer in $\tilde{G}$.
We have:

$$\begin{aligned} [N_{\tilde{G}}(H) : H] &= [N_{\tilde{G}}(H) : C_{\tilde{G}}(H)][C_{\tilde{G}}(H) : H] \\ &\leq [N_{\tilde{G}}(H) : C_{\tilde{G}}(H)][C_{\tilde{G}}(H) : H_0] \end{aligned} \tag{2.1}$$

By [38] Corollary 2(b), p. 43f, the first factor on the right hand side is finite. Thus we are left with showing the finiteness of the second one:
Combining the two equations

$$S < C_{\tilde{G}}(H)_0 < C_{\tilde{G}}(g)_0 \tag{2.2}$$

$$S < C_{\tilde{G}}(H)_0 < C_{\tilde{G}}(S)_0 \tag{2.3}$$

gives $S < C_{\tilde{G}}(H)_0 < C_{C_{\tilde{G}}(g)_0}(S)_0$. Now it follows from [37], Theorem 7.5, p. 51, Theorem 8.2, p. 52, and Corollary 9.4, p.60, that $C_{\tilde{G}}(g)_0$ is a reductive group and therefore $S = C_{C_{\tilde{G}}(g)_0}(S)_0$ by [13], Corollary A, p. 159. Therefore $S = C_{\tilde{G}}(H)_0$, which yields our result.
Now let G be reductive and let $C(G)$ be the (in general, positive dimensional) connected centre of G, and let $g \in \tilde{G}$ be again a semisimple element. Denote the conjugation with g by c_g. We are done, if we can show that the connected fixed point group $G_0^{c_g} = C_G(g)_0$ is reductive, because then we can repeat the proof as in the semisimple case above. Clearly, we have that c_g is an automorphism of G and therefore it has to stabilize $C(G)$. Hence c_g descends to an automorphism of the semisimple quotient group $\bar{G} = G/C(G)$ which we will denote again by c_g. Now using [37], Statement 4.5, p. 37, our exact sequence

$$1 \longrightarrow C(G) \longrightarrow G \longrightarrow \bar{G} \longrightarrow 1 \tag{2.4}$$

gives rise to an exact sequence

$$1 \to C(G)^{c_g} \to G^{c_g} \to \bar{G}^{c_g} \to ((1-c_g)(G) \cap C(G))/((1-c_g)(C(G))) \to 1. \tag{2.5}$$

By [37] Corollary 9.4, p. 60, $(\bar{G}^{c_g})_0$ is reductive. Since, in our case, $C(G)$ is a torus, the group $((1-c_g)(G) \cap C(G))/((1-c_g)(C(G)))$ is diagonalizable and hence abelian. Therefore $((\bar{G}^{c_g})_0, (\bar{G}^{c_g})_0) \subset (G^{c_g})_0/(C(G)^{c_g} \cap (G^{c_g})_0)$, where the first group is the semisimple part of $(\bar{G}^{c_g})_0$. Hence $(G^{c_g})_0/(C(G)^{c_g} \cap (G^{c_g})_0)$ has to be reductive and since $(C(G)^{c_g} \cap (G^{c_g})_0)$ is diagonalizable, $(G^{c_g})_0$ also has to be reductive. ♡.
Remark: This proof provides us with a construction recipe for a Cartan subgroup containing a given semisimple element. This will be exploited several times in the sequel.

Now, we prove a technical lemma, we will need quite often:

Lemma 2.3 *Let C be a Cartan subgroup of $\tilde{G}$ and $g \in C$ an element, such that gC_0 generates C/C_0. Then C_0 is a maximal torus in $C_G(g)_0$.*

Proof: Observe that g and C_0 generate C as an algebraic group. Assume that the statement of the lemma is false. Then, every maximal torus of $C_G(g)_0$ containing C_0 will centralize C. So, we get a contradiction to the fact, that C has finite index in its normalizer. ♡.

Proposition 2.2 *Let $\tilde{G}$, G as above. If C is a Cartan subgroup in $\tilde{G}$, then C_0 is regular torus in G.*

Proof: Clearly C_0 is a torus in G. Therefore $C_G(C_0)$ is reductive group. ($C_G(C_0)$ is connected, because it is a centralizer of a torus in a connected algebraic group G, see [13], Theorem 22.3, p. 140.) Hence we can get a decomposition $C_G(C_0) = Z\hat{G}$ with finite intersection, where Z is the centre of $C_G(C_0)$ and $\hat{G} = (C_G(C_0), C_G(C_0))$ is semisimple. Clearly $C_0 \subset Z$. We have to show that $\hat{G}$ is trivial. Let g be a generating element of C, then C_0 is a maximal torus in $C_G(g)_0$ by Lemma 2.3. Since conjugation with g leaves C_0 invariant, it also stabilizes $C_G(C_0)$ and therefore $\hat{G}$. Since, furthermore, C_0 is a maximal torus of $C_G(g)_0$, which is a reductive group by [37], loc cit, respectively the argument in the proof of Proposition 2.1 for G reductive, we have $C_0 = C_G(g)_0 \cap C_G(C_0) = C_{C_G(g)}(C_0)$. Therefore the conjugation map of g

$$c_g : \tilde{G} \to \tilde{G},\ h \mapsto c_g(h) = ghg^{-1} \tag{2.6}$$

has only finitely many fixed points on $\hat{G}$. Now applying [37], 10.12, p. 71, we conclude that $\hat{G}$ is solvable. Being also semisimple it has to be trivial.♡.

As a last general result on Cartan subgroups the following assertion holds:

Lemma 2.4 *Let $\tilde{G}$ be as above with G semisimple and simply connected. Then $C \cap G = C_0$, for every Cartan subgroup C.*

Proof: Let g be a generating element of C, then, by Lemma 2.3, C_0 is a maximal torus of $C_G(g)$. In this case, $C_G(g)$ is connected by [37], Proposition 8.2. Furthermore, $C \cap G \subset C_G(g)$ and hence $C \cap G \subset C_{C_G(g)}(C_0) = C_0$. ♡.

We now turn to the special situation of Section 1.3. From now on we consider $\tilde{G} = G \rtimes \Gamma$, where G is a semisimple algebraic group and Γ is a subgroup of the group of diagram automorphisms of the Dynkin diagram of G. We also restrict the characteristic of k as given in the beginning of Section 1.3. Furthermore, let $\tau \in \Gamma$ be a fixed diagram automorphism, from now on, and denote by $G\tau$ the connected component of $\tilde{G}$ containing τ. Our next aim is now to prove the following proposition:

Proposition 2.3 *Let $h, z \in G\tau$ be semisimple elements and C a Cartan subgroup, containing z, such that zC_0 is a generator of C/C_0, then h is G-conjugate into $C_0 z$.*

Before proving this result we need some preparations:
Let $z \in G\tau$ be a semisimple element as above and $T < B$ a pair of a maximal torus and a Borel subgroup of G and let C be a Cartan subgroup as in the claim of the proposition. Assume T and B to be stabilized by z, such that $C_0 = T_0^z$, where T^z denotes the z-fixed points of T and T_0^z its unit component. (The existence of $T < B$ can be proven as follows: By Proposition 2.2, we can take $T = C_G(C_0)$. Let B' be a Borel subgroup of $C_G(z)_0$ containing C_0. then by [37], Corollary 7.4, p. 50, we can find a Borel

subgroup $B > B'$ of G stabilized by z. Since $C_B(C_0)$ is a maximal torus of G, we get $T = C_B(C_0) \subset B$.)
First, we prove a lemma:

Lemma 2.5 *we keep the notations as above. Then, every element of Tz is T-conjugate to $T_0^z z$.*

Proof: Denote by $c_z : \tilde{G} \to \tilde{G}$, $c_z(g) = z\,g\,z^{-1}$ the conjugacy action with z on $\tilde{G}$. Then c_z is a semisimple automorphism of $\tilde{G}$. By the proof of Proposition 2.1, the group generated by z and T_0^z is a Cartan subgroup of $\tilde{G}$, whose unit component is T_0^z.
We have to show, that:

$$\forall h \in T \quad \exists h' \in T/T_0^z : \qquad h'\,h\,z\,h'^{-1} \in T_0^z z, \tag{2.7}$$

which is equivalent to the following equation:

$$\forall h \in T/T_0^z \quad \exists h' \in T/T_0^z : \qquad h' c_z(h'^{-1}) = h^{-1}, \tag{2.8}$$

where c_z operates on T/T_0^z in the obvious manner.
Since T_0^z is the unit component of a Cartan subgroup containing z and by property (ii) of the definition of Cartan subgroup, c_z can have only finitely many fixed points on T/T_0^z. Now we can apply [37], Theorem 10.1, p. 67, to our situation, yielding the surjectivity of the map $T/T_0^z \to T/T_0^z$ given by $h' \mapsto h' c_z(h'^{-1})$, thus proving our lemma. $\heartsuit$.
As a consequence of the previous lemma we get:

Corollary 2.1 *Every element of Tz is semisimple.*

Now we can prove the stated result:
Proof of Proposition 2.3: Let $C_0 < T < B$ be a maximal torus and a Borel subgroup stabilized by z as above, and T' and B' be a maximal torus and a Borel subgroup stabilized by h. After conjugating h with an element of G, we can assume, that $T = T'$ and $B = B'$. Since, furthermore, $C_0 = T_0^z$ we only have to show, by Lemma 2.5, that h is G-conjugate into Tz.
Now, hz^{-1} is an element in G stabilizing T and B. The first means $hz^{-1} \in N_G(T)$ thus exhibiting an automorphism of the corresponding root system $R(G;T)$. The latter means hz^{-1} stabilizes a basis thereof, whence $hz^{-1} \in T$, proving the proposition. $\heartsuit$.
Remark: This result suggests, that every element in $G\tau$ is G-conjugate into the shifted fixed point group $G_0^\tau \tau$. This result however is false and counterexamples for each choice of simple G and τ will be given in the appendix.

From this result we can now derive some consequences:

Proposition 2.4 *The map*
$\pi : \{$*Cartan subgroups of* $\tilde{G}\} \to \{$*cyclic subgroups of* $\Gamma\}$ *given by*
$\pi(C) = C/(C \cap G) = (C/C_0)/((C \cap G)/C_0)$ *induces*
(i) a bijection of $\tilde{G}$*-conjugacy classes of the former set and* Γ*-conjugacy classes of the latter set, as well as*
(ii) a bijection of G*-conjugacy classes of the former set and elements of the latter set.*

Proof: 1. Surjectivity: Any cyclic subgroup of Γ is generated by some element $\tau \in \Gamma$ acting semisimply on G. Therefore it is contained in a Cartan subgroup C of $\tilde{G}$ by Proposition 2.1, such that τC_0 generates C/C_0. Hence we have the surjectivity.
2. Claim: If C is a Cartan subgroup, $\gamma \in \Gamma$ a generator of $\pi(C) = C/(C \cap G)$, then there exists a generating element $z \in C$, with $zG = \gamma \in \tilde{G}/G = \Gamma$.
Proof of claim: By Lemma 2.2, we have $C = C_0 \times \mathbb{Z}/l\mathbb{Z}$ for a certain $l \in \mathbb{Z}$. Now we have $C/(C \cap G) = \mathbb{Z}/m\mathbb{Z}$, where clearly $m|l$. Let γ be a generator of $\mathbb{Z}/m\mathbb{Z}$. To prove the claim we end up with showing that given a surjective group homomorphism $\mathbb{Z}/l\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ the inverse image of a generator downstairs contains a generator upstairs, which can be found e.g. in [5], Section IV.4, Exercise 3. $\diamond$.
3. Injectivity: Let C, C' be two Cartan subgroups of $\tilde{G}$, whose images $\pi(C)$ and $\pi(C')$ are conjugate (i) respectively equal (ii) and let z be a generator of C. Then we can find $g \in \tilde{G}$, such that $g\,z\,g^{-1}G$ is a generator of $\pi(C')$, with $g = e$ in case (ii). By the claim above, we can find a generator z' of C', such that $g\,z\,g^{-1} \in z'G$. Since z, z' are semisimple, we can find, by Proposition 2.3, an element $g_0 \in G$ such that $g_0\,g\,z\,g^{-1}g_0^{-1} \in C_0'z'$. Therefore, C has to be conjugate to a subgroup of C' and vice versa by interchanging the roles of C and C' in the previous reasoning. In particular C_0 and C_0' are conjugate to subgroups of one another yielding $C_0 = C_0'$ by dimension reasons and irreducibility. Similarly, the number of connected components of C and C' must also coincide yielding the stated results. $\heartsuit$.

Corollary 2.2 *For* $\tilde{G} = G \rtimes \Gamma$ *as above, we have: If* C *is a Cartan subgroup of* $\tilde{G}$*, then* $C \cap G = C_0$.

Proof: Let $\tau \in \Gamma$ and $C(\tau)$ be a Cartan subgroup containing τ, such that $C(\tau)/C(\tau)_0$ is generated by $\tau C(\tau)_0$. Take, e.g. as in the proof of Proposition 2.1, for C the group generated by τ and a maximal torus of G_0^τ. Clearly, we have $C(\tau) \cap G = C(\tau)_0$. Let now C be an arbitrary Cartan subgroup of $\tilde{G}$. Then C is conjugate to a group of type $C(\tau)$ as above, by Proposition 2.4. Hence C and $C(\tau)$ share the same number of connected components and, again by Proposition 2.4, their quotient groups $C/(C \cap G)$ and $C(\tau)/(C(\tau) \cap G)$ have the same group order, which equals same number as the number of connected components, because it does for $C(\tau)$. $\heartsuit$.

Remark: This is a similar result as Lemma 2.4. Here, the group has to be a semidirect product of a semisimple algebraic group and a subgroup of the diagram automorphisms of its Dynkin diagram. In Lemma 2.4, the unit component had to be simply connected, but $\tilde{G}$ did not have to be a semidirect product.

Proposition 2.5 *Let C be a Cartan subgroup of $\tilde{G}$, and denote by C^* the set of elements of C which are mapped to a generator of $C/(C \cap G) = C/C_0$ under the quotient map. Then we have:*
(i) Two elements of C^* are conjugate under $\tilde{G}$ iff they are under $N_{\tilde{G}}(C)$, and:
(ii) Two elements of C^*, lying in the same connected component of C are conjugate under G iff they are under $N_G(C)$.

Proof: Let $x, y = g\,x\,g^{-1} \in C^*$ as in the claim of the proposition, with $g \in \tilde{G}$ for case (i) and $g \in G$ for case (ii). Then we have $x \in C \cap g^{-1}C\,g$ and even $x \in (g^{-1}C\,g)^*$, because C/C_0 and $g^{-1}C\,g/g^{-1}C_0\,g$ are conjugate in Γ, by Proposition 2.4. Therefore, we get the following disjoint union of algebraic varieties:

$$\begin{aligned} C &= C_0 \sqcup \ldots \sqcup x^{l-1}C_0 && (2.9) \\ g^{-1}C\,g &= g^{-1}C_0\,g \sqcup \ldots \sqcup x^{l-1}g^{-1}C_0\,g, && (2.10) \end{aligned}$$

where l is the number of connected components of C. Now C_0 and $g^{-1}C_0\,g$ are maximal tori in $C_G(x)_0$ by Lemma 2.3. Therefore we can find an element $h \in C_G(x)_0 < G$, such that $h\,g^{-1}C_0\;gh^{-1} = C_0$, which implies:

$$\begin{aligned} h\,g^{-1}C\,g\,h^{-1} &= h\,g^{-1}C_0\,g\,h^{-1} \sqcup \ldots \sqcup h\,x^{l-1}g^{-1}C_0\,g\,h^{-1} \\ &= C. && (2.11) \end{aligned}$$

Hence we must have $hg^{-1} \in N_{\tilde{G}}(C)$ in case (i) respectively $hg^{-1} \in N_G(C)$ in case (ii). Clearly we have $y = g\,x\,g^{-1} = g\,h^{-1}x\,h\,g^{-1}$. ♡.

If all cyclic subgroups of Γ have prime order, we get the following consequence:

Corollary 2.3 *Let $\tilde{G}$ be as above and C a Cartan subgroup of $\tilde{G}$ such that C/C_0 is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ where p is a prime number. (This holds, in particular, if G is simple of type $A\,D\,E$ and $\tilde{G}$ a nontrivial semidirect product of G with a subgroup of the diagram automorphisms of G.)*
Then two elements of C not lying in G are conjugate under $\tilde{G}$, (resp. G, if they also lie in the same connected component of $\tilde{G}$), iff they are under $N_{\tilde{G}}(C)$ (resp. $N_G(C)$).

Remark: If C is connected, which implies that it is a maximal torus of G, then (ii) gives us the well-known statement, that two elements of C are conjugate under G iff they are under the Weyl group.

2.2 A Density Result

In this section we prove the density of the set of semisimple elements of $G\tau$ in all of $G\tau$. Afterwards we derive some consequences thereof.

Proposition 2.6 *Let $\tau \in \Gamma$, $\tilde{G}$ as above, then the set of semisimple elements in $G\tau$ denoted by $(G\tau)_{s.s.}$ is a dense subset of $G\tau$.*

Proof: Let C be a Cartan subgroup of $\tilde{G}$, such that τC_0 is a generator of C/C_0 (i.e. C_0 is a maximal torus in G_0^τ by Lemma 2.3), then because of Proposition 2.3 it suffices to show, that $\bigcup_{g\in G} gC_0\tau g^{-1} = (G\tau)_{s.s.}$ contains an open (hence dense) subset of $G\tau$. This will be shown as follows:
We consider the conjugation map Φ:

$$\begin{aligned} \Phi : G/C_0 \times C_0 &\to G\tau \\ (gC_0, h) &\mapsto g\,h\tau\,g^{-1}. \end{aligned} \tag{2.12}$$

The dimensions of the image and preimage spaces are clearly equal. (They equal the dimension of G.) Since Φ is a morphism of algebraic varieties, its image $Im\,\Phi$ contains an open subset of its closure $\overline{Im\,\Phi}$, [11], Exercise 3.19, p. 94, [38], 1.13, Proposition 1, p. 14.
Apparently, we have $dim\,(Im\,\Phi) \leq dim\,G$. If we can find a point in $Im\,\Phi$, whose preimage is finite, we can apply the *dimension formula for morphisms of algebraic varieties*, stating that for a morphism $f : X \to Y$ of algebraic varieties and a connected component V of a fibre of f we have $dim\,V \geq dim\,X - dim\,Y$, found e.g. in [11], Exercise 3.22, p. 95, [38], 1.13, Lemma 2, p. 21. This will give us $dim\,(Im\,\Phi) \geq dim\,G$, and we are done.
So we have to find $t\tau \in C_0\tau$ such that $\Phi^{-1}(t\tau) \cong \{g \in G/C_0,\, g^{-1}t\tau\,g \in C_0\tau\}$ is finite.
Take $t\tau$ a generating element of C, then the assumption $g^{-1}t\tau\,g \in C_0\tau \subset C$ implies $g \in N_G(C)$ and hence gC_0 lies in the finite group $\mathcal{W}(C)$. ♡.
Remark: In the cases where the conjugation map $G \times G\tau \to G\tau$ $(g,h) \mapsto g\,h\,g^{-1}$ is separable, one gets a second proof of the proposition by showing that the differential of the map Φ, as defined in the proof above, is surjective at some point.

2.3 The Structure of the Outer Weyl Group

Here we want to give a description of the outer Weyl group $\mathcal{W}(C) = N_G(C)/C_0$. By Proposition 2.4, it suffices to take as a Cartan subgroup C one containing τ, such that τC_0 generates C/C_0. Then we shall denote $\mathcal{W}(C)$ by $\widetilde{\mathcal{W}}$. We can even achieve the following situation: τ stabilizes a maximal torus T of G and a certain Borel $B > T$, such that $C_0 = T_0^\tau$, the fixed point torus of T. Therefore τ acts on the root system $R(G;T)$ with basis $\Pi(B)$ as the diagram automorphism.
Now we can state the first result:

Lemma 2.6 *With the notation above we have a split exact sequence*

$$1 \longrightarrow (T/T_0^\tau)^\tau \longrightarrow \widetilde{\mathcal{W}} \xrightarrow{\varphi} \mathcal{W}^\tau \longrightarrow 1, \tag{2.13}$$

where the map φ is given by $\varphi : nT_0^\tau \mapsto nT$, for an element $n \in N_G(C)$.

Proof: 1. Exactness: Clearly, we have $(T/T_0^\tau)^\tau \subset \widetilde{\mathcal{W}}$, because t centralizes T_0^τ and $t\tau t^{-1}\tau^{-1} \in T_0^\tau$, for any representative t with $tT_0^\tau \in (T/T_0^\tau)^\tau$.
To prove exactness, we have to show the surjectivity and to determine the structure of the kernel of φ.
1.a. $Ker\,\varphi$: apparently we have $Ker\varphi = \{nT_0^\tau \in \widetilde{\mathcal{W}}, nT_0^\tau = nT = T\}$ yielding $n \in T$, for $n \in N_G(C), nT_0^\tau \in Ker\,\varphi$. For this choice of n, we must also have $n\tau n^{-1} \in \tau T_0^\tau$, which directly gives $nT_0^\tau = \tau n T_0^\tau \tau^{-1}$, proving our claim about $Ker\varphi$.
1.b. $Im\,\varphi$: Let $nT_0^\tau \in \widetilde{\mathcal{W}}$, then $\tau n \tau^{-1} \in nT_0^\tau \subset nT$. Since T is stabilized by τ we have $nT \in \mathcal{W}^\tau$.
If, on the other hand $w = nT \in \mathcal{W}^\tau$, we have to find an element $t \in T$, such that $nt \in N_G(C)$. Let $z = h\tau$, with $h \in T_0^\tau$, be a generating element for C. By the assumption on n, there exists $t' \in T$, such that $\tau^{-1} n \tau = nt'$. It follows that $z^{-1} n\, z = n\, t'\, t$, with $t = \tau^{-1} n^{-1} h^{-1} n\, h\tau \in T$. Clearly, the finite groups $(T/T_0^\tau)^\tau$ and $(T/T_0^\tau)^z$ coincide, therefore by [37], Theorem 10.1, p. 67, applied to conjugation with z^{-1} on (T/T_0^τ), we can find $\tilde{t} \in T$, such that $\tilde{t}\, z^{-1} \tilde{t}^{-1} z\, T_0^\tau = n\, t' t\, n^{-1} T_0^\tau$. A simple calculation, using the fact that z is a generator for C, shows $\tilde{t}n \in N_G(C)$, whence the surjectivity.
2. Sequence is split: By Theorem 1.1 and Proposition 1.1, we can describe $\mathcal{W}^\tau$ as the Weyl-group of the fixed point group G_0^τ, and T_0^τ is a maximal torus thereof by Lemma 2.3. Hence we get an embedding $\iota : \mathcal{W}^\tau \hookrightarrow \widetilde{\mathcal{W}}$. By the construction of $\mathcal{W}^\tau$ we clearly get $\varphi \circ \iota = id$. ♡.

Next, we want to understand the structure of the factor $(T/T_0^\tau)^\tau$ of $\mathcal{W}$ more thoroughly. Therefore we need some information about the tori T, T_0^τ and their respective character lattices.
Let S be any torus, i.e. a connected diagonalizable affine algebraic group, then denote by $\chi(S)$ the character lattice, i.e. the set of all algebraic group

homomorphisms $S \to k^*$. The following discussion uses basic facts about diagonalizable groups, found in e.g. [38], Section 2.6, [13], Chapter 16, or [2], Chapter 8.
Let T now be a maximal torus of G, $\chi(T)$ the corresponding character lattice, T_0^τ the unit component of the τ fixed point torus in G. Then, corresponding to the embedding $i : T_0^\tau \hookrightarrow T$, we have a projection $i^* : \chi(T) \to \chi(T_0^\tau)$ which is just given by taking the mean over each τ-orbit of each element $\mu \in \chi(T)$. Thus, by extending the notation of Chapter 1, we have $\chi(T_0^\tau) = {}^\tau\chi(T)^1$.
But there is also another lattice involved, the fixed point lattice $\chi(T)^\tau$ of the τ-action on $\chi(T)$. Corresponding to this lattice and the inclusion $\chi(T)^\tau \subset \chi(T)$, we have a torus denoted by T' and a quotient map $p : T \to T'$, such that $p^* : \chi(T)^\tau \to \chi(T)$ is just the inclusion. (Note that, for G simply connected, we have $\chi(T) = \Lambda(R(G;T))$, the weight lattice, and we are in the situation described in Chapter 1.) We now identify $T_0^\tau\tau$ with T_0^τ, (by multiplying with τ^{-1}). This identification becomes $\widetilde{\mathcal{W}}$-equivariant, if we let $\widetilde{\mathcal{W}}$ act on T_0^τ by the following rule: $w * t := n_w\, t\tau\, n_w^{-1}\tau^{-1}$, for all $t \in T_0^\tau$ and $w \in \widetilde{\mathcal{W}}$ with $n_w T_0^\tau = w$. Since we can realize the $\mathcal{W}^\tau$-part of $\widetilde{\mathcal{W}} = (T/T_0^\tau)^\tau \rtimes \mathcal{W}^\tau$ in G^τ, as in the proof of Lemma 2.6, $\mathcal{W}^\tau$ acts on T_0^τ in the usual way, but $(T/T_0^\tau)^\tau$ acts on T_0^τ by multiplication with $t\,\tau\, t^{-1}\tau^{-1}$.
We now give another description of T':

Lemma 2.7 *Let $\phi : T \to T$ be the group homomorphism $t \mapsto \phi(t) = t\,\tau\, t^{-1}\tau^{-1}$, then $T' \cong T/Im\,\phi$. The map ϕ is $\mathcal{W}^\tau$-equivariant.*

Proof: Denote the projection map $T \to T/Im\,\phi$ by π. Let μ be a character on the torus $T/Im\,\phi$, then $\pi^*(\mu)$ is a character on T being trivial on $Im\,\phi$. Therefore, we have $\pi^*(\mu)(\phi(t)) = 1$, for all $t \in T$, which is equivalent to $\pi^*(\mu)(\tau\, t\, \tau^{-1}) = \pi^*(\mu)(t)$, for all $t \in T$, which gives us $\pi^*(\mu) \in \chi(T)^\tau$.
On the other hand, let $\mu \in \chi(T)^\tau$. We define a character $\hat{\mu}$ on $T/Im\,\phi$ by $\hat{\mu}(t\,Im\,\phi) := \mu(t)$, which is well defined, because of the τ-invariance of μ. We have $\pi^*(\hat{\mu}) = \mu$. Therefore, the character lattices of $T/Im\,\phi$ and T' coincide (as sublattices of $\chi(T)$) and hence the tori must be isomorphic. The $\mathcal{W}^\tau$-equivariance is trivial. ♡.

For the next result, we need some further notation:
Let us consider the homomorphism $\nu : (T/T_0^\tau)^\tau \to T_0^\tau$ given by $tT_0^\tau \mapsto \nu(tT_0^\tau) = t\,\tau\, t^{-1}\tau^{-1}$. The image of ν will be denoted by H. This is a finite subgroup of T_0^τ. Note that ν need not be injective, its kernel is just given by T^τ/T_0^τ.
Now we can prove a first result:

Lemma 2.8 *Consider $\psi = p \circ i$:*

$$\begin{tikzcd} T \arrow[r, "p"] & T' \\ T_0^\tau \arrow[u, "i"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

Then $\psi : T_0^\tau \to T'$ is the quotient map of T_0^τ by the above defined action of $(T/T_0^\tau)^\tau$. It is $\mathcal{W}^\tau$-equivariant.

Proof: 1. ψ factors over the quotient map: Let $tT_0^\tau \in (T/T_0^\tau)^\tau$. Then, by the definition of the action of $(T/T_0^\tau)^\tau$ on T_0^τ given above, tT_0^τ acts on T_0^τ by left multiplication, independently of the representative t chosen, with $t\,\tau\,t^{-1}\tau^{-1} \in T_0^\tau \cap Im\,\phi$, where ϕ is the map defined in the previous lemma. By the same lemma, we get $p(tT_0^\tau * s) = p(s)$, for all $s \in T_0^\tau$ and $tT_0^\tau \in (T/T_0^\tau)^\tau$.
This shows that ψ factors over the quotient map $T_0^\tau \to T_0^\tau/(T/T_0^\tau)^\tau$. Therefore we have to show $ker\,\psi = H$ and the surjectivity of ψ.
2. Surjectivity of ψ: This is a consequence of Lemma 2.5 and the previous lemma:
Let $t \in T$ be a representative of an element in T', then we can find (by Lemma 2.5) an element $s \in T$, such that $t\,s\,\tau\,s^{-1}\tau^{-1} = t\,\phi(s)$ is an element of T_0^τ and also a representative of the same element of T' by the previous lemma.
3. $ker\,\psi = H$: Let $t \in Ker\,\psi = T_0^\tau \cap Im\,\phi$, i.e. we have $t = s\,\tau\,s^{-1}\tau^{-1}$ for an element $s \in T$. We conclude $s \in (T/T_0^\tau)^\tau$.
4. $\mathcal{W}^\tau$-equivariance: Since $H = ker\,\psi = T_0^\tau \cap Im\,\phi$, this follows from the $\mathcal{W}^\tau$-equivariance of ϕ, cf. Lemma 2.7. ♡.
Remark: In general, the sequence

$$1 \longrightarrow T_0^\tau \longrightarrow T \longrightarrow Im\,\phi \longrightarrow 1, \tag{2.14}$$

with ϕ as in Lemma 2.7, fails to be exact. It is exact iff $T^\tau = T_0^\tau$. However, in that case the sequence cannot be split, since $T_0^\tau \cap Im\,\phi = (T/T_0^\tau)^\tau$ will turn out to be nontrivial.

Next, we want to understand the structure of the group $(T/T_0^\tau)^\tau$ which, clearly, is a finite abelian group.
For simply connected and simple G we can directly calculate $(T/T_0^\tau)^\tau$:

Lemma 2.9 *If G is simply connected and simple, and if τ is a diagram automorphism of G as above, then we have:*

$$\text{If } \tau^2 = 1, \text{ then} \qquad (T/T_0^\tau)^\tau \cong (T/T^\tau)^\tau \cong (\mathbb{Z}/2\mathbb{Z})^{dimT - dimT_0^\tau} \tag{2.15}$$

$$\text{if } \tau^3 = 1, \text{ then} \qquad (T/T_0^\tau)^\tau \cong (T/T^\tau)^\tau \cong \mathbb{Z}/3\mathbb{Z}. \tag{2.16}$$

Proof: First, observe that G simply connected implies that G^τ is connected by [37], Theorem 8.2, p. 52. Therefore, $T^\tau = T_0^\tau$, because the latter is a maximal torus of G^τ.
By the exactness of the sequence of the preceding remark, we have $T/T^\tau \cong Im\,\phi$. This identification is even τ-equivariant. Therefore we can calculate on $Im\,\phi$.
Case 1: $\tau^2 = 1$*:* In this case we calculate $\tau(\phi(t)) = \tau\, t\, \tau\, t^{-1}\tau^{-1}\tau^{-1} = \phi(t^{-1})$. Hence τ acts on T/T^τ by inversion and we get $(T/T^\tau)^\tau = \{t \in Im\,\phi,\ t = t^{-1}\}$. Identifying the torus T/T^τ with a product of as many multiplicative groups k^* as it has dimension, we get the stated result.
Case 2: $\tau^3 = 1$*:* In this case G is $Spin_8(k)$ and G^τ is of type G_2. Hence we have $dim\,(T/T^\tau) = 2$. The automorphism τ is therefore a nontrivial automorphism of a two dimensional torus of order three, giving rise to a nontrivial automorphism of the corresponding character lattice $\chi(T/T^\tau) \cong \mathbb{Z}^2$. The isomorphism group of this lattice is $GL_2(\mathbb{Z})$ and since τ is of order three and $det(g) = \pm 1$, for all $g \in GL_2(\mathbb{Z})$, we must have $\tau \in SL_2(\mathbb{Z})$.
Now the image $\pi(\tau)$ of τ under the two-to-one cover $\pi : SL_2(\mathbb{Z}) \to PSL_2(\mathbb{Z})$ is again of order three. By the theory of the modular group , see e.g. [25], théorème VII. 1.1, $\pi(\tau)$ is conjugate in $PSL_2(\mathbb{Z})$ to an element of the form $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. Hence τ is conjugate in $SL_2(\mathbb{Z})$ to an element of the form $\pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ or $\pm\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, of which only $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ have order three. By replacing τ by τ^{-1} we can assume $\tau = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$.
Let now (t_1, t_2) be appropriate coordinates of $T/T^\tau \cong (k^*)^2$. Then τ acts by $\tau(t_1, t_2) = (t_1^{-1}t_2^{-1}, t_1)$.
A calculation now yields $(T/T^\tau)^\tau = \{(1,1), (\xi, \xi), (\xi^2, \xi^2)\} \cong \mathbb{Z}/3\mathbb{Z}$, for a primitive third root of unity ξ. ♡.
Remark: (i) In case, that G is semisimple and simply connected and that τ is a permutation of isomorphic simple normal subgroups of G, we obtain the following result: Let τ be of order n. Then τ acts as an isomorphism of order n on T/T^τ and we can conclude that $(T/T^\tau)^\tau \subset (\mathbb{Z}/n\mathbb{Z})^{dim\,T - dim\,T_0^\tau}$.
(ii) The result of the above theorem is also obtainable by direct case-by-case calculation as in [39], Lemma 4.10, p. 33, where the compact case is treated.

Next, we approach the general case:
Let H be the image of the map $\nu : (T/T_0^\tau)^\tau \to T_0^\tau$, as defined in the paragraph preceding Lemma 2.8. By Lemma 2.8 H is exactly the kernel of ψ.
Now we have:

Corollary 2.4 *If G is simple of type $A\,D\,E$, then the following statements hold:*
(i) We have identifications:

$$H \cong \chi(H) \cong \chi(T_0^\tau)/\chi(T')|_{T_0^\tau} \tag{2.17}$$

(ii) If G is simply connected, then $H \cong (T/T^\tau)^\tau$.

Proof: Part (ii) follows from the equality $T_0^\tau = T^\tau$, which holds for simply connected G by the proof of Lemma 2.9.
(i) For the second identification consider the exact sequence of diagonalizable groups

$$1 \longrightarrow H \longrightarrow T_0^\tau \longrightarrow T' \longrightarrow 1, \tag{2.18}$$

giving rise to an exact sequence of abelian groups, proving the second identification:

$$0 \longleftarrow \chi(H) \longleftarrow \chi(T_0^\tau) \longleftarrow \chi(T') \longleftarrow 0. \tag{2.19}$$

Now we want to prove the first identification. Since H is a finite abelian group and by the theory of finite groups as developed e.g. in [9, 26], we need to show that the group order of H is not divided by the characteristic of k. (Note that we imposed $char(k) \neq 2$ respectively $char(k) \neq 2, 3$, if $\tau^3 = 1$, in Section 1.3.) In fact, we will show that the order of H is a power of $ord\,\tau$: Let $p : V \to V^\tau$ be the projection of the real vector space V, in which our root and weight lattices are embedded, to its τ-invariant part as defined in Equation 1.2. Then we have $\chi(T_0^\tau) = p(\chi(T)) \subset {}^\tau\Lambda(R)^1$ and $\chi(T') \subset \Lambda(R')$. Furthermore, let $\hat{T}$ be a maximal torus of the simply connected group of the same type as G, thus $\chi(\hat{T}_0^\tau) = {}^\tau\Lambda(R)^1$. Let $\hat{T}'$ be a torus corresponding to $\Lambda(R')$ and $\hat{H}$ defined similarly to H with T replaced by $\hat{T}$. Then the commutative diagram of abelian groups with exact rows and injective columns

$$\begin{tikzcd}
0 & \chi(\hat{H}) \arrow[l] & \chi(\hat{T}_0^\tau) \arrow[l] & \chi(\hat{T}') \arrow[l] & 0 \arrow[l] \\
0 & \chi(H) \arrow[l] & \chi(T_0^\tau) \arrow[l] \arrow[u] & \chi(T') \arrow[l] \arrow[u] & 0 \arrow[l]
\end{tikzcd}$$

gives rise to commutative diagram of diagonalizable algebraic groups with exact rows and surjective columns

$$\begin{tikzcd}
1 \arrow[r] & \hat{H} \arrow[r] & \hat{T}_0^\tau \arrow[r] \arrow[d, "p"'] & \hat{T}' \arrow[r] \arrow[d, "p'"'] & 1 \\
1 \arrow[r] & H \arrow[r] & T_0^\tau \arrow[r] & T' \arrow[r] & 1.
\end{tikzcd}$$

(The surjectivity follows from the fact that images of algebraic groups under algebraic homomorphisms are closed, cf. e.g. [38], Section 1.13, Proposition

2, p. 20.) We have $p(\hat{H}) \subset H$, and by use of the Snake Lemma, we get an exact sequence

$$1 \longrightarrow ker\, p|_{\hat{H}} \longrightarrow ker\, p \longrightarrow ker\, p' \longrightarrow H/p(\hat{H}) \longrightarrow 1. \tag{2.20}$$

We now distinguish two cases: If τ is of order three, e.g. R is of type D_4, we have by Corollary 1.1 that $\hat{T}' = T'$ giving $H = p(\hat{H})$. By Lemma 2.9 and (ii) we have $\hat{H} \cong \mathbb{Z}/3\mathbb{Z}$ and we are done.
If τ is of order two we get, by Corollary 1.1, that $ker\, p'$ is either $\mathbb{Z}/2\mathbb{Z}$ or trivial and hence so is $H/p(\hat{H})$. Furthermore, by Lemma 2.9, $\hat{H}$ is a direct product of copies of $\mathbb{Z}/2\mathbb{Z}$. Therefore the group order of H also has to be a power of two. ♡.
Remark: (i) If G is not simple but semisimple and τ a diagram automorphism exchanging isomorphic simple normal subgroups of G, a similar result holds: The analogue of (ii) is still valid by the same reasoning. To achieve (i), we can similarly carry out the diagram argument using the Snake Lemma, if we impose further restriction on the characteristic of the base field k: In addition to the condition that $char(k)$ does not divide the order of τ we also need, that $char(k)$ does not divide the order of the fundamental group $\Lambda(R')/\mathbb{Z}(R')$.
(ii) In the case of $char(k) = 0$, we do not need Lemma 2.9 for the proof of the above corollary, because the isomorphism $H \cong \chi(H)$ always holds (note that we are dealing with finite abelian groups). Therefore, we get in this case, another proof of Lemma 2.9 by use of the above corollary and direct computation of ${}^{\tau}\Lambda(R)^1/\Lambda(R)^{\tau}$ by use of the set of generators of these lattices as exhibited in Section 1.2.

2.4 Representation Theory of $\tilde{G}$

Our next aim is to understand the representation theory of $\tilde{G}$ presupposing that of G. The relevant results may be found e.g. in [13], Chapter XI, or [38], Sections 3.3 and 3.4.
We start with a definition and a lemma about class functions, which will be useful later.

Definition 2.2 *Let be $\tilde{G}$ as above. A regular function $f \in k[G\tau]$ on the component $G\tau$ of $\tilde{G}$ defined by $\tau \in \Gamma$ will be called a class function, if it is invariant under conjugation by elements in the unit component G, i.e. $f(g\,h\,g^{-1}) = f(h)$, for all $h \in G\tau$, $g \in G$. The ring of all class functions will be denoted by $k[G\tau]^G$.*

Remark: This notion is a bit abusive, because class functions are usually functions on the whole group, invariant under conjugation of the whole group.

Now we can state a first lemma:

Lemma 2.10 *Let $f \in k[G\tau]^G$ be a class function, $g \in G\tau$, then $f(g) = f(g_s)$, where g_s denotes the semisimple part of g.*

Proof: First a remark: If $g \in G\tau$, then so is g_s. Note that $gG = \tau \in \Gamma = \tilde{G}/G$ and Γ consists only of semisimple elements. The unipotent part g_u of g must therefore lie in G, and hence the statement of the lemma makes sense.
Let $g = g_s g_u$ be the Jordan-decomposition of g. Since f is a class function we can assume, without loss of generality, that $g_s \in T_0^\tau \tau$ by Proposition 2.3. Clearly $g_u \in C_G(g_s)$ holds. By [37], Corollary 9.4, p. 60, the group $C_G(g_s)/C_G(g_s)_0$ is abelian and consists of semisimple elements. We even get $g_u \in C_G(g_s)_0$, a reductive group by [37], Corollary 9.4, p. 60. Then we can find a maximal torus $\bar{T}$ and a Borel subgroup $\bar{B} > \bar{T}$ of $C_G(g_s)_0$, such that $g_u \in \bar{B}$. Denote the corresponding basis of the root system $R(C_G(g_s)_0, \bar{T})$ by $\bar{\Pi}$. Since $\bar{\Pi}$ is a set of $\mathbb{Z}$-linear independent elements of the character lattice $\chi(\bar{T})$ we can find a one-parameter subgroup $\lambda : k^* \to \bar{T}$, i.e. an element $\lambda \in \chi(\bar{T})^*$, such that $\bar{\alpha}(\lambda) > 0$, for all $\bar{\alpha} \in \bar{\Pi}$. Then we have $\bar{\alpha}(\lambda) > 0$, for all $\bar{\alpha} \in R(C_G(g_s)_0, \bar{T})^+$, the set of positive roots in $R(C_G(g_s)_0, \bar{T})$.
Now it is a well known fact, that the unipotent radical $\bar{U} = \bar{B}_u$ of our Borel subgroup is isomorphic to an affine space having the cardinality l of $R(C_G(g_s)_0, \bar{T})^+$ as its dimension, see for instance [13], Proposition 28.2, p. 170. This isomorphism is given by $X_{\bar{\alpha}_1}(c_1) \times \ldots \times X_{\bar{\alpha}_l}(c_l) \mapsto (c_1, \ldots, c_l)$, where $X_{\bar{\alpha}_i} : k \to C_G(g_s)_0$ is the one-parameter additive root group corresponding to $\bar{\alpha}_i$, on which $t \in \bar{T}$ acts by $t\, X_{\bar{\alpha}_i}(c_i)\, t^{-1} = X_{\bar{\alpha}_i}(\bar{\alpha}_i(t) c_i)$.
Now, the conjugacy action of a one-parameter subgroup $\mu : k^* \to \bar{T}$ on $\bar{U}$ corresponds to an action $(c_1, \ldots, c_l) \mapsto \mu(s).(c_1, \ldots, c_l) = (s^{\bar{\alpha}_1(\mu)} c_1, \ldots, s^{\bar{\alpha}_l(\mu)} c_l)$ under the above identification with affine l-space. For our special choice of one-parameter group λ as above, the identity of $C_G(g_s)_0$, which corresponds to the zero point of affine l-space under the identification above, is in the closure of every $Im\, \lambda$ orbit in $\bar{U}$. Hence, for $g_u \in \bar{U}$, we get $g_s \in \overline{\lambda(s)\, g\, \lambda(s^{-1})_{s \in k^*}}$. Since f is a regular (hence continuous) class function, we get our result. ♡.

For describing the representation theory of $\tilde{G}$ we restrict to the situation, where $\Gamma < Aut(\Delta)$ is a cyclic group and τ is a generator thereof. Similarly to Mackey theory in the theory of finite group, outlined e.g. in [9, 26], we can extend irreducible representations from G to $\tilde{G}$. Here, we denote by $V(\lambda)$ the irreducible highest weight modules of G corresponding to a highest weight $\lambda \in \chi(T)$.
We have the following result:

Proposition 2.7 *Let G be simple:*
(i) Let $\lambda \in \chi(T)$ be a dominant character of G.
(a) If $\tau(\lambda) = \lambda$, then there exist $(ord\, \tau)$-many inequivalent irreducible $\tilde{G}$-representations on $V(\lambda)$, denoted by $\tilde{\rho}_1^\lambda, \ldots, \tilde{\rho}_{ord\, \tau}^\lambda$.

(b) If $\tau(\lambda) \neq \lambda$, then there exists up to equivalence one irreducible representation $\tilde{\rho}^\lambda$ of $\tilde{G}$ on $\bigoplus_{i=1}^{ord\,\tau} V(\tau^i(\lambda))$.
(ii) If $char(k) = 0$, every irreducible representation of $\tilde{G}$ is equivalent to one of type either (a) or (b) in (i).

Proof: This proposition is folklore in the representation theory of liner algebraic groups. Since we have not found a reference for this, we give a detailed proof:
(i) Let $V(\lambda)$ an irreducible G-module and denote the corresponding representation by ρ^λ. We can define another representation ρ^λ_τ of G on $V(\lambda)$ by twisting with τ:

$$\rho^\lambda_\tau(g) := \rho^\lambda(\tau\, g\, \tau^{-1}), \qquad \forall g \in G. \tag{2.21}$$

By Schur's Lemma and $\tau G \tau^{-1} = G$, we see that ρ^λ_τ is again an irreducible representation of G. Next, we investigate the action of ρ^λ_τ on the weight spaces $V(\lambda)_\mu$ of $V(\lambda)$. We get that $\rho^\lambda_\tau(t)|_{V(\lambda)_\mu}$ acts like $(\tau^{-1}(\mu))(t)\, id_{V(\lambda)_\mu}$ on the corresponding weight space. In particular, we see that $\tau^{-1}(\lambda)$ is the highest weight of the representation ρ^λ_τ. (Here, we also need, that τ stabilizes the Borel subgroup $B > T$, corresponding to our choice of simple roots.) Therefore, we get an intertwiner $S \in GL_G(\rho^\lambda_\tau, \rho^{\tau^{-1}(\lambda)})$, i.e. $S \in Aut(V(\lambda), V(\tau^{-1}(\lambda))$, unique up to a scalar $c \in k^*$, such that

$$S\, \rho^\lambda_\tau(g) = \rho^{\tau^{-1}(\lambda)}(g)\, S, \qquad \forall g \in G \tag{2.22}$$

holds. Now, we have to distinguish two cases:
(a) $\tau^{-1}(\lambda) = \lambda$: In this case we clearly have $S \in Aut(V(\lambda))$ and S operates on the (one-dimensional) weight space of the highest weight by a scalar $c \in k^*$. Let c be an $ord\,\tau$-th root of unity, then we define a map $\tilde{\rho}^\lambda_c : \tilde{G} \to Aut(V(\lambda))$ in the following way:

$$\begin{aligned} \tilde{\rho}^\lambda_c(g) &= \rho^\lambda(g), \quad \text{if } g \in G, & (2.23)\\ \tilde{\rho}^\lambda_c(\tau) &= S^{-1}. & (2.24) \end{aligned}$$

By use of the Equation 2.22, one easily calculates that $\tilde{\rho}^\lambda_c$ is a $\tilde{G}$-representation. For different c, c' these representations are non-isomorphic, because every isomorphism between $\tilde{\rho}^\lambda_c$ and $\tilde{\rho}^\lambda_{c'}$ has to be a multiple of the identity, (by Schur's Lemma and $\tilde{\rho}^\lambda_c|_G = \tilde{\rho}^\lambda_{c'}|_G = \rho^\lambda$).
(b) $\tau^{-1}(\lambda) \neq \lambda$: Let us denote the order of τ by p, and observe that in our case $p = 2, 3$, is a prime number. By Equation 2.22 we can find intertwiners $S_i \in GL_G(\rho_\tau^{\tau^{1-i}(\lambda)}, \rho^{\tau^{-i}(\lambda)})$, for all $i \in 1, .., p-1$. Inductively one proves now that $S_i ... S_1 \in GL_G(\rho^\lambda_{\tau^i}, \rho^{\tau^{-i}(\lambda)})$, in particular $S := S_{p-1} ... S_1 \in GL_G(\rho^\lambda_{\tau^{p-1}}, \rho^{\tau^{1-p}(\lambda)})$. Let us furthermore set $V := \bigoplus_{i=1}^p V(\tau^i(\lambda))$.

Now we define a map $\tilde{\rho}^\lambda : \tilde{G} \to Aut(V)$ by:

$$\tilde{\rho}^\lambda(g) = \begin{pmatrix} \rho^\lambda(g) & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \rho^{\tau^{1-p}(\lambda)}(g) \end{pmatrix}, \quad \text{if } g \in G, \tag{2.25}$$

$$\tilde{\rho}^\lambda(\tau) = \begin{pmatrix} 0 & S_1^{-1} & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & 0 & S_{p-1}^{-1} \\ S & \cdot & \cdot & \cdot & 0 \end{pmatrix} \tag{2.26}$$

One calculates easily, using Equation 2.22, that $\tilde{\rho}^\lambda$ is a representation of $\tilde{G}$. (In particular, one has to check $ord\,\tilde{\rho}^\lambda(\tau) = p$.)
Next, we show the irreducibility of $\tilde{\rho}^\lambda$, which will be done by use of Schur's Lemma. Therefore, observe that each $\rho^{\tau^{-i}(\lambda)}$, $i \in \{1, ..., p\}$ is an irreducible representation of G. Hence each element $x \in End_{\tilde{G}}(\bigoplus_{i=1}^p \rho^{\tau^i(\lambda)})$ has necessarily to be of the shape

$$x = \begin{pmatrix} a_1 E & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & a_p E \end{pmatrix}, \tag{2.27}$$

where $a_1, ..., a_p \in k$. Conjugating with $\tilde{\rho}^\lambda(\tau)$ gives:

$$\begin{pmatrix} 0 & S_1^{-1} & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & 0 & S_{p-1}^{-1} \\ S & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} a_1 E & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & a_p E \end{pmatrix} \begin{pmatrix} 0 & \cdot & \cdot & \cdot & S^{-1} \\ S_1 & 0 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & S_{p-1} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} a_2 E & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & a_1 E \end{pmatrix}$$

yielding $a_1 = a_2 = ... = a_p$ and hence $End_{\tilde{G}}(\tilde{\rho}^\lambda) \cong k$.
Since each S_i is unique only up to a constant of k^*, we still have to show that different choices of the S_i, $i \in \{1, ..., p-1\}$ lead to equivalent representations $\tilde{\rho}^\lambda$. Let $\tilde{\rho}_1^\lambda$ and $\tilde{\rho}_2^\lambda$ be two representations as constructed in Equations 2.25 and 2.26. Of course, these representations coincide on G. Now we can find

$c_1, ..., c_{p-1} \in k^*$, such that

$$\tilde{\rho}_1^\lambda(\tau) = \begin{pmatrix} 0 & S_1^{-1} & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & 0 & S_{p-1}^{-1} \\ S & \cdot & \cdot & \cdot & 0 \end{pmatrix} \quad \text{and} \tag{2.28}$$

$$\tilde{\rho}_2^\lambda(\tau) = \begin{pmatrix} 0 & c_1^{-1} S_1^{-1} & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & 0 & c_{p-1}^{-1} S_{p-1}^{-1} \\ c_1 ... c_{p-1} S & \cdot & \cdot & \cdot & 0 \end{pmatrix}. \tag{2.29}$$

Setting now $a_i := \Pi_{j=1}^{i-1} c_j$, $i \in \{1, ..., p\}$ and $U := diag(a_1\, E, ..., a_p\, E)$, we get $U \in GL_{\tilde{G}}(\tilde{\rho}_1^\lambda, \tilde{\rho}_2^\lambda)$, proving (b).
(ii) If ρ is an irreducible finite-dimensional $\tilde{G}$-representation on V then, since G is reductive, V decomposes into a direct sum of highest weight modules $V(\mu_i)$ for G. (Note that in characteristic zero all reductive groups are linearly reductive.) Now, besides ρ^{μ_i} also the τ^j-twisted representation, by (i) equivalent to $\rho^{\tau^{-j}(\mu_i)}$, has to appear for all $j \in \{1, ..., p\}$. Going through the proof of (i), ρ has to be isomorphic to a representation of (i)(a) or (b). Note that for $\tilde{\rho}_{(a)}^\lambda(\tau)$ being of order p we require $\rho(\tau)$ to be of the form either of Equation 2.24 or 2.26. ♡.

Remarks: (i) If G is semisimple and τ a permutation of isomorphic simple normal subgroups of G, a similar result holds in this case, too: For τ of prime order, the proof is exactly the same as above. If τ is not of prime order then, apart from the cases discussed in this proof, a third, intermediate case may occur: A dominant weight λ is not fixed by τ but has a nontrivial stabilizer Γ_λ.
In this case, a similar discussion as above leads to $|\Gamma_\lambda|$ many non-equivalent representations of $\tilde{G}$ on $V = \bigoplus V(\mu)$, where μ runs over weights in the Γ-orbit of λ. In this case, the representation matrix for any element of $\tilde{G}\tau$ has only trivial diagonal entries. We refrain from giving details.
(ii) In characteristic zero this result says, in particular, that every irreducible $\tilde{G}$-representation is induced by an irreducible representation of G. If we consider the more general situation of $\tilde{G}$ and G being arbitrary affine algebraic groups with $G < \tilde{G}$, the analogous result does not hold in general. An account on this is e.g. given in [8]. The statement holds, if we have e.g. that G is a parabolic subgroup of $\tilde{G}$ for connected $\tilde{G}$. For finite groups G and $\tilde{G}$ the question, which irreducible $\tilde{G}$-module is induced by an irreducible G-module is answered by Mackey theory, which is developed e.g. in [9], Chapter VII. The main reason, that in our situation the result holds is, that the quotient $\tilde{G}/G$ is cyclic, so in particular finite.

Our next task will be to calculate the characters $\tilde{X}^{\lambda}_{(a)}$ of the representations $\tilde{\rho}^{\lambda}_{(a)}$ of $\tilde{G}$ keeping the notation of the previous proposition.
Before we do this we need further preparations:
As in Chapter 1, denote the root system of G by R. Let T' be as defined in the paragraph preceding lemma 2.7, i.e. $\chi(T)^{\tau} = \chi(T')$. So we see that every multiplicative character on T, which is invariant under τ, can be interpreted as a character on T'. Furthermore, denote by G' a group having the root system R' of Section 1.2, whose maximal torus is T'. The requirement that T' is a maximal torus of G' fixes the fundamental group of G'. Then, the following lemma holds:

Lemma 2.11 *Let G, τ, T and $\chi(T)$ be as above. Then $\chi(T)^{\tau}$ is the character lattice $\chi(T')$ for a group G', a group having the root system R' of Section 1.2, whose maximal torus is T'.*
If G is simple we have:
G' is simply connected in the cases of A_{2n}, E_6 and in case D_4 with $\tau^3 = 1$ and either adjoint or simply connected in the other cases (A_{2n-1} and D_n with $\tau^2 = 1$).

Proof: This is just a rephrasing of Corollary 1.1 in terms of algebraic groups. $\heartsuit$.

Now, let us denote by X^{λ} the character of a G-representation with highest weight λ and by X'^{λ} that of G'.
With these preparations we can now state our result:

Proposition 2.8 *If $\tilde{X}^{\lambda}_{(a)}$ is the character of the irreducible $\tilde{G}$-representation $\tilde{\rho}^{\lambda}_{(a)}$, the following holds:*
(i) If $\tau(\lambda) = \lambda$, then we have:

$$\tilde{X}^{\lambda}_{(a)}|_G = X^{\lambda} \tag{2.30}$$

$$\tilde{X}^{\lambda}_{(a)}|_{G\tau^{\pm 1}} \equiv a^{\pm 1} X'^{\lambda}, \tag{2.31}$$

where the denoted equivalence $\equiv$ is given by considering the characters as elements in the group rings $\mathbb{Z}[\chi(T)^{\tau}] = \mathbb{Z}[\chi(T')]$.
(ii) If $\tau(\lambda) \neq \lambda$, then we have:

$$\tilde{X}^{\lambda}_{(a)}|_G = \frac{1}{|\Gamma_{\lambda}|} \sum_{i=1}^{ord\,\tau} X^{\tau^i(\lambda)} \tag{2.32}$$

$$\tilde{X}^{\lambda}_{(a)}|_{G\tau^{\pm 1}} = 0. \tag{2.33}$$

Proof: By the definition of the representations $\tilde{\rho}^{\lambda}_{(a)}$ in the two cases, (look at Equations 2.23 and 2.25 in the proof of Proposition 2.7, respectively),

we clearly get the statement for $\tilde{\rho}^{\lambda}_{(a)}|_G$. Since in the second case, $\tilde{\rho}^{\lambda}(g)$, for $g \in G\tau^{\pm 1}$, is a block-diagonal matrix with zeros on the diagonal, again, by the definition in Equation 2.26, we get $\tilde{X}^{\lambda}(g) = 0$, for $g \in G\tau^{\pm 1}$. The same conclusion holds for semisimple non-simple G and τ an exterior automorphism of G exchanging isomorphic simple normal subgroups of G by Remark (i) following Proposition 2.7.
Thus we are left to show the second part of (i):
By the definition of a, the scalar by which $\tilde{\rho}^{\lambda}_a(\tau)$ acts on the highest weight vector, we can assume without loss of generality that $a = 1$. By definition of the character, i.e. being a trace function, we have that $\tilde{\rho}^{\lambda}_a(g)$ is invariant under conjugation in $\tilde{G}$, for all $g \in \tilde{G}$. So, in particular, $\tilde{\rho}^{\lambda}_a|_{G\tau} \in k[G\tau]^G$ is a class function on $G\tau$. Therefore, using Lemma 2.10 and Proposition 2.3, it is completely determined by its values on $T_0^{\tau}\tau$.
Consider now the weight-space-decomposition of the G-module $V(\lambda)$:

$$V(\lambda) = \bigoplus_{\mu \in \chi(T)} V(\lambda)_{\mu}. \tag{2.34}$$

By the definition of the representer $\tilde{\rho}^{\lambda}_1(\tau)$ in Equation 2.24 and the discussion at the beginning of the proof of Proposition 2.7, we have that τ interchanges the weight spaces $V(\lambda)_{\mu}$ and $\tilde{\rho}^{\lambda}_1(\tau)(V(\lambda)_{\mu}) = V(\lambda)_{\tau(\mu)}$. On the other hand, elements $t \in T_0^{\tau}$ act on $V(\lambda)_{\mu}$ by $\mu(t)\, id_{V(\lambda)_{\mu}}$.
Therefore, we get the following formula:

$$\tilde{X}^{\lambda}(t\tau) = tr\, \tilde{\rho}^{\lambda}_1(t\tau) = \sum_{\mu \in \chi(T)^{\tau}} \mu(t)\, tr\, \tilde{\rho}^{\lambda}_1(\tau)|_{V(\lambda)_{\mu}}, \tag{2.35}$$

for all $t\tau \in T_0^{\tau}\tau$. (We can restrict to invariant weights because the corresponding representation matrix of $t\tau$ restricted to $\bigoplus_{i=1}^{ord\,\tau} V(\lambda)_{\tau^i(\mu)}$, a τ-stable subspace of $V(\lambda)$, for a weight $\mu \neq \tau(\mu)$ is zero on the diagonal.)
Hence we only need to calculate the traces of $\tilde{\rho}^{\lambda}_1(\tau)|_{V(\lambda)_{\mu}}$ for all $\mu \in \chi(T)^{\tau}$. This has already been carried out in [15], Chapter 9, Theorem 9, p. 30:

$$tr\, \tilde{\rho}^{\lambda}_1(t\tau)|_{V(\lambda)_{\mu}} = dim\, V'(\lambda)_{\mu}, \tag{2.36}$$

where, now, $\lambda, \mu \in \chi(T)^{\tau}$ are considered as weights of the group G' and $V'(\lambda)$ is the irreducible G'-module with highest weight λ.
Therefore we get

$$\tilde{X}^{\lambda}_1|_{T_0^{\tau}\tau} = \sum_{\mu \in \chi(T)^{\tau}} (dim\, V'(\lambda)_{\mu})\, \mu. \tag{2.37}$$

Since characters of semisimple connected algebraic groups G' are uniquely determined by their values on a maximal torus T', we get the stated result of our proposition. Since τ and τ^{-1} have identical properties, e.g. $T_0^{\tau} = T_0^{(\tau^{-1})}$

and τ acts on the highest weight space with the scalar 1 exactly when τ^{-1} does, we get the same result for restricting to $G\tau^{-1}$. ♡.
Remark: In the case of Kac-Moody algebras, traces of exterior automorphisms in representations have been determined in [10].

2.5 The Adjoint Quotient of $G\tau$

In this section we will show the isomorphism of the two quotients $G\tau /\!\!/ G$ and $T_0^\tau\tau/\widetilde{\mathcal{W}}$.
We keep the notation of the previous section. In particular, we require Γ to be cyclic and τ to be a generator of Γ. Using geometric invariant theory as e.g. developed in [22, 23, 32, 19], we define the categorical quotient $V/\!\!/G$ of the G-action on an affine variety V by setting $V/\!\!/G := \operatorname{Spec} k[V]^G$, the spectrum of the ring of G-invariant functions on V. Note that the existence of such a quotient is guaranteed, in the case of general reductive groups G, by [22], Theorem 1.1, p. 27.
In our situation, $k[T_0^\tau\tau]^{\widetilde{\mathcal{W}}}$ denotes the $\widetilde{\mathcal{W}}$-invariant functions on $T_0^\tau\tau$. Since $\widetilde{\mathcal{W}}$ is a finite group, we even get that $T_0^\tau\tau/\!\!/\widetilde{\mathcal{W}}$ is a geometric quotient, i.e. $T_0^\tau\tau/\!\!/\widetilde{\mathcal{W}}$ parameterizes exactly the $\widetilde{\mathcal{W}}$-orbits in $T_0^\tau\tau$, the set of which we will denote by $T_0^\tau\tau/\widetilde{\mathcal{W}}$, in the sequel.
Let G' be as defined in the paragraph preceding Proposition 2.8.
Now we can state our first major result:

Theorem 2.1 *(i) The inclusion map $i : T_0^\tau\tau \to G\tau$ gives rise to a commutative diagram*

$$\begin{array}{ccc} T_0^\tau\tau & \xrightarrow{\;i\;} & G\tau \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ T_0^\tau\tau/\widetilde{\mathcal{W}} & \xrightarrow{\;\tilde{i}\;} & G\tau/\!\!/G \end{array}$$

and an isomorphism $\tilde{i}$.
(ii) Furthermore, $G\tau/\!\!/G \cong G\tau/\!\!/\tilde{G}$.

Proof: (i) By the definition of the categorical quotient given above, the statement of the theorem amounts to showing that the restriction map i^*:

$$\begin{aligned} i^* : k[G\tau] &\to k[T_0^\tau\tau] \\ f &\mapsto f|_{T_0^\tau\tau} \end{aligned} \tag{2.38}$$

induces an isomorphism on the invariant subrings $k[G\tau]^G$, $k[T_0^\tau\tau]^{\widetilde{\mathcal{W}}}$. We proceed as follows:
1. $i^*(k[G\tau]^G) \subset k[T_0^\tau\tau]^{\widetilde{\mathcal{W}}}$: Since the action of $\widetilde{\mathcal{W}}$ on $T_0^\tau\tau$ is given by conjugation with a representative in $N_G(C)$, we clearly have $f|_{T_0^\tau\tau} \in k[T_0^\tau\tau]^{\widetilde{\mathcal{W}}}$, for all $f \in k[G\tau]^G$.
2. Injectivity: Let $f \in k[G\tau]^G$, such that $f|_{T_0^\tau\tau} = 0$. Then, since f is a class function, we have

$$f|_{\bigcup_{g\in G} g\,T_0^\tau\tau\,g^{-1}} = 0. \tag{2.39}$$

But f is a regular function and $\bigcup_{g\in G} g\,T_0^\tau\tau\,g^{-1}$ is a dense subset of $G\tau$ by Propositions 2.3 and 2.6. This yields $f = 0$.
3. Surjectivity: The idea of proof is the following: Take the collection of all of the characters $\tilde{X}_1^\lambda|_{G\tau}$, for dominant $\lambda \in \chi(T)^\tau$, as given in the previous section on the representation theory. Show that this collection will be a basis of $k[T_0^\tau\tau]^{\widetilde{\mathcal{W}}}$, if we restrict its elements to $T_0^\tau\tau$.
Using the $\widetilde{\mathcal{W}}$-equivariant isomorphism between T_0^τ and $T_0^\tau\tau$ as described in the paragraph preceding Lemma 2.7 and by Lemma 2.8 as well as the already proven injectivity of our restriction map $f \mapsto f|_{T_0^\tau\tau}$ for $f \in k[G\tau]^G$, we get the following sequence of rings:

$$k[G\tau]^G \hookrightarrow k[T_0^\tau\tau]^{\widetilde{\mathcal{W}}} \xrightarrow{\sim} k[T']^{\mathcal{W}^\tau}, \tag{2.40}$$

where the last isomorphism is given by

$$k[T_0^\tau\tau]^{\widetilde{\mathcal{W}}} \cong \left(k[T_0^\tau\tau]^{(T/T_0^\tau)^\tau}\right)^{\mathcal{W}^\tau} \cong k[T_0^\tau\tau/(T/T_0^\tau)^\tau]^{\mathcal{W}^\tau} \cong k[T']^{\mathcal{W}^\tau}, \tag{2.41}$$

using Lemma 2.6 for the first isomorphism and Lemma 2.8 for the last.
Now denote the composition of the two maps in Equation 2.40 by β.
Let $\tilde{X}_1^\lambda|_{G\tau}$, for $\lambda \in \chi(T)^\tau$, be the restriction of the character of the corresponding $\tilde{G}$-representation to $G\tau$. By Proposition 2.8, we have $\beta(\tilde{X}_1^\lambda|_{G\tau}) = X'^\lambda|_{T'}$, a character of G' as in Lemma 2.11, restricted to the maximal torus T' of G'. By the second remark after Lemma 1.4, we see that every restriction to T' of a character X'^λ of a G'-representation can be obtained as the image under β of a certain $\tilde{X}_1^\mu|_{G\tau}$ for suitable μ.
Hence we have:

$$\beta(\{\tilde{X}_1^\lambda|_{G\tau},\, \lambda \in \chi(T)^\tau \quad \text{dominant }\}) = \{X_1'^\lambda|_{T'},\, \lambda \in \chi(T') \quad \text{dominant }\}. \tag{2.42}$$

By [35], Lemma 6.3, p. 294, respectively [38], Section 3.4, Theorem 2, p. 87, respectively [3], Chapter `VI`, §3, the set on the right hand side is a basis for $k[T']^{\mathcal{W}^\tau}$, proving the surjectivity.
(ii) In the proof of (i), we have shown that the set of characters $\{\tilde{X}_1^\lambda|_{G\tau}, \lambda \in \chi(T)^\tau \quad \text{dominant }\}$ is a basis of $k[G\tau]^G$. Since all these elements, being characters, are invariant under conjugation with $\tilde{G}$ we get $k[G\tau]^G \subset k[G\tau]^{\tilde{G}}$. The inverse inclusion is clear. ♡.

For proving the surjectivity and statement (ii) above in the case of G simply connected and $char(k) = 0$, we can indicate a second proof avoiding the use of representation theory:
Second proof of step 3 and of (ii) above:
Surjectivity: As already indicated, assume $char(k) = 0$ and G simply connected. Similarly to the proof of Proposition 2.6, we consider the following map (note that here $T^\tau = T_0^\tau$):

$$\begin{aligned} \Phi : G/T^\tau \times T^\tau\tau &\rightarrow G\tau \\ (gT^\tau, h\tau) &\mapsto g\, h\tau\, g^{-1}. \end{aligned} \tag{2.43}$$

We can define a $\widetilde{\mathcal{W}}$-action on $G/T^\tau \times T^\tau\tau$ by

$$w.(gT^\tau, h\tau) := (g\, n_w^{-1}T^\tau, n_w\, h\tau\, n_w^{-1}), \tag{2.44}$$

where n_w is a representative of w in $N_G(< T^\tau, \tau >)$. (Recall that $< T^\tau, \tau >$ is a Cartan subgroup in the sense of Section 2.1). Apparently, this action is well defined and Φ is $\widetilde{\mathcal{W}}$-invariant. By Proposition 2.6 Φ is dominant. Hence Φ induces a map of the fields of rational functions of these varieties:

$$\Phi^* : k(G\tau) \longrightarrow k(G/T^\tau \times T^\tau\tau). \tag{2.45}$$

Since the transcendence degrees of these fields coincide, $k(G/T^\tau \times T^\tau\tau)$ is a finite field extension of $k(G\tau)$ which, in this case, is clearly separable. Denote the corresponding degree by n.
In this situation, there exists an open subset $U \subset G/T^\tau \times T^\tau\tau$ such that for all $u \in U$ the preimage $\Phi^{-1}(\Phi(u))$ contains exactly n elements, a well known fact in algebraic geometry, found e.g. in [19], `AI` 3.5, p. 251. Observe that Φ is G-equivariant, if we let G act on $G/T^\tau \times T^\tau\tau$ by left translation on the left factor and on $G\tau$ by conjugation. Hence we have:

$$g.\Phi^{-1}(\Phi(u)) = \Phi^{-1}(\Phi(g.u)). \tag{2.46}$$

Therefore, we can choose U to be G-stable without loss of generality. This means $U = G/T^\tau \times V$ where V is an open subset of $T^\tau\tau$.
Next, we want to show that n actually is the order of $\widetilde{\mathcal{W}}$:
To obtain this, we take $h\tau \in V$ and consider $C_G(h\tau)$. We clearly have $T^\tau < C_G(h\tau)$ and, furthermore:

$$C_G(h\tau)/T^\tau \times \{h\tau\} \subset \Phi^{-1}(\Phi(eT^\tau, h\tau)) = \Phi^{-1}(h\tau). \tag{2.47}$$

Since the latter is finite and since $C_G(h\tau)$ is connected, cf. [37], Theorem 8.2, p. 52, we conclude $C_G(h\tau) = T^\tau$. Let $(g\,T^\tau, h'\tau) \in \Phi^{-1}(h\tau)$, i.e. $g\,h'\tau\,g^{-1} = h\tau$. Then, by Proposition 2.5, we can find $g' \in N_G(< T^\tau, \tau >)$, such that $g'\,h\tau\,g'^{-1} = h'\tau$, giving $gg' \in C_G(h\tau) = T^\tau$. Hence $gT^\tau = g'^{-1}T^\tau \in \widetilde{\mathcal{W}}$,

proving that the cardinality of $\Phi^{-1}(h\tau)$ is the order of $\widetilde{\mathcal{W}}$.
Φ induces a map

$$\Phi' : \widetilde{\mathcal{W}}\backslash(G/T^\tau \times T^\tau\tau) \to G\tau. \tag{2.48}$$

By the $\widetilde{\mathcal{W}}$-invariance of Φ, we can choose the set V above, w.l.o.g., to be $\widetilde{\mathcal{W}}$-stable. Then, $\widetilde{\mathcal{W}}\backslash(G/T^\tau \times V) \subset \widetilde{\mathcal{W}}\backslash(G/T^\tau \times T^\tau\tau)$ is an open set, on which the fibre of Φ' consists of just one point. Therefore, $\widetilde{\mathcal{W}}\backslash(G/T^\tau \times T^\tau\tau)$ and $G\tau$ are birationally equivalent, yielding:

$$k(G\tau) \cong k(\widetilde{\mathcal{W}}\backslash(G/T^\tau \times T^\tau\tau)) = k(G/T^\tau \times T^\tau\tau)^{\widetilde{\mathcal{W}}}. \tag{2.49}$$

Observe, furthermore, that the G-action on $G/T^\tau \times T^\tau\tau$ commutes with the $\widetilde{\mathcal{W}}$-action so that this G-action descends to $\widetilde{\mathcal{W}}\backslash(G/T^\tau \times T^\tau\tau)$ and such that Φ' becomes G-equivariant.
Set:

$$pr : G/T^\tau \times T^\tau\tau \to T^\tau\tau, \tag{2.50}$$

the second projection, which clearly is $\widetilde{\mathcal{W}}$-equivariant. Take $f \in k[T^\tau\tau]^{\widetilde{\mathcal{W}}}$. Then we have $pr^*(f) \in k[G/T^\tau \times T^\tau\tau]^{\widetilde{\mathcal{W}}\times G}$ and, therefore, $\Phi^{*\,-1}(pr^*(f)) \in k(G\tau)^G$.
To complete the proof of the surjectivity, we have to show that $F := \Phi^{*\,-1}(pr^*(f))$ already is a regular function on $G\tau$ and that $\Phi^{*\,-1}(pr^*(f))|_{T^\tau\tau} = f$.
For the latter, observe: $F(h\tau) = (pr^*(f)((eT^\tau\tau, h\tau)) = f(h\tau)$, for all $h\tau \in T^\tau\tau$.
In our situation, G simply connected and $char(k) = 0$, the algebra $k[G] \cong k[G\tau]$ is a unique factorization domain. This follows from the fact that $Pic\, G$ is trivial by [17], Proposition 4.6, p. 74, which amounts to the unique factorization property of $k[G]$ by [11], Chapter I, Proposition 1.12A, p. 7. Therefore we can choose $H_1, H_2 \in k[G\tau]$ having no common divisor, such that $F = \frac{H_1}{H_2}$. Since F is G invariant we must have $\frac{g.H_1}{g.H_2} = F = \frac{H_1}{H_2}$ providing us with a group homomorphism

$$c : G \to k[G\tau]^*, \tag{2.51}$$

such that $g.H_i = c(g)H_i$. By use of [18], Proposition 1.2, p. 78, every element c of $k[G\tau]^* \cong k[G]^*$ with $c(e) = 1$ is a multiplicative character of G, giving $k[G\tau]^* = k^*$. Because of $G = (G, G)$ it follows, that G has no nontrivial characters. Hence we get $H_i \in k[G\tau]^G$.
Now, we have for arbitrary $x \in G\tau$:

$$H_1(x) = H_1(x_s) = H_1(\hat{x}_s) = F(\hat{x}_s)H_2(\hat{x}_s) = F(\hat{x}_s)H_2(x), \tag{2.52}$$

where x_s denotes the semisimple part of x and $\hat{x}_s$ is an element of $T^\tau\tau$ conjugate to x_s. The first and last equality follow from Lemma 2.10. Now we see that $H_2(x) = 0$ implies $H_1(x) = 0$, giving that each nontrivial divisor

of H_2 divides H_1, forcing H_2 to be a unit in $k[G\tau]$ yielding $F = H_1$ (up to a unit), and thus completing our proof.
(ii) Here we do not require any restrictions on the fundamental group of G and the characteristic of k:
The group Γ operates by conjugation on $T^\tau\tau$ and on $G\tau$, and the inclusion map i is Γ-equivariant. So are the quotient maps $T^\tau\tau \to T^\tau\tau/\widetilde{\mathcal{W}}$ and $G\tau \to G\tau/\!\!/G$. From the Γ-equivariance of i we get the equivariance of the isomorphism between these quotient spaces. But Γ acts trivially on $T^\tau\tau$ and therefore also on its quotient $T^\tau\tau/\widetilde{\mathcal{W}}$. ♡.

We get some consequences of the theorem:

Corollary 2.5 *The set $\{\tilde{X}_1^\lambda|_{G\tau}, \lambda \in \chi(T)^\tau \quad dominant\ \}$ of all restrictions to $G\tau$ of characters of $\tilde{G}$-representations $\tilde{\rho}_1^\lambda$ form a linear basis of the ring of class functions $k[G\tau]^G$.*

Let us denote the fundamental dominant weights of $\Lambda(R(G))$ by $\{\lambda_1, ..., \lambda_r\}$ and the fundamental dominant weights of $\Lambda(R(G))^\tau = \Lambda(R(G)')$ by $\{\lambda'_1, ..., \lambda'_s\}$. Then we have:

Corollary 2.6 *(i) If the group G' appearing in the paragraph preceding Proposition 2.8 is simply connected, i.e. $\chi(T)^\tau = \Lambda(R')$, then we have: $G\tau/\!\!/G \cong \mathbb{A}^{rkG'}$. (This happens, in particular, for G simply connected). Furthermore, the $\tilde{X}_1^{\lambda'_j}|_{G\tau}$, $j \in \{1, ..., s\}$ freely generate $k[G\tau]^G$ as k algebra.*
(ii) In particular the quotient $G\tau/\!\!/G$ is isomorphic to an affine space of dimension $dim\, T_0^\tau = dim\, T' = rk\, G'$ in the following cases:
(a) If G is of type A_{2n}, E_6 or D_4 with τ of order three.
(b) If G is simply connected of type A_{2n-1} or D_n with τ of order two.

Proof: (i) By the previous theorem and its proof we clearly have $G\tau/\!\!/G \cong T'/\mathcal{W}^\tau$. The statement now follows from statements in [35], Lemma 6.3, p. 294, respectively [38], Section 3.4, Theorem 2, p. 87 for G' simply connected and the equality $\beta(\tilde{X}_1^{\lambda'_j}|_{G\tau}) = X'^{\lambda'_j}|_{T'}$, $j \in \{1, ..., s\}$ (β as in the proof of the theorem above).
(ii) This is immediate by Lemma 2.11. ♡.

It may also happen, in the case where G is not simply connected of type A_{2n-1} or D_n with τ of order two, that the categorical quotient is an affine space. This happens whenever $\chi(T)^\tau = \Lambda(R')$. To figure out these cases, we need a thorough investigation of the action of τ on the fundamental group of G, which will be our next purpose. Here, we shall restrict to the case that G is simple:
Let $\hat{G}$ be the universal cover of G and denote by C_G the kernel of the quotient map $p : \hat{G} \to G$, as in Section 1.4. Since the automorphism τ also exists for $\hat{G}$, since the left multiplication with elements of C_G commutes with the adjoint action of $\hat{G}$ on $\hat{G}\tau$, and since the resulting action of $\hat{G}$ on $G\tau$ is just

the adjoint action of G on $G\tau$, we have the following commutative diagram:

$$\begin{tikzcd} \hat{G}\tau \arrow[r, "\hat{p}"] \arrow[d, "\hat{\pi}"'] & G\tau \arrow[d, "\pi"] \\ \mathbb{A}^s \cong \hat{G}\tau /\!\!/ \hat{G} \arrow[r, "\tilde{p}"] & G\tau /\!\!/ G, \end{tikzcd} \tag{2.53}$$

where $\hat{p} : \hat{G}\tau \to G\tau$ is the map induced by p on the exterior component, given by $g\tau \mapsto p(g)\tau$ and $\tilde{p}$ is induced by $\hat{p}$ on the quotient. Here we have set: $s := rk\, G'$. Furthermore, we assume that $char(k)$ does not divide the order of the centre of $\hat{G}$. Then we have the following identification:

$$C(\hat{G}) \cong \chi(C(\hat{G})) \cong \Lambda(R(\hat{G}))/\mathbb{Z}(R(\hat{G})), \tag{2.54}$$

where $\chi(C(\hat{G}))$ denotes the dual of $C(\hat{G})$. The explicit description of the quotient on the right side of the equation above can be found in [3], Planches, p. 250ff. We have:

Type $\hat{G}$	A_n	D_{2n}	D_{2n+1}	E_6
$C(\hat{G})$	$\mathbb{Z}/(n+1)\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$

With these preparations, we get the following explicit description of the quotient $G\tau /\!\!/ G$:

Proposition 2.9 *Under the additional assumption that $char(k)$ does not divide the order of $C(\hat{G})$, we have:*
(i) If G is any group of type A_n, E_6, or if G is $Spin_{2n}(k)$ or $SO_{2n}(k)$ with $\tau^2 = 1$, or if G is $Spin_8(k)$ or $PSO_8(k)$ with $\tau^3 = 1$, then $G\tau /\!\!/ G$ is an affine space of dimension $rk\, G^\tau$.
(ii) If G is of type $PSO_{4n}(k)$ and $\tau^2 = 1$, then we have:

$$G\tau /\!\!/ G \cong Z_n \times \mathbb{A}^{n-1}, \tag{2.55}$$

where $Z_n = \mathbb{A}^n/(\mathbb{Z}/2\mathbb{Z})$and the quotient is formed with respect to the diagonal action of $(\mathbb{Z}/2\mathbb{Z}) \cong \{\pm id_{\mathbb{A}^n}\}$ on $\mathbb{A}^n$.
(iii) If G is of type $PSO_{4n+2}(k)$ and $\tau^2 = 1$, then we have:

$$G\tau /\!\!/ G \cong Z_n \times \mathbb{A}^n, \tag{2.56}$$

with Z_n as in (ii).

Proof: Let us denote the fundamental dominant weights of $\Lambda(R(\hat{G}))$ by $\{\lambda_1, ..., \lambda_r\}$ and the fundamental dominant weights of $\Lambda(R(\hat{G}))^\tau = \Lambda(R(\hat{G})')$ by $\{\lambda'_1, ..., \lambda'_s\}$. From Corollary 2.6, we have the following explicit description of π:

$$\pi(g\tau) = (\tilde{X}_1^{\lambda'_1}(g\tau), ..., \tilde{X}_1^{\lambda'_s}(g\tau)). \tag{2.57}$$

By Schur's Lemma, we know that every element of $C(\hat{G})$ has to act on $V(\lambda)$ for every dominant $\lambda \in \Lambda(R(\hat{G}))$ by a multiple of the identity. It is a well known fact in the theory of linear algebraic groups, that $C(\hat{G}) \subset \hat{T}$, where $\hat{T}$ is any maximal torus of $\hat{G}$, e.g. see [13], Corollary 26.2B, p. 160. Hence for any $c \in C(\hat{G})$ and weight μ of $V(\lambda)$, i.e. $V(\lambda)_\mu \neq \{0\}$, we have

$$\lambda(c) = \mu(c). \tag{2.58}$$

But, since R is irreducible we have $\mathbb{Z}(\{\lambda - \mu, \mu \text{ weight of } V(\lambda)\}) = \mathbb{Z}(R)$, yielding $\lambda(c) = \bar{\lambda}(c)$, where $\bar{\lambda}$ is the class of λ in $\chi(C(\hat{G}))$. Therefore, we get the following action of $C(\hat{G})$ on $\hat{G}\tau /\!\!/ \hat{G} \cong \mathbb{A}^s$:

$$c.(a_1, ..., a_s) = (\bar{\lambda}'_1(c)a_1, ..., \bar{\lambda}'_s(c)a_s), \tag{2.59}$$

for all $c \in C(\hat{G})$ and $(a_1, ..., a_s) \in \mathbb{A}^s$. By use of the Diagram 2.53 we see that the quotient $G\tau /\!\!/ G$ is just the quotient of $\mathbb{A}^s$ by C_G.
Therefore, we only need to calculate the $\bar{\lambda}'_j(c)$ for all $j \in \{1, ..., s\}$ and $c \in C(\hat{G})$. Since we have $V(\lambda+\mu) \subset V(\lambda) \otimes V(\mu)$ as a subrepresentation for dominant λ, μ, we only need to calculate $\bar{\lambda}_i(c)$, $i \in \{1, ..., r\}$ and use the fact the λ'_j, $j \in \{1, ..., s\}$ are just sums over the τ-orbits of the λ_i, $i \in \{1, ..., r\}$.
By Corollary 2.6, we only need to investigate the cases D_n, $\tau^2 = 1$ and A_{2n+1}:
In both cases, we label the $\{\lambda_1, .., \lambda_r\}$ according to [3], Planches, p. 250ff.
1. G of type A_{2n+1}: By suitably labeling the $\{\lambda'_1, .., \lambda'_s\}$ (note that we have $r = 2n + 1$ and $s = n + 1$) we can achieve:

$$\begin{aligned} \lambda'_i &= \lambda_i + \lambda_{2n+2-i}, \quad i \neq n+1, & (2.60) \\ \lambda'_{n+1} &= \lambda_{n+1}. & (2.61) \end{aligned}$$

Calculating $\chi(C(\hat{G}))$, using the description of the λ_i, $i \in \{1, .., r\}$ in terms of the simple roots, as found in [3], Planches, p. 250, we see that $\bar{\lambda}_1$ is a generator of $\chi(C(\hat{G}))$ and that $\bar{\lambda}_i = i\,\bar{\lambda}_1$. Let ξ be the generator of $C(\hat{G})$ given by fixing $\bar{\lambda}_1(\xi)$ to be a given $(2n + 2)$-nd primitive root of unity, we calculate:

$$\begin{aligned} \bar{\lambda}'_i(\xi) &= 1, \quad i \neq n+1, & (2.62) \\ \bar{\lambda}'_{n+1}(\xi) &= -1. & (2.63) \end{aligned}$$

Therefore we get:
(i) $\tilde{p} : \hat{G}\tau /\!\!/ \hat{G} \to G\tau /\!\!/ G$ is an isomorphism, if C_G is generated by an even

power of ξ and
(ii) $\tilde{p} : \hat{G}\tau/\!\!/\hat{G} \to G\tau/\!\!/G$ is a quotient map by $\mathbb{Z}/2\mathbb{Z}$, if C_G is generated by an odd power of ξ. In the latter case $\mathbb{Z}/2\mathbb{Z}$ acts by multiplication in the $(n+1)$-st variable with ± 1. Since $\mathbb{A}^1/(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{A}^1$ (the identification is given by $[z] \mapsto z^2$, $[z]$ denoting an element of $\mathbb{A}^1/(\mathbb{Z}/2\mathbb{Z})$), we get the stated result.
2. G of type D_{2n+1}, $\tau^2 = 1$: By suitably labeling the $\{\lambda'_1, .., \lambda'_s\}$ (note that we have $r = 2n+1$ and $s = 2n$) we can achieve:

$$\lambda'_i = \lambda_i, \quad i \leq 2n-1, \tag{2.64}$$
$$\lambda'_{2n} = \lambda_{2n} + \lambda_{2n+1}. \tag{2.65}$$

We calculate $\chi(C(\hat{G}))$ using the description of the λ_i, $i \in \{1, .., r\}$ in terms of the simple roots, as found in [3], Planches, p. 256. We see that $\bar{\lambda}_{2n}$ is a generator of $\chi(C(\hat{G}))$, that $\bar{\lambda}_i = 2\bar{\lambda}_{2n}$, for $i < 2n$, odd, that $\bar{\lambda}_i = 0$, for $i < 2n$, even, and that $\bar{\lambda}_{2n+1} = 3\bar{\lambda}_{2n}$. Take a generator ξ of $C(\hat{G})$ such that $\bar{\lambda}_{2n}(\xi)$ is a given 4-th primitive root of unity. Then we have:

$$\bar{\lambda'}_i(\xi) = 1, \quad i, \text{ even}, \tag{2.66}$$
$$\bar{\lambda'}_i(\xi) = -1, \quad i, \text{ odd}. \tag{2.67}$$

Therefore we get:
(i) $\tilde{p} : \hat{G}\tau/\!\!/\hat{G} \to G\tau/\!\!/G$ is an isomorphism, if C_G is generated by ξ^2, which corresponds to the case $G = SO_{4n+2}(k)$ and
(ii) $\tilde{p} : \hat{G}\tau/\!\!/\hat{G} \to G\tau/\!\!/G$ is a quotient map by $\mathbb{Z}/2\mathbb{Z}$, if C_G is generated by ξ, which corresponds to the case $G = PSO_{4n+2}(k)$. In the latter case $\mathbb{Z}/2\mathbb{Z}$ acts by simultaneous multiplication with -1 in the coordinates having an odd label, which gives us the stated result.
3. G of type D_{2n}, $\tau^2 = 1$: By suitably labeling the $\{\lambda'_1, .., \lambda'_s\}$ (note that we have $r = 2n$ and $s = 2n-1$) we can achieve:

$$\lambda'_i = \lambda_i, \quad i \leq 2n-2, \tag{2.68}$$
$$\lambda'_{2n-1} = \lambda_{2n} + \lambda_{2n-1}. \tag{2.69}$$

Using the description of the λ_i, $i \in \{1, .., r\}$ in terms of the simple roots, as found in [3], Planches, p. 256, we see that we can choose $\bar{\lambda}_{2n}$ and $\bar{\lambda}_{2n-1}$ as generators of $\chi(C(\hat{G}))$. Furthermore we have $\bar{\lambda}_i = \bar{\lambda}_{2n} + \bar{\lambda}_{2n-1}$, for $i < 2n-1$, odd, and $\bar{\lambda}_i = 0$, for $i < 2n$, even. We choose generators ξ, η of $C(\hat{G})$ according to the rule $\bar{\lambda}_{2n}(\xi) = \bar{\lambda}_{2n-1}(\eta) = -1$ and $\bar{\lambda}_{2n-1}(\xi) = \bar{\lambda}_{2n}(\eta) = 1$. Then we see that the induced action of τ on $C(\hat{G})$ just interchanges ξ and η. Hence, the only τ-stable subgroup of $C(\hat{G})$ is $\{e, \xi\eta\}$ except for the trivial one and $C(\hat{G})$ itself. The group $G = \hat{G}/\{e, \xi\eta\}$ is $SO_{4n}(k)$. Now, we calculate:

$$\bar{\lambda'}_i(\xi) = \bar{\lambda'}_i(\eta) = 1, \quad i, \text{ even}, \tag{2.70}$$
$$\bar{\lambda'}_i(\xi) = \bar{\lambda'}_i(\eta) = -1, \quad i, \text{ odd}. \tag{2.71}$$

Therefore we get:
(i) $\tilde{p} : \hat{G}\tau /\!\!/ \hat{G} \to G\tau /\!\!/ G$ is an isomorphism, if C_G is generated by $\xi\eta$, which corresponds to the case $G = SO_{4n}(k)$ and
(ii) $\tilde{p} : \hat{G}\tau /\!\!/ \hat{G} \to G\tau /\!\!/ G$ is a quotient map by $\mathbb{Z}/2\mathbb{Z}$, if C_G is $C(\hat{G})$, which corresponds to the case $G = PSO_{4n}(k)$. In the latter case $\mathbb{Z}/2\mathbb{Z}$ acts by simultaneous multiplication with -1 in the coordinates having an odd label, which gives us the stated result. $\heartsuit$.

To conclude this chapter we want to illustrate our results by an example:
Example: We consider the case $G = SO_{2n}(k)$ and τ the exterior automorphism of $SO_{2n}(k)$, such that $\tilde{G} = O_{2n}(k)$.
As symmetric bilinear I form we choose the matrix $I = \left(\begin{smallmatrix} 0 & E_n \\ E_n & 0 \end{smallmatrix}\right)$. A particular maximal torus T with suitable parameterization can be given as follows:

$$T = \left\{ \begin{pmatrix} t_1 & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & t_n & & & \\ & & & & t_1^{-1} & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & & t_n^{-1} \end{pmatrix}, \quad t_i \in k^*, \quad i \in \{1, ..., n\} \right\}. \tag{2.72}$$

Furthermore, we denote by $\hat{T}$ the maximal torus of $Spin_{2n}(k)$, which is the inverse image of T under the two-to-one cover $Spin_{2n}(k) \to SO_{2n}(k)$. Let $\hat{T}$ be parameterized by $(\hat{t}_1, ..., \hat{t}_n)$, chosen in such a way that the one-parameter subgroup $\hat{t} \mapsto (1, .., 1, \hat{t}, 1, ..., 1)$, having nontrivial entries only in the i-th position, corresponds to $\check{\alpha}_i$, the dual root of the simple root α_i (for all $i \in \{1, ..., n\}$). Then, the covering map $\hat{T} \to T$ is given by the following equations:

$$t_1 = \hat{t}_1, \tag{2.73}$$
$$t_i = \hat{t}_{i-1}^{-1}\,\hat{t}_i, \quad 2 \leq i \leq n-2, \tag{2.74}$$
$$t_{n-1} = \hat{t}_{n-2}^{-1}\,\hat{t}_{n-1}\,\hat{t}_n \tag{2.75}$$
$$t_n = \hat{t}_{n-1}^{-1}\,\hat{t}_n. \tag{2.76}$$

We see that the kernel of this covering map is

$$\{(\hat{t}_1, ..., \hat{t}_n) \in \hat{T},\ \hat{t}_i = 1,\ i \in \{1, ..., n-2\},\ \hat{t}_{n-1} = \hat{t}_n = \pm 1\} \cong \mathbb{Z}/2\mathbb{Z} \tag{2.77}$$

The action of the exterior automorphism on $\hat{T}$ is given by interchanging the last two coordinates:

$$(\hat{t}_1, ..., \hat{t}_{n-2},\ \hat{t}_{n-1}\,\hat{t}_n) \mapsto (\hat{t}_1, ..., \hat{t}_{n-2},\ \hat{t}_n\,\hat{t}_{n-1}). \tag{2.78}$$

This induces the following τ action on T:

$$(t_1, ..., t_{n-1}\, t_n) \mapsto (t_1, ..., t_{n-1}\, t_n^{-1}). \tag{2.79}$$

We see that, in the $SO_{2n}(k)$ case, the action of τ can be implemented by conjugation with the following permutation matrix, which clearly is a matrix in $O_{2n}(k)\backslash SO_{2n}(k)$:

$$\tau = \left\{ \left(\begin{array}{cc|cc} E_{n-1} & & 0 & \\ & {\scriptstyle 0} & & {\scriptstyle 1} \\ \hline 0 & & E_{n-1} & \\ & {\scriptstyle 1} & & {\scriptstyle 0} \end{array} \right), \quad t_i \in k^*, \quad i \in \{1, ..., n\} \right\}. \tag{2.80}$$

Hence, we have $\tilde{G} = SO_{2n}(k) \rtimes \mathbb{Z}/2\mathbb{Z} \cong O_{2n}(k)$. Now it is an easy task to compute the τ fixed point tori:

$$\begin{aligned} \hat{T}^\tau = \hat{T}_0^\tau &= \{(\hat{t}_1, ..., \hat{t}_n) \in \hat{T},\, \hat{t}_{n-1} = \hat{t}_n\} && (2.81)\\ T^\tau &= \{(t_1, ..., t_n) \in T,\ t_n = \pm 1\} && (2.82)\\ T_0^\tau &= \{(t_1, ..., t_n) \in T,\ t_n = 1\}. && (2.83) \end{aligned}$$

Here we see that T^τ is not connected and that $\hat{T}^\tau$ is mapped to T_0^τ by the covering map $\hat{T} \to T$.
Furthermore, a simple calculation yields $(T/T_0^\tau)^\tau = T^\tau/T_0^\tau \cong \mathbb{Z}/2\mathbb{Z}$. This shows that the kernel of the map $(T/T_0^\tau)^\tau \to H$, as defined in the paragraph just preceding Corollary 2.4, is nontrivial. Using Theorem 2.1, we get:

$$G\tau /\!\!/ G \cong T_0^\tau \tau/\widetilde{\mathcal{W}} = T_0^\tau \tau/\mathcal{W}^\tau. \tag{2.84}$$

Now we could set up coordinates and directly calculate, that this quotient is an affine space to reproduce the result of Proposition 2.9. but we have a more convenient argument:
Let us denote the the tori $\hat{T}_0^\tau/(\hat{T}/\hat{T}_0^\tau)^\tau$ and $T_0^\tau = T_0^\tau/(T/T_0^\tau)^\tau$ by $\hat{T}'$ respectively T', following the notation of Section 2.3. Then, as in the proof of Corollary 2.4, we have a commutative diagram with exact rows:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & \hat{T}_0^\tau \arrow[r] \arrow[d, "p"] & \hat{T}' \arrow[r] \arrow[d, "p'"] & 1 \\ 1 \arrow[r] & 1 \arrow[r] & T_0^\tau \arrow[r] & T' \arrow[r] & 1. \end{tikzcd}$$

Furthermore, we have that the covering $\hat{T}_0^\tau \to T_0^\tau$ is two-to-one. Then, using the Snake Lemma we have an exact sequence:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow ker\, p' \longrightarrow 1. \tag{2.85}$$

This forces $ker\, p' = 1$. So we have:

$$G\tau /\!\!/ G \cong T_0^\tau \tau / \mathcal{W}^\tau = T' / \mathcal{W}^\tau = \hat{T}' / \mathcal{W}^\tau. \tag{2.86}$$

But by Corollary 2.6 we know that the right hand side is an affine space. (It would even have been easier to see directly $\chi(T)^\tau = \Lambda(R')$ which again gives the desired result using Corollary 2.6.)

Chapter 3

The Structure of the Fibres

In this chapter we want to describe the structure of the fibres of our quotient map $\pi : G\tau \to T_0^\tau/\widetilde{\mathcal{W}}$. We will show that they are isomorphic to associated fibre bundles over a homogeneous spaces of G modulo a centralizer of semisimple element with a fibre isomorphic to the unipotent variety of that centralizer. In the second part of this chapter, we investigate the centralizers of semisimple elements and, in the case of $char(k) = 0$, we will give a complete classification of them.

3.1 Description of the Fibres

Before entering the detailed description of the structure of the fibres, we need some preparations:
First, note that for each closed subgroup $H < G$ of an algebraic group G the quotient G/H together with the corresponding map $\pi : G \to G/H$ is a principal fibre bundle with fibre H, locally trivial in the etale topology, cf. [24], Section 2.5, Proposition 3, p. 1-12.

Definition 3.1 *Let G be an algebraic group, $H < G$ a closed subgroup and F be an affine reduced G variety. Then we call the quotient $(G \times F)/H = Spec\, k[G \times F]^H$ with respect to the action $h.(g, f) \mapsto (gh^{-1}, hf)$, for all $h \in H$, $g \in G$, $f \in F$, the associated fibre bundle over G/H with fibre F, and we denote it by $G \times^H F$.*
Elements of $G \times^H F$ will be denoted by $g * f$.

Remark: (i) Note that the quotient actually is a geometric quotient, since the H cosets in G are already closed.
(ii) By [24] the above bundles are locally trivial in the etale topology.

For standard results in the theory of associated bundles we refer to [24] and [27], Section 3.7.

Now we resume the notation of the previous chapter, i.e. G is semisimple and τ an exterior automorphism of the Dynkin diagram of G. Let $\pi : G\tau \to T_0^\tau \tau/\widetilde{\mathcal{W}}$ be the quotient map.

Theorem 3.1 *Let $t\tau$ be an element in $T_0^\tau \tau$ and $V(t\tau)$ the unipotent variety of $C_G(t\tau)$. Then the reduced fibre $\pi^{-1}(\pi(t\tau))_{red}$ of its image $\pi(t\tau)$ in $T_0^\tau/\widetilde{\mathcal{W}}$ is G-isomorphic to $G \times^{C_G(t\tau)} V(t\tau)$.*

Proof: First, note that for $x \in \pi^{-1}(\pi(t\tau))$ the element x_s is conjugate to $t\tau$: Let $x \in \pi^{-1}(\pi(t\tau))$, then $x_s \in \pi^{-1}(\pi(t\tau))$ by Lemma 2.10. By Proposition 2.3 and invariance of $\pi^{-1}(\pi(t\tau))$ under conjugation with elements of G we may assume, that $x_s \in T_0^\tau \tau$. Using Theorem 2.1, we see that x_s and $t\tau$ are conjugate under $\widetilde{\mathcal{W}}$.
Now we identify the conjugacy class $\mathcal{O}(t\tau)$ of $t\tau$ with $G/C_G(t\tau)$ by $gC_G(t\tau) \mapsto g\,t\tau\,g^{-1}$. (Cf. [2], Proposition 9.1, p. 128.) We construct a G-equivariant map

$$\begin{aligned} \psi : \pi^{-1}(\pi(t\tau))_{red} &\to G/C_G(t\tau) \cong \mathcal{O}(t\tau) \\ x &\mapsto x_s. \end{aligned} \tag{3.1}$$

Since there exists a polynomial $P \in k[t]$, such that $x_s = P(x)$ for all $x \in M_n(k)$, which only depends on the eigenvalues of x and its multiplicities, see [14], Proposition 4.2, p. 17, we have, using a faithful representation of $\tilde{G}$, that ψ is a morphism. (The coefficients of P are constant on $\pi^{-1}(\pi(t\tau))$.)
Now, Lemma 4 in [27], Section 3.7, p. 26, provides us with a G-isomorphism

$$\begin{aligned} \Phi : G \times^{C_G(t\tau)} \psi^{-1}(t\tau) &\to \pi^{-1}(\pi(t\tau)) \\ (g * u) &\mapsto g\,u\,g^{-1}. \end{aligned} \tag{3.2}$$

We have $\psi^{-1}(t\tau) = \{g \in G\tau, g_s = t\tau,\ g_u \in V(t\tau)\} = t\tau V(t\tau)$. So, identifying $V(t\tau)$ with $t\tau V(t\tau)$ just by right translation with $t\tau$, which is apparently $C_G(t\tau)$ equivariant, we get the stated result. ♡.
Remark: In Chapter 5, we will see that the schematic fibres of π are already reduced in the situation, where the group G', appearing in the paragraph preceding Proposition 2.8, is simply connected.

Next we want to draw some conclusions of the theorem above.
First observe that $C_G(t\tau)_0$ is reductive for $t\tau \in T_0^\tau \tau$ and that the quotient group $C_G(t\tau)/C_G(t\tau)_0$ is finite and consists only of semisimple elements by [37], Corollary 9.4, p. 60. Therefore, the unipotent variety $V(t\tau)$ of $C_G(t\tau)$ is already contained in $C_G(t\tau)_0$. Furthermore, by the proof of the theorem above, we see that two elements x, y in $\pi^{-1}(\pi(t\tau))$ sharing the same semisimple component are conjugate under G exactly when their unipotent parts x_u, y_u are under $C_G(x_s)$, a group, which itself is G-conjugate to $C_G(t\tau)$. Hence the orbit structure of $\pi^{-1}(\pi(t\tau))$ is already determined by the description of the $C_G(t\tau)$-conjugacy classes in $V(t\tau)$. Their properties

are given e.g. in [27], Section 3.10, Theorem, p. 31 or [38], Section 3.8, Theorem 1, p. 116.
We summarize them in a corollary:

Corollary 3.1 *Let G, τ be as above and $\pi^{-1}(\pi(t\tau))$ a fibre of the quotient map $\pi : G\tau \to T_0^\tau\tau/\widetilde{\mathcal{W}}$. Then we have:*
(i) The fibres $\pi^{-1}(\pi(t\tau))$ of π consist of finitely many conjugacy classes.
(ii) The fibres $\pi^{-1}(\pi(t\tau))$ are irreducible varieties of codimension equal to $dim\, T_0^\tau$.
(iii) Each fibre $\pi^{-1}(\pi(t\tau))$ contains exactly one conjugacy class consisting of semisimple elements, the so called semisimple orbit, which is the only closed orbit in $\pi^{-1}(\pi(t\tau))$ and which is contained in the closure of every other orbit of $\pi^{-1}(\pi(t\tau))$.
(iv) Each fibre $\pi^{-1}(\pi(t\tau))$ contains one open dense orbit corresponding to the regular unipotent class in $V(t\tau)$ having a complement of codimension two in $\pi^{-1}(\pi(t\tau))$.

Proof: (i) and the irreducibility of (ii) is clear by the reasoning and the references preceding the statement of the corollary. The uniqueness of the semisimple class of (iii) was already proved in the proof of the theorem above.
Because of (i) and the irreducibility of (ii) we must have a unique open dense orbit in $\pi^{-1}(\pi(t\tau))$. Denote the orbit of an element $x \in G\tau$ by $\mathcal{O}(x)$. Then we have $dim\,\mathcal{O}(x) = dim\, G/C_G(x)$ (which is also true in $char(k) = p > 0$, even though the two varieties need not be isomorphic) and $C_G(x) = C_{C_G(t\tau)}(x_u)$ where we may assume $x_s = t\tau$.
Now, $C_{C_G(t\tau)}(x_u)$ is minimal in dimension if x_u is regular unipotent. This, combined with the fact that the complement of the regular unipotent class in $C_G(t\tau)$ has codimension two in $V(t\tau)$, yields (iv).
Next observe that T_0^τ is a maximal torus in $C_G(t\tau)$ by Lemma 2.3. Hence the reductive rank $rk\, C_G(t\tau)$ of $C_G(t\tau)$ equals the dimension of T_0^τ, giving $dim\, C_G(t\tau u) = dim\, T_0^\tau$ for u regular unipotent in $C_G(t\tau)$, and thereby proving (ii).
We are left to prove that every orbit contains the semisimple one in its closure, which would prove (iii) because, due to the finiteness statement (i), there has to exist a closed orbit in $\pi^{-1}(\pi(t\tau))$.
Take $x \in \pi^{-1}(\pi(t\tau))$, w.o.l.g. $x_s = t\tau$. Then we can find a Borel subgroup $B(t\tau)$ of $C_G(t\tau)_0$ containing x_u and a maximal torus $S < B(t\tau)$. Proceeding as in the proof of Lemma 2.10, we can find a one-parameter multiplicative subgroup $\lambda : k^* \to S$, with $e \in \overline{\{\lambda(t)\, u\, \lambda(t)^{-1},\ t \in k^*\}}$, proving (iii). ♡.

Another conclusion is the following:

Corollary 3.2 *In the case, where $G\tau /\!\!/ G \cong \mathbb{A}^{dim\, T^\tau}$, an affine space, the quotient map $\pi : G\tau \to G\tau /\!\!/ G$ is flat.*

Proof: Since $G\tau$ and $G\tau /\!/ G$ are smooth, all their local rings are regular and therefore Cohen-Macaulay by [11], Chapter II, Theorem 8.21A, p. 184. By Corollary 3.1, all fibres of π have the same dimension. The statement follows now from [11], Chapter III, Exercise 10.9, p. 276. $\heartsuit$.

Analogously to the classical situation, we now introduce the notion of regular elements:

Definition 3.2 *Let x be an element of $G\tau$, then we call x regular, iff $\dim C_G(x) = rk\, G^\tau$ (which, by the corollary above, is the possible minimal dimension!)*

Remark: Since, by definition, the Cartan subgroup $C := < T_0^\tau, \tau >$ of $\tilde{G}$, has a generating element (see Section 2.1) and because of $C_G(< T_0^\tau, \tau >) = T^\tau$, we see that regular semisimple elements exist, the generating elements are.

To get a more thorough knowledge of the structure of the fibres we need to investigate the types of the centralizers of semisimple elements of $G\tau$, which will be done in the following section.
Before closing this section we give a description of the singularity structure of the reduced fibres:

Lemma 3.1 *Let G, τ, $\pi : G\tau \to T_0^\tau \tau / \widetilde{\mathcal{W}}$ be as above. Then we have the following description of the singularities of the reduced fibre $\pi^{-1}(\pi(t\tau))_{red}$:*

$$\left(\pi^{-1}(\pi(t\tau))_{red}\right)_{sing} \cong G \times^{C_G(t\tau)} \left(V(t\tau)_{sing}\right). \tag{3.3}$$

If furthermore $char(k) = 0$ or $char(k) > rk\, G^\tau + 1$, the nonsingular points of $\pi^{-1}(\pi(t\tau))_{red}$ are exactly the regular ones.

Proof: The second statement follows from the first one under the given restriction of the characteristic, by use of [27], Section 3.10, (vi) of the Theorem, p. 31, applied to the fibre over the unit element of the semisimple part of $C_G(t\tau)$.
To prove the first part of the statement, first recall that the bundle $G \times^{C_G(t\tau)} V(t\tau)$, as a bundle over $G/C_G(t\tau)$, is locally trivial in the etale topology. Hence for every $x \in G/C_G(t\tau)$ we find an open subset $U \subset C_G(t\tau)$ containing x and an etale morphism $p : \tilde{U} \to U$, such that

$$p^*(G \times^{C_G(t\tau)} V(t\tau)) \cong \tilde{U} \times V(t\tau). \tag{3.4}$$

Since p is etale and $G/C_G(t\tau)$ is smooth, we have that $\tilde{U}$ is also smooth. (Note that 'p etale' means that the completions of the local rings of points in $\tilde{U}$ and its image point under p in U are isomorphic, cf. e.g. [11], Chapter III, Exercise 10.4, p. 275; it follows from the Cohen Structure Theorem, found in [11], Chapter I, Theorem 5.4A, p. 34, that also their Zariski tangent spaces are isomorphic.) Hence we have:

$$\left(\tilde{U} \times V(t\tau)\right)_{sing} = \tilde{U} \times (V(t\tau)_{sing}). \tag{3.5}$$

If we denote the bundle projection $G \times^{C_G(t\tau)} V(t\tau) \to G/C_G(t\tau)$ by ν, we obtain, by the behaviour of etale morphisms under base change, cf. [11], Chapter III, Proposition 10.1, p. 268, that the induced map $p' : p^*(G \times^{C_G(t\tau)} V(t\tau)) \to \nu^{-1}(U)$ is also etale (note that here $p^*(G \times^{C_G(t\tau)} V(t\tau) \cong \tilde{U} \times_U \nu^{-1}(U))$. Now we use again the isomorphism of Zariski tangent spaces under etale morphisms to get the result. ♡.

Remark: In Chapter 5 we will obtain a sharper version of the second statement which is less restrictive on the characteristic.

3.2 Centralizers of Semisimple Elements

In this section we give a description of the root systems of centralizers of semisimple elements of the exterior component $G\tau$ in terms of certain subsystems of the folded root system ${}^\tau R^1$ of $R = R(G;T)$. In characteristic zero, we can give a complete classification of these as proper subsystems of the twisted affine root system corresponding to R and τ.

We keep the notations of Chapter 1 about the folded root system ${}^\tau R^1$ and the root system R'. First, we define a map d from the folded root system ${}^\tau R^1$ to R' by the following rules:

(i) If R is irreducible and not of type A_{2n} we set

$$d(\bar{\alpha}) = \bar{\alpha} \quad \text{for } \bar{\alpha} \text{ long in } {}^\tau R^1 \tag{3.6}$$

$$d(\bar{\alpha}) = (ord\,\tau)\,\bar{\alpha} \quad \text{for } \bar{\alpha} \text{ short in } {}^\tau R^1. \tag{3.7}$$

(ii) If R is of type A_{2n} we define

$$d(\bar{\alpha}) = 2\,\bar{\alpha} \quad \text{for } \bar{\alpha} \text{ long or intermediate in } {}^\tau R^1 \tag{3.8}$$

$$d(\bar{\alpha}) = 4\,\bar{\alpha} \quad \text{for } \bar{\alpha} \text{ short in } {}^\tau R^1. \tag{3.9}$$

(iii) If R is reducible and τ a permutation of irreducible parts of the same type, then we let Γ be the subgroup of $Aut(R)$ generated by τ and we define:

$$d(\bar{\alpha}) = \frac{(ord\,\tau)}{|\Gamma_\alpha|}\,\bar{\alpha}, \tag{3.10}$$

where $\alpha \in R$ is an element of $p^{-1}(\bar{\alpha})$. Here p is the projection map $p : R \to {}^\tau R^1$ defined in Equation 1.2.

Remarks: (i) The composition $d \circ p$ is the map $\alpha \mapsto \alpha'$ of Section 1.2, where $p : R \to {}^\tau R^1$ is the projection defined in Equation 1.2.

(ii) By the description of ${}^{\tau}R^1$ and R' in Section 1.2, we see that, in case (i), d is just the map dualizing the folded root system. In case (ii) the restriction of d to the root subsystem of type B_n in ${}^{\tau}R^1 = BC_n$, consisting of roots of intermediate and short length, dualizes this root subsystem of ${}^{\tau}R^1$. (Recall that this root subsystem is the root system of the group $C_G(\tau)_0$ by Proposition 1.1.)
(iii) Note, that in both cases the restriction of d to a reduced root subsystem of ${}^{\tau}R^1$ is injective, even though d in case (ii) is not. This follows from the fact that for each short root $\bar{\alpha}$ of ${}^{\tau}R^1$ we have $d(\bar{\alpha}) = d(2\,\bar{\alpha})$.
Furthermore, the restriction of d dualizes each root subsystem of ${}^{\tau}R^1$ in case (i). In case (ii), d dualizes each connected component of a root subsystem of ${}^{\tau}R^1$ containing a short root and acts as an isomorphism (multiplication by two) on each component of a root subsystem of ${}^{\tau}R^1$ containing no short root.

With these preparations we can state a first result:

Proposition 3.1 *Let G be simple and let $t\tau$ be a semisimple element of $G\tau$, w.l.o.g. $t\tau \in T_0^{\tau}\tau$, and $C_G(t\tau)_0$ its connected centralizer in G. Then we have:*
(i) $C_G(t\tau)_0$ is reductive and T_0^{τ} is a maximal torus of $C_G(t\tau)_0$. Furthermore, its root system $R(C_G(t\tau)_0)$ is a reduced root subsystem of the folded root system ${}^{\tau}R^1$.
(ii) $d(R(C_G(t\tau)_0))$ is $\mathbb{Z}$-closed in R'.

Proof: The reductiveness claim of (i) was already proved in [37], Corollary 9.4, p. 60. T_0^{τ} is a maximal torus of $C_G(t\tau)_0$ by Lemma 2.3.
The proof of the second part of (i) will be a reworking of parts of the proof of Theorem 8.2 in [37], which is essentially the background for Corollary 9.4 in [37].
Let B be a Borel subgroup of G and $T < B$ a maximal torus, such that T and B are stabilized by τ. Furthermore denote the unipotent radical of B by U. We want to compute the fixed point group $C_U(t\tau)$. To do this we need some preparations.
Let X_{α_i}, $i \in \{1, ..., n\}$, be the root groups as defined in the paragraph preceding Theorem 1.2.
We begin with a lemma on the commutator of root groups, proven e.g. in [2], §14.5, p. 184ff:

Lemma 3.2 *With the notation above we have:*

$$(X_\alpha(t), X_\beta(s)) = \prod_{\gamma} X_\gamma(P_\gamma(t, s)), \tag{3.11}$$

where γ runs over all elements of $R \cap (\mathbb{Z}^{>0}(\alpha,\, \beta))$ and where $P_\gamma(t, s) = C_\gamma t^a\, s^b$ is a monomial for $\gamma = a\,\alpha + b\,\beta$ and for a suitable element $C_\gamma \in k$ independent of s, t.

Next, recall that $n := dim\, U = |R^+|$ and that, by [13], Proposition 28.1, p. 170, there exists an isomorphism

$$\begin{aligned} \Phi : \mathbb{A}^n &\xrightarrow{\sim} U \\ (c_1, ..., c_n) &\mapsto \prod_{i=1}^{n} X_{\alpha_i}(c_i). \end{aligned} \tag{3.12}$$

Here we have $\{\alpha_1, ..., \alpha_n\} = R^+$, whose order is fixed. For us, it is appropriate to choose a specific order:
For different α', $\beta' \in R'^+ = d(p(R^+))$ we want that, between the root groups of different representatives α, $\hat{\alpha} \in p^{-1}(d^{-1}(\alpha'))$ in the above product, there appears no root group corresponding to a representative of $p^{-1}(d^{-1}(\beta'))$. Among the root groups of the representatives of $p^{-1}(d^{-1}(\alpha'))$ of $\alpha' \in R'^+ = d(p(R^+))$ we choose an order according to the following rule:
Choose a representative $\alpha \in R^+$ of $p^{-1}(d^{-1}(\alpha'))$ for $\alpha' \in R'^+$. By Lemma 1.2, we have to distinguish three cases:
(i) We have $\alpha = \tau(\alpha)$ and R is not of type A_{2n}. Then $p^{-1}(d^{-1}(\alpha')) = \{\alpha\}$ and $\alpha = \alpha'$. In this case we need no further ordering. We set

$$S_{\alpha'} = X_\alpha. \tag{3.13}$$

Furthermore, denote by R_0^+ the set of the representatives α chosen in this manner.
(ii) We have $\alpha \neq \tau(\alpha)$ and $\check{\alpha}(\tau(\alpha)) = 0$. Then $p^{-1}(d^{-1}(\alpha')) = \{\tau^i(\alpha),\, i \in \{1, ..., ord\, \tau\}\}$ and $\alpha' = \sum_{i=1}^{ord\, \tau} \tau^i(\alpha)$. Then we set

$$S_{\alpha'} = \prod_{i=1}^{ord\, \tau} X_{\tau^i(\alpha)}. \tag{3.14}$$

By the lemma above and Lemma 1.2, the different factors in the product commute.
Furthermore, denote by R_1^+ the set of the representatives α chosen in this way.
(iii) In the remaining case, choose $\alpha \in p^{-1}(d^{-1}(\alpha'))$ such that $\alpha \neq \tau(\alpha)$ and $\check{\alpha}(\tau(\alpha) \neq 0$.Then $p^{-1}(d^{-1}(\alpha')) = \{\alpha,\, \tau(\alpha),\, \alpha + \tau(\alpha)\}$ and $\alpha' = 2(\alpha + \tau(\alpha))$. Then we put:

$$S_{\alpha'} = X_\alpha\, X_{\tau(\alpha)}\, X_{\alpha+\tau(\alpha)}. \tag{3.15}$$

In this case, the rightmost factor commutes with each of the other two. By [36], Lemma 15, p. 22, we have $C_{\alpha+\tau(\alpha)} \in k^*$ in the commutator lemma above.
Furthermore, denote by R_2^+ the set of the representatives α chosen in this manner.
We impose no special order on the different $S_{\alpha'}$.
In addition, the set $R_0^+ \cup R_1^+ \cup R_2^+$ contains exactly one representative in

$p^{-1}(d^{-1}(\alpha'))$, for every $\alpha' \in R'^{+}$.
Clearly, $t\tau$ leaves the images of each of the parameterizations $S_{\alpha'} : \mathbb{A}^{|p^{-1}(d^{-1}(\alpha'))|} \to U$ invariant. Hence we have:

$$C_U(t\tau) = \prod_{\alpha' \in R'^{+}} (Im\, S_{\alpha'})^{t\tau}, \tag{3.16}$$

$(Im\, S_{\alpha'})^{t\tau}$ denoting the fixed point set of the $t\tau$ action on $Im\, S_{\alpha'} \subset U$. This will be calculated now: Again we have to distinguish three cases as above:
(i): $S_{\alpha'}(c)$ is a fixed point, iff $\alpha(t) = 1$ or $c = 0$.
(ii): In this case we calculate, using Theorem 1.2 and the commutativity of the root groups:

$$t\tau\, S_{\alpha'}(c_1, ..., c_{ord\,\tau})\,(t\tau)^{-1} = S_{\alpha'}(\alpha(t)\, c_{ord\,\tau}, ..., \alpha(t)\, c_{ord\,\tau - 1}), \tag{3.17}$$

if R is not of type A_{2n} and

$$t\tau\, S_{\alpha'}(c_1,\, c_2)\,(t\tau)^{-1} = S_{\alpha'}((-1)^{ht\,\alpha+1}\alpha(t)\, c_2,\, (-1)^{ht\,\alpha+1}\alpha(t)\, c_1), \tag{3.18}$$

if R is of type A_{2n}.
Here, we get a nontrivial fixed point iff $\alpha(t)$ is an $ord\,\tau$-th root of unity. In this case, we even get a one parameter family of fixed points

$$(Im\, S_{\alpha'})^{t\tau} = \{S_{\alpha'}(c, ..., \alpha^{ord\,\tau - 1}(t)c), \quad c \in k\}, \tag{3.19}$$

if R is not of type A_{2n}, and

$$(Im\, S_{\alpha'})^{t\tau} = \{S_{\alpha'}(c,\, (-1)^{ht\,\alpha+1}\alpha(t)c), \quad c \in k\}, \tag{3.20}$$

if R is of type A_{2n}.
(iii): In this case, we use again Theorem 1.2 and the above lemma on the commutator of root groups to get:

$$\begin{gathered} t\tau\, S_{\alpha'}(c_1,\, c_2,\, c_3)\,(t\tau)^{-1} = \\ S_{\alpha'}((-1)^{ht\,\alpha+1}\alpha(t)c_2,\, (-1)^{ht\,\alpha+1}\alpha(t)c_1,\, C_{\alpha+\tau(\alpha)}\, \alpha^2(t)\, c_1\, c_2 - \alpha^2(t)\, c_3). \end{gathered} \tag{3.21}$$

In this case, we get a nontrivial fixed point (again even a one-parameter family) in either of the two subcases:
(a) $\alpha^2(t) = 1$. Then the fixed point set is as follows:

$$(Im\, S_{\alpha'})^{t\tau} = \left\{ S_{\alpha'}(c,\, (-1)^{ht\,\alpha+1}\alpha(t)\, c,\, \frac{C_{\alpha+\tau(\alpha)}}{2}\, (-1)^{ht\,\alpha+1}\alpha(t)\, c^2), \quad c \in k \right\}. \tag{3.22}$$

But we may also have:
(b) $\alpha^2(t) = -1$. Then we get:

$$(Im\, S_{\alpha'})^{t\tau} = \{S_{\alpha'}(0,\, 0,\, c), \quad c \in k\}. \tag{3.23}$$

We summarize the above calculations:
(i) If R is not of type A_{2n}, we have:

$$C_U(t\tau) = \left\{ \prod_{\substack{\alpha\in R_0^+\\ \alpha(t)=1}} X_\alpha(c_\alpha) \prod_{\substack{\alpha\in R_1^+\\ \alpha^{ord\,\tau}(t)=1}} \prod_{i=1}^{ord\,\tau} X_{\tau^i(\alpha)}(\alpha^i(t)\,c_\alpha), \quad c_\alpha \in k \right\}. \tag{3.24}$$

(ii) If R is of type A_{2n} we have:

$$\begin{aligned} C_U(t\tau) = \Big\{ \prod_{\substack{\alpha\in R^+\\ \alpha=\tau(\alpha)\\ \alpha(t)=-1}} X_\alpha(c_\alpha) & \prod_{\substack{\alpha\in R_1^+\\ \alpha^2(t)=1}} X_\alpha(c_\alpha)\,X_{\tau(\alpha)}((-1)^{ht\,\alpha+1}\alpha(t)\,c_\alpha) \times \\ & \prod_{\substack{\alpha\in R_2^+\\ \alpha^2(t)=1}} \Big[X_\alpha(c_\alpha)\,X_{\tau(\alpha)}((-1)^{ht\,\alpha+1}\alpha(t)\,c_\alpha) \times \\ & X_{\alpha+\tau(\alpha)}\Big(\frac{C_{\alpha+\tau(\alpha)}}{2}\,(-1)^{ht\,\alpha+1}\alpha(t)\,c_\alpha^2\Big)\Big], \quad c_\alpha \in k\Big\}. \end{aligned}$$

In either case, $C_U(t\tau)$ is connected. By steps (1),(2),(4) and (7) of the proof of Theorem 8.2 of [37] (mind Lemma 9.2 in [37] if G is not simply connected), the knowledge of $C_U(t\tau)$ allows us to determine the root system $R(C_G(t\tau))$.
To summarize the calculations above:
We have obtained the positive roots of $R(C_G(t\tau))$ as a subset of ${}^\tau R^{1\,+} = p(R^+)$. (Note that in the above description we can replace $\alpha \in \bigcup_{j\in\{0,1,2\}} R_j^+$ by $p(\alpha) \in {}^\tau R^{1\,+}$, because T_0^τ is a maximal torus of $C_G(t\tau)$.)
Therefore we get (by using the description of the root lengths as given in Lemma 1.2):
If R is not of type A_{2n}, we have:

$$R(C_G(t\tau)) = \{\bar\alpha \in {}^\tau R^1,\ \bar\alpha(t) = 1,\ \text{if } \bar\alpha \text{ is long or } \bar\alpha^{ord\,\tau}(t) = 1 \text{ if } \bar\alpha \text{ is short}\}. \tag{3.25}$$

If R is of type A_{2n}, we have:

$$\begin{aligned} R(C_G(t\tau)) \;&=\; \{\bar\alpha \in {}^\tau R^1,\ \bar\alpha(t) = -1, \\ &\qquad \text{if } \bar\alpha \text{ is long, or } \bar\alpha^2(t) = 1 \text{ if } \bar\alpha \text{ is intermediate or short}\}. \end{aligned}$$

This proves (i).
To prove (ii) we apply just the map d to the root system $R(C_G(t\tau))$. Using the definition of d, we get in both cases:

$$d(R(C_G(t\tau))) = \{\alpha' \in R'\ \alpha'(t) = 1\}, \tag{3.26}$$

which is clearly $\mathbb{Z}$-closed in R'. ♡.
Remarks: (i) If G is semisimple and τ a permutation of isomorphic normal

simple subgroups of G, then the proposition remains valid, though we need some refinements in the proof:
We can proceed similarly as in the case where G is not of type A_{2n}. For $\alpha \in R(G)$ we have to substitute all sums and products over $ord\,\tau$ by the cardinality of the Γ orbit of α, where Γ is the subgroup of $Aut(R)$ generated by τ. All elements of the form $\tau^j(\alpha)$ have to be replaced by the elements of the Γ-orbit of α. Again, by the Remark following Lemma 1.2, the root groups X_α and X_β for different α, β with $\alpha' = \beta'$ commute. So we can use the same reasoning as in the case of the proof above where G is not of type A_{2n}.
Finally, we reach the following description of $R(C_G(t\tau)$:

$$R(C_G(t\tau)) = \{\bar{\alpha} \in {}^{\tau}R^1 \ \bar{\alpha}^{k(\bar{\alpha})}(t) = 1\}, \tag{3.27}$$

where $k(\bar{\alpha}) := \frac{ord\,\tau}{|\Gamma_\alpha|}$ is just the the cardinality of the Γ-orbit of α with $p(\alpha) = \bar{\alpha}$, where p is defined as in Equation 1.2. Again we get

$$d(R(C_G(t\tau))) = \{\alpha' \in R' \ \alpha'(t) = 1\}. \tag{3.28}$$

(ii) One can easily verify that the parameterizations of the sets $(Im\, S_{\alpha'})^{t\tau}$, as given in the proof above, are additive group homomorphisms in the parameter c, if these sets contain at least two elements. Therefore, these sets are the (images of the) root groups of $C_G(t\tau)$ and hence the corresponding parameterizations are the root groups.
(iii) Since by [2], Section 9.1, Proposition, p. 128, we have that $Lie(C_G(t\tau)) = \mathfrak{c}_{\mathfrak{g}}(t\tau)$, $t\tau$ being semisimple, where $\mathfrak{c}_{\mathfrak{g}}(t\tau)$ denotes the kernel of the adjoint action of $t\tau$ on $\mathfrak{g} = Lie\, G$, we would have been able to derive the same result using the Lie algebra and its Cartan decomposition. This should have been a bit simpler, for example we would not have to care about commutators, but in the next chapter we will make direct use of the description of the root groups of $C_G(t\tau)$ given here.

The next statement reduces our situation to that of automorphisms of G of finite order:

Theorem 3.2 *(i) Let $t\tau$ be in $T_0^\tau\tau$ as above, then there exists an automorphism σ of G of finite order, such that $R(C_G(t\tau))$ is $\mathbb{Q}$-closed in $R(C_G(\sigma))$.*
(ii) If G is simple and $char(k) = 0$, the Dynkin diagram of $R(C_G(t\tau))$ is a proper subdiagram of the Dynkin diagram of the twisted affine root system $R^{(ord\,\tau)}$, where R is the root system of G.
The Dynkin diagrams of the twisted affine root systems $R^{(ord\,\tau)}$ are given in the following table:

$\mathrm{A}_2^{(2)}$:

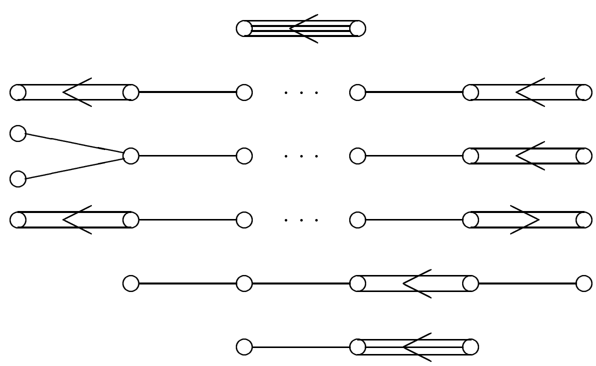
$\mathrm{A}_{2n}^{(2)} \quad (n \geq 2)$:

$\mathrm{A}_{2n-1}^{(2)} \quad (n \geq 3)$:

$\mathrm{D}_{n+1}^{(2)} \quad (n \geq 2)$:

$\mathrm{E}_6^{(2)}$:

$\mathrm{D}_4^{(3)}$:



Proof: (i) We consider the diagonalizable group $H := \overline{< (t\tau)^l, l \in \mathbb{Z} >}$. By Lemma 2.2 we have $H \cong S \times \mathbb{Z}/l\mathbb{Z}$, for a suitable $l \in \mathbb{Z}$ and a torus S. Now we can find an element σ of $\tilde{G}$ which is a generator of the $\mathbb{Z}/l\mathbb{Z}$ part of H. Since $t\tau S$ is a generator of $\mathbb{Z}/l\mathbb{Z} \cong H/S$ we can assume, w.o.l.g., that $\sigma \in G\tau$, yielding that $ord\,\tau$ is the smallest natural number n, such that $\sigma^n \in G$.
Now we have the the following chain of equations:

$$C_G(t\tau) = C_G(H) = C_G(S) \cap C_G(\sigma) = C_{C_G(\sigma)}(S). \tag{3.29}$$

Since σ is semisimple $C_G(\sigma)_0$ is reductive by Proposition 3.1 and $C_G(t\tau)_0 = C_{C_G(\sigma)}(S)_0$ is a Levi subgroup thereof (for Levi subgroups cf. e.g. [13], Section 30).
To prove (i) we are left to prove the following statement:
Claim: Let G be reductive with root system R, $S < G$ a torus and $C_G(S)$ its centralizer (which is reductive by [13], Corollary 26.2A, p. 159). Then the root system $R(C_G(S))$ is $\mathbb{Q}$-closed in R.
Proof of claim: Taking a maximal torus T of G containing S we have

$$R(C_G(S)) = \{\alpha \in R, \alpha(s) = 1, \ \forall\, s \in S\}. \tag{3.30}$$

Take $\beta \in R \cap \mathbb{Q}(R(C_G(S)))$. Then we can find $n \in \mathbb{N}$, such that $n\beta \in \mathbb{Z}(R(C_G(S)))$, forcing $(\beta(s))^n = 1$, for all $s \in S$. Hence $\beta(s)$ is an n-th root of unity.
Since $\beta|_S$ is a regular function on S and since it takes only discrete values, it must be constant by the connectedness of S. Hence $\beta(s) = \beta(1) = 1$, for all $s \in S$. $\diamondsuit$.
(ii) By [3], Chapter VI, §1, n°. 1.7, Proposition 24, p. 165, every basis of a $\mathbb{Q}$-closed root subsystem $\hat{R}$ of a root system R can be completed to a basis of R, i.e. the Dynkin diagram $\Delta(\hat{R})$ is a subdiagram of $\Delta(R)$ by removing certain points and all edges meeting these points.
The statement now follows from the classification of Dynkin diagrams of the centralizers of automorphisms of finite order for algebraic groups (resp.

their Lie algebras) in characteristic zero given in [16], Theorem 8.6 and Proposition 8.6, p. 136ff. ♡.

Remark: We can understand the second part of the theorem, except for the A_{2n}-case, as follows:
First, note that the set of irregular semisimple elements in $T_0^\tau \tau$ is described by the root system R'. Under the map $T_0^\tau \tau \to T'$, defined in Lemma 2.8, this set corresponds to the set of irregular elements of T', considered as maximal torus of the connected group G', defined in Section 2.4. Now, it is a well known fact that, in characteristic zero, the Dynkin diagram of a centralizer $C_{G'}(t')_0$ for $t' \in T'$ is a proper subdiagram of the affine non-twisted root system $R'^{(1)}$ of R'. (The proof is just analogously to our proof of part (ii) of the Theorem.) To obtain the root systems of our centralizers of elements in $T_0^\tau \tau$, we have to dualize, see the remark following the definition of the map d at the beginning of this section. Looking at the tables of affine Dynkin diagrams, found e.g. in [16], Tables Aff 1-3, p.54f, we realize that $R^{(ord\,\tau)}$ is exactly the dual of $R'^{(1)}$.

By the definition of regular elements in the previous section and this result, we get the following description of regular elements in $T_0^\tau \tau$:

Corollary 3.3 *(i) An element $t\tau \in T_0^\tau \tau$ is regular, iff $\alpha'(t) \neq 1$, for all $\alpha' \in R'$. (Here we consider the elements of R' as elements of $\mathbb{Z}({}^\tau R^1)$ via the map d.)*
(ii) An element $t\tau \in T\tau$ is regular, iff $\alpha'(t) \neq 1$, for all $\alpha' \in R'$. (Here we consider the elements of R' as elements of $\mathbb{Z}(R)$.)

Proof: (i) is clear by the results above.
(ii) follows from (i) by Lemma 2.5 in the following way:
For a singular $t\tau \in T\tau$ we can find, by Lemma 2.5, $\hat{t} \in T$ and $t'\tau \in T_0^\tau \tau$, such that $t\tau = \hat{t}\, t'\tau\, \hat{t}^{-1}$ and such that there exists $\alpha' \in R'$ with $\alpha'(t') = 1$, by (i). Then we have

$$\alpha'(t) = \alpha'(\hat{t}\, t'\tau\, \hat{t}^{-1}\, \tau^{-1}) = \alpha'(\hat{t})\, \alpha'(\tau\, \hat{t}^{-1}\, \tau^{-1})\, \alpha'(t') = \alpha'(t')\, \alpha'(\hat{t})\, (\tau^{-1}(\alpha'))(\hat{t}^{-1}) \tag{3.31}$$

The statement now follows from the τ invariance of α', cf. Section 1.2. ♡.

Chapter 4

Irregular Elements in the Exterior Component

In this chapter, we want to give a description of the set of irregular elements in $G\tau$. An element will turn out to be irregular, iff it is conjugate to an element $y \in B\tau$, such that there exists a simple root α' of R' with $\alpha'(y_s) = 1$ and y_u in the unipotent radical $U_{\alpha'}$ of a parabolic subgroup $P_{\alpha'}$. ($P_{\alpha'}$ will be defined in this chapter.) Furthermore, we will show that the set of semisimple irregular elements is dense in the set of all irregular elements.

First we have to prove some rather technical lemmas. We keep the notation from the previous chapters.
Let $B < G$ be a Borel subgroup of G and $T < B$ be a maximal torus, such that B and T are stabilized by τ. Furthermore, denote by U the unipotent radical of B.

Lemma 4.1 *With the notation above, we have:*
(i) $\bigcup_{g\in G} gB\tau g^{-1} = G\tau$.
(ii) For $x \in B\tau$ *we can find* $u \in U$, *such that* $u\, x_s\, u^{-1}$ *is an element in* $T\tau$.
(iii) For $x = t\tau u \in B\tau$ *with* $t\tau \in T\tau$, $u \in U$ *we have that the semisimple part of* x *is* U*-conjugate to* $t\tau$.

Proof: (i) Since $T_0^\tau \tau \subset B\tau$ we have by the density result, Lemma 2.6, that $\bigcup_{g\in G} gB\tau g^{-1}$ is a dense subset of $G\tau$. To prove (i) we merely need to prove that the latter is closed in $G\tau$:
Let B act on $G \times B\tau$ by $b.(g, \hat{b}\tau) := (g\, b^{-1}, b\,\hat{b}\tau\, b^{-1})$. Then we define a map Φ from the corresponding associated bundle $G \times^B B\tau$, cf. Section 3.1 and the references quoted there, to $G\tau$ as follows:

$$\Phi : G \times^B B\tau \;\to\; G\tau \tag{4.1}$$

$$g * b\tau \;\mapsto\; g\, b\tau\, g^{-1}. \tag{4.2}$$

By use of [27], Section 3.7, Lemma 1, p. 25, we get a commutative diagram:

$$\begin{tikzcd}
G \times^B B\tau \arrow[rr, "\Phi"] \arrow[dr, "\iota"'] & & G\tau \\
& G/B \times G\tau \arrow[ur, "p"'] &
\end{tikzcd} .$$

Here the closed embedding ι is defined by $\iota(g * b\tau) = (gB, g\, b\tau\, g^{-1})$ and p is the projection on the second factor.
Since G/B is a complete variety, cf. e.g. [13], Theorem 21.3, p. 134, the image $\Phi(G \times^B B\tau) = p(\iota(G \times^B B\tau)) \subset G\tau$ is closed.
(ii) We consider the closed non-connected subgroup $\tilde{B} := B \rtimes <\tau^l,\ l \in \mathbb{Z}>$ of $\tilde{G}$ and regard $x \in B\tau$ as an element thereof. Considering the projection map

$$\begin{aligned} p : \tilde{B} &\to \tilde{B}/B \cong <\tau^l,\ l \in \mathbb{Z}> \\ x &\mapsto xB, \end{aligned} \tag{4.3}$$

we see that $x_s \in B\tau$ and $x_u \in U < B$. Therefore, by [37], Theorem 7.5, p. 51, we can find a maximal torus $\hat{T}$ of B stabilized by x_s.
Now there exists an element $u \in U$, such that $u\,\hat{T}\,u^{-1} = T$, cf. [13], Theorem 19.3, p. 123. (Note, using the notation there, that $B^{\infty} = U$ for reductive G.) So w.o.l.g. we can assume $\hat{T} = T$. Then x_s operates on the root system $R(G;T)$, and stabilizes the basis corresponding to B. Hence x_s operates like a diagram automorphism on the corresponding Dynkin diagram Δ. Observe that the element $x_s\,\tau^{-1} \in G$ also has these properties forcing $x_s\,\tau^{-1} \in T$ and therefore proving (ii).
(iii) This is just a reformulation of (ii) using the fact that by [13], Theorem 19.3, p. 123, we have $B \cong T \ltimes U$. ♡.
Remarks: (i) The line of proof of (ii) is similar to that of Proposition 2.3.
(ii) Note that by use of Lemma 2.5 we can even achieve that x_s is conjugate in $T_0^{\tau}\tau$ in (ii), if we allow conjugation with elements in B rather than U.

As a summary of the lemma above and by Corollary 3.3, we have:

Corollary 4.1 *Let B and T be as above. If $x \in G\tau$ is irregular, then x is conjugate to $t\tau\, u$, where $t\tau \in T_0^{\tau}\tau$ and $u \in U$, such that $t\tau$ and u commute and such that there exists at least one element $\alpha' \in R'$ with $\alpha'(t) = 1$.*

Next we prove a lemma on Borel subgroups of $C_G(t\tau)_0$:

Lemma 4.2 *Let B be a Borel subgroup stabilized by τ and $t\tau \in T_0^{\tau}\tau$. Then $B \cap C_G(t\tau)_0$ is a Borel subgroup of $C_G(t\tau)_0$.*

Proof: Clearly, $B \cap C_G(t\tau)_0$ is solvable. Next we calculate $B \cap C_G(t\tau)_0$ directly. By [13], Theorem 19.3, p. 123, $B \cong T \ltimes U$ for each maximal torus T of B. We choose T as above, i.e. τ-stable. By the proof of Proposition 3.1 respectively the remark (i) following it, if G is semisimple and nonsimple and τ a permutation of its isomorphic normal, simple subgroups, we already know that $C_U(t\tau) \subset C_G(t\tau)_0$ by its connectedness and we already saw that its dimension is $\frac{1}{2}|R(C_G(t\tau)_0)|$. Furthermore we know that T_0^τ is a maximal torus of $C_G(t\tau)_0$, by Lemma 2.3. Therefore, we conclude $B \cap C_G(t\tau)_0 = T_0^\tau C_U(t\tau)$. This is a connected solvable subgroup of $C_G(t\tau)_0$ having maximal possible dimension. Hence it must be a Borel subgroup. $\heartsuit$.

Our next step is to construct certain parabolic subgroups of G corresponding to elements of a basis of R'. Let Π be the basis of R. Then, $\Pi' := d(p(\Pi)) = \{\alpha_1', ..., \alpha_s'\}$ is a basis of R', where d is the map defined at the beginning of Section 3.2. (Note, the composition $d \circ p$ is the map $\alpha \mapsto \alpha'$ of Section 1.2.)

For each $\alpha_i' \in \Pi'$ we associate a parabolic subgroup $P_{\alpha_i'}$ as follows:

Consider the set $\Psi_i := p^{-1}(d^{-1}(\alpha_i')) \subset R$ and define a subgroup $\mathcal{W}_i$ of the Weyl group $\mathcal{W}$ by $\mathcal{W}_i =< 1, s_\alpha,\, \alpha \in \Psi_i >$. Then we define $P_{\alpha_i'}$ by:

$$P_{\alpha_i'} := B\,\mathcal{W}_i\,B. \tag{4.4}$$

Furthermore set $U_{\alpha_i'} := R_u\,P_{\alpha_i'}$, the unipotent radical of $P_{\alpha_i'}$. To see that these groups are well defined as parabolic subgroups we have to show that $\mathcal{W}_i$ is already generated by the simple reflexions of $\mathcal{W}$ which lie in $\mathcal{W}_i$, cf. the theory of parabolic subgroups as developed in [13], Section 30:

To achieve this, we give a description of the sets Ψ_i using results of Section 1.2, especially Lemma 1.2. We have to distinguish three cases (here Γ denotes the subgroup of $Aut(R(G))$ generated by τ:

(i) $\alpha_i' = \alpha_i$ for an element $\alpha_i \in \Pi$. Then $\alpha_i = \tau(\alpha_i)$. It follows that $\Psi_i = \{\alpha_i\}$ and $\mathcal{W}_i = \{1,\, s_{\alpha_i}\} \cong \mathbb{Z}/2\mathbb{Z}$.

(ii) $\alpha_i' = \frac{1}{|\Gamma_{\alpha_i}|}\sum_{j=1}^{ord\,\tau}\tau^i(\alpha_i)$ for a suitable element $\alpha_i \in \Pi$. Then $\alpha_i \neq \tau(\alpha_i)$ and $\check{\alpha}_i(\tau(\alpha_i)) = 0$ implying $\Psi_i = \{\tau^j(\alpha_i),\, j \in \{1, ..., \frac{ord\,\tau\}}{|\Gamma_{\alpha_i}|}\}$ and $\mathcal{W}_i =$ $< 1,\, s_{\tau^j(\alpha_i)},\, j \in \{1, .., \frac{ord\,\tau}{|\Gamma_{\alpha_i}|}\} >\cong (\mathbb{Z}/2\mathbb{Z})^{\frac{ord\,\tau}{|\Gamma_{\alpha_i}|}}$.

(iii) $\alpha_i' = 2(\alpha_i + \tau(\alpha_i))$ for an element $\alpha_i \in \Pi$. Then $\alpha_i \neq \tau(\alpha_i)$ and $\check{\alpha}_i(\tau(\alpha_i)) \neq 0$ implying $\Psi_i = \{\alpha_i,\, \tau(\alpha_i),\, \alpha_i + \tau(\alpha_i)\}$ and $\mathcal{W}_i =$ $< 1,\, s_{\alpha_i},\, s_{\tau(\alpha_i)} >\cong \mathcal{S}_3$. (Note that in this case $w_{\alpha_i + \tau(\alpha_i)} = s_{\alpha_i}\, s_{\tau(\alpha_i)}\, s_{\alpha_i}$.)

We now get the description of the unipotent radicals of the $P_{\alpha_i'}$ as follows:

Lemma 4.3 *(i) We have*

$$U_{\alpha_i'} \cong \prod_{\substack{\beta \in R^+ \\ \beta \notin \Psi_i}} Im\,X_\beta, \tag{4.5}$$

where X_β is the root group corresponding to the root $\beta \in R$.
(ii) B normalizes $U_{\alpha'_i}$.
(iii) $P_{\alpha'_i}$ and $U_{\alpha'_i}$ are stabilized by τ.

Proof: (i) follows from [13], Section 30.2, p. 184f, and (ii) is clear by $B < P_{\alpha'_i}$. (iii) follows from the fact that τ stabilizes B, T and $\mathcal{W}_i$ and that $U_{\alpha'_i}$ is the unipotent radical of $P_{\alpha'_i}$. ♡.

Denote the kernel of $\alpha'_i \in \Pi'$ in T, respectively T_0^τ by $T_{\alpha'_i}$, respectively $T^\tau_{0\ \alpha'_i}$, and set furthermore

$$B_{\alpha'_i} \;:=\; T_{\alpha'_i} U_{\alpha_i} \cong T_{\alpha'_i} \ltimes U_{\alpha_i} \tag{4.6}$$

$$\bar{B}_{\alpha'_i} \;:=\; T^\tau_{0\ \alpha'_i} U_{\alpha_i} \cong T^\tau_{0\ \alpha'_i} \ltimes U_{\alpha_i}. \tag{4.7}$$

We will show that the sets $B_{\alpha'_i}\tau$, respectively $\bar{B}_{\alpha'_i}\tau$, which can be considered as exterior components of algebraic groups

$$\tilde{B}_{\alpha'_i} \;:=\; B_{\alpha'_i} \rtimes < \tau^j, j \in \mathbb{Z} > \quad \text{respectively} \tag{4.8}$$

$$\tilde{\bar{B}}_{\alpha'_i} \;:=\; \bar{B}_{\alpha'_i} \rtimes < \tau^j, j \in \mathbb{Z} >, \tag{4.9}$$

will describe the set of irregular elements of $G\tau$. The next two lemmas give a characterization of irregular elements in $G\tau$:

Lemma 4.4 *The elements of $B_{\alpha'_i}\tau$, respectively $\bar{B}_{\alpha'_i}\tau$, are irregular in $G\tau$.*

Proof: Let $x \in B_{\alpha'_i}\tau$ $(\bar{B}_{\alpha'_i}\tau) \subset B\tau$, then we have $x_s \in B_{\alpha'_i}\tau$ $(\bar{B}_{\alpha'_i})$ and $x_u \in U_{\alpha'_i}$. By Lemma 4.1.(ii) and the proof of Corollary 3.3.(ii), we can assume w.l.o.g. that

$$x_s \;=\; t\tau \;\in\; T^\tau_{0\ \alpha'_i}\tau \quad \text{and} \tag{4.10}$$

$$x_u \;=\; u \;\in\; U_{\alpha'_i}. \tag{4.11}$$

(Note that by Lemma 4.3.(ii) $U_{\alpha'_i}$ is stabilized by B). Apparently, $u \in C_{U_{\alpha'_i}}(t\tau)$.
By the description of the structure of the fibres of $\pi : G\tau \to T_0^\tau\tau/\widetilde{\mathcal{W}}$ given in Theorem 3.1 and by the definition of regularity of elements in Section 3.1, we only need to show that u is an irregular element of $C_G(t\tau)_0$.
By the description of $R(C_G(t\tau))$ in the proof of Proposition 3.1 there exists a (unique by the remarks preceding Proposition 3.1) root $\bar{\alpha}_i \in R(C_G(t\tau))$, such that $d(\bar{\alpha}_i) = \alpha'_i$. Furthermore by Remark (ii) following Proposition 3.1 and its proof we have a description of $(Im\, S_{\alpha'_i})^{t\tau}$ as an image of a root group of $C_G(t\tau))$.
By construction we have

$$(Im\, S_{\alpha'_i}) \cap U_{\alpha'_i} = \{e\}. \tag{4.12}$$

Let us set $s := dim\, T_0^\tau$. Observe that $t\tau\, U_{\alpha'_i}$ is a subset of $\bar{B}_{\alpha'_i}\tau$ of dimension $dim\, U_{\alpha'_i}$. Then $t\tau\, U_{\alpha'_i}$ is stabilized by $U_{\alpha'_i}$, T_0^τ and $(Im\, S_{\alpha'_i})^{t\tau}$. Stability under $U_{\alpha'_i}$ follows from the fact that $t\tau\, u'(t\tau)^{-1} \in U_{\alpha'_i}$, for all $u' \in U_{\alpha'_i}$. Observe that stability under T_0^τ and $(Im\, S_{\alpha'_i})^{t\tau}$ is clear.
Consider the (solvable) algebraic group generated by T_0^τ and $(Im\, S_{\alpha'_i})^{t\tau}$. Since $dim\,(Im\, S_{\alpha'_i})^{t\tau} = 1$, this group has dimension $s+1$. Now set:

$$G_{\alpha'_i} :=< T_0^\tau,\, (Im\, S_{\alpha'_i})^{t\tau} > \ltimes U_{\alpha'_i}. \tag{4.13}$$

This is an algebraic subgroup of $\tilde{G}$ of dimension $s + 1 + dim\, U_{\alpha'_i}$. Let us consider the morphism

$$\begin{aligned} \phi : G \times^{G_{\alpha'_i}} (t\tau\, U_{\alpha'_i}) \quad &\rightarrow \quad G\tau \\ g * t\tau\, u' \quad &\mapsto \quad g\, t\tau\, u'\, g^{-1}. \end{aligned} \tag{4.14}$$

The image of ϕ are exactly the G-conjugacy classes in $G\tau$, which meet $t\tau\, U_{\alpha'_i}$
Now, $dim\, G \times^{G_{\alpha'_i}} (t\tau\, U_{\alpha'_i}) = dim\, G - s - 1$. By Corollary 3.1 the regular classes in $G\tau$ have dimension $dim\, G - s$ which implies that the image of ϕ only contains irregular elements. $\heartsuit$.

Lemma 4.5 *If $x \in G\tau$ is irregular, then x is conjugate into one of the sets $B_{\alpha'_i}\tau$, respectively $\bar{B}_{\alpha'_i}\tau$, for a suitable simple root, $\alpha'_i \in R'$.*

Proof: Let x be as in the statement of the lemma. By use of Lemma 4.1, and Lemma 2.5, we can assume w.l.o.g. that:

$$x_s \;=\; t\tau \;\in\; T_0^\tau\tau \quad \text{and} \tag{4.15}$$

$$x_u \;=\; u \;\in\; U. \tag{4.16}$$

Since x is irregular, we can find, by Corollary 4.1, $\alpha' = d(\bar{\alpha}) \in R'$, such that $\alpha'(t) = 1$, where $\bar{\alpha} \in R(C_G(t\tau)) \subset {}^\tau R^1$. We have to show that, by conjugation, we can achieve α' simple and $u \in U_{\alpha'}$.
1. Claim: We can assume that $\bar{\alpha}$ fulfills both of the following properties:
(i) $\bar{\alpha}$ is simple in $R(C_G(t\tau))$ but not necessarily in ${}^\tau R^1$, i.e. $d(\bar{\alpha})(t) = 1$.
(ii) The $d(\bar{\alpha})$-part, i.e. the $(Im\, S_{d(\bar{\alpha})})^{t\tau}$-part of $u \in U$ with respect to the identification Φ of U with the product of the root groups in Equation 3.12, is trivial.
Proof of Claim: The existence of $\bar{\alpha}$ with these properties follows from the characterization of irregular elements in [35], Lemma 3.2 and Theorem 3.3, p. 286f. (Note that we use here the description of the root groups of $C_G(t\tau)$ as given in Remark (ii) following Proposition 3.1.) $\diamondsuit$.
From this it follows that the entire $Im\, S_{d(\bar{\alpha})}$-part of u has to be trivial with respect to our coordinates on U (i.e. those corresponding to Φ).

2. Claim 2: By conjugation with representatives, in $N_G(< T_0^\tau, \tau >)$, of elements of $\mathcal{W}^\tau$, we can achieve $\bar{\alpha}$ or $\frac{1}{2}\bar{\alpha}$ to be simple in ${}^\tau R^1$. (The last possibility appears only in the A_{2n}-case.)
Claim 2 clearly implies $d(\bar{\alpha})$ to be simple in R', whence the lemma.
Proof of Claim 2: This will be done by induction on $ht\,\bar{\alpha}$.
If $ht\,\bar{\alpha} = 1$ or $ht\,\bar{\alpha} = 2$ and $\bar{\alpha} = 2\,\bar{\alpha}_j$, for a simple $\bar{\alpha}_j \in {}^\tau R^1$, we are done.
Hence we assume now that $ht\,\bar{\alpha} \geq 2$ and that $\bar{\alpha}$ is not the double of a simple root in ${}^\tau R^1$. To carry out the induction, our strategy will be as follows:
Find a simple root $\bar{\alpha}_j$ of ${}^\tau R^1$ with $\check{\bar{\alpha}}_j(\bar{\alpha}) > 0$, which is not a root of $R(C_G(t\tau))$. Conjugate x with a representative $n_{d(\bar{\alpha}_j)} \in N_G(< T_0^\tau, \tau >)$ of the simple reflexion $w_{d(\bar{\alpha}_j)} \in \mathcal{W}^\tau$. Then, $w_{d(\bar{\alpha}_j)}(\bar{\alpha})$ is a simple root of $R(C_G(n_{d(\bar{\alpha}_j)}\,t\tau\,n_{d(\bar{\alpha}_j)}^{-1}))$ with smaller height than $\bar{\alpha}$ in ${}^\tau R^1$.
This will now be carried out in detail:
So, we assume now that $ht\,\bar{\alpha} \geq 2$ and that $\bar{\alpha}$ is not the double of a simple root in ${}^\tau R^1$. Then we can find a simple root $\bar{\alpha}_j \in {}^\tau R^1$, such that $\check{\bar{\alpha}}_j(\bar{\alpha}) > 0$. (Otherwise, we would obtain a contradiction to the fact that the strict convex cone of simple roots and the convex cone of the negatives of the fundamental weights only intersect at zero.) By [3] §1, n$^{\text{o}}$ 1.3, corollaire du théorème 1, p. 149, we have $\bar{\alpha} - \bar{\alpha}_j$ is an element in ${}^\tau R^{1\,+}$. By the simplicity of $\bar{\alpha}$ in $R(C_G(t\tau))$, we easily conclude that $\bar{\alpha}_j$ (respectively $2\,\bar{\alpha}_j$ in the A_{2n}-case) cannot be a root in $R(C_G(t\tau))$. (Otherwise, by the same reasoning as above we would end up with $\bar{\alpha} - \bar{\alpha}_j$ (resp. $\bar{\alpha} - 2\,\bar{\alpha}_j) \in R(C_G(t\tau))$.)
It therefore follows that, by our identification of U with the direct product of root groups via Φ in Equation 3.12 and by the definition of the $S_{\alpha'}$ in the proof of Proposition 3.1, the $S_{d(\bar{\alpha}_j)}$-part of u is trivial. Let us denote by $w_{d(\bar{\alpha}_j)}$ the corresponding reflexion in the Weyl group $\mathcal{W}^\tau$ (which need not be a reflexion in $\mathcal{W}$!). By the description of the fixed point Weyl group $\mathcal{W}^\tau$ in [7], Chapter 13, and our definition of the $\mathcal{W}_j$ in the discussion preceding Lemma 4.3, we see that $w_{d(\bar{\alpha}_j)}$ is the element of greatest length in $\mathcal{W}_j$ and the set $\Psi_j \subset R^+$ is exactly the set of positive roots of R made negative by $w_{d(\bar{\alpha}_j)}$ (considered as an element of $\mathcal{W}$.)
Therefore, we see:

$$u \in U \cap w_{d(\bar{\alpha}_j)}^{-1}\,U\,w_{d(\bar{\alpha}_j)} \subset U_{d(\bar{\alpha}_j)}. \tag{4.17}$$

Next we choose a representative $n_{d(\bar{\alpha}_j)} \in N_G(< T_0^\tau, \tau >)$ of $w_{d(\bar{\alpha}_j)}$. Then we have:

$$t'\tau := n_{d(\bar{\alpha}_j)}\,t\tau\,n_{d(\bar{\alpha}_j)}^{-1} \quad \in \quad T^\tau_{0\;d(w_{d(\bar{\alpha}_j)}(\bar{\alpha}))}\tau \tag{4.18}$$

$$u' := n_{d(\bar{\alpha}_j)}\,u\,n_{d(\bar{\alpha}_j)}^{-1} \quad \in \quad U_{d(\bar{\alpha}_j)} \subset U, \tag{4.19}$$

Observe that, in ${}^\tau R^1$, we have $ht\,(w_{d(\bar{\alpha}_j)}(\bar{\alpha})) < ht\,\bar{\alpha}$. Furthermore:

$$C_G(t'\tau) = n_{d(\bar{\alpha}_j)}\,C_G(t\tau)\,n_{d(\bar{\alpha}_j)}^{-1}. \tag{4.20}$$

From this and the fact that the $C_G(t\tau) \cap U \subset U_{d(\bar{\alpha}_j)}$ (recall that neither $\bar{\alpha}_j$ nor $2\,\bar{\alpha}_j$ are roots in $R(C_G(t\tau))$), with $U_{d(\bar{\alpha}_j)}$ being $w_{d(\bar{\alpha}_j)}$-invariant, we conclude that

$$\begin{aligned} U \cap C_G(t'\tau) &= U \cap n_{d(\bar{\alpha}_j)}\, C_G(t\tau)\, n_{d(\bar{\alpha}_j)}^{-1} \\ &= n_{d(\bar{\alpha}_j)}\, (n_{d(\bar{\alpha}_j)}^{-1}\, U\, n_{d(\bar{\alpha}_j)} \cap C_G(t\tau))\, n_{d(\bar{\alpha}_j)}^{-1} \\ &= n_{d(\bar{\alpha}_j)}\, (U \cap C_G(t\tau))\, n_{d(\bar{\alpha}_j)}^{-1} = n_{d(\bar{\alpha}_j)}\, U\, n_{d(\bar{\alpha}_j)}^{-1} \cap C_G(t'\tau) \end{aligned}$$

yielding

$$B \cap C_G(t'\tau) = n_{d(\bar{\alpha}_j)}\, B\, n_{d(\bar{\alpha}_j)}^{-1} \cap C_G(t'\tau). \tag{4.21}$$

Therefore, $w_{d(\bar{\alpha}_j)}(\bar{\alpha})$ must be a simple root of $C_G(t'\tau)$ with respect to the Borel subgroup $B \cap C_G(t'\tau)$. From this it follows that the $S_{d(w_{d(\bar{\alpha}_j)}(\bar{\alpha}))}$-part of u', with respect to the identification Φ of U as a product of its root groups is trivial. (Here we use, that the $Im\, X_\alpha$-part of an element of U, for a simple root α, is independent of the choice of an isomorphism between U and the product of its root groups. This follows directly from Lemma 3.2.)
Now, we can apply the induction assumption to get that either $\bar{\alpha}$ or $\frac{1}{2}\,\bar{\alpha}$ are simple roots in ${}^\tau R^1$ from which the simplicity of $d(\bar{\alpha})$ in R' follows. ♡.

As a summary of this we have:

Theorem 4.1 *(i) The set of irregular elements $(G\tau)_{irr}$ of $G\tau$ has the following shape:*

$$\begin{aligned} (G\tau)_{irr} &= \bigcup_{\substack{g \in G \\ \alpha' \in \Pi'}} g\, B_{\alpha'}\, \tau\, g^{-1} \\ &= \bigcup_{\substack{g \in G \\ \alpha' \in \Pi'}} g\, \bar{B}_{\alpha'}\, \tau\, g^{-1}. \end{aligned} \tag{4.22}$$

(ii) The set of semisimple elements of $(G\tau)_{irr}$ is dense in $(G\tau)_{irr}$.

Proof: (i) is just the combination of Lemmas 4.4 and 4.5.
(ii): We start with defining a set $J_{\alpha'}$, respectively $\bar{J}_{\alpha'}$, for $\alpha' \in \Pi$, as follows:

$$J_{\alpha'} := T_{\alpha'} \setminus \bigcup_{\substack{\beta' \in R'^+ \\ \beta' \neq \alpha'}} T_{\beta'} \tag{4.23}$$

$$\bar{J}_{\alpha'} := T^\tau_{0\ \alpha'} \setminus \bigcup_{\substack{\beta' \in R'^+ \\ \beta' \neq \alpha'}} T^\tau_{0\ \beta'}. \tag{4.24}$$

Apparently, these are open and dense subsets of $T_{\alpha'}$ respectively $T^{\tau}_{0\;\alpha'}$. Hence, $J_{\alpha'}\, U_{\alpha'}\tau$, respectively $\bar{J}_{\alpha'}\, U_{\alpha'}\tau$, are open and dense in $B_{\alpha'}\tau$, respectively $\bar{B}_{\alpha'}\tau$. By use of (i) we conclude that

$$\bigcup_{\substack{g\in G\\ \alpha'\in\Pi'}} g\, J_{\alpha'}\, U_{\alpha'}\tau\, g^{-1}, \quad \text{respectively} \quad \bigcup_{\substack{g\in G\\ \alpha'\in\Pi'}} g\, \bar{J}_{\alpha'}\, U_{\alpha'}\tau\, g^{-1}, \tag{4.25}$$

are dense in $(G\tau)_{irr}$. To complete the proof, we only need to show that every element x of $J_{\alpha'}\, U_{\alpha'}\tau$, respectively $\bar{J}_{\alpha'}\, U_{\alpha'}\tau$, is semisimple:
We have $x = t\tau\, u$ with $t \in T_{\alpha'}$, respectively $t \in T^{\tau}_{0\;\alpha'}$ and $u \in U_{\alpha'}$ By conjugation of $t\tau$ with an element of T we can assume w.l.o.g. $t \in T^{\tau}_{0\;\alpha'}$. Here, we use Lemma 2.5 together with the τ invariance of α' and the T-stability of $U_{\alpha'}$, cf. Lemma 4.3.
Applying Lemma 4.1 (iii) we have w.l.o.g. $t\tau = x_s$ and $u = x_u$. (Here we use that U stabilizes $U_{\alpha'}$, cf. Lemma 4.3.)
By construction we clearly see that $R(C_G(t\tau))$ is of type A_1. (A suitable choice of $d^{-1}(\alpha')$ and its negative being the only roots.) Therefore, we see that $C_G(t\tau)\cap U \subset Im\, S_{\alpha'}$, which amounts to

$$u \in C_G(t\tau)\cap U_{\alpha'} \subset Im\, S_{\alpha'} \cap U_{\alpha'} = \{e\}, \tag{4.26}$$

thus proving our assertion. ♡.

Chapter 5

The Steinberg Cross Section

In this chapter, we will construct a so called Steinberg Cross Section, a section with respect to the quotient map $\pi : G\tau \to G\tau /\!\!/ G$, in the case where the group G', appearing in the paragraph preceding Proposition 2.8, is simply connected. Furthermore, a differentiable characterization of regular elements and further properties of the fibres will be given.

We keep the notation of the previous chapters. Furthermore, let r be the rank of G, s the rank of G^τ which is the same as the rank of G', and let $\Pi = \{\alpha_1, ..., \alpha_r\}$ be a τ-stable basis of $R(G)$ which is defined with respect to a τ-stable maximal torus T of G contained in a τ-stable Borel subgroup B of G. Throughout this chapter, we assume G' to be simply connected, unless stated otherwise, i.e. $\mathbb{Z}(R)^\tau = \Lambda(R)^\tau$. Furthermore, let $\{\lambda_1, ..., \lambda_r\}$ be the fundamental dominant weights of $\Lambda(R)$ corresponding to Π and let $\{\lambda'_1, ..., \lambda'_s\}$ be the corresponding generators of $\Lambda(R)^\tau$, i.e. the λ'_i are the sums over the τ-orbit of the λ_j, cf. the proof of Lemma 1.4. Furthermore, denote the quotient map $G\tau \to G\tau /\!\!/ G$ by π.
We now define the notion of Steinberg cross section:

Definition 5.1 *A Steinberg cross section of the exterior component $G\tau$ of $\tilde{G}$ is a morphism $C : \mathbb{A}^s \to G\tau$, such that the following properties hold:*
(i) $Im\, C$ is closed in $G\tau$.
(ii) $\pi|_{Im\, C} : Im\, C \to G\tau /\!\!/ G$ is an isomorphism.
(iii) $Im\, C$ meets each fibre only in its regular orbit.

Next we want to construct a map $C : \mathbb{A}^s \to G\tau$, which has the property of a Steinberg cross section, according to the following recipe:
Take one representative α_i for each τ orbit of Π. By relabeling the elements of Π and the generators of $\Lambda(R)^\tau$ we can assume that $\alpha_1, ..., \alpha_s$ are our chosen representatives and that $\lambda'_i(\check{\alpha}_j) = \delta_{i\,j}$ for $i,\, j \in \{1, ..., s\}$.
Let X_{α_i}, $i \in \{1, ..., s\}$ be the corresponding root groups of G and $n_{\alpha_i} \in N_G(T)$ be a representative of the reflexion s_{α_i} in $\mathcal{W}$ corresponding to α_i, $i \in$

$\{1, ..., s\}$.
Now we define

$$\begin{aligned} C : \mathbb{A}^s &\to G\tau \\ (c_1, ..., c_s) &\mapsto \prod_{i=1}^{s} X_{\alpha_i}(c_i)\, n_{\alpha_i} \tau \end{aligned} \tag{5.1}$$

Remarks: (i) C depends on the choices we made, i.e. the order of the product in the definition of C and the choices of the representatives of the considered simple reflexions in $\mathcal{W}$.
(ii) Note that $C(0, ..., 0) = n_{\alpha_1}...n_{\alpha_s}\tau$ is a representative in $N_{\tilde{G}}(T)$ of a twisted Coxeter element, investigated in [31], Section 7. It has similar properties as the Coxeter element, e.g. it has no eigenvector of eigenvalue one on the vector space generated by the roots and is unique up to conjugation in the Weyl group.

Our aim is to prove that C has the properties listed in the above definition of a Steinberg cross section:

Lemma 5.1 *Keeping the notation above, $Im\, C$ is closed in $G\tau$ and isomorphic to $\mathbb{A}^s$.*

Proof: From the definition of root groups as given e.g. in [38], Section 3.2, it easily follows that for every $\alpha \in R$ and $w \in \mathcal{W}$, with $n_w \in N_G(T)$ being a representative of w, we have

$$n_w\, Im\, X_\alpha\, n_w^{-1} = Im\, X_{w(\alpha)}. \tag{5.2}$$

Using this property inductively, we easily get

$$Im\, C = \left(\prod_{i=1}^{s} Im\, X_{\beta_i} \right)\, n_{\alpha_1}...n_{\alpha_s}\tau, \tag{5.3}$$

where $\beta_i = (s_{\alpha_1}...s_{\alpha_{i-1}})(\alpha_i)$. Observe that $\{\beta_1, ..., \beta_s\}$ is just the set of positive roots made negative by $w := s_{\alpha_s}...s_{\alpha_1}$ or $\tau^{-1}s_{\alpha_s}...s_{\alpha_1}$, the inverse of our twisted Coxeter element. (Note that τ stabilizes the set R^+.)
Denoting by B^- the the opposite Borel subgroup with respect to B and by U^- its unipotent radical (B^- is the Borel subgroup corresponding to the basis $-\Pi$), we have by [13], Proposition 28.1, p. 170, the following equality

$$\prod_{i=1}^{s} Im\, X_{\beta_i} = U \cap w\, U^- w^{-1}. \tag{5.4}$$

The right hand side is closed in U and isomorphic to $\mathbb{A}^s$. $\heartsuit$.

Our next aim is to evaluate the quotient map on $Im\, C$. To carry out this, we need some preparations. Recall that, in our situation, the quotient map $\pi : G\tau \to T_0^\tau \tau/\widetilde{W}$ is given by the s-tuple of characters characters $\tilde{X}_1^{\lambda_i'}$ corresponding to the fundamental dominant weights of $\Lambda(R')$, restricted to $G\tau$, see Corollary 2.6:

$$\pi(g\tau) = (\tilde{X}_1^{\lambda_1'}(g\tau), ..., \tilde{X}_1^{\lambda_s'}(g\tau)), \quad \forall g \in G. \tag{5.5}$$

Furthermore, let

$$V(\lambda_i') = \bigoplus_{\mu \in \chi(T)} V(\lambda_i')_\mu \tag{5.6}$$

be the decomposition of the $\tilde{G}$-module $V(\lambda_i')$ in its weight spaces $V(\lambda_i')_\mu$. Denote the canonical injection $V(\lambda_i')_\mu \hookrightarrow V(\lambda_i')$ by ι_μ and the canonical projection $V(\lambda_i') \twoheadrightarrow V(\lambda_i')_\mu$ by π_μ. Then we clearly have:

$$\tilde{X}_1^{\lambda_i'}(g\tau) = \sum_{\mu \in \chi(T)} tr_{V(\lambda_i')_\mu} (\pi_\mu \, \tilde{\rho}_1^{\lambda_i'}(g\tau) \, \iota_\mu). \tag{5.7}$$

From now on we assume $g\tau \in Im\, C$. For our calculations we need a well known result on the action of the root groups with respect to the weight spaces, which may be found e.g. in [38], Section 3.3, Proposition 2, p. 80:

Lemma 5.2 *Let G be a reductive group, V a finite dimensional G-module and V_λ a weight space thereof. Then we have, for every $v \in V_\lambda$:*
(i) $X_\alpha(c).v = \sum_{i=0}^\infty c^i \, v_i$,
where $v_i \in V_{\lambda + i\,\alpha}$ is independent of c and $v_0 = v$.
Furthermore, the following holds:
(ii) $n_\alpha.v \in V_{s_\alpha(\lambda)} = V_{\lambda - \check{\alpha}(\lambda)\alpha}$,
(iii) $\tau.v \in V_{\tau(\lambda)}$.

We will prove now a sequel of lemmas to get property (ii) of the Steinberg cross section.

Lemma 5.3 *For $g\tau \in Im\, C$ we have:*

$$tr_{V(\lambda_i')_\mu} (\pi_\mu \, \tilde{\rho}_1^{\lambda_i'}(g\tau) \, \iota_\mu) \; = \; 0, \tag{5.8}$$

if $\tau(\mu) \neq \mu$.

Proof: By inductive application of Lemma 5.2, we get for every $v \in V(\lambda_i')_\mu$:

$$g\tau.v \in \bigoplus_{k_1,...,k_s \in \mathbb{Z}} V(\lambda_i')_{\tau(\mu) + \sum_j^s k_j \, \alpha_j}. \tag{5.9}$$

If we can prove the following assertion, we see that the corresponding trace of $g\tau$ in formula 5.8 vanishes:
Claim: If $\tau(\mu) \neq \mu$ we have $\mu \notin \tau(\mu) + \mathbb{Z}(\alpha_1, ..., \alpha_s)$.

Proof of Claim: Assume the contrary , i.e. $\tau(\mu) \neq \mu$ and $\mu \in \tau(\mu) + \mathbb{Z}(\alpha_1, ..., \alpha_s)$. Then we can find $p_j \in \mathbb{Z}$, $j \in \{1, ..., s\}$ such that

$$0 \neq \mu - \tau(\mu) = \sum_{j=1}^{s} p_j \, \alpha_j. \tag{5.10}$$

Furthermore, we easily see:

$$\sum_{i=1}^{ord\,\tau} \tau^i(\mu - \tau(\mu)) = 0. \tag{5.11}$$

Combining these two equations, we get:

$$0 = \sum_{i=1}^{ord\,\tau} \sum_{j=1}^{s} p_j \, \tau^i(\alpha_j). \tag{5.12}$$

By introducing $p'_i \in \mathbb{Z}, i \in \{1, ..., r\}$, by following the rule $p'_i = |\Gamma_{\alpha_i}| \, p_i$ and $p'_i = p'_j$, if α_i lies in the Γ-orbit of α_j, we can rewrite the above equation in the following way (recall that Γ is the subgroup of $Aut(R)$ generated by τ):

$$0 = \sum_{j=1}^{r} p'_j \, \alpha_j. \tag{5.13}$$

This implies $p'_j = 0$, for all $j \in \{1, ..., r\}$ which amounts to $p_j = 0$, for all $j \in \{1, ..., s\}$, a contradiction to the definition of the p_j, $j \in \{1, ..., s\}$. ♡.

As a direct consequence we get:

Corollary 5.1 *For $g\tau \in Im\, C$ we have:*

$$\tilde{X}_1^{\lambda'_i}(g\tau) = \sum_{\mu \in \Lambda(R')} tr_{V(\lambda'_i)_\mu} \, (\pi_\mu \, \tilde{\rho}_1^{\lambda'_i}(g\tau) \, \iota_\mu). \tag{5.14}$$

Next, we want to introduce an order relation $\dot{\geq}$ on the set of fundamental dominant weights $\{\lambda'_1, ..., \lambda'_s\}$ of $\Lambda(R')$ by the following rule:
$\lambda'_i \dot{\geq} \lambda'_j$, *iff there exists a dominant weight $\lambda' \in \Lambda(R')$, such that $\lambda' \prec \lambda'_i$ (in $\Lambda(R')$!), i.e. $\lambda'_i - \lambda'$ is a sum of positive roots and such that we have:*

$$\lambda' = \sum_{k=1}^{s} m_k \, \lambda'_k, \tag{5.15}$$

with $m_j > 0$.

Remark: Note, that by representation theory, found e.g. in [38], Sections 3.3 and 3.4, for all weight spaces $V(\lambda'_j)_\mu$, we have $\mu \prec \lambda'_j$.

First, we have to show that $\dot{\geq}$ actually is a partial order:

Lemma 5.4 *The set $\{\lambda'_1, ..., \lambda'_s\}$ with the relation $\dot{\geq}$ is a partially ordered set.*

Proof: We only need to show the transitivity and the antisymmetry of the order:
1. Transitivity: Let $\lambda'_i \dot{\geq} \lambda'_j$ and $\lambda'_j \dot{\geq} \lambda'_k$, i.e. we have λ', $\mu' \in \Lambda(R')$, such that:

$$\lambda'_i - \lambda' = \sum_{l=1}^{s} n_l \, \alpha'_l \tag{5.16}$$

$$\lambda'_j - \mu' = \sum_{l=1}^{s} n'_l \, \alpha'_l, \tag{5.17}$$

with n_l, $n'_l \in \mathbb{Z}^{\geq 0}$, for all $l \in \{1, ..., s\}$ and

$$\lambda' = \sum_{l=1}^{s} m_l \, \lambda'_l \tag{5.18}$$

$$\mu' = \sum_{l=1}^{s} m'_l \, \lambda'_l \tag{5.19}$$

with m_l, $m'_l \in \mathbb{Z}^{\geq 0}$, for all $l \in \{1, ..., s\}$, and m_j, $m'_k > 0$. We set:

$$\nu := \lambda'_i - \sum_{l=1}^{s} (n_l + m_j \, n'_l) \alpha'_l = \lambda' - m_j \sum_{l=1}^{s} n'_l \, \alpha'_l. \tag{5.20}$$

A calculation yields

$$\nu = \sum_{l=1}^{s} m''_l \, \lambda'_l \tag{5.21}$$

with $m''_j = m_j \, m'_j$ and $m''_l = m_l + m_j \, m'_l$ for $l \neq j$, yielding $m''_k > 0$.
2. Antisymmetry: We prove this by contradiction. Assume $\lambda'_i \dot{\geq} \lambda'_j$ and $\lambda'_j \dot{\geq} \lambda'_i$, with $i \neq j$. Since we already have the transitivity, we can consider the following situation:
There exist two sets of positive integers m_l, $l \in \{1, ..., s\}$ and n_k, $l \in \{1, ..., s\}$, each s-tuple being nontrivial, such that

$$\lambda'_i - \sum_{l=1}^{s} n_l \, \alpha'_l = \sum_{l=1}^{s} m_k \, \lambda'_k \tag{5.22}$$

with $m_i > 0$. This amounts to $(\sum_{l=1}^{s} m_k \, \lambda'_k) - \lambda'_i$ being a nontrivial element in the intersection of the the strict convex cone of negative linear combinations of positive roots and the strict convex cone of positive linear combinations of fundamental weights. But, by the table in [14], §13.2, p. 69, each fundamental weight is a positive $\mathbb{Q}$-linear combination of the positive roots, which shows that the two cones intersect only in the origin, giving a contradiction. $\heartsuit$.

Now we go back to the calculation of $\pi(g\tau)$ for $g\tau \in Im\, C$. To simplify the notation, we introduce $Y_{\alpha_i} : \mathbb{A} \to G$ by

$$Y_{\alpha_i}(c) := X_{\alpha_i}(c)\, n_{\alpha_i}. \tag{5.23}$$

Furthermore, we identify $g\tau \in \tilde{G}$ with its representer $\tilde{\rho}_1^{\lambda'_i}(g\tau)$ in the representation $\tilde{\rho}_1^{\lambda'_i}$. With this, we obtain a lemma:

Lemma 5.5 *For all $\mu \in \Lambda(R')$ we have:*

$$tr_{V(\lambda'_i)_\mu}\left(\pi_\mu \prod_{i=1}^{s} Y_{\alpha_1}(c_i)\, \tau\, \iota_\mu\right) = tr_{V(\lambda'_i)_\mu}\left(\prod_{i=1}^{s}(\pi_\mu\, Y_{\alpha_1}(c_i)\, \iota_\mu)\, \pi_\mu\, \tau\, \iota_\mu\right). \tag{5.24}$$

Proof: The introduction of a pair $\iota_\mu\, \pi_\mu$ directly in front of the the τ is justified by Lemma 5.3.
Now take $v \in V(\lambda'_i)_\mu$. By inductive use of Lemma 5.2 we find:

$$\prod_{i=1}^{s} Y_{\alpha_1}(c_i)\, \tau.v \;=\; \sum_{k_1,\dots,k_s=-\infty}^{\infty} v_{k_1,,\dots,k_s}(c_1,\dots,c_s), \tag{5.25}$$

where each $v_{k_1,,\dots,k_s}(c_1,\dots,c_s) \in V(\lambda'_i)_{\mu+\sum_{i=1}^{s} k_i\, \alpha_i}$ depends polynomially on the $c_1,\dots,c_s$. Therefore we have

$$\begin{aligned} \pi_\mu \prod_{i=1}^{s} Y_{\alpha_1}(c_i)\, \tau.v \;&=\; v_{0,\dots,0}(c_1,\dots,c_s) \\ &=\; \prod_{i=1}^{s}(\pi_\mu\, Y_{\alpha_1}(c_i)\, \iota_\mu)\, \pi_\mu\, \tau\, \iota_\mu.v. \end{aligned} \tag{5.26}$$

The last equality follows by an easy induction on s using the fact that the α_i, $i \in \{1,\dots,s\}$ are linearly independent. $\heartsuit$.

With these preparations, we can compute the trace of an element $C(c_1,\dots,c_s) \in G\tau$ on a weight space $V(\lambda'_i)_\mu$:

Lemma 5.6 *Let $\mu \in \Lambda(R')$ be a weight, assume $V(\lambda'_i)_\mu \neq 0$.*
Then, for $\mu = \sum_{j=1}^{s} m_j\, \lambda_j$, we have:

$$tr_{V(\lambda'_i)_\mu}\,(\pi_\mu\, C(c_1,\dots,c_s)\, \iota_\mu) = \begin{cases} a_\mu c_1^{m_1}\dots c_s^{m_s}, & \textit{if } \mu \textit{ is dominant,} \\ 0, & \textit{otherwise,} \end{cases} \tag{5.27}$$

for some $a_\mu \in k$.

Proof: To calculate the trace, we only need, because of Lemma 5.5, to determine the polynomial in c_j which appears in the expression $\pi_\mu Y_{\alpha_j}(c_j).v$ for $v \in V(\lambda'_i)_\mu$. By use of Lemma 5.2 and the fact that $m_j = \check{\alpha}_j(\mu)$, we get:

$$\pi_\mu Y_{\alpha_j}(c_j).v = \pi_\mu \sum_{k=0}^{\infty} c_j^k v_k, \tag{5.28}$$

where $v_k \in V(\lambda'_i)_{\mu+(k-m_j)\alpha_j}$. Hence, we have

$$\pi_\mu Y_{\alpha_j}(c_j).v = c_j^{m_j} v_{m_j}, \tag{5.29}$$

if $m_j \geq 0$.
Otherwise, we have

$$\pi_\mu Y_{\alpha_j}(c_j).v = 0. \tag{5.30}$$

Proceeding by induction on s, we get:

$$\pi_\mu C(c_1, ..., c_s).v = \prod_{j=1}^{s} c_j^{m_j} w, \tag{5.31}$$

with $w \in V(\lambda)_\mu$, if μ is dominant. Otherwise, we have:

$$\pi_\mu C(c_1, ..., c_s).v = 0 \tag{5.32}$$

In the first case, w depends only linearly on v by Lemma 5.2. Thus, $w = Av$, for a linear endomorphism A of $V(\lambda'_i)_\mu$. With $a_\mu := trA$ we get our statement. $\heartsuit$.

We now verify property (ii) of the definition of Steinberg cross section for our map C:

Proposition 5.1 *The quotient map $\pi : G\tau \to T_0^\tau \tau/\widetilde{\mathcal{W}} \cong \mathbb{A}^s$ induces an isomorphism of algebraic varieties*

$$\pi|_{Im\, C} : Im\, C \xrightarrow{\sim} T_0^\tau \tau/\widetilde{\mathcal{W}}. \tag{5.33}$$

Proof: To prove the proposition we shall prove the following claim:
Claim: Let λ'_i a fundamental dominant weight of $\Lambda(R')$. Then:

$$\tilde{X}_1^{\lambda'_i}(C(c_1, ..., c_s)) = a_{\lambda'_i} c_i + P_i(c_1, ..., c_s), \tag{5.34}$$

where $a_{\lambda'_i} \neq 0$ and where each P_i is a polynomial in those c_j with $\lambda'_j \dot{\leq} \lambda'_i$ and $\lambda'_j \neq \lambda'_i$.
Claim $\Rightarrow$ proposition: If we label the λ'_i, $i \in \{1, ..., s\}$, such that $\lambda'_j \dot{\leq} \lambda'_i$ implies $j \leq i$. Then we see that $\pi \circ C$ is clearly invertible by the shape of

Equation 5.34 $\diamondsuit$.
Proof of Claim: By Lemma 5.6, we need to show the statement about the variables appearing nontrivially in the polynomials P_i and the property $a_{\lambda'_i} \neq 0$:
We have, using Lemma 5.6:

$$\tilde{X}_1^{\lambda'_i}(C(c_1, ..., c_s)) \;=\; \sum_{\substack{\mu \in \Lambda(R') \\ \mu \text{ dominant}}} tr_{V(\lambda'_i)_\mu}\,(\pi_\mu\, C(c_1, ..., c_s)\, \iota_\mu). \tag{5.35}$$

By the remark about representation theory, following the definition of the partial order $\dot{\geq}$, and by Lemma 5.6, we clearly see that P_i is a polynomial in those c_j with $\lambda'_j \;\dot{\leq}\; \lambda'_i$. Similarly to the proof of the antisymmetry in Lemma 5.4, we can see that for every dominant $\mu = \sum_{j=1}^{s} m_j\, \lambda'_j \in \Lambda(R')$ with $\mu \prec \lambda'_i$ and $\mu \neq \lambda'_i$ we must have $m_i = 0$.
To show the nontriviality of $a_{\lambda'_i}$, we proceed as follows:
Take an element $v \in V(\lambda'_i)_{\lambda'_i}\backslash\{0\}$. Since $dim\, V(\lambda'_i)_{\lambda'_i} = 1$, cf. e.g. [13], Proposition 31.2, p. 189, this is a basis of $V(\lambda'_i)_{\lambda'_i}$. By the construction of our representations of $\tilde{G}$ in Section 2.4 we have $\tau.v = v$. Using Lemma 5.2 and the fact, that all weights μ of $\Lambda(R')$ having nontrivial weight spaces $V(\lambda'_i)_\mu$ in $V(\lambda'_i)$ have to fulfill $\mu \prec \lambda'_i$, we get:

$$Y_{\alpha_j}(c_j)|_{V(\lambda'_i)_{\lambda'_i}} = id_{V(\lambda'_i)_{\lambda'_i}}, \tag{5.36}$$

for $j \neq i$ and $c_j \in k$.
Therefore, it remains to show:

$$Y_{\alpha_i}(c_i)v = a\, c_i\, v + \text{ terms having zero } V(\lambda'_i)_{\lambda'_i}\text{-part, for } a \neq 0. \tag{5.37}$$

To obtain this, consider the subgroup $G_{\alpha_i} :=< Im\, X_{\alpha_i},\, Im\, X_{-\alpha_i} >$ of G of type A_1. Now, $V(\lambda'_i)_{\lambda'_i} \oplus V(\lambda'_i)_{\lambda'_i - \alpha_i}$ is an irreducible G_{α_i}-module. (Note, that the weights $\lambda'_i - k\,\alpha_i$ only have trivial weight spaces in $V(\lambda'_i)$ for $k \geq 2$. Otherwise, $s_{\alpha_i}(\lambda'_i - k\,\alpha_i) = \lambda'_i + (k-1)\,\alpha_i$ would have a nontrivial weight space in $V(\lambda'_i)$ for $k \geq 2$, which is absurd.) Then, G_{α_i} even has to be isomorphic to $SL_2(k)$ and $V(\lambda'_i)_{\lambda'_i} \oplus V(\lambda'_i)_{\lambda'_i - \alpha_i}$ to its natural representation. Using this isomorphism, we can choose $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.Then, $X_{\alpha_i}(c_i) = \begin{pmatrix} 1 & c_i \\ 0 & 1 \end{pmatrix}$. Since all different choices for n_{α_i} differ only by an element in T, and since all elements of T act on the weight spaces by multiplication with a non-zero constant, we can assume w.l.o.g. n_{α_i} to be in G_{α_i} and to have the form $n_{\alpha_i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, under the above isomorphism. We easily calculate:

$$\begin{pmatrix} 1 & c_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -c_i \\ -1 \end{pmatrix}. \tag{5.38}$$

The result now follows, with $a = -1$. $\heartsuit$.

We now prove our main result:

Theorem 5.1 *The map C is a Steinberg cross section of the exterior component $G\tau$.*

Proof: Properties (i) and (ii) have already been proven in Lemma 5.1 and Proposition 5.1. (iii) will be a consequence of Theorem 5.2. ♡.

Using the properties of C proven so far, we can give a characterization of regular elements by means of the derivative of the quotient map π in the case where G' as appearing in the paragraph preceding Proposition 2.8 is simply connected:

Theorem 5.2 *Let $g\tau$ be in $G\tau$. Then the three following properties are equivalent:*
(i) $g\tau$ is regular
(ii) The derivative of $(d\pi)_{g\tau}$ is surjective
(iii) $g\tau$ is G-conjugate to an element of $Im\, C$.

Proof: (iii) ⇒ (ii): This follows directly from Proposition 5.1 and

$$T_{\pi(g\tau)}(T_0^\tau\tau/\widetilde{\mathcal{W}}) = Im\,(d\pi|_{Im\,C})_{g\tau} \subset Im\,(d\pi)_{g\tau}. \tag{5.39}$$

(ii) ⇒ (i): We will show that for irregular $g\tau \in G\tau$ the differentials $(d\tilde{X}_1^{\lambda'_i}|_{G\tau})_{g\tau}$, $i \in \{1,...,s\}$ considered as elements of $T^*_{g\tau}(G\tau)$ are linearly dependent.
This will be carried out in several steps:
1. W.l.o.g. we can assume that $g\tau$ is irregular and semisimple.
Proof: Since the set of semisimple irregular elements in $g\tau$ is dense in $(G\tau)_{irr}$, cf. Theorem 4.1, we only need to show, that the set of elements in $G\tau$, for which the differentials $(d\tilde{X}_1^{\lambda'_i}|_{G\tau})_{g\tau}$, $i \in \{1,...,s\}$ are linearly dependent, is closed. (This holds on every smooth algebraic variety and a given set of global one-forms by a similar reasoning. Anyhow, we will give a proof in our situation, where the tangent bundle is trivial.)
Since $G\tau$ is isomorphic to G, as an algebraic variety and, since the cotangent bundle of G is trivial, so is that of $G\tau$. Let E_j^*, $j \in \{1,...,n\}$ with $n = dim\, G$ be a basis of the vector space of left invariant one forms on $G\tau$, which is isomorphic to $\mathfrak{g}^*$, the dual of $Lie\, G$. It is a well known fact, that the $(E_j^*)_{g\tau}$, $j \in \{1,...,n\}$ form a basis of $T^*_{g\tau}(G\tau)$.
Hence, we can find polynomial functions $C_{i\,j} \in k[G\tau]$, $i \in \{1,...,s\}$, $j \in \{1,...,n\}$, such that

$$d\tilde{X}_1^{\lambda'_i}|_{G\tau} = \sum_{j=1}^{n} C_{i\,j}\, E_j^*. \tag{5.40}$$

Now the set where all the $d\tilde{X}_1^{\lambda'_i}|_{G\tau}$, $i \in \{1,...,s\}$ are linearly dependent is exactly the set where all the $s \times s$ minors of the matrix $(C_{i\,j})_{i\in\underline{s},\, j\in\underline{n}}$ vanish. This set is clearly closed. ♢.
By Proposition 2.3 and the fact that, conjugating with an element of G, is

an automorphism of $G\tau$, we can even assume that $g\tau \in T\tau$, where T is a maximal torus stabilized by τ. We shall denote $g\tau$ by $t\tau$.

For the next step, we introduce some notation: Let $B < G$ be a Borel subgroup of G, stabilized by τ and containing T and let U be its unipotent radical and U^- the radical of the opposite Borel subgroup B^-. Set $\mathfrak{t} := T_{t\tau}(T\tau) \subset T_{t\tau}(G\tau)$. Recall that $k[G\tau]^G$ denotes the ring of class functions on $G\tau$. Then we have:

2. For every $F \in k[G\tau]^G$ the following statement holds:

$$(dF)_{t\tau} = 0 \Leftrightarrow (dF|_{T\tau})_{t\tau} = (dF)_{t\tau}|_{\mathfrak{t}} = 0. \tag{5.41}$$

Proof: In this proof we use properties of the Bruhat decomposition, see e.g. [13], Section 28:

Let $U^-\,T\,U\,\tau \subset G\tau$ be the big Bruhat cell, which is open in $G\tau$. Then we have an isomorphism, cf. [13] Proposition 28.5, p. 174:

$$\begin{array}{rcl} \Psi : \mathbb{A}^{|R|} \times (k^*)^r & \to & U^-\,T\,U\,\tau \\ ((u_\alpha)_{\alpha\in R^+},\,(v_\alpha)_{\alpha\in R^+},\,t_1,...,t_r) & \mapsto & \displaystyle\prod_{\alpha\prec 0} X_\alpha(u_{-\alpha}) \prod_{i=1}^{r} \nu_i(t_i)\tau \prod_{\alpha\succ 0} X_\alpha(v_\alpha). \end{array}$$

Here the X_α, $\alpha \in R$, are the root groups and the ν_i, $i \in \{1,...,r\}$ are one-parameter multiplicative groups spanning T. (The ν_i, $i \in \{1,...,r\}$ have to be chosen in such a manner that they form a basis of $\chi(T)^*$, the dual of the character lattice of T.)

By means of this isomorphism, we can write every element $Y \in T_{t\tau}(G\tau) = T_{t\tau}(U^-\,T\,U\tau)$ in a unique way as a sum $Y = Y_1 + Y_2$, such that we get:

$$\begin{array}{rcl} (d\Psi^{-1})_{t\tau}(Y_1) & = & \displaystyle\sum_{\alpha\in R^+} c_{-\alpha}\,(\partial_{v_\alpha})_{\Psi^{-1}(t\tau)} + d_\alpha\,(\partial_{u_\alpha})_{\Psi^{-1}(t\tau)} \in T_{\Psi^{-1}(t\tau)}\mathbb{A}^{|R|} \\ (d\Psi^{-1})_{t\tau}(Y_2) & = & \displaystyle\sum_{i=1}^{r} b_i\,(\partial_{t_i})_{\Psi^{-1}(t\tau)} \in T_{\Psi^{-1}(t\tau)}(k^*)^r. \end{array}$$

with b_i, $c_{-\alpha}$, $d_\alpha \in k$, $i \in \{1,...,r\}$, $\alpha \in R^+$.

We have to show that the following equation holds for every $Y \in T_{t\tau}(G\tau)$:

$$(dF)_{t\tau}(Y) = (dF)_{t\tau}(Y_2). \tag{5.42}$$

Now, we transfer the adjoint T_0^τ-action on $U^-\,T\,U\,\tau$ to $\mathbb{A}^{|R|} \times (k^*)^r$ via Ψ. Then this action looks as follows:

$$t'.((u_\alpha)_{\alpha\in R^+},\,(v_\alpha)_{\alpha\in R^+},\,t_1,...,t_r) = ((\alpha(t')^{-1}\,u_\alpha)_{\alpha\in R^+},\,(\alpha(t')\,v_\alpha)_{\alpha\in R^+},\,t_1,...,t_r), \tag{5.43}$$

with $t' \in T_0^\tau$.

Next we consider $\Psi^*(F)$, which can be uniquely separated in a sum $\Psi^*(F) = P_1+P_2$ of two polynomials P_1 and P_2, the first being a polynomial containing only powers of $t_i^{\pm 1}$, $i \in \{1, ..., r\}$, while each monomial appearing in the second contains at least a factor in the u_α or v_α, $\alpha \in R^+$. Now we let T_0^τ act on $\Psi^*(F)$. Since $\Psi^*(F)$ and P_1 are T_0^τ-invariant, P_2 is also. Since T_0^τ is a regular subtorus of T, by Proposition 2.2, no root $\alpha \in R$ is identically trivial on T_0^τ. Therefore, every monomial of P_2 must contain at least two factors from the set $\{u_\alpha \text{ or } v_\alpha,\ \alpha \in R^+\}$ which implies

$$(d(\Psi^{-1})^* P_2)_{t\tau}(Y) = 0, \tag{5.44}$$

(because of $\Psi^{-1}(t\tau) \in 0 \times (k^*)^r \subset \mathbb{A}^{|R|} \times (k^*)^r$). Therefore, from

$$(dF)_{t\tau}(Y) = (d(\Psi^{-1})^* P_1)_{t\tau}(Y) = (d(\Psi^{-1})^* P_1)_{t\tau}(Y_2) = (dF)_{t\tau}(Y_2), \tag{5.45}$$

we get our statement. $\diamondsuit$.

We can apply this to our characters $\tilde{X}_1^{\lambda'_i}|_{G\tau} \in k[G\tau]^G$, $i \in \{1, ..., s\}$ to see that the $(d\tilde{X}_1^{\lambda'_i}|_{G\tau})_{t\tau}$, $i \in \{1, ..., s\}$ are linearly dependent for $t\tau \in T\tau$, iff their restrictions to $T\tau$, i.e. the $(d\tilde{X}_1^{\lambda'_i}|_{T\tau})_{t\tau}$, $i \in \{1, ..., s\}$ are.

By Lemma 2.7 and Lemma 2.8, we have that $T\tau /\!\!/ T = T\tau/T = T'$, a maximal torus of G'. Let us denote the quotient map $T\tau \to T'$ by p and by $T_{\alpha'}$ the kernel of $\alpha' \in R'$ on T. We define $T'_{\alpha'}$ similarly. By the method of the proof of Lemma 2.7 we have $p^{-1}(T'_{\alpha'}) = T_{\alpha'}\tau$.

3. Now the following assertion holds:

$$p((T\tau)_{irr}) = T'_{irr} \qquad \textit{and} \qquad p^{-1}(T'_{irr}) = (T\tau)_{irr}, \tag{5.46}$$

$(T\tau)_{irr}$ and T'_{irr} denoting the set of irregular elements in $T\tau$ respectively T'.

Proof: This follows directly from Corollary 3.3 and the characterization of irregular elements in T' as given e.g. in [38], Section 3.5, Proposition 3, p. 96. $\diamondsuit$.

For the next step, recall that $k[T\tau]^T = k[T']$. In order to cause no confusion, we denote an element $F \in k[T\tau]^T$ by $\hat{F}$, if we consider it as regular function on T'. With this notation we clearly have $p^*(\hat{F}) = F$.

4. For every $F \in k[T\tau]^T$ we have:

$$(dF)_{t\tau} = 0 \ \Leftrightarrow \ (d\hat{F})_{p(t\tau)} = 0. \tag{5.47}$$

Proof: By Lemma 2.7 and [2], Chapter II, Proposition 6.5 p. 97, the quotient map p is a fibre bundle. Therefore, $(dp)_{t\tau}$ is surjective for every $t\tau \in T\tau$. The statement now follows from the chain rule

$$(dF)_{t\tau} = (d\hat{F})_{p(t\tau)} \circ (dp)_{t\tau}. \tag{5.48}$$

$\diamondsuit$.

We want to apply step 4 to $F = \tilde{X}_1^{\lambda'_i}|_{T\tau}$, $i \in \{1, ..., s\}$. Recall, that for $F = \tilde{X}_1^{\lambda'_i}|_{T\tau}$ we have $\hat{F} = X'^{\lambda'_i}|_{T'}$, cf. Proposition 2.8, $X'^{\lambda'_i}$ being the

character of the irreducible representation of G' with highest weight λ_i'.
Hence, the $(d\tilde{X}_1^{\lambda_i'}|_{G\tau})_{t\tau}$, $i \in \{1, ..., s\}$ are linearly dependent for $t\tau \in T\tau$, iff the $(dX'^{\lambda_i'}|_{T'})_{p(t\tau)}$, $i \in \{1, ..., s\}$ are for $p(t\tau) \in T'$.
5. The $(dX'^{\lambda_i'}|_{T'})_{t'}$, $i \in \{1, ..., s\}$ are linearly dependent for $t' \in T'$, if t' is irregular.
Proof: This is proven in [38], Section 3.8, Lemma p. 125, together with the identity of Weyl mentioned on [38], p. 127. (For a proof of the Weyl identity, cf. e.g. [14], Lemmas 24.1A and 24.1B, p. 136.) $\diamondsuit$.
Combining steps 3 and 5 yields (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iii): Let $g\tau \in G\tau$ be regular. Then, we can find $h\tau \in Im\, C$ with $\pi(h\tau) = \pi(g\tau)$, by Proposition 5.1. Since we already know that $h\tau$ is regular, the statement follows from Corollary 3.1. $\heartsuit$.

Now we can prove the reducedness of the fibres as promised in the remark following Theorem 3.1.

Proposition 5.2 *The quotient map $\pi : G\tau \to G\tau /\!/ G$ is flat and its fibres are reduced and normal.*

Proof: The flatness property of π was already proven in Corollary 3.2.
1. Reducedness: To show the reducedness of the schematic fibres, it suffices to show that they are regular in codimension zero and Cohen-Macaulay, by [21], Section 17.I, p. 125:
Take $y \in G\tau /\!/ G$. Then we have, for the corresponding schematic fibre:

$$\pi^{-1}(y) = V(\tilde{X}_1^{\lambda_1'}|_{G\tau} - y_1, ..., \tilde{X}_1^{\lambda_s'}|_{G\tau} - y_s). \tag{5.49}$$

Hence, $\pi^{-1}(y)$ is a complete intersection. By [11], Chapter II, Proposition 8.23, p. 186, it follows that $\pi^{-1}(y)$ is Cohen-Macaulay. If we choose $x \in \pi^{-1}(y)$ to be regular we have, by Theorem 5.2, that the $d(\tilde{X}_1^{\lambda_j'}|_{G\tau})_x$, $j \in \{1, ..., s\}$ are linearly independent. Hence $\mathcal{O}_{\pi^{-1}(y),\, x}$, the local ring of the schematic fibre, is a regular local ring. By Corollary 3.1, we know that the irregular elements in $\pi^{-1}(y)$ have codimension two, which implies that $\pi^{-1}(y)$ is regular in codimension zero. So we see that the schematic fibre is already reduced.
2. Normality: Since the set of regular elements of $\pi^{-1}(y)$ is open dense in $\pi^{-1}(y)$ with a complement of codimension two, see Corollary 3.1, and since these elements are nonsingular in $\pi^{-1}(y)$ (note that they are all conjugate to one another), we see that $\pi^{-1}(y)$ is regular in codimension one. By [11], Chapter II, Proposition 8.23, p. 186, the normality of $\pi^{-1}(y)$ follows. $\heartsuit$.

As a consequence of Theorem 5.2, we get the following description of the singular locus of the reduced fibres of the quotient map $\pi : G\tau \mapsto G\tau /\!/ G$. Here G' does not need to be simply connected. This is a sharper version of Lemma 3.1:

Corollary 5.2 *If $char(k)$ does not divide the order of the fundamental group of G, then for every $t\tau \in T_0^\tau \tau$, the nonsingular points of $\pi^{-1}(\pi(t\tau))_{red}$ are exactly the regular ones.*

Proof: By Theorem 5.2, the statement is true in case, that G is simply connected. If G is not simply connected, let us denote its universal cover by $\hat{G}$. As in Section 2.5, we denote the the kernel of the quotient map $p : \hat{G} \to G$ by C_G. Recall that we have the following commutative diagram:

$$\begin{tikzcd} \hat{G}\tau \arrow[r, "\hat{p}"] \arrow[d, "\hat{\pi}"] & G\tau \arrow[d, "\pi"] \\ \mathbb{A}^s \cong \hat{G}\tau /\!/ \hat{G} \arrow[r, "\tilde{p}"] & G\tau /\!/ G, \end{tikzcd} \tag{5.50}$$

where $\hat{p} : \hat{G}\tau \to G\tau$ is the map induced by p on the exterior component, given by $g\tau \mapsto p(g)\tau$ and $\tilde{p}$ is induced by $\hat{p}$ on the quotient. Apparently $\hat{p}$ maps the fibres of $\hat{\pi}$ onto those of π.
By our assumption on the characteristic of k, $\hat{p}$ is separable and hence etale because of the smoothness and dimension equality of G and $\hat{G}$, see [11], Chapter III, Proposition 10.4, p. 270. Therefore, the restriction of $\hat{p}$ to each fibre of $\hat{\pi}$ is etal. Hence, $\hat{p}$ maps the (non-)singular points of each fibre of $\hat{\pi}$ exactly to the (non-)singular points of the corresponding fibre of π.
Furthermore, we have, by [37], statement 4.5, p. 37, that $dim\, C_{\hat{G}}(\hat{g}\tau) = dim\, C_G(p(\hat{g})\tau)$. (Note that $C_G = ker\, p$ is finite). Now the result follows. ♡.

Chapter 6

Subregular Singularities

In this chapter, we shall establish a connection to singularity theory. To accomplish this, we start with a construction of a simultaneous resolution of all fibres of the quotient map $\pi : G\tau \to T_0^\tau \tau/\widetilde{\mathcal{W}}$, the so called "Grothendieck's simultaneous resolution". Then we construct semiuniversal deformations of subregular singularities in the fibres over those points in the quotient $T_0^\tau \tau/\widetilde{\mathcal{W}}$, of which the centralizer of a semisimple element in the corresponding fibre is simple of maximal possible rank. All the results of this chapter will be valid in the case where G is simple and τ acts without fixed points on the fundamental group of G and where G', as defined in Section 2.4, is simply connected. Furthermore, we need some restrictions on the characteristic of k.

6.1 Resolution of Individual Fibres

As a first preparatory result, we construct the resolution of a single fibre. First we state a result by Springer, found in [30], Theorem 1.4, or [27], Theorem 4.1, p. 43:

Proposition 6.1 *Let G be a reductive group, such that $char(k)$ does not divide the order of the fundamental group of G, B a Borel subgroup with unipotent radical U, and let V be the unipotent variety of G. Then the morphism*

$$\begin{array}{rcl} p : G \times^B U & \to & V \\ g * u & \mapsto & g\,u\,g^{-1} \end{array} \tag{6.1}$$

is a G-equivariant resolution of singularities of V.

Proof: See [27], p. 43. ♡.

Furthermore, we prove the following result about base change of G-equivariant resolutions:

Proposition 6.2 *Let G be an algebraic group, $H < G$ a closed subgroup and $\nu : \tilde{X} \to X$ an H equivariant resolution of singularities. Then:*

$$G \times^H \nu : G \times^H \tilde{X} \to G \times^H X \tag{6.2}$$

is, again, a resolution of singularities.

Proof: We have to show that $G \times^H \nu$ is proper and that it induces an isomorphism $\nu^{-1}(X_{reg}) \to X_{reg}$. Note that $G \times^H \tilde{X}$ is clearly smooth (it is an associated bundle with smooth fibre):
1. *Properness:* We have to show that $f := G \times^H \nu$ is universally closed. To do this, consider a morphism $\rho : Y \to G \times^H X$ and the corresponding Cartesian square:

$$\begin{tikzcd}
Y' \arrow[r] \arrow[d, "\hat{f}"'] & G \times^H \tilde{X} \arrow[d, "f"] \\
Y \arrow[r, "\rho"] & G \times^H X
\end{tikzcd} , \tag{6.3}$$

where $Y' := Y \times_{G \times^H X} (G \times^H \tilde{X})$ is the fibre product. Our aim is to show that $\hat{f}$ is closed. We consider the following commutative cube:

$$\begin{tikzcd}
& Z' \arrow[rr] \arrow[dl, "\tilde{\pi}'"'] \arrow[dd, "\tilde{f}"] & & G \times \tilde{X} \arrow[dl, "\tilde{\pi}"'] \arrow[dd, "id \times \nu"] \\
Y' \arrow[rr] \arrow[dd, "\hat{f}"'] & & G \times^H \tilde{X} \arrow[dd, "f"] & \\
& Z \arrow[rr] \arrow[dl, "\pi'"'] & & G \times X \arrow[dl, "\pi"] \\
Y \arrow[rr, "\rho"'] & & G \times^H X &
\end{tikzcd} , \tag{6.4}$$

where $Z' := Y' \times_{G \times^H \tilde{X}} (G \times \tilde{X})$ and $Z := Y \times_{G \times^H X} (G \times X)$, yielding Cartesian top and bottom faces. Here $\tilde{f}$ is given by $\tilde{f}(y, g * \tilde{x}, (g', \tilde{x}')) = (y, (g', \nu(\tilde{x}')))$, and π and $\tilde{\pi}$ denote quotient maps by the H-action. (Note

that $\tilde{f}$ is well defined, i.e. $Im\,\tilde{f} \subset Z$, because of $\rho(y) = f(g * \tilde{x}) = f(\tilde{\pi}(g', \tilde{x}')) = f(g' * \tilde{x}') = g' * \nu(\tilde{x}') = \pi(g', \nu(\tilde{x}')).$)
Next we define an H-action on Z' and Z in the following way:

$$\begin{aligned} h.(y,\, g * \tilde{x},\, (g',\, \tilde{x}')) &:= (y,\, g * \tilde{x},\, (g'\, h^{-1},\, h\, \tilde{x}')) && (6.5)\\ h.(y,\, (g',\, x)) &:= (y,\, (g'\, h^{-1},\, h\, x)), && (6.6)\end{aligned}$$

with $h \in H$, $(y,\, g * \tilde{x},\, (g',\, \tilde{x}')) \in Z'$ and $(y,\, (g',\, x)) \in Z$. Then $\tilde{f}$ becomes H-equivariant.
Now we define $\tilde{Z} := Z \times_{G \times X} (G \times \tilde{X})$, which becomes an H-variety with respect to the following action of H:

$$h.(y,\, (g,\, x),\, (g',\, \tilde{x}')) = (y,\, (g\, h^{-1},\, h\, x),\, (g'\, h^{-1},\, h\, \tilde{x}')), \qquad (6.7)$$

for all $h \in H$ and $(y,\, (g,\, x),\, (g',\, \tilde{x}')) \in \tilde{Z}$. Setting

$$\begin{aligned} \Phi : \tilde{Z} &\rightarrow Z' \\ \Phi(y,\, (g,\, x),\, (g',\, \tilde{x}')) &= (y,\, g' * \tilde{x}',\, (g',\, \tilde{x}')), && (6.8)\end{aligned}$$

for all $(y,\, (g,\, x),\, (g',\, \tilde{x}')) \in \tilde{Z}$, and

$$\begin{aligned} \Psi : Z' &\rightarrow \tilde{Z} \\ \Psi(y,\, g * \tilde{x},\, (g',\, \tilde{x}')) &= (y,\, (g',\, \nu(\tilde{x}')),\, (g',\, \tilde{x}')), && (6.9)\end{aligned}$$

for all $(y,\, g * \tilde{x},\, (g',\, \tilde{x}')) \in Z'$, we see that $\tilde{Z}$ and Z' are isomorphic as H-varieties. This implies that the back side of the cube is also Cartesian. (We believe this to be true for all commutative cubes with Cartesian top, bottom and front side. We gave this proof, because we have not found a reference.)
We are now interested in the categorical quotients $Z/\!\!/H$ and $Z'/\!\!/H$. Recall that we can consider

$$\begin{aligned} Z' &\subset Y' \times (G \times \tilde{X}) && (6.10)\\ Z &\subset Y \times (G \times X) && (6.11)\end{aligned}$$

as closed H-stable subsets, yielding that $Z/\!\!/H$ and $Z'/\!\!/H$ are closed subsets of $(Y \times (G \times X))/\!\!/H$ and $(Y' \times (G \times \tilde{X}))/\!\!/H$ respectively. Since H operates on the varieties of the right hand sides of Equations 6.10 and 6.11 only by acting on the second factor and, since the corresponding quotient of the respective second factor is geometrical, by [27], Section 3.7, p. 25, see also [22], remark, p. 8, it follows that also $(Y \times (G \times X))/\!\!/H$ and $(Y' \times (G \times \tilde{X}))/\!\!/H$ are geometrical quotients. They have the following explicit form:

$$\begin{aligned} (Y \times (G \times X))/H &= Y \times (G \times^H X) && (6.12)\\ (Y' \times (G \times \tilde{X}))/H &= Y' \times (G \times^H \tilde{X}), && (6.13)\end{aligned}$$

yielding:

$$Z/H = Y \times_{G\times^H X} (G \times^H X) \cong Y \tag{6.14}$$
$$Z'/H = Y' \times_{G\times^H \tilde{X}} (G \times^H \tilde{X}) \cong Y'. \tag{6.15}$$

(Observe that the isomorphism is just given by the embedding of the first factor as the graph of ρ, respectively the map induced by ρ into the product.) Therefore, we have identified $\tilde{\pi}'$ and π' as quotient morphisms under the H-actions.
By our hypothesis, ν is a proper morphism. Since properness is stable under base change by [11], Chapter II, Corollary 4.8, p. 102, we have that $id \times \nu$ is also proper. (We apply this property to the base change $pr_2 : G \times X \to X$ and use the isomorphism $(G \times X) \times_X \tilde{X} \cong G \times \tilde{X}$.) Applying the base change property again and exploiting the fact, that the back side of the cube is Cartesian, we get that $\tilde{f}$ is proper. Now choose a closed subset $A \subset Y'$. We have

$$\hat{f}(A) = \pi'(\tilde{f}(\tilde{\pi}'^{-1}(A))). \tag{6.16}$$

Now $\tilde{\pi}'^{-1}(A) \subset Z'$ is closed and H-stable. By the H-equivariance and properness of $\tilde{f}$ we get that $\tilde{f}(\tilde{\pi}'^{-1}(A)) \subset Z$ is closed and H-stable. Then the closedness of $\hat{f}(A)$ follows from the fact, that π' is a quotient map.
2. Isomorphism outside the singularity: Analogously to the proof of Lemma 3.1, we have:

$$(G \times^H X)_{sing} = G \times^H (X_{sing}). \tag{6.17}$$

Now the restriction $\nu|_{\tilde{X}\backslash\nu^{-1}(X_{sing})}$ of ν to $\tilde{X}\backslash\nu^{-1}(X_{sing})$ is an isomorphism:

$$\nu|_{\tilde{X}\backslash\nu^{-1}(X_{sing})} : \tilde{X}\backslash\nu^{-1}(X_{sing}) \xrightarrow{\sim} X\backslash X_{sing}. \tag{6.18}$$

From this, we apparently get an isomorphism:

$$(G \times^H \nu)|_{G\times^H\tilde{X}\backslash G\times^H(\nu^{-1}(X_{sing}))} = G \times^H \left(\nu|_{\tilde{X}\backslash(\nu^{-1}(X_{sing}))}\right), \tag{6.19}$$

whose inverse is just given by $G \times^H (\nu|_{\tilde{X}\backslash(\nu^{-1}(X_{sing}))})^{-1}$. ♡.

Now let G be a semisimple affine algebraic group and τ an exterior automorphism of the Dynkin diagram of G. We fix $t\tau \in T_0^\tau\tau$. Let us denote by $V(t\tau)$ the unipotent variety of $C_G(t\tau)$, by $B(t\tau)$ a Borel subgroup of $C_G(t\tau)$ and by $U(t\tau)$ its unipotent radical. Now, we assume that τ acts without fixed points on the fundamental group of G. By use of [37], Corollary 9.7, p. 61, this guarantees $C_G(t\tau)$ to be connected for $t\tau \in T_0^\tau\tau$.
Combining the two propositions above, we get, together with Theorem 3.1 and Corollary 5.2:

Corollary 6.1 *Let*

$$p_{t\tau} : C_G(t\tau)_0 \times^{B(t\tau)} U(t\tau) \to V(t\tau) \tag{6.20}$$

be the map induced by the conjugation map and assume that char(k) does not divide the order of the fundamental group of G and of $C_G(t\tau)$. Furthermore, assume that τ acts without fixed points on the fundamental group of G. Then

$$G \times^{C_G(t\tau)} p_{t\tau} : G \times^{B(t\tau)} U(t\tau) \to G \times^{C_G(t\tau)} V(t\tau) \tag{6.21}$$

is a resolution of the reduced fibre $\pi^{-1}(\pi(t\tau))_{red}$ of the quotient $\pi : G\tau \to G\tau /\!\!/ G$.

6.2 Grothendieck's Simultaneous Resolution

In this section, we want to construct a simultaneous resolution of the quotient map $\pi : G\tau \to G\tau/\!\!/G$. Therefore, we choose a Borel subgroup B of G and a maximal torus T thereof, which are both stabilized by τ. Here, we assume that τ acts without fixed points on the fundamental group of G. By the remark in the paragraph preceding Corollary 6.1, this guarantees $C_G(t\tau)$ to be connected for $t\tau \in T_0^\tau \tau$.
First, recall the definition of a simultaneous resolution, cf. [27], p. 45:

Definition 6.1 *A simultaneous resolution of a morphism $\pi : X \to Y$ of reduced varieties consists of a commutative diagram of morphisms of reduced varieties*

$$\begin{tikzcd} Z \arrow[r, "\Phi"] \arrow[d, "\Theta"'] & X \arrow[d, "\pi"] \\ T \arrow[r, "\Psi"] & Y, \end{tikzcd} \tag{6.22}$$

such that the following properties hold:
(i) Θ is smooth,
(ii) Ψ is finite and surjective,
(iii) Φ is proper,
(iv) for all $t \in T$, the morphism

$$\Phi_t : \Theta^{-1}(t) \to (\pi^{-1}(\Psi(t)))_{red} \tag{6.23}$$

is a resolution of singularities.

Before constructing the simultaneous resolution in our situation we need some preparations:
Let B act on $B\tau$ and T on $T\tau$ by conjugation. Then we have:

Lemma 6.1 *The embedding $T\tau \hookrightarrow B\tau$ induces an isomorphism $T\tau/\!\!/T \cong B\tau/\!\!/B$.*

Proof: We have to show, that the restriction map of the invariant rings

$$\begin{aligned} \Psi : k[B\tau]^B &\to k[T\tau]^T \\ f &\mapsto f|_{T\tau} \end{aligned} \tag{6.24}$$

is an isomorphism of rings:
1. Injectivity: The injectivity of Ψ follows from the following claim:
Claim: We have:

$$\bigcup_{b\in B} b\, T_0^\tau \tau\, b^{-1} \subset B\tau \tag{6.25}$$

is a dense subset.
Proof of claim: To prove this, we use Proposition 2.6.
Recall that, in its proof, we defined a map Φ:

$$\begin{aligned} \Phi : G/C_0 \times C_0 &\to G\tau \\ (gC_0, h) &\mapsto g\, h\tau\, g^{-1}. \end{aligned} \tag{6.26}$$

Now we denote by Θ its restriction to $B/T_0^\tau \times T_0^\tau \tau$:

$$\begin{aligned} \Theta : B/T_0^\tau \times T_0^\tau \tau &\to G\tau \\ (b\, T_0^\tau,\, t\tau) &\mapsto b\, t\tau\, b^{-1}. \end{aligned} \tag{6.27}$$

Clearly, $Im\, \Theta \subset B\tau$. Now, take $\Phi(e\, T_0^\tau,\, t\tau) \in B\tau$, where $t\tau$ is a topologically cyclic element of the Cartan subgroup $< T_0^\tau,\, \tau >$. By the proof of Proposition 2.6, we know, that $\Phi^{-1}(\Phi(e\, T_0^\tau,\, t\tau))$ is finite. Then, $\Theta^{-1}(\Phi(e\, T_0^\tau,\, t\tau))$ also has to be finite. The claim now follows from the dimension formula.$\diamondsuit$.
The injectivity of Ψ can now be proven as follows: Take $f \in k[B\tau]^B$ with $f|_{T\tau} = 0$. This implies:

$$f|_{\bigcup_{b\in B} b\, T_0^\tau \tau\, b^{-1}} = 0. \tag{6.28}$$

By the claim, f vanishes on a dense subset hence $f = 0$.
2. Surjectivity: Recall that $B \cong T \ltimes U$, cf. [13], Theorem 19.3, p. 123. Therefore, we have $B\tau \cong T\tau \times U$, as algebraic varieties, because U is τ-stable. Denote the projection on the first factor by π. Choose $f \in k[T\tau]^T$. We will show that $\pi^*(f) \in k[B\tau]^B$ and that $\pi^*(f)|_{T\tau} = f$. This yields the surjectivity of Ψ:
The second property, i.e. $\pi^*(f)|_{T\tau} = f$, is apparent by construction.
Therefore, we have to show the B-invariance of $\pi^*(f)$, for $f \in k[T\tau]^T$:
Let $b = t\, u \in B$ and $x = t'\tau\, u' \in B\tau$, with $t,\, t' \in T$ and $u,\, u' \in U$. We have to show that $\pi^*(f)(x) = \pi^*(f)(b\, x\, b^{-1})$. Now, a simple calculation yields, that $b\, x\, b^{-1} = t\, t'\, \tau\, t^{-1}\, \tau^{-1}\, \tau\, \hat{u}$, for a suitable $\hat{u} \in U$. Then we conclude:

$$\begin{aligned} \pi^*(f)(b\, x\, b^{-1}) = f(\pi(b\, x\, b^{-1})) = f(\pi(t\, t'\, \tau\, t^{-1}\, \tau^{-1}\, \tau\, \hat{u})) = \\ f(t\, t'\, \tau\, t^{-1}) = f(t'\, \tau) = f(\pi(x)) = \pi^*(f)(x). \end{aligned} \tag{6.29}$$

This proves the B-invariance of $\pi^*(f)$. $\heartsuit$.

Remark: The condition on the action of τ was not used to derive this lemma.

As a consequence we have:

Corollary 6.2 *With the notation of Section 2.3, the following holds:*

$$B\tau /\!\!/ B \cong T\tau/T \cong T_0^\tau/(T/T_0^\tau)^\tau \cong T'. \tag{6.30}$$

Proof: The first part follows from Lemma 6.1, and the other isomorphisms are given in Lemmas 2.7 and 2.8. $\heartsuit$.

Next, recall that for $t\tau \in T_0^\tau\tau$ the group $B(t\tau) := B \cap C_G(t\tau)_0$ is a Borel subgroup of $C_G(t\tau)_0$, by Lemma 4.2, and that $U(t\tau) = U \cap C_G(t\tau)_0$ is its unipotent radical. Furthermore, denote the quotient map $B\tau \to B\tau/\!\!/B$ by β. The following two lemmas summarize the properties of β:

Lemma 6.2 *β is smooth and $\beta = p \circ \pi$, where p is the quotient map $p : T\tau \to T'$, as in Lemma 2.8, and $\pi : B\tau \to T\tau$ is the projection of the first factor as defined in the proof of the surjectivity of Lemma 6.1. (Note that here we identify T with $T\tau$ via right multiplication with τ.)*

Proof: 1. $\beta = p \circ \pi$: First, observe that for $f \in k[B\tau]^B$ and $x \in B\tau$ we have $f(x) = f(x_s)$ analogously to Lemma 2.10. Therefore, we have $\beta(x) = \beta(x_s)$, for $x \in B\tau$.

Now take $x = b\tau = t\tau\, u \in B\tau$ with $t \in T$ and $u \in U$. By use of Lemma 4.1, x_s is conjugate to $t\tau$, such that $\beta(x) = p(t\tau)$, where we have used Lemma 6.1. Now, the statement is immediate.

2. Smoothness of β: Since p corresponds, under the identification of $T\tau$ with T, to the quotient map of T modulo a subtorus, see Lemma 2.7, we have that p is smooth. Furthermore, π is smooth because it is a projection map. The assertion now follows from the stability of smoothness under composition, see [11], Chapter III Proposition 10.1, p. 268. $\heartsuit$.

Remark: For this lemma we also do not need that τ acts on the fundamental group of G without fixed points.

Lemma 6.3 *In the situation described above, we have a B-equivariant isomorphism:*

$$\beta^{-1}(\beta(t\tau)) \cong B \times^{B(t\tau)} U(t\tau). \tag{6.31}$$

Proof: First we show that the above identification holds for the reduced fibre $\beta^{-1}(\beta(t\tau))_{red}$:

We already know, by the proof of the previous lemma, that for $f \in k[B\tau]^B$ and $x \in B\tau$ we have $f(x) = f(x_s)$.

For $x \in \beta^{-1}(\beta(t\tau))$ we can, by Lemma 4.1 and Lemma 2.5, assume that

$x_s \in T_0^\tau \tau$, up to conjugacy in B. We have $x_s \in \beta^{-1}(\beta(t\tau))$ and, therefore, x_s and $t\tau$ have to be conjugate under $(T/T_0^\tau)^\tau$, by Proposition 2.5. (Note that no representative of an element of $\mathcal{W}^\tau$ is contained in B.) Using our assumption on the action of τ on the fundamental group of G, we have that $C_G(t\tau)$ is connected and, therefore, that $C_B(t\tau) = B(t\tau)$, which follows from [37], Corollary 9.7, p. 61. By [2], Proposition 9.1, p. 128, we can identify the B-orbit of $t\tau$ with $B/B(t\tau)$. Using this identification, we define a map:

$$\begin{aligned} \hat{\Psi} : \beta^{-1}(\beta(t\tau)) &\rightarrow B/B(t\tau) && (6.32) \\ x &\mapsto x_s. && (6.33) \end{aligned}$$

If we consider the characteristic polynomial in a given, faithful representation of $\tilde{G}$, we see that its coefficients have to be constant for all $x \in \beta^{-1}(\beta(t\tau))$. (Recall that the semisimple part of a matrix X can be given as a polynomial in X only involving the coefficients of the characteristic polynomial, cf. [14], Proposition 4.2, p. 17.) We get that $\hat{\Psi}$ is an algebraic morphism. $\hat{\Psi}$ is clearly B-equivariant. Furthermore, we have:

$$\hat{\Psi}^{-1}(eB(t\tau)) = \{x \in \beta^{-1}(\beta(t\tau)),\ x_s = t\tau\} = U(t\tau). \qquad (6.34)$$

The statement now follows from [27], Section 3.7, Lemma 4, p. 26. Therefore we have:

$$\beta^{-1}(\beta(t\tau))_{red} \cong B \times^{B(t\tau)} U(t\tau). \qquad (6.35)$$

Next, we want to show that $\beta^{-1}(\beta(t\tau)$ is reduced:
Since β is smooth, it is flat and its schematic fibres are regular by [11], Chapter III, Theorem 10.2, p. 269. The regularity of the fibres of β implies that they are also Cohen-Macaulay, see [11], Chapter II, Theorem 8.21A, p. 184. Now, the reducedness of the fibres of β follows from [21], Section 17.I, p. 125. ♡.

To construct the simultaneous resolution of $\pi : G\tau \rightarrow G\tau /\!\!/ G$, we consider the space $G \times^B B\tau$ which, clearly, is a smooth variety. We define a map:

$$\begin{aligned} \Phi : G \times^B B\tau &\rightarrow G\tau && (6.36) \\ g * b\tau &\mapsto g\, b\tau\, b^{-1} && (6.37) \end{aligned}$$

By Lemma 4.1 and its proof, this map is surjective and proper.
We can construct a second map $\Theta : G \times^B B\tau \rightarrow B\tau /\!\!/ B \cong T'$ in the following way:
If we consider $B\tau /\!\!/ B$ as a B-variety with trivial B-action, we can define the following map:

$$\begin{array}{rcccccc} G \times^B \beta : & G \times^B B\tau & \longrightarrow & G \times^B (B\tau /\!\!/ B) & \xrightarrow{\sim} & G/B \times B\tau /\!\!/ B \\ & g * b\tau & \mapsto & g * [b\tau] & \mapsto & (gB, [b\tau]), \end{array} \qquad (6.38)$$

where $[b\tau]$ denotes a class in $B\tau /\!\!/ B$. (Note that the second isomorphism follows from [27], Section 3.7, Lemma 1, p. 25 with the trivial G-action on $B\tau /\!\!/ B$.) Then we set

$$\Theta := pr_2 \circ G \times^B \beta : G \times^B B\tau \to B\tau /\!\!/ B \cong T'. \tag{6.39}$$

We have the following property:

Lemma 6.4 *Θ is a smooth morphism.*

Proof: Since pr_2 is smooth as projection map and by [11], Chapter III, Proposition 10.1, p. 268, we only need to show, that $G \times^B \beta$ is smooth. Furthermore, since $B\tau$ and $G/B \times T'$ are smooth varieties, we only need to show, that $(d(G \times^B \beta))_x$ is surjective for every $x \in G \times^B B\tau$, by [11], Chapter III, Proposition 10.4, p. 270. Recall that, by [27], Section 3.7, the associated bundle $\nu : G \times^B B\tau \to G/B$ is locally trivial in the etale topology. So for every $y \in G/B$ we can find an open set $U \subset G/B$ and an etale morphism $\pi : U' \to U$, such that the following identity holds:

$$\pi^*(G \times^B B\tau) \cong U' \times B\tau. \tag{6.40}$$

For $x \in G \times^B B\tau$ we put $y = \nu(x)$.
Now we have a commutative diagram

$$\begin{array}{ccc} U' \times B\tau & \xrightarrow{\pi'} & \nu^{-1}(U) \\ \downarrow{\scriptstyle id\times\beta} & & \downarrow{\scriptstyle (G\times^B\beta)|_{\nu^{-1}(U)}} \\ U' \times T' & \xrightarrow{\pi\times id} & U \times T', \end{array} \tag{6.41}$$

where π' is induced from π by base change and is, therefore, again etale , see [11], Chapter III, Proposition 10.1, p. 268. For $x' \in \pi'^{-1}(x)$ we get an induced diagram:

$$\begin{array}{ccc} T_{x'}(U' \times B\tau) & \xrightarrow{(d\pi')_{x'}} & T_x(G \times^B B\tau) \\ \downarrow{\scriptstyle (d(id\times\beta))_{x'}} & & \downarrow{\scriptstyle (d(G\times^B\beta))_x} \\ T_{(id\times\beta)(x')}(U' \times T') & \xrightarrow{(d(\pi\times id)_{(id\times\beta)(x')})} & T_{G\times^B\beta(x)}(G/B \times T'), \end{array} \tag{6.42}$$

where the horizontal arrows are isomorphisms, because the corresponding morphisms are etale. The assertion now follows from the surjectivity of $(d(id \times \beta))_{x'}$, which follows from the smoothness of β. $\heartsuit$.

As a last preparation for our simultaneous resolution, we need the following lemma for quotients by regular finite group actions on varieties, which is standard in geometric invariant theory:

Lemma 6.5 *Let X be an affine variety, $\mathcal{W}$ a finite group, which acts regularly on X. Then the quotient map $\pi X \to X/\mathcal{W}$ is a finite morphism.*

Proof: Let $\pi^* : k[X]^{\mathcal{W}} \hookrightarrow k[X]$ be the inclusion map. We will show, that $k[X]$ is a finite $k[X]^{\mathcal{W}}$-module. Since both rings are coordinate rings of affine varieties, we clearly have that $k[X]$ is a finitely generated $k[X]^{\mathcal{W}}$-algebra. Therefore, we only need to show that every element of $k[X]$ is integral over $k[X]^{\mathcal{W}}$. If $f \in k[X]$, we choose

$$P(T) = \prod_{w \in \mathcal{W}} (T - w(f)) \in \left(k[X]^{\mathcal{W}}\right)[T]. \tag{6.43}$$

P is a monic and $P(f) = 0$. ♡.

With these preparations we can construct the simultaneous resolution:

Theorem 6.1 *Let G be as above, i.e. τ acts without fixed points on the fundamental group of G. Furthermore, assume that $char(k) > rk\,G + 1$. Then the following diagram gives a simultaneous resolution of $\pi : G\tau \to G\tau /\!\!/ G$:*

$$\begin{array}{ccc} G \times^B B\tau & \xrightarrow{\Phi} & G\tau \\ \downarrow{\scriptstyle\Theta} & & \downarrow{\scriptstyle\pi} \\ T' & \xrightarrow{\Psi} & T'/\mathcal{W}^\tau, \end{array} \tag{6.44}$$

where Φ and Θ are defined in the Equations 6.37 and 6.39, and where Ψ is the quotient map of T' by $\mathcal{W}^\tau$.

Proof: Recall that, by the paragraph preceding Corollary 6.1, the assumption on the fundamental group of G implies that all centralizers $C_G(t\tau)$ of semisimple elements $t\tau \in T_0^\tau \tau$ are connected.
The finiteness of Ψ was shown in Lemma 6.5. We have proven the smoothness of Θ in Lemma 6.4. The proof of Lemma 4.1 yields the properness of Φ. We therefore still have to show that Φ gives, fibrewise, a resolution of the reduced fibres and that the diagram is commutative. In this proof, we will denote the class of $t\tau \in T\tau$ in T' by $[t\tau]$.
1. Commutativity of the diagram: Take $x = g * t\tau u \in G \times^B B\tau$. Then we have:

$$\pi(\Phi(x)) = \pi(g\,t\tau u\,g^{-1}) = \pi(t\tau u) = \pi(t\tau), \tag{6.45}$$

where we have used Lemma 2.10 and that $(t\tau u)_s$ is U conjugate to $t\tau$ by Lemma 4.1 in the last step. On the other hand, we calculate:

$$\Psi(\Theta(x)) = \Psi(pr_2(gB, [t\tau])) = \Psi([t\tau]) = \pi(t\tau), \tag{6.46}$$

where the first equality follows from our definition of Θ and the description of β in Lemma 6.2 and the last equality follows from Lemma 2.8 and Lemma 2.7.
2. $\Phi|_{\Theta^{-1}([t\tau])} : \Theta^{-1}([t\tau]) \to \pi^{-1}(\Psi([t\tau]))_{red}$ *is a resolution of singularities:* W.l.o.g. we can assume that $t\tau \in T_0^\tau\tau$. Note, that our assumption on the characteristic implies $char(k) > rk\, C_G(t\tau)_0 + 1$. Hence $char(k)$ does not divide the fundamental group of $C_G(t\tau)_0$, for all $t\tau \in T_0^\tau\tau$.
Furthermore, we have:

$$\begin{aligned}
\Theta^{-1}([t\tau]) &= (G \times^B \beta)^{-1}(G/B \times \{[t\tau]\}) \\
&= G \times^B (\beta^{-1}([t\tau]) \\
&\cong G \times^B (B \times^{B(t\tau)} U(t\tau)) \\
&\cong G \times^{B(t\tau)} U(t\tau) \\
&\cong G \times^{C_G(t\tau)} (C_G(t\tau) \times^{B(t\tau)} U(t\tau)),
\end{aligned} \tag{6.47}$$

where all isomorphies are G-equivariant and where the third line follows from Lemma 6.3 while the forth and fifth follow from [27], Section 3.7, Lemma 2, p. 26. There, an explicit formula for the isomorphism $G \times^H (H \times^K F) \to G \times^K V$ for a chain of closed subgroups $K < H < G$ of an algebraic group and a K-variety V is presented. It is given by $g * h * v \mapsto gh * v$ and its inverse by $g * v \mapsto g * 1 * v$. Using this and our isomorphism in Lemma 6.3, we see that, under the identifications above, $\Phi|_{\Theta^{-1}([t\tau])}$ takes the following form:

$$g * (c * u) \mapsto g\, t\tau\, c\, u\, c^{-1}\, g^{-1}, \tag{6.48}$$

for every $g * (c * u) \in G \times^{C_G(t\tau)} (C_G(t\tau) \times^{B(t\tau)} U(t\tau))$.
Recall that by Theorem 3.1, we get a G-equivariant isomorphism

$$\pi^{-1}(\Psi([t\tau]))_{red} \cong G \times^{C_G(t\tau)} U(t\tau). \tag{6.49}$$

Applying this isomorphism, we calculate that, under the above identifications of the fibres of Θ and those of π, our map $\Phi|_{\Theta^{-1}([t\tau])}$ induces a map $\widehat{\Phi}$:

$$\begin{aligned}
\widehat{\Phi} : G \times^{C_G(t\tau)} \Big(C_G(t\tau) \times^{B(t\tau)} U(t\tau)\Big) &\to G \times^{C_G(t\tau)} V(t\tau) \\
g * (c * u) &\mapsto g * (c\, u\, c^{-1}),
\end{aligned} \tag{6.50}$$

which is exactly the resolution map from Lemma 6.1. ♡.

As a consequence we get:

Corollary 6.3 *If $g\tau \in G\tau$ is regular and semisimple, we have:*

$$|\Phi^{-1}(g\tau)| = |\mathcal{W}^\tau| \tag{6.51}$$

Proof: Under our assumption, we can find an element $t\tau \in T_0^\tau \tau$ which is G-conjugate to $g\tau$, by Proposition 2.3. Denote its image under π in $T'/\mathcal{W}^\tau$ by $\overline{t\tau}$. By Theorem 6.1, we have, for every $t' \in \Psi^{-1}(\overline{t\tau})$, that

$$\Phi|_{\Theta^{-1}(t')} : \Theta^{-1}(t') \to (\pi^{-1}(\overline{t\tau}))_{red} \tag{6.52}$$

is a resolution of singularities. Because of the regularity of $t\tau$, we conclude $C_G(t\tau) \cong T_0^\tau$, a torus, see Section 3.1. Hence, $(\pi^{-1}(\overline{t\tau}))_{red}$ is smooth, by Lemma 3.1, and $\Phi|_{\Theta^{-1}(t')}$ is an isomorphism.
Therefore we have:

$$|\Phi^{-1}(g\tau)| = |\Psi^{-1}(\overline{t\tau})|. \tag{6.53}$$

By our description of regular semisimple elements in Corollary 3.3, we see that $\alpha'([t\tau]) \neq 1$ for all roots $\alpha' \in R'$, where $[t\tau]$ denotes the class of $t\tau$ in T'. Therefore, we get $|\Psi^{-1}(\overline{t\tau})| = |\mathcal{W}^\tau|$. ♡.

6.3 Transversal Slices and Deformations

In this subsection, we will construct a transversal slice to a subregular element of $G\tau$, i.e an element x, whose centralizer $C_G(x)$ has dimension $dim\, T_0^\tau + 2$, the second minimal possibility, by Corollary 3.1. We shall investigate the singularities in this slice. This will only be carried out for G simple and specific fibres. We will also establish a link to the deformation theory of this singularity and provide a simultaneous resolution of the deformation. To achieve this, we need further restrictions on the characteristic.

First, we recall the definition of transversal slices to orbits, which is given e.g. in [27], Section 5.1, p. 60:

Definition 6.2 *Let G be an algebraic group and X be a G-variety. A transversal slice S to the G-orbit of a point $x \in X$ at x is a locally closed subvariety of X, such that the following properties hold:*
(i) $x \in S$,
(ii) the morphism $G \times S \to X$ given by $(g, s) \mapsto g\,s$ is smooth,
(iii) the dimension of S is the smallest possible, such that (i) and (ii) hold.

Without proof, we quote from [27], Section 5.1, Lemma 1, p. 60 or [20], Lemma III.1, p. 96, the following lemma on the existence of transversal slices:

Lemma 6.6 *Keeping the notation as above, we have: If x is a smooth point of an affine variety X, then a transversal slice S to the G orbit of a point $x \in X$ exists. Moreover, we can choose S to be stable under a linearly reductive or a finite subgroup of the stabilizer G_x.*

If G is reductive and the orbit $G.x$ is closed and if, in $char(k) > 0$, the orbit map $G \to G.x$ is separable, then there exists a much stronger version of the lemma, Luna's Slice Theorem, a proof of which can be found in [28], §4, Satz, p. 97, or [20], §III.1, Théorème du Slice, p. 97. The condition on the characteristic is discussed in [1], Section 7:

Theorem 6.2 *Let G be a reductive group, X an affine G-variety and $\pi : X \to X/\!\!/G$ the categorical quotient. Take a point $x \in X$, such that $G.x \subset X$ is closed and such that the orbit map $G \to G.x$ is separable and such that we can decompose $T_xX = T_xG.x \oplus U$ into subspaces, where U is stable under the induced action of the stabilizer G_x on T_xX. Then there exists a locally closed subvariety $S \subset X$, such that the following properties hold:*
(i) $x \in S$.
(ii) S is $H := G_x$-stable.
(iii) The map $G \times S \to X$ given by $(g, s) \mapsto g\,s$ induces an etale, G-equivariant morphism

$$\Psi : G \times^H S \to X, \tag{6.54}$$

which has an affine image.
(iv) The morphism induced by Ψ on the quotients

$$\Psi/\!\!/G : S/\!\!/H \cong (G \times^H S)/\!\!/G \to X/\!\!/G \tag{6.55}$$

is an etale morphism.
(v) The following diagram is Cartesian:

$$\begin{tikzcd} G \times^H S \arrow[r, "\Psi"] \arrow[d] & X \arrow[d] \\ S/\!\!/H \arrow[r, "\Psi/\!\!/G"] & X/\!\!/G, \end{tikzcd} \tag{6.56}$$

where the vertical arrows are the quotient maps.

Remark: The splitting of the tangent space T_xX is immediate in characteristic zero because, in this case, G_x is linearly reductive. It is reductive in any case.

After this digression on transversal slices, we return to our previous situation, e.g. G semisimple and τ an exterior automorphism. From now on we assume, unless explicitly stated otherwise, that the group G', as defined in Section 2.4, is simply connected. Recall that, by Proposition 5.2, the quotient morphism $\pi : G\tau \to G\tau/\!\!/G$ is flat and that its fibres are normal. Take $x \in G\tau$ and let S be a transversal slice to the G orbit of x, which exists by Lemma 6.6. Set

$$\sigma = \pi|_S : S \to T_0^\tau\tau/\widetilde{\mathcal{W}}. \tag{6.57}$$

Then, we have the following result:

Lemma 6.7 *Assume that char(k) does not divide the order of the fundamental group of G. Then the morphism σ is flat, its fibres are normal and $y \in S$ is nonsingular in $\sigma^{-1}(\sigma(y))$, iff it is a regular element of $G\tau$.*

Proof: This proof is analogous to the proof of the Lemma in [27], Section 5.2, p. 64f. ♡.

From now on, we assume, additionally, that τ has no fixed points on the fundamental group of G, because we want to apply the results of the previous section.
We consider the simultaneous resolution of $\pi : G\tau \to T'/\mathcal{W}^\tau$ as given in Theorem 6.1:

$$\begin{tikzcd} G \times^B B\tau \arrow[r, "\Phi"] \arrow[d, "\Theta"'] & G\tau \arrow[d, "\pi"] \\ T' \arrow[r, "\Psi"] & T'/\mathcal{W}^\tau. \end{tikzcd} \tag{6.58}$$

Now we set $S' := \Phi^{-1}(S) := (G \times^B B\tau) \times_{G\tau} S$ and $\Phi' := \Phi|_{S'}$ and $\Theta' := \Theta|_{S'}$. Then we have the following assertion:

Lemma 6.8 *Assume char$(k) > rk\, G + 1$. Then the following diagram is a simultaneous resolution of $\sigma : S \to T'/\mathcal{W}^\tau$:*

$$\begin{tikzcd} S' \arrow[r, "\Phi'"] \arrow[d, "\Theta'"'] & S \arrow[d, "\sigma"] \\ T' \arrow[r, "\Psi"] & T'/\mathcal{W}^\tau. \end{tikzcd} \tag{6.59}$$

Proof: This proof is in analogy to the proof of the Corollary in Section 5.3 of [27], p. 65f. ♡.

Our next aim is to establish a link to deformation theory. Therefore we restrict from now on to the case where G is simple, unless stated otherwise. Furthermore, we assume that τ acts without fixed points on the fundamental group of G. This implies that all centralizers of semisimple elements are connected by the remark preceding Corollary 6.1. In addition, assume that the group G', as defined in Section 2.4, is simply connected. This guarantees the quotient space $G\tau /\!\!/ G$ to be an affine space, by Corollary 2.6
We have the following statement:

Proposition 6.3 *Keep all the assumptions made above and assume additionally that char(k) is excellent for G, τ.*
Then we have:
For $t\tau \in T_0^\tau \tau$, we find an open subset $V \in C_G(t\tau)$ which is a transversal slice to $t\tau$ in $G\tau$ with respect to the G-action. Furthermore, it fulfills the properties of Theorem 6.2, i.e.:
(i) $x \in V t\tau$.
(ii) $V t\tau$ is $C_G(t\tau)$-stable.
(iii) The map $G \times V t\tau \to G\tau$ given by $(g, v\, t\tau) \mapsto g\, v\, t\tau\, g^{-1}$ induces an etale G-equivariant morphism

$$\Psi : G \times^{C_G(t\tau)} V t\tau \to G\tau, \tag{6.60}$$

which has an affine image.
(iv) The morphism induced by Ψ

$$\Psi /\!\!/ G : V t\tau /\!\!/ C_G(t\tau) \cong (G \times^{C_G(t\tau)} V t\tau)/\!\!/ G \to G\tau /\!\!/ G \tag{6.61}$$

is an etale morphism.
(v) The following diagram is Cartesian:

$$\begin{tikzcd} G \times^{C_G(t\tau)} V t\tau \arrow[r, "\Psi"] \arrow[d] & G\tau \arrow[d] \\ V t\tau /\!\!/ H \arrow[r, "\Psi /\!\!/ G"] & G\tau /\!\!/ G, \end{tikzcd} \tag{6.62}$$

where the vertical arrows are the quotient maps.

Proof: Note that, under our assumptions, the orbit of $t\tau$ is closed, by Corollary 3.1 and that the orbit maps of G on $G\tau$ are separable, by Theorem 1.3.
1. In order to apply Theorem 6.2, we have to show that there exists a $C_G(t\tau)$-stable splitting of the tangent space:

$$T_{t\tau} G\tau \cong T_{t\tau} G.t\tau \oplus Lie\,(C_G(t\tau)). \tag{6.63}$$

To prove this, we consider the maps:

$$\begin{array}{cccccc} \phi : & G & \longrightarrow & G\tau & \longrightarrow & G \\ & g & \mapsto & g\, t\tau\, g^{-1} & \mapsto & g\, t\tau\, g^{-1}\, (t\tau)^{-1}. \end{array} \tag{6.64}$$

Then we easily calculate $(d\phi(X))_e = X - Ad(t\tau)(X)$ which implies:

$$Im\,(d\phi)_e = (Ad(t\tau) - id)(\mathfrak{g}). \tag{6.65}$$

Since $Ad(t\tau) - id \in \mathfrak{gl}(\mathfrak{g})$ is diagonalizable, we have the following decomposition into eigenspaces $\mathfrak{g}_\lambda$ to the eigenvalues $\lambda \in k$:

$$\mathfrak{g} = \oplus_{\lambda \in k} \mathfrak{g}_\lambda. \tag{6.66}$$

Note that we have $\mathfrak{g}_0 = Lie\,(C_G(t\tau))$, [2], Proposition 9.1, p. 128. So we get

$$Ker\,(d\phi)_e = \mathfrak{g}_0 \tag{6.67}$$

$$Im\,(d\phi)_e = \oplus_{\lambda \in k^*} \mathfrak{g}_\lambda. \tag{6.68}$$

Since the action of $C_G(t\tau)$ commutes with the action of $t\tau$ on $\mathfrak{g}$, these eigenspaces of $Ad(t\tau)$ have to be $C_G(t\tau)$-stable. Since $T_{t\tau}(G.t\tau)$ is isomorphic to $Im\,(d\phi)_e$ under the right translation $r_{t\tau^{-1}} : G\tau \to G$, we get the desired decomposition of $T_{t\tau}G\tau$.
Now, we can apply Theorem 6.2 to get the existence of a transversal slice S with all the properties listed in the proposition.
To prove our proposition, we still have to show:
2. S can be chosen as an open subset of $C_G(t\tau)$: This is guaranteed by the reduction of the proof of Theorem 6.2 as given in [28], §4, Satz p. 97, to the fundamental lemma, see [28], §4, Fundamentallemma, p. 98, if we can show the following:
(i) The map

$$\begin{aligned} \psi : G \times^{C_G(t\tau)} C_G(t\tau)t\tau &\to G\tau \\ g * x &\mapsto g\,x\,g^{-1} \end{aligned} \tag{6.69}$$

is etale in $e * t\tau$.
(ii) The orbits $G * \{t\tau\} \subset G \times^{C_G(t\tau)} C_G(t\tau)t\tau$ and $G.t\tau \subset G\tau$ are closed in their respective ambient spaces.
(iii) The map

$$\psi|_{G*\{t\tau\}} : G * \{t\tau\} \to G\tau \tag{6.70}$$

is injective.
(i) follows by [11], Chapter III, Proposition 10.4, p. 270, from the fact that, by the calculations in *1.*, the tangent map $(d\psi)_{e*t\tau}$ is surjective and $G\tau$ and $G \times^{C_G(t\tau)} C_G(t\tau)t\tau$ are smooth and of equal dimension.
(ii) and (iii) are immediate by our assumptions, respectively the definition of associated fibre bundles. ♡.

From the fundamental lemma, [28], §4, Fundamentallemma, p. 98, we get two important consequences:

Corollary 6.4 *Under the assumptions of the Proposition 6.3, we have that $Vt\tau$ consists of complete fibres of the adjoint quotient*

$$\pi_{t\tau} : C_G(t\tau)t\tau \to C_G(t\tau)t\tau /\!\!/ C_G(t\tau). \tag{6.71}$$

In particular, we can consider $Vt\tau /\!\!/ C_G(t\tau)$ as a subset of $C_G(t\tau)t\tau /\!\!/ C_G(t\tau)$:

$$Vt\tau /\!\!/ C_G(t\tau) \subset C_G(t\tau)t\tau /\!\!/ C_G(t\tau) \tag{6.72}$$

and the nilpotent variety $V(t\tau)$ of $C_G(t\tau)$ is a subset of V.

By the classical results [35], Theorem 6.1 and Corollary 96.4, p. 294f, we know

$$C_G(t\tau)t\tau /\!\!/ C_G(t\tau) \cong T_0^\tau \tau / \widehat{\mathcal{W}}, \tag{6.73}$$

where $\widehat{\mathcal{W}} < \widetilde{\mathcal{W}}$ is the Weyl group of $C_G(t\tau)$.

Corollary 6.5 *With these notions and the assumptions of the Proposition 6.3, we consider the (ramified) covering map:*

$$\begin{array}{rcl} \Psi : T_0^\tau \tau / \widehat{\mathcal{W}} & \to & T_0^\tau \tau / \widetilde{\mathcal{W}} \\ \widehat{t'\tau} & \mapsto & \overline{t'\tau}, \end{array} \tag{6.74}$$

where $\widehat{t'\tau}$ denotes the class of $t'\tau$ in $T_0^\tau \tau / \widehat{\mathcal{W}}$ and $\overline{t'\tau}$ denotes the class of $t'\tau$ in $T_0^\tau \tau / \widetilde{\mathcal{W}}$. Then Ψ is etale in $\widehat{t\tau}$.

We can give a second, direct proof of this corollary:
Proof: Note that the ramification set R_Ψ of Ψ can easily be described as:

$$R_\Psi = \{\widehat{t'\tau} \in T_0^\tau \tau / \widehat{\mathcal{W}},\, \exists w \in \widetilde{\mathcal{W}} \backslash \widehat{\mathcal{W}},\, \widehat{w.t'\tau} = \widehat{t'\tau}\}. \tag{6.75}$$

Assume that Ψ is ramified in $\widehat{t\tau} \in T_0^\tau / \widehat{\mathcal{W}}$, then we would find $w \in \widetilde{\mathcal{W}} \backslash \widehat{\mathcal{W}}$ with $w.t\tau = t\tau$, which is absurd, since $C_G(t\tau)$ is connected by our assumption. $\heartsuit$.

Now we choose an open set $V \subset C_G(t\tau)$ as in Proposition 6.3 and define an embedding

$$\begin{array}{rcl} \iota : V & \hookrightarrow & G \times^{C_G(t\tau)} Vt\tau \\ u & \mapsto & e * u\,t\tau, \end{array} \tag{6.76}$$

as the fibre over $eC_G(t\tau) \in G/C_G(t\tau)$.
Furthermore, we define a map $\Phi : G \times^{C_G(t\tau)} Vt\tau \to V /\!\!/ C_G(t\tau)$ in the following way:
First recall that we have the isomorphism:

$$\begin{array}{rcl} G \times^{C_G(t\tau)} (Vt\tau /\!\!/ C_G(t\tau)) & \to & G/C_G(t\tau) \times Vt\tau /\!\!/ C_G(t\tau) \\ g * [vt\tau]' & \mapsto & (gC_G(t\tau),\, [vt\tau]'), \end{array} \tag{6.77}$$

where $[vt\tau]'$ denotes the class of $ut\tau$ in $Vt\tau /\!\!/ C_G(t\tau)$. (Note that $C_G(t\tau)$ acts trivially on $Vt\tau /\!\!/ C_G(t\tau)$.) Using this identification we set $\Phi := pr_2 \circ G \times^{C_G(t\tau)} \pi_{t\tau}|_V$, where $\pi_{t\tau} : C_G(t\tau) \to C_G(t\tau)t\tau /\!\!/ C_G(t\tau)$ is the adjoint

quotient of $C_G(t\tau)$.
Then, we apparently have the following commutative diagram:

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "\pi_{t\tau}|_V"'] & G \times^{C_G(t\tau)} (Vt\tau /\!\!/ C_G(t\tau)) \arrow[d, "\Phi"] \\ & Vt\tau /\!\!/ C_G(t\tau). \end{tikzcd} \tag{6.78}$$

Now, we take $x \in V$ and $S \subset V$, a transversal slice to the $C_G(t\tau)$-orbit of x at x, which exists by Lemma 6.6. (To be precise, we take a transversal slice in $C_G(t\tau)$ and intersect it with V.)
Then, we have the following commutative diagram:

$$\begin{tikzcd} S \arrow[r, "\iota'"] \arrow[drr, "\pi_{t\tau}|_S"'] & V \arrow[r, "\iota"] & G \times^{C_G(t\tau)} Vt\tau \arrow[r, "\psi|_{G\times^{C_G(t\tau)} Vt\tau}"] \arrow[d, "\Phi"] & G\tau \arrow[d, "\pi"] \\ & & Vt\tau /\!\!/ C_G(t\tau) \arrow[r, "\Psi|_{Vt\tau /\!\!/ C_G(t\tau)}"] & G\tau /\!\!/ G, \end{tikzcd} \tag{6.79}$$

where $\iota' : S \to V$ is the embedding, where ψ was defined in the proof of Proposition 6.3, Φ and ι in the last two paragraphs and Ψ in Corollary 6.5.
Now, we get the following result:

Lemma 6.9 *Keeping the assumptions of Proposition 6.3 we have: $St\tau$ is a transversal slice to the G-orbit of $xt\tau$ at the point $xt\tau$ in $G\tau$.*

Proof: We clearly have $xt\tau \in St\tau$.
Next, we show the smoothness of the map:

$$\begin{aligned} \mu : G \times St\tau &\to G\tau \\ (g,\, st\tau) &\mapsto g\, st\tau\, g^{-1}. \end{aligned} \tag{6.80}$$

Using the trivial action of $C_G(t\tau)$ on $St\tau$, observe that we have the following identification, by [27], Section 3.7, Lemma 2, p. 26:

$$G \times St\tau \to G \times^{C_G(t\tau)} (C_G(t\tau) \times St\tau), \tag{6.81}$$

given by $(g,\, st\tau) \to g * (e,\, st\tau)$, for all $(g,\, st\tau) \in G \times St\tau$. Its inverse is given by $g * (c,\, st\tau) \mapsto (gc,\, st\tau)$.
Now, let ϕ be the orbit map:

$$\begin{aligned} \phi : C_G(t\tau) \times St\tau &\to Vt\tau \\ (g,\, st\tau) &\mapsto g\, st\tau\, g^{-1}, \end{aligned} \tag{6.82}$$

which is smooth by assumption. Analogously to the proof of Lemma 6.4, where we have proven the smoothness of $G \times^B \beta$ as a morphism between smooth varieties under the assumption of β to be smooth, we get the smoothness of $G \times^{C_G(t\tau)} \phi$:

$$G \times^{C_G(t\tau)} \phi : G \times^{C_G(t\tau)} (C_G(t\tau) \times St\tau) \to G \times^{C_G(t\tau)} V t\tau. \tag{6.83}$$

But, by Proposition 6.3, we know that we have an etale map:

$$\begin{array}{rcl} \Psi : G \times^{C_G(t\tau)} V t\tau & \to & G\tau \\ g * ut\tau & \mapsto & g\, ut\tau\, g^{-1}. \end{array} \tag{6.84}$$

Observing that $\mu = \Psi \circ G \times^{C_G(t\tau)} \phi$, and using the identification between $G \times St\tau$ and $G \times^{C_G(t\tau)} (C_G(t\tau) \times St\tau)$, as above, the smoothness of μ follows from [11], Proposition 10.1, p. 268.
A simple calculation on dimensions gives

$$dim\, G\tau = dim\, St\tau + codim_{G\tau}\, St\tau, \tag{6.85}$$

from which the minimality property of $St\tau$ is immediate $\heartsuit$.

From now on, we consider elements $t\tau \in T_0^\tau \tau$, such that
(i) $C_G(t\tau)$ has maximal semisimple rank, i.e. the Dynkin diagram $\Delta(C_G(t\tau))$ of $R(C_G(t\tau))$ has $s = dim\, T_0^\tau$ points,
and such that
(ii) this Dynkin diagram $\Delta(C_G(t\tau))$ is connected.
Furthermore, we require $char(k) = 0$ or $char(k) > 4\, cox(G) - 2$, where $cox(G)$ is the Coxeter number of $R(G)$, the order of the Coxeter element, which we summarize in the following table, see [3], Planches, p. 250ff:

Type G	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$cox(G)$	$n+1$	$2n$	$2n$	$2n-2$	12	18	30	12	6

So, in particular, we have $char(k) > 4\, cox(C_G(t\tau)) - 2 > rk\,(C_G(t\tau)) + 1$.
If $char(k) = 0$, there exist the following possibilities for the type of $C_G(t\tau)$, see Theorem 3.2:

$\Delta(G)$	A_{2n}	A_{2n-1}	$D_{n+1},\, \tau^2 = 1$	E_6	$D_4,\, \tau^3 = 1$
$\Delta(C_G(t\tau))$	$B_n,\, C_n$	$C_n,\, D_n$	B_n	$C_4,\, F_4$	$A_2,\, G_2$

Keeping the same restriction on the characteristic, we define a simple singularity of type Δ, where X denotes a singular variety and the indicated groups are finite subgroups of $SL_2(k)$, see [27], Sections 6.1,6.2, p. 75ff:
1. Δ homogeneous:

X	Type (Δ)	Finite group
$\mathbb{A}^2/\mathbb{Z}_{n+1}$	A_n	$\mathbb{Z}_{n+1}$, cyclic group
$\mathbb{A}^2/\mathbb{D}_{n-2}$	D_n	$\mathbb{D}_{n-2}$, binary dihedral group
$\mathbb{A}^2/\mathbb{T}$	E_6	$\mathbb{T}$, binary tetrahedral group
$\mathbb{A}^2/\mathbb{O}$	E_7	$\mathbb{O}$, binary octahedral group
$\mathbb{A}^2/\mathbb{I}$	E_8	$\mathbb{I}$, binary icosahedral group

2. Δ inhomogeneous:
First, we define the associated homogeneous Dynkin diagram Δ_h by the following table:

Δ	B_n	C_n	F_4	G_2
Δ_h	A_{2n-1}	D_{n+1}	E_6	D_4

The space X for Δ will be the same as in the case for Δ_h but, additionally, we have a group $\Gamma \cong Aut(\Delta_h) < Aut(X)$ which acts freely on $X \backslash X_{sing}$. We have the following list, keeping the designations of the groups as above:

X	Type (Δ)	Γ
$\mathbb{A}^2/\mathbb{Z}_{2n}$	B_n	$\mathbb{D}_n/\mathbb{Z}_{2n} \cong \mathbb{Z}/2\mathbb{Z}$
$\mathbb{A}^2/\mathbb{D}_{n-1}$	C_n	$\mathbb{D}_{2(n-1)}/\mathbb{D}_{n-1} \cong \mathbb{Z}/2\mathbb{Z}$
$\mathbb{A}^2/\mathbb{T}$	F_4	$\mathbb{O}/\mathbb{T} \cong \mathbb{Z}/2\mathbb{Z}$
$\mathbb{A}^2/\mathbb{D}_2$	G_2	$\mathbb{O}/\mathbb{D}_2 \cong \mathcal{S}_3$

The singular spaces X with the naturally induced symmetry Γ will be denoted by (X, Γ).

To establish the bridge from our situation inside the group to these kind of singularities, we need some further notions:
We begin with a definition which holds in the more general context of Chapter 3, i.e. G semisimple and τ an exterior automorphism of G:

Definition 6.3 *An element $x \in G\tau$ is called subregular, iff $dim\, C_G(x) = dim\, T_0^\tau \tau + 2$.*

Remark: By Corollary 3.1, this is the second smallest possibility for the dimension of $dim\, C_G(x)$, for $x \in G\tau$.

We have the following description on the number of subregular G conjugacy classes in each fibre of $\pi : G\tau \to G\tau /\!\!/ G$:

Lemma 6.10 *If τ acts with no fixed points on the fundamental group of G, we have exactly as many subregular G-conjugacy classes in each fibre $\pi^{-1}(\pi(t\tau))$ of $\pi : G\tau \to G\tau /\!\!/ G$, for $t\tau \in T_0^\tau \tau$, as $\Delta(C_G(t\tau))$ has connected components.*

Proof: This follows directly from the description of the fibre structure in Theorem 3.1, which states that the subregular classes in $\pi^{-1}(\pi(t\tau))$ correspond to subregular unipotent classes in $C_G(t\tau)_0 = C_G(t\tau)$, together with Lemma 2 in [27], Section 5.4, p. 67f. (Note that $C_G(t\tau)_0 = C_G(t\tau)$ holds by [37], Corollary 9.7, p. 62.) ♡.

Now, we return to our previous situation: Fix a semisimple element $t\tau \in G\tau$, such that our requirements for the root system $R(C_G(t\tau))$ are fulfilled, as well as the restriction on the characteristic, e.g. $char(k) = 0$ or $char(k) > 4\,cox(G) - 2$. Furthermore, assume the conditions on τ to have no fixed points on the fundamental group of G and on G' to be simply connected.
Then let $u \in C_G(t\tau)$ be a (up to conjugacy unique) subregular unipotent element and let S be a transversal slice to the $C_G(t\tau)$ orbit of v at v in V as in Proposition 6.3. We have the following lemma:

Lemma 6.11 *Under our assumptions the following holds:*
$St\tau \cap \pi^{-1}(\pi(t\tau)) \cong S \cap V(t\tau)$ *has an isolated singularity of type* $\Delta(C_G(t\tau))$ *in* u. *Here,* $V(t\tau)$ *is the unipotent variety of* $C_G(t\tau)$.

Proof: This follows from [27], Section 8.4 Theorem, p. 129 for an analogous situation in $Lie(C_G(t\tau))$ together with the Comparison Theorem 3.15 in [27], p. 41. Note, that this Comparison Theorem also holds in $char(k) > 4\,cox(G) - 2$ because of our more general version of the Slice Theorem 6.2. ♡.

To get a connection with the deformation theory of the singularity $St\tau \cap \pi^{-1}(\pi(t\tau)) \cong S \cap V(t\tau)$ we have to choose S in a special way:
We consider the centralizer $C_{C_G(t\tau)}(u)$ for our subregular unipotent element in $u \in V(t\tau)$. By [27], Section 7.5, p. 114ff, we have:

$$C_{C_G(t\tau)}(u)/C_{C_G(t\tau)}(u)_0 \cong Aut(\Delta(C_G(t\tau))_h). \tag{6.86}$$

We choose S to be invariant under a section of the projection to the finite group $C_{C_G(t\tau)}(u) \to C_{C_G(t\tau)}(u)/C_{C_G(t\tau)}(u)_0$, where we also used Lemma 1 in [27], Section 7.6, p. 118.
Now we are ready to establish the connection to the deformation theory of these singularities. For an account on deformation theory of singularities with or without symmetry, we refer to [27], Section 2.
We have our main result of this chapter:

Theorem 6.3 *Under the assumptions above, we have:*
The map $\sigma := \pi|_{St\tau} : St\tau \to G\tau /\!\!/ G$ *induces a formal semiuniversal deformation of a simple singularity of type* $\Delta(C_G(t\tau))$.

Proof: We consider the following commutative diagram:

$$\begin{tikzcd} St\tau \arrow[r, "\iota"] \arrow[d, "\pi_{t\tau}|_{St\tau}"'] \arrow[dr, "\sigma"] & G\tau \arrow[d, "\pi"] \\ T_0^\tau\tau/\widehat{\mathcal{W}} \arrow[r, "\Psi"] & T_0^\tau\tau/\widetilde{\mathcal{W}}, \end{tikzcd} \tag{6.87}$$

where ι is the inclusion and where, as in Corollary 6.5, $\widehat{\mathcal{W}}$ is the Weyl group of $C_G(t\tau)$ and Ψ the corresponding (ramified) covering map. From Lemma 6.7 we have that σ is flat.
Now, we consider the corresponding completions in $ut\tau$, $\widehat{t\tau}$ respectively $\overline{t\tau}$, which we denote by a hat. The diagram then takes the form:

$$\begin{tikzcd} \widehat{(St\tau,\, ut\tau)} \arrow[r, "\hat{\iota}"] \arrow[d, "\widehat{\pi_{t\tau}|_{St\tau}}"'] \arrow[dr, "\widehat{\sigma}"] & \widehat{(G\tau,\, ut\tau)} \arrow[d, "\hat{\pi}"] \\ \widehat{(T_0^\tau\tau/\widehat{\mathcal{W}},\, \widehat{t\tau})} \arrow[r, "\widehat{\Psi}"] & \widehat{(T_0^\tau\tau/\widetilde{\mathcal{W}},\, \overline{t\tau})}. \end{tikzcd} \tag{6.88}$$

Since Ψ is etale in $\widehat{t\tau}$, by Corollary 6.5, $\widehat{\Psi}$ is an isomorphism, see [11], Chapter III, Exercise 10.4, p. 275.
By [27], Section 2.8, Corollary, p. 16, we have the following equivalence:

$$\pi|_{St\tau} \text{ is semiuniversal deformation} \Leftrightarrow \pi_{t\tau}|_{St\tau} \text{ is semiuniversal deformation.} \tag{6.89}$$

Now, the statement follows from [27], Section 8.7, Corollary, p. 137. $\heartsuit$.

Keeping the same notations as for the theorem above we get a simultaneous resolution of the semiuniversal deformation, using the statement and notations of Lemma 6.8:

Theorem 6.4 *Let $\sigma : St\tau \to T_0^\tau\tau/\widetilde{\mathcal{W}}$ be our semiuniversal deformation of the fibre $\sigma^{-1}(\pi(t\tau)) = St\tau \cap \pi^{-1}(\pi(t\tau))$, which is a singularity of type $\Delta(C_G(t\tau))$. Then the following diagram provides a simultaneous resolution of the semiuniversal deformation $\sigma : St\tau \to T_0^\tau\tau/\widetilde{\mathcal{W}}$:*

$$\begin{tikzcd} S' \arrow[r, "\Phi'"] \arrow[d, "\Theta'"'] & St\tau \arrow[d, "\sigma"] \\ T' \arrow[r, "\Psi"] & T_0^\tau\tau/\widetilde{\mathcal{W}}. \end{tikzcd} \tag{6.90}$$

Chapter 7

Conclusion

We consider a non-connected linear algebraic group $\tilde{G} = G \rtimes \Gamma$, where G is a semisimple linear algebraic group and where Γ is a finite subgroup of the group of diagram automorphisms of the Dynkin diagram $\Delta(G)$ of G. For every $\tau \in \Gamma$, we regard the corresponding component $G\tau$ of $\tilde{G}$ and investigate the adjoint action of G on $G\tau$. In particular, we are interested in the adjoint quotient $G\tau /\!\!/ G$.
Now, consider a slightly different situation: Let T be a τ stable maximal torus of G and T_0^τ the unit component of the group of its fixed points under τ. Define $\widetilde{\mathcal{W}} := N_G(< T_0^\tau, \tau >)/T_0^\tau$, the normalizer in G of the group generated by T_0^τ and τ modulo T_0^τ, which is a finite group by our results. We have shown that $\widetilde{\mathcal{W}}$ has the following structure:

$$\widetilde{\mathcal{W}} \cong (T/T_0^\tau)^\tau \rtimes \mathcal{W}^\tau,$$

where $\mathcal{W}^\tau$ is the fixed point Weyl group.
Using representation theory of $\tilde{G}$ in the case where Γ is cyclic, we deduced the following results:

- We have an isomorphism $G\tau /\!\!/ G \cong T_0^\tau \tau / \widetilde{\mathcal{W}}$.
- In certain cases, in particular for simply connected G, this quotient is an affine space.

Furthermore, we investigated the structure of the fibres of the quotient map $\pi : G\tau \to G\tau /\!\!/ G$. They have the following structure:

- Each fibre of π contains at least one element of $T_0^\tau \tau$ (by the results above). Then, we have the following description of the reduced fibre as an associated bundle:

  $$\pi^{-1}(\pi(t\tau))_{red} \cong G \times^{C_G(t\tau)} V(t\tau).$$

  Here $V(t\tau)$ is the unipotent variety of the reductive group $C_G(t\tau)$.

- Each fibre consists of finitely many conjugacy classes, in particular, there is one dense class (called the regular class) and one closed class (the one corresponding to the semisimple elements of this fibre). The whole structure of the fibre is gouverned by the structure of the unipotent variety of $C_G(t\tau)$.
- In characteristic zero, the Dynkin diagram of the centralizer of a semisimple element is a proper subdiagram of the twisted affine Dynkin diagram $R^{(ord\,\tau)}$.
- For simply connected G, the quotient map is flat and its fibres are reduced and normal.

We call an element in $G\tau$ regular, if its centralizer (in G) is of minimal possible dimension. Then the dense orbit of each fibre consists of regular elements.

In the case of simply connected G, we constructed a Steinberg cross section, a morphism $\sigma : T_0^\tau\tau/\widetilde{\mathcal{W}} \to G\tau$, meeting each fibre exactly once in its regular class. This cross section allowed us to show, that the differential $(d\pi)_x$, of the quotient map $\pi : G\tau \to G\tau/\!\!/G$ in a point x, is surjective if and only if x is a regular element.

Finally, we established a connection between the theory of simple singularities and simple algebraic groups by finding a singularity of type $\Delta(C_G(t\tau))$ in a transversal slice to the orbit of a subregular element of the fibre over $\pi(t\tau)$ inside the corresponding fibre, in the case where the Dynkin diagram $\Delta(C_G(t\tau))$ is connected. This discussion follows the strategy given by Peter Slodowy in [27]

In recent work, Stefan Helmke and Peter Slodowy managed to establish a similar connection between centrally extended holomorphic loop groups $\mathcal{L}G \rtimes \mathbb{C}^*$, for a simply connected simple group G, and elliptic singularities, [12]. A holomorphic loop group is the group of all holomorphic maps $\phi : \mathbb{C}^* \to G$. (In the semidirect product above $q \in \mathbb{C}^*$ acts on a loop $\phi \in \mathcal{L}G$ by $(q.\phi)(z) = \phi(q\,z)$.)

The crucial step in their construction was the interpretation of conjugacy classes in the loop group as isomorphism classes of holomorphic G-bundles over a certain elliptic curve E:

There exists a one-to-one correspondence between $\mathcal{L}G$-conjugacy classes in $\mathcal{L}G \times \{q\}$ for fixed value of q with $|q| < 1$ and isomorphism classes of principal G-bundles over $E = \mathbb{C}^*/\mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{C}^*$ by multiplication with powers of q. (This identification goes back to Looijenga, for details see [12].)

This interpretation gave them a thorough knowledge of the adjoint action of the loop group.

Furthermore, Brüchert has constructed a Steinberg cross section in the case of Kac Moody groups, [6].

If we consider now a non-connected loop group, i.e. a loop group $\mathcal{L}G$ where G is not simply connected, we expect similar results for its exterior components. The conjugacy classes in these exterior components should be interpreted as isomorphism classes of principal G-bundles over an elliptic curve E which are not topologically trivial. Using this, we think that we can understand the structure of exterior conjugacy classes for non-connected loop groups.

Appendix A

A No-Go Result

In this appendix, we want to show, that it is not possible to find for every element in $G\tau$ a G-conjugate in $G_0^\tau\tau$. The proof of this will be given by a case-by-case analysis. We restrict ourselves to the case, where G is simple.

Proposition A.1 *Let G be a simple affine algebraic group of type $A\,D\,E$ and τ a nontrivial diagram automorphism of G. Then we can find an element of $G\tau$, which is not G-conjugate into $G_0^\tau\tau$.*

Remark: By Proposition 2.3, the element in the statement of the proposition cannot be a semisimple one.
Proof: The proof will be given by contradiction.
So, let us assume that every element $x \in G\tau$ is G-conjugate to an element $y \in G_0^\tau\tau$. Because of Proposition 2.3 and the fact that every semisimple element of G_0^τ is G_0^τ-conjugate into a maximal torus thereof, we can assume w.l.o.g. that the semisimple parts x_s and y_s of x and y, respectively, are contained in $T_0^\tau\tau$ and that $y = g\,x\,g^{-1}$, yielding $g\,x_u\,g^{-1} \in G_0^\tau$. By use of Proposition 2.5, we can find an element $h \in C_G(x_s)$, such that $g\,h \in N_G(< T_0^\tau, \tau >)$. (Note that $< T_0^\tau, \tau >$ is a Cartan subgroup by the remark following Proposition 2.1.) We will show below, case-by-case, that we can find $x \in G\tau$ fulfilling the following properties:
(i) x is regular, i.e. x_u is regular in $C_G(x_s)_0$, by Corollary 3.1.
(ii) no regular unipotent element of $C_G(x_s)_0$ is contained in $C_G(x_s)_0 \cap G_0^\tau$.
This means that x_s is "less regular" in $G\tau$ than in $G_0^\tau\tau$.
Then, $x_u \notin G_0^\tau$ and $h\,x_u\,h^{-1} \notin G_0^\tau$, for all $h \in C_G(x_s)$. Hence, we can assume w.l.o.g. $g = n \in N_G(< T_0^\tau, \tau >)$.
Now, set:

$$p : N_G(< T_0^\tau, \tau >) \to \widetilde{\mathcal{W}} \cong \mathcal{W}^\tau \ltimes (T/T_0^\tau)^\tau, \tag{A.1}$$

the quotient map by T_0^τ, see Lemma 2.6. Then, we can find a decomposition (unique up to a factor in T_0^τ) for every $n \in N_G(< T_0^\tau, \tau >)$:

$$n = n'\,n'', \tag{A.2}$$

with $n'' \in p^{-1}((T/T_0^\tau)^\tau)$ and $n' \in p^{-1}(\mathcal{W}^\tau)$. Since we have, again by Lemma 2.6, that $p^{-1}(\mathcal{W}^\tau) \subset G_0^\tau$, we conclude that $n\,x\,n^{-1} \in G_0^\tau\tau$ implies $n''\,x\,n''^{-1} \in G_0^\tau\tau$. Therefore, we can consider, w.o.l.g., $n \in p^{-1}((T/T_0^\tau)^\tau)$. This will also be done by a case-by-case analysis. Recall that, by Section 2.3, $(T/T_0^\tau)^\tau$, acts on $T_0^\tau\tau$ by left translations, therefore by multiplication with elements in T_0^τ. Denote the corresponding subgroup of T_0^τ by H, as in Section 2.3.
Next we will carry out the indicated case-by-case arguments:
In this case-by-case analysis, we will make use of the explicit description of centralizers of elements in $T_0^\tau\tau$, as given in the proof of Proposition 3.1, as well as the description of irregular elements, as given in Theorem 4.1 and Corollary 3.3. Furthermore, we will use the notation introduced in Chapters 2-4, and we will fix a Borel subgroup B.
We have to distinguish three cases:
1. The groups of type A_{2n-1}, E_6 and D_n with $\tau^2 = 1$,
2. the case A_{2n},
3. the case D_4, $\tau^3 = 1$.

Case 1: A_{2n-1}, E_6 and D_n with $\tau^2 = 1$: In this case, note, that ${}^\tau R^1$ has a $\mathbb{Q}$-closed root subsystem of type B_2 and that its long roots are short roots in R'. Let $\bar{\alpha}$ and $\tilde{\bar{\alpha}}$ the simple roots of this B_2-root subsystem (They can be chosen to be also simple in ${}^\tau R^1$). Assume $\bar{\alpha}$ to be short and $\tilde{\bar{\alpha}}$ to be long. Then, $\bar{\beta} := \bar{\alpha} + \tilde{\bar{\alpha}}$ is also a short root. By the construction of ${}^\tau R^1$ and R' in Section 1.2, we have:
$$\bar{\beta} - \bar{\alpha} = \tilde{\bar{\alpha}} \in \mathbb{Z}(R'). \tag{A.3}$$
Set $\alpha' = d(\bar{\alpha})$ and $\beta' = d(\bar{\beta})$.
Then, we choose x in the following way:
$$\tau^{-1}x_s \in T^\tau_{0\,\alpha',\beta'} \setminus \bigcup_{\substack{\gamma' \in R'^+ \\ \gamma' \neq \alpha',\, \beta'}} T^\tau_{0\,\beta'}, \tag{A.4}$$
where we impose $\tilde{\bar{\alpha}}(\tau^{-1}x_s) = -1$ and $\bar{\alpha}(\tau^{-1}x_s) = 1$, whence $\bar{\beta}(\tau^{-1}x_s) = -1$. (Observe, that this implies $(2\bar{\alpha} + \tilde{\bar{\alpha}})(\tau^{-1}x_s) = -1$, where $2\bar{\alpha} + \tilde{\bar{\alpha}}$ is long.)
Now, we see that $C_G(x_s)_0$ is generated by $S_{\pm\alpha'}(c,\, c)$, for all $c \in k$, $S_{\pm\beta'}(c,\, -c)$, for all $c \in k$, and T_0^τ, and hence is of type $A_1 \times A_1$.
Now we choose $x_u = S_{\alpha'}(1,\, 1)S_{\beta'}(1,\, -1)$, yielding x_u regular unipotent in $C_G(x_s)_0$ and x regular in $G\tau$.
By Proposition 3.1 $C_{G_0^\tau}(x_s)_0 = (C_G(x_s)_0 \cap G_0^\tau)_0$ can be described as follows:
$$C_{G_0^\tau}(x_s)_0 = < T_0^\tau,\, \{S_{\alpha'}(c,\, c),\quad c \in k\},\, \{S_{-\alpha'}(c,\, c),\quad c \in k\} >, \tag{A.5}$$
a group of type A_1. Hence, $C_G(x_s)_0 \cap G_0^\tau$ contains no regular unipotent element of $C_G(x_s)_0$.

If we conjugate $x = x_s x_u$ with a representative $t \in T$ of an element in $(T/T_0^\tau)^\tau$ we get:

$$t\,x_u\,t^{-1} = S_{\alpha'}(\alpha(t),\, \alpha(t)\bar{\alpha}(t'))S_{\beta'}(\beta(t),\, -\beta(t)\bar{\beta}(t')), \tag{A.6}$$

where α, β are roots in R, that are mapped to $\bar{\alpha}$, $\bar{\beta} \in {}^{\tau}R^1$ by the projection p in Equation 1.2 and where $t' := t^{-1}\,\tau(t) \in H$. Recall that we have $\chi(H) = \chi(T_0^\tau)/\chi(T')$, by Corollary 2.4, and $\mathbb{Z}(R') \subset \chi(T')$, by Lemma 1.4. Therefore we conclude, using Equation A.3:

$$\bar{\alpha}|_H = \bar{\beta}|_H. \tag{A.7}$$

To achieve $t\,x_u\,t^{-1} \in G_0^\tau$ we need to find $t' \in H$ with $\bar{\alpha}(t') = 1$ and $\bar{\beta}(t') = -1$ by Equation A.6, which is absurd. $\diamondsuit$.

Case 2: A_{2n}: Here we choose an element x, such that we have:

$$\tau^{-1}x_s \in T^\tau_{0\;\alpha'} \setminus \bigcup_{\substack{\beta' \in R'^+ \\ \beta' \neq \alpha'}} T^\tau_{0\;\beta'}, \tag{A.8}$$

where α' is a long root in R'. We impose $\bar{\alpha}(\tau^{-1}x_s) = -1$, where $\bar{\alpha}$ is the unique long root in ${}^{\tau}R^1$ with $d(\bar{\alpha}) = \alpha'$.
Note, that $C_G(x_s)_0$ is generated by $S_{\alpha'}(0,\, 0,\, c)$, for all $c \in k$, $S_{-\alpha'}(0,\, 0,\, c)$, for all $c \in k$, and T_0^τ, and hence is of type A_1.
We take $x_u = S_{\alpha'}(0,\, 0,\, 1)$, yielding x_u regular unipotent in $C_G(x_s)_0$ and x regular in $G\tau$.
Furthermore, by the proof of proposition 3.1, x_s is regular in G_0^τ. Hence, $(C_G(x_s)_0 \cap G_0^\tau)$ contains no regular unipotent element of $C_G(x_s)$.
If we conjugate $x = x_s x_u$ with a representative $t \in T$ of an element in $(T/T_0^\tau)^\tau$ we get:

$$t\,x\,t^{-1} = t\,x_s\,t^{-1}\,t\,x_u\,t^{-1} \in T_0^\tau S_{\alpha'}(0,\, 0,\, k^*)\tau. \tag{A.9}$$

Again $t\,x_u\,t^{-1} = S_{\alpha'}(0,\, 0,\, \alpha(t)) \notin G_0^\tau$, where α is the root in R, which is mapped to $\bar{\alpha} \in {}^{1}R^{\tau}$ by the projection p in Equation 1.2. Hence we get $t\,x\,t^{-1} \notin G_0^\tau\tau$. $\diamondsuit$.

Case 3: D_4, $\tau^3 = 1$: Here we construct x in the following way:
First we choose ξ, a primitive third root of unity. Let $\{\bar{\alpha}_1,\, \bar{\alpha}_2\}$ be a basis of ${}^{\tau}R^1 = G_2$, such that $\bar{\alpha}_1$ is short and $\bar{\alpha}_2$ is long. Then, from the results of Section 1.2, we have:

$$\bar{\alpha}_2 \in \mathbb{Z}(R'). \tag{A.10}$$

Now, let us denote $\bar{\alpha} := \bar{\alpha}_1 + \bar{\alpha}_2$, and $\tilde{\bar{\alpha}} := 2\bar{\alpha}_1 + \bar{\alpha}_2$ and impose the following conditions on $\tau^{-1}x_s$:

$$\bar{\alpha}_1(\tau^{-1}x_s) = \xi \tag{A.11}$$
$$\bar{\alpha}_2(\tau^{-1}x_s) = \xi^2. \tag{A.12}$$

As a consequence, we get:

$$\bar{\alpha}(\tau^{-1}x_s) = 1 \tag{A.13}$$
$$\tilde{\bar{\alpha}}(\tau^{-1}x_s) = (3\bar{\alpha}_1 + 2\bar{\alpha}_2)(\tau^{-1}x_s) = \xi \tag{A.14}$$
$$(3\bar{\alpha}_1 + \bar{\alpha}_2)(\tau^{-1}x_s) = \xi^2. \tag{A.15}$$

Since $\bar{\alpha}_1$, $\bar{\alpha}$ and $\tilde{\bar{\alpha}}$ are the only positive short roots in ${}^{\tau}R^1$, we get, that $C_G(x_s)_0$ is generated by $S_{\pm d(\bar{\alpha}_1)}(c,\, \xi\, c,\, \xi^2\, c)$, for all $c \in k$, $S_{\pm d(\bar{\alpha})}(c,\, c,\, c)$, for all $c \in k$, $S_{\pm d(\tilde{\bar{\alpha}})}(c,\, \xi\, c,\, \xi^2\, c)$, for all $c \in k$, and T^{τ}, and hence is of type A_2. Now, we choose $x_u = S_{d(\bar{\alpha}_1)}(1,\, \xi,\, \xi^2) S_{d(\bar{\alpha})}(1,\, 1,\, 1) S_{d(\tilde{\bar{\alpha}})}(1,\, \xi,\, \xi^2)$, yielding x_u regular in $C_G(x_s)_0$ and x regular in $G\tau$.
By Proposition 3.1 $C_{G_0^{\tau}}(x_s)_0 = (C_G(x_s)_0 \cap G_0^{\tau})_0$ has the following description:

$$C_{G_0^{\tau}}(x_s)_0 =< T_0^{\tau},\, \{S_{d(\bar{\alpha})}(c,\, c,\, c), \quad c \in k\}, \{S_{-d(\bar{\alpha})}(c,\, c,\, c), \quad c \in k\} >, \tag{A.16}$$

a group of type A_1. We see that $C_G(x_s)_0 \cap G_0^{\tau}$ contains no regular unipotent element of $C_G(x_s)_0$.
If we conjugate $x = x_s x_u$ with a representative $t \in T$ of an element in $(T/T_0^{\tau})^{\tau}$ we get:

$$\begin{aligned} t\, x_u\, t^{-1} \;=\;& S_{d(\bar{\alpha}_1)}(\alpha_1(t),\, \xi\, \alpha_1(t)\, \bar{\alpha}_1(t'),\, \xi^2 \alpha_1(t) \bar{\alpha}_1(t'')) \times \\ & S_{d(\bar{\alpha})}(\alpha(t),\, \alpha(t)\, (\bar{\alpha})(t'),\, \alpha(t)\, (\bar{\alpha})(t'')) \times \\ & S_{d(\tilde{\bar{\alpha}})}(\tilde{\alpha}(t),\, \xi\, \tilde{\alpha}(t)\, (\tilde{\bar{\alpha}})(t'),\, \xi^2\, \tilde{\alpha}(t)\, (\tilde{\bar{\alpha}})(t'')), \end{aligned} \tag{A.17}$$

where α_1, α, $\tilde{\alpha}$ are a roots in R which are mapped to $\bar{\alpha}_1$, $\bar{\alpha}$, $\tilde{\bar{\alpha}} \in {}^{\tau}R^1$ by the projection p in Equation 1.2, where $t' := t^{-1}\, \tau^{-1}(t) \in H$ and where $t'' := t^{-1}\, \tau^{-2}(t) = \tau(\tau^{-1}(t^{-1})\, t) = t'^{-1}$. Again we have $\chi(H) = \chi(T_0^{\tau})/\chi(T')$, by Corollary 2.4, and $\mathbb{Z}(R') \subset \chi(T')$, by Lemma 1.4. Therefore, we conclude, using Equation A.10:

$$\bar{\alpha}|_H = \bar{\alpha}_1|_H \tag{A.18}$$
$$\tilde{\bar{\alpha}}|_H = 2\bar{\alpha}_1|_H. \tag{A.19}$$

To achieve $t\, x_u\, t^{-1} \in G_0^{\tau}$, we need to find $t' \in H$ with $\bar{\alpha}_1(t') = \xi^2$, $\bar{\alpha}(t') = 1$ and $\tilde{\bar{\alpha}}(t') = \xi^2$, by Equation A.17, which is absurd. ♡.

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Abstract

We consider a non-connected algebraic group $\tilde{G}$ with semisimple unit component G having the form $\tilde{G} = G \rtimes \Gamma$, where Γ is a nontrivial subgroup of the automorphisms of the Dynkin diagram $\Delta(G)$ of G. Now G acts on an exterior component $G\tau$, $\tau \neq e$ of $\tilde{G}$ by conjugation. In this work, we investigate the conjugacy classes of this action and the corresponding categorical quotient $G\tau /\!\!/ G$. If T is a maximal torus of G stabilized by τ and if T_0^τ is the unit component of the group of its fixed points, then we will show the isomorphism $G\tau /\!\!/ G \cong T_0^\tau \tau / \widetilde{\mathcal{W}}$, where $\widetilde{\mathcal{W}}$ is a finite extension of the fixed point Weyl group $\mathcal{W}^\tau$.
Furthermore this quotient turns out to be an affine space, if G is simply connected. In this case the quotient map is flat and its fibres are normal. An investigation of the fibres of the quotient map yields a description as an associated bundle: $\pi^{-1}(\pi(t\tau))_{red} \cong G \times^{C_G(t\tau)} V(t\tau)$, where $V(t\tau)$ is the unipotent variety of $C_G(t\tau)$. If G is simply connected, we can also construct a Steinberg cross section, a map $C : T_0^\tau \tau / \widetilde{\mathcal{W}} \to G\tau$, being a section to the quotient map. Our results resemble the results of Steinberg [35] or [38], which deal with the case $\tau = e$.
Finally, a link to deformation theory of simple singularities is established, analogously to the results of Slodowy [27].

Zusammenfassung

Wir betrachten eine nichtzusammenhängende, algebraische Gruppe $\tilde{G}$ mit halbeinfacher Einskomponente G, die ein semidirektes Produkt $\tilde{G} = G \rtimes \Gamma$ ist, wobei Γ eine nichttriviale Untergruppe der Diagrammautomorphismen des Dynkindiagramms $\Delta(G)$ von G ist. Dann operiert G auf einer äußeren Komponente $G\tau$, $\tau \neq e$ von $\tilde{G}$ durch Konjugation. In dieser Arbeit wollen wir die Konjugationsklassen dieser Operation und den zugehörigen kategorischen Quotienten $G\tau /\!\!/ G$ untersuchen. Ist T ein τ-stabiler, maximaler Torus von G und T_0^τ die Einskomponente der entsprechenden Fixpunktgruppe, dann werden wir zeigen, daß es eine Isomorphie $G\tau /\!\!/ G \cong T_0^\tau \tau / \widetilde{\mathcal{W}}$ gibt, wobei $\widetilde{\mathcal{W}}$ eine endliche Erweiterung der Fixpunkt-Weylgruppe $\mathcal{W}^\tau$ ist. Dieser Quotient ist desweiteren ein affiner Raum, falls G einfach zusammenhängend ist. In diesem Fall ist die Quotientenabbildung flach und ihre Fasern sind normal. Wie wir sehen werden, haben die Fasern die Struktur assoziierter Faserbündel: $\pi^{-1}(\pi(t\tau))_{red} \cong G \times^{C_G(t\tau)} V(t\tau)$, wobei $V(t\tau)$ die unipotente Varietät von $C_G(t\tau)$ ist. Im einfachzusammenhängenden Fall können wir sogar einen Steinberg-Querschnitt konstruieren, d.h. einen Morphismus $C : T_0^\tau \tau / \widetilde{\mathcal{W}} \to G\tau$, der ein Schnitt bezüglich der Quotientenabbildung ist. Unsere Ergebnisse sind analog zu denen von Steinberg, siehe [35] oder [38], wo der Fall $\tau = e$ behandelt wird.

Schließlich stellen wir noch eine Verbindung zur Theorie der einfachen Singularitäten her, analog zu den Ergebnissen von Slodowy [27].

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